CONSTANT CURVATURE SURFACES IN A PSEUDO-ISOTROPIC SPACE

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ABSTRACT. In this study, we deal with the local structure of curves and surfaces immersed in a pseudo-isotropic space \mathbb{I}_p^3 that is a particular Cayley-Klein space. We provide the formulas of curvature, torsion and Frenet trihedron in order for spacelike and timelike curves. The causal character of all admissible surfaces in \mathbb{I}_p^3 has to be timelike or lightlike up to its absolute. We introduce the formulas of Gaussian and mean curvature for timelike surfaces in \mathbb{I}_p^3 . As applications, we describe the surfaces of revolution which are the orbits of a plane curve under a hyperbolic rotation with constant Gaussian and mean curvature.

1. Introduction and preliminaries

Let $P(\mathbb{R}^3)$ be the projective 3-space and $(x_0: x_1: x_2: x_3)$ the homogenous coordinates. By a *quadric*, we mean a subset of points of $P(\mathbb{R}^3)$ described as zeros of a quadratic form associated with a non-zero symmetric bilinear form of $P(\mathbb{R}^3)$.

The Cayley-Klein 3-spaces can be defined in $P(\mathbb{R}^3)$ with an absolute figure, namely a sequence of quadrics and subspaces of $P(\mathbb{R}^3)$, see [12, 15, 25, 28]. We are interested in a particular Cayley-Klein space, the pseudo-isotropic 3-space \mathbb{I}_p^3 . Its absolute is composed of the quadruple $\{\omega, f_1, f_2, F\}$, where ω is the plane at infinity, f_1, f_2 two real lines in ω , F the intersection of f_1 and f_2 . In coordinate form, these arguments are given by

$$\omega: x_0 = 0, \ f_1: x_0 = x_1 = 0, \ f_2: x_0 = x_2 = 0, \ F(0:0:0:1).$$

For further details, see [8, 13, 17, 18].

We deal with an affine model of \mathbb{I}_p^3 via the coordinates $\left(x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}\right)$, $x_0 \neq 0$. The group of pseudo-isotropic motions is a six-parameter group given by

(2.1)
$$(x, y, z) \longmapsto (x', y', z') : \begin{cases} x' = a + qx, \\ y' = b + \frac{1}{q}y \ (q \neq 0), \\ z' = c + dx + ey + z, \end{cases}$$

where $a, b, c, d, e, q \in \mathbb{R}$. The pseudo-isotropic metric is introduced by the absolute, i.e. $ds^2 = dx^2 - dy^2$. Note that this metric can be also considered as $ds^2 = dxdy$ by standing x = (x + y)/2, x = (x - y)/2.

The investigation of curves and surfaces in 3-spaces is a classical field of study in differential geometry. In spite of the fact that the cyclides in \mathbb{I}_p^3 , i.e. algebraic

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surfaces of order 4, have been studied for many years; as far as we know, the local structure of curves and surfaces in \mathbb{I}_p^3 have not been established.

Indeed, we found motivation for this paper in B. Divjak's works ([8, 9, 20]), in which the author introduced the differential geometry of curves and surfaces in the pseudo-Galilean space as generalizing that of the Galilean space. Intending a similar approach for the isotropic geometry (for details, see [1, 2, 3, 5, 11, 14, 22, 23, 26, 27]), we are interested in the local theory of curves and surfaces in \mathbb{I}_n^3 .

The fact that the pseudo-isotropic metric is indefinite requires to introduce some basic notions (e.g. the causal character, the pseudo-angle, etc.) in \mathbb{I}_p^3 from the semi-Riemannian geometry (see Section 2). For detailed properties of such a geometry see [6, 16, 24].

In Section 3, it is suprisingly observed that each lightlike curve in \mathbb{I}_p^3 lies in the isotropic plane of the form $x \pm y = c, c \in \mathbb{R}$. As the local structures of the nonlightlike curves, the formulas in \mathbb{I}_p^3 analogous to the famous Frenet's formulas were

We get in Section 4 that each immersed admissible surface in \mathbb{I}_p^3 is timelike or lightlike. The formulas of the Gaussian and the mean curvatures for timelike surfaces are also introduced.

As several applications, in Section 5, we study and classify the surfaces of revolution, imposing some natural curvature conditions.

2. Basics in the sense of pseudo-isotropic geometry

The pseudo-isotropic scalar product between two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{I}_p^3$ can be defined as

$$\langle u, v \rangle = \begin{cases} u_3 v_3, & \text{if } u_1 = u_2 = v_1 = v_2 = 0, \\ u_1 v_1 - u_2 v_2, & \text{otherwise.} \end{cases}$$

A line is said to be *isotropic* (resp. *non-isotropic*) if its point at infinity is (resp. no) the absolute point F. Moreover, a plane is said to be *isotropic* (resp. nonisotropic) if its line at infinity containes (resp. does not) the absolute point F. In the affine model of \mathbb{I}_p^3 , the isotropic lines and planes are parallel to the z-axis. In the non-isotropic planes, the Lorentzian metric is basically used.

Let us consider the projection onto xy-plane given by

$$u = (u_1, u_2, u_3) \longmapsto \tilde{u} = (u_1, u_2, 0),$$

usually called top view. A nonzero vector u is said to be isotropic (resp. nonisotropic) if $\tilde{u} = 0$ (resp. $\tilde{u} \neq 0$). The zero vector is assumed to be non-isotropic. A non-isotropic vector $u \in \mathbb{I}_p^3$ is respectively called *spacelike*, *timelike* and *lightlike* (or null) if $\langle u, u \rangle > 0$ or u = 0, $\langle u, u \rangle < 0$ and $\langle u, u \rangle = 0$ ($u \neq 0$). The set of all lightlike vectors of \mathbb{I}_p^3 is called $lightlike\ cone$, i.e.,

$$\Lambda = \left\{ \left. (u_1, u_2, u_3) \in \mathbb{I}_p^3 \right| u_1^2 - u_2^2 = 0 \right\} - \left\{ 0 \in \mathbb{I}_p^3 \right\}.$$

Denote \mathcal{T} the set of all timelike vectors in \mathbb{I}_n^3 . For some $u \in \mathcal{T}$, the set given by

$$C(u) = \{ v \in T : \langle u, v \rangle < 0 \}$$

is called the *timelike cone* of \mathbb{I}_p^3 containing u.

The *pseudo-isotropic angle* of two timelike non-isotropic vectors $u, v \in \mathbb{I}_p^3$ lying in the same timelike-cone is defined as the Lorentzian angle between \tilde{u} and \tilde{v} , i.e.

$$\langle u, v \rangle = -\sqrt{-\langle u, u \rangle} \sqrt{-\langle v, v \rangle} \cosh \phi.$$

Note that all isotropic vectors are isotropically orthonogal to non-isotropic ones. Further, two non-isotropic vectors u, v in \mathbb{I}_p^3 are orthonogal if $\langle u, v \rangle = 0$.

3. Spacelike and timelike curves in \mathbb{I}_p^3

Let $\alpha(s) = (x(s), y(s), z(s))$ be a regular curve in \mathbb{I}_p^3 , i.e. $\alpha'(s) = \frac{d\alpha}{ds} \neq 0$ for all s. Then it is said to be *admissible* if $\alpha(s)$ has no isotropic osculating plane. An admissible curve $\alpha(s)$ in \mathbb{I}_p^3 is said to be *spacelike* (resp. *timelike*, *lightlike*) if $\alpha'(s)$ is spacelike (resp. timelike, lightlike) for all s. An easy compute shows that all lightlike curves lie in the isotropic plane of the form $x \pm y = c$, $c \in \mathbb{R}$.

Henceforth, we consider only spacelike and timelike admissible curves.

Now let $\alpha = \alpha\left(s\right)$ be a spacelike curve in \mathbb{I}_{p}^{3} parameterized by arc-length. Then we have

(3.1)
$$\langle \alpha', \alpha' \rangle = (x')^2 - (y')^2 = 1$$

and taking derivative of (3.1) gives

$$(3.2) x'x'' - y'y'' = 0.$$

Denote $T = \alpha'$ and call it tangent vector field. Since $T' = \alpha''$ is timelike in \mathbb{I}_p^3 we can define the following

$$\kappa = \sqrt{(y'')^2 - (x'')^2},$$

called *curvature* of α . Using (3.2), we get

(3.3)
$$\kappa = \frac{y''}{x'} \text{ or } \kappa = \frac{x''}{y'}.$$

Considering (3.1) and (3.2) into (3.3) we find

(3.4)
$$\kappa = \det\left(\tilde{\alpha}', \tilde{\alpha}''\right).$$

Define the normal vector field and torsion of α respectively as

(3.5)
$$N = \frac{1}{\kappa} T' \text{ and } \tau = \frac{\det\left(\alpha', \alpha'', \alpha'''\right)}{\kappa^2}, \ \kappa \neq 0.$$

Since B = (0, 0, 1) is isotropically orthogonal to T and N, we can take it as the binormal vector field of α .

From (3.5) we have

(3.6)
$$N' = \left(\frac{1}{\kappa}\right)'(x'', y'', z'') + \frac{1}{\kappa}(x''', y''', z''').$$

Put $N' = (n_1, n_2, n_3)$. Hence we write

$$(3.7) n_1 = \left(\frac{1}{\kappa}\right)' x'' + \frac{1}{\kappa} x'''.$$

Using (3.4) into (3.7) yields

(3.8)
$$n_1 = -\frac{x'}{\kappa^2} (x''y''' - x'''y'').$$

By taking derivative of (3.2) and considering into (3.8) we obtain

$$(3.9) n_1 = -\kappa x'.$$

Similar computations gives

$$(3.10) n_2 = -\kappa y'.$$

For the third component of N', we have

$$n_3 = \left(\frac{1}{\kappa}\right)' z'' + \frac{1}{\kappa} z'''.$$

It follows from (3.4) that

(3.11)
$$n_3 = \frac{1}{\kappa^2} \left\{ -\left(x'y''' - x'''y'\right) z'' + \left(x'y'' - x''y'\right) z''' \right\}.$$

By adding and substracting (x''y''' - y''x''')z' in (3.11) we conclude

(3.12)
$$n_3 = \frac{1}{\kappa^2} \left\{ \det \left(\alpha', \alpha'', \alpha''' \right) - \left(x'' y''' - x''' y'' \right) z' \right\}.$$

Taking derivative of (3.2) and considering into (3.12) implies

$$(3.13) n_3 = \tau - \kappa z'.$$

(3.9), (3.10) and (3.13) yield that $N' = -\kappa T + \tau B$. Thus we obtain the formulas analogous to these of Frenet as follows

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = 0. \end{cases}$$

By similar arguments, we can find the derivative formulas of the vector fields T, N, B for a timelike curve in \mathbb{I}^3_p as

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = 0, \end{cases}$$

where
$$\kappa = \sqrt{(x'')^2 - (y'')^2}$$
 and $\tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$.

Example 3.1. Consider a hyperbolic cylindrical curve in \mathbb{I}_p^3 given by (see [7])

(3.14)
$$\alpha(s) = (\cosh s, \sinh s, z(s)).$$

This is a timelike curve of arc-length in \mathbb{I}_p^3 with $\kappa(s) = 1$ and

(3.15)
$$\tau(s) = z'(s) - z'''(s).$$

If α has constant torsion τ_0 , then by solving (3.15) we find

$$z(s) = \tau_0 s + c_1 e^s - c_2 e^{-s} + c_3, \ c_1, c_2, c_3 \in \mathbb{R},$$

which gives the elemantary result:

Proposition 3.1. Let α be a hyperbolic cylindrical curve in \mathbb{I}_p^3 with constant torsion τ_0 . Then it is of the form

$$\alpha(s) = (\cosh s, \sinh s, \tau_0 s + c_1 e^s - c_2 e^{-s} + c_3),$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

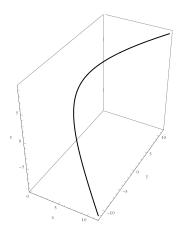


FIGURE 1. A hyperbolic cylindrical curve with $\tau_0 = 3$, $s \in (-\pi, \pi)$.

4. Timelike surfaces in \mathbb{I}_p^3

Let M be a surface immersed in \mathbb{I}_p^3 without isotropic tangent planes. Then we call such a surface admissible. Let T_xM be a non-isotropic tangent plane at a point $x \in M$. An admissible surface M is said to be timelike (resp. lightlike) if the induced metric g in T_xM for each $x \in M$ from \mathbb{I}_p^3 is non-degenerate of index 1 (resp. degenerate).

Henceforth, we will not consider lightlike surfaces.

Assume that M has a local parameterization in \mathbb{I}_n^3 as follows:

$$r: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}_p^3: (u_1, u_2) \longmapsto (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$$

for smooth real-valued functions x, y, z on a domain $D \subseteq \mathbb{R}^2$. Denote (g_{ij}) the matrical expression of g with respect to the basis $\{r_{u_1}, r_{u_2}\}$. Then we have

$$g_{ij} = \langle r_{u_i}, r_{u_j} \rangle, \ r_{u_i} = \frac{\partial r}{\partial u_i}, \ i, j = 1, 2.$$

It is easy to see that

$$\det(g_{ij}) = -(x_{u_1}y_{u_2} - x_{u_2}y_{u_1})^2.$$

The unit normal vector field of M is the isotropic vector $\xi = (0, 0, 1)$ since it is isotropically orthogonal to the tangent plane of M.

For the second fundamental form of M, we follow the similar way with Sachs (see [20], p. 155). Let r(s) be an arc-length curve on M and T its tangent vector. We can take a side tangential vector σ in T_xM such that $\{T,\sigma\}$ is a positive oriented base. Therefore we have a decomposition:

$$r'' = \frac{d^2r}{ds^2} = \kappa N = \kappa_g \sigma + \kappa_n \xi,$$

where N, κ_g and κ_n are the normal vector, geodesic and normal curvatures of r on M, respectively. Put $\sigma = a_1 r_{u_1} + a_2 r_{u_2}$. Due to $T = r_{u_1} \frac{du_1}{ds} + u_2 \frac{du_2}{ds}$ and $\langle T, \sigma \rangle = 0$, we get

$$a_1 = \theta \left(g_{12} \frac{du_1}{ds} + g_{22} \frac{du_2}{ds} \right), \ a_2 = -\theta \left(g_{11} \frac{du_1}{ds} + g_{12} \frac{du_2}{ds} \right),$$

where $\theta = \theta(u_1, u_2)$ is some nonzero smooth function. Then we achieve

$$1 = \det\left(\tilde{T}, \tilde{\sigma}\right) = -\sqrt{|\det\left(g_{ij}\right)|}\theta$$

and hence

$$\sigma = -\frac{1}{\sqrt{|\det(g_{ij})|}} \left[\left(g_{12} \frac{du_1}{ds} + g_{22} \frac{du_2}{ds} \right) r_{u_1} - \left(g_{11} \frac{du_1}{ds} + g_{12} \frac{du_2}{ds} \right) r_{u_2} \right].$$

Accordingly, we compute that

$$\kappa_n = \det\left(r', \sigma, r''\right) = \frac{1}{\sqrt{\left|\det\left(g_{ij}\right)\right|}} \det\left(r_{u_1}, r_{u_2}, r''\right)$$
$$= \frac{1}{\sqrt{\left|\det\left(g_{ij}\right)\right|}} \sum_{i,j=1}^{2} \det\left(r_{u_1}, r_{u_2}, r_{u_i u_j}\right) \left(\frac{du_i}{ds}\right) \left(\frac{du_j}{ds}\right),$$

which leads to the components of the second fundamental form given by

$$h_{ij} = \frac{\det(r_{u_1}, r_{u_2}, r_{u_i u_j})}{\sqrt{|\det(g_{ij})|}}, \ r_{u_i u_j} = \frac{\partial^2 r}{\partial u_i \partial u_j}, \ i, j = 1, 2.$$

Thus the Gaussian curvature and the mean curvature of M are respectively defined by

(4.1)
$$K = \frac{\det(h_{ij})}{\det(g_{ij})}$$

and

(4.2)
$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\det(g_{ij})}.$$

By permutation of the coordinates, two different types of graph surfaces appear up to the absolute of \mathbb{I}_p^3 . For a graph of the function $u=u\left(x,y\right)$, the formulas (4.1) and (4.2) reduce to

$$K = -u_{xx}u_{yy} + (u_{xy})^2, \ H = \frac{1}{2}(u_{xx} - u_{yy}).$$

Since the metric on the graph surface induced from \mathbb{I}_p^3 is $g = dx^2 - dy^2$, it always becomes a flat surface. So, its Laplacian turns to

$$\triangle = \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2}.$$

On the other side, the Gaussian and mean curvatures of the graph of $u=u\left(y,z\right)$ are given by

$$K = -\frac{u_{yy}u_{zz} - (u_{yz})^2}{(u_z)^4}, \ H = \frac{(u_z)^2 u_{yy} - 2u_y u_z u_{yz} + ((u_y)^2 - 1) u_{zz}}{2 (u_z)^3}.$$

5. Constant curvature surfaces of revolution in \mathbb{I}_p^3

Da Silva [8] provided via hyperbolic numbers that the pseudo isotropic motion given by $\bar{x}=px, \ \bar{y}=\frac{1}{p}y, \ p\neq 0$ is equivalent to the hyperpolic rotation (about z-axis) given by

(5.1)
$$\bar{x} = x \cosh \theta + y \sinh \theta, \ \bar{y} = x \sinh \theta + y \cosh \theta,$$
 where $\theta \in \mathbb{R}$.

Let $u \mapsto (u, 0, f(u))$ be a spacelike admissible curve lying in the isotropic xz-plane of \mathbb{I}_p^3 for a smooth function f. Rotating it around z-axis via hyperbolic rotations given by (5.1) we derive

$$(5.2) r(u,v) = (u \cosh v, u \sinh v, f(u)).$$

We call the rotating curve *profile curve*. If the profile curve is a timelike curve $u \longmapsto (0, u, f(u))$ lying in the isotropic yz-plane of \mathbb{I}_p^3 , then rotating it around \mathbf{z} -axis yields

$$(5.3) r(u,v) = (u \sinh v, u \cosh v, f(u)).$$

The surfaces given by (5.2) and (5.3) are called *surfaces of revolution* in \mathbb{I}_p^3 . The Gaussian curvature of these surfaces in \mathbb{I}_p^3 is

$$(5.4) K = \frac{f'f''}{u},$$

where
$$f'(u) = \frac{df}{du}$$
, etc.

Now we assume that it has nonzero constant Gaussian curvature K_0 in \mathbb{I}_p^3 . Then (5.4) can be rewritten as

(5.5)
$$f' = \sqrt{c_1 + K_0 u^2}, c_1 \in \mathbb{R}.$$

After integrating (5.5), we obtain

$$f(u) = \frac{u}{2}\sqrt{c_1 + K_0 u^2} + \frac{c_1}{2\sqrt{K_0}} \ln\left(2K_0 u + 2\sqrt{K_0}\sqrt{c_1 + K_0 u^2}\right) + c_2, \ c_2 \in \mathbb{R}$$

which implies the following result.

Theorem 5.1. Let M be a surface of revolution in \mathbb{I}_p^3 with nonzero constant Gaussian curvature K_0 . Then its profile curve is of the form (u, 0, f(u)), where

$$f\left(u\right) = \frac{u}{2}\psi\left(u\right) + \frac{c_1}{2\sqrt{K_0}}\ln\left|2\sqrt{K_0}\left(\sqrt{K_0}x + \psi\left(u\right)\right)\right|$$

for
$$\psi(u) = \sqrt{c_1 + K_0 u^2}, c_1, c_2 \in \mathbb{R}$$
.

We immediately have the following from (5.4).

Corollary 5.1. A surface of revolution is flat in \mathbb{I}_p^3 if and only if its profile curve is a non-isotropic line given by $(u, 0, c_1u + c_2)$, $c_1, c_2 \in \mathbb{R}$.

The mean curvature H of a surface of revolution M in \mathbb{I}_n^3 is

$$(5.6) H = \frac{1}{2} \left(\frac{f'}{u} + f'' \right).$$

Assume that M has constant mean curvature H_0 . After solving (5.6) we deduce

$$f(u) = \frac{H_0}{2}u^2 + c_1 \ln u + c_2, \ c_1, c_2 \in \mathbb{R}.$$

Therefore we have proved the following results.

Theorem 5.2. Let M be a surface of revolution in \mathbb{I}_p^3 with constant mean curvature H_0 . Then its profile curve is of the form (u,0,f(u)), where

$$f(u) = \frac{H_0}{2}u^2 + c_1 \ln u + c_2, \ c_1, c_2 \in \mathbb{R}.$$

Corollary 5.2. A surface of revolution is minimal in \mathbb{I}_p^3 if and only if its profile curve is a non-isotropic curve given by $(u,0,c_1 \ln u + c_2)$, $c_1,c_2 \in \mathbb{R}$.

Example 5.1. Take the surfaces of revolution in \mathbb{I}_p^3 parameterized

$$r(u, v) = (u \cosh v, u \sinh v, u), (u, v) \in [1, 2] \times [0, 1]$$

and

$$(u, v) = (u \cosh v, u \sinh v, \ln u + u^2), (u, v) \in [1, 2] \times [-1, 1].$$

The above first surface is flat and the second is a constant mean curvature surface of revolution, H = 2. We plot these as in Figure 2 and Figure 3, respectively.

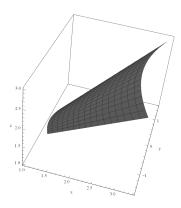


Figure 2. A flat surface of revolution, K = 0.

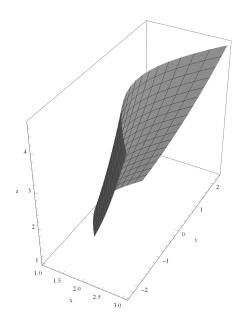


Figure 3. A constant curvature surface of revolution, H=2.

6. Surfaces of revolution with $H^2 = K$ in \mathbb{I}_n^3

Next we aim to classify the surfaces of revolution given by (5.2) in \mathbb{I}_p^3 that satisfy $H^2 = K$ which is the equality sign of the Euler inequality. For more generalizations of the famous inequality, see [4, 19, 20].

By considering the equalities (5.4) and (5.6), we have

(6.1)
$$\frac{1}{4} \left(\left(\frac{f'}{u} \right)^2 + 2 \frac{f'f''}{u} + (f'')^2 \right) = \frac{f'f''}{u}.$$

We can rewirte (6.1) as

$$\left(\frac{f'}{u} - f''\right)^2 = 0,$$

which implies

$$\frac{f'}{u} - f'' = 0.$$

After solving this, we obtain

$$f(u) = c_1 \frac{u^2}{2} + c_2$$

for $c_1, c_2 \in \mathbb{R}$. By comparing (5.2) with (6.2) we see that the surface of revolution can be given in explicit form

(6.3)
$$z = \frac{c_1}{2} (x^2 - y^2) + c_2,$$

which implies the following result.

Theorem 6.1. The surfaces of revolution given by (5.2) in \mathbb{I}_p^3 with $H^2 = K$ are only the spheres of parabolic type.

Example 6.1. Consider the sphere of parabolic type in \mathbb{I}_p^3 given via (6.3) such that $c_1 = 2$ and $c_2 = 0$. Then its curvatures become H = 2 and K = 4. We plot it as in Figure 4.

7. Acknowlodgements

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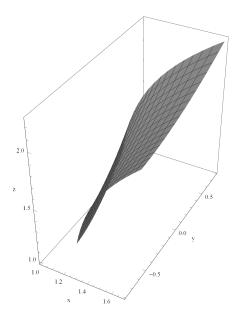


FIGURE 4. A surface of revolution with $H^2 = K$.

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