# GLOBAL REGULARITY OF THREE DIMENSIONAL DENSITY PATCHES FOR INHOMOGENEOUS INCOMPRESSIBLE VISCOUS FLOW

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ABSTRACT. Toward P.-L. Lions' open question in [20] concerning the propagation of regularity for density patch, we prove that the boundary regularity of the 3-D density patch persists by time evolution for inhomogeneous incompressible viscous flow, with the initial density given by  $(1-\eta)\mathbf{1}_{\Omega_0} + \mathbf{1}_{\Omega_0^c}$  for some small enough constant  $\eta$  and some  $W^{k+2,p}$  domain  $\Omega_0$ ,  $p \in ]3, \infty[$ , and with the initial velocity satisfying some smallness condition and appropriate conormal regularities.

Keywords: Inhomogeneous incompressible Navier-Stokes equations; density patch; striated distribution spaces

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# 1. Introduction

We consider the following inhomogeneous incompressible 3-D Navier-Stokes equations:

(1.1) 
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \rho(\partial_t v + v \cdot \nabla v) - \Delta v + \nabla \pi = 0, \\ \operatorname{div} v = 0, \\ (\rho, v)|_{t=0} = (\rho_0, v_0), \end{cases}$$

where  $\rho \in \mathbb{R}^+$ ,  $v \in \mathbb{R}^3$  and  $\pi \in \mathbb{R}$  stand for the density, velocity field and pressure of the fluid respectively. System (1.1) describes an incompressible fluid with variable density. Basic examples are mixture of incompressible and non reactant flows, models of rivers, fluids containing a melted substance, etc.

The existence and uniqueness issues of the solutions of (1.1) have been studied extensively. We just cite here among many others [3, 20, 24] for the construction of the weak solutions of (1.1), and the works [2, 3, 7, 9, 17] for the wellposedness results for the strong solutions. Recently progresses have been made: the smallness conditions on the density fluctuation was successfully removed in e.g. [1, 10], a small jump of the density function across some interface was permitted in e.g. [11, 15], and in the energy framework, [22] considered the positive density which is only assumed to be bounded from up and below.

In [20] P.-L. Lions proposed the following density patch problem: if the initial density  $\rho_0 = \mathbf{1}_D$  for some smooth domain D, then whether or not the boundary regularity of D will persist by time evolution? There are many recent progresses toward this problem: Danchin-Mucha [12] propagated the  $C^{1,\alpha}$  boundary regularity in the presence of vacuum. See [14, 13] for the propagation results of the lower-order Hölder boundary regularity. In [18, 19], the first author and P. Zhang considered this density patch problem with the boundary of any high-order regularity, in space dimension two, away from vacuum. More precisely in [18], the initial density  $\rho_0$  is taken of the following form

(1.2) 
$$\rho_0 = (1 - \eta) \mathbf{1}_{\Omega_0} + \mathbf{1}_{\Omega_0^c}, \quad 1 - \eta \in \mathbb{R}^+,$$

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where  $|\eta|$  is a sufficiently small constant, and  $\Omega_0$  is some simply connected  $W^{k+2,p}(\mathbb{R}^2)$ ,  $k \geq 1$ ,  $p \in ]2, 4[$  bounded domain in  $\mathbb{R}^2$ . Let  $X_0 \in W^{k+1,p}(\mathbb{R}^2)$  be the divergence-free tangential vector field of  $\partial \Omega_0^{-1}$  and denote by  $\partial_{X_0} f \stackrel{\text{def}}{=} X_0 \cdot \nabla f = \text{div}(fX_0)$  the derivative of f in the direction of  $X_0$ . Then they proved the following persistence result of the boundary regularity:

**Theorem 1.1.** Given initial density  $\rho_0$  of (1.2) with  $|\eta|$  sufficiently small, and initial velocity  $v_0$  of (1.1) satisfies the following conormal regularities for some  $\epsilon \in ]0,1[$ :

$$(1.3) v_0 \in W^{1,p}(\mathbb{R}^2) \quad \text{with} \quad \partial_{X_0}^{\ell} v_0 \in W^{1-\frac{\ell}{k}\epsilon,p}(\mathbb{R}^2), \quad \ell = 1, \cdots, k.$$

Then the Cauchy problem (1.1)-(1.2)-(1.3) has a unique global solution  $(\rho, v)$  such that

$$\rho(t,x) = (1-\eta)\mathbf{1}_{\Omega(t)} + \mathbf{1}_{\Omega(t)^c}, \quad \text{with} \quad \Omega(t) = \psi(t,\Omega_0),$$

where  $\psi$  is the flow of v, and  $\Omega(t)$  remains a simply connected bounded  $W^{k+2,p}(\mathbb{R}^2)$  domain.

The smallness condition on the jump  $\eta$  was successfully removed in [19], by using time weighted energy estimates. There, the initial velocity  $v_0$ , together with its conormal derivatives  $\partial_{X_0}^{\ell} v_0$ , is taken in the energy spaces. Precisely, there exists some  $s_0 \in ]0,1[$  and some  $\epsilon_1 \in ]0, s_0[$ , such that

$$v_0 \in L^2 \cap \dot{B}_{2,1}^{s_0}, \quad \partial_{X_0}^{\ell} v_0 \in L^2 \cap \dot{B}_{2,1}^{s_{\ell}} \quad \text{for} \quad s_{\ell} \stackrel{\text{def}}{=} s_0 - \epsilon_1 \ell/k, \ \ell = 1, \cdots, k.$$

The purpose of this paper is to extend Theorem 1.1 to three dimensional case, for the density patch problem with small jump (1.2) but not in the finite-energy framework.

One of the main tools we use in this paper is the so-called conormal (or striated) distribution spaces, which have been successfully used by J.-Y. Chemin [5, 6] for studying the evolution of the boundary regularity of the 2-D vortex patch problem for Euler equations (see also [4]). Then the subsequent works [8] considered the viscous case and the works [16, 23] extended the results to the 3-D case.

There are two obvious difficulties when considering 3-D case. One comes from the fact that, unlike 2-D case, the global well-posedness of 3-D Navier-Stokes equations remains still unknown to us. In fact, it is one of the most challenging open problems in fluid mechanics. So here we choose to assume the following smallness condition on the initial velocity

$$(1.4) ||v_0||_{\dot{B}_{p_1,r}^{-1+\frac{3}{p_1}}(\mathbb{R}^3)} \le c_0, ext{ for some } 1 < p_1 < 3 ext{ and } 1 < r < \min\{\frac{2p_1}{3(p_1-1)}, \frac{4p}{2p-3}, 3\},$$

where  $c_0$  is some sufficiently small constant, and  $\dot{B}_{p_1,r}^{-1+\frac{3}{p_1}}$  is Besov space, see Definition 1.2 below. Under this smallness condition on the initial data, we have the following proposition to guarantee the existence of global weak solutions to (1.1).

**Proposition 1.1** (a particular case of Theorem 1.1 in [15]). Let the initial data  $(\rho_0, v_0)$  satisfy (1.2) and (1.4). If  $|\eta|$ ,  $c_0$  are sufficiently small, then (1.1) has a global weak solution  $(\rho, v)$ , and there exists a positive constant C such that

The other difficulty is that, the boundary of a two dimensional domain is one dimensional curve, whose tangent space can be spanned by the tangent vector (e.g. by  $X_0$  given above). But the boundary of a three dimensional domain is a two dimensional surface, whose tangent

<sup>&</sup>lt;sup>1</sup>Let  $g_0 \in W^{k+2,p}(\mathbb{R}^2)$  such that  $\partial \Omega_0 = g_0^{-1}(0)$  and  $\nabla g_0$  does not vanish on  $\partial \Omega_0$ . Then we can choose  $X_0 = \nabla^{\perp} g_0$ .

space has dimension two, hence in order to propagate the boundary regularity, we need to discuss differentiations in different directions, which makes the problem more complicated.

To deal with this difficulty, we here follow [16] to select "good" tangent vector directions to work with. We first adopt the following definition of admissible systems introduced in [16]:

**Definition 1.1** (Definition 3.1 of [16]). Any system  $W = \{W_1, \dots, W_N\}$  composed of N continuous vector fields is said to be admissible if the function

$$[\mathcal{W}]^{-1} \stackrel{\text{def}}{=} \left(\frac{2}{N(N-1)} \sum_{\mu < \nu} |W_{\mu} \wedge W_{\nu}|^2\right)^{-\frac{1}{4}}$$

is bounded. Here, for any two vector fields  $X=(X^1,X^2,X^3)^T$ ,  $Y=(Y^1,Y^2,Y^3)^T$ , the wedge product  $X\wedge Y$  is defined as  $X\wedge Y=\left(X^2Y^3-X^3Y^2,X^3Y^1-X^1Y^3,X^1Y^2-X^2Y^1\right)^T$ .

Now we can select five *divergence-free* tangential vector fields for the two dimensional boundary according to the following result, which is Proposition 3.2 of [16]:  $^2$ 

**Proposition 1.2.** For any  $W^{k+2,p}(\mathbb{R}^3)$  two dimensional compact submanifold  $\Sigma$  of  $\mathbb{R}^3$ , we can find an admissible system consisting of five  $W^{k+1,p}(\mathbb{R}^3)$ , divergence-free vector fields tangent to  $\Sigma$ .

Before we go to the statement of density patch problem in three space dimension, let us introduce the following notations which will be used freely in the following context. For any system  $\mathcal{W} = \{W_1, \cdots, W_N\}$  composed of N continuous vector fields, and for any multi-index  $\alpha = (\alpha_1, \cdots, \alpha_m)$  of length m with  $\alpha_1, \cdots, \alpha_m \in \{1, \cdots, N\}$ , we denote

$$\partial_{\mathcal{W}}^{\alpha} = \partial_{\mathcal{W}}^{(\alpha_1, \dots, \alpha_m)} = \partial_{W_{\alpha_1}} \dots \partial_{W_{\alpha_m}}.$$

We emphasize that the order of differentiation is important. We furthermore denote

$$\partial_{\mathcal{W}}^m = (\partial_{\mathcal{W}}^\alpha)_\alpha$$

to be an  $N^m$  dimensional vector, where  $\alpha$  takes over all the multi-index with length m and with elements taking integer values between 1 and N.

For a Banach space B, we shall use the shorthand  $||u||_{L^p_T B}$  for the norm  $||||u(t,\cdot)||_B||_{L^p(0,T)}$ , and we will simply write  $v \in B$  for the vector-valued field  $v \in B^3$ . The notation C stands for some real positive constant which may be different in each occurrence.

Now we can state the density patch problem in  $\mathbb{R}^3$ . Let us take the initial density  $\rho_0$  of the form (1.2) where  $\Omega_0$  is a simply connected bounded domain of  $\mathbb{R}^3$ , such that its boundary  $\partial\Omega_0$  is  $W^{k+2,p}(\mathbb{R}^3)$ ,  $k \geq 1$ ,  $p \in ]3, \infty[$ , two dimensional compact submanifold in  $\mathbb{R}^3$ . By Proposition 1.2, we can find an admissible system as follows

(1.6) 
$$\mathcal{X}_0 = \{X_{1,0}, X_{2,0}, X_{3,0}, X_{4,0}, X_{5,0}\}, \text{ with } X_{i,0} \text{ tangent to } \partial\Omega_0, \text{ and } X_{i,0} \in W^{k+1,p}(\mathbb{R}^3), k \ge 1, p \in ]3, \infty[, \text{ div } X_{i,0} = 0, i = 1, \cdots, 5.$$

We assume the following (conormal) regularities on  $v_0$  for some  $\varepsilon \in ]0,1[$ 

(1.7) 
$$v_0 \in W^{1,p}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3), \quad \partial_{\chi_0}^{\ell} v_0 \in W^{1-\frac{\ell}{k}\varepsilon,p}(\mathbb{R}^3), \quad \ell = 1, \dots, k.$$

In the following, we consider the propagation of the boundary regularity for the 3-D density patch problem. The main result of this paper states as follows:

<sup>&</sup>lt;sup>2</sup>Proposition 3.2 in [16] concerns the framework of Hölder spaces, however the proof also works in the framework of Sobolev spaces.

**Theorem 1.2.** Let the initial data  $(\rho_0, v_0)$  satisfy (1.2)-(1.4)-(1.6)-(1.7), with  $|\eta|, c_0$  sufficiently small. Then (1.1) has a unique global solution  $(\rho, v)$  such that

$$\rho(t,x) = (1-\eta)\mathbf{1}_{\Omega(t)} + \mathbf{1}_{\Omega(t)^c}, \text{ with } \Omega(t) = \psi(t,\Omega_0),$$

where  $\psi$  is the flow of v, and  $\Omega(t)$  remains a simply connected bounded  $W^{k+2,p}(\mathbb{R}^3)$  domain.

**Remark 1.1.** Results in the same spirit as Theorem 1.1 and Theorem 1.2, have been obtained recently by R. Danchin and X. Zhang, see [13]. Their strategy is to assume that  $\rho$  belongs to some multiplier space, which allows them to propagate lower-order Hölder regularity.

Recall in [18] that in the two dimensional case, if the velocity of the flow  $v \in L^1_{loc}(\mathbb{R}^+, W^{2,p})$ , then the tangential vector field X is transported by the flow from  $X_0$  as follows

$$\partial_t X + v \cdot \nabla X = X \cdot \nabla v, \quad X|_{t=0} = X_0,$$

and to prove the persistence of  $W^{k+2,p}(\mathbb{R}^2)$  regularity of the domain  $\Omega_0$  is equivalent to show that the tangential vector field X satisfies the following (conormal) regularities:

$$X, \partial_X X, \cdots, \partial_X^{k-1} X \in W^{2,p}(\mathbb{R}^2), \text{ where div } X = 0 \text{ and } \partial_X f \stackrel{\text{def}}{=} X \cdot \nabla f = \text{div } (fX).$$

Here we consider the system  $\mathcal{X}(t) = \{X_1(t), \dots, X_5(t)\}$  satisfying for any  $1 \le i \le 5$ ,

(1.8) 
$$\begin{cases} \partial_t X_i + v \cdot \nabla X_i = X_i \cdot \nabla v, \\ X_i(0, x) = X_{i,0}(x), \end{cases}$$

where the initial system  $\mathcal{X}_0 = \{X_{0,1}, \dots, X_{0,5}\}$  is given by (1.6), and we aim to control  $\|(\partial_{\mathcal{X}}^{\ell}X_i)(t,\cdot)\|_{W^{2,p}}$ . The above choice of  $\mathcal{X}$  ensures that the directional derivatives  $\partial_{X_i}$  and the material derivative  $(\partial_t + v \cdot \nabla)$  commute with each other, and hence by view of (1.1),  $\partial_{X_i}^{\ell}\rho$  satisfy the free transport equations

$$(\partial_t + v \cdot \nabla)(\partial_{X_i}^{\ell} \rho) = 0, \quad \forall \ell = 1, \dots, k, \quad \forall i = 1, \dots, 5,$$

with the initial condition  $\partial_{X_{i,0}}^{\ell} \rho_0 \equiv 0$ , which is guaranteed by the choice of  $\rho_0$  and  $\mathcal{X}_0$ , see (1.2) and (1.6). Therefore we can derive that

(1.9) 
$$\partial_{\mathcal{X}}^{\ell} \rho \equiv 0, \quad \forall \ell = 1, \cdots, k,$$

which means that the tangential regularity for  $\rho$  propagates for all the times. On the other side,  $X_i(t)$  should remain divergence-free all the time since div  $X_i$  satisfies the free transport equation by (1.8):

$$\begin{cases} \partial_t (\operatorname{div} X_i) + v \cdot \nabla (\operatorname{div} X_i) = 0, \\ \operatorname{div} X_i|_{t=0} = \operatorname{div} X_{i,0} = 0. \end{cases}$$

Moreover, we have the following proposition:

**Proposition 1.3.** Under the assumptions of Theorem 1.2, the system (1.1) and (1.8) has a unique global solution  $(v, \rho, \mathcal{X})$  satisfying for any  $i \in \{1, \dots, 5\}, \ \ell \in \{1, \dots, k\}$  that

$$(1.10) v \in L^1_{\text{loc}}(\mathbb{R}^+, W^{2,p}(\mathbb{R}^3)), \quad \partial^{\ell-1}_{\mathcal{X}} X_i \in L^{\infty}_{\text{loc}}(\mathbb{R}^+; W^{2,p}(\mathbb{R}^3)), \quad \text{div } X_i = 0.$$

The proof of this proposition will be postponed to the next section. In the following, we shall present the proof of Theorem 1.2 by assuming Proposition 1.3.

Proof of Theorem 1.2. For the unique global solution  $(v, \rho)$  of (1.1) given by Proposition 1.3, let  $\psi(t, x)$  be the flow associated with the velocity field v, that is,

(1.11) 
$$\begin{cases} \partial_t \psi(t, x) = v(t, \psi(t, x)), \\ \psi(0, x)) = x, \end{cases}$$

then by view of (1.10), we have

$$\psi(t,\cdot) - Id \in L^{\infty}_{loc}(\mathbb{R}^+; W^{2,p}),$$

and hence  $\Omega(t) \stackrel{\text{def}}{=} \psi(t, \Omega_0)$  is a  $W^{2,p}(\mathbb{R}^3)$  domain. Moreover, the first equation of (1.1) gives

(1.12) 
$$\rho(t,x) = (1-\eta)\mathbf{1}_{\Omega(t)} + \mathbf{1}_{\Omega(t)^c}, \quad 1-\eta \in \mathbb{R}^+.$$

On the other side, due to the Finite Covering Theorem, there exists a finite number of charts  $\{\mathbf{V}^{\beta}\}_{\beta=1}^{m}$  covering the two dimensional  $W^{k+2,p}(\mathbb{R}^{3})$  compact submanifold  $\partial\Omega_{0}$  such that we can parametrize anyone of them, say  $\mathbf{V}^{1}$ , as follows

$$\phi_1: \mathbf{U}^1 \to \mathbf{V}^1$$
, via  $(r,s) \mapsto \phi_1(r,s)$ ,  $\phi_1 \in W^{k+2,p}(\mathbf{U}^1)$ ,

where  $\mathbf{U}^1$  is an open set on  $\mathbb{R}^2$  and  $\mathbf{V}^1$  is  $\partial\Omega_0$ -open set in  $\mathbb{R}^3$ . In order to show that  $\partial\Omega(t) = \psi(t,\partial\Omega_0) \in W^{k+2,p}, k \geq 1$ , it suffices to show (without loss of generality)

$$\partial_r^{k_1} \partial_s^{k_2} \psi(t, \phi_1(r, s)) \in L_{\text{loc}}^{\infty}(W^{2, p}(\mathbf{U}^1)), \ \forall k_1 + k_2 = k.$$

Hence we only need to verify that

(1.13) 
$$(\partial_{Y_0}^{k_1} \partial_{Z_0}^{k_2} \psi)(t, \cdot) \in L^{\infty}_{loc}(W^{2,p}(\mathbf{V}^1)), \ \forall k_1 + k_2 = k,$$

where the tangent vector fields  $Y_0, Z_0 \in W^{k+1,p}(\mathbf{V}^1; \mathbb{R}^3)$  are defined by

$$Y_0(\phi_1(r,s)) = \partial_r \phi_1(r,s), \quad Z_0(\phi_1(r,s)) = \partial_s \phi_1(r,s).$$

Indeed, a direct calculation gives for any  $(r, s) \in \mathbf{U}^1$ ,

$$\partial_r \psi(t, \phi_1(r, s)) = Y_0(\phi_1(r, s)) \frac{\partial \psi(t, \phi_1(r, s))}{\partial x} = (\partial_{Y_0} \psi)(t, \phi_1(r, s)),$$
  
$$\partial_s \psi(t, \phi_1(r, s)) = Z_0(\phi_1(r, s)) \frac{\partial \psi(t, \phi_1(r, s))}{\partial x} = (\partial_{Z_0} \psi)(t, \phi_1(r, s)),$$

and hence by an induction argument we achieve

$$\partial_r^{k_1} \partial_s^{k_2} \psi(t, \phi_1(r, s)) = (\partial_{Y_0}^{k_1} \partial_{Z_0}^{k_2} \psi)(t, \phi_1(r, s)), \text{ with } \phi_1 \in W^{k+2, p}(\mathbf{U}^1).$$

Noting that the initial system  $\mathcal{X}_0$  given in (1.6) is admissible, so for any  $(\widetilde{r}, \widetilde{s}) \in \mathbf{U}^1$ , without loss of generality, we can assume

$$(1.14) |X_{1,0} \wedge X_{2,0}|(\phi_1(\widetilde{r},\widetilde{s})) \ge \min_{(\widetilde{r},\widetilde{s}) \in \mathbf{U}^1} ([\mathcal{X}_0](\phi_1(\widetilde{r},\widetilde{s})))^2 > 0.$$

Then the fact that  $X_{i,0}$  is continuous since  $W^{k+1,p}(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3)$ , guarantees the existence of a  $\partial\Omega_0$ -open set  $\mathbf{V}_0 \subset \mathbf{V}^1$  containing  $\phi_1(\tilde{r}, \tilde{s})$  such that

(1.15) 
$$\inf_{x \in \mathbf{V}_0} |X_{1,0} \wedge X_{2,0}|(x) > 0.$$

Thus we can decompose  $Y_0$ ,  $Z_0$  as a linear combination of  $X_{1,0}$  and  $X_{2,0}$  on  $V_0$ , namely

$$Y_0 = c_1 X_{1,0} + c_2 X_{2,0}, \quad Z_0 = d_1 X_{1,0} + d_2 X_{2,0},$$

where the coefficients are defined by

$$c_{i} = \frac{(Y_{0}, X_{i,0})|X_{j,0}|^{2} - (Y_{0}, X_{j,0})(X_{1,0}, X_{2,0})}{|X_{1,0} \wedge X_{2,0}|^{2}},$$

$$d_{i} = \frac{(Z_{0}, X_{i,0})|X_{j,0}|^{2} - (Z_{0}, X_{j,0})(X_{1,0}, X_{2,0})}{|X_{1,0} \wedge X_{2,0}|^{2}}, \quad i, j = 1, 2, i \neq j.$$

Hence without loss of generality, to prove (1.13) reduces to prove

$$(1.16) \qquad (\partial_{c_1 X_{1,0} + c_2 X_{2,0}}^{k_1} \partial_{d_1 X_{1,0} + d_2 X_{2,0}}^{k_2} \psi)(t, \cdot) \in L^{\infty}_{loc}(W^{2,p}(\mathbf{V}_0)), \ \forall k_1 + k_2 = k.$$

A new problem needed to consider is that the differential may act on the coefficients  $c_1, c_2, d_1, d_2$ . But thanks to the fact  $X_{1,0}, X_{2,0}, Y_0, Z_0 \in W^{k+1,p}(\mathbf{V}^1)$ ,  $k \geq 1$ , p > 3, and (1.15), we know that the coefficients  $c_i, d_i, i = 1, 2$  belong to  $W^{k+1,p}(\mathbf{V}_0), k \geq 1$ . Hence it suffices to show

(1.17) 
$$\partial_{(X_{1,0},X_{2,0})}^k \psi \in L^{\infty}_{loc}(W^{2,p}(\mathbf{V}_0)).$$

To do this, let us recall the definition (1.11) of the stream function  $\psi$ , thus the vector field  $X_i(t,x)$  defined by (1.8) can be written as

(1.18) 
$$X_i(t,x) = (\partial_{X_{i,0}}\psi)(t,\psi^{-1}(t,x)), \quad i = 1, 2.$$

Hence for any function  $f(t,x) \equiv g(t,\psi^{-1}(t,x))$ , there holds

$$\begin{split} (\partial_{X_i} f)(t,x) &= (\partial_{X_{i,0}} \psi)(t,\psi^{-1}(t,x)) \cdot \nabla \psi^{-1}(t,x) \cdot (\nabla g)(t,\psi^{-1}(t,x)) \\ &= X_{i,0}(t,\psi^{-1}(t,x)) \cdot (\nabla \psi)(t,\psi^{-1}(t,x)) \cdot \nabla \psi^{-1}(t,x) \cdot (\nabla g)(t,\psi^{-1}(t,x)) \\ &= X_{i,0}(t,\psi^{-1}(t,x)) \cdot (\nabla g)(t,\psi^{-1}(t,x)) \\ &= (\partial_{X_{i,0}} g)(t,\psi^{-1}(t,x)), \quad i = 1,2. \end{split}$$

Applying the above formula repeatedly on (1.18) yields that

$$(\partial_{(X_1,X_2)}^{\alpha(k-1)} X_{\alpha_k})(t,x) = (\partial_{(X_{1,0},X_{2,0})}^{\alpha(k)} \psi)(t,\psi^{-1}(t,x)),$$

for any multi-index  $\alpha(k) = (\alpha_1, \dots, \alpha_k)$  with  $\alpha_1, \dots, \alpha_k \in \{1, 2\}$ , and  $\alpha(k-1)$  denotes  $(\alpha_1, \dots, \alpha_{k-1})$ . Hence in order to prove (1.17), we only need to show

(1.19) 
$$\partial_{(X_1,X_2)}^{k-1}(X_1,X_2) \in L^{\infty}_{loc}\left(W^{2,p}(\mathbf{V}(t))\right), \quad \mathbf{V}(t) \stackrel{\text{def}}{=} \psi(t,\mathbf{V}_0),$$

which is guaranteed by (1.10). This completes the proof of this theorem.

At the end of this section, we recall some basic facts on Littlewood-Paley theory for readers' convenience. For  $u \in \mathcal{S}'(\mathbb{R}^d)$ , where  $\mathcal{S}'$  stands for tempered distribution space, we set

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u$$
 and  $\dot{S}_j u = \chi(2^{-j}D)u$ ,  $\forall j \in \mathbb{Z}$ ,

where  $\chi, \varphi$  are smooth functions supported in the ball  $\mathcal{B} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$  and the ring  $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  respectively, such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for}\quad \xi\neq 0,\quad\text{and}\quad \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1\quad\text{for}\quad \xi\in\mathbb{R}^d\,.$$

Then we can define the homogeneous Besov space  $\dot{B}_{p,r}^s$  with  $s < \frac{d}{p}$  as follows.

**Definition 1.2.** Let  $(p,r) \in [1,+\infty]^2$ ,  $s < \frac{d}{p}$  and  $u \in \mathcal{S}'_h(\mathbb{R}^d)$ , which means that u is in  $\mathcal{S}'(\mathbb{R}^d)$  and satisfies  $\lim_{j \to -\infty} ||\dot{S}_j u||_{L^{\infty}} = 0$ . We define

$$\dot{B}^s_{p,r}(\mathbb{R}^d) \stackrel{def}{=} \left\{ u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \|u\|_{\dot{B}^s_{p,r}} < \infty \right\}, \quad \text{where} \quad \|u\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \left\| \left( 2^{js} \|\dot{\Delta}_j u\|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

### 2. The proof of Proposition 1.3

The purpose of this section is to present the proof of Proposition 1.3. For any smooth enough solution  $(v, \rho, \mathcal{X})$  of (1.1) and (1.8), we consider

(2.1) 
$$J_0(t) \stackrel{\text{def}}{=} 1 + \|(\partial_t v, \Delta v, \nabla \pi)\|_{L_t^{r_0}(L^p)} + \|\nabla v\|_{L_t^{\sigma_1}(L^\infty) \cap L_t^{s_0}(L^p)} + \|v\|_{L_t^\infty(L^3) \cap L_t^{\sigma_2}(L^\infty)},$$
 and the following inductively defined quantities  $J_\ell(t)$ ,  $\ell = 1, \dots, k$ ,

(2.2) 
$$J_{\ell}(t) \stackrel{\text{def}}{=} J_{\ell-1}(t) + \|(\partial_{t}\partial_{\mathcal{X}}^{\ell}v, \Delta\partial_{\mathcal{X}}^{\ell}v, \nabla\partial_{\mathcal{X}}^{\ell}\pi)\|_{L_{t}^{r_{\ell}}(L^{p})} + \|\nabla\partial_{\mathcal{X}}^{\ell}v\|_{L_{t}^{r_{\ell}}(L^{\infty})\cap L_{t}^{s_{\ell}}(L^{p})} + \|\partial_{t}\partial_{\mathcal{X}}^{\ell-1}\mathcal{X}\|_{L_{t}^{s_{\ell}}(W^{1,p})} + \|\partial_{\mathcal{X}}^{\ell-1}\mathcal{X}\|_{L_{t}^{\infty}(W^{2,p})},$$

where  $\mathcal{X}$  is defined by (1.8), and  $r_0$ ,  $s_0$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $r_\ell$ ,  $s_\ell$  can be taken freely as long as

(2.3) 
$$r_{0} \in ]1, 2[, \quad s_{0} \in ]2, \infty[, \quad \sigma_{1} \in ]\frac{2p}{p+3}, \frac{2p}{3}[, \quad \sigma_{2} \in ]\frac{4p-6}{p}, \infty[, \quad \text{and} \quad r_{\ell} \in ]1, \frac{2k}{k+\ell\varepsilon}[, \quad s_{\ell} \in ]2, \frac{2k}{\ell\varepsilon}[, \quad \ell = 1, \cdots, k.$$

Then the main ingredient of proving Proposition 1.3 is the following A priori estimates:

$$(2.4) J_0(t) \leq C_0, J_{\ell}(t) \leq \mathcal{H}_{\ell+1}(t) \stackrel{\text{def}}{=} C_0 \underbrace{\exp \cdots \exp}_{\ell+1 \text{ times}}(C_0 t), \forall \ell = 1, \cdots, k, \forall t \in \mathbb{R}^+.$$

Here and in all that follows,  $C_0$  denotes some positive constant which depends only on the initial data and may vary from lines to lines in the following context. We remark that (2.4) is not the explicit bound, the huge number of exponentials is only a technical artefact.

We shall prove (2.4) in the following Subsections 2.1, 2.2 and 2.3 for the case  $\ell = 0$ ,  $\ell = 1$  and  $\ell \geq 2$  respectively. And Subsection 2.4 is devoted to the proof of Proposition 1.3.

2.1. The proof of (2.4) for  $\ell = 0$ . Let us first state the following a priori estimate.

**Lemma 2.1.** Assume the same hypothesis as in Theorem 1.2. Then for the global weak solution given in Proposition 1.1, there exists a positive constant  $C_0$  such that

*Proof.* It suffices to prove (2.5) for smooth solutions of the equation (1.1) <sup>3</sup>. Taking  $L^2(\mathbb{R}^3)$  inner product between the momentum equation in (1.1) with v|v| gives

(2.6) 
$$\frac{1}{3} \frac{d}{dt} \|\rho^{\frac{1}{3}} v(t)\|_{L^3}^3 - \int_{\mathbb{R}^3} \Delta v \cdot v |v| dx + \int_{\mathbb{R}^3} \nabla \pi \cdot v |v| dx = 0.$$

Integrating by parts, we have

(2.7) 
$$-\int_{\mathbb{R}^3} \Delta v \cdot v |v| dx = \int_{\mathbb{R}^3} |\nabla v|^2 |v| dx + \int_{\mathbb{R}^3} \nabla |v| \cdot \nabla v \cdot v dx$$
$$\geq \frac{1}{2} \int_{\mathbb{R}^3} \nabla |v| \cdot \nabla |v|^2 dx = \left\| \frac{2}{3} \nabla |v|^{\frac{3}{2}} \right\|_{L^2}^2.$$

Applying the three dimensional interpolation inequality that

$$||f||_{L^q(\mathbb{R}^3)} \le C||f||_{L^2(\mathbb{R}^3)}^{\frac{3}{q} - \frac{1}{2}} ||\nabla f||_{L^2(\mathbb{R}^3)}^{\frac{3}{2} - \frac{3}{q}} \text{ for } \forall q \in [2, 6],$$

<sup>&</sup>lt;sup>3</sup>Indeed, we can take approximated smooth solutions  $(\rho_n, v_n, \pi_n)$  of the equation (1.1) accompanied with the mollified initial data  $(1 + S_n(\rho_0 - 1), S_n v_0)$  such that (2.5) holds uniformly in n. Then a passage to the limit implies Lemma 2.1.

and Young's inequality, we obtain for  $r \in [1, 3]$ 

$$\begin{split} \int_{\mathbb{R}^{3}} \nabla \pi \cdot v |v| dx &\leq \|\nabla \pi\|_{L^{\frac{3r}{3r-2}}} \||v|^{\frac{3}{2}}\|_{L^{2r}}^{\frac{4}{3}} \\ &\leq \|\nabla \pi\|_{L^{\frac{3r}{3r-2}}} \||v|^{\frac{3}{2}}\|_{L^{2}}^{2 \cdot \frac{3-r}{3r}} \|\nabla |v|^{\frac{3}{2}}\|_{L^{2}}^{2 \cdot \frac{r-1}{r}} \\ &\leq \frac{1}{9} \|\nabla |v|^{\frac{3}{2}}\|_{L^{2}}^{2} + C \|\nabla \pi\|_{L^{\frac{3r}{3r-2}}}^{r} \||v|^{\frac{3}{2}}\|_{L^{2}}^{2} + C \|\nabla \pi\|_{L^{\frac{3r}{3r-2}}}^{r}. \end{split}$$

Substituting (2.7), (2.8) into (2.6), we get

$$\frac{1}{3}\frac{d}{dt} \left\| \rho^{\frac{1}{3}} v(t) \right\|_{L^{3}}^{3} + \frac{1}{3} \left\| \nabla |v|^{\frac{3}{2}} \right\|_{L^{2}}^{2} \leq C \| \nabla \pi \|_{L^{\frac{3r}{3r-2}}}^{r} \|v\|_{L^{3}}^{3} + C \| \nabla \pi \|_{L^{\frac{3r}{3r-2}}}^{r}.$$

Then we use (1.12), Gronwall's inequality, and the estimate (1.5) to achieve (2.5).

We will also use the following two lemmas, which can be proved exactly along the same line of the proofs of Lemma 4.1, Lemma 4.2 in [18], and Lemma 7.3 in [21]. Hence we just sketch their proofs for the readers' convenience.

**Lemma 2.2.** Let  $p \in [\frac{3}{2}, \infty[, r \in ]1, 2[, s \in ]2, \infty]$  and  $q \in [\frac{4p}{2p+3}, \frac{4p}{3}[$ . Let  $v_0 \in W^{1,p}$  and  $v_L(t) \stackrel{\text{def}}{=} e^{t\Delta}v_0$ . Then there exists some positive constant C such that

*Proof.* The fact that  $v_0 \in W^{1,p}$  ensures

$$\nabla^2 v_0 \in \dot{B}_{p,\infty}^{-1} \cap \dot{B}_{p,\infty}^{-2} \hookrightarrow \dot{B}_{p,1}^{-1-\sigma}, \quad \nabla v_0 \in \dot{B}_{p,\infty}^{0} \cap \dot{B}_{p,\infty}^{-1} \hookrightarrow \dot{B}_{p,1}^{-\sigma} \hookrightarrow \dot{B}_{2p,1}^{-\sigma-\frac{3}{2p}}, \quad \forall \sigma \in ]0,1[.$$

Hence by the characterisation of the Besov spaces with negative index, we know  $\forall \sigma \in ]0,1[$ ,

$$||t^{\frac{1}{2} + \frac{\sigma}{2} - \frac{1}{r}} e^{t\Delta} \nabla^{2} v_{0}||_{L^{r}(\mathbb{R}^{+}; L^{p})} + ||t^{\frac{\sigma}{2} - \frac{1}{s}} e^{t\Delta} \nabla v_{0}||_{L^{s}(\mathbb{R}^{+}; L^{p})} + ||t^{\frac{\sigma}{2} + \frac{3}{4p} - \frac{1}{q}} e^{t\Delta} \nabla v_{0}||_{L^{q}(\mathbb{R}^{+}; L^{2p})}$$

$$= ||\nabla^{2} v_{0}||_{\dot{B}_{p,r}^{-1-\sigma}} + ||\nabla v_{0}||_{\dot{B}_{p,s}^{-\sigma}} + ||\nabla v_{0}||_{\dot{B}_{2p,q}^{-\sigma}} \leq C||v_{0}||_{W^{1,p}}.$$

Using this with  $\sigma = \frac{1}{r} - \frac{1}{2}$  and  $\frac{1}{r}$  respectively, we deduce

$$\|\Delta v_L\|_{L^r(\mathbb{R}^+;L^p)} \leq \|t^{\frac{1}{2}(\frac{1}{2}-\frac{1}{r})}e^{t\Delta}\nabla^2 v_0\|_{L^r((0,1);L^p)} + \|t^{\frac{1}{2}(1-\frac{1}{r})}e^{t\Delta}\nabla^2 v_0\|_{L^r((1,\infty);L^p)} \leq C\|v_0\|_{W^{1,p}}.$$
And the other two terms in the left hand side of (2.9) can be estimated similarly.  $\square$ 

**Lemma 2.3.** For any  $p \in \left[\frac{3}{2}, \infty\right[, r \in ]1, 2[$ , and q given by  $\frac{1}{q} = \frac{1}{r} - \left(\frac{1}{2} - \frac{3}{4p}\right)$ , there exists some positive constant C such that

(2.10) 
$$\left\| \int_0^t \Delta e^{(t-t')\Delta} f(t') dt' \right\|_{L^r_T(L^p)} + \left\| \int_0^t \nabla e^{(t-t')\Delta} f(t') dt' \right\|_{L^{\frac{2r}{2-r}}_T(L^p)}$$

$$+ \left\| \int_0^t \nabla e^{(t-t')\Delta} f(t') dt' \right\|_{L^q_T(L^{2p})} \le C \|f\|_{L^r_T(L^p)}, \quad \forall T \in \mathbb{R}^+,$$

*Proof.* To prove (2.10) it suffices to write

$$e^{(t-t')\Delta}f(t') = \frac{1}{\sqrt{t-t'}}K(\frac{\cdot}{\sqrt{t-t'}}) * f(t',\cdot),$$

with K denoting the inverse Fourier transform of  $e^{-|\xi|^2}$ , and then to take Young's inequality first in the space variable and then in the time variable.

Now we are in a position to prove  $J_0(t) \leq C_0$ . As we are considering perturbations of the reference density 1, it is convenient to set  $a \stackrel{\text{def}}{=} 1/\rho - 1$  so that System (1.1) translates into

(2.11) 
$$\begin{cases} \partial_t a + v \cdot \nabla a = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t v + v \cdot \nabla v + (1+a)(\nabla \pi - \Delta v) = 0, \\ \operatorname{div} v = 0, \\ (a, v)|_{t=0} = (a_0, v_0). \end{cases}$$

For any  $\lambda > 0$ , and any function g(t), we denote

$$g_{\lambda}(t) \stackrel{\text{def}}{=} g(t) \exp\left(-\lambda \int_{0}^{t} V(t') dt'\right), \quad \text{with } V(t) \stackrel{\text{def}}{=} \|v(t)\|_{L^{2p}}^{\frac{4p}{2p-3}} \ge 0.$$

Then by virtue of (2.11),  $v_{\lambda}$  satisfies

$$(2.12) \partial_t v_\lambda + \lambda V(t) v_\lambda = (\partial_t v)_\lambda = \Delta v_\lambda + (-v \cdot \nabla v_\lambda + a \Delta v_\lambda - (1+a) \nabla \pi_\lambda),$$

which can also be written in the form

$$(2.13) \quad v_{\lambda}(t) = e^{t\Delta}v_{0,\lambda} + \int_0^t e^{-\lambda \int_{t'}^t V(t'') dt''} e^{(t-t')\Delta} \Big( -v \cdot \nabla v_{\lambda} + a\Delta v_{\lambda} - (1+a)\nabla \pi_{\lambda} \Big) (t') dt'.$$

Taking space divergence to (2.12) gives

$$\Delta \pi_{\lambda} = -\operatorname{div}(v \cdot \nabla v_{\lambda}) + \operatorname{div}(a(\Delta v_{\lambda} - \nabla \pi_{\lambda})),$$

from which and the fact that

$$||a||_{L^{\infty}} \le ||a_0||_{L^{\infty}} = \frac{|\eta|}{1 - |\eta|}$$

since a satisfies a free transport equation in (2.11), we infer

In view of (2.13), we get, by applying Lemma 2.2, Lemma 2.3 and (2.14), that

$$\begin{split} \|\Delta v_{\lambda}\|_{L_{t}^{r_{0}}(L^{p})} + \|\nabla v_{\lambda}\|_{L_{t}^{2r_{0}}(L^{p})} + \|\nabla v_{\lambda}\|_{L_{t}^{q_{0}}(L^{2p})} \\ \leq & \|\Delta v_{L}\|_{L_{t}^{r_{0}}(L^{p})} + \|\nabla v_{L}\|_{L_{t}^{2r_{0}}(L^{p})} + \|\nabla v_{L}\|_{L_{t}^{q_{0}}(L^{2p})} \\ & + C \bigg( \int_{0}^{t} e^{-\lambda r_{0} \int_{t'}^{t} V(t'') dt''} \Big( \|v \cdot \nabla v_{\lambda}(t')\|_{L^{p}}^{r_{0}} + \|a(t')\|_{L^{\infty}}^{r_{0}} \|\Delta v_{\lambda}(t')\|_{L^{p}}^{r_{0}} \\ & + \Big( 1 + \|a(t')\|_{L^{\infty}}^{r_{0}} \Big) \|\nabla \pi(t')\|_{L^{p}}^{r_{0}} \Big) dt' \bigg)^{\frac{1}{r_{0}}} \\ \leq & C \bigg( \|v_{0}\|_{W^{1,p}} + |\eta| \|\Delta v_{\lambda}\|_{L_{t}^{r_{0}}(L^{p})} + \Big( \int_{0}^{t} e^{-\lambda r_{0} \int_{t'}^{t} V(t'') dt''} \|v \cdot \nabla v_{\lambda}(t')\|_{L^{p}}^{r_{0}} dt' \bigg)^{\frac{1}{r_{0}}} \bigg), \end{split}$$

where  $q_0$  is taken to be  $\frac{1}{q_0} = \frac{1}{r_0} - (\frac{1}{2} - \frac{3}{4p})$ . Thus we can use Hölder's inequality to get

$$\left(\int_{0}^{t} e^{-\lambda r_{0} \int_{t'}^{t} V(t'') dt''} \|v \cdot \nabla v_{\lambda}(t')\|_{L^{p}}^{r_{0}} dt'\right)^{\frac{1}{r_{0}}} \\
\leq \left(\int_{0}^{t} e^{-\frac{4p}{2p-3}\lambda \int_{t'}^{t} V(t'') dt''} \|v(t')\|_{L^{2p}}^{\frac{4p}{2p-3}} dt'\right)^{\frac{2p-3}{4p}} \|\nabla v_{\lambda}\|_{L_{t}^{q_{0}}(L^{2p})} \\
\leq C\lambda^{-\frac{2p-3}{4p}} \|\nabla v_{\lambda}\|_{L_{t}^{q_{0}}(L^{2p})}.$$

We take  $|\eta|$  small enough and  $\lambda$  large enough to achieve

By use of the estimates (1.5) and (2.5), we deduce from the interpolation inequality that

$$(2.16) \qquad \left( \int_0^t V(t') \, dt' \right)^{\frac{2p-3}{4p}} = \|v\|_{L_t^{\frac{4p}{2p-3}}(L^{2p})} \le C \|v\|_{L_t^{\infty}(L^3)}^{1-\theta} \|\Delta v\|_{L_t^r(L^{\frac{3r}{3r-2}})}^{\theta} \le C_0,$$

where  $\theta = \frac{(2p-3)r}{4p} \in ]0,1[$ . Thus we deduce from (2.15) that

(2.17) 
$$\|\Delta v\|_{L_{t}^{r_{0}}(L^{p})} + \|\nabla v\|_{L_{t}^{2r_{0}}(L^{p})} + \|\nabla v\|_{L_{t}^{q_{0}}(L^{2p})}$$

$$\leq C\|v_{0}\|_{W^{1,p}} \cdot e^{\lambda \int_{0}^{t} V(t') dt'} \leq C\|v_{0}\|_{W^{1,p}} \cdot e^{\lambda C_{0}} \leq C_{0}.$$

This together with (2.14) and (2.16) ensures that

and hence we deduce from the velocity equation in (2.11) that

Moreover, for any p > 3,  $r_0 \in ]1, 2[$ , there holds

It is easy to observe that when  $r_0$  varies from 1 to 2, we have  $\sigma_1 \stackrel{\text{def}}{=} \frac{2pr_0}{2p+3r_0-pr_0} \in \left] \frac{2p}{p+3}, \frac{2p}{3} \right[$ . Similarly, we deduce that for  $\sigma_2 \stackrel{\text{def}}{=} \frac{2p-3}{p} \cdot \frac{2r_0}{2-r_0} \in \left] \frac{4p-6}{p}, \infty \right[$ , there holds

$$||v||_{L_{t}^{\sigma_{2}}(L^{\infty})} \leq C||v||_{L_{t}^{\infty}(L^{3})}^{\frac{p-3}{2p-3}} ||\nabla v||_{L_{t}^{\frac{2r_{0}}{2p-r_{0}}}(L^{p})}^{\frac{p}{2p-3}} \leq C_{0}.$$

By view of the definition (2.1), we deduce  $J_0(t) \leq C_0$  from the estimates (2.5), (2.17), (2.18), (2.19), (2.20), (2.21), which completes the proof of (2.4) for  $\ell = 0$ .

2.2. The proof of (2.4) for  $\ell = 1$ . We first deduce from the equation (1.8) and the proved fact  $J_0(t) \leq C_0$  that for any  $p \in ]3, \infty[$ ,

$$(2.22) ||X_i(t)||_{L^p \cap L^\infty} \le ||X_{i,0}||_{L^p \cap L^\infty} \exp\left(\int_0^t ||\nabla v(t')||_{W^{1,p}} dt'\right) \le \mathcal{H}_1(t).$$

Then by applying  $\partial_k$ , k = 1, 2, 3 to equation (1.8), we obtain

$$\partial_t \partial_k X_i + v \cdot \nabla \partial_k X_i + \partial_k v \cdot \nabla X_i = \partial_k X_i \cdot \nabla v + X_i \cdot \nabla \partial_k v$$

Then the standard energy estimates for transport equation leads to

$$(2.23) \quad \|\nabla X_i(t)\|_{L^p} \le \left(\|\nabla X_{i,0}\|_{L^p} + \|X_i\|_{L^{\infty}_t(L^{\infty})}\|\nabla^2 v\|_{L^1_t(L^p)}\right) \exp\left(2\int_0^t \|\nabla v(t')\|_{L^{\infty}} dt'\right)$$

$$\le \mathcal{H}_1(t).$$

Similarly, by applying the operator  $\Delta$  to (1.8), we get

$$\partial_t \Delta X_i + v \cdot \nabla \Delta X_i + 2 \sum_{j=1}^3 \partial_j v \cdot \nabla \partial_j X_i + \Delta v \cdot \nabla X_i = \Delta \partial_{X_i} v.$$

Noting that  $p \in ]3, \infty[$ , thus we can achieve, by using the standard  $L^p$  energy estimate and the interpolation inequality  $||f||_{L^{\infty}} \leq C||f||_{L^p}^{1-\frac{3}{p}}||\nabla f||_{L^p}^{\frac{3}{p}} \leq C(||f||_{L^p} + ||\nabla f||_{L^p})$ , that

$$\frac{d}{dt} \|\Delta X_{i}(t)\|_{L^{p}} \leq 2\|\nabla v(t)\|_{L^{\infty}} \|\nabla^{2} X_{i}(t)\|_{L^{p}} + \|\Delta v(t)\|_{L^{p}} \|\nabla X_{i}(t)\|_{L^{\infty}} + \|\Delta \partial_{X_{i}} v(t)\|_{L^{p}} 
\leq C \Big( (\|\nabla v(t)\|_{L^{p}} + \|\Delta v(t)\|_{L^{p}}) \|\Delta X_{i}(t)\|_{L^{p}} 
+ \|\Delta v(t)\|_{L^{p}} (\|\nabla X_{i}(t)\|_{L^{p}} + \|\Delta X_{i}(t)\|_{L^{p}}) + \|\Delta \partial_{X_{i}} v(t)\|_{L^{p}} \Big).$$

Applying Gronwall's inequality, together with (2.23) and  $J_0(t) \leq C_0$ , leads to

$$\|\Delta X_{i}(t)\|_{L^{p}} \leq C \Big( \|\Delta X_{i,0}\|_{L^{p}} + \|\Delta v\|_{L^{1}_{t}(L^{p})} \|\nabla X_{i}\|_{L^{\infty}_{t}(L^{p})} + \|\Delta \partial_{X_{i}}v\|_{L^{1}_{t}(L^{p})} \Big)$$

$$\times \exp\Big( C \Big( \|\nabla v\|_{L^{1}_{t}(L^{p})} + \|\Delta v\|_{L^{1}_{t}(L^{p})} \Big) \Big)$$

$$\leq \mathcal{H}_{1}(t) \Big( 1 + \|\Delta \partial_{X_{i}}v\|_{L^{1}_{t}(L^{p})} \Big).$$

Next we shall focus on the estimate of  $\|\Delta \partial_{X_i} v\|_{L^1_t(L^p)}$ . To do this, we first apply the operator  $\partial_{X_i}$  to the velocity equation in (2.11) to get the equation for  $\partial_{X_i} v$ :

(2.25) 
$$\partial_t(\partial_{X_i}v) + v \cdot \nabla(\partial_{X_i}v) + (1+a)(\nabla\partial_{X_i}\pi - \Delta\partial_{X_i}v) = F_1(v,\pi,(i)), \text{ with } F_1(v,\pi,(i)) \stackrel{\text{def}}{=} (1+a)(\nabla X_i \cdot \nabla \pi - \Delta X_i \cdot \nabla v - 2\nabla X_i : \nabla^2 v),$$

A direct calculation shows that  $(\partial_{X_i}\pi)$  satisfies (noticing that div  $X_i=0$ )

$$\operatorname{div}\left((1+a)\nabla\partial_{X_{i}}\pi\right) = \operatorname{div}\left(-\partial_{t}(\partial_{X_{i}}v) - v \cdot \nabla(\partial_{X_{i}}v) + \Delta\partial_{X_{i}}v + a\Delta\partial_{X_{i}}v + F_{1}\right)$$

$$= \operatorname{div}\left(-\partial_{t}X_{i} \cdot \nabla v - \partial_{t}v \cdot \nabla X_{i} - v \cdot \nabla(\partial_{X_{i}}v) + \Delta X_{i} \cdot \nabla v + 2\nabla X_{i} : \nabla^{2}v + \Delta v \cdot \nabla X_{i} + a\Delta\partial_{X_{i}}v + F_{1}(v, \pi, (i))\right) \stackrel{\text{def}}{=} \operatorname{div}G,$$

which implies

$$\nabla \partial_{X_i} \pi = \nabla \Delta^{-1} \operatorname{div} \left( G - a \nabla \partial_{X_i} \pi \right).$$

Then by using the fact that Riesz transform is  $L^p$ , 1 bounded, we obtain

$$(2.26) (1 - C||a_0||_{L^{\infty}}) ||\nabla \partial_{X_i} \pi||_{L^p} \le C||G||_{L^p}.$$

For any  $r_1 \in \left]1, \frac{2k}{k+\varepsilon}\right[$ , we can find some  $r_0 \in ]r_1, 2[$  and  $\sigma_1 \in \left]r_1, \frac{2p}{3}\right[$ . Then we can use the equations (1.8), together with the estimates (2.22), (2.23),  $J_0(t) \leq C_0$ , to get

$$\left\| \left( \partial_{t} X_{i} \cdot \nabla v, v \cdot \nabla (\partial_{X_{i}} v) \right) \right\|_{L_{t}^{r_{1}}(L^{p})} + \left\| \left( \partial_{t} v \cdot \nabla X_{i}, \nabla X_{i} : \nabla^{2} v \right) \right\|_{L_{t}^{r_{1}}(L^{p})} \\
\lesssim \left\| \nabla v \right\|_{L_{t}^{\sigma_{1}}(L^{\infty})} \left( \left\| X_{i} \right\|_{L_{t}^{\infty}(L^{\infty})} \left\| \nabla v \right\|_{L_{t}^{\frac{\sigma_{1}r_{1}}{\sigma_{1} - r_{1}}(L^{p})}} + \left\| \nabla X_{i} \right\|_{L_{t}^{\infty}(L^{p})} \left\| v \right\|_{L_{t}^{\frac{\sigma_{1}r_{1}}{\sigma_{1} - r_{1}}(L^{\infty})}} \\
+ \left\| X_{i} \right\|_{L_{t}^{\infty}(L^{\infty})} \left\| \nabla^{2} v \right\|_{L_{t}^{r_{0}}(L^{p})} \left\| v \right\|_{L_{t}^{\frac{r_{0}r_{1}}{r_{0} - r_{1}}(L^{\infty})}} + \left\| \left( \partial_{t} v, \Delta v \right) \right\|_{L_{t}^{r_{1}}(L^{p})} \left\| \nabla X_{i} \right\|_{L_{t}^{\infty}(L^{\infty})} \\
\lesssim \mathcal{H}_{1}(t) \left( 1 + \left\| \nabla X_{i} \right\|_{L_{t}^{\infty}(L^{\infty})} \right).$$

And Similarly we have

Then we achieve, by substituting (2.27) and (2.28) into (2.26), and choosing  $||a_0||_{L^{\infty}}$  to be sufficiently small, that

$$\|\nabla \partial_{X_{i}} \pi\|_{L_{t}^{r_{1}}(L^{p})} \leq \mathcal{H}_{1}(t) \left(1 + \|\nabla X_{i}\|_{L_{t}^{\infty}(L^{\infty})}\right) + \|a_{0}\|_{L^{\infty}} \|\Delta \partial_{X_{i}} v\|_{L_{t}^{r_{1}}(L^{p})}$$

$$+ \left(\int_{0}^{t} \|\nabla v(t')\|_{L^{\infty}}^{r_{1}} \|\Delta X_{i}(t')\|_{L^{p}}^{r_{1}} dt'\right)^{\frac{1}{r_{1}}}.$$

On the other hand, we can get from (2.25) that

$$(2.30) \qquad (\partial_{X_i}v)(t) = e^{t\Delta}(\partial_{X_{i,0}}v_0) + \int_0^t e^{(t-t')\Delta} \Big(-v \cdot \nabla(\partial_{X_i}v) + a\Delta\partial_{X_i}v - (1+a)\nabla\partial_{X_i}\pi + F_1(v,\pi,(i))\Big)(t') dt'.$$

Similarly as Lemma 2.2, we can prove the following estimate:

$$(2.31) \|\Delta e^{t\Delta} \partial_{X_{i,0}}^{\ell} v_0\|_{L_t^{r_{\ell}}(L^p)} + \|\nabla e^{t\Delta} \partial_{X_{i,0}}^{\ell} v_0\|_{L_t^{s_{\ell}}(L^p)} \le C \|\partial_{X_{i,0}}^{\ell} v_0\|_{W^{1-\frac{\ell}{k}\varepsilon,p}}, \quad \forall \ell = 1, \cdots, k.$$

Using this, we infer from (2.30) that

$$\|\partial_t \partial_{X_i} v\|_{L_t^{r_1}(L^p)} + \|\Delta \partial_{X_i} v\|_{L_t^{r_1}(L^p)} \le \|\partial_{X_{i,0}} v_0\|_{W^{1-\frac{\varepsilon}{k},p}} + C\Big(\|v \cdot \nabla(\partial_{X_i} v)\|_{L_t^{r_1}(L^p)} + \|a_0\|_{L^\infty} \|\Delta \partial_{X_i} v\|_{L_t^{r_1}(L^p)} + \|\nabla \partial_{X_i} \pi\|_{L_t^{r_1}(L^p)} + \|F_1\|_{L_t^{r_1}(L^p)}\Big).$$

Hence by virtue of (2.27)-(2.29), we get, by taking  $||a_0||_{L^{\infty}}$  to be sufficiently small, that

$$(2.32) \qquad \|\partial_t \partial_{X_i} v\|_{L_t^{r_1}(L^p)} + \|\Delta \partial_{X_i} v\|_{L_t^{r_1}(L^p)} \\ \leq \mathcal{H}_1(t) \left(1 + \|\nabla X_i\|_{L_t^{\infty}(L^{\infty})}\right) + C \left(\int_0^t \|\nabla v(t')\|_{L^{\infty}}^{r_1} \|\Delta X_i(t')\|_{L^p}^{r_1} dt'\right)^{\frac{1}{r_1}}.$$

Substituting the above estimate into (2.24), and using (2.23), gives rise to

$$\begin{split} \|\Delta X_i\|_{L_t^{\infty}(L^p)} &\leq \mathcal{H}_1(t) \Big(1 + \|\nabla X_i\|_{L_t^{\infty}(L^p)}^{1 - \frac{2}{p}} \|\Delta X_i\|_{L_t^{\infty}(L^p)}^{\frac{2}{p}} + \Big(\int_0^t \|\nabla v(t')\|_{L^{\infty}}^{r_1} \|\Delta X_i(t')\|_{L^p}^{r_1} dt'\Big)^{\frac{1}{r_1}} \Big) \\ &\leq \mathcal{H}_1(t) + \frac{1}{2} \|\Delta X_i\|_{L_t^{\infty}(L^p)} + \mathcal{H}_1(t) \Big(\int_0^t \|\nabla v(t')\|_{L^{\infty}}^{r_1} \|\Delta X_i(t')\|_{L^p}^{r_1} dt'\Big)^{\frac{1}{r_1}}, \end{split}$$

which gives

$$\|\Delta X_i\|_{L_t^{\infty}(L^p)}^{r_1} \leq \mathcal{H}_1(t) + \mathcal{H}_1(t) \int_0^t \|\nabla v(t')\|_{L^{\infty}}^{r_1} \|\Delta X_i(t')\|_{L^p}^{r_1} dt'.$$

Then using Gronwall's inequality, together with the fact  $J_0(t) \leq C_0$ , we obtain

(2.33) 
$$\|\Delta X_i\|_{L_t^{\infty}(L^p)}^{r_1} \le \mathcal{H}_1(t) \exp\left(\mathcal{H}_1(t) \int_0^t \|\nabla v(t')\|_{L^{\infty}}^{r_1} dt'\right) \le \mathcal{H}_2(t).$$

This together with (2.29) and (2.32) leads to

By now, we have completed the proof of (2.4) for  $\ell = 1$ .

2.3. The proof of (2.4) for  $\ell \geq 2$ . First, we would like to point out that, the Section 6 of [18] gives the corresponding proof for 2-D case, but the estimates there are in fact independent of spatial dimension. Thus all we need to do here is to write clear the differences in notations, as we need to consider taking  $\ell \geq 2$  order derivatives in different tangential directions.

Before proceeding, we state the following commutator estimates which can be viewed as generalizations of Lemma 6.1 and Remark 6.1 in [18].

**Lemma 2.4.** For any  $\ell \in \{1, ..., k\}$  and any system of vector fields  $\mathcal{W} = \{W_1, ..., W_N\}$ , let  $\alpha(\ell) = (\alpha_1, ..., \alpha_\ell)$  be a multi-index of length  $\ell$  with indices taking value in  $\{1, ..., N\}$ , and we denote  $\widehat{\alpha}(i) = (\alpha_{\ell-i+1}, ..., \alpha_{\ell})$  for  $i = 0, ..., \ell$ . Then there exists a positive constant C such that  $\forall r_{\ell} \in \left]1, \frac{2k}{k+\ell \varepsilon}\right[, s_{\ell} \in \left]2, \frac{2k}{\ell \varepsilon}\right[, \forall X \in \mathcal{W}, \forall 1 \leq i \leq \ell,$ 

$$(2.35) \qquad \begin{aligned} \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X - \nabla \partial_{\mathcal{W}}^{\alpha(\ell)} X \|_{L_{t}^{\infty}(W^{1,p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla^{2} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X - \nabla^{2} \partial_{\mathcal{W}}^{\alpha(\ell)} X \|_{L_{t}^{\infty}(L^{p})} \\ + \|\partial_{\mathcal{W}}^{\alpha(i)} \partial_{t} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X - \partial_{t} \partial_{\mathcal{W}}^{\alpha(\ell)} X \|_{L_{t}^{s_{\ell}}(W^{1,p})} \leq C J_{\ell}^{\ell+1}, \end{aligned}$$

and when  $i \neq \ell$ , there holds

$$\begin{aligned} \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v - \nabla \partial_{\mathcal{W}}^{\alpha(\ell)} v \|_{L_{t}^{r_{\ell-1}}(L^{\infty}) \cap L_{t}^{s_{\ell-1}}(L^{p})} \\ (2.36) \qquad & + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla^{2} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v - \nabla^{2} \partial_{\mathcal{W}}^{\alpha(\ell)} v \|_{L_{t}^{r_{\ell-1}}(L^{p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} \pi - \nabla \partial_{\mathcal{W}}^{\alpha(\ell)} \pi \|_{L_{t}^{r_{\ell-1}}(L^{p})} \\ & + \|\partial_{\mathcal{W}}^{\alpha(i)} \partial_{t} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v - \partial_{t} \partial_{\mathcal{W}}^{\alpha(\ell)} v \|_{L_{t}^{r_{\ell-1}}(L^{p})} \leq C J_{\ell-1}^{\ell+1}, \end{aligned}$$

when  $i = \ell$ , there holds

$$(2.37) \begin{aligned} \|\partial_{\mathcal{W}}^{\alpha(\ell)} \nabla v - \nabla \partial_{\mathcal{W}}^{\alpha(\ell)} v + \partial_{\mathcal{W}}^{\alpha(\ell-1)} \nabla X_{\alpha_{\ell}} \cdot \nabla v \|_{L_{t}^{r_{\ell-1}}(L^{\infty}) \cap L_{t}^{s_{\ell-1}}(L^{p})} \leq C J_{\ell-1}^{\ell+1}, \\ \|\partial_{\mathcal{W}}^{\alpha(\ell)} \Delta v - \Delta \partial_{\mathcal{W}}^{\alpha(\ell)} v + \partial_{\mathcal{W}}^{\alpha(\ell-1)} \Delta X_{\alpha_{\ell}} \cdot \nabla v + 2 \partial_{\mathcal{W}}^{\alpha(\ell-1)} \nabla X_{\alpha_{\ell}} : \nabla^{2} v \|_{L_{t}^{r_{\ell-1}}(L^{p})} \leq C J_{\ell-1}^{\ell+1}, \\ \|\partial_{\mathcal{W}}^{\alpha(\ell)} \nabla \pi - \nabla \partial_{\mathcal{W}}^{\alpha(\ell)} \pi + \partial_{\mathcal{W}}^{\alpha(\ell-1)} \nabla X_{\alpha_{\ell}} \cdot \nabla \pi \|_{L_{t}^{r_{\ell-1}}(L^{p})} \leq C J_{\ell-1}^{\ell+1}, \\ \|\partial_{\mathcal{W}}^{\alpha(\ell)} \partial_{t} v - \partial_{t} \partial_{\mathcal{W}}^{\alpha(\ell)} v + \partial_{\mathcal{W}}^{\alpha(\ell-1)} \partial_{t} X_{\alpha_{\ell}} \cdot \nabla v \|_{L_{t}^{r_{\ell-1}}(L^{p})} \leq C J_{\ell-1}^{\ell+1}. \end{aligned}$$

Moreover, it follows from (2.2), (2.35), (2.36) and (2.37) that for any  $0 \le i \le \ell$ ,

$$(2.38) \qquad \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X\|_{L_{t}^{\infty}(W^{1,p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla^{2} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X\|_{L_{t}^{\infty}(L^{p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \partial_{t} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} X\|_{L_{t}^{s_{\ell+1}}(W^{1,p})} \leq C J_{\ell+1}^{\ell+1}, \quad \text{and}$$

$$(2.39) \qquad \begin{aligned} \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v\|_{L_{t}^{r_{\ell}}(L^{\infty}) \cap L_{t}^{s_{\ell}}(L^{p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla^{2} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v\|_{L_{t}^{r_{\ell}}(L^{p})} \\ + \|\partial_{\mathcal{W}}^{\alpha(i)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} \pi\|_{L_{t}^{r_{\ell}}(L^{p})} + \|\partial_{\mathcal{W}}^{\alpha(i)} \partial_{t} \partial_{\mathcal{W}}^{\widehat{\alpha}(\ell-i)} v\|_{L_{t}^{r_{\ell}}(L^{p})} \leq CJ_{\ell}^{\ell+1}. \end{aligned}$$

*Proof.* Firstly, for any  $X,Y\in\mathcal{W}$ , it is easy to observe that

$$\|\partial_{Y}\nabla X - \nabla\partial_{Y}X\|_{L_{t}^{\infty}(W^{1,p})} + \|\partial_{Y}\nabla^{2}X - \nabla^{2}\partial_{Y}X\|_{L_{t}^{\infty}(L^{p})} + \|\partial_{Y}\partial_{t}X - \partial_{t}\partial_{Y}X\|_{L_{t}^{s_{1}}(W^{1,p})}$$

$$\leq C\|\nabla X\|_{L_{t}^{\infty}(W^{1,p})} (\|\nabla Y\|_{L_{t}^{\infty}(W^{1,p})} + \|\partial_{t}Y\|_{L_{t}^{s_{1}}(W^{1,p})}) \leq CJ_{1}^{2}.$$

This shows that (2.35) holds for  $\ell=1$ . It is also easy to see that (2.36) and (2.37) hold trivially for  $\ell=1$ . Hence Lemma 2.4 holds for k=1.

Let us now assume that (2.35)-(2.39) hold for  $\ell \leq j-1$  with  $j \leq k$ . We are going to prove that (2.35)-(2.37) also hold for  $\ell = j$ , which will imply immediately (2.38)-(2.39) for  $\ell = j$ .

In the following, for  $n \leq m$  and for the m-length multi-index  $(l_1, \dots, l_m)$  such that

$$(l_1, \dots, l_m) \in L_m^n \stackrel{\text{def}}{=} \{(l_1, \dots, l_m) \mid l_1 < \dots < l_n, \ l_{n+1} < \dots < l_m, \ \{l_1, \dots, l_m\} = \{1, \dots, m\} \},$$
  
we denote  $\alpha^l(n) = (\alpha_{l_1}, \dots, \alpha_{l_n})$  and  $\widehat{\alpha}^l(m-n) = (\alpha_{l_{n+1}}, \dots, \alpha_{l_m}).$ 

For any positive integer  $i \leq j-1$ , a direct calculation gives

$$\partial_{\mathcal{W}}^{\alpha(i+1)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(j-i-1)} X - \nabla \partial_{\mathcal{W}}^{\alpha(j)} X = \sum_{m=0}^{i} \partial_{\mathcal{W}}^{\alpha(m)} [\partial_{X_{\alpha_{m+1}}}, \nabla] \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X$$
$$= \sum_{m=0}^{i} \partial_{\mathcal{W}}^{\alpha(m)} (\nabla X_{\alpha_{m+1}} \cdot \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X),$$

where  $[\cdot,\cdot]$  stands for the standard commutator. Then the induction assumptions give

$$\|\partial_{\mathcal{W}}^{\alpha(i+1)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(j-i-1)} X - \nabla \partial_{\mathcal{W}}^{\alpha(j)} X \|_{L_{t}^{\infty}(W^{1,p})}$$

$$(2.40) \quad \leq C \sum_{m=0}^{i} \sum_{n=0}^{m} \sum_{(l_{1}, \cdots, l_{m}) \in L_{m}^{n}} \|\partial_{\mathcal{W}}^{\alpha^{l}(n)} \nabla X_{\alpha_{m+1}} \|_{L_{t}^{\infty}(W^{1,p})} \|\partial_{\mathcal{W}}^{\widehat{\alpha}^{l}(m-n)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X \|_{L_{t}^{\infty}(W^{1,p})}$$

$$\leq C \sum_{n=0}^{i} \sum_{n=0}^{m} J_{n+1}^{n+1} J_{j-n}^{j-n} \leq C J_{j}^{j+1}.$$

We follow the same lines as above to obtain

$$\begin{split} \|\partial_{\mathcal{W}}^{\alpha(i+1)} \nabla^2 \partial_{\mathcal{W}}^{\widehat{\alpha}(j-i-1)} X - \nabla^2 \partial_{\mathcal{W}}^{\alpha(j)} X \|_{L_t^{\infty}(L^p)} &= \|\sum_{m=0}^i \partial_{\mathcal{W}}^{\alpha(m)} [\partial X_{\alpha_{m+1}}, \nabla^2] \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X \|_{L_t^{\infty}(W^{1,p})} \\ &\leq C \sum_{m=0}^i \sum_{n=0}^m \sum_{(l_1, \cdots, l_m) \in L_m^n} \left( \|\partial_{\mathcal{W}}^{\alpha^l(n)} \nabla^2 X_{\alpha_{m+1}} \|_{L_t^{\infty}(L^p)} \|\partial_{\mathcal{W}}^{\widehat{\alpha}^l(m-n)} \nabla \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X \|_{L_t^{\infty}(L^{\infty})} \right. \\ & + \|\partial_{\mathcal{W}}^{\alpha^l(n)} \nabla X_{\alpha_{m+1}} \|_{L_t^{\infty}(L^{\infty})} \|\partial_{\mathcal{W}}^{\widehat{\alpha}^l(m-n)} \nabla^2 \partial_{\mathcal{W}}^{\widehat{\alpha}(j-m-1)} X \|_{L_t^{\infty}(L^p)} \right) \\ &\leq C \sum_{m=0}^i \sum_{n=0}^m J_{n+1}^{n+1} J_{j-n}^{j-n} \leq C J_j^{j+1}, \quad \text{and} \\ & \|\partial_{\mathcal{W}}^{\alpha(i+1)} \partial_t \partial_{\mathcal{W}}^{\widehat{\alpha}(j-i-1)} X - \partial_t \partial_{\mathcal{W}}^{\alpha(j)} X \|_{L_t^{s_j}(W^{1,p})} \leq C J_j^{j+1}. \end{split}$$

These two estimates together with (2.40) guarantee that (2.35) holds for  $\ell = j$ .

The same argument to achieve (2.35) for  $\ell = j$  yield (2.36) and (2.37) for  $\ell = j$ . We complete the proof of Lemma 2.4 by the induction argument.

Next, we introduce the corresponding term to  $F_{\ell}(v,\pi)$  given in Lemma 6.2 of [18]. For any multi-index  $\alpha(\ell) = (\alpha_1, \dots, \alpha_{\ell})$ , we apply the operator  $\partial_{\mathcal{X}}^{\alpha(\ell-1)}$  to (2.25) for  $\partial_{X_{\alpha_{\ell}}} v$  to get

$$(2.41) \partial_t \partial_{\mathcal{X}}^{\alpha(\ell)} v + v \cdot \nabla \partial_{\mathcal{X}}^{\alpha(\ell)} v - (1+a) \left( \Delta \partial_{\mathcal{X}}^{\alpha(\ell)} v - \nabla \partial_{\mathcal{X}}^{\alpha(\ell)} \pi \right) \stackrel{\text{def}}{=} F_{\ell}(v, \pi, \alpha(\ell)).$$

Here  $F_{\ell}(v, \pi, \alpha(\ell))$  is given by induction

$$F_{\ell}(v,\pi,\alpha(\ell)) = \partial_{X_{\alpha_1}} F_{\ell-1}(v,\pi,\widehat{\alpha}(\ell-1)) + F_{1}(\partial_{\mathcal{X}}^{\widehat{\alpha}(\ell-1)}v,\partial_{\mathcal{X}}^{\widehat{\alpha}(\ell-1)}\pi,(\alpha_1)),$$

and hence by view of the definition of  $F_1$  in (2.25), and (1.9), we arrive at

$$F_{\ell}(v,\pi,\alpha(\ell)) = \sum_{i=0}^{\ell-1} \partial_{\mathcal{X}}^{\alpha(\ell-1-i)} F_{1}(\partial_{\mathcal{X}}^{\widehat{\alpha}(i)} v, \partial_{\mathcal{X}}^{\widehat{\alpha}(i)} \pi, (\alpha_{\ell-i}))$$

$$= (1+a) \sum_{i=0}^{\ell-1} \partial_{\mathcal{X}}^{\alpha(\ell-1-i)} \Big( \nabla X_{\alpha_{\ell-i}} \cdot \nabla \partial_{\mathcal{X}}^{\widehat{\alpha}(i)} \pi - \Delta X_{\alpha_{\ell-i}} \cdot \nabla \partial_{\mathcal{X}}^{\widehat{\alpha}(i)} v - 2 \nabla X_{\alpha_{\ell-i}} : \nabla^{2} \partial_{\mathcal{X}}^{\widehat{\alpha}(i)} v \Big).$$

Then exactly along the same line as the proofs of Lemma 6.2, 6.3 and Proposition 6.1 in [18], by using Lemma 2.4 instead of Lemma 6.1 and Remark 6.1 used there, we achieve a similar estimate as follows

$$\begin{split} &\|(\partial_t \partial_{\mathcal{X}}^{\alpha(\ell)} v, \Delta \partial_{\mathcal{X}}^{\alpha(\ell)} v, \nabla \partial_{\mathcal{X}}^{\alpha(\ell)} \pi)\|_{L_t^{r_\ell}(L^p)} + \|\nabla \partial_{\mathcal{X}}^{\alpha(\ell)} v\|_{L_t^{r_\ell}(L^\infty) \cap L_t^{s_\ell}(L^p)} + \|\partial_{\mathcal{X}}^{\alpha(\ell)} v\|_{L_t^{s_\ell}(L^\infty) \cap L_t^\infty(L^p)} \\ &+ \|\partial_t \partial_{\mathcal{X}}^{\alpha(\ell-1)} \mathcal{X}\|_{L_t^{s_\ell}(W^{1,p})} + \|\partial_{\mathcal{X}}^{\alpha(\ell-1)} \mathcal{X}\|_{L_t^\infty(W^{2,p})} \leq \mathcal{H}_{\ell+1}(t), \quad \forall \ell = 2, \cdots, k, \quad \forall t \in \mathbb{R}^+, \end{split}$$

which clearly implies (2.4) for  $\ell \geq 2$  since the choice of  $\alpha(\ell)$  is arbitrary.

2.4. The proof of Proposition 1.3. The existence of a weak solution  $(\rho, v)$  to (1.1) is guaranteed by Proposition 1.1. Moreover, the previous part of this section has shown that, this solution satisfies the energy estimate (2.4), which obviously implies that  $v \in L^1_{loc}(\mathbb{R}^+, W^{2,p}(\mathbb{R}^3))$ . In particular,  $v \in L^1_{loc}(\mathbb{R}^+; \text{Lip})$ , thus the uniqueness of the solutions can be proved by using Lagrangian formulation of (1.1) as in [11, 15]. We omit the details here.

The fact that v is in  $L^1_{loc}(\mathbb{R}^+; \text{Lip})$  also implies the existence and uniqueness of the solution to (1.8) in  $L^{\infty}_{loc}(\mathbb{R}^+; W^{2,p})$ , by using the classical theory on transport equations. Then the energy estimate (2.4) implies that, this solution  $X_i$  satisfies the conormal regularity in (1.10). This completes the proof of Proposition 1.3.

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## References

- [1] H. Abidi, G. Gui and P. Zhang, On the wellposedness of 3—D inhomogeneous Navier-Stokes equations in the critical spaces, *Arch. Ration. Mech. Anal.*, **204** (2012), 189–230.
- [2] H. Abidi and M. Paicu, Existence globale pour un fluide inhomogène, Ann. Inst. Fourier, 57 (2007), 883–917.
- [3] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov, *Boundary value problems in mechanics of non-homogeneous fluids*, Studies in Mathematics and its Applications 22, North-Holland Publishing Co., Amsterdam, 1990.
- [4] A.-L. Bertozzi and P. Constantin, Global regularity for vortex patches, Comm. Math. Phys., 152 (1993), 19-28.
- [5] J.-Y. Chemin, Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnel, *Invent. Math.*, **103** (1991), 599–629.
- [6] J.-Y. Chemin, Persistance de structures géométriques dans les fluides incompressibles bidimensionnels, Ann. Sci. École Norm. Sup., 26 (1993), 517-542.
- [7] H. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations, 28(2003), 1183–1201.
- [8] R. Danchin, Poches de tourbillon visqueuses, J. Math. Pures Appl. (9), 76 (1997), 609-647.
- [9] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A*, **133** (2003), 1311–1334.

- [10] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids, Adv. Differential Equations, 9 (2004), no. 3-4, 353-386.
- [11] R. Danchin and P. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, *Comm. Pure Appl. Math.*, **65** (2012), 1458-1480.
- [12] R. Danchin and P.-B. Mucha, The incompressible Navier-Stokes equations in vacuum, arXiv:1705.06061.
- [13] R. Danchin and X. Zhang, On the persistence of Hölder regularity patches of density for the inhomogeneous Navier-Stokes equations, arXiv: 1612.00282.
- [14] F. Gancedo and E. Garcia-Juarez, Global regularity of 2D density patches for inhomogeneous Navier-Stokes, arXiv:1612.08665.
- [15] J. Huang, M. Paicu and P. Zhang, Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity, Arch. Ration. Mech. Anal., 209 (2013), 631–682.
- [16] P. Gamblin and X. Saint Raymond, On three-dimensional vortex patches, Bull. Soc. math. France, 123 (1995), 375–424.
- [17] O.A. Ladyzhenskaya and V.A. Solonnikov, Unique solvability of an initial- and boundary-value problem for viscous incompressible nonhomogeneous fluids, J. Soviet. Math., 9(5) (1978), 697-749.
- [18] X. Liao and P. Zhang, On the global regularity of two-dimensional density patch for inhomogeneous incompressible viscous flow, *Arch. Ration. Mech. Anal.*, **220** (2016), 937-981.
- [19] X. Liao and P. Zhang, Global regularities of two-dimensional density patch for inhomogeneous incompressible viscous flow with general density, arXiv:1604.07922.
- [20] P.-L. Lions, Mathematical topics in fluid mechanics. Vol. 1. Incompressible models. Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.
- [21] P. G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman & Hall/CRC Research Notes in Mathematics, 431, 2002.
- [22] M. Paicu, P. Zhang and Z. Zhang, Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density, Comm. Partial Differential Equations, 38 (2013), 1208-1234.
- [23] P. Zhang and Q. Qiu, Propagation of higher-order regularities of the boundaries of 3-D vortex patches, Chinese Ann. Math. Ser. A, 18 (1997), 381–390.
- [24] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal., 21 (1990), 1093–1117.
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