EQUIVARIANT ETA FORMS AND EQUIVARIANT DIFFERENTIAL K-THEORY

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ABSTRACT. In this paper, we construct an analytic model of equivariant differential K-theory for closed manifolds with almost free action, which is a ring valued functor with the usual properties of a differential extensions of a cohomology theory, using the equivariant Bismut-Cheeger eta forms with the equivariant version of spectral sections developed by Melrose-Piazza. In fact, it could also be regarded as an analytic model of differential K-theory for closed orbifolds. Furthermore, we construct a well-defined push-forward map in equivariant differential K-theory and prove the properties of it.

In order to do these, we extend the Melrose-Piazza spectral section to the equivariant case, introduce the equivariant version of higher spectral flow for arbitrary dimensional fibers and use them to prove the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms.

0. INTRODUCTION

By the de Rham theory, the de Rham cohomology of a smooth manifold can be represented by differential forms, thus getting the global information by means of local data. In a similar way, a generalized differential cohomology theory gives a way to combine the cohomological information with differential geometric objects. An important case is the differential K-theory.

The differential K-theory is partly motivated by the study of D-branes in theoretical physics introduced by Witten [36] in 1998. He points out that D-branes carry Ramond-Ramond charges that massless Ramond-Ramond fields are differential forms, and that the D-brane charges should be understood in terms of K-theory. Various models of differential K-theory for manifolds have been proposed (see e.g. Bunke-Schick [12], Freed-Lott [19], Hopkins-Singer [20] and Simons-Sullivan [34]). We note that in the model of Bunke-Schick [12], the differential K-theory is constructed using the geometric families and Bismut-Cheeger eta forms with taming.

Until now, the equivariant version of the differential K-theory is not well understood yet. When the group action is finite, the equivariant differential K-theory is studied by Szabo-Valentino [35] and Ortiz [30]. In [14], Bunke and Schick extend their model to the orbifold case using the language of stacks, which could be regarded as a model of equivariant differential K-theory for the almost free action.

Inspired by the model of Bunke and Schick [14], as a parallel version, in this paper, we construct a purely analytic model of equivariant differential K-theory for closed manifolds with almost free action using the local index technique developed by [9]. Furthermore, we prove that the push-forward map is well-defined in our construction, which is a problem proposed in [12, 14] and the main motivation for this new construction. This model is a direct generalization of [12] without using the language of stacks and could also be regarded as an analytic model of differential K-theory for closed orbifolds. Note that when restricted to the non-equivariant case, our construction is a little different from that in [12] although they are isomorphic. We use the analytic tools: spectral sections and higher spectral flows instead of the taming and Kasporov KK-theory in [12].

The key tools in our model are the equivariant Bismut-Cheeger eta forms [6] with the equivariant version of Melrose-Piazza spectral sections [28, 29] and Dai-Zhang higher spectral flows [16].

In [28, 29], in order to prove the family index theorem for manifolds with boundary, Melrose and Piazza define the spectral section for a family of self-adjoint (resp. odd $\mathbb{Z}_2$-graded self-adjoint) 1st-order elliptic pseudodifferential operators on a family of odd (resp. even) dimensional manifolds when the family index of the operators vanishes. In [16], using the spectral section, Dai and Zhang introduce the higher spectral flow for a family of Dirac type operators on a family of odd dimensional manifolds.

In this paper, in order to construct the analytic model of the equivariant differential K-theory, we extend the spectral section, the higher spectral flow and the eta form to the equivariant case. Especially, we introduce the equivariant higher spectral flow for a family of even dimensional manifolds. Furthermore, we prove the anomaly formula and the functoriality of equivariant eta forms using the language of equivariant higher spectral flow, which is an analogue of the results in [11, 23] and using the techniques in [15, 24, 25, 26]. Note that the proof of the funtoriality of equivariant eta forms is highly nontrivial. It is a question proposed in [14].

Let $\pi : W \to B$ be a proper submersion of smooth closed manifolds with oriented fibers Z. Let $TZ = TW/B$ be the relative tangent bundle to the fibers Z with Riemannian metric g^{TZ} and $T^H W$ be a horizontal subbundle of TW, such that $TW = T^H W \oplus TZ$. Let $o \in o(TZ)$ be an orientation of TZ. Let ∇^{TZ} be the Euclidean connection on TZ defined in (1.7). We assume that TZ has a Spin^c structure. Let L_Z be the complex line bundle associated with the Spin^c structure of TZ with a Hermitian metric h^{L_Z} and a Hermitian connection ∇^{L_Z}. Let (E, h^E) be a $\mathbb{Z}_2$-graded Hermitian vector bundle with a Hermitian connection ∇^E. Let G be a compact Lie group which acts on W such that $\pi \circ g = g \circ \pi$ for any $g \in G$. We assume that the G-action preserves everything. We denote by the family of G-equivariant geometric data $\mathcal{F} = (W, L_Z, E, o, T^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E))$ an equivariant geometric family. Let $D(\mathcal{F})$ be the fiberwise Dirac operators of $\mathcal{F}$. Then we have the family index map $\operatorname{ind}(D(\mathcal{F})) \in K_G^*(B)$, where $* = 0$ or 1 corresponds to the dimensions of fibers Z are even or odd.

Let $\mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$) be the set of equivalent classes of isomorphic equivariant geometric families such that the dimension of all fibers are even (resp. odd). We denote by $\mathcal{F} \sim \mathcal{F}'$ if $\operatorname{ind}(D(\mathcal{F})) = \operatorname{ind}(D(\mathcal{F}'))$. The following proposition is proved in [14]. It could be regarded as a geometric version of the equivariant Atiyah-Jänich theorem.

Proposition 0.1. *We have the ring isomorphism*

$$(0.1) \qquad\qquad \mathrm{F}_G^*(B)/\sim \simeq K_G^*(B).$$

Let D be a family of 1st-order pseudodifferential operator on $\mathcal{F}$ along the fibers, which is self-adjoint, fiberwisely elliptic and commutes with the G-action and the principal symbol of which is the same as that of $D(\mathcal{F})$. Furthermore, if $\mathcal{F}$ is even (resp. odd), we assume that D anti-commutes (resp. commutes) with the $\mathbb{Z}_2$-grading. As in [16], we call such D is an equivariant B-family on $\mathcal{F}$ (see Definition 2.1). If $\operatorname{ind}(D) = 0 \in K_G^*(B)$ and at least one component of the fiber has nonzero dimension, there exists an equivariant spectral section P and a smooth operator A_P associated with P, such that $D + A_P$ is an invertible equivariant B-family (see Proposition 2.3). Let P, Q be equivariant spectral sections, we could define the difference $[P - Q] \in K_G^*(B)$ (see (2.5) and (2.11)).

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^1(B)$ (resp. $\mathrm{F}_G^0(B)$) which have the same topological structure, that is, the only differences between them are metrics and connections. Let D_0, D_1 be two equivariant

B-families on $\mathcal{F}$, $\mathcal{F}'$ respectively. Let Q_0, Q_1 be equivariant spectral sections with respect to D_0, D_1 respectively. We define the equivariant higher spectral flow $\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) to be an element in $K_G^0(B)$ (resp. $K_G^1(B)$) in Definition 2.5 and 2.6. Note that when $\mathcal{F}$ is odd, it is the direct extension of higher spectral flow in [16]; when $\mathcal{F}$ is even, it is defined by adding an additional dimension.

Moreover, besides the equivariant geometric family, we could also represent the element of equivariant K-group as equivariant higher spectral flow.

Proposition 0.2. *For any $x \in K_G^0(B)$ (resp. $K_G^1(B)$), there exists $\mathcal{F} \in \mathrm{F}_G^1(B)$ (resp. $\mathrm{F}_G^0(B)$) and equivariant spectral sections P, Q with respect to $D(\mathcal{F})$, such that $[P - Q] = x$.*

From this point of view, the equivariant higher spectral flow here is the same as the term $\mathrm{ind}((\mathcal{E} \times I)_{bt})$ in [14, 2.5.8], which uses the KK-theory. This enable us to replace the techniques of KK-theory in [14] by that of equivariant higher spectral flow, which is purely analytic.

Let D be an equivariant B-family on $\mathcal{F}$. A perturbation operator with respect to D is a family of bounded pseudodifferential operators A such that $D + A$ is an invertible equivariant B-family on $\mathcal{F}$, which is a generalization of A_P.

Note that if at least one component of the fibers of $\mathcal{F}$ has nonzero dimension, a perturbation operator exists with respect to D if and only if $\mathrm{ind}(D(\mathcal{F})) = 0 \in K_G^*(B)$.

If **the G-action on B is trivial**, for any $g \in G$, we can define an equivariant eta form $\tilde{\eta}_g(\mathcal{F}, A)$ with respect to a perturbation operator A in Definition 2.11. If the equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ have the same topological structure, we prove the anomaly formula as follows.

Theorem 0.3. *Assume that the G-action on B is trivial. Let $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}_G^*(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be the APS projections (see Proposition 2.3) with respect to $D(\mathcal{F}) + A$, $D(\mathcal{F}') + A'$ respectively. For any $g \in G$, modulo exact forms, we have*

$$(0.2) \quad \tilde{\eta}_g(\mathcal{F}', A') - \tilde{\eta}_g(\mathcal{F}, A) = \int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge \mathrm{ch}_g(E, \nabla^E)$$

$$+ \int_{Z^g} \mathrm{Td}_g(\nabla'^{TY}, \nabla'^{L_Y}) \wedge \widetilde{\mathrm{ch}}_g(\nabla^E, \nabla'^E) + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F}) + A, P), (D(\mathcal{F}') + A', P')\}\right),$$

where Z^g is the fixed point set of g on the fibers Z and the characteristic forms and Chern-Simons forms are defined in Section 2.

Note that when $\mathcal{F}$, $\mathcal{F}' \in \mathrm{F}_G^0(B)$, the proof of the anomaly formula uses a special case of functoriality of equivariant eta forms which is highly nontrivial.

Let $\pi : V \to B$ be an equivariant proper submersion with closed oriented equivariant Spin^c fibers Y. We assume that V is closed and G acts trivially on B. Then an equivariant geometric family $\mathcal{F}_X$ over V induces an equivariant geometric family $\mathcal{F}_Z$ over B (see (1.25)). For any $g \in G$, let Y^g and Z^g be the fixed point sets of g on the fibers Y and Z respectively. We obtain the functoriality of equivariant eta forms.

Theorem 0.4. *Let A_Z and A_X be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then for $T \geq 1$ large enough and any $g \in G$, modulo exact forms, we have*

$$(0.3) \quad \widetilde{\eta}_g(\mathcal{F}_Z, A_Z) = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \widetilde{\eta}_g(\mathcal{F}_X, A_X)$$
$$- \int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z}) \wedge \mathrm{ch}_g(E, \nabla^E)$$
$$+ \mathrm{ch}_g(\mathrm{sf}_G\{(D(\mathcal{F}_Z) + A_Z, P), (D(\mathcal{F}_{Z,T}) + 1\widehat{\otimes}TA_X, P')\}),$$

where $\mathcal{F}_{Z,T}$ is the equivariant geometric family defined in (2.68), $\nabla^{TY,TX}$ is defined in (2.69) and P, P' are the associated APS projections respectively.

In the last section, inspired by [12, 14, 30], we use the results above to define the equivariant differential K-theory for the closed manifolds with the almost free action and study the properties of it.

Essential to our definition is that when the group action is almost free, $K_G^*(B) \otimes \mathbb{R}$ is isomorphic to the delocalized cohomology $H_{deloc,G}^*(B, \mathbb{R})$ defined in (3.7), which is the cohomology of complex $(\Omega_{deloc,G}^*(B, \mathbb{R}), d)$ of differential forms on the disjoint union of the fixed point set of a representative element in the conjugacy classes. Furthermore, we could define $\widetilde{\eta}_G(\mathcal{F}, A) \in \Omega_{deloc,G}^*(B, \mathbb{R})/\mathrm{Im}d$ when the group action is almost free on B.

A cycle for an equivariant differential K-theory class over B is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F} \in \mathrm{F}_G^*(B)$ and $\rho \in \Omega_{deloc,G}^*(B, \mathbb{R})/\mathrm{Im}\, d$. The cycle $(\mathcal{F}, \rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega_{deloc,G}^{\mathrm{odd}}(B, \mathbb{R})/\mathrm{Im}\, d$ (resp. $\rho \in \Omega_{deloc,G}^{\mathrm{even}}(B, \mathbb{R})/\mathrm{Im}\, d$). Two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho = \rho'$. Let $\widehat{\mathrm{IC}}_G^0(B)$ (resp. $\widehat{\mathrm{IC}}_G^1(B)$) denote the set of isomorphism classes of even (resp. odd) cycles over B. Let $\mathcal{F}^{\mathrm{op}}$ be the equivariant geometric family reversing the $\mathbb{Z}_2$-grading of E in $\mathcal{F}$, which implies that $\mathrm{ind}(D(\mathcal{F}^{\mathrm{op}})) = -\mathrm{ind}(D(\mathcal{F}))$. We call two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ paired if

$$(0.4) \qquad\qquad \mathrm{ind}(D(\mathcal{F})) = \mathrm{ind}(D(\mathcal{F}')),$$

and there exists a perturbation operator A with respect to $D(\mathcal{F} + \mathcal{F}'^{\mathrm{op}})$ such that

$$(0.5) \qquad\qquad \rho - \rho' = \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A).$$

Let $\sim$ denote the equivalence relation generated by the relation "paired".

Definition 0.5. The equivariant differential K-theory $\widehat{K}_G^0(B)$ (resp. $\widehat{K}_G^1(B)$) is the group completion of the abelian semigroup $\widehat{\mathrm{IC}}_G^0(B)/\sim$ (resp. $\widehat{\mathrm{IC}}_G^1(B)/\sim$).

Let $\pi_Y : V \to B$ be an equivariant submersion of closed smooth G-manifolds with closed Spin^c fiber Y. We assume that the G-action on B is almost free. Thus, so is the action on V. As in [12], we define the equivariant differential K-orientation with respect to π_Y in Definition 3.7 and the map $\widehat{\pi}_Y! : \widehat{\mathrm{IC}}_G^*(V) \to \widehat{\mathrm{IC}}_G^*(B)$ in (3.22). Then we prove that

Theorem 0.6. *The map $\widehat{\pi}_Y! : \widehat{K}_G^*(V) \to \widehat{K}_G^*(B)$ is well-defined.*

By Theorem 0.3 and 0.4, in Section 3, we also prove that our model is a ring valued functor with the usual properties of a differential extensions of a cohomology. Finally, we explain that this model could be naturally regarded as a model of differential K-theory for orbifolds.

Note that there is no adiabatic limit in Theorem 0.4. So in non-equivariant case, our proofs of these properties simplify that in [12].

This paper is organized as follows.

In Section 1, we give a geometric description of equivariant K-theory. In Section 2, we extend the Melrose-Piazza spectral section to the equivariant case, introduce the equivariant Dai-Zhang higher spectral flow for arbitrary dimensional fibers and use them to obtain the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms. In Section 3, we construct an analytic model for equivariant differential K-theory and prove some properties.

To simplify the notations, we use the Einstein summation convention in this paper.

In the whole paper, we use the superconnection formalism of Quillen [32]. If A is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in A$, then we will note $[a, b]$ as the supercommutator of a, b. If B is another $\mathbb{Z}_2$-graded algebra, we will note $A\widehat{\otimes}B$ as the $\mathbb{Z}_2$-graded tensor product.

If $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded space, we denote by

$$(0.6) \qquad \mathrm{Tr}_s[A] = \mathrm{Tr}\,|_{E_+}[A] - \mathrm{Tr}\,|_{E_-}[A].$$

For a fiber bundle $\pi : W \to B$, we will often use the integration of the differential forms along the fibers Z in this paper. Since the fibers may be odd dimensional, we must make precise our sign conventions. If α is a differential form on W which in local coordinates is given by

$$(0.7) \qquad \alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \wedge \beta(x)dx^1 \wedge \cdots \wedge dx^n,$$

we set

$$(0.8) \qquad \int_Z \alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \int_Z \beta(x)dx^1 \wedge \cdots \wedge dx^n.$$

1. Equivariant K-theory

In this section, we give a geometric description of equivariant K-theory for compact Lie groups, which could be regarded as an analogue of de Rham theory, and define the push-forward map of it. The setting in this section is the equivariant extension of those in [12].

1.1. Clifford algebra.
Let V^n be an oriented Euclidean space, such that $\dim V^n = n$, with orthonormal basis $\{e_i\}_{1 \le i \le n}$. Let $C(V^n)$ be the complex Clifford algebra of V^n defined by the relations

$$(1.1) \qquad e_i e_j + e_j e_i = -2\delta_{ij}.$$

To avoid ambiguity, we denote by $c(e_i)$ the element of $C(E)$ corresponding to e_i. We consider the group Spin_n^c as a multiplicative subgroup of the group of units of $C(E)$. For the definition and the properties of the group Spin_n^c, see [21, Appendix D].

For $n = 2k$, even, up to isomorphism, $C(V^n)$ has a unique irreducible module, $\mathcal{S}_n$, which has dimension 2^k and is $\mathbb{Z}_2$-graded. Let τ_V be the $\mathbb{Z}_2$-grading of V. If $n = 2k - 1$ is odd, $C(V^n)$ has two inequivalent irreducible modules, each of dimension 2^{k-1}. However, they are equivalent when restricted to Spin_n^c. We still denote it by $\mathcal{S}_n$.

Let W^m be another real inner product space with orthonormal basis $\{f_p\}$. Then as Clifford algebras,

$$(1.2) \qquad C(W^m \oplus V^n) \simeq C(W^m)\widehat{\otimes}C(V^n).$$

Let $\mathcal{S}_W$ and $\mathcal{S}_V$ be the spinors respectively.

If either m or n is even, we simply assume m is even, the spinor $\mathcal{S}_{W\oplus V} = \mathcal{S}_W\widehat{\otimes}\mathcal{S}_V$ is $\mathcal{S}_W \otimes \mathcal{S}_V$ with Clifford action $c(f_p)\widehat{\otimes}1 := c(f_p) \otimes 1$, $1\widehat{\otimes}c(e_i) := \tau_W \otimes c(e_i)$.

If m and n are all odd, the spinor $\mathcal{S}_{W\oplus V} = \mathcal{S}_W\widehat{\otimes}\mathcal{S}_V$ is $\mathcal{S}_W \otimes \mathcal{S}_V \otimes \mathbb{C}^2$. Let

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

The Clifford actions are defined by $c(f_p)\widehat{\otimes}1 := c(f_p) \otimes 1 \otimes \Gamma_1$, $1\widehat{\otimes}c(e_i) := 1 \otimes c(e_i) \otimes \Gamma_2$. The $\mathbb{Z}_2$-grading of $\mathcal{S}_{W \oplus V}$ is

$$\tau_{W \oplus V} := \mathrm{Id}_{\mathcal{S}_W} \otimes \mathrm{Id}_{\mathcal{S}_V} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1.2. **Equivariant geometric family.** In this subsection, we introduce the equivariant geometric family.

Let $\pi : W \to B$ be a smooth proper submersion of smooth closed manifolds with closed fibers Z (maybe non-connected). Let $B = \sqcup_i B_i$ be a finite disjoint union of closed connected manifolds. Let W_i be the restriction of W on B_i. Let $W_i = \sqcup_j W_{ij}$ be a finite disjoint union of closed connected manifolds. Let Z_{ij} be the fibers of the submersions restricted on W_{ij}. We note here that the dimension of Z_{ij} might be zero. In the sequel, we will often omit the subscript i, j.

Let $TZ = TW/B$ be the relative tangent bundle to the fibers Z. We assume that TZ is orientable with an orientation $o \in o(TZ)$. Let $T_\pi^H W$ be a horizontal subbundle of TW such that

$$(1.3) \qquad\qquad\qquad\qquad TW = T_\pi^H W \oplus TZ.$$

The splitting (1.3) gives an identification

$$(1.4) \qquad\qquad\qquad\qquad T_\pi^H W \cong \pi^* TB.$$

If there is no ambiguity, we will omit the subscript π in $T_\pi^H W$. Let P^{TZ} be the projection

$$(1.5) \qquad\qquad\qquad P^{TZ} : TW = T^H W \oplus TZ \to TZ.$$

Let g^{TZ}, g^{TB} be Riemannian metrics on TZ, TB. We equip $TW = T^H W \oplus TZ$ with the Riemannian metric

$$(1.6) \qquad\qquad\qquad\qquad g^{TW} = \pi^* g^{TB} \oplus g^{TZ}.$$

Let ∇^{TW}, ∇^{TB} be the Levi-Civita connections on (W, g^{TW}), (B, g^{TB}). Set

$$(1.7) \qquad\qquad\qquad\qquad \nabla^{TZ} = P^{TZ} \nabla^{TW} P^{TZ}.$$

Then ∇^{TZ} is a Euclidean connection on TZ. By [5, Theorem 1.9], we know that ∇^{TZ} only depends on $(T^H W, g^{TZ})$.

Let $C(TZ)$ be the Clifford algebra bundle of (TZ, g^{TZ}), whose fiber at $x \in W$ is the Clifford algebra $C(T_x Z)$ of the Euclidean space $(T_x Z, g^{T_x Z})$.

We make the assumption that TZ has a Spin^c structure. Then there exists a complex line bundle L_Z over W such that $\omega_2(TZ) = c_1(L_Z) \mod (2)$. Let $\mathcal{S}(TZ, L_Z)$ be the fundamental complex spinor bundle for (TZ, L_Z), which has a smooth action of $C(TZ)$ (cf. [21, Appendix D.9]). Locally, the spinor $\mathcal{S}(TZ, L_Z)$ may be written as

$$(1.8) \qquad\qquad\qquad \mathcal{S}(TZ, L_Z) = \mathcal{S}(TZ) \otimes L_Z^{1/2},$$

where $\mathcal{S}(TZ)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on TZ, and $L_Z^{1/2}$ is the (possibly non-existent) square root of L_Z. Let h^{L_Z} be the Hermitian metric on L_Z and ∇^{L_Z} be the Hermitian connection on (L_Z, h^{L_Z}). Let $h^{\mathcal{S}_Z}$ be the Hermitian metric on $\mathcal{S}(TZ, L_Z)$ induced by g^{TZ} and h^{L_Z} and $\nabla^{\mathcal{S}_Z}$ be the connection on $\mathcal{S}(TZ, L_Z)$ induced by ∇^{TZ}

and ∇^{L_Z} from (1.8). Then $\nabla^{\mathcal{S}_Z}$ is a Hermitian connection on $(\mathcal{S}(TZ, L_Z), h^{\mathcal{S}_Z})$. Moreover, it is a Clifford connection associated with ∇^{TZ}, i.e., for any $U \in TW$, $V \in \mathscr{C}^\infty(W, TZ)$,

$$(1.9) \qquad \left[\nabla_U^{\mathcal{S}_Z}, c(V) \right] = c\left(\nabla_U^{TZ} V \right).$$

If $n = \dim Z$ is even, the spinor $\mathcal{S}(TZ, L_Z)$ is $\mathbb{Z}_2$-graded and the action of TZ exchanges the $\mathbb{Z}_2$-grading. In the following, we often simply denote the spinor by $\mathcal{S}_Z$.

Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle over W with Hermitian metric h^E, for which E_+ and E_- are orthogonal, and ∇^E a Hermitian connection on (E, h^E). Set

$$(1.10) \qquad \nabla^{\mathcal{S}_Z \widehat{\otimes} E} := \nabla^{\mathcal{S}_Z} \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla^E.$$

Then $\nabla^{\mathcal{S}_Z \widehat{\otimes} E}$ is a Hermitian connection on $(\mathcal{S}_Z \widehat{\otimes} E, h^{\mathcal{S}_Z} \otimes h^E)$.
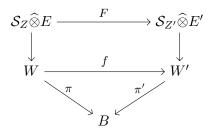
Let G be a compact Lie group which acts on W such that for any $g \in G$, $\pi \circ g = g \circ \pi$. We assume that the action of G preserves the splitting (1.3), the Spin^c structure of TZ and g^{TZ}, h^{L_Z}, ∇^{L_Z} are G-invariant. We assume that E is a G-equivariant $\mathbb{Z}_2$-graded complex vector bundle and h^E, ∇^E are G-invariant.

Definition 1.1. An equivariant geometric family $\mathcal{F}$ over B is a family of G-equivariant geometric data

$$(1.11) \qquad \mathcal{F} = (W, L_Z, E, o, T^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E)$$

described as above. We call the equivariant geometric family $\mathcal{F}$ is even (resp. odd) if for any connected component of fibers, the dimension of it is even (resp. odd).

Definition 1.2. Let $\mathcal{F}$ and $\mathcal{F}'$ be two equivariant geometric families over B. An isomorphism $\mathcal{F} \overset{\sim}{\to} \mathcal{F}'$ consists of the following data:

$$
\begin{array}{ccc}
\mathcal{S}_Z \widehat{\otimes} E & \xrightarrow{\quad F \quad} & \mathcal{S}_{Z'} \widehat{\otimes} E' \\
\downarrow & & \downarrow \\
W & \xrightarrow{\quad f \quad} & W' \\
& \searrow{\scriptstyle \pi} \quad \swarrow{\scriptstyle \pi'} & \\
& B &
\end{array}
$$

where

1. f is a diffeomorphism commuting with the G-action such that $\pi' \circ f = \pi$,
2. f preserves the G-invariant orientation and Spin^c structure of the relative tangent bundle,
3. F is an equvariant bundle isomorphism over f preserving the grading of the vector bundle and the spinor (if it has),
4. f preserves the horizontal subbundle and the vertical metric,
5. F preserves the metrics and the connections.

If only the first three conditions hold, we say that $\mathcal{F}$ and $\mathcal{F}'$ have **the same topological structure**.

Let $\mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$) be the set of equivalent classes of even (resp. odd) equivariant geometric families.

For two equivariant geometric families $\mathcal{F}, \mathcal{F}'$, we can form their sum $\mathcal{F} + \mathcal{F}'$ over B as a new equivariant geometric family. The underlying fibration with closed fibers of the sum is $\pi \sqcup \pi' : W \sqcup W' \to B$, where $\sqcup$ is the disjoint union. The remaining structures of $\mathcal{F} + \mathcal{F}'$ are

induced in the obvious way. Let $F_G^*(B) = F_G^0(B) \oplus F_G^1(B)$ be the set of equivalent classes of equivariant geometric families. It is a additive semigroup.

Let $f : B \times B' \to B$ be the projection onto the first part. For any $\mathcal{F} \in \mathrm{F}_G^*(B)$, we could construct the pullback $f^*\mathcal{F}$ in a natural way.

Definition 1.3. The opposite family $\mathcal{F}^{\mathrm{op}}$ of an equivariant geometric family $\mathcal{F}$ is obtained by reversing the $\mathbb{Z}_2$-grading of E.

1.3. **Equivariant K-Theory.** In this subsection, we give some examples of the equivariant geometric families and get a geometric description of equivariant K-theory.

Recall that [33] the equivariant K-group $K_G^0(B)$ is the Grothendieck group of the equivalent classes of equivariant vector bundles over B. Let $i : B \to B \times S^1$ be a G-equivariant inclusion map. It is well known that if the G-action on S^1 is trivial,

$$(1.12) \qquad K_G^1(B) \simeq \ker\left(i^* : K_G^0(B \times S^1) \to K_G^0(B)\right).$$

Let $\{e_i\}$ be a local orthonormal frame of TZ. Let $D(\mathcal{F})$ be the fiberwise Dirac operator

$$(1.13) \qquad D(\mathcal{F}) = c(e_i)\nabla_{e_i}^{\mathcal{S}_Z \widehat{\otimes} E}$$

associated with $\mathcal{F} \in \mathrm{F}_G^*(B)$. Then the G-action commutes with $D(\mathcal{F})$. Thus the classical construction of Atiyah-Singer assigns to this family its equivariant (analytic) index $\mathrm{ind}(D(\mathcal{F})) \in K_G^0(B)$ (resp. $K_G^1(B)$) when $\mathcal{F} \in \mathrm{F}_G^0(B)$ (resp. $\mathrm{F}_G^1(B)$) [2, 3].

Let $K_G^*(B) = K_G^0(B) \oplus K_G^1(B)$. Since

$$(1.14) \qquad \mathrm{ind}(D(\mathcal{F} + \mathcal{F}')) = \mathrm{ind}(D(\mathcal{F})) + \mathrm{ind}(D(\mathcal{F}')) \in K_G^*(B),$$

the equivariant index defines a semigroup homomorphism

$$(1.15) \qquad \begin{aligned} \mathrm{ind} : \mathrm{F}_G^*(B) &\to K_G^*(B), \\ \mathcal{F} &\mapsto \mathrm{ind}(D(\mathcal{F})). \end{aligned}$$

Furthermore, for $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$, we can form their cup product $\mathcal{F} \cup \mathcal{F}'$ over B. The underlying fibration with closed fibers of the cup product is $\pi \cup \pi' : W \times_B W' \to B$. The vector bundle now is $(\mathcal{S}_Z \widehat{\otimes} \mathcal{S}_{Z'}) \widehat{\otimes} E \widehat{\otimes} E'$. Here the spinor $\mathcal{S}_Z \widehat{\otimes} \mathcal{S}_{Z'}$ is constructed as in Section 1.1. The remaining structures of $\mathcal{F} \cup \mathcal{F}'$ are induced in the obvious way. It is well-known that

$$(1.16) \qquad \mathrm{ind}(D(\mathcal{F} \cup \mathcal{F}')) = \mathrm{ind}(D(\mathcal{F})) \cup \mathrm{ind}(D(\mathcal{F}')) \in K_G^*(B).$$

Example 1.4. a) Let (E, h^E) be an equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle over B with a G-invariant Hermitian connection ∇^E. Then (E, h^E, ∇^E) can be regarded as an even equivariant geometric family $\mathcal{F}$ for $Z = pt$. In this case, $D(\mathcal{F}) = 0$ and $\mathrm{ind}(D(\mathcal{F})) = [E_+] - [E_-] \in K_G^0(B)$.

b) We choose $\mathcal{F}$ as in a). Let $W' = B \times S^2$ and γ be the canonical nontrivial complex line bundle on $S^2 = \mathbb{C}P^1$. Then γ can be naturally extended on W'. Thus (W', γ) with canonical metrics, connections and the standard orientation $o \in o(S^2)$ forms an even geometric family $\mathcal{F}'$ over B. In this case, since $\mathrm{ind}(D_{S^2}^\gamma) = 1$, where $D_{S^2}^\gamma$ is the Dirac operator on S^2 associated with γ, from (1.16), we could get that $\mathrm{ind}(D(\mathcal{F} \cup \mathcal{F}')) = \mathrm{ind}(D(\mathcal{F})) \in K_G^0(B)$.

c) Let $B = S^1 = \mathbb{R}/\mathbb{Z}$, $W = S^1 \times S^1$ and π be the natural projection on the first part. We consider the Hermitian line bundle (L, h^L) which is obtained by identifying $(\theta = 0, t, v) \in [0, 1] \times S^1 \times \mathbb{C}$ and $(\theta = 1, t, \exp(-2\pi t\sqrt{-1})v) \in [0, 1] \times S^1 \times \mathbb{C}$. The line bundle L is naturally endowed a Hermitian connection $\nabla^L = d + 2\pi(\theta - 1/2)\sqrt{-1}dt$. We choose the $\mathbb{Z}_2$-grading of L such that $L_+ = L$ and $L_- = 0$. Then we get an odd geometric family $\mathcal{F}^L$ after choosing the

natural geometric data $(o^L \in o(TS^1), T_\pi^H(S^1 \times S^1), g^{TS^1})$. In fact, $\mathrm{ind}(D(\mathcal{F}^L))$ is a generator of $K^1(S^1) \simeq \mathbb{Z}$.

d) Let $\mathcal{F} \in \mathrm{F}_G^*(B)$. Let p_1 and p_2 be the natural projection onto the first and second part of $B \times S^1$ respectively. We choose $\mathcal{F}^L$ as in c). Then $p_1^*\mathcal{F} \cup p_2^*\mathcal{F}^L$ is an equivariant geometric family over $B \times S^1$. From the proof of [8, Theorem 2.10], for $\mathcal{F} \in \mathrm{F}_G^1(B)$, there exists inclusion $i : B \to B \times S^1$ such that $i^* \mathrm{ind}(D(p_1^*\mathcal{F} \cup p_2^*\mathcal{F}^L)) = 0$. Moreover, as a element of $K_G^1(B)$ in the sense of (1.12), by [29, Proposition 6], we have

$$(1.17) \qquad \mathrm{ind}(D(\mathcal{F})) = \mathrm{ind}(D(p_1^*\mathcal{F} \cup p_2^*\mathcal{F}^L)).$$

In fact, we could prove (1.16) by (1.17). This example is essential in our construction of higher spectral flow for even case.

We denote by $\mathcal{F} \sim \mathcal{F}'$ if $\mathrm{ind}(D(\mathcal{F})) = \mathrm{ind}(D(\mathcal{F}'))$. It is an equivalent relation and compatible with the semigroup structure. So $\mathrm{F}_G^*(B)/\sim$ is a semigroup and the map

$$(1.18) \qquad \mathrm{ind} : \mathrm{F}_G^*(B)/\sim \longrightarrow K_G^*(B)$$

is an injective semigroup homomorphism.

By Definition 1.3, we have

$$(1.19) \qquad \mathrm{ind}(D(\mathcal{F}^{\mathrm{op}})) = -\mathrm{ind}(D(\mathcal{F})).$$

After defining $-\mathcal{F} := \mathcal{F}^{\mathrm{op}}$, the semigroup $\mathrm{F}_G^*(B)/\sim$ can be regarded as an abelian group. So, by (1.19), the equivariant index map in (1.18) is a group homomorphism. Furthermore, by (1.16), the equivariant index map in (1.18) is a ring homomorphism. The following proposition is proved in [14]. We sketch the proof for completion.

Proposition 1.5. *The equivariant index map* ind *in (1.18) is surjective. In other words, we have the $\mathbb{Z}_2$-graded ring isomorphism*

$$(1.20) \qquad \mathrm{F}_G^*(B)/\sim \, \simeq K_G^*(B).$$

Proof. When $* = 0$, we can get the proposition directly from Example 1.4 a) or b).

When $* = 1$, by [31, Theorem 2.8.8] (see also [14, Section 2.5.8]), for any $[F] \in K_G^1(B)$, there exists a finite dimensional G-representation V, such that $[F]$ can be represented as a G-invariant unitary element of $\mathrm{End}(V) \otimes \mathscr{C}^0(B, \mathbb{C})$. For $(b, t, v) \in B \times [0, 1] \times V$, the relation $(b, 0, v) \sim (b, 1, F(b)v)$ makes a G-equivariant vector bundle W over $B \times S^1$. Let $U = B \times S^1 \times V$ be the G-equivariant trivial bundle. Then $[W] - [U] \in K_G^0(B \times S^1)$ corresponds to $[F] \in K_G^1(B)$ under the isomorphism (1.12). Moreover, after taking the natural geometric data, we get an odd equivariant geometric family $\mathcal{F}$ over B with fiber S^1 and $\mathbb{Z}_2$-graded equivariant vector bundle $W \oplus U$. As in [10, Section 4.2.3], we can get $\mathrm{ind}(D(\mathcal{F})) = [F]$.

The proof of Proposition 1.5 is complete. $\qquad \square$

Remark 1.6. Note that if we replace the Spin^c condition of the geometric family by the general Clifford module condition (which is the setting in [12]) or the Spin condition, Proposition 1.5 also holds. Since we don't use the language of Clifford modules here, our definition of $\mathcal{F}^{\mathrm{op}}$ in Definition 1.3 is simper than that in [12]. In fact, in the sense of (1.20), they are the same.

1.4. **Push forward map.** In this subsection, we define the push-forward map of equivariant K-theory using the equivariant geometric families.

Let $\pi_Y : V \to B$ be a G-equivariant submersion of smooth closed manifolds with closed Spin^c fibers Y.

Definition 1.7. An equivariant K-orientation of π_Y is an equivariant Spin^c structure of TY. We call the equivariant K-orientation is even (resp. odd) if for any connected component of fibers Y, the dimension of it is even (resp. odd). Let $\mathcal{O}_G^0(\pi_Y)$ (resp. $\mathcal{O}_G^1(\pi_Y)$) be the set of even (resp. odd) equivariant K-orientations.

If π_Y has an equivariant K-orientation $\mathcal{O}_Y \in \mathcal{O}_G^*(\pi_Y)$, we will use Proposition 1.5 to define the push-forward map of equivariant K-groups $\pi_Y! : K_G^{*'}(V) \to K_G^{*'+*}(B)$ as follows.

Let $\pi_X : W \to V$ be a G-equivariant submersion of smooth closed manifolds with closed Spin^c fibers X and

(1.21) $$\mathcal{F}_X = (W, L_X, E, o_X, T_{\pi_X}^H W, g^{TX}, h^{L_X}, \nabla^{L_X}, h^E, \nabla^E)$$

be a G-equivariant geometric family in $F_G^{*'}(V)$. Then $\pi_Z := \pi_Y \circ \pi_X : W \to B$ is a smooth submersion with closed Spin^c fibers Z. Then we have the diagram of smooth fibrations:

$$
\begin{array}{ccc}
X \longrightarrow Z \longrightarrow W & & \\
\Big\downarrow \quad \pi_X\Big\downarrow \quad \searrow^{\pi_Z} & & \\
Y \longrightarrow V \xrightarrow{\;\pi_Y\;} B. &
\end{array}
$$

Set $T_{\pi_X}^H Z := T_{\pi_X}^H W \cap TZ$. Then we have the splitting of smooth vector bundles over W,

(1.22) $$TZ = T_{\pi_X}^H Z \oplus TX,$$

and

(1.23) $$T_{\pi_X}^H Z \cong \pi_X^* TY.$$

Then we can get a G-invariant orientation $o_Y \cup o_X \in o(TZ)$. Set

(1.24) $$L_Z := \pi_X^* L_Y \otimes L_X.$$

Take the geometric data $(T_{\pi_Z}^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z})$ of π_Z such that $T_{\pi_Z}^H W \subset T_{\pi_X}^H W$, $g^{TZ} = \pi_X^* g^{TY} \oplus g^{TX}$, $h^{L_Z} = \pi_X^* h^{L_Y} \otimes h^{L_X}$ and $\nabla^{L_Z} = \pi_X^* \nabla^{L_Y} \otimes 1 + 1 \otimes \nabla^{L_X}$. We get a new equivariant geometric family over B

(1.25) $$\mathcal{F}_Z := (W, L_Z, E, o_Y \cup o_X, T_{\pi_Z}^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E).$$

In the sense of (1.20), we will prove that

Theorem 1.8. *For equivariant K-orientation $\mathcal{O}_Y \in \mathcal{O}_G^*(\pi_Y)$ fixed, the push-forward map*

(1.26)
$$\pi_Y! : K_G^{*'}(V) \to K_G^{*'+*}(B),$$
$$[\mathcal{F}_X] \quad \mapsto [\mathcal{F}_Z]$$

is a well-defined group homomorphism and independent of the geometric data $(T_{\pi_Z}^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z})$ of π_Z.

Proof. From the definition and the homotopy invariance property of the equivariant family index, the push-forward map in (1.26) is a group homomorphism and independent of the geometric data $(T_{\pi_Z}^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z})$ of π_Z.

We will prove the well-defined property in Lemma 2.15 later. $\qquad\square$

Let $\pi_U : B \to S$ be a G-equivariant submersion of smooth closed manifolds with closed Spin^c fibers U and an equivariant K-orientation $\mathcal{O}_U$. Then $\pi_A := \pi_U \circ \pi_Y : V \to S$ is a G-equivariant submersion with an equivariant K-orientation constructed by $\mathcal{O}_Y$ and $\mathcal{O}_U$. From the construction of the push-forward map and Theorem 1.8, the following theorem is obvious.

Theorem 1.9. *We have the equality of homomorphisms* $K_G^*(V) \to K_G^*(S)$

$$\pi_A! = \pi_U! \circ \pi_Y!. \tag{1.27}$$

2. Equivariant higher spectral flows and equivariant eta forms

In this section, we extend the Melrose-Piazza spectral section to the equivariant case, introduce the equivariant version of Dai-Zhang higher spectral flow for arbitrary dimensional fibers and use them to obtain the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms.

2.1. Equivariant spectral section and equivariant higher spectral flow.

In this subsection, we define the equivariant version of spectral section and higher spectral flow and give a model of equivariant K-theory using the equivariant higher spectral flow.

Definition 2.1. (compare with [16, Definition 1.6]) Let $\mathcal{F} \in \mathrm{F}_G^*(B)$. We call an operator D is an equivariant B-family on $\mathcal{F}$ if it is a 1st-order pseudodifferential operator on $\mathcal{F}$ along the fiber, which is self-adjoint, fiberwisely elliptic and commutes with the G-action, such that
  (a) its principal symbol is given by that of $D(\mathcal{F})$;
  (b) it preserves the $\mathbb{Z}_2$-grading of E when the fiber is odd dimensional;
  (c) it anti-commutes with the $\mathbb{Z}_2$-grading of $\mathcal{S}_Z \widehat{\otimes} E$ when the fiber is even dimensional.

Note that the fiberwise Dirac operator $D(\mathcal{F})$ is an equivariant B-family. In fact, for any equivariant B-family D, we have $\mathrm{ind}(D) = \mathrm{ind}(D(\mathcal{F})) \in K_G^*(B)$.

Definition 2.2. (compare with [28, Definition 1] and [29, Definition 1]) The equivariant Melrose-Piazza spectral section is a family of self-adjoint pseudodifferential projections P with respect to an equivariant B-family D, which commutes with the G-action, such that
  (a) for some smooth function $f : B \to \mathbb{R}$ (depending on P) and every $b \in B$,

$$D_b u = \lambda u \Longrightarrow \begin{cases} P_b u = u, & \text{if } \lambda > f(b); \\ P_b u = 0, & \text{if } \lambda < -f(b); \end{cases} \tag{2.1}$$

  (b) if $\mathcal{F}$ is odd, P commutes with the $\mathbb{Z}_2$-grading of E;
  (c) if $\mathcal{F}$ is even,

$$\tau P + P\tau = \tau, \tag{2.2}$$

where τ is the $\mathbb{Z}_2$-grading of $\mathcal{S}_Z \widehat{\otimes} E$.

The following proposition is the natural equivariant extension of the results in [28, 29].

Proposition 2.3. *Let* $\mathcal{F} \in \mathrm{F}_G^*(B)$ *and* D *be an equivariant B-family on* $\mathcal{F}$.
  (i) (compare with [28, Proposition 1] *and* [29, Proposition 2]*) If there exists an equivariant spectral section with respect to* D *on* $\mathcal{F} \in \mathrm{F}_G^0(B)$ *(resp.* $\mathrm{F}_G^1(B)$*), then* $\mathrm{ind}(D) = 0 \in K_G^0(B)$ *(resp.* $K_G^1(B)$*). Conversely, If* $\mathrm{ind}(D) = 0 \in K_G^0(B)$ *(resp.* $K_G^1(B)$*) and at least one component of the fibers has the nonzero dimension, there exists an equivariant spectral section with respect to* D.
  (ii) (compare with [28, Proposition 2]*) For* $\mathcal{F} \in \mathrm{F}_G^1(B)$*, given equivariant spectral sections* P, Q *with respect to* D*, there exists an equivariant spectral section* R *with respect to* D *such that* $PR = R$ *and* $QR = R$*. We say that* R *majors* P, Q.
  (iii) (compare with [28, Lemma 8] *and* [29, Lemma 1]*) If there is an equivariant spectral section* P *with respect to* D*, then there is a family of self-adjoint equivariant smoothing operators*

A_P *(when the dimension of the fibers are zero, it descends to an endmorphism) with range in a finite sum of eigenspaces of D such that $D + A_P$ is an invertible equivariant B-family and P is the Atiyah-Patodi-Singer projection (we often simply denoted by APS projection later) onto the positive part of the spectrum of $D + A_P$.*

Proof. The proofs of these properties are almost the same as those in [28, 29]. We sketch the proof here.

Case 1: Let $\mathcal{F} \in \mathrm{F}_G^1(B)$.

Let $Q = D/(1 + D^2)^{1/2}$. Then Q is bounded, commutes with the G-action and $\mathrm{ind}(Q) = \mathrm{ind}(D) \in K_G^1(B)$. Since the Hilbert bundle is trivial, we can regard Q as a continuous family of self-adjoint bounded operators $\{Q_b\}_{b \in B}$ on $L^2(Z, \mathcal{S}_Z \widehat{\otimes} E)$.

(i) Assume that there exists an equivariant spectral section P of D. Let $K = (1 + D^2)^{-1/2}$. Then Q is in the same G-homotopy classes of $P(Q + rK)P + (1 - P)(Q - rK)(1 - P)$ for $r > 0$. When r is large enough, for any $b \in B$, $P_b(Q_b + rK_b)P_b$ is positive and $(1 - P_b)(Q_b - rK_b)(1 - P_b)$ is negative. Since $\mathrm{ind}(Q)$ is G-homotopy invariant, we have $\mathrm{ind}(Q) = 0 \in K_G^1(B)$.

If $\mathrm{ind}(Q) = 0 \in K_G^1(B)$, then Q is G-homotopic to constant. The proof of [28, Proposition 1] provides a process to construct a spectral section. All the operators in the process commute with the G-action.

(ii) We consider the operators PDP on the range of P. By the definition of the spectral section, there exists $N > 0$, such that all but the first N eigenfunctions of PDP are eigenfunctions of D. When $M > 0$ is large enough, using the method in the construction of the equivariant spectral section in the proof of (i) (compare with [22, Proposition 3]), we could find an equivariant subbundle of the range of P contains the first N eigenfunctions and be contained in the span of the first M eigenfunctions. Let R be the orthogonal projection on the complement of the subbundle and extend to be zero on $I - P$. Then R is an equvariant spectral section such that $PR = R$. If the integer N is chosen large enough, then the projection R will have range contained in the intersection of the ranges of any two given spectral sections P and Q. So $QR = R$.

(iii) Let $\mathcal{P}_{\lambda \in [a_1, a_2], b}(D_b)$ be the span of the eigenfunctions of D_b corresponding to the eigenvalues $\lambda \in [a_1, a_2]$. Since B is compact, we can choose $s > 0$, such that P is an equivariant spectral section for $f(b) \equiv s$. By the proof of (ii), we can choose equivariant spectral sections Q, R, such that for any $b \in B$, $Q_b = 0$ on $\mathcal{P}_{\lambda \leq s, b}(D_b)$ and $R_b = I$ on $\mathcal{P}_{\lambda \geq -s, b}(D_b)$. Then the operator

$$(2.3) \qquad \widetilde{D} = QDQ + sPR(I - Q) + (I - R)D(I - R) - s(I - P)R(I - Q).$$

is an invertible equivariant B-family. We take $A_P = \widetilde{D} - D$.

Case 2: Let $\mathcal{F} \in \mathrm{F}_G^0(B)$.

Since D is odd, the equivariant K^1 index of the whole self-adjoint family D vanishes (see [29, Proposition 2]). Following the same process in the proof of (i) in the odd case, we could obtain that there exists an equivariant spectral section P' in the odd sense, which means that it is a equivariant spectral section without the condition (2.2).

(iii) By the proof of (ii) for the odd case, if at least one component of the fiber has the nonzero dimension, we could choose P' such that $P'DP'$ is positive on the range of P. Then the operator

$$(2.4) \qquad A_P = P - P' - \tau(P - P')\tau + P'DP' + \tau P'\tau D\tau P'\tau - D$$

satisfies all the conditions. If the dimensions of all the fibers are zero, we could take $A_P = P - \tau P\tau - D$.

(i) Assume that $\mathrm{ind}(D) = 0 \in K_G^0(B)$ and at least one component of the fiber has the nonzero dimension. As the proof of (ii) for the odd case, for $r > 0$ fixed, we can choose an equivariant spectral section P' in the odd sense such that $P' = 0$ on $\mathcal{P}_{\lambda \leq r}(D)$. From (2.2), we have $\tau P' \tau = 0$ on $\mathcal{P}_{\lambda \geq -r}(D)$. Let $N = \ker(P' + \tau P' \tau)$. Then N is a finite dimensional equivariant vector bundle over B. We split the vector bundle by $N = N_+ \oplus N_-$. Then $\mathrm{ind}(D) = [N_+] - [N_-] \in K_G^0(B)$. The assumption $\mathrm{ind}(D) = 0 \in K_G^0(B)$ implies that there exists a vector bundle U such that $N_+ \oplus U \simeq N_- \oplus U$.

We choose another equivariant spectral section P'' in the odd sense such that $\mathrm{Range}(P' - P'')$ is an equivariant vector bundle whose rank is large enough. Let $N' = \ker(P'' + \tau P'' \tau)$ and $N' = N'_+ \oplus N'_-$. Let $W_\pm$ be the vector bundles such that $N'_\pm = N_\pm \oplus W_\pm$. Then D_+ induces an isomorphism between W_+ and W_-. Since the rank of $W_\pm$ is large enough, there exist subbundles $U_+ \subset W_+$ and $U_- \subset W_-$ such that $U_+ \simeq U_- \simeq U$. So $N'_+ \simeq N'_-$ as vector bundles. Since $\mathrm{ind}(D) = 0 \in K_G^0(B)$, the isomorphism is G-equivariant. Let $\phi : N'_+ \to N'_-$ be an equivariant unitary bundle isomorphism. Using this to write operators on N' as 2×2 matrices

$$P_N = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

we obtain an equivariant spectral section $P = P'' + P_N$.

The other direction follows from (iii) easily.

The proof of Proposition 2.3 is complete.

$\square$

Remark 2.4. From the proof of Proposition 2.3 (i) in the even case, we note that for any $\mathcal{F} \in \mathrm{F}_G^*(B)$, even for the zero dimensional case, there exists an equivariant spectral section for $\mathcal{F} + \mathcal{F}^{\mathrm{op}}$.

If $\mathcal{F}$ is odd and R, P are two equivariant spectral sections with respect to an equivariant B-family D such that $PR = R$, then $\mathrm{coker}\{P_b R_b : \mathrm{Im}(R_b) \to \mathrm{Im}(P_b)\}_{b \in B}$ forms an equivariant vector bundle over B, denoted by $[P - R]$. Hence for any two equivariant spectral sections P, Q, the difference element $[P - Q]$ can be defined as an element in $K_G^0(B)$ as follows:

$$(2.5) \qquad [P - Q] := [P - R] - [Q - R] \in K_G^0(B),$$

where R is any equivariant spectral section as in Proposition 2.3 (ii) such that $PR = R$ and $QR = R$. We note that the class in (2.5) is independent of the choice of R.

From the definition in (2.5), we can obtain that if P_1 , P_2 , P_3 are equivariant spectral sections with respect to D, then

$$(2.6) \qquad [P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K_G^0(B).$$

Recall that in Definition 1.2, $\mathcal{F} \simeq \mathcal{F}'$ if they satisfy five conditions. If only the first three conditions hold, we say that $\mathcal{F}$ and $\mathcal{F}'$ have the same topological structure.

Note that a horizontal subbundle on W is simply a splitting of the exact sequence

$$(2.7) \qquad 0 \to TZ \to TW \to \pi^* TB \to 0.$$

As the space of the splitting map is affine, it follows that any pair of equivariant horizontal subbundles can be connected by a smooth path of equivariant horizontal distributions.

Assume that $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$ have the same topological structure. Let $r \in [0, 1]$ parametrize a smooth path $\{T_{\pi,r}^H W\}_{r \in [0,1]}$ such that $T_{\pi,0}^H W = T_\pi^H W$ and $T_{\pi,1}^H W = T_\pi'^H W$. Similarly, let g_r^{TZ}, $h_r^{L_Z}$ and h_r^E be the G-invariant metrics on TZ, L_Z and E, depending smoothly on $r \in [0,1]$, which coincide with g^{TZ}, h^{L_Z} and h^E at $r = 0$ and with g'^{TZ}, h'^{L_Z} and h'^E at $r = 1$. By the

same reason, we can choose G-invariant Hermitian connection $\nabla_r^{L_Z}$ and ∇_r^E on L_Z and E, such that $\nabla_0^E = \nabla^E$, $\nabla_1^E = \nabla'^E$, $\nabla_0^{L_Z} = \nabla^{L_Z}$, $\nabla_1^{L_Z} = \nabla'^{L_Z}$.

Let $\widetilde{B} = [0,1] \times B$ and $\mathrm{pr} : \widetilde{B} \to B$ be the projection. We consider the bundle $\widetilde{\pi} : \widetilde{W} := [0,1] \times W \to \widetilde{B}$ together with the canonical projection $\mathrm{Pr} : \widetilde{W} \to W$. Then $T_{\widetilde{\pi}}^H \widetilde{W}_{(r,\cdot)} = \mathbb{R} \times T_{\pi,r}^H W$ defines an equivariant horizontal subbundle of $T\widetilde{W}$, and $T\widetilde{Z} := \mathrm{Pr}^* TZ$, $\widetilde{L_Z} := \mathrm{Pr}^* L_Z$ and $\widetilde{E} := \mathrm{Pr}^* E$ are naturally equipped with metrics $g^{T\widetilde{Z}}$, $h^{\widetilde{L_Z}}$, $h^{\widetilde{E}}$ and connections $\nabla^{\widetilde{L_Z}}$, $\nabla^{\widetilde{E}}$. Then the fiberwise G-action can be naturally extended to $\widetilde{\pi} : \widetilde{W} \to \widetilde{B}$ such that G acts as identity on $[0,1]$ and $g^{T\widetilde{Z}}$, $h^{\widetilde{L_Z}}$, $h^{\widetilde{E}}$, $\nabla^{\widetilde{L_Z}}$, $\nabla^{\widetilde{E}}$ are G-invariant. Thus, we get two equivariant geometric families

$$(2.8) \qquad \mathcal{F}_r = (W, L_Z, E, o, T_{\pi,r}^H W, g_r^{TZ}, h_r^{L_Z}, \nabla_r^{L_Z}, h_r^E, \nabla_r^E)$$

and

$$(2.9) \qquad \widetilde{\mathcal{F}} = (\widetilde{W}, \widetilde{L_Z}, \widetilde{E}, o, T_{\widetilde{\pi}}^H \widetilde{W}, g^{T\widetilde{Z}}, h^{\widetilde{L_Z}}, \nabla^{\widetilde{L_Z}}, h^{\widetilde{E}}, \nabla^{\widetilde{E}})$$

such that $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_1 = \mathcal{F}'$.

Now we consider a continuous family of operators D_r on $\mathcal{F}_r$ for $r \in [0,1]$ such that for any $r \in [0,1]$, D_r is an equivariant B-family on $\mathcal{F}_r$. Assume that the index of $D(\mathcal{F})$ vanishes. Then the homotopy invariance of index implies that the indice of D_r vanish. Let Q_0, Q_1 be equivariant spectral sections with respect to D_0, D_1 respectively. If we consider the total family $\widetilde{D} = \{D_r\}$ parametrized by $B \times I$, then there is a total equivariant spectral section $\widetilde{P} = \{P_r\}$. Let P_r be the restriction of $\widetilde{P}$ over $B \times \{r\}$. Thus we have the natural equivariant extension of [16, Definition 1.5]

Definition 2.5. If $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^1(B)$, the equivariant Dai-Zhang higher spectral flow $\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K_G^0(B)$ defined by

$$(2.10) \qquad \mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\} = [Q_0 - P_0] - [Q_1 - P_1] \in K_G^0(B).$$

From (2.6), we know that this definition is independent of the total equivariant spectral section $\widetilde{P}$.

In the following, we define the equivariant higher spectral flow for the even case.

Let $\mathcal{F} \in \mathrm{F}_G^0(B)$. Let D be an equivariant B-family on $\mathcal{F}$. We assume that there exists an equivariant spectral section P with respect to D. Let A_P be the smooth operator associated with P. We take the odd equivariant geometric family $p_1^* \mathcal{F} \cup p_2^* \mathcal{F}^L$ as in Example 1.4 d).

Let τ be the $\mathbb{Z}_2$-grading of the $\mathcal{S}_Z \widehat{\otimes} E$ in $\mathcal{F}$. Let $D_L = \tau \otimes D(\mathcal{F}^L)$ on L over $p_1^* W \times_{B \times S^1} p_2^*(S^1 \times S^1)$ (compare with the notation of Clifford algebra in Section 1.1).

Let $D_P = (D + A_P) \otimes 1 + D_L$. Then D_P is an equivariant $B \times S^1$-family on the odd geometric family $p_1^* \mathcal{F} \cup p_2^* \mathcal{F}^L$ and commutes with the group action. Since D and A_P anti-commutes with τ, we have $[(D + A_P) \otimes 1, D_L] = 0$. So $D_P^2 = ((D + A_P) \otimes 1 + D_L)^2 = (D + A_P)^2 \otimes 1 + D_L^2 > 0$. It implies that D_P is invertible. Let P' be the APS projection for D_P. Then P' is an equivariant spectral section with respect to the equivariant $B \times S^1$-family D_P.

Similarly, Let Q be another equivariant spectral section of D, we can get the equivariant spectral section Q' with respect to the equivariant $B \times S^1$-family D_Q as above. Since $p_1^* \mathcal{F} \cup p_2^* \mathcal{F}^L \in \mathrm{F}_G^1(B)$, from Definition 2.5, we could get $\mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\} \in K_G^0(B \times S^1)$.

Now we consider the Example 1.4 c) more explicitly. It is easy to calculate that for $\theta \in [0,1)$ fixed, the eigenvalues of $D(\mathcal{F}^L)$ are $\lambda_k(\theta) = 2\pi k + 2\pi(\theta - 1/2)$, $k \in \mathbb{Z}$. So for $\theta \in S^1$, $\theta \neq 1/2$, we have $D_L^2 > 0$. Thus for any $s \in [0,1]$, $\theta \neq 1/2$, restricted on $B \times \{\theta\}$, $(1 - s)D_P + s D_Q$ is invertible. From Definition 2.5, it means that for $\theta \neq 1/2$, $\mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\}|_{B \times \{\theta\}} =$

$0 \in K_G^0(B \times \{\theta\})$. In the sense of (1.12), we have $\mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\} \in K_G^1(B)$. We define

$$(2.11) \qquad\qquad [P - Q] := \mathrm{sf}_G\{(D_P, P'), (D_Q, Q')\} \in K_G^1(B).$$

The idea for this construction comes from [29, Proposition 4]. We note that when the group G is trivial, this definition is equivalent to that there.

Similarly, if P_1, P_2, P_3 are equivariant spectral sections with respect to D, then

$$(2.12) \qquad\qquad [P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K_G^1(B).$$

Now we extend the difference $[P - Q]$ to the equivariant higher spectral flow. Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^0(B)$ which have the same topological structure and D_0, D_1 be two equivariant B-families on $\mathcal{F}$, $\mathcal{F}'$ respectively. Let Q_0, Q_1 be equivariant spectral sections with respect to D_0, D_1 respectively. Let $D(r)$, $r \in [0, 1]$ be a continuous curve of equivariant B-families on $\mathcal{F}_r$ such that $D(0) = D_0 + A_{Q_0}$ and $D(1) = D_1 + A_{Q_1}$. Let $\widetilde{D} = \{D(r) \otimes 1 + D^L\}$ parametrized by $B \times S^1 \times I$. Since $\mathrm{ind}(D(0) \otimes 1 + D^L) = 0 \in K_G^1(B \times S^1)$, we have $\mathrm{ind}(\widetilde{D}) = 0 \in K_G^1(B \times S^1 \times I)$. Let $\widetilde{P} = \{P(r)\}_{r \in [0,1]}$ be an equivariant spectral section with respect to $\widetilde{D}$. When restricted on $B \times \{\theta\} \times I$ for $\theta \neq 1/2$, $\widetilde{D}|_{B \times \{\theta\} \times I}$ is invertible. Let $\{P'(r)_\theta\}_{r \in [0,1]}$ be the APS projection of $\widetilde{D}|_{B \times \{\theta\} \times I}$. Then $P'(0)_\theta = Q_0'|_{B \times \{\theta\}}$ and $P'(1)_\theta = Q_1'|_{B \times \{\theta\}}$. Since $[P'(r)_\theta - P(r)|_{B \times \{\theta\} \times I}]$ is an equivariant vector bundle over $B \times \{\theta\} \times I$, we have $([Q_0' - P(0)] - [Q_1' - P(1)])|_{B \times \{\theta\}} = 0 \in K_G^0(B \times \{\theta\})$. It implies that

$$(2.13) \quad \mathrm{sf}_G\{(D_0 \otimes 1 + D^L, Q_0'), (D_1 \otimes 1 + D^L, Q_1')\} = [Q_0' - P(0)] - [Q_1' - P(1)] \in K_G^1(B).$$

Definition 2.6. The equivariant higher spectral flow $\mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs (D_0, Q_0), (D_1, Q_1) is an element in $K_G^1(B)$ defined by

$$(2.14) \qquad \mathrm{sf}_G\{(D_0, Q_0), (D_1, Q_1)\} := \mathrm{sf}_G\{(D_0 \otimes 1 + D^L, Q_0'), (D_1 \otimes 1 + D^L, Q_1')\} \in K_G^1(B).$$

Note that when $\mathcal{F} = \mathcal{F}'$, $D_0 = D_1$, the equivariant higher spectral flow is $[Q_0 - Q_1]$. From (2.6), we know that this definition is independent of the total equivariant spectral section $\widetilde{P}$.

Now we will represent the element of equivariant K-group as equivariant higher spectral flow. The proof of the following proposition is constructive.

Proposition 2.7. *For any $x \in K_G^0(B)$ (resp. $K_G^1(B)$), there exist $\mathcal{F} \in \mathrm{F}_G^1(B)$ (resp. $\mathrm{F}_G^0(B)$) and equivariant spectral sections P, Q with respect to $D(\mathcal{F})$, such that $[P - Q] = x$.*

Proof. Any element of $K_G^0(B)$ could be represented as an equivariant virtual bundle $E_+ - E_-$. Let $\pi : B \times S^1 \to B$ be the projection on the first part. Let $\mathcal{F}_0 = (B \times S^1, \pi^*(E_\pm), o, T^H(B \times S^1), g^{TS^1}, \pi^* h^{E_\pm}, \pi^* \nabla^{E_\pm}) \in \mathrm{F}_G^1(B)$, where $o \in o(TS^1)$, g^{TS^1} are the canonical orientation and metric on S^1 and $T^H(B \times S^1) = TB \times S^1$. Let ∂_t be the generator of TS^1. Then $D(\mathcal{F}_0) = -\sqrt{-1}\partial_t \cdot \mathrm{Id}_{E_\pm}$. We could calculate that the eigenvalues of $D(\mathcal{F}_0)$ are $\lambda_k = k$ for $k \in \mathbb{Z}$. We denote by $P_{\lambda \geq k}$ the orthogonal projection onto the union of the eigenspaces of $\lambda \geq k$. Then for any k, $P_{\lambda \geq k}$ is an equivariant spectral section of $D(\mathcal{F}_0)$. In particular, we have $[P_{\lambda \geq k} - P_{\lambda \geq k+1}] = [E_+] - [E_-] \in K_G^0(B)$.

For any $x \in K_G^1(B)$, as in the proof of Proposition 1.5, there exists a finite dimensional G-representation V, such that x can be represented as a G-invariant unitary element $F \in \mathrm{End}(V) \otimes \mathscr{C}^0(B, \mathbb{C})$. Let $\mathcal{F}_1 = (B, E_+ = E_- = B \times V, h^{E_\pm}, \nabla^{E_\pm}) \in F_G^0(B)$, where $h^{E_\pm}$ and $\nabla^{E_\pm}$ are trivial. Set

$$A_0 = \begin{pmatrix} 0 & F(b)^* \\ F(b) & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let P_i be the orthogonal projection onto the positive part of the spectrum of A_i for $i = 0, 1$. From Definition 2.2, we know that P_0 and P_1 are equivariant spectral sections with respect to $D(\mathcal{F}_1) = 0$ on $\mathcal{F}_1$. Let $D_i = A_i \otimes 1 + \tau \otimes D(\mathcal{F}^L)$ on $p_1^* \mathcal{F}_1 \cup p_2^* \mathcal{F}^L$ and P_i' be the APS projection of D_i. Let $D_s = (1-s)D_0 + sD_1$ for $s \in [0, 1]$. We claim that

$$(2.15) \qquad \mathrm{sf}_G\{(D_0, P_0'), (D_1, P_1')\} = [W] - [U] \in K_G^0(B \times S^1),$$

where W and U are bundles constructed in the proof of Proposition 1.5. Then from the proof of Proposition 1.5, (2.11) and (2.15), we obtain Proposition 2.7 in the odd case.

We prove the claim (2.15) constructively. Let $\lambda_{b,i}$ be the eigenvalues of $F(b)$ on V with eigenvectors $v_{b,i}$. Then $\overline{\lambda}_{b,i}$ is the eigenvalues of $F^*(b)$ on V with eigenvectors $v_{b,i}$. Let $v_{b,i}^{\pm}$ be the corresponding vector in $E_{\pm}$. Let $s_k(\theta)$ be the eigenvector of $\lambda_k(\theta)$ with respect to $D(\mathcal{F}^L)$. Let $\lambda_{s,b,i,k}(\theta) = \sqrt{\lambda_k(\theta)^2 + s^2 + (1-s)^2 + s(1-s)(\lambda_{b,i} + \overline{\lambda}_{b,i})}$. It is easy to calculate that $\{\lambda_{s,b,i,k}\}$ are the set of nonnegative eigenvalues of D_s. Let

$$u_{s,b,i}^{(1)}(\theta) = (v_{b,i}^- + (\lambda_{s,b,i,1}(\theta) - \lambda_1(\theta))^{-1}((1-s)\overline{\lambda}_{b,i} + s)v_{b,i}^+) \otimes s_1(\theta),$$
$$\text{for } 0 \le \theta \le 1/2,$$

$$u_{s,b,i}^{(2)}(\theta) = (v_{b,i}^+ + (\lambda_{s,b,i,-1}(\theta) + \lambda_{-1}(\theta))^{-1}((1-s)\lambda_{b,i} + s)v_{b,i}^-) \otimes s_{-1}(\theta),$$
$$\text{for } 1/2 \le \theta \le 1,$$

$$(2.16) \qquad u_{s,b,i}^{(3)}(\theta) = \begin{cases} (v_{b,i}^- + (\lambda_{s,b,i,0}(\theta) - \lambda_0(\theta))^{-1} \cdot ((1-s)\overline{\lambda}_{b,i} + s)v_{b,i}^+) \otimes s_0(\theta), \\ \qquad\qquad \text{for } 0 \le \theta \le 1/2, \lambda_{s,b,i,0}(\theta) \ne 0, \\ v_{b,i}^- \otimes s_0(\theta), \qquad\qquad \text{for } 0 \le \theta \le 1/2, \lambda_{s,b,i,0}(\theta) = 0, \end{cases}$$

$$u_{s,b,i}^{(4)}(\theta) = \begin{cases} (v_{b,i}^+ + (\lambda_{s,b,i,0}(\theta) + \lambda_0(\theta))^{-1} \cdot ((1-s)\lambda_{b,i} + s)v_{b,i}^-) \otimes s_0(\theta), \\ \qquad\qquad \text{for } 1/2 \le \theta \le 1, \lambda_{s,b,i,0}(\theta) \ne 0. \\ v_{b,i}^+ \otimes s_0(\theta), \qquad\qquad \text{for } 1/2 \le \theta \le 1, \lambda_{s,b,i,0}(\theta) = 0 \ . \end{cases}$$

Then we can calculate that $u_{s,b,i}^{(1)}$ and $u_{s,b,i}^{(2)}$ are eigenfunctions of $\lambda_{s,b,i,1}(\theta)$ with respect to D_s and $u_{s,b,i}^{(3)}$ and $u_{s,b,i}^{(4)}$ are that of $\lambda_{s,b,i,0}(\theta)$. Choose $\chi(\theta) \in \mathcal{C}^{\infty}([0, 1/2])$ with $\chi(\theta) = 1/2$ at $\theta = 0$ and $\chi(\theta) = 0$ at $\theta = 1/2$. Let

$$(2.17) \qquad \begin{aligned} u_{s,b,i}^{(5)}(\theta) &= \chi(\theta)u_{s,b,i}^{(1)} + (1 - \chi(\theta))u_{s,b,i}^{(3)}, \quad \text{for } 0 \le \theta \le 1/2, \\ u_{s,b,i}^{(6)}(\theta) &= \chi(1 - \theta)u_{s,b,i}^{(2)} + (1 - \chi(1 - \theta))u_{s,b,i}^{(4)}, \quad \text{for } 1/2 \le \theta \le 1. \end{aligned}$$

We have $u_{s,b,i}^{(5)}(0) = u_{s,b,i}^{(6)}(1)$. So $\mathrm{span}\{u_{s,b,i}^{(5)}\}$ and $\mathrm{span}\{u_{s,b,i}^{(6)}\}$ make a trivial equivariant vector bundle over $B \times (S^1 \backslash \{1/2\}) \times [0, 1]$. Thus the relations $u_{s,b,i}^{(5)}(1/2) \sim u_{s,b,i}^{(6)}(1/2)$ makes an equivariant vector bundle $\widetilde{W}$ over $B \times S^1 \times [0, 1]$. Let $\widetilde{R}$ be the orthogonal projection onto the sum of $\widetilde{W}$ and the eigenspaces with non-positive eigenvalues of $\widetilde{D} = \{D_s\}$. Then $\widetilde{Q} = 1 - \widetilde{R}$ is an equivariant spectral section with respect to $\widetilde{D}$. It is easy to see that $[P_0' - \widetilde{Q}|_{s=0}] = W$ and $[P_1' - \widetilde{Q}|_{s=1}] = U$. So we obtain the claim (2.15).

The proof of Proposition 2.7 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Note that the proof of Proposition 2.7 in odd case gives a nontrivial example of equivariant higher spectral flow for even dimensional fibers and an example of equivariant spectral section without the spectral gap.

Remark 2.8. If $G = \{e\}$, there is a stronger version of Proposition 2.7 (see [29, Proposition 12]): for any $x \in K^0(B)$ (resp. $K^1(B)$), $\mathcal{F} \in \mathrm{F}^1(B)$ (resp. $\mathrm{F}^0(B)$) and spectral section P with respect to $D(\mathcal{F})$, there exists spectral section Q with respect to $D(\mathcal{F})$, such that $[P - Q] = x$.

2.2. **Equivariant local family index theorem.** In this subsection, we use the notations in Section 1.2 and describe the equivariant local index theorem for $\mathcal{F} \in \mathrm{F}_G^*(B)$ when **the G-action on B is trivial**.

For $b \in B$, let $\mathscr{E}_b$ be the set of smooth sections over Z_b of $\mathcal{S}_Z \widehat{\otimes} E_b$. As in [5], we will regard $\mathscr{E}$ as an infinite dimensional fiber bundle over B.

Let $^0\nabla^{TW}$ be the connection on $TW = T^HW \oplus TZ$ defined by

$$(2.18) \qquad ^0\nabla^{TW} = \pi^*\nabla^{TB} \oplus \nabla^{TZ}.$$

Then $^0\nabla^{TW}$ preserves the metric g^{TW} in (1.6). Set

$$(2.19) \qquad S = \nabla^{TW} - {}^0\nabla^{TW}.$$

If $V \in TB$, let $V^H \in T_\pi^H W$ be its horizontal lift in $T_\pi^H W$ so that $\pi_* V^H = V$. For any $V \in TB$, $s \in \mathscr{C}^\infty(B, \mathscr{E}) = \mathscr{C}^\infty(W, \mathcal{S}_Z \widehat{\otimes} E)$, by [7, Proposition 1.4], the connection

$$(2.20) \qquad \nabla_V^{\mathscr{E}, \mathcal{U}} s := \nabla_{V^H}^{\mathcal{S}_Z \widehat{\otimes} E} s - \frac{1}{2} \langle S(e_i)e_i, V^H \rangle s$$

preserves the L^2-product on $\mathscr{E}$.

Let $\{f_p\}$ be a local orthonormal frame of TB and $\{f^p\}$ be its dual. We denote by $\nabla^{\mathscr{E}, \mathcal{U}} = f^p \wedge \nabla_{f_p}^{\mathscr{E}, \mathcal{U}}$. Let T be the torsion of $^0\nabla^{TW}$. Then $T(f_p^H, f_q^H) \in TZ$. We denote by

$$(2.21) \qquad c(T) = \frac{1}{2} c\left(T(f_p^H, f_q^H)\right) f^p \wedge f^q \wedge .$$

By [5, (3.18)], the (rescaled) Bismut superconnection

$$(2.22) \qquad \mathbb{B}_u : \mathscr{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathscr{E}) \to \mathscr{C}^\infty(B, \Lambda(T^*B) \widehat{\otimes} \mathscr{E})$$

is defined by

$$(2.23) \qquad \mathbb{B}_u = \sqrt{u} D(\mathcal{F}) + \nabla^{\mathscr{E}, \mathcal{U}} - \frac{1}{4\sqrt{u}} c(T).$$

Obviously, the Bismut superconnection $\mathbb{B}_u$ commutes with the G-action. Moreover, $\mathbb{B}_u^2$ is a 2-order elliptic differential operator along the fibers Z. Let $\exp(-\mathbb{B}_u^2)$ be the family of heat operators associated with the fiberwise elliptic operator $\mathbb{B}_u^2$. From [4, Theorem 9.50], we know that $\exp(-\mathbb{B}_u^2)$ is a smooth family of smoothing operators.

If P is a trace class operator acting on $\Lambda(T^*B) \widehat{\otimes} \mathrm{End}(\mathscr{E}_Z)$ which takes value in $\Lambda(T^*B)$, we use the convention that if $\omega \in \Lambda(T^*B)$,

$$(2.24) \qquad \mathrm{Tr}_s[\omega P] = \omega \, \mathrm{Tr}_s[P].$$

We denote by $\mathrm{Tr}_s^{\mathrm{odd/even}}[P]$ the part of $\mathrm{Tr}_s[P]$ which takes value in odd or even forms. Set

$$(2.25) \qquad \widetilde{\mathrm{Tr}}[P] = \begin{cases} \mathrm{Tr}_s[P], & \text{if } \dim Z \text{ is even;} \\ \mathrm{Tr}_s^{\mathrm{odd}}[P], & \text{if } \dim Z \text{ is odd.} \end{cases}$$

We assume that G acts trivially on B. Take $g \in G$. Let W^g be the fixed point set of g on W. Set

$$(2.26) \qquad \mathrm{ch}_g(E, \nabla^E) = \mathrm{Tr}_s\left[g \exp\left(\frac{\sqrt{-1}}{2\pi} R^E|_{W^g}\right)\right].$$

Let $\mathrm{ch}_g(E) \in H^{even}(W^g, \mathbb{C})$ denote the cohomology class of $\mathrm{ch}_g(E, \nabla^E)$. When $Z = pt$, it descends to the equivariant Chern character map

$$(2.27) \qquad\qquad \mathrm{ch}_g : K_G^0(B) \longrightarrow H^{even}(B, \mathbb{C}).$$

By (1.12), for $x \in K_G^1(B)$, we can regard x as an element x' in $K_G^0(B \times S^1)$. The odd equivariant Chern character map

$$(2.28) \qquad\qquad \mathrm{ch}_g : K_G^1(B) \longrightarrow H^{odd}(B, \mathbb{C})$$

is defined by

$$(2.29) \qquad\qquad \mathrm{ch}_g(x) = \int_{S^1} \mathrm{ch}_g(x').$$

We adopt the sign notation in the integral as in (0.8).

Set

$$(2.30) \qquad\qquad \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) = \mathrm{Tr}_s \left[g \exp\left(\frac{\sqrt{-1}}{4\pi} R^{L_Z} |_{W^g} \right) \right].$$

Let ∇ be a Euclidean connection on (TZ, g^{TZ}). We denote by

$$(2.31) \qquad\qquad \mathrm{Td}_g(\nabla, \nabla^{L_Z}) := \widehat{\mathrm{A}}_g(TZ, \nabla) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}).$$

For the definitions of the characteristic form $\widehat{\mathrm{A}}_g$, see [23, (1.44)]. Let $\mathrm{Td}_g(TZ, L_Z) \in H^{even}(W^g, \mathbb{C})$ denote the cohomology class of $\mathrm{Td}_g(\nabla, \nabla^{L_Z})$.

For $\alpha \in \Omega^i(B)$, set

$$(2.32) \qquad\qquad \psi_B(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}} \right)^{\frac{i}{2}} \cdot \alpha, & \text{if } i \text{ is even;} \\ \frac{1}{\sqrt{\pi}} \left(\frac{1}{2\pi\sqrt{-1}} \right)^{\frac{i-1}{2}} \cdot \alpha, & \text{if } i \text{ is odd.} \end{cases}$$

We state the equivariant family local index theorem [23, Theorem 1.2] here. Note that $\pi : W^g \to B$ is a fiber bundle with closed fiber Z^g. From [23, Proposition 1.1], Z^g is naturally oriented.

Theorem 2.9. *For any $u > 0$ and $g \in G$, the differential form $\psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathbb{B}_u^2)] \in \Omega^*(B, \mathbb{C})$ is closed and its cohomology class represents $\mathrm{ch}_g(\mathrm{ind}(D(\mathcal{F}))) \in H^*(B, \mathbb{C})$. As $u \to 0$, we have*

$$(2.33) \qquad\qquad \lim_{u \to 0} \psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathbb{B}_u^2)] = \int_{Z^g} \mathrm{Td}_g(\nabla^{TZ}, \nabla^{L_Z}) \wedge \mathrm{ch}_g(E, \nabla^E).$$

To simplify the notations, we set

$$(2.34) \qquad\qquad \mathrm{FLI}_g(\mathcal{F}) = \int_{Z^g} \mathrm{Td}_g(\nabla^{TZ}, \nabla^{L_Z}) \wedge \mathrm{ch}_g(E, \nabla^E).$$

So Theorem 2.9 says that for $\mathcal{F} \in \mathrm{F}_G^{0/1}(B)$

$$(2.35) \qquad\qquad [\mathrm{FLI}_g(\mathcal{F})] = \mathrm{ch}_g(\mathrm{ind}(D(\mathcal{F}))) \in H^{even/odd}(B, \mathbb{C}).$$

When $\mathcal{F}$ is the equivariant geometric family in Example 1.4 a), the equivariant family local index theorem degenerates to the equivariant Chern-Weil theory:

$$(2.36) \qquad\qquad \lim_{u \to 0} \psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathbb{B}_u^2)] = \psi_B \mathrm{Tr}_s[g \exp(-\nabla^{E,2})] = \mathrm{ch}_g(E, \nabla^E).$$

In this case, $\mathrm{FLI}_g(\mathcal{F}) = \mathrm{ch}_g(E, \nabla^E) = \mathrm{ch}_g(E_+, \nabla^{E_+}) - \mathrm{ch}_g(E_-, \nabla^{E_-})$.

2.3. **Equivariant eta form.** In this subsection, we also assume that G acts trivially on B. We define the equivariant Bismut-Cheeger eta form with perturbation operator.

Definition 2.10. Let D be an equivariant B-family on $\mathcal{F}$. A perturbation operator with respect to D is a family of bounded pseudodifferential operators A such that $D + A$ is an invertible equivariant B-family on $\mathcal{F}$.

Note that if there exists an equivariant spectral section with respect to D, the smooth operator associated with it is a perturbation operator.

In this subsection, we assume that there exists a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$. It will imply that $\mathrm{ind}(D(\mathcal{F})) = 0$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be a cut-off function such that

$$(2.37) \qquad \chi(u) = \begin{cases} 0, & \text{if } u < 1; \\ 1, & \text{if } u > 2. \end{cases}$$

Let A be a perturbation operator with respect to $D(\mathcal{F})$. For $g \in G$, set

$$(2.38) \qquad \mathbb{B}'_u = \mathbb{B}_u + \sqrt{u}\chi(\sqrt{u})A.$$

Since $\chi(\sqrt{u}) = 0$ when $u \to 0$, by (2.34),

$$(2.39) \qquad \lim_{u \to 0} \psi_B \widetilde{\mathrm{Tr}}[g \exp(-(\mathbb{B}'_u)^2)] = \mathrm{FLI}_g(\mathcal{F}) \in \Omega^*(B, \mathbb{C}).$$

Since $\chi(\sqrt{u}) = 1$ when $u \to +\infty$, from [4, Theorem 9.19], we have

$$(2.40) \qquad \lim_{u \to +\infty} \psi_B \widetilde{\mathrm{Tr}}\left[g \exp\left(-(\mathbb{B}'_u)^2\right)\right] = 0.$$

If $\alpha \in \Lambda(T^*(\mathbb{R}_+ \times B))$, we can expand α in the form

$$(2.41) \qquad \alpha = du \wedge \alpha_0 + \alpha_1, \quad \alpha_0, \alpha_1 \in \Lambda(T^*B).$$

Set

$$(2.42) \qquad [\alpha]^{du} = \alpha_0.$$

Definition 2.11. For any $g \in G$, modulo exact forms, the equivariant eta form with perturbation operator A is defined by

$$(2.43) \quad \tilde{\eta}_g(\mathcal{F}, A) = -\int_0^\infty \left\{ \psi_B \widetilde{\mathrm{Tr}}\left[g \exp\left(-\left(\mathbb{B}'_u + du \wedge \frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du$$

$$\in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).$$

The regularities of the integral in the right hand side of (2.43) are proved in [23, Section 1.4]. As in [23, (1.81)], we have

$$(2.44) \qquad d\tilde{\eta}_g(\mathcal{F}, A) = \mathrm{FLI}_g(\mathcal{F}).$$

From the proof of the anomaly formula in [23, Theorem 1.7], it is easy to see that the value of $\tilde{\eta}_g(\mathcal{F}, A)$ in $\Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$ is independent of the choice of the cut-off function. Similarly, If A_P and A'_P are two smooth operators associated with the same equivariant spectral section P, we have $\tilde{\eta}_g(\mathcal{F}, A_P) = \tilde{\eta}_g(\mathcal{F}, A'_P) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$. In this case, we often simply denote it by $\tilde{\eta}_g(\mathcal{F}, P)$.

Note that the taming in [10, 12, 14] is a perturbation operator. So the definition here is also a generalization of eta forms there.

If the fiber Z is connected, we could calculate the equivariant eta form explicitly:

$$(2.45) \quad \tilde{\eta}_g(\mathcal{F}, A) = \begin{cases} \int_0^\infty \frac{1}{\sqrt{\pi}} \psi_B \operatorname{Tr}_s^{\mathrm{even}} \left[g \frac{\partial \mathbb{B}'_u}{\partial u} \exp(-(\mathbb{B}'_u)^2) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathcal{F} \text{ is odd}; \\ \int_0^\infty \frac{1}{2\sqrt{-1}\sqrt{\pi}} \psi_B \operatorname{Tr}_s \left[g \frac{\partial \mathbb{B}'_u}{\partial u} \exp(-(\mathbb{B}'_u)^2) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathcal{F} \text{ is even and } \dim Z > 0. \\ \int_0^\infty \frac{\sqrt{-1}}{2\pi} \operatorname{Tr}_s \left[g \frac{\partial \nabla^E_u}{\partial u} \exp\left(-\frac{(\nabla^E_u)^2}{2\pi\sqrt{-1}} \right) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } \dim Z = 0, \end{cases}$$

where $\nabla^E_u = \nabla^E + \sqrt{u}\chi(\sqrt{u})A$.

When $\dim Z = 0$, the equivariant geometric family degenerates to the case of Example 1.4 a). As in [27, Definition B.5.3], from (2.36) and (2.44), the equivariant eta form in this case is just the equivariant transgression between $\mathrm{ch}_g(E, \nabla^E)$ and 0.

Furthermore, by changing the variable (see also [23, Remark 1.4]), we could get another form of equivariant eta form:

$$(2.46) \qquad \tilde{\eta}_g(\mathcal{F}, A) = -\int_0^\infty \left\{ \psi_B \widetilde{\operatorname{Tr}} \left[g \exp\left(-\left(\mathbb{B}'_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du.$$

Let $(Z', g^{TZ'})$ be a Spin^c manifold with even dimension and $(E', h^{E'})$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle over Z' with Hermitian connection $\nabla^{E'}$. Let $\mathrm{pr}_2 : B \times Z' \to Z'$ be the projection onto the second part. Then all the bundles and geometric data above could be pulled back on $B \times Z'$. Thus the fiber bundle $\mathrm{pr}_1 : B \times Z' \to B$, which is the projection onto the first part, and the structures pulled back by pr_2 form a geometric family $\mathcal{F}'$ with fibers Z'. We assume that the group action on $\mathcal{F}'$ is trivial. In this case, $\mathrm{ind}(D(\mathcal{F}'))$ is a constant integer. For $\mathcal{F} \in \mathrm{F}_G^*(B)$, let A be a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$. Then $A \widehat{\otimes} 1$ is a perturbation operator with respect to $D(\mathcal{F} \cup \mathcal{F}')$ on $\mathcal{F} \cup \mathcal{F}'$.

Lemma 2.12. *For $g \in G$, we have*

$$(2.47) \qquad\qquad \tilde{\eta}_g(\mathcal{F} \cup \mathcal{F}', A \widehat{\otimes} 1) = \tilde{\eta}_g(\mathcal{F}, A) \cdot \mathrm{ind}(D(\mathcal{F}')).$$

Proof. We denote by $\mathrm{Tr}\,|_{\mathcal{F}}$ the trace operator on $\mathcal{F}$. Then

(2.48) $\quad \tilde{\eta}_g(\mathcal{F} \cup \mathcal{F}', A\widehat{\otimes}1)$

$$= -\int_0^\infty \left\{ \psi_B \,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\cup\mathcal{F}'} \left[g \exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1 + 1\widehat{\otimes}uD(\mathcal{F}') + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} du$$

$$= \int_0^\infty \left\{ \psi_B \,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\cup\mathcal{F}'} \left[g(1\widehat{\otimes}D(\mathcal{F}')) \exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1 \right)^2 - \left(1\widehat{\otimes}uD(\mathcal{F}') \right)^2 \right) \right] \right\} du$$

$$- \int_0^\infty \left\{ \psi_B \,\widetilde{\mathrm{Tr}}|_{\mathcal{F}\cup\mathcal{F}'} \left[g \exp\left(-\left(\mathbb{B}'_{u^2}\widehat{\otimes}1 + du \wedge \frac{\partial}{\partial u} \right)^2 - 1\widehat{\otimes}u^2 D(\mathcal{F}')^2 \right) \right] \right\}^{du} du$$

$$= \int_0^\infty \left\{ \psi_B \,\widetilde{\mathrm{Tr}}|_{\mathcal{F}} \left[g \exp\left(-\left(\mathbb{B}'_{u^2} \right)^2 \right) \right] \cdot \mathrm{Tr}_s\,|_{\mathcal{F}'} \left[D(\mathcal{F}') \exp\left(-u^2 D(\mathcal{F}')^2 \right) \right] \right\} du$$

$$- \int_0^\infty \left\{ \psi_B \,\widetilde{\mathrm{Tr}}|_{\mathcal{F}} \left[g \exp\left(-\left(\mathbb{B}_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \cdot \mathrm{Tr}_s\,|_{\mathcal{F}'} \left[\exp\left(-u^2 D(\mathcal{F}')^2 \right) \right] \right\}^{du} du$$

From the definition of $\mathcal{F}'$, we have

(2.49)
$$\mathrm{Tr}_s\,|_{\mathcal{F}'} \left[D(\mathcal{F}') \exp\left(-\left(u^2 D(\mathcal{F}')^2 \right) \right) \right] = 0,$$
$$\mathrm{Tr}_s\,|_{\mathcal{F}'} \left[\exp\left(-u^2 D(\mathcal{F}')^2 \right) \right] = \mathrm{ind}(D(\mathcal{F}')).$$

So we get $\tilde{\eta}_g(\mathcal{F}\cup\mathcal{F}', A\widehat{\otimes}1) = \tilde{\eta}_g(\mathcal{F}, A)\cdot\mathrm{ind}(D(\mathcal{F}'))$. The proof of Lemma 2.12 is complete. $\quad\square$

2.4. Anomaly formula.

In this subsection, we will study the anomaly formula of the equivariant eta forms for two odd equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ with the same topological structure. In this subsection, we also assume that G acts on B trivially.

Lemma 2.13. *(compare with* [28, Proposition 17]*) Assume that $\mathcal{F} \in \mathrm{F}_G^1(B)$. Let A be a family of bounded pseudodifferential operator on $\mathcal{F}$ such that $D(\mathcal{F}) + A$ is an equivariant B-family. Let P, Q be two equivariant spectral sections with respect to $D(\mathcal{F}) + A$. Let A_P, A_Q be smooth operators associated with P, Q. For any $g \in G$, modulo exact forms, we have*

(2.50) $\qquad \tilde{\eta}_g(\mathcal{F}, A + A_P) - \tilde{\eta}_g(\mathcal{F}, A + A_Q) = \mathrm{ch}_g([P - Q]) \in H^{even}(B, \mathbb{C}).$

Proof. The proof is the natural equivariant extension of that of [28, Proposition 17]. We sketch it here for the completion.

From (2.6), we only need to prove the lemma when Q majorizes P. Let $\widetilde{\mathcal{F}}$ be the equivariant geometric family defined in (2.9) such that $\mathcal{F}_r = \mathcal{F}$ for any $r \in [0, 1]$. Let $\widetilde{\mathbb{B}}_u$ be the Bismut superconnection associated with $\widetilde{\mathcal{F}}$. Choose the smooth operators A_P, A_Q as in (2.4). Set $A_r = A + A_P + r(A_Q - A_P)$ and $\widetilde{\mathbb{B}}'_u|_{(u,r)} = \widetilde{\mathbb{B}}_u|_{(u,r)} + \sqrt{u}\chi(\sqrt{u})A_r$. Let Π be the orthogonal projection onto $[P - Q]$. Then $\mathrm{Range}(\Pi) = [P - Q]$ is an equivariant vector bundle over $\widetilde{B}$ and $\nabla^\Pi := \Pi \circ \nabla^{\mathscr{E},\mathcal{U}} \circ \Pi$ is a G-invariant connection on $\mathrm{Range}(\Pi)$. From (2.3), (2.23) and (2.38), we could calculate that

(2.51) $\qquad\qquad \Pi \circ (D(\mathcal{F}) + A + A_r) \circ \Pi = s(1 - 2r)\Pi.$

So when u large,

(2.52) $\qquad\qquad \Pi \circ \widetilde{\mathbb{B}}'^2_{u^2} \circ \Pi = u^2 s^2 (1 - 2r)^2 - 2usdr \wedge + \Pi \left(\nabla^{\mathscr{E},\mathcal{U}} \right)^2 \Pi + O(u^{-1}).$

As in the proof of [4, Theorem 9.19] (or following the process in [23, Section 4]), with respect to the splitting $\mathrm{Range}(\Pi) \oplus \mathrm{Range}(\Pi)^{\perp}$, when $u \to +\infty$, we can write $\left[\exp\left(-\widetilde{\mathbb{B}}'^{\,2}_{u^2}\right)\right]^{dr}$ as

$$(2.53) \qquad \left[\exp\left(-\widetilde{\mathbb{B}}'^{\,2}_{u^2}\right)\right]^{dr} = \begin{pmatrix} 2use^{-u^2s^2(1-2r)^2}\exp\left(-(\nabla^{\Pi})^2\right)+O(u^{-1}) & O(u^{-1}) \\ O(u^{-1}) & O(u^{-2}) \end{pmatrix}.$$

Set

$$r_1(u,r) = \left\{\psi_B \,\mathrm{Tr}^{\mathrm{odd}}_s\left[g\exp\left(-\widetilde{\mathbb{B}}'^{\,2}_{u^2}\right)\right]\right\}^{dr}\bigg|_{(u,r)}.$$

Then from [23, (1.95)], we have

$$(2.54) \quad \tilde{\eta}_g(\mathcal{F}, A+A_P) - \tilde{\eta}_g(\mathcal{F}, A+A_Q) = -\lim_{u\to+\infty}\int_0^1 r_1(u,r)dr$$

$$= \frac{1}{\sqrt{\pi}}\lim_{u\to+\infty}\int_0^1 2use^{-u^2s^2(1-2r)^2}dr\cdot\psi_B\,\mathrm{Tr}[g\exp(-(\nabla^{\Pi})^2)] = \mathrm{ch}_g([P-Q]).$$

The uniformly convergence condition needed in [23, (1.95)] relies on (2.53).

The proof of Lemma 2.13 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From [27, Theorem B.5.4], modulo exact forms, the Chern-Simons forms

$$(2.55) \qquad \begin{aligned} \widetilde{\mathrm{Td}}_g(\nabla^{TZ},\nabla^{L_Z},\nabla'^{TZ},\nabla'^{L_Z}) &:= -\int_0^1[\mathrm{Td}_g(\nabla^{T\widetilde{Z}},\nabla^{\widetilde{L_Z}})]^{ds}ds, \\ \widetilde{\mathrm{ch}}_g(\nabla^E,\nabla'^E) &:= -\int_0^1[\mathrm{ch}_g(\widetilde{E},\nabla^{\widetilde{E}})]^{ds}ds, \end{aligned}$$

do not depend on the choices of the objects with $\sim$. Moreover,

$$(2.56) \qquad \begin{aligned} d\widetilde{\mathrm{Td}}_g(\nabla^{TZ},\nabla^{L_Z},\nabla'^{TZ},\nabla'^{L_Z}) &= \mathrm{Td}_g(\nabla'^{TZ},\nabla'^{L_Z}) - \mathrm{Td}_g(\nabla^{TZ},\nabla^{L_Z}), \\ d\widetilde{\mathrm{ch}}_g(\nabla^E,\nabla'^E) &= \mathrm{ch}_g(E,\nabla'^E) - \mathrm{ch}_g(E,\nabla^E). \end{aligned}$$

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^*_G(B)$ which have the same topological structure. By (2.35), we have $[\mathrm{FLI}_g(\mathcal{F})] = [\mathrm{FLI}_g(\mathcal{F}')] \in H^*(B,\mathbb{C})$. Set

$$(2.57) \quad \widetilde{\mathrm{FLI}}_g(\mathcal{F},\mathcal{F}') = \int_{Z^g}\widetilde{\mathrm{Td}}_g(\nabla^{TZ},\nabla^{L_Z},\nabla'^{TZ},\nabla'^{L_Z})\wedge\mathrm{ch}_g(E,\nabla^E)$$

$$+ \int_{Z^g}\mathrm{Td}_g(\nabla'^{TZ},\nabla'^{L_Z})\wedge\widetilde{\mathrm{ch}}_g(\nabla^E,\nabla'^E) \in \Omega^*(B,\mathbb{C})/d\Omega^*(B,\mathbb{C}).$$

From (2.56), we have

$$(2.58) \qquad\qquad\qquad d\widetilde{\mathrm{FLI}}_g(\mathcal{F},\mathcal{F}') = \mathrm{FLI}_g(\mathcal{F}') - \mathrm{FLI}_g(\mathcal{F}).$$

Using Lemma 2.13, we can get the anomaly formula for odd case as follows. The even case will be proved later.

Proposition 2.14. *Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}^1_G(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be the APS projections with respect to $D(\mathcal{F}) + A$, $D(\mathcal{F}') + A'$ respectively. For any $g \in G$, modulo exact forms, we have*

$$(2.59) \quad \tilde{\eta}_g(\mathcal{F}',A') - \tilde{\eta}_g(\mathcal{F},A) = \widetilde{\mathrm{FLI}}_g(\mathcal{F},\mathcal{F}') + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F}')+A',P'),(D(\mathcal{F})+A,P)\}\right).$$

Proof. Let $\widetilde{\mathcal{F}}$ be the equivariant geometric family defined in (2.9). Let $D_r = D(\mathcal{F}_r) + (1 - r)A + rA'$ and $\widetilde{D} = \{D_r\}_{r \in [0,1]}$ on $\widetilde{\mathcal{F}}$. Since the equivariant index of $\mathcal{F}$ vanishes, the homotopy invariance of the equivariant index bundle implies that the equivariant indices of each D_r and $\widetilde{D}$ vanish. If we consider the total family $\widetilde{\mathcal{F}}$, then from Proposition 2.3(i), there exists a total equivariant spectral section $\widetilde{P}$ with respect to $\widetilde{D}$. Let P_r be the restriction of $\widetilde{P}$ over $\{r\} \times B$. Let A_{P_r} be a smooth operator associated with P_r. Following the proof of [23, Theorem 1.7], we can get

$$(2.60) \qquad \tilde{\eta}_g(\mathcal{F}', A' + A_{P_1}) - \tilde{\eta}_g(\mathcal{F}, A + A_{P_0}) = \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}').$$

Then Proposition 2.14 follows from Lemma 2.13, (2.10) and (2.60). $\qquad\square$

2.5. Functoriality of equivariant eta forms.
In this subsection, we will study the functoriality of the equivariant eta forms and use it to prove the anomaly formula of equivariant eta forms for even equivariant geometric families. In this subsection, we use the notations in Section 1.4.

Let ∇ and ∇' be Euclidean connections on (TZ, g^{TZ}) and ∇^{L_Z} and ∇'^{L_Z} be Hermitian connections on (L_Z, h^{L_Z}). Similarly as (2.34), we define

$$(2.61) \qquad \mathrm{FLI}_g(\nabla, \nabla^{L_Z}) := \int_{Z^g} \mathrm{Td}_g(\nabla, \nabla^{L_Z}) \wedge \mathrm{ch}_g(E, \nabla^E).$$

As in (2.55) and (2.56), there exists a well-defined equivariant Chern-Simons form $\widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$ such that

$$(2.62) \qquad d\,\widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) = \mathrm{Td}_g(\nabla', \nabla'^{L_Z}) - \mathrm{Td}_g(\nabla, \nabla^{L_Z}).$$

Set

$$(2.63) \qquad \widetilde{\mathrm{FLI}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) := \int_{Z^g} \widetilde{\mathrm{Td}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) \wedge \mathrm{ch}_g(E, \nabla^E).$$

From (2.62), we have

$$(2.64) \qquad d\,\widetilde{\mathrm{FLI}}_g(\nabla, \nabla^{L_Z}, \nabla', \nabla'^{L_Z}) = \mathrm{FLI}_g(\nabla', \nabla'^{L_Z}) - \mathrm{FLI}_g(\nabla, \nabla^{L_Z}).$$

Lemma 2.15. *If* $\mathrm{ind}(D(\mathcal{F}_X)) = 0 \in K_G^{*'}(V)$, *then* $\mathrm{ind}(D(\mathcal{F}_Z)) = 0 \in K_G^{*'+*}(B)$. *Therefore, we get the well-defined property of the push-forward map in Theorem 1.8.*

Proof. Let

$$(2.65) \qquad g_T^{TZ} = \pi_X^* g^{TY} \oplus \frac{1}{T^2} g^{TX}.$$

We denote the Clifford algebra bundle of TZ with respect to g_T^{TZ} by $C_T(TZ)$. If $U \in TV$, let $U^H \in T_{\pi_X}^H W$ be the horizontal lift of U, so that $\pi_{X,*}(U^H) = U$. Let $\{e_i\}$, $\{f_p\}$ be local orthonormal frames of (TX, g^{TX}), (TY, g^{TY}). Then $\{f_{p,1}^H\} \cup \{Te_i\}$ is a local orthonormal frame of (TZ, g_T^{TZ}). We define a Clifford algebra isomorphism

$$(2.66) \qquad \mathcal{G}_T : C_T(TZ) \to C(TZ)$$

by

$$(2.67) \qquad \mathcal{G}_T(c(f_{p,1}^H)) = c(f_{p,1}^H), \quad \mathcal{G}_T(c_T(Te_i)) = c(e_i).$$

Recall that $\pi_1^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X$ is defined in Section 1.1. Under this isomorphism, we can consider $((\pi_1^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X) \widehat{\otimes} E, h^{\pi_1^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X} \otimes h^E)$ as a self-adjoint Hermitian equivariant Clifford module of $C_T(TZ)$. Let ∇_T^{TZ} be the connection associated with $(T_{\pi_Z}^H W, g_T^{TZ})$ as in (1.7).

Then

$$(2.68) \qquad \mathcal{F}_{Z,T} = (W, L_Z, E, o_Y \cup o_X, T_{\pi_Z}^H W, g_T^{TZ}, , h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E)$$

is an equivariant geometric family over B and $\mathcal{F}_Z = \mathcal{F}_{Z,1}$.

If $\mathrm{ind}(D(\mathcal{F}_X)) = 0 \in K_G^*(V)$ and at least one component of X has nonzero dimension, from Proposition 2.3 (i), there exists a perturbation operator A_X such that $\ker(D(\mathcal{F}_X) + A_X) = 0$. We extend A_X to a pseudodifferential operator acting on $\mathcal{C}^\infty(W, (\pi_1^* \mathcal{S}_Y \widehat{\otimes} \mathcal{S}_X) \widehat{\otimes} E)$ the same way as the extension of $c(e_i)$ in Section 1.1. We denote it by $1 \widehat{\otimes} A_X$. As in [23, Lemma 4.3], there exists $T' \geq 1$, such that when $T \geq T'$, we have $\ker(D(\mathcal{F}_{Z,T}) + 1 \widehat{\otimes} T A_X) = 0$. So by the homotopy invariance of the equivariant index, for any $T \geq 1$, we have $\mathrm{ind}(D(\mathcal{F}_{Z,T})) = 0$.

If $\dim X = 0$, it is obvious.

The proof of Lemma 2.15 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that when A_X is smooth along the fibers X, $1 \widehat{\otimes} A_X$ is not a smooth operator along the fibers Z. This is the reason for us to define the eta form for bounded perturbation operator instead of smooth operator in [10, 12, 16, 28].

Recall that in (1.22), $TZ = T_{\pi_X}^H Z \oplus TX$. Let $\nabla^{TY,TX}$ be the connection on TZ defined by

$$(2.69) \qquad\qquad \nabla^{TY,TX} = \pi_X^* \nabla^{TY} \oplus \nabla^{TX}$$

as in (2.18).

The following technical lemma is a modification of the main result in [23]. The proof of it will be left to the next subsection.

Lemma 2.16. *Let $T' \geq 1$ be the constant taking in the proof of Lemma 2.15. Let A_X be a perturbation operator with respect to $D(\mathcal{F}_X)$. Then modulo exact forms, for $T \geq T'$, we have*

$$(2.70)$$
$$\widetilde{\eta}_g(\mathcal{F}_{Z,T}, 1 \widehat{\otimes} T A_X) = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \widetilde{\eta}_g(\mathcal{F}_X, A_X) - \widetilde{\mathrm{FLI}}_g \left(\nabla_T^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right).$$

Using Lemma 2.16, we could extend the anomaly formula Proposition 2.14 to the general case.

Theorem 2.17. *Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$ which have the same topological structure. Let A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$ and P, P' be the APS projections with respect to $D(\mathcal{F}) + A$, $D(\mathcal{F}') + A'$ respectively. For any $g \in G$, modulo exact forms, we have*

$$(2.71) \quad \widetilde{\eta}_g(\mathcal{F}', A') - \widetilde{\eta}_g(\mathcal{F}, A) = \widetilde{\mathrm{FLI}}_g(\mathcal{F}, \mathcal{F}') + \mathrm{ch}_g \left(\mathrm{sf}_G\{(D(\mathcal{F}') + A', P'), (D(\mathcal{F}) + A, P)\} \right).$$

Proof. We only need to prove the even case. We will add an additional dimension as in Example 1.4 d) such that the new family is odd.

Let $L \to S^1 \times S^1$ be the Hermitian line bundle in Example 1.4 c) with ∇^L constructed there. Let $\mathrm{pr}_2 : B \times S^1 \times S^1 \to S^1 \times S^1$ be the natural projection. Then all the bundles and geometric data could be pulled back on $B \times S^1 \times S^1$. Thus the fiber bundle $\mathrm{pr}_1 : B \times S^1 \times S^1 \to B$, which is the projection onto the first part, and the structures pulled back by pr_2 make a geometric family $\mathcal{F}_0$. In this case, $\mathrm{ind}(D(\mathcal{F}_0)) = 1$. The key observation is

$$(2.72) \qquad\qquad\qquad p_1!(p_1^* \mathcal{F} \cup p_2^* \mathcal{F}^L) = \mathcal{F} \cup \mathcal{F}_0.$$

Since $A\widehat{\otimes}1_{\mathcal{F}_0}$ is a perturbation operator of $D(\mathcal{F}\cup\mathcal{F}_0)$, we could choose $T'=1$ in Lemma 2.16. So by Lemma 2.12 and 2.16, we have

$$(2.73)\quad \tilde{\eta}_g(\mathcal{F},A) = \tilde{\eta}_g(\mathcal{F}\cup\mathcal{F}_0, A\widehat{\otimes}1_{\mathcal{F}_0})$$

$$= \int_{S^1}\tilde{\eta}_g(p_1^*\mathcal{F}\cup p_2^*\mathcal{F}^L, A\widehat{\otimes}1_{\mathcal{F}^L}) - \widetilde{\mathrm{FLI}}_g\left(\nabla^{T(Z\times S^1)},\nabla^{L_Z},\nabla^{TZ,TS^1},\nabla^{L_Z}\right).$$

Note that $D(p_1^*\mathcal{F}\cup p_2^*\mathcal{F}^L) = D(\mathcal{F})\otimes 1 + D^L$. By Proposition 2.14, the construction of the equivariant higher spectral flow for even case and (2.29), we have

$$(2.74)\quad \tilde{\eta}_g(\mathcal{F}',A') - \tilde{\eta}_g(\mathcal{F},A) = \int_{S^1}\left\{\tilde{\eta}_g(p_1^*\mathcal{F}'\cup p_2^*\mathcal{F}^L, A'\widehat{\otimes}1_{\mathcal{F}^L}) - \tilde{\eta}_g(p_1^*\mathcal{F}\cup p_2^*\mathcal{F}^L, A\widehat{\otimes}1_{\mathcal{F}^L})\right\}$$

$$+ \int_{S^1}\int_{Z^g}\left\{\widetilde{\mathrm{Td}}_g\left(\nabla^{T(S^1\times Z')},\nabla^{L_{Z'}},\nabla^{TS^1,TZ'},\nabla^{L_{Z'}}\right) - \widetilde{\mathrm{Td}}_g\left(\nabla^{T(S^1\times Z)},\nabla^{L_Z},\nabla^{TS^1,TZ},\nabla^{L_Z}\right)\right\}$$

$$= \int_{S^1}\int_{Z^g}\widetilde{\mathrm{Td}}_g\left(\nabla^{TS^1,TZ},\nabla^{L_Z},\nabla^{TS^1,TZ'},\nabla^{L_{Z'}}\right)$$

$$+ \int_{S^1}\mathrm{ch}_g\left(\mathrm{sf}_G\{(D(p_1^*\mathcal{F}'\cup p_2^*\mathcal{F}^L) + A'\widehat{\otimes}1, P_0'), (D(p_1^*\mathcal{F}\cup p_2^*\mathcal{F}^L) + A\widehat{\otimes}1, P_0)\}\right)$$

$$= \int_{Z^g}\widetilde{\mathrm{Td}}_g\left(\nabla^{TZ},\nabla^{L_Z},\nabla^{TZ'},\nabla^{L_{Z'}}\right) + \mathrm{ch}_g\left(\mathrm{sf}_G\{(D(\mathcal{F}') + A', P'), (D(\mathcal{F}) + A, P)\}\right),$$

where P_0, P_0' are the associated APS projections respectively. Note that in order to adapt the sign convention (0.8), the sign in the beginning of the second line of (2.74) is alternated.

The proof of Theorem 2.17 is complete.

$\square$

Using Theorem 2.17, we could write the Lemma 2.16 as a more elegant form.

Theorem 2.18. *Let A_Z and A_X be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then modulo exact forms, for $T\geq 1$ large enough, we have*

$$(2.75)\quad \tilde{\eta}_g(\mathcal{F}_Z, A_Z) = \int_{Y^g}\mathrm{Td}_g(\nabla^{TY},\nabla^{L_Y})\wedge\tilde{\eta}_g(\mathcal{F}_X, A_X) - \widetilde{\mathrm{FLI}}_g\left(\nabla^{TZ},\nabla^{L_Z},\nabla^{TY,TX},\nabla^{L_Z}\right)$$

$$+ \mathrm{ch}_g(\mathrm{sf}_G\{(D(\mathcal{F}_Z) + A_Z, P), (D(\mathcal{F}_{Z,T}) + 1\widehat{\otimes}TA_X, P')\}),$$

where P and P' are the associated APS projections respectively.

From Theorem 2.17 and 2.18, we could extend Lemma 2.12 to the general case.

Theorem 2.19. *(compare with [12, (24)]) Let $\mathcal{F}, \mathcal{F}'\in \mathrm{F}_G^*(B)$. Let A and A' be the perturbation operators with respect to $D(\mathcal{F})$ and $D(\mathcal{F}\cup\mathcal{F}')$. Then there exists $x\in K_G^*(B)$, such that*

$$(2.76)\qquad \tilde{\eta}_g(\mathcal{F}\cup\mathcal{F}', A') = \tilde{\eta}_g(\mathcal{F},A)\wedge\mathrm{FLI}_g(\mathcal{F}') + \mathrm{ch}_g(x).$$

Proof. Here we use a trick in [12] similarly as (2.72). Let $\pi': W'\to B$ be the submersion in $\mathcal{F}'$. We could get the pullback $\pi'^*\mathcal{F}$. Let $\pi'^*\mathcal{F}\otimes E'$ be the equivariant geometric family which is obtained from $\pi'^*\mathcal{F}$ by twisting with $\delta^*(\mathcal{S}_{Z'}\otimes E')$, where $\delta: W\times_B W'\to W'$. Then we have

$$(2.77)\qquad \mathcal{F}\cup\mathcal{F}' \simeq \pi'!(\pi'^*\mathcal{F}\otimes E').$$

Since the fibers of $\pi'^*W\to B$ is $Z'\times Z$, the fiberwised connection $\nabla^{T(Z'\times Z)} = \nabla^{TZ',TZ}$. So Theorem 2.19 follows from Theorem 2.18.

The proof of Theorem 2.19 is complete.

$\square$

Remark 2.20. When the parameter space B is a point, $\dim Z$ is odd, letting $A = P_{\ker D}$ be the orthogonal projection onto the kernel of $D(\mathcal{F})$, which we simply denote by D

$$(2.78) \quad \tilde{\eta}_g(\mathcal{F}, A) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr} \left[g(D + (u\chi(u))' P_{\ker D}) \exp(-(uD + u\chi(u)P_{\ker D})^2) \right] du$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr} \left[g(D + (u\chi(u))' P_{\ker D}) \exp(-u^2 D - u^2 \chi(u)^2 P_{\ker D}) \right] du$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr} \left[gD \exp(-u^2 D^2) \right] du + \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr} \left[g(u\chi(u))' P_{\ker D} \exp(-u^2 \chi(u)^2 P_{\ker D}) \right] du$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2} \mathrm{Tr}[gD \exp(-uD^2)] du + \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \exp(-u^2) du \cdot \mathrm{Tr}[g P_{\ker D}]$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2} \mathrm{Tr}[gD \exp(-uD^2)] du + \frac{1}{2} \mathrm{Tr}[g P_{\ker D}],$$

which is just the usual **equivariant reduced eta invariant** [18]. So Theorem 2.17 and 2.18 naturally degenerate to the case of equivariant reduced eta invariants and the higher spectral flow degenerates to the canonical spectral flow. Even for this case, our results are not found in the literatures.

2.6. **Proof of Lemma 2.16.** The proof of Lemma 2.16 is almost the same as the proof of [23, Theorem 2.4] and Assumption 2.1 and 2.3 in [23] naturally hold in our case.

Let $T' \geq 1$ be the constant taking in the proof of Lemma 2.15. For $T \geq T'$, let $\mathbb{B}_{u,T}$ be the Bismut superconnection associated with equivariant geometric family $\mathcal{F}_{Z,T}$. Let

$$(2.79) \qquad \widehat{\mathbb{B}}|_{(T,u)} = \mathbb{B}_{u^2,T} + uT\chi(uT)(1\widehat{\otimes}A_X) + dT \wedge \frac{\partial}{\partial T} + du \wedge \frac{\partial}{\partial u}.$$
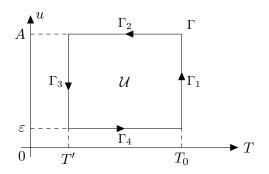
We define $\beta_g = du \wedge \beta_g^u + dT \wedge \beta_g^T$ to be the part of $\psi_B \widetilde{\mathrm{Tr}}[g \exp(-\widehat{\mathbb{B}}^2)]$ of degree one with respect to the coordinates (T, u), with functions $\beta_g^u, \beta_g^T : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B, \mathbb{C})$.

Comparing with [23, Proposition 3.2], there exists a smooth family $\alpha_g : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B, \mathbb{C})$ such that

$$(2.80) \qquad \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = dT \wedge du \wedge d^S \alpha_g.$$

Take ε, A, T_0, $0 < \varepsilon \leq 1 \leq A < \infty$, $T' \leq T_0 < \infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_{+,T} \times \mathbb{R}_{+,u}$.

The contour Γ is made of four oriented pieces $\Gamma_1, \cdots, \Gamma_4$ indicated in the above picture. For $1 \leq k \leq 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and (2.80),

$$(2.81) \qquad \sum_{k=1}^{4} I_k^0 = \int_{\partial \mathcal{U}} \beta_g = \int_{\mathcal{U}} \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = d^S \left(\int_{\mathcal{U}} \alpha_g dT \wedge du \right).$$

The following theorems are the analogues of Theorem 3.3-3.6 in [23]. We will sketch the proofs in the next subsection.

Theorem 2.21. *i) For any $u > 0$, we have*

$$(2.82) \qquad \lim_{T \to \infty} \beta_g^u(T, u) = 0.$$

ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 1$, we have

$$(2.83) \qquad |\beta_g^u(T, u)| \leq C.$$

iii) We have the following identity:

$$(2.84) \qquad \lim_{T \to +\infty} \int_1^{\infty} \beta_g^u(T, u) du = 0.$$

Theorem 2.22. *We have the following identity:*

$$(2.85) \qquad \lim_{u \to +\infty} \int_1^{\infty} \beta_g^T(T, u) dT = 0.$$

We know that $\widehat{A}_g(TZ, \nabla)$ only depends on $g \in G$ and $R := \nabla^2$. So we can denote it by $\widehat{A}_g(R)$. Let $R_T^{TZ} := (\nabla_T^{TZ})^2$. Set

$$(2.86) \qquad \gamma_\Omega(T) = - \left. \frac{\partial}{\partial b} \right|_{b=0} \widehat{A}_g \left(R_T^{TZ} + b \frac{\partial \nabla_T^{TZ}}{\partial T} \right).$$

By a standard argument in Chern-Weil theory, we know that

$$(2.87) \qquad \frac{\partial}{\partial T} \widetilde{\widehat{A}}_g(TZ, \nabla_{T'}^{TZ}, \nabla_T^{TZ}) = -\gamma_\Omega(T).$$

Proposition 2.23. *When $T \to +\infty$, we have $\gamma_\Omega(T) = O(T^{-2})$. Moreover, modulo exact forms on W^g, we have*

$$(2.88) \qquad \widetilde{\widehat{A}}_g(TZ, \nabla_{T'}^{TZ}, \nabla^{TY, TX}) = - \int_{T'}^{+\infty} \gamma_\Omega(T) dT.$$

Let $\mathbb{B}_{X,T}$ be the Bismut superconnection associated with the equivariant geometric family $\mathcal{F}_X$. Set

$$(2.89) \qquad \gamma_1(T) = \left\{ \psi_{V^g} \widetilde{\mathrm{Tr}}|_{V^g} \left[g \exp \left(- \left(\mathbb{B}_{X,T^2}|_{V^g} + T \chi(T) A_X|_{V^g} + dT \wedge \frac{\partial}{\partial T} \right)^2 \right) \right] \right\}^{dT}.$$

Then

$$(2.90) \qquad \widetilde{\eta}_g(\mathcal{F}_X, A_X) = - \int_0^{\infty} \gamma_1(T) dT.$$

Theorem 2.24. *i) For any $u > 0$, there exist $C > 0$ and $\delta > 0$ such that, for $T \geq T'$, we have*

$$(2.91) \qquad |\beta_g^T(T, u)| \leq \frac{C}{T^{1+\delta}}.$$

ii) For any $T > 0$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \beta_g^T(T\varepsilon^{-1}, \varepsilon) = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \gamma_1(T). \tag{2.92}$$

iii) There exists $C > 0$ such that for $\varepsilon \in (0, 1/T']$, $\varepsilon T' \leq T \leq 1$,

$$\varepsilon^{-1} \left| \beta_g^T(T\varepsilon^{-1}, \varepsilon) + \int_{Z^g} \gamma_\Omega(T\varepsilon^{-1}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \right| \leq C. \tag{2.93}$$

iv) There exist $\delta \in (0, 1]$, $C > 0$ such that, for $\varepsilon \in (0, 1]$, $T \geq 1$,

$$\varepsilon^{-1}|\beta_g^T(T\varepsilon^{-1}, \varepsilon)| \leq \frac{C}{T^{1+\delta}}. \tag{2.94}$$

Now using the theorems above, we can prove Lemma 2.16.

By (2.81), we know that

$$\int_\varepsilon^A \beta_g^u(T_0, u)du - \int_{T'}^{T_0} \beta_g^T(T, A)dT - \int_\varepsilon^A \beta_g^u(T', u)du + \int_{T'}^{T_0} \beta_g^T(T, \varepsilon)dT = I_1 + I_2 + I_3 + I_4 \tag{2.95}$$

is an exact form. We take the limits $A \to \infty$, $T_0 \to \infty$ and then $\varepsilon \to 0$ in the indicated order. Let I_j^k, $j = 1, 2, 3, 4$, $k = 1, 2, 3$ denote the value of the part I_j after the kth limit. By [17, §22, Theorem 17], $d\Omega(B)$ is closed under uniformly convergence on B. Thus,

$$\sum_{j=1}^4 I_j^3 \equiv 0 \bmod d\,\Omega^*(B, \mathbb{C}). \tag{2.96}$$

Since the definition of the equivariant eta form does not depend on the cut-off function in the definition, from (2.43), we obtain that modulo exact forms,

$$I_3^3 = \tilde{\eta}_g(\mathcal{F}_{Z,T'}, T'(1\widehat{\otimes}A_X)). \tag{2.97}$$

Furthermore, by Theorem 2.22, we get

$$I_2^2 = I_2^3 = 0. \tag{2.98}$$

From Theorem 2.21, we have

$$I_1^3 = 0. \tag{2.99}$$

Finally, using Theorem 2.24, we get

$$I_4^3 = -\int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \tilde{\eta}_g(\mathcal{F}_X, A_X) + \widetilde{\mathrm{FLI}}_g\left(\nabla_{T'}^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z}\right) \tag{2.100}$$

as follows: We write

$$\int_{T'}^{+\infty} \beta_g^T(T, \varepsilon)dT = \int_{\varepsilon T'}^{+\infty} \varepsilon^{-1} \beta_g^T(T\varepsilon^{-1}, \varepsilon)dT. \tag{2.101}$$

Convergence of the integrals above is granted by (2.91). Using (2.92), (2.94) and Proposition 2.23, we get

$$\lim_{\varepsilon \to 0} \int_1^{+\infty} \varepsilon^{-1} \beta_g^T(T\varepsilon^{-1}, \varepsilon)dT = \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \int_1^{+\infty} \gamma_1(T)dT \tag{2.102}$$

and

$$(2.103) \quad \lim_{\varepsilon \to 0} \int_{\varepsilon T'}^{1} \varepsilon^{-1} \left[\beta_g^T(T\varepsilon^{-1}, \varepsilon) dT + \int_{Z^g} \gamma_\Omega(T\varepsilon^{-1}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \right] dT$$

$$= \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y}) \wedge \int_0^1 \gamma_1(T) dT.$$

The remaining part of the integral yields by (2.93)

$$(2.104) \quad \lim_{\varepsilon \to 0} \int_{\varepsilon T'}^{1} \varepsilon^{-1} \int_{Z^g} \gamma_\Omega(T\varepsilon^{-1}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) dT$$

$$= \int_{Z^g} \int_{T'}^{+\infty} \gamma_\Omega(T) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) dT = -\widetilde{\mathrm{FLI}}_g \left(\nabla_{T'}^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right).$$

These four equations for I_k^3, $k = 1, 2, 3, 4$, and (2.50) imply Lemma 2.16.

2.7. Proofs of Theorem 2.21-2.24.

Since $\ker(D(\mathcal{F}_X) + A_X) = 0$, the proofs of Theorem 2.21-2.24 in our case are much easier. We only need to replace D^X and D_T^Z somewhere in [23] by $D(\mathcal{F}_X) + A_X$ and $D(\mathcal{F}_{Z,T}) + 1 \widehat{\otimes} T A_X$ and take care in the local index computation in the proof of Theorem 2.24 ii). In this subsection, we sketch the local index part here.

Set

$$(2.105) \qquad \mathcal{B}'_{\varepsilon, T/\varepsilon} = (\mathbb{B}_{\varepsilon^2, T/\varepsilon} + T\chi(T)A_X)^2 + \varepsilon^{-1} dT \wedge \left. \frac{\partial(\mathbb{B}_{\varepsilon^2, T'} + \varepsilon T' \chi(\varepsilon T') A_X)}{\partial T'} \right|_{T' = T\varepsilon^{-1}}.$$

By the definition of $\beta_g^T(T, \varepsilon)$, we have

$$(2.106) \qquad \varepsilon^{-1} \beta_g^T(T/\varepsilon, \varepsilon) = \left\{ \psi_S \widetilde{\mathrm{Tr}}[g \exp(-\mathcal{B}'_{\varepsilon, T/\varepsilon})] \right\}^{dT}.$$

Let S_X be the tensor in (2.19) with respect to π_X. Let $\{e_i\}$, $\{f_p\}$ and $\{g_\alpha\}$ be the locally orthonormal basis of TX, TY and TB and $\{f_{p,1}^H\}$ and $\{g_{\alpha,3}^H\}$ be the corresponding horizontal lifts. Precisely, by (2.23), we have

$$(2.107) \quad \varepsilon^{-1} \left. \frac{\partial(\mathbb{B}_{\varepsilon^2, T'} + \varepsilon T' \chi(\varepsilon T') A_X)}{\partial T'} \right|_{T' = T\varepsilon^{-1}} = D^X + \chi(T)A_X + T\chi'(T)A_X$$

$$- \frac{1}{8T^2}(\langle \varepsilon^2 [f_{p,1}^H, f_{q,1}^H], e_i \rangle c(e_i) c(f_{p,1}^H) c(f_{q,1}^H)$$

$$+ 4\varepsilon \langle S_X(g_{\alpha,3}^H) e_i, f_{p,1}^H \rangle c(e_i) c(f_{p,1}^H) g_3^\alpha \wedge + \langle [g_{\alpha,3}^H, g_{\beta,3}^H], e_i \rangle c(e_i) g^\alpha \wedge g^\beta \wedge).$$

As in (2.105), we set

$$(2.108) \qquad \mathcal{B}''_{T^2}|_{V^g} = (\mathbb{B}_{X,T^2}|_{V^g} + T\chi(T)A_X)^2 + dT \wedge \left. \frac{\partial(\mathbb{B}_{X,T^2} + T\chi(T)A_X)}{\partial T} \right|_{V^g}.$$

Then by (2.89), we have

$$(2.109) \qquad \gamma_1(T) = \left\{ \psi_{V^g} \widetilde{\mathrm{Tr}}[g \exp(-\mathcal{B}''_{T^2}|_{V^g})] \right\}^{dT}.$$

As the same process in [23, Section 6], we could localize the problem near $\pi_X^{-1}(V^g)$ and define the operator $\mathcal{B}'_{\varepsilon, T/\varepsilon}$ to a neighborhood of $\{0\} \times X_{y_0}$ in $T_{y_0}Y \times X_{y_0}$.

Let d^V, d^W be the distance functions on V, W associated to g^{TV}, g^{TW}. Let Inj^V, Inj^W be the injective radius of V, W. In the sequel, we assume that given $0 < \alpha < \alpha_0 < \inf\{\mathrm{Inj}^V, \mathrm{Inj}^W\}$

are chosen small enough so that if $y \in V$, $d^V(g^{-1}y, y) \le \alpha$, then $d^V(y, V^g) \le \frac{1}{4}\alpha_0$, and if $z \in W$, $d^W(g^{-1}z, z) \le \alpha$, then $d^W(z, W^g) \le \frac{1}{4}\alpha_0$. Let $\rho : T_{y_0}Y \to [0,1]$ be a smooth function such that

$$(2.110) \qquad \rho(U) = \begin{cases} 1, & |U| \le \alpha_0/4; \\ 0, & |U| \ge \alpha_0/2. \end{cases}$$

Let Δ^{TY} be the ordinary Laplacian operator on $T_{y_0}Y$.

Set

$$(2.111) \qquad L^1_{\varepsilon,T} = (1 - \rho^2(U))(-\varepsilon^2 \Delta^{TY} + T^2(D^X + A_X)^2_{y_0}) + \rho^2(U)\mathcal{B}'_{\varepsilon, T/\varepsilon}.$$

For $(U, x) \in N_{Y^g/Y, y_0} \times X_{y_0}$, $|U| < \alpha_0/4$, $\varepsilon > 0$, set

$$(2.112) \qquad (S_\varepsilon s)(U, x) = s(U/\varepsilon, x).$$

Put

$$(2.113) \qquad L^2_{\varepsilon,T} := S_\varepsilon^{-1} L^1_{\varepsilon,T} S_\varepsilon.$$

Let $\dim T_{y_0}Y^g = l'$ and $\dim N_{Y^g/Y, y_0} = 2l''$. Let $\{f_1, \cdots, f_{l'}\}$ be an orthonormal basis of $T_{y_0}Y^g$ and let $\{f_{l'+1}, \cdots, f_{l'+2l''}\}$ be an orthonormal basis of $N_{Y^g/Y, y_0}$. For $\alpha \in \mathbb{C}(f^p \wedge i_{f_p})_{1 \le p \le l'}$, let $[\alpha]^{max} \in \mathbb{C}$ be the coefficient of $f^1 \wedge \cdots \wedge f^{l'}$ in the expansion of α. Let R_ε be a rescaling such that

$$(2.114) \qquad \begin{aligned} R_\varepsilon(c(e_i)) &= c(e_i), \\ R_\varepsilon(c(f^H_{p,1})) &= \frac{f^{p,H}_1 \wedge}{\varepsilon} - \varepsilon\, i_{f^H_{p,1}}, \quad \text{for } 1 \le p \le l', \\ R_\varepsilon(c(f^H_{p,1})) &= c(f^H_{p,1}), \quad \text{for } l' + 1 \le p \le l' + 2l''. \end{aligned}$$

Then R_ε is a Clifford algebra homomorphism. Set

$$(2.115) \qquad L^3_{\varepsilon,T} = R_\varepsilon(L^2_{\varepsilon,T}).$$

Corresponding to [23, Lemma 3.4], from (2.106), (2.107) and (2.108), we have

Lemma 2.25. *When $\varepsilon \to 0$, the limit $L^3_{0,T} = \lim_{\varepsilon \to 0} L^3_{\varepsilon,T}$ exists and*

$$(2.116) \qquad L^3_{0,T}|_{V^g} = -\left(\partial_p + \frac{1}{4}\langle R^{TY}|_{V^g}U, f^H_{p,1}\rangle\right)^2 + \frac{1}{2}R^{L_Y}|_{V^g} + \mathcal{B}''_{T^2}|_{V^g}.$$

So all the computations here are the same as [23, Section 6].

3. Equivariant differntial K-theory

In this section, we assume that the the G-action on B is almost free (with finite stabilizers only). With this action, we construct an analytic model of equivariant differential K-theory and prove some properties using the results in Section 2.

3.1. Equivariant differential K-theory.
In this subsection, we construct an analytic model of equivariant differential K-theory. When $G = \{e\}$, this construction is the same as that in [12].

Let E be a G-vector bundle over B. Then its restriction to B^g is acted on fibrewisely by g for $g \in G$. So it decomposes as a direct sum of subbundles E_ζ for each eigenvalue ζ of g. Set $\phi_g(E) := \sum \zeta E_\zeta$. Then it induces a homomorphism (for K^1_G replace B by $B \times S^1$)

$$(3.1) \qquad \phi_g : K^*_G(B) \otimes \mathbb{C} \longrightarrow [K^*(B^g) \otimes \mathbb{C}]^{C_G(g)},$$

where $\mathrm{C}_G(g)$ is the centralizer of g in G. Let (g) be the conjugacy class of $g \in G$. For $g, g' \in (g)$, there exists $h \in G$, such that $g' = h^{-1}gh$. Furthermore, the map

$$(3.2) \qquad\qquad h : B^{g'}/\mathrm{C}_G(g') \to B^g/\mathrm{C}_G(g)$$

is a homeomorphism . So $[K^*(B^g) \otimes \mathbb{C}]^{\mathrm{C}_G(g)} \simeq [K^*(B^{g'}) \otimes \mathbb{C}]^{\mathrm{C}_G(g')}$. By [1, Corollary 3.13], we know that the additive decomposition

$$(3.3) \qquad\qquad \phi = \oplus_{(g),g \in G}\phi_g : K_G^*(B) \otimes \mathbb{C} \to \bigoplus_{(g),g \in G} [K^*(B^g) \otimes \mathbb{C}]^{\mathrm{C}_G(g)}$$

is an isomorphism, where (g) ranges over the conjugacy classes of G.

Since B, G are compact and the group action is with finite stabilizers, the direct sum in (3.3) only has finite terms and does not depend the choice of the element in (g).

From (3.2), we also know that the map h^* induces an isomorphism

$$(3.4) \qquad\qquad h^* : [\Omega^*(B^g, \mathbb{C})]^{\mathrm{C}_G(g)} \to [\Omega^*(B^{g'}, \mathbb{C})]^{\mathrm{C}_G(g')}.$$

We denote by

$$(3.5) \qquad\qquad \Omega_{deloc,G}^*(B, \mathbb{C}) := \bigoplus_{(g),g \in G} \left\{ [\Omega^*(B^g, \mathbb{C})]^{\mathrm{C}_G(g)} \right\}$$

the set of delocalized differential forms, where $\{\cdot\}$ denotes the isomorphic class in sense of (3.4). The definition above does not depend on the choice of $g \in (g)$. It is easy to see that the exterior differential operator d preserves $\Omega_{deloc,G}^*(B, \mathbb{C})$. We denote by the delocalized de Rham cohomology $H_{deloc,G}^*(B, \mathbb{C})$ the cohomology of the differential complex $(\Omega_{deloc,G}^*(B, \mathbb{C}), d)$.

Then from (3.1) and (3.3), the equivariant Chern character isomorphism can be naturally defined by

$$\mathrm{ch}_G : K_G^*(B) \otimes \mathbb{C} \xrightarrow{\simeq} H_{deloc,G}^*(B, \mathbb{C}),$$
$$(3.6) \qquad\qquad \mathcal{K} \quad \mapsto \bigoplus_{(g),g \in G} \{\mathrm{ch}(\phi_g(\mathcal{K}))\}.$$

We note that $\mathrm{ch}(\phi_g(\mathcal{K})) = \mathrm{ch}_g(\mathcal{K})$ is $\mathrm{C}_G(g)$-invariant by the definition.

Observe that the fixed-point set for g-action coincides with that for g^{-1}-action. Set (compare with [30, (1)])

$$(3.7) \qquad H_{deloc,G}^*(B, \mathbb{R}) = \{c = \oplus_{(g),g \in G}\{c_g\} \in H_{deloc,G}^*(B, \mathbb{C}) : \forall g \in G, c_{g^{-1}} = \overline{c_g}\}.$$

Let $\Omega_{deloc,G}^*(B, \mathbb{R}) \subset \Omega_{deloc,G}^*(B, \mathbb{C})$ be the ring of forms $\omega = \oplus_{(g),g \in G}\{\omega_g\}$, such that $\forall g \in G, \omega_{g^{-1}} = \overline{\omega_g}$. Then $H_{deloc,G}^*(B, \mathbb{R})$ is the cohomology of the differential complex $(\Omega_{deloc,G}^*(B, \mathbb{R}), d)$. From (3.6), we have the isomorphism

$$(3.8) \qquad\qquad \mathrm{ch}_G : K_G^*(B) \otimes \mathbb{R} \xrightarrow{\simeq} H_{deloc,G}^*(B, \mathbb{R}).$$

Definition 3.1. (compare with [12, Definition 2.4]) A cycle for an equivariant differential K-theory class over B is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F} \in \mathrm{F}_G^*(B)$ and $\rho \in \Omega_{deloc,G}^*(B, \mathbb{R})/\mathrm{Im}\, d$. The cycle $(\mathcal{F}, \rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega_{deloc,G}^{odd}(B, \mathbb{R})/\mathrm{Im}\, d$ (resp. $\rho \in \Omega_{deloc,G}^{even}(B, \mathbb{R})/\mathrm{Im}\, d$). Two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho = \rho'$. Let $\widehat{\mathrm{IC}}_G^0(B)$ (resp. $\widehat{\mathrm{IC}}_G^1(B)$) denote the set of isomorphism classes of even (resp. odd) cycles over B with a natural abelian semi-group structure by $(\mathcal{F}, \rho) + (\mathcal{F}', \rho') = (\mathcal{F} + \mathcal{F}', \rho + \rho')$.

For $\mathcal{F} \in \mathrm{F}_G^*(B)$, we assume that there exists a perturbation operator A with respect to $D(\mathcal{F})$. For any $g \in G$, by Definition 2.11, the equivariant eta form restricted on the fixed point set of g is $\mathrm{C}_G(g)$-invariant, that is, $\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}) \in [\Omega^*(B^g, \mathbb{C})]^{\mathrm{C}_G(g)}$. Let h^* be map in (3.4). Since the perturbation operator A is equivariant, from Definition 2.11, we have

$$(3.9) \qquad \widetilde{\eta}_{g'}(\mathcal{F}|_{B^{g'}}, A|_{B^{g'}}) = h^* \widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}).$$

Since $\widetilde{\eta}_{g^{-1}}(\mathcal{F}|_{B^g}, A|_{B^g}) = \overline{\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g})}$, the following definition is well-defined.

Definition 3.2. The delocalized eta form $\widetilde{\eta}_G(\mathcal{F}, A)$ is defined by

$$(3.10) \qquad \widetilde{\eta}_G(\mathcal{F}, A) = \bigoplus_{(g), g \in G} \{\widetilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g})\} \in \Omega_{deloc, G}^*(B, \mathbb{R}).$$

By the same process, we can define

$$(3.11) \qquad \mathrm{FLI}_G(\mathcal{F}) = \bigoplus_{(g), g \in G} \{\mathrm{FLI}_g(\mathcal{F})\} \in \Omega_{deloc, G}^*(B, \mathbb{R}).$$

Let $\mathcal{F} \in \mathrm{F}_G^*(B)$ and A be a perturbation operator with respect to $D(\mathcal{F})$. Then by Definition 1.3, there exists a perturbation operator A^{op} with respect to $D(\mathcal{F}^{\mathrm{op}})$ such that

$$(3.12) \qquad \widetilde{\eta}_G(\mathcal{F}^{\mathrm{op}}, A^{\mathrm{op}}) = -\widetilde{\eta}_G(\mathcal{F}, A).$$

Let $\mathcal{F}, \mathcal{F}' \in \mathrm{F}_G^*(B)$, A, A' be perturbation operators with respect to $D(\mathcal{F})$, $D(\mathcal{F}')$. By Definition 2.11, we have

$$(3.13) \qquad \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}', A \sqcup_B A') = \widetilde{\eta}_G(\mathcal{F}, A) + \widetilde{\eta}_G(\mathcal{F}', A').$$

As in [12, page 19], from Remark 2.4 , we know that for any $\mathcal{F} \in \mathrm{F}_G^*(B)$, there exists a perturbation operator A with respect to $D(\mathcal{F} + \mathcal{F}^{\mathrm{op}})$ such that

$$(3.14) \qquad \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}^{\mathrm{op}}, A) = 0.$$

Definition 3.3. (compare with [12, Definition 2.10]) We call two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ **paired** if $\mathrm{ind}(D(\mathcal{F})) = \mathrm{ind}(D(\mathcal{F}'))$, and there exists a perturbation operator A with respect to $D(\mathcal{F} + \mathcal{F}'^{\mathrm{op}})$ such that

$$(3.15) \qquad \rho - \rho' = \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A).$$

Let $\sim$ denote the equivalence relation generated by the relation "paired".

From (3.12), (3.13) and (3.14), we have

Lemma 3.4. *(compare with* [12, Lemma 2.11, 2.12]*) The relation "paired" is symmetric, reflexive and compatible with the semigroup structure on* $\widehat{\mathrm{IC}}_G^*(B)$.

Definition 3.5. (compare with [12, Definition 2.14]) The equivariant differential K-theory $\widehat{K}_G^0(B)$ (resp. $\widehat{K}_G^1(B)$) is the group completion of the abelian semigroup $\widehat{\mathrm{IC}}_G^{\mathrm{even}}(B)/\sim$ (resp. $\widehat{\mathrm{IC}}_G^{\mathrm{odd}}(B)/\sim$).

Remark 3.6. Note that in [30, Section 3.3], the author constructed a geometric model of equivariant differential K^0 theory under the finite group action. From (2.45) and the explanations after it, we could easily see that when G is finite, the definition of $\widehat{K}_G^0$ in that paper is isomorphic to that of ours.

If $(\mathcal{F}, \rho) \in \widehat{\mathrm{IC}}^*_G(B)$, we denote by $[\mathcal{F}, \rho] \in \widehat{K}^*_G(B)$ the corresponding class in equivariant differential K-theory. From (3.12)-(3.14), for any $[\mathcal{F}, \rho], [\mathcal{F}', \rho'] \in \widehat{K}^i_G(B)$, $i = 0, 1$, we have

$$(3.16) \qquad [\mathcal{F}, \rho] = [\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, \rho - \rho'] + [\mathcal{F}', \rho'].$$

So every element of $\widehat{K}^*_G(B)$ can be represented in the form $[\mathcal{F}, \rho]$. Furthermore, we have $-[\mathcal{F}, \rho] = [\mathcal{F}^{\mathrm{op}}, -\rho]$.

3.2. Push-forward map.

In this subsection, we construct a well-defined push-forward map in equivariant differential K-theory and prove the functoriality of it using the theorems in Section 2. This solves a question proposed in [12] when $G = \{e\}$. We use the notations in Section 1.4.

Let $\pi_Y : V \to B$ be an equivariant submersion of closed smooth G-manifolds with closed Spin^c fiber Y. We assume that the G-action on B is almost free. Thus, so is the action on V.

For $g \in G$, the fixed point set V^g is the total space of the fiber bundle $\pi_Y|_{V^g} : V^g \to B^g$ with Y^g. Since the pull back isomorphism h^* in (3.4) commutes with the integral along the fiber, for $\alpha = \oplus_{(g), g \in G}\{\alpha_g\} \in \Omega^*_{deloc,G}(V, \mathbb{R})$, the integral

$$(3.17) \qquad \int_{Y,G} \alpha := \bigoplus_{(g), g \in G} \left\{ \int_{Y^g} \alpha_g \right\} \in \Omega^*_{deloc,G}(B, \mathbb{R})$$

does not depend on $g \in (g)$. So it defines an integral map

$$(3.18) \qquad \int_{Y,G} : \Omega^*_{deloc,G}(V, \mathbb{R}) \to \Omega^*_{deloc,G}(B, \mathbb{R}).$$

Consider the set $\widehat{\mathcal{O}}^*_G(\pi_Y)$ of equivariant geometric data $\widehat{o}_Y = (T^H_{\pi_Y}V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$, where $\sigma_Y \in \Omega^{odd}_{deloc,G}(V)/\mathrm{Im}d$.

Let

$$(3.19) \qquad \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) = \bigoplus_{(g), g \in G} \{\mathrm{Td}_g(\nabla^{TY}, \nabla^{L_Y})\} \in \Omega^*_{deloc,G}(V, \mathbb{R}).$$

Let $\widehat{o}'_Y = (T'^H_{\pi_Y}V, g'^{TY}, \nabla'^{L_Y}, \sigma'_Y) \in \widehat{\mathcal{O}}^*_G(\pi_Y)$ be another equivariant tuple with the same equivariant K-orientation in Definition 1.7. As in (2.62), from [27, Theorem B.5.4], we can construct the Chern-Simons form $\widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \in \Omega^{odd}_{deloc,G}(V)/\mathrm{Im}d$ such that

$$(3.20) \qquad d\,\widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) = \mathrm{Td}_G(\nabla'^{TY}, \nabla'^{L_Y}) - \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}).$$

We introduce a relation $\widehat{o}_Y \sim \widehat{o}'_Y$ as in [12]. Two equivariant tuples $\widehat{o}_Y$, $\widehat{o}'_Y$ are related if and only if

$$(3.21) \qquad \sigma'_Y - \sigma_Y = \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}),$$

where we mark the objects associated with the second tuple by $'$.

Definition 3.7. (compare with [12, Definition 3.5]) The set of equivariant differential K-orientations is the set of equivalence classes $\widehat{\mathcal{O}}^*_G(\pi_Y))/\sim$.

We now start with the construction of the push-forward map $\widehat{\pi}_Y! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$ for a given equivariant differential K-orientation which extends Theorem 1.8 to the differential case.

For $[\mathcal{F}_X, \rho] \in \widehat{K}_G^*(V)$, let $\mathcal{F}_Z$ be the equivariant geometric family defined in (1.25). We define

$$(3.22) \quad \widehat{\pi}_Y!([\mathcal{F}_X, \rho]) = \left[\mathcal{F}_Z, \int_{Y,G} \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) \wedge \rho - \widetilde{\mathrm{FLI}}_G \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right) \right.$$
$$\left. + \int_{Y,G} \sigma_Y \wedge (\mathrm{FLI}_G(\mathcal{F}_X) - d\rho) \right] \in \widehat{K}_G^*(B),$$

where $\widetilde{\mathrm{FLI}}_G \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right) = \bigoplus_{(g),g \in G} \widetilde{\mathrm{FLI}}_g \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right) \in \Omega_{deloc,G}^*(B, \mathbb{R})$.

Theorem 3.8. *(compare with [12, Lemma 3.14]) The map $\widehat{\pi}_Y! : \widehat{K}_G^*(V) \to \widehat{K}_G^*(B)$ in (3.22) is well-defined.*

Proof. Let $(\mathcal{F}_X, \rho)$, $(\mathcal{F}_X', \rho')$ be two cycles over V. By (3.22), we have

$$(3.23) \qquad\qquad \widehat{\pi}_Y!(\mathcal{F}_X, \rho) - \widehat{\pi}_Y!(\mathcal{F}_X', \rho') = \widehat{\pi}_Y!(\mathcal{F}_X \sqcup \mathcal{F}_X'^{\mathrm{op}}, \rho - \rho').$$

If $(\mathcal{F}_X, \rho)$ is paired with $(\mathcal{F}_X', \rho')$, there exists a perturbation operator A, such that

$$(3.24) \qquad\qquad \rho - \rho' = \widetilde{\eta}_G(\mathcal{F}_X \sqcup \mathcal{F}_X'^{\mathrm{op}}, A).$$

So we only need to prove that if there exists a perturbation operator A_X with respect to $D(\mathcal{F}_X)$, $\widehat{\pi}_Y!([\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, A_X)]) = 0 \in \widehat{K}_G^*(B)$.

From (3.22), we have

$$(3.25) \quad \widehat{\pi}_Y!([\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, A_X)]) = \left[\mathcal{F}_Z, \int_{Y,G} \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) \wedge \widetilde{\eta}_G(\mathcal{F}_X, A_X)) \right.$$
$$\left. - \widetilde{\mathrm{FLI}}_G \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right) + \int_{Y,G} \sigma_Y \wedge (\mathrm{FLI}_G(\mathcal{F}_X) - d\widetilde{\eta}_G(\mathcal{F}_X, A_X)) \right].$$

From Proposition 2.3 (iii) and Lemma 2.15, there exists a perturbation operator A_Z with respect to $D(\mathcal{F}_Z)$. By (2.44), (3.13), (3.17) and Theorem 2.18, there exists $x \in K_G^*(B)$ such that

$$(3.26) \qquad\qquad \widehat{\pi}_Y!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, A_X)) = [\mathcal{F}_Z, \widetilde{\eta}_G(\mathcal{F}_Z, A_Z) - \mathrm{ch}_G(x)].$$

From Proposition 2.7, there exist $\mathcal{F} \in \mathrm{F}_G^*(B)$ and equivariant spectral sections P, Q with respect to $D(\mathcal{F})$, such that $[P - Q] = x$. Let A_P, A_Q be the perturbation operators associated with P, Q. From Theorem 2.17, we have

$$(3.27) \qquad\qquad \mathrm{ch}_G(x) = \widetilde{\eta}_G(\mathcal{F}, A_P) - \widetilde{\eta}_G(\mathcal{F}, A_Q).$$

By (3.12), (3.26), (3.27) and Definition 3.3, we have

$$(3.28) \quad \widehat{\pi}_Y!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, A_X)) = \left[\mathcal{F}_Z, \widetilde{\eta}_G(\mathcal{F}_Z, A_Z) - \widetilde{\eta}_G(\mathcal{F}, A_P) - \widetilde{\eta}_G(\mathcal{F}^{\mathrm{op}}, A_Q^{\mathrm{op}}) \right]$$
$$= [\mathcal{F} + \mathcal{F}^{\mathrm{op}}, 0] = [\mathcal{F}, 0] - [\mathcal{F}, 0] = 0 \in \widehat{K}_G^*(B).$$

Then from Theorem 1.8, we complete the proof of Theorem 3.8. $\qquad\square$

Here our construction of $\widehat{\pi}_Y!$ involve an explicit choice of a representative $\widehat{o}_Y = (T_{\pi_Y}^H V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$ of the equivariant differential K-orientation. In fact, it does not depend on the choice.

Lemma 3.9. *(compare with [12, Lemma 3.17]) The homomorphism $\widehat{\pi}_Y! : \widehat{K}_G^*(V) \to \widehat{K}_G^*(B)$ only depend on the equivariant differential K-orientation.*

Proof. Let $\widehat{o}_Y = (T^H_{\pi_Y}V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$, $\widehat{o}'_Y = (T'^H_{\pi_Y}V, g'^{TY}, \nabla'^{L_Y}, \sigma'_Y)$ be two representatives of an equivariant differential K-orientation. We will mark the objects associated with the second representative by $'$. From (2.63), we could get

(3.29)
$$\widetilde{\mathrm{FLI}}_G(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z}) = \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge \mathrm{FLI}_G(\mathcal{F}_X).$$

Then from (3.20), (3.21) and (3.29), we have

(3.30) $\quad \widehat{\pi}'_Y!([\mathcal{F}_X, \rho]) - \widehat{\pi}_Y!([\mathcal{F}_X, \rho]) = [\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z,$

$$\int_{Y,G} \left(\mathrm{Td}_G(\nabla'^{TY}, \nabla'^{L_Y}) - \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) \right) \wedge \rho - \widetilde{\mathrm{FLI}}_G \left(\nabla'^{TZ}, \nabla'^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z} \right)$$

$$+ \widetilde{\mathrm{FLI}}_G \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{TY,TX}, \nabla^{L_Z} \right) + \int_{Y,G} (\sigma'_Y - \sigma_Y) \wedge (\mathrm{FLI}_G(\mathcal{F}_X) - d\rho) \Big]$$

$$= \Big[\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z, \int_{Y,G} d\,\widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge \rho$$

$$- \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge d\rho + \int_{Y,G} \widetilde{\mathrm{Td}}_G(\nabla^{TY}, \nabla^{L_Y}, \nabla'^{TY}, \nabla'^{L_Y}) \wedge \mathrm{FLI}_G(\mathcal{F}_X)$$

$$+ \widetilde{\mathrm{FLI}}_G \left(\nabla^{TZ}, \nabla^{L_Z}, \nabla'^{TZ}, \nabla'^{L_Z} \right) - \widetilde{\mathrm{FLI}}_G \left(\nabla^{TY,TX}, \nabla^{L_Z}, \nabla'^{TY,TX}, \nabla'^{L_Z} \right) \Big]$$

$$= [\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z, \widetilde{\mathrm{FLI}}_G \left(\mathcal{F}_Z, \mathcal{F}'_Z \right)].$$

By Proposition 2.3 (iii) and Lemma 2.15, there exists a perturbation operator A with respect to $D(\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z)$. By Theorem 2.17 and (3.14), there exists $x \in K^*_G(B)$ such that

(3.31) $\qquad \widetilde{\mathrm{FLI}}_G \left(\mathcal{F}_Z, \mathcal{F}'_Z \right) = \widetilde{\mathrm{FLI}}_G \left(\mathcal{F}_Z + \mathcal{F}^{\mathrm{op}}_Z, \mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z \right) = -\widetilde{\eta}_G(\mathcal{F}'_Z + \mathcal{F}^{\mathrm{op}}_Z, A) + \mathrm{ch}_G(x).$

Following the same process in (3.26)-(3.28), we have $\widehat{\pi}'_Y!([\mathcal{F}_X, \rho]) = \widehat{\pi}_Y!([\mathcal{F}_X, \rho])$.

The proof of Lemma 3.9 is complete. $\qquad\qquad\square$

We now discuss the functoriality of the push-forward maps with respect to the composition of fiber bundles. Let $\pi_Y : V \to B$ with fiber Y be as in the above subsection together with a representative of an equivariant differential K-orientation $\widehat{o}_Y = (T^H_{\pi_Y}V, g^{TY}, \nabla^{L_Y}, \sigma_Y)$. Let $\pi_U : B \to S$ be another equivariant proper submersion with closed fibers U together with a representative of an equivariant differential K-orientation $\widehat{o}_U = (T^H_{\pi_U}B, g^{TU}, \nabla^{L_U}, \sigma_U)$.

Let $\pi_A := \pi_U \circ \pi_Y : V \to S$ be the composition of two submersions with fibers A. Let $T^H_{\pi_A}V$ be a horizontal subbundle associated with π_A. We assume that $T^H_{\pi_A}V \subset T^H_{\pi_Y}V$. Set $g^{TA} = \pi^*_Y g^{TU} \oplus g^{TY}$, $\nabla^{L_A} = \pi^*_Y \nabla^{L_U} \otimes \nabla^{L_Y}$.

Definition 3.10. (compare with [12, Definition 3.21]) We define the composite $\widehat{o}_A = \widehat{o}_U \circ \widehat{o}_Y$ of the representatives of equivariant differential K-orientations of π_Y and π_U by

(3.32) $$\widehat{o}_A := (T^H_{\pi_A}V, g^{TA}, \nabla^{L_A}, \sigma_A),$$

where

(3.33) $\quad \sigma_A := \sigma_Y \wedge \pi^*_Y \mathrm{Td}_G(\nabla^{TU}, \nabla^{L_U}) + \mathrm{Td}_G(\nabla^{TY}, \nabla^{L_Y}) \wedge \pi^*_Y \sigma_U$

$$+ \widetilde{\mathrm{Td}}_G(\nabla^{TA}, \nabla^{L_A}, \nabla^{TU,TY}, \nabla^{L_A}) - d\sigma_Y \wedge \pi^*_Y \sigma_U.$$

Theorem 3.11. *(compare with* [12, Theorem 3.23]*) We have the equality of homomorphisms* $\widehat{K}_G^*(V) \to \widehat{K}_G^*(S)$

$$(3.34) \qquad\qquad\qquad \widehat{\pi}_A! = \widehat{\pi}_U! \circ \widehat{\pi}_Y!.$$

Proof. The topological part of Theorem 3.11 is just Theorem 1.9 and the differential part follows from a direct calculation using (3.22) and (3.33). □

3.3. **Cup product.**

In this subsection, we construct the cup product of equivariant differential K-theory in our model as in [12, 14] and prove the desired properties.

Let $[\mathcal{F}, \rho] \in \widehat{K}_G^i(B)$ and $[\mathcal{F}', \rho'] \in \widehat{K}_G^*(B)$, where $i = 0, 1$. We define (compare with [12, Definition 4.1])

$$(3.35) \qquad [\mathcal{F}, \rho] \cup [\mathcal{F}', \rho'] := [\mathcal{F} \cup \mathcal{F}', (-1)^i \mathrm{FLI}_G(\mathcal{F}) \wedge \rho' + \rho \wedge \mathrm{FLI}_G(\mathcal{F}') - (-1)^i d\rho \wedge \rho'].$$

Proposition 3.12. *(compare with* [12, Proposition 4.2, 4.5]*) (i) The product is well defined. It turns* $B \mapsto \widehat{K}_G^*(B)$ *into a functor from closed smooth G-manifolds to unital graded commutative rings. The unit is simply given by* $[\mathcal{F}, 0]$*, where $\mathcal{F}$ is the equivariant geometric family in Example 1.4 a) such that E_+ is 1 dimensional trivial representation and $E_- = 0$.*

(ii) The product is associative.

(iii) Let $\pi_U : B \to S$ be an equivariant proper submersion with an equivariant differential K-orientation. For $x \in \widehat{K}_G^*(B)$ and $y \in \widehat{K}_G^*(S)$, we have

$$(3.36) \qquad\qquad\qquad \widehat{\pi}_U!(\pi_U^* y \cup x) = y \cup \widehat{\pi}_U!(x).$$

Proof. The product is obviously biadditive.

Let $f : B_1 \to B_2$ be a G-equivariant smooth map. Let $\mathcal{F} \in \mathrm{F}_G^*(B_2)$. Then we can naturally define $f^*\mathcal{F} \in \mathrm{F}_G^*(B_1)$. We only need to be careful with the the pull back of the horizontal subbundle. Let F be the natural map from f^*W to W. The new horizontal subbundle $T_{f^*\pi}^H(f^*W)$ is chosen by the condition that $dF(T_{f^*\pi}^H(f^*W)) \subseteq T_\pi^H(W)$. (We note that the chosen of the new horizontal subbundle is not unique. But it is unique in the differential K-theory level.) Let A be a perturbation operator with respect to $D(\mathcal{F})$. Then f^*A is a perturbation operator with respect to $D(f^*\mathcal{F})$. Moreover, from Definition 2.11, we have

$$(3.37) \qquad\qquad\qquad \widetilde{\eta}_G(f^*\mathcal{F}, f^*A) = f^*\widetilde{\eta}_G(\mathcal{F}, A).$$

From Definition 3.3, we can get a well defined pullback map $f^* : \widehat{K}_G^*(B_2) \to \widehat{K}_G^*(B_1)$. Evidently, $\mathrm{Id}_B^* = \mathrm{Id}_{\widehat{K}_G(B)}$. Let $f' : B_0 \to B_1$ be another equivariant smooth map. We could get $f'^* f^* = (f \circ f')^* : \widehat{K}_G^*(B_2) \to \widehat{K}_G^*(B_0)$. It is obvious that the product is natural with respect to pull-backs.

From Theorem 2.19 and a direct calculation, we could get the product is compatible with the equivalent relation in differential K-theory. Other properties are the direct extension of the discussions in [12, page 47-50]. □

Theorem 3.13. *(compare with* [12, Section 3,4]*) The equivariant differential K-theory $\widehat{K}_G$ is a functor $B \to \widehat{K}_G(B)$ from the category of closed smooth G-manifolds with almost free action to unital $\mathbb{Z}_2$-graded commutative rings together with well-defined transformations*

(1) $R : \widehat{K}_G^*(B) \to \Omega_{deloc,G,cl}^*(B, \mathbb{R})$ (curvature);

(2) $I : \widehat{K}_G^*(B) \to K_G^*(B)$ (underlying K_G group);

(3) $a : \Omega_{deloc,G}^*(B, \mathbb{R})/\mathrm{Im}\, d \to \widehat{K}_G(X)$ (action of forms),

where $\Omega_{deloc,G,cl}^*(B, \mathbb{R})$ denote the set of closed delocalized differential forms, such that

(1) The following diagram commutes

$$\begin{array}{ccc} \widehat{K}^*_G(B) & \xrightarrow{\quad I \quad} & K^*_G(B) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_G} \\ \Omega^*_{deloc,G,cl}(B,\mathbb{R}) & \xrightarrow{de\ Rham} & H^*_{deloc,G}(B,\mathbb{R}). \end{array}$$

(2)

(3.38)
$$R \circ a = d.$$

(3) a is of degree 1.

(4) For $x, y \in \widehat{K}^*_G(B)$ and $\alpha \in \Omega^*_{deloc,G}(B,\mathbb{R})/\mathrm{Im}d$, we have

(3.39)
$$R(x \cup y) = R(x) \wedge R(y), \quad I(x \cup y) = I(x) \cup I(y), \quad a(\alpha) \cup x = a(\alpha \wedge R(x)).$$

(5) The following sequence is exact:

(3.40)
$$K^{*-1}_G(B) \xrightarrow{\mathrm{ch}_G} \Omega^{*-1}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\,d \xrightarrow{a} \widehat{K}^*_G(B) \xrightarrow{I} K^*_G(B) \longrightarrow 0.$$

Proof. We define the natural transformation

(3.41)
$$I : \widehat{K}^*_G(B) \to K^*_G(B)$$

by

(3.42)
$$I([\mathcal{F}, \rho]) := \mathrm{ind}(D(\mathcal{F})).$$

From Definition 3.3, the transformation I is well defined.

Let a be a parity-reversing natural transformation

(3.43)
$$a : \Omega^{\mathrm{even/odd}}_{deloc,G}(B,\mathbb{R})/\mathrm{Im}\,d \to \widehat{K}^{1/0}_G(B)$$

by

(3.44)
$$a(\rho) := [\emptyset, -\rho],$$

where $\emptyset$ is the empty geometric family.

We define a transformation

(3.45)
$$R : \widehat{\mathrm{IC}}^*_G(B) \to \Omega^*_{deloc,G,cl}(B,\mathbb{R})$$

by

(3.46)
$$R((\mathcal{F}, \rho)) := \mathrm{FLI}_G(\mathcal{F}) - d\rho.$$

If $(\mathcal{F}', \rho')$ is paired with $(\mathcal{F}, \rho)$, there exists a perturbation operator A with respect to $D(\mathcal{F} + \mathcal{F}'^{\mathrm{op}})$, such that $\rho - \rho' = \widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A)$. From (2.44) and (3.15), we have

(3.47)
$$R((\mathcal{F}, \rho)) = \mathrm{FLI}_G(\mathcal{F}) - d\rho = \mathrm{FLI}_G(\mathcal{F}) - d\rho' - d\widetilde{\eta}_G(\mathcal{F} + \mathcal{F}'^{\mathrm{op}}, A)$$
$$= \mathrm{FLI}_G(\mathcal{F}) - d\rho' - \mathrm{FLI}_G(\mathcal{F}) + \mathrm{FLI}_G(\mathcal{F}') = R((\mathcal{F}', \rho')).$$

Since R is additive, it descends to $\widehat{\mathrm{IC}}^*_G(B)/\sim$ and finally to the map $R : \widehat{K}^*_G(B) \to \Omega^*_{deloc,G,cl}(B,\mathbb{R})$. Let $f : B_1 \to B_2$ be a G-equivariant smooth map. It follows from $\mathrm{FLI}_G(f^*\mathcal{F}) = f^*\mathrm{FLI}_G(\mathcal{F})$ that R is natural.

From (3.44) and (3.46), we have

(3.48)
$$R \circ a = d.$$

By (2.35), the diagram commutes.

The formulas in (3.39) follow from straight calculations using the definitions.

At last, we prove the exactness of the sequence (3.40).

The surjectivity of I follows from Proposition 1.5.

Next, we show the exactness at $\widehat{K}_G^*(B)$. It is obvious that $I \circ a = 0$. For a cycle $(\mathcal{F}, \rho)$, if $I([\mathcal{F}, \rho]) = 0$, we have $\mathrm{ind}(D(\mathcal{F})) = 0$. By Example 1.4 b), we could take $\mathcal{F}$ such that at least one component of the fiber has the nonzero dimension. So there exists a perturbation operator A with respect to $D(\mathcal{F})$ from Proposition 2.3. By (3.15), we have

$$(3.49) \qquad [\mathcal{F}, \rho] = a(\widetilde{\eta}_G(\mathcal{F}, A) - \rho).$$

Finally, We prove the exactness at $\Omega_{deloc,G,cl}^{*-1}(B, \mathbb{R})/\mathrm{Im}\, d$. Following the same process in (3.26)-(3.28), for any $x \in K_G^*(B)$,

$$(3.50) \qquad a \circ \mathrm{ch}_G(x) = (\emptyset, \tilde{\eta}_G(\mathcal{F}, A_Q) - \tilde{\eta}_G(\mathcal{F}, A_P)) = [\mathcal{F}, 0] - [\mathcal{F}, 0] = 0.$$

If $a(\rho) = 0$, for any equivariant geometric family $\mathcal{F}$ with a perturbation operator A with respect to $D(\mathcal{F})$, by Definition 3.3 and (3.49), we have

$$(3.51) \qquad [\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A) - \rho] = a(\rho) = 0 = [\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A)].$$

So by Definition 3.5, there exists another cycle $(\mathcal{F}', \rho')$, such that $(\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho) \sim (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A))$. Since $\sim$ is generated by "paired", we have the cycles $\{(\mathcal{F}_i, \rho_i)\}_{0 \le i \le r}$ such that for any $1 \le i \le r$, $(\mathcal{F}_i, \rho_i)$ is paired with $(\mathcal{F}_{i-1}, \rho_{i-1})$, $(\mathcal{F}_0, \rho_0) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho)$ and $(\mathcal{F}_r, \rho_r) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A))$. By Definition 3.3, for any $1 \le i \le r$, there exists a perturbation operator A_i with respect to $D(\mathcal{F}_{i-1} + \mathcal{F}_i^{\mathrm{op}})$ such that $\rho_{i-1} - \rho_i = \tilde{\eta}_G(\mathcal{F}_{i-1} + \mathcal{F}_i^{\mathrm{op}}, A_i)$. Let A_i' ($0 \le i \le r$) be the perturbation operator with respect to $D(\mathcal{F}_i + \mathcal{F}_i^{\mathrm{op}})$ taken in (3.14). Therefore, by Theorem 2.17, (3.13) and (3.14), there exists $x \in K_G^{(}B)$, such that

$$(3.52) \quad -\rho = \sum_{i=1}^{r}(\rho_{i-1} - \rho_i) = \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_1^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_r^{\mathrm{op}}, A_1 \sqcup_B \cdots \sqcup_B A_r)$$

$$= \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_r^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_{r-1}^{\mathrm{op}}, A_1 \sqcup_B \cdots \sqcup_B A_r)$$

$$- \tilde{\eta}_G(\mathcal{F}_0 + \mathcal{F}_r^{\mathrm{op}} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_{r-1}^{\mathrm{op}}, A_0' \sqcup_B \cdots \sqcup_B A_{r-1}') = \mathrm{ch}_G(x).$$

The proof of Theorem 3.13 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The direct extension of [12, Proposition 3.19 and Lemma 3.20] show that the pullback map and the exact sequence (3.40) are compatible with the push-forward maps.

Remark 3.14. If the group G is trivial, all the models of differential K-theory are isomorphic (see e.g. [13]). For equivariant case, the uniqueness is an open question.

3.4. **Differential K-theory for orbifolds.** Let $\mathcal{X}$ be a closed orbifold (effective orbifold in some literatures). There exist a closed smooth manifold B and a compact Lie group G such that $\mathcal{X}$ is diffeomorphic to a quotient for a smooth, effective, and almost free G-action on B (see [1, Theorem 1.23]).

Let $K_{orb}^0(\mathcal{X})$ be the orbifold K-group of the compact orbifold $\mathcal{X}$ defined as the Grothendieck ring of the equivalent classes of orbifold vector bundles over $\mathcal{X}$. Since $\mathcal{X}$ is an orbifold, $\mathcal{X} \times S^1$ is an orbifold. Moreover, $i : \mathcal{X} \to \mathcal{X} \times S^1$ is a morphism in the category of orbifolds. As in (1.12), we define the orbifold K^1 group $K_{orb}^1(\mathcal{X}) = \ker(i^* : K_{orb}^0(\mathcal{X} \times S^1) \to K_{orb}^0(\mathcal{X}))$.

Let $p : B \to B/G$ be the projection. Then from [1, Proposition 3.6], it induces an isomorphism $p^* : K_{orb}^*(\mathcal{X}) \to K_G^*(B)$.

So we can consider the orbifold K-theory as a special case of equivariant K-theory. Note that if the orbifold $\mathcal{X}$ can be presented in two differential ways as a quotient, say $B'/G' \simeq \mathcal{X} \simeq B/G$, the proposition shows that $K_{G'}^*(B') \simeq K_{orb}^*(\mathcal{X}) \simeq K_G^*(B)$. Furthermore, from the definition of

the differential structure on orbifolds, we know that $\Omega^*_{deloc,G}(B,\mathbb{R})/\operatorname{Im}d \simeq \Omega^*_{deloc,G'}(B',\mathbb{R})/\operatorname{Im}d$. From the exact sequence in (3.40) and five lemma, we have

$$(3.53) \qquad\qquad \widehat{K}^*_{G'}(B') \simeq \widehat{K}^*_G(B).$$

Therefore, this model of equivariant differential K-theory for almost free action could be regarded as a model of differential K-theory for orbifolds.

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