SMOOTH FANO POLYTOPES WHOSE EHRHART POLYNOMIAL HAS A ROOT WITH LARGE REAL PART

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ABSTRACT. The symmetric edge polytopes of odd cycles (del Pezzo polytopes) are known as smooth Fano polytopes. In this paper, we show that if the length of the cycle is 127, then the Ehrhart polynomial has a root whose real part is greater than the dimension. As a result, we have a smooth Fano polytope that is a counterexample to the two conjectures on the roots of Ehrhart polynomials.

Introduction

Let $d \geq 3$ be an integer and A_d , the $(d+1) \times (2d+1)$ matrix

$$A_d = \begin{pmatrix} 0 & 1 & & & -1 & -1 & & & 1 \\ 0 & -1 & \ddots & & & & 1 & \ddots & & \\ \vdots & & \ddots & 1 & & & \ddots & -1 & \\ 0 & & & -1 & 1 & & & 1 & -1 \\ \hline 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

In the present paper, we study the convex hull $Conv(A_d)$ of A_d . The matrix A_d is the centrally symmetric configuration [6] and $Conv(A_d)$ is called the symmetric edge polytope of the cycle of length d. From the results in [3, 5], we have

Proposition. The polytope $Conv(A_d)$ is a Gorenstein Fano polytope (reflexive polytopes) of dimension d-1. In addition, $Conv(A_d)$ is a smooth Fano polytope if and only if d is odd.

Here, we first construct the reduced Gröbner basis \mathcal{G} of I_{A_d} . Next, using \mathcal{G} , we compute the Ehrhart polynomial and the h-vector of $\operatorname{Conv}(A_d)$. Finally, we study the roots of the Ehrhart polynomial when d is odd. We show that the Ehrhart polynomial of $\operatorname{Conv}(A_{127})$ has a root whose real part is greater than $\dim(\operatorname{Conv}(A_{127}))$. This is a counterexample to the conjectures given in [1, 5].

1. Gröbner bases of toric ideals

Let $\mathcal{R}_d = K[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}, s]$ be the Laurent polynomial ring over a field K and let $K[z, X, Y] = K[z, x_1, \dots, x_d, y_1, \dots, y_d]$ be the polynomial ring over K. We define the ring homomorphism $\pi: K[z, X, Y] \to \mathcal{R}_d$ by setting $\pi(x_i) = t_i t_{i+1}^{-1} s$, $\pi(y_i) = t_i^{-1} t_{i+1} s$ for $1 \leq i \leq d$ (here we set $t_{d+1} = t_1$) and $\pi(z) = s$. The toric

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ideal I_{A_d} is ker (π) . Let < be the reverse lexicographic order on K[z,X,Y] with the ordering $z < y_d < x_d < \dots < y_1 < x_1$. For $d \ge 3$, let $[d] = \{1, \dots, d\}$ and $k = \lceil \frac{d}{2} \rceil$.

Theorem 1.1. The reduced Gröbner basis of I_{A_d} with respect to < consists of

$$(1) x_i y_i - z^2 (1 \le i \le d)$$

(1)
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(2)
$$\prod_{l=1}^k x_{i_l} - z \prod_{l=1}^{k-1} y_{j_l} \quad ([d] = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{k-1}\})$$

(3)
$$\prod_{l=1}^{k} y_{j_l} - z \prod_{l=1}^{k-1} x_{i_l} \quad ([d] = \{i_1, \dots, i_{k-1}\} \cup \{j_1, \dots, j_k\})$$

if d is odd and

$$(4) x_i y_i - z^2 (1 \le i \le d)$$

(4)
$$x_{i}y_{i} - z^{2} \qquad (1 \le i \le d)$$
(5)
$$\prod_{l=1}^{k} x_{i_{l}} - y_{d} \prod_{l=1}^{k-1} y_{j_{l}} \quad ([d-1] = \{i_{1}, \dots, i_{k}\} \cup \{j_{1}, \dots, j_{k-1}\})$$

(6)
$$\prod_{l=1}^{k} y_{j_l} - x_d \prod_{l=1}^{k-1} x_{i_l} \quad ([d-1] = \{i_1, \dots, i_{k-1}\} \cup \{j_1, \dots, j_k\})$$

if d is even. The initial monomial of each binomial is the first monomial.

Proof. Let \mathcal{G} be the set of all binomials above. It is easy to see that $\mathcal{G} \subset I_{A_d}$ and that the initial monomial of each binomial in \mathcal{G} is the first monomial. Let $\operatorname{in}(\mathcal{G}) =$ $\langle \operatorname{in}_{<}(g) \mid g \in \mathcal{G} \rangle$. Suppose that d is odd and that \mathcal{G} is not a Gröbner basis of I_{A_d} . Then, there exists an irreducible binomial $(0 \neq)$ $f = u - v \in I_{A_d}$ such that neither u nor v belongs to in(\mathcal{G}). Let $u = z^{\alpha} \prod_{l=1}^{m} x_{i_l}^{p_l} \prod_{l=1}^{n} y_{j_l}^{q_l}$ and $v = z^{\alpha'} \prod_{l=1}^{m'} x_{i_l'}^{p_l'} \prod_{l=1}^{n'} y_{j_l'}^{q_l'}$, where $0 < p_l, q_l, p'_l, q'_l \in \mathbb{Z}$ for all l and $\mathcal{I} = \{i_1, \ldots, i_m\}, \ \mathcal{J} = \{j_1, \ldots, j_n\}, \ \mathcal{I}' = \{j_1, \ldots, j_m\}, \ \mathcal{I}' = \{j_1, \ldots$ $\{i_1',\ldots,i_{m'}'\},~\mathcal{J}'=\{j_1',\ldots,j_{n'}'\}$ are subsets of [d] with the cardinality m,~n,~m',and n, respectively. Since neither u nor v is divided by $x_i y_i$, we have $\mathcal{I} \cap \mathcal{J} =$ $\mathcal{I}' \cap \mathcal{J}' = \emptyset$. In addition, since neither u nor v is divided by the initial monomials of binomials (2) and (3), it follows that $m, n, m', n' \leq k - 1$. Moreover, since f is irreducible, we have $\mathcal{I} \cap \mathcal{I}' = \mathcal{J} \cap \mathcal{J}' = \emptyset$. Let $p = \sum_{l=1}^{m} p_l, \ q = \sum_{l=1}^{n} q_l,$ $p' = \sum_{l=1}^{m'} p'_l, \ q' = \sum_{l=1}^{n'} q'_l$. Then, $\pi(u) = s^{\alpha+p+q} \prod_{l=1}^{m} (t_{i_l} t_{i_l+1}^{-1})^{p_l} \prod_{l=1}^{n} (t_{j_l}^{-1} t_{j_l+1})^{q_l},$ $\pi(v) = s^{\alpha'+p'+q'} \prod_{l=1}^{m'} (t_{i'_l} t_{i'_l+1}^{-1})^{p'_l} \prod_{l=1}^{n'} (t_{j'_l}^{-1} t_{j'_l+1})^{q'_l},$ where we set $t_{d+1} = t_1$. Since $\pi(u) = t_1$ $\pi(v)$, it follows that $\pi(u') = \pi(v')$, where $u' = z^{\alpha+2q} \prod_{l=1}^m x_{i_l}^{p_l} \prod_{l=1}^{n'} x_{j'_l}^{q'_l}$ and $v' = x_{i_l}^{q'_l} \prod_{l=1}^{n'} x_{j'_l}^{q'_l}$ $z^{\alpha'+2q'}\prod_{l=1}^{m'}x_{i'_l}^{p'_l}\prod_{l=1}^nx_{j_l}^{q_l}$. Thus, g=u'-v' belongs to I_{A_d} . Since g belongs to K[z,X], g belongs to the toric ideal I_B , where B is the matrix consisting of the first d+1columns of A_d . In addition, by virtue of $m, n, m', n' \leq k - 1$, we have $|\mathcal{I} \cup \mathcal{J}'| \leq k - 1$ 2(k-1) < d, $|\mathcal{I}' \cup \mathcal{J}| \le 2(k-1) < d$. Thus, neither u' nor v' is divided by $x_1 \cdots x_d$. Since $g \in I_B = \langle x_1 \cdots x_d - z^d \rangle$, we have g = 0, that is, u' = v'. Then, from $\mathcal{I} \cap \mathcal{J} = \mathcal{I}' \cap \mathcal{J}' = \mathcal{I} \cap \mathcal{I}' = \mathcal{J} \cap \mathcal{J}' = \emptyset$, we have $(\mathcal{I} \cup \mathcal{J}') \cap (\mathcal{I}' \cup \mathcal{J}) = \emptyset$. Hence, $\prod_{l=1}^m x_{i_l}^{p_l} \prod_{l=1}^{n'} x_{j_l'}^{q_l'}$ and $\prod_{l=1}^{m'} x_{i_l'}^{p_l'} \prod_{l=1}^n x_{j_l}^{q_l}$ have no common variables. Since u' = v', we have m = n = m' = n' = 0. Hence, $u = z^{\alpha}$ and $v = z^{\alpha'}$. Since f is a homogeneous binomial, this is a contradiction. Thus, \mathcal{G} is a Gröbner basis of I_{A_d} . It is trivial that \mathcal{G} is reduced. The case when d is even is analyzed by a similar argument. \square

2. Ehrhart polynomials and roots

For $0 \le i \le d$, let $r_d(i)$ denote the number of squarefree monomials in K[X,Y] of degree i that do not belong to the initial ideal $\operatorname{in}_{<}(I_{A_d})$ and let $s_d(i+1)$ denote the number of squarefree monomials in K[z,X,Y] of degree (i+1) that are divided by z and do not belong to $\operatorname{in}_{<}(I_{A_d})$. For example, $r_d(0) = s_d(1) = 1$ and $r_d(d) = 0$.

Lemma 2.1. For $0 \le i \le d-1$, we have $r_d(i) = \binom{d}{i} \sum_{\ell=1}^{d-i} \binom{i}{k-\ell}$ and $s_d(i+1) = r_d(i)$. In particular, $r_d(i) = \binom{d}{i} 2^i$ for $0 \le i \le k-1$.

Proof. Since the variable z does not appear in the initial monomials of the binomials in Theorem 1.1, $u \notin \operatorname{in}_{<}(I_{A_d})$ if and only if $z \ u \notin \operatorname{in}_{<}(I_{A_d})$ for any squarefree monomial $u \in K[X,Y]$. Thus, $s_d(i+1) = r_d(i)$ for $0 \le i \le d-1$. Suppose that d is odd. Then, from Theorem 1.1, $r_d(i)$ is the number of monomials $\prod_{i \in \mathcal{I}} x_i \prod_{j \in \mathcal{J}} y_j$ where $\mathcal{I}, \mathcal{J} \subset [d], \mathcal{I} \cap \mathcal{J} = \emptyset, |\mathcal{I} \cup \mathcal{J}| = i$ and $|\mathcal{I}|, |\mathcal{J}| \le k-1$. Since the number of subsets $\mathcal{I}, \mathcal{J} \subset [d]$ such that $\mathcal{I} \cap \mathcal{J} = \emptyset, |\mathcal{I} \cup \mathcal{J}| = i$ and $|\mathcal{I}| = \lambda$ is $\binom{d}{\lambda, i-\lambda, d-i} = \binom{d}{d-i}\binom{i}{\lambda} = \binom{d}{i}\binom{i}{\lambda}$, it follows that $r_d(i) = \sum_{\ell=1}^{d-i} \binom{d}{i}\binom{i}{k-\ell} = \binom{d}{i}\sum_{\ell=1}^{d-i} \binom{i}{k-\ell}$ for $0 \le i \le d-1$. If d is even, then the proof is similar.

It is known [9, Chapter 8] that $\operatorname{in}_{<}(I_{A_d}) = \sqrt{\operatorname{in}_{<}(I_{A_d})}$ is the Stanley–Reisner ideal of a regular unimodular triangulation Δ of $\operatorname{Conv}(A_d)$. Thus, $r_d(i) + r_d(i+1)$ is the number of *i*-dimensional faces of Δ . From Lemma 2.1 and [7, Theorem 1.4], the Hilbert polynomial of $K[z, X, Y]/I_{A_d}$ can be computed as follows:

Theorem 2.2. The Ehrhart polynomial of $Conv(A_d)$ is $\sum_{i=0}^{d-1} r_d(i) {m \choose i}$. Moreover, the normalized volume of $Conv(A_d)$ equals $k {d \choose k}$.

Let $(h_0^{(d)}, h_1^{(d)}, \dots, h_{d-1}^{(d)})$ be the *h*-vector of $Conv(A_d)$. Note that $h_0^{(d)} = 1$. Since $Conv(A_d)$ is Gorenstein, we have $h_j^{(d)} = h_{d-1-j}^{(d)}$ for each $0 \le j \le d-1$. Thus, it is enough to study $h_j^{(d)}$ for $1 \le j \le k-1$.

Theorem 2.3. For $1 \le j \le k-1$, we have

$$h_j^{(d)} = (-1)^j \sum_{i=0}^j (-2)^i \binom{d}{i} \binom{d-i-1}{j-i} = \left\{ \begin{array}{ll} 2^{d-1} & j=k-1 \ and \ d \ is \ odd, \\ h_j^{(d-1)} + h_{j-1}^{(d-1)} & otherwise. \end{array} \right.$$

Proof. By Lemma 2.1 and a well-known expression ([7, p. 58]), one can show the first equality and that $h_j^{(d)}$ is the coefficient of u^j in the expansion of $2^d(u+1)^d(u+2)^{-d+j}$. The second equality follows from this fact and the identity given in [8, p.148].

Finally, we study the roots of the Ehrhart polynomial when d is odd. In this case, $Conv(A_d)$ is a smooth Fano polytope of dimension d-1. Since $Conv(A_d)$ is a Gorenstein Fano polytope, the roots of the Ehrhart polynomial are symmetrically

distributed in the complex plane with respect to the line Re(z) = -1/2. Here, Re(z) is the real part of $z \in \mathbb{C}$. The following conjectures are given in [1, 5]:

Conjecture 2.4 ([1]). All roots α of Ehrhart polynomials of D dimensional lattice polytopes satisfy $-D \leq \text{Re}(\alpha) \leq D - 1$.

Conjecture 2.5 ([5]). All roots α of Ehrhart polynomials of D dimensional Gorenstein Fano polytopes satisfy $-D/2 \leq \text{Re}(\alpha) \leq D/2 - 1$.

Using the software packages Maple, Mathematica, and Maxima, we computed the largest real part of roots of the Ehrhart polynomial of $Conv(A_d)$:

d	$\dim(\operatorname{Conv}(A_d))$	the largest real part	
35	34	16.35734046	a counterexample to Conjecture 2.5
125	124	123.5298262	a counterexample to Conjecture 2.4
127	126	126.5725840	greater than its dimension

Remark 2.6. It was shown [2] that Conjecture 2.4 is true for $D \leq 5$. Recently, a simplex (not a Fano polytope) that does not satisfy the condition "Re(α) $\leq D-1$ " in Conjecture 2.4 was presented in [4]. Our polytope Conv(A_{125}) is the first example satisfying neither " $-D \leq \text{Re}(\alpha)$ " nor "Re(α) $\leq D-1$ " in Conjecture 2.4.

References

- [1] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, in "Integer Points in Polyhedra Geometry, Number theory, Algebra, Optimization," Contemp. Math. 374 (2005), 15–36.
- [2] B. Braun and M. Develin, Ehrhart polynomial roots and Stanley's non-negativity theorem, in "Integer Points in Polyhedra Geometry, Number Theory, Representation Theory, Algebra, Optimization, Statistics," Contemp. Math. 452 (2008), 67–78.
- [3] A. Higashitani, Smooth Fano polytopes arising from finite directed graphs, preprint. arXiv:1103.2202v1 [math.CO]
- [4] A. Higashitani, Counterexamples of the conjecture on roots of Ehrhart polynomials, preprint. arXiv:1106.4633v2 [math.CO]
- [5] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, Roots of Ehrhart polynomials arising from graphs, J. Algebraic Combinatorics, in press.
- [6] H. Ohsugi and T. Hibi, Centrally symmetric configurations of integer matrices, preprint. arXiv:1105.4322v1 [math.AC].
- [7] R. P. Stanley, Combinatorics and commutative algebra, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996.
- [8] R. P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1999.
- [9] B. Sturmfels, "Gröbner bases and convex polytopes," Amer. Math. Soc., Providence, RI, 1996.

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