WELL-POSEDNESS AND CONTINUITY PROPERTIES OF THE FORNBERG-WHITHAM EQUATION IN BESOV SPACES

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ABSTRACT. In this paper, we prove well-posedness of the Fornberg-Whitham equation in Besov spaces $B_{2,r}^s$ in both the periodic and non-periodic cases. This will imply the existence and uniqueness of solutions in the aforementioned spaces along with the continuity of the data-to-solution map provided that the initial data belongs to $B_{2,r}^s$. We also establish sharpness of continuity on the data-to-solution map by showing that it is not uniformly continuous from any bounded subset of $B_{2,r}^s$ to $C([-T,T];B_{2,r}^s)$. Furthermore, we prove a Cauchy-Kowalevski type theorem for this equation that establishes the existence and uniqueness of real analytic solutions and also provide blow-up criterion for solutions.

1. Introduction

We consider the Cauchy problem for the Fornberg-Whitham (FW) equation

$$\begin{cases} u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}u u_{xxx} - \frac{3}{2}u u_x + u_x = 0\\ u(x,0) = u_0(x), \end{cases}$$
 (1.1)

when $x \in \mathbb{R}$ or \mathbb{T} and $t \in \mathbb{R}$. This equation was first written down in 1967 by Whitham [31] and again by Whitham and Fornberg [11] as a model for breaking waves. We show that the above Cauchy problem for the FW equation is well-posed in both periodic and non-periodic Besov spaces $B_{2,r}^s$ for particular parameter values of r and s. Furthermore, we demonstrate that the data-to-solution map is not uniformly continuous from any bounded subset of $B_{2,r}^s$ to $C([-T,T];B_{2,r}^s)$ for the aforementioned parameters, however it is Hölder continuous in the $B_{2,r}^\alpha$ topology for any $\alpha < s$. We also establish the existence and uniqueness of real analytic solutions provided that the initial data is real analytic as well as formulating blow-up criterion of solutions. Our results hold in both periodic and non-periodic spaces, unless otherwise stated.

The FW equation can also be written in its non-local form

$$u_t + \frac{3}{2}uu_x = (1 - \partial_x^2)^{-1}\partial_x u,$$
 (1.2)

where $\mathcal{F}[(1-\partial_x^2)^{-1}f] = \frac{1}{1+\xi^2}\widehat{f}(\xi)$ for any test function f. If we write the FW equation in this form, we see that it resembles a special case of a family of nonlinear wave equations

$$u_t + auu_x = \mathcal{L}(u),$$

where $a \in \mathbb{R}$ and \mathscr{L} is a linear operator with constant coefficients, which has been studied by multiple authors. In both [31] and [11], the FW equation (1.2) was compared with the Kortewegde Vries (KdV) equation

$$u_t + 6uu_x = -\partial_x^3 u$$

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which was first derived by Boussinesque [3] in 1877 and then by Korteweg and de Vries [25] in 1895.

Since 1834 with John Scott Russell's observation of solitary waves, we have known of their physical existence and it wasn't until the advent of the KdV equation via Boussinesque, Korteweg and de Vries that we were finally able to see their mathematical existence as well. In fact, we have in the non-periodic case, that the solitary wave solutions of the KdV equation are of the form

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right),$$

where the constant c is the wave speed. Unfortunately, while we now had a model that generated Russell's solitary wave solutions, it was only appropriate for long wave behavior and not for short wave phenomena close to shore lines, which exhibit the property of wave breaking. Furthermore, it did not produce solitary waves of greatest height with a sharp peaked crest which are now called peakons. The goal now had become one of seeking out a model that generated such solutions that not only maintained dispersive qualities like KdV, but also exhibited the aforementioned property of wave breaking.

In 1967, while Whitham [31] was exploring applications to water waves he wrote down the FW equation (1.2) and noted that it produced the so-called peakon solutions with maximum amplitude of 8/9. Then in 1978, Whitham and Fornberg [11] researched both numerical and theoretical results on the FW equation and wrote down the explicit peakon solutions

$$u(x,t) = \frac{8}{9}e^{-\frac{1}{2}|x-\frac{4}{3}t|},$$

along with noting the wave breaking properties. In fact, 8/9 was found as a limiting behavior of the amplitude of the above exponential peakon. From then on, many authors have researched and found other nonlinear wave equations that resemble that of FW.

In 1993, Camassa and Holm [4] produced the equation which now bears their names (Camassa-Holm equation (CH))

$$(1 - \partial_x^2)u_t = uu_{xxx} + 2u_x u_{xx} - 3uu_x \tag{1.3}$$

which appears in the context of hereditary symmetries studied by Fokas and Fuchssteiner [12]. This equation was first written explicitly and derived from the Euler equations by Camassa and Holm in [4], where they also found its "peakon" traveling wave solutions. These are solutions with a discontinuity in the first spacial derivative at its crest. The simplest one in the non-periodic case is of the form $u_c(x,t) = ce^{-|x-t|}$, where c is a positive constant. The CH equation possesses many other remarkable properties such as infinitely many conserved quantities, a bi-Hamiltonian structure and a Lax pair. For more information about how CH arises in the context of hereditary symmetries we refer to [12]. Concerning it's physical relevance, we refer the reader to the works by Johnson [23], [24] and Constantine and Lannes [6].

In 1998 the following equation appeared

$$(1 - \partial_x^2)u_t = uu_{xxx} + 3u_x u_{xx} - 4uu_x. \tag{1.4}$$

Degasperi and Procesi [10] discovered this equation as one of the three equations to satisfy asymptotic integrability conditions in the following family of equations

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_{1,n} u^2 + c_{2,n} u_x^2 + c_3 u u_{xx})_x, \tag{1.5}$$

where $\alpha, c_0, c_{1,n}, c_{2,n}, c_3 \in \mathbb{R}$ are constants. Other integrable members of the family (1.5) are the CH and the KdV equations. Also, like the CH equation (1.3), the DP equation (1.4) possesses peakon solutions of the form $u_c(x,t) = ce^{-|x-ct|}$.

Recently, Vladimir Novikov [27], in addition to several integrable equations with quadratic nonlinearities like the CH and DP, generated about 10 integrable equations with cubic nonlinearities while investigating the integrability of Camassa-Holm type equations of the form

$$(1 - \partial_x^2)u_t = P(u, u_x, u_{xx}, ...), \tag{1.6}$$

where P is a polynomial of u and its derivatives. One of these CH-type equations with cubic nonlinearities, which happened to be a new integrable equation, is the following

$$(1 - \partial_x^2)u_t = u^2 u_{xxx} + 3u u_x u_{xx} - 4u^2 u_x. (1.7)$$

Equation (1.7) is now called the Novikov equation (NE) and also possesses many properties exhibited by Camassa-Holm type equations. One of which is the existence of peakon solutions of the form $u_c(x,t) = \sqrt{c}e^{-|x-ct|}$. It is important to mention that the aforementioned equations are integrable; they possess infinitely many conserved quantities, an infinite hierarchy of quasi-local symmetries, a Lax pair and a bi-Hamiltonian structure. In contrast, the FW equation is not integrable.

Our first result is concerned with the well-posedness for the FW equation (1.1) in Besov spaces in both periodic and non-periodic cases. We base this result primarily off of the methodology presented in Danchin's paper on the CH equation [7]. The result is stated as follows.

Theorem 1.1. If $s > \frac{3}{2}$, $r \in (1, \infty)$ or $s \geq 3/2$, r = 1 and the initial data $u_0 \in B_{2,r}^s$, then there exists a lifetime T > 0 and a unique solution $u \in C([-T, T]; B_{2,r}^s)$ of the FW initial value problem, which depends continuously on the initial data. Additionally, we have the solution size and lifespan estimates

$$||u||_{B_{2,r}^s} \le 2||u_0||_{B_{2,r}^s}, \quad and \quad T \le \frac{c}{||u_0||_{B_{2,r}^s}},$$
 (1.8)

where c is a positive constant depending only upon s. Furthermore, when $r = \infty$, there exists a unique solution $u \in C([-T,T]; B_{2,\infty}^{s'})$, for any s' < s, which depends continuously upon the initial data and satisfies the same lifespan and energy estimates in $B_{2,\infty}^{s'}$.

It is important to note that when r=2, this corresponds to the Sobolev space H^s where well-posedness has been shown by Yin [32] by applying Kato's semigroup approach. Well-posedness in Sobolev spaces H^s for s>3/2 was also shown by Holmes [21] where he employed a Galerkin type approximation argument as in Taylor [28]. The same methodology was used by Himonas and Holliman for the CH equation [16], the DP equation [14] and the Novikov equation [15].

From our well-posedness result, we are also able to demonstrate that the dependence on the initial data is sharp, as summarized in the following theorem.

Theorem 1.2. If $s \geq \frac{3}{2}$ and $r \in [1, \infty]$, then the data-to-solution map is not uniformly continuous from any bounded subset of $B_{2,r}^s(\mathbb{T})$ to $C([-T,T];B_{2,r}^s(\mathbb{T}))$.

To demonstrate this sharpness of continuity, sequences of approximate solutions are constructed. Then the actual solutions are found by solving the Cauchy problem with initial data given by the approximate solutions at time t=0. The error produced by solving the Cauchy problem using approximate solutions is shown to be inconsequential. This was motivated by the works of Himonas and Holliman [14, 15, 16], Himonas and Kenig [17], Himonas, Kenig and Misiolek [18],

and Himonas, Misiołek and Ponce [20], Thompson [29] and Grayshan [13] on CH-type equations. Furthermore, Holmes investigated this result for the FW equation in Sobolev spaces in [21].

Following our proof of Theorem 1.1, we shall expand our range of continuity properties by considering weaker topologies on $B_{2,r}^s$. Although the data-to-solution map is not uniformly continuous on $B_{2,r}^s$, we will show that this map is Hölder continuous if we choose a properly weakened topology. This is summarized in the following result.

Theorem 1.3. If $s \ge \frac{3}{2}$, $r \in [1, \infty]$ and $u_0 \in B_{2,r}^s$, then for any $s-1 \le \alpha < s$, the data-to-solution map is Hölder continuous from any bounded subset of $B_{2,r}^{\alpha}$ to $C([-T, T]; B_{2,r}^{\alpha})$.

We also prove the uniqueness and existence of real analytic solutions. Methodology employed here follows that of Himonas and Misiołek in [19] where they show analyticity of solutions for the Camassa-Holm equation (1.3). This is given by the following result.

Theorem 1.4. If the initial data u_0 is analytic on \mathbb{T} then there exists an $\varepsilon > 0$ and a unique solution u(x,t) to the Cauchy problem for the FW equation (1.2) that is analytic in both x and t on \mathbb{T} for all t in $(-\varepsilon, \varepsilon)$.

Finally, we prove the following blow-up criterion. The proof of which will follow a similar argument to one presented in [7].

Theorem 1.5. Let $u_0 \in B_{p,r}^s$ as in Theorem 1.1 and u be the corresponding solution to the Cauchy problem (1.1). Assume that T^* is the maximal time of existence of the solution to the aforementioned Cauchy problem. If

$$T^* < \infty$$
, then $\int_0^{T^*} \|u_x\|_{L^\infty} = +\infty$.

The paper is structured as follows. In Section two, we provide some preliminary results and lemmas that are used throughout the rest of the document. In sections three and four, we prove the well-posedness result as stated in Theorem 1.1 along with Theorem 1.3. Then in Section five, we prove nonuniform dependence upon the initial data as stated in Theorem 1.2. Finally, in Section 6 and 7 we prove analyticity of solutions and blow-up criterion as stated in Theorem 1.4 and Theorem 1.5 respectively.

2. Preliminaries

In this section, we will recall some conclusions on the properties of Littlewood-Paley decomposition, the Besov spaces and the theory of the transport equation which can be seen in [5], [7, 8, 9], [30].

Lemma 2.1. (Littlewood-Paley decomposition). There exists a couple of smooth radial functions (χ, φ) valued in [0,1] such that χ is supported in the ball $B = \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ and φ is supported in the ring $C = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Moreover,

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1$$

and

$$supp \ \varphi(2^{-q} \cdot) \cap supp \ \varphi(2^{-q'} \cdot) = \emptyset, \quad if \ |q - q'| \ge 2,$$
$$supp \ \chi(\cdot) \cap supp \ \varphi(2^{-q} \cdot) = \emptyset, \quad if \ |q| \ge 1.$$

Then for $u \in \mathcal{S}'(\mathbb{R})$ the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{split} &\Delta_q u = 0, \quad if \quad q \leq -2, \\ &\Delta_{-1} u = \chi(D) u = \mathscr{F}_x^{-1} \chi \mathscr{F} u, \\ &\Delta_q u = \varphi(2^{-q}D) = \mathscr{F}_x^{-1} \varphi(2^{-q}\xi) \mathscr{F} u, \quad if \quad q \geq 0. \end{split}$$

Thus $u = \sum_{q \in \mathbb{Z}} \Delta_q u$ in $\mathcal{S}'(\mathbb{R})$.

Remark. The low frequency cut-off S_q is defined by

$$S_q u = \sum_{p=-1}^{q-1} \Delta u = \chi(2^{-q}D)u = \mathscr{F}_x^{-1} \chi(2^{-q}\xi) \mathscr{F}_x u, \quad \forall q \in \mathbb{N}.$$

We can see that

$$\Delta_p \Delta_q u \equiv 0, \quad if \quad |p - q| \ge 2,$$

$$\Delta_q (S_{p-1} u \Delta_p v) \equiv 0, \quad if \quad |p - q| \ge 5, \quad \forall u, v \in \mathcal{S}'(\mathbb{R})$$

as well as

$$\|\Delta_q u\|_{L^p} \le \|u\|_{L^p}, \quad \|S_q u\|_{L^p} \le C\|u\|_{L^p}, \quad \forall p \in [1, \infty]$$

with the aid of Young's Inequality, where C is a positive constant independent of q.

Definition 2.1. (Besov Spaces) Let $s \in \mathbb{R}$, $p, r \in [1, \infty]$ and $u \in S'(\mathbb{R})$. Then we define the Besov space of functions as

$$B_{p,r}^s = B_{p,r}^s(\mathbb{R}) = \{ u \in S'(\mathbb{R}) : ||u||_{B_{p,r}^s} < \infty \},$$

where

$$||u||_{B_{p,r}^s} \doteq \begin{cases} \left(\sum_{q \geq -1} (2^{sq} ||\Delta_q u||_{L^p})^r \right)^{1/r} & \text{if } 1 \leq r < \infty \\ \sup_{q \geq -1} 2^{sq} ||\Delta_q u||_{L^p} & \text{if } r = \infty. \end{cases}$$

In particular, $B_{p,r}^{\infty} = \bigcap_{s \in \mathbb{R}} B_{p,r}^s$.

Lemma 2.2. Let $s \in \mathbb{R}$, $1 \le p, r, p_j, r_j \le \infty$, j = 1, 2, then

- (1) Topological properties: $B_{p,r}^s$ is a Banach space which is continuously emedded in $\mathcal{S}'(\mathbb{R})$.
- (2) Density: C_c^{∞} is dense in $B_{p,r}^s \iff p,r \in [1,\infty)$.
- (3) Embedding: $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-n(\frac{1}{p_1} \frac{1}{p_2})}$, if $p_1 \leq p_2$ and $r_1 \leq r_2$. $B_{p_2,r_2}^{s_2} \hookrightarrow B_{p_2,r_1}^{s_1}$ locally compact if $s_1 < s_2$.
- (4) Algebraic properties: $\forall s > 0$, $B_{p,r}^s \cap L^\infty$ is a Banach algebra. $B_{p,r}^s$ is a Banach algebra $\iff B_{p,r}^s \hookrightarrow L^\infty \iff s > \frac{1}{p}$ or $(s \ge \frac{1}{p} \text{ and } r = 1)$. In particular, $B_{2,1}^{1/2}$ is continuously embedded in $B_{2,\infty}^{1/2} \cap L^\infty$ and $B_{2,\infty}^{1/2} \cap L^\infty$ is a Banach algebra.
- (5) 1-D Moser-type estimates:
 - (i) For s > 0,

$$||fg||_{B_{p,r}^s} \le C(||f||_{B_{p,r}^s} ||g||_{L^\infty} + ||f||_{L^\infty} ||g||_{B_{p,r}^s}).$$

(ii)
$$\forall s_1 \leq \frac{1}{p} < s_2(s_2 \geq \frac{1}{p} \text{ if } r = 1) \text{ and } s_1 + s_2 > 0, \text{ we have}$$

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) Complex interpolation:

$$||f||_{B^{\theta s_1}_{p,r}+(1-\theta)s_2} \le ||f||^{\theta}_{B^{s_1}_{p,r}}||g||^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}, \quad \forall \theta \in [0,1].$$

(7) Real interpolation: $\forall \theta \in (0,1), s_1 > s_2, s = \theta s_1 + (1-\theta)s_2$, there exists a constant C such that

$$||u||_{B^{s}_{p,1}} \leq \frac{C(\theta)}{s_1 - s_2} ||u||_{B^{s_1}_{p,\infty}}^{\theta} ||u||_{B^{s_2}_{p,\infty}}^{1 - \theta}, \quad \forall u \in B^{s_1}_{p,\infty}.$$

In particular, for any $\theta \in (0,1)$, we have that

$$\|u\|_{B^{1/2}_{2,1}} \leq \|u\|_{B^{\frac{3}{2}-\theta}_{2,1}} \leq C(\theta) \|u\|^{\theta}_{B^{1/2}_{2,\infty}} \|u\|^{1-\theta}_{B^{3/2}_{2,\infty}}.$$

(8) Fatou lemma: If $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n \to u$ in $\mathcal{S}'(\mathbb{R})$, then $u \in B^s_{p,r}$ and $\|u\|_{B^s_{p,r}} \le \liminf_{n \to \infty} \|u_n\|_{B^s_{p,r}}$.

Let f be a solution to the following linear transport equation

$$\begin{cases} \partial_t f + v \partial_x f = F, \\ f(x, 0) = f_0(x). \end{cases}$$
 (2.1)

The following proposition can be found in [7], Proposition A.1.

Proposition 2.1. Let $s > \frac{3}{2}$, $r \in (1, \infty)$ or $s \ge 3/2$, r = 1 and let v be a function such that $\partial_x v \in L^1((0,T); B^{s-1}_{2,r})$. Also, let $f_0 \in B^s_{2,r}$, $F \in L^1((0,T); B^s_{2,r})$ and suppose $f \in L^{\infty}((0,T); B^s_{2,r}) \cap C([0,T]; S')$ is a solution to (2.1). Then for some C = C(s) and $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{s-1}_{2,r}} d\tau$ we have

$$||f(t)||_{B_{2,r}^s} \le e^{CV(t)} \left(||f_0||_{B_{2,r}^s} + C \int_0^t e^{-CV(\tau)} ||F(\tau)||_{B_{2,r}^s} d\tau \right). \tag{2.2}$$

Moreover, the above inequality holds when $r = \infty$ and for all s' < s.

For the sake of brevity, from now on we will denote our Besov space cut-off functions $\varphi \doteq \varphi_q$. All other cut-off functions φ are assumed to be of Schwartz class.

3. Local Well-posedness on the Torus

Well posedness is broken into four steps. First we show existence of a solution on lifepspan T and energy estimate, then we show that the solution is unique, and finally we show that the solution depends continuously upon the initial data.

3.1. Existence and lifespan.

Theorem 3.1. Suppose $s > \frac{3}{2}$, $r \in (1, \infty)$ or $s \geq 3/2$, r = 1 and $u_0 \in B_{2,r}^s$. Then there exists a time

$$T \approx \frac{1}{\|u_0\|_{B_{2r}^s}},$$

such that the Cauchy problem for the FW equation has a solution $u \in C([0,T]; B_{2,r}^s) \cap C^1([0,T]; B_{2,r}^{s-1})$ and the solution has the bound

$$\sup_{t \in [0,T]} \|u(t)\|_{B_{2,r}^s} \le 2\|u_0\|_{B_{2,r}^s}.$$

To show existence of a solution, we will find a sequence of approximate solutions from which we may extract a sub-sequence which converges to a solution of FW.

3.1.1. Proof of Theorem 3.1. We have from the Besov space embedding theorem that $B_{2,r}^s \hookrightarrow C^1$ (in particular, in this case for all $\sigma \geq s-1$, $B_{2,r}^{\sigma}$ is an algebra). We will consider the following sequence of smooth functions $\{u_n(x,t)\}_{n\geq 0}$, (where the subscript denotes an indexation) with the initial function $u_0(x,t) = 0$ solving the Cauchy problem for the linear transport equation

$$\begin{cases} \partial_t u_{n+1} + \frac{3}{2} u_n \partial_x u_{n+1} = \Lambda^{-1} [\partial_x u_n] \\ u_{n+1}(x,0) = J_{n+1} u_0(x), \end{cases}$$
(3.1)

where J_{n+1} is a mollifier.

Our first task is to show that the solutions to the above linear transport equation are uniformly bounded on a common lifespan. Applying the first estimate from Proposition 2.1, we have

$$||u^{n+1}(t)||_{B_{2,r}^s} \le e^{CU_n(t)} \left(||u_0||_{B_{2,r}^s} + C \int_0^t e^{-CU_n(\tau)} ||\Lambda^{-1}[\partial_x u_n](\tau)||_{B_{2,r}^s} d\tau \right), \tag{3.2}$$

where $U_n(t) = \frac{3}{2} \int_0^t \|\partial_x u_n(\tau)\|_{B_2^{s-1}} d\tau$. We estimate have

$$\|\Lambda^{-1}[\partial_x u_n]\|_{B^s_{2,r}} \le \|u_n\|_{B^s_{2,r}},\tag{3.3}$$

from which we conclude

$$||u_{n+1}(t)||_{B_{2,r}^s} \le e^{CU_n(t)} \left(||u_0||_{B_{2,r}^s} + C \int_0^t e^{-CU_n(\tau)} ||u_n(\tau)||_{B_{2,r}^s} d\tau \right), \tag{3.4}$$

Lemma 3.1 (Lifespan and Energy). There exists a minimum lifespan

$$T = \frac{1}{4C\|u_0\|_{B_{2,r}^s}},$$

such that for all $n \in \mathbb{Z}^+$ and for all $t \in [0,T]$

$$||u_n(t)||_{B_{2,r}^s} \le \frac{||u_0||_{B_{2,r}^s}}{1 - 2C||u_0||_{B_{2,r}^s}} \le 2||u_0||_{B_{2,r}^s}.$$

Proof. We will prove this claim by induction. The base case is trivially true, therefore, plugging the inductive hypothesis into (3.4) we obtain

$$||u_{n+1}(t)||_{B_{2,r}^s} \le e^{CU_n(t)} ||u_0||_{B_{2,r}^s} + C \int_0^t e^{CU_n(t)} e^{-CU_n(\tau)} \frac{||u_0||_{B_{2,r}^s}}{1 - 2C ||u_0||_{B_{2,r}^s}^k \tau} d\tau.$$
(3.5)

By the definition of U_n , and the inductive hypothesis

$$U_n(t) \le \int_0^t \frac{\|u_0\|_{B_{2,r}^s}}{1 - 2C\|u_0\|_{B_{2,r}^s}} d\tau = \frac{-1}{2C} \ln(1 - 2C\|u_0\|_{B_{2,r}^s} t)$$

Using the above estimate we obtain the estimate

$$e^{CU_n(t)} \le \frac{1}{(1 - 2C||u_0||_{B_2^s} t)^{1/2}}.$$
 (3.6)

Similarly, we estimate the difference

$$U_n(t) - U_n(\tau) \le \frac{-1}{2C} \ln(1 - 2C \|u_0\|_{B_{2,r}^s} t) + \frac{1}{2C} \ln(1 - 2C \|u_0\|_{B_{2,r}^s} \tau),$$

which yields the estimate

$$e^{CU_n(t)}e^{-CU_n(\tau)} \le \left(\frac{1 - 2C\|u_0\|_{B_{2,r}^s}\tau}{1 - 2C\|u_0\|_{B_{2,r}^s}t}\right)^{1/2}.$$
(3.7)

Using inequality (3.6) on the first term, and estimate (3.7) on the second term, we obtain (3.5) is bounded by

$$||u_{n+1}(t)||_{B_{2,r}^s} \le \frac{||u_0||_{B_{2,r}^s}}{1 - 2C||u_0||_{B_{s-1}^s}^k t},$$
(3.8)

which completes the inductive step.

Next we will extract a subsequence of $\{u_n\}$ which converges to a solution of the FW equation.

Step 1. Our first step will be to show that there exists a sub-sequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges in $L^{\infty}([0,T];B^s_{2,r})$. Since the family $\{u_n\}$ is bounded (by the previous energy estimate) in $C([0,T];B^s_{2,r})$, we may apply the Fatou property in Besov spaces (See Danchin Theorem 2.72), and conclude that there is a sub-sequence, $\{u_{n_j}\}$, which converges to a limit u, and

$$||u||_{B_{2,r}^s} \le C \liminf_{n_j \to \infty} ||u_{n_j}||_{B_{2,r}^s}.$$

Step 2. Next we will employ Ascoli's theorem, to conclude that our element u is in $C([0,T]; B_{2,r}^{s-1})$. Since the uniform limit of continuous functions is continuous, if we show that the sequence u_n is equitontinuous in $C([0,T]; B_{2,r}^{s-1})$, we show that there exists a limit point $u \in C([0,T]; B_{2,r}^{s-1})$. Let t_1 and t_2 be in [0,T]. By the mean value theorem,

$$||u_n(t_1) - u_n(t_2)||_{B_{2,r}^{s-1}} \le |t_1 - t_2| \sup_{t \in [0,T]} ||\partial_t u_n||_{B_{2,r}^{s-1}}.$$

We have

$$\|\partial_t u_n\|_{B_{2,r}^{s-1}} \le \frac{3}{2} \|u_{n-1}\partial_x u_n\|_{B_{2,r}^{s-1}} + \|\Lambda^{-1}[\partial_x u]\|_{B_{2,r}^{s-1}}$$
(3.9)

Using $B_{2,r}^{s-1}$ is an algebra, we find

$$\|\partial_t u_n\|_{B_{2,r}^{s-1}} \le \frac{3}{2} \|u_{n-1}\|_{B_{2,r}^{s-1}} \|\partial_x u_n\|_{B_{2,r}^{s-1}} + \|u_{n-1}\|_{B_{2,r}^{s-1}}$$

Hence,

$$||u_n(t_1) - u_n(t_2)||_{B_{2,r}^{s-1}} \le |t_1 - t_2| 2 \left(||u_0||_{B_{2,r}^s} + ||u_0||_{B_{2,r}^s}^2 \right).$$

Thus, u_n is equicontinuous and we conclude that there exists a limit point of u_{n_j} , u, such that $u \in C([0,T]; B_{2,r}^{s-1})$.

Step 3. We now show that the solution $u \in C([0,T]; B_{2,r}^s)$. Since $u \in L^{\infty}([0,T]; B_{2,r}^s)$, we must show that that for every fixed $t \in (0,T)$

$$\lim_{\tau \to t} ||u(\tau) - u(t)||_{B_{2,r}^s} = 0.$$

We will show this limit by applying the Dominated convergence theorem. By definition of the Besov space norm, and using the solution is mean-zero we have

$$||u(\tau) - u(t)||_{B_{2,r}^s}^r = \sum_{q=0}^{\infty} 2^{sqr} \left(\sum_{n \neq 0} |\varphi_q(n) \left(\widehat{u}(n,\tau) - \widehat{u}(n,t) \right)|^2 \right)^{r/2}$$
(3.10)

$$= \sum_{q=0}^{\infty} 2^{sqr} \left(\sum_{n \neq 0} \varphi_q^2(n) \left(\widehat{u}^2(n,\tau) - 2\widehat{u}(n,\tau) \widehat{u}(n,t) + \widehat{u}^2(n,t) \right) \right)^{r/2}$$
(3.11)

$$= \sum_{q=0}^{\infty} 2^{sqr} \left(\|\varphi_q \widehat{u}(\tau)\|_{\ell^2}^2 + \|\varphi_q \widehat{u}(t)\|_{\ell^2}^2 - 2\langle \varphi_q \widehat{u}(\tau), \varphi_q \widehat{u}(t) \rangle_{\ell^2} \right)^{r/2}.$$
 (3.12)

First notice that the function being summed is dominated

$$2^{sqr} \left(\|\varphi_q\widehat{u}(\tau)\|_{\ell^2}^2 + \|\varphi_q\widehat{u}(t)\|_{\ell^2}^2 - 2\langle\varphi_q\widehat{u}(\tau),\varphi_q\widehat{u}(t)\rangle_{\ell^2}\right)^{r/2} \leq 2^{sqr} \left(4 \sup_{t' \in [0,T]} \|\varphi_q\widehat{u}(t')\|_{\ell^2}^2\right)^{r/2},$$

which is q-summable since $u \in L^{\infty}([0,T]; B_{2,r}^s)$. Thus, if for every q, we show the two limits (where we are considering functions of q)

$$\langle \varphi_q \widehat{u}(\tau), \varphi_q \widehat{u}(t) \rangle_{\ell^2} \xrightarrow{\tau \to t} \|\varphi_q \widehat{u}(t)\|_{\ell^2}^2$$
 (3.13)

$$\|\varphi_q \widehat{u}(\tau)\|_{\ell^2} \xrightarrow{\tau \to t} \|\varphi_q \widehat{u}(t)\|_{\ell^2}, \tag{3.14}$$

then we may conclude that for each q

$$\lim_{\tau \to t} 2^{sqr} \left(\|\varphi_q \widehat{u}(\tau)\|_{\ell^2}^2 + \|\varphi_q \widehat{u}(t)\|_{\ell^2}^2 - 2\langle \varphi_q \widehat{u}(\tau), \varphi_q \widehat{u}(t) \rangle_{\ell^2} \right)^{r/2} = 0,$$

and we may conclude that by the Dominated convergence theorem for series

$$\lim_{\tau \to t} \|u(\tau) - u(t)\|_{B_{2,\tau}^s} = 0.$$

Proof of (4.4). We have

$$D_1 = \langle \varphi_q \widehat{u}(\tau), \varphi_q \widehat{u}(t) \rangle_{\ell^2} - \|\varphi_q \widehat{u}(t)\|_{\ell^2}^2 = \langle \varphi_q(\widehat{u}(\tau) - \widehat{u}(t)), \varphi_q \widehat{u}(t) \rangle_{\ell^2}, \tag{3.15}$$

and therefore, the Cauchy-Schwarz inequality yields

$$D_1 \le \|\varphi_q(\widehat{u}(\tau) - \widehat{u}(t))\|_{\ell^2} \|\varphi_q\widehat{u}(t)\|_{\ell^2}$$
(3.16)

$$= \|\Delta_{\sigma}(u(\tau) - u(t))\|_{L^{2}} \|\Delta_{\sigma}u(t)\|_{L^{2}}.$$
(3.17)

Summing over q, and using the definition of Besov spaces, the above is bounded by

$$D_1 \le \sum_{q=0}^{\infty} \|\Delta_q(u(\tau) - u(t))\|_{L^2} \sum_{q=0}^{\infty} \|\Delta_q u(t)\|_{L^2}$$
(3.18)

$$= \|u(\tau) - u(t)\|_{B_{2,1}^0} \|u(t)\|_{B_{2,1}^0}. \tag{3.19}$$

Since s-1>0, we may apply the Besov space embedding theorem, to conclude

$$D_1 \le \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}} \|u(t)\|_{B_{2,r}^s} \le 2\|u_0\|_{B_{2,r}^s} \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}}.$$
 (3.20)

From Step 2, we know that $\lim_{\tau \to t} \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}} = 0$, and we complete the proof of (4.4).

Proof of (4.5). We see that

$$D_2 \doteq \|\varphi_q \widehat{u}(\tau)\|_{\ell^2}^2 - \|\varphi_q \widehat{u}(t)\|_{\ell^2}^2 \tag{3.21}$$

$$=\sum_{n\neq 0}\varphi_q^2(n)\widehat{u}^2(n,\tau)-\sum_{n\neq 0}\varphi_q^2(n)\widehat{u}^2(n,t) \tag{3.22}$$

$$= \sum_{n \neq 0} \varphi_q(n) \left(\widehat{u}(n,\tau) - \widehat{u}(n,t)\right) \varphi_q(n) \left(\widehat{u}(n,\tau) + \widehat{u}(n,t)\right). \tag{3.23}$$

Applying the Cauchy Schwarz inequality yields

$$D_2 \le \|\varphi_q(\widehat{u}(\tau) - \widehat{u}(t))\|_{\ell^2} \|\varphi_q(\widehat{u}(\tau) + \widehat{u}(t))\|_{\ell^2}$$
(3.24)

$$= \|\Delta_q(u(\tau) - u(t))\|_{L^2} \|\Delta_q(u(\tau) + u(t))\|_{L^2}. \tag{3.25}$$

We sum over q to obtain

$$D_2 \le \|u(\tau) - u(t)\|_{B_{2,1}^0} \|u(\tau) + u(t)\|_{B_{2,1}^0}. \tag{3.26}$$

Using the embedding theorem for Besov spaces, as well as the solution size estimate for the second term yields

$$D_2 \le 4 \|u_0\|_{B_{2,r}^s} \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}}.$$
(3.27)

Taking the limit as $\tau \to t$, of the above we conclude

$$\lim_{\tau \to t} D_2 = 0 \Longrightarrow \|\varphi_q \widehat{u}(\tau)\|_{\ell^2} \xrightarrow{\tau \to t} \|\varphi_q \widehat{u}(t)\|_{\ell^2}, \tag{3.28}$$

which completes the proof of continuity; i.e. $u \in C([0,T]; B_{2,r}^s)$.

Step 4. We now have enough information about u to show that it is a classical solution of the FW equation:

$$\partial_t u = -\frac{3}{2}u\partial_x u + \Lambda^{-1}[\partial_x u].$$

First, we see that $u_n \to u$ in $C([0,T];C^1)$ since $B_{2,r}^s \hookrightarrow C^1$, therefore the distributional derivatives, $\partial_t u_n \to \partial_t u$; i.e.

$$\int_{0}^{T} u_{n}(t)\partial_{t}\phi(t)dt \to \int_{0}^{T} u(t)\partial_{t}\phi(t)dt$$

for $\phi \in \mathcal{S}([0,T])$ and for x almost everywhere. Now we will verify that the limit of $\partial_t u_n$ is

$$\partial_t u_n \to -\frac{3}{2} u \partial_x u + \Lambda^{-1}[\partial_x u] \in C([0, T]; C),$$
 (3.29)

which shows that u is classically differentiable in t and is a classical solution of FW. Indeed, letting $||u_n||_{C([0,T];B_{2r}^s)} < \rho$, we have

$$\|\Lambda^{-1}[\partial_x u_n] - \Lambda^{-1}[\partial_x u]\|_{C([0,T];C)} \le C_1(\rho)\|u_n - u\|_{C([0,T];B_{p,r}^s)} \to 0$$
(3.30)

$$||u_n \partial_x u_{n+1} - u \partial_x u||_{C([0,T];C)} \le C_2(\rho) ||u_n - u||_{C([0,T];B^s_{2,r})} \to 0,$$
(3.31)

where we used the Besov space embedding theorem $(B_{p,r}^{s-1} \hookrightarrow C)$. Thus, taking the limit as $n \to \infty$ we obtain (4.18).

Since u is differentiable in t, and satisfies the FW equation, we can measure in what space $\partial_t u$ exists in. We observe from inequality (4.1), by replacing $\partial_t u_n$ with $\partial_t u$, that $u \in C^1([0,T]; B_{2,r}^{s-1})$, thus we have shown existence of a solution to the Cauchy problem for the FW equation

$$u \in C([0,T];B^s_{2,r}) \cap C^1([0,T];B^{s-1}_{2,r})$$

3.2. Uniqueness. The proof of uniqueness follows from the following energy estimate. Next proposition illustrates how a change in initial data reflects on the solution. Uniqueness and continuity of the solution map follow from this result.

Proposition 3.1. For $1 < r < \infty$ and s > 3/2, or r = 1 and $s \ge 3/2$; given two solutions $u, v \in C([0,T]; B_{2,r}^s) \cap C^1([0,T]; B_{2,r}^{s-1})$ of FW with initial data $u_0, v_0 \in B_{2,r}^s$ we have

$$\sup_{t \in [0,T]} \|u(t) - v(t)\|_{B_{2,r}^{s-1}} \le \|u_0 - v_0\|_{B_{2,r}^{s-1}}$$
(3.32)

Proof. Let w = u - v. Then

$$\partial_t w + \frac{3}{2} v \partial_x w = -\frac{3}{2} (u+v) w \partial_x u + \Lambda^{-1} \left[\partial_x w \right]. \tag{3.33}$$

This is a transport equation, and by Proposition 2.1, followed by the algebra property for Besov spaces, we have

$$||w||_{B_{2,r}^{s-1}} \le ||w_0||_{B_{2,r}^{s-1}} e^{c \int_0^t I(\tau) d\tau} + C \int_0^t e^{c \int_\tau^t I(\tau') d\tau'} ||w(\tau)||_{B_{2,r}^{s-1}} d\tau$$
(3.34)

where $I(t) = \|\partial_x v\|_{B_{2,r}^{s-1}}$, from which we obtain

$$\frac{d}{dt} \left(\|w\|_{B_{2,r}^{s-1}} e^{-c \int_0^t I(\tau) d\tau} \right) \le C e^{-c \int_0^t I(\tau) d\tau} \|w(t)\|_{B_{2,r}^{s-1}},$$

with $||w||_{B_{2,r}^{s-1}}|_{t=0} = ||w_0||_{B_{2,r}^{s-1}}$. By Gronwall's inequality we obtain

$$||w||_{B_{2,r}^{s-1}} \le ||w_0||_{B_{2,r}^{s-1}} e^{Rt}.$$

3.3. Continuous dependence. Let $u_0 \in B_{2,r}^s$ and let $\{u_{0,m}\}_{m=0}^{\infty}$ be a sequence in $B_{2,r}^s$ such that $u_{0,m} \to u_0$ in $B_{2,r}^s$. If u is the corresponding solution to the FW equation with initial data u_0 , and u_m are the solutions corresponding to the initial data $u_{0,m}$, then we will show

$$\lim_{m \to \infty} ||u_m - u||_{C([0,T]; B_{2,r}^s)} = 0.$$

This is equivalent to showing that for any $\delta > 0$, there exists an integer N > 0, such that if m > N

$$||u_m - u||_{C([0,T];B_{2,n}^s)} < \delta. \tag{3.35}$$

To do this, we will add and subtract u^{ε} and u_{m}^{ε} ; solutions corresponding to mollified initial data $J_{1/\varepsilon}u_{0}$ and $J_{1/\varepsilon}u_{0,m}$. Applying the triangle, we see that to prove inequality (3.35) it suffices to show

$$||u_m - u_m^{\varepsilon}||_{C([0,T];B_{2,r}^s)} < \delta/3 \tag{3.36}$$

$$\|u_m^{\varepsilon} - u^{\varepsilon}\|_{C([0,T];B_{2r}^s)} < \delta/3 \tag{3.37}$$

$$||u^{\varepsilon} - u||_{C([0,T];B_{2r}^s)} < \delta/3.$$
 (3.38)

3.3.1. Estimating $u^{\varepsilon} - u$ and $u_m - u_m^{\varepsilon}$. The estimates for the first and last terms are identical. Therefore, we will show the estimate

$$||u^{\varepsilon} - u||_{C([0,T];B_{2,r}^s)} < \delta/3.$$

We add and subtract u_n , the approximate solution to the transport equation, equation (3.1).

$$||u^{\varepsilon} - u||_{C([0,T];B_{2,r}^s)} < ||u^{\varepsilon} - u_n||_{C([0,T];B_{2,r}^s)} + ||u_n - u||_{C([0,T];B_{2,r}^s)}$$

We know that $u = \lim_{n \to \infty} u_n$ in $C([0,T]; B_{2,r}^s)$, thus by choosing n sufficiently large,

$$||u_n - u||_{C([0,T];B_{2,n}^s)} < \delta/6.$$

For the first term, define u_n^{ε} to be the solution to the transport equation (3.1) with initial data $J_n J_{1/\varepsilon} u_0(x)$. Then we have

$$||u^{\varepsilon} - u_n||_{C([0,T];B_{2,r}^s)} \le ||u^{\varepsilon} - u_n^{\varepsilon}||_{C([0,T];B_{2,r}^s)} + ||u_n^{\varepsilon} - u_n||_{C([0,T];B_{2,r}^s)}$$

From our proof of existence of solutions to the Cauchy problem for the FW equation, we know that $u_n^{\varepsilon} \to u^{\varepsilon}$. Thus, by choosing n sufficiently large,

$$||u^{\varepsilon} - u_n||_{C([0,T];B_{2,r}^s)} \le \delta/12 + ||u_n^{\varepsilon} - u_n||_{C([0,T];B_{2,r}^s)}$$

Let $v_n^{\varepsilon} = u_n^{\varepsilon} - u_n$. Then by linearity v_n^{ε} is a solution to the transport equation, with initial data $v_n^{\varepsilon}(x,0) = J_n J_{1/\varepsilon} u_0(x) - J_n u_0(x)$, we may apply the estimate found in Proposition 1 to obtain

$$||v_n^{\varepsilon}(t)||_{B_{2r}^s} \le ||J_n J_{1/\varepsilon} u_0(x) - J_n u_0(x)||_{B_{2r}^s} < \delta/12$$
(3.39)

by choosing $1/\varepsilon$ sufficiently large.

3.3.2. Estimating $u_m^{\varepsilon} - u^{\varepsilon}$. We will now prove $\|u_m^{\varepsilon} - u^{\varepsilon}\|_{C([0,T];B_{2,r}^s)} < \delta/3$. For notational convenience, we will suppress the ε superscripts. However, we note that u and u_m are in $C([0,T];B_{2,r}^{s+1})$ since the mollified initial data is $B_{2,r}^{s+1}$. We let $v = u_m - u$, and $w = u_m + u$ to obtain

$$\partial_t v + w \partial_x v = -v \partial_x w + \Lambda^{-1} [\partial_x v].$$

Using the energy estimate, (3.32), we obtain

$$||v||_{B_{2,r}^s} \lesssim ||v_0||_{B_{2,r}^s} = ||u_m^{\varepsilon}(0) - u^{\varepsilon}(0)||_{B_{2,r}^s} \leq ||u_m(0) - u(0)||_{B_{2,r}^s}$$
(3.40)

Taking m > N sufficiently large, we obtain

$$\|u_m^{\varepsilon} - u^{\varepsilon}\|_{B_{2,n}^s} < \delta/3,\tag{3.41}$$

which completes the proof of continuous dependence for the data-to-solution map.

3.4. Hölder dependence in a weaker topology. The proof of uniqueness is found by showing the data-to-solution map is Lipshiz continuous from $B_{2,r}^{s-1}$ to $C([-T,T];B_{2,r}^{s-1})$. By interpolation, we can obtain that the data-to-solution map is Hölder continuous if $s-1 < \sigma < s$ with Hölder exponent $s-\sigma$.

Corollary 3.1. Let r > 1 and s > 3/2 or r = 1 and $s \ge 3/2$. If $\sigma \in \mathbb{R}$ such that $s - 1 < \sigma < s$, then the data-to-solution map is Hölder continuous from $B_{2,r}^{\sigma}$ to $C([0,T]; B_{2,r}^{\sigma})$.

Proof. The proof of this follows directly from the energy estimate in $B_{s,r}^{s-1}$ and interpolation. \square

4. Local Well-posedness on the Real Line

Here we can duplicate the proofs for uniqueness and continuity of the flow map as well as the preliminary estimates and lifespan without any difficulty. The trouble lies in the existence of a solution on the real line. Furthermore, we must change our Friedrichs mollifiers to be non-periodic and thus we fix $j \in \mathcal{S}(\mathbb{R})$ such that $\hat{j}(0) = 1$. Then we may define $j_m(x) = mj(mx)$. Therefore, we have our Friedrichs mollifier to be J_m such that

$$J_m f \doteq j_m \star f$$
.

Now we may proceed in the proof of existence.

Step 1. Our first step will be to show that there exists a sub-sequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges in $L^{\infty}([0,T]; B^s_{2,r})$. Since the family $\{u_n\}$ is bounded (by the previous energy estimate)

in $C([0,T]; B_{2,r}^s)$, we may apply the Fatou property in Besov spaces (See Danchin Theorem 2.72), and conclude that there is a sub-sequence, $\{u_{n_i}\}$, which converges to a limit u, and

$$||u||_{B_{2,r}^s} \le C \liminf_{n_j \to \infty} ||u_{n_j}||_{B_{2,r}^s}.$$

Taking the supremum over t we obtain $u \in L^{\infty}([0,T]; B_{2,r}^s)$.

Relabeling $\{u_{n_j}\}$ as $\{u_n\}$ we will refine this sequence to converge to $u \in C([0,T]; B_{2,r}^{s-1})$. Also, throughout the rest of the proof for existence, we shall apply a cut-off function $\varphi \in \mathcal{S}(\mathbb{R})$ to each of our terms to guarantee convergence in particular topologies along the real line.

Step 2. Next we will employ Ascoli's theorem, to conclude that our element u is in $C([0,T]; B_{2,r}^{s-1})$. First we take note of the following theorem that can be found in Danchin [9].

Theorem 4.1. Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\varphi \in C_c^{\infty}(\mathbb{R})$, the map $u \mapsto \varphi u$ is compact from $B_{p,r}^{s+\varepsilon}$ to $B_{p,r}^s$.

This tells us that the set $U(t) = \{\varphi u_n(t)\}$ is precompact in $B_{2,r}^{s-1}$. Since the uniform limit of continuous functions is continuous, if we show that the sequence φu_n is equicontinuous in $C([0,T];B_{2,r}^{s-1})$, we show that there exists a limit point $\varphi u \in C([0,T];B_{2,r}^{s-1})$. Let t_1 and t_2 be in [0,T]. By the mean value theorem,

$$\|\varphi u_n(t_1) - \varphi u_n(t_2)\|_{B_{2,r}^{s-1}} \le |t_1 - t_2| \sup_{t \in [0,T]} \|\partial_t u_n\|_{B_{2,r}^{s-1}}.$$

We have

$$\|\partial_t u_n\|_{B_{2,r}^{s-1}} \le \frac{3}{2} \|u_{n-1}\partial_x u_n\|_{B_{2,r}^{s-1}} + \|\Lambda^{-1}[\partial_x u]\|_{B_{2,r}^{s-1}}$$

$$\tag{4.1}$$

Using $B_{2,r}^{s-1}$ is an algebra, we find

$$\|\partial_t u_n\|_{B^{s-1}_{2,r}} \le \frac{3}{2} \|u_{n-1}\|_{B^{s-1}_{2,r}} \|\partial_x u_n\|_{B^{s-1}_{2,r}} + \|u_{n-1}\|_{B^{s-1}_{2,r}}$$

Hence,

$$\|\varphi u_n(t_1) - \varphi u_n(t_2)\|_{B_{2,r}^{s-1}} \le |t_1 - t_2| 2\left(\|u_0\|_{B_{2,r}^s} + \|u_0\|_{B_{2,r}^s}^2\right).$$

Thus, φu_n is equicontinuous and we conclude that there exists a limit point of φu_{n_j} , φu , such that $\varphi u \in C([0,T]; B_{2,r}^{s-1})$.

Step 3. We now show that the solution $u \in C([0,T]; B_{2,r}^s)$. Since $u \in L^{\infty}([0,T]; B_{2,r}^s)$, we must show that that for every fixed $t \in (0,T)$

$$\lim_{\tau \to t} \|u(\tau) - u(t)\|_{B_{2,r}^s} = 0.$$

We will show this limit by applying the Dominated convergence theorem. By definition of the Besov space norm we have

$$||u(\tau) - u(t)||_{B_{2,r}^s}^r = \sum_{q \ge -1} 2^{sqr} ||\Delta_q(u(\tau) - u(t))||_{L^2}^r$$
(4.2)

$$= \sum_{q>-1} 2^{sqr} [\|\Delta_q u(\tau)\|_{L^2}^2 + \|\Delta_q u(t)\|_{L^2}^2 - 2\langle \Delta_q u(\tau), \Delta_q u(t) \rangle_{L^2}]^r.$$
 (4.3)

First notice that the function being summed is dominated

$$2^{sqr} [\|\Delta_q u(\tau)\|_{L^2}^2 + \|\Delta_q u(t)\|_{L^2}^2 - 2\langle \Delta_q u(\tau), \Delta_q u(t) \rangle_{L^2}]^r \leq 2^{sqr} \left(4 \sup_{t' \in [0,T]} \|\Delta_q u(t')\|_{L^2}^2\right)^r,$$

which is q-summable since $u \in L^{\infty}([0,T]; B_{2,r}^s)$. Thus, if for every q, we show the two limits (where we are considering functions of q)

$$\langle \Delta_q u(\tau), \Delta_q u(t) \rangle_{L^2} \xrightarrow{\tau \to t} \|\Delta_q u(t)\|_{L^2}$$
 (4.4)

$$\|\Delta_{q}u(\tau)\|_{L^{2}} \xrightarrow{\tau \to t} \|\Delta_{q}u(t)\|_{L^{2}},\tag{4.5}$$

then we may conclude that for each q

$$\lim_{\tau \to t} 2^{sqr} [\|\Delta_q u(\tau)\|_{L^2}^2 + \|\Delta_q u(t)\|_{L^2}^2 - 2\langle \Delta_q u(\tau), \Delta_q u(t) \rangle_{L^2}]^r = 0,$$

and we may conclude that by the Dominated convergence theorem for series

$$\lim_{\tau \to t} \|u(\tau) - u(t)\|_{B_{2,r}^s} = 0.$$

Proof of (4.4). We have

$$D_1 = \langle \Delta_q u(\tau), \Delta_q u(t) \rangle_{L^2} - \|\Delta_q u(t)\|_{L^2}^2 = \langle \Delta_q (u(\tau) - u(t)), \Delta_q u(t) \rangle_{L^2}, \tag{4.6}$$

and therefore, the Cauchy-Schwarz inequality yields

$$D_1 \le \|\Delta_q(u(\tau) - u(t))\|_{L^2} \|\Delta_q u(t)\|_{L^2}. \tag{4.7}$$

Summing over q, and using the definition of Besov spaces, the above is bounded by

$$D_1 \le \sum_{q=0}^{\infty} \|\Delta_q(u(\tau) - u(t))\|_{L^2} \sum_{q=0}^{\infty} \|\Delta_q u(t)\|_{L^2}$$
(4.8)

$$= \|u(\tau) - u(t)\|_{B_{2,1}^0} \|u(t)\|_{B_{2,1}^0}. \tag{4.9}$$

Since s-1>0, we may apply the Besov space embedding theorem, to conclude

$$D_1 \le \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}} \|u(t)\|_{B_{2,r}^s} \le 2\|u_0\|_{B_{2,r}^s} \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}}. \tag{4.10}$$

From Step 2, we know that $\lim_{\tau \to t} \|u(\tau) - u(t)\|_{B_{2,r}^{s-1}} = 0$, and we complete the proof of (4.4).

Proof of (4.5). We see that

$$D_2 \doteq \|\Delta_q u(\tau)\|_{L^2}^2 - \|\Delta_q u(t)\|_{L^2}^2 \tag{4.11}$$

$$= \int_{\mathbb{R}} (\Delta_q u(\tau))^2 - (\Delta_q u(t))^2 dx \tag{4.12}$$

$$= \int_{\mathbb{R}} \Delta_q(u(\tau) + u(t)) \Delta_q(u(\tau) - u(t)) dx. \tag{4.13}$$

Applying the Cauchy Schwarz inequality yields

$$D_2 \le \|\Delta_q(u(\tau) - u(t))\|_{L^2} \|\Delta_q(u(\tau) + u(t))\|_{L^2}. \tag{4.14}$$

We sum over q to obtain

$$D_2 \le \|u(\tau) - u(t)\|_{B_{2,1}^0} \|u(\tau) + u(t)\|_{B_{2,1}^0}. \tag{4.15}$$

Using the embedding theorem for Besov spaces, as well as the solution size estimate for the second term yields

$$D_2 \le 4||u_0||_{B_{2,r}^s}||u(\tau) - u(t)||_{B_{2,r}^{s-1}}. (4.16)$$

Taking the limit as $\tau \to t$, of the above we conclude

$$\lim_{\tau \to t} D_2 = 0 \Longrightarrow \|\Delta_q u(\tau)\|_{L^2} \xrightarrow{\tau \to t} \|\Delta_q u(t)\|_{L^2},\tag{4.17}$$

which completes the proof of continuity; i.e. $u \in C([0,T]; B_{2,r}^s)$.

Step 4. We now have enough information about u to show that it is a classical solution of the FW equation:

$$\partial_t u = -\frac{3}{2}u\partial_x u + \Lambda^{-1}[\partial_x u].$$

First, we see that $\varphi u_n \to \varphi u$ in $C([0,T];C^1)$ since $B_{2,r}^s \hookrightarrow C^1$, therefore the distributional derivatives, $\partial_t \varphi u_n \to \partial_t \varphi u$; i.e.

$$\int_0^T \varphi u_n(t) \partial_t \phi(t) dt \to \int_0^T \varphi u(t) \partial_t \phi(t) dt$$

for $\phi \in \mathcal{S}([0,T])$ and for x almost everywhere. Now we will verify that the limit of $\partial_t \varphi u_n$ is

$$\partial_t \varphi u_n \to -\varphi \frac{3}{2} u \partial_x u + \varphi \Lambda^{-1} [\partial_x u] \in C([0, T]; C),$$
 (4.18)

which shows that u is classically differentiable in t and is a classical solution of FW. Indeed, letting $\|\varphi u_n\|_{C([0,T];B_{2,r}^s)} < \rho$, we have

$$\|\varphi\Lambda^{-1}[\partial_x u_n] - \varphi\Lambda^{-1}[\partial_x u]\|_{C([0,T];C)} \le C_1(\rho)\|\varphi u_n - \varphi u\|_{C([0,T];B_{n,r}^s)} \to 0 \tag{4.19}$$

$$\|\varphi u_n \partial_x u_{n+1} - \varphi u \partial_x u\|_{C([0,T];C)} \le C_2(\rho) \|\varphi u_n - \varphi u\|_{C([0,T];B^s_{2,r})} \to 0, \tag{4.20}$$

where we used the Besov space embedding theorem $(B_{p,r}^{s-1} \hookrightarrow C)$. Thus, taking the limit as $n \to \infty$ we obtain (4.18).

Since u is differentiable in t, and satisfies the FW equation, we can measure in what space $\partial_t u$ exists in. We observe from inequality (4.1), by replacing $\partial_t \varphi u_n$ with $\partial_t \varphi u$, that $u \in C^1([0,T]; B_{2,r}^{s-1})$, thus we have shown existence of a solution to the Cauchy problem for the FW equation

$$u \in C([0,T];B^s_{2,r}) \cap C^1([0,T];B^{s-1}_{2,r}).$$

5. Nonuniform dependence of the data-to-solution map on the torus

In this section, we show that the data-to-solution map is not uniformly continuous. We will use the method of approximate solutions, and set

$$u^{n}(x,t) = \frac{2}{3}n^{-1} + n^{-s}\cos(nx - t)$$
 and $v^{n} = -\frac{2}{3}n^{-1} + n^{-s}\cos(nx + t)$.

Now let u_n and v_n denote the actual solutions to the FW equation (1.1) with u^n and v^n as initial data respectively. We shall show the following properties concerning u^n , v^n and the corresponding solutions u_n and v_n .

- (1) $\lim_{n\to\infty} \|u_{0,n} v_{0,n}\|_{B_2^s} = 0$
- (2) $||u_n(t)||_{B_{2n}^s} + ||v_n(t)||_{B_{2n}^s} \le C \quad \forall \quad 0 \le t \le T$
- (3) $||u_n(t) u^n(t)||_{B_{2,r}^s} + ||v_n(t) v^n(t)||_{B_{2,r}^s} \le n^{-\alpha}, \quad \alpha > 0, \quad \forall \quad 0 \le t \le T$
- (4) $||u_n(t) v_n(t)||_{B_{2n}^s} > \sin(t), \quad \forall \quad 0 < t \le T.$

We will provide the details when $r = \infty$. In the case $r < \infty$, the proof is similar to the result contained in [22].

5.1. Convergence of initial data and common lifespan. Taking the difference, we see that the first property is trivially satisfied. From our well–posedness result, the second property is satisfied if the initial data is bounded independent of n. We have

$$||u_{0,n}||_{B_{2,\infty}^{\gamma}} \le n^{-1} + ||n^{-s}\cos(nx)||_{B_{2,\infty}^{\gamma}}$$
(5.1)

$$= n^{-1} + \sup_{q} n^{-s} 2^{\gamma q} \|\Delta_q \cos(nx)\|_{L^2}, \tag{5.2}$$

Letting $\delta_n(m)$ be the indicator function, taking a value of 1 if m=n and 0 otherwise, we have

$$\|\Delta_q \cos(nx)\|_{L^2} = \frac{\pi}{2} \left(\sum_{m=-\infty}^{\infty} \varphi_q^2(m) \left| \delta_n(m) + \delta_{-n}(m) \right|^2 \right)^{1/2} = \frac{\pi}{2} \left(2\varphi_q^2(n) \right)^{1/2}, \tag{5.3}$$

where we used that φ_q is symmetric. By the definition of φ_q , we have

$$\varphi_q(n) = \begin{cases} 1 & \text{if } \frac{1}{\ln(2)} \ln(\frac{3}{8}n) \le q \le \frac{1}{\ln(2)} \ln(\frac{4}{3}n) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we obtain

$$||u_{0,n}||_{B_{2,\infty}^{\gamma}} \approx n^{-1} + n^{\gamma - s}.$$
 (5.4)

5.2. Error bounds for $u_n - u^n$. We have that the difference between the exact and approximate solutions, $\mathcal{E} = u_n - u^n$ satisfy

$$\begin{cases} \partial_t \mathcal{E} + \frac{3}{2} u^n \partial_x \mathcal{E} = -\frac{3}{2} \mathcal{E} \partial_x u_n + \Lambda^{-1} \left[\partial_x \mathcal{E} \right] - R \\ \mathcal{E}(x, 0) = 0, \end{cases}$$
 (5.5)

where the residue $R = \partial_t u^n + \frac{3}{2} u^n \partial_x u^n - \Lambda^{-1} [\partial_x u^n]$. We shall show that for all $0 < t < T \lim_{n \to \infty} \|\mathcal{E}(t)\|_{B^s_{2,r}} = 0$, which we shall accomplish by showing that for $\frac{1}{2} < \gamma < s - 1$, $\lim_{n \to \infty} \|\mathcal{E}(t)\|_{B^{\gamma}_{2,r}} = 0$. Then we shall interpolating between a particular $s_1 < s$ and $s_2 > s$.

Since the error \mathcal{E} satisfies the transport equation (5.5), for γ satisfying $s-1>\gamma>1/2$ we may apply Proposition 2.1 to obtain

$$e^{-CV(t)} \|\mathcal{E}\|_{B_{2,\infty}^{\gamma}} \le C \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{2,\infty}^{\gamma}} d\tau,$$
 (5.6)

where

$$V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{\gamma-1}_{2,\infty}} d\tau = \int_0^t \|\partial_x u^n\|_{B^{\gamma-1}_{2,\infty}} d\tau \lesssim n^{\gamma-s} t$$

and

$$F(t) = -\frac{3}{2}\mathcal{E}\partial_x u_n + \Lambda^{-1} \left[\partial_x \mathcal{E}\right] - R.$$

We differentiate (5.6) with respect to t and use

$$||F(\tau)||_{B_{2,\infty}^{\gamma}} = ||-\frac{3}{2}\mathcal{E}\partial_x u_n + \Lambda^{-1}[\partial_x \mathcal{E}] - R||_{B_{2,\infty}^{\gamma}} \lesssim (1 + n^{\gamma + 1 - s})||\mathcal{E}||_{B_{2,\infty}^{\gamma}} + ||R||_{B_{2,\infty}^{\gamma}},$$

to obtain the inequality

$$\frac{d}{dt} \left[e^{-CV(t)} \| \mathcal{E} \|_{B_{2,\infty}^{\gamma}} \right] \le C e^{-CV(t)} (1 + n^{\gamma + 1 - s}) \| \mathcal{E} \|_{B_{2,\infty}^{\gamma}} + C e^{-CV(t)} \| R \|_{B_{2,\infty}^{\gamma}}. \tag{5.7}$$

Using the bound on V and setting $y=e^{-CV(t)}\|\mathcal{E}\|_{B^{\gamma}_{2,\infty}}$, we obtain the ODE

$$\frac{dy}{dt} \le C(1 + n^{\gamma + 1 - s})y + C_{s,\gamma}e^{Cn^{\gamma - s}t} ||R||_{B_{2,\infty}^{\gamma}}.$$
(5.8)

Since $\gamma + 1 < s$ we obtain

$$\|\mathcal{E}\|_{B_{2,\infty}^{\gamma}} \approx y \lesssim \|R\|_{B_{2,\infty}^{\gamma}},\tag{5.9}$$

i.e. the difference between actual and approximate solutions are bounded by the size of the residue, R, for the entire lifespan of the solutions. The following lemma yields the size estimate for the residue.

Lemma 5.1. For all $t \in \mathbb{R}$ and $0 < \gamma < s$, we have $||R||_{B_{2,\infty}^{\gamma}} < n^{-\alpha}$ for some $\alpha = \alpha(s,\gamma) > 0$.

Proof. Recall, the residue is defined as

$$R = \partial_t u^n + \frac{3}{2} u^n \partial_x u^n - \Lambda^{-1} \partial_x u^n.$$

By the definition of the approximate solution $u^n(x,t)$, we have

$$\partial_t u^n(x,t) = n^{-s} \sin(nx-t), \qquad \partial_x u^n(x,t) = -n^{1-s} \sin(nx-t)$$

Therefore,

$$\partial_t u^n + \frac{3}{2} u^n \partial_x u^n = n^{-s} \sin(nx - t) + \frac{3}{2} \left(\frac{2}{3} n^{-1} + n^{-s} \cos(nx - t) \right) (-n^{1-s} \sin(nx - t))$$
$$= -\frac{3}{2} n^{1-2s} \cos(nx - t) \sin(nx - t).$$

Using $\cos(x)\sin(x) = \frac{1}{2}\sin(2x)$ we obtain

$$||R||_{B_{2,\infty}^{\gamma}} \lesssim n^{1-2s} n^{\gamma} + n^{1-s} n^{\gamma-2} \lesssim n^{-\alpha},$$
 (5.10)

where

$$\alpha = \begin{cases} s + 1 - \gamma & \text{if } s \ge 2\\ 2s - 1 - \gamma & \text{if } s < 2. \end{cases}$$

$$(5.11)$$

From the estimate in inequality (5.4), the solution size estimate $||u_n(t)||_{B_{2,\infty}^{s_2}} \le 2||u_n(0)||_{B_{2,\infty}^{s_2}}$, and repeated application of the algebra property, we have for any $s_2 > s$

$$\|\mathcal{E}(t)\|_{B_{2\infty}^{s_2}} \lesssim n^{s_2-s}.$$
 (5.12)

By applying the interpolation lemma for Besov spaces, we obtain

$$\|\mathcal{E}(t)\|_{B_{2,\infty}^s} \le \|\mathcal{E}(t)\|_{B_{2,\infty}^{\gamma}}^{\frac{s_2-s}{s_2-\gamma}} \|\mathcal{E}(t)\|_{B_{2,\infty}^{s_2-\gamma}}^{\frac{s-\gamma}{s_2-\gamma}} \lesssim n^{-\beta}, \tag{5.13}$$

where

$$\beta = \begin{cases} \frac{s_2 - s}{s_2 - \gamma} & \text{if } s \ge 2\\ \frac{(s_2 - s)(s - 1)}{s_2 - \gamma} & \text{if } s < 2. \end{cases}$$

$$(5.14)$$

5.3. Nonuniform convergence. We now show that u_n and v_n stay bounded away from each other for any $t \neq 0$.

$$||u_n - v_n||_{B_{2,\infty}^s} \ge ||u^n - v^n||_{B_{2,\infty}^s} - ||u_n - u^n||_{B_{2,\infty}^s} - ||v^n - v_n||_{B_{2,\infty}^s}.$$

$$(5.15)$$

From (5.13) the second two terms tend towards zero as $n \to \infty$, therefore,

$$\lim_{n \to \infty} \inf \|u_n - v_n\|_{B_{2,\infty}^s} \ge \lim_{n \to \infty} \inf \|u^n - v^n\|_{B_{2,\infty}^s}
\ge \lim_{n \to \infty} \inf n^{-s} \|\cos(nx - t) - \cos(nx + t)\|_{B_{2,\infty}^s}
\approx \lim_{n \to \infty} \inf n^{-s} \|\sin(nx)\|_{B_{2,\infty}^s} |\sin(t)| \approx |\sin(t)| > 0.$$

6. Analyticity of Solutions

In this section, we search for real analytic solutions of the periodic Cauchy problem for the FW equation (1.1). The classical Cauchy-Kowalevski theorem does not apply to equation (1.1). However, a contraction argument on a scale of Banach spaces can be used for the nonlocal form of this equation (1.2) to prove Theorem 1.4.

Remark 6.1: The contraction argument used in the proof of Theorem 1.4 is on a decreasing scale of Banach spaces. We give the statement of this abstract theorem here as stated in [1] and [26] for the convenience of the reader:

Given a Cauchy problem

$$\partial_t u = F(t, u(t)), \quad u(0) = 0, \tag{6.1}$$

and a decreasing scale of Banach spaces $\{X_s\}_{0 < s < 1}$ so that for any s' < s we have $X_s \subset X_{s'}$ and $||| \cdot |||_{s'} \le ||| \cdot |||_s$, let T, R, and C be positive numbers, suppose that F satisfies the following conditions:

- (1) If for 0 < s' < s < 1 the function $t \mapsto u(t)$ is holomorphic in |t| < T and continuous on $|t| \le T$ with values in X_s and $\sup_{|t| \le T} |||u(t)|||_s < R$, then $t \mapsto F(t, u(t))$ is a holomorphic function on |t| < T with values in $X_{s'}$.
- (2) For any $0 < s' < s \le 0$ and any $u, v \in X_s$ with $|||u|||_s < R$, $|||v|||_s < R$,

$$\sup_{|t| < T} |||F(t, u) - F(t, v)|||_{s'} \le \frac{C}{s - s'} |||u - v|||_{s}.$$

(3) There exists M > 0 such that for any 0 < s < 1,

$$\sup_{|t| \le T} |||F(t,0)|||_s \le \frac{M}{1-s}.$$

Then there exists a $T_0 \in (0,T)$ and a unique function u(t), which for every $s \in (0,1)$ is holomorphic in $|t| < (1-s)T_0$ with values in X_s , and is a solution to the initial value problem (6.1).

For s > 0 let E_s be defined as

$$E_s = \left\{ u \in C^{\infty}(\mathbb{T}) : |||u|||_s = \sup_{k>0} \frac{\|\partial_x^k u\|_{B_{2,1}^{3/2}} s^k}{k!/(k+1)^2} < \infty \right\}.$$

Let X_s be the space E_s . We use the decreasing scale of Banach spaces X_s with s > 0 for the contraction argument. We take note that the above space is similar to the one introduced in [19]

where $B_{2,1}^{3/2}$ is replaced by H^2 . It is not difficult to check that E_s equipped with the norm $||| \cdot |||_s$ is a Banach space and that for any 0 < s' < s, E_s is continuously included in $E_{s'}$ with

$$|||u|||_{s'} < |||u|||_{s}$$
.

We now include three properties of the spaces E_s that we shall use in the proof of Theorem 1.4. Since we have the following from Besov spaces

$$\|fg\|_{B^{3/2}_{2,1}} \leq C(\|f\|_{B^{3/2}_{2,1}}\|g\|_{B^{1/2}_{2,1}} + \|g\|_{B^{3/2}_{2,1}}\|f\|_{B^{1/2}_{2,1}})$$

and

$$\|\Lambda^{\sigma}\partial_x^k f\|_{B_2^{3/2}} \le C\|f\|_{B_2^{3/2+k+\sigma}}$$

for $\sigma \in \mathbb{R}$ and k a non-negative integer, these three properties follow directly from analogous ones found in [19] just by replacing H^2 with $B_{2,1}^{3/2}$. We therefore omit the details for the sake of brevity and refer the reader to [19].

The first may be seen as a ring property for E_s when 0 < s < 1: For any $u, v \in E_s$ there is a constant c > 0 that is independent of s such that

$$|||uv|||_s \le c|||u|||_s|||v|||_s.$$

The second property is an estimate for the differential operator ∂_x : For 0 < s' < s < 1 we have

$$|||u_x|||_{s'} \le \frac{C}{s-s'}|||u|||_s$$

for all $u \in E_s$. The third property is a pair of estimates for the operator $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$: For 0 < s < 1 we have

$$|||\Lambda^{-2}u|||_{s'} \le |||u|||_s$$
 and $|||\Lambda^{-2}\partial_x u|||_{s'} \le |||u|||_s$

for all $u \in E_s$. Now we proceed to the proof of Theorem 1.4.

Proof. Note that it is sufficient to verify conditions (1) and (2) in the abstract Cauchy-Kowalevski theorem for

$$F(u) = -\frac{3}{2}uu_x + \Lambda^{-2}\partial_x u$$

since F(u) does not depend on t explicitly. We find that for 0 < s' < s < 1 we have

$$|||F(u)|||_{s'} \le \frac{C}{s-s'}|||u|||_s^2 + |||u|||_s,$$

hence condition (1) holds. To verify the second condition we estimate that

$$|||F(u) - F(v)|||_{s'} \le C_R \left(|||u - v|||_s + \frac{1}{s - s'}|||u - v|||_s \right)$$

for any $u, v \in E_s$. Therefore, condition (2) is satisfied.

7. Blow-up Criterion

Here we prove Theorem 1.5.

Proof. Applying Δ_q to (1.2) yields

$$\left(\partial_t + \frac{3}{2}u\partial_x\right)\Delta_q u = \left[\frac{3}{2}u, \Delta_q\right]\partial_x u + \Delta_q \Lambda^{-1}\partial_x u. \tag{7.1}$$

From (2.54) on p. 112 in [2], we have that

$$\left\| 2^{sq} \left\| \left[\frac{3}{2} u, \Delta_q \right] \partial_x u \right\|_{L^2} \right\|_{L^r} \le C \|u_x\|_{L^\infty} \|u\|_{B^s_{2,r}}. \tag{7.2}$$

We also have that

$$\|\Lambda^{-1}\partial_x u\|_{B^s_{2,r}} \le \|u\|_{B^s_{2,r}}. (7.3)$$

Going along the lines of the proof of Proposition A.1 in [7] and by using (7.2) and (7.3) we obtain

$$||u||_{B_{2,r}^s} \le ||u_0||_{B_{2,r}^s} + C \int_0^t [1 + ||u_x||_{L^\infty}] ||u||_{B_{2,r}^s} d\tau \tag{7.4}$$

Applying Gronwall's inequality to (7.4) yields

$$||u(t)||_{B_{2,r}^s} \le ||u_0||_{B_{2,r}^s} \exp\left[C \int_0^t [1 + ||u_x||_{L^\infty}] d\tau\right].$$
 (7.5)

Assume that T^* is the maximal time of existence of the solution to the Cauchy problem to (1.2). If $T^* < \infty$ we claim that

$$\int_{0}^{T^{*}} \|u_{x}\|_{L^{\infty}} d\tau = +\infty. \tag{7.6}$$

We prove the claim (7.6) by contradiction. If (7.6) is invalid, then from (7.5) we know that $||u(T^*)||_{B_{2,r}^s} < \infty$ which contradicts with the fact that T^* is the maximal time of existence to the Cauchy problem to (1.2). We have concluded the proof of Theorem 1.5.

References

- [1] M.S. Baouendi, C. Goulaouic, Sharp estimates for analytic pseudodifferential operators and application to Cauchy problems, J. Differential Equations 48 (1983), no. 2, 241–268.
- [2] H. Bahouri, J. Chemin, R. Danchin, Foruier Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin, Heidelberg, 2011.
- [3] J. Boussinesq, Essai sur la théorie des eaux courantes, Mémoires présentés par divers savants à l'Académie des Sciences 23 (1877), no. 1, 1–680.
- [4] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993), 1661–1664.
- [5] J.Y. Chemin, Localization in Fourier space and Navier-Stokes, Phase Space Analysis of Partial Differential Equations, CRM series, Scuola Norm. Sup., Pisa, (2004), 53–136.
- [6] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Arch. Ration. Mech. Anal. 192 (2009), no. 1, 165–186.
- [7] R. Danchin, A few remarks on the Camassa-Holm equation, Differential and Integral Equations 14 (2001), 953–988.
- [8] R. Danchin, A note on well-posedness for the Camassa-Holm equation, J. Differential Equations 192 (2003), 429-444.
- [9] R. Danchin, Fourier Analysis Method for PDEs, Lecture Notes 14, 2005.
- [10] A. Degasperis and M. Procesi, Asymptotic integrability symmetry and perturbation theory, (Rome, 1998), World Sci. Publ. (1999), 23–37.
- [11] G. Fornberg and G.B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, Philos. Trans. Roy. Soc. London Ser. A 289 (1978), no. 1361, 373–404.
- [12] B. Fuchssteiner and A. Fokas, Symplectic structures, their Bäklund transformations and hereditary symmetries, Phys. D 4 (1981/82), no. 1, 47–66.
- [13] K. Grayshan, Continuity properties of the data-to-solution map for the periodic b-family equation, Differential and Integral Equations 25 (2012), no. 1-2, 1-20.
- [14] A. Himonas and C. Holliman, On well-posedness of the Degasperis-Procesi equation, Nonlinearity 25 (2012), 449–479.
- [15] A. Himonas and C. Holliman, The Cauchy problem for the Novikov equation, Discrete Contin. Dyn. Syst. 31 (2011), no. 2, 469–488.

- [16] A. Himonas and C. Holliman, *The Cauchy problem for a generalized Camassa-Holm equation*, Advances in Differential Equations **19** (2014), no. 1-2, 161–200.
- [17] A. Himonas and C. Kenig, Non-uniform dependence on initial data for the CH equation on the line, Differential and Integral Equations 22 (2009), no. 3–4, 201–224.
- [18] A. Himonas, C. Kenig and G. Misiołek, *Non-uniform dependence for the periodic CH equation*, Comm. Partial Differential Equations **35** (2010), no. 6, 1145–1162.
- [19] A. Himonas and G. Misiołek, Analyticity of the Cauchy problem for an integrable evolution equation, Math. Ann. 327 (2003), no. 3, 575–584.
- [20] A. Himonas, G. Misiolek and G. Ponce, Non-uniform continuity in H¹ of the solution map of the CH equation, Asian J. Math. 11 (2007), no. 1, 141–150.
- [21] J. Holmes, Well-Posedness of the Fornberg-Whitham equation on the circle, J. Differential Equations 260 (2016), no.12, 8530–8549
- [22] J. Holmes and F. Tiglay, The Cauchy problem for the periodic Hunter-Saxton equation in Besov spaces, Preprint.
- [23] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, J. Fluid Mech. 457 (2002), 63–82.
- [24] R.S. Johnson, The Camassa-Holm equation for water waves moving over a shear flow, Fluid Dynam. Res. 33 (2003), 97–111.
- [25] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philos. Mag. 39 (1895), 422–443.
- [26] T. Nishida, A note on a theorem of Nirenberg, J. Differential Geometry 12 (1977), no. 4, 629-633.
- [27] V. Novikov, Generalizations of the Camassa-Holm equation, J. Phys. A. 42 (2009), no. 34, 342002, 14 pp.
- [28] M. Taylor, Pseudodifferential Operators and Nonlinear PDE, Birkhauser, Boston 1991.
- [29] R. C. Thompson, The periodic Cauchy problem for the 2-component Camassa-Holm system, Differential and Integral Equations 26 (2013), no. 1-2, 155–182.
- [30] M. Vishik, Hydrodynamics in Besov spaces, Arch. Ration. Mech. Anal. 145 (1998), 197-214.
- [31] G.B. Whitham, Variational methods and applications to water waves, Proc. of the Royal Soc. **299** (1967), no. 1456, 6–25.
- [32] A. Yin, Well-posedness and blowup phenomena for a class of nonlinear third-order partial differential equations, Houston J. Math. 31 (2005), no. 3, 961–972.

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