Direct Images in Non Abelian Hodge Theory

R. Donagi, T. Pantev and C. Simpson

Abstract

In this paper we explain how non-abelian Hodge theory allows one to compute the L^2 cohomology or middle perversity higher direct images of harmonic bundles and twistor D-modules in a purely algebraic manner. Our main result is a new algebraic description for the fiberwise L^2 cohomology of a tame harmonic bundle or the corresponding flat bundle or tame polystable parabolic Higgs bundle. Specifically we give a formula for the Dolbeault version of the L^2 pushforward in terms of a modification of the Dolbeault complex of a Higgs bundle which takes into account the monodromy weight filtration in the normal directions of the horizontal parabolic divisor. The parabolic structure of the higher direct image is obtained by analyzing the V-filtration at a normal crossings point. We prove this algebraic formula for semistable families of curves.

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1 Introduction

The Non Abelian Hodge Correspondence (NAHC) on a variety X is the equivalence

$$\begin{pmatrix} \text{semisimple local} \\ \text{systems on } X \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{polystable Higgs bundles on} \\ X \text{ with vanishing Chern} \\ \text{classes } c_1 = c_2 = 0 \end{pmatrix}$$

given by Hitchin's equations. Here a local system L is a representation of the fundamental group of X, while a Higgs bundle on X is a pair (E,φ) where E is a vector bundle on X and the Higgs field $\varphi \in \Gamma(X, \operatorname{End}(E) \otimes \Omega^1_X)$ self-commutes, in the sense that

$$\varphi \wedge \varphi = 0 \in \Gamma(X, \operatorname{End}(E) \otimes \Omega_X^2),$$

so there is a *Dolbeault complex*:

$$DOL(X, (E, \varphi)) := \left[E \xrightarrow{\varphi} E \otimes \Omega_X^1 \xrightarrow{\varphi} E \otimes \Omega_X^2 \xrightarrow{\varphi} \dots \right]. \tag{1}$$

There is a richer object called a harmonic bundle, reviewed in section 2.2, which naturally determines both a Higgs bundle (E,φ) and a local system L. In fact, it determines a family of λ -connections for all $\lambda \in \mathbb{C}$, with the Higgs bundles arising at $\lambda = 0$ and the local system, or equivalently a flat connection, at $\lambda = 1$. The NAHC ("Kobayashi-Hitchin correspondence" in Mochizuki's terminology) says that this sets up an equivalence between Higgs bundles, local systems, and harmonic bundles. The equivalence, in the compact case, is due in one direction to Hitchin [7] for one dimensional X and to Simpson [19] in higher dimension. The other direction is provided by Donaldson's appendix to Hitchin's paper for curves, and Corlette [6] in all dimensions.

A version of the NAHC on non compact spaces $X \setminus D$, where X is compact and $D \subset X$ is a normal crossing divisor, was established in [18] for one dimensional X, in [2] for X of arbitrary dimension with smooth D, and in [12] in general. The setup involves a local system L and a Higgs bundle (E,φ) that are defined on $X \setminus D$ and carry order of growth filtrations along the components of D. More precisely, L is a polystable filtered local system with vanishing parabolic Chern classes, and (E,φ) is a parabolic Higgs bundle, consisting of a locally abelian parabolic vector bundle E with vanishing parabolic Chern classes, together

with a Higgs field φ that is logarithmic with respect to the parabolic structure along D. The theorem states that both types of data are equivalent to a *tame* harmonic bundle on $X \setminus D$. We refer to section 2.2 for more details.

Natural operations such as pullback and direct image with respect to a morphism $f: X \to Y$ are well defined for harmonic bundles and local systems, and commute with the NAHC. Pullback for Higgs bundles is also well defined and commutes with the NAHC. The main goal of the present work is to give an algebraic definition of direct image for Higgs bundles and to show that it too commutes with the NAHC.

Consider the case that X is compact and Y is a point. The direct image of a Higgs bundle (E, φ) on X should be an object of the bounded derived category of coherent sheaves on the point, so it should be described by a complex of vector spaces. For this we take the Dolbeault complex (1). This is the correct choice in the sense that if (E, φ) corresponds to L by the NAHC then there is a natural isomorphism:

$$Rf_*L \xrightarrow{\cong} \mathbb{R}\Gamma(X, \mathrm{DOL}(X, (E, \varphi))).$$

For example, the trivial local system $L = \mathbb{C}$ corresponds to the trivial vector bundle $E = \mathcal{O}$ along with the the null Higgs field $\varphi = 0$. In this case the above isomorphism is the ordinary Hodge theorem: $Rf_*\mathbb{C}$ gives the de Rham cohomology $H^{\bullet}_{DR}(X,\mathbb{C})$, while $DOL(X,(E,0)) = \bigoplus_i \Omega^i_X[-i]$ are the holomorphic forms on X, which give the Dolbeault cohomology $H^{\bullet}_{Dol}(X,\mathbb{C}) = \bigoplus_{p,q} H^p(X,\Omega^q_X)$.

The analogous result in the open case, when X is one dimensional, was established by Zucker [23] for variations of Hodge structure. In this work we extend these results to families of spaces and to general local systems.

Our main result is Theorem 3.6. We consider a map $f: X \to Y$ from a smooth projective surface X to a smooth projective curve Y, with a reduced divisor $Q \subset Y$, and a (reduced) simple normal crossings divisor $D = D_V + D_H \subset X$, consisting of horizontal and vertical parts: D_H is etale over Y, while $D_V = f^{-1}(Q)$. Starting from a Higgs bundle (E, φ) on $X \setminus D$, we construct in (2) the L^2 Dolbeault complex $DOL_{L^2}^{par}(X/Y, E_{\alpha(a)})$. This construction is purely algebraic. The theorem asserts that the direct images $F_a^i := \mathbb{R}^i f_* \left(DOL_{L^2}^{par}(X/Y, E_{\alpha(a)}) \right)$ are the correct direct images for the Higgs bundle (E, φ) , in the sense that they fit together into the parabolic Higgs bundles \underline{F}^i that correspond under NAHC to the direct images of

¹Our notational convention will be that a parabolic structure is denoted by an underlined letter; it consists of a collection of sheaves indexed by the parabolic levels, but the letters for these component sheaves are not underlined.

the various objects (harmonic bundle and filtered local system) corresponding under NAHC to the original Higgs bundle (E, φ)

As discussed in Section 7.2 at the end, one can envision analytic arguments going in the direction of the proof. But the essential ingredient would be to know that the higher direct image sheaves F_a^i are locally free. That should be viewed as some kind of strictness property, not easy to obtain "by hand".

It is therefore natural to use Sabbah's theory of twistor \mathcal{D} -modules [17], generalizing the technique of Saito [15] for Hodge modules. This allows furthermore to leave the analytic considerations in the background as they are already treated within the context of the general theory.

Our strategy of proof may be described, in general terms, as follows: Sabbah has already proven a compatibility between V-filtrations and higher direct images. The V-filtration uses in an intrinsic way the structure of module over the ring \mathcal{R} obtained by using the Rees construction of the standard filtration of \mathcal{D}_X .

The required strictness property is reflected in the Decomposition Theorem of Sabbah and Mochizuki [17, 10].

In order to pass between the \mathscr{R} -module picture and the parabolic Higgs picture, we need to understand the V-filtration and nearby cycles functors in a relative way over the λ -line $\mathscr{A} := \mathbb{A}^1$.

Let us discuss this first on Y. The \mathscr{R} -module $\widetilde{\mathscr{F}}^i$, defined as the direct image of the relative de Rham complex for \mathscr{R} -modules, has a V-filtration along the divisor Q. On the base curve we could restrict to an open disk so we may think of Q as consisting of a single point, in particular we use only a single real weight a. We have the filtration by subsheaves

$$V_{a-1}\widetilde{\mathscr{F}}^i\subset\widetilde{\mathscr{F}}^i.$$

These are $V_0\mathscr{R}$ -modules, where the latter ring is the one associated to the sheaf of rings of differential operators that is Koszul dual to the logarithmic cotangent complex $\Omega_Y^{\bullet}(\log Q)$.

In the range a < 1 these sheaves fit together to form a structure of parabolic bundle. That is to say they are locally free $\mathcal{O}_{Y \times \mathscr{A}}$ -modules organized in a filtration satisfying the axioms of a parabolic structure. One should extend by periodicity to the case $a \geq 1$.

The restriction of this parabolic structure to any λ is the parabolic bundle associated to the harmonic bundle associated to the middle higher direct image. This comes from Saito-Mochizuki-Sabbah's theory. The reader may refer to [17, Chapter 5] for a review of these relationships.

In order to prove our main theorem, we would like to show that the natural isomorphism of bundles on Y-Q extends to give an isomorphism

$$V_{a-1}\widetilde{\mathscr{F}}^i|_{Y\times\{0\}}\cong F_a^i.$$

The strategy to show this is to use the compatibility of V-filtrations with higher direct images: we can define a V-filtration on \mathscr{E} over $\mathscr{X} = X \times \mathscr{A}$, and Sabbah shows [17, Theorem 3.1.8] following Saito [15, Proposition 3.3.17] that the higher direct image of the de Rham complex made from a level of this V-filtration, is the corresponding level of the V-filtration on the higher direct image.

A first question for obtaining information on the Dolbeault fiber $\lambda=0$ is that the axiomatic characterization of the V-filtration uses the full \mathscr{R} -module structure over the λ -line \mathscr{A} , so it is not a priori well-defined just in terms of Dolbeault data. This aspect was pointed out to us rather early on by Sabbah and Mochizuki. However, it turns out that Saito and Mochizuki have proven useful expressions for the V-filtration viewed as being generated by certain subsheaves closely related to the parabolic structure. This is stated in Proposition 4.4. Furthermore, we furnish our own proof of how to get from the weaker to the stronger version of the generation statement, in Corollary 6.4 and Lemma 6.7. These expressions are well-defined in the restriction to $\lambda=0$.

There are now two difficulties. Near a horizontal divisor (but that is automatically away from the singularities of f), we would like to further reduce from the de Rham complex for an $\mathscr{R}_{\mathscr{X}}$ -module which is not $\mathcal{O}_{\mathscr{X}}$ -coherent, to an L^2 de Rham complex. For this, we can just quote the relevant statement in Sabbah, using the weight filtration given by Corollary 3.2.

The main difficulty is the following: at a singular point of f, corresponding to the intersection of two vertical divisor components, the map f is not smooth so Saito and Sabbah do not directly define the V-filtration along f = 0 (here we are assuming that Y is a disk and the singular point is at the origin y = 0).

Rather, in order to define the V-filtration for the function f, they tell us to take the graph embedding and use the V-filtration for the new coordinate function. We therefore have to prove a statement about compatibility between the complex obtained from the graph construction, and the L^2 parabolic Dolbeault complex. It is this compatibility statement that will be our main result.

Here is an outline of the paper.

Section 2 is preparatory, and includes a review of some of the relevant issues from non-abelian Hodge theory. We start in subsection 2.1 by describing our basic geometric situation. In subsection 2.2 we review the non-abelian Hodge correspondence in both the compact and non-compact cases. We also explain the Nilpotence Hypothesis (2.2) that will be assumed throughout. There and in the next subsection we give the main definitions: harmonic bundle (wild, tame, tame with trivial filtrations), λ -connection. A central theme of this work will be the comparison of harmonic bundles with \mathcal{R} -modules. In subsection 2.4 we define the relevant \mathcal{R} -modules, especially the minimal extension.

In section 3 we state and discuss the main theorem. We start in section 3.1 by constructing the monodromy weight filtration $W(\operatorname{Gr}_{k,b}(\underline{E})) := W(N_{k,b})$ associated to a parabolic Higgs bundle (\underline{E}, φ) . In section 3.2, these weight filtrations are used to construct the L^2 parabolic Dolbeault complex, in equation (2). This allows us, in section 3.3, to state the Main Theorem 3.6.

In section 4, as a step towards the proof of the Main Theorem 3.6, we formulate its analogue, Theorem 4.1, on the level of \mathscr{R} -modules and their V-filtrations. This result, in the \mathscr{R} -module context, is already known, by Sabbah's work. So this leads us to the comparison problem of relating these two points of view, in order to deduce our main theorem from Theorem 4.1. In a little more detail: we start in subsection 4.1 with a brief review of the Kashiwara-Malgrange V-filtration of a \mathcal{D} -module with respect to a map or a subscheme. While the smooth case is straightforward, the singular case requires factoring through a graph construction, as we recall in section 4.1. In section 4.2 we define the de Rham complex on \mathscr{X} in terms of this graph factorization, take its higher direct image to $\mathscr{Y} = Y \times \mathscr{A}$, and then state and prove Theorem 4.1 about compatibility with the V-filtration. Finally, in subsection 4.4, we approach the comparison problem by constructing a map u(a) (cf. Lemma 4.5) from the parabolic de Rham complex to the new de Rham complex. The proof of the Main Theorem 3.6 is then reduced, in Proposition 4.8, to Theorem 4.6 which asserts that the comparison maps u(a), $u_0(a)$ are indeed quasi-isomorphisms. This is proved over the next two sections.

In section 5, after some preliminaries and reductions, the desired quasi-isomorphism is established over smooth points, thus focusing our attention, in Theorem 5.9, on the double points. Some further reductions lead us to consider, in Proposition 5.13, a graded version $gr_a(u_0)$ described in equation 13.

The needed local calculations are then completed in section 6. These include a Tensor Product Formula and explicit calculations with Koszul complexes.

In the final section we consider several improvements and extensions. The first two subsections provide some details on the proofs of parts 3 and 5 of Theorem 3.6. We start by looking at the absolute Dolbeault complex on X and use it to define the Gauss-Manin Higgs field on the higher direct images. Next we look at the analytical aspects of the direct image harmonic bundle. This includes a sketch of our original strategy for proving the theorem, involving the study of the family of L^2 cohomology spaces. The remaining sections consider extensions of our basic geometric setup to allow maps between various higher dimensional spaces.

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2 Preliminaries

This section is preparatory, and includes a review of some of the relevant issues from non-abelian Hodge theory. We start by describing our geometric situation, and then give the main definitions: harmonic bundle (wild, tame, tame with trivial filtrations), λ -connection, minimal $\mathcal{R}_{\mathcal{X}}$ -module.

2.1 The underlying geometry

Suppose we are given a smooth projective surface X with a map $f: X \to Y$ to a smooth projective curve. Suppose we are given a (reduced) simple normal crossings divisor $D \subset X$. Suppose given a reduced divisor $Q \subset Y$ so Q is just a finite collection of points q_i . We assume that D decomposes as

$$D = D_V + D_H$$

into two simple normal crossings divisors meeting transversally, called the *vertical* and *horizontal* divisors respectively. We assume that

$$D_V = f^{-1}(Q)$$

as a divisor (that is to say, the fibers of f over the points q_k are reduced with simple normal crossings), and that D_H is etale over Y (so it is a disjoint union of smooth components, not intersecting each other but they can intersect D_V).

When we would like to consider an irreducible component of a vertical divisor we write $D_{v(i)}$, and when we would like to consider an irreducible component of a horizontal divisor we write $D_{h(j)}$. We use D_k to refer to any one of these divisors. Each D_k is smooth and irreducible:

$$D_V = \sum_{i=1}^{n_v} D_{v(i)},$$

$$D_H = \sum_{j=1}^{n_h} D_{h(j)},$$

so that

$$D = \sum_{i} D_{v(i)} + \sum_{j} D_{h(j)} = \sum_{k} D_{k}.$$

We assume that f is smooth away from D_V , so the fibers $f^{-1}(q_l)$ are the only singular fibers. Each one of these fibers will, in general, be composed of several irreducible components $D_{v(i)}$. However, since we will rather quickly reduce to the consideration of a neighborhood of only one q_i , we don't need to introduce additional notation for that.

It follows from our etaleness assumption that D_H is entirely contained in the smooth locus of f.

We next consider the " λ -line"

$$\mathscr{A} := \mathbb{A}^1$$

with coordinate denoted λ . This is the base of the Deligne moduli space of λ -connections which is the first coordinate chart of the twistor \mathbb{P}^1 .

Denote by script letters the spaces

$$\mathscr{X} := X \times \mathscr{A}, \quad \mathscr{Y} := Y \times \mathscr{A},$$

with their divisors denoted

$$\mathscr{D} = \mathscr{D}_V + \mathscr{D}_H = D_V \times \mathscr{A} + D_H \times \mathscr{A}$$

and $\mathcal{Q} = Q \times \mathcal{A}$. The corresponding map is denoted

$$f_{\mathscr{X}}:\mathscr{X}\to\mathscr{Y}$$

or just by f if there is no confusion.

Similarly bundles over $\mathscr X$ or $\mathscr Y$ will be denoted by script letters when possible. Their fibers over $\lambda=0$ will be denoted by regular letters and the fiber over an arbitrary λ will be denoted functionally. Thus if $\mathscr E$ is a bundle on $\mathscr X$ we obtain bundles $E(\lambda)$ on $X(\lambda)=X$, and E:=E(0).

We have the sheaf of rings $\mathscr{R}_{\mathscr{X}}$ over \mathscr{X} defined by doing the Rees construction to the filtration of \mathcal{D}_X by order of differential operators (see Subsection 2.4 below). Denote its restriction to fibers by $R_X(\lambda)$, so we have

$$R_X := R_X(0) = \operatorname{Sym}^*(T_X)$$

whereas for any $\lambda \neq 0$,

$$R_X(\lambda) \cong \mathcal{D}_X$$
.

2.2 Harmonic bundles

In this subsection we review the non-abelian Hodge correspondence in both the compact and non-compact cases. We also explain the Nilpotence Hypothesis (2.2) on the residues of the Higgs field that will be assumed throughout.

Recall that a harmonic bundle over X-D consists of the data $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', h)$ where \mathcal{L} is a \mathcal{C}^{∞} bundle over X-D, with a hermitian metric h, having operators

$$\mathbf{D}' = \partial + \overline{\varphi}, \ \mathbf{D}'' = \overline{\partial} + \varphi: \ \mathcal{L} \to \mathcal{A}^1(\mathcal{L})$$

such that ∂ and φ are of type (1,0) and $\overline{\partial}, \overline{\varphi}$ are of type (0,1). Put $\mathbf{D} := \mathbf{D}' + \mathbf{D}''$. These are subject to the following conditions:

- 1. $\partial + \overline{\partial}$ is an h-unitary connection;
- 2. $\varphi + \overline{\varphi}$ is h-self-adjoint;
- 3. $(\mathbf{D}'')^2 = 0$ so that $E^o = (\mathcal{L}, \overline{\partial})$ is a holomorphic bundle² and $\varphi : E^o \to E^o \otimes \Omega^1_{X-D}$ is a holomorphic Higgs field; and
- 4. $\mathbf{D}^2 = 0$, so that $L := \mathcal{L}^{\mathbf{D}}$ is a local system (i.e. a locally constant sheaf of finite-dimensional \mathbb{C} -vector spaces).

In the compact case with empty divisor (D = 0), recall that the local system L is semisimple; the Higgs bundle (E^o, φ) is polystable with vanishing rational Chern classes; and for either one of these kinds of objects there exists a unique harmonic bundle associated to it. The metric is unique up to rescaling on each direct factor. This sets up the nonabelian Hodge correspondence between Higgs bundles and local systems, pioneered by Hitchin [7].

In the noncompact case, that is to say in the presence of a nontrivial divisor at infinity D that we are assuming has normal crossings, more needs to be said about the asymptotics of the harmonic bundle near the divisor. For dimension 1 that was the subject of [18]. In the higher dimensional case this discussion was initially extended over the smooth points of D by Biquard [2]. The existence of a pluriharmonic metric on a local system with monodromy eigenvalues of norm 1, was done by Jost and Zuo [8]. The full correspondence in general was done by Takuro Mochizuki in a series of works including [10, 12], and was later extended to cover even wild ramifications that we don't consider here.

This is the starting point for our investigation of higher direct images. We shall make some simplifying assumptions and now review the basic outlines. Consider a smooth point $p \in D_i$ of one of the divisor components, and let z_i be the coordinate function defining D_i near p. Let $\{r(t)\}_{t\in(0,1)}$ be a ray going towards p, with $|z_i(r(t))| = t$. If $\{u(t) \in \mathcal{L}_{r(t)}\}$ is a flat section of the local system L over the ray, we can look at the growth rate of $||u(t)||_{h(r(t))}$ with respect to the harmonic metric. We say that u has polynomial growth (respectively, sub-polynomial growth) along the ray, if for some (respectively all) b > 0 we have:

$$||u(t)||_{h(r(t))} \le Ct^{-b}.$$

The harmonic bundle is said to be *tame* if all its flat sections have polynomial growth along rays. It is said to be *tame with trivial filtrations* if flat sections have sub-polynomial growth along rays.

²In order to avoid confusion we sometimes use a superscript ()^o to denote objects such as E^o on the complement X - D.

Tameness is equivalent to the condition that the eigen-one-forms of the Higgs field φ are multivalued sections of the logarithmic cotangent bundle. Triviality of the filtrations on the local system is then equivalent to the condition that the limiting values of the eigen-one-forms (these will be the residues of the logarithmic Higgs field) are purely imaginary.

Suppose $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', \varphi)$ is a tame harmonic bundle. Using the order of growth defines a collection of filtrations on the restrictions of the local system L to punctured neighborhoods of each of the divisor components. Let $j: X - D \hookrightarrow X$. If $\eta = \{\eta_i\}$ is a parabolic level, that is to say $\eta_i \in \mathbb{R}$ for each i corresponding to a divisor component D_i , then L_{η} is the subsheaf of $j_*(L)$ consisting of sections that have growth rate $\leq Ct^{-\eta_i - \epsilon}$ for any $\epsilon > 0$, along rays going towards smooth points of D_i . The collection $\underline{L} = \{L_{\eta}\}$ is the filtered local system associated to $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', \varphi)$. The condition of trivial filtrations is equivalent to $L_0 = j_*(L)$ and $L_{\eta} = j_!(L)$ for $\eta_i < 0$.

Similarly, the Higgs bundle (E^o, φ) extends to a parabolic sheaf (\underline{E}, φ) defined as follows. For a parabolic level $\beta = \{\beta_i\}$ with $\beta_i \in \mathbb{R}$, let $E_\beta \subset j_*(E^o)$ be the subsheaf of holomorphic sections that locally near smooth points of D_i have growth rate $\leq C|z_i|^{-\beta_i-\epsilon}$ for any $\epsilon > 0$.

Our convention is that the parabolic levels (and weights) are indexed by $\beta = (\dots, \beta_k, \dots)$ with one $\beta_k \in \mathbb{R}$ for each divisor component D_k . The parabolic filtrations are increasing: $E_{\alpha} \subseteq E_{\beta}$ if $\alpha_i \leq \beta_i$, and if we let δ^k denote the index with all values 0 except for 1 at position k, then

$$E_{\beta+\delta^k}=E_{\beta}(\mathscr{D}_k).$$

In our notation for the parabolic bundle, we don't underline when there is a subscript: a notation of the form E_{β} with a subscript denotes one of the bundles in the structure of the parabolic bundle.

The following collects some of Mochizuki's main results:

Theorem 2.1 (Non-compact non-abelian Hodge correspondence). If $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', \varphi)$ is a tame harmonic bundle, then the filtered local system is locally abelian: it is locally an extension of standard rank 1 filtered local systems. The parabolic sheaf \underline{E} is a locally abelian parabolic bundle, and the Higgs field φ is logarithmic in the sense that for any parabolic weight β we have

$$\varphi: E_{\beta} \to E_{\beta} \otimes \Omega^1_X(\log D).$$

The filtered local system, and the parabolic logarithmic Higgs bundle, are both polystable objects with vanishing Chern classes. Furthermore, any polystable filtered local system or parabolic logarithmic Higgs bundle with vanishing Chern classes comes from an essentially

unique harmonic bundle (the metric is unique up to scaling on each irreducible direct factor). This sets up one to one correspondences between the three kinds of objects.

Proof. See [12].
$$\Box$$

We recall here the correspondence between residual data for filtered local systems and parabolic logarithmic Higgs bundles. This was given in the table of [18, p. 720] for the case of curves, and it works the same way at smooth points of D in higher dimensions. Notice however that we are using the convention that filtrations are increasing, so there is a sign change: our η_i and β_i have opposite signs with respect to [18].

Denote by $m_i \in \mathbb{C}^*$ an eigenvalue of the monodromy transformation of L around D_i , in the η_i -graded piece for the filtration. This generalized eigenspace corresponds to a generalized eigenspace of the same dimension, with eigenvalue $r_i \in \mathbb{C}$, for the residue of the Higgs field φ acting on the β_i -graded piece of the parabolic structure along D_i . The correspondence is:

	Local system $(\lambda = 1)$	Higgs $(\lambda = 0)$
Grade	$\eta_i = 2 \mathrm{Re}(r_i)$	$\beta_i = \operatorname{Arg}(m_i)/2\pi$
Eigenvalue	$m_i = e^{4\pi \operatorname{Im}(r_i) + 2\pi \sqrt{-1}\beta_i}$	$r_i = \eta_i / 2 + \frac{\sqrt{-1}}{4\pi} \log m_i $

The following hypothesis, in effect throughout the paper, will considerably simplify many parts of the discussion. We note that local systems coming from geometry certainly satisfy this hypothesis, and it leaves a wide lattitude for the choice of local system.

Hypothesis 2.2 (Nilpotence). Throughout the present paper, we shall assume that the filtered local system has trivial filtrations ($\eta_i = 0$) and the eigenvalues of the monodromy are complex numbers of norm 1 ($m_i \in S^1 \subset \mathbb{C}^*$). This corresponds for the parabolic logarithmic Higgs bundle to the hypothesis that the residue of the Higgs field has trivial eigenvalues ($r_i = 0$). The correspondence becomes:

	Local system $(\lambda = 1)$	Higgs $(\lambda = 0)$
Grade	$\eta_i = 0$	$\beta_i = \operatorname{Arg}(m_i)/2\pi$
Eigenvalue	$m_i = e^{2\pi\sqrt{-1}\beta_i}$	$r_i = 0$

2.3 The λ -connections

One can extend the construction of Theorem 2.1 to obtain a twistor family of parabolic logarithmic λ -connections, interpolating between the connection and the Higgs field.

Consider a harmonic bundle $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', h)$.

Our Nilpotence Hypothesis 2.2 is in effect; this makes it so that the whole family of parabolic logarithmic λ -connections associated to the harmonic bundle, has a collection of residual data that varies in a nice way. Complications such as were illustrated in [17, Figure 3.A], leading for example to the need for choosing small open neighborhoods in the λ -line, are avoided.

For any $\lambda \in \mathcal{A}$, we have a vector bundle with λ -connection $(E^o(\lambda), \nabla(\lambda))$ on X - D where

$$E^{o}(\lambda) = (\mathcal{L}, \overline{\partial} + \lambda \overline{\varphi}), \ \nabla(\lambda) = \lambda \partial + \varphi.$$

At $\lambda = 0$ this specializes to the Higgs bundle $(E^o(0), \nabla(0)) = (E^o, \varphi)$. These bundles fit together to form a holomorphic vector bundle $\mathscr{E}^o(-)$ over $\mathscr{X} - \mathscr{D} = (X - D) \times \mathbb{A}^1$, with a relative λ -connection $\nabla(-)$. Denote again by $j : \mathscr{X} - \mathscr{D} \hookrightarrow \mathscr{X}$ the inclusion.

Theorem 2.3. Under the Nilpotence Hypothesis 2.2, if we set $\mathscr{E}_{\beta} \subset j_*(E^o(-))$ to be the subsheaf of sections that have order of growth near \mathscr{D}_i bounded by $C|z_i|^{-\beta_i-\epsilon}$ for any $\epsilon > 0$, then these fit together into a locally abelian parabolic bundle $\underline{\mathscr{E}} = \{\mathscr{E}_{\beta}\}$ and $\nabla(-)$ extends to a relative parabolic logarithmic λ -connection

$$\nabla : \mathscr{E}_{\beta} \to \mathscr{E}_{\beta} \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega^{1}_{\mathscr{X}/\mathscr{A}}(\log \mathscr{D}).$$

The fiber over $\lambda = 1$ corresponds to the local system L.

Proof. See [10] (Proposition 1.6 and many others) for the basic regularity properties of the filtered holomorphic bundles. See [11] and [12] for the Kobayashi-Hitchin correspondences between Higgs and flat bundles. \Box

The Nilpotence Hypothesis 2.2 tells us, in this context, that the eigenvalues of the residue of the connection $\nabla(\lambda)$ on the β_i -graded piece of the parabolic bundle are equal to $\lambda\beta_i$. This formula is one of the things that is much simplified by our Nilpotence Hypothesis.

(That could be seen by calculating directly for rank one systems, or one may refer to [17, Lemma 3.3.4(7)] as discussed in the remarks on page 17 below.)

As was the case for \underline{E} , the parabolic levels are indexed by $\beta = (\ldots, \beta_k, \ldots)$ with one level for each divisor component. The parabolic filtrations are increasing: $\mathscr{E}_{\alpha} \subseteq \mathscr{E}_{\beta}$ if $\alpha_i \leq \beta_i$, and

$$\mathscr{E}_{\beta+\delta^k} = \mathscr{E}_{\beta}(\mathscr{D}_k).$$

2.4 The \mathscr{R} -module \mathscr{E}

A central theme of this work will be the comparison of $\mathscr{R}_{\mathscr{X}}$ -modules and harmonic bundles. In this subsection we define the relevant $\mathscr{R}_{\mathscr{X}}$ -modules, especially the minimal extension.

Mochizuki shows in [10] that, starting from a harmonic bundle, we obtain a pure twistor \mathcal{D} -module in the sense of Sabbah [17]. This generalizes Saito's construction of a pure Hodge module out of a variation of Hodge structure [15, Théorème 5.4.3].

Recalling various parts of Sabbah's theory will take up much of our exposition, and the reader will have to refer to [17] for many details. Here is the starting point.

The sheaf of rings \mathcal{D}_X of differential operators on X has a filtration F^{\bullet} by order of the operator: $F^k\mathcal{D}_X$ consists of differential operators of order $\leq k$. Thus, $F^0\mathcal{D}_X = \mathcal{O}_X$. The Rees construction for the filtration F^{\bullet} produces the sheaf of rings $\mathscr{R}_{\mathscr{X}}$ over $\mathscr{X} = X \times \mathscr{A}$. Concretely, consider the sheaf of rings

$$\operatorname{pr}_X^*(\mathcal{D}_X) = \mathcal{D}_{\mathscr{X}/\mathscr{A}}$$

on ${\mathscr X}$ and let

$$\mathscr{R}_{\mathscr{X}} := \sum_{i=0}^{\infty} \lambda^{i} \operatorname{pr}_{X}^{*} F^{i} \mathcal{D}_{X} \subset \operatorname{pr}_{X}^{*} (\mathcal{D}_{X})$$

be the subsheaf of $\mathcal{O}_{\mathscr{X}}$ -algebras which is generated in local coordinates by

$$\partial_i := \lambda \frac{\partial}{\partial x_i}.$$

Notice that $F^1\mathscr{R}_{\mathscr{X}}$ is still a direct sum $\mathcal{O}_{\mathscr{X}} \oplus T_{\mathscr{X}/\mathscr{A}}$, with the operators ∂_i being a basis for the relative tangent bundle $T_{\mathscr{X}/\mathscr{A}}$. However, this copy of the tangent bundle acts through derivations on $\mathcal{O}_{\mathscr{X}}$ only after multiplying by λ .

According to our convention, we denote the restriction to fibers of this sheaf of rings by $R_X(\lambda)$, so we have

$$R_X := R_X(0) = \operatorname{Sym}^*(T_X)$$

whereas for any $\lambda \neq 0$,

$$R_X(\lambda) \cong \mathcal{D}_X$$
.

A λ -connection on a quasicoherent sheaf over \mathscr{X} is the same as a structure of $\mathscr{R}_{\mathscr{X}}$ -module. Our parabolic collection of sheaves with logarithmic λ -connections yields a structure of $\mathscr{R}_{\mathscr{X}}$ -module on the union

$$\widetilde{\mathscr{E}}:=\bigcup_{\beta}\mathscr{E}_{\beta}.$$

This is almost, but not quite, the $\mathscr{R}_{\mathscr{X}}$ -module associated to our local system in the theory of Saito-Sabbah-Mochizuki.

It is too big, not being finitely generated over \mathcal{R} because there is no derivation at $\lambda = 0$ that could increase the pole order. For this reason, Saito and Sabbah introduce the following definition, see [17, Definition 3.4.7].

Definition 2.4. The minimal extension \mathscr{E} is the smallest $\mathscr{R}_{\mathscr{X}}$ -submodule $\mathscr{E} \subset \widetilde{\mathscr{E}}$ containing the \mathscr{E}_{β} for $\beta_i < 1$.

We recall that a notation of the form \mathscr{E}_{β} with a subscript denotes one of the bundles in the structure of the parabolic bundle, whereas \mathscr{E} without subscript is the $\mathscr{R}_{\mathscr{X}}$ -module.

The union of parabolic components $\widetilde{\mathscr{E}}$ that we defined first, is the localization of \mathscr{E} along \mathscr{D} , obtained by tensoring with functions with poles along \mathscr{D} .

Caution: If $\beta_i < 1$ then $\mathscr{E}_{\beta} \subset \mathscr{E}$. But if some $\beta_i \geq 1$ then \mathscr{E}_{β} is not necessarily contained in \mathscr{E} but only in $\widetilde{\mathscr{E}}$. The $\mathscr{E}_{\beta} \cap \mathscr{E}$ are characterized as the submodules obtained by applying the appropriate number of derivatives to $\mathscr{E}_{\beta'}$ with $\beta'_i < 1$.

Note that for a given $\widetilde{\mathscr{E}}$ there can be several possible holonomic sub- $\mathscr{R}_{\mathscr{X}}$ -modules $\mathscr{E} \subset \widetilde{\mathscr{E}}$ giving the same localization. One first restricts to submodules that are *strictly specializable* [17, Definitions 3.3.8, Proposition 3.3.11]. However, these can have subobjects or quotients supported on the divisor.

Remark 2.5. Recall that Sabbah defines the notion of strictly S-decomposable \mathcal{R} -module [17, Definition 3.5.1]. This is a strictly specializable one satisfying the conditions of [17, 3.11(e)] (that we don't recopy here) about the morphisms can and var. Then \mathcal{E} has strict supports if it is strictly S-decomposable and has no submodules supported in smaller dimension [17, Definition 3.5.3]. Now, the minimal extension may be characterized as the unique extension with strict supports.

In the other direction, the characterization of [17, Corollary 3.5.5] says that an \mathcal{R} -module is strictly S-decomposable if and only if it is a direct sum of minimal extensions of smooth ones on locally closed subvarieties.

We will not need to calculate with the formulas for minimal extension; these would come into the discussion of our quasiisomorphism along horizontal divisors, but Sabbah has already done what we need and we will just be quoting Sabbah's calculation in Proposition 5.8 below. Along the vertical divisors the localized module $\widetilde{\mathscr{E}}$ is just as well adapted (if not sometimes better) for consideration of the V-filtrations.

Recall that the fibers over $\lambda = 0$ are to be denoted by regular letters: thus we get an R_X -module E and a parabolic Higgs bundle E on X.

One of our main concerns will be the relationship between the parabolic Higgs bundle (resp. parabolic bundle with λ -connection) and the R_X -module (resp. $\mathscr{R}_{\mathscr{X}}$ -module):

	$\lambda = 0$	all λ
Parabolic bundle (section 2.2)	$\underline{E} = \{E_{\beta}\}$	$\underline{\mathscr{E}} = \{\mathscr{E}_{\beta}\}$
$R_X/\mathscr{R}_{\mathscr{X}}$ -module (section 2.4)	E	E

Remarks: With the Nilpotence Hypothesis 2.2 in effect, the residual data vary in a nice way in the family of λ -connections associated to our $\mathcal{R}_{\mathscr{X}}$ -module \mathscr{E} . In the fiber over an arbitrary λ , the bundle $\mathscr{E}_{\beta}(\lambda)$ has a parabolic λ -connection. Along a smooth point of some divisor D_k , we obtain the associated-graded $Gr_{k,\beta_k}(\mathscr{E}_{\beta}(\lambda))$ with the residue endomorphism of the λ -connection. Hypothesis 2.2 implies that the eigenvalues of the residue endomorphism on the β_k -graded piece are of the form $\lambda\beta_k$.

To put this fact into perspective, the reader may want to look at the notations in [17, p. 17]. Here is a short guide to the correspondence of notations: what we are denoting λ is denoted there by z. The complex numbers α indexing the pieces of the nearby cycles functors are real in our case, indeed $\alpha = \beta_k$ for the β_k -th graded piece we are looking at. If α is real then $\alpha \star z = z\alpha$ in the notation of [17, p. 17], corresponding to $\lambda \beta_k$ for us. Similarly, $\ell_{z_0}(\alpha) = \alpha$.

In [17, Lemma 3.3.4(6)] with our hypothesis that α is real, $\ell_{z_0}(\alpha) = a$ is equivalent to $\alpha = a$ and the gr_a^V is the same as just $\psi_{t,\alpha}$. This gives a simple identification between the nearby cycles functor and the associated-graded of the V-filtration that we'll consider below.

In view of the simplification $\alpha \star z = \lambda \beta_k$, the formula of [17, Lemma 3.3.4(7)] says that the eigenvalue of the residue of our λ -connection on the β_k graded piece of the parabolic filtration, is $\lambda \beta_k$.

3 Main setup and results

In this section we state the main theorem and make some preliminary constructions and reductions. We start in section 3.1 by constructing the monodromy weight filtration $W(Gr_{k,b}(\underline{E})) := W(N_{k,b})$ associated to a parabolic Higgs bundle (\underline{E}, φ) . In section 3.2, these weight filtrations are used to construct the L^2 parabolic Dolbeault complex, in equation (2). This allows us, in section 3.3, to state the Main Theorem 3.6.

3.1 Local study of the parabolic Higgs bundle

Let (\underline{E}, φ) be a parabolic Higgs bundle satisfying the Nilpotence Hypothesis 2.2. Along a component D_k of the parabolic divisor D, our \underline{E} determines a family $\operatorname{Gr}_{k,b}(\underline{E})$, indexed by real b, of parabolic vector bundles on D_k . Their parabolic structure is along the divisor $D_{\cap k} := \overline{(D-D_k)} \cap D_k$ where D_k meets the other components. These come with endomorphisms $N_{k,b}$ induced by the residues of the Higgs field φ . In this subsection we discuss the monodromy weight filtration $W(\operatorname{Gr}_{k,b}(\underline{E})) := W(N_{k,b})$. Subsequently, these will be used to construct the L^2 parabolic Dolbeault complex.

Consider one of the components D_k of the divisor. Let $D_{\cap k} := \overline{(D - D_k)} \cap D_k$ be the divisor on D_k induced by the other components of D, and put $D_k^* := D_k - D_{\cap k}$.

For a real number b, let $\operatorname{Gr}_{k,b}(\underline{E})$ be the parabolic bundle on D_k , with respect to the divisor $D_{\cap k}$, defined as follows. The indexing of its parabolic structure is by the irreducible components, which in our case of 1-dimensional D_k are just points in the zero-dimensional divisor $p \in D_{\cap k}$. Note that for each such point there is an index $j \neq k$ such that $p \in D_{jk} = D_j \cap D_k$. Given a parabolic weight vector α for these indices, then we obtain a bundle $E_{\alpha,b}^{(D_k)}$ in a neighborhood of D_k (say, a tubular neighborhood in the usual topology). Near a point $p \in D_{jk}$, the divisor D_j contains a piece³ transverse to D_k at p and we can use the parabolic structure of \underline{E} with weight α_p for such a piece of D_j , and weight b along D_k .

Define

$$\operatorname{Gr}_{k,b}(\underline{E})_{\alpha} := E_{\alpha,b}^{(D_k)} / E_{\alpha,b-\epsilon}^{(D_k)}.$$

Assuming that the original parabolic bundle was locally abelian (i.e. locally a direct sum of parabolic line bundles), then $Gr_{k,b}(\underline{E})$ will be a locally abelian parabolic bundle on D_k with

³Notice here that there could in principle be several different points $p \in D_{jk}$; but the local pieces of D_j intersected with the tubular neighborhood are disjoint so they can be assigned different weights. Alternately one could assume by further blow-up that two divisor components intersect in at most a single point.

respect to the divisor $D_{\cap k}$.

The Higgs field

$$\varphi: E_{\beta} \to E_{\beta} \otimes \Omega^1_X(\log D)$$

induces a map

$$N_{k,b} := \operatorname{res}_b(\varphi) : \operatorname{Gr}_{k,b}(\underline{E}) \to \operatorname{Gr}_{k,b}(\underline{E}).$$

It is a map of parabolic bundles on D_k since φ respects the parabolic structure of \underline{E} . Our assumption 2.2 is that this residue is nilpotent.

Along smooth points of D_k (which we assume irreducible) we have the following basic fact, a special case of the more general result of Mochizuki that we'll refer to below, but in this case it goes back to Biquard [2].

Lemma 3.1. If $p, p' \in D_k^*$ then $N_{k,b}(p)$ and $N_{k,b}(p')$ are conjugate as nilpotent endomorphisms of a vector space.

Corollary 3.2. Over D_k^* for any real number b, there is a weight filtration $W(N_{k,b})$ of the vector bundle $Gr_{k,b}(\underline{E})$ with respect to the endomorphism $N_{k,b}$, a filtration by strict subbundles such that the nilpotent endomorphism gives isomorphisms of bundles in the usual way, in particular the restriction of this filtration to any point p is the weight filtration of $N_{k,b}(p)$.

Proof. We have a vector bundle $V := \operatorname{Gr}_{k,b}(\underline{E})$ over a variety D_k^* , and a nilpotent endomorphism $N \in End(V)$ whose values at all points of D_k are conjugate to each other by Lemma 3.1. The claim is that the vector spaces W_ℓ arising from the monodromy weight filtration of N form vector subbundles of V.

Without loss of generality, we may assume (by restricting to an open subset) that V is trivial, $V = V_0 \otimes \mathcal{O}_{D_k^*}$. Then N gives a map $X \to G := \operatorname{GL}(V)$. Our assumption is that the image of this map is contained in a single (nilpotent) orbit $O \cong G/S \subset G$, where S is the stabilizer. This means that we may as well replace: D_k^* by O, V by $V_0 \otimes \mathcal{O}_O$, and N by (the restriction from G to O of) the tautological endomorphism. The group G now acts transitively on O = G/S preserving V, N, so each of the W_ℓ is now a homogeneous vector bundle on O. The original W_ℓ on D_k^* are the pullbacks, so they still form vector bundles. \square

This property extends to the normal crossings points too. Suppose $p \in D_{\cap k}$ is one of the intersection points with some other D_j so $p \in D_{jk}$. Fix b, α and consider the vector bundle

$$V := \operatorname{Gr}_{k,b}(\underline{E})_{\alpha}$$

over D_k .

This bundle has a parabolic filtration at the point p, that is to say we have a filtration $0 = F_{-m} \subset \cdots \subset F_0 = V(p)$ of the fiber at p, and there are parabolic weights attached to the pieces. Call them $a_{-m} < \cdots < a_0$, and we attach the weight a_{-i} to the filtration element F_{-i}/F_{-i-1} . Notice that a_{-m} doesn't actually occur here but it satisfies $a_{-m} = a_0 - 1$. These weights come from the parabolic weights of \underline{E} along D_j . Precisely, the top weight is $a_0 = \alpha_j$, the weight attached to D_j in our global weight α . Then the other ones are some of the parabolic weights attached to D_j (they don't include ones for which the graded piece for $\beta_k = b$ along D_k vanishes).

The bundle V also has the nilpotent endomorphism $N:V\to V$, and N preserves the filtration, because of the condition that φ acts on the full parabolic bundle E.

Mochizuki shows the following version of Lemma 3.1 at the crossing points:

Proposition 3.3. Let $Gr^F(V(p)) := \bigoplus_{i=0}^{m-1} F_{-i}/F_{-i-1}$. Then the pair consisting of $Gr^F(V(p))$ together with the induced endomorphism $Gr^F(N(p))$ on this graded vector space, has the same isomorphism type as any (V(p'), N(p')) for $p' \in D_k^*$ a smooth point of the divisor D.

Proof. This is [10, Lemma 12.34]. The basic idea is that for $\lambda \neq 0$ the independence of the point is due to the fact that the module with connection splits into a direct sum of pieces according to the eigenvalues of the monodromy. Then, the main result [10, Lemma 12.33], due to strictness of the nilpotent map with respect to a limiting mixed twistor structure, says that the conjugacy classes are independent of λ . The independence of the conjugacy class as a function of p at $\lambda = 0$ then follows.

We need to strengthen this somewhat:

Lemma 3.4. In the above situation, (V(p), N(p)) is also isomorphic to (V(p'), N(p')).

Proof. Inclusion of orbit closures induces a partial ordering on the set of nilpotent conjugacy classes in a given finite dimensional vector space. On the one hand, (V(p), N(p)) is a limit of the family of (V(p'), N(p')) as $p' \to p$, and by Lemma 3.1, the conjugacy class of (V(p'), N(p')) is independent of p', so (V(p), N(p)) is in the orbit closure of (V(p'), N(p')) for a fixed p'. On the other hand, $(Gr^F(V(p)), Gr^F(N(p)))$ is in the orbit closure of (V(p), N(p)), as is seen by the Rees construction. But by Proposition 3.3, $(Gr^F(V(p)), Gr^F(N(p)))$ is isomorphic, in the category of vector spaces with nilpotent endomorphism, to (V(p'), N(p')) for any $p' \in D_k^*$. So (V(p'), N(p')), (V(p), N(p)) are in each other's orbit closure, and therefore they are conjugate.

Corollary 3.5. The weight filtration extends as a strict filtration on the bundle V over all of D_k . Furthermore, this weight filtration induces on $Gr^F(V(p))$ the weight filtration of $Gr^F(N(p))$.

Proof. By Lemma 3.1 away from the intersection points, and Lemma 3.4 at the intersection points, the conjugacy classes of N(p) acting on V(p) are the same for all points $p \in D_k$. The argument of Corollary 3.2 gives a weight filtration by strict vector subbundles of V, defined all along D_k . It is clearly the weight filtration of N(p') for smooth points $p' \in D_k^*$.

At an intersection point p, the filtration is also by construction the weight filtration of N(p). In order to show that it induces the weight filtration of $Gr^F(N(p))$, consider the Rees construction for the filtration F. This gives an analogous situation of a family indexed by a curve (the affine line), in which we know that the special and general points have the same isomorphism type. Applying Corollary 3.2 we get a canonical extension of the weight filtration along the Rees line. Since it is canonical it is \mathbb{G}_{m} -invariant so the filtration on the special fiber is the associated-graded for F of the filtration on the general fiber, that is to say the filtration on the special fiber is the one induced on $Gr^F(V(p))$. We get that this induced filtration is the weight filtration of $Gr^F(N(p))$.

As a result, we obtain weight filtrations denoted $W(Gr_{k,b}(\underline{E})) := W(N_{k,b})$ of the parabolic vector bundles $Gr_{k,b}(\underline{E})$ over D_k , with parabolic structure along $D_{\cap k}$. These are filtrations by strict parabolic subbundles.

3.2 The L^2 parabolic Dolbeault complex

In this subsection we use our weight filtrations to define an L^2 parabolic Dolbeault complex (2) coming from our parabolic logarithmic Higgs bundle. It plays a central role in our main result, theorem 3.6. As we point out below, one can similarly define a de Rham complex for the parabolic logarithmic λ -connection.

The definition will be algebraic, involving the weight filtrations we constructed in subsection 3.1 along the horizontal divisor components. But let us start by considering the analytic motivation.

Consider an open fiber $X_y^o = (X - D)_y$ for $y \in Y - Q$. Give it a metric that is asymptotically the Poincaré metric near puncture points. We are interested in the L^2 cohomology of the harmonic bundle $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', h)$ restricted to this fiber, as shall be discussed in more detail in Subsection 7.2 later. Basically, this is the cohomology of the complex of forms

with coefficients in \mathcal{L} that are L^2 , and whose derivative is L^2 . There are both de Rham cohomologies with differential \mathbf{D} , and Dolbeault cohomologies with differential \mathbf{D}'' .

In the case of coefficients in a variation of Hodge structure, these cohomology spaces were considered by Zucker [23] who proved that they are finite-dimensional and that the de Rham and Dolbeault cohomologies are isomorphic, both being isomorphic to the space of harmonic forms (recall from the Kähler identities that $\Delta_{\mathbf{D}} = 2\Delta_{\mathbf{D}''}$ and this holds for harmonic bundles too). But in fact, Zucker's theory also applies to tame harmonic bundles satisfying the Nilpotence Hypothesis 2.2, indeed locally near a puncture point they are asymptotically the same as the standard local models for variations of Hodge structure [18], so the estimates needed by Zucker hold. The theory is treated in detail by Sabbah [17, Section 6.2].

For our current purposes, we would like to have a calculation of the cohomology space in algebraic terms using the parabolic Higgs bundle. To get there, the main fact is that the L^2 Dolbeault cohomology is isomorphic to the hypercohomology of the complex on X_y consisting of holomorphic forms whose restriction to X_y^o is in L^2 and whose differential is in L^2 . So, we would like to write down the resulting complex in algebraic terms.

First a general estimate tells us that a holomorphic section of $E|_{X_y^o}$ (resp. $E|_{X_y^o} \otimes \Omega^1_{X_y^o}$) can be in L^2 only if it extends to a section of the 0-th component of the parabolic structure E_0 (resp. $E_0 \otimes \Omega^1_{X_y}(\log D_y)$). Furthermore, if it is in E_a (resp. $E_a \otimes \Omega^1_{X_y}(\log D_y)$) for a < 0 then it is automatically L^2 .

Suppose e is a section of $(E_0)_y$ near a point p on the horizontal divisor component $D_{k,y}$; it projects to $\operatorname{gr}_{k,0}(e) \in \operatorname{Gr}_{k,0}(\underline{E})_y$. Suppose this projection is in W_ℓ but not $W_{\ell-1}$. Then, denoting by z a coordinate on X_y vanishing at p, the norm of e is asymptotically

$$|e| \sim |\log |z||^{\ell/2}$$
.

Calculations with the Poincaré metric for the norms of sections or holomorphic 1-forms, done in [23, Proposition 4.4], tell us the following. First, e is in L^2 if and only if $m \leq 0$. Then similarly, a section $e^{\frac{dz}{z}}$ of $E_0 \otimes \Omega^1_{X_y}(\log D_y)$ is in L^2 if and only if

$$\operatorname{gr}_{k,0}(e) \in W_{-2}\operatorname{Gr}_{k,0}(\underline{E}).$$

These observations give us an algebraic description of the complex of holomorphic L^2 forms on the fibers X_y .

Proceed to define a complex on X/Y as follows. For any divisor component D_k , and any

multiindex β , put

$$\operatorname{Gr}_{k,\beta_k}(E_\beta) := \operatorname{Gr}_{k,\beta_k}(\underline{E})_{\beta(\cap k)}$$

where $\beta(\cap k)$ consists of the parabolic weights of β for the components of $D_{\cap\beta}$. We have explicitly

$$\operatorname{Gr}_{k,\beta_k}(E_\beta) = E_\beta / E_{\beta - \epsilon \delta_k}$$

where δ_k is the multiindex with 1 in position k and 0 elsewhere.

In the previous subsection we defined the weight filtration $W(\operatorname{Gr}_{k,\beta_k}(\underline{E}))$ of the parabolic bundle $\operatorname{Gr}_{k,\beta_k}(\underline{E})$ on D_k . By assigning parabolic weights $\beta(\cap k)$ on $D_{\cap k}$ this gives a weight filtration of the bundle $\operatorname{Gr}_{k,\beta_k}(E_\beta)$, and we call that $W(\operatorname{Gr}_{k,\beta_k}(E_\beta))$.

Let us now denote by

$$W(k, E_{\beta}) \subset E_{\beta}$$

the pullback of the weight filtration $W(Gr_{k,\beta_k}(E_\beta))$ over D_k , to a filtration of E_β by locally free subsheaves, via the map

$$E_{\beta} \to \operatorname{Gr}_{k,\beta_k}(E_{\beta}).$$

Let $W(H, E_{\beta})$ denote the weight filtration obtained by using $W(h(j), E_{\beta})$ along each horizontal component $D_{h(j)}$. More precisely, we use the weight filtration as we have defined in the previous subsection on the parabolic bundle $\operatorname{Gr}_{k,\beta_k}(\underline{E})$, and take the resulting weight filtration on the piece $\operatorname{Gr}_{k,\beta_k}(E_{\beta}) = \operatorname{Gr}_{k,\beta_k}(\underline{E})_{\beta(\cap k)}$ of this parabolic bundle.

For any real number a, let $\alpha(a)$ denote the parabolic weight for the divisor D determined by using weight a along the vertical components and weight 0 along the horizontal components. We then obtain the levels of the horizontal weight filtrations

$$W_{\ell}(H, E_{\alpha(a)}) \subset E_{\alpha(a)}.$$

Notice that along the horizontal divisor components $D_{h(j)}$ we have $\alpha(a)_{h(j)} = 0$ so the horizontal weight filtrations come from filtrations on the parabolic weight zero graded pieces $Gr_{h(j),0}(E_{\alpha(a)})$.

We may now define our relative L^2 parabolic Dolbeault complex:

$$DOL_{L^{2}}^{par}(X/Y, E_{\alpha(a)}) := \left[W_{0}(H, E_{\alpha(a)}) \xrightarrow{\varphi} W_{-2}(H, E_{\alpha(a)}) \otimes_{\mathcal{O}_{X}} \Omega_{X/Y}^{1}(\log D)\right], \qquad (2)$$

where $\Omega^1_{X/Y}(\log D) = \Omega^1_X(\log D)/f^*\Omega^1_Y(\log Q)$ is the sheaf of relative logarithmic one forms along the fibers of f. To make the notation less cumbersome we still write φ for the projection of the Higgs field to the relative logarithmic forms $\Omega^1_{X/Y}(\log D)$.

3.3 The Main Theorem

We can now state the main theorem of this paper.

Theorem 3.6. Let

$$F_a^i := \mathbb{R}^i f_* \left(\mathrm{DOL}_{L^2}^{\mathrm{par}}(X/Y, E_{\alpha(a)}) \right).$$

- 1. The F_a^i are locally free, and fit together as a varies into a parabolic bundle \underline{F}^i .
- 2. Formation of the higher direct images is compatible with base-change, in other words $F_a^i(y)$ is the cohomology of the fiber over $y \in Y$.
- 3. The parabolic bundle \underline{F}^i has a Higgs field θ given by the usual Gauss-Manin construction (section 7.1), making it into a parabolic Higgs bundle.
- 4. This parabolic Higgs bundle on (Y,Q) is the one associated to the middle perversity higher direct image (of degree i=0,1,2) of the local system underlying our original harmonic bundle.
- 5. More specifically, over Y Q the bundle \underline{F}^i has a harmonic metric given by the L^2 metric on cohomology classes in the fibers, and the parabolic Higgs structure is the one associated to this harmonic metric.

Clearly we are interested mostly in the case i = 1. It is useful to include i = 0, 2 because these facilitate using an Euler characteristic argument to control the dimension jumps at i = 1.

We note here that one can also define a relative version over the λ -line, denoted as an L^2 de Rham complex:

$$DR_{L^{2}}^{\mathrm{par}}(\mathscr{X}/\mathscr{Y},\mathscr{E}_{\alpha(a)}) := \left[W_{0}(H,\mathscr{E}_{\alpha(a)}) \xrightarrow{\nabla} W_{-2}(H,\mathscr{E}_{\alpha(a)}) \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega^{1}_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}). \right]$$

The restriction to $\lambda = 0$ is well-behaved since the complex is flat over \mathscr{A} and it gives back the Dolbeault complex (2):

$$DR_{L^2}^{\text{par}}(\mathcal{X}/\mathcal{Y}, \mathcal{E}_{\alpha(a)})|_{X(0)} = \text{DOL}_{L^2}^{\text{par}}(X/Y, E_{\alpha(a)}).$$
(3)

See [17, Theorem 6.2.4]. Part (2) of that statement gives only a quasiisomorphism between the restriction of the L^2 de Rham complex to X(0), and the L^2 Dolbeault complex. The restriction to X(0) corresponds to the case $z_0 = 0$ in Sabbah's notation. In our situation, the restriction map between complexes is an isomorphism. Indeed, the additional terms that occur in the proof of [17, Theorem 6.2.4] concern the case of non-real values of the parameter α , whereas we are assuming that the eigenvalues of the monodromy transformations are roots of unity so that the parameters α that occur are only real. In the terms for α real (taking into account the shift of 1 from Sabbah's $\alpha = -1$ to our $\alpha = 0$), the description of the L^2 complex in terms of the weight filtration coincides with what we have said above, both before and after restricting to X(0) i.e. $z_0 = 0$.

The relative de Rham version of our main theorem is the following statement:

Theorem 3.7. Let

$$\mathscr{F}_a^i := \mathbb{R}^i f_* \left(\mathrm{DR}_{L^2}^{\mathrm{par}} (\mathscr{X}/\mathscr{Y}, \mathscr{E}_{\alpha(a)}) \right).$$

The \mathscr{F}_a^i are locally free and the higher direct images are compatible with base-change. These fit together into a parabolic bundle $\underline{\mathscr{F}}^i$ with logarithmic λ -connection on \mathscr{Y} relative to \mathscr{A} . It is the parabolic λ -connection associated to the harmonic bundle of Theorem 3.6.

4 The de Rham complex

As a step towards the proof of the Main Theorem 3.6, we formulate its analogue, Theorem 4.1, on the level of \mathcal{R} -modules and their V-filtrations. This result, in the $\mathcal{R}_{\mathcal{X}}$ -module context, is already known, by Sabbah's work. So this leads us to the comparison problem of relating these two points of view, in order to deduce our main theorem from Theorem 4.1.

In a little more detail: we start in subsection 4.1 with a brief review of the Kashiwara-Malgrange V-filtration of a \mathcal{D} -module with respect to a map or a subscheme. While the smooth case is straightforward, the singular case requires factoring through a graph construction, as we recall in section 4.1. In subsection 4.2 we define the de Rham complex on \mathscr{X} in terms of this graph factorization, take its higher direct image to \mathscr{Y} , and then state and prove Theorem 4.1 about compatibility with the V-filtration.

In subsection 4.3, we look briefly at the notion of multi-V-filtration with respect to several smooth divisors meeting transversally. This structure gives back our parabolic structures. It will be useful to have a formulation of the filtered objects involved in the parabolic structures, in the language of V-filtrations.

Finally, in subsection 4.4, we approach the comparison problem by constructing a map (cf. Lemma 4.5) from the parabolic de Rham complex to the new de Rham complex. The

proof of the Main Theorem 3.6 is then reduced, in Proposition 4.8, to Theorem 4.6 which asserts that the comparison map is indeed a quasi-isomorphism.

4.1 The V-filtration

Consider first a smooth map $p: P \to Y$, with y a local coordinate on Y vanishing at a point $q \in Y$, and x a coordinate along the fibers, so x, y are local coordinates on P. (Or $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_k)$ could be coordinate systems). In these local coordinates, the sheaf of rings \mathcal{D}_P of differential operators on P looks like

$$\mathcal{D}_P = \mathbb{C}[x, y, \partial_x, \partial_y]$$

The V-filtration on \mathcal{D}_P with respect to the smooth map p, or with respect to the smooth subvariety $Z := p^{-1}(q) = \{y = 0\}$) defined by the ideal I := (y), is given by:

$$V_j \mathcal{D}_P := \left\{ \xi \in \mathcal{D}_P \mid \xi I^i \subset I^{i-j} \text{ for all } i \in \mathbb{Z} \right\},$$

where by convention $I^k = \mathcal{O}_P$ for all $k \leq 0$. In local coordinates, $V_j \mathcal{D}_P$ is spanned over \mathcal{O}_P by the expressions $\partial_x^a y^b \partial_y^c$ with $c - b \leq j$. In particular, $V_0 \mathcal{D}_P$ is the sheaf of rings of differential operators Koszul dual to the logarithmic cotangent complex $\Omega_X^{\bullet}(\log Z)$, i.e. generated by the log tangent vectors $\partial_x, y \partial_y$.

The V-filtration (still with respect to the smooth map p or subvariety Z) on a left- \mathcal{D}_{P} module M is an increasing filtration of M by coherent $V_0\mathcal{D}_{P}$ -submodules V_a . This is subject
to several axioms [3, 4], the main one being that on

$$\operatorname{Gr}_a(M) := V_a(M)/V_{a-\epsilon}(M),$$

the first order operator $\sum_k \partial_{y_k} y_k$ acts with generalized eigenvalue -a, that is to say $\sum_k \partial_{y_k} y_k + a$ is nilpotent. Equivalently⁴, the vector field $\sum_k y_k \partial_{y_k}$ acts there with generalized eigenvalue -(a+1).

It is fairly straightforward to verify that the V-filtration is uniquely characterized by the axioms. Kashiwara and Malgrange show that if M is regular holonomic and quasi-unipotent, its V-filtration exists.

⁴This notation is shifted by 1 from what one finds in \mathcal{D} -module literature [3, 4], but is commonly used in [17] and other relevant references. The relationship with the parabolic structure is shifted as shall be discussed further in the next section. For these reasons we usually include the shift in the notation and consider V_{a-1} .

We can include λ -connections, i.e. replace \mathcal{D} -modules by \mathscr{R} -modules, by working instead with $p: \mathscr{P} \to \mathscr{Y}$. Only minor changes are needed, e.g. the generalized eigenvalue becomes $-\lambda a$ instead of -a.

The interesting case for the present paper is when X is a smooth complex variety but the map $f: X \to Y$ is arbitrary. Equivalently, the subscheme $Z \subset X$ defined by the ideal $I = (f_1, \ldots, f_k)$ is allowed to be arbitrarily singular, where the f_i are coordinates of the map f. In that case, let $P := X \times Y$ be the product and set $\mathscr{P} := \mathscr{X} \times_{\mathscr{A}} \mathscr{Y}$. We have the map $g: \mathscr{X} \to \mathscr{P}$ given by the graph of f, that is to say

$$g(x, \lambda) := ((x, \lambda), (f(x), \lambda)).$$

Let $p: \mathscr{P} \to \mathscr{Y}$ be the projection, and let $\mathscr{X}_q \cong \mathscr{X}$ denote the fiber over $q \in Q$ of the projection.

We may assume that there is only one singular point q, and indeed that Y is just an open disc. In coordinate notation we shall denote by t the coordinate on \mathscr{P} pulled back from the coordinate function of the disk by the projection p. We can use interchangeably the notation $\mathscr{X}_0 = \mathscr{X}_q$ for the fiber at t = 0.

From our $\mathscr{R}_{\mathscr{X}}$ -module \mathscr{E} we get an $\mathscr{R}_{\mathscr{P}}$ -module $g_{+}(\mathscr{E})$ supported along the image of g. We then get the V-filtration $V_{a-1}(g_{+}(\mathscr{E}))$.

4.2 The de Rham complex

We can form the de Rham complex of $g_+(\mathcal{E})$ on \mathscr{P} relative to \mathscr{Y} , in the world of \mathscr{R} -modules, and take the higher direct image to \mathscr{Y} . The relative de Rham complex is

$$DR(\mathscr{P}/\mathscr{Y}, g_{+}\mathscr{E}) := \begin{bmatrix} g_{+}(\mathscr{E}) & \longrightarrow g_{+}(\mathscr{E}) \otimes_{\mathcal{O}_{\mathscr{P}}} \Omega^{1}_{\mathscr{P}/\mathscr{Y}} & \longrightarrow g_{+}(\mathscr{E}) \otimes_{\mathcal{O}_{\mathscr{P}}} \Omega^{2}_{\mathscr{P}/\mathscr{Y}} \end{bmatrix}.$$

$$0 \qquad 1 \qquad 2$$

Note that this is a somewhat different kind of object than the L^2 parabolic Dolbeault or the L^2 parabolic de Rham complex considered above: it doesn't a priori bring the parabolic structure into play, rather relying on the $\mathscr{R}_{\mathscr{X}}$ -module structure of \mathscr{E} . Our comparison problem is to relate these two points of view, in order to deduce our main theorem from the following result already known by Sabbah's work in the $\mathscr{R}_{\mathscr{X}}$ -module context.

Let

$$\mathscr{F}^i := \mathbb{R}^i f_* \left(\mathrm{DR}(g_+ \mathscr{E}) \right).$$

This $\mathscr{R}_{\mathscr{Y}}$ -module was also denoted by $\mathcal{H}^i f_{\dagger} \mathscr{E}$ in [17, Remark 1.4.8].

Let $\overline{\mathscr{F}}^i$ denote the minimal extension to Y of $\mathscr{F}^i|_{\mathscr{Y}-\mathscr{Q}}$. The localization $\widetilde{\mathscr{F}}^i$ (inverting the equations of the points of Q) is the same as the localization of $\overline{\mathscr{F}}^i$.

On the other hand, consider the following local systems G^i on Y - Q: the fiber of G^i at a point $y \in Y - Q$ is the cohomology of the middle perversity extension $j_{X_y,*}(L|_{X_y-D_{H,y}})$ on the fiber X_y . Equivalently, it is the L^2 cohomology of the harmonic bundle restricted to $X_y - D_{H,y}$.

Theorem 4.1. With the above notations, and using the V-filtration:

1. $\overline{\mathscr{F}}^i$ is an $\mathscr{R}_{\mathscr{Y}}$ -module on \mathscr{Y} , with strict supports (Remark 2.5), in other words it is the minimal extension (Definition 2.4) of its restriction to the open subset. In particular, the V-filtrations at any point $q \in Q$ satisfy

$$V_{b-1}(\overline{\mathscr{F}}^i) = V_{b-1}(\mathscr{F}^i) = V_{b-1}(\widetilde{\mathscr{F}}^i)$$

for b < 1, and $V_{b-1}(\overline{\mathscr{F}}^i)$ are locally free.

- 2. $\overline{\mathscr{F}}^i$ is the $\mathscr{R}_{\mathscr{Y}}$ -module corresponding to the harmonic bundle associated to the local system G^i .
- 3. In particular, the parabolic Higgs bundle whose components near a point $q \in Q$ are:

$$F_b^i := V_{b-1}(\widetilde{\mathscr{F}}^i)(0),$$

is the parabolic Higgs bundle associated to G^i .

Proof. Sabbah proves the decomposition theorem for the higher direct images of a pure twistor \mathcal{D} -module. Mochizuki shows that our original harmonic bundle on X gave rise to a pure twistor \mathcal{D} -module, of which \mathscr{E} is the part over the chart \mathscr{A} for the twistor line. We obtain the statement that \mathscr{F}^i decomposes as a direct sum of $\mathscr{R}_{\mathscr{Y}}$ modules with strict support (2.5). Therefore, $\overline{\mathscr{F}}^i$ is a direct summand of \mathscr{F}^i . This gives the properties of the V-filtrations at points $q \in Q$. By the discussion of [17, Chapter 5], $\overline{\mathscr{F}}^i$ corresponds to a local system on Y - Q, and again applying this correspondence in the fibers, that local system is G^i . That correspondence gives the statement (3).

The V-filtration of the relative de Rham complex, with respect to the function

$$p: \mathscr{P} \to \mathscr{Y}$$

is given by:

$$DR(V_{a-1}(g_{+}\mathscr{E})) := \left[V_{a-1}(g_{+}\mathscr{E}) \to V_{a-1}(g_{+}\mathscr{E}) \otimes \Omega^{1}_{\mathscr{P}/\mathscr{Y}} \to V_{a-1}(g_{+}\mathscr{E}) \otimes \Omega^{2}_{\mathscr{P}/\mathscr{Y}} \right].$$

The compatibility between V-filtrations and higher direct images says:

Proposition 4.2. We have

$$V_{a-1}(\widetilde{\mathscr{F}}^i) = \mathbb{R}^i p_* DR(V_{a-1}(g_+\mathscr{E})).$$

Furthermore, these bundles are strict with respect to specialization on the λ -line so

$$V_{a-1}(\widetilde{\mathscr{F}}^i)|_{Y(0)} = \mathbb{R}^i p_* \left(DR(V_{a-1}(g_+\mathscr{E}))|_{P(0)} \right).$$

Proof. See Sabbah [17, Theorem 3.1.8] generalizing Saito [15, Proposition 3.3.17], but in turn this main property goes back to the original work of Kashiwara-Malgrange on the V-filtrations for \mathcal{D} -modules.

In view of this proposition, we would like to compare

$$DR(V_{a-1}(g_+\mathscr{E}))|_{P(0)}$$
 and $DOL_{L^2}^{par}(E_{\alpha(a)})$.

4.3 The multi-V-filtration and the parabolic structure

Let us consider, on the other hand, the multi-V-filtration obtained by combining the individual V-filtrations along the divisor components. This is easier because the divisor components are smooth so we don't need to use the graph embedding.

For each component D_i , we obtain a filtration

$$V_{a-1}^i(\mathscr{E})\subset \mathscr{E}.$$

It may be defined locally on open subsets where D_i is the zero-locus of a function. Now let us intersect them: if β is a parabolic weight vector then we can use each β_i along the component D_i . Denote by 1 the multi-index whose components are 1. Then we put

$$V_{\beta-1}^{\text{multi}}(\mathscr{E}) := \bigcap_{i} V_{\beta_{i}-1}^{i}(\mathscr{E}) \subset \mathscr{E}.$$

These submodules are closely related to the parabolic structure. However, that relationship only holds for negative values of the weight, because of the constraint imposed by the notion of "minimal extension". This restriction can be removed by passing to the localized sheaf $\widetilde{\mathscr{E}}$ of sections with poles along \mathscr{D} . Note that that sheaf is no longer of finite type over \mathscr{R} , however as Sabbah notes [17, Section 3.4], the V-filtration still works and we obtain as above

$$V_{\beta-1}^{\mathrm{multi}}(\widetilde{\mathscr{E}}) \subset \widetilde{\mathscr{E}}.$$

Proposition 4.3. Suppose $\beta_i < 1$, then there is a natural identification between the multi-V-filtration and the parabolic structure of \mathcal{E} :

$$V_{\beta-1}^{\mathrm{multi}}(\mathscr{E}) \cong \mathscr{E}_{\beta}.$$

Notice here that the shift by 1 is not present in the parabolic structure.

After localizing this extends to all values of β :

$$V_{\beta-1}^{\mathrm{multi}}(\widetilde{\mathscr{E}}) \cong \mathscr{E}_{\beta}.$$

In particular, for any β we have $\mathscr{E}_{\beta} \subset \widetilde{\mathscr{E}}$, and for $\beta_i < 1$ we have $\mathscr{E}_{\beta} \subset \mathscr{E}$.

Proof. See [17, Lemma 3.4.1]. Then for the identification with the parabolic filtration of the parabolic Higgs bundle, see [17, Corollary 5.3.1] in the one-dimensional case. The shift between the parabolic index and the index for the V-filtration was noted in [17, Section 5], passing through his introduction of an upper-indexed decreasing filtration that already incorporates the shift. There is a change of direction in the parabolic filtrations: Sabbah considers a decreasing parabolic filtration. This comparison has been extended to the higher dimensional case in Chapter 15 of [10] (cf 15.1.2 for the shift).

Unfortunately, the multi-V-filtration doesn't enter into the theory of Saito-Sabbah about compatibility with direct images. We need instead to look at the V-filtration with respect to the full vertical divisor $\mathcal{D}_V = f^{-1}(Q)$ defined above. Along smooth points of \mathcal{D}_V this is the same as the multi-V-filtration because there is only one index. The difficulty occurs at a normal crossing point, and will be the subject of our main work to follow.

4.4 Comparison of the two de Rham complexes

In order to compare the two complexes we would like to have a map between them. However, they are not complexes on the same space.

Consider \mathscr{E} as a quasicoherent sheaf of $\mathcal{O}_{\mathscr{X}}$ -modules on \mathscr{X} and as such take the direct image $g_*(\mathscr{E})$ to \mathscr{P} . There is an inclusion of sheaves of $\mathcal{O}_{\mathscr{P}}$ -modules

$$g_*(\mathscr{E}) \hookrightarrow g_+(\mathscr{E}).$$

Locally if t is a coordinate on a neighborhood in Y and we let ∂_t be the corresponding tangent vector field, then we can write

$$g_{+}(\mathscr{E}) = g_{*}(\mathscr{E})[\partial_{t}] \tag{4}$$

(see [17, Remark 1.4.7]). The above inclusion is just the term with the 0-th power of ∂_t .

If we let $\mathscr{E}_{\alpha(a)}$ denote the $\alpha(a)$ piece in the parabolic bundle with λ -connection \mathscr{E}_* then for a < 1 we have a morphism

$$\mathscr{E}_{\alpha(a)} \to \mathscr{E}$$
.

We get the composition

$$q_*(\mathscr{E}_{\alpha(a)}) \to q_*(\mathscr{E}) \to q_+(\mathscr{E}).$$

Now recall that $g_+(\mathscr{E})$ is an $\mathscr{R}_{\mathscr{P}}$ -module, in particular it has an action of $V_0\mathscr{R}_{\mathscr{P}}$ which is the V-filtration along the divisor \mathscr{X}_q (here we fix a point $q \in Q$ which we think of as the origin t = 0 of a disc).

Near the horizontal divisors we will be importing later the discussion of the L^2 complex from Sabbah's Section 6.2, so for the moment we work locally away from the horizontal divisors.

Saito-Mochizuki's description of the V-filtration may be summed up in the following proposition:

Proposition 4.4. Away from the horizontal divisors, for a < 1 the V-filtration V_{a-1} of the module $g_+(\mathcal{E})$ along \mathcal{X}_q is the sub- $V_0\mathcal{R}_{\mathscr{P}}$ -module generated by the image of

$$g_*(\mathscr{E}_{\alpha(a)}) \to g_+(\mathscr{E}).$$

In fact, more strongly it is the sub- $q^{-1}\mathcal{R}_{\mathscr{X}}$ -module generated by this image where $q: \mathscr{P} \to \mathscr{X}$ is the first projection.

Proof. Mochizuki gives a review in [13, 12.2.2], referring to [10, Section 16.1] and going back to Saito's [16, Theorem 3.4]. Note that they phrase the statement in terms of the multi-V-filtration considered before, then we use the compatibility of Proposition 4.3.

The stronger statement that $V_{a-1}g_+(\mathscr{E})$ is obtained from $g_*(\mathscr{E}_{\alpha(a)})$ just by the action of $\mathscr{R}_{\mathscr{X}}$ is what is actually stated and shown in the references. In order to clarify the property that this generates a $V_0\mathscr{R}_{\mathscr{P}}$ -module, we'll show ourselves that the two generation expressions are the same in Section 6 below.

In particular we obtain a map

$$g_*(\mathscr{E}_{\alpha(a)}) \to V_{a-1}(g_+(\mathscr{E})).$$
 (5)

We would like to use this to obtain a map of de Rham complexes. Recall that

$$DR_{L^2}^{\mathrm{par}}(\mathscr{X}/\mathscr{Y},\mathscr{E}_{\alpha(a)}) := \left[W_0(H,\mathscr{E}_{\alpha(a)}) \longrightarrow W_{-2}(H,\mathscr{E}_{\alpha(a)}) \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega^1_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}) \right].$$

Locally away from the horizontal divisor this may be written more simply as

$$\left[\mathscr{E}_{\alpha(a)} \longrightarrow \mathscr{E}_{\alpha(a)} \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega^1_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})\right].$$

Apply g_* to this complex, to get a complex whose terms are coherent $\mathcal{O}_{\mathscr{P}}$ -modules. The differential is still well-defined, since it comes from an action of vector fields tangent to the image $\mathscr{G} = g(\mathscr{X})$ of the graph embedding. We get a complex that we would like to map to the de Rham complex $DR(\mathscr{P}/\mathscr{Y}, V_{a-1}(g_+(\mathscr{E})))$ on \mathscr{P} relative to \mathscr{Y} .

If Y is a disk with coordinate t, $\Omega^1_{\mathscr{Y}}$ is trivialized with generator denoted dt. Wedge with dt gives maps of sheaves of differentials

$$\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}) \xrightarrow{\wedge dt} \Omega^{i+1}_{\mathscr{X}}.$$

In local coordinates, t = xy and the above map for i = 1 is given by

$$\frac{dx}{x} \mapsto \frac{dx}{x} \wedge dt = dx \wedge dy.$$

Recall here that $\Omega^1_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})$ is locally free of rank 1 on \mathscr{X} .

Take the product with a copy of \mathscr{Y} . Introduce the notation $T:=Y\times Y$ and $\mathscr{T}:=\mathscr{Y}\times_{\mathscr{A}}\mathscr{Y}$. We have a map

$$(f,1): \mathscr{P} \to \mathscr{T}.$$

Let t_1 and t_2 denote the two coordinates on \mathscr{T} . Over \mathscr{P} the previous multiplication map, interpreted in the first variable of $\mathscr{P} = \mathscr{X} \times_{\mathscr{A}} \mathscr{Y}$, gives the multiplication map

$$\Omega^{i}_{\mathscr{P}/\mathscr{T}}(\log \mathscr{D} \times_{\mathscr{A}} \mathscr{Y}) \stackrel{\wedge dt_{1}}{\longrightarrow} \Omega^{i+1}_{\mathscr{P}/\mathscr{Y}},$$

where $\mathscr{D} \times_{\mathscr{A}} \mathscr{Y} \subset \mathscr{P}$ is the divisor induced by $\mathscr{D} \subset \mathscr{X}$.

Using this together with the previous map (5) gives a map between complexes that may be described as follows.

Rewriting the terms, we claim that

$$g_*(\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})) = g_*(\mathcal{O}_{\mathscr{X}}) \otimes_{\mathcal{O}_{\mathscr{P}}} \Omega^i_{\mathscr{P}/\mathscr{T}}(\log \mathscr{D} \times_{\mathscr{A}} \mathscr{Y}),$$

where $\mathscr{D} \times_{\mathscr{A}} \mathscr{Y} \subset \mathscr{P}$ is the induced divisor. To see this use the first projection $q: \mathscr{P} \to \mathscr{X}$ and note

$$\Omega^{i}_{\mathscr{P}/\mathscr{T}}(\log \mathscr{D} \times_{\mathscr{A}} \mathscr{Y}) = q^{*}(\Omega^{i}_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})).$$

Clearly

$$g_*(\mathcal{O}_{\mathscr{X}}) \otimes_{\mathcal{O}_{\mathscr{P}}} q^*(\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})) = g_*(\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}))$$

since g is a section of the projection.

Using the above formula gives

$$g_*\left(\mathscr{E}_{\alpha(a)}\otimes_{\mathscr{O}_{\mathscr{X}}}\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log\mathscr{D})\right)=\left(g_*\mathscr{E}_{\alpha(a)}\right)\otimes_{\mathscr{O}_{\mathscr{P}}}\Omega^i_{\mathscr{P}/\mathscr{T}}(\log\mathscr{D}\times_{\mathscr{A}}\mathscr{Y}).$$

Then use

$$\Omega^i_{\mathscr{P}/\mathscr{T}}(\log \mathscr{D} \times_{\mathscr{A}} \mathscr{Y}) \xrightarrow{\wedge dt_1} \Omega^{i+1}_{\mathscr{P}/\mathscr{Y}}$$

to get a map

$$g_*\left(\mathscr{E}_{\alpha(a)}\otimes_{\mathcal{O}_{\mathscr{X}}}\Omega^i_{\mathscr{X}/\mathscr{Y}}(\log\mathscr{D})\right)\longrightarrow g_*(\mathscr{E}_{\alpha(a)})\otimes_{\mathcal{O}_{\mathscr{P}}}\Omega^{i+1}_{\mathscr{P}/\mathscr{Y}}.$$

Compose furthermore with

$$(g_*\mathscr{E}_{\alpha(a)}) \to V_{a-1}(g_+(\mathscr{E}))$$

to obtain a map

$$g_* \left(\mathscr{E}_{\alpha(a)} \otimes_{\mathcal{O}_{\mathscr{X}}} \Omega^i_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}) \right) \longrightarrow V_{a-1}(g_+(\mathscr{E})) \otimes_{\mathcal{O}_{\mathscr{D}}} \Omega^{i+1}_{\mathscr{D}/\mathscr{Y}}.$$
 (6)

Another way of looking at things is as follows: let $T(\mathcal{G}/\mathcal{Y})$ be the sheaf of tangent vector fields on the graph $\mathcal{G} := g(\mathcal{X}) \subset \mathcal{P}$, relative to \mathcal{Y} . Thinking that $\mathcal{G} \cong \mathcal{X}$, these tangent vector fields are just the tangent vector fields on \mathcal{X} relative to \mathcal{Y} , and that bundle is dual to $\Omega^1_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D})$. Now, this sheaf of tangent vector fields acts on $g_*(\mathcal{E}_{\alpha(a)})$ and (away from the horizontal divisors) the parabolic de Rham complex $g_*DR_{L^2}^{\mathrm{par}}(\mathcal{X}/\mathcal{Y}, \mathcal{E}_{\alpha(a)})$ is just the de Rham complex (or perhaps more accurately the "Spencer complex" [17, §0.6]) associated to this action. On the other hand, the relative tangent vectors to the graph map to the tangent vectors of \mathcal{P}/\mathcal{Y} :

$$T(\mathscr{G}/\mathscr{Y}) \to T(\mathscr{P}/\mathscr{Y})|_{\mathscr{G}}.$$

This map induces the desired map on de Rham complexes.

The expression (4) for $g_+(\mathscr{E})$, if made in the world of left modules, requires a choice of coordinate on Y. To get an invariant expression in terms of left modules (we thank T. Mochizuki for asking for that), let $\mathscr{G} = g(\mathscr{X}) \subset \mathscr{P}$ coming from $G = g(X) \subset P$ denote the graph divisor, and recall that the notation (* \mathscr{G}) means functions with arbitrary finite order of pole along \mathscr{G} . Let

$$\mathcal{O}_{\mathscr{P}}({}^{\lambda} * \mathscr{G}) \subset \mathcal{O}_{\mathscr{P}}(*\mathscr{G})$$

denote the sub- $\mathscr{R}_{\mathscr{P}}$ -module generated by the functions with a pole of order 1. It is the minimal extension (Definition 2.4) of $\mathcal{O}_{\mathscr{P}-\mathscr{G}}$, and may be written down as the sheaf of sections that are sums of λ^{m-1} times functions with poles of order m. Use the notation $({}^{\lambda}*\mathscr{G})$ for tensoring with $\mathcal{O}_{\mathscr{P}}({}^{\lambda}*\mathscr{G})$.

Recall also that $q: \mathscr{P} \to \mathscr{X}$ denotes the first projection. The canonical expression is

$$g_{+}(\mathscr{E}) = \frac{q^{*}(\mathscr{E})(^{\lambda} * \mathscr{G})}{q^{*}(\mathscr{E})}.$$
 (7)

The map (5) depends on a choice of section of $\mathcal{O}_{\mathscr{P}}(^{\lambda} * \mathscr{G})/\mathcal{O}_{\mathscr{P}}$ having a pole of order one, that depends on our choice of coordinate on Y.

Writing down a canonical map will in fact give the beginning of our map of complexes. Let

$$\zeta_{\mathscr{G}} \in \frac{\Omega^{1}_{\mathscr{P}/\mathscr{Y}}(\log \mathscr{G})}{\Omega^{1}_{\mathscr{P}/\mathscr{Y}}} \subset \frac{\Omega^{1}_{\mathscr{P}/\mathscr{Y}}(^{\lambda} * \mathscr{G})}{\Omega^{1}_{\mathscr{P}/\mathscr{Y}}}$$

be the unique section given by a logarithmic differential with residue 1 along \mathscr{G} . Multiplication by $\zeta_{\mathscr{G}}$ gives a map

$$q^*(\mathscr{E})|_{\mathscr{G}} = g_*(\mathscr{E}) \xrightarrow{\cdot \zeta_{\mathscr{G}}} \frac{q^*(\mathscr{E}) \otimes \Omega^1_{\mathscr{P}/\mathscr{Y}}({}^{\lambda} * \mathscr{G})}{q^*(\mathscr{E}) \otimes \Omega^1_{\mathscr{P}/\mathscr{Y}}} = g_+(\mathscr{E}) \otimes \Omega^1_{\mathscr{P}/\mathscr{Y}}. \tag{8}$$

For the second step of the complex, consider the map of multiplication by $\zeta_{\mathscr{G}}$,

$$q^*(\Omega^1_{\mathscr{X}}) = \Omega^1_{\mathscr{P}/\mathscr{Y}} \xrightarrow{\wedge \zeta_{\mathscr{G}}} \frac{\Omega^2_{\mathscr{P}/\mathscr{Y}}(^{\lambda} * \mathscr{G})}{\Omega^2_{\mathscr{P}/\mathscr{Y}}}.$$

We claim that it factors through and extends to a map

$$q^*(\Omega^1_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D})) \xrightarrow{\wedge \zeta_{\mathscr{G}}} \frac{\Omega^2_{\mathscr{P}/\mathscr{Y}}(^{\lambda} * \mathscr{G})}{\Omega^2_{\mathscr{P}/\mathscr{Y}}}.$$

Let us calcuate this locally, with Y a disk having coordinate t_2 and X a product of two disks with coordinates x, y. The map is $t_1 = f(x, y) = xy$ so the graph is given by the

equation $xy - t_2$. Note that $dt_1 = ydx + xdy$ so

$$\Omega^1_{\mathscr{X}/\mathscr{Y}} = \frac{\Omega^1_{\mathscr{X}}}{\langle ydx + xdy \rangle}.$$

On the other hand, P has coordinates x, y, t_2 and

$$\zeta_{\mathscr{G}} = \frac{d(xy - t_2)}{xy - t_2} = d\log(t_1 - t_2).$$

The factorization part of the claim is equivalent to the equation

$$(ydx + xdy) \wedge \frac{d(xy - t_2)}{xy - t_2} = 0 \mod dt_2.$$

$$(9)$$

This equation is clear since, modulo dt_2 it just becomes $d(xy) \wedge d(xy)/(xy - t_2) = 0$. The extension part of the claim asks that

$$\frac{dx}{x} \wedge \frac{d(xy - t_2)}{xy - t_2} \in \Omega^2_{\mathscr{P}/\mathscr{Y}}({}^{\lambda} * \mathscr{G}).$$

Again modulo dt_2 the expression becomes

$$\frac{dx}{x} \wedge \frac{ydx + xdy}{xy - t_2} = \frac{dx \wedge dy}{xy - t_2}$$

which is indeed in $\Omega^2_{\mathscr{P}/\mathscr{Y}}(^{\lambda}*\mathscr{G})$. The same holds for multiplication by dy/y. This completes the proof of the claim.

We now get the map of the second term in our complex:

$$q^{*}(\mathscr{E} \otimes \Omega^{1}_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}))|_{\mathscr{G}} = g_{*}(\mathscr{E} \otimes \Omega^{1}_{\mathscr{X}/\mathscr{Y}}(\log \mathscr{D}))$$

$$\xrightarrow{\wedge \zeta_{\mathscr{G}}} \frac{q^{*}(\mathscr{E}) \otimes \Omega^{2}_{\mathscr{P}/\mathscr{Y}}(^{\lambda} * \mathscr{G})}{q^{*}(\mathscr{E}) \otimes \Omega^{2}_{\mathscr{D}/\mathscr{Y}}} = g_{+}(\mathscr{E}) \otimes \Omega^{2}_{\mathscr{P}/\mathscr{Y}}. \tag{10}$$

Lemma 4.5. The above maps (6) for i = 0, 1 are the same as the maps (8) and (10) respectively. These maps are compatible with the differentials in the complexes, so they define a map of complexes

$$u(a):g_*(DR_{L^2}^{\mathrm{par}}(\mathscr{X}/\mathscr{Y},\mathscr{E}_{\alpha(a)}))\to DR(\mathscr{P}/\mathscr{Y},V_{a-1}(g_+(\mathscr{E}))).$$

It is shifted, i.e. it sends the degrees 0,1 in the first one, to the degrees 1,2 in the second one.

Proof. For the compatibility statement note that $W_{\ell}(H, \mathcal{E}_{\alpha(a)}) \subset \mathcal{E}$. The maps (8) and (10) are defined on \mathcal{E} but they restrict to maps defined on the appropriate $W_{\ell}(H, \mathcal{E}_{\alpha(a)})$. Conserving the notations t_1 and t_2 for the two maps from \mathscr{P} to \mathscr{Y} , with $t_1 = xy$ and $xy - t_2 = t_1 - t_2$ being the equation of the graph, the identification between (7) and (4) is

$$\frac{e}{(t_1 - t_2)} \mapsto e. \tag{11}$$

The maps (6) for i = 0, 1 are $e \mapsto e \wedge dt_1$ whereas (8) and (10) are $e \mapsto e \wedge \frac{dt_1}{t_1 - t_2}$. These are the same after the identification (11).

The differential $\zeta_{\mathscr{G}} = d \log(t_1 - t_2)$ is closed, so wedging with it commutes with the exterior derivative. Commutativity of the square comprising this map of complexes can be seen more explicitly in the calculations of the next sections.

Remark: Recall that the above general discussion was happening away from the horizontal divisors, in order to lighten the notation and not include the weight filtrations and so forth. At the horizontal divisors, the map from the L^2 complex to the \mathcal{R} -module de Rham complex exists, by Sabbah's discussion of [17, Section 6.2.a, Lemma 6.2.2]. This needs to be interpreted in the following way. In the general theory, if we try to define the de Rham complex using the graph embedding for a smooth function, we get a different complex but one that is quasiisomorphic. The map expressing this quasiisomorphism will go from the de Rham complex on the smaller variety, pushed forward to the graph, towards the de Rham complex of the graph embedding (we will be doing a version of this at the normal crossing points, below). We should compose this map, with the map from the L^2 complex to the full de Rham complex given in [17, Section 6.2.a], in order to obtain the map from our L^2 Dolbeault complex to $DR(\mathcal{P}/\mathcal{Y}, V_{a-1}g_+(\mathcal{E}))$. This will coincide with the map we are considering here away from the horizontal divisors.

Our main result is:

Theorem 4.6. The map of complexes u(a) in Lemma 4.5 is a quasiisomorphism.

Corollary 4.7. The higher direct images from \mathscr{X} to \mathscr{Y} of the complex $DR_{L^2}^{par}(\mathscr{X}/\mathscr{Y}, \mathscr{E}_{\alpha(a)})$, or of any of its restrictions to $X(\lambda)$, notably including the Dolbeault complex at $\lambda = 0$, are locally free and compatible with base-change.

Proof. We know from Saito-Sabbah-Mochizuki that this is true for $DR(\mathscr{P}/\mathscr{Y}, V_{a-1}(g_+(\mathscr{E})))$, then apply the quasiisomorphism of the theorem.

Proposition 4.8. Theorem 3.6 follows from Theorem 4.6.

Proof. Parts 1 and 2 of Theorem 3.6 follow from Theorem 4.1 once we identify the two de Rham complexes using the quasiisomorphism of Theorem 4.6. Part 3 follows from the discussion of [17, Chapter 5] relating \mathcal{R} -modules on a curve with parabolic Higgs bundles. The Higgs field may also be constructed in the usual way, starting from the absolute Dolbeault complex on X (rather than the relative one on X/Y that we are looking at)—see Section 7.1 for more details.

Part 4 comes from part 2 of Theorem 4.1.

For part 5, the calculation in [20] applies to the L^2 metric on a family of open complete curves (we are using the Poincaré metrics at the punctures of the fiber curves). Here some analytic considerations are needed that are left to the reader: the spaces of harmonic forms, all having the same dimension, fit together into a bundle and we can differentiate harmonic forms just as was done for the compact case in [20]. That argument shows that the L^2 metric is harmonic. Norm estimates show one direction of compatibility with the parabolic structure, and the other direction then follows because the degree of the parabolic bundle is zero. This discussion will be expanded upon in Section 7.2 below.

5 Quasiisomorphism

Put

$$\mathscr{Q}(a) := DR^{\mathrm{par}}_{L^2}(\mathscr{X}/\mathscr{Y},\mathscr{E}_{\alpha(a)})$$

and

$$Q(a) := \mathscr{Q}(a)|_{\lambda=0} = \mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)}).$$

Recall that $\mathcal{Q}(a)$ is a complex of vector bundles on \mathcal{X} and Q(a) is a complex of vector bundles on X. The differentials of $\mathcal{Q}(a)$ are differential operators, but on Q(a) they are \mathcal{O}_X -linear. We have

$$\mathcal{Q}(a-1) = t\mathcal{Q}(a) = \mathcal{Q}(a)(-D_V) \subset \mathcal{Q}(a)$$

where t is the coordinate defining the fiber over t = 0 in Y. We have $\mathcal{Q}(a) \subset \mathcal{Q}(a')$ when $a \leq a'$. The same hold for the Q(a).

The graph embedding $g: X \to P$ induces complexes $g_*(\mathcal{Q}(a))$ on \mathscr{P} and $g_*(Q(a))$ on P.

On the other hand, put

$$\mathscr{K}(a) := DR(\mathscr{P}/\mathscr{Y}, V_{a-1}(g_{+}(\mathscr{E})))$$

and

$$K(a) := \mathscr{K}(a)|_{\lambda=0}$$

These are complexes of quasicoherent sheaves on \mathscr{P} and P respectively. For a < 1 these collections have the same properties as the $\mathscr{Q}(a)$ and Q(a).

Our map of complexes given in Lemma 4.5

$$u(a): g_*(\mathcal{Q}(a)) \to \mathcal{K}(a)$$

induces on $\lambda = 0$ the map

$$u_0(a): g_*(Q(a)) \to K(a).$$
 (12)

Consider the projection maps $p: \mathscr{P} \to \mathscr{Y}$ and similarly (denoted by the same letter) $p: P \to Y$. Note that $p \circ g = f$ is our original map.

5.1 Higher direct image of K(a)

For $\mathcal{K}(a)$ and K(a) which are the de Rham complexes of \mathcal{R} -modules the general theory of Saito-Sabbah gives the properties we state in the following proposition.

Proposition 5.1. For a < 1, we have a decomposition

$$\mathbb{R}p_*(\mathscr{K}(a)) \cong \bigoplus_{i=0}^2 \mathscr{F}^i(a)[-i]$$

where

$$\mathscr{F}^i(a) := \mathbb{R}^i p_*(\mathscr{K}(a)),$$

and these terms are vector bundles. Furthermore again assuming a, a' < 1 if $a \le a'$ then $\mathscr{F}^i(a) \hookrightarrow \mathscr{F}^i(a')$. In particular $\mathscr{F}^i(a-1) = t\mathscr{F}^i(a)$.

Let

$$F^i(a) := \mathbb{R}^i p_*(K(a)).$$

Then $F^i(a)$ is also equal to the restriction of $\mathscr{F}^i(a)$ to $\lambda = 0$, in particular these are vector bundles. We again have the decomposition

$$\mathbb{R}p_*(K(a)) \cong \bigoplus_{i=0}^2 F^i(a)[-i]$$

and $F^i(a-1) = tF^i(a)$ for a < 0, with $F^i(a) \subset F^i(a')$ when $a \le a' < 0$.

Proof. The first part comes from the decomposition theorem of Sabbah and Mochizuki. The other parts restate Theorem 4.1 and gather some of the usual properties of the V-filtration.

We now recall the main property calculating the parabolic Higgs bundle associated to the harmonic bundle associated to the higher direct image.

Scholium 5.2. In the global situation over Y, the collection of $F^i(a)$ for a < 1 coincides with the parabolic Higgs bundle over Y associated to the harmonic bundle associated to the local system $R^i f_{*!}(L)$.

Proof. This restates Theorem 4.1.

5.2 Higher direct image of Q(a)

Our complexes $\mathcal{Q}(a)$ and Q(a) are complexes of vector bundles. Recall that $\mathbb{R}f_* = \mathbb{R}p_* \circ g_*$ on these complexes since g_* is a closed embedding so it is acyclic. We have the following standard property.

Proposition 5.3. The complexes $\mathbb{R}f_*(\mathcal{Q}(a))$ and $\mathbb{R}f_*(Q(a))$ are perfect complexes on \mathscr{Y} and Y respectively.

Proof. This is by a standard argument using the fact that f is flat and that the complexes are complexes of vector bundles with differential operators as maps.

Put

$$\mathscr{C}(a) := \operatorname{Cone}(u(a)), \quad C(a) := \operatorname{Cone}(u_0(a)).$$

These are complexes on \mathscr{P} and P respectively, supported on the graph of f which is the image of g. Their terms are quasicoherent sheaves on \mathscr{P} or P respectively.

Lemma 5.4. The terms of all of our complexes are flat over \mathscr{Y} and Y respectively.

Proof. There is no λ -torsion, then after restricting to $\lambda = 0$ there is no t-torsion.

5.3 The quasiisomorphism over smooth points

The first basic result is:

Lemma 5.5. If $x \in X$ is a point where f is smooth, then u(a) and $u_0(a)$ are quasiisomorphisms at g(x) = (x, f(x)).

Proof. This is by the independence of the V-filtration under choice of how to calculate it (by a graph embedding or not); our complex $\mathcal{Q}(a)$ is just the de Rham complex of the V_{a-1} piece, at a smooth point of f.

Corollary 5.6. The complexes $\mathcal{C}(a)$ and C(a) are cohomologically supported at the singular points of f, so this support is finite over \mathcal{Y} (resp. Y).

Corollary 5.7. For any a, if we can show that the map $\mathbb{R}p_*(u_0(a))$ is a quasiisomorphism then C(a) and $\mathscr{C}(a)$ are acyclic, hence u(a) and $u_0(a)$ are quasiisomorphisms.

Proof. Suppose C(a) is not acyclic. Then since it is supported at a finite set it would imply that $\mathbb{R}p_*(u_0(a))$ not be a quasiisomorphism. From our hypothesis we therefore conclude that C(a) is acyclic. Now it follows that $u_0(a)$ is a quasiisomorphism.

Consider the map $\mathbb{R}p_*(u(a))$. It is a map between perfect complexes on \mathscr{Y} , and its restriction to Y is $\mathbb{R}p_*(u_0(a))$, which we are assuming is a quasiisomorphism. Therefore it is a quasiisomorphism over an open set in \mathscr{A} containing $\lambda = 0$. As before, from the finiteness statement of Corollary 5.6 we conclude that u(a) is a quasiisomorphism over that open set.

We now claim that for any semisimple local system L and any $\lambda \neq 0$, if we consider the corresponding parabolic λ -connection $(\underline{E}(\lambda), \nabla)$ then the map $u(a)(\lambda)$ is a quasiisomorphism. We sketch an argument as follows. Look at a 1-parameter family of local systems L_t such that the $t\lambda$ -connection corresponding to L_t is $(\underline{E}(\lambda), t\nabla)$. We know that $(\underline{E}(\lambda), t\nabla)$ approach a limiting polystable parabolic Higgs bundle (one can use the same technique as in [21]). For this limiting Higgs bundle the map $u_0(a)$ is a quasiisomorphism as we have seen above, and by semicontinuity (using perfectness of the source and target of the higher direct image, and the global to local argument coming from Corollary 5.6 as above), it follows that the map $u_t(a)$ corresponding to $(\underline{E}(\lambda), t\nabla)$ is a quasiisomorphism for t near to 0. However, the maps $u_t(a)$ are all the same as our original u(a) up to scaling, so this shows that the original u(a) is a quasiisomorphism.

5.4 The quasiisomorphism near horizontal divisors

Next we quote the following main result from Saito and Sabbah. It basically goes back to Zucker's paper [23]. This result was of course the motivation for introducing the L^2 de Rham complexes.

Proposition 5.8. Suppose $x \in X$ is an intersection point of a horizontal and vertical divisor component. Then u(a) and $u_0(a)$ are quasiisomorphisms at g(x) = (x, f(x)).

Proof. This is Proposition 6.2.4 of [17]. \Box

5.5 The quasiisomorphism at double points—statement

In this subsection we state the result of our main calculation.

Theorem 5.9. Suppose $x \in X$ is an intersection point of two vertical divisor components. Then the map $u_0(a)$ of (12) is a quasiisomorphism at g(x) = (x, f(x)).

The proof will occupy the several upcoming sections.

Corollary 5.10. The maps u(a) and $u_0(a)$ are quasiisomorphisms, that is to say this gives the proof of Theorem 4.6.

Proof. The statement of Theorem 5.9 gives the quasiisomorphism at crossing points of vertical components. On smooth points use Lemma 5.5, and at points where the horizontal divisor meets the vertical divisor, use Proposition 5.8.

Corollary 5.11. For any a we have

$$\mathbb{R}f_*(Q(a)) = \bigoplus R^i f_*(Q(a)).$$

These terms are vector bundles. For $a \leq -2$ we have $R^i f_*(Q(a)) = F^i(a)$. The vector bundles $R^i f_*(Q(a))$ fit together into a parabolic Higgs bundle, and this is the parabolic Higgs bundle associated to the local system $R^i f_{*!}(L)$.

5.6 Reduction to t = 0

We point out here, using the above properties, that it suffices to consider the restriction to the fiber over t = 0. Let $Z := f^{-1}(0) \subset X$. Since we are localizing to a disk in Y with only one singular point, we have in fact $Z = D_V$. Let $\mathscr{Z} := Z \times \mathscr{A}$ as usual. Note that $Z = g(X) \cap (X \times \{0\})$.

Proposition 5.12. In order to prove Theorem 5.9 it suffices to prove that the map

$$u_0(a)|_{X\times\{0\}}: g_*(Q(a))|_{X\times\{0\}} \to K(a)|_{X\times\{0\}}$$

is a quasiisomorphism near any double point of $Z \subset X \times \{0\}$.

Proof. The complexes Q(a) and K(a) are flat over Y, hence so are their higher direct image complexes. Let

$$B(a) := \operatorname{Cone} \left(\mathbb{R} f_*(Q(a)) \xrightarrow{\mathbb{R} p_*(u_0(a))} \mathbb{R} p_*(K(a)) \right).$$

It is again a complex of sheaves flat over Y. The exact triangle containing B(a) restricts to an exact triangle over the point t = 0 and indeed

$$B(a)|_{t=0} = \operatorname{Cone}\left(\mathbb{H}^{\bullet}(Q(a)|_{Z}) \to \mathbb{H}^{\bullet}(K(a)|_{X \times \{0\}})\right).$$

Suppose we know the statement that $u_0(a)|_{X\times\{0\}}$ is a quasiisomorphism. Then it follows (using the above arguments at smooth points of Z as well as at points of $D_H \cap Z$) that $B(a)|_{t=0}$ is exact. On the other hand, both $\mathbb{R}f_*(Q(a))$ and $\mathbb{R}p_*(K(a))$ are perfect complexes as we have pointed out above. Therefore the cone B(a) on the map between them is a perfect complex. It is cohomologically supported at t=0. Now we may conclude by using the following property of perfect complexes: a perfect complex which is cohomologically supported at a point, and whose restriction to that point is acyclic, is acyclic. Therefore, B(a) is acyclic and $\mathbb{R}p_*(u_0(a))$ is a quasiisomrphism. By Corollary 5.7 this will show Theorem 5.9, completing our reduction.

We may further reduce using the parabolic structure. Let ε be strictly smaller than the difference of any two parabolic weights. For any parabolic weight a we have the map of complexes

$$\operatorname{gr}_a(u_0): g_*(Q(a)/Q(a-\varepsilon)) \to K(a)/K(a-\varepsilon).$$
 (13)

Proposition 5.13. In order to prove Theorem 5.9, it suffices to show that for any parabolic weight a, the map $gr_a(u_0)$ of (13) is a quasiisomorphism near double points of Z.

Proof. Use the previous Proposition 5.12. Both the source and target of the map $u_0(a)|_{X\times\{0\}}$ have filtrations such that the graded quotients are respectively $g_*(Q(a')/Q(a'-\varepsilon))$ and $K(a')/K(a'-\varepsilon)$, for the parabolic weights $a-1 < a' \le a$. Hence, if we know that the $\operatorname{gr}_{a'}(u_0)$ are quasiisomorphisms, it will follow that $u_0(a)|_{X\times\{0\}}$ is a quasiisomorphism; then Proposition 5.12 leads to Theorem 5.9.

6 Proof at a normal crossing

We now turn to the proof of Theorem 5.9, using the reductions above. By proposition 5.13 we would like to obtain a quasiisomorphism for the graded pieces of the parabolic structure.

6.1 The V-filtration at a normal crossing

As the remaining problem is to treat an intersection of two vertical divisor components, let us restrict to a local situation. Thus we may assume that Y is a disk with coordinate t, and the singular fiber is t = 0. On $\mathscr{P} = \mathscr{X} \times_{\mathscr{A}} \mathscr{Y}$ we think of t as being the coordinate of the second factor \mathscr{Y} . Let ∂_t denote the vector field generating the \mathscr{Y} -direction of the tangent bundle of \mathscr{P} . Recall that it acts on functions by the derivation $\lambda \partial/\partial t$.

Consider our $\mathscr{R}_{\mathscr{X}}$ -module \mathscr{E} . We look at $g_{+}(\mathscr{E})$ which is an $\mathscr{R}_{\mathscr{P}}$ -module on \mathscr{P} . It is supported on the graph $q(\mathscr{X})$.

We may identify sheaves on $\mathscr P$ supported on $g(\mathscr X)$, with their pullbacks to $\mathscr X$ via g^{-1} . This yields on $\mathscr X$ the $g^{-1}\mathscr R_{\mathscr P}$ -module $g^{-1}g_+(\mathscr E)$.

Lemma 6.1. We may write

$$g^{-1}g_{+}(\mathscr{E}) = \mathscr{E}[\partial_{t}].$$

The action of $g^{-1}\mathcal{R}_{\mathscr{P}}$ is given by the formulae of [16, Equations (3.4.3)].

Proof. It is possible to change coordinates on \mathscr{P} such that the image of the graph is just a coordinate hyperplane. We can therefore write explicitly the pushforward g_+ as follows:

$$g_{+}(\mathscr{E}) = \bigoplus_{i=0}^{\infty} \partial_{t}^{i} \cdot (g_{*}(\mathscr{E})).$$

The statement of the lemma is given by pulling back to \mathscr{X} along g^{-1} .

The action of elements of $\mathscr{R}_{\mathscr{P}}$ is easy to calculate in the changed coordinate system, in particular it means that the actions of ∂_t and $\mathcal{O}_{\mathscr{P}}$ are just the usual ones. On the other hand, one must use a change of variables formula to get the action of the vector fields on the original factor \mathscr{X} . This is done by Saito [16, Theorem 3.4] and Mochizuki [10, Section 16.1] [13, 12.2.2].

Recall that $V_0 \mathscr{R}_{\mathscr{P}} \subset \mathscr{R}_{\mathscr{P}}$ is the sheaf of subrings generated by the tangent vector fields tangent to the fiber t = 0; it contains in particular the tangent vector field $t\partial_t$. And, the

V-filtration of $g_{+}(\mathscr{E})$ is characterized as the increasing filtration of this module by finitely generated sub- $V_0\mathscr{R}_{\mathscr{P}}$ -modules $V_{b-1}(g_{+}(\mathscr{E}))$, such that on

$$Gr_{b-1}(g_{+}(\mathscr{E})) = V_{b-1}(g_{+}(\mathscr{E}))/V_{v-\epsilon-1}(g_{+}(\mathscr{E})),$$

the vector field $t\partial_t$ acts with generalized eigenvalue $-\lambda b$, that is to say $t\partial_t + \lambda b$ is nilpotent.

Proposition 6.2. Assume b < 1. The V-filtration of $g_+(\mathscr{E})$ is determined as follows: $V_{b-1}(g_+(\mathscr{E}))$ is the $V_0\mathscr{R}_{\mathscr{P}}$ -submodule of $g_+(\mathscr{E})$ generated by $g_*(\mathscr{E}_{b,b})$.

Proof. This restates the first part of Proposition 4.4. We'll describe the argument for the second part of that proposition in two steps, in 6.4 and 6.7 below. \Box

Let $q: \mathscr{P} \to \mathscr{X}$ denote the projection. We have an inclusion of sheaves of rings

$$q^{-1}\mathscr{R}_{\mathscr{X}}\subset V_0\mathscr{R}_{\mathscr{P}}.$$

Denote by s the section of $V_0 \mathscr{R}_{\mathscr{P}}$ corresponding to $t\partial_t$. We obtain the sheaf of rings

$$q^{-1}\mathscr{R}_{\mathscr{X}}[s] \subset V_0\mathscr{R}_{\mathscr{P}}.$$

Note that $g^{-1}q^{-1}\mathcal{R}_{\mathscr{X}}=\mathcal{R}_{\mathscr{X}}$. This gives an inclusion of sheaves of rings on \mathscr{X} ,

$$\mathscr{R}_{\mathscr{X}}[s] \subset g^{-1}V_0\mathscr{R}_{\mathscr{P}}.$$

We have $g^{-1}g_*(\mathscr{E}) = \mathscr{E}$ and it has a natural map to $g^{-1}g_+(\mathscr{E})$ corresponding to the inclusion of the degree 0 part of $\mathscr{E}[\partial_t]$ in the expression of Lemma 6.1.

A basic fact for our calculations is the following lemma.

Lemma 6.3. Given a collection of sections of $g^{-1}g_*(\mathscr{E})$, the $g^{-1}V_0\mathscr{R}_{\mathscr{P}}$ -submodule of $g^{-1}g_+(\mathscr{E})$ that they generate is the same as the $\mathscr{R}_{\mathscr{X}}[s]$ -submodule they generate.

Proof. Let x, y be the coordinates on X with g(x, y) = xy. A section of the ring $g^{-1}V_0\mathcal{R}_{\mathscr{P}}$ may be written as

$$r = \sum_{i,j,k} (t\partial_t)^i \partial_x^j \partial_y^k f_{ijk}(t,x,y).$$

Setting $a_{ijk}(x,y) := f_{ijk}(xy,x,y)$ we have (for some function u_{ijk})

$$a_{ijk}(x,y) - f_{ijk}(t,x,y) = u_{ijk}(t,x,y)(t-xy).$$

If e is a section of $g^{-1}g_*(\mathscr{E})$ then (t-xy)e=0. Hence $f_{ijk}(t,x,y)\cdot e=a_{ijk}(x,y)\cdot e$, therefore

$$r \cdot e = \sum_{i,j,k} (t\partial_t)^i \partial_x^j \partial_y^k a_{ijk}(x,y) \cdot e.$$

This expression is in the submodule generated by e under the action of $\mathscr{R}_{\mathscr{X}}[s]$.

Corollary 6.4. The V-filtration of $g_+(\mathcal{E})$ is determined in negative degrees as follows: for b < 1,

$$g^{-1}V_{b-1}g_{+}(\mathscr{E}) \subset g^{-1}g_{+}(\mathscr{E}) = \mathscr{E}[\partial_{t}]$$

is the $\mathscr{R}_{\mathscr{X}}[s]$ -submodule generated by $g^{-1}g_*(\mathscr{E}_{b,b}) = \mathscr{E}_{b,b}$.

Proof. Combine the previous proposition and lemma.

We now look more closely at the action of $\mathscr{R}_{\mathscr{X}}[s]$. As was explained in [17, Section 3.4], it is useful to localize by inverting t = xy. Sabbah denoted the localization with a tilde and we conserve that notation.

Notice that xy never acts invertibly on the module \mathscr{E} ; this is somewhat different from the case of \mathcal{D} -modules where we can have a holonomic module in which xy is invertible. Because of multiplication of the derivatives by λ , a finitely generated $\mathscr{R}_{\mathscr{X}}$ -module will not have xy acting invertibly. However, as Sabbah points out, one may make this localization if we are interested in the V-filtration.

Let
$$\widetilde{\mathscr{E}} := \mathscr{E}[(xy)^{-1}]$$
. Then

$$g_{+}(\widetilde{\mathscr{E}}) = g_{+}(\mathscr{E})[t^{-1}].$$

Notice that $\mathscr{E} \subset \widetilde{\mathscr{E}}$ is a submodule so

$$g_{+}(\mathscr{E}) \subset g_{+}(\widetilde{\mathscr{E}})$$

is a sub- $\mathcal{R}_{\mathscr{P}}$ -module.

We also have the following relationship with the parabolic structure:

$$\widetilde{\mathscr{E}} = \bigcup_{a,b} \mathscr{E}_{a,b}.$$

Proposition 6.5. The (non-finite type) $\mathscr{R}_{\mathscr{P}}$ -module $g_{+}(\widetilde{\mathscr{E}})$ also admits a V-filtration characterized by the same properties as in the holonomic case. For b < 1 we have

$$V_{b-1}g_+(\mathscr{E}) = V_{b-1}g_+(\widetilde{\mathscr{E}}).$$

We have

$$g^{-1}g_{+}(\widetilde{\mathscr{E}}) = \widetilde{\mathscr{E}}[s],$$

although one must be careful that the action of $\mathscr{R}_{\mathscr{X}}$ includes terms in s as referred to in Lemma 6.1.

For any b, $g^{-1}V_{b-1}g_{+}(\widetilde{\mathcal{E}})$ is the sub- $\mathscr{R}_{\mathscr{X}}[s]$ -module of $\widetilde{\mathscr{E}}[s]$ generated by $\mathscr{E}_{b,b}$.

Proof. This is the same as the previous statements, noting that t = xy is invertible on $\widetilde{\mathscr{E}}$, allowing us to go between ∂_t and $s = t\partial_t$ by multiplying or dividing by t.

We can now restrict to $\lambda = 0$. This restriction is a quotient, dividing everything by the submodules generated by λ . The other operations that have intervened above, namely taking localization and taking submodules generated by something, are all of the form colimits. Therefore, all these operations commute.

Recall that E denotes the $R_{X,0}$ -module over X. It is the restriction of \mathscr{E} to $\lambda = 0$. We have $g_+(E)$ the restriction of $g_+(\mathscr{E})$, with the formula

$$g^{-1}(g_+(E)) = E[\partial_t].$$

Let \widetilde{E} be obtained by inverting xy on E. Again, it is the restriction of $\widetilde{\mathscr{E}}$ to $\lambda = 0$. We have

$$g^{-1}(g_{+}(\widetilde{E})) = \widetilde{E}[s]$$

where as before $s = t\partial_t$.

Proposition 6.6. The V-filtrations $V_{b-1}g_+(E)$ (resp. $V_{b-1}g_+(\widetilde{E})$) are the restrictions to $\lambda = 0$ of the $V_{b-1}g_+(\mathscr{E})$ (resp. $V_{b-1}g_+(\widetilde{\mathscr{E}})$). We have

$$V_{b-1}(q_{+}(E)) = V_{b-1}(q_{+}(\widetilde{E}))$$
 for $b < 1$

and for any b, $V_{b-1}(g_+(\widetilde{E}))$ is the submodule of $\widetilde{E}[s]$ generated by $g_*E_{b,b}$ under the action of the ring $R_{X,0}[s]$.

6.2 Towards explicit calculations

After the discussion from above, we are in the following situation. We have that X is a product of disks, and D is the union of the two coordinate lines D_1 and D_2 . Use coordinates (x, y) on X, with D_1 given by y = 0 and D_2 by x = 0.

From now on we identify sheaves on P supported along g(X) with sheaves on X via g^{-1} and g_* . We work with sheaves on X.

We are given a module \widetilde{E} over the ring of functions with poles along D, call it $\mathcal{O}_X[x^{-1}, y^{-1}]$. It has submodules denoted by $E_{a,b}$ which are locally free over \mathcal{O}_X and give \widetilde{E} when localized by inverting x and y. Define

$$\psi_{b,b} := E_{b,b}/E_{b-\epsilon,b-\epsilon}.$$

The Higgs field on \widetilde{E} is given by two sections φ_x and φ_y of $End(\widetilde{E})$, having logarithmic poles with respect to each of the submodules $E_{a,b}$. In particular, we have

$$\varphi_x(E_{b,b}) \subset x^{-1}E_{b,b}, \quad \varphi_y(E_{b,b}) \subset y^{-1}E_{b,b}$$

and the same for $b - \epsilon$. It follows that $x\varphi_x$ and $y\varphi_y$ act on $\psi_{b,b}$.

We have

$$V_{b-1}(g_+\widetilde{E}) \subset g_+(\widetilde{E}) = \widetilde{E}[s]$$

is the submodule generated by $E_{b,b}$ under the operations of s and the basis vector fields ∂_x and ∂_y . These act according to the formulae of [16, Equations (3.4.3)] as explained above. In the present situation we are restricting to $\lambda = 0$ so there is no differentiation: the actions of all the vector fields commute and they act trivially on functions. Recall, however, that in order to characterize the V-filtration one needs to use the full $\mathscr{R}_{\mathscr{X}}$ -module $g_+(\widetilde{\mathscr{E}})$ with differentiation. After the characterization as a submodule generated by $\mathscr{E}_{b,b}$ we can then restrict to $\lambda = 0$ and have the same characterization there.

In view of commutativity, the actions of the vector fields are easier to write down: ∂_x and ∂_y act respectively by endomorphisms

$$A_x := \varphi_x + s/x$$
, and $A_y := \varphi_y + s/y$.

The same statement holds for $V_{b-\epsilon-1}$.

By definition the module of nearby cycles is

$$\Psi_{b-1} := V_{b-1}(\widetilde{E}[s])/V_{b-\epsilon-1}(\widetilde{E}[s]).$$

Notice that we keep here the subscript b-1 in order to conform to the usual practice in the theory of \mathcal{D} -modules. For brevity on the other hand we used the notation $\psi_{b,b}$ without the -1's. This shouldn't cause too much confusion as either expression may be considered as some kind of notation.

The following lemma completes our review of the second part of Proposition 4.4, which we recall is due to Saito and Mochizuki. We felt it would be useful to give here a proof adapted specifically to the Dolbeault case.

Lemma 6.7. The submodule $V_{b-1}(\widetilde{E}[s]) \subset \widetilde{E}[s]$ can also be characterized as the submodule generated by $E_{b,b}$ under just \mathcal{O}_X and the operations A_x and A_y .

Proof. Recall that $E_{b,b}$ is preserved by the logarithmic Higgs field, so it is stable under the operations $x\varphi_x$ and $y\varphi_y$. The same is therefore true of $E_{b,b}[s]$. Thus xA_x also preserves $E_{b,b}[s]$ and

$$s^k E_{b,b} = (x\varphi_x - xA_x)^k E_{b,b} \subset E_{b,b}[s].$$

Now $V_{b-1}(\widetilde{E}[s])$ is the sub- \mathcal{O}_X -module of $\widetilde{E}[s]$ generated by $E_{b,b}$ under the operations s, A_x, A_y . Hence, it is also the submodule generated by $E_{b,b}[s]$ under the operations A_x, A_y . From the previous formula we see that it is generated from $E_{b,b}$ by A_x, A_y .

6.3 The tensor product formula

Let the ring $\mathcal{O}_X[u,v]$ act on $\widetilde{E}[s]$ by setting the action of u equal to A_x and the action of v equal to A_y . Define

$$\varphi_{\log} := x\varphi_x - y\varphi_y,$$

acting on \widetilde{E} , hence also on $\widetilde{E}[s]$; and consider the endomorphism of $\widetilde{E}[s]$ given by

$$A_{\log} := xA_x - yA_y = xu - yv.$$

The action of this endomorphism is equal to the action of φ_{\log} . In other words, the element $xu - yv - \varphi_{\log}$ acts by 0 on $\widetilde{E}[s]$.

Let w := xu - yv so we have a map $\mathcal{O}_X[w] \to \mathcal{O}_X[u,v]$, and we note that for any $\mathcal{O}_X[w]$ -module M we have

$$M \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v] = \frac{M[u,v]}{(xu - yv - w)M[u,v]}.$$

If furthermore M is an \mathcal{O}_D -module, equivalent to saying that xyM = 0, then we may also write

$$M \otimes_{\mathcal{O}_D[w]} \mathcal{O}_D[u,v] = \frac{M[u,v]}{(xu - yv - w)M[u,v]}.$$

We clearly have a map $E_{b,b} \to V_{b-1}(\widetilde{E}[s])$ giving

$$\psi_{b,b} \to \Psi_{b-1}$$
,

and on the other hand the operations A_x and A_y on Ψ_{b-1} give an $\mathcal{O}_D[u,v]$ -module structure. The action of xu - yv coincides with the action of φ_{\log} on $\psi_{b,b}$, so we obtain a natural map

$$\psi_{b,b} \otimes_{\mathcal{O}_D[w]} \mathcal{O}_D[u,v] \to \Psi_{b-1}. \tag{14}$$

The following proposition gives a formula for Ψ_{b-1} :

Theorem 6.8 (Tensor product formula). The map (14) is an isomorphism.

Proof. From the above discussion, we may also say that we would like to show that the map

$$\frac{\psi_{b,b}[u,v]}{(xu - yv - \varphi_{\log})\psi_{b,b}[u,v]} \to \Psi_{b-1}.$$
(15)

is an isomorphism.

Notice that by the definition of V_{b-1} , the map

$$E_{b,b}[u,v] \to V_{b-1}(\widetilde{E}[s])$$

is surjective. Hence it follows that the map (15) is surjective. We would like to show that it is injective.

The right hand side of (15) is

$$\frac{E_{b,b}[u,v]}{E_{b-\epsilon,b-\epsilon}[u,v] + (xu - yv - \varphi_{\log})E_{b,b}[u,v]}.$$

We need to show that if $a \in E_{b,b}[u,v]$ and the image of a in $\widetilde{E}[s]$ is in $V_{b-\epsilon-1}(\widetilde{E}[s])$, that is to say if the image of a is in the image of $E_{b-\epsilon,b-\epsilon}[u,v]$, then

$$a \in E_{b-\epsilon,b-\epsilon}[u,v] + (xu - yv - \varphi_{\log})E_{b,b}[u,v].$$

Our first claim is that the map

$$\frac{\widetilde{E}[u,v]}{(xu - yv - \varphi_{\log}) \cdot \widetilde{E}[u,v]} \to \widetilde{E}[s]$$
(16)

is injective. In this situation x and y are invertible and we can write $s = xu - x\varphi_x = yv - y\varphi_y$. Consider the following change of variables: put $u' := x(u - \varphi_x)$ and $v' := y(v - \varphi_y)$ acting on \widetilde{E} . These formulas determine a map

$$\widetilde{E}[u',v'] \to \widetilde{E}[u,v]$$

by sending

$$\sum (u')^i (v')^j e_{ij} \text{ to } \sum (x(u-\varphi_x))^i (y(v-\varphi_y))^j e_{ij}.$$

This map is an isomorphism. The composed map

$$\widetilde{E}[u',v'] \to \widetilde{E}[u,v] \to \widetilde{E}[s]$$

maps the action of u' to the action of $x(A_x - \varphi_x) = s$ and also the action of v' to the action of $y(A_y - \varphi_y) = s$, that is to say it sends u' and v' to s. The kernel is therefore the submodule $(u' - v')\widetilde{E}[u, v]$. Transporting back by the isomorphism, we see that the kernel of $\widetilde{E}[u, v] \to \widetilde{E}[s]$ is generated by $x(u - \varphi_x) - y(v - \varphi_y) = xu - yv - \varphi_{\log}$. This proves the claim that (16) is injective.

We next note that (16) is surjective. Indeed, we saw above that its image is the same as the image of $\widetilde{E}[u',v'] \to \widetilde{E}[s]$ and this map is visibly surjective since u' and v' map to s.

We have shown that (16) is an isomorphism, in particular we may replace $\tilde{E}[s]$ by the left hand side of this map. Therefore we may consider the map

$$E_{b,b}[u,v] \to \widetilde{E}[u,v]/(xu - yv - \varphi_{\log})\widetilde{E}[u,v] \cong \widetilde{E}[s].$$
(17)

The image is $V_{b-1}(\widetilde{E}[s])$. Our second claim is that the kernel of (17) is equal to $(xu - yv - \varphi_{\log})E_{b,b}[u,v]$. This claim is equivalent to the statement

$$E_{b,b}[u,v] \cap (xu - yv - \varphi_{\log})\widetilde{E}[u,v] = (xu - yv - \varphi_{\log})E_{b,b}[u,v].$$

To prove this statement, choose a basis for $E_{b,b}$ so we may write $E_{b,b} \cong \mathcal{O}_X^r$. Write $\varphi_{\log} = B_{ij}$ in terms of this basis, with $B_{ij} \in \mathcal{O}_X$. In this notation, $\widetilde{E}[u,v] = \mathcal{O}_X[x^{-1},y^{-1}][u,v]^r$. Suppose we have a vector (f_i) here, so $f_i \in \mathcal{O}_X[x^{-1},y^{-1}][u,v]$. Suppose that

$$(xu - yv)f_i + \sum_j B_{ij}f_j \in \mathcal{O}_X[u, v].$$

Write

$$f_i = x^{-a}y^{-b} \sum_{k,l} g_i^{kl} u^k v^l.$$

Assume that a, b are chosen so that $g_i^{kl} \in \mathcal{O}_X$ but at least one of them is nonzero along each of the components of D, i.e. a and b are the smallest possible. Suppose one of a or b is > 0, that is to say some f_i has a pole. We will obtain a contradiction. Suppose for example a > 0. Then restrict the g_i^{kl} to (x = 0), and consider the terms of maximal k + l, call \hat{g}_i

the sum of these terms of the form $g_i^{kl}u^kv^l$. After multiplying by (xu-yv) we get terms with strictly bigger degree in u, v, but the restriction of the term $xu\hat{g}_i$ to (x=0) vanishes; the restriction of $yv\hat{g}_i$ is nonzero. But it has strictly bigger degree in u, v than any possible term in the restriction to (x=0) of $\sum_j B_{ij}g_j$. Here $g_i := \sum_{k,l} g_i^{kl}u^kv^l$. It follows that the restriction of

$$(xu - yv)g_i + \sum_j B_{ij}g_j$$

to (x=0) is nonvanishing, but since $f_i = x^{-a}y^{-b}g_i$ with a > 0 this contradicts the hypothesis that $(xu - yv)f_i + \sum_j B_{ij}f_j \in \mathcal{O}_X[u,v]$. We conclude that $a \leq 0$ and similarly $b \leq 0$, in other words our section (f_i) is in $\mathcal{O}_X^r[u,v] = E_{b,b}[u,v]$. This proves the second claim.

The corollary of the second claim is the formula

$$V_{b-1}(E[s]) = E_{b,b}[u, v]/(xu - yv - \varphi_{\log})E_{b,b}[u, v].$$

This may also be written as

$$V_{b-1}(\widetilde{E}[s]) = E_{b,b} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v].$$

The same holds for $b + \epsilon$. But now the statement of the proposition follows: we have a right exact sequence

$$E_{b-\epsilon,b-\epsilon} \to E_{b,b} \to \psi_{b,b} \to 0$$

and tensor product is right exact, so we get the exact sequence

$$E_{b-\epsilon,b+\epsilon} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v] \to E_{b,b} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v] \to \psi_{b,b} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v] \to 0,$$

In view of the previous formula for $V_{b-1}(E[s])$ and the same for $b-\epsilon$, this may be written

$$V_{b-\epsilon-1}(\widetilde{E}[s]) \to V_{b-1}(\widetilde{E}[s]) \to \psi_{b,b} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v] \to 0,$$

in other words

$$\Psi_{b-1} = V_{b-1}(\widetilde{E}[s])/V_{b-\epsilon-1}(E[s]) = \psi_{b,b} \otimes_{\mathcal{O}_X[w]} \mathcal{O}_X[u,v].$$

This proves the tensor product formula of the proposition.

6.4 Consequence for the Koszul complexes

In general suppose Z is a scheme or analytic space and \mathcal{V} is a vector bundle. Let $\operatorname{Sym}^*(\mathcal{V}^{\vee})$ denote the symmetric algebra on the dual vector bundle \mathcal{V}^{\vee} . A $\operatorname{Sym}^*(\mathcal{V}^{\vee})$ -module coherent over Z is the same thing as a coherent sheaf \mathcal{F} on Z together with a morphism

$$\phi: \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{V}$$

such that the induced map $\phi \wedge \phi : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_Z} \bigwedge^2 \mathcal{V}$ is zero. The case of a Higgs bundle is when $\mathcal{V} = \Omega_Z^1$. In our case, the structure of logarithmic Higgs bundles of E_β over Z = X corresponds to $\mathcal{V} = \Omega_{X/Y}^1(\log D_V)$.

In general given $(\mathcal{V}, \mathcal{F}, \phi)$ as above we get the Koszul complex

$$\mathbf{Kosz}(\mathcal{V}, \mathcal{F}, \phi) := \left[\dots \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} \bigwedge^i \mathcal{V} \xrightarrow{\wedge \phi} \mathcal{F} \otimes_{\mathcal{O}_Z} \bigwedge^{i+1} \mathcal{V} \longrightarrow \dots \right]. \tag{18}$$

If we are working with modules over a ring rather than quasicoherent sheaves on a scheme or space, we shall use the same notation.

Locally at a crossing point of vertical divisors we have

$$Q(b) = \mathrm{DOL}_{L^2}^{\mathrm{par}}(X/Y, E_{\alpha(b)}) = \mathbf{Kosz}(\Omega_{X/Y}^1(\log D), E_{b,b}, \varphi).$$

The Koszul complex is compatible with quotients in the module variable, since tensoring with the locally free sheaves $\bigwedge^i \mathcal{V}$ is exact. It follows that

$$Q(b)/Q(b-\epsilon) = \mathbf{Kosz}(\Omega^{1}_{X/Y}(\log D), \psi_{b,b}, \varphi)$$
(19)

locally near a crossing point of two vertical divisor components because $\psi_{b,b} = E_{b,b}/E_{b-\epsilon,b-\epsilon}$. On the other hand,

$$K(b) = \mathbf{Kosz}(\Omega^1_{P/Y}, V_{b-1}(g_+(\mathscr{E})), --),$$

so again

$$K(b)/K(b-\epsilon) = \mathbf{Kosz}(\Omega^{1}_{P/Y}, \Psi_{b-1}, --).$$
(20)

We didn't give a name to the Higgs field for the sheaf Ψ_{b-1} on P/Y.

The $\psi_{b,b}$ and Ψ_{b-1} are \mathcal{O}_D -modules supported on $D \subset X$. Therefore we may consider the Koszul complexes as being constructed over the divisor Z := D, in the neighborhood of a normal crossing point of the vertical divisor.

In view of the tensor product formula of the previous subsection, let us consider an \mathcal{O}_D -module M with action of an endomorphism φ , and set

$$N := M[u, v]/(xu - yv - \varphi)M.$$

In the previous notations, $M = \psi_{b,b}$ and $N = \Psi_{b-1}$. We have shortened φ_{\log} to just φ here.

Over D we have two distinct vector bundles that can be used to define Koszul complexes. The first V_1 is the restriction to D of $\Omega^1_{X/Y}(\log D_V)$. It has rank 1 and a local generator for V_1^* is $x\partial_x - y\partial_y$. The operator $\varphi = \varphi_{\log}$ is an action of $S(V_1^*)$ on M and we get the Koszul complex $\mathbf{Kosz}(V_1, M, \varphi)$.

On the other hand, V_2 is the vector bundle Ω_X^1 restricted to D, of rank 2 with local generators ∂_x and ∂_y . We have $S(V_2^*) = \mathcal{O}_D[u, v]$, and the $\mathcal{O}_D[u, v]$ -module structure of N corresponds to a map

$$\psi: N \to N \otimes V_2$$
.

We get the Koszul complex $\mathbf{Kosz}(V_2, N, \psi)$.

The map of Koszul complexes that we would like to consider is the vertical map between the two horizontal complexes:

$$\mathbf{Kosz}(V_1, M, \varphi) = M \xrightarrow{\varphi} M$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow$$

$$\mathbf{Kosz}(V_2, N, \psi) = N \rightarrow N^{\oplus 2} \rightarrow N$$

$$(21)$$

where the differentials on the bottom are $n \mapsto (vn, un)$ and $(n, n') \mapsto un - vn'$. The middle vertical map sends m to (xm, ym) and the right vertical map is just the standard inclusion. In N we have $(xu - yv)m = \varphi m$ so the square commutes.

The upper complex is the Koszul complex for M with the action of φ , whereas the lower complex is the Koszul complex for N with its action of u and v.

Theorem 6.9 (Koszul Quasiisomorphism theorem). Suppose M is an \mathcal{O}_D -module with action of $\varphi: M \to M$ as above. We suppose that M has no elements annihilated by both x and y, for example suppose it has no sections supported in dimension 0. Then the vertical map of complexes (21) is a quasiisomorphism.

Proof. Filter N by submodules $N_{\leq a}$ where $N_{\leq a}$ is the image of the polynomials of degree $\leq a$ in u, v, under the surjection $M[u, v] \to N$. Each Koszul complex is turned into a filtered complex by shifting the filtrations so that the differential precisely preserves the filtrations.

To be precise, let $C^0 \to C^1 \to C^2$ denote the lower complex $\mathbf{Kosz}(V_2, N, \psi)$ and define the increasing filtration $U.C^{\cdot}$ by

$$U_a C^0 := N_{\leq a}, \quad U_a C^1 := (N_{\leq a+1})^{\oplus 2}, \quad U_a C^2 := N_{\leq a+2}.$$

Let $D^1 \to D^2$ denote the upper complex $\mathbf{Kosz}(V_1, M, \varphi)$ and define

$$U_{-2}D^1 := 0, \quad U_{-1}D^1 := M,$$

and

$$U_{-3}D^2 := 0, \quad U_{-2}D^2 := M.$$

The associated-graded of the complex D is

$$\operatorname{Gr}_{-1}(D^{\cdot}) = 0 \to M \to 0,$$

$$\operatorname{Gr}_{-2}(D^{\cdot}) = 0 \to 0 \to M.$$

In all, the associated-graded complex is isomorphic to D^{\cdot} itself but with zero as differential. We claim that the associated-graded of the complex C^{\cdot} is isomorphic to the complex

$$\tilde{N} \to \tilde{N}^{\oplus 2} \to \tilde{N}$$

where

$$\tilde{N} := M[u, v]/(xu - yv)M[u, v]$$

and the differentials are defined in the same way as before.

It suffices to see that on each piece, let us say on C^0 for example.

Consider the exact sequence

$$M[u,v] \stackrel{xu-yv-\varphi}{\longrightarrow} M[u,v] \to N \to 0.$$

Define filtrations on the pieces by setting $U_aM[u,v]$ equal to the polynomials of degree $\leq a$ in u,v, in the middle; on the left, shift by one, using the filtration $U_a':=U_{a-1}$. On the right $N_{\leq a}$ is the image of U_a from the middle. We claim that the exact sequence of associated-graded pieces is still exact. On the right, the map is clearly surjective because of the definition of the filtration on N. In the middle, suppose we have an element $f \in U_aM[u,v]$ which maps to zero in $N_{\leq a}/N_{\leq a-1}$. This means that $f \in (xu - yv - \varphi)M[u,v] + U_{a-1}M[u,v]$. We would like to show that $f \in (xu - yv - \varphi)U_{a-1}M[u,v]$ (it is somewhat similar to the proof of the tensor formula (Proposition 6.8) in the previous section). Write

$$f = (xu - yv - \varphi)g + h$$

with $g \in M[u,v]$ and $h \in U_{a-1}M[u,v]$. Suppose $g \in U_cM[u,v]$ and that c is the smallest such, so the projection of g in U_c/U_{c-1} is nonzero. We assume c > a-1 and would like to deduce a contradiction. Write

$$g = \sum_{k+l \le c} g^{kl} u^k v^l$$

with $g^{kl} \in M$. Some g^{kl} is nonzero for a k+l=c.

Our situation is that xyM = 0, and if xm = 0 and ym = 0 then m = 0. It follows that the map

$$M \to (M/xM)/\mathrm{tors} \oplus (M/yM)/\mathrm{tors}$$

is injective, where "tors" denotes the x and y torsion. Indeed, if m is such that its projection into M/xM and M/yM are torsion, then $x^nm \in yM$ and writing $x^nm = ym'$ we have $x^{n+1}m = 0$; similarly in the other direction we get some $y^{n+1}m = 0$, but then m = 0 by the hypothesis.

We may therefore assume for example that the projection of some g^{kl} into $M_1 := (M/xM)/\text{tors}$ is nonzero.

Now the projection of f into

$$U_{c+1}M_1[u,v]/U_cM_1[u,v]$$

(extending the notation U. in the natural way) is equal to the projection of -yvg here. Indeed, xug projects to zero and also h projects to zero because it is in U_{a-1} and a-1 < c. However, the projection of yg^{kl} is nonzero because, by hypothesis M_1 contains no elements annihilated by y. Therefore, the projection of f into the above graded quotient of level c+1 is nonzero. This contradicts the hypothesis that $f \in U_aM[u,v]$ but c+1>a.

This proves the claim, which says that the sequence of associated-graded pieces is still exact. This claim says, in other terms, that

$$\operatorname{Gr}(C^0) \cong \tilde{N}.$$

Similarly for the other terms of the complex with the appropriate shift of indices. We obtain the statement that the complex Gr(C) is just the same Koszul complex but for the module \tilde{N} constructed using $\varphi = 0$.

Give the upper Koszul complex $\mathbf{Kosz}(V_1, M, \varphi)$ a filtration compatibly with the lower one $\mathbf{Kosz}(V_2, N, \psi)$, so that the differential vanishes on the associated-graded pieces. The associated-graded of this complex is just the same complex but with zero as differential: $M \stackrel{0}{\to} M$. To show a quasiisomorphism, it suffices to show a filtered quasiisomorphism.

We have seen above that the associated-graded complexes are the same ones, but for the endomorphism $\varphi = 0$. This reduces the theorem to the case $\varphi = 0$. That will be the subject of the calculations in the next section.

Note that we need to consider all possible module types for M. This amounts to looking at $M = \mathcal{O}_D$ and $M = \mathcal{O}_D/x\mathcal{O}_D$ and $M = \mathcal{O}_D/y\mathcal{O}_D$. These calculations, done in the next section, will complete the proof of the theorem.

6.5 Calculation

We now finish the calculations needed for the above proof. Since the question is local, we can assume we are in a simplified global situation of an affine variety consisting of two crossed lines in \mathbb{A}^2 . Thus, work with the coordinate ring:

$$A := \mathbb{C}[x, y]/(xy)$$

of our variety $D \subset \mathbb{A}^2$, and the algebra over it:

$$N := A[u, v]/(xu - vy).$$

Our module M is just M = A. We set up the Koszul complex K with respect to u, v:

$$N \to N^2 \to N$$
.

The first map sends 1 to $ve_1 + ue_2$, and the second map sends e_1 to u and e_2 to -v. The question is to calculate the cohomology of K.

We will use the grading:

$$N = \bigoplus_{i=0}^{\infty} N_i,$$

where N_i is the A-submodule of N involving monomials of degree i in u, v. The Koszul sequence K is now the direct sum of graded pieces K_i :

$$0 \to N_{i-1} \to N_i^{\oplus 2} \to N_{i+1} \to 0.$$

Proposition 6.10. For $i \geq 1$, the Koszul complex K is exact.

Proof. For exactness on the ends, note that the two individual maps

$$N_{i-1} \to N_i$$

given by multiplication with u, v respectively are both injective.

- —Hence $N_{i-1} \to N_i^{\oplus 2}$ is injective, so the cohomology on left vanishes.
- —Every monomial in N_{i+1} is the product of either u or v with a monomial in N_i , so the cohomology on the right vanishes.

The following lemma says that K_i has no cohomology in the middle either.

Lemma 6.11. The map $N_{i-1} \to \ker(N_i^{\oplus 2} \to N_{i+1})$ given by multiplication by (v, u) is an isomorphism.

Proof. The claim is essentially combinatorial - we check it by matching where monomials go. With z := xu = yv, N_i has a monomial basis consisting of 4 blocks:

$$u^{i}, u^{i-1}v, \dots, v^{i}$$

$$yu^{i}, zu^{i-1}, zu^{i-2}v, \dots, zv^{i-1}, xv^{i}$$

$$y^{2}u^{i}, y^{3}u^{i}, \dots$$

$$x^{2}v^{i}, x^{3}v^{i}, \dots$$

Multiplication by either u or v sends each block to the corresponding block of N_{i+1} , so it suffices to check the claim on each of the four blocks separately. For each of the first two blocks, the two multiplication maps are injective, and differ from each other by a shift of one place, so pairs in the kernel are precisely the images of basis elements in the corresponding two blocks of N_{i-1} . For basis elements in the third block, multiplication by v vanishes, while multiplication by v sends them bijectively to basis elements in the third block of v of v in the kernel again consists of the basis elements in the third block of v in the kernel iff v iff there is some v in this third block sends v in the hard v in the kernel iff v in the kernel if v in the kernel iff v in the kernel if v in the kernel iff v in the kernel iff v in the kernel if v in the kernel if

This completes the proof of the proposition.

This leaves the sequences K_i for i = 0 and i = -1. For i = 0 the sequence is

$$0 \to A^{\oplus 2} \to N_1 \to 0.$$

This is clearly surjective (everything in N_1 is divisible by u or v or both), and the kernel is the submodule of $A^{\oplus 2}$ generated (over A) by (x, y). So the cohomology is isomorphic to A, in the middle.

Finally, for i = -1, the sequence is $0 \to 0 \to A$, so the cohomology is A on the right.

So this confirms that the cohomology is A, that is to say O_D , occurring in two places, thus completing the proof of Therem 6.9 in this case.

We need to consider one additional case. The ring A is, as before,

$$A := \mathbb{C}[x, y]/(xy)$$

but now we also have the A-module:

$$M := A/(x) = \mathbb{C}[x,y]/(x) = \mathbb{C}[y]$$

(on which x acts as 0), and we set:

$$N := M[u, v]/(xu - yv)M = M[u, v]/(yv)M = \mathbb{C}[y, u, v]/(yv)$$

As before, we set up the Koszul complex K with respect to u, v:

$$N \to N^2 \to N$$

We let e_1, e_2 be the generators of the two copies of N in the middle. The first map sends 1 to $-ve_1 + ue_2$, and the second map sends e_1 to u and e_2 to v. The question is to calculate its cohomology. Again, we use the grading:

$$N = \bigoplus_{i=0}^{\infty} N_i,$$

where N_i is the A-submodule of N involving monomials of degree i in u, v. The Koszul sequence K is again the direct sum of graded pieces K_i :

$$0 \to N_{i-1} \to N_i^{\oplus 2} \to N_{i+1} \to 0.$$

For $i \geq 1$, we see that K_i is exact. More precisely:

• of the two individual maps

$$N_{i-1} \to N_i$$

given by multiplication with u, v respectively, the first is injective, the second is not.

- hence $N_{i-1} \to N_i^{\oplus 2}$ is still injective, so no cohomology on left.
- every monomial in N_{i+1} is the product of either u or v with a monomial in N_i , so there is no cohomology on the right.
- by matching where monomials go we see that the kernel of $N_i^{\oplus 2} \to N_{i+1}$ is generated as a vector space by: $-vu^{i-1}e_1+u^ie_2, -v^2u^{i-2}e_1+vu^{i-1}e_2, \dots, -v^ie_1+v^{i-1}ue_2, yu^ie_2, y^2u^ie_2, y^3u^ie_2, \dots,$ so as an M-module it is generated by $-vu^{i-1}e_1+u^ie_2, -v^2u^{i-2}e_1+vu^{i-1}e_2, \dots, -v^ie_1+v^{i-1}ue_2$, and all these generators are clearly in the image of N_{i-1} . So no cohomology in the middle either.
- This leaves the sequences K_i for i = 0 and i = -1. For i = 0 the sequence is

$$0 \to M^{\oplus 2} \to N_1 \to 0.$$

This is clearly surjective (everything in N_1 is divisible by u or v), and the kernel is the submodule yMe_2 generated (over M) by ye_2 . So the cohomology is isomorphic to M, in the middle.

• Finally, for i = -1, the sequence is $0 \to 0 \to M$, so the cohomology is M on the right.

This completes the calculation of the Koszul cohomology and shows that the two complexes are quasi isomorphic as claimed, thus completing the proof of Theorem 6.9.

6.6 Completion of proofs

In this section we go back and see how this result leads to the statement we were originally looking for.

In the previous section we have completed the proof of the Koszul quasiisomorphism theorem 6.9.

The Tensor product formula of Theorem 6.8 tells us that if we set M equal to the module $\psi_{b,b}$ then Ψ_{b-1} corresponds to the module denoted by N in Theorem 6.9.

By (19) the Koszul complex for M is the same as the complex denoted $Q(b)/Q(b-\epsilon)$, on the left of the morphism (13) above Proposition 5.13. Similarly by (20), the Koszul complex for N is the same as the complex denoted $K(b)/K(b-\epsilon)$ on the right of the morphism (13)) above Proposition 5.13.

The map of Koszul complexes is the same as the map $gr_a(u_0)$ in (13) (setting a = b). Therefore, the Koszul quasiisomorphism theorem 6.9 tells us the quasiisomorphism asked for in Proposition 5.13. From that proposition, we now obtain the proof of Theorem 5.9.

Now Corollary 5.10 gives the proof of Theorem 4.6, and in turn Proposition 4.8 gives the proof of the main Theorem 3.6.

7 Further considerations

In this section we look in more detail at several aspects. The first two subsections provide some details on the proofs of parts 3 and 5 of Theorem 3.6.

We start by looking at the absolute Dolbeault complex on X and use it to define the Gauss-Manin Higgs field on the higher direct images. Next we look at the analytical aspects of the direct image harmonic bundle. This includes a sketch of our original strategy for proving the theorem, involving the study of the family of L^2 cohomology spaces.

Then in the last two subsections we explore generalizations to higher dimensional cases.

7.1 The absolute complex and Gauss-Manin

Recall that the Gauss-Manin connection on the relative algebraic de Rham cohomology of a vector bundle with connection, comes from an exact sequence of complexes. Whereas the higher direct image bundle is defined using the relative de Rham complex, the exact sequence needed to define the Gauss-Manin connection uses the absolute de Rham complex. Similarly, in our case, in order to construct the Higgs field on the parabolic bundle \underline{F}^i , we should look at the absolute Dolbeault complex on X.

In order to prepare for the generalized situation to be considered in subsection 7.3 later, let us consider the case when (X, D) is of arbitrary dimension and let $f: (X, D) \to (Y, Q)$ be a smooth split semistable family of curves (see Definition 7.3). In this case we define the absolute L^2 Dolbeault complex as follows. For every $i \ge 0$ we have a short exact sequences of forms on X:

$$0 \longrightarrow f^*\Omega^i_Y(\log Q) \longrightarrow \Omega^i_X(\log D) \longrightarrow \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^{i-1}_Y(\log Q) \longrightarrow 0,$$

where $\Omega^1_{X/Y}(\log D) = \Omega^1_X(\log D)/f^*\Omega^1_Y(\log Q)$ is the relative logarithmic dualizing sheaf of

f. Consider the tensor product of this sequence with $W_0(H, E_{\alpha(a)})$:

$$\begin{array}{c}
0\\
\downarrow\\W_0(H,E_{\alpha(a)})\otimes f^*\Omega^i_Y(\log Q)\\
\downarrow\\W_0(H,E_{\alpha(a)})\otimes \Omega^i_X(\log D)\\
\downarrow\\W_0(H,E_{\alpha(a)})\otimes \Omega^1_{X/Y}(\log D)\otimes f^*\Omega^{i-1}_Y(\log Q)\\
\downarrow\\0
\end{array}$$

and pull it back by the natural map

$$W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^{i-1}_Y(\log Q) \to W_0(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^{i-1}_Y(\log Q).$$

This gives us a new object

$$W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_X^i(\log D)\right)$$

which fits into an extension:

$$\begin{array}{c} 0 \\ \downarrow \\ W_0(H,E_{\alpha(a)}) \otimes f^*\Omega^i_Y(\log Q) & \longrightarrow W_0(H,E_{\alpha(a)}) \otimes f^*\Omega^i_Y(\log Q) \\ \downarrow \\ W_{-2,0}\left(H,E_{\alpha(a)} \otimes \Omega^i_X(\log D)\right) & \longrightarrow W_0(H,E_{\alpha(a)}) \otimes \Omega^i_X(\log D) \\ \downarrow \\ W_{-2}(H,E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^{i-1}_Y(\log Q) & \longrightarrow W_0(H,E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^{i-1}_Y(\log Q) \\ \downarrow \\ 0 & \downarrow \\ 0 & 0 \end{array}$$

and we get a well defined absolute complex

$$\operatorname{DOL}_{L^{2}}^{\operatorname{par}}(X, E_{\alpha(a)}) := \begin{bmatrix} W_{0}(H, E_{\alpha(a)}) & 0 \\ \downarrow \wedge \varphi & \\ W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_{X}^{1}(\log D)\right) & 1 \\ \downarrow \wedge \varphi & \\ \vdots & \\ \downarrow \wedge \varphi & \\ \vdots & \\ \downarrow \wedge \varphi & \\ W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_{X}^{2}(\log D)\right) & \vdots \\ \downarrow \wedge \varphi & \\ W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_{X}^{d_{X}}(\log D)\right) & d_{X} \end{bmatrix}$$

This absolute complex maps naturally onto the vertical Dolbeault L^2 complex

$$\mathrm{DOL}^{\mathrm{par}}_{L^{2}}(X/Y, E_{\alpha(a)}) := \left[\begin{array}{c} W_{0}(H, E_{\alpha(a)}) \\ \downarrow \wedge \varphi \\ W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^{1}_{X/Y}(\log D) \end{array} \right] \quad 0$$

that we previously considered.

Using this map we get a short exact sequence of complexes

$$0 \qquad (22)$$

$$DOL_{L^{2}}^{par}(X/Y, E_{\alpha(a)})[-1] \otimes f^{*}\Omega_{Y}^{1}(\log Q)$$

$$DOL_{L^{2}}^{par}(X, E_{\alpha(a)})/I^{2}(E_{\alpha(a)})$$

$$DOL_{L^{2}}^{par}(X/Y, E_{\alpha(a)})$$

$$0$$

where as usual the subcomplexes $I^k(E_{\alpha(a)})$ are defined inductively:

$$I^{0}(E_{\alpha(a)}) = \mathrm{DOL}_{L^{2}}^{\mathrm{par}}(E_{\alpha(a)})$$

$$I^{k+1}(E_{\alpha(a)}) = \mathrm{image} \left[I^{k}(E_{\alpha}) \otimes f^{*}\Omega_{Y}^{1}(\log Q) \longrightarrow \mathrm{DOL}_{L^{2}}^{\mathrm{par}}(E_{\alpha(a)}) \right].$$

By the usual construction [20] the push forward of this sequence by f yields a connecting homomorphism

$$\mathbb{R}^{i} f_{*} \mathrm{DOL}^{\mathrm{par}}_{L^{2}}(X/Y, E_{\alpha(a)}) \xrightarrow{\theta} \mathbb{R}^{i+1} f_{*} \left(\mathrm{DOL}^{\mathrm{par}}_{L^{2}}(X/Y, E_{\alpha(a)})[-1] \otimes \Omega^{1}_{Y}(\log Q) \right)$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\underline{F}^{i}_{a} \otimes f^{*} \Omega^{1}_{Y}(\log Q)$$

which is a tame Higgs field on the parabolic bundle $\underline{F}^{i}_{\bullet}$.

7.2 Analytic considerations

This subsection treats the analytic family of L^2 cohomology spaces. These considerations are certainly present in the theory of Saito-Sabbah-Mochizuki used above, but they are used in a somewhat roundabout way: Saito and Sabbah used adapted versions of Zucker's theory [23] in order to take the higher direct image along a family of curves.

Our original approach to our question was to look at the L^2 metric on cohomology and give some estimates (to be described below) on the order of growth of holomorphic sections of the higher direct image bundles F_a^1 . Using Poincaré duality can give estimates in the other direction. This way of thinking can almost lead to a proof of the main theorem. However, we need to know the local freeness of the higher direct images, which is part 1 of Theorem 3.6. This local freeness is, fundamentally speaking, a consequence of strictness for mixed Hodge or twistor structures. That strictness is encapsulated in the theory of Saito-Sabbah-Mochizuki. Once we know local freeness, one can either appeal to the full statement of Sabbah's theorem, as we have done to identify F_a^1 as the parabolic structure associated to the higher direct image local system, or alternatively the analytic considerations treated in this subsection can also give that identification. In either case, we still need the present subsection in order to get part 5 of Theorem 3.6 about the metric.

Choose a Kähler metric on X-D that has the local behavior of the Poincaré metric along the components of D, and of the product of Poincaré metrics at crossing points. For each $y \in Y-Q$ we get a quasiprojective curve $X_y^o := X_y - D_{H,y}$ and the induced metric is equivalent to the Poincaré metric at the puncture points (i.e. the points of $D_{H,y}$).

Now, if $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', h)$ is a harmonic bundle on X-D we obtain its restriction $(\mathcal{L}_y, \mathbf{D}'_y, \mathbf{D}''_y, h)$ to X_y^o . Define

$$\mathbf{Har}^1(X_y^o, \mathcal{L}_y) \subset A^1(X_y^o, \mathcal{L}_y)$$

to be the space of harmonic 1-forms with coefficients in \mathcal{L}_y . These are the forms in the kernel of the Laplacian. Recall [19] that the \mathbf{D}_y -laplacian coincides with twice the \mathbf{D}_y' or \mathbf{D}_y'' laplacians.

The analogue of Zucker's theory [23], for which we may refer to [17, Section 6.2], tells us that this space is naturally isomorphic to the cohomology group $H^1(X_y, j_{y,*}(L_y))$ where $j_y: X_y^o \hookrightarrow X_y$ is the inclusion and $L_y:=\mathcal{L}_y^{D_y}$ is the restriction of our global local system to X_y^o . Recall that in this case of sheaves on curves, $j_{y,*}(L_y)$ is the middle perversity extension and $H^1(X_y, j_{y,*}(L_y))$ is the middle perversity intersection cohomology group of L_y with respect to the compactification X_y of X_y^o .

These cohomology spaces vary in a local system, in particular they have the same dimension. We use without proof the corollary that the family of spaces of harmonic forms fits together into a \mathcal{C}^{∞} vector bundle denoted

$$\mathcal{H} = \mathbf{Har}^1((X-D)/(Y-Q), \mathcal{L})$$

over Y - Q, whose fibers are the $\mathbf{Har}^1(X_y^o, \mathcal{L}_y)$.

Furthermore, we also use without proof, the extension of the calculations in [20], that were for the case of compact fibers, showing that the naturally defined operators $\mathbf{D}'_{\mathcal{H}}, \mathbf{D}''_{\mathcal{H}}, \mathbf{D}_{\mathcal{H}} = \mathbf{D}'_{\mathcal{H}} + \mathbf{D}''_{\mathcal{H}}$, and the L^2 metric $h_{\mathcal{H}}$ (obtained by using the L^2 inner product on each fiber $\mathcal{H}_y = \mathbf{Har}^1(X_y^o, \mathcal{L}_y)$) are \mathcal{C}^{∞} and fit together to give a structure of harmonic bundle on \mathcal{H} over Y - Q. The underlying flat bundle of this harmonic bundle is $R^1 f_*(j_* L)$ where here $j: X - D \hookrightarrow X - D_V$ is the inclusion into the partial compactification with the horizontal divisors.

Another way of saying the previous paragraph is that the L^2 metric on the fibers of the local system $R^1f_*(j_*L)$, obtained from the identifications of these fibers with the $\mathbf{Har}^1(X_y^o, \mathcal{L}_y)$, is a harmonic metric over Y - Q.

We would like to understand the asymptotic behavior of the harmonic bundle $(\mathcal{H}, \mathbf{D}'_{\mathcal{H}}, \mathbf{D}''_{\mathcal{H}}, h_{\mathcal{H}})$ near a point $q \in Q$.

Proposition 7.1. This harmonic bundle is a tame harmonic bundle corresponding to a filtered local system with trivial filtrations in the terminology of [18] applied to the curve Y - Q.

Proof. Choose $q \in Q$ and consider a coordinate t on a neighborhood of q with t(q) = 0. Along the ray $t \in \mathbb{R}_{>0}$ suppose we are given a family of cohomology classes $\overline{\xi}(t) \in H^1(X_t, j_{t,*}(L_t))$.

We would like to estimate the function

$$t \mapsto \|\overline{\xi}(t)\|_{h_{\mathcal{H}}(t)}.$$

The norm of the cohomology class is by definition the norm of its harmonic representative, and this is the minimum of the norms of all representatives. Thus, given a family of representatives $\xi(t)$ for the classes $\overline{\xi}(t)$, we get

$$|\overline{\xi}(t)|_{h_{\mathcal{H}}(t)} \le ||\xi(t)||_{L^2, X_t^o}.$$

We get a family of representatives by choosing a lift of the radial vector field over the ray, into X, and flowing a representative on the fiber over t = 1 towards the singular fiber. The norm on the local system has sub-polynomial growth as we approach the divisor, and a standard choice of vector field satisfies a sub-polynomial estimate just as in the case of scalar coefficients. These calculations (which are not detailed here) give

$$\|\xi(t)\|_{L^2, X_t^o} \le C_{\epsilon} t^{-\epsilon}$$

for any $\epsilon > 0$, so

$$|\overline{\xi}(t)|_{h_{\mathcal{H}}(t)} \le C_{\epsilon} t^{-\epsilon}.$$

This says that the norms of our cohomology classes have sub-polynomial growth.

Poincaré duality for intersection cohomology says that starting from the dual local system L^* leads to the dual vector bundle \mathcal{H}^* , and the resulting L^2 metric on \mathcal{H}^* is the dual of $h_{\mathcal{H}}$. The same estimate holds for the dual, so we get that flat sections of \mathcal{H}^* also have sub-polynomial growth. We conclude that the harmonic bundle $(\mathcal{H}, \mathbf{D}'_{\mathcal{H}}, \mathbf{D}''_{\mathcal{H}}, h_{\mathcal{H}})$ is tame [18].

This tame harmonic bundle corresponds to a parabolic Higgs bundle $\underline{G}^1 = \{G_a^1\}$ on Y, with parabolic structure on the divisor Q. Theorem 3.6 identifies this parabolic structure bundle with the parabolic bundle $\underline{F}^1 = \{F_a^1\}$ obtained by higher direct image of the L^2 parabolic Dolbeault complex. Our proof used Sabbah's theory in full to identify \underline{G}^1 . We indicate here a different proof of part of that, the present proof being useful in order to fix the identification as stated in part 5 of Theorem 3.6.

The fact that $\Delta_{\mathbf{D}_y} = 2\Delta_{\mathbf{D}_y''}$ gives an isomorphism between L^2 Dolbeault cohomology and L^2 de Rham cohomology on each fiber. The analogue of Zucker's theory for our case [17, 23]

tells us that this L^2 cohomology is the same as the hypercohomology of $\mathrm{DOL}_{L^2}^{\mathrm{par}}(X/Y,\underline{E})$ on each of the fibers (as discussed in section 3.2). In other words, we get an isomorphism

$$\mathcal{H}_y \cong F_a^1(y)$$

for any $y \in Y - Q$, and (as usual, by some analytic considerations that we don't treat here) these fit together to give an isomorphism of \mathcal{C}^{∞} bundles

$$\mathcal{H} \cong F_a^1|_{Y-Q}. \tag{23}$$

The holomorphic structure $\overline{\partial}_{\mathcal{H}}$ (the (0,1) component of $\mathbf{D}''_{\mathcal{H}}$) corresponds to the holomorphic structure of $F_a^1|_{Y=Q}$. The Higgs field $\varphi_{\mathcal{H}}$ corresponds to the Higgs field on F_a^1 given by the Gauss-Manin construction with the absolute Dolbeault complex discussed in the previous subsection.

On the other hand, $(\mathcal{H}, \mathbf{D}''_{\mathcal{H}})$ is also isomorphic to $G_a^1|_{Y-Q}$ since the latter is by definition the parabolic bundle associated to $(\mathcal{H}, \mathbf{D}'_{\mathcal{H}}, \mathbf{D}''_{\mathcal{H}}, h_{\mathcal{H}})$. Therefore

$$F_a^1|_{Y-Q} \cong G_a^1|_{Y-Q}$$

as holomorphic Higgs bundles on Y-Q. We would like to show that this extends to an isomorphism of parabolic bundles, in other words $F_a^1 \cong G_a^1$.

In the parabolic Higgs bundle associated to the harmonic bundle the piece G_a^1 of parabolic weight a at a point $q \in Q$ is the sheaf of holomorphic sections whose norm is less than $|z|^{-a-\epsilon}$ for any $\epsilon > 0$. The identification $F_a^1 \cong G_a^1$ is therefore equivalent to the following statement.

Theorem 7.2. A holomorphic section of $F_a^1|_{Y=Q}$ in a neighborhood of $q \in Q$ is in F_a^1 if and only if the section of \mathcal{H} corresponding to it by (23), has norm bounded by $|z|^{-a-\epsilon}$ for any $\epsilon > 0$.

Proof. Recall that the parabolic structure on the bundle E along D_V is characterized by a similar norm estimate. Using this together with the definition of F_a^i and a local estimate for the size of forms on the degenerating curves X_y as $y \to q$ (the same as in the constant coefficient case, see [5] for example) we conclude one direction of the statement: any section of F_a^1 has norm bounded by $|z|^{-a-\epsilon}$ for any $\epsilon > 0$. (In fact with more work one could obtain a more precise estimate of the form $|z|^{-a}|\log|z||^k$ for some k.) This proves that

$$F_a^1 \subset G_a^1$$
.

One way to conclude from here would be to calculate by Riemann-Roch the parabolic degree of \underline{F} . That will of course turn out to be 0, and since \underline{G} also has parabolic degree 0 these imply that the two parabolic structures are the same.

One may alternatively proceed, as in the previous proof of tameness, by using Poincaré duality. Let ${}^{\vee}E$ denote the dual bundle assoicated to the dual local system ${}^{\vee}L$, and let ${}^{\vee}H$, ${}^{\vee}F_a^1$ and ${}^{\vee}G_a^1$ be the resulting objects. Intersection cohomology and L^2 cohomology of ${}^{\vee}E$ on the fibers X_y are compatible with duality, so the pairing

$$(\eta,\xi)\mapsto \int_X \eta \wedge \xi$$

induces a perfect pairing

$$\mathcal{H} \times {}^{\vee}\mathcal{H} \to \mathcal{C}^{\infty}_{Y-O}$$
.

It is holomorphic in the Dolbeault realization, and by looking at parabolic growth rates we get a pairing

$$I_G: G_a^1 \otimes {}^{\vee}G_{-a}^1 \to \mathcal{O}_Y.$$
 (24)

This gives a morphism of parabolic sheaves

$$\underline{G}^1 \to ({}^{\vee}\underline{G})^{\vee}$$

that is an isomorphism over Y_Q . Since both have degree 0 it follows that it is an isomorphism of parabolic bundles, in other words the pairing (24) is a perfect pairing of vector bundles on Y for any a.

The parabolic bundle ${}^{\vee}\underline{E}$ is dual to \underline{E} , and this extends to the L^2 Dolbeault complex. For that, it is convenient to use the alternate version of the L^2 Dolbeault complex built using terms W_1 and W_{-1} instead of W_0 and W_{-2} . These two versions are quasiisomorphic, as Zucker observed [23] (the same reasoning holds in the twistor case). We get

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)}) \cong \mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, {}^{\vee}\!E_{\alpha(-a)})^{\vee} \otimes \omega_{X/Y}[-1].$$

Now, duality for the morphism $f: x \to Y$ gives a perfect pairing

$$I_F: F_a^1 \otimes {}^{\vee}F_{-a}^1 \to \mathcal{O}_Y.$$

Here is where we appeal to the results of the calculations in the main part of the paper, that show part 1 of Theorem 3.6: the higher direct image sheaves $F_a^1 \otimes {}^{\vee}F_{-a}^1$ are bundles, so the duality pairing is a perfect pairing of locally free sheaves.

These pairings coincide over Y-Q. Thus, the inclusions $F_a^1 \subset G_a^1$ and ${}^{\vee}F_a^1 \subset {}^{\vee}G_a^1$ give a commutative diagram

$$\begin{array}{cccc} F_a^1 \otimes {}^{\vee}\!F_{-a}^1 & \stackrel{I_F}{\to} & \mathcal{O}_Y \\ \downarrow & & \downarrow & \\ G_a^1 \otimes {}^{\vee}\!G_{-a}^1 & \stackrel{I_G}{\to} & \mathcal{O}_Y \end{array}$$

with perfect pairings on the top and the bottom. It now follows that $F_a^1 \to G_a^1$ is an isomorphism. We show that it induces an injection on fibers over a point $q \in Q$. If η is a section of F_a^1 nonvanishing at q, then there is a section ξ of F_{-a}^1 such that $I_F(\eta \otimes \xi)(q) \neq 0$. But if η maps to a section of G_a^1 vanishing at q it would imply that $I_G(\eta \otimes \xi)(q) = 0$, contradicting the commutativity of the diagram. This shows that $F_a^1(q) \hookrightarrow G_a^1(q)$. This holds at all points of Q. Since both bundles have the same rank and the map is an isomorphism on Y - Q this shows that it is an isomorphism over all of Y. This completes the proof of the theorem.

In conclusion, the above proof provides our basic compatibility of Theorem 3.6, provided we know that the higher direct image sheaves F_a^1 are locally free.

We didn't see how to prove this local freeness property in general. One could plan to use a strategy based on Steenbrink's argument [22]. It would take place in explicit normal crossings situations using a double complex whose terms come from multiple intersections of the divisor components. It should be possible to develop Steenbrink's approach for twistor connections, using the strictness property of mixed twistor structures, but we didn't do that. Indeed, Steenbrink's argument was never developed to its full potential, because the advent of Saito's theory of Hodge modules provided a very general and more powerful method. We have taken that route by appealing to [17] for the proof of local freeness of the F_a^1 , but then we also get the calculation of the higher direct image as part of the same package.

7.3 Semistable families over higher dimensional base

In this section we take note that our main theorems lead rather directly to the corresponding statements in the case of families of curves over a higher dimensional base.

Definition 7.3. We say that a morphism $f:(X,D)\to (Y,Q)$ is a split smooth semistable family of curves if:

(1) X and Y are smooth projective varieties,

- (2) D and Q are reduced divisors with simple normal crossings,
- (3) all the fibers of f are reduced curves,
- (4) we have a decomposition $D = D_V + D_H$ where $D_V = f^{-1}(Q)$ and D_H is a disjoint union of components mapping locally isomorphically to Y,
- (5) the map f is smooth away from D_V , and
- (6) for $y \in Q$, the curve $X_y := f^{-1}(y)$ is a semistable curve with only nodes. (It follows that the nodes are distinct from the points marked by the components of D_H .)

Suppose $f:(X,D)\to (Y,Q)$ is a split smooth semistable family. We use notational conventions analogous to those in effect up until now.

Suppose L is a local system on X-D such that the eigenvalues of the monodromy around all components of D are in $S^1 \subset \mathbb{C}^{\times}$. Define as before the associated harmonic bundle $(\mathcal{L}, \mathbf{D}', \mathbf{D}'', h)$, the associated parabolic bundle with λ -connection $\underline{\mathscr{E}} = \{\mathscr{E}_{\beta}\}$, and the associated $\mathscr{R}_{\mathscr{X}}$ -module \mathscr{E} . Suppose a is a parabolic weight for the divisor Q, and define the parabolic weight $\alpha(a)$ by associating the weight a_i to any component D_j of D_V mapping to the component Q_i of Q. The weight of $\alpha(a)$ associated to components of D_H is 0. The 0-th associated graded pieces along the horizontal divisor components $D_{h(j)} \subset D_H$ are defined as previously, and they vary in a locally constant family by [10]. Define the L^2 parabolic de Rham and Dolbeault complexes

$$DR_{L^2}^{par}(\mathscr{X}/\mathscr{Y};\mathscr{E}_{\alpha(a)}), \quad DOL_{L^2}^{par}(X/Y;E_{\alpha(a)})$$

by the same formulas (3) and (2) using the weight filtrations on horizontal complexes as before.

On the other hand, we have the \mathcal{R} -module de Rham complex

$$DR(\mathcal{X}/\mathcal{Y}; g_{+}(\mathcal{E}))$$

defined as before using the graph embedding $g: \mathscr{X} \to \mathscr{P} := \mathscr{X} \times_{\mathscr{A}} \mathscr{Y}$.

Theorem 7.4. In the above situation, the higher direct images

$$\mathscr{F}_a^i := \mathbb{R}^i f_* DR_{L^2}^{\mathrm{par}}(\mathscr{X}/\mathscr{Y};\mathscr{E}_{\alpha(a)})$$

are locally free on \mathscr{Y} and they fit together to form a parabolic vector bundle with λ -connection. Their restriction to $\lambda = 0$ is

$$\mathscr{F}_a^i(0) = F_a^i = \mathbb{R}^i f_* DOL_{L^2}^{\text{par}}(X/Y; E_{\alpha(a)})$$

and these fit together to form a parabolic Higgs bundle. These parabolic Higgs bundles and parabolic bundles with λ -connection are those associated to the local system G^i of middle perversity higher direct images of L to Y-Q. The higher direct image (under the projection $p: \mathscr{P} \to \mathscr{Y}$)

$$\mathscr{F}^i := \mathbb{R}^i p_* DR(\mathscr{X}/\mathscr{Y}; g_+(\mathscr{E}))$$

are strictly S-decomposable $\mathscr{R}_{\mathscr{Y}}$ -modules (see Remark 2.5) whose piece of strict support \mathscr{Y} is equal to the main chart of the pure twistor \mathcal{D} -module associated to G^i .

The identification of the theorem is functorial, in particular the component sheaves of the parabolic structure reflect the growth rate of the L^2 harmonic metric on the local system G^i .

Proof. If $C \subset Y$ is a curve immersed into Y and transverse to each boundary divisor Q_i individually, then $X \times_V C$ is a smooth surface mapping to C by a split smooth semistable map. Hence, the considerations of the main part of the paper apply. Notice that we may choose such curves passing through any multiple intersection of Q (since we only asked transversality to each component). We obtain that the ranks of the higher direct image sheaves \mathscr{F}_a^i or F_a^i are constant over Y. Since these are cohomology sheaves of perfect complexes it follows from semicontinuity that they are locally free, and the higher direct image is compatible with base change. Now, notice that we know from [10] that the local system G^i corresponds to a harmonic bundle, which in turn corresponds to a parabolic Higgs bundle and parabolic λ -connection. We have an identification with F_a^i or \mathscr{F}_a^i over Y-Q (resp. $\mathscr{Y}-\mathscr{Q}$). But now, these bundes over the open set have two extensions to parabolic bundles on Y with respect to Q, namely on one hand the F_a^i (resp. \mathscr{F}_a^i), and on the other hand the parabolic bundles associated to G^i . For any smooth embedded curve C transverse to the Q_i , the restrictions of the two parabolic bundles to C coincide. It follows that the two parabolic bundles are the same.

7.4 Higher dimensional families over a curve

One may similarly ask the question of how to generalize our result to the case of higher dimensional fibers. Note first of all that a higher dimensional map can be decomposed into a series of 1-dimensional fibrations, using alterations [1]. Therefore, in principle Theorem 7.4 can be applied inductively to obtain some approximation of the higher direct image. This is an approximation because there may be extra terms along the way coming from the birational transformations and finite coverings involved in making the required alterations.

It is therefore natural to ask for a global formula. The semistable reduction theorem of Abramovich and Karu in the case of higher dimensional base and higher dimensional fibers involves reduction to toric singularities [1]. It is an interesting and important question to understand how to calculate in this situation, but that would go way beyond the scope of the methods that we are discussing here.

We can, nonetheless, ask about the case of higher dimensional fibers over a 1-dimensional base. This case also presents a certain collection of difficulties, and we are not able to state a theorem about it at the present time. Let us review some of these difficulties and discuss what might be done.

Suppose first of all that there are no horizontal divisors, in other words the map $X - D \to Y - Q$ is proper. In this case, we don't need to consider the intersection cohomology or L^2 cohomology on the fibers. Suppose L is a local system on X - D whose monodromy eigenvalues are in $S^1 \subset \mathbb{C}^{\times}$, provided with trivial filtrations, and let (\underline{E}, φ) be the parabolic logarithmic Higgs bundle associated to it by [12]. We would like to obtain the parabolic Higgs bundle on Y associated to the local system $R^i f_*(L)$ on Y - Q.

Given a parabolic weight a (at a point $q \in Q$ in the base), let $\alpha(a)$ be the parabolic weight on D obtained by assigning a to each divisor component. We have a bundle $E_{\alpha(a)}$ on X with logarithmic Higgs field

$$E_{\alpha(a)} \stackrel{\varphi}{\to} E_{\alpha(a)} \otimes \Omega^1_X(\log D)$$

inducing the relative Higgs field with values in

$$\Omega^1_{X/Y}(\log D) := \Omega^1_X(\log D) / f^*\Omega^1_Y(\log Q).$$

The relative Dolbeault complex is

$$DOL(X/Y, E_{\alpha(a)}) := \left[\dots \xrightarrow{\varphi} E_{\alpha(a)} \otimes \Omega^{i}_{X/Y}(\log D) \xrightarrow{\varphi} \dots \right].$$

The generalization of Theorem 3.6 to this case should say that

$$F_a^i := \mathbb{R}^i f_* \mathrm{DOL}(X/Y, E_{\alpha(a)})$$

are locally free over Y, compatible with base change, and they fit together to form a parabolic Higgs bundle \underline{F}^i ; and this is the parabolic Higgs bundle associated to $R^i f_*(L)$.

In this case, the proof should in principle follow the same outline as what we have done here. One would need to identify the V-filtration for the graph embedding, in the case of a normal crossing of several divisors. This should be an algebraic problem similar to the one we have treated here for the crossing of two divisors, perhaps only requiring a more general notation.

The more difficult case is when there is a horizontal divisor with normal crossings. The first question is how to define the appropriate Dolbeault complex. It is natural to conjecture that it should be the complex consisting of holomorphic E-valued forms that are in L^2 (for the Poincaré metric near D), such that their derivatives are in L^2 . Jost and Zuo have proven this theorem in the case of variations of Hodge structure [9].

Next, it would be good to have an algebraic description in terms of weight filtrations. This is bound to be considerably more complicated than in the case of relative dimension 1, because there are several different weight filtrations that interact in a subtle way.

Assuming a good understanding of these issues, the resulting algebraic structures will then interact with the parabolic weights along the vertical divisors, at points of $D_H \cap D_V$. Identifying the V-filtration will be a significant question since normal crossing points of D_V can touch points (or even crossing points) of D_H . One of the crucial simplifications in our arguments of the present paper was that the crossing points of D_V were disjoint from D_H and the two aspects could be treated separately.

This rough overview highlights the difficulties that would be involved in obtaining directly a generalized formula for families of higher relative dimension.

The other path is to decompose a morphism f into a sequence of one-dimensional fibrations, by making alterations at each stage, and applying Theorem 7.4 inductively. This has the advantage of being currently accessible, however in practice it will require the undoubtedly complicated investigation of what happens under the alterations that occur in the middle of the process.

Note however that if we are in the special situation where the horizontal divisors are smooth over Y and meet only smooth points of the vertical divisors all these difficulties disapear and we can give an algebraic description of the L^2 Dolbeault complex. We give this

description next.

Suppose $f: X \to Y$ is a morphism between smooth projective varieties. Suppose also that $D \subset X$ and $Q \subset Y$ are reduced simple normal crossings divisors such that D decomposes as $D = D_H + D_V$, with $D_V = f^{-1}(Q)$ scheme theoretically, and that f is smooth away from D_V . Additionally we assume that each component of D_H is smooth over Y and that $D_H \cup D_V$ is contained in the smooth locus of D_V . Note that these assumptions in particular imply that D_H is a disjoint union of smooth connected components, i.e. D_H embeds in the normalization D^{ν} of D as a union of connected components of D^{ν} .

Let $(\underline{E}_{\bullet}, \varphi)$ be a tame parabolic Higgs bundle on (X, D) and let a be a parabolic level along Q. By definition a is an assignment of a real number to each irreducible component of Q. As before we will write $\alpha(a)$ for the parabolic level along D which assigns 0 to each horizontal component of D, while to a vertical component of D it assigns the value of a on the image of this vertical component under f. As before we consider the level $\alpha(a)$ representative $E_{\alpha(a)}$ of the parabolic bundle \underline{E}_{\bullet} and the horizontal weight filtration $W_{\ell}(H, E_{\alpha(a)})$ of $E_{\alpha(a)}$.

Explicitly, for each component $D_{h(j)}$ of D_H we have the associated graded bundle

$$\operatorname{Gr}_{h(j),0}(E_{\alpha(a)}) = E_{\alpha(a)}/E_{\alpha(a)-\epsilon\delta^{h(j)}}$$

and the induced nilpotent endomorphism $\operatorname{res}_{D_{h(j)}}\varphi$ of $\operatorname{Gr}_{h(j),0}(E_{\alpha(a)})$. This gives rise to an associated monodromy weight filtration $W_{\ell}\left(\operatorname{Gr}_{h(j),0}(E_{\alpha(a)})\right)$. Set

$$\operatorname{Gr}_{H,0}(E_{\alpha(a)}) = \bigoplus_{j} \operatorname{Gr}_{h(j),0}(E_{\alpha(a)}),$$

$$W_{\ell}(\operatorname{Gr}_{H,0}(E_{\alpha(a)})) = \bigoplus_{j} W_{\ell}\left(\operatorname{Gr}_{h(j),0}(E_{\alpha(a)})\right).$$

Since in our setup all $D_{h(j)}$ are disjoint, we can view $Gr_{H,0}(E_{\alpha(a)})$ as a torsion sheaf on X equipped with a surjective sheaf map

$$E_{\alpha(a)} \to \operatorname{Gr}_{H,0}(E_{\alpha(a)}).$$

Following the pattern in section 3.1 we define $W_{\ell}(H, E_{\alpha(a)})$ as the preimage of the monodromy weight filtration $W_{\ell}(Gr_{D_H,0})$ associated to the action of the nilpotent $res_{D_H}\varphi$.

For each form degree $i \ge 0$ we have⁵ a residue map

$$\operatorname{res}_{D_H}: \Omega_X^i(\log D) \longrightarrow \Omega_{D_H}^{i-1},$$

⁵The standard Poincare residue $\operatorname{res}_D: \Omega^i_X(\log D) \longrightarrow \Omega^{i-1}_{D^{\nu}}$ maps logarithmic *i*-forms on (X,D) to holomorphic i-1 forms on the normalization D^{ν} . Since $D_H \subset D^{\nu}$ we can compose res_D with the projection $\Omega^{i-1}_{D^{\nu}} \twoheadrightarrow \Omega^{i-1}_{D_H}$ to get a residue map res_{D_H} .

and after tensoring with $W_0(H, E_{\alpha(a)})$, a residue map

$$\operatorname{res}_{D_H}: W_0(H, E_{\alpha(a)}) \otimes \Omega_X^i(\log D) \longrightarrow W_0(H, E_{\alpha(a)})_{|D_H} \otimes \Omega_{D_H}^{i-1}.$$

Similarly, for every $i \geq 0$ we have a residue map on f-relative logarithmic forms

$$\operatorname{res}_{D_H}: W_0(H, E_{\alpha(a)}) \otimes \Omega^i_{X/Y}(\log D) \longrightarrow W_0(H, E_{\alpha(a)})_{|D_H} \otimes \Omega^{i-1}_{D_H/Y}.$$

Pulling back the subbundles

$$W_{-2}(H, E_{\alpha(a)})_{|D_H} \otimes \Omega_{D_H}^{i-1} \subset W_0(H, E_{\alpha(a)})_{|D_H} \otimes \Omega_{D_H}^{i-1}$$
$$W_{-2}(H, E_{\alpha(a)})_{|D_H} \otimes \Omega_{D_H/Y}^{i-1} \subset W_0(H, E_{\alpha(a)})_{|D_H} \otimes \Omega_{D_H/Y}^{i-1}$$

by these residue maps yields locally free subsheaves

$$W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_X^i(\log D)\right) \subset W_0(H, E_{\alpha(a)}) \otimes \Omega_X^i(\log D),$$

$$W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_{X/Y}^i(\log D)\right) \subset W_0(H, E_{\alpha(a)}) \otimes \Omega_{X/Y}^i(\log D).$$

By construction these subsheaves are preserved by φ and so we get absolute and relative parabolic Dolbeault complexes:

$$DOL_{L^{2}}^{par}(X, E_{\alpha(a)}) := \begin{bmatrix} W_{0}(H, E_{\alpha(a)}) & 0 \\ \downarrow \land \varphi & \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X}^{1}(\log D) \right) & 1 \\ \downarrow \land \varphi & \\ \vdots & \vdots & \vdots \\ \downarrow \land \varphi & \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X}^{2}(\log D) \right) & \vdots \\ \downarrow \land \varphi & \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X}^{d_{X}}(\log D) \right) & d_{X} \end{bmatrix}$$

$$(25)$$

and

$$DOL_{L^{2}}^{par}(X/Y, E_{\alpha(a)}) := \begin{bmatrix} W_{0}(H, E_{\alpha(a)}) & 0 \\ \downarrow \land \varphi \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X/Y}^{1}(\log D) \right) & 1 \\ \downarrow \land \varphi \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X/Y}^{2}(\log D) \right) & 2 \\ \vdots & \vdots \\ \downarrow \land \varphi \\ W_{-2,0} \left(H, E_{\alpha(a)} \otimes \Omega_{X/Y}^{d_{X/Y}}(\log D) \right) & d_{X} - d_{Y} \end{bmatrix}$$

$$(26)$$

Remark 7.5. Note that if we choose local coordinates z_i on X so that a component $D_{h(j)}$ of D_H is given by the equation $z_1 = 0$, then on this local chart $\Omega_X^p(\log D)$ decomposes as

$$\Omega_X^p(\log D) = \wedge^p(\bigoplus_{i \ge 2} \mathcal{O} dz_i) \oplus \frac{dz_1}{z_1} \wedge (\wedge^{p-1}(\bigoplus_{i \ge 2} \mathcal{O} dz_i).$$

Using this decomposition and the definition of $W_{-2,0}$ we get an identification

$$W_{-2,0} (H, E_{\alpha(a)} \otimes \Omega_X^i (\log D))$$

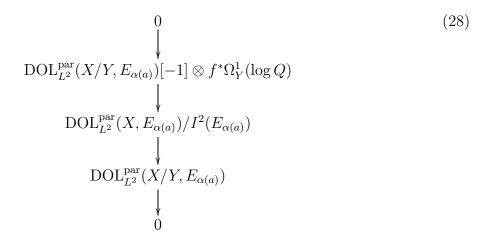
$$= \begin{bmatrix} W_0(H, E_{\alpha(a)}) \otimes \wedge^p(\bigoplus_{i \geq 2} \mathcal{O} dz_i) \\ \bigoplus \\ (W_{-2}(H, E_{\alpha(a)}) + z_1 W_0(H, E_{\alpha(a)})) \otimes \frac{dz_1}{z_1} \wedge (\wedge^{p-1}(\bigoplus_{i \geq 2} \mathcal{O} dz_i) \end{bmatrix}.$$

By the definition of a parabolic bundle we have that $z_1W_0(H, E_{\alpha(a)})) \subset E_{\alpha(a)-\delta^{h(j)}}$ and so under the natural map $W_0(H, E_{\alpha(a)})) \twoheadrightarrow W_0(\operatorname{Gr}_{h(j),0})$ the subsheaf $z_1W_0(H, E_{\alpha(a)}))$ maps to zero. This implies that $z_1W_0(H, E_{\alpha(a)})) \subset W_{-2}(H, E_{\alpha(a)})$ and hence

$$W_{-2,0}\left(H, E_{\alpha(a)} \otimes \Omega_X^i(\log D)\right) = \begin{bmatrix} W_0(H, E_{\alpha(a)}) \otimes \wedge^p(\bigoplus_{i \geq 2} \mathcal{O} dz_i) \\ \bigoplus \\ W_{-2}(H, E_{\alpha(a)}) \otimes \frac{dz_1}{z_1} \wedge (\wedge^{p-1}(\bigoplus_{i \geq 2} \mathcal{O} dz_i) \end{bmatrix}. \tag{27}$$

This formula implies immediately that in the case when $f: X \to Y$ is of relative dimension one, the complexes $\mathrm{DOL}_{L^2}^{\mathrm{par}}(X, E_{\alpha(a)})$ and $\mathrm{DOL}_{L^2}^{\mathrm{par}}(X/Y, E_{\alpha(a)})$ we just defined coincide with the complexes defined in section 7.1.

Tautologically we again get a short exact sequence of complexes



with $I^k(E_{\alpha(a)})$ defined inductively as in section 7.1.

Again pushing forward this short exact sequence by f will give rise to a parabolic Higgs sheaf on Y. If (\underline{E}, φ) comes from a harmonic bundle we expect that this push forward will correspond via the NAHC with the L^2 push forward of this harmonic bundle. This can be verified in two important special cases.

First, Remark 7.5 and Theorem 7.4 imply that this holds if f is of relative dimension one. Second, the analytic considerations in the section 7.2 imply that this statement holds in the case when Y is a point, i.e. when we are dealing with the global cohomology of a harmonic bundle on the complement of a smooth divisor.

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Ron Donagi, University of Pennsylvania, donagi@math.upenn.edu

Tony Pantev, University of Pennsylvania, tpantev@math.upenn.edu

Carlos Simpson, CNRS, Université Côte d'Azur, LJAD, Carlos.SIMPSON@unice.fr