# Existence of Hamiltonians for Some Singular Interactions on Manifolds

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#### Abstract

The existence of the Hamiltonians of the renormalized point interactions in two and three dimensional Riemannian manifolds and that of a relativistic extension of this model in two dimensions are proven. Although it is much more difficult, the proof of existence of the Hamiltonian for the renormalized resolvent for the non-relativistic Lee model can still be given. To accomplish these results directly from the resolvent formula, we employ some basic tools from the semigroup theory.

#### 1 Introduction

Typical field theory problems require a concept known as renormalization, which is a way of rendering infinite quantities to finite values to get physically meaningful results. This is a very hard problem, and it would be interesting to find some simple models in which the ideas can be tested in depth and a mathematically sound description can be presented as much as possible. This will illuminate the underlying mathematical and physical ideas in more complicated models.

There are indeed some simple models introduced in the past. One of them is the Dirac-delta potentials in quantum mechanics which was first studied rigorously by Berezin and Fadeev [1] and later discussed extensively by Albeverio et al [2, 3]. These works show that Dirac-delta potential can be understood from the self-adjoint extension point of view, hence all could be made mathematically sound. Many body version of this problem on  $\mathbb{R}^2$  is known as the formal non-relativistic limit of the  $\lambda \phi^4$  scalar field theory in (2+1) dimensions. All these are extensively discussed first in the unpublished thesis of J. Hoppe [4]. A similar model is the non-relativistic Lee model, which exhibits an additive divergence. We are not aware of a mathematically rigorous discussion of this model. Physically the relativistic simplified version of the Lee model is more important and there is quite of a bit of work from a nonperturbative point of view to understand the physics behind it (see the references in [5]). The approach we follow is introduced in [11] by Rajeev, and recently we have introduced the generalizations of these models on to manifolds [6, 7, 8, 9, 10]. The rigorous understanding of the existence of the Hamiltonian left

aside in our previous works. We would like to address this issue in the present work. There is a general approach which is exposed in the excellent book by Albeverio and Kurasov [3], and it should be applicable in the Dirac-delta functions case for the manifolds. However, we will use an alternative approach. The advantage we have is the following, the self-adjoint extension point of view becomes complicated when we discuss field theories, it is usually hard to give a meaning to operator valued distributions and their extension theory is even more delicate. The other alternative which uses resolvent convergence of cut-off Hamiltonians is problematic when we use other regularization schemes, e.g. the powerful dimensional regularization. This is why we want to utilize a direct approach. In the problems that we deal with, the resulting operator is not given but instead the resolvent is renormalized.

To answer the existence we use the following theorem taken from semi-group theory. Let  $\Delta$  be a subset of the complex plane. A family J(E),  $E \in \Delta$  of bounded linear operators on the Hilbert space  $\mathcal{H}$  under consideration, which satisfies the resolvent identity

$$J(E_1) - J(E_2) = (E_1 - E_2)J(E_1)J(E_2)$$
(1)

for  $E_1$ ,  $E_2 \in \Delta$  is called a pseudo resolvent on  $\Delta$  [12].

The following corollary (Corollary 9.5 in [12]) gives the condition for which there exists a densely defined closed linear operator A such that J(E) is the resolvent family of A: Let  $\Delta$  be an unbounded subset of  $\mathbb C$  and J(E) be a pseudo resolvent on  $\Delta$ . If there is a sequence  $E_k \in \Delta$  such that  $|E_k| \to \infty$  as  $k \to \infty$  and

$$\lim_{k \to \infty} -E_k J(E_k) x = x , \qquad (2)$$

for all  $x \in \mathcal{H}$ , then J(E) is the resolvent of a unique densely defined closed operator A. As we will see, the family satisfies  $J(E)^{\dagger} = J(E^*)$  so it is a holomorphic family of type (A) in the sense of Kato [13]. Hence, it defines a self-adjoint operator.

Let us mention the possibility of using the results from [3] in the case of Dirac-delta functions. In the approach of [3] we consider an operator A with a dense domain, and consider the singular perturbation by an element  $\phi$  in some dual space, formally:

$$A^{-} = A + \lambda \langle \phi, . \rangle \phi , \qquad (3)$$

here the bracket refers to dual pairing. The interesting case is when we have  $\phi \in \mathcal{H}_{-2}(A)$ , where

$$\phi \in \mathcal{H}_{-2}(A) \quad \text{if} \quad \left| \left| \frac{1}{1 + |A|} \phi \right| \right| < \infty \tag{4}$$

and ||.|| refers to the usual norm in the Hilbert space. Then the theorem in [3] states that the operator  $A^-$  provides a self-adjoint extension with a new domain. In our case,

$$\left|\left|\frac{1}{1 + (-\nabla_g^2)} \delta_g(a, .)\right|\right| = \int_0^\infty ds \ s \ e^{-s} \ K_s(a, a; g) < \infty \tag{5}$$

thus we have the same type of singular perturbation—so called form unbounded one. It is interesting to see how the conditions on Ricci curvature we found will arise in this approach.

In our presentation, we do not follow a formal writing style since the paper is rather long and has many technical details, hopefully this makes reading more enjoyable. Various operator identities that we use can be proved rigorously in the compact case, but they require some more work in the noncompact case and it can be done by using the spectral theorem for the Laplace-Beltrami operator. We refrain from completing these arguments since they are more or less standard in operator theory.

## 2 Point Interactions in Two and Three Dimensional Riemannian Manifolds

We adopt the natural units  $\hbar=1$  in the non-relativistic models discussed in this paper for simplicity. In [8], after the renormalization we have found the resolvent kernel corresponding to the Hamiltonian for the N point interactions (Dirac-delta interactions) in two and three dimensional Riemannian manifolds as

$$R(x,y|E) = R_0(x,y|E) + \sum_{i,j=1}^{N} R_0(x,a_i|E)\Phi_{ij}^{-1}(E)R_0(a_j,y|E) , \qquad (6)$$

where

$$\Phi_{ij}(E) = \begin{cases} \int_0^\infty dt \ K_t(a_i, a_i; g) \ \left(e^{-t\mu_i^2} - e^{tE}\right) & \text{if } i = j \\ -\int_0^\infty dt \ K_t(a_i, a_j; g) \ e^{tE} & \text{if } i \neq j \ . \end{cases}$$
 (7)

Here  $\Re(E) < 0$  and  $K_t(x, y; g)$  is the heat kernel on the Riemannian manifold, which is defined as the fundamental solution to the heat equation

$$\frac{1}{2m}\nabla_g^2 K_t(x, y; g) = \frac{\partial K_t(x, y; g)}{\partial t} . \tag{8}$$

In order to show that the resolvent kernel given in the equation (6) corresponds to a unique densely defined closed operator H, we need to first prove that it satisfies the resolvent identity, i.e,

$$R(x,y|E_1) - R(x,y|E_2) = (E_1 - E_2) \int_{\mathcal{M}} d_g^D z \ R(x,z|E_1) R(z,y|E_2) \ . \tag{9}$$

A detailed proof as well as all properties of the heat kernel that we use in this paper, are given in our previous work [8] and the relevant literature is also given there. Here we will just give the main idea of the proof for the completeness of this paper. If we substitute the equation (6) into the equation (9), we obtain

$$R_{0}(x,y|E_{1}) - R_{0}(x,y|E_{2}) + \sum_{i,j=1}^{N} R_{0}(x,a_{i}|E_{1})\Phi_{ij}^{-1}(E_{1})R_{0}(a_{j},y|E_{1})$$

$$- \sum_{i,j=1}^{N} R_{0}(x,a_{i}|E_{2})\Phi_{ij}^{-1}(E_{2})R_{0}(a_{j},y|E_{2})$$

$$= (E_{1} - E_{2}) \int_{\mathcal{M}} d_{g}^{D} z \left[ R_{0}(x,z|E_{1})R_{0}(z,y|E_{2}) + \sum_{i,j=1}^{N} R_{0}(x,z|E_{1})R_{0}(z,a_{i}|E_{2})\Phi_{ij}^{-1}(E_{2})R_{0}(a_{j},y|E_{2}) + \sum_{i,j=1}^{N} R_{0}(x,a_{i}|E_{1})\Phi_{ij}^{-1}(E_{1})R_{0}(a_{j},z|E_{1})R_{0}(z,y|E_{2}) + \sum_{i,j=1}^{N} \sum_{r,l=1}^{N} R_{0}(x,a_{i}|E_{1})\Phi_{ij}^{-1}(E_{1})R_{0}(a_{j},z|E_{1})$$

$$\times R_0(z, a_r | E_2) \Phi_{rl}^{-1}(E_2) R_0(a_l, y | E_2)$$
(10)

The term  $R_0(x, y|E_1) - R_0(x, y|E_2)$  equals to the first term in the right hand side of the equation above since the free resolvent kernel  $R_0(x, y|E)$  must satisfy the resolvent identity (9). If we add and subtract the terms

$$\sum_{i,j=1}^{N} R_0(x, a_i | E_1) \Phi_{ij}^{-1}(E_1) R_0(a_j, y | E_2)$$
(11)

$$\sum_{i,j=1}^{N} R_0(x, a_i|E_1) \Phi_{ij}^{-1}(E_2) R_0(a_j, y|E_2)$$
(12)

to the remaining terms in the equation above and rearrange, one can complete the proof for the resolvent identity (9) by showing that the difference of the principal matrix  $\Phi_{ij}(E_2) - \Phi_{ij}(E_1)$  equals to the difference of free resolvent kernels  $R_0(a_i, a_j | E_1) - R_0(a_i, a_j | E_2)$ . It is easy to show this by using the formula expressing the free resolvent kernel as a Laplace transformation of the heat kernel and semigroup property of the heat kernel following a change of variable for the time variable in the heat kernel [8]. The equation (2) requires the following condition to complete the second part of the proof

$$||E_k R(E_k)f + f|| \to 0 , \qquad (13)$$

as  $k \to \infty$ , where f belongs to the Hilbert space  $\mathcal{H} = L^2(\mathcal{M})$  and the norm is the usual  $L^2(\mathcal{M})$  norm. Let us choose the sequence  $E_k = -k|E_0|$ , where  $E_0$  is below the lower bound  $E_*$  on the ground state energy which has been found in [8]. Then, we must show that

$$|||E_k|R(E_k)f - f|| \to 0$$
, (14)

as  $k \to \infty$ . Using the equation (6) and separating the free part, we get

$$|||E_{k}|R(E_{k})f - f|| \leq |||E_{k}|R_{0}(E_{k})f - f|| + |E_{k}|||R_{0}(E_{k})\Phi^{-1}(E_{k})R_{0}(E_{k})f||.$$
(15)

It is well known that the first part of the sum converges to zero as  $k \to \infty$ , that is, the free resolvent corresponds to a densely defined closed operator (Laplacian). Moreover, the Laplacian on geodesically complete Riemannian manifolds is essentially self-adjoint in  $L^2(\mathcal{M})$  [14, 15]. Therefore, we are going to investigate only the second term in two and three dimensions separately. Two dimensional analysis has been already worked out in [8] and we will just review it here and then give the detailed proof for the three dimensional case. Since the norm of an operator is smaller than its Hilbert-Schmidt norm:  $||A|| \leq \text{Tr}^{1/2}(A^{\dagger}A)$  with  $A = R_0(E_k)\Phi^{-1}(E_k)R_0(E_k)$ , we have

$$|E_{k}|||R_{0}(E_{k})\Phi^{-1}(E_{k})R_{0}(E_{k})f||$$

$$\leq |E_{k}|\left[\sum_{i,j,r,l=1}^{N}\int_{\mathcal{M}}d_{g}^{D}x\ R_{0}(a_{i},x|E_{k})R_{0}(x,a_{l}|E_{k})\right]$$

$$\times \int_{\mathcal{M}}d_{g}^{D}y\ R_{0}(a_{j},y|E_{k})R_{0}(y,a_{r}|E_{k})|\Phi_{ij}^{-1}(E_{k})||\Phi_{rl}^{-1}(E_{k})|\right]^{1/2}.(16)$$

Let us first consider the diagonal case l = i and r = j for the terms inside the bracket above.

$$|E_{k}| \left[ \sum_{i,j=1}^{N} \int_{\mathcal{M}} d_{g}^{D} x \, R_{0}(a_{i}, x | E_{k}) R_{0}(x, a_{i} | E_{k}) \right] \times \int_{\mathcal{M}} d_{g}^{D} y \, R_{0}(a_{j}, y | E_{k}) R_{0}(y, a_{j} | E_{k}) |\Phi_{ij}^{-1}(E_{k})| |\Phi_{ji}^{-1}(E_{k})| \right]^{1/2}$$

$$\leq |E_{k}| \left[ N^{2} \max_{1 \leq i \leq N} \alpha_{i}(E_{k}) \, \max_{1 \leq j \leq N} \alpha_{j}(E_{k}) \, \max_{1 \leq i,j \leq N} |\Phi_{ij}^{-1}(E_{k})|^{2} \right]^{1/2} , \qquad (17)$$

where we have defined  $\alpha_i(E_k) = \int_{\mathcal{M}} d_g^D y \ R_0(a_i, y|E_k) R_0(y, a_i|E_k)$  for simplicity. It is easy to see that

$$\int_{\mathcal{M}} d_g^D x \ R_0(a_i, x | E_k) R_0(x, a_l | E_k) = \int_0^\infty \int_0^\infty dt_1 \, dt_2 \ K_{t_1 + t_2}(a_i, a_l; g) e^{-(t_1 + t_2)|E_k|} 
= \int_0^\infty dt \ t \ K_t(a_i, a_l; g) e^{-t|E_k|}$$
(18)

by using the fact that the free resolvent kernel is just the Laplace transform of the heat kernel. The upper bound of the heat kernel was given in [16, 17] and summarized in [8] for compact (with bounded Ricci) and Cartan-Hadamard manifolds [9]. We shall use the notation for the dimensionless constants coming from the bounds of the heat kernel as C with subscripts for simplicity since the exact form of these constants do not play any role here. The upper bound of the heat kernel for compact (with bounded Ricci) and Cartan-Hadamard manifolds is given in the following form

$$K_{t}(x, y; g) \leq \begin{cases} \left[ \frac{C_{1}}{V(\mathcal{M})} + \frac{C_{2}}{(t/2m)^{D/2}} \right] \exp\left(-\frac{md^{2}(x, y)}{C_{3}t}\right) & \text{for compact manifolds} \\ \frac{C_{4}}{(t/2m)^{D/2}} \exp\left(-\frac{md^{2}(x, y)}{C_{5}t}\right) & \text{for Cartan-Hadamard manifolds} \end{cases}$$
(19)

where  $V(\mathcal{M})$  is the volume of the manifold and d(x,y) is the geodesic distance between the point x and y. Then, on-diagonal upper bound of the equation (18) for compact manifolds (with bounded Ricci) becomes

$$\max_{1 \le i \le N} \alpha_i(E_k) \le \frac{C_1}{V(\mathcal{M})|E_k|^2} + C_6(2m)^{\frac{D}{2}} |E_k|^{\frac{D}{2} - 2}, \tag{20}$$

where  $C_6 = C_2\Gamma(2-D/2)$ . For Cartan-Hadamard manifolds, we get

$$\max_{1 \le i \le N} \alpha_i(E_k) \le C_4 (2m)^{\frac{D}{2}} |E_k|^{\frac{D}{2} - 2}.$$
(21)

We have also

$$\max_{1 \le i,j \le N} |\Phi_{ij}^{-1}|^2 \le \max_{1 \le i \le N} \sum_{j=1}^N |\Phi_{ij}^{-1}|^2 = \max_{1 \le i \le N} (\Phi^{-1}(E_k)\Phi^{-1}(E_k))_{ii} \le \rho(\Phi^{-2}(E_k)) 
\le ||\Phi^{-2}(E_k)|| \le ||\Phi^{-1}(E_k)||^2$$
(22)

where we have used  $\Phi^{\dagger}(E_k) = \Phi(E_k)$  for  $E_k \in \mathbb{R}$  and  $\rho$  is the spectral radius.

In order to find the upper bound for the norm of the inverse principal matrix, we first decompose the principal matrix into two positive matrices

$$\Phi = D - K \tag{23}$$

where D and K stand for the on-diagonal and the off-diagonal part of the principal matrix, respectively. Then, it is easy to see  $\Phi = D(1 - D^{-1}K)$ . The principal matrix is invertible if and only if  $(1 - D^{-1}K)$ , and  $(1 - D^{-1}K)$  has an inverse if the matrix norm satisfies  $||D^{-1}K|| < 1$ . Then, we can write the inverse of  $\Phi$  as a geometric series

$$\Phi^{-1} = (1 - D^{-1}K)^{-1}D^{-1} 
= (1 + (D^{-1}K) + (D^{-1}K)^{2} + ...) D^{-1},$$
(24)

and the norm has the following upper bound

$$||\Phi^{-1}|| = ||(1 - D^{-1}K)^{-1}D^{-1}|| \le ||(1 - D^{-1}K)^{-1}|| ||D^{-1}|| \le \frac{1}{1 - ||D^{-1}K||} ||D^{-1}||.$$
(25)

Since we are not concerned with the sharp bounds on the norm of  $\Phi^{-1}$  for this problem, we can choose  $|E_k|$  sufficiently large such that  $||D^{-1}K|| < 1/2$  without loss of generality and get

$$||\Phi^{-1}(E_k)|| \le 2||D^{-1}(E_k)||$$
 (26)

Whenever  $D^{-1} = \text{diag}(\Phi_{11}^{-1}, \Phi_{22}^{-1}, \dots, \Phi_{NN}^{-1})$ , then

$$||D^{-1}|| = \max_{1 \le i \le N} |\Phi_{ii}^{-1}|. \tag{27}$$

The lower bound of the diagonal principal matrix for compact and Cartan-Hadamard manifolds, which was given in [8], leads to the upper bound of the inverse of the diagonal part of the principal matrix. Hence, we find

$$\max_{1 \le i \le N} |\Phi_{ii}^{-1}(E_k)| \le \begin{cases} C_7(2m)^{-1} \ln^{-1} \left(|E_k|/\mu^2\right) & \text{if } D = 2\\ C_8(2m)^{-3/2} \left[|E_k|^{1/2} - \mu\right]^{-1} & \text{if } D = 3 \end{cases},$$
(28)

for compact manifolds and

$$\max_{1 \le i \le N} |\Phi_{ii}^{-1}(E_k)| \le \begin{cases} C_9(2m)^{-1} \ln^{-1} \left( \frac{|E_k| + \xi}{\mu^2 + \xi} \right) & \text{if } D = 2\\ C_{10}(2m)^{-3/2} \left[ (|E_k| + \xi)^{1/2} - (\mu^2 + \xi)^{1/2} \right]^{-1} & \text{if } D = 3 \end{cases},$$
(29)

for Cartan-Hadamard manifolds. Here  $\xi$  is a positive constant and defined in [8].

If we substitute the results (20), (21) and (28), (29) into (17) for D=2, and take the limit  $k \to \infty$ , the result goes to zero. Since the norm is always positive, we prove

$$|||E_k|R(E_k)f - f|| \to 0 \tag{30}$$

as  $k \to \infty$ . For the off-diagonal terms, we do not have to make a separate detailed analysis since one can easily show that these terms are essentially exponentially suppressed by the factor  $e^{-\sqrt{2m|E_k|}d(a_i,a_j)}$  due to the upper bounds of the modified Bessel functions which are given in [8]. Therefore, all off-diagonal terms exponentially vanish when we take the limit  $k \to \infty$ , which is enough for our purposes.

On the other hand, this proof does not work in the three dimensional case as one can easily see. In three dimensions, estimating the operator norm by the Hilbert-Schmidt norm is not a good way. Instead we will return to the second term in (15), and show that

$$|E_{k}| \left[ \int_{\mathcal{M}} d_{g}^{3}x \sum_{i,j,r,l=1}^{N} R_{0}(x,a_{i}|E_{k}) \Phi_{ij}^{-1}(E_{k}) \int_{\mathcal{M}} d_{g}^{3}z R_{0}(a_{j},z|E_{k}) f^{*}(z) \right] \times R_{0}(x,a_{r}|E_{k}) \Phi_{rl}^{-1}(E_{k}) \int_{\mathcal{M}} d_{g}^{3}y R_{0}(a_{l},y|E_{k}) f(y)$$
(31)

goes to zero as  $k \to \infty$  for any  $f \in L^2(\mathcal{M})$ . From our previous argument, we know that the inverse of the principal matrix  $\Phi$  satisfies for three dimensional compact and Cartan-Hadamard manifolds:

$$\max_{1 \le i,j \le N} |\Phi_{ij}^{-1}(E_k)| \le \frac{C_{11}(2m)^{-3/2}}{|E_k|^{1/2}} , \tag{32}$$

where we define all the constant terms coming from the bounds of the heat kernel as  $C_{11}$  and ignore the term in the denominator for large values of  $|E_k|$  for simplicity. Moreover, we can combine the two resolvents with the common variable x in the equation (31). As a result, we can express this combination as in the equation (18) and the diagonal upper bounds of it for D=3 has been given in the equations (20) and (21) for compact and Cartan-Hadamard manifolds, respectively. Once again, we skip the detailed calculations for the off-diagonal terms  $(i \neq r)$  in the above sum since they are exponentially suppressed by the factor  $e^{-\sqrt{2m|E_k|}d(a_i,a_r)}$ . We always concentrate on the least convergent part in the terms. Once we have achieved our goal for these terms, we are done.

Therefore, it is sufficient to deal with only the diagonal term (r = i) in the equation (31). It is easy to show that it is smaller than the following term

$$N^{3/2}|E_{k}| \left[ \max_{1 \leq i \leq N} \alpha_{i}(E_{k}) \right]^{1/2} \left[ \max_{1 \leq i,j \leq N} |\Phi_{ij}^{-1}(E_{k})| \max_{1 \leq i,l \leq N} |\Phi_{il}^{-1}(E_{k})| \right]^{1/2} \times \left[ \max_{1 \leq j \leq N} \left( \int_{\mathcal{M}} d_{g}^{3} z \ R_{0}(a_{j}, z|E_{k})|f(z)| \right) \max_{1 \leq l \leq N} \left( \int_{\mathcal{M}} d_{g}^{3} y \ R_{0}(y, a_{l}|E_{k})|f(y)| \right) \right]^{1/2}.$$
(33)

Using the equations (20), (21) and (32) in the above equation, we get the upper bound of (33) for three dimensional compact manifolds

$$N^{3/2}|E_{k}| \left[ \frac{C_{1}}{V(\mathcal{M})|E_{k}|^{2}} + \frac{C_{6}(2m)^{\frac{3}{2}}}{|E_{k}|^{1/2}} \right]^{1/2} \left[ \frac{C_{11}(2m)^{-3/2}}{|E_{k}|^{1/2}} \right] \times \left[ \max_{1 \leq j \leq N} \left( \int_{\mathcal{M}} d_{g}^{3}z \ R_{0}(a_{j}, z|E_{k})|f(z)| \right) \max_{1 \leq l \leq N} \left( \int_{\mathcal{M}} d_{g}^{3}y \ R_{0}(y, a_{l}|E_{k})|f(y)| \right) \right]^{1/2}$$
(34)

and for three dimensional Cartan-Hadamard manifolds

$$N^{3/2}|E_{k}| \left[ \frac{C_{4}(2m)^{\frac{3}{2}}}{|E_{k}|^{1/2}} \right]^{1/2} \left[ \frac{C_{11}(2m)^{-3/2}}{|E_{k}|^{1/2}} \right] \times \left[ \max_{1 \leq j \leq N} \left( \int_{\mathcal{M}} d_{g}^{3}z \ R_{0}(a_{j}, z|E_{k})|f(z)| \right) \max_{1 \leq l \leq N} \left( \int_{\mathcal{M}} d_{g}^{3}y \ R_{0}(y, a_{l}|E_{k})|f(y)| \right) \right]^{1/2}$$
(35)

All these imply that the term

$$\int_{\mathcal{M}} d_g^3 y \ R_0(a_j, y | E_k) |f(y)| \tag{36}$$

must decay at least faster than  $|E_k|^{-1/4}$ . We now recall that the free resolvent kernel is just the Laplace transform of the heat kernel

$$R_0(a_j, y | E_k) = \int_0^\infty dt \ e^{-t|E_k|} K_t(a_j, y; g) , \qquad (37)$$

so that we can find an upper bound for it by using the equation (19) and evaluating the integrals over t

$$R_{0}(a_{j}, y | E_{k}) \leq \frac{mC_{12}}{d(a_{j}, y)} \exp \left[-2\left(\frac{md^{2}(a_{j}, y) | E_{k}|}{C_{3}}\right)^{1/2}\right] + \frac{C_{13}d(a_{j}, y)\sqrt{m}}{V(\mathcal{M})\sqrt{|E_{k}|}} \left[1 + \left(\frac{C_{3}}{md^{2}(a_{j}, y) | E_{k}|}\right)^{1/2}\right] \exp \left[-2\left(\frac{md^{2}(a_{j}, y) | E_{k}|}{C_{3}}\right)^{1/2}\right]$$
(38)

for three dimensional compact manifolds and

$$R_0(a_j, y|E_k) \le \frac{mC_{14}}{d(a_j, y)} \exp\left[-2\left(\frac{md^2(a_j, y)|E_k|}{C_5}\right)^{1/2}\right],$$
 (39)

for three dimensional Cartan-Hadamard manifolds. Here we have used the upper bound of the modified Bessel function  $K_1(x)$  given in [8].

For simplicity, let us first consider the generic term which is common for both compact and Cartan-Hadamard manifolds and keep the inverse volume term aside for the moment. Then, we find for the generic term

$$\int_{\mathcal{M}} d_g^3 y \ R_0(a_j, y | E_k) |f(y)| \le m C_{12} \int_{\mathcal{M}} d_g^3 y \ \exp\left[-2\left(\frac{m d^2(a_j, y) |E_k|}{C_3}\right)^{1/2}\right] \frac{|f(y)|}{d(a_j, y)} \ . \tag{40}$$

We now divide the integration region into two pieces

$$\int_{B_{\delta}(a_{j})} d_{g}^{3} y \exp \left[-2 \left(\frac{m d^{2}(a_{j}, y) |E_{k}|}{C_{3}}\right)^{1/2}\right] \frac{|f(y)|}{d(a_{j}, y)} + \int_{\mathcal{M} \setminus B_{\delta}(a_{j})} d_{g}^{3} y \exp \left[-2 \left(\frac{m d^{2}(a_{j}, y) |E_{k}|}{C_{3}}\right)^{1/2}\right] \frac{|f(y)|}{d(a_{j}, y)}, \tag{41}$$

where  $B_{\delta}(a_i)$  is the geodesic ball of radius  $\delta$  centered at  $a_i$ . It is easily seen that

$$\int_{\mathcal{M}\setminus B_{\delta}(a_{j})} d_{g}^{3}y \exp\left[-2\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right] \frac{|f(y)|}{d(a_{j},y)} \\
\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right] \int_{\mathcal{M}\setminus B_{\delta}(a_{j})} d_{g}^{3}y \exp\left[-\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right] |f(y)| \\
\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right] \int_{\mathcal{M}} d_{g}^{3}y \exp\left[-\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right] |f(y)| \\
\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right] \left[\int_{\mathcal{M}} d_{g}^{3}y \exp\left[-2\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right] \right]^{1/2} ||f||, \quad (42)$$

where we have used the fact that  $d(a_j, y) \ge \delta$  for all j and  $y \in \mathcal{M} \setminus B_{\delta}(a_j)$  in the second line. We then find an upper bound in terms of the norm of the function f(x) by using Cauchy-Schwartz inequality in the last line.

For compact manifolds, it is a simple matter to find the upper bound to the above integral

$$\int_{\mathcal{M}\setminus B_{\delta}(a_{j})} d_{g}^{3}y \exp\left[-2\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right] \frac{|f(y)|}{d(a_{j},y)}$$

$$\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^{2}|E_{k}|}{C_{2}}\right)^{1/2}\right] \left[V(\mathcal{M}) \sup_{y\in\mathcal{M}} \left(\exp\left[-2\left(\frac{md^{2}(a_{j},y)|E_{k}|}{C_{3}}\right)^{1/2}\right]\right)\right]^{1/2} ||f||$$

$$\leq \frac{1}{\delta} \exp\left[-\left(\frac{m\delta^{2}|E_{k}|}{C_{2}}\right)^{1/2}\right] V^{1/2}(\mathcal{M})||f||, \qquad (43)$$

due to the fact that the volume of a compact manifold is finite. For non-compact manifolds, it is useful to consider the above integral in the Riemann normal coordinates near one of the centers  $a_i$ . We further assume that the radius of the ball  $\delta$  is less than the injectivity radius  $\operatorname{inj}(a_i)$ . Let us recall that in Gaussian spherical coordinates, the volume integral of a function h on a D dimensional Riemannian manifold  $\mathcal{M}$  can be written as

$$\int_{\mathcal{M}} d_g^D x \ h(x) = \int_{\mathbb{S}^{D-1}} d\Omega \int_0^{\rho_{\Omega}} dr \ r^{D-1} J(r, \theta) h(r, \theta) \ . \tag{44}$$

Here  $\theta = (\theta_1, \dots \theta_{D-1})$  denotes the direction in the tangent space around a point that we choose, and  $\rho_{\Omega}$  refers to distance to the cut locus of the point in the direction  $\theta$ . Hence, we get:

$$\int_{\mathcal{M}} d_g^3 y \exp\left[-2\left(\frac{md^2(a_j, y)|E_k|}{C_3}\right)^{1/2} \\
= \int_{\mathbb{S}^2} d\Omega \int_0^{\rho_{\Omega}} dr \ r^2 J(r, \theta) \exp\left[-2\left(\frac{mr^2|E_k|}{C_3}\right)^{1/2}\right]. \tag{45}$$

To proceed further we assume that  $\mathcal{M}$  has Ricci tensor bounded from below by  $K_1$ , i.e.  $\operatorname{Ric}(\underline{\cdot},\underline{\cdot}) > K_1$  g(.,.) everywhere and the sectional curvature is bounded from above by  $K_2$  on  $\overline{B_{\delta}(a_i)}$ . The upper bound on the sectional curvature is automatically satisfied, since there are a finite number of Dirac-delta centers and because we take the metric to be  $\mathcal{C}^{\infty}(\mathcal{M})$ . Had we considered a random distribution of Dirac-delta function centers, in which case they could have been located at arbitrarily distant points where the sectional curvature could have been

unbounded, we would have had to constrain the sectional curvature from above. Then Bishop-Gunther volume comparison theorems state that the Jacobian factor of the Gaussian spherical coordinates in D dimensions satisfies [18, 19],

$$\frac{\operatorname{sn}_{K_2}^{D-1}(r)}{r^{D-1}} \le J(r,\theta) \le \frac{\operatorname{sn}_{K_1}^{D-1}(r)}{r^{D-1}} , \tag{46}$$

where

$$\operatorname{sn}_{K}(r) = \begin{cases} \frac{\sin(\sqrt{K}r)}{\sqrt{K}} & \text{if } K > 0\\ r & \text{if } K = 0\\ \frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}} & \text{if } K < 0 \end{cases}$$

$$\tag{47}$$

Then the upper bound of the equation (45) becomes

$$\int_{\mathbb{S}^{2}} d\Omega \int_{0}^{\infty} dr \frac{\sinh^{2}(\sqrt{-K_{1}}r)}{(-K_{1})} \exp\left[-2\left(\frac{mr^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right] 
= \frac{1}{|K_{1}|^{3/2}} \int_{\mathbb{S}^{2}} d\Omega \int_{0}^{\infty} dr' \sinh^{2}(r') \exp\left[-2\left(\frac{mr'^{2}|E_{k}|}{C_{3}|K_{1}|}\right)^{1/2}\right] 
\leq \frac{\pi}{|K_{1}|^{3/2}} \int_{0}^{\infty} dr' \exp\left[-2r'\left(\left(\frac{m|E_{k}|}{C_{3}|K_{1}|}\right)^{1/2} - 1\right)\right] 
= \frac{\pi}{2|K_{1}|^{3/2}\left[\left(\frac{m|E_{k}|}{C_{3}|K_{1}|}\right)^{1/2} - 1\right]}.$$
(48)

as long as  $\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} \geq 1$ . Since we are interested in the limit  $k \to \infty$  it is satisfied for sufficiently large values of  $|E_k|$ . Therefore, equation (42) is smaller than

$$\frac{\left(\frac{\pi}{2}\right)^{1/2}}{\left(\delta|K_1|^{3/4}\right)} \exp\left[-\left(\frac{m\delta^2|E_k|}{C_3}\right)^{1/2}\right] \left[\left(\frac{m|E_k|}{C_3|K_1|}\right)^{1/2} - 1\right]^{-1/2} ||f||.$$
(49)

If we choose  $\delta = (m\sqrt{R})^{-1/3}|E_k|^{-1/3}$ , where R is an appropriate scale coming from the Ricci tensor around a point, where Ricci tensor is non-zero. The prefactor multiplying the exponent goes to infinity whereas the exponent decays rapidly. In fact, it decays fast enough to make the expression as a whole go to zero as  $k \to \infty$ .

Let us go back to the first integral in the equation (41) and write it in the Gaussian spherical coordinates:

$$\int_{\mathbb{S}^2} d\Omega \int_0^{\delta} dr \ r^2 J(r, \theta) \exp\left[-2\left(\frac{mr^2 |E_k|}{C_3}\right)^{1/2}\right] \frac{|f(r, \theta)|}{r} \ . \tag{50}$$

Let us now make the observation that there are constants  $A_+, A_-$ , which depend only on  $\delta$  and  $K_i$ 's such that,

$$A_{-}(K_i, K_j) \le \frac{\operatorname{sn}_{K_i}(r)}{\operatorname{sn}_{K_i}(r)} \le A_{+}(K_i, K_j)$$
 (51)

for  $r \in [0, \delta]$ . For this part of the integral we use the following characterization of essential supremum: let us define

$$\Lambda(\epsilon) = \mu(\{r \in [0, \delta] | |r^{3/2}F(r)| > \epsilon\}), \tag{52}$$

where  $\mu$  is the standard Lebesgue measure. Then we have

Essup 
$$|r^{3/2}F(r)| = \inf_{\epsilon} \{\epsilon | \Lambda(\epsilon) = 0\}$$
. (53)

Let us use now  $F(r) = \int_{\mathbb{S}^2} d\Omega |f(r,\theta)|$ , and using Bishop-Gunther bound for the first part as,

$$\int_{0}^{\delta} dr \ r^{3/2} \int_{\mathbb{S}^{2}} d\Omega \ |f(r,\theta)| \frac{\exp\left[-2\left(\frac{mr^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right]}{r^{1/2}} \frac{\operatorname{sn}_{K_{1}}^{2}(r)}{r^{2}}$$
(54)

which is smaller than;

$$A_{+}^{2}(K_{1},0) \int_{0}^{\delta} dr \ r^{3/2} \int_{\mathbb{S}^{2}} d\Omega \ |f(r,\theta)| \frac{\exp\left[-2\left(\frac{mr^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right]}{r^{1/2}}$$

$$\leq A_{+}^{2}(K_{1},0) \left(\operatorname{Essup}_{r\in[0,\delta]}|r^{3/2}F(r)|\right) \left(\int_{0}^{\delta} dr \ \frac{\exp\left[-2\left(\frac{mr^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right]}{r^{1/2}}\right)$$

$$\leq \left(\operatorname{Essup}_{r\in[0,\delta]}|r^{3/2}F(r)|\right) \frac{A_{+}^{2}(K_{1},0)}{2(m/C_{3})^{1/4}|E_{k}|^{1/4}}$$
(55)

If we take the limit  $\delta = (m\sqrt{R})^{-1/3}|E_k|^{-1/3} \to 0$ , we claim that the essential-supprenum goes to zero. To see this, we observe by Markov inequality [20] that

$$\Lambda(\epsilon) \leq \frac{1}{\epsilon} \int_{0}^{\delta} dr \, |r^{3/2}F(r)| 
\leq \frac{1}{\epsilon} \left[ \int_{0}^{\delta} dr \, r \right]^{1/2} \left[ \int_{0}^{\delta} dr \, r^{2} \left( \int_{\mathbb{S}^{2}} d\Omega \, |f(r,\theta)| \right)^{2} \right]^{1/2} 
\leq \frac{1}{\epsilon} \frac{\delta}{\sqrt{2}} \left[ \int_{0}^{\delta} dr \, \frac{r^{2}}{\operatorname{sn}_{K_{2}}^{2}(r)} \operatorname{sn}_{K_{2}}^{2}(r) \int_{\mathbb{S}^{2}} d\Omega \, |f(r,\theta)|^{2} \int_{\mathbb{S}^{2}} d\Omega \right]^{1/2} 
\leq \frac{1}{\epsilon} \frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_{+}(0, K_{2}) \left[ \int_{0}^{\delta} dr \, \operatorname{sn}_{K_{2}}^{2}(r) \int_{\mathbb{S}^{2}} d\Omega \, |f(r,\theta)|^{2} \right]^{1/2} 
\leq \frac{1}{\epsilon} \frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_{+}(0, K_{2}) \left[ \int_{0}^{\delta} dr \, r^{2} \int_{\mathbb{S}^{2}} d\Omega \, J(r,\theta) |f(r,\theta)|^{2} \right]^{1/2} 
\leq \frac{1}{\epsilon} \frac{\delta}{\sqrt{2}} (4\pi)^{1/2} A_{+}(0, K_{2}) ||f|| .$$
(56)

For any  $\epsilon > 0$ , our choice of  $\delta$  implies that we can make  $\Lambda(\epsilon) = 0$ . Thus, the infimum goes to zero in the limit as  $k \to \infty$ . As a result we see that the equation (31) is smaller than

$$C_{15} \left[ \frac{A_{+}^{2}(K_{1},0)}{2(1/C_{3})^{1/4}} \left( \underset{r \in [0,\delta]}{\text{Essup}} |r^{3/2}F(r)| \right) + |E_{k}|^{1/4} m^{1/4} \frac{\exp\left[ -\left(\frac{m\delta^{2}|E_{k}|}{C_{3}}\right)^{1/2}\right]}{\left(\delta|K_{1}|^{3/4}\right)} \left(\frac{\pi}{2}\right)^{1/2} \left[ \left(\frac{m|E_{k}|}{C_{3}|K_{1}|}\right)^{1/2} - 1 \right]^{-1/2} ||f|| \right],$$
(57)

and it goes to zero as  $k \to \infty$ . The repeated application of the same analysis leads us to the same conclusion for the other terms coming from the inverse volume term which has been omitted for simplicity. Indeed, all these terms decay with  $|E_k|$  faster than the result that we have obtained above. This completes the proof of the existence of the Hamiltonian for point interactions in three dimensional Riemannian manifolds.

### 3 Relativistic Point Interactions on Two Dimensional Riemannian Manifolds

The resolvent for this system have been found in [9] and it is given by

$$R(E) = R_0(E) + R_0(E)b^{\dagger}\Phi^{-1}(E)bR_0(E) , \qquad (58)$$

where

$$b^{\dagger} = \sum_{i=1}^{N} \phi^{(-)}(a_i) \chi_i \tag{59}$$

and

$$\Phi(E) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \int_{0}^{\infty} ds \ e^{-s^{2}/4} \int_{0}^{\infty} du \ \left( e^{s\mu_{i}\sqrt{u}} - e^{sE\sqrt{u}} \right) e^{-um^{2}} K_{u}(a_{i}, a_{i}; g) \chi_{i}^{\dagger} \chi_{i} 
- \frac{1}{\sqrt{\pi}} \sum_{\substack{i,j \ (i \neq j)}} \int_{0}^{\infty} ds \ e^{-s^{2}/4} \int_{0}^{\infty} du \ e^{sE\sqrt{u}} e^{-um^{2}} K_{u}(a_{i}, a_{j}; g) \chi_{i}^{\dagger} \chi_{j}$$
(60)

Here  $\phi^{(-)}(x)$  is the positive frequency part of the bosonic field and  $a_i$  stands for the position of one of the N Dirac-delta function potential centers and  $\mu_i$  is the bound state energy for the single delta center at  $a_i$ . The operator  $\chi_i$ , called angel operator, was first introduced for this purpose by Rajeev in [11] and it obeys orthofermionic algebra.  $K_t(x, y; g)$  is the heat kernel on the Riemannian manifold, which is defined as the fundamental solution to the heat equation

$$\nabla_g^2 K_t(x, y; g) = \frac{\partial K_t(x, y; g)}{\partial t} \,. \tag{61}$$

We would like to first check whether R(E) satisfies the resolvent identity (1). If we put the form of the resolvent in second quantized form,

$$R(E) = R_0(E) + R_0(E)b^{\dagger}\Phi^{-1}(E)bR_0(E)$$
(62)

into the above resolvent identity (1) and we simplify by purely algebraic operations, to arrive at the following identity,

$$\Phi_{ij}(E_1) - \Phi_{ij}(E_2) + b_i(R_0(E_1) - R_0(E_2))b_j^{\dagger} = 0 , \qquad (63)$$

where we stripped off the angels and wrote everything in terms of explicit matrix indices, and thus  $\Phi(E) = \Phi_{ij}(E)\chi_i^{\dagger}\chi_j$  and also  $b_i = \phi^{(+)}(a_i)$  and similarly for  $b_j^{\dagger}$ . Let us now verify the above identity, we note that

$$\Phi_{ij}(E_1) - \Phi_{ij}(E_2) = \frac{1}{\sqrt{\pi}} \int_0^\infty ds \ e^{-s^2/4} \int_0^\infty du \ [e^{sE_2\sqrt{u}} - e^{sE_1\sqrt{u}}] e^{-um^2} K_u(a_i, a_j; g) \ . \tag{64}$$

Let us work out the other term, acting on *no particle Fock space*, this is the same calculation we have done for the renormalized term. For simplicity, we present the calculation in a formal eigenfunction expansion of the Laplace operator (which is rigorously valid for only compact manifolds)

$$b_{i}(R_{0}(E_{1}) - R_{0}(E_{2}))b_{j}^{\dagger} = \sum_{\sigma} f_{\sigma}^{*}(a_{i}) \frac{a_{\sigma}}{\sqrt{\omega_{\sigma}}} \left[ \frac{1}{H_{0} - E_{1}} - \frac{1}{H_{0} - E_{2}} \right] \sum_{\lambda} f_{\lambda}(a_{j}) \frac{a_{\lambda}^{\dagger}}{\sqrt{\omega_{\lambda}}}$$

$$= \sum_{\sigma} f_{\sigma}^{*}(a_{i}) \frac{a_{\sigma}}{\sqrt{\omega_{\sigma}}} \sum_{\lambda} f_{\lambda}(a_{j}) \frac{a_{\lambda}^{\dagger}}{\sqrt{\omega_{\lambda}}} \left[ \frac{1}{H_{0} + \omega_{\lambda} - E_{1}} - \frac{1}{H_{0} + \omega_{\lambda} - E_{2}} \right]$$

$$= \sum_{\sigma} \sum_{\lambda} f_{\sigma}^{*}(a_{i}) f_{\lambda}(a_{j}) \left[ \frac{a_{\lambda}^{\dagger}}{\sqrt{\omega_{\lambda}}} \frac{a_{\sigma}}{\sqrt{\omega_{\sigma}}} + \frac{\delta_{\sigma\lambda}}{\sqrt{\omega_{\lambda}}\sqrt{\omega_{\sigma}}} \right] \left[ \frac{1}{\omega_{\lambda} - E_{1}} - \frac{1}{\omega_{\lambda} - E_{2}} \right]$$

$$= \sum_{\lambda} f_{\lambda}^{*}(a_{i}) f_{\lambda}(a_{j}) \frac{1}{\omega_{\lambda}} \left[ \frac{1}{\omega_{\lambda} - E_{1}} - \frac{1}{\omega_{\lambda} - E_{2}} \right]$$

$$= \int_{0}^{\infty} ds \ s \sum_{\lambda} f_{\lambda}^{*}(a_{i}) f_{\lambda}(a_{j}) \int_{0}^{1} d\zeta \ e^{-s\omega_{\lambda}} \left[ e^{s\zeta E_{1}} - e^{s\zeta E_{2}} \right]$$

$$= \int_{0}^{\infty} ds \ s \int_{0}^{1} d\zeta \ \frac{s}{2\sqrt{\pi}} \int_{0}^{\infty} du \ e^{-s^{2}/4 - m^{2}u} K_{u}(a_{i}, a_{j}; g) \left[ e^{s\sqrt{u}\zeta E_{1}} - e^{s\sqrt{u}\zeta E_{2}} \right]$$

$$= \int_{0}^{\infty} ds \ s \int_{0}^{1} d\zeta \ \frac{s}{2\sqrt{\pi}} \int_{0}^{\infty} du \ e^{-s^{2}/4 - m^{2}u} K_{u}(a_{i}, a_{j}; g) \left[ e^{s\sqrt{u}\zeta E_{1}} - e^{s\sqrt{u}\zeta E_{2}} \right]$$

$$= \int_{0}^{\infty} ds \ \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} du \ e^{-s^{2}/4 - m^{2}u} K_{u}(a_{i}, a_{j}; g) \left[ e^{s\sqrt{u}E_{1}} - e^{s\sqrt{u}E_{2}} \right]. \tag{65}$$

After calculating the  $\zeta$  integral, we performed an integration by parts over the variable s. Hence we have found the required result (63).

Similar to the previous problem, let us choose the sequence  $E_k = -k|E_0| = -|E_k|$ , where  $E_0$  is sufficiently below the lower bound  $E_*$  on the ground state energy which has been found in [9] and negative. We now want to show (30). Substituting the resolvent equation, written in the second quantized language, in this expression we obtain the following:

$$\lim_{k \to \infty} ||E_k[(H_0 - E_k)^{-1} + (H_0 - E_k)^{-1}\phi^{(-)}(a_i)\Phi_{ij}^{-1}(E_k)\phi^{(+)}(a_j)(H_0 - E_k)^{-1}]f + f|| = 0. \quad (66)$$

The free resolvent, that is the first term in the above equation,

$$\lim_{k \to \infty} ||E_k R_0(E_k) f + f|| = \lim_{k \to \infty} ||E_k (H_0 - E_k)^{-1} f + f|| = 0$$
(67)

already satisfies the resolvent equation hence we should look only into the second part. To see this, note that by the triangle inequality

$$\lim_{k \to \infty} ||E_k R(E_k) f + f|| \le \lim_{k \to \infty} \left( ||E_k (H_0 - E_k)^{-1} f + f|| + ||E_k [(H_0 - E_k)^{-1} \phi^{(-)} (a_i) \Phi_{ij}^{-1} (E_k) \phi^{(+)} (a_j) (H_0 - E_k)^{-1}] f|| \right)$$

$$\le \lim_{k \to \infty} ||E_k [(H_0 - E_k)^{-1} \phi^{(-)} (a_i) \Phi_{ij}^{-1} (E_k) \phi^{(+)} (a_j) (H_0 - E_k)^{-1}] f|| . \tag{68}$$

We choose a one-particle wave function of the form given below. Even though, this is the most general one-particle wave function one can write down, it does not include multi-particle wave

functions. However, due to the mutually non-interacting nature of the particles involved, the total Hamiltonian appearing in the resolvent will be a sum of n identical, individual Hamiltonians in the case of a n-particle state and therefore will decay faster than in the one-particle case.

$$|\psi\rangle = \int_{\mathcal{M}} d_g^2 x \ \psi(x) \phi^{(-)}(x) |0\rangle = \sum_{\sigma} \hat{\psi}(\sigma) \frac{a_{\sigma}^{\dagger}}{\sqrt{\omega_{\sigma}}} |0\rangle \ . \tag{69}$$

A direct computation now reveals that,

$$\langle \psi | \psi \rangle = \sum_{\sigma} \frac{|\hat{\psi}(\sigma)|^2}{\omega_{\sigma}} \,. \tag{70}$$

We verify that the limit

$$\lim_{k \to \infty} |E_k| || \sum_{i=1}^N \sum_{j=1}^N (H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j) (H_0 + |E_k|)^{-1} |\psi\rangle ||$$
 (71)

$$\leq \lim_{k \to \infty} |E_k| |\sum_{i=1}^N \sum_{j=1}^N |\Phi_{ij}^{-1}(E_k)| ||(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \phi^{(+)}(a_j) (H_0 + |E_k|)^{-1} |\psi\rangle|| ||, \qquad (72)$$

converges to zero. An explicit computation reveals that

$$\phi^{(+)}(a_j)(H_0 + |E_k|)^{-1}|\psi\rangle = \sum_{\sigma} \frac{f_{\sigma}(a_j)\hat{\psi}(\sigma)}{(\omega_{\sigma} + |E_k|)\omega_{\sigma}}|0\rangle.$$
 (73)

The action of  $(H_0 + |E_k|)^{-1}\phi^{(-)}(a_i)$  onto this expression leads to

$$(H_0 + |E_k|)^{-1}\phi^{(-)}(a_i) \sum_{\sigma} \frac{f_{\sigma}(a_j)\hat{\psi}(\sigma)}{(\omega_{\sigma} + |E_k|)\omega_{\sigma}} |0\rangle = \sum_{\sigma} \frac{f_{\sigma}(a_j)\hat{\psi}(\sigma)}{(\omega_{\sigma} + |E_k|)\omega_{\sigma}} \sum_{\sigma'} \frac{f_{\sigma'}(a_i)}{(\omega_{\sigma'} + |E_k|)} \frac{a_{\sigma'}^{\dagger}}{\sqrt{\omega_{\sigma'}}} |0\rangle .$$

$$(74)$$

We have now a term like  $|\Phi_{ij}^{-1}(E_k)| ||F(a_i, a_j|E_k)||$ , which satisfies,

$$\left| \left| \sum_{i,j=1}^{N} |\Phi_{ij}^{-1}(E_k)| \right| |F(a_i, a_j | E_k)| \right| \right| \leq \left[ \text{Tr} \left| \Phi^{-1}(E_k) \right|^2 \right]^{1/2} \left[ \text{Tr} \left| |F(E_k)| \right|^2 \right]^{1/2}.$$
 (75)

This can be used in the above norm, and we get after one more use of the Cauchy-Schwartz inequality,

$$\lim_{k \to \infty} |E_k| ||(H_0 + |E_k|)^{-1} \phi^{(-)}(a_i) \Phi_{ij}^{-1}(E_k) \phi^{(+)}(a_j) (H_0 + |E_k|)^{-1} |\psi\rangle||$$

$$\leq N|E_k| \max_{1 \leq i, j \leq N} |\Phi_{ij}^{-1}(E_k)| \left[ \sum_{i=1}^N \sum_{\sigma} \frac{|f_{\sigma}(a_i)|^2}{(\omega_{\sigma} + |E_k|)^2 \omega_{\sigma}} \right] \left[ \sum_{\tau} \frac{|\hat{\psi}(\tau)|^2}{\omega_{\tau}} \right]^{1/2}, \tag{76}$$

where we have used

$$\sum_{\sigma} \frac{|f_{\sigma}(a_j)\hat{\psi}(\sigma)|}{(\omega_{\sigma} + |E_k|)\omega_{\sigma}} \le \left[\sum_{\sigma} \frac{|f_{\sigma}(a_j)|^2}{(\omega_{\sigma} + |E_k|)^2\omega_{\sigma}}\right]^{1/2} \left[\sum_{\sigma} \frac{|\hat{\psi}(\tau)|^2}{\omega_{\tau}}\right]^{1/2}.$$
 (77)

We recall that by choosing k sufficiently large we can make the off-diagonal elements as small as we like, while the diagonal elements increase. Therefore, without repeating the arguments of the previous section for sufficiently large values of  $|E_k|$ , we can show that

$$\max_{1 \le i,j \le N} |\Phi_{ij}^{-1}(E_k)| \le ||\Phi_{ij}^{-1}(E_k)|| \le 2 \max_{1 \le i \le N} |\Phi_{ii}^{-1}(E_k)|$$
(78)

where

$$\max_{1 \le i \le N} |\Phi_{ii}^{-1}(E_k)| \le \frac{C_{16}}{\ln(|E_k|/(m - \mu_i^{min}))}. \tag{79}$$

and the constant  $C_{16}$  depends on the class of manifolds under consideration. In the above equation m should be superseded on Cartan-Hadamard manifolds by its counterpart  $m_{CH}$ , as defined in reference [9]. Listed below are the values of this constant for compact Riemannian manifolds with positive Ricci curvature, for flat space and for Cartan-Hadamard manifolds.

$$C_{16} = \begin{cases} 2\pi & \text{for flat and compact manifolds} \\ \frac{2\pi}{c(\delta)} & \text{for Cartan-Hadamard manifolds} \end{cases}$$
 (80)

Thus, essentially we are faced with the sum/integral:

$$I = \sum_{\sigma} \frac{|f_{\sigma}(a)|^2}{\omega_{\sigma}(\omega_{\sigma} + |E_k|)^2} . \tag{81}$$

We will work this out: First we recall that

$$\frac{1}{\omega_{\sigma}(\omega_{\sigma} + |E_k|)^2} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 d\zeta \, \frac{\zeta}{[\omega_{\sigma} + \zeta |E_k|]^3} \,. \tag{82}$$

Let us now use the exponential form for the integrand;

$$\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 d\zeta \, \frac{\zeta}{[\omega_\sigma + \zeta |E_k|]^3} = \frac{\Gamma(3)}{2\Gamma(2)\Gamma(1)} \int_0^1 d\zeta \, \zeta \int_0^\infty ds \, s^2 e^{-s\omega_\sigma} e^{-\zeta s|E_k|} \,. \tag{83}$$

and use subordination for  $\omega_{\sigma}$ ,

$$e^{-s\omega_{\sigma}} = \frac{s}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} e^{-s^{2}/4u - m^{2}u} e^{-\lambda(\sigma)u} , \qquad (84)$$

where  $\lambda(\sigma)$  is the eigenvalue of the Laplacian defined in [9]. If we combine the last exponential with  $|f_{\sigma}(a)|^2$  terms, we get the heat kernel at the same points,  $K_u(a, a; g)$  and collecting them, we find

$$I = \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_{0}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} \int_{0}^{1} \mathrm{d}\zeta \, \zeta \int_{0}^{\infty} \mathrm{d}s \, s^{3} \, e^{-s^{2}/4u - m^{2}u} K_{u}(a, a; g) e^{-\zeta s|E_{k}|}$$

$$= \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_{0}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} \int_{0}^{\infty} \mathrm{d}s \, s^{3} \, e^{-s^{2}/4u - m^{2}u} K_{u}(a, a; g) \int_{0}^{1} \mathrm{d}\zeta \, \zeta e^{-\zeta s|E_{k}|}$$

$$= \frac{\Gamma(3)}{4\sqrt{\pi}\Gamma(2)\Gamma(1)} \int_{0}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} \int_{0}^{\infty} \mathrm{d}s \, s^{3} \, e^{-s^{2}/4u - m^{2}u} K_{u}(a, a; g) \frac{1}{s^{2}|E_{k}|^{2}} \left[1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|}\right]$$

$$\leq \frac{\Gamma(3)}{4\sqrt{\pi}|E_{k}|^{2}\Gamma(2)\Gamma(1)} \int_{0}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} \int_{0}^{\infty} \mathrm{d}s \, s \, e^{-s^{2}/4u - m^{2}u} \left[\frac{C_{17}}{A(\mathcal{M})} + \frac{C_{18}}{u}\right] \left[1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|}\right]$$

$$= \frac{1}{|E_{k}|^{2}} \int_{0}^{\infty} \mathrm{d}u \left[\frac{C_{19}}{A(\mathcal{M})u^{3/2}} + \frac{C_{20}}{u^{5/2}}\right] \int_{0}^{\infty} \mathrm{d}s \, s \, e^{-s^{2}/4u - m^{2}u} \left[1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|}\right]$$

$$= \frac{1}{|E_{k}|^{2}} \int_{0}^{\infty} \mathrm{d}s \, s \left[1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|}\right] \int_{0}^{\infty} \mathrm{d}v \left[\frac{C_{19}m}{A(\mathcal{M})v^{3/2}} + \frac{C_{20}m^{3}}{v^{5/2}}\right] e^{-(ms)^{2}/4v - v} . \tag{85}$$

where  $A(\mathcal{M})$  is the area of the manifold. Here the most divergent contribution comes from the last term in the above expression, so we first analyze this term. By inspecting the following integral representation of the modified Bessel function  $K_{3/2}(v)$  [21],

$$K_{3/2}(ms) = \frac{1}{2} \left(\frac{ms}{2}\right)^{3/2} \int_0^\infty \frac{\mathrm{d}v}{v^{3/2+1}} e^{-(ms)^2/4v-v} , \qquad (86)$$

we obtain the following

$$I_2 = \frac{m^{3/2} C_{21}}{|E_k|^2} \int_0^\infty \frac{\mathrm{d}s}{\sqrt{s}} K_{3/2}(ms) \left[ 1 - e^{-s|E_k|} - s|E_k|e^{-s|E_k|} \right]. \tag{87}$$

We now use another integral representation of the modified Bessel function  $K_{3/2}(x)$  [21];

$$K_{3/2}(ms) = \frac{\Gamma(2)2^{3/2}s^{3/2}}{\sqrt{\pi}m^{3/2}} \int_0^\infty dt \, \frac{\cos(mt)}{(t^2 + s^2)^2} \,. \tag{88}$$

As a result we see that

$$I_{2} = \frac{C_{22}}{|E_{k}|^{2}} \int_{0}^{\infty} ds \, s \int_{0}^{\infty} dr \, s \frac{\cos(msr)}{(r^{2}s^{2} + s^{2})^{2}} \left[ 1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|} \right]$$

$$\leq \frac{C_{22}}{|E_{k}|^{2}} \int_{0}^{\infty} \frac{ds}{s^{2}} \left[ 1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|} \right] \int_{0}^{\infty} \frac{dr}{(r^{2} + 1)^{2}}$$

$$\leq \frac{C_{23}}{|E_{k}|^{2}} \int_{0}^{\infty} \frac{ds}{s^{2}} \left[ 1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|} \right]$$

$$\leq \frac{C_{23}}{|E_{k}|} \int_{0}^{\infty} \frac{ds}{s^{2}} \left[ 1 - e^{-s} - se^{-s} \right]. \tag{89}$$

We now note that the integral

$$\int_0^\infty \frac{\mathrm{d}s}{s^2} \Big[ 1 - e^{-s} - se^{-s} \Big] \tag{90}$$

is actually convergent. The first term instead becomes,

$$I_{1} = \frac{2\sqrt{\pi}C_{24}}{|E_{k}|^{2}} \int_{0}^{\infty} ds \left[1 - e^{-s|E_{k}|} - s|E_{k}|e^{-s|E_{k}|}\right] e^{-ms}$$

$$\leq \frac{C_{25}}{|E_{k}|^{3}} \int_{0}^{\infty} ds \left[1 - e^{-s} - se^{-s}\right] e^{-ms/|E_{k}|}$$

$$\leq \frac{C_{26}}{A(\mathcal{M})|E_{k}|^{3}}.$$
(91)

Hence,

$$\sum_{\sigma} \frac{|f_{\sigma}(a)|^2}{\omega_{\sigma}(\omega_{\sigma} + |E_k|)^2} \le \frac{C_{27}}{|E_k|} + \frac{C_{26}}{A(\mathcal{M})|E_k|^3}$$
(92)

is shown. As a result we see that

$$\lim_{k \to \infty} ||[1 - |E_k|R(E_k)]|\psi\rangle|| \le \lim_{k \to \infty} |E_k| \left[ \frac{C_{28}}{|E_k|\ln(|E_k|)} + \frac{C_{29}}{A(\mathcal{M})|E_k|^3 \ln(|E_k|)} \right] \to 0, \quad (93)$$

which proves that our formula defines a densely defined closed operator.

## 4 Non-Relativistic Lee Model in Two and Three Dimensional Riemannian Manifolds

#### 4.1 The Lower Bound on the Ground State Energy

After the renormalization of the model in [7], the principal operator was given explicitly in three dimensions. We can similarly extend the calculations given in three dimensions to the two dimensional case [10], so that we have

$$\Phi(E) = H_0 - E + \mu + \lambda^2 \int_0^\infty dt \ K_t(a, a; g) \left[ e^{-t(m-\mu)} - e^{-t(H_0 + m - E)} \right] 
- \lambda^2 \int_0^\infty dt \int_{\mathcal{M}^2} d_g^D x \, d_g^D y \ K_t(x, a; g) K_t(y, a; g) \, \phi_g^{\dagger}(x) e^{-t(H_0 + 2m - E)} \phi_g(y) , \qquad (94)$$

where D=2,3 and  $\mu$  is the experimentally measured bound state energy of the system consisting of a boson and the attractive fermion at the center. In this section we will restrict E to the real axis. In order to give the proof that the energy E is bounded from below, we split the principal operator as

$$\Phi(E) = K(E) - U(E) , \qquad (95)$$

such that

$$K(E) = H_0 - E + \mu \,, \tag{96}$$

and

$$U(E) = U_1(E) + U_2(E) = -\lambda^2 \int_0^\infty dt \ K_t(a, a; g) \left[ e^{-t(m-\mu)} - e^{-t(H_0 + m - E)} \right]$$
  
+  $\lambda^2 \int_0^\infty dt \int_{\mathcal{M}^2} d_g^D x \, d_g^D y \ K_t(x, a; g) K_t(y, a; g) \, \phi_g^{\dagger}(x) \, e^{-t(H_0 + 2m - E)} \, \phi_g(y) \ .$  (97)

It follows immediately that  $K(E) \ge nm - E + \mu$ , so it is a positive definite operator from our assumption  $E < nm + \mu$ . Due to the positivity of the heat kernel and since the difference of the two exponentials is a positive operator, the first integral term  $U_1(E)$  is a negative operator. We thus remark that

$$U(E) \le U_2(E) \ . \tag{98}$$

This clearly forces

$$\Phi(E) \ge K(E) - U_2(E) , \qquad (99)$$

or rewriting it as

$$\Phi(E) \ge K(E)^{1/2} \left(1 - \tilde{U}_2(E)\right) K(E)^{1/2} , \qquad (100)$$

where  $\tilde{U}_2(E) = K(E)^{-1/2} U_2(E) K(E)^{-1/2}$  and K(E),  $U_2(E)$  are positive operators (so is  $\tilde{U}_2(E)$ ). It must be emphasized that the unique square root of the positive self-adjoint operators K(E) are well defined for all real values of E below  $\mu$ . We will now show that by choosing E sufficiently small it is always possible to make the operator  $\Phi(E)$  strictly positive, hence it becomes invertible, and has no zeros beyond this particular value of E (in the last section, the self-adjointness will be further clarified). Therefore, if we impose

$$||\tilde{U}_2(E)|| < 1 , (101)$$

then the principal operator  $\Phi(E)$  becomes strictly positive. For Cartan-Hadamard manifolds, we have obtained in [7]

$$||\tilde{U}_2(E)|| \le n C_{30} m^{D/2} \frac{\lambda^2 \Gamma(2)}{\Gamma(1/2)^2} (nm + \mu - E)^{\frac{D}{2} - 2} \Gamma(2 - \frac{D}{2}) \left[ \frac{\sqrt{\pi} \Gamma(1 - \frac{D}{4})}{\Gamma(\frac{3}{2} - \frac{D}{4})} \right]^2.$$
 (102)

Then the strict positivity of the principal operator (101) implies a lower bound for the ground state energy

$$E_{gr} \ge nm + \mu - \left(nC_{31}\lambda^2 m^{D/2}\right)^{\frac{1}{2-\frac{D}{2}}},$$
 (103)

where

$$C_{31} = C_{30} \frac{\pi \Gamma(2) \Gamma(1 - \frac{D}{4})^2 \Gamma(2 - \frac{D}{2})}{\Gamma(\frac{1}{2})^2 \Gamma(\frac{3}{2} - \frac{D}{4})^2} . \tag{104}$$

For the compact manifolds with Ricci curvature bounded from below by  $-K \geq 0$ , we have similarly obtained

$$||\tilde{U}_{2}(E)|| \leq n \frac{\lambda^{2} \Gamma(2)}{\Gamma(1/2)^{2}} \left[ \frac{4}{V(\mathcal{M})\mu^{\frac{D}{2}}} + \frac{4A'^{1/2}m^{D/4}\pi^{1/2}\Gamma(2 - \frac{D}{4})\Gamma(1 - \frac{D}{4})}{\mu^{\frac{D}{4}}V(\mathcal{M})^{1/2}\Gamma(\frac{3}{2} - \frac{D}{4})} + \frac{A'm^{D/2}\pi\Gamma(2 - \frac{D}{2})\Gamma(1 - \frac{D}{4})^{2}}{\Gamma(\frac{3}{2} - \frac{D}{4})^{2}} \right] \frac{1}{(nm + \mu - E)^{2 - \frac{D}{2}}},$$
(105)

so the lower bound of the ground state energy was found

$$E_{ar} \ge nm + \mu - \left(n\lambda^2 C_{32}\right)^{\frac{1}{2-\frac{D}{2}}}$$
, (106)

where

$$C_{32} = \frac{\Gamma(2)}{\Gamma(1/2)^2} \left[ \frac{4}{V(\mathcal{M})\mu^{\frac{D}{2}}} + \frac{4A'^{1/2}m^{D/4}\pi^{1/2}\Gamma(2 - \frac{D}{4})\Gamma(1 - \frac{D}{4})}{\mu^{\frac{D}{4}}V(\mathcal{M})^{1/2}\Gamma(\frac{3}{2} - \frac{D}{4})} + \frac{A'm^{D/2}\pi\Gamma(2 - \frac{D}{2})\Gamma(1 - \frac{D}{4})^2}{\Gamma(\frac{3}{2} - \frac{D}{4})^2} \right].$$
(107)

Therefore, the lower bounds on the ground state energies for different classes of manifolds (103) and (106) are of almost the same form up to a constant factor, so the form of the lower bound has a general character.

### 4.2 Existence of the Hamiltonian for the Lee Model in Two and Three Dimensional Riemannian Manifolds

The explicit formula for the resolvent of the Hamiltonian in terms of the inverse of the principal operator  $\Phi^{-1}(E)$  is given in [7, 10] by

$$R(E) = \frac{1}{H - E} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} , \qquad (108)$$

where

$$\alpha = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^{\dagger} \Phi^{-1}(E) b \frac{1}{H_0 - E}$$

$$\beta = -\Phi^{-1}(E) b \frac{1}{H_0 - E}$$

$$\gamma = -\frac{1}{H_0 - E} b^{\dagger} \Phi^{-1}(E)$$

$$\delta = \Phi^{-1}(E)$$

$$b = \lambda \phi_g(a) . \tag{109}$$

Let us check that the resolvent identity  $R(E_1) - R(E_2) = (E_1 - E_2)R(E_1)R(E_2)$  is satisfied, that is, we must have

$$\begin{pmatrix}
\alpha(E_{1}) - \alpha(E_{2}) & \gamma(E_{1}) - \gamma(E_{2}) \\
\beta(E_{1}) - \beta(E_{2}) & \delta(E_{1}) - \delta(E_{2})
\end{pmatrix}$$

$$= (E_{1} - E_{2}) \begin{pmatrix}
\alpha(E_{1})\alpha(E_{2}) + \gamma(E_{1})\beta(E_{2}) & \alpha(E_{1})\gamma(E_{2}) + \gamma(E_{1})\delta(E_{2}) \\
\beta(E_{1})\alpha(E_{2}) + \delta(E_{1})\beta(E_{2}) & \beta(E_{1})\gamma(E_{2}) + \delta(E_{1})\delta(E_{2})
\end{pmatrix} . (110)$$

We first consider the first diagonal element of the above matrix. Using the fact that free resolvent satisfies the resolvent identity, we get

$$R_{0}(E_{1})b^{\dagger}\Phi^{-1}(E_{1})\left[\Phi(E_{1}) - \Phi(E_{2}) + b(R_{0}(E_{1}) - R_{0}(E_{2}))b^{\dagger} + E_{1} - E_{2}\right]\Phi^{-1}(E_{2})bR_{0}(E_{2}) = 0$$
(111)

Let us look at the term in the square bracket more closely. By using the explicit expression of the principal operator (94), this term becomes

$$\lambda^{2} \int_{0}^{\infty} dt \ K_{t}(a, a; g) \left[ e^{-t(H_{0} + m - E_{2})} - e^{-t(H_{0} + m - E_{1})} \right]$$

$$+ \lambda^{2} \int_{0}^{\infty} dt \int_{\mathcal{M}^{2}} d_{g}^{D} x \, d_{g}^{D} y \ K_{t}(x, a; g) K_{t}(y, a; g) \, \phi_{g}^{\dagger}(x) \left[ e^{-t(H_{0} + 2m - E_{2})} - e^{-t(H_{0} + 2m - E_{1})} \right] \phi_{g}(y)$$

$$+ \lambda^{2} \phi_{g}(a) \left[ (H_{0} - E_{1})^{-1} - (H_{0} - E_{2})^{-1} \right] \phi_{g}^{\dagger}(a) .$$

$$(112)$$

One can shift the operator  $\phi_g^{\dagger}(x)$  to the left

$$\frac{1}{H_0 - E} \phi_g^{\dagger}(x) = \int_{\mathcal{M}} d_g^D x' \, \phi_g^{\dagger}(x') \int_0^{\infty} dt \, e^{-t(H_0 + m - E)} \, K_t(x, x'; g) \,, \tag{113}$$

and shift the operator  $\phi_g(x)$  to the right

$$\phi_g(x) \frac{1}{H_0 - E} = \int_M d_g^D x' \int_0^\infty dt \ e^{-t(H_0 + m - E)} K_t(x, x'; g) \phi_g(x') , \qquad (114)$$

which we have also used in [7] for the renormalization. The last term in the equation (112) can be normal ordered as

$$\lambda^{2} \int_{0}^{\infty} dt \ K_{t}(a, a; g) \left[ e^{-t(H_{0} + m - E_{1})} - e^{-t(H_{0} + m - E_{2})} \right]$$

$$+ \lambda^{2} \int_{0}^{\infty} dt \int_{\mathcal{M}^{2}} d_{g}^{D} x \, d_{g}^{D} y \ K_{t}(x, a; g) K_{t}(y, a; g) \, \phi_{g}^{\dagger}(x) \left[ e^{-t(H_{0} + 2m - E_{1})} - e^{-t(H_{0} + 2m - E_{2})} \right] \phi_{g}(y)$$

$$(115)$$

Then we prove that

$$\Phi(E_1) - \Phi(E_2) + b(R_0(E_1) - R_0(E_2))b^{\dagger} + E_1 - E_2 = 0.$$
(116)

The other term in the matrix equality (110)

$$\gamma(E_1) - \gamma(E_2) = (E_1 - E_2) \left[ \alpha(E_1)\gamma(E_2) - \gamma(E_1)\delta(E_2) \right]$$
(117)

can be written as

$$-R_0(E_1)b^{\dagger}\Phi^{-1}(E_1)\left[\Phi(E_1)-\Phi(E_2)+b(R_0(E_1)-R_0(E_2))b^{\dagger}+E_1-E_2\right]\Phi^{-1}(E_2)=0, (118)$$

due to (116). Similarly, the other terms can be put into the following forms

$$\Phi^{-1}(E_1) \left[ \Phi(E_1) - \Phi(E_2) + b(R_0(E_1) - R_0(E_2))b^{\dagger} + E_1 - E_2 \right] \Phi^{-1}(E_2)bR_0(E_2) = 0$$

$$\Phi^{-1}(E_1) \left[ \Phi(E_1) - \Phi(E_2) + b(R_0(E_1) - R_0(E_2))b^{\dagger} + E_1 - E_2 \right] \Phi^{-1}(E_2) = 0 , \qquad (119)$$

and they are all satisfied thanks to the equality (116). Hence, we prove that the resolvent identity is satisfied.

Recall that the resolvent for the Lee model is defined in the following Fock space  $\mathcal{F}_{\mathcal{B}}^{(n+1)}(\mathcal{H}) \otimes \chi_{+} \oplus \mathcal{F}_{\mathcal{B}}^{(n)}(\mathcal{H}) \otimes \chi_{-}$ , for any given  $n \in \mathbb{N}$ , and  $\chi_{\pm}$  is the spin states. In matrix form, we have  $R(E) : \mathcal{F}_{\mathcal{B}}^{(n+1)}(\mathcal{H}) \oplus \mathcal{F}_{\mathcal{B}}^{(n)}(\mathcal{H}) \to \mathcal{F}_{\mathcal{B}}^{(n+1)}(\mathcal{H}) \oplus \mathcal{F}_{\mathcal{B}}^{(n)}(\mathcal{H})$ . Then we must show that

$$||E_k R(E_k)|f\rangle + |f\rangle|| = |||E_k R(E_k)|f\rangle - |f\rangle|| \to 0, \qquad (120)$$

as  $k \to \infty$ . Here  $|f\rangle \in \mathcal{F}_{\mathcal{B}}^{(n+1)}(\mathcal{H}) \oplus \mathcal{F}_{\mathcal{B}}^{(n)}(\mathcal{H})$  and the norm is taken with respect to  $\mathcal{F}_{\mathcal{B}}^{(n+1)}(\mathcal{H}) \oplus \mathcal{F}_{\mathcal{B}}^{(n)}(\mathcal{H})$ . Let us decompose the vector  $|f\rangle$  as

$$\begin{pmatrix} |f^{(n+1)}\rangle \\ |f^{(n)}\rangle \end{pmatrix} \tag{121}$$

where

$$|f^{(n)}\rangle = \int_{\mathcal{M}^n} d_g^D x_1 \dots d_g^D x_n \ f(x_1, x_2, \dots, x_n) |x_1, x_2, \dots, x_n\rangle \ .$$
 (122)

So we have

$$\left| \left( |E_{k}|\alpha(E_{k}) | |E_{k}|\gamma(E_{k}) | |f^{(n+1)}\rangle | |f^{(n+1)}\rangle | |E_{k}|\beta(E_{k}) | |E_{k}|\delta(E_{k}) | |f^{(n+1)}\rangle | |E_{k}|\gamma(E_{k})|f^{(n)}\rangle | \right) \right| \\
= \left[ |||E_{k}|\alpha(E_{k})||f^{(n+1)}\rangle - |f^{(n+1)}\rangle + |E_{k}|\gamma(E_{k})||f^{(n)}\rangle ||^{2} \right] \\
+ |||E_{k}|\beta(E_{k})||f^{(n+1)}\rangle + |E_{k}|\delta(E_{k})||f^{(n)}\rangle - |f^{(n)}\rangle ||^{2} \right] \\
\leq \left[ \left( |||E_{k}|\alpha(E_{k})||f^{(n+1)}\rangle - |f^{(n+1)}\rangle || + |||E_{k}|\gamma(E_{k})||f^{(n)}\rangle || \right)^{2} \\
+ \left( |||E_{k}|\beta(E_{k})||f^{(n+1)}\rangle || + |||E_{k}|\delta(E_{k})||f^{(n)}\rangle - |f^{(n)}\rangle || \right)^{2} \right]^{1/2}, \tag{123}$$

since  $||A + B|| \le ||A|| + ||B||$ . We shall investigate each norm separately. Let us first consider the term  $|||E_k|\beta(E_k)|f^{(n+1)}\rangle||$ 

$$||\lambda|E_k|\Phi^{-1}(E_k)\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle|| \le \lambda|E_k|||\Phi^{-1}(E_k)||||\phi_g(a)\frac{1}{H_0+|E_k|}|f^{(n+1)}\rangle||. (124)$$

Using the formula (114) for  $E = -|E_k|$  and x = a, we get

$$||\phi_{g}(a)\frac{1}{H_{0} + |E_{k}|}|f^{(n+1)}\rangle|| \leq ||\int_{\mathcal{M}} d_{g}^{D}x \int_{0}^{\infty} dt \ e^{-t|E_{k}|}K_{t}(x, a; g)\phi_{g}(x)|f^{(n+1)}\rangle||$$

$$\leq \left[\int_{\mathcal{M}} d_{g}^{D}x \left(\int_{0}^{\infty} dt \ e^{-t|E_{k}|}K_{t}(x, a; g)\right)^{2}\right]^{1/2} \sqrt{n+1}|||f^{(n+1)}\rangle||.$$
(125)

Let us first consider the compact manifolds. Then, one can take the integral over the variable t by the help of the upper bound of the heat kernel (19) for compact manifolds and obtain

$$||\phi_{g}(a)\frac{1}{H_{0} + |E_{k}|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle||$$

$$\times \left[ \int_{\mathcal{M}} d_{g}^{D}x \left( \frac{md^{2}(x,a)}{|E_{k}|} \right) \left( \frac{C_{33}}{V(\mathcal{M})} K_{1} \left( 2d(x,a) \sqrt{m|E_{k}|/C_{3}} \right) + C_{34} \left( \frac{m|E_{k}|}{d^{2}(x,a)} \right)^{D/4} K_{\frac{D}{2}-1} \left( 2d(x,a) \sqrt{m|E_{k}|/C_{3}} \right) \right)^{2} \right]^{1/2}.$$

$$(126)$$

We now choose Riemann normal coordinates around the point a, assuming that  $\delta < \operatorname{inj}(a)$ . Then, we split the integration region into the two parts as  $\int_{\mathcal{M}} = \int_{B_{\delta}(a)} + \int_{\mathcal{M}\setminus B_{\delta}(a)}$ . Expressing

the first integral in the Gaussian spherical coordinates, we get

$$||\phi_{g}(a)\frac{1}{H_{0} + |E_{k}|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle||$$

$$\times \left[\int_{\mathbb{S}^{D-1}} d\Omega \int_{0}^{\delta} dr \ r^{D+1} \left(\frac{mA_{+}^{D-1}(K_{1},0)}{|E_{k}|}\right) \left[\frac{C_{33}}{V(\mathcal{M})}K_{1}\left(2r\sqrt{m|E_{k}|/C_{3}}\right)\right]^{2} + C_{34} \left(\frac{m|E_{k}|}{r^{2}}\right)^{D/4} K_{\frac{D}{2}-1} \left(2r\sqrt{m|E_{k}|/C_{3}}\right)\right]^{2}$$

$$+ \int_{\mathcal{M}\backslash B_{\delta}(a)} d_{g}^{D} x \left(\frac{md^{2}(x,a)}{|E_{k}|}\right) \left[\frac{C_{33}}{V(\mathcal{M})}K_{1}\left(2d(x,a)\sqrt{m|E_{k}|/C_{3}}\right)\right]^{2} + C_{34} \left(\frac{m|E_{k}|}{d^{2}(x,a)}\right)^{D/4} K_{\frac{D}{2}-1} \left(2d(x,a)\sqrt{m|E_{k}|/C_{3}}\right)\right]^{1/2} ,$$

$$(127)$$

where we have used the equations (46) and (51). Let us now consider the first integral. It is smaller than the following expression

$$\int_{\mathbb{S}^{D-1}} d\Omega \int_{0}^{\infty} dr \ r^{D+1} \left( \frac{m A_{+}^{D-1}(K_{1}, 0)}{|E_{k}|} \right) \left[ \frac{C_{33}}{V(\mathcal{M})} K_{1} \left( 2r \sqrt{m|E_{k}|/C_{3}} \right) + C_{34} \left( \frac{m|E_{k}|}{r^{2}} \right)^{D/4} K_{\frac{D}{2}-1} \left( 2r \sqrt{m|E_{k}|/C_{3}} \right) \right]^{2}.$$
(128)

One can evaluate the integrals [22]

$$\int_{0}^{\infty} dr \ r^{D+1} K_{1}^{2}(ar) = \frac{\sqrt{\pi} \Gamma(1 + \frac{D}{2}) \Gamma(2 + \frac{D}{2}) \Gamma(\frac{D}{2})}{4a^{D+2} \Gamma(\frac{3+D}{2})}$$

$$\int_{0}^{\infty} dr \ r K_{\frac{D}{2}-1}^{2}(ar) = \frac{\pi(D-2) \csc(\pi D/2)}{4a^{2}}$$

$$\int_{0}^{\infty} dr \ r^{\frac{D}{2}+1} K_{1}(ar) K_{\frac{D}{2}-1}(ar) = \frac{2^{\frac{D}{2}} \Gamma(\frac{D}{2})}{(D+2)a^{\frac{D}{2}+2}},$$
(129)

where  $a \in \mathbb{R}^+$  and D = 2, 3. Then the upper bound of the first integral in (127) becomes

$$\frac{mA_{+}^{D-1}(K_{1},0)}{|E_{k}|} \left( \frac{C_{35}}{V^{2}(\mathcal{M})} (m|E_{k}|)^{-\frac{(D+2)}{2}} + C_{36}(m|E_{k}|)^{\frac{D}{2}-1} + \frac{C_{37}}{V(\mathcal{M})} (m|E_{k}|)^{-1} \right). \tag{130}$$

For Cartan-Hadamard manifolds, we do not repeat the analysis above because the upper bound of the heat kernel for Cartan-Hadamard manifolds given in the equation (19) corresponds to removing the volume term from the one for the compact manifolds. As a result, we get the upper bound of the first term in the equation (127) for Cartan-Hadamard manifolds

$$\frac{mC_{38}}{|E_k|}(m|E_k|)^{\frac{D}{2}-1}. (131)$$

Let us now consider the second term in the equation (127) for compact and Cartan-Hadamard manifolds. Due to the upper bounds of the Bessel functions used in [8], we find for compact manifolds

$$\int_{\mathcal{M}\backslash B_{\delta}(a)} d_{g}^{D} x \left(\frac{md^{2}(x,a)}{|E_{k}|}\right) \left[\frac{C_{33}}{V(\mathcal{M})} K_{1} \left(2d(x,a)\sqrt{m|E_{k}|/C_{3}}\right) + C_{34} \left(\frac{m|E_{k}|}{d^{2}(x,a)}\right)^{D/4} K_{\frac{D}{2}-1} \left(2d(x,a)\sqrt{m|E_{k}|/C_{3}}\right)\right]^{2} \\
\leq \int_{\mathcal{M}\backslash B_{\delta}(a)} d_{g}^{D} x \left(\frac{md^{2}(x,a)}{|E_{k}|}\right) \left[\frac{C_{33}}{V(\mathcal{M})} \exp\left(-d(x,a)\sqrt{m|E_{k}|/C_{3}}\right) \right] \\
\times \left(\frac{1}{2d(x,a)\sqrt{m|E_{k}|/C_{3}}} + \frac{1}{2}\right) + C_{34} \left(\frac{m|E_{k}|}{d^{2}(x,a)}\right)^{D/4} \frac{2 \exp\left(-\frac{2d(x,a)}{(4-D)}\sqrt{m|E_{k}|/C_{3}}\right)}{(2d(x,a)\sqrt{m|E_{k}|/C_{3}})^{(4-D)/2}}\right]^{2}.$$
(132)

Since  $d(x, a) \geq \delta$  for all  $x \in \mathcal{M} \setminus B_{\delta}(a)$ , the upper bound of the above equation is

$$\exp\left(-\delta\sqrt{m|E_k|/C_3}\right) \int_{\mathcal{M}\backslash B_{\delta}(a)} d_g^D x \left(\frac{md^2(x,a)}{|E_k|}\right) \left[\frac{C_{33}}{V(\mathcal{M})} \exp\left(-\frac{d(x,a)}{2}\sqrt{m|E_k|/C_3}\right) \times \left(\frac{1}{2\delta\sqrt{m|E_k|/C_3}} + \frac{1}{2}\right) + C_{34} \left(\frac{m|E_k|}{\delta^2}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_k|/C_3})^{(4-D)/2}} \times \exp\left(\left(\frac{d(x,a)}{2} - \frac{2d(x,a)}{(4-D)}\right)\sqrt{m|E_k|/C_3}\right)\right]^2.$$

$$(133)$$

For compact manifolds, we have a simplification. This upper bound above is smaller than

$$\exp\left(-\delta\sqrt{m|E_{k}|/C_{3}}\right) \int_{\mathcal{M}} d_{g}^{D} x \left(\frac{md^{2}(x,a)}{|E_{k}|}\right) \left[\frac{C_{33}}{V(\mathcal{M})} \left(\frac{1}{2\delta\sqrt{m|E_{k}|/C_{3}}} + \frac{1}{2}\right) + C_{34} \left(\frac{m|E_{k}|}{\delta^{2}}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_{k}|/C_{3}})^{(4-D)/2}}\right]^{2}.$$
(134)

Due to the fact the geodesic distance between any two points on the manifold and the volume of the manifold is finite, that is,  $d(x, a) \leq d_{max}(a) = \max_x d(x, a)$ , the upper bound to the above integral can easily be found as

$$\left(\frac{md_{max}^{2}(a)}{|E_{k}|}\right) \exp\left(-\delta\sqrt{m|E_{k}|/C_{3}}\right) V(\mathcal{M}) \left[\frac{C_{33}}{V(\mathcal{M})} \left(\frac{1}{2\delta\sqrt{m|E_{k}|/C_{3}}} + \frac{1}{2}\right) + C_{34} \left(\frac{m|E_{k}|}{\delta^{2}}\right)^{D/4} \frac{2}{(2\delta\sqrt{m|E_{k}|/C_{3}})^{(4-D)/2}}\right]^{2}.$$
(135)

For Cartan-Hadamard manifolds, we similarly find

$$\leq C_{39} m^{D-1} |E_{k}|^{D-3} \exp\left(-\delta \left(m|E_{k}|/C_{5}\right)^{1/2}\right) \int_{\mathcal{M}\backslash B_{\delta}(a)} d_{g}^{D} x \frac{\exp\left[2\left(\frac{d(x,a)}{2} - \frac{2d(x,a)}{(4-D)}\right)\sqrt{m|E_{k}|/C_{5}}\right]}{d^{2}(x,a)} \\
\leq \frac{C_{39} m^{D-1} |E_{k}|^{D-3} \exp\left(-\delta \sqrt{m|E_{k}|/C_{5}}\right)}{\delta^{2}} \int_{\mathcal{M}} d_{g}^{D} x \exp\left[-d(x,a)\left(\frac{4}{4-D} - 1\right)\sqrt{m|E_{k}|/C_{5}}\right], \tag{136}$$

where we have used  $d(x, a) \geq \delta$  for all  $x \in \mathcal{M} \setminus B_{\delta}(a)$ . Let us write the above integral in Gaussian spherical coordinates as we did in Section 2,

$$\int_{\mathbb{S}^{D-1}} d\Omega \int_0^{\rho_\Omega} dr \ r^{D-1} J(r,\theta) \exp\left[-r\left(\frac{4}{4-D}-1\right) \sqrt{m|E_k|/C_5}\right]. \tag{137}$$

To proceed further we assume that  $\mathcal{M}$  has Ricci tensor bounded from below by  $K_1$ . As a result of this and using the equations (46) and (47), the upper bound to the equation (136) becomes

$$\leq \frac{C_{40}m^{D-1}|E_{k}|^{D-3}\exp\left(-\delta\sqrt{m|E_{k}|/C_{5}}\right)}{\delta^{2}(-K_{1})^{(D-1)/2}} \int_{0}^{\infty} dr \sinh^{D-1}(\sqrt{-K_{1}}r) \times \exp\left[-r\left(\frac{4}{4-D}-1\right)\sqrt{m|E_{k}|/C_{5}}\right].$$
(138)

Since  $\sinh^{D-1}(x) \leq e^{(D-1)x}/2^{D-1}$ , we can take the integral and get

$$\frac{C_{41}m^{D-1}|E_k|^{D-3}e^{-\delta\sqrt{m|E_k|/C_5}}}{(-K_1)^{(D-1)/2}\delta^2\left[\left(\frac{4}{4-D}-1\right)\sqrt{m|E_k|/C_5}-(D-1)\sqrt{-K_1}\right]},$$
(139)

as long as  $\left[\left(\frac{4}{4-D}-1\right)\sqrt{m|E_k|/C_5}-(D-1)\sqrt{-K_1}\right]\geq 0$ . But this is always satisfied for sufficiently large values of  $|E_k|$ .

Therefore, if we combine the results (130) and (135) we obtain for compact manifolds that

$$||\phi_{g}(a)\frac{1}{H_{0} + |E_{k}|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle|| \left[\frac{mA_{+}^{D-1}(K_{1},0)}{|E_{k}|}\left(\frac{C_{33}}{V^{2}(\mathcal{M})}(m|E_{k}|)^{-\frac{(D+2)}{2}}\right)\right] + C_{34}(m|E_{k}|)^{\frac{D}{2}-1} + \frac{C_{35}}{V(\mathcal{M})}(m|E_{k}|)^{-1}\right) + \left(\frac{md_{max}^{2}(a)}{|E_{k}|}\right) \exp\left(-\delta\sqrt{m|E_{k}|/C_{3}}\right) \times V(\mathcal{M})\left(\frac{C_{33}}{V(\mathcal{M})}\left(\frac{1}{2\delta\sqrt{m|E_{k}|/C_{3}}} + \frac{1}{2}\right) + C_{34}\left(\frac{m|E_{k}|}{\delta^{2}}\right)^{D/4}\frac{2}{(2\delta\sqrt{m|E_{k}|/C_{3}})^{(4-D)/2}}\right)^{2}\right]^{1/2},$$

$$(140)$$

and the results (131) and (139) for Cartan-Hadamard manifolds give

$$||\phi_{g}(a)\frac{1}{H_{0}+|E_{k}|}|f^{(n+1)}\rangle|| \leq \sqrt{n+1}|||f^{(n+1)}\rangle|| \left[\frac{mC_{38}}{|E_{k}|}(m|E_{k}|)^{\frac{D}{2}-1}\right] + \frac{C_{41}m^{D-1}|E_{k}|^{D-3}e^{-\delta\sqrt{m|E_{k}|/C_{5}}}}{(-K_{1})^{(D-1)/2}\delta^{2}\left(\left(\frac{4}{4-D}-1\right)\sqrt{m|E_{k}|/C_{5}}-(D-1)\sqrt{-K_{1}}\right)}\right]^{1/2}.$$
(141)

We are now going to find an upper bound of the inverse norm of the principal operator. In order to do this, let us recall that we split the principal operator when we try find the lower bound of the ground state energy. We now split the principal operator in the following way:  $\Phi = (K - U_1) - U_2$ , where  $U_1$  and  $U_2$  are defined exactly as before. Then, we have

$$\Phi^{-1} = (K - U_1)^{-1/2} \left[ 1 - (K - U_1)^{-1/2} U_2 (K - U_1)^{-1/2} \right]^{-1} (K - U_1)^{-1/2} . \tag{142}$$

Let us substitute the identity operator  $K^{1/2}K^{-1/2}$  between the operators  $(K - U_1)^{-1/2}$  and  $U_2$ . Hence,

$$\Phi^{-1} = (K - U_1)^{-1/2} [1 - X]^{-1} (K - U_1)^{-1/2} , \qquad (143)$$

where we have defined  $X = (K - U_1)^{-1/2} K^{1/2} \tilde{U}_2 K^{1/2} (K - U_1)^{-1/2}$  for simplicity. Here the following operator can be written as an infinite geometric sum

$$[1 - X]^{-1} = \sum_{l=0}^{\infty} X^{l} , \qquad (144)$$

as long as ||X|| < 1. This leads to

$$||[1-X]^{-1}|| \le [1-||X||]^{-1}$$
 (145)

Since  $-U_1$  is a positive operator,  $(K-U_1)^{-1/2} \leq K^{-1/2}$ . Then, we have

$$||X|| \le ||\tilde{U}_2|| \ . \tag{146}$$

If we make  $|E_k|$  sufficiently large then  $||\tilde{U}_2|| \leq 1/2$  and

$$[1 - ||X||]^{-1} \le 2. (147)$$

As a result of this, we get

$$||\Phi^{-1}(E_k)|| \le 2||K^{-1/2}(E_k)||^2 \le \frac{2}{|E_k|},$$
 (148)

where we have used

$$K^{-1/2}(E_k) = (H_0 + \mu + |E_k|)^{-1/2} \le \frac{1}{|E_k|^{1/2}}.$$
 (149)

Then, substituting the equations (130) and (135) for compact or substituting the equations (131) and (139) for Cartan-Hadamard manifolds into the equation (127) and using the above

upper bound for the inverse principal operator, and taking the limit as  $k \to \infty$ , we eventually obtain

$$|E_k| ||\Phi^{-1}(E_k)|| ||\phi_g(a) \frac{1}{H_0 + |E_k|} |f^{(n+1)}\rangle|| \to 0.$$
 (150)

Let us consider the other terms in the equation (123) now:

$$|||E_{k}|\alpha(E_{k})|f^{(n+1)}\rangle - |f^{(n+1)}\rangle|| \le ||\left(\frac{|E_{k}|}{H_{0} + |E_{k}|} - 1\right)|f^{(n+1)}\rangle|| + \lambda^{2}|E_{k}||\frac{1}{H_{0} + |E_{k}|}\phi_{g}^{\dagger}(a)|||\Phi^{-1}(E_{k})||||\phi_{g}(a)\frac{1}{H_{0} + |E_{k}|}|f^{(n+1)}\rangle||$$
(151)

The *upper* bound for the norm  $||\frac{1}{H_0+|E_k|}\phi_g^{\dagger}(a)||$  can be similarly found, and comes out to be the same as the one for  $||\phi_g(a)\frac{1}{H_0+|E_k|}||$  (the norms are different in general). As a result of this, we obtain

$$||E_k|\alpha(E_k)|f^{(n+1)}\rangle - |f^{(n+1)}\rangle|| \to 0,$$
 (152)

as  $k \to \infty$ . Similarly, the following term

$$|||E_k|\gamma(E_k)|f^{(n)}\rangle|| \le \lambda |E_k| ||\frac{1}{H_0 + |E_k|} \phi_g^{\dagger}(a)|f^{(n)}\rangle|| ||\Phi^{-1}(E_k)||$$
(153)

and

$$|||E_k|\beta(E_k)|f^{(n+1)}\rangle|| \le \lambda |E_k| ||\phi_g(a)\frac{1}{H_0 + |E_k|}|f^{(n+1)}\rangle|| ||\Phi^{-1}(E_k)||$$
(154)

both vanishes as  $k \to \infty$ . Moreover, we have

$$||(|E_{k}|\delta(E_{k})-1)|f^{(n)}\rangle|| = ||[|E_{k}|\Phi^{-1}(E_{k})-1]|f^{(n)}\rangle||$$

$$= ||[|E_{k}|K^{-1/2}(E_{k})[1+(1-\tilde{U}(E_{k}))^{-1}-1]K^{-1/2}(E_{k})-1]|f^{(n)}\rangle||$$

$$\leq ||(|E_{k}|K^{-1}(E_{k})-1)|f^{(n)}\rangle|| + |||E_{k}|K^{-1/2}(E_{k})[(1-\tilde{U}(E_{k}))^{-1}-1]K^{-1/2}(E_{k})|f^{(n)}\rangle||$$

$$= ||(|E_{k}|K^{-1}(E_{k})-1)|f^{(n)}\rangle|| + |||E_{k}|K^{-1/2}(E_{k})(1-\tilde{U}(E_{k}))^{-1}\tilde{U}(E_{k})K^{-1/2}(E_{k})|f^{(n)}\rangle||, (155)$$

where we have used the fact that the factor  $(1 - \tilde{U}(E_k))^{-1}$  can be considered as an infinite geometric sum. The first term goes to zero as  $k \to \infty$  since

$$\left| \left| \left( \frac{|E_{k}|}{H_{0} + |E_{k}| + \mu} - 1 \right) |f^{(n)}\rangle \right| \right| = \left| \left| \left( \frac{|E_{k}| + \mu - \mu}{H_{0} + |E_{k}| + \mu} - 1 \right) |f^{(n)}\rangle \right| \right|$$

$$\leq \left| \left| \left( \frac{|E_{k}| + \mu}{H_{0} + |E_{k}| + \mu} - 1 \right) |f^{(n)}\rangle \right| \right| + \left| \left| \left( \frac{\mu}{H_{0} + |E_{k}| + \mu} \right) |f^{(n)}\rangle \right| \right|$$

$$\leq \left| \left| \left( \frac{|E_{k}| + \mu}{H_{0} + |E_{k}| + \mu} - 1 \right) |f^{(n)}\rangle \right| \right| + \frac{\mu}{|E_{k}|} \left| \left| \left| f^{(n)}\rangle \right| \right|,$$

$$(156)$$

where the term containing  $H_0$  vanishes as  $k \to \infty$  because it is the free resolvent and the second part clearly goes to zero. This shows that the first term in the equation (155) vanishes in the limit. As for the second term, it is smaller than

$$|E_k|||K^{-1/2}(E_k)|| \left[ (1-||\tilde{U}(E_k)||)^{-1} \right] \left[ ||\tilde{U}_1(E_k)|| + ||\tilde{U}_2(E_k)|| \right] ||K^{-1/2}(E_k)|| |||f^{(n)}\rangle|| . (157)$$

Since  $m > \mu$  for bound states, one can easily see that

$$||\tilde{U}_{1}(E_{k})|| \leq C_{42}\lambda^{2}||(H_{0} + m + |E_{k}|)^{-1/2} \int_{0}^{\infty} dt \ K_{t}(a, a; g) \left[e^{-t(m-\mu)} - e^{-t(H_{0} + m + |E_{k}|)}\right] \times (H_{0} + m + |E_{k}|)^{-1/2}||$$

$$\leq \begin{cases} C_{43}\lambda^{2}||(H_{0} + m + |E_{k}|)^{-1} \ln\left(\frac{H_{0} + m + |E_{k}|}{m - \mu}\right)|| & \text{for } D = 2 \\ C_{44}\lambda^{2}||(H_{0} + m + |E_{k}|)^{-1/2}|| & \text{for } D = 3 \end{cases}.$$

$$(158)$$

Here we use the fact that the operator in the parenthesis, which we call A(s) is positive, and for a positive family, if two integrable functions satisfy  $0 \le f(s) \le g(s)$ , then  $\int ds \ f(s)A(s) \le \int ds \ g(s)A(s)$ . Moreover, for positive operators, order relation implies the same ordering for the norms of the operators. For simplicity, we have also disregarded the more convergent in  $|E_k|$  coming from the volume terms of upper bound of the heat kernel for compact manifolds. We now note that

$$||\frac{1}{H_0 + |E_k| + m} \ln \left( \frac{H_0 + |E_k| + m}{m - \mu} \right)|| = ||\int_0^1 \frac{\mathrm{d}t}{[H_0 + |E_k| + m]t + m - \mu}||$$

$$\leq ||\int_0^1 \frac{\mathrm{d}t}{[H_0 + |E_k| + m]t^2 + m - \mu}||$$

$$\leq ||\frac{1}{\sqrt{m - \mu}[H_0 + |E_k| + m]^{1/2}}||\int_0^\infty \frac{\mathrm{d}s}{s^2 + 1}$$

$$\leq \frac{C_{45}}{\sqrt{m - \mu}|E_k|^{1/2}} .$$

$$(159)$$

Hence, we get

$$||\tilde{U}_1(E_k)|| \le \begin{cases} \frac{C_{46}\lambda^2}{\sqrt{m-\mu}|E_k|^{1/2}} & \text{for } D=2\\ \frac{C_{47}\lambda^2}{|E_k|^{1/2}} & \text{for } D=3 \end{cases}$$
 (160)

Using the results (102) and (105) for  $E = -|E_k|$  with the above analysis, we finally obtain

$$|E_k|||K^{-1}(E_k)|| ||\tilde{U}(E_k)|| ||(1 - \tilde{U}(E_k))^{-1}|| |||f^{(n)}\rangle|| \to 0,$$
 (161)

as  $k \to \infty$  and this completes the proof that our renormalized formula corresponds to the resolvent of a densely defined closed operator.

We will further show that  $\Phi(E)$  is a holomorphic self-adjoint family of type (A) in the sense of Kato [13]. This will in turn justify the claim that the resolvent corresponds to a self-adjoint operator. To prove this we will use the theorem given by R. Wüst [23]. First, we define a holomorphic family of type (A) as follows: Let  $G \subset \mathbb{C}$  be domain and L(z) be a family of closed linear operators, acting on a Hilbert space  $\mathcal{H}$ ,  $\{L(z)|z\in G\}$ . If

- 1) the domain  $\mathcal{D}(L(z)) = \mathcal{D}$  is independent of  $z \in G$
- 2) for any  $f \in \mathcal{D}$ , and  $g \in \mathcal{H}$  then  $\langle g|L(z)|f\rangle$  is holomorphic in G, (162)

then this is a holomorphic family of type (A).

An operator which is a holomorphic family of type (A) is a *self-adjoint* holomorphic family of type (A) if

1) G is a symmetric domain of the complex plane relative to the real axis.

2) 
$$\mathcal{D}$$
 is dense in  $\mathcal{H}$ 

3) 
$$\mathcal{D}(L(z)^{\dagger}) = \mathcal{D}(L(z^*))$$
 for all  $z \in G$ . (163)

**Theorem**(Wüst): Let G be a symmetric domain of complex plane relative to the real axis, and L(z) is a holomorphic family of type (A) defined on G. Assume

1) 
$$\mathcal{D}$$
 is dense in  $\mathcal{H}$ 

2) 
$$\mathcal{D}(L(z)^{\dagger}) \supset \mathcal{D}(L(z^*))$$
. (164)

Let

$$M = \{ z \in G | L(z)^{\dagger} = L(z^*) \} . \tag{165}$$

If M is not the empty set then it is the whole domain G. This implies that the family is a self-adjoint holomorphic family of type (A).

Let us consider our case. We choose the domain G as

$$G = \{ E \in \mathbb{C} | \Re(E) < \mu \} , \tag{166}$$

which is symmetric with respect to the real axis. The principal operator  $\Phi(E)$  given explicitly in (94) formally satisfies the relation  $\Phi(E)^{\dagger} = \Phi(E^*)$  so this implies  $\mathcal{D}(\Phi(E)^{\dagger}) \supset \mathcal{D}(\Phi(E^*))$ . Let us assume that the family is holomorphic for now. Note that a densely defined holomorphic family of operators satisfying the formal relation  $\Phi(E)^{\dagger} = \Phi(E^*)$  is closable. Proof: Let us consider the common domain  $\mathcal{D}$ , and choose  $|g_l\rangle \in \mathcal{D} \to 0$  as  $l \to \infty$ . We further assume that  $\Phi(E)|g_l\rangle$  converges to some  $|g(E)\rangle$ . Then we have

$$\langle f|\Phi(E)g_l\rangle = \langle \Phi^{\dagger}(E)f|g_l\rangle = \langle \Phi(E^*)f|g_l\rangle \to 0 ,$$
 (167)

for any  $|f\rangle \in \mathcal{D}$  as  $l \to \infty$ . This implies that

$$\Phi(E)|g_l\rangle \to |g(E)\rangle = 0. \tag{168}$$

This is the requirement of closure. Of course, we must establish this closure uniformly, that is, we need to show that for every sequence  $|g_l\rangle$  converging to some element, if  $\Phi(E_0)|g_l\rangle$  converges for one  $E_0$  inside the region G, then it converges for all  $E \in G$ . Hence, we can define a unique closure over G, the closures having a common domain  $\mathcal{D}^-$ . Once we determine a common domain for the family  $\Phi(E)$ , we will prove that there is indeed a closure over a common domain.

For the family  $\Phi(E)$ , we choose  $\mathcal{D} = \mathcal{D}(H_0)$ . It is well known that if  $\mathcal{M}$  is a geodesically complete manifold then Laplacian defined on  $\mathcal{M}$  is a closed, densely defined self-adjoint operator [14, 15]. Then, the operator  $H_0 = \int_{\mathcal{M}} d_g^D x \; \phi_g^{\dagger}(x) (-\frac{1}{2m} \nabla_g^2) \phi_g(x)$  defined over the direct products of n copies of the Hilbert space  $L^2(\mathcal{M})$  is also a densely defined (essentially) self-adjoint operator. Moreover, the finite direct sum of such operators will preserve this property.

The first term  $H_0 - E + \mu$  is obviously defined over this domain  $\mathcal{D}(H_0)$  and it is a closed operator. Note that the other term can be defined by the spectral theorem,

$$\int_0^\infty ds \ K_s(a, a; g) \left[ e^{-s(m-\mu)} - e^{-s(H_0 + m - E)} \right]$$
 (169)

and it is a positive operator when E is real and  $\Re(E) < \mu$ . Its domain of definition includes  $\mathcal{D}(H_0)$  when  $\Re(E) < \mu$ . To see this we will use a different integral representation,

$$\int_0^\infty ds \ K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(H_0 - E + m)}]$$

$$= \int_0^\infty ds \ s \ K_s(a, a; g) \int_0^1 du \ e^{-su(H_0 + \mu - E) - s(m - \mu)} (H_0 - E + \mu) \ . \tag{170}$$

When the operator acts on an element  $|f^{(n)}\rangle$  in the domain of  $H_0$ , the norm of the resulting vector is smaller than,

$$\int_0^\infty ds \ s \ K_s(a, a; g) \int_0^1 du \ ||e^{-su(H_0 + \mu - E) - s(m - \mu)}||||(H_0 - E + \mu)|f^{(n)}\rangle|| \ . \tag{171}$$

Now we can estimate the following factor using the bounds on the heat kernels given in (19),

$$\int_{0}^{\infty} ds \ s K_{s}(a, a; g) \int_{0}^{1} du ||e^{-su(H_{0} + \mu - E) - s(m - \mu)}|| 
\leq \int_{0}^{1} du \int_{0}^{\infty} ds \ s \left[ \frac{C_{1}}{V(\mathcal{M})} + \frac{C_{2}}{(s/2m)^{D/2}} \right] e^{-sunm} e^{-su(\mu - \Re(E)) - s(m - \mu)} .$$
(172)

Thus we show that

$$\left| \left| \int_0^\infty ds \ K_s(a, a; g) \left[ e^{-s(H_0 - E + m)} - e^{-s(m - \mu)} \right] \right| \le F(\mu - \Re(E)) \left( \left| \left| H_0 \right| f^{(n)} \right\rangle \right| + \left| \mu - E \right| \left| \left| \left| f^{(n)} \right\rangle \right| \right), \tag{173}$$

where

$$F(\mu - \Re(E)) = \frac{C_1}{(nm + \mu - \Re(E))V(\mathcal{M})} \left( \frac{1}{m - \mu} + \frac{1}{(n+1)m - \Re(E)} \right) + \frac{C_2(2m)^{D/2}\Gamma(2 - \frac{D}{2})}{(nm + \mu - \Re(E)(\frac{D}{2} - 1))} \left[ ((n+1)m - \Re(E))^{\frac{D}{2} - 1} - (m-\mu)^{\frac{D}{2} - 1} \right].$$
(174)

As a result the domain of this operator family includes  $\mathcal{D}(H_0)$ . In fact, by the spectral theorem the operators so defined are closed, when we restrict them to a smaller domain, i. e. to  $\mathcal{D}(H_0)$  they remain closed. So the sum of the two pieces,  $H_0 + \mu - E$  and the term above, defined over  $\mathcal{D}(H_0)$  is closed, since they were already closed operators defined over a common domain.

The last part requires more work, for this we will first show that U(E) is relatively bounded with respect to  $H_0$  hence its domain includes  $\mathcal{D}(H_0)$ . Moreover, if we have a holomorphic family of operators defined over a dense domain, then they are preclosed, that is we can define the closure of this family, as we have shown. It is easy to see that

$$||U(E)H_0^{-1}H_0|f^{(n)}\rangle|| \le ||U(E)H_0^{-1}|| ||H_0|f^{(n)}\rangle||,$$
 (175)

where the first norm can be estimated by exactly the same method developed in [7]. So we are giving the result in order not to repeat the similar calculations, for  $n \ge 1$ ,

$$||U(E)H_0^{-1}|| \leq C_{48}n \int_0^\infty ds \int_0^1 du \ s \ K_{us}^{1/2}(a,a;g) K_s^{1/2}(a,a;g) e^{-snm} e^{su\Re(E)} \ . \tag{176}$$

After using the upper bound of the heat kernel given in (19) and defining new variables  $p = C_1/V(\mathcal{M})$  and  $q(s) = C_2/(s/2m)^{D/2}$  we get

$$||U(E)H_0^{-1}|| \le C_{48}n \int_0^\infty ds \int_0^1 du \ s \ e^{-snm} e^{su\Re(E)} \left[ p^2 + pq(s) + \frac{pq(s)}{u^{3/2}} + \frac{q(s)^2}{u^{3/2}} \right]$$

$$+ \frac{2pq(s)}{u^{3/4}} - \frac{2pq(s)}{u^{3/4}} \right]^{1/2}$$

$$\leq C_{48}n \int_0^\infty ds \int_0^1 du \ s \ e^{-snm} e^{su\Re(E)} \left[ \left( p + \frac{q(s)}{u^{3/4}} \right) + \sqrt{pq(s)} \left( 1 + \frac{1}{u^{3/2}} - \frac{2}{u^{3/4}} \right) \right] . (177)$$

Taking the s and u integral, we obtain

$$||U(E)H_0^{-1}|| \le \frac{nC_{49}}{V(\mathcal{M})(nm-\mu)^2} + \frac{n(2m)^{D/2}C_{50}}{(nm-\mu)^{2-D/2}} + \frac{n(2m)^{D/4}C_{51}}{\sqrt{V(\mathcal{M})}(nm-\mu)^{2-D/4}}, \quad (178)$$

since  $\Re E < \mu$ . Thus we choose the domain of U(E) as  $\mathcal{D}(H_0)$ , and now the family is closable over this domain. However, as a result of the closure, the domains for different values of E may become different. In fact, this does not happen, as we will see.

Now we show that we can perform the closure *uniformly*, as a result of the following: for any  $E_1, E_2 \in G \Phi(E_1) - \Phi(E_2)$  becomes a bounded operator. A short computation shows that,

$$||\Phi(E_1) - \Phi(E_2)|| \le |E_1 - E_2| \left[ 1 + (n+1)\lambda^2 \int_0^\infty ds \ sK_s(a, a; g) e^{-snm} e^{-s(m-\mu)} \right]. \tag{179}$$

If  $|g_l\rangle \in \mathcal{D}$  is convergent to a vector  $|f\rangle$ , and assume that  $\Phi(E_1)|g_l\rangle$  converges to  $|g(E_1)\rangle$  for one  $E_1$ , then we set  $\Phi(E_1)|f\rangle = |g(E_1)\rangle$  to define the closure at point  $E_1$ . Then, for any  $E_2$ , we have

$$||\Phi(E_{2})|g_{l}\rangle - \Phi(E_{2})|f\rangle|| = ||\Phi(E_{2})|g_{l}\rangle - |g(E_{1})\rangle - [\Phi(E_{2}) - \Phi(E_{1})]|f\rangle||$$

$$= ||[\Phi(E_{2}) - \Phi(E_{1})]|g_{l}\rangle + \Phi(E_{1})|g_{l}\rangle - |g(E_{1})\rangle - [\Phi(E_{2}) - \Phi(E_{1})]|f\rangle||$$

$$< ||[\Phi(E_{2}) - \Phi(E_{1})]|| ||g_{l}\rangle - |f\rangle|| + ||\Phi(E_{1})|g_{l}\rangle - |g(E_{1})\rangle|| \mapsto 0,$$
(180)

and this shows that whenever  $|g_l\rangle$  converges to  $|f\rangle$  and  $\Phi(E_1)|g_l\rangle$  converges to  $|g(E_1)\rangle$ , we have  $\Phi(E_2)|g_l\rangle$  becomes convergent and the resulting vector is exactly equal to  $\Phi(E_2)|f\rangle$  as it should be for the requirements of the closure. Hence the sum of all these three parts will make a holomorphic family  $\Phi(E)$  with a dense common domain  $\mathcal{D}(H_0)$ . Moreover, the sum is closable over a dense common domain which we call  $\mathcal{D}(H_0)^-$ .

We would now make holomorphicity more precise, up to now we have not actually made use of it. To prove that the family is holomorphic we will refer to the following theorem, which is stated in a slightly simplified form according to our needs and the proof of which can be found in [24]: Assume X is a measure space with a  $\sigma$ -finite measure  $\nu$  defined on it, let I be a measurable subset of X. Let G be a open domain of the complex plane. Consider a function  $\gamma: I \times G \mapsto \mathbb{C}$  such that

1) 
$$\gamma(x,.) \in L^1(X,|\nu|)$$

2)  $\gamma(.,z)$  is holomorphic in G

3) 
$$\int_{I} |\mathrm{d}\nu| |\gamma(x,z)|$$
 is bounded on all compact subsets of  $G$ . (181)

Then the function

$$\Gamma(z) = \int_I d\nu \ \gamma(x, z)$$
 is holomorphic in  $G$ . (182)

To use the above theorem, let us write our family in the following form  $\Phi(E) = V(E)(H_0 - E + \mu) = [1 + V_1(E) + V_2(E)](H_0 - E + \mu)$ , where

$$V_1(E) = \lambda^2 (H_0 - E + \mu)^{-1} \int_0^\infty ds \ K_s(a, a; g) [e^{-s(m-\mu)} - e^{-s(H_0 - E + m)}]$$

$$V_2(E) = \lambda^2 \int_0^\infty ds \ \phi_g^{\dagger}(x) K_s(x, a; g) e^{-s(H_0 - E + 2m)} K_s(a, y; g) \phi_g(y) (H_0 - E + \mu)^{-1} . (183)$$

By using our previous estimate in (166) and (167) we see that, the first term  $V_1(E)$  is indeed uniformly bounded in  $\Re E < \mu$ . Hence, the integrals  $\langle f^{(n)}|V_1(E)|g^{(n)}\rangle$  for all functions f,g in the Fock space are absolutely convergent. Moreover, any such matrix element satisfies all the other conditions on holomorphicity and integrability. The second part, again using ideas very similar to the previous estimates, can be written as

$$V_2(E) = \lambda^2 \int_0^\infty ds \ s \int_0^1 du \ \phi_g^{\dagger}(K_{su}(.,a;g)) e^{-s(H_0 - E + m + (1-u)\mu + mu)} \phi_g(K_s(.,a;g)) \ . \tag{184}$$

This integrand as a function of E is holomorphic in E for  $\Re E < \mu$  and it is absolutely integrable for any  $\Re E < \mu$  on  $[0, \infty) \times [0, 1]$ . Similarly, it can be shown that the following bound holds

$$|\langle f^{(n)}|V_2(E)|g^{(n)}\rangle| \le n \frac{C_{52}}{(nm - \Re E)^{D/2-1}} |||f^{(n)}\rangle\rangle||||g^{(n)}\rangle||,$$
 (185)

which clearly shows that for  $\Re E < \mu$  is uniformly bounded everywhere, hence on compact subsets as well. Hence, applying the theorem stated above we see that the resulting function  $V_2(E)$  is holomorphic for  $\Re E < \mu$ . There is one subtle point about the closure operation, but this is also solved by the following observation. Let us consider the limit of  $\Phi(E)|g_l\rangle$  as  $l \to \infty$  as a function of E for any convergent  $|g_l\rangle$  sequence in the closure operation. If this sequence is uniformly convergent on compact subsets the limit is a holomorphic function (by an application of Morera's theorem). Note that this family  $\Phi(E)|g_l\rangle$  is norm bounded by a constant multiple of a simple function given in the equation (171),  $|E_1 - E_2|$ . This function itself is uniformly bounded on compact sets centered around any given point  $E_1$ , hence the sequence of functions  $\Phi(E)|g_l\rangle$  is a uniformly convergent sequence. This shows that the closure remains a holomorphic function of E for  $\Re E < \mu$  as required. Thus we complete the proof that the closure of the family  $\Phi(E)$  over  $D(\mathcal{H}_0)^-$  is holomorphic for  $\Re(E) < \mu$ .

Now we are ready to apply the theorem of Wüst. If we choose  $E \in \mathbb{R}$  and sufficiently small  $E < E_*$ , then U(E) has relative bound with respect to K(E) which is less than 1. Hence, by Kato-Rellich theorem [13] of perturbations of self-adjoint operators, U(E) will be self-adjoint for  $E < E_*$ . By the theorem of Wüst, the family is self-adjoint everywhere as desired. This result is important to establish that the spectrum only lies along the real axis, and justifies our search for the lower bound of energy and shows that the resulting operator is self-adjoint as it should be.

#### 5 Conclusion

In this paper, we have proven that for the three models that we have constructed, namely non-relativistic point interactions in two and three dimensional Riemannian manifolds, relativistic point interactions in two dimensional Riemannian manifolds and non-relativistic Lee model in two and three dimensional Riemannian manifolds, the Hamiltonian after renormalization is a densely defined self-adjoint operator.

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