Non-degenerated groundstates in the antiferromagnetic Ising model on triangulations

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Abstract

A triangulation is an embedding of a graph into a closed Riemann surface so that each face boundary is a 3-cycle of the graph. In this work, groundstate degeneracy in the anti-ferromagnetic Ising model on triangulations is studied. We show that for every fixed closed Riemann surface Ω , there are vertex-increasing sequences of triangulations of Ω with a non-degenerated groundstate. In particular, we exhibit geometrically frustrated systems with a non-degenerated groundstate.

1 Introduction

The Ising model is one of the most studied models of interacting particles in statistical physics. This model has been strongly linked to the study of discrete mathematics [9, 5]. In this sense, tools and techniques developed in the discrete setting have shown to be very useful to deal with the solution of problems related to the Ising model and vice versa.

Typically, to study the Ising model, particles are located at the vertices of a graph and the type of interaction between them is determined by the existence and weight of edges in the graph. In this notes, we explore the Ising model where particles and their interaction describe triangulations of closed Riemann surfaces with edge-weight equal to -1.

A triangulation of a closed Riemann surface Ω , or simply a triangulation, is an embedding of a graph in Ω so that each face boundary is a 3-cycle of the graph. Throughout this work, the closed Riemann surface Ω will be specified just in needed cases.

Let's introduce the main ingredients of the Ising model. Given a triangulation T, a state of the Ising model on T is a function σ that assigns to each vertex of T a value from the set $\{+1,-1\}$. In other words, $\sigma \in \{+1,-1\}^{|V(T)|}$, where V(T) denotes the set of vertices of T. Values +1 and -1 are usually called *spins*. Then, we also say that a state on T is a *spin-assignment* on T. For each state σ , the *energy* or *Hamiltonian* of the Ising model on a triangulation T is defined

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by $H(\sigma) = -\sum_{uv \in E(T)} J_{uv} \sigma_u \sigma_v$, where E(T) denotes the set of edges of T and for each uv in E(T) the parameter J_{uv} is called *coupling constant*. The main purpose of a coupling constant J_{uv} is to specify the type of interaction between vertices u and v. In general, this constant may vary from positive to negative depending on the characteristics of the system to be studied. It can also be randomly chosen, like in the case of spin-glasses.

The antiferromagnetic variant of the Ising model takes coupling constant equal to -1 for all edges of the graph. Then, given a triangulation T and a state σ , the energy of σ in the antiferromagnetic Ising model is given by

$$H(\sigma) = \sum_{uv \in E(T)} \sigma_u \sigma_v. \tag{1}$$

Many mathematical problems naturally arise from the Ising model. One of them is the study of states that provide the minimum possible energy for the system. Those states are usually known as groundstates. Related to the study of groundstates, is the groundstate degeneracy, which by definition corresponds to the number of different groundstates that a system supports. If the groundstate degeneracy of a system is greater than two (i.e. more than one pair of groundstates exist), it is said that the groundstate is degenerated. Otherwise, it is called non-degenerated. The groundstate degeneracy has been vastly studied [10, 8], being of great physical interest, because (among others) it determines the entropy of the system, and characterizing entropy's behaviour helps to understand physical phenomena associated to order and stability of the system [14].

Let T be a triangulation and $\sigma \in \{+1, -1\}^{|V(T)|}$ be a state of the antiferromagnetic Ising model on T. It is said that $uv \in E(T)$ is frustrated by σ or that σ frustrates $uv \in E(T)$ if $\sigma_u = \sigma_v$. Observe that each state on T frustrates at least one edge of each face boundary of the triangulation since each face boundary is 3-cycle.

This feature (every state frustrates at least one edge of each face boundary) is known as geometrical frustration. The understanding of order and stability of geometrically frustrated systems, is one of the main questions that condensed matter physicists face to explain. It is expected that systems which exhibit geometrical frustration lead to highly degenerated groundstates with a non-zero entropy at zero temperature (see [11]). In other words, groundstate degeneracy in a geometrically frustrated system is typically exponentially large as a function of the number of vertices of the underlying graph. Indeed, groundstate degeneracy of any plane triangulation is exponential in the number of vertices, since groundstate degeneracy of plane triangulations is twice the number of perfect matchings of cubic bridgeless planar graphs (see [4, 7]).

Surprisingly, in this work it is shown that there are triangulations of closed Riemann surfaces with an arbitrary number of vertices and a non-degenerated antiferromagnetic groundstate. More precisely, we establish the next result.

Theorem 1 Let Ω be a fixed closed Riemann surface with positive genus (g > 0). Then, for every n > 0 there is a triangulation T of Ω with $n \leq |V(T)|$ so that T has a non-degenerated antiferromagnetic groundstate.

In particular, when the closed Riemann surface Ω is a torus, we have the following.

Theorem 2 For every n > 0 there is a toroidal triangulation T with $n \leq |V(T)|$ so that T has a non-degenerated antiferromagnetic groundstate.

2 Non-degenerated groundstates in triangulations

2.1 Preliminaries

Throughout this work, it will be needed to consider triangulations of closed Riemann surfaces with some removed faces (with holes) so that each hole is circumscribed by a 3-cycle. Triangulations of this type will be called punctured triangulations (see for example Figure 3(b); triangles in grey depict holes). In general, every term defined for triangulations is naturally adapted to punctured triangulations. However, there are some facts that hold only for triangulations; they will be properly specified. We now introduce definitions for both triangulations and punctured triangulations.

Let T be a (punctured) triangulation. Recall that every spin-assignment on T frustrates at least one edge of each face boundary of T. Notice that a spin-assignment is a groundstate if it has the smallest possible number of frustrated edges (see equation 1). A spin-assignment σ on T is said to be satisfying if σ frustrates exactly one edge of each face boundary of T.

Let T be a triangulation. Obviously any satisfying spin-assignment on T is a ground-state. The converse is true for plane triangulations [7]. Nevertheless, the equivalence does not hold in general because a satisfying spin-assignment doesn't need to exist. However, note that when satisfying spin-assignments exist, then every groundstate corresponds to a satisfying spin-assignment. The situation is more complicated if T is a punctured triangulation, since distinct satisfying spin-assignments on T may provide distinct antiferromagnetic energy. In Figure 1 an example is shown.

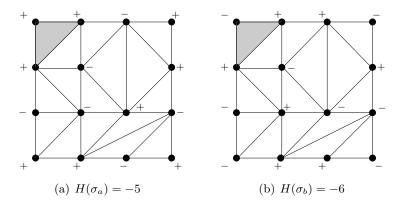


Figure 1: Consider the depicted punctured toroidal triangulation T (triangle in grey represents a hole). In both pictures (a) and (b) a satisfying spin-assignment on T is sketched; σ_a and σ_b respectively. However, the energy of σ_a is greater than energy of σ_b .

Observe that given a (punctured) triangulation T, a spin-assignment σ on T is satisfying if and only if $-\sigma$ is also a satisfying spin-assignment on T. We shall refer to this fact as sign symmetry. We will use it in order to reduce the number of cases that need to be analysed in the proofs. Moreover, if T admits exactly two satisfying spin-assignments σ and $-\sigma$, we say that T admits a unique pair of satisfying spin-assignments.

If σ assigns the same spin on all vertices of a subgraph H of T (respectively all elements of $S \subseteq V(T)$), we say that H (respectively a subset S) is monochromatic under σ . Similarly, we say that an edge is monochromatic (respectively non-monochromatic) under σ if σ assigns the same (respectively distinct) spins on both ends of the edge. In other words, an edge is monochromatic under σ if and only if it is frustrated by σ . Monochromatic and non-monochromatic faces are defined analogously depending on whether or not its face boundary is either monochromatic or non-monochromatic. Then, a spin-assignment σ to T is satisfying if and only if every face of T is non-monochromatic under σ . An edge e in E(T) will be called serious if and only if e is monochromatic under every satisfying spin-assignment on T.

In what follows, serious edges are depicted as thicker lines and holes of punctured triangulations are depicted as grey areas.

2.2 Non-degenerated groundstates in toroidal triangulations

This section is devoted to prove Theorem 2 and to discuss some features of toroidal triangulations with a non-degenerated groundstate. Here, we show a strategy to construct toroidal triangulations with a non-degenerated groundstate, where a groundstate is a satisfying spin-assignment. Nevertheless, we strongly believe there are many other ways to construct them and also that the class of toroidal triangulations with a non-degenerated groundstate is not small. The proof of Theorem 2 is constructive and it is based on a simple idea. However, it is far from being trivial to find the concrete triangulations to make that idea work.

Next, we give definitions and an overview of the proof. Let T be a toroidal triangulation that admits exactly one pair of satisfying spin-assignments (recall that when a satisfying spin-assignment exists, satisfying spin-assignments are identical to groundstates) and σ denote a satisfying spin-assignment on T. Let \tilde{F} be a non-empty set of faces of T. We say that the pair (T,σ) is invariant under removal of \tilde{F} if the punctured triangulation \tilde{T} obtained by removing all faces contained in the set \tilde{F} , has exactly one pair of satisfying spin-assignments. In particular, $\tilde{\sigma}$ is a satisfying spin-assignment on \tilde{T} if and only $\tilde{\sigma} \in \{+\sigma, -\sigma\}$. If such a set of faces \tilde{F} exists, we say that T is a supporting triangulation, \tilde{F} will be called a removable set of faces of T and \tilde{F} will be referred to as the supporting punctured triangulation associated to T and \tilde{F} ; when T and \tilde{F} are clear (or implicit) from the context we just say that \tilde{T} is a supporting punctured triangulation. Clearly, \tilde{T} is an embedding of a graph in a torus with $|\tilde{F}|$ triangular holes. Observe that the existence of a removable set of faces is not trivial: one could remove from T a non-empty set of faces in such a way that the obtained triangulation has more satisfying spin-assignments than T.

Let T be a supporting triangulation, \tilde{F} be a removable set of faces of T, σ be a satisfying spin-assignment on T and \tilde{T} be the supporting punctured triangulation associated to T and \tilde{F} .

The face boundaries of the faces in \tilde{F} (in other words, the 3-cycles circumscribing the holes of \tilde{T}), will be referred to as the *expandable cycles* of \tilde{T} . The set of edges contained in the expandable cycles which are monochromatic under σ will be called the *fundamental edges* of \tilde{T} . Notice that each expandable cycle contains exactly one fundamental edge, since each expandable cycle is non-monochromatic under σ . Clearly, fundamental edges of \tilde{T} are serious.

Assume that a supporting triangulation T exists. Then, we know that a supporting punctured triangulation \tilde{T} associated to T admits exactly one pair of satisfying spin-assignments, say σ , $-\sigma$ and all expandable cycles of \tilde{T} are non-monochromatic under σ with its fundamental edges monochromatic under σ . In our construction, each hole of \tilde{T} (circumscribed by an expandable cycle) will be covered by a plane triangulation in such a way that the number of vertices of the toroidal triangulation increases and the number of satisfying spin-assignments keeps constant. To achieve this task, it is needed a plane triangulation Δ with an arbitrary number of vertices and satisfying the following condition: if $\{xyz\}$ is the face boundary of the outer face of Δ , then there is a unique pair of satisfying spin-assignments on Δ such that at least one edge from the set of edges $\{xy,yz,xz\}$ is monochromatic; say xy. We refer to Δ as an augmenting triangulation always exists, since plane triangulations are duals of cubic bridgeless planar graphs and every perfect matching of a cubic bridgeless planar graph G corresponds to the set of monochromatic edges under a satisfying spin-assignment on G^* . Then, an augmenting triangulation is the dual of a cubic bridgeless planar graph with a specified edge contained in exactly one perfect matching.

Next, we show that supporting triangulations exist. In Subsection 2.2.2 we will describe a family of augmenting triangulations. Finally, in Subsection 2.2.3 we formalize the proof of Theorem 2.

2.2.1 Supporting triangulations

The minimal triangulations of a surface are those that have every edge in a noncontractible 3-cycle. A splitting of a vertex v replaces the vertex v by two vertices v_1 and v_2 connected by a new edge v_1v_2 , and replaces each edge vu incident to v either by the edge v_1u or by v_2u . It is well-known that every triangulation of a given surface Ω , may be generated by a sequence of vertex-splittings from a minimal triangulation of Ω . In general, the set of minimal triangulations is finite for every fixed surface [1, 2]. In particular, there are 21 minimal toroidal triangulations (see $[12, \S 5.4]$).

It is natural and potentially useful to look for supporting triangulations in the set of minimal toroidal triangulations. On one hand its reduced number of vertices allow to verify the required properties manually, on the other hand, it strongly indicates a possible way to describe the whole set of toroidal triangulations with a non-degenerated groundstate since every toroidal triangulation may be generated from a minimal one.

We exhibit the subset of minimal triangulations of the torus which have the property of being a supporting triangulation (see Figure 2). However, some minimal triangulations of the torus which are not supporting can become supporting after some slight modifications, namely,

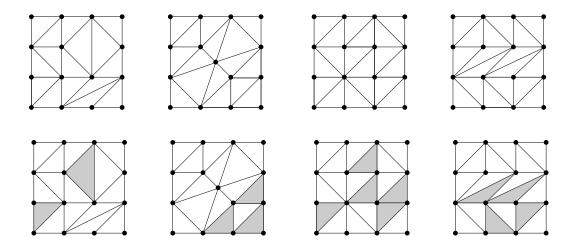


Figure 2: In the first row, minimal triangulation of the torus which are supporting. Below each supporting triangulation an associated supporting punctured triangulation is depicted (regions in grey represent the holes generated by the removal of a removable set of faces of each supporting triangulation).

after edge-flippings and vertex-splittings. Nevertheless, this analysis is not the aim of this work and will be studied separately.

In the first row of Figure 2, all minimal toroidal triangulations which are supporting triangulations are depicted (in each picture opposite sides have to be identified). In the second row of Figure 2 are depicted the supporting punctured triangulations obtained from the removal of a set of removable faces of each supporting triangulation above. We will formally prove supportability only for one of the triangulation depicted in Figure 2. The same proof ideas can be easily applied for the remaining cases.

Proposition 3 Let T be the toroidal triangulation depicted in Figure 3(a) and \tilde{F} be the subset of faces of T with face boundary the cycles $\{uvw\}$ and $\{\tilde{u}\tilde{v}\tilde{w}\}$. Then, triangulation T is a supporting triangulation and \tilde{F} is a removable set of faces of T.

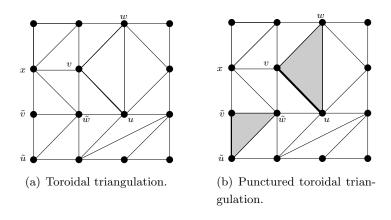


Figure 3: The supporting triangulation and punctured triangulation of Proposition 3.

Proof: Let \tilde{T} be the punctured triangulation obtained by removing from T all faces contained in \tilde{F} (see Figure 3(b)). To see that T is a supporting triangulation and \tilde{F} is a removable set of faces of T, it is enough to prove that \tilde{T} has exactly one pair of satisfying spin-assignments and that both expandable cycles of \tilde{T} are non-monochromatic under a satisfying spin-assignment on \tilde{T} .

To do that, by sign symmetry it suffices to verify that exactly one of the following three initial configurations can be extended to a satisfying spin-assignment on \tilde{T} (vertex labels as in Figure 3(b)): [1] when the cycle $\{uv\tilde{w}\}$ is assigned spin ++-, [2] when the cycle $\{uv\tilde{w}\}$ is assigned spin +--. Then, it is necessary to check that both expandable cycles are non-monochromatic under such satisfying spin-assignment.

In Figures 4(a) and 4(b) the case when uv is non-monochromatic is worked out; a subindex i accompanying a + or - sign indicates that the spin is forced by the spin-assignments with smaller indices in order for the assignment to be satisfiable — if spins assigned on the vertices of a face boundary are forced to be all of the same sign, then no satisfying assignment can exist under the given initial conditions. This establishes that when the cycle $\{uv\tilde{w}\}$ is assigned +-+ or +--, there is no satisfying spin-assignment extension on \tilde{T} .

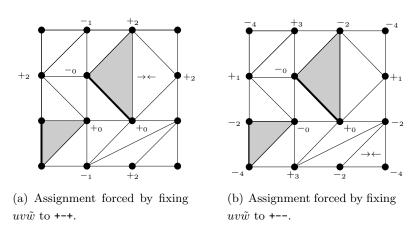
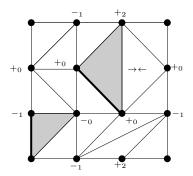
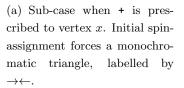


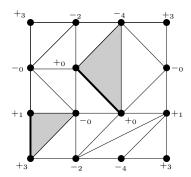
Figure 4: Case when uv is non-monochromatic. Forced monochromatic faces are labelled by $\rightarrow \leftarrow$.

In Figure 5, the case when edge uv is monochromatic is studied. In this situation, two subcases arise, depending on whether or not the spin +1 is assigned to vertex x (vertex labels as in Figure 3(b)) — each subcase is dealt in the same way as the previous situation and worked out separately in Figures 5(a) and 5(b). This shows that there exist a unique satisfying spin-assignment on \tilde{T} .

Finally, notice that both expandable cycles of \tilde{T} are non-monochromatic under the unique satisfying spin-assignment on \tilde{T} (depicted in Figure 5(b)).







(b) Sub-case when – is prescribed to vertex x. Initial spin-assignment can be extended to a unique satisfying spin-assignment on \tilde{T} .

Figure 5: The case when $\{uv\tilde{w}\}\$ is prescribed spin ++-.

2.2.2 Augmenting triangulations

We already mentioned that an augmenting triangulation always exists. In this subsection we just show an easy way to construct a vertex-increasing family of such triangulations.

Let $\Delta_0 = \{x, y_0, z\}$ be a plane triangle. For $i \geq 1$, let Δ_i be the plane triangulation obtained by applying the following rule to Δ_{i-1} : (1) insert a new vertex y_i in the outer face of Δ_{i-1} , and (2) connect the new vertex y_i to each vertex in the outer face of Δ_{i-1} so that the outer face of the new plane triangulation has face boundary $\{xy_iz\}$ (See Figure 6). Clearly, the number of vertices of Δ_n is n+3. We will see that every triangulation from the collection $\{\Delta_j\}_{j>0}$ is an augmenting triangulation with fundamental edge xy_j . This type of triangulations belongs to the set of stack triangulations (see [6]) and more families of augmenting triangulations can be easily found in that set.

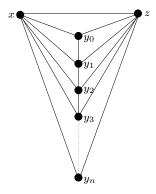


Figure 6: Construction of Δ_n .

Theorem 4 Let n > 0 and $\{xy_nz\}$ the face boundary of the outer face of Δ_n (see Figure 6). There exist a unique pair of satisfying spin-assignments on Δ_n so that edge xy_n is monochro-

matic.

Proof: By sign symmetry, it suffices to prove that if spin ++- is prescribed to $\{xy_nz\}$, then there is a unique satisfying spin-assignment extension on Δ_n . We will proceed by induction on n. If n=1 it is trivial to check that the result holds. Let n>1. If we prescribe spin ++- to $\{xy_nz\}$, then in order to have a satisfying spin-assignment on Δ_n the vertex y_{n-1} is forced to have spin -. It implies that in any extension to a satisfying spin-assignment on Δ_n , the 3-cycle $\{x,y_{n-1},z\}$ has spin +--. Then, by induction hypothesis uniqueness holds.

2.2.3 Proof of Theorem 2

Let T be a supporting triangulation, \tilde{F} be a set of removable faces of T and \tilde{T} be the supporting punctured triangulation associated to T and \tilde{F} . Consider $|\tilde{F}| = t$ and let $\{\Delta^i\}_{i \in [t]}$ by a collection of augmenting triangulations. Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ denote the expandable cycles of \tilde{T}

Let \mathcal{T} denote the triangulation obtained by gluing together the supporting punctured triangulation \tilde{T} and the collection of augmenting triangulations $\{\Delta^i\}_{i\in[t]}$ in the following way: take the collection of augmenting triangulations $\{\Delta^i\}_{i\in[t]}$ and for each $i\in[t]$, identify the boundary face of the outer face of Δ^i with the expandable cycle \mathcal{C}_i of \tilde{T} in such a way that the fundamental edge the augmenting triangulation coincides with the fundamental edge of \mathcal{C}_i .

It follows from the construction that the toroidal triangulation \mathcal{T} has a unique pair of ground-states. This finishes the proof of Theorem 2.

2.3 Proof of Theorem 1

The proof of Theorem 1 is relatively straightforward from the construction made for proving Theorem 2. Unlike the case of triangulations of the torus, in Theorem 1 the genus of the closed Riemann surface may be arbitrarily large. However, the described supporting punctured triangulations can perform the task of increasing genus and at the same time keeping all properties of existence and uniqueness of satisfying spin-assignments in such a way that the same construction made in Subsection 2.2.3 works. Next, we add some new definitions and show how to deal with the construction for proving Theorem 1.

Let T be a supporting triangulation and \tilde{F} be a set of removable faces of T. If \tilde{F} contains at least two faces f_1 , f_2 such that its face boundaries C_1 , C_2 don't share any vertex, we say that the supporting punctured triangulation \tilde{T} associated to T and \tilde{F} is a *connector* and that C_1 and C_2 are its *connection cycles*. Clearly, a connector exists. Indeed, the supporting punctured triangulation depicted in Figure 3(b) is a connector. Properties and names from supporting punctured triangulations are transferred to connectors.

We are ready to describe the construction. Let g be a integer positive number, $\{\tilde{T}_i\}_{i\in[g]}$ be a collection of connectors and t_i+2 be the number of expandable cycles of connector \tilde{T}_i for each $i\in[g]$ (clearly, $t_i\geq 0$ for all i). Moreover, let \mathcal{C}_{2i-1} , \mathcal{C}_{2i} denote the connection cycles of \tilde{T}_i for each $i\in[g]$.

Let $\tilde{\mathcal{T}}$ denote the punctured triangulation of a surface of genus g obtained by gluing together the collection of connectors $\{\tilde{T}_i\}_{i\in[g]}$ in the following way: for each $j\in[g-1]$ identify the connection cycle \mathcal{C}_{2j} of \tilde{T}_j with the connection cycle $\mathcal{C}_{2(j+1)-1}$ of \tilde{T}_{j+1} in such a way that the fundamental edge of \mathcal{C}_{2j} coincides with the fundamental edge of $\mathcal{C}_{2(j+1)-1}$.

It is routine to check that $\tilde{\mathcal{T}}$ has exactly one pair of satisfying spin-assignments and that each hole of $\tilde{\mathcal{T}}$ is non-monochromatic under any satisfying state on $\tilde{\mathcal{T}}$.

To conclude, the Theorem 1 follow directly from applying the same construction presented in Subsection 2.2.3, using $\tilde{\mathcal{T}}$ instead of a supporting punctured triangulation.

3 Final Comments

The strategy to construct triangulations presented in this notes may be (easily) extended to construct triangulations of a fixed surface with n vertices and groundstate degeneracy f(n), where f(n) is a function depending on n which can be either constant or polynomial on n; it can be reached by taking instead of an augmenting triangulation, a plane triangulation with the required property (which always exists — see for example [6]).

We believe that this work leaves many doors open and unanswered questions. First, we think that it is of particular relevance to find a complete description of triangulations with a non-degenerated groundstate. Also, in this context, we strongly believe that it is of great interest to study the following question: what is the groundstate degeneracy of random triangulations provided with the antiferromagnetic Ising model?. This will help to understand the behaviour of geometrically frustrated systems.

On the other hand, the problem of spin glasses has attracted considerable attention over recent years. Both, in solid physics and in statistical physics (for instance see [13]). In the Ising spin glass model, coupling constants are randomly distributed. Usually, each coupling constant is set randomly to either +1 or -1 with equal probability. The case of the antiferromagnetic Ising model is the critical case when the coupling constant is set to -1 with probability equal to one. In this context, next goal would be to study the Ising model on toroidal triangulations with this probability less than one and very close to one.

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