ON YOUNG SYSTEMS

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ABSTRACT. In this article, we study differential equations driven by continuous paths with with bounded p-variation for $1 \leq p < 2$ (Young systems). The most important class of examples of theses equations is given by stochastic differential equations driven by fractional Brownian motion with Hurst index $H > \frac{1}{2}$. We give a formula type Itô-Kunita-Ventzel and a substitution formula adapted to Young integral. It allows us to give necessary conditions for existence of conserved quantities and symmetries of Young systems. We give a formula for the composition of two flows associated to Young sistems and study the Cauchy problem for Young partial differential equations.

1. INTRODUCTION

The aim of this article is to study differential equations driven by continuous paths with with bounded p-variation for $1 \leq p < 2$ from a symmetry viewpoint. In recent years there has been much interest in this type of equations, see Y. Hu and D. Nualart [9], M. Gubinelli A. Lejay and S. Tindel [7], A. Lejay [13], X. Li and T. Lyons [14], D. Nualart and R. Rascanu [17], A. Ruzmaikina [19]. This subject is a particular case of Rough Paths, when $p < 2$. In fact the Theory of Rough Paths, developed by T. Lyons and colaborators (see [5], [15], [16] and references), cover the case $p \geq 2$.

We are interested in the composition of flows of of differential equations driven by continuous paths with bounded p-variation (Young systems), we obtain a formula type Itô-Kunita-Ventzel. We use this formula in order to obtain necessary conditions for existence of conserved quantities and symmetries. We study the composition of two flows and the Cauchy problem, via the characteristic method, for first order Young partial differential equations.

The plan of exposition is as follows: In section 2 we give some preliminaires on p-variation paths and Young integration. We prove a formula type Itô-Kunita-Ventzel and a substitution formula adapted to Young integral.

In section 3, we apply ours formulae in order to establish necessary conditions for conserved quantities and symmetries of Young systems. We give an adaptation of the H. Kunita result about decomposition of solutions of stochastic differential equations to Young systems, see [10]. Finally, we study the Cauchy problem, via the characteristic method, for first order Young partial differential equations of the following type

$$du_t = \sum_{j=1}^{n} F^j(t, x, u_t, Du_t) dX_t^j$$

where $(X^1, \cdots, X^n)$ is a path with bounded p-variation.

There are various different approaches to the Cauchy problem for Young partial differential equations (see [7] for a semigroup approach and [2], [4] and [8] for rough paths and SPDE). We are strongly influenced by the method of characteristics as developed by H. Kunita [11]. In fact, we prove existence and uniqueness for Young partial differential equations under adapted hypoteses, following the ideas of H. Kunita.

It is clear that our results extend naturally to stochastic differential equations driven by a fractional Brownian motion with Hurst index $H > \frac{1}{2}$, where stochastic integrals are changed by Young integrals, see [5], [15] and [16].

2. PRELIMINAIRES

Let E and V be Banach spaces. We denote by $\mathcal{P}([a, b])$ the set of all partitions $D = \{a = t_0 < \cdots < t_{k-1} < t_k = b\}$ of an interval $[a, b]$. Let $C^k([0, T], E), k \in \mathbb{N}$ denote the set of C^k-class paths of $[0, T]$ in E.

Definition 2.1. Let $p \in (0, \infty)$. The p-*variation* of a path $X \colon [0, T] \to E$ on the subinterval $[a, b]$ of $[0, T]$ is defined by

$$(2.1) \qquad \|X\|_{p,[a,b]} = \big(\sup_{D \in \mathcal{P}([a,b])} \sum_{t_i \in D} (\|X_{t_{i+1}} - X_{t_i}\|^p) \big)^{\frac{1}{p}}.$$

We say that a path $X \colon [0, T] \to E$ is of *finite p-variation* if $\|X\|_{p,[0,T]} < \infty$.

If $p \in (0, 1)$ and $X \colon [0, T] \to E$ is a continuous path of p-variation then $X(t) = X(0)$ for all $t \in [0, T]$, since

$$d(X_t, X_0) \leq \sum_{t_i \in D} d(X_{t_{i+1}}, X_{t_i}) \leq \max_i d(X_{t_{i+1}}, X_{t_i})^{1-p} \|X\|_{p,[0,T]}^p$$

for all $D \in \mathcal{P}([0, T])$.

We denote by $\mathcal{V}^p([0, T], E)$ the set of all continuous paths of finite p-variation from $[0, T]$ to E. If $1 \leq p \leq q < \infty$ then

$$(2.2) \qquad \|X\|_{q,[0,T]} \leq \|X\|_{p,[0,T]}$$

for each $X \colon [0, T] \to E$. In particular,

$$(2.3) \qquad \mathcal{V}^1([0, T], E) \subset \mathcal{V}^p([0, T], E) \subset \mathcal{V}^q([0, T], E) \subset C([0, T], E).$$

We observe that the set $\mathcal{V}^p([0, T], E)$ becomes a Banach space provided with the norm

$$(2.4) \qquad \|X\|_{\mathcal{V}^p([0,T],E)} = \|X\|_{p,[0,T]} + \sup_{t \in [0,T]} \|X_t\|$$

called *p-variation norm*. We also have the *p-variation metric*

$$(2.5) \qquad \bar{d}_p(X, Y) = \|X - Y\|_{\mathcal{V}^p([0,T],E)}$$

in $\mathcal{V}^p([0, T], E)$ induced by the p-variation norm. We denote by $\mathcal{V}_0^p([0, T], E)$ the subspace of $\mathcal{V}^p([0, T], E)$ consisting of paths starting at $0 \in E$.

Let E and V be Banach spaces. Let $X \colon [0, T] \to E$ and $Z \colon [0, T] \to \mathcal{L}(E, V)$ be continuous paths. The Riemann-Stieltjes integral of Z with respect to X is defined as the limit

$$(2.6) \qquad \lim_{\substack{|D| \to 0 \\ D \in \mathcal{P}([0,T])}} \sum_{s_i \in D} Z_{s_i}(X_{s_{i+1}} - X_{s_i})$$

and is denoted by $\int_0^t Z_s\, dX_s$. L. C. Young presented the sufficient conditions for the existence of Riemann-Stieltjes integrals. More precisely, he proved that the integral $\int_0^t Z_s\, dX_s$ exists when X has finite p-variation, Z has finite q-variation and is valid the condition $(1/p) + (1/q) > 1$. This result is known as Young's theorem. We also have that the path W given by $W(\cdot) = \int_0^\cdot Z_s\, dX_s$ has the same variation of the integrator X, that is, W has finite p-variation. We refer the reader to the paper [20] by L. C. Young and also [15]. Based on Young's Theorem, we say that a Riemann-Stieltjes integral $\int_0^t Z_s\, dX_s$ is an integral in the Young sense if there exist $p, q \in [1, \infty)$ such that $X \in \mathcal{V}^p([0,T], E)$, $Z \in \mathcal{V}^q([0,T], \mathcal{L}(E,V))$ and $\theta = \frac{1}{p} + \frac{1}{q} > 1$. In this case holds the following Young-Loeve estimative,

$$(2.7) \qquad \| \int_s^t Z_r dX_r - Z_s(X_t - X_s) \| \leq C_{p,q} \|Z\|_{q,[s,t]} \|X\|_{p,[s,t]}$$

where $C_{p,q} = \frac{1}{1-2^{1-\theta}}$.

We also have that

$$(2.8) \qquad \int_s^t Z_r dX_r = \lim_{\substack{|D| \to 0 \\ D \in \mathcal{P}([s,t])}} \sum_{s_i \in D} Z_{s_i^*}(X_{s_{i+1}} - X_{s_i})$$

where $s_i^* \in [s_i, s_{i+1}]$.

Definition 2.2. A path $F : [0,T] \to V$ is Holder continuous with exponent $\alpha \geq 0$, or simply α-Holder, if

$$\|F\|_{\alpha;H} = \sup_{s \neq t} \frac{\|F(x) - F(y)\|}{|t-s|^\alpha} < \infty.$$

Let $C_H^\alpha([0,T];V)$ denote the set of α-Holder paths of V.

We observe that $C_H^\alpha([0,T];V) \subset \mathcal{V}^{\frac{1}{\alpha}}([0,T],V)$. In fact,

$$\|F\|_{\frac{1}{\alpha},[s,t]}^{\frac{1}{\alpha}} \leq \|F\|_{\alpha;H}^{\frac{1}{\alpha}} |t-s|.$$

Now, we prove a generalization of the fundamental theorem of calculus in the context of Young integration.

Lemma 2.3. *Let* $X \in \mathcal{V}^p([0,T], V)$ *and* $g : [0,T] \times V \to W$ *be a continuous function twice continuously differentiable in relation to* V *(* $1 \leq p \leq 2$*). Let* $h \in C(V, C_H^{\frac{1}{q}}([0,T], L(W,U)))$ *and* $Z \in \mathcal{V}^p([0,T], W)$ *(* $\frac{1}{p} + \frac{1}{q} > 1$*) such that*

$$g_t(x) = g_0(x) + \int_0^t h_s(x) dZ_s$$

where the integral is in the Young sense. Then

$$(2.9) \qquad g_t(X_t) = g_0(X_0) + \int_0^t h_s(X_s) dZ_s + \int_0^t D_x g_s(X_s) dX_s.$$

Proof. Let $D = \{0 = t_0 < \cdots < t_{k-1} < t_k = T\} \in \mathcal{P}([0,T])$. We write,

$$\sum_{t \geq t_i \in D} g_{t_{i+1}}(X_{t_{i+1}}) - g_{t_i}(X_{t_i}) = \sum_{t \geq t_i \in D} (g_{t_{i+1}}(X_{t_{i+1}}) - g_{t_i}(X_{t_{i+1}}))$$

$$(2.10) \qquad \qquad \qquad + \sum_{t \geq t_i \in D} (g_{t_i}(X_{t_{i+1}}) - g_{t_i}(X_{t_i})).$$

It follows from definitions that

$$
\begin{aligned}
g_{t_{i+1}}(X_{t_{i+1}}) - g_{t_i}(X_{t_{i+1}}) &= \int_{t_i}^{t_{i+1}} h_s(X_{t_{i+1}}) dZ_s \\
&= \int_{t_i}^{t_{i+1}} (h_s(X_{t_{i+1}}) - h_{t_i}(X_{t_{i+1}})) dZ_s + h_{t_i}(X_{t_{i+1}})(Z_{t_{i+1}} - Z_{t_i}).
\end{aligned}
$$

Taking norm and applying the Young-Loeve estimative (2.7), gives

$$
\| \sum_{t \geq t_i \in D} \int_{t_i}^{t_{i+1}} (h_s(X_{t_{i+1}}) - h_{t_i}(X_{t_{i+1}})) dZ_s \| \leq \sum_{t \geq t_i \in D} C_{p,q} \|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]}.
$$

Let $\theta = \frac{1}{p} + \frac{1}{q}$ and $E(D) = \max_{t_i \in D}(\|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]})^{1-\frac{1}{\theta}}$. Applying the Holder inequality ($\frac{1}{p\theta} + \frac{1}{q\theta} = 1$) and some elementary calculations, we have that

$$
\begin{aligned}
\sum_{t \geq t_i \in D} \|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]} &\leq E(D)(\sum_{t \geq t_i \in D} (\|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]})^{\frac{1}{\theta}} \\
&\leq E(D)(\sum_{t \geq t_i \in D} \|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]}^q)^{\frac{1}{q\theta}} (\sum_{t \geq t_i \in D} \|Z\|_{p,[t_i,t_{i+1}]}^p)^{\frac{1}{p\theta}} \\
&\leq E(D)\|Z\|_{p,[0,T]}^{\frac{1}{\theta}} (\sum_{t \geq t_i \in D} \|h.(X_{t_{i+1}})\|_{q,[t_i,t_{i+1}]}^q)^{\frac{1}{q\theta}} \\
&\leq E(D)\|Z\|_{p,[0,T]}^{\frac{1}{\theta}} (\sum_{t \geq t_i \in D} \|h.(X_{t_{i+1}})\|_{\frac{1}{q},H}^q (t_{i+1} - t_i))^{\frac{1}{q\theta}} \\
&\leq E(D)\|Z\|_{p,[0,T]}^{\frac{1}{\theta}} (\sup_s \|h.(X_s)\|_{\frac{1}{q},H}^q T)^{\frac{1}{q\theta}}.
\end{aligned}
$$

Taking limit and using (2.8),

$$
(2.11) \qquad \lim_{|D| \to 0} \sum_{t \geq t_i \in D} (g_{t_{i+1}}(X_{t_{i+1}}) - g_{t_i}(X_{t_{i+1}})) = \int_0^t h_s(X_s) dZ_s,
$$

because $\lim_{|D| \to 0} E(D) = 0$.

We claim that

$$
(2.12) \qquad \lim_{|D| \to 0} \sum_{t \geq t_i \in D} (g_{t_i}(X_{t_{i+1}}) - g_{t_i}(X_{t_i})) = \int_0^t D_x g_s(X_s) dX_s.
$$

In fact, by Taylor's theorem

$$
\begin{aligned}
g_{t_i}(X_{t_{i+1}}) - g_{t_i}(X_{t_i}) &= D_x g_{t_i}(X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i}) + \frac{1}{2} D_x^2 g_{t_i}(X_{t_i} + s_i(X_{t_{i+1}} - X_{t_i})) \\
&\quad \cdot (X_{t_{i+1}} - X_{t_i})^2
\end{aligned}
$$

where $s_i \in (0,1)$. Taking norm,

$$
\| \sum_{t \geq t_i \in D} D_x^2 g_{t_i}(X_{t_i} + s_i(X_{t_{i+1}} - X_{t_i})) \cdot (X_{t_{i+1}} - X_{t_i})^2 \| \leq K \sum_{t \geq t_i \in D} \|X_{t_{i+1}} - X_{t_i}\|^2
$$

where $K = \max\{\|D_x^2 g_s(X_s + r(X_t - X_s))\| : 0 \leq s \leq t \leq T$ and $r \in [0,1]\}$. Since $X \in \mathcal{V}^p([0,T], V)$ with $1 \leq p < 2$ it follows that $Dg(X) \in \mathcal{V}^p([0,T], L(V,W))$. Combining the above estimative and definitions we have (2.12).

Finally, from the continuity of g,

$$(2.13) \qquad g_t(X_t) - g_0(X_0) = \lim_{|D| \to 0} \sum_{t \geq t_i \in D} g_{t_{i+1}}(X_{t_{i+1}}) - g_{t_i}(X_{t_i}).$$

Taking limit in (2.10) and then substituing (2.11), (2.12) and (2.13), we obtain

$$g_t(X_t) = g_0(X_0) + \int_0^t h_s(X_s)dZ_s + \int_0^t D_x g_s(X_s)dX_s.$$

$\square$

Corollary 2.4. *Let $g : V \to W$ be a twice differentiable function and $Z \in \mathcal{V}^p([0,T], V)$ $(1 \leq p < 2)$. Then $g(Z) \in \mathcal{V}^p([0,T], W)$, $Dg(Z) \in \mathcal{V}^p([0,T], L(V,W))$ and for all $0 \leq t \leq T$,*

$$g(Z_t) - g(Z_0) = \int_0^t Dg(Z_r)dZ_r.$$

In particular,

$$(2.14) \qquad dg(Z_t) = Dg(Z_t)dZ_t.$$

The following substitution formula holds.

Lemma 2.5. *Let $Z \in \mathcal{V}^p([0,T], V)$, $f \in \mathcal{V}^q([0,T], Hom(V,W))$ and $g \in \mathcal{V}^l([0,T], Hom(W,U))$ where $\frac{1}{q}, \frac{1}{l} > 1 - \frac{1}{p}$. Then for all $0 \leq s \leq t \leq T$,*

$$(2.15) \qquad \int_s^t g_r dY_r = \int_s^t g_r \circ f_r dZ_r$$

where $Y_t = \int_0^t f(Z_r)dZ_r$.

Proof. Let $D = \{0 = t_0 < ... < t_{k-1} < t_k = T\} \in \mathcal{P}([0,T])$. Then

$$\sum_{t \geq t_i \in D} g_{t_i}(Y_{t_{i+1}} - Y_{t_i}) - \sum_{t \geq t_i \in D} g_{t_i} f_{t_i}(Z_{t_{i+1}} - Z_{t_i}) = \sum_{t \geq t_i \in D} g_{t_i} \int_{t_i}^{t_{i+1}} (f_r - f_{t_i})dZ_r.$$

Taking norm and applying the Young-Loeve estimative (2.7), gives

$$\begin{aligned}
\| \sum_{t \geq t_i \in D} g_{t_i} \int_{t_i}^{t_{i+1}} (f_r - f_{t_i})dZ_r \| &\leq \sum_{t \geq t_i \in D} \|g_{t_i}\| \| \int_{t_i}^{t_{i+1}} (f_r - f_{t_i})dZ_r \| \\
&\leq \sum_{t \geq t_i \in D} K C_{p,q} \|f\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]}.
\end{aligned}$$

Let $\theta = \frac{1}{p} + \frac{1}{q}$ and $L(D) = \max_{t_i \in D}(\|f\|_{q,[t_i,t_{i+1}]}\|Z\|_{p,[t_i,t_{i+1}]})^{1-\frac{1}{\theta}}$. Applying the Holder inequality ($\frac{1}{p\theta} + \frac{1}{q\theta} = 1$), we have that

$$\begin{aligned}
\sum_{t \geq t_i \in D} \|f\|_{q,[t_i,t_{i+1}]} \|Z\|_{p,[t_i,t_{i+1}]} &\leq L(D)(\sum_{t \geq t_i \in D} (\|f\|_{q,[t_i,t_{i+1}]}\|Z\|_{p,[t_i,t_{i+1}]})^{\frac{1}{\theta}} \\
&\leq L(D)(\sum_{t \geq t_i \in D} \|f\|_{q,[t_i,t_{i+1}]}^q)^{\frac{1}{q\theta}} \cdot (\sum_{t \geq t_i \in D} \|Z\|_{p,[t_i,t_{i+1}]}^p)^{\frac{1}{p\theta}} \\
&\leq L(D)\|f\|_{q,[0,T]}^{\frac{1}{\theta}})\|Z\|_{p,[0,T]}^{\frac{1}{\theta}}.
\end{aligned}$$

Combining the three above inequalities we obtain that

$$(2.16) \qquad \| \sum_{t \geq t_i \in D} g_{t_i}(Y_{t_{i+1}} - Y_{t_i}) - \sum_{t \geq t_i \in D} g_{t_i} f_{t_i}(Z_{t_{i+1}} - Z_{t_i}) \| \leq \tilde{K}L(D).$$

From the continuity of $\|f\|_{q,[s,t]} \, \|Z\|_{p,[s,t]}$ we have that

$$(2.17) \qquad\qquad\qquad \lim_{|D| \to 0} L(D) = 0.$$

Combining (2.16) and (2.17) we have that

$$\lim_{|D| \to 0} \| \sum_{t \geq t_i \in D} g_{t_i}(Y_{t_{i+1}} - Y_{t_i}) - \sum_{t \geq t_i \in D} g_{t_i} f_{t_i}(Z_{t_{i+1}} - Z_{t_i}) \| = 0.$$

$\square$

3. Young systems

Let E_1 and E_2 be Banach spaces, $p \in [1,2)$ and f be a function from E_1 to $L(E_2, E_1)$ which is γ-Holder continuous, $\gamma \in (0,1]$. We call such a function a $Lip(\gamma)$-vector field from E_1 to E_2. For $Y \in \mathcal{V}^p([0,T], E_1)$ we have that $f(Y)$ belongs to $\mathcal{V}^{\frac{p}{\gamma}}([0,T], L(E_2, E_1))$ and

$$\|f(Y)\|_{\frac{p}{\gamma}} \leq \|f\|_{\gamma,H} \|Y\|_p^{\gamma}.$$

We suppose that $\gamma + 1 > p$. From Young's theorem it is clear that for $X \in \mathcal{V}^p([0,T], E_2)$ there exists the Young integral

$$\int_0^t f(Y_s) dX_s.$$

Let $X \in \mathcal{V}^p([0,T], E_2)$ and f be a $Lip(\gamma)$-vector field from E_1 to E_2 with $\gamma \in (0,1]$. Given an initial condition $y_0 \in E_1$, we understand that a *trajectory* is a path Y of finite p-variation in E_1 which is the solution starting at y_0 of an equation of type

$$(3.1) \qquad\qquad\qquad dY = f(Y) \, dX$$

in the sense that $Y(t) = y_0 + \int_0^t f(Y(s)) \, dX_s$, for all $t \in [0,T]$.

We will call the equation (3.1) by Young equation. The Young equation $dY = f(Y) \, dX$ admits solution starting at $y_0 \in E_2$. In order of obtain uniqueness we need a stronger regularity assumption on f, for example we can assume that f is a $Lip(1+\gamma)$-vector field, this is, f is continuously differentiable and its derivative is a γ-Holder continuous function from from E_1 to $L(E_1 \otimes E_2, E_1)$. Moreover, if $I_f(y_0, X)$ denotes a solution starting at y_0 then the mapping $(y_0, X) \mapsto I_f(y_0, X)$ is an diffeomorphism.

In the case that E_1 and E_2 are finite dimensional spaces, we have similar results on existence and uniquenness of solutions for the Young equation driven by time dependent fields

$$(3.2) \qquad\qquad\qquad dY_s = \sum_{i=1}^{n} f_i(s, Y_s) \, dX_s^i$$

where $f_i : [0,T] \times E_2 \to E_2$ are the vector fields γ-Holder with $1 + \gamma > 1$ in space and uniformly of finite q-variation in time with $\frac{1}{p} + \frac{1}{q} > 1$, see [13].

We refer the reader to [5], [13], [14], [15] and [16] for more information about existence and uniquennes solutions of Young equations.

3.1. **Symmetries and invariants.** The theory of conserved quantities (first integrals) and symmetry (invariant under transformation) for dynamical systems must been one of the most important subjets in applied mathematics, see [1], [6] and [18]. Hence, it is natural to formulate theses notions for Young systems. In this subsection we consider only homogeneous Young systems.

Definition 3.1. A function $F \in C^2(E_1; E_3)$ is a conserved quantity of (3.1) if for each solution Y of (3.1) we have that $F(Y(t)) = F(Y(0))$ for all $t \in [0, T]$.

The following necessary condition for a function be a conserved quantity of (3.1) is an immediate consequence of Lemmas 2.3 and 2.5.

Corollary 3.2. *Let $F \in C^2(E_1; E_3)$ such that $DF \cdot f = 0$. Then F is a conserved quantity of (3.1).*

We formulate the notion of symmetry for Young systems in an analogous way to that in differentiable dynamical systems. We are interest in Lie point time independent symmetries.

Definition 3.3. A transformation $\Phi \in C^2(E_1; E_1)$ is a simmetry of (3.1) if for each solution Y of (3.1) we have that $\Phi(Y)$ is also a solution of (3.1).

Proposition 3.4. *Let $\Phi \in C^1(E_1; E_1)$ such that $f \circ \Phi = D\Phi \cdot f$. Then Φ is a symmetry of (3.1).*

Proof. Applying the Lemma 2.3 and Proposition 2.5 we have that,

$$
\begin{aligned}
\Phi(Y_t) - \Phi(Y_0) &= \int_0^t D\Phi(Y_s)f(Y_s)dX_s \\
&= \int_0^t f(\Phi(Y_s))dX_s.
\end{aligned}
$$

$\square$

Let $\{e_1, \cdots, e_n\}$ be a basis of E_2. Thus $X_s = \sum_{i=1}^n X_s^i e_i$ and we can write the Young equation (3.1) as

$$
(3.3) \qquad\qquad dY_s = \sum_{i=1}^n f_i(Y_s)\, dX_s^i
$$

where $f_i : E_1 \to E_1$ are the vector fields given by $f_i(y) = f(y)(e_i)$ for $i = 1, \cdots, n$.

Proposition 3.5. *Let Y be a solution of (3.3). Then for all $F \in C^2(E_1; E_3)$,*

$$
(3.4) \qquad\qquad F(Y_t) = F(Y_0) + \sum_{i=1}^n \int_0^t f_i F(Y_s)dX_s^i.
$$

Proof. Applying the Lemma 2.3 and Proposition 2.5 we have that,

$$
\begin{aligned}
F(Y_t) - F(Y_0) &= \int_0^t DF(Y_s)f(Y_s)dX_s \\
&= \int_0^t DF(Y_s)\sum_{i=1}^n f(Y_s)(e_i)dX_s^i \\
&= \sum_{i=1}^n \int_0^t DF(Y_s)f_i(Y_s)dX_s^i \\
&= \sum_{i=1}^n \int_0^t f_iF(Y_s)dX_s^i.
\end{aligned}
$$

$\square$

Definition 3.6. A vector field $g \in C^2(E_1; E_1)$ is an infinitesimal symmetry of (3.1) if its flow Φ_t is a flow of symmetries of (3.1).

The following Corollaries provide conditions for a transformation be a conserved quantity or a symmetry in terms of the vector fields that are driven the Young equation.

Corollary 3.7. *Let $F \in C^1(E_1; E_3)$ such that $f_iF = 0$ for $i = 1, \cdots, n$. Then F is a conserved quantity of (3.3).*

Corollary 3.8. *Let $\Phi \in C^1(E_1; E_1)$ such that $\Phi_* f_i = f_i$ for $i = 1, \cdots, n$. Then Φ is a symmetry of (3.3).*

Proposition 3.9. *Let $g \in C^2(E_1; E_1)$ be a vector field such that $[g, f_i] = 0$ for $i = 1, \cdots, n$. Then g is an infinitesimal symmetry of (3.3).*

Proof. Let Φ_t be the flow of g. Then

$$\partial_t \Phi_t^* f_i = \partial_s|_{s=0} \Phi_{t+s}^* f_i = \Phi_t^*[g, f_i] = 0$$

so $\Phi_t^* f_i$ is constant in t. Thus $f_i = \Phi_t^* f_i$.

$\square$

The following Theorem is an adaptation of the H. Kunita results about decomposition of solutions of stochastic differential equations to Young systems, see [10].

Theorem 3.10. *Let $p \in [1, 2)$, $p < \gamma$, $U \in \mathcal{V}^p([0, T], E_0)$, $X \in \mathcal{V}^p([0, T], E_1)$, $f \in \mathrm{Lip}^\gamma(E, L(E_0, E))$ and $g \in \mathrm{Lip}^\gamma(E, L(E_1, E))$. Let V and Y be solutions of $dV = f(V)dU$ and $dY = g(Y)dX$. Then $Z = Y \circ V$ satisfies*

$$dZ = g(Z)dX + Y_* f(Z)dU.$$

Proof. By assumption,

$$(3.5) \qquad Y_t(x) = x + \int_0^t g(Y_s(x))dX_s.$$

Combining Lemma 2.3 and Proposition 2.5 we have that

$$
\begin{aligned}
Z_t &= Y_t(V_t) \\
&= x + \int_0^t g(Y_s(V_s))dX_s + \int_0^t D_x Y_s(V_s)dV_s \\
&= x + \int_0^t g(Z_s)dX_s + \int_0^t D_x Y_s(V_s)f(V_s)dU_s \\
&= x + \int_0^t g(Z_s)dX_s + \int_0^t (D_x Y_s \cdot f) \circ Y_s^{-1}(Z_s)dU_s \\
&= x + \int_0^t g(Z_s)dX_s + \int_0^t (Y_s)_* f(Z_s)dU_s
\end{aligned}
$$

$\square$

3.2. **First order Young partial differential equations.** In this section we deal with a class of evolution first order differential equations driven by a path with finite p-variation, with $p \in [1,2)$. Let $X \in \mathcal{V}^p([0,T], \mathbb{R}^n)$, $\phi : \mathbb{R}^d \to \mathbb{R}$ and $F^j : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \cdots, n$. We will consider the following equation

(3.6)
$$
\begin{cases}
du_t = \sum_{j=1}^n F^j(t, x, u_t, Du_t)dX_t^j \\
u_0 = \phi.
\end{cases}
$$

We assume that F^j are continuous, continuously differentiable in the first variavel and for each t, $F^j(t, \cdot) \in \mathcal{C}^{3,\alpha}(\mathbb{R}^{2d+1})$.

Definition 3.11. Given $\phi \in \mathcal{C}^1(\mathbb{R}^d)$ a local field $u_t(x)$, $x \in \mathbb{R}^d$ $t \in [0, T(x))$ with values in $\mathbb{R}$ is called a local solution of (3.6) with the initial condition $u_0 = \phi$, if $0 < T(x) \le T$ and

$$
u(t, x) = \phi(x) + \sum_{j=1}^n \int_0^t F^j(r, x, u(r, x), D_x u(r, x))dX_r^j
$$

for all (t, x) such that $t < T(x)$.

We use the following notations $F_{x_i} = D_{x_i}F$, $F_{p_i} = D_{p_i}F$, $F_x = (F_{x_1}, \cdots, F_{x_d})$ and $F_p = (F_{p_1}, \cdots, F_{p_d})$.

The characteristic Young system associated with (3.6) is defined by

(3.7)
$$
\begin{cases}
da_t &= -\sum_{j=1}^n F_p^j(t, a_t, b_t, c_t)dX_t^j \\
db_t &= \sum_{j=1}^n \{F^j(t, a_t, b_t, c_t) - F_p^j(t, a_t, b_t, c_t) \cdot c_t\}dX_t^j \\
dc_t &= \sum_{j=1}^n \{F_x^j(t, a_t, b_t, c_t) + F_u^j(t, a_t, b_t, c_t)b_t\}dX_t^j
\end{cases}
$$

Given $(x, u, p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ there is an unique solution $(a_t(x, u, p), b_t(x, u, p), c_t(x, u, p))$ starting from (x, u, p) at time $t = 0$ with time life $[0, T(x, u, p))$.

Theorem 3.12. *Let u be a local solution of (3.6) such that $u(t, \cdot) \in C^3(\mathbb{R}^d)$ for all $t \in [0, T]$. Assume that a_t solves the equation*

(3.8)
$$
da_t = -\sum_{j=1}^n F_p^j(t, a_t, b_t, c_t)dX_t^j
$$

where $b_t = u(t, a_t)$ and $c_t = D_x u(t, a_t)$. Then (a_t, b_t, c_t) solves the characteristic system (3.7).

Proof. By Lemma 2.3,

$$u(t, a_t) = \phi(a_0) + \sum_{j=1}^{n} \int_0^t F^j(r, a_r, u(r, a_r), D_x u(r, a_r)) dX_r^j$$

$$+ \int_0^t D_x u(r, a_r) da_r.$$

Thus

$$(3.9) \qquad b_t = \phi(a_0) + \sum_{j=1}^{n} \int_0^t F^j(r, a_r, b_r, c_r) dX_r^j + \int_0^t D_x u(r, a_r) da_r.$$

From Lemma 2.5 and (3.8), we have

$$\int_0^t D_x u(r, a_r) da_r = -\sum_{j=1}^{n} \int_0^t D_x u(r, a_r) F_p^j(r, a_r, b_r, c_r) dX_r^j$$

$$(3.10) \qquad\qquad = -\sum_{j=1}^{n} \int_0^t F_p^j(r, a_r, b_r, c_r) \cdot c_t dX_r^j.$$

Combining (3.9) with (3.10) we obtain

$$(3.11) \qquad b_t = \phi(a_0) + \sum_{j=1}^{n} \int_0^t \{F^j(r, a_r, b_r, c_r) - F_p^j(r, a_r, b_r, c_r) \cdot c_t\} dX_r^j.$$

Our next goal is to determine the equation for c_t. We observe that

$$u_{x_i}(t, x) = \phi_{x_i}(x) + \sum_{j=1}^{n} \int_0^t D_{x_i} F^j(r, x, u(r, x), D_x u(r, x)) dX_r^j$$

$$= \phi_{x_i}(x) + \sum_{j=1}^{n} \int_0^t \{F_{x_i}^j(r, x, u(r, x), D_x u(r, x))$$

$$+ F_u^j(r, x, u(r, x), D_x u(r, x)) u_{x_i}(r, x)$$

$$+ \sum_{l=1}^{d} F_{p_l}^j(r, x, u(r, x), D_x u(r, x)) u_{x_l x_i}(r, x)\} dX_r^j.$$

By Lemma 2.3 and definitions,

$$c_t^i = \phi_{x_i}(a_0) + \sum_{j=1}^{n} \int_0^t \{F_{x_i}^j(r, a_r, b_r, c_r) + F_u^j(r, a_r, b_r, c_r) c_r^i$$

$$(3.12) \qquad + \sum_{l=1}^{d} F_{p_l}^j(r, a_r, b_r, c_r) u_{x_l x_i}(r, a_r)\} dX_r^j + \int_0^t D_x u_{x_i}(r, a_r) da_r$$

where $c_t^i = u_{x_i}(t, a_t)$.

From Lemma 2.5 and (3.8), we have

$$\int_0^t D_x u_{x_i}(r, a_r) da_r = -\sum_{j=1}^{n} \int_0^t D_x u_{x_i}(r, a_r) F_p^j(r, a_r, b_r, c_r) dX_r^j$$

$$(3.13) \qquad\qquad - \sum_{j=1}^{n} \int_0^t \sum_{l=1}^{d} F_{p_l}^j(r, a_r, b_r, c_r) u_{x_l x_i}(r, a_r) dX_r^j.$$

Combining (3.12) with (3.13) we obtain

$$(3.14) \qquad c_t^i = \phi_{x_i}(a_0) + \sum_{j=1}^{n} \int_0^t \{F_{x_i}^j(r, a_r, b_r, c_r) + F_u^j(r, a_r, b_r, c_r)c_r^i\}dX_r^j.$$

This is

$$(3.15) \qquad c_t = \phi_{x_i}(a_0) + \sum_{j=1}^{n} \int_0^t \{F_x^j(r, a_r, b_r, c_r) + F_u^j(r, a_r, b_r, c_r)c_r\}dX_r^j.$$

$\square$

Following H. Kunita [11] we define $\overline{a}_t(x) = a_t(x, \phi(x), D\phi(x))$, $\overline{b}_t(x) = b_t(x, \phi(x), D\phi(x))$ and $\overline{c}_t(x) = c_t(x, \phi(x), D\phi(x))$ for $t \in [0, \overline{T}(x)$ where $\overline{T}(x) = T(x, \phi(x), D\phi(x))$. We observe that $\overline{a}_t : \{x : \overline{T}(x) > t\} \to \mathbb{R}^d$ is not a diffeomorphism in general, since $D\overline{a}_t(x)$ can be singular at some $t < \overline{T}(x)$. We define

$$(3.16) \qquad \tau(x) = \inf\{t > 0 : \det D\overline{a}_t(x) = 0\} \wedge \overline{T}(x)$$

and its adjoint is given by

$$(3.17) \qquad \sigma(y) = \inf\{t > 0 : y \notin \overline{a}_t(\{\tau > t\})\}.$$

The proofs of the following two Lemmas are an easy adaptation of Lemma 2.1 and Lemma 3.3 of [11].

Lemma 3.13. *The application* $\overline{a}_t : \{x : \overline{T}(x) > t\} \to \mathbb{R}^d$ *is a diffeomorphism. For* $t < \sigma(y)$, *the inverse* $\overline{a}_t^{-1}$ *satisfies*

$$(3.18) \qquad d\overline{a}_t^{-1}(y) = \sum_{j=1}^{n} D\overline{a}_t(\overline{a}_t^{-1}(y))^{-1} F_p^j(t, y, \overline{b}_t \circ \overline{a}_t^{-1}(y), \overline{c}_t \circ \overline{a}_t^{-1}(y))dX_t^j.$$

Lemma 3.14. *For* $i = 1, \cdots, d$

$$\begin{aligned} D_{x_i}\overline{b}_t &= \overline{c}_t \cdot D_{x_i}\overline{a}_t \\ D_{x_i}\overline{b}_t \circ \overline{a}_t^{-1} &= \overline{c}_t^i \circ \overline{a}^{-1}. \end{aligned}$$

Theorem 3.15. *Let* $\phi \in C^3(\mathbb{R}^d)$. *Then* $u(t, x) = \overline{b}_t(\overline{a}_t^{-1}(x))$, $[0, \sigma(x))$ *is a local solution of (3.6).*

Proof. By Lemma 2.3,

$$(3.19) \qquad d\overline{b}_t \circ \overline{a}_t^{-1} = d\overline{b}_t(\overline{a}_t^{-1}) + D_x\overline{b}_t(\overline{a}_t^{-1})d\overline{a}_t^{-1}.$$

From (3.7) we have

$$(3.20) \qquad d\overline{b}_t(\overline{a}_t^{-1}) = \sum_{j=1}^{n} \{F^j(t, \cdot, \overline{b}_t \circ \overline{a}_t^{-1}, \overline{c}_t \circ \overline{a}_t^{-1}) \\ - F_p^j(t, \cdot, \overline{b}_t \circ \overline{a}_t^{-1}, \overline{c}_t \circ \overline{a}_t^{-1})\overline{c}_t \circ \overline{a}_t^{-1}\}dX_t^j$$

By Lemma 2.5 and Lemma 3.13,

$$(3.21) \qquad D_x\overline{b}_t(\overline{a}_t^{-1})d\overline{a}_t^{-1} = \sum_{j=1}^{n} F_p^j(t, \cdot, \overline{b}_t \circ \overline{a}_t^{-1}, \overline{c}_t \circ \overline{a}_t^{-1}) \\ D_x\overline{b}_t(\overline{a}_t^{-1})D_x\overline{a}_t(\overline{a}_t^{-1})^{-1}dX_t^j.$$

From Lemma 3.2 and definitions,

$$(3.22) \qquad D_x u_t = D_x(\overline{b}_t \circ \overline{a}_t^{-1}) = \overline{c}_t \circ \overline{a}_t^{-1} = D_x \overline{b}_t(\overline{a}_t^{-1}) D_x \overline{a}_t(\overline{a}_t^{-1})^{-1}.$$

Combining (3.19), (3.20), (3.21) and (3.22) we conclude that

$$du_t = \sum_{j=1}^{n} F^j(t, \cdot, \overline{b}_t \circ \overline{a}_t^{-1}, \overline{c}_t \circ \overline{a}_t^{-1}) dX_t^j = \sum_{j=1}^{n} F^j(t, \cdot, u_t, D_x u_t) dX_t^j.$$

$\square$

Theorem 3.16. *Let u be a local solution of (3.6), where $\phi \in C^3(\mathbb{R}^d)$ such that $u(t, \cdot) \in C^2(\mathbb{R}^d)$ for all $t \in [0, T(x))$. Then $u(t, x) = \overline{b}_t(\overline{a}_t^{-1}(x))$ for $t \in [0, T(x) \wedge \sigma(x))$.*

Proof. It is an easy consequence of Theorem 3.12. $\square$

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