# UNIQUENESS OF CLOSED SELF-SIMILAR SOLUTIONS TO THE GAUSS CURVATURE FLOW

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ABSTRACT. We show the uniqueness of strictly convex closed smooth self-similar solutions to the  $\alpha$ -Gauss curvature flow with  $(1/n) < \alpha < 1 + (1/n)$ . We introduce a Pogorelov type computation, and then we apply the strong maximum principle. Our work combined with earlier works on the Gauss Curvature flow imply that the  $\alpha$ -Gauss curvature flow with  $(1/n) < \alpha < 1 + (1/n)$  shrinks a strictly convex closed smooth hypersurface to a round sphere.

# 1. INTRODUCTION

We recall that given  $\alpha > 0$ , an one-parameter family of immersions  $F: M^n \times [0,T) \to \mathbb{R}^{n+1}$  is a solution of the  $\alpha$ -Gauss curvature flow, if for each  $t \in [0,T)$ ,  $F(M^n,t) = \Sigma_t$  is a complete convex hypersurface embedded in  $\mathbb{R}^{n+1}$ , and  $F(\cdot,t)$  satisfying

$$\frac{\partial}{\partial t}F(p,t)=K^{\alpha}(p,t)\vec{n}(p,t).$$

where K(p,t) and  $\vec{n}(p,t)$  are the Gauss curvature and the interior unit vector of  $\Sigma_t$  at the point F(p,t), respectively. In particular, if  $\alpha=1$  we call the immersion  $F:M^n\times [0,T)\to \mathbb{R}^{n+1}$  a solution of the Gauss curvature flow.

We consider a closed strictly convex smooth self-similar solution to the  $\alpha$ -Gauss curvature flow for  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ . Since a closed self-similar solution  $\Sigma$  is a shrinking solution, there exists an immersion  $F: M^n \to \mathbb{R}^{n+1}$  such that  $F(M^n) = \Sigma$  and the following holds

$$(*^{\alpha})$$
  $K^{\alpha}(p) = -\langle F(p), \vec{n}(p) \rangle.$ 

In [7] W. Firey introduced the Gauss curvature flow  $\alpha = 1$  and showed (assuming the existence and regularity of the flow) that a convex closed and centrally symmetric solution in  $\mathbb{R}^3$  contracts to a point and becomes a round sphere after rescaling. He also conjectured that the same result holds true without the symmetry assumption.

In [10] K. Tso established the existence and uniqueness of the Gauss curvature flow  $\alpha = 1$  in  $\mathbb{R}^{n+1}$  and showed that the flow contracts a closed, smooth and strictly convex hypersurface to a point in finite time. In [6] B. Chow extended Tso's result to the  $\alpha$ -Gauss curvature flow for all  $\alpha > 0$  in  $\mathbb{R}^{n+1}$ .

In [5] E. Calabi showed that if  $\alpha = \frac{1}{n+2}$ , closed self-similar solutions are ellipsoids. On the other hand, B. Chow proved in [6] that if  $\alpha = \frac{1}{n}$ , a strictly convex closed solution converges to a round sphere after normalizing the enclosed volume, which implies that the strictly convex closed self-similar solution is the unit sphere.

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In [1] B. Andrews proved Firey's conjecture, showing that the Gauss curvature flow  $\alpha=1$  and n=2 contracts a weakly convex hypersurface in  $\mathbb{R}^3$  to a round sphere. Also, in [3] B. Andrews and X. Chen established the same convergence result for  $\alpha \in (\frac{1}{2}, 1)$  and n=2. The proof of B. Andrew's result in [1] is based on a beautiful pinching estimate which unfortunately does not generalize in higher dimensions.

Recently, P. Guan and L. Ni [8] obtained the convergence of a centrally symmetric solution of the Gauss curvature flow  $\alpha=1$  to a sphere and in [4] they extended the same result to  $\alpha\geqslant 1$  jointly with B. Andrews. The convergence of the Gauss curvature flow  $\alpha=1$  to the sphere without any symmetry assumption in higher dimensions has remained an open question.

On the other hand it follows from the works [2, 4, 8, 9], that if  $\alpha > \frac{1}{n+2}$ , then a strictly convex closed solution to the  $\alpha$ -Gauss curvature flow converges to a strictly convex smooth closed self-similar solution after normalizing the enclosed volume. Thus the convergence of the  $\alpha$ -Gauss curvature flow to the sphere for  $\alpha > \frac{1}{n+2}$  is reduced to the classification of convex smooth closed self-similar solutions.

In this work we show that if  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$  then the only strictly convex smooth and closed self-similar solution of the  $\alpha$ -Gauss curvature flow is the round sphere.

**Theorem 1.1** (Uniqueness of closed self-similar solutions). Given  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ , the unit n-sphere is the unique closed strictly convex smooth solution to  $(*^{\alpha})$ .

As we discussed above, the results in [2, 4, 8, 9] combined with Theorem 1.1 imply the convergence of the  $\alpha$ -Gauss curvature flow to the round sphere, which in particular proves the higher dimensional Firey's conjecture.

**Theorem 1.2.** Let  $\Sigma_t$  be a strictly convex, closed and smooth solution to the  $\alpha$ -Gauss curvature flow with  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ ,  $n \geq 2$ . Then, there exists a finite time T at which the surface  $\Sigma_t$  converges after rescaling to the round sphere.

Discussion of the proof: In [6], B. Chow used the quantity  $HK^{-\frac{1}{n}}$  as a subsolution to obtain the convergence of the  $\alpha$ -Gauss curvature flow to the sphere when  $\alpha = \frac{1}{n}$ . The third order terms of the evolution equation of  $HK^{-\frac{1}{n}}$  are controlled by the concavity of the  $K^{\frac{1}{n}}$  operator. Also, the evolution equation has no reaction term, because  $HK^{-\frac{1}{n}}$  is a homogeneous of degree 0 function.

In this paper, we use the quantity  $w(p) \coloneqq K^{\alpha} \lambda_{\min}^{-1}(p) - \frac{n\alpha - 1}{2n\alpha} |F|^2(p)$ , where  $\lambda_{\min}$  is the smallest principal curvature. The second order terms in the equation of  $\mathcal{L}(K^{\alpha}\lambda_{\min}^{-1})$  can be controlled by terms that appear in the equation of  $\frac{n\alpha - 1}{2n\alpha} \mathcal{L}|F|^2$ , where  $\mathcal{L}$  is the linearized elliptic operator given in Notation 2.1. Hence, we only need to control the third order terms of the equation of  $\mathcal{L}w$ . To deal with the third order terms, we adopt a Pogorelov type estimate with  $\lambda_{\min}^{-1}$  replaced by  $(b^{1i}g_{ij}b^{j1})^{\frac{1}{2}}$ , where  $\{b^{ij}\}$  is the inverse matrix of  $\{h_{ij}\}$  and at a point where  $\lambda_{\min} = b^{11}$ . This is the main calculation in our work and will be done in the proof of Theorem 3.2, where we will show that if w(p) attains its maximum at a point  $F(p_0)$ , then the point  $F(p_0)$  is an *umbilical point*.

In section 4 we will use the strong maximum principle to establish our uniqueness result, Theorem 1.1. To this end, we need to introduce the quantity  $f(p) := K^{\alpha} \sum_{i=1}^{n} \lambda_{i}^{-1}(p) - \frac{n\alpha - 1}{2\alpha} |F|^{2}(p)$  and first show in Proposition 4.1 that if it attains its maximum at a point  $F(p_{0})$ , then the point  $F(p_{0})$  is also an umbilical point (notice that  $\lambda_{1}, \dots, \lambda_{n}$  denote as usual the principal curvatures). This is an immediate consequence of Theorem 3.2. Then, we will apply the strong maximum principle on F(p) and prove our uniqueness result. In the Pogorelov type estimate on F(p) we can diagonalize the second fundamental form F(p) at one given point (the maximum point). The reason we need to use the quantity F(p) is that in this case we can diagonalize F(p) at each point.

Remark 1.3 (Pogorelov estimate on powers of a matrix). Pogorelov type estimates in our context have been frequently applied in the past by using  $b^{11}$ , the first entry of a matrix  $A^{-1} := \{b^{ij}\}$ . However, if one applies the Pogorelov estimate for  $b^{11}K^{\alpha} - \frac{n\alpha-1}{2n\alpha}|F|^2$ , one can obtain the result of Theorem 3.2 only for  $\alpha \in (\frac{1}{n}, \frac{1}{2}]$ . In this work, by using instead  $(b^{1i}g_{ij}b^{j1})^{\frac{1}{2}}$ , the root of the first entry of the square  $A^{-2}$  of the matrix  $A^{-1}$ , we are able to extend the result of Theorem 3.2 to the range of exponents  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ , which includes the classical case of the Gauss curvature flow  $\alpha = 1$ .

One can apply a similar Pogorelov type estimate using the *m*-th root of the first entry of  $A^{-m}$ , with large  $m \in \mathbb{N}$  (depending on *n*) and extend our result to the range of exponents  $\alpha \in (\frac{1}{n}, 1 + (\frac{n-1}{n})^{\frac{1}{2}})$ . Notice that if  $\alpha = 1 + (\frac{n-1}{n})^{\frac{1}{2}}$ , then we have  $I_1 = 0$ , where  $I_1 = \frac{n\alpha - 1}{n\alpha} + 1 - \alpha$  is given in the proof of Theorem 3.2.

Since our goal of this paper is to prove Firey's conjecture in higher dimensions, we provide the proof of the uniqueness of closed self-shrinkers to the  $\alpha$ -Gauss curvature flow for  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$  by using  $A^{-2}$ .

## 2. Preliminaries

Notation 2.1. For reader's convenience, we summarize the notation as follows.

- (i) We recall the metric  $g_{ij} = \langle F_i, F_j \rangle$ , where  $F_i := \nabla_i F$ , and its inverse matrix  $g^{ij}$  of  $g_{ij}$ , namely  $g^{ij}g_{jk} = \delta^i_k$ . Also, we use the notation  $F^i = g^{ij}F_j$ .
- (ii) For a strictly convex smooth hypersurface  $\Sigma$ , we denote by  $b^{ij}$  inverse matrix of its *second fundamental form*  $h_{ij}$ , namely  $b^{ij}h_{ik} = \delta^i_{\iota}$ .
- (iii) We denote by  $\mathcal{L}$  the *linearized* operator

$$\mathcal{L} = \alpha K^{\alpha} b^{ij} \nabla_i \nabla_j$$

Also,  $\langle \ , \ \rangle_{\mathcal{L}}$  denotes the associated inner product  $\langle \nabla f, \nabla g \rangle_{\mathcal{L}} := \alpha K^{\alpha} b^{ij} \nabla_i f \nabla_j g$ , where f, g are differentiable functions on  $M^n$ , and  $\| \cdot \|_{\mathcal{L}}$  denotes the  $\mathcal{L}$ -norm given by the inner product  $\langle \ , \ \rangle_{\mathcal{L}}$ .

- (iv) We denote as usual by H and  $\lambda_{\min}$  the *mean curvature* and the *smallest principal curvature*, respectively.
- (v) We will use in the sequel the functions  $f: M^n \to \mathbb{R}$  and  $w: M^n \to \mathbb{R}$  defined by

$$f(p) = \left(K^{\alpha}b^{ij}g_{ij} - \frac{n\alpha - 1}{2\alpha}|F|^2\right)(p), \qquad w(p) = \left(K^{\alpha}\lambda_{\min}^{-1} - \frac{n\alpha - 1}{2n\alpha}|F|^2\right)(p).$$

**Proposition 2.2.** Given a strictly convex smooth solution  $F: M^n \to \mathbb{R}^{n+1}$  of  $(*^{\alpha})$ , the following hold

$$\nabla_i b^{jk} = -b^{jl} b^{km} \nabla_i h_{lm},$$

$$(2.2) \qquad \mathcal{L}|F|^2 = 2\alpha K^{\alpha} b^{ij} (g_{ij} - h_{ij} K^{\alpha}) = 2\alpha K^{\alpha} b^{ij} g_{ij} - 2n\alpha K^{2\alpha},$$

(2.3) 
$$\nabla_i K^{\alpha} = h_{ij} \langle F, F^j \rangle,$$

(2.4) 
$$\mathcal{L}K^{\alpha} = \langle F, \nabla K^{\alpha} \rangle + n\alpha K^{\alpha} - \alpha K^{2\alpha}H,$$

(2.5) 
$$\mathcal{L}b^{pq} = K^{-\alpha}b^{pr}b^{qs}\nabla_{r}K^{\alpha}\nabla_{s}K^{\alpha} + \alpha K^{\alpha}b^{pr}b^{qs}b^{ij}b^{km}\nabla_{r}h_{ik}\nabla_{s}h_{jm} + \langle F, \nabla b^{pq} \rangle - b^{pq} - (n\alpha - 1)g^{pq}K^{\alpha} + \alpha K^{\alpha}Hb^{pq}.$$

*Proof.* From  $\nabla_i(b^{jk}h_{kl}) = \nabla_i\delta_l^j = 0$ , we can derive  $h_{kl}\nabla_i b^{jk} = -b^{jk}\nabla_i h_{kl}$ . Hence, we have (2.1) by  $\nabla_i b^{jm} = b^{lm}h_{kl}\nabla_i b^{jk} = -b^{lm}b^{jk}\nabla_i h_{kl}.$ 

Also, by definition  $\mathcal{L} := \alpha K^{\alpha} b^{ij} \nabla_i \nabla_j$  we have

$$\mathcal{L}|F|^2 = 2\alpha K^{\alpha}b^{ij}\langle F_i, F_i\rangle + 2\alpha K^{\alpha}b^{ij}\langle F, \nabla_i \nabla_j F \vec{n}\rangle = 2\alpha K^{\alpha}b^{ij}g_{ij} + 2\alpha K^{\alpha}b^{ij}\langle F, h_{ij}\vec{n}\rangle.$$

Thus, the given equation  $(*^{\alpha})$  implies (3.5).

Equation (2.3) can be simply obtained by differentiating ( $*^{\alpha}$ )

$$\nabla_i K^{\alpha} = h_{ik} \langle F, F^k \rangle.$$

Differentiating the equation above again we obtain

$$\nabla_{i}\nabla_{j}K^{\alpha} = \nabla_{i}h_{jk}\langle F, F^{k}\rangle + h_{ij} + h_{ik}h_{j}^{k}\langle F, \vec{n}\rangle = \langle F, \nabla h_{ij}\rangle + h_{ij} - h_{ik}h_{j}^{k}K^{\alpha}.$$

On the other hand, (2.1) and direct differentiation yield

$$\nabla_i \nabla_j K^{\alpha} = \nabla_i (\alpha K^{\alpha} b^{pq} \nabla_j h_{pq}) = \alpha K^{\alpha} b^{pq} \nabla_i \nabla_j h_{pq} + \alpha^2 K^{\alpha} b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} - \alpha K^{\alpha} b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq}.$$

Observing

$$\nabla_{i}\nabla_{j}h_{pq} = \nabla_{i}\nabla_{p}h_{jq} = \nabla_{p}\nabla_{i}h_{jq} + R_{ipjm}h_{q}^{m} + R_{ipqm}h_{j}^{m}$$

$$= \nabla_{p}\nabla_{q}h_{ij} + (h_{ij}h_{pm} - h_{im}h_{jp})h_{q}^{m} + (h_{iq}h_{pm} - h_{im}h_{pq})h_{i}^{m}$$

we obtain

$$\alpha K^{\alpha} b^{pq} \nabla_i \nabla_j h_{pq} = \alpha K^{\alpha} b^{pq} \nabla_p \nabla_q h_{ij} + \alpha K^{\alpha} H h_{ij} - n \alpha K^{\alpha} h_{im} h_j^m = \mathcal{L} h_{ij} + \alpha K^{\alpha} H h_{ij} - n \alpha K^{\alpha} h_{im} h_j^m.$$

Combining the equations above yields

(2.6) 
$$\mathcal{L}h_{ij} = -\alpha^2 K^{\alpha} b^{rs} b^{pq} \nabla_i h_{rs} \nabla_j h_{pq} + \alpha K^{\alpha} b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq} + \langle F, \nabla h_{ij} \rangle + h_{ij} + (n\alpha - 1) h_{ik} h_i^k K^{\alpha} - \alpha K^{\alpha} H h_{ij}.$$

We now observe

$$\mathcal{L}K^{\alpha} = \alpha K^{\alpha}b^{ij}\nabla_{i}(\alpha K^{\alpha}b^{pq}\nabla_{j}h_{pq})$$

$$= \alpha^{3}K^{2\alpha}b^{ij}b^{pq}b^{rs}\nabla_{i}h_{rs}\nabla_{j}h_{pq} - \alpha^{2}K^{2\alpha}b^{ij}b^{pr}b^{qs}\nabla h_{rs}\nabla_{j}h_{pq} + \alpha K^{\alpha}b^{pq}\mathcal{L}h_{pq}$$

which gives (2.4), since

$$\mathcal{L}K^{\alpha} = \alpha K^{\alpha}b^{ij}(\langle F, \nabla h_{ij} \rangle + h_{ij} + (n\alpha - 1)h_{ik}h_{i}^{k}K^{\alpha} - \alpha K^{\alpha}Hh_{ij}) = \langle F, \nabla K^{\alpha} \rangle + n\alpha K^{\alpha} - \alpha K^{2\alpha}H.$$

Finally, by using (2.1), we can derive

$$\mathcal{L}b^{pq} = \alpha K^{\alpha}b^{ij}\nabla_{i}(-b^{pr}b^{qs}\nabla_{j}h_{rs}) = 2\alpha K^{\alpha}b^{ij}b^{pk}b^{rm}b^{qs}\nabla_{i}h_{km}\nabla_{j}h_{rs} - b^{pr}b^{qs}\mathcal{L}h_{rs}.$$

Applying (2.6) yields

$$\mathcal{L}b^{pq} = \alpha^2 K^{\alpha} b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ij} \nabla_s h_{km} + \alpha K^{\alpha} b^{pr} b^{qs} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} + \langle F, \nabla b^{pq} \rangle - b^{pq} - (n\alpha - 1) g^{pq} K^{\alpha} + \alpha K^{\alpha} H b^{pq}.$$

Thus,  $\nabla K^{\alpha} = \alpha K^{\alpha} b^{ij} \nabla h_{ij}$  gives the desired result.

# 3. Pogorelov type computation

We consider the function  $w: M^n \to \mathbb{R}$  given by

$$w(p) := \left(K^{\alpha} \lambda_{\min}^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2\right)(p).$$

We will employ in this section a Pogorelov type computation to show that the maximum point of w(p) is an umbilical point. We begin with the following standard observation which we include here for the reader's convenience.

**Proposition 3.1** (Euler's formula). Let  $\Sigma$  be a smooth strictly convex hypersurface and  $F: M^n \to \mathbb{R}^{n+1}$  be a smooth immersion with  $F(M^n) = \Sigma$ . Then, given a coordinate chart  $\varphi: U(\subset \mathbb{R}^n) \to M^n$  of a point  $p \in \varphi(U)$ , the following holds for each  $i \in \{1, \dots, n\}$ 

$$\sum_{j=1}^{n} \frac{b^{ij}b_{j}^{i}(p)}{g^{ii}(p)} \leqslant \frac{1}{\lambda_{\min}^{2}(p)}.$$

*Proof.* For a fixed point  $p \in M^n$ , we choose an orthonormal basis  $\{E_1, \dots, E_n\}$  of  $T\Sigma_{F(p)}$  such that  $L(E_j) = \lambda_j E_j$ , where L is the Weingarten map and  $\lambda_1, \dots, \lambda_n$  are the principal curvatures of  $\Sigma$  at p. Given a chart  $(\varphi, U)$  of  $p \in \varphi(U) \subset M^n$ , we denote by  $\{a_{ij}\}$  the matrix satisfying  $F_i(p) := \nabla_i F(p) = a_{ij} E_j$  and by  $\{c_{ij}\}$  the diagonal matrix diag $(\lambda_1, \dots, \lambda_n)$ . We also denote by  $\{a^{ij}\}$  and  $\{c^{ij}\}$  the inverse matrices of  $\{a_{ij}\}$  and  $\{c_{ij}\}$ , respectively.

We observe  $g_{ij}(p) = \langle F_i, F_j \rangle(p) = \langle a_{ik}E_k, a_{jl}E_l \rangle = a_{ik}a_{jk}$ . Also, we can obtain  $F^i(p) = a^{ji}E_j$  by  $a^{ji} = a^{jk}\langle F_k(p), F^i(p) \rangle = \langle a^{jk}a_{kl}E_l, F^i(p) \rangle = \langle E_j, F^i(p) \rangle$ . So, we have  $g^{ij}(p) = \langle F^i, F^j \rangle(p) = a^{ki}a^{kj}$ . In addition,  $LF_i(p) = h_{ij}(p)F^j(p)$  implies

$$a^{mi}E_{m} = F^{i}(p) = b^{ij}(p)h_{jk}(p)F^{k}(p) = b^{ij}(p)LF_{j}(p)$$
$$= b^{ij}(p)L(a_{jk}E_{k}) = b^{ij}(p)a_{jk}L(E_{k}) = b^{ij}(p)a_{jk}\lambda_{k}E_{k} = b^{ij}(p)a_{jk}c_{km}E_{m}.$$

Hence, we have  $b^{in}(p) = b^{ij}(p)a_{ik}c_{km}c^{ml}a^{ln} = a^{mi}c^{ml}a^{ln}$ , and thus the following holds

$$b^{1r}g_{rs}b^{s1}(p) = a^{i1}c^{ij}a^{jr}a_{rk}a_{sk}a^{m1}c^{ml}a^{ls} = a^{i1}c^{ij}\delta_k^j\delta_k^la^{m1}c^{ml} = a^{i1}c^{ij}a^{m1}c^{mj} = \sum_j(a^{j1})^2\lambda_j^{-2}$$

$$\leq \sum_i(a^{j1})^2\lambda_{\min}^{-2} = \lambda_{\min}^{-2}\sum_{k,j}\langle a^{k1}E_k, a^{j1}E_j\rangle = \lambda_{\min}^{-2}\langle F^1(p), F^1(p)\rangle = \lambda_{\min}^{-2}g^{11}(p),$$

which is the desired result for i = 1 and we can obtain the same result for each  $i \in \{1, \dots, n\}$ .

We will now show that one of the Pogorelov type expressions of the function w plays a role as a subsolution of  $(*^{\alpha})$  at a given maximum point, to imply that the maximum point of w(p) is an umbilical point.

**Theorem 3.2** (Pogorelov type computation). Let  $\Sigma$  be a strictly convex smooth closed solution of  $(*^{\alpha})$  for an exponent  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ . Assume that  $F : M^n \to \mathbb{R}^{n+1}$  is a smooth immersion such that  $F(M^n) = \Sigma$ , and the continuous function w(p) attains its maximum at a point  $p_0$ . Then,  $F(p_0)$  is an umbilical point and  $\nabla |F|^2(p_0) = 0$  holds.

*Proof.* We begin by choosing a coordinate chart  $(U,\varphi)$  of  $p_0 \in \varphi(U) \subset M^n$  such that the covariant derivatives  $\{\nabla_i F(p_0) := \partial_i (F \circ \varphi)(\varphi^{-1}(p_0))\}_{i=1,\cdots,n}$  form an orthonormal basis of  $T\Sigma_{F(p_0)}$  satisfying

$$g_{ij}(p_0) = \delta_{ij}, \qquad h_{ij}(p_0) = \delta_{ij}\lambda_i(p_0), \qquad \lambda_1(p_0) = \lambda_{\min}(p_0),$$

which guarantees  $b^{11}(p_0) = \lambda_{\min}^{-1}(p_0)$  and  $g^{11}(p_0) = 1$ . Next, we define the function  $\bar{w}: \varphi(U) \to \mathbb{R}$  by

$$\overline{w}(p) \coloneqq K^{\alpha} \left( \frac{b^{1i} g_{ij} b^{j1}}{g^{11}} \right)^{\frac{1}{2}}(p) - \frac{n\alpha - 1}{2n\alpha} |F|^2(p).$$

Then, by Proposition 3.1 we have

$$\bar{w}(p) \leqslant w(p) \leqslant w(p_0) = \bar{w}(p_0),$$

which means that  $\bar{w}$  attains its maximum at  $p_0$ .

We will now calculate  $\mathcal{L}\bar{w} := \alpha K^{\alpha} b^{ij} \nabla_i \nabla_j \bar{w}$  at the point  $p_0$ . First we derive the following equation from (2.5)

$$\mathcal{L}\left(b_p^1b^{p1}\right) = 2\alpha K^{\alpha}b^{ij}\nabla_i b^{p1}\nabla_j b_p^1 + 2K^{-\alpha}b_p^1b^{pr}b^{1s}\nabla_r K^{\alpha}\nabla_s K^{\alpha} + 2\alpha K^{\alpha}b_p^1b^{pr}b^{1s}b^{ij}b^{km}\nabla_r h_{ik}\nabla_s h_{jm} + \langle F, \nabla(b_p^1b^{p1})\rangle - 2b_p^1b^{p1} - 2(n\alpha - 1)K^{\alpha}b^{11} + 2\alpha K^{\alpha}Hb_p^1b^{p1}.$$

Thus, we obtain

$$\begin{split} \mathcal{L} \left( \frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} &= -\frac{\alpha K^\alpha b^{ij} \nabla_i (b_p^1 b^{1p}) \nabla_j (b_q^1 b^{1q})}{4 (b_r^1 b^{r1})^{\frac{3}{2}} (g^{11})^{\frac{1}{2}}} + \frac{\alpha K^\alpha b^{ij} \nabla_i b^{p1} \nabla_j b_p^1}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} \\ &+ \frac{b_p^1 b^{pr} b^{1s} \nabla_r K^\alpha \nabla_s K^\alpha}{K^\alpha (b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} + \frac{\alpha K^\alpha b_p^1 b^{pr} b^{1s} b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm}}{(b_q^1 b^{q1} g^{11})^{\frac{1}{2}}} \\ &+ \left\langle F, \nabla \left( b_p^1 b^{p1} / g^{11} \right)^{\frac{1}{2}} \right\rangle - \left( \frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}} - \frac{(n\alpha - 1) K^\alpha b^{11}}{(b_p^1 b^{p1} g^{11})^{\frac{1}{2}}} + \alpha K^\alpha H \left( \frac{b_p^1 b^{p1}}{g^{11}} \right)^{\frac{1}{2}}. \end{split}$$

Combining this with (2.4) yields

$$(3.1) \qquad \mathcal{L}\bar{w} = -\frac{n\alpha - 1}{2n\alpha} \mathcal{L}|F|^{2} + 2\left\langle\nabla K^{\alpha}, \nabla\left(\frac{b_{p}^{1}b^{p1}}{g^{11}}\right)^{\frac{1}{2}}\right\rangle_{\mathcal{L}} - \frac{\alpha K^{2\alpha}b^{ij}b_{p}^{1}b_{q}^{1}\nabla_{i}b^{1p}\nabla_{j}b^{1q}}{(b_{r}^{1}b^{r1})^{\frac{3}{2}}(g^{11})^{\frac{1}{2}}} + \frac{\alpha K^{2\alpha}b^{ij}\nabla_{i}b^{p1}\nabla_{j}b_{p}^{1}}{(b_{q}^{1}b^{q1}g^{11})^{\frac{1}{2}}} + \frac{b_{p}^{1}b^{pr}b^{1s}\nabla_{r}K^{\alpha}\nabla_{s}K^{\alpha}}{(b_{q}^{1}b^{q1}g^{11})^{\frac{1}{2}}} + \frac{\alpha K^{2\alpha}b_{p}^{1}b^{pr}b^{1s}b^{ij}b^{km}\nabla_{r}h_{ik}\nabla_{s}h_{jm}}{(b_{q}^{1}b^{q1}g^{11})^{\frac{1}{2}}} + \left\langle F, \nabla\left(K^{\alpha}\left(b_{p}^{1}b^{p1}/g^{11}\right)^{\frac{1}{2}}\right)\right\rangle + (n\alpha - 1)K^{\alpha}\left(\frac{b_{p}^{1}b^{p1}}{g^{11}}\right)^{\frac{1}{2}} - \frac{(n\alpha - 1)K^{2\alpha}b^{11}}{(b_{p}^{1}b^{p1}g^{11})^{\frac{1}{2}}}.$$

Observe that

$$2\left\langle \nabla K^{\alpha}, \nabla \left(\frac{b_{p}^{1}b^{p1}}{g^{11}}\right)^{\frac{1}{2}}\right\rangle_{\mathcal{L}} = 2\alpha K^{\alpha}b^{ij}(g^{11})^{-\frac{1}{2}}\left(b_{q}^{1}b^{q1}\right)^{-\frac{1}{2}}b_{p}^{1}\nabla_{i}K^{\alpha}\nabla_{j}b^{p1},$$

and

$$\nabla \left( K^{\alpha} \left( b_p^1 b^{p1} / g^{11} \right)^{\frac{1}{2}} \right) = \nabla \overline{w} + \frac{n\alpha - 1}{2n\alpha} \nabla |F|^2.$$

Hence, applying the equations above, (2.2) and  $\nabla \bar{w}(p_0) = 0$  to (3.1) yields that the following holds at the maximum point  $p_0$ 

$$(3.2) \quad 0 \geqslant 2\alpha K^{\alpha} \sum_{i=1}^{n} b^{ii} \nabla_{i} K^{\alpha} \nabla_{i} b^{11} - \alpha K^{2\alpha} \sum_{i=1}^{n} b^{ii} h_{11} |\nabla_{i} b^{11}|^{2} + \alpha K^{2\alpha} \sum_{j,p} b^{jj} h_{11} |\nabla_{j} b^{p1}|^{2} + |b^{11} \nabla_{1} K^{\alpha}|^{2}$$

$$+ \alpha K^{2\alpha} (b^{11})^{2} \sum_{i,j} b^{ii} b^{jj} |\nabla_{1} h_{ij}|^{2} + \frac{n\alpha - 1}{2n\alpha} \langle F, \nabla |F|^{2} \rangle + (n\alpha - 1) K^{\alpha} (b^{11} - \frac{1}{n} \sum_{i=1}^{n} b^{ii}).$$

By (2.1), the second and third terms on the right hand side of the inequality above (3.2) satisfy

$$\begin{split} &-\sum_{i=1}^{n}b^{ii}h_{11}|\nabla_{i}b^{11}|^{2}+\sum_{j,p}b^{jj}h_{11}|\nabla_{j}b^{p1}|^{2}=-\sum_{i=1}^{n}b^{ii}(b^{11})^{3}|\nabla_{i}h_{11}|^{2}+\sum_{j,p}b^{jj}b^{11}(b^{pp})^{2}|\nabla_{j}h_{p1}|^{2}\\ &=\sum_{j=1}^{n}\sum_{p\neq 1}b^{jj}b^{11}(b^{pp})^{2}|\nabla_{j}h_{p1}|^{2}\geqslant\sum_{p\neq 1}(b^{11}b^{pp})^{2}|\nabla_{p}h_{11}|^{2}=\sum_{p\neq 1}(b^{pp}h_{11})^{2}|\nabla_{p}b^{11}|^{2}. \end{split}$$

Also, by (2.1) the fifth term on the right hand side of (3.2) satisfies

$$(b^{11})^2 \sum_{i,j} b^{ii} b^{jj} |\nabla_1 h_{ij}|^2 \geqslant (b^{11})^4 |\nabla_1 h_{11}|^2 + 2 \sum_{i \neq 1} (b^{11})^3 b^{ii} |\nabla_i h_{11}|^2 = |\nabla_1 b^{11}|^2 + 2 \sum_{i \neq 1} b^{ii} h_{11} |\nabla_i b^{11}|^2.$$

Furthermore, we have

$$\alpha K^{2\alpha} |\nabla_1 b^{11}|^2 + 2\alpha K^\alpha b^{11} \nabla_1 K^\alpha \nabla_1 b^{11} \geqslant -\alpha |b^{11} \nabla_1 K^\alpha|^2.$$

Hence, by applying the inequalities above, we can reduce (3.2) to

$$(3.3) 0 \ge 2\alpha \sum_{i \ne 1} b^{ii} \nabla_i K^{\alpha} \left( K^{\alpha} \nabla_i b^{11} \right) + \alpha \sum_{p \ne 1} (b^{pp} h_{11})^2 |K^{\alpha} \nabla_p b^{11}|^2 + 2\alpha \sum_{i \ne 1} b^{ii} h_{11} |K^{\alpha} \nabla_i b^{11}|^2$$
$$+ (1 - \alpha) |b^{11} \nabla_1 K^{\alpha}|^2 + \frac{n\alpha - 1}{2n\alpha} \langle F, \nabla |F|^2 \rangle + (n\alpha - 1) K^{\alpha} \left( b^{11} - \frac{1}{n} \sum_{i=1}^n b^{ii} \right).$$

We now employ (2.3) to obtain the following at the point  $p_0$ 

(3.4) 
$$b^{ii}\nabla_i K^{\alpha} = b^{ii}h_{ii}\langle F, F^i \rangle = \langle F, F^i \rangle.$$

In addition, at the point  $p_0$ ,  $\nabla_i \overline{w}(p_0) = 0$  yields

$$K^{\alpha}\nabla_{i}b^{11} = -b^{11}\nabla_{i}K^{\alpha} + \frac{n\alpha - 1}{2n\alpha}\nabla_{i}|F|^{2} = -b^{11}h_{ii}\langle F, F^{i}\rangle + \frac{n\alpha - 1}{n\alpha}\langle F, F_{i}\rangle = (\beta - \theta_{i})\langle F, F_{i}\rangle,$$

where  $\theta_i = b^{11} h_{ii}(p_0)$  and  $\beta = \frac{n\alpha - 1}{n\alpha}$ . We also have

$$(3.5) \qquad \langle F, \nabla | F |^2 \rangle := \langle F, (\nabla_i | F |^2) F^i \rangle = \langle F, F^i \rangle (\nabla_i | F |^2) = 2 \langle F, F_i \rangle \langle F, F^i \rangle.$$

Hence, we can rewrite (3.3) as

$$(3.6) 0 \geqslant \sum_{i \neq 1} \langle F, F_i \rangle^2 J_i + \langle F, F_1 \rangle^2 I_1 + (n\alpha - 1) K^{\alpha} \left( b^{11} - \frac{1}{n} \sum_{i=1}^n b^{ii} \right),$$

where

$$I_1 = rac{nlpha-1}{nlpha} + 1 - lpha, \qquad \qquad J_i = 2lpha \Big(eta - heta_i\Big) + lpha \Big( heta_i^{-2} + 2 heta_i^{-1}\Big) ig(eta - heta_iig)^2 + eta.$$

We observe that  $I_1 > 0$  holds, and also  $J_i$  satisfies

$$J_{i} = 2\alpha\beta - 2\alpha\theta_{i} + \alpha\beta^{2}\theta_{i}^{-2} + 2\alpha\beta^{2}\theta_{i}^{-1} - 2\alpha\beta\theta_{i}^{-1} - 4\alpha\beta + \alpha + 2\alpha\theta_{i} + \beta$$

$$= \alpha(1 - \beta) + \beta(1 - \alpha) + 2\alpha\beta(\beta - 1)\theta_{i}^{-1} + \alpha\beta^{2}\theta_{i}^{-2} = \frac{1}{n} + \beta(1 - \alpha) - \frac{2\beta}{n}\theta_{i}^{-1} + \alpha\beta^{2}\theta_{i}^{-2}$$

$$= \beta(1 - \alpha) + \frac{1}{n} + \alpha\left(\beta\theta_{i}^{-1} - \frac{1}{n\alpha}\right)^{2} - \frac{1}{n^{2}\alpha} \geqslant \beta(1 - \alpha) + \frac{1}{n}\left(\frac{n\alpha - 1}{n\alpha}\right) = \beta(1 - \alpha + \frac{1}{n}) > 0.$$

Since we have  $b^{11}(p_0) = \lambda_{\min}^{-1}(p_0) \ge \lambda_i^{-1}(p_0) \ge b^{ii}(p_0)$  and  $\langle F, F_i \rangle^2(p_0) \ge 0$  for all  $i \in \{1, \dots, n\}$ , the inequality (3.6) and  $I_1, I_i > 0$  give the desired result.

## 4. STRONG MAXIMUM PRINCIPLE

In this section, we will show how Theorem 3.2 can be modified to give us the proof of our main result, Theorem 1.1. To this end, we will introduce the new geometric, chart-independent quantity

$$f(p) = \left(K^{\alpha}b^{ij}g_{ij} - \frac{n\alpha - 1}{2\alpha}|F|^{2}\right)(p)$$

and apply the strong maximum principle. If we use w(p),  $h_{ij}$  can be diagonalized only at one given point. However, if we employ f(p), we can diagonalize  $h_{ij}$  at each point. We begin with the following observation which simply follows from Theorem 3.2.

**Proposition 4.1.** Let  $\Sigma$  be a strictly convex smooth closed solution of  $(*^{\alpha})$  for an exponent  $\alpha \in (\frac{1}{n}, 1 + \frac{1}{n})$ . Assume that  $F: M^n \to \mathbb{R}^{n+1}$  is a smooth immersion such that  $F(M^n) = \Sigma$ , and the continuous function f(p) attains its maximum at a point  $p_0$ . Then,  $F(p_0)$  is an umbilical point and  $\nabla |F|^2(p_0) = 0$  holds.

*Proof.* We observe  $b^{ij}g_{ij}(p) = \sum_{i=1}^n \lambda_i^{-1}(p)$ , where  $\lambda_1(p), \dots, \lambda_n(p)$  are the principal curvatures of  $\Sigma$  at F(p). Therefore, we have  $f(p) \leq n w(p)$ . However, if  $w(p_0) = \max_{p \in M^n} w(p)$ , then  $f(p_0) = n w(p_0)$  holds, because  $F(p_0)$  is an umbilical point by Theorem 3.2. Hence, we have

$$f(p) \leqslant n w(p) \leqslant \max_{p \in M^n} n w(p) = \max_{p \in M^n} f(p).$$

Thus, if f attains its maximum at a point  $p_0$ , then w also attains its maximum at  $p_0$ , and thus we can obtain the desired result by Theorem 3.2.

We will now employ the strong maximum principle to prove Theorem 1.1.

*Proof of Theorem 1.1.* We define a set  $M_f \subset M^n$  by

$$M_f = \{ p \in M^n : f(p) = \max_{M^n} f \}.$$

Since f(p) is a continuous function defined on a closed manifold  $M^n$ , f attains its maximum, and thus  $M_f$  is not an empty set. We now define the continuous function  $\Lambda: M^n \to \mathbb{R}$  and the open set  $V \subset M^n$  by

$$\Lambda(p) = \sum_{i,j} \left( \frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} \right)^2(p), \qquad V = \left\{ p \in M^n : \Lambda(p) < \left( \frac{10}{9} - \frac{9}{10} \right)^2 \right\}.$$

We now begin by combining (2.4) and (2.5) to obtain

$$\mathcal{L}\left(K^{\alpha}b^{pq}\right) = 2\langle \nabla K^{\alpha}, \nabla b^{pq} \rangle_{\mathcal{L}} + b^{pr}b^{qs}\nabla_{r}K^{\alpha}\nabla_{s}K^{\alpha} + \alpha K^{2\alpha}b^{pr}b^{qs}b^{ij}b^{km}\nabla_{r}h_{ik}\nabla_{s}h_{jm} + \langle F, \nabla(K^{\alpha}b^{pq})\rangle + (n\alpha - 1)K^{\alpha}(b^{pq} - g^{pq}K^{\alpha}).$$

Therefore, we can derive the following from (2.2) and  $\nabla g_{pq} = 0$ 

$$\mathcal{L}f = 2g_{pq}\langle \nabla K^{\alpha}, \nabla b^{pq} \rangle_{\mathcal{L}} + b^{pr}b_{p}^{s}\nabla_{r}K^{\alpha}\nabla_{s}K^{\alpha} + \alpha K^{2\alpha}b^{pr}b_{p}^{s}b^{ij}b^{km}\nabla_{r}h_{ik}\nabla_{s}h_{jm} + \langle F, \nabla (K^{\alpha}b^{pq}g_{pq}) \rangle.$$

By using (3.5), we can obtain

$$\langle F, \nabla (K^{\alpha}b^{pq}g_{pq}) \rangle = \langle F, \nabla f \rangle + \frac{n\alpha - 1}{2\alpha} \langle F, \nabla |F|^2 \rangle = \langle F, \nabla f \rangle + (n - \alpha^{-1}) \langle F, F_i \rangle \langle F, F^i \rangle.$$

Hence, we have

$$(4.1) \qquad \mathcal{L}f - \langle F, \nabla f \rangle = 2\alpha (b^{ij}\nabla_{i}K^{\alpha})(K^{\alpha}g_{pq}\nabla_{j}b^{pq}) + b^{pr}b_{p}^{s}\nabla_{r}K^{\alpha}\nabla_{s}K^{\alpha} + \alpha K^{2\alpha}b^{pr}b_{p}^{s}b^{ij}b^{km}\nabla_{r}h_{ik}\nabla_{s}h_{jm} + (n - \alpha^{-1})\langle F, F_{i}\rangle\langle F, F^{i}\rangle.$$

Given a fixed point  $p_0 \in V$ , we choose an orthonormal frame at  $F(p_0)$  satisfying

$$g_{ij}(p_0) = \delta_{ij}, \qquad h_{ij}(p_0) = \lambda_i(p_0)\delta_{ij}.$$

Then, at the point  $p_0$ , we can rewrite (4.1) as

$$(4.2) \qquad \mathcal{L}f - \langle F, \nabla f \rangle = 2\alpha \sum_{i,j} (b^{ii} \nabla_i K^{\alpha}) (K^{\alpha} \nabla_i b^{jj}) + \sum_i |b^{ii} \nabla_i K^{\alpha}|^2$$

$$+ \alpha K^{2\alpha} \sum_{i,j,k} (b^{ii})^2 b^{jj} b^{kk} |\nabla_i h_{jk}|^2 + (n - \alpha^{-1}) \sum_i \langle F, F_i \rangle^2.$$

Since  $p_0 \in V$  and the definition of V guarantees that  $b^{ii}h_{jj}(p_0) \ge \frac{9}{10}$ , by using (2.1) we can derive

$$\begin{split} \alpha K^{2\alpha} \sum_{i,j,k} (b^{ii})^2 b^{jj} b^{kk} |\nabla_i h_{jk}|^2 \geqslant & \alpha \sum_i |K^{\alpha} \nabla_i b^{ii}|^2 + 2\alpha \sum_{i \neq j} b^{jj} h_{ii} |K^{\alpha} \nabla_j b^{ii}|^2 + \alpha \sum_{i \neq j} (b^{ii} h_{jj})^2 |K^{\alpha} \nabla_i b^{jj}|^2 \\ \geqslant & \alpha \sum_i |K^{\alpha} \nabla_i b^{ii}|^2 + \frac{5}{2} \alpha \sum_{i \neq j} |K^{\alpha} \nabla_i b^{jj}|^2. \end{split}$$

We also have

$$\alpha \sum_{i} |K^{\alpha} \nabla_{i} b^{ii}|^{2} + 2\alpha \sum_{i} (b^{ii} \nabla_{i} K^{\alpha}) (K^{\alpha} \nabla_{i} b^{ii}) \geqslant -\alpha \sum_{i} |b^{ii} \nabla_{i} K^{\alpha}|^{2}$$

and

$$\frac{5}{2}\alpha\sum_{i\neq j}|K^{\alpha}\nabla_{i}b^{jj}|^{2}+2\alpha\sum_{i\neq j}(b^{ii}\nabla_{i}K^{\alpha})(K^{\alpha}\nabla_{i}b^{jj})\geqslant -\frac{2}{5}\alpha\sum_{i\neq j}|b^{ii}\nabla_{i}K^{\alpha}|^{2}=-\frac{2}{5}\alpha(n-1)\sum_{i}|b^{ii}\nabla_{i}K^{\alpha}|^{2}.$$

Applying the inequalities above and (3.4) to (4.2) yields

$$\mathcal{L}f - \langle F, \nabla f \rangle \geqslant \left( (1 - \alpha) - \frac{2}{5}\alpha(n - 1) + (n - \alpha^{-1}) \right) \sum_{i} \langle F, F_{i} \rangle^{2}$$
$$= \frac{1}{5\alpha} \left( -(2n + 3)\alpha^{2} + 5(n + 1)\alpha - 5 \right) \sum_{i} \langle F, F_{i} \rangle^{2}.$$

Let us consider the function  $y(\alpha) = -(2n+3)\alpha^2 + 5(n+1)\alpha - 5$ . Then, we have

$$y(1+1/n) = 3n - 2 - (3/n) - (3/n^2) \ge 0,$$
  $y(1/n) = (3/n) - (3/n^2) \ge 0,$ 

which implies  $y(\alpha) \ge 0$  for  $\alpha \in \left[\frac{1}{n}, 1 + \frac{1}{n}\right]$ . Therefore, on V the following holds

$$\mathcal{L} f - \langle F, \nabla f \rangle \geqslant 0.$$

Notice that  $\mathcal{L}f - \langle F, \nabla f \rangle$  is a chart-independent function. Hence, the Hopf maximum principle and  $M_f \subset V$  show that  $M_f = V$ . However,  $M_f$  is a closed set and V is an open set by the continuity of f and  $\Lambda$ , respectively. So, we conclude that  $M_f = M^n$ , and thus Proposition 4.1 gives the desired result.

#### REFERENCES

- [1] B. Andrews. Gauss curvature flow: the fate of the rolling stones. *Inventiones mathematicae*, 138(1):151–161, 1999.
- [2] B. Andrews. Motion of hypersurfaces by Gauss curvature. Pacific Journal of Mathematics, 195(1):1–34, 2000.
- [3] B. Andrews and X. Chen. Surfaces moving by powers of Gauss curvature. *Pure and Applied Mathematics Quarterly*, 8(4):825–834, 2012.
- [4] B. Andrews, P. Guan, and L. Ni. Flow by the power of the Gauss curvature. Advances in Mathematics, 299:174–201, 2016.
- [5] E. Calabi. Complete affine hyperspheres I. In *Convegno di Geometria Differenziale (INDAM, Rome, 1971), Symposia Mathematica*, volume 10, pages 19–38, 1972.
- [6] B. Chow. Deforming convex hypersurfaces by the *n*-th root of the Gaussian curvature. *Journal of Differential Geometry*, 22(1):117–138, 1985.
- [7] W. J. Firey. Shapes of worn stones. *Mathematika*, 21(1):1–11, 1974.
- [8] P. Guan and L. Ni. Entropy and a convergence theorem for Gauss curvature flow in high dimension. *Journal of the European Mathematical Society*. to appear.
- [9] L. Kim and K.-A. Lee.  $\alpha$ -Gauss curvature flows. arXiv:1306.1100, 2013.

[10] K. Tso. Deforming a hypersurface by its Gauss-Kronecker curvature. *Communications on Pure and Applied Mathematics*, 38(6):867–882, 1985.

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