# Free arrangements and coefficients of characteristic polynomials

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#### **Abstract**

Ziegler showed that free arrangements have free restricted multiarrangements (multirestrictions). After Ziegler's work, several results concerning "reverse direction", namely characterizing freeness of an arrangement via that of multirestriction, have appeared. In this paper, we prove that the second Betti number of the arrangement plays a crucial role.

# 1 Introduction

Let V be a vector space of dimension  $\ell$  over a field  $\mathbb{K}$ . Fix a system of coordinate  $(z_1, \ldots, z_{\ell})$  of  $V^*$ . We denote by  $S = S(V^*) = \mathbb{K}[z_1, \ldots, z_{\ell}]$  the symmetric algebra. A hyperplane arrangement  $\mathcal{A} = \{H_1, \ldots, H_n\}$  is a finite collection of hyperplanes in V.

Freeness of an arrangement is a key notion which connects arrangement theory with algebraic geometry and combinatorics. There are several ways to prove freeness, e. g. using Saito's criterion [Sa], addition-deletion theorem [T], etc. In [Z], Ziegler proved that the multirestriction  $(\mathcal{A}^{H_0}, m^{H_0})$  of a free arrangement  $\mathcal{A}$  is also free (see §2 for details). The converse is not true in general. However Schulze [Sc] recently proved that if the dimension is  $\ell \leq 4$  (or  $\ell \geq 5$  under tameness assumption), freeness of  $\mathcal{A}$  is characterized in terms of multirestriction and characteristic polynomials. The purpose of this paper is to give a stronger characterization of freeness for any dimension. Namely,

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we characterize the freeness in terms of the multirestriction and the second coefficients of characteristic polynomials (without posing any conditions on dimension or tameness).

This paper is organized as follows. In §2, we recall basic facts on characteristic polynomials for both simple and multiarrangements. In §3, we recall results from [Yo1, Yo2, Sc], which will be used in the proof of the main result. In §4 we formulate a new combinatorial technique. Fix a hyperplane  $H_0 \in \mathcal{A}$ . Then we can associate two arrangements: deconing  $\mathbf{d}_{H_0}\mathcal{A}$  and the restriction  $\mathcal{A}^{H_0}$ . We define a natural map  $\rho: L(\mathbf{d}_{H_0}\mathcal{A}) \to L(\mathcal{A}^{H_0})$  of their intersection posets. The map will be used in the proof of main result. In §5, we state and prove the main result. In §6, we prove several related results by localizing our main result.

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# 2 Characteristic polynomials

In this section, we summarize several facts on the characteristic polynomials.

# 2.1 For simple arrangements

Let  $\mathcal{A}$  be an arrangement of affine hyperplanes in an affine space V of  $\dim_{\mathbb{K}} V = \ell$ . Let  $L(\mathcal{A})$  be the set of nonempty intersections of elements of  $\mathcal{A}$ . Define a partial order on  $L(\mathcal{A})$  by  $X \leq Y \iff X \supseteq Y$ , where  $X, Y \in L(\mathcal{A})$ . Then  $L(\mathcal{A})$  is a ranked poset with rank  $X = \operatorname{codim} X$ . Denote the set of  $X \in L(\mathcal{A})$  of rank r by  $L_r(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = r\}$ .

Let  $\mu: L(\mathcal{A}) \to \mathbb{Z}$  be the Möbius function of  $L(\mathcal{A})$  defined by  $\mu(V) = 1$ , and for X > V by the recursion  $\sum_{Y \leq X} \mu(Y) = 0$ . The characteristic polynomial of  $\mathcal{A}$  is defined as  $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X} \in \mathbb{Z}[t]$ . Set the coefficients of the characteristic polynomial  $b_k(\mathcal{A})$  as  $\chi(\mathcal{A}, t) = t^{\ell} - b_1(\mathcal{A})t^{\ell-1} + b_2(\mathcal{A})t^{\ell-2} - \cdots + (-1)^{\ell}b_{\ell}(\mathcal{A})$ . The following Local-Global formula is straightforward.

## Proposition 2.1 (Local-Global formula for Betti numbers)

Let  $\mathcal{A}$  be an affine arrangement. Then

(2.1) 
$$b_k(\mathcal{A}) = \sum_{X \in L_k(\mathcal{A})} b_k(\mathcal{A}_X),$$

where  $A_X = \{ H \in A \mid H \supset X \}.$ 

We call  $\mathcal{A}$  central if  $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$ . In this case, the characteristic polynomial  $\chi(\mathcal{A}, t)$  is divisible by (t - 1). We denote  $\chi_0(\mathcal{A}, t) = \frac{1}{t - 1} \chi(\mathcal{A}, t)$ .

Given a nonempty central arrangement  $\mathcal{A}$  and a hyperplane  $H_0 \in \mathcal{A}$ . Choose coordinates  $z_1, \ldots, z_\ell$  of V so that  $H_0 = \{z_\ell = 0\}$ . Let  $H'_0 = \{z_\ell = 1\}$  be an affine hyperplane parallel to  $H_0$ . The deconing  $\mathbf{d}_{H_0}\mathcal{A}$  of  $\mathcal{A}$  with respect to  $H_0$  is the affine arrangement  $H'_0 \cap \mathcal{A}$  on  $H'_0$ . The deconing  $\mathbf{d}_{H_0}\mathcal{A}$  is an affine arrangement of rank  $(\ell - 1)$  whose characteristic polynomial satisfies  $\chi(\mathbf{d}_H\mathcal{A}, t) = \chi_0(\mathcal{A}, t)$ .

## 2.2 For multiarrangements

Let  $\mathcal{A}$  be a central arrangement. A map  $m: \mathcal{A} \to \mathbb{Z}_{\geq 0}$  is called a multiplicity. We define the S-module  $D(\mathcal{A}, m)$  for a multiarrangement  $(\mathcal{A}, m)$  by

$$D(\mathcal{A}, m) = \{ \delta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \delta(\alpha_H) \in (\alpha_H^{m(H)}), \forall H \in \mathcal{A} \},$$

where  $\alpha_H$  is a linear form such that  $\ker \alpha_H = H$  for each hyperplane  $H \in \mathcal{A}$ . A multiarrangement  $(\mathcal{A}, m)$  is called free with exponents  $(d_1, \ldots, d_\ell)$  if  $D(\mathcal{A}, m)$  is a free S-module with a homogeneous basis  $\delta_1, \ldots, \delta_\ell \in D(\mathcal{A}, m)$  such that  $\deg \delta_i = d_i$ .

Let  $\Omega_V^1 = \bigoplus_{i=1}^{\ell} S \cdot dx_i$  and  $\Omega_V^p = \bigwedge^p \Omega_V^1$ . We define an S-module  $\Omega^p(\mathcal{A}, m)$  of a multiarrangement  $(\mathcal{A}, m)$  by

$$\Omega^{p}(\mathcal{A}, m) = \left\{ \omega \in \frac{1}{Q} \Omega_{V}^{p} \mid \frac{d\alpha_{H}}{\alpha_{H}^{m(H)}} \wedge \omega \in \frac{1}{Q} \Omega_{V}^{p+1} \right\},\,$$

where  $Q = Q(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$ . Next we recall the characteristic polynomial of a multiarrangement  $(\mathcal{A}, m)$  [ATW]. Recall that for a finitely generated graded S-module  $M = \bigoplus_{d \in \mathbb{Z}} M_d$ , the Hilbert series  $P(M, x) \in \mathbb{Z}[x^{-1}][[x]]$  of M is defined by

$$P(M,x) = \sum_{d \in \mathbb{Z}} (\dim_{\mathbb{K}} M_d) x^d.$$

For a multiarrangement (A, m), we define

$$\Phi((\mathcal{A}, m); x, t) = \sum_{p=0}^{\ell} P(\Omega^{p}(\mathcal{A}, m), x) (t(1-x) - 1)^{p} \in \mathbb{Z}[x^{\pm 1}, t].$$

## Definition 2.2 ([ATW])

The characteristic polynomial of (A, m) is defined as follows.

$$\chi((\mathcal{A}, m), t) = \lim_{x \to 1} \Phi((\mathcal{A}, m); x, t) \in \mathbb{Z}[t].$$

Define the integer  $\sigma_i(\mathcal{A}, m) \in \mathbb{Z}$  by

$$\chi((A, m), t) = t^{\ell} - \sigma_1(A, m)t^{\ell-1} + \sigma_2(A, m)t^{\ell-2} - \dots + (-1)^{\ell}\sigma_{\ell}(A, m).$$

#### Remark 2.3

To be precise, the characteristic polynomial  $\chi((A, m), t)$  was first defined by using the dual module  $D^p(A, m)$  in [ATW]. Set

$$\psi((A, m), t, q) = \sum_{p=0}^{\ell} P(D^{p}(A, m), q)(t(q-1) - 1)^{p},$$

and it was defined that  $\chi((A, m), t) := (-1)^{\ell} \lim_{q \to 1} \psi((A, m), t, q)$ . However, this is equivalent to Definition 2.2. It is proved by checking the following two facts.

- (i) "Local-global formula" (Prop. 2.5 below) holds for both  $\Phi((A, m); 1, t)$  and  $\psi((A, m); 1, t)$ .
- (ii) The constant terms coincide.

The assertion (i) can be proved in a similar way with [ATW]. To verify the assertion (ii), we have to prove that  $\Phi((\mathcal{A},m);1,0) = \sum_{p=0}^{\ell} (-1)^p P(\Omega^p(\mathcal{A},m),x)$  and  $\psi((\mathcal{A},m);1,0) = \sum_{p=0}^{\ell} (-1)^{\ell-p} P(D^p(\mathcal{A},m),x)$  are equal, which is proved by using the isomorphism  $D^p(\mathcal{A},m) \xrightarrow{\simeq} \Omega^{\ell-p}(\mathcal{A},m)[-\deg Q] : \delta \longmapsto \iota_{\delta} \left(\frac{dx_1 \wedge \cdots \wedge dx_{\ell}}{Q}\right)$ . By (i), every coefficient of  $\chi((\mathcal{A},m),t)$  can be interpreted as the sum constant terms of certain localized subarrangements. The constant terms of these localizations coincides thanks to (ii).

#### Remark 2.4

It is not known whether or not  $\sigma_i(A, m) \geq 0$  holds. In [A], it is proved under the assumption of tameness.

Let  $X \in L(\mathcal{A})$ . Recall that  $\mathcal{A}_X$  is the set of all hyperplanes which contains X. The restricted multiplicity is denoted by  $m_X = m|_{\mathcal{A}_X}$ .

## Proposition 2.5 ([ATW])

Let (A, m) be a central multiarrangement. Then

(2.2) 
$$\sigma_k(\mathcal{A}, m) = \sum_{X \in L_k(\mathcal{A})} \sigma_k(\mathcal{A}_X, m_X).$$

Let  $X \in L_2(\mathcal{A})$ . Since  $(\mathcal{A}_X, m_X)$  is a rank two multiarrangement, it is free. We denote the exponents by  $(e_1(X), e_2(X), 0, \dots, 0)$ . Using these data,  $\sigma_1(\mathcal{A}, m)$  and  $\sigma_2(\mathcal{A}, m)$  are expressed as follows.

(2.3) 
$$\sigma_1(\mathcal{A}, m) = \sum_{H \in \mathcal{A}} m(H),$$
$$\sigma_2(\mathcal{A}, m) = \sum_{X \in L_2(\mathcal{A})} e_1(X) \cdot e_2(X).$$

If (A, m) is free with exponents  $(d_1, \ldots, d_\ell)$ , then

$$\chi((\mathcal{A}, m), t) = \prod_{i=1}^{\ell} (t - d_i).$$

# 3 Freeness via multirestriction

Multiarrangements naturally appear as restriction of simple arrangements. Let  $\mathcal{A}$  be a central arrangement. Fix a hyperplane  $H_0 \in \mathcal{A}$ . Let  $Q(\mathcal{A}')$  be the defining equation of the deleted arrangement  $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ . Ziegler's multirestriction is a multiarrangement on  $H_0$  defined by the equation  $Q(\mathcal{A}')|_{H_0}$ . We denote it by  $(\mathcal{A}^{H_0}, m^{H_0})$ . We have  $Q(\mathcal{A}^{H_0}, m^{H_0}) = Q(\mathcal{A}')|_{H_0}$ .

# Proposition 3.1 ([Z])

Let  $\mathcal{A}$  be a free arrangement. Let  $H_0 \in \mathcal{A}$  and let  $(\mathcal{A}^{H_0}, m^{H_0})$  be the Ziegler restriction. Then

(3.1) 
$$\chi_0(A, t) = \chi((A^{H_0}, m^{H_0}), t).$$

In other words,

$$(3.2) b_k(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_k(\mathcal{A}^{H_0}, m^{H_0}),$$

for  $k = 1, ..., \ell - 1$ .

In general, (3.1) does not hold. However to characterize the freeness, the formula (3.1) plays an important role as follows.

## Proposition 3.2 ([Yo2])

Let  $\ell = 3$ . Let  $\mathcal{A}$  be a central arrangement in  $\mathbb{K}^3$  and  $H_0 \in \mathcal{A}$ .

- (1)  $b_2(\mathbf{d}_{H_0}\mathcal{A}) \geq \sigma_2(\mathcal{A}^{H_0}, m^{H_0}).$
- (2)  $\mathcal{A}$  is free if and only if  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0})$ .

When  $\ell = 3$ , the condition  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0})$  is equivalent to the coincidence of characteristic polynomials:

(3.3) 
$$\chi(\mathbf{d}_{H_0}\mathcal{A}, t) = \chi((\mathcal{A}^{H_0}, m^{H_0}), t).$$

The result above has been generalized by M. Schulze as follows.

## Proposition 3.3 ([Sc])

Let  $\mathcal{A}$  be a central arrangement in  $\mathbb{K}^{\ell}$  and fix  $H_0 \in \mathcal{A}$ . Suppose that  $\ell = 4$  or  $\ell \geq 5$  with (weakly) tameness assumption (i.e., pdim  $\Omega^p(\mathcal{A}^{H_0}, m^{H_0}) \leq p$ ,  $\forall p$ ). Then  $\mathcal{A}$  is free if and only if

- $(\mathcal{A}^{H_0}, m^{H_0})$  is free, and
- the relation (3.3) holds.

#### Definition 3.4

 $H_0 \in \mathcal{A}$ .  $\mathcal{A}$  is said to be locally free along  $H_0$  if  $\mathcal{A}_X$  is free for all  $X \in L(\mathcal{A})$  with  $0 \neq X \subset H_0$ .

# Proposition 3.5 ([Yo1])

Let  $\mathcal{A}$  be a central arrangement in  $\mathbb{K}^{\ell}$  and fix  $H_0 \in \mathcal{A}$  with  $\ell \geq 4$ . Then  $\mathcal{A}$  is free if and only if

- $(\mathcal{A}^{H_0}, m^{H_0})$  is free, and
- $\mathcal{A}$  is locally free along  $H_0$ .

In  $\S 5$ , we generalize Proposition 3.2 and 3.3 to higher dimensions.

# 4 Combinatorial restriction map

Let  $\mathcal{A}$  be a central arrangement in  $V = \mathbb{K}^{\ell}$ . Fix  $H_0 \in \mathcal{A}$ . Recall that  $\mathbf{d}_{H_0}\mathcal{A}$  is an affine arrangement in  $H'_0$ . Hence we may consider  $X \in L_k(\mathbf{d}_{H_0}\mathcal{A})$  to be an affine subspace of V of dimension  $(\ell - k - 1)$ . (Note that by definition,  $X \in L_k(\mathbf{d}_{H_0}\mathcal{A})$  is codimension k in  $H'_0$ .) Then X generates a  $(\ell - k)$ -dimensional linear subspace  $\mathbb{K}X \subset V$ . By taking intersection with  $H_0$ , we obtain an  $(\ell - k - 1)$ -dimensional linear subspace  $\mathbb{K}X \cap H_0 \in L_k(\mathcal{A}^{H_0})$  of  $H_0$ . We denote the map by  $\rho$ 

$$\rho: L(\mathbf{d}_{H_0}\mathcal{A}) \longrightarrow L(\mathcal{A}^{H_0}): X \longmapsto \mathbb{K}X \cap H_0$$

which preserves the rank and the order of posets.

The map  $\rho$  is compatible with the localization in the following manner. Let  $X \in L(\mathcal{A})$  with  $X \subset H_0$ . Then  $H_0 \in \mathcal{A}_X \subset \mathcal{A}$ , and the following diagram commutes.

(4.1) 
$$L(\mathbf{d}_{H_0}\mathcal{A}) \xrightarrow{\rho} L(\mathcal{A}^{H_0})$$

$$\uparrow \qquad \uparrow$$

$$L(\mathbf{d}_{H_0}\mathcal{A}_X) \xrightarrow{\rho_X} L(\mathcal{A}_X^{H_0}).$$

Furthermore the vertical maps are full, that is,  $Y_1 \in L(\mathbf{d}_{H_0}\mathcal{A}_X), Y_2 \in L(\mathbf{d}_{H_0}\mathcal{A})$  with  $Y_2 \leq Y_1$ , then  $Y_2 \in L(\mathbf{d}_{H_0}\mathcal{A}_X)$ . This implies that if  $Y_1 \in L(\mathbf{d}_{H_0}\mathcal{A}_X)$ , then the value Möbius function of  $Y_1$  in  $L(\mathbf{d}_{H_0}\mathcal{A}_X)$  is equal to that in  $L(\mathbf{d}_{H_0}\mathcal{A})$ .

# 5 Main results

### Theorem 5.1

Let  $\mathcal{A}$  be a central arrangement in  $V = \mathbb{K}^{\ell}$  (with  $\ell \geq 3$ ) and  $H_0 \in \mathcal{A}$ .

- (1)  $b_2(\mathbf{d}_{H_0}\mathcal{A}) \ge \sigma_2(\mathcal{A}^{H_0}, m^{H_0}).$
- (2) The equality  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0})$  holds if and only if  $\mathcal{A}_X$  is free for all  $X \in L_3(\mathcal{A})$  with  $X \subset H_0$ . (We may say that  $\mathcal{A}$  is locally free in codimension three along  $H_0$ .)
- (3) Assume that  $(\mathcal{A}^{H_0}, m^{H_0})$  is free. Then the following are equivalent.
  - (i) A is free.
  - (ii)  $\chi(\mathbf{d}_{H_0}\mathcal{A}, t) = \chi((\mathcal{A}^{H_0}, m^{H_0}), t).$
  - (iii)  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0}).$

(iv) A is locally free in codimension three along  $H_0$ .

*Proof.* We first prove (1) and (2). By using  $L_2(\mathbf{d}_{H_0}\mathcal{A}) = \bigsqcup_{X \in L_2(\mathcal{A}^{H_0})} \rho^{-1}(X)$ , we have

$$b_{2}(\mathbf{d}_{H_{0}}\mathcal{A}) = \sum_{Y \in L_{2}(\mathbf{d}_{H_{0}}\mathcal{A})} \mu(Y)$$

$$= \sum_{X \in L_{2}(\mathcal{A}^{H_{0}})} \left( \sum_{Y \in \rho^{-1}(X)} \mu(Y) \right)$$

$$= \sum_{X \in L_{2}(\mathcal{A}^{H_{0}})} b_{2}(\mathbf{d}_{H_{0}}(\mathcal{A}_{X})).$$

Suppose  $X \in L_2(\mathcal{A}^{H_0})$ . Then  $X \in L_3(\mathcal{A})$  with  $X \subset H_0$ , and  $\mathcal{A}_X$  is a rank 3 central arrangement. Hence from Proposition 3.2, we have

$$(5.2) b_2(\mathbf{d}_{H_0}(\mathcal{A}_X)) \ge e_1(X) \cdot e_2(X),$$

where  $(e_1(X), e_2(X))$  is the exponents of  $(\mathcal{A}_X^{H_0}, m_X^{H_0})$  as in (2.3). Hence we have

(5.3) 
$$b_2(\mathbf{d}_{H_0}\mathcal{A}) \ge \sum_{X \in L_2(\mathcal{A}^{H_0})} e_1(X) \cdot e_2(X) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0}).$$

Thus (1) is proved. Furthermore, in (5.2), the equality holds if and only if  $\mathcal{A}_X$  is free. Hence the equality  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0})$  holds if and only if  $\mathcal{A}_X$  is free for all  $X \in L_2(\mathcal{A}^{H_0})$ . Thus we have (2).

Next we prove (3). First, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)  $\Longleftrightarrow$ (iv) is obvious (from (2)). We shall prove (iv) $\Longrightarrow$ (i) by induction on  $\ell$ . By definition,  $\mathcal{A}_X$  is free for all  $X \in L_3(\mathcal{A})$  with  $X \subset H_0$ . If  $\ell = 3$ , then  $L_3(\mathcal{A}) = \{0\}$ ,  $\mathcal{A}_X = \mathcal{A}$  for  $X \in L_3(\mathcal{A})$  and there is nothing to prove. Let  $\ell \geq 4$ . We will use Proposition 3.5. It suffices to show that  $\mathcal{A}$  is locally free along  $H_0$ . Let  $Z \in L(\mathcal{A})$  with  $0 \neq Z \subset H_0$ . Then  $\mathcal{A}_Z$  has rank at most  $(\ell - 1)$  since  $Z \neq 0$ . Since  $(\mathcal{A}_Z^{H_0}, m_Z^{H_0})$  is a localization of  $(\mathcal{A}^{H_0}, m^{H_0})$ , it is free. It is easily checked that  $\mathcal{A}_Z$  satisfies (iii). Hence by the inductive assumption,  $\mathcal{A}_Z$  is free. Consequently,  $\mathcal{A}$  is locally free along  $H_0$ .

The following are immediate.

#### Corollary 5.2

Let  $\mathcal{A}$  and  $H_0 \in \mathcal{A}$  be as above. Suppose that  $(\mathcal{A}^{H_0}, m^{H_0})$  is free with exponents  $(d_1, \ldots, d_{\ell-1})$ . Then the inequality

$$b_2(\mathbf{d}_{H_0}\mathcal{A}) \ge \sum_{1 \le i < j \le \ell-1} d_i d_j$$

holds. Furthermore, A is free if and only if the equality holds.

#### Corollary 5.3

Let  $A_1$  and  $A_2$  be central arrangements. Fix  $H_1 \in A_1$  and  $H_2 \in A_2$ . Assume that

- $A_1$  is free,
- $(A_1^{H_1}, m^{H_1}) \simeq (A_2^{H_2}, m^{H_2})$ , and
- $b_2(\mathcal{A}_1) = b_2(\mathcal{A}_2)$ .

Then  $A_2$  is also free.

In Theorem 5.1 (3) and Corollary 5.2, we can not drop the assumption that the multirestriction  $(\mathcal{A}^{H_0}, m^{H_0})$  is free. Indeed, there exists non-free arrangement  $\mathcal{A}$  such that  $\chi(\mathbf{d}_{H_0}\mathcal{A}, t) = \chi((\mathcal{A}^{H_0}, m^{H_0}), t)$ .

#### Example 5.4

Let  $A_1$  be a central arrangement in  $\mathbb{C}^4$  defined by

$$x(x-w)y(y-w)(x+y+z)(x-y+z)zw.$$

Let  $H_0 = \{w = 0\}$ . Then  $\mathbf{d}_{H_0} \mathcal{A}_1$  is an affine arrangement in  $\mathbb{C}^3$  defined by

$$x(x-1)y(y-1)(x+y+z)(x-y+z)z$$

whose characteristic polynomial is  $\chi(\mathbf{d}_{H_0}\mathcal{A}_1,t)=t^3-7t^2+18t-17$ . On the other hand, the multirestriction  $(\mathcal{A}_1^{H_0},m^{H_0})$  is defined by  $x^2y^2(x+y+z)(x-y+z)z$ . The characteristic polynomial is  $\chi((\mathcal{A}_1^{H_0},m^{H_0}),t)=t^3-7t^2+18t-17$  ([ATW]) and we have  $\sigma(\mathcal{A}_1^{H_0},m^{H_0})=b_2(\mathbf{d}_{H_0}\mathcal{A}_1)=18$ . However, since the characteristic polynomial does not factor, both  $\mathcal{A}_1$  and  $(\mathcal{A}_1^{H_0},m^{H_0})$  are nonfree.

#### Remark 5.5

Recall that the arrangement  $\mathcal{A}$  is called formal if every linear dependence  $t_1\alpha_1 + \cdots + t_n\alpha_n = 0$  of defining equations is a linear combination of three terms dependences  $t_i\alpha_i + t_j\alpha_j + t_k\alpha_k = 0$ , in other words, linear dependences are generated by codimension two flats  $L_2(\mathcal{A})$ . In [Yu], Yuzvinsky proved that free arrangements are formal. Theorem 5.1 shows that the freeness of  $\mathcal{A}$  is characterized by combinatorial structures in codimension two and a multirestriction. Our results seem to have some relations with formality. However it is not clear yet.

## 6 Related results

In general,  $\chi(\mathbf{d}_{H_0}\mathcal{A},t)$  and  $\chi((\mathcal{A}^{H_0},m^{H_0}),t)$  are not equal. However, under some assumptions on locally freeness, they are almost equal (they are equal except for the constant terms). We will give two different proofs for the following result.

#### Theorem 6.1

If  $\mathcal{A}$  is locally free along  $H_0$ , then  $\chi(\mathbf{d}_{H_0}\mathcal{A},t) - \chi((\mathcal{A}^{H_0},m^{H_0}),t) \in \mathbb{Z}$ .

## 6.1 First proof

Let  $1 \leq k \leq \ell - 2$ . We shall prove  $b_k(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_k(\mathcal{A}^{H_0}, m^{H_0})$ . From (2.1), we have

(6.1) 
$$b_k(\mathbf{d}_{H_0}\mathcal{A}) = \sum_{X \in L_k(\mathcal{A}^{H_0})} \left( \sum_{Y \in \rho^{-1}(X)} b_k((\mathbf{d}_{H_0}\mathcal{A})_Y) \right).$$

Since A is locally free along  $H_0$ ,  $A_X$  is free. Hence

(6.2) 
$$\sigma_k(\mathcal{A}_X^{H_0}, m_X^{H_0}) = b_k((\mathbf{d}_{H_0}\mathcal{A}_X)) = \sum_{Y \in \rho^{-1}(X)} b_k((\mathbf{d}_{H_0}\mathcal{A})_Y).$$

From local-global formula, we have  $b_k(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_k(\mathcal{A}^{H_0}, m^{H_0})$ .

# 6.2 Second proof

We first recall restriction maps for logarithmic forms following [Sc, Yo2]. Let us fix coordinates  $z_1, \ldots, z_\ell$  of V so that  $H_0 = \{z_\ell = 0\}$  (as in §2.1).

A logarithmic differential form  $\omega \in \Omega^p(\mathcal{A})$  can be expressed as

$$\omega = \omega_1 + \frac{dz_\ell}{z_\ell} \wedge \omega_2,$$

where  $\omega_1, \omega_2$  are rational differential forms generated by  $dz_1, \ldots, dz_{\ell-1}$ . We can define the restriction map  $\operatorname{res}_{H_0}^p : \Omega^p(\mathcal{A}) \longrightarrow \Omega^p(\mathcal{A}^{H_0}, m^{H_0})$  by

$$\omega_1 + \frac{dz_\ell}{z_\ell} \wedge \omega_2 \longmapsto \omega_1|_{H_0}.$$

The image of the map  $\operatorname{res}_{H_0}^p$  is denoted by  $\operatorname{res}_{H_0}^p(\Omega^p(\mathcal{A})) = M^p \subset \Omega^p(\mathcal{A}^{H_0}, m^{H_0})$ , and its cokernel by  $C^p$ . We have the exact sequence

$$0 \longrightarrow M^p \longrightarrow \Omega^p(\mathcal{A}^{H_0}, m^{H_0}) \longrightarrow C^p \longrightarrow 0.$$

## Proposition 6.2 ([Yo2])

If A is free, then  $\operatorname{res}_{H_0}^p$  is surjective.

Define  $\Phi(C^{\bullet}; x, y)$  to be

$$\Phi(C^{\bullet}; x, y) = \sum_{p=0}^{\ell-1} P(C^p, x) y^p.$$

Then we have (see [Sc])

(6.3)  

$$\chi((\mathcal{A}^{H_0}, m^{H_0}); t) - \chi_0(\mathcal{A}, t) = \lim_{x \to 1} \Phi(C^{\bullet}; x, t(1 - x) - 1)$$

$$= \lim_{x \to 1} \sum_{p=0}^{\ell-1} P(C^p, x) (t(1 - x) - 1)^p$$

$$= \lim_{x \to 1} \sum_{p=0}^{\ell-1} P(C^p, x) \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} t^k (1 - x)^k.$$

$$= \lim_{x \to 1} \sum_{k=0}^{\ell-1} \binom{p}{k} t^k (1 - x)^k \left(\sum_{p > k} (-1)^{p-k} P(C^p, x)\right).$$

Now we assume that  $\mathcal{A}$  is locally free along  $H_0$ . Then the cokernel of the restriction map  $C^p$  is supported on the origin  $0 \in H_0$ . Therefore  $\dim_{\mathbb{K}} C^p$  is finite dimensional, and the Hilbert series  $P(C^p, x)$  is a (Laurent) polynomial. Hence  $\lim_{x\to 1} (1-x)^k P(C^p, x) = 0$  if  $k \geq 1$ . Thus we have

$$\chi((\mathcal{A}^{H_0}, m^{H_0}); t) - \chi_0(\mathcal{A}, t) = \sum_{p=0}^{\ell-1} (-1)^p \dim_{\mathbb{K}} C^p.$$

#### Corollary 6.3

Let  $\ell \geq 4$ . Assume that the multirestriction  $(\mathcal{A}^{H_0}, m^{H_0})$  is locally free, i.e., for any  $0 \neq X \subset H_0$ ,  $(\mathcal{A}_X^{H_0}, m_X^{H_0})$  is free. Then the following are equivalent.

(i) A is locally free along  $H_0$ .

(ii)  $\chi(\mathbf{d}_{H_0}\mathcal{A}, t) - \chi((\mathcal{A}^{H_0}, m^{H_0}), t) \in \mathbb{Z}.$ 

(iii)  $b_2(\mathbf{d}_{H_0}\mathcal{A}) = \sigma_2(\mathcal{A}^{H_0}, m^{H_0}).$ 

The proof is similar to that of Theorem 5.1.

#### Example 6.4

Let  $A_2$  be a central arrangement in  $\mathbb{C}^4$  defined by

$$x(x-w)y(y-w)(x+y+z)(x-y+z)(z-w)w.$$

(Note that the 7-th hyperplane is different from  $A_1$  in Example 5.4.) Let  $H_0 = \{w = 0\}$ . Then  $\mathbf{d}_{H_0}A_2$  is an affine arrangement in  $\mathbb{C}^3$  defined by

$$x(x-1)y(y-1)(x+y+z)(x-y+z)(z-1)$$

whose characteristic polynomial is  $\chi(\mathbf{d}_{H_0}\mathcal{A}_2,t)=t^3-7t^2+18t-19$ . On the other hand, the multirestriction  $(\mathcal{A}_1^{H_0},m^{H_0})$  is defined by  $x^2y^2(x+y+z)(x-y+z)z$ , which is the same arrangement with Example 5.4. The characteristic polynomial is  $\chi((\mathcal{A}_2^{H_0},m^{H_0}),t)=t^3-7t^2+18t-17$ . Since  $(\mathcal{A}_2^{H_0},m^{H_0})$  is rank three, hence locally free, and  $\sigma(\mathcal{A}_2^{H_0},m^{H_0})=b_2(\mathbf{d}_{H_0}\mathcal{A}_2)=18$ ,  $\mathcal{A}_2$  is locally free along  $H_0$ .

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