On Properties of the Support of Capacity-Achieving Distributions for Additive Noise Channel Models with Input Cost Constraints

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Abstract

We study the classical problem of characterizing the channel capacity and its achieving distribution in a generic fashion. We derive a simple relation between three parameters: the input-output function, the input cost function and the noise probability density function, one which dictates the type of the optimal input. In Layman terms we prove that the support of the optimal input is bounded whenever the cost grows faster than a "cut-off" rate equal to the logarithm of the noise PDF evaluated at the input-output function. Furthermore, we prove a converse statement that says whenever the cost grows slower than the "cut-off" rate, the optimal input has necessarily an unbounded support. In addition, we show how the discreteness of the optimal input is guaranteed whenever the triplet satisfy some analyticity properties. We argue that a suitable cost function to be imposed on the channel input is one that grows similarly to the "cut-off" rate.

Our results are valid for any cost function that is super-logarithmic. They summarize a large number of previous channel capacity results and give new ones for a wide range of communication channel models, such as Gaussian mixtures, generalized-Gaussians and heavy-tailed noise models, that we state along with numerical computations.

Keywords: Channel capacity, memoryless channels, non-linear channels, input cost function, logarithmic cost, heavy-tailed noise, alpha-stable, Middleton class B, Gaussian mixtures, convex optimization, Karush-Kuhn-Tucker conditions, discrete inputs.

I. INTRODUCTION

In communication systems design, a key engineering objective is to build systems that operate close to channel capacity. Needless to say that this quantity, as defined by Shannon [1], [2] in his pioneering work, is the cutoff value which delimits the achievable region for "reliable" communications. Clearly, the channel capacity and how it can be achieved are intimately related to the channel model. Despite the well-known capacity results for discrete memoryless channels, closed-form capacity expressions are rarely found in the literature for continuous ones. The most well-understood –and perhaps important– continuous channel is the linear Additive White Gaussian Channel

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(AWGN) subjected to an average power constraint. This AWGN model was studied by Shannon and may be seen as an instance of the generic real, deterministic and memoryless discrete-time additive noise model of the form:

$$Y = f(X) + N,$$

where $Y \in \mathbb{R}$ is the channel output, and the channel input $X \in \mathcal{X} \in \mathbb{R}$ satisfies an average cost constraint of the form $\mathsf{E}\left[\mathcal{C}\left(|X|\right)\right] \leq A$, for some $A > 0$. Naturally X is assumed to be independent of the additive noise N.

In the literature, multiple instances of such channel models were investigated by making variations to the Shannon setup in the following aspects:

- *The input-output relationship $f(\cdot)$*: While Shannon considered a deterministic linear input-output relationship, many studies assumed a non-deterministic relationship [3]–[6] or generally a non-linear deterministic one [7].

- *The input constraint or cost function $\mathcal{C}(\cdot)$*: One of the main reasons of the popularity of the second moment constraint $\mathsf{E}\left[X^2\right]$ –which corresponds to a cost function $\mathcal{C}(x) = x^2$, is that it represents the average power of the discrete time transmitted signal which is equal to the average power of the corresponding white continuous process assuming that the transmitted signals are square integrable. Nevertheless, other input constraints were studied starting with Smith [8] who considered peak power constraints and a combination of peak and average power constraints. More recently, the capacity of Gaussian Channels with duty cycle and average power constraints was studied in [9] .

- *The noise distribution*: Though Gaussian statistics of the noise can be motivated by the Central Limit Theorem (CLT), it also has an appealing property of being the worst case noise from an entropy perspective among finite second moment Random Variables (RV)s. Nevertheless, Non-Gaussian average power constrained communication channels have some applications and their channel capacities were investigated under a general setup in the work of Das [10] where the noise is assumed to have a finite second moment, a condition that was not imposed on the non-Gaussian noise distributions in [11].

- *Combinations of more than one aspect* were also considered in the literature. Smith [8] extended his capacity results for the peak power constrained Gaussian channel to non-Gaussian ones where the noise statistics are Gaussian like. Later, Tchamkerten [12] considered a scalar additive channel whose input is amplitude constrained and for which the additive noise is assumed to satisfy some general properties however not necessarily having a finite second moment. Lately, Fahs and Abou-Faycal [7], [13] investigated non-linear Gaussian channels under a general setup of input constraints such as even moments, compact support constraints and a combination of both types. Finally, channel capacity under Fractional Order Moments (FOM) of the form $\mathsf{E}\left[|X|^r\right] \leq A$, for some $A > 0$, $r > 0$ was characterized under a symmetric alpha-stable additive noise [14] or when the noise has two components, an alpha-stable component and a Gaussian one [15].

Nearly, for all the cited models above, and whenever the noise Probability Density Function (PDF) is assumed to have an analytical extendability property, the optimal input is proven to be of a discrete nature and in most cases with a finite number of mass points. Additionally, channel capacity could not be written in closed-form. In this sense, the linear AWGN channel and some "equivalent" channels [7] seem to be an exception, along with

a few channel models such as the additive exponential noise channel under a mean constraint with non-negative inputs [16] and recently the Cauchy channel under a logarithmic constraint [17]. For these channel models, the optimal input distribution is found to be of the same nature of the noise and capacity is described in closed-form.

One is tempted to study whether there is a general relation between the input-output function $f(\cdot)$, the input cost function $\mathcal{C}(\cdot)$, and the noise PDF $p_N(\cdot)$ that governs the type of the capacity-achieving input. In this work, we conduct this study for general types of the considered channel whereby "general" we mean the input-output relationship may not be linear but required nevertheless to satisfy some rather mild conditions. Additionally, instead of formulating the problem in terms of the average power constraint or other moments constraints, we use generic input cost functions that are also required to satisfy some technical conditions. We emphasize that our results cover all cost functions which are "super-logarithmic" which is a rather very large set. When it comes to the noise statistics, the noise is assumed to be absolutely continuous with respect to the Lebesgue measure with positive and continuous PDFs that are with or without monotonic tails and have a finite logarithmic-type of moments. Two conditions are however imposed on the noise PDF and are subsequently presented. The first guarantees the finiteness of the noise differential entropy. The second concerns the tail behavior of a lower envelope to the noise PDF. These two conditions are "easily satisfied" such as whenever the PDF has a dominant exponential or a dominant polynomial component. Despite the apparent long list of requirements, we emphasize that the considered functions $f(\cdot)$, input costs $\mathcal{C}(\cdot)$ and noise PDFs cover the vast majority of the known models found in the literature.

Our main contributions are fourfold:

1- Our study provides new capacity results for a multitude of communication channels. We showcase some of them:

   – *Gaussian mixtures* and *generalized Gaussian* noise distributions are commonly used in the literature [18]–[21], however no previous channel capacity studies were conducted for these types of noise models. The application of our results to such channels is presented in Sections VI-B and VI-C respectively.

   – Many communication channels are suitably modeled as *impulsive channels* where the statistics of the noise have, for example, an *alpha-stable* distribution or a composite *alpha-stable* plus *Gaussian* such as telephone noise [22], audio noise signals [23] and Multiple Access Interference (MAI) [24]–[26]. More recently [27]–[30], the performance of new receivers, mitigation and diversity techniques were investigated when such impulsive statistics were used as models of additive noise in MAI networks. In Section VI-D.1, we characterize and compute for a generic cost constraint $\mathcal{C}(\cdot)$ the channel capacity for the alpha-stable and the composite noise channels: *alpha-stable* plus *Gaussian*, which is commonly referred to as the Middleton class B when the stable component is symmetric [31], [32].

2- The results stated in Theorems 1 and 2 generalize those of Das [10], who made similar statements for a linear channel whenever $\mathcal{C}(x) = x^2$ and whenever the noise is restricted, among other things, to have a finite second moment. The generalization is one to generic possibly non-linear channels, generic cost functions and noise distributions. They are in line with all the previous channel-capacity results presented earlier. In addition, when

having an analyticity property of the noise PDF, they recover literally most of the known discreteness results for cost constrained deterministic channels. These results are stated in Theorem 3.

3- Our methodology also provides capacity results even when the input is subjected only to a support (such as a peak power constraint), or to a combination of support and cost constraints. In fact, the results stated in Theorem 4 corroborate those found in [12] for channels whose input is amplitude constrained and where the noise has a finite r-th moment constraint for some $r > 0$. However, Theorem 4 is in some sense more general as it holds whenever the input has a compact support for *all* noise distributions that have a finite "super-logarithmic" moment.

4- When applied to monotonically-tailed noise PDFs for example, our main results - stated in Theorems 1 and 2, imply that whenever $\mathcal{C}(x) = \omega \left(\ln \left[\frac{1}{p_N(f(x))} \right] \right)^*$, the support[†] of the capacity-achieving input is necessarily bounded. In addition, we state and prove a converse statement that states that whenever $\mathcal{C}(x) = o \left(\ln \left[\frac{1}{p_N(f(x))} \right] \right)$, the optimal input is necessarily unbounded. These results state that –for monotonically noise PDFs, there exists a threshold growth rate for the cost function which constitutes the transition between bounded and unbounded optimal inputs. Indeed, for an optimal input to be unbounded, a "necessary condition" for the cost function is to be at most $\Theta \left(\ln \left[\frac{1}{p_N(f(x))} \right] \right)$. This condition is satisfied by the Gaussian channel under an average power constraint, the exponential channel under a mean moment constraint and the Cauchy channel under a logarithmic constraint for which a Gaussian, exponentially tailed and a Cauchy input are respectively optimal [7], [16], [17].

On a related note, we argue that this study does provide insights on what is a suitable measure of signal strength. Though this question is not crucial when the additive noise has a finite second moment due to the natural power measure provided by the second moment, it seems of great importance when dealing with heavy tailed noise distributions having infinite second moments. For these types of channels, since the noise has an infinite second moment, using the second moment as a measure of signal strength is absurd for evaluating the Signal-to-Noise Ratio (SNR) for example. In fact, for heavy-tailed noise models, and more specifically for the alpha-stable class, a general theory of stable signal processing based on Fractional Lower Order Moments (FLOM) ($\mathcal{C}(|x|) = |x|^r$, $0 < r < 2$) was presented in [33]; The "stable theory" was in accordance with the fact that second order methods and linear estimation theory were no longer suitable for infinite variance additive noise channels and new criteria based on the dispersion of alpha-stable RVs and FLOM were investigated. The stable theory was also used in the treatment of various detection and estimation problems [34], [35], and the performance of optimum receivers were investigated in [36].

However, one can argue that moments of the form $\mathsf{E}\left[|X|^r\right]$, $r > 0$ do not provide a suitable strength measure for variables having heavy tailed distributions, simply because no single value of r can be found appropriate [37]. This

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*In this work, we say that $f(x) = \omega \left(g(x) \right)$ if and only if $\forall \kappa > 0, \exists c > 0$ such that $|f(x)| \geq \kappa |g(x)|, \forall |x| \geq c$. Equivalently, we say that $g(x) = o \left(f(x) \right)$. We say that $f(x) = \Omega \left(g(x) \right)$ if and only if $\exists \kappa > 0, c > 0$ such that $|f(x)| \geq \kappa |g(x)|, \forall |x| \geq c$. Equivalently, we say that $g(x) = O \left(f(x) \right)$. We say that $f(x) = \Theta \left(g(x) \right)$ if and only if $f(x) = O \left(g(x) \right)$ and $f(x) = \Omega \left(g(x) \right)$.

[†]We define the support of a RV as being the set of its points of increase i.e. $\{x \in \mathbb{R} : \Pr(x - \eta < X < x + \eta) > 0 \text{ for all } \eta > 0\}$.

conclusion is further supported when making the following reasoning on the additive noise channels considered in this work:

- Let $\mathsf{E}\left[\mathcal{C}_0(|X|)\right]$ be a measure of the average signal strength where $\mathcal{C}_0(|x|)$ is some positive, lower semi-continuous, non-decreasing function of $|x|$ and let $p_N(x)$ be the noise PDF which is assumed to have a monotonic tail.

- Whenever $\mathcal{C}_0(|x|) = \omega\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$, one can always find a cost function $\mathcal{C}(|x|)$ such that $\mathcal{C}(|x|)$ is both $\omega\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$ and $o\left(\mathcal{C}_0(|x|)\right)$.

- Now, since $\mathcal{C}(|x|) = \omega\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$, the channel capacity under an input constraint of the form $\mathsf{E}\left[\mathcal{C}(|X|)\right] \leq A$, $A > 0$ is achieved by a bounded input by virtue of Theorem 1. On the other hand, since $\mathcal{C}(|x|) = o\left(\mathcal{C}_0(|x|)\right)$, then there exists a distribution function satisfying the cost constraint with "signal strength" $\mathsf{E}\left[\mathcal{C}_0(|X|)\right]$ equal to ∞.

- Hence, in the input space of distribution functions, there exist distributions having possibly infinite strength while the capacity is achieved by a distribution which has a finite one since its support is bounded.

- This non-intuitive conclusion is only possible under the choice of a strength measure that is $\omega\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$.

By this reasoning, suitable signal strength measures should be at most $\Theta\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$. Said differently, depending on the noise, measures of the form $\Theta\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$ are more appropriate. This boils down to $\Theta(f^2(x))$ under the Gaussian noise and to $\Theta\left(\ln\left[f(x)\right]\right)$ for polynomially-tailed additive noise. The latter condition comes in accordance with the work of Gonzalez et al. [37] who presented a new approach for dealing with heavy-tailed noise environments. After presenting the shortcomings of the FLOM approach, they presented a "general" unit of strength-measure based on logarithmic moments where they motivated its usage within the framework of estimation and filtering under impulsive noise.

The remainder of this paper is organized as follows. Section II presents the generic channel model along with all the assumptions made in our study. Preliminary lemmas concerning lower and upper bounds on some quantities of interest are stated and proven in Section III. In Section IV, we discuss the Karush-Kuhn-Tucker (KKT) theorem and in Section V the main results of this paper are presented as Theorems 1, 2, 3 and 4. Examples and numerical evaluations of channel capacity and its achieving input distributions are presented in Section VI where the application of the four theorems is explored for Gaussian mixtures, Generalized Gaussians and impulsive noise. Finally, Section VII concludes the paper.

II. A GENERIC CHANNEL MODEL

We consider a generic memoryless real discrete-time noisy communication channel where the noise is additive and where the input and output are possibly non-linearly related as follows:

$$Y_i = f(X_i) + N_i, \tag{1}$$

where i is the time index. We denote by $Y_i \in \mathbb{R}$ the channel output at time i. The input at time i is denoted X_i and is assumed to have an alphabet $\mathcal{X} \subseteq \mathbb{R}$. The channel's input is distorted according to the deterministic and possibly

non-linear function $f(x)$. Additionally, the communication channel is subjected to an additive noise process that is independent of the input. The variables $\{N_i\}_i$ are also assumed to be Independent and Identically Distributed (IID) RVs.

Finally, we subject the input to an average cost constraint of the form: $\mathsf{E}\left[\mathcal{C}\left(|X_i|\right)\right] \leq A$, for some $A \in \mathbb{R}^{+*}$ where $\mathcal{C}(\cdot)$ is some cost function:

$$\mathcal{C} : \mathbb{R}^+ \longrightarrow \mathbb{R}.$$

Accordingly, we define for $A > 0$

$$\mathcal{P}_A = \left\{ \text{ Probability distributions } F \text{ of } X : \int \mathcal{C}\left(|x|\right) dF(x) \leq A \right\}, \tag{2}$$

the set of all distribution functions satisfying the average cost constraint.

Given that the channel model is stationary and memoryless, the capacity-achieving statistics of X_i are also memoryless (IID), therefore we suppress the time index and write

$$Y = f(X) + N, \tag{3}$$

where the noise is absolutely continuous with respect to the Lebesgue measure and is assumed to have a PDF $p_N(\cdot)$. This implies that the channel transition probability density function is given by

$$p_{Y|X}(y|x) = p_N(y - f(x)), \quad y \in \mathbb{R}, \ x \in \mathcal{X}. \tag{4}$$

We characterize the tail behavior of $p_N(\cdot)$ by considering the following positive functions which are non-increasing for $x \geq 0$ and non-decreasing for $x < 0$:

$$T_{\mathrm{l}}(x) = \begin{cases} \inf_{0 \leq t \leq x} p_N(t) & x \geq 0 \\ \inf_{x \leq t \leq 0} p_N(t) & x < 0, \end{cases} \qquad T_{\mathrm{u}}(x) = \begin{cases} \sup_{t \geq x} p_N(t) & x \geq 0 \\ \sup_{t \leq x} p_N(t) & x < 0. \end{cases}$$

Considering the tail behavior of $T_{\mathrm{l}}(x)$ and $T_{\mathrm{u}}(x)$ instead of $p_N(x)$ allows us to include in our analysis PDFs which do not possess a monotonic tail. For those that do, $p_N(x)$, $T_{\mathrm{l}}(x)$ and $T_{\mathrm{u}}(x)$ will be identical for large values of $|x|$.

The main results of this work are based on relating the tail behavior of $\mathcal{C}(\cdot)$ to that of $T_{\mathrm{l}}(\cdot)$ and $T_{\mathrm{u}}(\cdot)$ in order to characterize the capacity-achieving input distributions of channel (3). More explicitly we prove that, whenever $\mathcal{C}\left(|x|\right) = \omega\left(\ln\left[\frac{1}{T_{\mathrm{l}}(f(x))}\right]\right)$, the optimal input has necessarily a bounded support. Furthermore, we prove a converse statement: whenever $\mathcal{C}\left(|x|\right) = o\left(\ln\left[\frac{1}{T_{\mathrm{u}}\left(f(x)\right)}\right]\right)$, the capacity-achieving input is not bounded.

When the noise PDF has a monotonic tail, our results infer that cost functions which are $\Theta\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$ form somehow a "transition" between bounded and unbounded optimal inputs. For example, whenever the noise is Gaussian, the "transitional" cost is of the form $\Theta\left(f^2(x)\right)$. The discreteness –and hence the finiteness of the number of mass points of the optimal input in the bounded case– is a direct consequence of the analyticity properties of $p_N(\cdot)$ and $\mathcal{C}(\cdot)$ whenever these properties exist.

A. Assumptions

In this work, we make the following assumptions:

- The function $f(\cdot)$:

  C1- The function is continuous.

  C2- The absolute value of the function $|f(\cdot)|$ is a non-decreasing function of $|x|$ and $|f(x)| \to +\infty$ as $|x| \to +\infty$.

- The cost function $\mathcal{C}(\cdot)$:

  C3- The cost function is lower semi-continuous and non-decreasing. Without Loss of Generality (WLOG) we assume that $\mathcal{C}(0) = 0$: if it were not, define $\mathcal{C}_0(|x|) = \mathcal{C}(|x|) - \mathcal{C}(0)$ and adjust the input space under the cost $\mathcal{C}_0(|x|)$ to $\mathcal{P}_{A-\mathcal{C}(0)}$. Note that necessarily $A - \mathcal{C}(0) \geq 0$.

  C4- $\mathcal{C}(|x|) = \omega(\ln|f(x)|)$.

- The noise PDF $p_N(\cdot)$:

  C5- The PDF is positive and continuous on $\mathbb{R}$. Note that this automatically implies that $p_N(\cdot)$ is upper bounded.

  C6- There exits a non-decreasing function

  $$\mathcal{C}_N : \mathbb{R}^+ \longrightarrow \mathbb{R}$$

  such that $\mathcal{C}_N(|x|) = \omega(\ln|x|)$, and

  $$\mathsf{E}_N[\mathcal{C}_N(|N|)] = L_N < \infty.$$

  This necessarily implies that $\mathsf{E}_N[\ln(1 + |N|)] < \infty$. Note that, for example, the above condition holds true for any noise PDF whose tail is faster than $\frac{1}{x(\ln x)^3}$.

  Since from an information theoretic perspective, the general channel model (1) is invariant with respect to output scaling, we consider WLOG that the noise PDF is less than "1" for technical reasons. Furthermore, the boundedness of $p_N(\cdot)$ along with the fact that it has a finite logarithmic moment insure that its differential entropy exists and is finite $h(N) < \infty$ (see [38, Proposition 1]).

  Restrictions C1 to C6 are "technical" in the sense that they represent sufficient conditions for the existence of a solution to the capacity problem as defined in [8] and enables the formulation of the Karush Kuhn Tucker (KKT) conditions as being necessary and sufficient for optimality of the input probability distribution.

- The lower and upper bounds $T_\mathrm{l}(\cdot)$ and $T_\mathrm{u}(\cdot)$:

  Note that by definition, $0 < T_\mathrm{l}(x) \leq p_N(x) \leq T_\mathrm{u}(x) \leq 1$ for all $x \in \mathbb{R}$. We assume that $T_\mathrm{l}(\cdot)$ and $T_\mathrm{u}(\cdot)$ satisfy the following properties:

  C7- The function $L(x) = \ln\left[\frac{1}{T_\mathrm{l}(x)}\right]$ which is positive, non-decreasing for $x \geq 0$ and non-increasing in $x < 0$, satisfies the following inequality:

  $$L(x + y) \leq \kappa_\mathrm{l}(L(x) + L(y)), \tag{5}$$

  for some positive constant κ_l, whenever $|x|, |y|$ are sufficiently large.

We note that functions that satisfy condition C7 define a convex set. In fact, let $f(x)$, $g(x)$ be two positive, non-decreasing functions on $\mathbb{R}^+$ non-increasing on $\mathbb{R}^{-*}$. Let $\alpha \in [0,1]$ and define $h = \alpha f + (1-\alpha)g$. The function $h(x)$ is positive, having the same monotonic properties. Then, whenever there exists κ_{f} and $\kappa_{\mathrm{g}} > 0$ for which f and g satisfy condition C7, we have

$$h(x+y) = \alpha f(x+y) + (1-\alpha)g(x+y) \leq \kappa_{\mathrm{h}}(h(x) + h(y)),$$

where $\kappa_{\mathrm{h}} = \max\{\kappa_{\mathrm{f}}; \kappa_{\mathrm{g}}\} > 0$.

We clarify that condition C7 is for example satisfied by all noise distribution functions where $T_{\mathrm{l}}(x)$ is any linear combinations of:

$$T_{\mathrm{l}}(x) = \Theta\left(s(x)e^{r(x)}\right) \qquad T_{\mathrm{l}}(x) = \Theta\left(\frac{s(x)}{r(x)}\right),$$

where

$$r(x) = |x|^a \underbrace{\log \ldots \log(|x|)}_{\beta \text{ times}}, \qquad s(x) = |x|^{a'} \underbrace{\log \ldots \log(|x|)}_{\beta' \text{ times}},$$

and where the parameters a, $a' \in \mathbb{R}^+$, and β, $\beta' \in \mathbb{N}$, chosen so that $T_{\mathrm{l}}(x)$ is positive, its total integral is no greater than one, and conserves its monotonic behavior[‡]. The fact that these two general types satisfy condition C7 is based on the following basic identities [39]:

- For all x, y and $r \in \mathbb{R}$,

$$|x+y|^r \leq \max\{1; 2^{r-1}\}\left(|x|^r + |y|^r\right).$$

- For any $x_0 > 0$, there exist $y_0 > 0$ such that

$$|x| + |y| \leq |xy|^p, \text{ for some } p > 1 \text{ whenever } |x| > x_0, |y| > y_0.$$

Finally, we also assume that

C8- The integral $-\displaystyle\int_{-\infty}^{+\infty} T_{\mathrm{u}}(x) \ln T_{\mathrm{l}}(x) \, dx$ exists and is finite.

Note that whenever the tail of $p_N(\cdot)$ is monotone, condition C8 is not necessary and boils down to saying that noise differential entropy is finite which is a byproduct of properties C5 and C6 of the noise PDF.

When it comes to conditions C5 through C8 –and specifically C7 and C8–, they are satisfied by a rather large class of noise probability functions that includes most of the known probability models such as Gaussian, generalized Gaussian, generalized t, alpha-stable distributions and all of their possible mixtures.

III. Preliminaries

In this section we establish some preliminary results that are needed in subsequent sections: we derive lower and upper bounds on the output probability and a quantity of interest presented hereafter.

[‡]The values $\beta = 0$ and $\beta' = 0$ imply that respectively $r(x)$ and $s(x)$ have no logarithmic component.

We start by noting that for channel (3), the existence of a positive, continuous transition PDF such as in (4), implies the existence for any input distribution F of an induced output probability density function $p_Y(y) = p(y; F)$ which is also continuous (hence upper-bounded) [7] and is given by:

$$p_Y(y; F) = p(y; F) = \int p_N(y - f(x)) \, dF_X(x) \leq 1. \tag{6}$$

Furthermore, equation (6) along with the fact that $f(\cdot)$ is continuous insures that the property that $p_N(\cdot)$ is bounded away from zero on compact subsets of $\mathbb{R}$ is conserved as well for $p_Y(y; F)$. This in turns implies that $p_Y(\cdot)$ is also positive on $\mathbb{R}$.

A. Bounds on $p(y; F)$

In what follows, we derive upper and lower bounds on the output probability distribution induced by an input distribution F.

Lemma 1. *Let $y_0 > 0$ be sufficiently large. For an input distribution F, the PDF $p(y; F)$ of the output of channel (3) is lower bounded by*

$$p(y; F) \geq \begin{cases} \dfrac{T_l(y - y_0)}{2} & y \leq -y_0 \\ \dfrac{T_l(y + y_0)}{2} & y \geq y_0, \end{cases}$$

Proof: Given an input probability distribution F, we define the following:

- We denote by d_F a positive constant such that $\Pr(|X| \leq d_F) \geq \frac{1}{2}$.
- We denote by $f_{\max} = \sup_{|x| \leq d_F} |f(x)|$, the existence of which is guaranteed by the assumption that $f(\cdot)$ is continuous on $\mathbb{R}$.

Let $y_0 > f_{\max}$. In what follows, we only present in detail the case $y \geq y_0$ as the proof in the other range follows similar steps.

$$p_Y(y; F) \geq \int_{x: |x| \leq d_F} p_N(y - f(x)) \, dF(x)$$

$$\geq \int_{x: |x| \leq d_F} T_l(y - f(x)) \, dF(x) \tag{7}$$

$$\geq \frac{1}{2} T_l(y + f_{\max}) \geq \frac{1}{2} T_l(y + y_0), \tag{8}$$

where equation (7) is due to the fact that $T_l(\cdot)$ is a lower bound on $p_N(\cdot)$ by definition and inequalities (8) are justified since $T_l(\cdot)$ is non-increasing on the considered interval. ∎

We also derive an upper bound on the output law whenever the input is bounded within $[-B, B]$ for some $B > 0$:

Lemma 2. *For an input distribution F that has a bounded support within $[-B, B]$ for some $B > 0$, the PDF $p(y; F)$ of the output of channel (3) is upper bounded by*

$$p(y; F) \leq \begin{cases} T_u\left(y + y_0^B\right) & y \leq -y_0^B \\ T_u\left(y - y_0^B\right) & y \geq y_0^B, \end{cases}$$

for any large-enough y_0^B.

Proof: Let $f_{\max}^B = \sup_{[-B;B]} |f(x)|$, the existence of which is guaranteed by the fact that $f(\cdot)$ is continuous on $\mathbb{R}$. Also let $y_0^B \geq f_{\max}^B$. For $y \geq y_0^B$, since $T_u(\cdot)$ is an upper bound on $p_N(\cdot)$, we have,

$$
\begin{aligned}
p(y; F) &= \int p_N(y - f(x)) \, dF(x) \\
&= \int_{-B}^{B} p_N(y - f(x)) \, dF(x) \qquad (9) \\
&\leq \int_{-B}^{B} T_u(y - f(x)) \, dF(x) \\
&\leq T_u(y - f_{\max}^B) \leq T_u(y - y_0^B) \qquad (10)
\end{aligned}
$$

where equations (10) are due to the fact that $T_u(x)$ is non-increasing on the positive semi-axis. A similar derivation yields the result for $y \leq -y_0^B$. ∎

We emphasize that this upper bound on $p(y; F)$ is only possible under the assumption that the support of F is bounded (as seen in equation (9)).

B. Bounds on $i(x; F)$

In this section we analyze the function of interest

$$
\begin{aligned}
i(x; F) &= -\int_{-\infty}^{+\infty} p_N(y - x) \ln p_Y(y; F) \, dy \qquad (11) \\
&= -\int_{-\infty}^{+\infty} p_N(y) \ln p_Y(y + x; F) \, dy.
\end{aligned}
$$

Lemma 3. *For any probability distribution F,*

$$
i(x; F) = O\left(\ln\left[\frac{1}{T_l(x)}\right]\right).
$$

Proof: Consider a large-enough y_0 so that Lemma 1 holds, and let x be such that $x > y_0$. For a probability distribution F on the input we compute,

$$
i(x; F) = -\int_{-\infty}^{+\infty} p_N(y) \ln p_Y(y + x; F) \, dy = I_1 + I_2 + I_3,
$$

where the interval of integration is divided into three sub-intervals: $(-\infty, -x - y_0)$, $[-x - y_0, y_0]$, $(y_0, +\infty)$.

We study the growth rate in x of the integral terms I_1, I_2 and I_3 function of the rate of decay of $T_l(\cdot)$.

Using Lemma 1,

$$I_1 = - \int_{-\infty}^{-x-y_0} p_N(y) \ln p_Y(y+x; F) \, dy$$

$$\leq - \int_{-\infty}^{-x-y_0} p_N(y) \ln \left[\frac{T_1(y+x-y_0)}{2} \right] dy = \int_{-\infty}^{-x-y_0} p_N(y) \ln \left[\frac{2}{T_1(y+x-y_0)} \right] dy$$

$$\leq \ln 2 + \kappa \int_{-\infty}^{-x-y_0} p_N(y) \left(\ln \left[\frac{1}{T_1(y)} \right] + \ln \left[\frac{1}{T_1(x)} \right] + \ln \left[\frac{1}{T_1(-y_0)} \right] \right) dy \qquad (12)$$

$$\leq \ln 2 + \kappa \ln \left[\frac{1}{T_1(-y_0)} \right] + \kappa \ln \left[\frac{1}{T_1(x)} \right] + \kappa \int_{-\infty}^{+\infty} p_N(y) \ln \left[\frac{1}{T_1(y)} \right] dy \qquad (13)$$

$$\leq 2\kappa \ln \left[\frac{1}{T_1(x)} \right],$$

for some positive κ and for $x > y_0$ large-enough. Equation (12) is due to property C7 since both x and y_0 are large enough and so is $|y|$. The integral term in (13) is finite by property C8 and the last equation is valid since $\ln \left[\frac{1}{T_1(x)} \right]$, which is positive, is increasing to $+\infty$.

Similarly,

$$I_3 = - \int_{y_0}^{\infty} p_N(y) \ln p_Y(y+x; F) \, dy$$

$$\leq - \int_{y_0}^{\infty} p_N(y) \ln \left[\frac{T_1(y+x+y_0)}{2} \right] dy = \int_{y_0}^{\infty} p_N(y) \ln \left[\frac{2}{T_1(y+x+y_0)} \right] dy$$

$$\leq \ln 2 + \kappa \int_{y_0}^{\infty} p_N(y) \left(\ln \left[\frac{1}{T_1(y)} \right] + \ln \left[\frac{1}{T_1(x)} \right] + \ln \left[\frac{1}{T_1(y_0)} \right] \right) dy$$

$$\leq 2\kappa \ln \left[\frac{1}{T_1(x)} \right],$$

As for I_2,

$$I_2 = - \int_{-x-y_0}^{y_0} p_N(y) \ln p_Y(y+x; F) \, dy$$

$$= - \int_{-x-y_0}^{-x+y_0} p_N(y) \ln p_Y(y+x; F) \, dy - \int_{-x+y_0}^{y_0} p_N(y) \ln p_Y(y+x; F) \, dy$$

$$\leq \sup_{|y| \leq y_0} \ln \left[\frac{1}{p_Y(y; F)} \right] + \int_{-x+y_0}^{y_0} p_N(y) \ln \left[\frac{2}{T_1(y+x+y_0)} \right] dy$$

$$\leq \sup_{|y| \leq y_0} \ln \left[\frac{1}{p_Y(y; F)} \right] + \ln 2 + \ln \left[\frac{1}{T_1(x+2y_0)} \right] \qquad (14)$$

$$\leq \sup_{|y| \leq y_0} \ln \left[\frac{1}{p_Y(y; F)} \right] + \ln 2 + \kappa \ln \left[\frac{1}{T_1(x)} \right] + \kappa \ln \left[\frac{1}{T_1(2y_0)} \right] \qquad (15)$$

$$\leq 2\kappa \ln \left[\frac{1}{T_1(x)} \right].$$

The supremum is finite since it is taken over a compact set where $p_Y(y)$ (which is less than one) is positive, continuous and hence positively lower bounded. Equation (14) is due to the fact that $\ln \left[\frac{1}{T_1(\cdot)} \right]$ is non-decreasing on the positive axis, equation (15) is given by property C7 since both x and y_0 are large enough and the last equation is justified since $\ln \left[\frac{1}{T_1(x)} \right]$ is increasing to $+\infty$ as $|x| \to +\infty$.

A similar procedure can be adopted to prove this result when $x \to -\infty$ by adjusting the intervals of integration to the following: $(-\infty, -y_0)$, $[-y_0, -x + y_0]$, $(-x + y_0, +\infty)$ where $x < -y_0$ such that $|x|$ is large enough. This would imply that for any probability distribution F, $i(x; F) = I_1 + I_2 + I_3 = O\left(\ln\left[\frac{1}{T_\mathrm{l}(x)}\right]\right)$. ∎

We also derive a lower bound whenever the input is bounded within $[-B, B]$ for some $B > 0$:

Lemma 4. *For an input distribution F that has a bounded support within $[-B, B]$ for some $B > 0$,*

$$i(x; F) = \Omega\left(\ln\left[\frac{1}{T_u(x)}\right]\right).$$

Proof: We proceed in a manner akin to the proof of Lemma 3: For an input distribution F that has a bounded support within $[-B, B]$ for some $B > 0$, we consider a large-enough y_0^B so that Lemma 2 holds, and let x be such that $x > y_0^B$.

$$
\begin{aligned}
i(x; F) &= -\int_{-\infty}^{\infty} p_N(y) \ln p(y + x; F)\, dy \\
&\geq -\int_{y_0^B}^{+\infty} p_N(y) \ln p(y + x; F)\, dy \\
&\geq \int_{y_0^B}^{+\infty} p_N(y) \ln\left[\frac{1}{T_\mathrm{u}\left(y + x - y_0^B\right)}\right] dy \quad (16) \\
&\geq \left(1 - F_N(y_0^B)\right) \ln\left[\frac{1}{T_\mathrm{u}(x)}\right] > 0. \quad (17)
\end{aligned}
$$

In order to write equation (16) we use the upper bound in Lemma 2. Equation (17) is justified since $\ln\left[\frac{1}{T_\mathrm{u}(\cdot)}\right]$ is non-decreasing on the non-negative semi-axis and the end result is positive since the support of N is $\mathbb{R}$. A similar analysis may be conducted for the case when $x < -y_0^B < 0$. ∎

IV. THE KARUSH-KUHN-TUCKER (KKT) THEOREM

The capacity of channel (1) is the supremum of the mutual information $I(\cdot)$ between the input X and output Y over all input probability distributions F that meet the constraint $\mathcal{P}_A$:

$$C = \sup_{F \in \mathcal{P}_A} I(F) = \sup_{F \in \mathcal{P}_A} \iint p_N(y - f(x)) \ln\left[\frac{p_N(y - f(x))}{p(y; F)}\right] dy\, dF(x). \quad (18)$$

Conditions C1 to C6 guarantee that this optimization problem is well-defined and that its solution –the capacity– is finite and is achievable [40, Theorem 2]. Indeed, the conditions are sufficient for $\mathcal{P}_A$ to be convex and compact [40, Theorem 3] and for $I(\cdot)$ to be concave and continuous (in the weak sense [41, Sec.III.7]) [40, Theorems 4,5].

When dealing with constrained optimization problems, the Lagrangian theorem [42] is a useful tool as it transforms the problem to an unconstrained one when some convexity conditions are satisfied by the objective function and the constraints. In our problem these conditions are satisfied as the mutual information is concave and the cost is linear - and hence convex. The theorem states that there exists a non-negative parameter ν_A such that the optimization

problem (18) can be written as:

$$C = \sup_{F \in \mathcal{P}_A} I(F) = \sup_F \left\{ I(F) - \nu_A \big(\mathsf{E}_F \left[\mathcal{C} \left(|X| \right) \right] - A \big) \right\} \tag{19}$$

$$= I(F^*) - \nu_A \, \mathsf{E}_{F^*} \left[\mathcal{C} \left(|X| \right) \right] + \nu_A A,$$

where the last equality is true since the solution is finite and achievable by an optimal F^*. Furthermore,

$$\nu_A \left(\mathsf{E}_{F^*} \left[\mathcal{C} \left(|X| \right) \right] - A \right) = 0.$$

For every positive A, denote by $C(A)$ the capacity of the channel under the constraint $F \in \mathcal{P}_A$, and consider the function $C(A)$ for $A > 0$. The significance of the Lagrange parameter ν_A is addressed in the following Lemma.

Lemma 5. *Whenever for some positive A the parameter $\nu(A) = 0$, then $C(A') = C(A)$ for all $A' \geq A$.*

Proof: We start by noting that the channel capacity $C(A)$ is a non-decreasing function of A, due to the fact $\mathcal{P}_A \subseteq \mathcal{P}_{A'}$, for $0 < A \leq A'$. Now assume that $\nu(A) = 0$ for some $A > 0$. For this value of A, equation (19) becomes

$$C = \sup_{F \in \mathcal{P}_A} I(F) = \sup_F \left\{ I(F) - \nu_A \big(\mathsf{E}_F \left[\mathcal{C} \left(|X| \right) \right] - A \big) \right\} = \sup_F I(F),$$

which is a maximal value over all probability distributions irrespective of the constraint. This observation along with the fact that $C(A)$ is non-decreasing establish the result. ∎

In our setup, a value of $\nu(A) = 0$ can be ruled out. Said differently, the cost constraint in equation (18) is binding. The argument we make is similar to the one used in [3]: we consider a family of input signals composed of N discrete levels with equal probabilities at locations $\{1, L, L^2, \cdots, L^{2^{N-2}}\}$. When L increases, the probability of error of a minimum probability of error receiver goes to zero, which implies by Fano's inequality that the mutual information approaches $\ln(N)$. Therefore, as $A \to \infty$, the achievable rates in our setup are arbitrarily large and $C(A)$ increases to infinity; a fact that is not possible if $\nu(A)$ were equal to zero for some A by Lemma 5. This conclusion is corroborated by the fact that the capacity for general memoryless continuous-input, continuous-output channels is achieved by a boundary input for unbounded input cost functions [43].

Whenever weak (Gateaux) differentiability is guaranteed, one can further write necessary and sufficient conditions on the maximum achieving distribution; conditions that are commonly referred to as the KKT conditions [42]. More formal statements on the theory of convex optimization are summarized in Appendix C in [7]. The KKT approach was used previously in many studies [1], [3], [4], [6], [8], [9], [11], [12], [44]–[47] in order to solve the capacity problem and for the purpose of this work, we follow similar steps. We indeed prove in Appendix I the weak differentiability of $I(\cdot)$ at any optimal input F^* and proceeding as in [3], we write the KKT conditions as being necessary and sufficient conditions for the optimal input to satisfy. These conditions state that an input RV X^* with probability distribution F^* achieves the capacity C of an average cost constrained channel if and only if there exists $\nu \geq 0$ such that,

$$\nu(\mathcal{C}\left(|x| \right) - A) + C + H + \int p_N \left(y - f(x) \right) \ln p(y; F^*) \, dy = \nu(\mathcal{C}\left(|x| \right) - A) + C + H - i(f(x); F^*) \geq 0, \tag{20}$$

for all x in $\mathbb{R}$, with equality if x is a point of increase of F^*, and where H is the entropy of the noise.

V. MAIN RESULTS

Theorem 1. *Whenever* $\mathcal{C}\left(|x|\right) = \omega\left(\ln\left[\frac{1}{T_l(f(x))}\right]\right)$, *the support of the capacity-achieving input of channel (3) is compact.*

Proof: We consider the necessary and sufficient conditions of optimality (20), and we study the behavior of the expression function of the variable x as its magnitude goes to infinity.

These conditions state that for the optimal input X^*, condition (20) is satisfied with equality for any point of increase x_0 of the capacity-achieving distribution $F^* \in \mathcal{P}_A$. For such an x_0 we obtain,

$$\nu(\mathcal{C}\left(|x_0|\right) - A) + C + H = i(f(x_0); F^*).$$

If these points of increase of X^* take arbitrarily large values, $|f(x_o)| \to +\infty$ since $|x_o| \to +\infty$. Using Lemma 3, $i(f(x_o); F) = O\left(\ln\left[\frac{1}{T_l(f(x_o))}\right]\right)$, and therefore

$$\nu(\mathcal{C}\left(|x_0|\right) - A) + C + H = O\left(\ln\left[\frac{1}{T_l\left(f(x_0)\right)}\right]\right),$$

which is a contradiction whenever $\mathcal{C}\left(|x|\right) = \omega\left(\ln\left[\frac{1}{T_l(f(x))}\right]\right)$ unless $\nu = 0$. This has been ruled out in Section IV, which implies that the support of X^* is bounded. Finally, we note that the support is always closed, as its complement is open. Therefore, X^* is compactly supported. ∎

A. A converse theorem

Now we make use of the upper bound on the noise PDF. In this section, we state and prove a converse formulation of Theorem 1. Indeed we prove that whenever $\mathcal{C}\left(|x|\right) = o\left(\ln\left[\frac{1}{T_u\left(f(x)\right)}\right]\right)$, the capacity-achieving input is not bounded.

Theorem 2. *Whenever* $\mathcal{C}\left(|x|\right) = o\left(\ln\left[\frac{1}{T_u\left(f(x)\right)}\right]\right)$, *the support of the capacity-achieving input of channel (3) is unbounded.*

Proof: Suppose that the optimal input X^* with distribution function F^* has a bounded support within $[-B, B]$ for some $B > 0$. The KKT conditions imply that there exists $\nu \geq 0$ such that,

$$\nu(\mathcal{C}\left(|x|\right) - A) + C + H + \int p_N\left(y - f(x)\right) \ln p(y; F^*)\, dy \geq 0,$$

for all x in $\mathbb{R}$, with equality if x is a point of increase of F^*. Using Lemma 3, the integral term $i(f(x); F^*) = \Omega\left(\ln\left[\frac{1}{T_u\left(f(x)\right)}\right]\right)$ and hence, equation (20) necessarily implies that,

$$\nu(\mathcal{C}\left(|x|\right) - A) + C + H = \Omega\left(\ln\left[\frac{1}{T_u\left(f(x)\right)}\right]\right),$$

which is impossible whenever $\mathcal{C}\left(|x|\right) = o\left(\ln\left[\frac{1}{T_u\left(f(x)\right)}\right]\right)$. ∎

B. Discreteness

In what follows, we further characterize the capacity-achieving input statistics when the cost function, the noise PDF and the channel distortion function have an additional analyticity property. This property guarantees the type of the optimal bounded input to be a discrete one, and hence with a finite number of mass points by virtue of compactness. This characterization permits to proceed to numerical computations in order to compute channel capacity and find the achieving input.

In this section, let $\eta > 0$ denote a positive scalar and let $\mathcal{S}_\eta = \{z \in \mathbb{C} : |\Im(z)| < \eta\}$ be a horizontal strip in the complex domain. We adopt in this section an alternative definition of $T_\mathrm{u}(x)$:

$$
T_\mathrm{u}(x) = \begin{cases} \sup\limits_{\zeta \in \mathcal{S}_\eta : \Re(\zeta) \geq x} |p_N(\zeta)| & x \geq 0 \\ \sup\limits_{\zeta \in \mathcal{S}_\eta : \Re(\zeta) \leq x} |p_N(\zeta)| & x < 0, \end{cases} \tag{21}
$$

and we assume that the following condition holds: The integral $-\displaystyle\int_{-\infty}^{+\infty} T_\mathrm{u}(x) \ln T_\mathrm{l}(x) \, dx$ exists and is finite. Note that this condition is similar to C8 but it is function of a redefined $T_\mathrm{u}()$. One may think of the condition as more restrictive. However, this strengthened condition is needed only to establish discreteness. In the remainder of this document we will refer to this condition as "the strengthened-C8". We present hereafter, a lemma that guarantees the analyticity of $i(\cdot; F)$ on $\mathcal{S}_\eta$:

Lemma 6. *Whenever there exists an $\eta > 0$ such that $p_N(\cdot)$ admits an analytic extension on $\mathcal{S}_\eta$, the function $i(\cdot; F) : \mathcal{S}_\eta \to \mathbb{C}$ defined by:*

$$
z \to i(z; F) = -\int_{-\infty}^{\infty} p_N(y - z) \ln p(y; F) \, dy, \tag{22}
$$

is analytic.

Proof: To prove this lemma, we will make use of Morera's theorem:

a) We start first by proving the *continuity* of $i(\cdot; F)$. In fact, let $\rho > 0$, z_0 and $z \in \mathcal{S}_\eta$ such that $|z - z_0| \leq \rho$,

$$
\lim_{z \to z_0} i(z; F) = -\lim_{z \to z_0} \int p_N(y - z) \ln p(y; F) \, dy
$$

$$
= -\int \lim_{z \to z_0} p_N(y - z) \ln p(y; F) \, dy \tag{23}
$$

$$
= -\int p_N(y - z_0) \ln p(y; F) \, dy = i(z_0; F). \tag{24}
$$

Equation (24) is justified by $p_N(y - z)$ being a continuous function of z on $\mathcal{S}_\eta$ by virtue of its analyticity and equation (23) by Lebesgue's Dominated Convergence Theorem (DCT). Indeed, in what follows we find an integrable function $r(y)$ such that,

$$
\left| p_N(y - z) \ln p(y; F) \right| = -\left| p_N(y - z) \right| \ln p(y; F) \leq r(y),
$$

for all $z \in \mathcal{S}_\eta$ such that $|z - z_0| \leq \rho$ and for all $y \in \mathbb{R}$. We upper bound first $|p_N(y - z)|$: let y_0 be large enough so that Lemma 1 holds

- If $y \leq -(y_0 + |\Re(z_0)| + \rho)$, then $y \leq -y_0 + \Re(z_0) - \rho$ (where y_0 has been defined in Lemma 1) and

$$|p_N(y-z)| \leq T_{\mathrm{u}}(y - \Re(z)) \leq \max_{\zeta \in \mathcal{S}_\eta : |\zeta - z_0| \leq \rho} T_{\mathrm{u}}(y - \Re(\zeta)) = T_{\mathrm{u}}(y - \Re(z_0) + \rho),$$

where the last equality is due to the fact that for $x \leq 0$, $T_{\mathrm{u}}(x)$ is non-decreasing, and for $\zeta \in \mathcal{S}_\eta; |\zeta - z_0| \leq \rho$, $(y - \Re(\zeta)) \leq (y - \Re(z_0) + \rho) < 0$.

- Similarly, for $y \geq (y_0 + |\Re(z_0)| + \rho) \geq (y_0 + \Re(z_0) + \rho)$,

$$\left| p_N(y-z) \right| \leq T_{\mathrm{u}}(y - \Re(z_0) - \rho).$$

Next, using Lemma 1 we also upper bound $-\ln p(y; F)$ to obtain:

$$r(y) = \begin{cases} T_{\mathrm{u}}(y - \Re(z_0) + \rho) \ln\left[\dfrac{2}{T_{\mathrm{l}}(y - y_0)}\right] & y \leq -(y_0 + |\Re(z_0)| + \rho) \\[2mm] -M \ln M' & |y| < y_0 + |\Re(z_0)| + \rho \\[2mm] T_{\mathrm{u}}(y - \Re(z_0) - \rho) \ln\left[\dfrac{2}{T_{\mathrm{l}}(y + y_0)}\right] & y \geq y_0 + |\Re(z_0)| + \rho, \end{cases}$$

where

$$M = \max_{\{|y| \leq (y_0 + |\Re(z_0)| + \rho)\}} \max_{\{\zeta \in \mathcal{S}_\eta : |\zeta - z_0| \leq \rho\}} |p_N(y - \zeta)| \quad \& \quad M' = \min_{\{|y| \leq (y_0 + |\Re(z_0)| + \rho)\}} p_Y(y; F).$$

Note that M is finite since $p_N(\cdot)$ is analytic and the maximization is taken over a compact set, and $0 < M' < 1$, since $p_Y(\cdot; F)$ is positive, continuous and less than 1. Properties C7 and strengthened-C8 insure the integrability of $r(y)$ which concludes the proof of continuity of $i(z; F)$.

b) To continue the proof of analyticity, we need to integrate $i(\cdot; F)$ on the boundary $\partial \Delta$ of a compact triangle $\Delta \subset \mathcal{S}_\eta$. We denote by $|\Delta|$ its perimeter, $\eta_0 = \min_{z \in \partial \Delta} \Re(z)$, $\eta_1 = \max_{z \in \partial \Delta} \Re(z)$ and $\phi = y_0 + \max\{|\eta_0|, |\eta_1|\}$. By similar arguments as above, we have

$$\int_{\mathbb{R}} \int_{\partial \Delta} |p_N(y - z)| |\ln p(y; F)| dz\, dy$$
$$\leq |\Delta| M'' \int_{|y| \leq \phi} |\ln p(y; F)| dy + |\Delta| \int_{y \leq -\phi} T_{\mathrm{u}}(y - \eta_0) \ln\left[\frac{2}{T_{\mathrm{l}}(y - y_0)}\right] dy + |\Delta| \int_{y \geq \phi} T_{\mathrm{u}}(y - \eta_1) \ln\left[\frac{2}{T_{\mathrm{l}}(y - y_0)}\right] dy < \infty,$$

where

$$M'' = \max_{y : |y| \leq \phi} \max_{\xi \in \partial \Delta} |p_N(y - \xi)| < \infty.$$

Using Fubini's theorem to interchange the order of integration,

$$\int_{\partial \Delta} i(z; F) dz = -\int_{\partial \Delta} \int_{\mathbb{R}} p_N(y - z) \ln p(y; F)\, dy\, dz = -\int_{\mathbb{R}} \int_{\partial \Delta} p_N(y - z) \ln p(y; F)\, dz\, dy$$
$$= -\int_{\mathbb{R}} \ln p(y; F) \int_{\partial \Delta} p_N(y - z)\, dz\, dy = 0, \tag{25}$$

where (25) is justified by the fact that $p_N(y - z)$ is analytic for all $z \in \mathcal{S}_\eta$ and $y \in \mathbb{R}$. Equation (25) in addition to the continuity of $i(\cdot; F)$ insure its analyticity on $\mathcal{S}_\eta$. ∎

Theorem 3. *Assume there exists an $\eta > 0$ such that $p_N(x)$ is analytically extendable on $\mathcal{S}_\eta$, and let $\mathcal{I}$ be an unbounded closed interval of $\mathbb{R}^\S$. The capacity-achieving input of channel (3) is compactly supported and discrete with finite number of mass points on $\mathcal{I}$, whenever the following conditions hold:*

- $\mathcal{C}\left(|x|\right) = \omega\left(\ln\left[\frac{1}{T_l(f(x))}\right]\right)$.
- *The restrictions of $f(x)$ and $\mathcal{C}\left(|x|\right)$ on $\mathcal{I}$ admit analytic extensions to $\mathcal{I} \times \mathbb{R}$, denoted $f_{\mathcal{I}}(\cdot)$ and $\mathcal{C}_{\mathcal{I}}(\cdot)$ respectively.*
- *The inverse map $f_{\mathcal{I}}^{-1}(\cdot)$ of $f_{\mathcal{I}}(\cdot)$ conserves connectedness.*

Before we prove the theorem, we note that a necessary condition for analytical extendability is to have $\mathcal{C}\left(|x|\right)$ an explicit function of the variable x on $\mathcal{I}$ which can be possibly realized when $\mathcal{I}$ is for example a subset of either $\mathbb{R}^+$ or $\mathbb{R}^-$.

Proof: We start by setting some notation and making a few remarks:

- Define $\mathcal{J}$ to be the image of interval $\mathcal{I}$ by $f_{\mathcal{I}}(\cdot)$.

  Since by analyticity $f_{\mathcal{I}}(\cdot)$ is continuous, then $\mathcal{J}$ is an interval of $\mathbb{R}$ because $f_{\mathcal{I}}(\cdot)$ is identical to $f(\cdot)$ on $\mathcal{I}$, and is real valued.

- Let $\mathcal{J}_\eta = \{z \in \mathcal{S}_\eta : \Re(z) \in \mathcal{J}\}$ and define $\mathcal{I}_\eta = f_{\mathcal{I}}^{-1}(\mathcal{J}_\eta)$, the inverse image of $\mathcal{J}_\eta$ by $f_{\mathcal{I}}(\cdot)$.

  Note that since $\mathcal{J}$ is an interval, $\mathcal{J}_\eta$ is connected and so is $\mathcal{I}_\eta$ by virtue of the properties of $f_{\mathcal{I}}^{-1}(\cdot)$. Additionally, since $f_{\mathcal{I}}(\mathcal{I}) = \mathcal{J}$ then $\mathcal{I} \subset \mathcal{I}_\eta$.

  In what follows, we work using the induced topology on $\mathcal{I}_\eta$. Under this topology, $\mathcal{I}_\eta$ is both open and closed.

We proceed with the proof and assume that the optimal input X^* with distribution function F^* has at least one point of increase in $\mathcal{I}$ for otherwise the result becomes trivial. *Assume that the points of increase of F^* in $\mathcal{I}$ are accumulating*, and let

$$s(z) = \nu\left(\mathcal{C}_{\mathcal{I}}(z) - A\right) + C + H - i(f_{\mathcal{I}}(z); F^*).$$

By the result of Lemma 6, $i(f_{\mathcal{I}}(z); F^*)$ is analytic on $\mathcal{I}_\eta$ since it is the composition of two analytic functions: $f_{\mathcal{I}}(\cdot)$ on $\mathcal{I}_\eta$ and $i(\cdot; F^*)$ on $\mathcal{J}_\eta = f_{\mathcal{I}}(\mathcal{I}_\eta) \subset \mathcal{S}_\eta$. This implies that the function $s(z)$ is analytic on $\mathcal{I}_\eta$. Since by assumption the points of increase of F^* have an accumulation point on $\mathcal{I}$ then by the KKT conditions, $s(z)$ has accumulating zeros on $\mathcal{I} \subset \mathcal{I}_\eta$, which necessarily implies by the identity Theorem [48, sec. 66] that $s(\cdot)$ is identically null on $\mathcal{I}_\eta$, since $\mathcal{I}_\eta$ is open and connected. Therefore,

$$\nu(\mathcal{C}\left(|x|\right) - A) + C + H = -\int p_N(y) \ln p(y - f(x); F^*)\, dy, \qquad \forall x \in \mathcal{I}.$$

Since $\mathcal{I}$ is unbounded, this equality is impossible for large values of x by the result of Theorem 1 unless $\nu = 0$ which is non sensible. This leads to a contradiction and rules out the assumption of having an accumulation point on $\mathcal{I}$. Since $\mathbb{R}$ is Lindelof, X^* is necessarily discrete on $\mathcal{I}$. Additionally, since the support of X^* is compact and $\mathcal{I}$ is closed in $\mathbb{R}$, X^* has necessarily a finite number of mass points on $\mathcal{I}$. ∎

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§We consider that $\mathbb{R}$ is both closed and open.

Theorem 4. *Assume there exists an $\eta > 0$, such that $p_N(x)$ is analytically extendable on $\mathcal{S}_\eta$, and let $\mathcal{I}$ be an unbounded closed interval of $\mathbb{R}$. Whenever the input is constrained to have a compact support $\mathcal{X}$, the capacity-achieving input is discrete with a finite number of mass points on $\mathcal{X} \cap \mathcal{I}$ if the following holds:*

- *The restriction of $f(x)$ on $\mathcal{X} \cap \mathcal{I}$ admits an analytic extension to $\mathcal{I} \times \mathbb{R}$, denoted $f_{\mathcal{I}}(\cdot)$.*
- *The inverse map $f_{\mathcal{I}}^{-1}(\cdot)$ of $f_{\mathcal{I}}(\cdot)$ conserves connectedness.*

Before proving the theorem, we note that the condition that the support of $\mathcal{X}$ is compact is a generalization of the peak power constraint. Also it makes sense to consider sets $\mathcal{X}$ that are not discrete, for otherwise the problem is ill defined.

Proof: We first note that the KKT conditions are valid under the setup of the compactly-supported input constraint: Indeed, the input space is compact in the weak topology and convex (see [7], [8]). Also note that there exists a cost function $\mathcal{C}(|x|)$ the tail of which is $\omega(\ln|f(x)|)$ and such that $\sup_{\mathcal{X}} \mathcal{C}(|x|) = A$, for some $A > 0$.

Now, for any $F \in \mathcal{P}_{\mathcal{X}}$ –the set of all input distributions having a compact support $\mathcal{X}$, we have $\int \mathcal{C}(|x|)\, dF(x) \leq A$ which implies that $\mathcal{P}_{\mathcal{X}} \subset \mathcal{P}_A$. Since the mutual information is finite, continuous and weakly differentiable on $\mathcal{P}_A$ whenever $\mathcal{C}(|x|) = \omega(\ln|f(x)|)$ (see Appendix I) then it is as such on $\mathcal{P}_{\mathcal{X}}$. Under this setup, the KKT conditions state that an input RV X^* with CDF F^* achieves the capacity C of a compact-support constrained channel if and only if,

$$C + H + \int p_N(y - f(x)) \ln p(y; F^*)\, dy \geq 0, \qquad \forall x \in \mathcal{X},$$

with equality if x is a point of increase of F^*, and where H is the entropy of the noise. By virtue of the analyticity conditions, the function $s(z) = C + H + \int p_N(y - f_{\mathcal{I}}(z)) \ln p(y; F^*)\, dy$ would also be analytic on $\mathcal{I}$. The assumption that the points of increase of X^* on $\mathcal{X} \cap \mathcal{I}$ have an accumulation point is impossible since it will lead by the identity theorem to $s(x) = 0$ on $\mathcal{I}$ which is impossible since $i(x; F^*) = -\int p_N(y - f(x)) \ln p(y; F^*) = \Omega\left(\ln\left[\frac{1}{T_{\mathrm{u}}(f(x))}\right]\right)$ (see the proof of Theorem 2), which increases to ∞. Therefore, X^* is necessarily discrete on $\mathcal{X} \cap \mathcal{I}$. The finiteness of the number of mass points is a direct consequence of the compactness of $\mathcal{X} \cap \mathcal{I}$. ∎

Note. A similar statement to that of Theorem 4 may be made whenever, in addition to a compact support constraint, there is also a cost constraint satisfying the conditions of Theorem 3 with tail behavior either $\omega\left(\ln\left[\frac{1}{T_{\mathrm{l}}(f(x))}\right]\right)$ or $o\left(\ln\left[\frac{1}{T_{\mathrm{u}}(f(x))}\right]\right)$.

Before moving to giving some concrete examples to our general theorems, we would like to state that some conditions were only considered for either the sake of the clarity of the proofs, or for conserving the general aspect of the results. Many such conditions could be relaxed while conserving some or all of the found conclusions. For example,

- The notions of ω, Ω, o and O used in this document are defined as $|x| \to +\infty$, i.e., in such a way to capture a symmetric rate of decay for both tails. However, one can only consider left or right tail behaviors separately. The results of boundedness and discreteness could be given in terms of each tail where for example for non-symmetric noise PDFs or non-symmetric cost functions, the optimal input could only be bounded on one of

the semi-axis.

- For Theorems 1 and 2, the assumption that $p_N(\cdot)$ is positive could be relaxed to one sided noise PDFs. These theorems are still valid on one side of the axis.

- The proven theorems –stated in terms of $T_l(x)$ and $T_u(x)$– could be stated in terms of any two functions having the same properties and providing lower and upper bounds on $p_N(x)$ for large values of $|x|$.

VI. APPLICATIONS OF THE THEOREMS AND NUMERICAL RESULTS

In this section we apply our results to a variety of specific channels of interest that fit under the general framework presented previously. For those channels that have been previously studied in the literature, we verify our results –in the form of Theorems 1, 2, 3 and 4, and for the other models we state some new results. We note that in all the examples presented subsequently the considered functions $f(\cdot)$ and the cost constraints satisfy the general conditions C1 through C4 in Section II-A. The noise distributions are absolutely continuous with positive, continuous PDFs with tails that have "at least" a polynomial decay and hence satisfying the assumptions C5 and C6. Finally, in all the provided examples the noise PDFs possess a monotonic tail and a finite differential entropy and therefore, condition C8 is satisfied. It remains to check for each example condition C7 and possibly the strengthened-C8.

For the purpose of verifying condition C7, we note that one can use $p_N(x)$ instead of $T_l(x)$ since they are identical at large values of $|x|$. When it comes to discreteness, whenever $|x|$ is large enough the function $T_u(x)$ defined in (21) becomes

$$T_u(x) = \sup_{\{z: \quad \Re(z) = x \ \& \ |\Im(z)| < \eta\}} |p_N(z)|,$$

because $|p_N(z)|$ is decreasing with $|\Re(z)|$ at large values for all the given examples.

For each model we consider in what follows, we will check whether the appropriate conditions are satisfied, state the results –specialized to the channel at hand, and compare with the known results in the literature.

A. The Gaussian Model

For a Gaussian noise distribution with mean zero and variance σ^2, the PDF is $p_N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ and we write $N \sim \mathcal{N}(0, \sigma^2)$.

Checking the conditions: Condition C7 is validated as follows: for large values of $|x|$ and $|y|$,

$$
\begin{aligned}
L(x+y) &= \ln\left[\frac{1}{p_N(x+y)}\right] = \ln\sqrt{2\pi\sigma^2} + \frac{(x+y)^2}{2\sigma^2} \\
&\leq 2\left(\ln\sqrt{2\pi\sigma^2} + \frac{x^2}{2\sigma^2} + \ln\sqrt{2\pi\sigma^2} + \frac{y^2}{2\sigma^2}\right) - 3\ln\sqrt{2\pi\sigma^2} = \kappa_1\left(L(x) + L(y)\right),
\end{aligned}
$$

where $\kappa_1 > 2$. When it comes to discreteness, let $p_N(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$, be an analytic extension of $p_N(x)$ to the complex plane, where $z = x + jy$. The magnitude of $p_N(z)$ is

$$|p_N(z)| = \frac{1}{\sqrt{2\pi\sigma^2}} \left| e^{-\frac{z^2}{2\sigma^2}} \right| = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2-y^2}{2\sigma^2}},$$

and is decreasing in $x = \Re(z)$. Therefore, $T_u(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2-\eta^2}{2\sigma^2}} = e^{\frac{\eta^2}{2\sigma^2}} p_N(x)$.

Checking for the strengthened-C8, the integral $-\int_{-\infty}^{+\infty} T_{\mathrm{u}}(x) \ln T_{\mathrm{l}}(x)\, dx = e^{\frac{\eta^2}{2\sigma^2}} h(N)$ which is finite because the noise differential entropy $h(N)$ is finite.

The following theorem is a specialization of Theorems 1 and 2 for this specific Gaussian case:

Theorem 5. *Whenever $\mathcal{C}(|x|) = o\left(f(x)^2\right)$, the support of the capacity-achieving input of channel (3) when $N \sim \mathcal{N}\left(0, \sigma^2\right)$ is unbounded.*

Whenever $\mathcal{C}(|x|) = \omega\left(f(x)^2\right)$, the support of the capacity-achieving input of channel (3) when $N \sim \mathcal{N}\left(0, \sigma^2\right)$ is compact. Furthermore, the optimal input is discrete with finite number of mass points whenever $\mathcal{C}(\cdot)$ and $f(\cdot)$ satisfy the analyticity and connectedness conditions of Theorem 3.

Under a compact support constraint, the optimal input is also discrete with finite number of mass points whenever $f(\cdot)$ satisfies the conditions of Theorem 4.

<u>Previous work:</u> A possibly non-linear ($f(x) = x^n$, $n \in \mathbb{N}^*$) Gaussian channel under an even moment constraint ($\mathcal{C}(|x|) = x^{2k}$) was considered in [7] as a core channel model from which results on multiple non-linear channel models were derived. The authors applied a standard Hilbert space decomposition using Hermite polynomials as bases and proved that, for $n < 2k$, the capacity-achieving distribution has the following behavior:

- Whenever $n = k$
  - if n is odd, the optimal input F^* is absolutely continuous.
  - if n is even, F^* is discrete with no accumulation points.
- Whenever $n < k$, F^* is discrete with finite number of mass points.
- Whenever $k < n < 2k$, F^* is discrete with no accumulation points.

We point out that while the results stated in Theorem 5 do not cover the limiting case $n = k$ –which corresponds to the case $\mathcal{C}(|x|) = \theta\left(f^2(x)\right)$, the result for the case "$n < k$" is identical. Whenever $k < n$, Theorem 5 states that the support of F^* is not bounded; a conclusion that could not be reached in [7].

B. Gaussian Mixtures

Gaussian mixtures are widely used as more tractable models to some non-Gaussian noise statistics [18], [19]. One approach in dealing with such distributions is based on the observation that in limiting cases Gaussian mixtures are nearly Gaussian and they are simplified accordingly. The PDF of a Gaussian mixture RV is:

$$p_N(x) = \sum_{i=1}^{n} \alpha_i\, p_{N_i}(x),$$

where $n \in \mathbb{N}^*$ and for $1 \leq i \leq n$,

- $N_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$ are Gaussian RVs with mean μ_i and variances $\sigma_i^2 \neq 0$. We assume WLOG that $\sigma_1 \geq \cdots \geq \sigma_n$.
- $0 \leq \alpha_i \leq 1$, and $\sum_{i=1}^{n} \alpha_i = 1$.

Before proceeding, we note that the rate of decay of this noise PDF is

$$
\begin{aligned}
\ln\left[\frac{1}{p_N(x)}\right] &= \ln\left[\frac{\frac{\sqrt{2\pi\sigma_1^2}}{\alpha_1}e^{\frac{(x-\mu_1)^2}{2\sigma_1^2}}}{1+\sum_{i=2}^{n}\frac{\alpha_i\sigma_1}{\alpha_1\sigma_i}e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}+\frac{(x-\mu_1)^2}{2\sigma_1^2}}}\right] \\
&= \ln\left[\frac{\sqrt{2\pi\sigma_1^2}}{\alpha_1}\right]+\frac{(x-\mu_1)^2}{2\sigma_1^2}-\ln\left[1+\sum_{i=2}^{n}\frac{\alpha_i\sigma_1}{\alpha_1\sigma_i}e^{-x^2\left(\frac{1}{2\sigma_i^2}-\frac{1}{2\sigma_1^2}\right)+x\left(\frac{\mu_i}{\sigma_i^2}-\frac{\mu_1}{\sigma_1^2}\right)-\left(\frac{\mu_i^2}{2\sigma_i^2}-\frac{\mu_1^2}{2\sigma_1^2}\right)}\right] \\
&= \ln\left[\frac{\sqrt{2\pi\sigma_1^2}}{\alpha_1}\right]+\frac{(x-\mu_1)^2}{2\sigma_1^2}-\Theta\left(\sum_{i=2}^{n}\frac{\alpha_i\sigma_1}{\alpha_1\sigma_i}e^{-x^2\left(\frac{1}{2\sigma_i^2}-\frac{1}{2\sigma_1^2}\right)+x\left(\frac{\mu_i}{\sigma_i^2}-\frac{\mu_1}{\sigma_1^2}\right)-\left(\frac{\mu_i^2}{2\sigma_i^2}-\frac{\mu_1^2}{2\sigma_1^2}\right)}\right) \\
&= \Theta(x^2).
\end{aligned}
$$

Checking the conditions: Since in Section II-A, we proved that condition C7 defines a convex set of functions, then by the results of the Gaussian model, each $p_{N_i}(\cdot)$ satisfies condition C7 and so does $p_N(\cdot)$. To study discreteness, we let $p_N(z) = \sum_{i=1}^{n}\alpha_i\,p_{N_i}(z)$, be an analytic extension of $p_N(x)$ on the complex plane. Since

$$
|p_N(z)| \leq \sum_{i=1}^{n}\alpha_i\,|p_{N_i}(z)| = \sum_{i=1}^{n}\frac{\alpha_i}{\sqrt{2\pi\sigma_i^2}}e^{-\frac{(x-\mu_i)^2-y^2}{2\sigma_i^2}},
$$

then, for a large-enough $|x|$,

$$
T_{\mathrm{u}}(x) = \sup_{\{z:\ \Re(z)=x\,\&\,|\Im(z)|<\eta\}}|p_N(z)| \ \leq\ e^{\frac{\eta^2}{\sigma_n^2}}\sum_{i=1}^{n}\frac{\alpha_i}{\sqrt{2\pi\sigma_i^2}}e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}} = e^{\frac{\eta^2}{\sigma_n^2}}\,p_N(x),
$$

which implies that strengthened C8 is valid of the finiteness of

$$
-\int_{-\infty}^{+\infty}T_{\mathrm{u}}(x)\ln T_{\mathrm{l}}(x)\ dx,
$$

as $h(N)$ is finite by virtue of the fact that N has a finite variance $\sigma^2 = \sum_{i=1}^{n}\alpha_i\sigma_i^2$.

Specializing the results to the channel at hand, we can state the following:

Theorem 6. *Whenever $\mathcal{C}(|x|) = o\left(f(x)^2\right)$, the support of the capacity-achieving input of channel (3) when N is a Gaussian mixture is unbounded.*

Whenever $\mathcal{C}(|x|) = \omega\left(f(x)^2\right)$, the support of the capacity-achieving input of channel (3) when N is a Gaussian mixture is compact. Furthermore, the optimal input is discrete with finite number of mass points whenever $\mathcal{C}(\cdot)$ and $f(\cdot)$ satisfy the analyticity and connectedness conditions of Theorem 3.

Under a compact support constraint, the optimal input is also discrete with finite number of mass points whenever $f(\cdot)$ satisfies the conditions of Theorem 4.

Previous work: To our knowledge, a formal analysis of Gaussian mixtures channels has not been conducted before, and hence Theorem 6 states a new previously unknown result. We note that since the transitional rate of decay is $\theta\left(f^2(x)\right)$, the capacity of the linear ($f(x) = x$) Gaussian mixtures channel under an average power constraint ($\mathcal{C}(|x|) = x^2$) is not in the scope of this work. However, in [11] it was shown that, except for Gaussian noise, the capacity of the linear average power constrained channel is achieved by discrete statistics for all noise distributions satisfying certain conditions, ones that are indeed satisfied by Gaussian mixtures.

In Figure 1, we plot the numerically-computed capacity of a sample Gaussian mixture channel. The results of [11] were used and an optimal discrete input distribution that satisfies the necessary and sufficient KKT condition was sought. The numerical computations were conducted using Matlab.
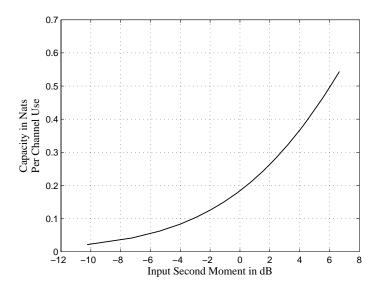

Fig. 1. Capacity of the linear channel under the Gaussian mixture noise $p_N(x) = 0.5\,p_{N_1}(x) + 0.5\,p_{N_2}(x)$ where $N_1 \sim \mathcal{N}(0,1)$ and $N_2 \sim \mathcal{N}(0,4)$.

C. Generalized Gaussian

Generalized Gaussians [49] are viewed as a class of distributions generalizing the well-known Laplacian and Gaussian distributions. Additive noise is often assumed to be a generalized Gaussian RV in order to model the impulsive¶ nature of noise in communication channels [50]–[53]. In other instances, these models were considered for the ultra-wide band multiple access interference plus noise [20], [21].

Generalized Gaussians have exponentially decaying PDFs given by:

$$p_N(x) = \frac{a}{2b\Gamma\left(\frac{1}{a}\right)} e^{-\left(\frac{|x-\mu|}{b}\right)^a},$$ (26)

where $\Gamma(\cdot)$ is the Gamma function, $a \in \mathbb{R}^{+*}$ is a shape parameter, $b \in \mathbb{R}^{+*}$ is a scale parameter and $\mu \in \mathbb{R}$ is a location parameter. In the remainder of this section, we will assume WLOG that the location parameter μ is equal to zero.

¶By impulsive it is meant that extreme values of the noise signal are observed very frequently (i.e., with notable amount of probability).

Checking the conditions: Condition C7 is satisfied. Indeed,

$$
\begin{aligned}
L(x+y) &= \ln\left[\frac{1}{p_N(x+y)}\right] = \ln\left[\frac{2b\Gamma\left(\frac{1}{a}\right)}{a}\right] + \frac{|x+y|^a}{b} \\
&\leq \ln\left[\frac{2b\Gamma\left(\frac{1}{a}\right)}{a}\right] + \max\{1; 2^{a-1}\}\frac{|x|^a + |y|^a}{b} \\
&= \max\{1; 2^{a-1}\}\left[L(x) + L(y)\right] + \min\{0; 1 - 2^{a-1}\}\ln\left[\frac{2b\Gamma\left(\frac{1}{a}\right)}{a}\right] \\
&\leq \kappa_{\mathrm{l}}\left[L(x) + L(y)\right],
\end{aligned}
$$

for some $\kappa_{\mathrm{l}} > \max\{1; 2^{a-1}\}$ for large-enough values of $|x|$ and $|y|$.

One can therefore state the following theorem:

Theorem 7. *Whenever $\mathcal{C}\left(|x|\right) = o\left(|f(x)|^a\right)$, the support of the capacity-achieving input of channel (3) when N is a generalized Gaussian RV (26) is unbounded.*

Whenever $\mathcal{C}\left(|x|\right) = \omega\left(|f(x)|^a\right)$, the support of the capacity-achieving input of channel (3) when N is a generalized Gaussian RV (26) is compact.

Previous work: To our knowledge, no previous information theoretic work has appeared regarding this channel model. For the linear channel under an average power constraint for instance, the optimal input of channel (3) is bounded whenever the noise is a generalized Gaussian with parameter $a < 2$.

D. Polynomially-Tailed Distributions

Gaussian mixtures and generalized Gaussians are considered by many researchers to fail to capture the "impulsiveness" of the noise. This failure is due to several reasons, the most important of which is that they do not possess the algebraic behavior of heavy-tailed noise distributions encountered in typical communication channels [54]. One family of such distributions, the "generalized Cauchy" [49], is found to be reasonable in modeling the amplitude of atmospheric impulse noise [55]. In this document, we refer by "polynomially-tailed" noise distributions to all distributions satisfying

$$
p_N(x) = \Theta\left(\frac{1}{|x|^{1+\alpha}}\right), \quad \text{for some } \alpha > 0,
$$

which include among others: the Gamma, Pareto (one sided) and alpha-stable distributions.

Checking the conditions: In order to proceed, we use the "obvious" lower and upper bounds on $p_N(x)$ for large values of $|x|$ instead of $p_N(x)$ itself and we state the corresponding theorems accordingly. These bounds are of the form $\frac{\zeta_{\mathrm{l}}}{|x|^{1+\alpha}}$ and $\frac{\zeta_{\mathrm{u}}}{|x|^{1+\alpha}}$, for some ζ_{l} and $\zeta_{\mathrm{u}} > 0$. We prove now that condition C7 is satisfied; Let

$$
L(x) = \ln\left[\frac{|x|^{1+\alpha}}{\zeta_{\mathrm{l}}}\right] = (1 + \alpha)\ln|x| - \ln\zeta_{\mathrm{l}},
$$

which implies that for large-enough $|x|$ and $|y|$,

$$
\begin{aligned}
L(x+y) = (1+\alpha)\ln|x+y| - \ln\zeta_{\mathrm{l}} &\leq (1+\alpha)\ln\left[|x|+|y|\right] - \ln\zeta_{\mathrm{l}} \\
&\leq p(1+\alpha)\left[\ln|x| + \ln|y|\right] - \ln\zeta_{\mathrm{l}} \\
&= p\left[(1+\alpha)\ln|x| - \ln\zeta_{\mathrm{l}} + (1+\alpha)\ln|y| - \ln\zeta_{\mathrm{l}}\right] + (2p-1)\ln\zeta_{\mathrm{l}} \\
&\leq 2p\left[(1+\alpha)\ln|x| - \ln\zeta_{\mathrm{l}} + (1+\alpha)\ln|y| - \ln\zeta_{\mathrm{l}}\right] \\
&= 2p\left[L(x) + L(y)\right],
\end{aligned}
$$

where $p > 1$. Consequently, the following holds:

Theorem 8. *Whenever* $\mathcal{C}\left(|x|\right) = \omega\left(\ln|f(x)|\right)$*, the support of the capacity-achieving input of channel (3) when* N *is polynomially-tailed is compact.*

For example, for a linear channel subjected to an additive polynomially-tailed noise, the optimal input has a bounded support for any cost function that is super logarithmic (i.e., $\omega\left(\ln|x|\right)$) such as the average power constraint.

Note that the other "range" $\mathcal{C}\left(|x|\right) = o\left(\ln|f(x)|\right)$ is outside the scope of this work as condition C4 will not be satisfied. When it comes to discreteness and strengthened-C8, it depends on the analyticity property of the specific $p_N(\cdot)$ under consideration.

The remaining part of this Section is dedicated to two important types of polynomially decaying distributions, for which we prove that the discreteness results of Theorem 3 apply.

1) Non-Totally Skewed Alpha-Stable and their Mixtures: The term "stable" is used because, under some constraints, these distributions are closed under convolution. The stable distributions, which are a subset of that of infinitely divisible distributions, are the only laws that have the captivating property of being the resultant of a limit of normalized sums of IID RVs. This result is referred to as the Generalized Central Limit Theorem (GCLT), a property that constitutes one of the main reasons behind the adoption of Gaussian statistics for noise models in communication channels.

Though the Gaussian distribution is one of the stable laws, it represents the exception: it is unique in the sense that it is the only one that has a finite variance and an exponential tail; All others have an infinite variance and a polynomial tail. A complete literature on the theory of stable distributions can be found in [56]–[59]. In this document we use the term "alpha-stable" to refer to stable variables *other than the Gaussian*. Although only few alpha-stable RVs have closed form densities (namely the Cauchy and the Lévy laws), these distributions are well characterized in the Fourier domain: The characteristic function[||] of an alpha-stable RV is given by:

$$
\phi(t) = \exp\left[i\delta t - \gamma^\alpha\left[1 - i\beta\operatorname{sgn}(t)\Phi(t)\right]|t|^\alpha\right], \qquad \left(0 < \alpha < 2 \quad -1 \leq \beta \leq 1 \quad \gamma \in \mathbb{R}^{+*} \quad \delta \in \mathbb{R}\right),
$$

[||]The characteristic function $\phi(t)$ of a distribution function $F(x)$ is defined by:

$$
\phi(t) = \int_{\mathbb{R}} e^{itx}\, dF(x).
$$

where $\operatorname{sgn}(t)$ is the sign of t, and the function $\Phi(\cdot)$ is given by:

$$\Phi(t) = \begin{cases} \tan\left(\dfrac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ -\dfrac{2}{\pi}\ln|t| & \alpha = 1. \end{cases}$$

The constant α is called the "characteristic exponent", β is the "skewness" parameter, γ is the "scale" parameter (γ^α is often called the "dispersion") and δ is the "location" parameter. Such a RV will be denoted $N \sim \mathcal{S}(\alpha, \beta, \gamma, \delta)$.

In what follows, we limit our analysis to non-totally skewed alpha-stable variables, i.e., ones for which $|\beta| \neq 1$.

<u>Checking the conditions:</u> For non-totally skewed laws, both the right and the left tails are polynomially decaying as $\Theta\left(\frac{1}{|x|^{\alpha+1}}\right)$ (see [60, Th.1.12, p.14]), and Theorem 8 holds. Furthermore, whenever $\alpha \geq 1$ the alpha-stable variables are analytically extendable, to the whole complex plane when $\alpha > 1$ and to some horizontal strip when $\alpha = 1$ [61, theorem 2.3.1 p. 48 and remark 1 p. 49]. We check in what follows the strengthened-C8. We derive in Appendix II a novel bound on the rate of decay of the complex extension of the alpha-stable PDF when $\alpha \geq 1$: For small-enough $\eta > 0$, there exist $\kappa > 0$ and $n_0 > 0$ such that

$$|p_N(z)| \leq \frac{\kappa}{|\Re(z)|^{\alpha+1}}, \quad \forall z \in \mathcal{S}_\eta : |\Re(z)| \geq n_0. \tag{27}$$

This bound insures the validity of Theorems 3 and 4 whenever the conditions on $\mathcal{C}(\cdot)$ and $f(\cdot)$ are satisfied, and hence the following theorem is valid:

Theorem 9. *Whenever $\mathcal{C}(|x|) = \omega(\ln|f(x)|)$, the support of the capacity-achieving input of channel (3) when $N \sim \mathcal{S}(\alpha, \beta, \gamma, \delta)$ is a non-totally skewed alpha-stable variable is compact.*

Whenever $\alpha \geq 1$, the optimal input is discrete with finite number of mass points whenever $\mathcal{C}(\cdot)$ and $f(\cdot)$ satisfy the analyticity and connectedness conditions of Theorem 3.

Under a compact support constraint, the optimal input is also discrete with finite number of mass points whenever $f(\cdot)$ satisfies the conditions of Theorem 4.

Note that by virtue of the fact that condition C7 defines a convex set, the results presented here for one alpha-stable variable are valid for any convex combinations of them.

<u>Previous work:</u> The capacity of the additive linear channel was considered in [14], where the noise is modeled as symmetric alpha-stable ($\beta = 0$) for the range $\alpha \geq 1$. Subjected to a fractional r-th moment constraint, $\mathsf{E}\left[|X|^r\right] \leq a$, $a > 0$ and $r > 1$, the optimal input was found to be achieved by discrete statistics. Theorem 9 generalizes this result to cover the non totally-skewed alpha-stable family and generic input cost functions that are "super-logarithmic". As a direct application of Theorem 9, it can be seen that the result of [14] also holds for $|\beta| \neq 1$ (not necessarily equal to 0) and for the range $r \leq 1$.

We use a specialized numerical MATLAB package [7] to search for the positions of the optimal points and their respective probabilities whenever the optimal input is discrete. In Figure 2, we plot the capacity of channel (3) whenever $f(x) = x$, $\mathcal{C}(|x|) = x^2$ and $N \sim \mathcal{S}(\alpha, 0, 1, 0)$ for $\alpha = 1, 1.2, 1.5$ and 1.8. The capacity curves clearly shows that as α gets bigger the capacity is higher. This is in accordance with the fact that the lower the value of α, the distribution becomes heavier.
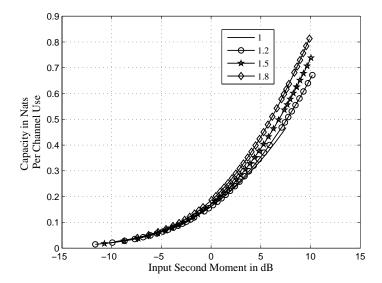
Fig. 2. Capacity of the linear channel subject to symmetric "standard" alpha-stable noise $N \sim \mathcal{S}(\alpha, 0, 1, 0)$ for various values of the characteristic exponent α.

2) Composite noise: Gaussian + Alpha-Stable: Recently, a compound noise model was adopted to capture potentially different sources of noise: a Gaussian model for the thermal noise and an alpha-stable model for the potential MAI, as is the case for ad-hoc self configuring networks with applications in CDMA networks [29], and in the general context of ultra wideband technologies [26]. Further information on the subject can be found in [62]–[64]. This noise model is widely known as the Middleton class B model [31], [32]. We consider hence the following additive noise $N = N_1 + N_2$, where

- $N_1 \sim \mathcal{S}(\alpha, \beta, \gamma, \delta)$, which represents the effect of the MAI, assumed a non totally-skewed alpha-stable RV.
- $N_2 \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is a Gaussian RV that models the effect of thermal noise.

Checking the conditions: It has been proved in [15, Appendix I] that $p_N(x)$ is polynomially-tailed which implies that Theorem 8 holds for the compound noise model. In order to apply Theorems 3 and 4 for the channels impaired by the composite noise N, we use the fact that its PDF is analytically extendable on $\mathbb{C}$ (for all values of $0 < \alpha < 2$) and therefore on $\mathcal{S}_\eta$ [15, Appendix I], and check the strengthened-C8:

$$
\begin{aligned}
T_{\mathrm{u}}\left(x\right) = \sup_{|\Im(z)| < \eta} |p_N(z)| &\leq \sup_{|\Im(z)| < \eta} \frac{1}{\sqrt{2\pi\sigma^2}} \int \left| e^{-\frac{(z-t)^2}{2\sigma^2}} \right| p_{N_1}(t)\, dt \\
&\leq \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\eta^2}{2\sigma^2}} \int e^{-\frac{(x-t)^2}{2\sigma^2}} p_{N_1}(t)\, dt = e^{\frac{\eta^2}{2\sigma^2}} p_N(x),
\end{aligned}
$$

which implies

$$
-\int_{-\infty}^{+\infty} T_{\mathrm{u}}\left(x\right) \ln T_{\mathrm{l}}\left(x\right)\, dx \leq -e^{\frac{\eta^2}{\sigma^2}} \int_{-\infty}^{+\infty} p_N(x) \ln p_N(x)\, dx = e^{\frac{\eta^2}{\sigma^2}} h(N) < \infty.
$$

The following theorem therefore holds:

Theorem 10. *Whenever $\mathcal{C}(|x|) = \omega(\ln|f(x)|)$, the support of the capacity-achieving input of channel (3) when $N = N_1 + N_2$ is compact. The optimal input is discrete with finite number of mass points whenever $\mathcal{C}(\cdot)$ and $f(\cdot)$ satisfy the analyticity and connectedness conditions of Theorem 3.*

Under a compact support constraint, the optimal input is also discrete with finite number of mass points whenever $f(\cdot)$ satisfies the conditions of Theorem 4.

<u>Previous work:</u> The capacity of the additive channel subjected to the compound noise $N = N_1 + N_2$ was characterized in [15] by the authors.

We plot in Figure 3 the capacity of the linear channel under an input second-moment constraint whenever $N_1 \sim \mathcal{N}(0,1)$ and $N_2 \sim \mathcal{S}(\alpha, 0, 1, 0)$ for the values of $\alpha = 1$ and 1.5 where the optimal input at 7.27dB was found to have 16 and 18 mass points respectively.

We note that the composite noise channel cannot be approximated by a Gaussian channel because the overall noise will be heavy tailed whenever the stable noise is present. Indeed, the composite noise here has infinite variance. If one where to ignore the presence of a "mild" stable noise component such as $N \sim \mathcal{S}(1, 0, 0.1, 0)$ and assume the additive noise to have only a Gaussian component, $\frac{1}{2}\ln\left(1 + \frac{\mathsf{E}[X^2]}{\sigma^2}\right) = 0.356$ nats at 0.16 dB. This is to be compared with the capacity of the composite channel which is only 0.298 nats/channel-use.
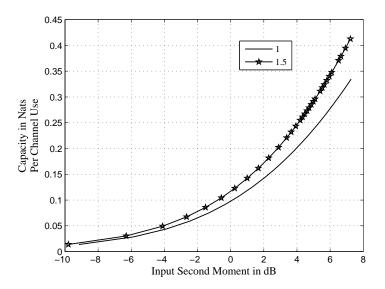

Fig. 3. Capacity of the linear channel under the composite noise: a standard Gaussian & a standard alpha-stable for $\alpha = 1$ & 1.5.

VII. CONCLUSION

We studied the problem of characterizing the capacity and its achieving distributions for additive noise channels of the form $Y = f(X) + N$, where the input is subjected to an input cost constraint of the form $\mathsf{E}[\mathcal{C}(|X|)] \leq A$, $A > 0$.

We proved that the type of the optimal input is intimately related to the growth rate at infinity of the functions $f(x)$, $\mathcal{C}(|x|)$ and $\frac{1}{p_N(x)}$ through a simple relationship. Indeed, for monotonically tailed noise density functions whenever $\mathcal{C}(|x|) = \omega\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$, the support of the optimal input is necessarily bounded. Conversely, if $\mathcal{C}(|x|) = o\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$, the support is unbounded. Similar statements are true for non-monotonically tailed PDFs with replacing $p_N(\cdot)$ by well chosen lower and upper envelopes whose tails are monotone. Furthermore, whenever some analyticity properties are satisfied by the triplet, the discrete nature of the optimal distribution is guaranteed. Discreteness holds also if additional input compact support constraints are imposed.

These results are very broad; They are consistent with a multitude of previously know capacity results, and provide solutions for a multitude of new channel models. The generalization is one to many: generic input-output functions, generic cost functions and generic noise PDFs which include a large number of well-known noise models such as the Gaussian, generalized-Gaussian, alpha-stable and their mixtures. Interestingly, the results hold for all cost functions that are $\omega(\ln|x|)$ where it is guaranteed that the channel capacity exists and is finite.

The main idea behind the proofs of the theorems is the characterization of the behavioral pattern of the KKT equation at infinity after providing lower and upper bounds on some quantities of interest. A key property is the subadditivity of the logarithm of the inverse of a lower bound on the noise PDF at large values. This was referred to as property C7 and is satisfied by all noise PDFs whose tail has a dominant polynomial or exponential component.

A direct implication of the results concerns the question on what are suitable power measures of the input signals of a communication channel. Though the question seems to be absurd when dealing with noise models with finite second moment where the natural power measure would be the standard average power –which corresponds the cost function $\mathcal{C}(|x|) = x^2$, defining a power measure when the noise second moment is infinite is deemed crucial. This is due to the fact that the natural signal-to-noise ratio would be equal to zero. Based on our results, suitable average power measures should correspond to cost functions which are "at most" $\Theta\left(\ln\left[\frac{1}{p_N(f(x))}\right]\right)$ since otherwise capacity will be achieved by an input having a finite power while the input space contains distributions having an infinite one. Hence, each channel has its suitable average power measure resulting from a suitable cost function. For example, a suitable cost behaves "at most" like $\Theta(x^2)$ for the linear Gaussian channel and has a logarithmic growth for channels with polynomially tailed additive noise.

APPENDIX I

WEAK DIFFERENTIABILITY OF $I(\cdot)$ AT F^*

Theorem 11. *Let F^* be an optimal input distribution. Under a cost constraint $\int \mathcal{C}(|X|)\,dF(x) \leq A$, $A > 0$, the mutual information $I(F)$ between the input and the output of channel (3) is weakly differentiable at F^*.*

Before proceeding to the proof, we note that the existence of an optimal F^* and the finiteness of the solution are insured as per the discussion in Section IV.

Proof: Let θ be a number in $[0,1]$, $(F^*, F) \in \mathcal{P}_A \times \mathcal{P}_A$ and define $F_\theta = (1-\theta)F^* + \theta F$. The weak derivative

of $I(.)$ at F^* in the direction of F is defined as,

$$I'(F^*, F) \triangleq \lim_{\theta \to 0^+} \frac{I(F_\theta) - I(F^*)}{\theta},$$

whenever the limit exists. For simplicity, we denote by

$$t(x) = i(x; F^*),$$

where $i(x; F)$ is given by equation (11), and we prove

$$
\begin{aligned}
I'(F^*, F) &= -\int p(y; F) \ln p(y; F^*) \, dy \;-\; h_Y(F^*) \\
&= \int t(f(x)) \, dF(x) \;-\; h_Y(F^*),
\end{aligned}
$$

where by Tonelli, the interchange is valid as long as the integral term is finite which we prove next. Using L'Hôpital's rule,

$$
\begin{aligned}
I'(F^*, F) = \lim_{\theta \to 0^+} \frac{I(F_\theta) - I(F^*)}{\theta} &= \lim_{\theta \to 0^+} \frac{h_Y(F_\theta) - h_Y(F^*)}{\theta} \\
&= \lim_{\theta \to 0^+} - \left[\int p(y; F_\theta) \ln p(y; F_\theta) \, dy \right]',
\end{aligned}
\tag{28}
$$

where the derivative is with respect to θ. In order to evaluate $\left[\int p(y; F_\theta) \ln p(y; F_\theta) \, dy \right]'$ we use the definition of the derivative

$$
\begin{aligned}
&\left[\int p(y; F_\theta) \ln p(y; F_\theta) \, dy \right]' \\
&= \lim_{h \to 0} \left[\frac{\int p(y; F_{\theta+h}) \ln p(y; F_{\theta+h}) \, dy}{h} - \frac{\int p(y; F_\theta) \ln p(y; F_\theta) \, dy}{h} \right],
\end{aligned}
$$

where by the limit we mean that both, the limit as h goes to 0^+ and the limit as h goes to 0^- exist and are equal. In what follows, we only provide detailed evaluations as h goes to 0^+ since those when h goes to 0^- are similar. Using the mean value theorem, for some $0 \le c(h) \le h$,

$$
\begin{aligned}
&\lim_{h \to 0^+} \left[\frac{\int p(y; F_{\theta+h}) \ln p(y; F_{\theta+h}) \, dy}{h} - \frac{\int p(y; F_\theta) \ln p(y; F_\theta) \, dy}{h} \right] \\
&= \lim_{h \to 0^+} \int \left. [p(y; F_\theta) \ln p(y; F_\theta)]' \right|_{\theta+c(h)} dy.
\end{aligned}
$$

Now, since $p(y; F_\theta) = p(y; F^*) + \theta \left[p(y; F) - p(y; F^*) \right]$,

$$
\begin{aligned}
&\lim_{h \to 0^+} \int \left. [p(y; F_\theta) \ln p(y; F_\theta)]' \right|_{\theta+c(h)} dy \\
&= \lim_{h \to 0^+} \int [p(y; F) - p(y; F^*)] \ln p(y; F_{\theta+c(h)}) \, dy + \int [p(y; F) - p(y; F^*)] \, dy \\
&= \int \lim_{h \to 0^+} [p(y; F) - p(y; F^*)] \ln p(y; F_{\theta+c(h)}) \, dy \tag{29} \\
&= \int [p(y; F) - p(y; F^*)] \ln p(y; F_\theta) \, dy, \tag{30}
\end{aligned}
$$

where (30) is due to the fact that $c(h) \to 0$ as $h \to 0$ and that $p(y; F_\theta)$ is continuous in θ by virtue of its linearity, and (29) is due to DCT. Indeed,

$$\left| [p(y; F) - p(y; F^*)] \ln p(y; F_{\theta+c(h)}) \right| \leq \left(p(y; F) + p(y; F^*) \right) \left| \ln p(y; F_{\theta+c(h)}) \right|,$$

and

$$p(y; F_{\theta+c(h)}) = [1 - \theta - c(h)] \, p(y; F^*) + [\theta + c(h)] \, p(y; F)$$

$$\geq [1 - \theta - c(h)] \, p(y; F^*) \geq \frac{1}{2} \, p(y; F^*),$$

whenever $\theta + c(h) \leq \frac{1}{2}$, which is true since both θ and $c(h)$ are arbitrarily small. Therefore, since $0 < p(y; F) < 1$ for all F

$$\left| [p(y; F) - p(y; F^*)] \ln p(y; F_{\theta+c(h)}) \right| \leq - \left(p(y; F) + p(y; F^*) \right) \ln \left[\frac{1}{2} \, p(y; F^*) \right].$$

Since $h_Y(F) = - \int p(y; F) \ln p(y; F) \, dy$ is finite for all F in $\mathcal{P}_A$ [40, Theorem 2], $-p(y; F^*) \ln p(y; F^*)$ is integrable. It remains to prove that $-p(y; F) \ln p(y; F^*)$ is integrable to justify (29) and hence (30). To this end, we will proceed by choosing first a specific $F(\cdot)$, namely

$$F_s(x) = \left(1 - \frac{B_s}{\mathcal{C}(x_s)} \right) u(x)^\P + \frac{B_s}{\mathcal{C}(x_s)} u(x - x_s),$$

for some $x_s > 0$ such that $\mathcal{C}(x_s) > 0$ and where $(0 <) B_s < \min \{ A; \mathcal{C}(x_s) \}$. We note that $F_s \in \mathcal{P}_A$ since $\mathcal{C}(0) = 0$ and hence $\int \mathcal{C}(|x|) dF_s = B_s \leq A$. If F_s were the input distribution, it would induce the following output

$$p(y; F_s) = \left(1 - \frac{B_s}{\mathcal{C}(x_s)} \right) p_N(y) + \frac{B_s}{\mathcal{C}(x_s)} p_N(y - f(x_s)). \tag{31}$$

Equation (31) along with lemma 1 and properties C7 and C8 show that $-p(y; F_s) \ln p(y; F^*)$ is integrable and (30) is justified for $F \equiv F_s$. Hence,

$$I'(F^*, F_s) = \lim_{\theta \to 0^+} - \left[\int p(y; F_\theta^*) \ln p(y; F_\theta^*) \, dy \right]'$$

$$= \lim_{\theta \to 0^+} \int [p(y; F_s) - p(y; F^*)] \ln p(y; F_\theta^*) \, dy = \int t(f(x)) dF_s(x) \; - \; h_Y(F^*).$$

where the interchange between the limit and integral sign is justified in an identical fashion as done to validate (30).

Now, since F^* is optimal, necessarily $I'(F^*, F_s) \leq 0$ (see Appendix C in [7]), which implies that

$$\int t(f(x)) \, dF_s(x) \leq h_Y(F^*).$$

Plugging in the expression of $F_s(x)$ yields,

$$\left(1 - \frac{B_s}{\mathcal{C}(x_s)} \right) t(f(0)) + \frac{B_s}{\mathcal{C}(x_s)} t(f(x_s)) \leq h_Y(F^*) \Leftrightarrow \quad t(f(x_s)) \leq \frac{h_Y(F^*) - t(f(0))}{B_s} \mathcal{C}(x_s) + t(f(0)). \tag{32}$$

The above equation is valid for any $x_s > 0$ (such that $\mathcal{C}(x_s) > 0$) and therefore for all $|x| \geq x_s$ since $\mathcal{C}(|x|)$ is non-decreasing in $|x|$. we proceed by writing

$$\int t(f(x)) \, dF = \int_{|x| \leq x_s} t(f(x)) \, dF + \int_{|x| > x_s} t(f(x)) \, dF.$$

¶where $u(x)$ denotes the Heaviside unit step function.

As for the first integral term, we have:

$$\int_{|x| \leq x_s} t(f(x)) \, dF$$

$$= -\int_{|x| \leq x_s} \int p_N(y - f(x)) \ln p(y; F^*) \, dy \, dF$$

$$= -\int_{|x| \leq x_s} \int_{|y| \geq y_0} p_N(y - f(x)) \ln p(y; F^*) \, dy \, dF$$

$$- \int_{|x| \leq x_s} \int_{|y| \leq y_0} p_N(y - f(x)) \ln p(y; F^*) \, dy \, dF \tag{33}$$

Using lemma 1 and property C7, the first term of equation (33) is finite. As for the second term, it is finite by the fact that $p_{(}y; F^*)$ is positive and continuous hence achieves a positive minimum on compact subsets of $\mathbb{R}$. When it comes to the range $|x| > x_s$, we use the upper bound in (32) which gives:

$$\int_{|x| > x_s} t(f(x)) \, dF \quad \leq \quad \int_{|x| > x_s} \left(\frac{h_Y(F^*) - t(f(0))}{B_u} \mathcal{C}(|x|) + t(0) \right) dF$$

$$\leq \quad \frac{h_Y(F^*) - t(f(0))}{B_u} A + t(f(0)),$$

which is finite.

In conclusion,

$$- \int p(y; F) \ln p(y; F^*) \, dy = \int t(f(x)) \, dF < \infty,$$

and $I'(F^*, F) = \int t(f(x)) \, dF - h_Y(F^*), \forall F \in \mathcal{P}_A.$ ■

Cost

The mapping from $\mathcal{F}$ to $\mathbb{R}$:

$$\mathcal{T}(F) = \int \mathcal{C}(|x|) \, dF - A$$

is weakly differentiable on $\mathcal{P}_A$ as well. In fact,

$$\mathcal{T}'(F^*, F) = \mathcal{T}(F) - \mathcal{T}(F^*),$$

which is finite, since $-A < \mathcal{T}(F) \leq 0$ for all $F \in \mathcal{P}_A$.

APPENDIX II

RATE OF DECAY OF $S(\alpha, \beta, \gamma, \delta)$ ON THE HORIZONTAL STRIP

We study in this appendix the rate of decay of alpha-stable distributions $S(\alpha, \beta, \gamma, \delta)$ on the horizontal strip $\mathcal{S}_\eta = \{z \in \mathbb{C} : |\Im(z)| < \eta\}$ where η is a small-enough positive number.

We prove in this appendix that $|p_N(z)| = O\left(\frac{1}{|\Re(z) + \delta|^{\alpha + 1}}\right)$ as $|\Re(z)| \to \infty$, whenever $N \sim S(\alpha, \beta, \gamma, \delta)$ and $z \in \mathcal{S}_\eta$. The study is limited to the case: $\alpha \in [1, 2)$, $\beta \in] -1, 1[$, $\gamma \in \mathbb{R}^{+*}$ and $\delta \in \mathbb{R}$.

Before we proceed, we first prove the following Lemma:

Lemma 7. *Whenever $N \sim S(\alpha, \beta, \gamma, \delta)$, where $\alpha \in [1, 2)$, $\beta \in\;]-1, 1[$, $\gamma \in \mathbb{R}^{+*}$ and $\delta \in \mathbb{R}$, $p_N(\cdot)$ can be formally extended on $\mathcal{S}_\eta = \{z \in \mathbb{C} : |\Im(z)| < \eta\}$ as*

$$p_N(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izt} \phi(t) dt. \tag{34}$$

Proof: By definition,

$$p_N(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \phi(t) dt,$$

where

$$
\begin{aligned}
\phi(t) &= \exp\left[i\delta t - \gamma^\alpha \left[1 - i\beta\,\mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha\right] \\
\Phi(t) &= \begin{cases} \tan\left(\dfrac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ -\dfrac{2}{\pi}\ln|t| & \alpha = 1. \end{cases}
\end{aligned}
$$

Let $p_N(z)$ be the extension of $p_N(x)$ on $\mathbb{C}$. It is known that $p_N(z)$ is analytic on $\mathcal{S}_\eta$ (see [58] for example) . Now, define

$$q(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} \phi(t) dt,$$

for all $z = (x + iy) \in \mathbb{C}$. If we establish that $q(z)$ is analytic on $\mathcal{S}_\eta$ then by the identity theorem, $p_N(z) = q(z)$, for all $z \in \mathcal{S}_\eta$. We start by proving the continuity of $q(z)$:

$$
\begin{aligned}
\lim_{z \to z_0} q(z) &= \lim_{z \to z_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} \phi(t) dt \\
&= \frac{1}{2\pi} \int \lim_{z \to z_0} e^{-izt} \phi(t) dt \\
&= \frac{1}{2\pi} \int e^{-iz_0 t} \phi(t) dt = q(z_0).
\end{aligned} \tag{35}
$$

where the interchange in (35) is justified by DCT since:

$$\left| e^{-izt} \phi(t) \right| \leq e^{yt - |\gamma t|^\alpha},$$

which is integrable on $\mathcal{S}_\eta$ since η is small-enough and chosen so that $|y| < \eta \leq \gamma^\alpha$. Now, let $\Delta \subset \mathcal{S}_\eta$ be a compact triangle and denote by $\partial\Delta$ its boundary and $|\Delta|$ its perimeter. We obtain

$$
\begin{aligned}
\int_{\partial\Delta} q(z) dz &= \frac{1}{2\pi} \int_{\partial\Delta} \int_{\mathbb{R}} e^{-izt} \phi(t)\, dt\, dz \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\partial\Delta} e^{-izt} \phi(t)\, dz\, dt \\
&= \int_{\mathbb{R}} \phi(t) \int_{\partial\Delta} e^{-izt} dz = 0,
\end{aligned} \tag{36}
$$

where the last equation is due to the fact that e^{-izt} is entire. The interchange in (36) is valid by Fubini since

$$\frac{1}{2\pi} \int_{\partial\Delta} \int_{\mathbb{R}} \left| e^{-izt} \phi(t) \right|\, dt\, dz \leq \frac{1}{2\pi} \int_{\partial\Delta} \int_{\mathbb{R}} e^{yt - |\gamma t|^\alpha}\, dt\, dz < \frac{|\Delta|}{2\pi} \int_{\mathbb{R}} e^{yt - |\gamma t|^\alpha}|\, dt < \infty.$$

By applying Morera's Theorem [48, sec. 53], $q(z)$ is analytic on $\mathcal{S}_\eta$ and the result is established. ∎

Note that equation (34) shows that $p_N(z) = p_{N'}(z - \delta)$ where $N' \sim S(\alpha, \beta, \gamma, 0)$. Therefore, and without loss of generality, we restrict our analysis in the remainder of this section to $p_N(z)$, for $N \sim S(\alpha, \beta, \gamma, 0)$.

For $z = (x + iy)$,

$$p_N(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt - \gamma^\alpha \left[1 - i\beta \, \mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt + yt - \gamma^\alpha \left[1 - i\beta \, \mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt - \gamma^\alpha \left[1 - i\beta \, \mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha} \sum_{n=0}^{\infty} \frac{y^n}{n!} t^n \, dt$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{y^n}{n!} \int_{-\infty}^{\infty} t^n e^{-ixt - \gamma^\alpha \left[1 - i\beta \, \mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha} dt. \tag{37}$$

The interchange in (37) is justified by DCT. Indeed,

$$\left| \sum_{n=0}^{N} \frac{y^n}{n!} t^n e^{-ixt - \gamma^\alpha \left[1 - i\beta \, \mathrm{sgn}(t)\Phi(t)\right]|t|^\alpha} \right| \leq \sum_{n=0}^{\infty} \frac{|y|^n}{n!} |t|^n e^{-|\gamma t|^\alpha} = e^{|y||t| - |\gamma t|^\alpha},$$

which is integrable for $|y| < \eta \ (\leq \gamma^\alpha)$ and $\alpha \geq 1$. Now we proceed to studying the rate of decay in two separate cases.

A. Rate of Decay for $1 < \alpha < 2$:

In this case $\Phi(t)$ is a constant and it is equal to $\Phi(t) = \Phi = \tan\left(\frac{\pi\alpha}{2}\right)$. Then, using equation (37), we obtain by the change of variable $u = \gamma t$

$$p_N(z) = \frac{1}{2\pi\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{y}{\gamma}\right)^n \int_{-\infty}^{\infty} t^n e^{-i\frac{x}{\gamma}t - [1 - i\beta \, \mathrm{sgn}(t)\Phi]|t|^\alpha} dt$$

$$= \frac{1}{2\pi\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{y}{\gamma}\right)^n T_n\left(-\frac{x}{\gamma}; \beta\right), \tag{38}$$

where $T_n(x; \beta)$ is a function defined as $T_n(x; \beta) \doteq \int_{-\infty}^{\infty} t^n e^{ixt - [1 - i\beta \, \mathrm{sgn}(t)\Phi]|t|^\alpha} dt$.** Define $k_1 = (1 - i\beta\Phi)$ and denote by $\overline{k}_1 = (1 + i\beta\Phi)$ its conjugate. In what follows, we study the behavior of the function $T_n(x; \beta)$.

For $n \geq 1$ and $x > 0$, we have

$$x^{n+\alpha+1} T_n(x; \beta) = x^{n+\alpha} \left[\int_0^{\infty} x t^n e^{ixt - k_1 t^\alpha} dt + (-1)^n \int_0^{\infty} x t^n e^{-ixt - \overline{k}_1 t^\alpha} dt \right]$$

$$= -i\, x^{n+\alpha} \left[\int_0^{\infty} t^n \left(e^{ixt - k_1 t^\alpha} \right)' dt + k_1 \, \alpha \int_0^{\infty} t^{n+\alpha-1} e^{ixt - k_1 t^\alpha} dt \right.$$

$$\left. + (-1)^{n-1} \int_0^{\infty} t^n \left(e^{-ixt - \overline{k}_1 t^\alpha} \right)' dt + (-1)^{n-1} \overline{k}_1 \, \alpha \int_0^{\infty} t^{n+\alpha-1} e^{-ixt - \overline{k}_1 t^\alpha} dt \right]$$

$$= in x^{n+\alpha} \left[\int_0^{\infty} t^{n-1} e^{ixt - k_1 t^\alpha} dt + (-1)^{n-1} \int_0^{\infty} t^{n-1} e^{-ixt - \overline{k}_1 t^\alpha} dt \right]$$

$$- i\alpha x^{n+\alpha} \left[k_1 \int_0^{\infty} t^{n+\alpha-1} e^{ixt - k_1 t^\alpha} dt + (-1)^{n-1} \overline{k}_1 \int_0^{\infty} t^{n+\alpha-1} e^{-ixt - \overline{k}_1 t^\alpha} dt \right] \tag{39}$$

$$= i\, n\, x^{n+\alpha} T_{n-1}(x; \beta) - i\alpha \left[k_1 S_n(x; k_1) + (-1)^{n-1} \overline{k}_1 \overline{S}_n(x; \overline{k}_1) \right], \tag{40}$$

**Note that $T_n(-x; \beta) = (-1)^n T_n(x; -\beta)$ and that $p_N^{(n)}(x) = \frac{1}{2\pi} \frac{(-i)^n}{\gamma^{n+1}} T_n(-\frac{x}{\gamma}; \beta) = \frac{1}{2\pi} \frac{i^n}{\gamma^{n+1}} T_n(\frac{x}{\gamma}; -\beta)$, $n \in \mathbb{N}^*$.

where equation (39) is obtained by integration by parts and regrouping, and where $\overline{S_n}(\cdot;\cdot)$ is the complex conjugate of $S_n(\cdot;\cdot)$ defined as,

$$S_n(x;k_1) = x^{n+\alpha}\int_0^\infty t^{n+\alpha-1}e^{ixt-k_1t^\alpha}dt = c\int_0^\infty e^{iv^c-k_1\zeta v^{\alpha c}}dv,$$

where $c = \frac{1}{n+\alpha}\,(>0)$, $\zeta = x^{-\alpha}\,(>0)$ and the change of variable is $v = (xt)^{n+\alpha}$. As $x\to\infty$, $\zeta\to 0^+$ and hence

$$\lim_{x\to+\infty}S_n(x;k_1) = c\lim_{\zeta\to 0^+}\int_0^\infty e^{iv^c-k_1\zeta v^{\alpha c}}\,dv = c\lim_{\zeta\to 0^+}\int_0^\infty \lim_{\theta\to 0}e^{iv^c e^{ic\theta}-k_1\zeta v^{\alpha c}e^{i\alpha c\theta}+i\theta}\,dv$$

$$= c\lim_{\zeta\to 0^+}\lim_{\theta\to 0}\int_0^\infty e^{iv^c e^{ic\theta}-k_1\zeta v^{\alpha c}e^{i\alpha c\theta}+i\theta}\,dv \tag{41}$$

$$= c\lim_{\theta\to 0}\lim_{\zeta\to 0^+}\int_0^\infty e^{iv^c e^{ic\theta}-k_1\zeta v^{\alpha c}e^{i\alpha c\theta}+i\theta}\,dv \tag{42}$$

$$= c\lim_{\theta\to 0}\int_0^\infty e^{iv^c e^{ic\theta}+i\theta}\,dv \tag{43}$$

$$= c\lim_{\theta\to 0}\lim_{R\to\infty,\rho\to 0}\int_{L_1} e^{iz^c}\,dz,$$

where $z = ve^{i\theta}$ and $L_1 = \{z\in\mathbb{C}: z = ve^{i\theta},\, 0 < \rho \leq v \leq R\}$. Equation (41) is justified by DCT since:

$$\left|e^{iv^c e^{ic\theta}-k_1\zeta v^{\alpha c}e^{i\alpha c\theta}+i\theta}\right| \leq e^{-v^c\sin(c\theta)-\zeta v^{\alpha c}[\cos(\alpha c\theta)+\beta\Phi\sin(\alpha c\theta)]} \leq e^{-\frac{\zeta}{2}v^{\alpha c}},$$

for small-enough θ, and the upper-bound is integrable since c and ζ are positive. The last inequality is justified by virtue that $\sin(c\theta) > 0$ and $\left[\cos(\alpha c\theta) + \beta\tan\frac{\alpha\pi}{2}\sin(\alpha c\theta)\right] > \frac{1}{2}$ for small positive θ. Similarly, (43) is justified because the integrand in (42) is $O(e^{-v^c\sin c\theta})$ as $\zeta\to 0^+$ which is also integrable. The interchange between the two limits in (42) is valid by the preceding argument as long as the result in (43) is finite. To evaluate the limit of $\int_{L_1} e^{iz^c}dz$ as $R\to\infty$, $\rho\to 0$, we use contour integration over $\mathcal{C}$ shown in Figure 4.
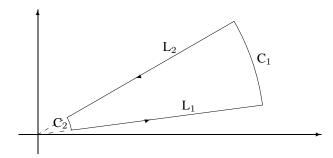

Fig. 4.   The contour $\mathcal{C}$.

The arcs C_1 and C_2 are of radius R, and ρ respectively and are between angles θ and $\varphi \overset{\triangle}{=} \frac{\pi}{2c} \bmod 2\pi$. Note that since we are interested in the limit as θ goes to zero, we can always choose it small enough in order to have the contour counter-clockwise. Finally, L_2 is a line connecting the extremities of the arcs.

Now since $f(z) = e^{iz^c}$ is analytic on and inside $\mathcal{C}$ (by choosing an appropriate branch cut in the plane), by

Cauchy's Theorem [65, p.111 Sec.2.2],

$$0 = \oint_{\mathcal{C}} f(z)\,dz = \int_{L_1} f(z) + \int_{C_1} f(z) + \int_{L_2} f(z) + \int_{C_2} f(z).$$

On C_1, we have:

$$\lim_{R\to\infty} \left| \int_{C_1} f(z)dz \right| = \lim_{R\to\infty} \left| \int_\theta^\varphi iRe^{i\phi}e^{iR^c e^{ic\phi}}\,d\phi \right| \le \lim_{R\to\infty} \int_\theta^\varphi Re^{-R^c \sin(c\phi)}d\phi = \int_\theta^\varphi \lim_{R\to\infty} Re^{-R^c \sin(c\phi)}d\phi = 0,$$

where the interchange is valid because $Re^{-R^c \sin(c\phi)}$ is decreasing as $0 < c\theta \le c\phi \le \frac{\pi}{2}$. Similarly, on C_2,

$$\lim_{\rho\to 0} \left| \int_{C_2} f(z)dz \right| = \lim_{\rho\to 0} \left| \int_\theta^\varphi i\rho e^{i\phi}e^{i\rho^c e^{ic\phi}}\,d\phi \right| \le \lim_{\rho\to 0} \int_\theta^\varphi \rho e^{-\rho^c \sin(c\phi)}d\phi = \int_\theta^\varphi \lim_{\rho\to 0} \rho e^{-\rho^c \sin(c\phi)}d\phi = 0,$$

where we justify the interchange by virtue of the fact that $\rho e^{-\rho^c \sin(c\phi)}$ is bounded for small values of ρ. It remains to evaluate the integral on L_2 where $z = te^{i\frac{\pi}{2c}}$,

$$\lim_{R\to\infty,\rho\to 0} \int_{L_2} f(z)dz = -\int_0^\infty e^{i\frac{\pi}{2c}}e^{it^c e^{i\frac{\pi}{2}}}\,dt = -e^{i\frac{\pi}{2c}}\int_0^\infty e^{-t^c}\,dt = -e^{i\frac{\pi}{2c}}\frac{1}{c}\Gamma\left(\frac{1}{c}\right).$$

In conclusion,

$$\lim_{R\to\infty,\rho\to 0} \int_{L_1} f(z)\,dz = e^{i\frac{\pi}{2c}}\frac{1}{c}\Gamma\left(\frac{1}{c}\right),$$

which implies that

$$\lim_{x\to+\infty} S_n(x;k_1) = e^{i\frac{\pi}{2}(n+\alpha)}\Gamma(n+\alpha),$$

and by (40), we can write for $n \ge 1$

$$\lim_{x\to+\infty} \left[x^{n+\alpha+1}T_n(x;\beta) - i\,nx^{n+\alpha}T_{n-1}(x;\beta) \right]$$
$$= W_n(\beta) \;\hat{=}\; -i\alpha\Gamma(n+\alpha)\left[k_1 e^{i\frac{\pi}{2}(n+\alpha)} + (-1)^{n-1}\overline{k}_1 e^{-i\frac{\pi}{2}(n+\alpha)} \right],$$

which implies that $U_n(\beta) \;\hat{=}\; \lim_{x\to+\infty} x^{n+\alpha+1}T_n(x;\beta)$ is a well defined quantity because

$$U_0(\beta) = \lim_{x\to+\infty} \left[x^{\alpha+1}T_0(x;\beta) \right] = 2\pi\gamma \lim_{x\to+\infty} \left[x^{\alpha+1}p_N(-\gamma x) \right],$$

exists –and is non zero for $\beta \ne 1$ and $U_0(1) = 0$, and

$$U_n(\beta) = inU_{n-1}(\beta) + W_n(\beta) = n!\left[i^n\, U_0(\beta) + \sum_{k=0}^{n-1} \frac{i^k}{(n-k)!}W_{n-k}(\beta) \right].$$

Furthermore, for $n \ge 0$,

$$
\begin{aligned}
|U_n(\beta)| &\le n!\left[|U_0(\beta)| + \sum_{k=0}^{n-1} \frac{|W_{n-k}(\beta)|}{(n-k)!} \right] \le n!\left[|U_0(\beta)| + 2\alpha|k_1|\sum_{k=0}^{n-1} \frac{\Gamma(n+\alpha-k)}{(n-k)!} \right] \\
&\le n!\left[|U_0(\beta)| + 4|k_1|\sum_{k=0}^{n-1} \frac{\Gamma(n+2-k)}{(n-k)!} \right] \qquad\qquad (44)\\
&= n!\left[|U_0(\beta)| + 4|k_1|\sum_{k=0}^{n-1} (n+1-k) \right] = 2\,n!\left(|k_1|n^2 + 3|k_1|n + \frac{|U_0(\beta)|}{2} \right),
\end{aligned}
$$

where equation (44) is justified using the fact that $0 < \alpha < 2$ and $\Gamma(\alpha+l)$ is increasing in $\alpha > 0$ for $l \in \mathbb{N}^*$.

Now using equation (38),

$$
\begin{aligned}
\lim_{x \to \infty} x^{\alpha+1} \left| p_N(z) \right| &= \frac{1}{2\pi\gamma} \lim_{x \to \infty} x^{\alpha+1} \left| \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{y}{\gamma} \right)^n T_n \left(-\frac{x}{\gamma}; \beta \right) \right| \\
&= \frac{1}{2\pi\gamma} \left| \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{y}{\gamma} \right)^n \lim_{x \to \infty} x^{\alpha+1} T_n \left(-\frac{x}{\gamma}; \beta \right) \right| \\
&\leq \frac{1}{2\pi\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{y}{\gamma} \right|^n \lim_{x \to \infty} x^{\alpha+1} \left| T_n \left(\frac{x}{\gamma}; -\beta \right) \right| \\
&\leq \frac{1}{2\pi\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{y}{\gamma} \right|^n \lim_{x \to \infty} x^{n+\alpha+1} \left| T_n \left(\frac{x}{\gamma}; -\beta \right) \right| = \frac{1}{2\pi\gamma} \sum_{n=0}^{\infty} \frac{1}{n!} \left| \frac{y}{\gamma} \right|^n \gamma^{n+\alpha+1} \left| U_n(-\beta) \right| \\
&\leq \frac{\gamma^\alpha}{\pi} \sum_{n=0}^{\infty} |y|^n \left(|k_1| n^2 + 3|k_1| n + \frac{|U_0(-\beta)|}{2} \right),
\end{aligned}
\tag{45}
$$

which is finite because $|y| < \eta$ which is small-enough (and assumed to be less than one), and where we used the fact that $f(x) = |x|$ is continuous. The interchange in (45) is valid because the end result is finite.

In conclusion, $\lim_{x \to +\infty} x^{\alpha+1} \left| p_N(z) \right| < \infty$ which concludes our proof.

B. Rate of Decay for $\alpha = 1$:

In this case, $\Phi(t) = -\frac{2}{\pi} \log |t|$ is a function of t. According to equation (37) and for $z = x + iy$,

$$
p_N(z) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{y^n}{n!} \int_{-\infty}^{\infty} t^n e^{-ixt - \gamma \left[1 - i\beta \, \mathrm{sgn}(t) \Phi(t) \right] |t|} \, dt.
\tag{46}
$$

Once more, we study the behavior of the integral $I_n(x) = \int_{-\infty}^{\infty} t^n e^{-ixt - \gamma(1 - i\beta \, \mathrm{sgn}(t) \Phi)|t|} \, dt$ as $x \to \infty$ for $n \geq 0$.

We note that $I_0(x) = 2\pi p_N(x; 1, \beta, \gamma, 0)$ which is $\Theta\left(\frac{1}{x^2}\right)$. For $n \geq 1$,

$$
\begin{aligned}
I_n(x) &= \int_{-\infty}^{+\infty} t^n e^{-ixt - \gamma(1 - i\beta \operatorname{sgn}(t)\Phi)|t|} \, dt \\
&= \int_0^{+\infty} t^n e^{-ixt - \gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \, dt + \int_{-\infty}^0 t^n e^{-ixt - \gamma\left(1 - i\frac{2}{\pi}\beta \log(-t)\right)(-t)} \, dt \\
&= \int_0^{+\infty} e^{-ixt} t^n e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \, dt + (-1)^n \int_0^{+\infty} e^{ixt} t^n e^{-\gamma\left(1 - i\frac{2}{\pi}\beta \log(t)\right)t} \, dt \\
&= \left[-\frac{1}{ix} e^{-ixt} t^n e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \right]_0^{+\infty} + (-1)^n \left[\frac{1}{ix} e^{ixt} t^n e^{-\gamma\left(1 - i\frac{2}{\pi}\beta \log(t)\right)t} \right]_0^{+\infty} \\
&\quad + \frac{1}{ix} \int_0^{+\infty} e^{-ixt} \left[nt^{n-1} - \gamma t^n - i\frac{2}{\pi}\beta\gamma t^n - i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \, dt \\
&\quad + \frac{(-1)^{n+1}}{ix} \int_0^{+\infty} e^{ixt} \left[nt^{n-1} - \gamma t^n + i\frac{2}{\pi}\beta\gamma t^n + i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 - i\frac{2}{\pi}\beta \log(t)\right)t} \, dt \qquad (47) \\
&= \frac{1}{ix} \int_0^{+\infty} e^{-ixt} \left[nt^{n-1} - \gamma t^n - i\frac{2}{\pi}\beta\gamma t^n - i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \, dt \\
&\quad + \frac{(-1)^{n+1}}{ix} \int_0^{+\infty} e^{ixt} \left[nt^{n-1} - \gamma t^n + i\frac{2}{\pi}\beta\gamma t^n + i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 - i\frac{2}{\pi}\beta \log(t)\right)t} \, dt \\
&= \left[\frac{1}{x^2} e^{-ixt} \left[nt^{n-1} - \gamma t^n - i\frac{2}{\pi}\beta\gamma t^n - i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t} \right]_0^{+\infty} \\
&\quad - \frac{1}{x^2} \int_0^{+\infty} e^{-ixt} g_n(t) \, dt \\
&\quad + (-1)^{n+1} \left[-\frac{1}{x^2} e^{ixt} \left[nt^{n-1} - \gamma t^n + i\frac{2}{\pi}\beta\gamma t^n + i\frac{2}{\pi}\beta\gamma t^n \log(t) \right] e^{-\gamma\left(1 - i\frac{2}{\pi}\beta \log(t)\right)t} \right]_0^{+\infty} \\
&\quad + \frac{(-1)^{n+1}}{x^2} \int_0^{+\infty} e^{ixt} h_n(t) \, dt \qquad (48) \\
&= \frac{1}{x^2} \left((-1)^{n+1} \int_0^{+\infty} e^{ixt} h_n(t) \, dt - \int_0^{+\infty} e^{-ixt} g_n(t) \, dt \right) \qquad (49)
\end{aligned}
$$

where equations (47) and (48) are due to integration by parts. The functions $g_n(\cdot)$ and $h_n(\cdot)$, $n \geq 1$ are defined on $\mathbb{R}^{+*}$ and are given by:

$$
\begin{aligned}
g_n(t) &= \left[n(n-1)t^{n-2} - 2n\gamma t^{n-1} + \left(\gamma^2 - \frac{4}{\pi^2}\beta^2\gamma^2 + i\frac{4}{\pi}\beta\gamma^2\right)t^n + \left(-\frac{8}{\pi^2}\beta^2\gamma^2 + i\frac{4}{\pi}\beta\gamma^2\right)t^n \log(t) \right. \\
&\quad \left. - i\frac{2}{\pi}(2n+1)\beta\gamma t^{n-1} - i\frac{4}{\pi}n\beta\gamma t^{n-1} \log(t) - \frac{4}{\pi^2}\beta^2\gamma^2 t^n \log^2(t) \right] e^{-\gamma\left(1 + i\frac{2}{\pi}\beta \log(t)\right)t}. \qquad (50)
\end{aligned}
$$

The term $n(n-1)t^{n-2}$ is equal to zero when $n = 1$ and $h_n(t)$ is deduced from $g_n(t)$ by replacing β by $-\beta$. The functions $g_n(t)$, $h_n(t)$ are $\mathbb{L}^1(\mathbb{R}^+)$ functions and hence by Riemann-Lebesgue [66, p.3 sec.2 th.1] their $\mathbb{L}^1(\mathbb{R}^+)$ Fourier transforms are $o(1)$. Therefore equation (49) is $o(\frac{1}{x^2})$. Equivalently, $I_n(x) = o(\frac{1}{x^2})$ as $x \to \infty$ for all

$n \geq 1$. Now using equation (46) we obtain:

$$\lim_{x \to \infty} 2\pi x^2 |p_N(z)|$$

$$= \lim_{x \to \infty} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} x^2 I_n(x) \right|$$

$$= \lim_{x \to \infty} \left| 2\pi x^2 p_N(x) + \sum_{n=1}^{\infty} \frac{y^n}{n!} \left((-1)^{n+1} \int_0^{+\infty} e^{ixt} h_n(t)\, dt - \int_0^{+\infty} e^{-ixt} g_n(t)\, dt \right) \right|$$

$$\leq \lim_{x \to \infty} 2\pi x^2 p_N(x) + \lim_{x \to \infty} \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \left(\int_0^{+\infty} |h_n(t)|\, dt + \int_0^{+\infty} |g_n(t)|\, dt \right)$$

$$= \lim_{x \to \infty} 2\pi x^2 p_N(x) + \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \int_0^{+\infty} (|h_n(t)| + |g_n(t)|)\, dt$$

$$= \lim_{x \to \infty} 2\pi x^2 p_N(x) + \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{|y|^n}{n!} (|h_n(t)| + |g_n(t)|)\, dt \tag{51}$$

The interchange in (51) is valid since:

$$\sum_{n=1}^{\infty} \frac{|y|^n}{n!} (|h_n(t)| + |g_n(t)|)$$

$$\leq \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \left[A_1 n(n-1)t^{n-2} + A_2 t^n + A_3 t^n |\log(t)| + A_4 (2n+1) t^{n-1} \right.$$

$$\left. + A_5 n t^{n-1} |\log(t)| + A_6 t^n \log^2(t) \right] e^{-\gamma t}$$

$$\leq e^{-\gamma t} \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \left[A_1 n(n-1)t^{n-2} + (A_2 + A_3 |\log(t)| + A_6 \log^2(t)) t^n \right.$$

$$\left. + n(3A_4 + A_5 |\log(t)|) t^{n-1} \right] \tag{52}$$

$$\leq e^{-\gamma t} \left[A_1 y^2 e^{|y|t} + (A_2 + A_3 |\log(t)| + A_6 \log^2(t))(e^{|y|t} - 1) + |y|(3A_4 + A_5) e^{|y|t} \right]$$

$$\leq e^{-(\gamma - |y|)t} \left[A_2 + (3A_4 + A_5)|y| + A_1 y^2 + A_3 |\log(t)| + A_6 \log^2(t) \right]$$

which is integrable on $[0, +\infty[$ since $|y| < \eta\, (< \gamma)$. The A_is, $1 \leq i \leq 6$ are positive constants function of β, γ and can be derived from the expression of $g_n(t)$ (equation 50) and from that of $h_n(t)$ accordingly after taking the norm of each term in those expressions. To write equation (52), we used the obvious inequality $2n + 1 \leq 3n$ whenever $n \geq 1$. Back to (51),

$$\lim_{x \to \infty} 2\pi x^2 |p_N(z)| \leq \lim_{x \to \infty} 2\pi x^2 p_N(x) + \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{|y|^n}{n!} (|h_n(t)| + |g_n(t)|)\, dt$$

$$\leq \lim_{x \to \infty} 2\pi x^2 p_N(x) + \int_0^{+\infty} l(t)\, dt$$

where $l(t) = e^{-(\gamma - |y|)t} \left[A_2 + (3A_4 + A_5)|y| + A_1 y^2 + A_3 |\log(t)| + A_6 \log^2(t) \right]$. Since $\lim_{x \to \infty} 2\pi x^2 p_N(x)$ and $\int_0^{+\infty} l(t)\, dt$ are both finite and non zero when $|y| < \eta\, (< \gamma)$, then $0 \leq \lim_{x \to \infty} 2\pi x^2 |p_N(z)| < \infty$ and $|p_N(z)| = O\left(\frac{1}{|\Re(z)|^2} \right)$ as $\Re(z) \to \infty$ whenever $z \in \mathcal{S}_\eta$.

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