# Boundary homogenization for a triharmonic intermediate problem

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#### Abstract

We consider the triharmonic operator subject to homogeneous boundary conditions of intermediate type on a bounded domain of the N-dimensional Euclidean space. We study its spectral behaviour when the boundary of the domain undergoes a perturbation of oscillatory type. We identify the appropriate limit problems which depend on whether the strength of the oscillation is above or below a critical threshold. We analyse in detail the critical case which provides a typical homogenization problem leading to a strange boundary term in the limit problem.

### 1 Introduction

Given a sufficiently regular bounded domain  $\Omega$  in  $\mathbb{R}^N$  with  $N \geq 2$  and  $f \in L^2(\Omega)$ , we consider the following boundary value problem

$$\begin{cases}
-\Delta^3 u + u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \\
\frac{\partial^3 u}{\partial \nu^3} = 0, & \text{on } \partial\Omega,
\end{cases} \tag{1}$$

where  $\nu$  denotes the unit outer normal to  $\partial\Omega$ . The variational weak formulation of problem (1) reads

$$\int_{\Omega} \left( D^3 u : D^3 \varphi + u \varphi \right) dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega), \tag{2}$$

in the unknown  $u \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$ . Here  $D^3v = \{\frac{\partial^3 v}{\partial x_i \partial x_j \partial_k}\}_{i,j,k=1,2,3}$  denotes the set of all derivatives of order three of a function v and  $D^3u:D^3\varphi=\sum_{i,j,k=1}^3 \frac{\partial^3 u}{\partial x_i \partial x_j \partial_k} \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial_k}$  is the usual Frobenius product. Moreover,  $W^{k,2}(\Omega)$  denotes the Sobolev space of functions in  $L^2(\Omega)$  with weak derivatives in  $L^2(\Omega)$  up to order k endowed with its standard norm, and  $W_0^{k,2}(\Omega)$  the closure in  $W^{k,2}(\Omega)$  of the space  $C_c^\infty(\Omega)$  of smooth functions with compact support in  $\Omega$ . We note that the first two boundary conditions in (1) are encoded in the condition  $u \in W_0^{2,2}(\Omega)$ , while the third boundary condition in (1) is the natural boundary condition arising from integrating by parts the left-hand side in (2), see e.g., [14, Chp. 1, Prop. 2.4].

Recall that in the classical Dirichlet problem for the triharmonic operator, the boundary condition  $\frac{\partial^3 u}{\partial \nu^3} = 0$  in (1) is replaced by the condition  $\frac{\partial^2 u}{\partial \nu^2} = 0$  and the corresponding weak problem can be formulated exactly as in (2) with  $W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$  replaced by  $W_0^{3,2}(\Omega)$ . We note that using the energy space  $W^{3,2}(\Omega)$  in (2) rather than  $W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$  would lead to a Neumann-type boundary value problem in the same spirit of standard Neumann problems for the Laplace or the biharmonic operator. Thus, since the energy space  $W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$  used in (2) satisfies the inclusions  $W_0^{3,2}(\Omega) \subset W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega) \subset W^{3,2}(\Omega)$ , we refer to problem (1) as an intermediate problem. For an introduction to the theory of polyharmonic operators we refer to the extensive monograph [11].

By standard operator theory, problems (1) and (2) can be recast in the form

$$H_{\Omega}u = f,\tag{3}$$

where  $H_{\Omega}$  is a positive self-adjoint operator densely defined in  $L^2(\Omega)$  such that the domain  $\operatorname{Dom}(H_{\Omega}^{1/2})$  of its square root  $H_{\Omega}^{1/2}$  is  $W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$  and such that  $< H_{\Omega}^{1/2}u, H_{\Omega}^{1/2}\varphi>_{L^2(\Omega)} = \int_{\Omega} D^3u: D^3\varphi + u\varphi dx$  for all  $u, \varphi \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$ . Moreover,  $u \in \operatorname{Dom}(H_{\Omega})$  if and only if  $u \in \operatorname{Dom}(H_{\Omega}^{1/2})$  and there exists  $f \in L^2(\Omega)$  such that  $< H_{\Omega}^{1/2}u, H_{\Omega}^{1/2}\varphi>_{L^2(\Omega)} = < f, \varphi>_{L^2(\Omega)}$  for all  $\varphi \in \operatorname{Dom}(H_{\Omega}^{1/2})$ , in which case  $H_{\Omega}u = f$ . If  $\Omega$  is sufficiently regular (a Lipschitz continuous boundary is enough) then the resolvent  $H_{\Omega}^{-1}$  is compact, hence the spectrum of  $H_{\Omega}$  is discrete. Formally, the operator  $H_{\Omega}$  can be identified with the classical operator  $-\Delta^3 + \mathbb{I}$  subject to the boundary conditions in (1)

In this paper, we continue the analysis addressed in [2, 3] for the case of the biharmonic operator, and we study the compact convergence of the resolvent operators  $H_{\Omega_{\epsilon}}^{-1}$  defined on suitable families of domains  $\{\Omega_{\epsilon}\}_{\epsilon>0}$  approaching a fixed domain  $\Omega$  as  $\epsilon \to 0$ . As in [2, 3], in order to simplify the setting, we suppose that  $\Omega = W \times (-1,0)$  where W is a sufficiently regular bounded domain of  $\mathbb{R}^{N-1}$ ,  $\Omega_{\epsilon} = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < g_{\epsilon}(\bar{x})\}$  where  $g_{\epsilon}(\bar{x}) = \epsilon^{\alpha} b(\bar{x}/\epsilon)$  for all  $\bar{x} \in W$ , and b is a Y-periodic smooth function with  $Y = (-1/2, 1/2)^{N-1}$ .

Compact convergence is a standard notion in functional analysis and is equivalent to the convergence in operator norm in the case of self-adjoint operators defined on a fixed Hilbert space. In our case, the underlying Hilbert space is the space  $L^2(\Omega_{\epsilon})$  which depends on  $\epsilon$ . This leads to a number of technical difficulties which can be overcome by using the notion of  $\mathcal{E}$ -compact convergence where  $\mathcal{E}$  denotes an operator which allows to pass from the reference Hilbert space  $L^2(\Omega)$  to the other Hilbert spaces  $L^2(\Omega_{\epsilon})$ . In our setting,  $\mathcal{E}$  is just the extension-by-zero operator which can be thought as an operator from  $L^2(\Omega)$  to  $L^2(\Omega_{\epsilon})$  defined by  $\mathcal{E}u = u_0|\Omega_{\epsilon}$  for all  $u \in L^2(\Omega)$ , where  $u_0$  is the function defined by u in u and zero outside u. For the convenience of the reader we recall the following definition (see e.g., [1] and the references therein).

**Definition 1.** i) We say that  $v_{\epsilon} \in L^{2}(\Omega_{\epsilon})$   $\mathcal{E}$ -converges to  $v \in L^{2}(\Omega)$  if  $||v_{\epsilon} - \mathcal{E}v||_{L^{2}(\Omega_{\epsilon})} \to 0$  as  $\epsilon \to 0$ . We write this as  $v_{\epsilon} \stackrel{\mathcal{E}}{\to} v$ . ii) The family of bounded linear operators  $B_{\epsilon} \in \mathcal{L}(L^{2}(\Omega_{\epsilon}))$   $\mathcal{E}\mathcal{E}$ - converges to  $B \in \mathcal{L}(L^{2}(\Omega))$  if  $B_{\epsilon}v_{\epsilon} \stackrel{\mathcal{E}}{\to} Bv$  whenever  $v_{\epsilon} \stackrel{\mathcal{E}}{\to} v$ . We write this as  $B_{\epsilon} \stackrel{\mathcal{E}\mathcal{E}}{\to} B$ . iii) The family of bounded linear and compact operators  $B_{\epsilon} \in \mathcal{L}(L^{2}(\Omega_{\epsilon}))$   $\mathcal{E}$ -compact converges to  $B \in \mathcal{L}(L^{2}(\Omega))$  if  $B_{\epsilon} \stackrel{\mathcal{E}\mathcal{E}}{\to} B$  and for any family of functions  $v_{\epsilon} \in L^{2}(\Omega_{\epsilon})$  with  $\|v_{\epsilon}\|_{L^{2}(\Omega_{\epsilon})} \leq 1$  there exists a subsequence, denoted by  $v_{\epsilon}$  again, and a function  $w \in L^{2}(\Omega)$  such that  $B_{\epsilon}v_{\epsilon} \stackrel{\mathcal{E}}{\to} w$ . We write  $B_{\epsilon} \stackrel{\mathcal{C}}{\to} B$ .

We note that the  $\mathcal{E}$ -compact convergence of the resolvent operators  $H_{\Omega_{\epsilon}}^{-1}$  implies not only the convergence of the solutions  $u_{\epsilon}$  of the Poisson problems  $H_{\Omega_{\epsilon}}u_{\epsilon}=f$  but also the convergence of the eigenvalues and eigenfunctions of the operators  $H_{\Omega_{\epsilon}}$ .

Our main result is the following theorem which can be considered as the triharmonic analogue of [3, Theorem 7.3] concerning a problem somewhat close to the so-called Babuska Paradox for the biharmonic operator (see also the important contributions to this subject in [12, 13]). Here and in the sequel the part of the boundary of  $\Omega$  given by  $W \times \{0\}$  is denoted by W.

**Theorem 1.** With the notation above, the following statements hold true.

- (i) [Spectral stability] If  $\alpha > 3/2$ , then  $H_{\Omega_{\epsilon}}^{-1} \stackrel{\mathcal{C}}{\to} H_{\Omega}^{-1}$ .
- (ii) [Strange term] If  $\alpha = 3/2$ , then  $H_{\Omega_{\epsilon},I}^{-1} \stackrel{\mathcal{C}}{\to} \hat{H}_{\Omega}^{-1}$ , where  $\hat{H}_{\Omega}$  is the operator  $-\Delta^3 + \mathbb{I}$  with intermediate boundary conditions on  $\partial\Omega \setminus W$  and the following boundary conditions on  $W: u = \frac{\partial u}{\partial x_n} = \frac{\partial^3 u}{\partial x_n^3} K \frac{\partial^2 u}{\partial x_n^2} = 0$ , where the factor K is given by

$$K = \int_{Y \times (-\infty,0)} |D^3 V|^2 dy = -\int_Y \left( \Delta \left( \frac{\partial^2 V}{\partial y_N^2} \right) + 2\Delta_{N-1} \left( \frac{\partial^2 V}{\partial y_N^2} \right) \right) b(\bar{y}) d\bar{y}, \tag{4}$$

 $\Delta_{N-1}$  denotes the Laplacian in the first N-1 variables, and V is a function, Y-periodic in the variables  $\bar{y}$ , satisfying the following microscopic problem

$$\begin{cases} \Delta^3 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial V}{\partial y_N}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3}(\bar{y}, 0) = 0, & \text{on } Y. \end{cases}$$

(iii) [Degeneration] If  $0 < \alpha < 3/2$  and b is non-constant, then  $H_{\Omega_{\epsilon}}^{-1} \xrightarrow{\mathcal{C}} H_{\Omega,D}^{-1}$ , where  $H_{\Omega,D}$  is the operator  $-\Delta^3 + \mathbb{I}$  with intermediate boundary conditions on  $\partial \Omega \setminus W$  and classical Dirichlet boundary conditions on W, namely  $u = \frac{\partial u}{\partial x_n} = \frac{\partial^2 u}{\partial x_n^2} = 0$  on W.

We note that the analysis of the cases  $\alpha \leq 3/2$  is in spirit of the paper [9] which is devoted to the Navier-Stokes system. For recent results concerning domain perturbation problems for higher order operators we refer to [4, 5, 6, 7, 8].

## 2 Proof of Theorem 1

In this section we provide the proof the Theorem 1. For simplicity, we shall always assume that  $b \geq 0$ , hence  $\Omega \subset \Omega_{\epsilon}$  for all  $\epsilon > 0$ .

Let  $f_{\epsilon} \in L^{2}(\Omega_{\epsilon})$  and  $f \in L^{2}(\Omega)$  be such that  $||f_{\epsilon}||_{L^{2}(\Omega_{\epsilon})}$  is uniformly bounded and  $f_{\epsilon} \rightharpoonup f$  in  $L^{2}(\Omega)$  as  $\epsilon \to 0$ . Let  $v_{\epsilon} \in W^{3,2}(\Omega_{\epsilon}) \cap W_{0}^{2,2}(\Omega_{\epsilon})$  be such that

$$H_{\Omega_{\epsilon},I}v_{\epsilon} = f_{\epsilon}, \qquad (5)$$

for all  $\epsilon > 0$  small enough. We plan to pass to the limit in (5) as  $\epsilon \to 0$  and prove that the limit problem is as in Theorem 1. Clearly,  $\|v_{\epsilon}\|_{W^{3,2}(\Omega_{\epsilon})} \leq M$  for all  $\epsilon > 0$  sufficiently small, hence, possibly passing to a subsequence, there exists  $v \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$  such that  $v_{\epsilon} \rightharpoonup v$  in  $W^{3,2}(\Omega)$  and  $v_{\epsilon} \to v$  in  $L^2(\Omega)$ . In order to use the weak formulation of problem (5), we need to define a suitable test function in  $\Omega_{\epsilon}$  starting from a given test function in  $\Omega$ . Following the approach in [3], this is done by means of an appropriate pullback operator. Namely, we consider a diffeomorphism  $\Phi_{\epsilon}$  from  $\Omega_{\epsilon}$  to  $\Omega$  defined by  $\Phi_{\epsilon}(\bar{x}, x_N) = (\bar{x}, x_N - h_{\epsilon}(\bar{x}, x_N))$  for all  $(\bar{x}, x_N) \in \Omega_{\epsilon}$  where

$$h_{\epsilon}(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\epsilon, \\ g_{\epsilon}(\bar{x}) \left(\frac{x_N + \epsilon}{g_{\epsilon}(\bar{x}) + \epsilon}\right)^4, & \text{if } -\epsilon \le x_N \le g_{\epsilon}(\bar{x}). \end{cases}$$

The map  $\Phi_{\epsilon}$  is a diffeomorphism of class  $C^3$ , even though the highest order derivatives may not be uniformly bounded as  $\epsilon \to 0$ . Note in particular that there exists a constant c > 0 independent of  $\epsilon$  such that for all  $\epsilon > 0$  small enough we have

$$|h_{\epsilon}| \le c\epsilon^{\alpha}, \quad \left|\frac{\partial h_{\epsilon}}{\partial x_{i}}\right| \le c\epsilon^{\alpha-1}, \quad \left|\frac{\partial^{2} h_{\epsilon}}{\partial x_{i} \partial x_{j}}\right| \le c\epsilon^{\alpha-2}, \quad \left|\frac{\partial^{3} h_{\epsilon}}{\partial x_{i} \partial x_{j} \partial x_{k}}\right| \le c\epsilon^{\alpha-3}.$$

Then we consider the pullback operator  $T_{\epsilon}$  from  $L^2(\Omega)$  to  $L^2(\Omega_{\epsilon})$  defined by  $T_{\epsilon}\varphi = \varphi \circ \Phi_{\epsilon}$ , for all  $\varphi \in L^2(\Omega)$ .

 $\varphi \circ \Phi_{\epsilon}$ , for all  $\varphi \in L^{2}(\Omega)$ . Let  $\varphi \in W^{3,2}(\Omega) \cap W_{0}^{2,2}(\Omega)$  be fixed. Since  $T_{\epsilon}\varphi \in W^{3,2}(\Omega_{\epsilon}) \cap W_{0}^{2,2}(\Omega_{\epsilon})$ , by (5) we get

$$\int_{\Omega_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x + \int_{\Omega_{\epsilon}} v_{\epsilon} T_{\epsilon} \varphi \, \mathrm{d}x = \int_{\Omega_{\epsilon}} f_{\epsilon} T_{\epsilon} \varphi \, \mathrm{d}x, \tag{6}$$

and passing to the limit as  $\epsilon \to 0$  we have that

$$\int_{\Omega_{\epsilon}} v_{\epsilon} T_{\epsilon} \varphi \, \mathrm{d}x \to \int_{\Omega} v \varphi \, \mathrm{d}x, \quad \int_{\Omega_{\epsilon}} f_{\epsilon} T_{\epsilon} \varphi \, \mathrm{d}x \to \int_{\Omega} f \varphi \, \mathrm{d}x. \tag{7}$$

Now we consider the first integral in the left-hand side of (6) and set  $K_{\epsilon} = W \times (-1, -\epsilon)$ . By splitting the integral in three terms corresponding to  $\Omega_{\epsilon} \setminus \Omega$ ,  $\Omega \setminus K_{\epsilon}$  and  $K_{\epsilon}$  and by arguing as in [3, Section 8.3] one can show that

$$\int_{K_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x \to \int_{\Omega} D^3 v : D^3 \varphi \, \mathrm{d}x, \quad \int_{\Omega_{\epsilon} \setminus \Omega} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x \to 0, \tag{8}$$

as  $\epsilon \to 0$ . Hence, it remains to analyse the behaviour of the term  $\int_{\Omega \setminus K_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x$ . We distinguish now the three cases.

Case  $\alpha > 3/2$ . In this case, one can prove that  $\int_{\Omega \setminus K_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, dx \to 0$ . Thus, by combining the previous limit relations, we get  $\int_{\Omega} D^3 v : D^3 \varphi \, dx + \int_{\Omega} v \varphi \, dx = \int_{\Omega} f \varphi \, dx$ , which proves statement (i).

Case  $\alpha=3/2$ . In this case, the problem is more complicated and the proof of statement (ii) will follow from Theorems 2 and 4. The proofs of such theorems are based on the unfolding method (see e.g., [10]). We recall now a few notions from homogenization theory. For any  $k\in\mathbb{Z}^{N-1}$  and  $\epsilon>0$  we consider the small cell  $C^k_{\epsilon}=\epsilon(k+Y)$ , where as above  $Y=\left(-\frac{1}{2},\frac{1}{2}\right)^{N-1}$ . Let  $I_{W,\epsilon}=\{k\in\mathbb{Z}^{N-1}:C^k_{\epsilon}\subset W\}$  and  $\widehat{W}_{\epsilon}=\bigcup_{k\in I_{W,\epsilon}}C^k_{\epsilon}$ . We set  $Q_{\epsilon}=\widehat{W}_{\epsilon}\times(-\epsilon,0)$  and we split again the remaining integral in two summands, namely

$$\int_{\Omega \setminus K_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x = \int_{\Omega \setminus (K_{\epsilon} \cup Q_{\epsilon})} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x + \int_{Q_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \varphi \, \mathrm{d}x. \quad (9)$$

Arguing as in [3, Section 8.3] we get that  $\int_{\Omega\setminus (K_\epsilon\cup Q_\epsilon)} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \to 0$  as  $\epsilon \to 0$ . Thus, it remains to study the limiting behaviour of the last summand in the right hand-side of (9) and this is done by unfolding it. We recall the following

**Definition 2.** Let u be a measurable real-valued function defined in  $\Omega$ . For any  $\epsilon > 0$  sufficiently small the unfolding  $\hat{u}$  of u is the function defined on  $\widehat{W}_{\epsilon} \times Y \times (-1/\epsilon, 0)$  by  $\hat{u}(\bar{x}, \bar{y}, y_N) = u\left(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon y, \epsilon y_N\right)$ , for almost all  $(\bar{x}, \bar{y}, y_N) \in \widehat{W}_{\epsilon} \times Y \times (-1/\epsilon, 0)$ , where  $\left[\frac{\bar{x}}{\epsilon}\right]$  denotes the integer part of the vector  $\bar{x}\epsilon^{-1}$  with respect to Y, i.e.,  $\bar{x}\epsilon^{-1} = k$  if and only if  $\bar{x} \in C_{\epsilon}^k$ .

The unfolding operator allows to 'unfold' integrals by means of the well-known exact integration formula which in our case can be written as

$$\int_{\widehat{W}_{\epsilon} \times (a,0)} u(x) dx = \epsilon \int_{\widehat{W}_{\epsilon} \times Y \times (a/\epsilon,0)} \hat{u}(\bar{x},y) d\bar{x} dy, \tag{10}$$

for any  $a \in [-1,0[$ . This formula will be essential in computing the limit of  $\int_{Q_{\epsilon}} D^3 v_{\epsilon}$ :  $D^3 T_{\epsilon} \varphi \, \mathrm{d} x$  as  $\epsilon \to 0$ . Before doing this, we need two technical lemmas. By  $w_{Per_Y}^{3,2}(Y \times (-\infty,0))$  we denote the space of functions u in  $W_{loc}^{3,2}(\mathbb{R}^{N-1} \times (-\infty,0))$  which are Y-periodic in the first N-1 variables and such that  $\|D^{\eta}u\|_{L^2(Y \times (-\infty,0))} < \infty$  for all  $|\eta| = 3$ .

**Lemma 1.** Let  $v_{\epsilon} \in W^{3,2}(\Omega)$  with  $||v_{\epsilon}||_{W^{3,2}(\Omega)} < M$ , for all  $\epsilon > 0$ . Let  $V_{\epsilon}(\bar{x}, y) = \hat{v_{\epsilon}}(\bar{x}, y) - P(\hat{v_{\epsilon}}(\bar{x}, y))$  for all  $(\bar{x}, y) \in \widehat{W_{\epsilon}} \times Y \times (-1/\epsilon, 0)$ , where the operator P is defined by

$$P(w(\bar{x}, y)) = \int_{Y} \left( w(\bar{x}, \bar{y}, 0) - \sum_{|\eta|=2} \int_{Y} D_{\bar{y}}^{\eta} w(\bar{x}, \bar{y}, 0) \, d\bar{y} \, \frac{\bar{y}^{\eta}}{\eta!} \right) d\bar{y}$$

$$+ \int_{Y} \nabla_{y} w(\bar{x}, \bar{y}, 0) \, d\bar{y} \cdot y + \sum_{|\eta|=2} \int_{Y} D_{y}^{\eta} w(\bar{x}, \bar{y}, 0) \, d\bar{y} \, \frac{y^{\eta}}{\eta!} \, .$$

Then there exists  $\hat{v} \in L^2(W, w_{\text{Per}_Y}^{3,2}(Y \times (-\infty, 0)))$  such that

(a) 
$$\frac{D_y^{\eta}V_{\epsilon}}{\epsilon^{5/2}} \rightharpoonup D_y^{\eta}\hat{v}$$
 in  $L^2(W \times Y \times (d,0))$  as  $\epsilon \to 0$ , for all  $d < 0$  and  $\eta \in \mathbb{N}^N$ ,  $|\eta| \le 2$ .

(b) 
$$\frac{D_y^{\eta}V_{\epsilon}}{\epsilon^{5/2}} = \frac{D_y^{\eta}\hat{v}_{\epsilon}}{\epsilon^{5/2}} \rightharpoonup D_y^{\eta}\hat{v}$$
 in  $L^2(W \times Y \times (-\infty, 0))$  as  $\epsilon \to 0$ , for all  $\eta \in \mathbb{N}^N$ ,  $|\eta| = 3$ ,

where it is understood that functions  $V_{\epsilon}$ ,  $D_y^{\eta}V_{\epsilon}$  are extended by zero in the whole of  $W \times Y \times (-\infty, 0)$  outside their natural domain of definition  $\widehat{W}_{\epsilon} \times Y \times (-1/\epsilon, 0)$ .

Proof. The proof is similar to the one in [3, Lemma 8.9]. Using formula (10), one can easily prove that  $\|\epsilon^{-5/2}D_y^{\eta}V_{\epsilon}\|_{L^2(W\times Y\times (-\infty,0))}$  is uniformly bounded with respect to  $\epsilon$  for all  $|\eta|=3$ . Note that the operator P is a projector on the space of polynomials of the second degree in the variable y. Thus, a standard argument exploiting a Poincaré-Wirtinger-type inequality implies the existence of a real-valued function  $\hat{v}$  defined on  $W\times Y\times (-\infty,0)$  which admits weak derivatives up to the third order locally in the variable y, such that statements (a) and (b) hold. In order to prove the periodicity of  $\hat{v}$  in  $\bar{y}$ , we can apply to  $D^2V_{\epsilon}$  an argument similar to the one contained in Lemma 4.3 in [9] to obtain that  $\nabla_y \hat{v}$  is periodic. Then we find out that  $\hat{v}$  is also periodic because  $\int_V \nabla_y \hat{v}(\bar{x}, \bar{y}, 0) d\bar{y} = 0$ , being this true for all the functions  $V_{\epsilon}$ .  $\square$ 

**Lemma 2.** For all  $y \in Y \times (-1,0)$  and i,j,k = 1,...,N the functions  $\hat{h}_{\epsilon}(\bar{x},y)$ ,  $\widehat{\frac{\partial h_{\epsilon}}{\partial x_{i}}(\bar{x},y)}$ ,  $\widehat{\frac{\partial^{2} h_{\epsilon}}{\partial x_{i}\partial x_{j}}(\bar{x},y)}$  and  $\widehat{\frac{\partial^{3} h_{\epsilon}}{\partial x_{i}\partial x_{j}\partial x_{k}}}(\bar{x},y)$  are independent of  $\bar{x}$ . Moreover,  $\hat{h}_{\epsilon}(\bar{x},y) = O(\epsilon^{3/2})$ ,  $\widehat{\frac{\partial h_{\epsilon}}{\partial x_{i}}(\bar{x},y)} = O(\epsilon^{1/2})$  as  $\epsilon \to 0$ ,

$$\epsilon^{1/2} \frac{\widehat{\partial^2 h_{\epsilon}}}{\partial x_i \partial x_j} (\bar{x}, y) \to \frac{\partial^2 (b(\bar{y})(y_N + 1)^4)}{\partial y_i \partial y_j},$$

as  $\epsilon \to 0$ , for all i, j = 1, ..., N, uniformly in  $y \in Y \times (-1, 0)$ , and

$$\epsilon^{3/2} \frac{\widehat{\partial^3 h_\epsilon}}{\partial x_i \partial x_j \partial x_k} (\bar{x}, y) \to \frac{\partial^3 (b(\bar{y})(y_N + 1)^4)}{\partial y_i \partial y_j \partial y_k},$$

as  $\epsilon \to 0$ , for all i, j, k = 1, ..., N, uniformly in  $y \in Y \times (-1, 0)$ .

*Proof.* It is a matter of easy but lengthy calculations, which can be carried out as in [3, Lemma 8.27].

Now we are ready to prove the following

**Theorem 2.** Let M be a positive real number. Let  $f_{\epsilon} \in L^{2}(\Omega_{\epsilon})$ ,  $||f_{\epsilon}||_{L^{2}(\Omega_{\epsilon})} < M$  for all  $\epsilon > 0$  and  $f \in L^{2}(\Omega)$  be such that  $f_{\epsilon} \rightharpoonup f$  in  $L^{2}(\Omega)$ . Let  $v_{\epsilon} \in W^{3,2}(\Omega_{\epsilon}) \cap W_{0}^{2,2}(\Omega_{\epsilon})$  be the solutions to  $H_{\Omega_{\epsilon}}v_{\epsilon} = f_{\epsilon}$ . Then, up to a subsequence, there exists  $v \in W^{3,2}(\Omega) \cap W_{0}^{2,2}(\Omega)$  and  $\hat{v} \in L^{2}(W, w_{Per_{Y}}^{3,2}(Y \times (-\infty, 0)))$  such that  $v_{\epsilon} \rightharpoonup v$  in  $W^{3,2}(\Omega)$ ,  $v_{\epsilon} \rightarrow v$  in  $L^{2}(\Omega)$ , statements (a) and (b) in Lemma 1 hold, and such that for each  $\varphi \in W^{3,2}(\Omega) \cap W_{0}^{2,2}(\Omega)$  the following holds

$$\int_{\Omega} D^{3}v : D^{3}\varphi + u\varphi \,dx - 3 \int_{W} \int_{Y \times (-1,0)} D_{y}^{2} \left(\frac{\partial \hat{v}}{\partial y_{N}}\right) : D_{y}^{2} (b(\bar{y})(1+y_{N})^{4}) \,dy \,\frac{\partial^{2}\varphi}{\partial x_{N}^{2}} (\bar{x},0) d\bar{x} 
- \int_{W} \int_{Y \times (-1,0)} y_{N} (D_{y}^{3}(\hat{v}) : D^{3} (b(\bar{y})(1+y_{N})^{4}) \,dy \,\frac{\partial^{2}\varphi}{\partial x_{N}^{2}} (\bar{x},0) d\bar{x} = \int_{\Omega} f\varphi \,dx. \tag{11}$$

*Proof.* By using formula (10), one can write  $\int_{Q_{\epsilon}} D^3 v_{\epsilon} : D^3(T_{\epsilon}\varphi) dx$  as an integral over  $\widehat{W}_{\epsilon} \times Y \times (-1,0)$  for suitable combinations of derivatives of  $\widehat{v}_{\epsilon}$  and  $\widehat{T}_{\epsilon}\varphi$ . Then using Lemmas 1, 2, one can prove that the limit of  $\int_{Q_{\epsilon}} D^3 v_{\epsilon} : D^3(T_{\epsilon}\varphi) dx$  as  $\epsilon \to 0$  equals

$$-3 \int_{W} \int_{Y \times (-1,0)} D_{y}^{2} \left(\frac{\partial \hat{v}}{\partial y_{N}}\right) : D_{y}^{2} (b(\bar{y})(1+y_{N})^{4}) \, dy \, \frac{\partial^{2} \varphi}{\partial x_{N}^{2}} (\bar{x},0) d\bar{x}$$
$$- \int_{W} \int_{Y \times (-1,0)} y_{N} (D_{y}^{3}(\hat{v}) : D^{3} (b(\bar{y})(1+y_{N})^{4}) \, dy \, \frac{\partial^{2} \varphi}{\partial x_{N}^{2}} (\bar{x},0) d\bar{x}. \quad (12)$$

This combined with (7) and (8) allows to pass to the limit in (6) and obtain the validity of (11).

The term (12) appearing in (11) plays the role of the so-called strange term, typical of many homogenization problems. We plan to write it in a more explicit way in order to complete the proof of statement (ii) in Theorem 1. To do so, we characterise  $\hat{v}$  as the solution to a suitable boundary value problem by proceeding as follows. Let  $\psi \in C^{\infty}(\overline{W} \times \overline{Y} \times ]-\infty,0]$  be such that  $\sup \psi \subset C \times \overline{Y} \times [d,0]$  for some compact set  $C \subset W$  and  $d \in ]-\infty,0[$  and such that  $\psi(\bar{x},\bar{y},0)=\nabla \psi(\bar{x},\bar{y},0)=0$  for all  $(\bar{x},\bar{y})\in W\times Y$ . Assume also  $\psi$  to be Y-periodic in the variable  $\bar{y}$ . We set  $\psi_{\epsilon}(x)=\epsilon^{\frac{5}{2}}\psi\left(\bar{x},\frac{\bar{x}}{\epsilon},\frac{x_N}{\epsilon}\right)$ , for all  $\epsilon>0$ ,  $x\in W\times ]-\infty,0[$ . Then, by the Y-periodicity and the vanishing conditions imposed on  $\psi$ ,  $T_{\epsilon}\psi_{\epsilon}\in W^{3,2}(\Omega_{\epsilon})\cap W_0^{2,2}(\Omega_{\epsilon})$  for sufficiently small  $\epsilon>0$ , hence we can use it in the weak formulation of the problem in  $\Omega_{\epsilon}$ , getting

$$\int_{\Omega_{\epsilon}} D^3 v_{\epsilon} : D^3 T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x + \int_{\Omega_{\epsilon}} v_{\epsilon} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x = \int_{\Omega_{\epsilon}} f_{\epsilon} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x. \tag{13}$$

Passing to the limit in (13) yields the limit problem for  $\hat{v}$ .

**Theorem 3.** Let  $\hat{v} \in L^2(W, w_{Per_Y}^{3,2}(Y \times (-\infty, 0)))$  be the function from Theorem 2. Then

$$\int_{W\times Y\times(-\infty,0)} D_y^3 \hat{v}(\bar{x},y) : D_y^3 \psi(\bar{x},y) d\bar{x} dy = 0, \tag{14}$$

for all  $\psi \in L^2(W, w_{Per_Y}^{3,2}(Y \times (\infty, 0)))$  such that  $\psi(\bar{x}, \bar{y}, 0) = \nabla \psi(\bar{x}, \bar{y}, 0) = 0$  on  $W \times Y$ . Moreover, for any  $i, j = 1, \ldots, N-1$ , we have

$$\frac{\partial^2 \hat{v}}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) = \frac{\partial b}{\partial y_i}(\bar{y}) \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0) \quad \text{on } W \times Y,$$
 (15)

and

$$\frac{\partial^2 \hat{v}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0) = 0 \quad \text{on } W \times Y.$$
 (16)

*Proof.* By approximation we can assume that  $\psi$  is smooth, with support described as above. Then it is easy to see that  $\int_{\Omega_{\epsilon}} v_{\epsilon} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x$ ,  $\int_{\Omega_{\epsilon}} f_{\epsilon} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x$ ,  $\int_{\Omega_{\epsilon} \setminus \Omega} D^{3} v_{\epsilon}$ :  $D^{3} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x \to 0$  as  $\epsilon \to 0$ . Moreover, a slight modification of [3, Lemma 8.47] combined with Lemma 1 yields  $\int_{\Omega} D^{3} v_{\epsilon}$ :  $D^{3} T_{\epsilon} \psi_{\epsilon} \, \mathrm{d}x \to \int_{W \times Y \times (-\infty,0)} D_{y}^{3} \hat{v}(\bar{x},y)$ :  $D_{y}^{3} \psi(\bar{x},y) \, \mathrm{d}\bar{x} \, \mathrm{d}y$ . Thus, passing to the limit in (13) we obtain (14). Differentiating

the equality  $\nabla v_{\epsilon}(\bar{x}, g_{\epsilon}(\bar{x})) = 0$  which holds for all  $\bar{x} \in W$ , we get that for any i, j = 1, ..., N - 1

$$\frac{\partial^2 v_{\epsilon}}{\partial x_i \partial x_j} (\bar{x}, g_{\epsilon}(\bar{x})) + \frac{\partial^2 v_{\epsilon}}{\partial x_i \partial x_N} (\bar{x}, g_{\epsilon}(\bar{x})) \frac{\partial g_{\epsilon}(\bar{x})}{\partial x_j} = 0, \quad \text{for all } \bar{x} \in W .$$
 (17)

Hence, setting  $V_{\epsilon}^{ij} = \left(0, \dots, 0, -\frac{\partial^2 v_{\epsilon}}{\partial x_i \partial x_N}, 0, \dots, 0, \frac{\partial^2 v_{\epsilon}}{\partial x_i \partial x_j}\right)$ , for all  $i = 1, \dots, N, j = 1, \dots, N-1$  we get  $V_{\epsilon}^{ij} \cdot \nu_{\epsilon} = 0$ , on  $\Gamma_{\epsilon}$ , where  $\Gamma_{\epsilon}$  is the part of the boundary of  $\Omega_{\epsilon}$  given by the graph of  $g_{\epsilon}$  and  $\nu_{\epsilon}$  is the unit outer normal to  $\Gamma_{\epsilon}$ . We note that by Lemma 1

$$\frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial y_k} \left( \frac{\widehat{\partial^2 v_{\epsilon}}}{\partial x_i \partial x_j} \right) \rightharpoonup \frac{\partial^3 \widehat{v}}{\partial y_i \partial y_j \partial y_k}$$

in  $L^2(W\times Y\times]-\infty,0[)$  as  $\epsilon\to 0$ . Applying [9, Lemma 4.3], we get that  $\frac{\partial^2\hat{v}}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)=\frac{\partial b}{\partial y_j}(\bar{y})\frac{\partial^2 v}{\partial x_N\partial x_i}(\bar{x},0)$  on  $W\times Y$  for all  $i=1,\ldots,N,\ j=1,\ldots,N-1,$  and since  $v\in W^{3,2}(\Omega)\cap W^{2,2}_0(\Omega)$  this implies the validity of (15) and (16).

The two scale problem (14) can be written in a more explicit way by separation of variables. We need the following lemma.

**Lemma 3.** There exists  $V \in w_{Pery}^{3,2}(Y \times (-\infty,0))$  satisfying the equation

$$\int_{Y \times (-\infty,0)} D^3 V : D^3 \psi \, \mathrm{d}y = 0,$$

for all the test-functions  $\psi \in w^{3,2}_{Per_Y}(Y \times (-\infty,0))$  such that  $\psi(\bar{y},0) = 0 = \nabla \psi(\bar{y},0)$  on Y, and satisfying the boundary conditions

$$\begin{cases} V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial V}{\partial y_N}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y. \end{cases}$$

Function V is unique up to a sum of a monomial in  $y_N$  of the form  $ay_N^2$ . Moreover,  $V \in W_{Per_Y}^{6,2}(Y \times (d,0))$  for any d < 0 and it satisfies the equation  $\Delta^3 V = 0$ , in  $Y \times (d,0)$  subject to the boundary condition  $\frac{\partial^3 V}{\partial y_N^3}(\bar{y},0) = 0$  on Y. Finally,

$$\int_{Y\times(-1,0)} 3D_y^2 \left(\frac{\partial V}{\partial y_N}\right) : D_y^2(b(\bar{y})(1+y_N)^4) + y_N(D_y^3V : D^3(b(\bar{y})(1+y_N)^4) \, \mathrm{d}y = K.$$
(18)

where K is as in Theorem 1.

Proof. The first part of the lemma can be proved by standard direct methods of the calculus of variations and regularity theory, as in [3]. We now prove (18). Let  $\phi$  be the real-valued function defined on  $Y \times ]-\infty,0]$  by  $\phi(y)=y_Nb(\bar{y})(1+y_N)^4$  if  $-1 \le y_N \le 0$  and  $\phi(y)=0$  if  $y_N<-1$ . Then  $\phi \in W^{3,2}(Y \times (-\infty,0)), \ \phi(\bar{y},0)=0$  and  $\nabla \phi(\bar{y},0)=(0,0,\ldots,0,b(\bar{y}))$ . Note that the function  $\psi=V-\phi$  is a suitable

test-function in equation  $\int_{Y\times(-\infty,0)} D^3V: D^3\psi\,\mathrm{d}y=0$ . By plugging it in we get

$$\int_{Y\times(-\infty,0)} |D^{3}V|^{2} dy = \int_{Y\times(-1,0)} D^{3}V : D^{3}\phi dy$$

$$= 3 \int_{Y\times(-1,0)} D_{y}^{2} \left(\frac{\partial V}{\partial y_{N}}\right) : D_{y}^{2}(b(\bar{y})(1+y_{N})^{4}) dy$$

$$+ \int_{Y\times(-1,0)} y_{N}(D_{y}^{3}V : D^{3}(b(\bar{y})(1+y_{N})^{4}) dy. \quad (19)$$

Thus the left-hand side of (18) equals  $\int_{Y\times(-\infty,0)} |D^3V|^2 dy$ . The second equality in (4) can be proved by integrating repeatedly by parts.

**Theorem 4.** Let V be the function defined in Lemma 3. Let  $v, \hat{v}$  be as in Theorem 2. Then up to the sum of monomials of the type  $a(\bar{x})y_N^2$ , we have that

$$\hat{v}(\bar{x}, y) = V(y) \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0). \tag{20}$$

In particular, the strange term in (12) equals  $-K \int_W \frac{\partial^2 v}{\partial x_N^2}(\bar{x},0) \frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x},0) d\bar{x}$  where K is as in (1).

Proof. The function  $\hat{v}(\bar{x},y) = V(y) \frac{\partial^2 v}{\partial x_N^2}(\bar{x},0)$  satisfies problem (14) subject to the boundary conditions (15), (16). Since the solution to such problem is unique up to the sum of monomials of order 2 in  $y_N$ , we conclude that the first part of the statement holds. The second part of the statement follows by replacing  $\hat{v}(\bar{x},y)$  by  $V(y) \frac{\partial^2 v}{\partial x_N^2}(\bar{x},0)$  in (12) and using (18).

It is now clear that combining Theorem 2 with Theorem 4 provides the proof of statement (ii) of Theorem 1.

Case  $\alpha < 3/2$ . In this case, it is not necessary to use the operator  $T_{\epsilon}$  defined above, because it turns out that the limit energy space is not  $W^{3,2}(\Omega) \cap W^{2,2}_0(\Omega)$  but  $W^{3,2}(\Omega) \cap W^{3,2}_0(\Omega) \cap W^{3,2}_{0,W}(\Omega)$  where  $W^{3,2}_{0,W}(\Omega)$  is defined as the closure in  $W^{3,2}(\Omega)$  of the  $C^{\infty}$ -functions which vanish in a neighbourhood of W. (Note that  $W^{3,2}_{0,W}(\Omega)$  can be equivalently defined as the space of those functions  $u \in W^{3,2}(\Omega)$  such  $D^{\alpha}u = 0$  on W for all  $|\alpha| \leq 2$ .) In fact, since  $\alpha < 3/2$  and the solution  $v_{\epsilon}$  of problem (5) satisfies (17), by [9] the vector fields  $V^{ij}_{\epsilon}$  defined above converge weakly in  $W^{1,2}_{0,W}(\Omega)$ . This implies by [9, Lemma 4.3, Theorem 5.1] that  $D^{\alpha}v = 0$  on W for all  $|\alpha| = 2$ , which gives  $v \in W^{3,2}_{0,W}(\Omega)$ . Let  $\varphi \in W^{3,2}(\Omega) \cap W^{2,2}_0(\Omega) \cap W^{3,2}_{0,W}(\Omega)$  be fixed. Let  $\varphi_0$  be the function obtained by extending  $\varphi$  by zero outside  $\Omega$ . It is straightforward to prove that  $\varphi_0 \in W^{3,2}(\Omega_{\epsilon}) \cap W^{2,2}_0(\Omega_{\epsilon})$  hence it is possible to test  $\varphi_0$  in the weak formulation of problem (5) to obtain  $\int_{\Omega_{\epsilon}} D^3 v_{\epsilon} : D^3 \varphi_0 \, \mathrm{d}x + \int_{\Omega_{\epsilon}} v_{\epsilon} \varphi_0 \, \mathrm{d}x = \int_{\Omega_{\epsilon}} f_{\epsilon} \varphi_0 \, \mathrm{d}x$ . By passing to the limit in this equality as  $\epsilon \to 0$ , one easily obtain that  $\int_{\Omega} D^3 v : D^3 \varphi \, \mathrm{d}x + \int_{\Omega} v \varphi \, \mathrm{d}x = \int_{\Omega} f \varphi \, \mathrm{d}x$  which concludes the proof of statement (iii).

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