BOUNDARY ε-REGULARITY IN OPTIMAL TRANSPORTATION

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ABSTRACT. We develop an ε-regularity theory at the boundary for a general class of Monge-Ampère type equations arising in optimal transportation. As a corollary we deduce that optimal transport maps between Hölder densities supported on C^2 uniformly convex domains are $C^{1,\alpha}$ up to the boundary, provided that the cost function is a sufficient small perturbation of the quadratic cost $-x \cdot y$.

1. INTRODUCTION

Let f and g be two probability densities supported respectively on two bounded domains X and Y in $\mathbb{R}^n$. Let $c : X \times Y \to \mathbb{R}$ be a cost function. The optimal transport problem is about finding a map $T : X \to Y$ among all transport maps minimizing the transportation cost

$$\int_X c(x, T(x)) f(x) dx,$$

where the term "transport map" means $T_\sharp f = g$. Existence and uniqueness of optimal transport maps under mild conditions are now well understood, see for instance [6] and [21]. The regularity theory of optimal transport map with quadratic cost $c(x, y) = -x \cdot y$ [1] has been developed by Caffarelli [1, 2, 3, 4, 5] (see also [34]). Since this paper is concerned with the regularity theory up to the boundary, we state here Caffarelli's global regularity result:

Theorem 1.1. [5] *Suppose that* $0 < f \in C^\alpha(\overline{X})$ *and* $0 < g \in C^\alpha(\overline{Y})$, *where* X *and* Y *are uniformly convex bounded domains of class* C^2. *Then the optimal transport map* T *associated to the cost* $c(x, y) = -x \cdot y$ *is of class* $C^{1,\alpha}$ *up to the boundary of* X.

For general costs, Ma, Trudinger, and Wang [30] found the so called "MTW condition", which guarantees the smoothness of the optimal transport map provided the densities are smooth and the domains satisfy some suitable convexity conditions. Their condition reads as follows:

$$(1.1) \qquad \sum_{i,j,k,l,p,q,r,s} c^{p,q}(c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}\xi_i\xi_j\eta_k\eta_l \geq 0 \qquad \text{in } X \times Y$$

for all $\xi, \eta \in \mathbb{R}^n$ satisfying $\xi \perp \eta$, where lower indices before (resp. after) the comma indicates derivatives with respect to x (resp. y) (so for instance $c_{i,j} = \frac{\partial^2 c}{\partial x_i \partial y_j}$), $(c^{i,j})$ is the inverse of $(c_{i,j})$, and all derivatives are evaluated at $(x, y) \in X \times Y$. Later, Loeper [27] showed that MTW condition is actually a necessary condition for the optimal transport map to be smooth for any positive smooth densities. After the breakthroughs in [30, 27], many works have been

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[1] Usually one refers to $c(x, y) = |x-y|^2/2$ as "quadratic cost". However it is well-known that the costs $|x-y|^2/2$ and $-x \cdot y$ are completely equivalent (see for instance [10, Section 3.1]), thus we will use the term "quadratic cost" to refer to the latter.

devoted to the regularity theory of optimal transport map under MTW condition, to cite some see [16, 25, 32, 33, 17, 28, 29, 26, 18, 24, 20, 19, 14, 15].

By now, regularity of optimal transport maps under the MTW condition is well understood. However, several interesting costs do not satisfy this condition, for instance $c(x,y) = \frac{1}{p}|x - y|^p$ does not satisfy MTW condition when $p \in (1,2) \cup (2,\infty)$, and actually the class of costs satisfying the MTW condition is very restricted.

Recently, De Philippis and Figalli [9] obtained a partial regularity result for optimal transport problem with general cost without assuming neither the MTW condition nor any convexity on the domains. The key result in [9] consists in the interior versions of our Theorems 2.1 and 2.2. Roughly speaking they prove that, given any $0 < \beta < 1$, if there are interior points $x_0 \in X, y_0 \in Y$ such that the cost function is sufficiently close $-x \cdot y$ in C^2 near (x_0, y_0), the densities are sufficiently close to 1 in C^0 (resp. the densities are C^α) near x_0 and y_0 respectively, and the potential function u is sufficiently close to $\frac{1}{2}|x|^2$ in C^0 near x_0, then u is $C^{1,\beta}$ (resp. $C^{2,\alpha}$) in a neighbourhood of x_0. Exploiting that semiconvex functions are twice differentiable almost everywhere, De Philippis and Figalli used this result to show that optimal maps are always smooth outside a closed set of measure zero (see the proof of [9, Theorem 1.3] for more details).

In this paper we prove the analogue of De Philippis and Figalli's result when x_0 is on the boundary of X. As an application we show that optimal transport maps between Hölder densities supported on $C^{2,\alpha}$ uniformly convex domains are $C^{1,\alpha}$ in the interior and $C^{1,\alpha'}$ up to the boundary for some $\alpha' \in (0, \alpha)$, provided that the cost function is a sufficient small perturbation of the quadratic cost $-x \cdot y$. This allows us to improve a recent result of Caffarelli, Gonzáles, and Nguyen [7], where they prove that the optimal transport map is of class $C^{1,\alpha}$ *strictly* in the interior of X (more precisely, as the size of the perturbation on the cost goes to zero, the transport map is $C^{1,\alpha}$ in larger and larger domains which invade X, see [7, Theorem 1.1]).

We note that, in our case, to obtain an almost everywhere regularity of transport maps on the boundary (as done in [9] for the interior case) we should prove that at almost every point on the boundary (with respect to the $(n - 1)$-dimensional Hausdorff measure) our assumptions are satisfied. This seems to be a very delicate issue and it will be investigate in future works.

The proof of our ε-regularity result follows the lines of [9], but compared to the interior case the boundary regularity presents many new additional difficulties, and several new ideas have to be introduced to overcome them.

Indeed, first of all, notice that one of the key steps in the proof of the interior regularity result of De Philippis and Figalli [9] is to construct a smooth approximating solution to the original problem, which is based on solving an optimal transportation problem with cost $-x \cdot y$ and constant densities. In their case, the condition that the potential function u is close to $\frac{1}{2}|x|^2$ ensures that the approximating solution is smooth. But when we are around a point on the boundary of the domain one cannot expect such approximating solution to be smooth. To handle this we find a new approximation argument by using a suitable symmetrization trick, where we first slightly modify the domain and then we reflect it around a boundary point with respect to the tangent hyperplane.

Second, for proving $C^{2,\alpha}$ regularity, the comparison principle plays very important role. Recall that the proof of $C^{2,\alpha}$ regularity of solutions to Monge-Ampère type equation is usually

based on an iteration argument. In the interior case, at each iteration, the potential function u solves a Monge-Ampère type equation in a sub-level set of u with Dirichlet boundary data. Then, one can construct an approximating solution v which solves the standard Monge-Ampère equation with constant right hand side in a small convex neighborhood of the the sub-level set of u. The comparison principle is used to compare the difference between u and v. For the interior case, the comparison principle can be proved more or less in a standard way. However, around a point on the boundary of the domain, the Dirichlet data of the solutions (both the original solution and the approximating solution) are not under control. Luckily, in our case, we can prove that near x_0 the optimal map sends the boundary of source onto the boundary of the target, and this property allows us to show that the Neumann data of the original solution and the approximating solution are very close. Hence, we are able to use comparison principle for Monge-Ampère equation with mixed boundary data.

The rest of the paper is organized as follows. In section 2 we introduce some notation and preliminaries, and state the main results. Section 3 is devoted to the proof of Corollaries 2.3 and 2.4. Section 4 contains the proof of Theorem 2.1, and in the last section we prove Theorem 2.2.

2. Preliminaries and main results

First, we introduce some conditions which should be satisfied by the cost. Let X and Y be two bounded open subsets of $\mathbb{R}^n$.

(**C0**) The cost function is of class C^3 with $\|c\|_{C^3(X \times Y)} < \infty$.
(**C1**) For any $x \in X$, the map $Y \ni y \mapsto D_x c(x, y) \in \mathbb{R}^n$ is injective.
(**C2**) For any $y \in Y$, the map $X \ni x \mapsto D_y c(x, y) \in \mathbb{R}^n$ is injective.
(**C3**) $\det(D_{xy} c)(x, y) \neq 0$ for all $(x, y) \in X \times Y$.

A function $u : X \to \mathbb{R}$ is said c-*convex* if it can be written as

$$u(x) = \sup_{y \in Y} \{-c(x, y) + \lambda_y\} \tag{2.1}$$

for some family of constants $\{\lambda_y\}_{y \in Y} \subset \mathbb{R}$. Note that (**C0**) and (2.1) imply that a c-convex function is semiconvex, namely, there exists some constant K depending only on $\|c\|_{C^2(X \times Y)}$ such that $u + K|x|^2$ is convex. One immediate consequence of the semiconvexity is that u is twice differentiable almost everywhere.

It is well known that (**C0**) and (**C1**) ensure that there exists a unique optimal transport map, and there exists a c-convex function u such that the optimal map is a.e. uniquely characterized in terms of u (and for this reason we denote it by T_u) by the relation

$$- D_x c(x, T_u(x)) = \nabla u(x) \qquad \text{for a.e. } x. \tag{2.2}$$

As explained for instance in [9, Section 2] (see also [10]), the transport condition $(T_u)_{\#}f = g$ implies that u solves at almost every point the Monge-Ampère type equation

(2.3)
$$\det\Big(D^2 u(x) + D_{xx}c\big(x, \text{c-exp}_x(\nabla u(x))\big)\Big) = \big|\det\big(D_{xy}c\big(x, \text{c-exp}_x(\nabla u(x))\big)\big)\big| \frac{f(x)}{g(\text{c-exp}_x(\nabla u(x)))},$$

where c-exp denotes the *c-exponential map* defined as

(2.4) for any $x \in X$, $y \in Y$, $p \in \mathbb{R}^n$, $\text{c-exp}_x(p) = y \iff p = -D_x c(x, y)$.

Notice that, with this notation, $T_u(x) = \text{c-exp}_x(\nabla u(x))$.

For a c-convex function, analogous to the subdifferential for convex function, we can talk about its c-subdifferential: If $u : X \to \mathbb{R}$ is a c-convex function as above, the *c-subdifferential* of u at x is the (nonempty) set

$$\partial_c u(x) := \big\{y \in \overline{Y} : u(z) \geq -c(z, y) + c(x, y) + u(x) \quad \forall z \in X\big\}.$$

We also define *Frechet subdifferential* of u at x as

$$\partial^- u(x) := \{p \in \mathbb{R}^n : u(z) \geq u(x) + p \cdot (z - x) + o(|z - x|)\}.$$

It is easy to check that

$$y \in \partial_c u(x) \implies -D_x c(x, y) \in \partial^- u(x).$$

In the following, we use the notation

(2.5)
$$S_u(h) := \{x \in X : u(x) < h\}$$

to denote the sub-level set of a function u. In fact, there is a more general concept of c-sub-level set of a c-convex function, namely, one can define

$$S(u, h, x_0, y_0) := \{x \in X : u(x) < c(x, y_0) - c(x_0, y_0) + u(x_0) + h\},$$

where $x_0 \in X$ and $y_0 \in \partial_c u(x_0)$. In this paper, we will always perform some transformation so that $x_0 = \mathbf{0}$, $y_0 = \mathbf{0}$, $u(\mathbf{0}) = 0$, $c(x, \mathbf{0}) = 0$, and the notation $S_u(h)$ will be enough for our purpose.

To state our main results we need to introduce some more notation.

We denote $x' := (x^1, \cdots, x^{n-1}) \in \mathbb{R}^{n-1}$, and $x = (x', x^n) \in \mathbb{R}^n$. Given two domains $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n$, we are going to assume that there exist two functions $P, Q : \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^2 satisfying $P(\mathbf{0}) = Q(\mathbf{0}) = 0$, $\nabla P(\mathbf{0}) = \nabla Q(\mathbf{0}) = \mathbf{0}$, and

(2.6)
$$\{x^n > P(x')\} \cap B_{1/2} \subset \mathcal{C}_1 \subset \{x^n > P(x')\} \cap B_2, \quad \{y^n > Q(y')\} \cap B_{1/2} \subset \mathcal{C}_2 \subset \{y^n > Q(y')\} \cap B_2.$$

Note that (2.6) implies that

(2.7)
$$\mathcal{C}_1 \subset \{x^n \geq -2\|P\|_{C^2}\}, \quad \mathcal{C}_2 \subset \{y^n \geq -2\|Q\|_{C^2}\}.$$

In the following K_1, K_2, K_3, and K, are always used to denote some universal constants whose value may change depending on the context. In the next theorems we implicitly assume that $\mathcal{C}_1 \subset X$ and $\mathcal{C}_2 \subset Y$, so that the cost is defined and satisfies **(C0)**-**(C3)** on $\mathcal{C}_1 \times \mathcal{C}_2$.

Theorem 2.1. *Let f, g be two densities supported in $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, let P and Q be as in (2.6), and let $u : \mathcal{C}_1 \to \mathbb{R}$ be a c-convex function such that $\partial_c u(\mathcal{C}_1) \subset B_2$ and $(T_u)_\sharp f = g$ (see (2.2)). Then, for every $\beta \in (0, 1)$ there exist constants $\delta_0, \eta_0 > 0$ such that the following holds: if*

$$(2.8) \qquad \|P\|_{C^2} + \|Q\|_{C^2} \leq \delta_0,$$

$$(2.9) \qquad \|f - \mathbf{1}\|_{L^\infty(\mathcal{C}_1)} + \|g - \mathbf{1}\|_{L^\infty(\mathcal{C}_2)} \leq \delta_0,$$

$$(2.10) \qquad \|c(x, y) + x \cdot y\|_{C^2(\mathcal{C}_1 \times \mathcal{C}_2)} \leq \delta_0,$$

and

$$(2.11) \qquad \left\| u - \frac{1}{2}|x|^2 \right\|_{L^\infty(\mathcal{C}_1)} \leq \eta_0,$$

then $u \in C^{1,\beta}(\overline{\mathcal{C}_1 \cap B_{\rho_0}})$ for sufficiently small ρ_0.

Theorem 2.2. *Let $u, f, g, \eta_0, \delta_0$ be as in Theorem 2.1, and assume in addition $f \in C^\alpha(B_{1/2} \cap \{x^n \geq P(x')\})$, $g \in C^\alpha(B_{1/2} \cap \{y^n \geq Q(y')\})$. There exist small constants $\eta_1 \leq \eta_0$ and $\delta_1 \leq \delta_0$ such that, if*

$$\|P\|_{C^2} + \|Q\|_{C^2} \leq \delta_1,$$

$$(2.12) \qquad \|f - \mathbf{1}\|_{L^\infty(\mathcal{C}_1)} + \|g - \mathbf{1}\|_{L^\infty(\mathcal{C}_2)} \leq \delta_1,$$

$$(2.13) \qquad \|c(x, y) + x \cdot y\|_{C^2(\mathcal{C}_1 \times \mathcal{C}_2)} \leq \delta_1,$$

and

$$(2.14) \qquad \left\| u - \frac{1}{2}|x|^2 \right\|_{L^\infty(\mathcal{C}_1)} \leq \eta_1,$$

then there exists $\rho_1 > 0$ small such that the following holds: for any point $z \in B_{\rho_1} \cap \{x_n = P(x')\}$ there exists a second order polynomial p such that,

$$|u(x) - p(x)| \leq C|x - z|^{2+\alpha'} \qquad \forall x \in B_{\rho_1}(z) \cap \{x_n \geq P(x')\},$$

where $C > 0$ and $\alpha' \in (0, \alpha]$ are constants depending only on $\delta_0, \eta_0, \alpha, \|f\|_{C^\alpha}, \|g\|_{C^\alpha}$. Moreover, there exists $\rho_2 > 0$ small such that $u \in C^{2,\alpha}_{\text{loc}}(\mathcal{C}_1 \cap B_{\rho_2}) \cap C^{2,\alpha'}(\overline{\mathcal{C}_1 \cap B_{\rho_2}})$.

As a corollary of the two theorems above, we can easily recover (and improve) the results from [7].

Corollary 2.3. *Suppose X and Y are two C^2 uniformly convex bounded domains in $\mathbb{R}^n$. Suppose λ_1 and λ_2 are two positive constants such that $\int_X \lambda_1 = \int_Y \lambda_2$. Assume f and g are two nonnegative densities satisfying*

$$(2.15) \qquad \int_X f(x)dx = \int_Y g(y)dy, \quad \|f - \lambda_1\|_{C^0(X)} + \|g - \lambda_2\|_{C^0(Y)} \leq \delta$$

*for some $\delta > 0$. Let u be the c-convex function associated to the optimal transport problem between f and g with cost $c(x, y)$ (see (2.2)), where c satisfies (**C0**)-(**C3**) and*

$$(2.16) \qquad \|c + x \cdot y\|_{C^2(X \times Y)} \leq \delta.$$

Then, for every $\beta \in (0,1)$ there exists $\bar{\delta} > 0$, depending only on β, n, λ_1, and λ_2, such that $u \in C^{1,\beta}(\overline{X})$ provided $\delta \leq \bar{\delta}$.

In particular, if $c(x,y) = \frac{1}{p}|x-y|^p$ with $p > 1$:

• There exists $\bar{\delta} > 0$, depending only on β, n, p, $\operatorname{diam}(X)$, $\operatorname{diam}(Y)$, λ_1, and λ_2, such that if $\operatorname{dist}(X,Y) \geq 1/\bar{\delta}$ then $u \in C^{1,\beta}(\overline{X})$.

• Let $R > 0$. There exists $\bar{\delta} > 0$ depending only on β, n, p, $\operatorname{diam}(X)$, $\operatorname{diam}(Y)$, R, λ_1, and λ_2, such that if $2 - \bar{\delta} \leq p \leq 2 + \bar{\delta}$ and $\operatorname{dist}(X,Y) \geq R$, then $u \in C^{1,\beta}(\overline{X})$.

Corollary 2.4. *In Corollary 2.3, assume that the condition (2.15) is replaced by*

$$(2.17) \qquad \int_X f(x)dx = \int_Y g(y)dy, \qquad 0 < f \in C^\alpha(\overline{X}), \quad 0 < g \in C^\alpha(\overline{Y}),$$

and that X, Y are of class $C^{2,\alpha}$. Then there exists $\bar{\delta} > 0$, depending only on β, n, $\inf f$, $\inf g$, $\|f\|_{C^\alpha}$, and $\|g\|_{C^\alpha}$, such that $u \in C^{2,\alpha}_{\mathrm{loc}}(X) \cap C^{2,\alpha'}(\overline{X})$ for some $\alpha' \in (0,\alpha)$, provided $\|c(x,y) + x \cdot y\|_{C^2(X \times Y)} \leq \bar{\delta}$.

In particular, if $c(x,y) = \frac{1}{p}|x-y|^p$ with $p > 1$:

• There exists $\bar{\delta} > 0$, depending only on β, n, p, $\operatorname{diam}(X)$, $\operatorname{diam}(Y)$, $\inf f$, $\inf g$, $\|f\|_{C^\alpha}$, and $\|g\|_{C^\alpha}$, such that if $\operatorname{dist}(X,Y) \geq 1/\bar{\delta}$ then $u \in C^{2,\alpha}_{\mathrm{loc}}(X) \cap C^{2,\alpha'}(\overline{X})$ for some $\alpha' \in (0,\alpha)$.

• Let $R > 0$. There exists $\bar{\delta} > 0$ depending only on β, n, p, $\operatorname{diam}(X)$, $\operatorname{diam}(Y)$, R, $\inf f$, $\inf g$, $\|f\|_{C^\alpha}$, and $\|g\|_{C^\alpha}$, such that if $|p-2| \leq \bar{\delta}$ and $\operatorname{dist}(X,Y) \geq R$ then $u \in C^{2,\alpha}_{\mathrm{loc}}(X) \cap C^{2,\alpha'}(\overline{X})$ for some $\alpha' \in (0,\alpha)$.

Remark 2.5. In Corollary 2.4, if in addition X, Y are of class C^∞, $f \in C^\infty(\overline{X})$, and $g \in C^\infty(\overline{Y})$, then $u \in C^\infty(\overline{X})$. This follows from the standard regularity theory for linear uniformly elliptic equation with oblique boundary condition, for instance see [22, Theorem 6.31]. The second part of the corollary follows as in the proof of Corollary 2.3.

3. Proof of the corollaries

Proof of Corollary 2.3. Without loss of generality we assume $\lambda_1 = \lambda_2 = 1$ (the general case being completely analogous). Let v be the c-convex function associated to the optimal transport problem between $\mathbf{1}_X$ and $\mathbf{1}_Y$ with cost $-x \cdot y$. Recall that v is of class $C^{2,\alpha}$ up to the boundary (see Theorem 1.1).

Given a point $x_0 \in \partial X$, let $y_0 = \nabla v(x_0)$. By [7, Proposition 2.1], after an affine transformation and a translation of coordinates we can assume that $x_0 = y_0 = \mathbf{0}$, $X \subset \{x^n \geq 0\}$, $Y \subset \{y^n \geq 0\}$, $D^2 v(\mathbf{0}) = \mathrm{Id}$, and (up to subtracting a constant) $v(\mathbf{0}) = u(\mathbf{0}) = 0$.

Now, by [7, Proposition 4.1] (see also [9, Lemma 4.1]) we have that

$$(3.1) \qquad \qquad \|u - v\|_{L^\infty(X)} \leq \omega(\delta) \to 0 \quad \text{as } \delta \to 0.$$

Since $D^2 v(\mathbf{0}) = \mathrm{Id}$, and v is of class $C^{2,\alpha}$ up to the boundary, for $h > 0$ small the sub-level sets of v (recall the notation (2.5)) satisfy

$$X \cap B_{\frac{2\sqrt{h}}{3}} \subset S_v(h) \subset X \cap B_{\frac{3\sqrt{h}}{2}},$$

and

$$Y \cap B_{\frac{2\sqrt{h}}{3}} \subset \partial^- v(S_v(h)) \subset Y \cap B_{\frac{3\sqrt{h}}{2}}.$$

By (2.16) and (3.1) it is easy to check that, for $h > 0$ fixed, provided δ is sufficiently small u also satisfies similar properties as follows:

$$(3.2) \qquad X \cap B_{\sqrt{h}/2} \subset S_u(h) \subset X \cap B_{2\sqrt{h}},$$

and

$$(3.3) \qquad Y \cap B_{\sqrt{h}/2} \subset \partial_c u(S_u(h)) \subset Y \cap B_{2\sqrt{h}}.$$

Then, we perform the change of variables

$$\begin{cases} \tilde{x} := \frac{x}{\sqrt{h}} \\ \tilde{y} := \frac{y}{\sqrt{h}} \end{cases}$$

and we set

$$\tilde{c}(\tilde{x}, \tilde{y}) := \frac{1}{h} c(\sqrt{h}\tilde{x}, \sqrt{h}\tilde{y}), \qquad \tilde{u}(\tilde{x}) := \frac{1}{h} u(\sqrt{h}\tilde{x}).$$

Note that, after this change of variables, X (resp. Y) becomes $\frac{1}{\sqrt{h}} X$ (resp. $\frac{1}{\sqrt{h}} Y$). Hence the boundary part $\partial(\frac{1}{\sqrt{h}} X) \cap B_2$ (resp. $\partial(\frac{1}{\sqrt{h}} Y) \cap B_2$) becomes flatter and flatter as $h \to 0$, and in particular it will satisfy (2.8) provided h is small enough. Combining this with (2.16), (3.1), (3.2), and (3.3), we see that $\tilde{u}$ satisfies all the conditions in Theorem 2.1, hence, $\tilde{u}$ is $C^{1,\beta}$ in a neighborhood of $\mathbf{0}$.

When initially x_0 is in the interior, the argument is similar, the only difference is that instead of using our Theorem 2.1 we use its interior version by De Philippis and Figalli (see [9, Theorem 4.3]). Then the proof of the first statement is completed by a standard covering argument.

In the case when $c(x, y) = \frac{1}{p}|x - y|^p$ with $p > 1$ is suffices to observe that, in both cases, after subtracting $\frac{1}{2}|x|^2 + \frac{1}{2}|y|^2$ to c (that does not change the optimal transport problem, see [10, Section 3.1] or [9] for more comments on this point) one has

$$\|c + x \cdot y\|_{C^2(X \times Y)} \to 0 \qquad \text{as } \bar{\delta} \to 0$$

(see [7] for more details). Hence, since c is smooth when $x \neq y$, the result follows from the first part of the corollary. $\qquad\square$

Proof of Corollary 2.4. We only need to slightly modify the proof of Corollary 2.3. Let v be the potential function to the optimal transport problem from $f\mathbf{1}_X$ to $g\mathbf{1}_Y$ with cost $-x \cdot y$. Since f and g are of class C^α, Caffarelli's boundary $C^{2,\alpha}$ estimate still applies. Using the same argument as in the proof of Corollary 2.3, we still have (2.16) (when the cost is $c(x, y) = \frac{1}{p}|x - y|^p$ with $p > 1$), (3.1), (3.2), (3.3), and flatness of the boundary. Hence all the conditions in Theorem 2.2 are satisfied. Therefore u is $C^{2,\alpha}_{\mathrm{loc}} \cap C^{2,\alpha'}$ with $\alpha' \in (0, \alpha)$ in a small neighborhood of x_0, for any x_0 on the boundary of X. Combining this with the interior $C^{2,\alpha}$ result of [9, Theorem 5.3] we conclude that u is $C^{2,\alpha}$ in the interior of X and $C^{2,\alpha'}$ up to the boundary, provided δ_0 is sufficiently small. $\qquad\square$

4. Proof of the main theorems

Proof of Theorem 2.1. We divide the proof into several steps.

• *Step 1: A first change of variables.* For $x_0 \in \mathcal{C}_1 \cap B_{\rho_0}$ with $\rho_0 \ll 1$ to be chosen, take $y_0 \in \partial_c u(x_0)$. Then we perform a change of variables $\bar{x} := x - x_0$, $\bar{y} := y - y_0$, and we define

$$(4.1) \qquad \bar{c}(\bar{x}, \bar{y}) := c(x, y) - c(x, y_0) - c(x_0, y) + c(x_0, y_0),$$

$$(4.2) \qquad \bar{u}(\bar{x}) := u(x) + c(x, y_0) - c(x_0, y_0) - u(x_0),$$

$$\bar{f}(\bar{x}) := f(\bar{x} + x_0), \qquad \bar{g} := g(\bar{y} + y_0).$$

First we show that, in the new coordinates,

$$(4.3) \qquad \|\bar{c}(\bar{x}, \bar{y}) + \bar{x} \cdot \bar{y}\|_{C^2(\mathcal{C}_1 \times \mathcal{C}_2)} \le 4\delta_0 =: \tilde{\delta}_0,$$

$$(4.4) \qquad \|\bar{u}(\bar{x}) - 1/2|\bar{x}|^2\|_{L^\infty(\mathcal{C}_1)} \le K(\sqrt{\eta_0} + \delta_0) =: \tilde{\eta}_0.$$

For this, notice that (4.3) follows from (2.10) easily, so we only need to verify (4.4). To this aim, we define

$$p_{x_0} := -D_x c(x_0, y_0) \in \partial^- u(x_0).$$

We claim that, for any direction $\mathbf{e}$, if $x_0 + t\mathbf{e} \in \mathcal{C}_1 \cap B_{1/2}$ for $0 \le t \le \sqrt{\eta_0}$ then $(p_{x_0} - x_0) \cdot \mathbf{e} \le K\sqrt{\eta_0}$ for some universal constant K. Notice that u is semiconvex, namely, there exists a constant C (depending only on $\|c\|_{C^2}$) such that $w(x) := u(x) - \frac{1}{2}|x|^2 + C|x - x_0|^2$ is convex. Since $p_{x_0} - x_0 \in \partial^- w(x_0)$, by convexity and (2.11) we have

$$\begin{aligned} (p_{x_0} - x_0) \cdot \mathbf{e} &\le \frac{w(x_0 + \sqrt{\eta_0}\mathbf{e}) - w(x_0)}{\sqrt{\eta_0}} \\ &\le \frac{2\eta_0 + C\eta_0}{\sqrt{\eta_0}} = (2 + C)\sqrt{\eta_0}. \end{aligned}$$

Hence the claim follows with $K := C + 2$.

We now notice that, by (2.10),

$$|p_{x_0} - y_0| \le \|D_x c + \mathrm{id}\|_{L^\infty(\mathcal{C}_1 \times \mathcal{C}_2)} \le \delta_0,$$

therefore,

$$(4.5) \qquad (y_0 - x_0) \cdot \mathbf{e} = (y_0 - p_{x_0}) \cdot \mathbf{e} + (p_{x_0} - x_0) \cdot \mathbf{e} \le K\sqrt{\eta_0} + \delta_0.$$

Now we consider two cases.

- *Case 1:* $d(x_0, \{x^n = P(x')\}) \ge 2\sqrt{\eta_0} + 2\delta_0$. In this case, we can use any $\mathbf{e} \in \mathbb{S}^n$ in (4.5), hence $|y_0 - x_0| \le K\sqrt{\eta_0} + \delta_0$.

- *Case 2:* $d(x_0, \{x^n = P(x')\}) \le 2\sqrt{\eta_0} + 2\delta_0$. For this case, we can still apply (4.5) with $\mathbf{e}$ satisfying $\mathbf{e} \cdot (0, \cdots, 0, 1) \ge 1/2$, provided δ_0 is small. Combining this with the fact that $y^n \ge -2\delta_0$ (see (2.7) and (2.8)), we also have $|y_0 - x_0| \le K(\sqrt{\eta_0} + \delta_0)$, where K needs to be enlarged by a universal constant.

Hence, in both cases

$$(4.6) \qquad |y_0 - x_0| \le K(\sqrt{\eta_0} + \delta_0),$$

and using (2.10) and (2.11) we get

$$
\begin{aligned}
\left| \bar{u}(\bar{x}) - \frac{1}{2}|\bar{x}|^2 \right| &= \left| u(x) - c(x, y_0) + c(x_0, y_0) - u(x_0) - \frac{1}{2}|x - x_0|^2 \right| \\
&\leq \left| u(x) - \frac{|x|^2}{2} \right| + \left| u(x_0) - \frac{|x_0|^2}{2} \right| \\
&\quad + |c(x, y_0) + x \cdot x_0| + |c(x_0, y_0) + x_0 \cdot x_0| \\
&\leq 2\eta + (|x| + |x_0|)|y_0 - x_0| + 2\delta_0 \leq K(\sqrt{\eta_0} + \delta_0)
\end{aligned}
$$

for some universal constant K, as desired. This concludes the proof of (4.4).

Recall that by assumption $|x_0| \leq \rho_0$ and (4.6) holds, hence (provided ρ_0, η_0, and δ_0 are sufficiently small) we have $B_3(x_0) \supset B_2$, $B_3(y_0) \supset B_2$, $B_{1/3}(x_0) \subset B_{1/2}$, $B_{1/3}(y_0) \subset B_{1/2}$. Also, in these new coordinates, the lower part of boundary of $\mathcal{C}_1 \cap B_{1/3}$ (resp. $\mathcal{C}_2 \cap B_{1/3}$) is defined by $\bar{P}$ (resp. $\bar{Q}$), and the graph of $\bar{P}$ (resp. $\bar{Q}$) is only a translation of the graph of P (resp. Q) by x_0 (resp. y_0) Notice that, since x_0 and y_0 are not necessarily boundary points, it is not true in general that $\bar{P}(\mathbf{0}) = \bar{Q}(\mathbf{0}) = 0$ nor that $\nabla \bar{P}(\mathbf{0}) = \nabla \bar{Q}(\mathbf{0}) = \mathbf{0}$.

• *Step 2: $\bar{u}$ is close to a strictly convex solution of the Monge-Ampère equation.* In this step, we approximate $\bar{u}$ by the solution of an optimal transport problem with quadratic cost. This step consists of two lemmas: Lemma 4.1 is about the construction of the approximating solution, and Lemma 4.2 is devoted to the smoothness of the approximating solution. .

Lemma 4.1. *Let $\delta > 0$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two closed sets such that*

$$(4.7) \qquad B_{1/K} \cap \{x^n \geq -d_i + \delta\} \subset \mathcal{C}_i \subset B_K \cap \{x^n \geq -d_i\}, \text{ for } i = 1, 2,$$

where $0 \leq d_i < \frac{1}{10}$. Let $\tilde{\mathcal{C}}_i := \mathcal{C}_i \cup (B_{1/K} \cap \{x^n \geq -d_i\})$. Suppose f and g are two densities supported respectively in $\mathcal{C}_1$ and $\mathcal{C}_2$, and $u : \mathcal{C}_1 \to \mathbb{R}$ is a c-convex functions such that $\partial_c u(\mathcal{C}_1) \subset B_K \cap \{x^n \geq -d_2\}$ and $(T_u)_\sharp f = g$. Let $\varrho > 0$ be such that $|\tilde{\mathcal{C}}_1| = |\varrho \tilde{\mathcal{C}}_2|$ (where $\varrho \tilde{\mathcal{C}}_2$ denotes the dilation of $\tilde{\mathcal{C}}_2$ with respect to the origin), and let v be a convex function such that $(\nabla v)_\sharp \mathbf{1}_{\tilde{\mathcal{C}}_1} = \mathbf{1}_{\varrho \tilde{\mathcal{C}}_2}$ and $v(\mathbf{0}) = u(\mathbf{0})$. Then there exists an increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on K and satisfying $\omega(\delta) \geq \delta$ and $\omega(0^+) = 0$, such that, if

$$(4.8) \qquad \|f - \mathbf{1}\|_{L^\infty(\mathcal{C}_1)} + \|g - \mathbf{1}\|_{L^\infty(\mathcal{C}_2)} \leq \delta$$

and

$$(4.9) \qquad \|c(x, y) + x \cdot y\|_{C^2(\mathcal{C}_1 \times \mathcal{C}_2)} \leq \delta,$$

then

$$\|u - v\|_{L^\infty(\mathcal{C}_1 \cap B_{1/K})} \leq \omega(\delta).$$

Proof. The proof of this lemma is similar to the proof of [9, Lemma 4.1]. For reader's convenience, we include the details here. We prove the lemma by contradiction. Assume the lemma is false. Then there exists $\epsilon_0 > 0$, a sequence of closed sets $\mathcal{C}_1^m$, $\mathcal{C}_2^m$ satisfying (4.7), $0 \leq d_i^m \leq 1/10$ for $i = 1, 2$, functions f_m, g_m satisfying (4.8) with $\delta = 1/m$, and costs c_m converging in C^2 to $-x \cdot y$, such that

$$u_m(\mathbf{0}) = v_m(\mathbf{0}) = 0, \text{ and } \|u_m - v_m\|_{L^\infty(\mathcal{C}_1^m \cap B_{1/K})} \geq \epsilon_0,$$

where u_m and v_m are as in the statement. Note that after passing to a subsequence we can assume $d_i^m \to d_i^\infty$ as $m \to \infty$, for $i = 1, 2$. Now we extend u_m and v_m to B_K as

$$u_m(x) := \sup_{z \in \mathcal{C}_1^m, y \in \partial_{c_m} u_m(z)} \{u_m(z) - c_m(x, y) + c_m(z, y)\},$$

$$v_m(x) := \sup_{z \in \mathcal{C}_1^m, p \in \partial^- v_m(z)} \{v_m(z) + p \cdot (x - z)\}.$$

Note that $(T_{u_m})_\sharp f_m = g_m$ gives that $\int f_m = \int g_m$, so it follows from (4.7) and (4.8) that

$$\varrho_m = \left(\frac{|\tilde{\mathcal{C}}_1^m|}{|\tilde{\mathcal{C}}_2^m|} \right)^{1/n} \to 1 \quad \text{as } m \to \infty,$$

which implies that $\partial^- v_m(B_K) \subset B_{\varrho_m K} \subset B_{2K}$. Thus, since the C^1 norm of c_m is uniformly bounded, we deduce that both u_m and v_m are uniformly Lipschitz. By the assumption that $u_m(\mathbf{0}) = v_m(\mathbf{0}) = 0$, passing to a subsequence, we have that u_m and v_m uniformly converge inside B_K to u_∞ and v_∞ respectively, where

$$(4.10) \qquad u_\infty(\mathbf{0}) = v_\infty(\mathbf{0}) = 0 \quad \text{and} \quad \|u_\infty - v_\infty\|_{L^\infty(B_{1/K} \cap \{x^n \geq -d_1^\infty\})} \geq \epsilon_0.$$

Moreover we have that f_m (resp. g_m) weak-$*$ converges in L^∞ to some density f_∞ (resp. g_∞). Also, since $\varrho_m \to 1$, by (4.8) we have that $\mathbf{1}_{\mathcal{C}_1}$ (resp. $\mathbf{1}_{\varrho \tilde{\mathcal{C}}_2}$) weak-$*$ converges in L^∞ to f_∞ (resp. g_∞). Note that by (4.8) and the fact that $\mathcal{C}_1^m \supset B_{1/K} \cap \{x^n \geq -d_i^m + \delta\}$, we also have that $f_\infty \geq \mathbf{1}_{B_{1/K} \cap \{x^n \geq -d_1^\infty\}}$.

Now, we apply [35, Theorem 5.20] to deduce that both ∇u_∞ and ∇v_∞ are optimal transport maps for the cost $-x \cdot y$ sending f_∞ onto g_∞. By uniqueness of optimal map, we have that $\nabla u_\infty = \nabla v_\infty$ almost everywhere inside $B_{1/K} \cap \{x^n \geq -d_1^\infty\} \subset \text{spt } f_\infty$, hence (since $u_\infty(\mathbf{0}) = v_\infty(\mathbf{0}) = 0$) $u_\infty = v_\infty$ in $B_{1/K} \cap \{x^n \geq -d_1^\infty\}$, contradicting to (4.10). $\qquad \square$

Denote $\tilde{\mathcal{C}}_1^+ := \mathcal{C}_1 \cup (B_{1/3} \cap \{\bar{x}^n \geq -x_0^n - 2\tilde{\delta}_0\})$ and $\tilde{\mathcal{C}}_2^+ := \mathcal{C}_2 \cup (B_{1/3} \cap \{\bar{y}^n \geq -y_0^n - 2\tilde{\delta}_0\})$ (notice that by (2.7) and (2.8), the inclusions $B_{1/3} \cap \{\bar{x}^n \geq -x_0^n + 2\tilde{\delta}_0\} \subset \mathcal{C}_1 \subset B_3 \cap \{\bar{x}^n \geq -x_0^n - 2\tilde{\delta}_0\}$ and $B_{1/3} \cap \{\bar{y}^n \geq -x_0^n + 2\tilde{\delta}_0\} \subset \mathcal{C}_2 \subset B_3 \cap \{\bar{y}^n \geq -x_0^n - 2\tilde{\delta}_0\}$ hold).

Now let ϱ be such that $|\tilde{\mathcal{C}}_1^+| = |\varrho\tilde{\mathcal{C}}_2^+|$ (where $\varrho\tilde{\mathcal{C}}_2^+$ is the dilation of $\tilde{\mathcal{C}}_2^+$ with respect to the origin), and let v be a convex function such that $\nabla v_\sharp \mathbf{1}_{\tilde{\mathcal{C}}_1^+} = \mathbf{1}_{\tilde{\mathcal{C}}_2^+}$ and $v(\mathbf{0}) = \bar{u}(\mathbf{0}) = 0$. By (4.3) and Lemma 4.1

$$(4.11) \qquad \|\bar{u} - v\|_{L^\infty(\{\bar{x}^n \geq \bar{P}\} \cap B_{1/3})} \leq \omega(\tilde{\delta}_0),$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\omega(\delta) \geq \delta$ and $\omega(0^+) = 0$.

Next, we use a symmetrization trick. Let $\tilde{\mathcal{C}}_1^-$ (resp. $(\varrho\tilde{\mathcal{C}}_2)^-$) be the reflection of $\tilde{\mathcal{C}}_1^+$ (resp. $\varrho\tilde{\mathcal{C}}_2^+$) with respect to the hyperplane $\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\}$ (resp. $\{\bar{y}^n = \varrho(-y_0^n - 2\tilde{\delta}_0)\}$), and denote $\tilde{\mathcal{C}}_1 := \tilde{\mathcal{C}}_1^+ \cup \tilde{\mathcal{C}}_1^-$ and $\tilde{\mathcal{C}}_2 := \varrho\tilde{\mathcal{C}}_2^+ \cup (\varrho\tilde{\mathcal{C}}_2)^-$. Let $\tilde{v}$ be the convex potential of the optimal transportation from $\mathbf{1}_{\tilde{\mathcal{C}}_1}$ to $\mathbf{1}_{\tilde{\mathcal{C}}_2}$, with $\tilde{v}(\mathbf{0}) = 0$. Then $\tilde{v}$ enjoys the following nice properties.

Lemma 4.2. $\tilde{v}|_{\tilde{\mathcal{C}}_1^+} = v$, and $\tilde{v}$ is smooth inside $B_{1/10}$ with $\|\tilde{v}\|_{C^3(B_{1/10})} \leq K$.

Proof. To prove the claim, it is more convenient to translate both coordinates $\bar{x}$ and $\bar{y}$ so that both the center of $B_{1/3} \cap \{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\}$ and the center of $B_{1/3} \cap \{\bar{y}^n = \varrho(-y_0^n - 2\tilde{\delta}_0)\}$) are located at the origin. For simplicity, we still use $\bar{x}$ and $\bar{y}$ to denote the new variables.

Then, the first part of the claim follows because the uniqueness of optimal transport map implies that it must be symmetric. Indeed, being the densities symmetric with respect to the hyperplanes $\{\bar{x}^n = 0\}$ and $\{\bar{y}^n = 0\}$, if $T = (T', T^n) : \mathbb{R}^n \to \mathbb{R}^n$ is optimal for the cost $-x \cdot y$ then the map $\hat{T}(\bar{x}', \bar{x}^n) := \left(T'(\bar{x}', -\bar{x}^n), -T^n(\bar{x}', -\bar{x}^n)\right)$ is still optimal (as it has the same transportation cost of T), so by uniqueness $\hat{T} = T$. This proves that $\nabla \tilde{v}|_{\tilde{\mathcal{C}}_1^+} = \nabla v$, and because $\tilde{v}(\mathbf{0}) = v(\mathbf{0})$ the result follows.

For the second part of the claim notice that, by (4.4) and (4.11),

$$\|\tilde{v} - 1/2|\bar{x}|^2\|_{L^\infty(B_{1/3} \cap \{|\bar{x}^n| \geq \delta_0\})} \leq \omega(\tilde{\delta}_0) + \tilde{\eta}_0.$$

Now by convexity of v one can easily show that $\partial^- \tilde{v}(B_{1/10}) \subset B_{1/6}$, so arguing as in [13] we have that $\tilde{v}$ is smooth inside $B_{1/8}$, with $\|\tilde{v}\|_{C^3(B_{1/8})} \leq K$. Recalling that $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$ with $|x_0| \leq \rho_0$ and $|y_0| \leq \rho_0 + K\left(\sqrt{\tilde{\eta}_0} + \delta_0\right)$ (see (4.6)), provided ρ_0, δ_0, η_0 are sufficiently small we see that, in the original $(\bar{x}, \bar{y})$ coordinates, $\tilde{v}$ is smooth inside $B_{1/10}$ with $\|\tilde{v}\|_{C^3(B_{1/10})} \leq K$. $\quad\square$

Next, we compute the hessian of $\tilde{v}$ at $\hat{x}_1 = (0, \cdots, 0, -x_0^n - 2\tilde{\delta}_0)$ and the origin. First note that the C^3 bound of $\tilde{v}$ implies that $\mathrm{Id}/K \leq D^2\tilde{v} \leq K\mathrm{Id}$. By symmetry we have that $\nabla_n \tilde{v}$ is constant on $\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\}$, which implies

(4.12) $$\tilde{v}_{ni}(\hat{x}_1) = 0, \text{ for } 1 \leq i \leq n - 1.$$

Since $|x_0| \leq \rho_0$, by the C^3 bound of $\tilde{v}$ we have that

(4.13) $$\tilde{v}_{ij}(\mathbf{0}) = \tilde{v}_{ij}(\hat{x}_1) + O(\rho_0) + O(\tilde{\delta}_0), 1 \leq i, j \leq n,$$

so, in particular, $\tilde{v}_{ni}(\mathbf{0}) = O(\rho_0) + O(\tilde{\delta}_0)$ for all $i = 1, \ldots, n - 1$.

• *Step 3: Initial step for the iteration.* In the next Lemma we will show that there exists an affine transformation such that $\bar{u} \circ A$ satisfies all the properties in the list:
1. both the sub-level set $\{\bar{u} \leq h_0\}$ and its image are comparable to $B_{\sqrt{h_0}}$;
2. $\bar{u}$ will be very close to $\frac{|\bar{x}|^2}{2}$;
3. both $\|A\|$ and $\|A^{-1}\|$ are bounded by some universal constant.

Lemma 4.3. *For every $\tilde{\eta}_0$ small, there exist small positive constants h_0, $\tilde{\delta}_0$ for which the following holds: there exists a symmetric matrix A satisfying $\|A\|$, $\|A^{-1}\| \leq K_1$ and $\det A = 1$ such that*

$$A\left(B_{\sqrt{\frac{h_0}{3}}}(\mathbf{0})\right) \cap \{\bar{x}^n \geq \bar{P}\} \subset S_{\bar{u}}(h_0) := \{\bar{u} \leq h_0\} \subset A\left(B_{\sqrt{3h_0}}(\mathbf{0})\right) \cap \{\bar{x}^n \geq \bar{P}\},$$

$$A^{-1}\left(B_{\sqrt{\frac{h_0}{3}}}(\mathbf{0})\right) \cap \{\bar{y}^n \geq \bar{Q}\} \subset \partial_{\bar{c}}\bar{u}(S_{\bar{u}}(h_0)) \subset A^{-1}\left(B_{\sqrt{3h_0}}(\mathbf{0})\right) \cap \{\bar{y}^n \geq \bar{Q}\}.$$

Moreover

$$\left\|\bar{u} - \frac{1}{2}|A^{-1}\bar{x}|^2\right\|_{L^\infty\left(A(B_{\sqrt{3h_0}}(\mathbf{0})) \cap \{\bar{x}^n \geq \bar{P}\}\right)} \leq \tilde{\eta}_0 h_0,$$

and $A^{-1}(\boldsymbol{e}_n)$ is parallel to $A(\boldsymbol{e}_n)$.

Proof. First we estimate the norm of $\nabla\tilde{v}(\mathbf{0})$. We claim that

$$(4.14) \qquad |\nabla\tilde{v}(\mathbf{0})| \leq K_2\sqrt{\omega(\tilde{\delta}_0)},$$

where K_2 is a universal constant. To prove (4.14), it is enough to show that

$$-K_2\sqrt{\omega(\tilde{\delta}_0)} \leq \tilde{v}_n(\mathbf{0}) \leq K_2\sqrt{\omega(\tilde{\delta}_0)} \quad \text{and} \quad -\nabla\tilde{v}(\mathbf{0})\cdot\mathbf{e} \leq K_2\sqrt{\omega(\tilde{\delta}_0)},$$

for any unit vector $\mathbf{e}$ satisfying $\mathbf{e}\cdot\mathbf{e}_n \geq 1/2$.

Since $\bar{u}$ is semiconvex and $\tilde{v}$ is smooth inside $B_{1/10}$ with $\|\tilde{v}\|_{C^3(B_{1/10})} \leq K$, there exists a universal constant K_2 such that $\bar{u} - \tilde{v} + K_2|\bar{x}|^2$ is a convex function inside $B_{1/10}\cap\{\bar{x}^n \geq P(\bar{x}')\}$. Then by convexity, the fact that $\mathbf{0}$ is a minimum point for $\bar{u}$, and (4.11), we get

$$\begin{aligned}
-\tilde{v}_n(\mathbf{0}) &\leq& \frac{(\bar{u} - \tilde{v} + K_2|\bar{x}|^2)|_{(\mathbf{0}+\sqrt{\omega(\tilde{\delta}_0)}\mathbf{e}_n)} - (\bar{u} - \tilde{v} + K_2|\bar{x}|^2)|_{\mathbf{0}}}{\sqrt{\omega(\tilde{\delta}_0)}} \\
&\leq& \frac{(K_2+2)\omega(\tilde{\delta}_0)}{\sqrt{\omega(\tilde{\delta}_0)}} = (K_2+2)\sqrt{\omega(\tilde{\delta}_0)},
\end{aligned}$$

which implies $(-K_2-2)\sqrt{\omega(\tilde{\delta}_0)} \leq \tilde{v}_n(\mathbf{0})$. By replacing $\mathbf{e}_n$ with unit vector $\mathbf{e}$ satisfying $\mathbf{e}\cdot\mathbf{e}_n \geq 1/2$ in the above computation, we also have

$$(4.15) \qquad -\nabla\tilde{v}(\mathbf{0})\cdot\mathbf{e} \leq K_2\sqrt{\omega(\tilde{\delta}_0)}.$$

Finally, we prove the upper bound on $\tilde{v}_n(\mathbf{0})$. Denote by d_1 the distance between $\mathbf{0}$ and $\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\}$.

When $d_1 \geq \sqrt{\omega(\tilde{\delta}_0)}$, the proof is the same to the above proof of the lower bound on $\tilde{v}_n(\mathbf{0})$ simply replacing $-\mathbf{e}_n$ with $\mathbf{e}_n$.

When $d_1 \leq \sqrt{\omega(\tilde{\delta}_0)}$ we use that $\nabla\tilde{v}$ maps $\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\}$ onto $\{\bar{y}^n = \rho(-y_0^n - 2\tilde{\delta}_0\}$ (this follows by the symmetric of $\tilde{v}$, see the proof of Lemma 4.2) to deduce that $\tilde{v}_n(0, 0, \cdots, -d_1) \leq 0$, hence

$$\begin{aligned}
\tilde{v}_n(\mathbf{0}) &=& \tilde{v}_n(0, 0, \cdots, -d_1) + \left(\int_0^1 \tilde{v}_{nn}(0, 0, \cdots, -td_1)\, dt\right)d_1 \\
&\leq& K_2 d_1 \leq K_2\sqrt{\omega(\tilde{\delta}_0)},
\end{aligned}$$

concluding the proof of (4.14).

By Lemma 4.2 and Taylor expansion we have

$$(4.16) \qquad \tilde{v}(\bar{x}) = \nabla\tilde{v}(\mathbf{0})\cdot\bar{x} + \frac{1}{2}D^2\tilde{v}(\mathbf{0})\bar{x}\cdot\bar{x} + O(|\bar{x}|^3),$$

where we used $\tilde{v}(\mathbf{0}) = \bar{u}(\mathbf{0}) = 0$, hence by (4.11) and (4.14) we get

$$\left\|\bar{u} - \frac{1}{2}D^2\tilde{v}(\mathbf{0})\bar{x}\cdot\bar{x}\right\|_{L^\infty\left(E(4h_0)\cap\{\bar{x}^n\geq\bar{P}\})\right)} \leq \omega(\tilde{\delta}_0) + K_2\sqrt{\omega(\tilde{\delta}_0)}\sqrt{h_0} + K_3 h_0^{\frac{3}{2}},$$

where K_2, K_3 are universal constants, and

$$E(h_0) := \left\{ \bar{x} : \frac{1}{2} D^2 \tilde{v}(\mathbf{0}) \bar{x} \cdot \bar{x} \leq h_0 \right\}.$$

So if initially we take $\tilde{\delta}_0$, h_0 small, we have

$$\left\| \bar{u} - \frac{1}{2} D^2 \tilde{v}(\mathbf{0}) \bar{x} \cdot \bar{x} \right\|_{L^\infty \left(E(4h_0) \cap \{\bar{x}^n \geq \bar{P}\} \right)} \leq \frac{1}{2} \tilde{\eta}_0 h_0.$$

Denote $A_1 := [D^2 \tilde{v}(\mathbf{0})]^{-1/2}$. By (4.13) and Lemma 4.2 we see that the angle between $A_1^{-1}(\mathbf{e}_n)$ and $A_1(\mathbf{e}_n)$ is bounded by $O(\rho_0)$. Then, it is easy to find a symmetric matrix A, with $\|A - A_1\| = O(\rho_0) + O(\tilde{\delta}_0)$, such that $A^{-1}(\mathbf{e}_n)$ is parallel to $A(\mathbf{e}_n)$. In particular, by choosing ρ_0 sufficiently small we obtain

$$\left\| \bar{u} - \frac{1}{2} |A^{-1} \bar{x}|^2 \right\|_{L^\infty \left(A(B_{\sqrt{4h_0}}(\mathbf{0})) \cap \{\bar{x}^n \geq \bar{P}\} \right)} \leq \tilde{\eta}_0 h_0.$$

We now perform a normalization using A: Let

$$\begin{cases} \tilde{x} := A^{-1} \bar{x} \\ \tilde{y} := A \bar{y} \end{cases}$$

and

$$\tilde{c}(\tilde{x}, \tilde{y}) := \bar{c}(A\tilde{x}, A^{-1}\tilde{y}),$$

$$\tilde{u} := \bar{u}(A\tilde{x}).$$

Note that, since $\mathrm{Id}/K \leq A \leq K\mathrm{Id}$, we have

$$(4.17) \qquad \|\tilde{c} + \tilde{x} \cdot \tilde{y}\|_{C^2} \leq K \tilde{\delta}_0$$

and

$$(4.18) \qquad \left\| \tilde{u} - \frac{1}{2} |\tilde{x}|^2 \right\|_{L^\infty \left(B_{\sqrt{4h_0}}(\mathbf{0}) \cap A^{-1}\{\bar{x}^n \geq \bar{P}\} \right)} \leq \tilde{\eta}_0 h_0.$$

Let us denote by d_1' (resp. d_2') the distance between $\mathbf{0}$ and the hyperplane $A^{-1}(\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\})$ (resp. $A(\{\bar{y}^n = \varrho(-y_0^n - 2\tilde{\delta}_0)\})$). Since by construction $A D^2 \tilde{v}(\mathbf{0}) A = 1 + O(\rho_0) + O(\tilde{\delta}_0)$, it follows by Lemma 4.2 that the hessian of the function $\tilde{v}(Ax)$ is equal to $(1 + o(1))\mathrm{Id}$ inside $B_{K\sqrt{h_0}}$ for some fixed large K, where $o(1) \to 0$ as $\rho_0, h_0, \tilde{\delta}_0 \to 0$. Since $\nabla(\tilde{v} \circ A)$ maps $A^{-1}(\{\bar{x}^n = -x_0^n - 2\tilde{\delta}_0\})$ onto $A(\{\bar{y}^n = \varrho(-y_0^n - 2\tilde{\delta}_0)\})$, we deduce that

$$-d_2' = \partial_n [\tilde{v}(A\bar{x})]|_{\mathbf{0}} - (1 + o(1)) d_1',$$

so, using (4.14) and the fact that $|x_0| \leq \rho_0 \ll \sqrt{h_0}$,

$$(4.19) \qquad |d_1' - d_2'| \leq K_2 \sqrt{\omega(\tilde{\delta}_0)} + o(1) \sqrt{h_0}.$$

By (4.18) and an argument similar to the proof of (4.14), one obtains that $\tilde{u}$ (resp. $\nabla \tilde{u}$) is close to $\tilde{v} \circ A$ (resp. $\nabla[\tilde{v} \circ A]$) and, exactly as in the interior case (see [9, Proof of Theorem 4.3, Step 3]), we get

$$B_{\sqrt{\frac{1}{2} h_0}}(\mathbf{0}) \cap A^{-1}\{\bar{x}^n \geq \bar{P}\} \subset S_{\tilde{u}}(h_0) \subset B_{\sqrt{3h_0}}(\mathbf{0}) \cap A^{-1}\{\bar{x}^n \geq \bar{P}\},$$

and
$$\partial_{\tilde{c}}\tilde{u}(S_{\tilde{u}}(h_0)) \subset B_{\sqrt{3h_0}}(\mathbf{0}) \cap A\{\bar{y}^n \geq \bar{Q}\}.$$

Now let $\tilde{u}^{\tilde{c}} : B_{\sqrt{4h_0}}(\mathbf{0}) \cap A\{\bar{y}^n \geq \bar{Q}\} \to \mathbb{R}$ be a $\tilde{c}^*$-convex function defined by

$$\tilde{u}^{\tilde{c}}(\tilde{y}) := \sup_{\tilde{x} \in B_{\sqrt{4h_0}}(\mathbf{0}) \cap A^{-1}\{\bar{x}^n \geq \bar{P}\}} \{-\tilde{c}(\tilde{x}, \tilde{y}) - \tilde{u}(\tilde{x})\},$$

where $\tilde{c}^*(x, y) := \tilde{c}(y, x)$. Then by (4.17), (4.18), (4.19), we have

$$\left\| \tilde{u}^{\tilde{c}} - \frac{1}{2}|\tilde{y}|^2 \right\|_{B_{\sqrt{4h_0}}(\mathbf{0}) \cap A\{\bar{y}^n \geq \bar{Q}\}} \quad \leq \quad \tilde{\eta}_0 h_0 + K\tilde{\delta}_0 + \left(K_2\sqrt{\omega(\tilde{\delta}_0)} + o(1)\sqrt{h_0} \right)\sqrt{h_0}$$

$$\leq \quad 2\tilde{\eta}_0 h_0,$$

provided $\tilde{\delta}_0$, ρ_0, and $\sqrt{h_0}$ are small enough. Also, similarly to above,

$$\partial_{\tilde{c}^*}\tilde{u}^{\tilde{c}}\left(B_{\sqrt{\frac{1}{3}h_0}}(\mathbf{0}) \cap A\{\bar{y}^n \geq \bar{Q}\} \right) \subset B_{\sqrt{\frac{1}{2}h_0}}(\mathbf{0}) \cap A^{-1}\{\bar{x}^n \geq \bar{P}\} \subset S_{\tilde{u}}(h_0).$$

Therefore $B_{\sqrt{\frac{1}{3}h_0}}(\mathbf{0}) \cap A\{\bar{y}^n \geq \bar{Q}\} \subset \partial_{\tilde{c}}\tilde{u}(S_{\tilde{u}}(h_0))$, and translating back to the $\bar{x}, \bar{y}$ coordinates this completes the proof of Lemma 4.3. $\qquad\square$

• *Step 4: The iteration argument.* We begin by noticing that by construction $\bar{d}_1 := \text{dist}(\mathbf{0}, B_{1/3} \cap \{\bar{x}^n = \bar{P}\}) \leq \rho_0$.

Up to an affine transformation we can assume $D_{xy}\bar{c}(\mathbf{0}, \mathbf{0}) = \text{Id}$. We now perform a change of variable: Let

$$\begin{cases} \tilde{x} := \frac{1}{\sqrt{h_0}}A^{-1}\bar{x} \\ \tilde{y} := \frac{1}{\sqrt{h_0}}A\bar{y} \end{cases}$$

and

$$c_1(\tilde{x}, \tilde{y}) := \frac{1}{h_0}\bar{c}(\sqrt{h_0}A\tilde{x}, \sqrt{h_0}A^{-1}\tilde{y}),$$

$$u_1 := \frac{1}{h_0}\bar{u}(\sqrt{h_0}A\tilde{x}),$$

where A is from Lemma 4.3. Note that, since $A^{-1}(\mathbf{e}_n)$ is parallel to $A(\mathbf{e}_n)$, after the transformation, we have that $\{\bar{x}^n = \bar{P}(\bar{x}')\}$ (resp. $\{\bar{y}^n = \bar{Q}(\bar{y}')\}$) becomes $A^{-1}\{\bar{x}^n = \bar{P}(\bar{x}')\}$ (resp. $A\{\bar{y}^n = \bar{Q}(\bar{y}')\}$), and after a rotation of coordinates it can be written as $\{\tilde{x}^n = P_1(\tilde{x}')\}$ (resp. $\{\tilde{y}^n = Q_1(\tilde{y}')\}$). Since $(\|A\| + \|A^{-1}\|)\sqrt{h_0} \ll 1$, we can ensure that $\|P_1\|_{C^2} + \|Q_1\|_{C^2} \leq \tilde{\delta}_0$.

We also define

$$f_1(\tilde{x}) := \bar{f}(\sqrt{h_0}A\tilde{x}), \quad g_1(\tilde{y}) := \bar{g}(\sqrt{h_0}A^{-1}\tilde{y}).$$

Since $\det(A) = 1$, we have that $(T_{u_1})_\sharp f_1 = g_1$. Moreover, defining

$$\mathcal{C}_1^{(1)} := S_{u_1}(1), \quad \mathcal{C}_2^{(1)} := \partial_{c_1}u_1(S_{u_1}(1)),$$

it follows by Lemma 4.3 that

$$(4.20) \qquad\qquad B_{1/3} \cap \{\tilde{x}^n \geq P_1(\tilde{x}')\} \subset \mathcal{C}_1^{(1)} \subset B_3 \cap \{\tilde{x}^n \geq P_1(\tilde{x}')\},$$

$$(4.21) \qquad\qquad B_{1/3} \cap \{\tilde{y}^n \geq Q_1(\tilde{x}')\} \subset \mathcal{C}_2^{(1)} \subset B_3 \cap \{\tilde{y}^n \geq Q_1(\tilde{x}')\}.$$

Now it is easy to check that $u_1, c_1, f_1, g_1, \mathcal{C}_1^{(1)}, \mathcal{C}_2^{(1)}$ satisfy all the conditions for Lemma 4.3. Therefore, we can apply Lemma 4.3 to u_1 and we can find a matrix A_1 satisfying $\|A_1\|, \|A_1^{-1}\| \leq K_1$, $\det(A_1) = 1$,

$$(4.22) \quad A_1 \left(B_{\sqrt{\frac{h_0}{3}}}(\mathbf{0}) \right) \cap \{\tilde{x}^n \geq P_1\} \subset S_{u_1}(h_0) := \{u_1 \leq h_0\} \subset A_1 \left(B_{\sqrt{3h_0}}(\mathbf{0}) \right) \cap \{\tilde{x}^n \geq P_1\},$$

$$(4.23) \quad A_1^{-1} \left(B_{\sqrt{\frac{h_0}{3}}}(\mathbf{0}) \right) \cap \{\tilde{y}^n \geq Q_1\} \subset \partial_{c_1} u_1(S_{u_1}(h_0)) \subset A_1^{-1} \left(B_{\sqrt{3h_0}}(\mathbf{0}) \right) \cap \{\tilde{y}^n \geq Q_1\},$$

$$\left\| u_1 - \frac{1}{2}|A_1^{-1}\tilde{x}|^2 \right\|_{L^\infty \left(A_1(B_{\sqrt{3h_0}}(\mathbf{0})) \cap \{\tilde{x}^n \geq P_1\} \right)} \leq \tilde{\eta}_0 h_0,$$

and $A_1^{-1}(\mathbf{e}_n)$ is parallel to $A_1(\mathbf{e}_n)$. Note that after a rotation we can just assume $A_1^{-1}(\mathbf{e}_n)$ and $A_1(\mathbf{e}_n)$ are in $\mathbf{e}_n$ direction.

Now, we finally fix $\rho_0 := 10K\sqrt{h_0}$ (where K is constant in Lemma 4.2). As long as $\mathrm{dist}(\mathbf{0}, B_{1/3} \cap \{\tilde{x}^n = P_k\}) \leq \rho_0$, , we can continue the iteration, hence we only need to consider two cases.

-*Case 1*: At step $k+1$, $d(\mathbf{0}, B_{1/3} \cap \{\tilde{x}^n = P_{k+1}\}) > \rho_0$. Here we assume $k+1$ is the smallest among all such integers.

-*Case 2*: The iteration goes on forever. Note that case 2 only happens when $\mathbf{0}$ is on the boundary of $\mathcal{C}_1$.

For Case 1, at step $k+1$, after an affine transformation $M_k := A_{k-1} \cdots A_1$ we have that

$$u_{k+1} := \frac{1}{h_0^k} u_1(h_0^{k/2} M_k \tilde{x}), \qquad c_{k+1} := \frac{1}{h_0^k} c_1(h_0^{k/2} M_k \tilde{x}, h_0^{k/2} M_k'^{-1} \tilde{y}),$$

$$f_{k+1} := f_1(h_0^{k/2} M_k \tilde{x}), \qquad g_{k+1} := g_1(h_0^{k/2} M_k'^{-1} \tilde{y})$$

satisfy the same conditions as u_1, c_1, f_1, g_1 with exactly the same constants (here and in the sequel, M_k' denotes the transpose of M_k). Since $\mathrm{dist}(\mathbf{0}, B_{1/3} \cap \{\tilde{x}^n = P_{k+1}\}) \geq \rho_0 = 10K\sqrt{h_0}$, by doing one more rescaling we obtain that $\mathrm{dist}(\mathbf{0}, B_{1/3} \cap \{\tilde{x}^n = P_{k+2}\}) \geq 1/K'$ for some $K' > 0$ universal, so we have reduced ourselves to the interior problem as the one studied in [9]. In particular, by [9, Theorem 4.3] we obtain that u is $C^{1,\beta}$ at $\mathbf{0}$.

For Case 2, with the same notation as in Case 1 we have that, for each $k \geq 1$,

$$(4.24) \qquad\qquad \mathrm{Id}/K_1^k \leq M_k \leq K_1^k \mathrm{Id}, \quad \det M_k = 1,$$

$$(4.25) \qquad M_k \left(B_{\frac{1}{3} h_0^{k/2}} \right) \cap \{\tilde{x}^n \geq P_1\} \subset S_{u_1}(h_0^k) \subset M_k \left(B_{3h_0^{k/2}} \right) \cap \{\tilde{x}^n \geq P_1\}.$$

$$(4.26) \qquad M_k'^{-1} \left(B_{\frac{1}{3} h_0^{k/2}} \right) \cap \{\tilde{y}^n \geq Q_1\} \subset \partial_{c_1} u_1 \left(S_{u_1}(h_0^k) \right) \subset M_k'^{-1} \left(B_{3h_0^{k/2}} \right) \cap \{\tilde{y}^n \geq Q_1\}.$$

By (4.24) we have

$$(4.27) \qquad\qquad B_{\frac{1}{3}\left(\frac{\sqrt{h_0}}{K_1}\right)^k} \cap \{\tilde{x}^n \geq P_1\} \subset S_{u_1}(h_0^k) \subset B_{3(K_1\sqrt{h_0})^k} \cap \{\tilde{x}^n \geq P_1\}$$

and

$$(4.28) \qquad B_{\frac{1}{3}\left(\frac{\sqrt{h_0}}{K_1}\right)^k} \cap \{\tilde{x}^n \geq Q_1\} \subset \partial_{c_1} u_1\left(S_{u_1}(h_0^k)\right) \subset B_{3(K_1\sqrt{h_0})^k} \cap \{\tilde{y}^n \geq Q_1\},$$

so defining $r_0 := \frac{\sqrt{h_0}}{3K_1}$ we obtain, for $\beta < 1$ fixed,

$$\|u_1\|_{L^\infty\left(B_{r_0^k} \cap \{\tilde{x}^n \geq P_1\}\right)} \leq h_0^k = (3K_1 r_0)^{2k} \leq r_0^{(1+\beta)k},$$

provided h_0 (and so r_0) is sufficiently small. This implies the $C^{1,\beta}$ regularity of u_1 at $\mathbf{0}$, which means that u is $C^{1,\beta}$ at x_0. Since $x_0 \in \mathcal{C}_1 \cap B_{\rho_0}$ is arbitrary, this concludes the proof of the theorem. $\qquad\square$

Remark 4.4. Under the conditions of Theorem 2.1, the following useful property holds: there exists $\rho_1 \leq \rho_0$ such that $T_u(B_{\rho_1} \cap \{x^n = P(x')\}) \subset \{y^n = Q(y')\}$. Indeed, let

$$u^c(y) = \sup_{x \in \mathcal{C}_1} \{-c(x, y) - u(x)\}.$$

By (2.8), (2.10), and (2.11) it is easy to check that $\|u^c(y) - \frac{1}{2}|y|^2\|_{C^0(B_{1/2} \cap \{y^n > Q(y')\})} \to 0$ as $\delta_0 \to 0$. Hence, when δ_0 is sufficiently small, by restricting to a smaller domain we can still apply Theorem 2.1 to u^c obtaining that u^c is $C^{1,\beta}$ in $B_{\rho_0'} \cap \{y^n \geq Q(y')\}$ for some $\rho_0' > 0$. Let T_{u^c} be the optimal transport map from $\mathcal{C}_2$ to $\mathcal{C}_1$. It is well known that $Du^c(y) = D_y c(T_{u^c}(y), y)$, and T_{u^c} is the inverse of T_u in an almost everywhere sense. Since u (resp. u^c) is $C^{1,\beta}$ in $B_{\rho_0} \cap \{x^n \geq P(x')\}$ (resp. $B_{\rho_0'} \cap \{y^n \geq Q(y')\}$) we deduce that both T_u and T_{u^c} are continuous near $\mathbf{0}$, being one the inverse of the other, T_u is a homeomorphism from $B_{\rho_1} \cap \{x^n \geq P(x')\}$ to $T_u(B_{\rho_1} \cap \{x^n \geq P(x')\})$ for any ρ_1 sufficiently small. From this fact it is easy to conclude that $T_u(B_{\rho_1} \cap \{x^n = P(x')\}) \subset \{y^n = Q(y')\}$, as desired.

5. $C^{2,\alpha}$ REGULARITY

Below we still use P and Q to denote two C^2 functions defined on $\mathbb{R}^{n-1}$ such that $P(\mathbf{0}) = Q(\mathbf{0}) = 0$, $\nabla P(\mathbf{0}) = \nabla Q(\mathbf{0}) = \mathbf{0}$, and

$$(5.1) \qquad \|P\|_{C^2} + \|Q\|_{C^2} \leq \delta.$$

$co[E]$ is used to denote the convex hull of a set E, and $\mathcal{N}_r(E)$ is used to denote the r-neighborhood of E. S^- denotes the reflection of S with respect to the hyperplane $\{x^n = 0\}$, and $\tilde{S} := S \cup S^-$.

Lemma 5.1. *(Comparison principle) Let u be a c-convex function of class C^1 inside the set $S := \{u < 1\}$, and assume that $u(\mathbf{0}) = 0$,*

$$(5.2) \qquad B_{1/K} \cap \{x^n \geq P(x')\} \subset S \subset B_K \cap \{x^n \geq P(x')\},$$

$$(5.3) \qquad B_{1/K} \cap \{y^n \geq Q(y')\} \subset \partial_c u(S) \subset B_K \cap \{y^n \geq Q(y')\},$$

and

$$(5.4) \qquad \partial_c u\left(\partial S \cap \{x^n = P(x')\}\right) \subset B_K \cap \{y^n = Q(y')\}.$$

Let f, g be two densities such that

$$\|f/\lambda_1 - 1\|_{L^\infty(S)} + \|g/\lambda_2 - 1\|_{L^\infty(S)} \leq \epsilon$$

for some constants $\lambda_1, \lambda_2 \in (1/2, 2)$ *and* $\epsilon \in (0, 1/4)$, *and assume that* $(T_u)_\sharp f = g$. *Furthermore, suppose that*

$$(5.5) \qquad \|c + x \cdot y\|_{C^2(S \times \partial_c u(S))} \leq \delta.$$

Then there exist a universal constant $\gamma \in (0, 1)$ *and* $\delta_1 = \delta_1(K) > 0$ *small, such that the following holds: Let* v *be the solution of*

$$(5.6) \qquad \begin{cases} \det(D^2 v) = \lambda_1/\lambda_2 & \text{in } \mathcal{N}_{\delta^\gamma}(co[\tilde{S}]), \\ v = 1 & \text{on } \partial\left(\mathcal{N}_{\delta^\gamma}(co[\tilde{S}])\right) \end{cases}$$

Then

$$(5.7) \qquad \|u - v\|_{L^\infty(S)} \leq C_K(\epsilon + \delta^{\gamma/n}) \quad \text{provided } \delta \leq \delta_1,$$

where C_K *is a constant independent of* $\lambda_1, \lambda_2, \epsilon,$ *and* δ.

Note that in our application of the comparison principle in the proof of Theorem 2.2, condition (5.4) follows from Remark 4.4.

Proof. First of all we notice that, for any $\gamma \in (0, 1)$, we have that

$$(5.8) \qquad \text{dist}\left(x, \partial\left(\mathcal{N}_{\delta^\gamma}(co[\tilde{S}])\right)\right) \leq C'_K \delta^\gamma, \text{ for any } x \in \partial S \cap \{x^n > P(x')\}.$$

Indeed, this follows by the very same argument as the one at the beginning of the proof of [9, Proposition 5.2], where the same estimate is proved in a similar situation.

Now, by standard interior estimates for solution of the Monge-Ampère equation with constant right hand side, we also have

$$(5.9) \qquad \text{osc}_S v \leq C''_K,$$

$$(5.10) \qquad 1 - C''_K \delta^{\gamma/n} \leq v \leq 1 \quad \text{on } \partial S \cap \{x^n > P(x')\},$$

$$(5.11) \qquad C''_K \delta^{-\frac{(n-1)\gamma}{\tau}} \geq D^2 v \geq \delta^{\gamma/\tau} \text{Id}/C''_K \quad \text{in } co[\tilde{S}],$$

for some $\tau > 0$ universal and some constant C''_K depending only on K. For any point x on $\partial S \cap \{x^n = P(x')\}$, by (5.1), (5.4), and (5.5) we have

$$(5.12) \qquad \left|\frac{\partial u}{\partial x^n}\right| \leq C''_K \delta,$$

Since $\mathcal{N}_{\delta^\gamma}(co[\tilde{S}])$ is symmetric with respect to $\{x^n = 0\}$, we see that v satisfies $v(x', x^n) = v(x', -x^n)$, which implies $\frac{\partial v}{\partial x^n} = 0$ on $co[\tilde{S}] \cap \{x^n = 0\}$. Now, for $x = (x', x^n) \in \partial S \cap \{x^n = P(x')\}$, we take the point $z := (x', 0) \in co[\tilde{S}] \cap \{x^n = 0\}$, by (5.11) we have that

$$(5.13) \qquad 0 \leq \frac{\partial v}{\partial x^n}(x) \leq \frac{\partial v}{\partial x^n}(z) + C''_K \delta^{-\frac{(n-1)\gamma}{\tau}} \delta \leq C''_K \delta^{\gamma/n},$$

where we choose $\gamma \leq \frac{\tau}{4}$ small so that $\gamma/n \leq 1 - \frac{(n-1)\gamma}{\tau}$.

Let us define

$$v^+ := (1 + 4\epsilon + 2\sqrt{\delta})v - 4\epsilon - 2\sqrt{\delta} + 4C''_K \delta^{\gamma/n}(x^n - K),$$

and

$$v^- := \left(1 - 4\epsilon - \frac{\sqrt{\delta}}{2}\right)v + 4\epsilon + \frac{\sqrt{\delta}}{2} + 4C_K''\delta^{\gamma/n}(-x^n + K + 1).$$

First, it is easy to check that $v^- \geq u \geq v^+$ on $\partial S \cap \{x^n > P(x')\}$. Also, by (5.12) and (5.13) we have

$$\frac{\partial v^+}{\partial x^n} > \frac{\partial u}{\partial x^n} > \frac{\partial v^-}{\partial x^n} \tag{5.14}$$

on $\partial S \cap \{x^n = P(x')\}$. To prove the lemma, we need only to show that $v^- \geq u \geq v^+$ on S. In fact, if $u \geq v^+$ fails, then $\max(v^+ - u) > 0$ is achieved at some point $z \in S$. If $z \in \partial S \cap \{x^n = P(x')\}$, then we move the graph of v^+ down and then lift it up, it will touch the graph of u at point $(z, u(z))$ from below, which is contradict to (5.14). Therefore, z must be an interior point of S. Hence, we can find a number $\eta > 0$ such that $\{v^+ - \eta - u \geq 0\} \subset\subset S$, and using a maximum principle argument for the equations (2.3) and (5.6) we can reach a contradiction as same as the proof of [9, Proposition 5.2]. The other part is similar. $\square$

Proof of Theorem 2.2. Fixed a point $x_0 \in B_{\rho_1/2} \cap \{x^n = P(x')\}$, take $y_0 := \text{c-exp}_{x_0}(\nabla u(x_0)) \in \partial_c u(x_0)$ (notice that u is C^1 at the boundary by Theorem 2.1). Up to a change of variable as in the proof of Theorem 2.1, we can assume that $x_0 = y_0 = \mathbf{0}$, $u \geq 0$, $u(\mathbf{0}) = 0$, and $D_{xy}c(\mathbf{0},\mathbf{0}) = \text{Id}$.
    We set $S_h := S_u(h) = \{u \leq h\}$.

• *Step 1: $C^{1,1}$ estimate at $\mathbf{0}$.* We show that

$$B_{\sqrt{h}/K} \cap \{x^n \geq P(x')\} \subset S_h \subset B_{K\sqrt{h}} \cap \{x^n \geq P(x')\} \quad \forall\, h \leq h_1 \tag{5.15}$$

for some h_1 and K universal.
    To prove this fact we begin by observing that, by (2.14), for any $h_1 > 0$ we can choose $\eta_1 = \eta_1(h_1) > 0$ small enough such that (5.15) holds for S_{h_1} with $K = 2$. Hence, assuming without loss of generality that $\delta_1 \leq 1$, we have that

$$B_{\sqrt{\frac{h_1}{3}}} \subset \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(co[\tilde{S}_{h_1}]) \subset B_{3\sqrt{h_1}},$$

where γ is the exponent from Lemma 5.1, and $\tilde{S}_{h_1}$ is constructed in the same way as $\tilde{S}$ in Lemma 5.1.
    Let v_1 solve the following Monge-Ampère equation

$$\begin{cases} \det(D^2 v) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(co[\tilde{S}_{h_1}]), \\ v = 1 & \text{on } \partial\left(\mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(co[\tilde{S}_{h_1}])\right). \end{cases} \tag{5.16}$$

Since $B_{1/3} \subset \mathcal{N}_{\delta_1^\gamma \sqrt{h_1}}(co[\tilde{S}_{h_1}])/\sqrt{h_1} \subset B_3$, by standard Pogorelov estimates we have that $|D^2 v_1(\mathbf{0})| \leq M$, where $M > 0$ is a large universal constant.
    Now we recall a useful fact for the standard Monge-Ampère equation. Let w be a solution of

$$\begin{cases} \det(D^2 w) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } Z, \\ w = 1 & \text{on } \partial Z, \end{cases} \tag{5.17}$$

where $\mathbf{0} \in Z$ is a convex set, $-1 < \inf w \le w(\mathbf{0}) < 1/2$, and $|D^2 w(\mathbf{0})| \le M + 1$. Then there exists a large universal $\bar{K}$ such that $B_{1/\bar{K}} \subset Z \subset B_{\bar{K}}$. For reader's convenience, we give the proof below.

Proof. By John's lemma we can find a matrix A with $A(\mathbf{0}) = \mathbf{0}$ and $\det(A) = 1$ such that $B_{1/C_1}(z) \subset A(Z) \subset B_{C_1}(z)$, where C_1 is a universal constant. Then it is easy to check that $\bar{w}(x) = w(A^{-1}x)$ solves

(5.18)
$$\begin{cases} \det(D^2 \bar{w}) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } A(Z), \\ \bar{w} = 1 & \text{on } \partial(A(Z)). \end{cases}$$

Since $-1 < \inf \bar{w} \le \bar{w}(\mathbf{0}) < 1/2$, it follows that $\text{dist}(\mathbf{0}, \partial A(Z)) > c$, where $c > 0$ is a universal constant, hence $B_c \subset A(Z) \subset B_{C_1}$. By interior estimates for the standard Monge-Ampère equation, we have that $|D^2 \bar{w}(\mathbf{0})| \le C_2$ for some universal constant C_2. Since $D^2 \bar{w}(\mathbf{0}) = A^{-1} D^2 w(\mathbf{0}) A^{-1}$, $|D^2 \bar{w}(\mathbf{0})| \le C_2$ and $|D^2 w(\mathbf{0})| \le M + 1$, it follows from (5.17) and (5.18) that $\|A\|, \|A^{-1}\| \le C_3$ for some universal constant C_3 (recall that by assumption $f(\mathbf{0})/g(\mathbf{0})$ is bounded away from zero and infinity). Since $A^{-1}(B_c) \subset Z \subset A^{-1}(B_{C_1})$ and $A(\mathbf{0}) = \mathbf{0}$, it follows that there exists a universal constant $\bar{K}$ such that $B_{1/\bar{K}} \subset Z \subset B_{\bar{K}}$, as desired. $\square$

Now, we prove by induction that (5.15) holds with $K := 2\bar{K}$. Let $h_k := h_1 2^{-k}$. If $h = h_1$ then we already know that (5.15) holds with $K = 2$. Assume that (5.15) holds with $h = h_k$ and $K = 2\bar{K}$, and we will show that it holds with $h = h_{k+1}$. For this, for any $k \in \mathbb{N}$ let v_k be the solution of

(5.19)
$$\begin{cases} \det(D^2 v_k) = f(\mathbf{0})/g(\mathbf{0}) & \text{in } \mathcal{N}_{\delta^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_k}]), \\ v_k = 1 & \text{on } \partial\left(\mathcal{N}_{\delta^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_1}])\right), \end{cases}$$

and $\delta_k := \|c(x,y) + x \cdot y\|_{C^2\left(S_{h_k} \times T_u(S_{h_k})\right)} + \|P_k\|_{C^2} + \|Q_k\|_{C^2}$, where $P_k = \frac{1}{\sqrt{h_k}} P(\sqrt{h_k} x')$ and $Q_k = \frac{1}{\sqrt{h_k}} Q(\sqrt{h_k} y')$. Then it is easy to see that $\|P_k\|_{C^2} + \|Q_k\|_{C^2} \le C \sqrt{h_k}$. By the $C^{1,\beta}$ regularity of u (which implies that T_u is $C^{0,\beta}$) we also have

$$\|c(x,y) + x \cdot y\|_{C^2\left(S_{h_k} \times T_u(S_{h_k})\right)} \le C\left(\text{diam}(S_{h_k}) + \text{diam}(T_u(S_{h_k}))\right) \le C h_k^{\beta/2},$$

which implies in particular that

(5.20)
$$\delta_k \le C h_k^{\beta/2}.$$

Let us consider the rescaled functions

$$\bar{u}_k(x) := u(\sqrt{h_k} x)/h_k, \quad \bar{v}_k(x) := v_k(\sqrt{h_k} x)/h_k$$

and notice that, by the inductive hypothesis,

(5.21)
$$B_{\frac{1}{2K}} \cap \{x^n \ge P_k(x')\} \subset \bar{S}_k := \{\bar{u}_k \le 1\} \subset B_{2\bar{K}} \cap \{x^n \ge P_k(x')\}.$$

Note that by (4.25), (4.26) we have that there exists an affine transformation L_k such that both $L_k(\bar{S}_k)$ and $L_k'^{-1}\left(\partial_c u(\bar{S}_k)\right)$ are universally comparable to half-balls. By (5.21), we see that $\bar{S}_k$ is already comparable to a half-ball, hence L_k satisfies $\|L_k\|, \|L_k^{-1}\| \le K'$ for some universal

constant K', which implies that $\partial_c u(\bar{S}_k) = L_k'\big(L_k'^{-1}\big(\partial_c u(\bar{S}_k)\big)\big)$ is also universally comparable to a half-ball, that is, there exists a universal constant $\bar{K}' \geq \bar{K}$ such that

$$B_{\frac{1}{2\bar{K}'}} \cap \{y^n \geq Q_k(y')\} \subset \partial_c u(\bar{S}_k) \subset B_{2\bar{K}'} \cap \{y^n \geq Q_k(y')\}.$$

This estimate and (5.21) allow us to apply Lemma 5.1 and deduce that

$$(5.22) \qquad \|\bar{u}_k - \bar{v}_k\|_{L^\infty(\bar{S}_k)} \leq C_{\bar{K}'}\bigg(\underset{S_{h_k}}{\operatorname{osc}} f + \underset{T_u(S_{h_k})}{\operatorname{osc}} g + \delta_k^{\gamma/n}\bigg) \leq C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}}.$$

By (5.21) we see that when h_1 is small $\{\bar{v}_k \leq 1\}$ has "good shape", namely

$$(5.23) \qquad B_{\frac{1}{3K}\sqrt{h_k}} \subset \mathcal{N}_{\delta_k^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_k}]) \subset B_{3\bar{K}\sqrt{h_k}}.$$

Now we show that

$$(5.24) \qquad 2C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}} > \inf_{\mathcal{N}_{\delta^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_k}])} \bar{v}_k \geq -2C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}},$$

provided h_1 is small enough. By convexity and symmetry, we know that $\displaystyle\inf_{\mathcal{N}_{\delta^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_k}])} \bar{v}_k$ is achieved at $\{x^n = 0\}$. Let us denote

$$\hat{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1} := \{x^n = 0\}$$

as the standard projection, and for any $(z', 0) \in \hat{\pi}\left(\{\bar{u}_k = 1\} \cap \{x^n = P_k(x')\}\right)$ we denote the corresponding point on $\{\bar{u}_k = 1\} \cap \{x^n = P_k(x')\}$ as $z := (z', z^n)$. By (5.13), (5.20), and (5.22), we have that

$$\bar{v}_k(z', 0) \geq \bar{u}_k(z) - C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}} - C''_{\bar{K}} \delta_k^{\gamma/n} \sqrt{h_k} \geq 1/2,$$

provided h_1 small. Also, by (5.22) we also have $\bar{v}_k(\mathbf{0}) \leq C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}}$ (recall that $u(\mathbf{0}) = 0$, so $\displaystyle\inf_{\mathcal{N}_{\delta^\gamma \sqrt{h_k}}(co[\tilde{S}_{h_k}])} \bar{v}_k$ is achieved inside $\hat{\pi}\left(\{\bar{S}_k \cap \{x^n = P_k(x')\}\right)$. Again by (5.13), (5.20), and (5.22), we have

$$\bar{v}_k(z', 0) \geq \bar{u}_k(z) - C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}} - C'' \delta_k^{\gamma/n} \sqrt{h_k} \geq -2C_{\bar{K}'} h_k^{\frac{\alpha\beta\gamma}{2n}}$$

for any $(z', 0) \in \hat{\pi}\left(\{\bar{S}_k \cap \{x^n = P_k(x')\}\right)$. Hence (5.24) follows from the above discussion easily. Note that (5.24) implies that $\mathbf{0}$ is almost the minimum point of $\bar{v}_k$.

Now, as in the proof of [9, Theorem 5.3], by standard estimates on the sections of solutions to the Monge-Ampère equation it follows that the shapes of $\{\bar{v}_k \leq 1\}$ and $\{\bar{v}_k \leq 1/2\}$ are comparable, and in addition sections are well included into each other: hence, thanks to (5.23) there exists a universal constant $L > 1$ such that

$$B_{1/L\bar{K}} \subset \{\bar{v}_k \leq 1/2\} \subset B_{L\bar{K}}, \quad \operatorname{dist}\left(\{\bar{v}_k \leq 1/4\}, \partial\{\bar{v}_k \leq 1/2\}\right) \geq 1/(L\bar{K}).$$

Using again (5.22) we have that, if h_1 is sufficiently small,

$$B_{1/(2L\bar{K})} \cap \{x^n \geq P_k(x')\} \subset \{\bar{u}_k \leq 1/2\} \subset B_{2L\bar{K}} \cap \{x^n \geq P_k(x')\},$$

$$(5.25) \qquad \operatorname{dist}\left(\{\bar{u}_k \leq 1/4\}, \{\bar{u}_k = 1/2\}\right) \geq 1/(2L\bar{K}),$$

This allows us to apply Lemma 5.1 to $\bar{u}_{k+1}$ to get

$$(5.26) \qquad \|\bar{u}_{k+1} - \bar{v}_{k+1}\|_{L^\infty(\bar{S}_{k+1})} \leq C_{2L\bar{K}} \left(\operatorname*{osc}_{S_{h_{k+1}}} f + \operatorname*{osc}_{T_u(S_{h_{k+1}})} g + \delta_k^{\gamma/n} \right) \leq C_{2L\bar{K}} h_k^{\frac{\alpha\beta\gamma}{2n}}.$$

Hence, combining (5.22) and (5.26),

$$
\begin{aligned}
\|v_k - v_{k+1}\|_{L^\infty(S_{h_{k+1}})} &\leq \|v_k - u\|_{L^\infty(S_{h_k})} + \|u - v_{k+1}\|_{L^\infty(S_{h_{k+1}})} \\
&= h_k \|\bar{u}_k - \bar{v}_k\|_{L^\infty(\bar{S}_k)} + h_{k+1} \|\bar{u}_{k+1} - \bar{v}_{k+1}\|_{L^\infty(\bar{S}_{k+1})} \\
(5.27) \qquad &\leq C(C_{\bar{K}'} + C_{2L\bar{K}}) h_k^{1+\sigma},
\end{aligned}
$$

where $\sigma = \frac{\alpha\beta\gamma}{2n}$.

Now, we denote

$$S_{h_{k+1}}^+ := S_{h_{k+1}} \cup \left\{ (x', x^n) : |x^n| \leq C h_k, \, x' \in \hat{\pi}\left(\{ S_{h_{k+1}} \cap \{ x^n = P(x') \} \} \right) \right\},$$

and we also denote $S_{h_{k+1}}^-$ as the reflection of $S_{h_{k+1}}^+$ with respect to $\{x^n = 0\}$. By scaling back (5.25), we have that $\operatorname{dist}\left(S_{h_{k+2}}, \partial(S_{h_{k+1}}^+ \cup S_{h_{k+1}}^-) \right) \geq \sqrt{h_k}/(4L\bar{K})$. Then by (5.13) and (5.27) we have $\|v_k - v_{k+1}\|_{L^\infty\left(S_{h_{k+1}}^+ \cup S_{h_{k+1}}^-\right)} \leq C_{\bar{K}',2L\bar{K}} h_k^{1+\sigma}$, with $C_{\bar{K}',2L\bar{K}}$ is a constant depending only on $\bar{K}'$ and $2L\bar{K}$, provided δ_1 is small. Hence, we can apply the classical Pogorelov and Schauder estimates to get

$$(5.28) \qquad \|D^2 v_k - D^2 v_{k+1}\|_{L^\infty(S_{h_{k+2}})} \leq C'_{\bar{K}',2L\bar{K}} h_k^\sigma,$$

$$(5.29) \qquad \|D^3 v_k - D^3 v_{k+1}\|_{L^\infty(S_{h_{k+2}})} \leq C'_{\bar{K}',2L\bar{K}} h_k^{\sigma-1/2},$$

where $C'_{\bar{K}',2L\bar{K}}$ is a constant depending only on $\bar{K}'$ and $2L\bar{K}$. Since by the inductive assumption (5.15) holds with $K = 2\bar{K}$ for $h = h_j$ with $j = 1, \cdots, k$, we can apply (5.28) to v_j to get

$$
\begin{aligned}
(5.30) \qquad |D^2 v_{k+1}|(\mathbf{0}) &\leq D^2 v_1(\mathbf{0}) + \sum_{j=1}^{k} |D^2 v_j(\mathbf{0}) - D^2 v_{j+1}(\mathbf{0})| \\
(5.31) \qquad &\leq M + C'_{\bar{K}',2L\bar{K}} h_1^\sigma \sum_{j=0}^{k} 2^{-j\sigma} \\
(5.32) \qquad &\leq M + \frac{C'_{\bar{K}',2L\bar{K}}}{1 - 2^{-\sigma}} h_1^\sigma \leq M + 1,
\end{aligned}
$$

provided we choose h_1 small enough. By the definition of $\bar{K}$ it follows that also $S_{h_{k+1}}$ satisfies (5.15), concluding the proof of inductive step.

● *Step 2: $C^{2,\sigma'}$ estimate at $\mathbf{0}$.* We now prove that u is $C^{2,\sigma'}$ at the origin with $\sigma' := 2\sigma$, that is, there exists a sequence of paraboloids p_k such that

$$(5.33) \qquad \sup_{B_{r_0^k/C}} |u - p_k| \leq C r_0^{k(2+\sigma')}$$

for some $r_0, C > 0$.

Let v_k be as in the previous step, and let p_k be their second order Taylor expansion at $\mathbf{0}$:

$$p_k(x) := v_k(\mathbf{0}) + \nabla v_k(\mathbf{0}) \cdot x + \frac{1}{2}D^2 v_k(\mathbf{0})x \cdot x.$$

By (5.15) we have

$$(5.34) \qquad \|v_k - p_k\|_{L^\infty\left(B(\sqrt{h_{k+2}}/K)\right)} \leq \|v_k - p_k\|_{L^\infty(S_{h_{k+2}})} \leq C\|D^3 v_k\|_{L^\infty(S_{h_{k+2}})}h_k^{3/2}.$$

In addition, applying (5.29) to v_j with $j = 1, \cdots, k$ and recalling that $h_k = h_1 2^{-k}$ and $2\sigma < 1$, we get

$$(5.35) \qquad \|D^3 v_k\|_{L^\infty(S_{h_{k+2}})} \quad \leq \quad \|D^3 v_1\|_{L^\infty(S_{h_3})} + \sum_{j=1}^{k}\|D^3 v_j - D^3 v_{j+1}\|_{L^\infty(S_{h_{j+2}})}$$

$$(5.36) \qquad\qquad\qquad\qquad \leq \quad C\left(1 + \sum_{j=1}^{k}h_j^{\sigma-1/2}\right) \leq Ch_k^{\sigma-1/2}.$$

Hence, combining (5.15), (5.22), (5.34), and (5.35), we get

$$\|u - p_k\|_{L^\infty\left(B(\sqrt{h_{k+2}}/K)\right)} \leq \|v_k - p_k\|_{L^\infty(S_{h_{k+2}})} + \|v_k - u\|_{L^\infty(S_{h_{k+2}})} \leq Ch_k^{1+\sigma},$$

so (5.33) follows with $r_0 = 1/\sqrt{2}$ and $\sigma' = 2\sigma$.

• *Step 3: $C^{2,\alpha'}$ regularity near the boundary.* Since (recall the beginning of the proof of the theorem) the point $\mathbf{0}$ represented an an arbitrary point in $B_{\rho_1/2} \cap \{x^n = P(x')\}$, By Step 2 we know that (5.33) holds at any point on $B_{\rho_1/2} \cap \{x^n = P(x')\}$.

Set $\alpha' := \sigma'/2$. Given $\rho > 0$ let $\Omega_\rho := \{x \in \mathcal{C}_1 \cap B_{\rho_1/4} : d(x, \partial\Omega) \leq \rho\}$. We want to show that if $\rho \ll \rho_1$ is sufficiently small, then $u \in C^{2,\alpha}_{\text{loc}}(\Omega_\rho)$ and $\|u\|_{C^{2,\alpha'}(\Omega_\rho)} \leq C$. To prove this, given $x_1 \in \Omega_\rho$ denote $d := \text{dist}(x_1, \partial\mathcal{C}_1)$ and assume with no loss of generality that $d = \text{dist}(x_1, \mathbf{0})$. Since u is pointwise $C^{2,\sigma'}$ at $\mathbf{0}$ (see Step 2), after an affine transformation and change of variables similar to (4.1) and (4.2) we can assume that $\|u - \frac{1}{2}|x|^2\|_{L^\infty(B_{4d})} \leq Cd^{2+\sigma'}$. Then, we perform the blow up

$$\begin{cases} \tilde{x} := \frac{1}{4d}x \\ \tilde{y} := \frac{1}{4d}y, \end{cases}$$

$c_1(\tilde{x}, \tilde{y}) := \frac{1}{16d^2}c(4d\tilde{x}, 4d\tilde{y})$, and $u_1(\tilde{x}) := \frac{1}{16d^2}u(4d\tilde{x})$. In the new coordinates $B_{1/6}(\tilde{x}_1)$ lies in the interior of the domain $\widetilde{\Omega}$, where $\tilde{x}_1 := \frac{1}{4d}x_1$ and $\widetilde{\Omega} := \Omega_{2\rho}/4d$. It is immediate to check that $\|u_1(\tilde{x}) - \frac{1}{2}|\tilde{x}|^2\|_{B_{1/6}(\tilde{x}_1)} \leq Cd^{\alpha'}$, and $\|c(\tilde{x}, \tilde{y}) + \tilde{x} \cdot \tilde{y}\|_{B_{1/6}(\tilde{x}_1) \times B_{1/6}(\tilde{y}_1)} \to 0$ as $d \to 0$, where $\tilde{y}_1 \in \partial_{c_1} u_1(\tilde{x}_1)$. Hence, if ρ (and therefore d) is sufficiently small we can apply [9, Theorem 5.3] to deduce that $\tilde{u} \in C^{2,\alpha}(B_{1/7}(\tilde{x}_1))$, with a universal bound. In particular $|D^2\tilde{u}(\tilde{x}_1)| \leq C$, thus $|D^2 u(x_1)| \leq C$. This proves that u is uniformly $C^{1,1}$ inside Ω_ρ, which implies that (2.3) becomes uniformly elliptic there. Writing $\hat{u}(\tilde{x}) := \frac{1}{16d^{2+\alpha}}\left[u(x) - \frac{1}{2}|x|^2\right]$, it is easy to check that $\hat{u}$ solves a uniformly elliptic equation of the form

$$G\left(\tilde{x}, \nabla\hat{u}(\tilde{x}), D^2\hat{u}(\tilde{x})\right) = 0$$

where $G(x, 0, 0) = 0$ and $\|G(\tilde{x}, \cdot, \cdot)\|_{C^{0,\alpha}(B_{1/6}(\tilde{x}_0))} \leq C$. Hence, by standard elliptic regularity for fully-nonlinear elliptic equations we deduce that $\|\hat{u}\|_{C^{2,\alpha'}(B_{1/7}(\tilde{x}_1))} \leq \|\hat{u}\|_{C^{2,\alpha}(B_{1/7}(\tilde{x}_1))} \leq C$. Going back to the original coordinates, we deduce that

$$\|u\|_{C^{2,\alpha}(B_{d/7}(x_1))} \leq Cd^{\alpha'-\alpha}, \qquad \|u\|_{C^{2,\alpha'}(B_{d/7}(x_1))} \leq C.$$

Because of the arbitrariness of x_1, the first estimate proves that u is of class $C^{2,\alpha}$ in the interior of Ω_ρ, while the second estimate combined with the fact that (5.33) holds at every boundary point allows one to prove by standard arguments (see for instance the proof of [31, Proposition 2.4]) the $C^{2,\alpha'}$ regularity of u in the whole Ω_ρ. $\qquad\square$

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