Semiregular Polytopes and Amalgamated C-groups

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Abstract

In the classical setting, a convex polytope is said to be semiregular if its facets are regular and its symmetry group is transitive on vertices. This paper studies semiregular abstract polytopes, which have abstract regular facets, still with combinatorial automorphism group transitive on vertices. We analyze the structure of the automorphism group, focusing in particular on polytopes with two kinds of regular facets occurring in an "alternating" fashion. In particular we use group amalgamations to prove that given two compatible n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, there exists a universal abstract semiregular $(n+1)$-polytope which is obtained by "freely" assembling alternate copies of $\mathcal{P}$ and $\mathcal{Q}$. We also employ modular reduction techniques to construct finite semiregular polytopes from reflection groups over finite fields.

Key Words: semiregular polytope; abstract polytope; group amalgamation; reflection groups; modular reduction

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1   Introduction

In the classical setting, a convex n-polytope $\mathcal{P}$ is said to be *uniform* if its facets are uniform and its symmetry group is transitive on vertices (see Coxeter [4, 6], Johnson [16]). To start this inductive definition in a pleasant way, we agree that uniform polygons should be regular. The same definition can be transferred to the abstract (combinatorial) setting. But *all* polygons are combinatorially regular, so one soon suspects that the abstract uniform polytopes form a huge, perhaps untamable class of strange objects. Indeed, the abstract uniform polytopes generalize the abstract regular polytopes, which in a sense are 'maximally' symmetric and which are already quite abundant.

In this paper, we will generalize regularity more modestly and focus on *semiregular* polytopes $\mathcal{P}$, which have regular facets, still with automorphism group $\Gamma(\mathcal{P})$ transitive on vertices (see Coxeter [6, 7, 8]). All of the classical Archimedean solids, for example, are convex semiregular 3-polytopes. Perhaps the best-known non-regular example is the *cuboctahedron*, which can be obtained either by truncating a cube to its edge midpoints, or by assembling squares and equilateral triangles, two of each placed alternately around each vertex.

This sort of behaviour also appears in the familiar tiling $\mathcal{T}$ of Euclidean 3-space by regular octahedra and tetrahedra, the beginning of which is displayed in Figure 1. In fact, $\mathcal{T}$ is an infinite semiregular 4-polytope. Our main concern in this paper will be abstract semiregular polytopes like this, with two kinds of regular facets occurring in an 'alternating' fashion. The essential features of our construction are contained in Theorem 4.7.

Later, in Theorem 5.5, we prove the existence of a universal abstract semiregular (n+1)-polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, which is obtained by "freely" assembling alternate copies of two compatible n-polytopes $\mathcal{P}$ and $\mathcal{Q}$. Finally, in Section 6, we employ modular reduction techniques to construct finite semiregular polytopes from reflection groups over finite fields.

2   Abstract Polytopes and their Automorphism Groups

An abstract n-polytope $\mathcal{P}$ has some of the key combinatorial properties of the face lattice of a convex n-polytope; in general, however, $\mathcal{P}$ need not be a lattice, need not be finite, need not have any familiar geometric realization. Let us summarize some general definitions and results, referring to McMullen & Schulte [19] for details. An *abstract n-polytope* $\mathcal{P}$ is a partially ordered set with properties **A**, **B** and **C** below.

A: $\mathcal{P}$ has a strictly monotone rank function with range $\{-1, 0, \ldots, n\}$.

An element $F \in \mathcal{P}$ with $\mathrm{rank}(F) = j$ is called a *j-face*; often F_j will indicate a j-face. Moreover, $\mathcal{P}$ has a unique least face F_{-1} and unique greatest face F_n. Each maximal chain or *flag* in $\mathcal{P}$ therefore contains $n + 2$ faces, so that n is the number of *proper* faces

in each flag. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags in $\mathcal{P}$. Naturally, faces of ranks 0, 1 and $n-1$ are called *vertices*, *edges* and *facets*, respectively.

B: Whenever $F < G$ with $\operatorname{rank}(F) = j - 1$ and $\operatorname{rank}(G) = j + 1$, there are exactly two j-faces H with $F < H < G$.

For $0 \leqslant j \leqslant n - 1$ and any flag Φ, there thus exists a unique *adjacent* flag Φ^j, differing from Φ in just the face of rank j . With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the *flag graph* for $\mathcal{P}$. If $F \leqslant G$ are incident faces in $\mathcal{P}$, we call

$$G/F := \{H \in \mathcal{P} \,|\, F \leqslant H \leqslant G\} \,.$$

a *section* of $\mathcal{P}$.

C: $\mathcal{P}$ is *strongly flag–connected*, that is, the flag graph for each section is connected.

It follows that G/F is a $(k - j - 1)$-polytope in its own right, if $F \leqslant G$ with $\operatorname{rank}(F) = j \leqslant k = \operatorname{rank}(G)$. In particular, F_n/F is the *co-face* of a face F in $\mathcal{P}$, and its rank $n - 1 - \operatorname{rank}(F)$ is the *co-rank* of F. For example, if F is a vertex, then the section F_n/F is called the *vertex-figure* over F. Likewise, it is useful to think of the k-face G as having the structure of the k-polytope G/F_{-1}.

  The *automorphism group* $\Gamma(\mathcal{P})$ consists of all order-preserving bijections on $\mathcal{P}$. We say $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flag set $\mathcal{F}(\mathcal{P})$. In this case we may choose any one flag $\Phi \in \mathcal{F}(\mathcal{P})$ as *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j, for $0 \leqslant j \leqslant n - 1$. Each ρ_j has period 2. From [19, 2B] we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*, meaning that it has the following properties **SC1** and **SC2**:

SC1: $\Gamma(\mathcal{P})$ is generated by $\{\rho_0, \ldots, \rho_{n-1}\}$. These involutory generators satisfy the commutativity relations typical of a Coxeter group with string diagram, namely

$$(\rho_j \rho_k)^{p_{jk}} = 1, \text{ for } 0 \leqslant j \leqslant k \leqslant n - 1, \tag{1}$$

where $p_{jj} = 1$ and $p_{jk} = 2$ whenever $|j - k| > 1$.

SC2: $\Gamma(\mathcal{P})$ satisfies the *intersection condition*

$$\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle, \text{ for any } I, J \subseteq \{\rho_0, \ldots, \rho_{n-1}\} \,. \tag{2}$$

The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [19, 2E].

  The periods $p_j := p_{j-1,j}$ in (1) satisfy $2 \leqslant p_j \leqslant \infty$ and are assembled into the *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$ for the regular polytope $\mathcal{P}$. We note again that every 2-polytope or *polygon* $\{p_1\}$ is automatically abstractly regular; its automorphism group is the dihedral group $\mathbb{D}_{2p_1}$ of order $2p_1$.

  There are various ways to relax symmetry and thereby broaden the class of groups $\Gamma(\mathcal{P})$. As we suggested in §1, the class of uniform polytopes is far too broad for our purposes here. Thus we restrict our reach somewhat with

Definition 2.1. *An abstract polytope $\mathcal{P}$ is* semiregular *if it has regular facets and its automorphism group $\Gamma(\mathcal{P})$ is transitive on vertices* [18, p. 77].

Every regular polytope is clearly semiregular. In this paper we focus our investigation of semiregular polytopes on a particularly interesting subclass derived from Wythoff's construction described in the next section. We might call these *alternating semiregular polytopes*, since they have facets of possibly two distinct types appearing in alternating fashion around faces of co-rank 2.

3 A Concrete (Geometrical) Version of Wythoff's Construction

Let us take a closer look at the uniform tessellation $\mathcal{T}$ mentioned earlier. As a combinatorial object, this tessellation of $\mathbb{R}^3$ is an abstract semiregular 4-polytope whose combinatorial automorphism group can be identified with its geometric symmetry group (see Figure 1).
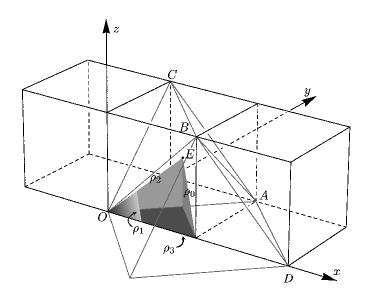

Figure 1: The birth of the 4-polytope $\mathcal{T}$, a semiregular tessellation of $\mathbb{R}^3$.

A simple way to describe $\mathcal{T}$ is to first imagine Euclidean space $\mathbb{R}^3$ tiled as usual by unit cubes. (Although this tiling is itself a regular 4-polytope, for now it will serve mainly as scaffolding for $\mathcal{T}$.) Each cube has two inscribed regular tetrahedra. Pick one in each cube, starting with the tetrahedron with vertices $O = (0,0,0), A = (1,1,0), B = (1,0,1), C = (0,1,1)$ in the standard unit cube, then alternating thereafter as one passes between adjacent cubes. We thus get the tetrahedral facets of $\mathcal{T}$; the octahedral facets tile what is

left of $\mathbb{R}^3$. Every vertex of $\mathcal{T}$ is surrounded by six octahedra and eight tetrahedra; indeed, each vertex-figure is a cuboctahedron. (Compare [5, §4.7], where Coxeter describes $\mathcal{T}$ as a *quasiregular* tessellation, with modified Schläfli symbol $\{3, \frac{3}{4}\}$. The usage of the term "quasiregular" in [5] implies the local alternating behaviour we focus on in this paper.)

Notice that the Euclidean symmetry group $\Gamma(\mathcal{T})$ contains the face-centred cubic lattice generated by translations τ_1, τ_2, τ_3 along the edges OA, OB, OC of the base tetrahedron. The point group stabilizing vertex O is the octahedral group of order 48 generated by reflections ρ_1, ρ_2, ρ_3, whose mirrors are indicated in Figure 1, along with the mirror for a fourth reflection ρ_0. These four reflections generate $\Gamma(\mathcal{T})$, and their mirrors enclose a tetrahedral fundamental region for the action of $\Gamma(\mathcal{T})$ on $\mathbb{R}^3$. In fact, $\Gamma(\mathcal{T})$ is the (infinite) Coxeter group of type $\widetilde{B}_3$.

It is convenient to blur the distinction between the affine reflections ρ_j and their abstract counterparts in a presentation of $\Gamma(\mathcal{T})$ as the affine Coxeter group $\widetilde{B}_3$ with Coxeter diagram

$$\tag{3}$$

Recall from [15, §6.5] that $\Gamma(\mathcal{T})$ has these defining relations on its standard generators:
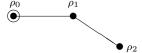
$$\rho_j^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_2\rho_3)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = (\rho_1\rho_3)^4 = 1 \ . \tag{4}$$

Evidently $\Gamma(\mathcal{T})$ is not a string C-group. Indeed, $\mathcal{T}$ is not regular, for it has two kinds of facets and two flag orbits. Nevertheless, $\Gamma(\mathcal{T})$ does satisfy the intersection condition (2), as does any Coxeter group [15, Theorem 5.5]. This fact reappears in Definition 4.2 below.

The ringed node of the diagram in (3) is an ingenious decoration invented by Coxeter [4] and is meant to encode *Wythoff's construction* for $\mathcal{T}$. The essential idea is that each j-face of $\mathcal{T}$ lies in the same $\Gamma(\mathcal{T})$-orbit as a special j-face F_j, whose stabilizer $\Sigma(F_j)$ is a certain parabolic subgroup of $\Gamma(\mathcal{T})$. This $\Sigma(F_j)$ is generated by the ρ_k's corresponding to nodes in a subdiagram, which in turn consists of a connected 'active' part, which has j nodes including the ringed node, and a 'passive' part induced on all nodes not connected to the active part. For example, there is one base vertex $F_0 = O$ in Figure 1; it has empty active part and is fixed (passively) by $\Sigma(F_0) = \langle \rho_1, \rho_2, \rho_3 \rangle$. The base edge F_1 has vertices O and $A = (O)\rho_0$, the image of O under ρ_0; the active part of $\Sigma(F_1) = \langle \rho_0, \rho_2, \rho_3 \rangle$ is $\langle \rho_0 \rangle$. The base equilateral triangle F_2 has vertices OAB and $\Sigma(F_2) = \langle \rho_0, \rho_1 \rangle$ is totally active. Finally, there are two special 3-faces, $F_3 = F_3^{(1)}, F_3^{(2)}$ (say), namely an octahedral facet $F_3^{(1)}$ with $\Sigma(F_3^{(1)}) = \langle \rho_0, \rho_1, \rho_3 \rangle$, and a tetrahedral facet $F_3^{(2)}$ with $\Sigma(F_3^{(2)}) = \langle \rho_0, \rho_1, \rho_2 \rangle$. In this 'geometrical' version of Wythoff's construction, each basic face F_j is the convex hull of the $\Sigma(F_j)$-orbit of O. Observe that general j-faces in the $\Gamma(\mathcal{T})$-orbit of F_j correspond exactly to the right cosets of $\Sigma(F_j)$ in $\Gamma(\mathcal{T})$.

We note that the description above can be adjusted somewhat to accommodate other sorts of Coxeter diagrams, with arbitrary sets of ringed nodes [4]. (Active parts may

then become disconnected.) However, the regular case works as expected: just ring one terminal node in a string diagram [19, 1B]. Consider, for example, this subdiagram extracted from the diagram in (3):

The subgroup $\langle \rho_0, \rho_1, \rho_2 \rangle$ of $\widetilde{B}_3$ is, in fact, the A_3 Coxeter group (isomorphic to the symmetric group $\mathbb{S}_4$); and Wythoff's construction now produces a regular tetrahedron, such as $OABC$ in Figure 1.
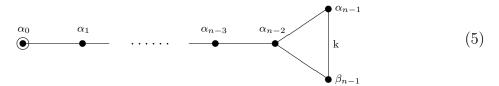
Let us summarize the discussion above in

Example 3.1. The 4-polytope $\mathcal{T}$ defined by the diagram in (3) is a semiregular tessellation of $\mathbb{R}^3$ by regular octahedra and tetrahedra. Its symmetry group $\Gamma(\mathcal{T})$ is the Coxeter group $\widetilde{B}_3$.

We refer to [3, 27] for a much broader look at the concrete geometrical aspects of Wythoff's construction. Below we shall pursue instead an abstract, that is to say, combinatorial generalization of the construction, motivated by the above example, though still tailored to our immediate needs.

4 An Abstract (Combinatorial) Version of Wythoff's Construction

Suppose that $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is a group generated by involutions which satisfy the commutativity relations implicit in the *tail-triangle diagram*

(5)

The label 'k' indicates that $\alpha_{n-1}\beta_{n-1}$ has period k, for some $k = 2, \ldots, \infty$. However, all other periods of products of two 'adjacent' generators are unspecified for the moment. Then the group Γ is called a *tail-triangle group*. We have already encountered such a diagram in (3), in which $k = 2$ for the non-adjacent nodes labelled ρ_2 and ρ_3. When $n = 2$ there is no "tail" and Γ is *triangle group* with a *triangle diagram*; this is consistent with standard terminology (see [17]). We allow the degenerate (base) case $n = 1$ when $\Gamma = \langle \alpha_0, \beta_0 \rangle$ is just the dihedral group $\mathbb{D}_{2k}$.

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Suppose also that Γ satisfies the intersection condition on its distinguished subgroups. In other words, for all subsets $I, J \subseteq \{\alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1}\}$, we assume that

$$\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle. \tag{6}$$

Thus Γ is a C-group (see [19, 2E]).

It follows that the subgroups

$$\Gamma_n^{\mathcal{P}} := \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle$$

and

$$\Gamma_n^{\mathcal{Q}} := \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$$

are string C-groups, indeed automorphism groups for regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Definition 4.1. *We call a group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ represented by a tail-triangle diagram as in (5) and satisfying the intersection property (6) a* tail-triangle C-group.

We can now describe a combinatorial version of the Wythoff construction implied by ringing the node labelled α_0 in (5). The actual details follow fairly closely those for the regular case described in [19, 2E]. Anticipating Theorem 4.7, let us denote the resulting $(n+1)$-polytope $\mathcal{S}$, or $\mathcal{S}(\Gamma)$, if we wish to emphasize the underlying group. Keep in mind that any such group comes with a specified list of generators, arranged as in (5).

Definition 4.2. The semiregular (n+1)-polytope $\mathcal{S} = \mathcal{S}(\Gamma)$.

Suppose that the group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is a tail-triangle C-group. Take the improper faces of $\mathcal{S}$ to be two distinct copies Γ_{-1} and Γ_{n+1} of Γ. Next, for $0 \leqslant j \leqslant n-2$, define the j-faces of $\mathcal{S}$ to be all right cosets in Γ of

$$\Gamma_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle.$$

The $(n-1)$-faces, or ridges, of $\mathcal{S}$ are all right cosets of

$$\Gamma_{n-1} := \langle \alpha_0, \ldots, \alpha_{n-2} \rangle.$$

Finally, the n-faces, or facets, of $\mathcal{S}$ are all right cosets of either $\Gamma_n^{\mathcal{P}}$ or $\Gamma_n^{\mathcal{Q}}$. When there is no risk of confusion we use Γ_n to denote either of these subgroups.

Finally we define what turns out to be a partial order on the set of all such faces by taking

$$\Gamma_j \nu < \Gamma_k \mu$$

whenever $-1 \leqslant j < k \leqslant n+1$ and $\Gamma_j \nu \cap \Gamma_k \mu \neq \emptyset$, for $\mu, \nu \in \Gamma$.

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Remarks. There is a reason for indicating strict inequality '$<$' here. For $j = k = n$, we must explicitly forbid cosets $\Gamma_n^{\mathcal{P}}\mu$ and $\Gamma_n^{\mathcal{Q}}\nu$ from being incident. (Occasionally these are non-disjoint; but distinct facets must never be incident. But for that glitch, we could say that $\mathcal{S}$ is a thin coset geometry.) Anyway, considering this we naturally agree that $\Gamma_j\nu \leqslant \Gamma_k\mu$ if and only if $\Gamma_j\nu < \Gamma_k\mu$ or (when $j = k$) $\Gamma_j\nu = \Gamma_k\mu$ (cf. Lemma 4.3(b) below).

Before we move on, note that (in contrast to standard notation for string C-groups) Γ_j is here not always equal to the subgroup of Γ generated by all generators but the jth (if applicable).

We will now prove that $\mathcal{S}$ is an abstract polytope. We begin with a few lemmas concerning a tail-triangle group Γ.

Lemma 4.3. (a) *The distinguished subgroups $\langle J \rangle$ are pairwise distinct. In particular, the subgroups $\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma_n^{\mathcal{P}}, \Gamma_n^{\mathcal{Q}}$ are mutually distinct and never equal Γ itself.*

(b) *Suppose $\Gamma_j\mu = \Gamma_k\nu$, for $\mu, \nu \in \Gamma$ and $0 \leqslant j \leqslant k \leqslant n$. Then $j = k$ and $\Gamma_j = \Gamma_k$. Furthermore, cosets $\Gamma_n^{\mathcal{P}}\mu$ and $\Gamma_n^{\mathcal{Q}}\nu$ can never be equal.*

(c) *Let $0 \leqslant j_1 < j_2 < \ldots < j_m \leqslant n$. Suppose that $\mu_{j_1}, \ldots, \mu_{j_m} \in \Gamma$ with*

$$\Gamma_{j_i}\mu_{j_i} \cap \Gamma_{j_{i+1}}\mu_{j_{i+1}} \neq \emptyset \,,$$

for $1 \leqslant i < m$. (Recall that Γ_n denotes exactly one of $\Gamma_n^{\mathcal{P}}$ or $\Gamma_n^{\mathcal{Q}}$.) Then there exists some common $\mu \in \Gamma$ such that $\Gamma_{j_i}\mu_{j_i} = \Gamma_{j_i}\mu$ for $1 \leqslant i \leqslant m$.

Proof. Part (a) follows from (6) just as in [19, 2E]. For part (b) simply note that $\Gamma_j\mu = \Gamma_k\nu$ forces $\Gamma_j = \Gamma_k$ (as with any two subgroups of a group).

Part (c) is proved just as in [11, Lemma 2.2]. The cases $m = 1, 2$ are trivial (even when $j_1 < j_2 = n$), so assume $m \geqslant 3$. By induction we have γ such that $\Gamma_{j_i}\mu_{j_i} = \Gamma_{j_i}\gamma$ for $2 \leqslant i \leqslant m$, as well as some λ such that $\Gamma_{j_i}\mu_{j_i} = \Gamma_{j_i}\lambda$ for $i = 1, 2$. From the overlap at j_2 we have $\lambda = \tau\gamma$ for some $\tau \in \Gamma_{j_2}$. But $j_2 < j_3 \leqslant n$, so that removal of the node labelled α_{j_2} must disconnect the graph in (5). Thus $\tau = \tau'\tau'' = \tau''\tau'$, where $\tau' \in \Gamma_{j_i}$ for all $i = 2, \ldots, m$, and $\tau'' \in \Gamma_{j_1} \cap \Gamma_{j_2}$. Then $\mu = (\tau'')^{-1}\lambda = \tau'\gamma$ is the desired common coset representative. $\square$

Lemma 4.4. $[\mathcal{S}, \leqslant]$ *is a partially ordered set with rank function*

$$\begin{aligned} \mathrm{rank} : \mathcal{S} &\to \{-1, \ldots, n+1\} \\ \Gamma_j\mu &\mapsto j \end{aligned}$$

Proof. Keeping Lemma 4.3 in mind, it is clear that we need only show that '$\leqslant$' is transitive. But that follows at once from the common coset representative μ constructed for part (c) of Lemma 4.3. $\square$

By Lemma 4.3(c), every flag of $\mathcal{S}$ can be written as

$$[\Gamma_0\mu, \Gamma_1\mu, \ldots, \Gamma_{n-1}\mu, \Gamma_n\mu] \,,$$

for some $\mu \in \Gamma$ (suppressing the two improper faces). Again we emphasize that Γ_n refers to exactly one of $\Gamma_n^{\mathcal{P}}$ or $\Gamma_n^{\mathcal{Q}}$, so that there are two competing *base flags*

$$\Phi^{\mathcal{P}} := [\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma_n^{\mathcal{P}}] \text{ and } \Phi^{\mathcal{Q}} := [\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma_n^{\mathcal{Q}}] .$$

Clearly there is a right action of Γ on the flag set $\mathcal{F}(\mathcal{S})$, and each flag Ψ is equivalent to one of $\Phi^{\mathcal{P}}$ or $\Phi^{\mathcal{Q}}$ under this action; on the other hand, $\Phi^{\mathcal{P}}$ cannot be Γ-equivalent to $\Phi^{\mathcal{Q}}$, by Lemma 4.3(b). (We later investigate when we can merge these two flag orbits into one by extending Γ.)

Now suppose $K = \{j_1, \ldots, j_m\}$ is some set of ranks satisfying $0 \leqslant j_1 < \ldots < j_m \leqslant n$. Each flag $\Psi = \Phi^{\mathcal{P}}\mu$ contains a chain $\Psi_K := [\Gamma_{j_1}\mu, \ldots, \Gamma_{j_m}\mu]$ of type K (with $\Gamma_{j_m} = \Gamma_n^{\mathcal{P}}$ if $j_m = n$). Likewise, each flag $\Psi = \Phi^{\mathcal{Q}}\mu$ contains a suitable chain Ψ_K.

Notice that the Γ-stabilizer of a basic chain $[\Gamma_{j_1}, \ldots, \Gamma_{j_m}]$ is simply

$$\Gamma_K := \Gamma_{j_1} \cap \ldots \cap \Gamma_{j_m}.$$

By the intersection condition, this is a standard subgroup of Γ. However, a detailed description of its generators is a bit more involved than for regular polytopes [19, Lemma 2E9]. In particular, as hinted earlier, here Γ_K is not simply the subgroup of Γ generated by all generators but those with $j \notin K$. With this notation in place we have
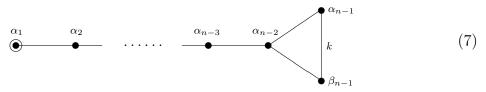
Lemma 4.5. (a) *The group Γ acts on $\mathcal{F}(\mathcal{S})$ and the Γ-stabilizer of the chain Ψ_K is $\mu^{-1}\Gamma_K\mu$.*

(b) *Γ is transitive on all chains of type K if $n \notin K$, and is transitive on all chains of type K contained in flags in either of the two flag orbits.*

(c) *If $K \supseteq \{0, \ldots, n-1\}$, then Γ_K is trivial.*

(d) *The action of Γ on $\mathcal{F}(\mathcal{S})$ is faithful, and we may consider Γ to be a subgroup of $\Gamma(\mathcal{S})$.*

Proof. Part (a) follows easily from our earlier observations. Part (b) follows at once from Lemma 4.3(c). For part (c) we check that $\Gamma_0 \cap \ldots \cap \Gamma_{n-1}$ is trivial. In part (d) we need only observe that each $\gamma \in \Gamma$ does indeed induce an order preserving bijection on $\mathcal{S}$. $\square$

Lemma 4.6. (a) *Let F be any facet of $\mathcal{S}$. Then the section F/Γ_{-1} is isomorphic to $\mathcal{P}$ or to $\mathcal{Q}$.*

(b) *Let $n \geqslant 2$, and let F be any vertex of $\mathcal{S}$. Then the vertex-figure Γ_{n+1}/F is isomorphic to the ranked poset $\widehat{\mathcal{S}}$ constructed in like manner by deleting the left node in (5) and transferring the ring to the node labelled α_1:*

$$\tag{7}$$

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(c) (The base case) *When $n = 1$ the diagram for Γ becomes*

$$\begin{array}{c} \odot\ \alpha_0 \\ \Big|\ k \\ \odot\ \beta_0 \end{array} \qquad\qquad (8)$$

which describes the polygon $\mathcal{S} = \{2k\}$.

Proof. In part (a) we may assume without loss of generality that $F = \Gamma_n^{\mathcal{P}}$. Since $\Gamma_{n-1} \subset \Gamma_n^{\mathcal{P}}$, an $(n-1)$-face $\Gamma_{n-1}\mu \leqslant \Gamma_n^{\mathcal{P}}$ if and only if $\mu \in \Gamma_n^{\mathcal{P}}$. By Lemma 4.3(c), this means that all j-faces $\Gamma_j\mu$ incident with $\Gamma_n^{\mathcal{P}}$ are likewise represented by certain $\mu \in \Gamma_n^{\mathcal{P}}$. On the other hand, a basic j-face of $\mathcal{P}$ can be identified with

$$G_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1} \rangle = \Gamma_j \cap \Gamma_n^{\mathcal{P}}.$$

We therefore have a well-defined bijection

$$\begin{aligned} \eta : \mathcal{P} &\to \Gamma_n^{\mathcal{P}}/\Gamma_{-1} \\ G_j\mu &\mapsto \Gamma_j\mu\,, \end{aligned}$$

for $\mu \in \Gamma_n^{\mathcal{P}}$, $0 \leqslant j \leqslant n-1$. It is easy to check that η is order-preserving. Now suppose that $\Gamma_j\mu \cap \Gamma_k\nu \neq \emptyset$, for $j < k$ and $\mu, \nu \in \Gamma_n^{\mathcal{P}}$. By Lemma 4.3(c) we may assume $\mu = \nu$; clearly $G_j\mu \cap G_k\mu \neq \emptyset$. Thus η^{-1} is also order-preserving.

Now we turn to part (b). We may assume that $F = \Gamma_0$. Notice that Γ_0 is nothing more than the group described by the diagram (7) and that this group still satisfies the intersection condition on its own standard subgroups. Thus the proper k-faces of the ranked poset $\widehat{\mathcal{S}}$ are cosets of suitable basic subgroups. In fact, by the intersection condition for Γ we find that these basic subgroups are

$$\Gamma_{0,k} := \Gamma_0 \cap \Gamma_{k+1}\,,$$

for $0 \leqslant k \leqslant n-1$. (Thus $\Gamma_{0,n-1}$ comes in two varieties, as determined by the vertex-figures of $\mathcal{P}$ and $\mathcal{Q}$.)

Once more using Lemma 4.3(c) we find for $j \geqslant 1$ that any j-face $\Gamma_j\mu$ incident with Γ_0 is represented by some $\mu \in \Gamma_0$. We may therefore define

$$\begin{aligned} \lambda : \Gamma_{n+1}/\Gamma_0 &\to \widehat{\mathcal{S}} \\ \Gamma_j\mu &\mapsto \Gamma_{0,j-1}\,\mu = (\Gamma_0 \cap \Gamma_j)\mu\,, \end{aligned}$$

for $\mu \in \Gamma_0$, $1 \leqslant j \leqslant n$ (again ignoring improper faces). As before, it is easy to check that λ is a poset isomorphism.

In part (c), the group $\Gamma = \langle \alpha_0, \beta_0 \rangle$ is the dihedral group $\mathbb{D}_{2k}$. There are two basic facets (here edges), namely $\Gamma_1^{\mathcal{P}} = \langle \alpha_0 \rangle$ and $\Gamma_1^{\mathcal{Q}} = \langle \beta_0 \rangle$, each incident with the basic vertex $\Gamma_0 = \{1\}$. One checks that these faces are arranged as indicated in (9) below (taking $n = 1$). $\qquad \square$

Theorem 4.7. *The abstract Wythoff's construction described in Definition 4.2 and summarized in diagram* (5) *defines a semiregular* $(n+1)$-*polytope* $\mathcal{S}$. *Its facets are isomorphic to* $\mathcal{P}$ *or* $\mathcal{Q}$, *with* k *of each of these occurring alternately around each face* $\mathcal{R}$ *of co-rank 2.* (*Thus each 2-section* $\mathcal{S}/\mathcal{R}$ *is a* $2k$-*gon.*) *The face-wise* Γ-*stabilizer of any ridge of* $\mathcal{S}$ *is trivial. Finally, each vertex-figure of* $\mathcal{S}$ *is isomorphic to the semiregular n-polytope* $\widehat{\mathcal{S}}$ *defined by the diagram in* (7).

Proof. We adapt the methods of [19, 2E]. Polytope property **A** is already in place, so we move to the 'diamond condition' **B**. For $0 \leqslant j \leqslant n$, we must consider incident faces of ranks $j-1$ and $j+1$. By Lemma 4.5(b) we can take these to be Γ_{j-1} and Γ_{j+1} in the basic chain

$$\Phi_K = [\Gamma_{-1}, \ldots, , \Gamma_{j-1}, \Gamma_{j+1}, \ldots, \Gamma_n, \Gamma_{n+1}],$$

with $K := \{0, 1, \ldots, n+1\} \setminus \{j\}$.

If $j \leqslant n-2$, the stabilizer of this chain is given by $\Gamma_K = \langle \alpha_j \rangle = \{1, \alpha_j\}$. This implies that the distinct faces Γ_j and $\Gamma_j \alpha_j$ are the only j-faces incident with Γ_{j-1} and Γ_{j+1}.

If $j = n-1$, then $\Gamma_K = \langle \alpha_{n-1} \rangle$ (resp. $\langle \beta_{n-1} \rangle$) if $\Gamma_n = \Gamma_n^{\mathcal{P}}$ (resp. $\Gamma_n = \Gamma_n^{\mathcal{Q}}$), and we proceed similarly.

If $j = n$, then $\Gamma_K = \{1\}$. However, $\Gamma_{n-1} = \Gamma_n^{\mathcal{P}} \cap \Gamma_n^{\mathcal{Q}}$, so $\Gamma_{n-1} \cap \Gamma_n^{\mathcal{P}} \mu \neq \emptyset$ forces $\Gamma_n^{\mathcal{P}} \mu = \Gamma_n^{\mathcal{P}}$. Thus $\Gamma_n^{\mathcal{P}}$ and $\Gamma_n^{\mathcal{Q}}$ are the two distinct facets incident with the ridge Γ_{n-1}.

Next let us examine the structure of the 2-section $\mathcal{S}/\mathcal{R}$. Without loss of generality we can assume that $\mathcal{R} = \Gamma_{n-2}$. It is easy to check that the $(n-1)$-faces in this section are all $\Gamma_{n-1}\mu$, with μ in the dihedral group $\mathbb{D}_{2k} := \langle \alpha_{n-1}, \beta_{n-1} \rangle$. Since $\mathbb{D}_{2k} \cap \Gamma_{n-1} = \{1\}$ (by the intersection condition), there are exactly $2k$ such cosets. Similarly, the n-faces come in two types: k each of types $\Gamma_n^{\mathcal{P}} \mu$ and $\Gamma_n^{\mathcal{Q}} \mu$, again taking $\mu \in \mathbb{D}_{2k}$. Thus each $\Gamma_{n-1} \mu$ meets exactly one n-face of each type, namely $\Gamma_n^{\mathcal{P}} \mu$ and $\Gamma_n^{\mathcal{Q}} \mu$. Likewise, $\Gamma_n^{\mathcal{P}} \mu$ (resp. $\Gamma_n^{\mathcal{Q}} \mu$) meets just $\Gamma_{n-1} \mu$ and $\Gamma_{n-1} \alpha_{n-1} \mu$ (resp. $\Gamma_{n-1} \mu$ and $\Gamma_{n-1} \beta_{n-1} \mu$). We see that this section has the structure of a $2k$-gon, as indicated in

$$\cdots \underset{\Gamma_{n-1}\beta_{n-1}\alpha_{n-1}}{\overset{\Gamma_n^{\mathcal{P}}\beta_{n-1}\alpha_{n-1}}{\bullet}} - - \underset{\Gamma_{n-1}\alpha_{n-1}}{\overset{\Gamma_n^{\mathcal{Q}}\alpha_{n-1}}{\bullet}} - \underset{\Gamma_{n-1}}{\overset{\Gamma_n^{\mathcal{P}}}{\bullet}} - - \underset{\Gamma_{n-1}\beta_{n-1}}{\overset{\Gamma_n^{\mathcal{Q}}}{\bullet}} - \underset{\Gamma_{n-1}\alpha_{n-1}\beta_{n-1}}{\overset{\Gamma_n^{\mathcal{P}}\beta_{n-1}}{\bullet}} \overset{\Gamma_n^{\mathcal{Q}}\alpha_{n-1}\beta_{n-1}}{} - - \cdots \quad (9)$$

Now we can verify the strong connectedness property **C**. We must show that any section F/G of $\mathcal{S}$ is connected. If $\mathrm{rank}(G) \geqslant 0$ or $\mathrm{rank}(F) \leqslant n$, this follows from a standard inductive argument based on Lemma 4.6 and the fact that $\mathcal{P}$ and $\mathcal{Q}$, being polytopes, are themselves strongly connected. To complete the induction we need only show that $\mathcal{S}$ is itself connected. For this it will suffice to show that any facet $\Gamma_n^{\mathcal{P}} \mu$ can be connected to $\Gamma_n^{\mathcal{P}}$, say, by a sequence of consecutively incident facets and ridges. If $\mu = \beta_{n-1}$, we observe such a sequence in (9) above. (Of course, facets of type $\Gamma_n^{\mathcal{Q}}$ will appear in the sequence.) Translating by any $\gamma \in \Gamma$ we get a sequence connecting $\Gamma_n^{\mathcal{P}} \gamma$ to $\Gamma_n^{\mathcal{P}} \beta_{n-1} \gamma$.

Now consider a general facet $\Gamma_n^{\mathcal{P}}\mu$, and write μ as a word of minimal length in the generators, say

$$\mu = \rho_1 \cdots \rho_m, \quad \text{with} \quad \rho_j \in \{\alpha_0, \ldots, \alpha_{n-1}, \beta_{n-1}\}.$$

Suppose $m = 1$. If $\mu = \rho_1 = \alpha_j$ for some j, then $\Gamma_n^{\mathcal{P}}\mu = \Gamma_n^{\mathcal{P}}\alpha_j = \Gamma_n^{\mathcal{P}}$ and there is nothing to prove. If $\mu = \rho_1 = \beta_{n-1}$, then we have already observed the required sequence in (9) above.

We proceed inductively, assuming the existence of a suitable sequence connecting $\Gamma_n^{\mathcal{P}}$ to $\Gamma_n^{\mathcal{P}}\gamma$ whenever γ has length $m-1$ in the generators. Let $\mu = \rho_1 \cdots \rho_m$, so that $\mu = \rho_1\gamma$ for $\gamma = \rho_2 \cdots \rho_m$. If $\rho_1 = \alpha_j$, then as before $\Gamma_n^{\mathcal{P}}\mu = \Gamma_n^{\mathcal{P}}\gamma$, which by induction can in fact be joined to $\Gamma_n^{\mathcal{P}}$. On the other hand, if $\rho_1 = \beta_{n-1}$, then $\Gamma_n^{\mathcal{P}}\mu = \Gamma_n^{\mathcal{P}}\beta_{n-1}\gamma$ is linked as in (9) to $\Gamma_n^{\mathcal{P}}\gamma$, which in turn is linked to $\Gamma_n^{\mathcal{P}}$ by the induction hypothesis.

This completes the proof. $\qquad\square$

Remarks. In the particularly interesting special issue when the diagram in (5) is a star with three branches (that is $n = 3$ and $k = 2$), there generally are three non-equivalent ways in which the diagram yields semiregular polytopes of rank 4. For example, the affine Coxeter group $\widetilde{B}_3$ represented by the diagram in (3) actually gives rise to two semiregular 3-polytopes, namely the semiregular tessellation of $\mathbb{R}^3$ with tetrahedral and octahedral tiles shown in Figure 1, as well as the standard cubical tessellation of $\mathbb{R}^3$; the latter is geometrically regular, by our next proposition, and $\widetilde{B}_3$ is a subgroup of index 2 in its full symmetry group $\widetilde{C}_3$. Here, the cubical tessellation can be derived in two ways from the diagram; on other words, two of the three semiregular polytopes associated with the diagram actually are isomorphic. In Section 6 we describe examples where the three semiregular polytopes are mutually non-isomorphic. At the other extreme, for the finite Coxeter group D_4, all three semiregular polytopes are mutually isomorphic; in fact, each is regular and isomorphic to the 4-cube.

Recall that an abstract polytope is called a 2-*orbit polytope* if its automorphism group has precisely two flag orbits. (See [13]; for general structure results about the groups of 2-orbit polytopes see also [14].)

Proposition 4.8. *Suppose $\mathcal{S}$ is the semiregular $(n + 1)$-polytope constructed from the tail-triangle diagram in (5).*

(a) *Then $\mathcal{S}$ is a regular polytope if and only if Γ admits a group automorphism induced by the diagram symmetry which swaps α_{n-1} and β_{n-1} in (5), while fixing the remaining α_j's. In this case $\mathcal{P} \simeq \mathcal{Q}$, say with Schläfli type $\{p_1, \ldots, p_{n-1}\}$, and $\mathcal{S}$ is regular of type $\{p_1, \ldots, p_{n-1}, 2k\}$; moreover, $\Gamma(\mathcal{S}) \simeq \Gamma \rtimes C_2$.*

(b) *If $\mathcal{S}$ is not regular, then $\mathcal{S}$ is a 2-orbit polytope and $\Gamma(\mathcal{S}) \simeq \Gamma$. In particular, this is so if the facets $\mathcal{P}$ and $\mathcal{Q}$ are non-isomorphic (as is the case, for example, if $\alpha_{n-2}\,\alpha_{n-1}$ and $\alpha_{n-2}\,\beta_{n-1}$ have different periods).*

Proof. Each flag of $\mathcal{S}$ is equivalent under Γ to exactly one of the base flags

$$\Phi^{\mathcal{P}} := [\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma_n^{\mathcal{P}}], \quad \text{or} \quad \Phi^{\mathcal{Q}} := [\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma_n^{\mathcal{Q}}]$$

(relative to Γ). Thus $\mathcal{S}$ has two flag orbits under Γ, and one or two flag orbits under its full automorphism group $\Gamma(\mathcal{S})$. In particular, $\mathcal{S}$ is regular if and only if $\Gamma(\mathcal{S})$ has just one flag orbit, and then Γ has index 2 in $\Gamma(\mathcal{S})$.

Now suppose $\mathcal{S}$ is regular and $\Phi := \Phi^{\mathcal{P}}$ denotes the base flag of $\mathcal{S}$ relative to the full group $\Gamma(\mathcal{S})$. Then $\Gamma(\mathcal{S}) := \langle \gamma_0, \ldots, \gamma_n \rangle$, where $\gamma_0, \ldots, \gamma_n$ are the distinguished generators. Identifying Γ with a subgroup of $\Gamma(\mathcal{S})$ and inspecting the action of its generators on Φ, we find that $\gamma_j = \alpha_j$ for $j \leqslant n-1$. Hence, if $j \leqslant n-2$ then

$$\gamma_n \alpha_j \gamma_n = \gamma_n \gamma_j \gamma_n = \gamma_j = \alpha_j.$$

Moreover, conjugation in $\Gamma(\mathcal{S})$ by γ_n takes the distinguished generators $\gamma_0, \ldots, \gamma_n$ relative to $\Phi = \Phi^{\mathcal{P}}$ to the distinguished generators relative to the adjacent flag $\Phi^n = \Phi^{\mathcal{Q}}$ of $\mathcal{S}$. Arguing as before we see from the action on $\Phi^{\mathcal{Q}}$ that $\gamma_n \gamma_{n-1} \gamma_{n-1} = \beta_{n-1}$ and therefore

$$\gamma_n \alpha_{n-1} \gamma_n = \gamma_n \gamma_{n-1} \gamma_n = \beta_{n-1}.$$

Hence conjugation in $\Gamma(\mathcal{S})$ by γ_n induces an involutory group automorphism on Γ corresponding to the horizontal diagram symmetry. Clearly, $\Gamma(\mathcal{S}) \simeq \Gamma \rtimes C_2$, with $C_2 = \langle \gamma_n \rangle$. The remaining claims of part (a) are obvious.

Conversely, suppose that an automorphism τ of Γ is induced by the diagram symmetry which swaps α_{n-1} and β_{n-1}, while fixing the remaining α_j's. Then, in the terminology of [19, 8A], a twisting operation applied with Γ and τ recovers the polytope $\mathcal{S}$ and proves that it is regular. The details are routine.

Now suppose $\mathcal{S}$ is not regular (for example, this occurs if the two facet types $\mathcal{P}$ and $\mathcal{Q}$ are not isomorphic). Then both $\Gamma(\mathcal{S})$ and its subgroup Γ have two flag orbits, so $\Gamma(\mathcal{S}) \simeq \Gamma$ and $\mathcal{S}$ is a 2-orbit $(n+1)$-polytope (of type $2_{\{0,\ldots,n-1\}}$, in the terminology of [13]). $\quad\square$

Remarks. We note for part (b) that $\Gamma = \Gamma(\mathcal{S})$ can certainly hold even when $\mathcal{P} \simeq \mathcal{Q}$. There are already instances of this behaviour when $n = 2$, for we need only truncate a regular polyhedron of type $\{p, p\}$ which is *not* self-dual. A quick check of Hartley's Census [12] reveals several instances. The smallest such polyhedron is flat with 72 flags and Schläfli type $\{6, 6\}$. However, for ease of description we choose instead the following

Example 4.9. Let $\mathcal{M}$ to be the regular polyhedron $\{6, 6\} * 240a$ in the Census [12]. The corresponding string C-group Γ has order 240 and can be generated by permutations

$$\alpha_1 = (2, 3)(4, 5), \ \alpha_0 = (1, 2), \ \beta_1 = (2, 4)(3, 5)(6, 7)$$

associated to the diagram

$$\tag{10}$$

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We construct $\mathcal{M}$ by ringing the top node and its dual $\mathcal{M}^*$ by ringing the bottom node. If, however, we ring the middle node, then we obtain a semiregular polyhedron $\mathcal{S}$ with 480 flags. This $\mathcal{S}$ has Schläfli type $\{6,6\}$, but it cannot be regular and $\Gamma(\mathcal{S}) \simeq \Gamma$ has order 240. Intuitively, we construct $\mathcal{S}$ by truncating $\mathcal{M}$ to its edge midpoints.

Example 4.10. The Coxeter group B_3 of order 48 has diagram

$$\begin{array}{c} \bullet\, \alpha_1 \\ 4 \nearrow \quad | \\ \alpha_0\, \circledcirc \qquad | \, 3 \\ \quad | \\ \bullet\, \beta_1 \end{array} \qquad\qquad (11)$$

The geometrical version of Wythoff's construction gives the cube $\{4,3\}$, which is a convex regular solid, of course. The abstract construction is subtly different. Since $\alpha_0\beta_1$ has period 2, each edge of the cube must be replaced by a digon, so that we do indeed get a semiregular polytope with square and digonal faces and with hexagonal vertex-figures. In the convex setting, these digons collapse to line segments and the vertex-figures to equilateral triangles, as expected. (A nice spherical model is obtained by inscribing the cube in its circumsphere; the six face planes cut the sphere in the 24 arcs needed for edges.)

In the same way we can disturb any of the classical convex regular polytopes, indeed, any regular polytope whatsoever.

The tessellation $\mathcal{T}$ described in Example 3.1 and Figure 1 has octahedra among its facets, and cuboctahedral vertex-figures. Both have spherical type and can be projectified by antipodal identifications. This is accomplished by adjoining to (4) the first or second relation in (12) below. We choose both for the next example (see [21] for the other possibilities):

Example 4.11. *The tomotope.* Define a group Γ by adjoining to (4) the two relations which create hemioctahedra and hemicuboctahedra:

$$(\rho_0\rho_1\rho_3)^3 = (\rho_2\rho_1\rho_3)^3 = 1 \,. \qquad\qquad (12)$$

Using GAP [10], it is easy to check that Γ now has finite order 96, yet still satisfies the intersection condition (6). Thus, by Theorem 4.7, we obtain a finite semiregular 4-polytope $\mathcal{T}_{hh}$, which in [21] was called the *tomotope* (as a small gift for Tomaž Pisanski).

The tomotope has 4 vertices, 12 edges, 16 triangles, and 4 tetrahedra and 4 hemioctahedra. Two of each kind of facet alternate around each edge. Since Γ acts faithfully on edges $1,\ldots,12$ (say), we have this permutation representation:

$$\begin{aligned}
\rho_0 &= (5,10)(6,9)(7,12)(8,11)\,, \\
\rho_1 &= (1,6)(2,5)(3,8)(4,7)\,, \\
\rho_2 &= (5,9)(6,10)(7,11)(8,12)\,, \\
\rho_3 &= (5,8)(6,7)(9,12)(10,11)\,.
\end{aligned}$$

It is also possible to obtain Γ by reducing the crystallographic group $\widetilde{B}_3$ modulo 2. (This fact resurfaces in Section 6 below.)

Our main purpose in [21] was to investigate the regular covers of $\mathcal{T}_{hh}$. Since the epimorphism $\widetilde{B}_3 \to \Gamma$ induces a 2-covering $\mathcal{T} \to \mathcal{T}_{hh}$ of semiregular polytopes [19, 2D], we may construct $\mathcal{T}_{hh}$ by making suitable identifications in $\mathcal{T}$ (see below). But putting that aside, we find that the tomotope has the peculiar property, impossible in lower ranks, of having infinitely many distinct, finite, minimal *regular* covers.

To visualize the tomotope $\mathcal{T}_{hh}$ imagine a core octahedron with 8 tetrahedra glued to its faces, suggesting the *stella octangula*. Next imagine this complex inscribed in a $2 \times 2 \times 2$ cube and from that make toroidal-type identifications for the vertices and edges lying in the boundary of the cube. Finally, we further identify antipodal faces of all ranks to get $\mathcal{T}_{hh}$. In Figure 2 you can see the 4 vertices, $4 = 8/2$ tetrahedra and 1 hemioctahedron in the core. The other three hemioctahedra are red, yellow and green, and 'run around' the belts of those colours. For example, before making identifications, we may split a red octahedron into four sectors around a vertical axis of symmetry, then fit these into four red slots, two of which are visible in Figure 2. In this way we fill out the $2 \times 2 \times 2$ cube before the final antipodal identifications.
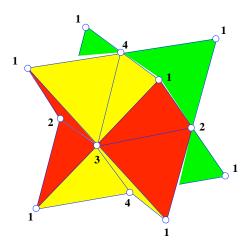

Figure 2: The tomotope $\mathcal{T}_{hh}$.

The proof of the intersection property for tail-triangle C-groups can often be reduced to the consideration of only a small number of cases. For $i = -1, 0, \ldots, n-2$, define $\Gamma_i^+ := \langle \alpha_{i+1}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$.

Lemma 4.12. *Suppose that* $\Gamma = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ *is a tail-triangle group with* $n \geqslant 2$, *and suppose that its subgroups* $\Gamma_n^1 := \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle$, $\Gamma_n^2 := \langle \alpha_0, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$ *and* $\Gamma_0 := \langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ *are C-groups. Then* Γ *is a tail-triangle C-group if and only if* $\Gamma_n^1 \cap \Gamma_n^2 = \langle \alpha_0, \dots, \alpha_{n-2} \rangle$ *and both* $\Gamma_i^+ \cap \Gamma_n^1 = \langle \alpha_{i+1}, \dots, \alpha_{n-1} \rangle$ *and* $\Gamma_i^+ \cap \Gamma_n^2 = \langle \alpha_{i+1}, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$ *for* $i = 0, \dots, n-2$.

Proof. The case $n = 2$ is trivial, since the stated conditions on Γ_n^1, Γ_n^2 and Γ_0^+ are just saying that any two of the three distinguished dihedral subgroups intersect as required. Now suppose $n \geqslant 3$ and $I, J \subseteq \{\alpha_0, \dots, \alpha_{n-1}, \beta_{n-1}\}$. We must show that $\langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle$. Clearly, one inclusion is obvious, and the other holds trivially if I or J is the full generating set. Hence we may exclude the latter possibility from now on.

First consider the case when $\{\alpha_0, \dots, \alpha_{n-2}\} \not\subseteq I, J$. Let i (resp. j) denote the largest integer k with $k \leqslant n-2$ and $\alpha_k \notin I$ (resp. $\alpha_k \notin J$). Assume that $i \leqslant j$. Then $I = I_0 \cup I_1$ and $J = J_0 \cup J_1$, where

$$I_0 \subseteq \{\alpha_0, \dots, \alpha_{i-1}\}, \quad \{\alpha_{i+1}, \dots, \alpha_{n-2}\} \subseteq I_1 \subseteq \{\alpha_{i+1}, \dots, \alpha_{n-1}, \beta_{n-1}\}$$

and similarly

$$J_0 \subseteq \{\alpha_0, \dots, \alpha_{j-1}\}, \quad \{\alpha_{j+1}, \dots, \alpha_{n-2}\} \subseteq J_1 \subseteq \{\alpha_{j+1}, \dots, \alpha_{n-1}, \beta_{n-1}\}.$$

By the commutativity relations implicit in the underlying diagram, $\langle I \rangle \simeq \langle I_0 \rangle \times \langle I_1 \rangle$ and $\langle J \rangle \simeq \langle J_0 \rangle \times \langle J_1 \rangle$. (It will emerge below that these and similar products are direct.)

Now suppose $\mu \in \langle I \rangle \cap \langle J \rangle$. Then $\mu = \lambda_0 \lambda_1$ with $\lambda_0 \in \langle I_0 \rangle$ and $\lambda_1 \in \langle I_1 \rangle$, and similarly $\mu = \nu_0 \nu_1$ with $\nu_0 \in \langle J_0 \rangle$ and $\nu_1 \in \langle J_1 \rangle$. But $i \leqslant j$, so both λ_1 and ν_1 are elements in $\langle \alpha_{i+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle$ related by $\nu_1 = \xi \lambda_1$ with $\xi := \nu_0^{-1} \lambda_0 \in \langle \alpha_0, \dots, \alpha_{j-1} \rangle$ (and $\xi = \nu_1 \lambda_1^{-1} \in \langle \alpha_{i+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle$). Then by our assumptions $\xi \in \Gamma_i^+ \cap \Gamma_n^1 = \langle \alpha_{i+1}, \dots, \alpha_{n-1} \rangle$, and hence also $\xi \in \langle \alpha_0, \dots, \alpha_{j-1} \rangle \cap \langle \alpha_{i+1}, \dots, \alpha_{n-1} \rangle = \langle \alpha_{i+1}, \dots, \alpha_{j-1} \rangle$ since Γ_n^1 is a C-group. (Replacing I, J by I_0, I_1 we now see why $\langle I_0 \rangle \cap \langle I_1 \rangle = \langle 1 \rangle$.) Returning to our proof, we note that it suffices to show that $\lambda_0, \lambda_1 \in \langle I \cap J \rangle$.

First consider $\nu_0 = \lambda_0 \xi^{-1}$ as an element of the direct product

$$\langle \alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{j-1} \rangle \simeq \langle \alpha_0, \dots, \alpha_{i-1} \rangle \times \langle \alpha_{i+1}, \dots, \alpha_{j-1} \rangle,$$

bearing in mind here that $\lambda_0 \in \langle \alpha_0, \dots, \alpha_{i-1} \rangle$ and $\xi \in \langle \alpha_{i+1}, \dots, \alpha_{j-1} \rangle$. But

$$\nu_0 \in \langle J_0 \rangle \cap \langle \alpha_0, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{j-1} \rangle = \langle J_0 \setminus \{\alpha_i\} \rangle,$$

once again by the intersection property in Γ_n^1. Since the factorization of the element ν_0 of $\langle J_0 \setminus \{\alpha_i\} \rangle$ as an element of the direct product is unique, we must also have $\lambda_0, \xi \in \langle J_0 \rangle$; in fact, we even have $\xi \in \langle J_0 \setminus \{\alpha_0\} \rangle$. But then $\lambda_0 \in \langle I_0 \rangle \cap \langle J_0 \rangle = \langle I_0 \cap J_0 \rangle \leqslant \langle I \cap J \rangle$, by the intersection property in Γ_n^1.

Similarly we look at the factorization $\lambda_1 = \xi^{-1} \nu_1$ in the direct product

$$\langle \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle \simeq \langle \alpha_{i+1}, \dots, \alpha_{j-1} \rangle \times \langle \alpha_{j+1}, \dots, \alpha_{n-1}, \beta_{n-1} \rangle,$$

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remembering here that $\xi \in \langle \alpha_{i+1}, \ldots, \alpha_{j-1} \rangle$ and $\nu_1 \in \Gamma_{j+1}^+$. Now

$$\lambda_1 \in \langle I_1 \rangle \cap \langle \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle = \langle I_1 \setminus \{\alpha_j\} \rangle,$$

by the intersection property of Γ_0. But the factorization of the element λ_1 of $\langle I_1 \setminus \{\alpha_j\} \rangle$ as an element of the direct product is unique, so in fact $\xi, \nu_1 \in \langle I_1 \rangle$. Hence,

$$\nu_1 \in \langle I_1 \rangle \cap \langle J_1 \rangle = \langle I_1 \cap J_1 \rangle \leqslant \langle I \cap J \rangle,$$

by the intersection property of Γ_0. Moreover,

$$\xi \in \langle I_1 \rangle \cap \langle J_0 \setminus \{\alpha_0\} \rangle = \langle I_1 \cap (J_0 \setminus \{\alpha_0\}) \rangle \leqslant \langle I \cap J \rangle,$$

once again by the intersection property of Γ_0. Thus $\lambda_1 \in \langle I \cap J \rangle$, as required. This completes the proof for the case when $\{\alpha_0, \ldots, \alpha_{n-2}\} \not\subseteq I, J$.

Now suppose that one of the sets, say J, contains $\{\alpha_0, \ldots, \alpha_{n-2}\}$. Then either $J \subseteq \{\alpha_0, \ldots, \alpha_{n-1}\}$ and $\langle J \rangle \leqslant \Gamma_n^1$, or $J = \{\alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1}\}$ and $\langle J \rangle = \Gamma_n^2$.

If also I contains $\{\alpha_0, \ldots, \alpha_{n-2}\}$, then either $I \subseteq J$ or $J \subseteq I$, or one of I, J is $\{\alpha_0, \ldots, \alpha_{n-1}\}$ and the other is $\{\alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1}\}$. In the former case the intersection condition holds trivially, and in the latter case by our assumptions on $\Gamma_n^1 \cap \Gamma_n^2$.

Now suppose that $\{\alpha_0, \ldots, \alpha_{n-2}\} \not\subseteq I$ and that $\mu \in \langle I \rangle \cap \langle J \rangle$. As above let i denote the largest integer k with $k \leqslant n-2$ and $\alpha_k \notin I$, let $I = I_0 \cup I_1$, and let $\mu = \lambda_0 \lambda_1$ with $\lambda_0 \in \langle I_0 \rangle$ and $\lambda_1 \in \langle I_1 \rangle$. But $I_0 \subseteq J$, so clearly $\lambda_0 \in \langle I \cap J \rangle$ and $\lambda_1 = \lambda_0^{-1}\mu \in \langle J \rangle$.

First suppose that $J \subseteq \{\alpha_0, \ldots, \alpha_{n-1}\}$. Then

$$\lambda_1 \in \langle I_1 \rangle \cap \langle J \rangle \leqslant \Gamma_i^+ \cap \Gamma_n^1 = \langle \alpha_{i+1}, \ldots, \alpha_{n-1} \rangle,$$

by our assumption on the rightmost intersection. Hence, if $\alpha_{n-1} \in I_1, J$ then necessarily $\{\alpha_{i+1}, \ldots, \alpha_{n-1}\} = I_1 \cap J$ and hence $\lambda_1 \in \langle I \cap J \rangle$; bear in mind that $\{\alpha_{i+1}, \ldots, \alpha_{n-2}\} \subseteq I_1$ by the definition of i. Now, if $\alpha_{n-1} \notin J$ then

$$\lambda_1 \in \langle J \rangle \cap \langle \alpha_{i+1}, \ldots, \alpha_{n-1} \rangle = \langle \alpha_{i+1}, \ldots, \alpha_{n-2} \rangle = \langle I_1 \cap J \rangle \leqslant \langle I \cap J \rangle$$

by the intersection property of Γ_n^1. On the other hand, if $\alpha_{n-1} \notin I_1$ then

$$\lambda_1 \in \langle I_1 \rangle \cap \langle \alpha_{i+1}, \ldots, \alpha_{n-1} \rangle \leqslant \Gamma_n^2 \cap \Gamma_n^1 = \langle \alpha_0, \ldots, \alpha_{n-2} \rangle,$$

and therefore also

$$\lambda_1 \in \langle \alpha_0, \ldots, \alpha_{n-2} \rangle \cap \langle \alpha_{i+1}, \ldots, \alpha_{n-1} \rangle = \langle \alpha_{i+1}, \ldots, \alpha_{n-2} \rangle = \langle I_1 \cap J \rangle \leqslant \langle I \cap J \rangle,$$

once again by the intersection property of Γ_n^1.

Finally, if $J = \{\alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1}\}$ then

$$\lambda_1 \in \langle I_1 \rangle \cap \langle J \rangle \leqslant \Gamma_i^+ \cap \Gamma_n^2 = \langle \alpha_{i+1}, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle.$$

Two possibilities can occur for I_1. If $\beta_{n-1} \in I_1$, then $\{\alpha_{i+1}, \ldots, \alpha_{n-2}, \beta_{n-1}\} = I_1 \cap J$ and hence $\lambda_1 \in \langle I \cap J \rangle$. However, if $\beta_{n-1} \notin I_1$ then

$$\lambda_1 \in \langle I_1 \rangle \cap \langle \alpha_{i+1}, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle \leqslant \Gamma_n^1 \cap \Gamma_n^2 = \langle \alpha_0, \ldots, \alpha_{n-2} \rangle,$$

and therefore also

$$\lambda_1 \in \langle \alpha_{i+1}, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle \cap \langle \alpha_0, \ldots, \alpha_{n-2} \rangle = \langle \alpha_{i+1}, \ldots, \alpha_{n-2} \rangle = \langle I_1 \cap J \rangle,$$

now by the intersection property in Γ_n^2.

Thus in either case $\lambda_1 \in \langle I \cap J \rangle$, and since $\lambda_0 \in \langle I \cap J \rangle$, we also have $\mu \in \langle I \cap J \rangle$. This completes the proof. $\qquad \square$

Remarks. The intersection conditions explicitly mentioned in Lemma 4.12 can often be further reduced. For example, when $n = 3$, the three conditions for the mutual intersections of 3-generator subgroups imply the remaining conditions, provided these subgroups are C-groups.

5   A Universal Construction

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes with isomorphic facets $\mathcal{K}$ and imagine that we have unlimited copies of each. Pick a base copy of $\mathcal{P}$, say, and along each of its facets attach a distinct copy of $\mathcal{Q}$. Now only certain facets of the new $\mathcal{Q}$'s are exposed, so to them we attach distinct $\mathcal{P}$'s. Continue this in 'alternating' fashion. Presumably we may then adjoin an infinite number of copies of $\mathcal{P}$'s and $\mathcal{Q}$'s about each ridge of each copy. But does this process really make sense and do we get an $(n + 1)$-polytope $\mathcal{S}$ from it? If so, what is $\Gamma(\mathcal{S})$? Does it matter if instead we begin with a base copy of $\mathcal{Q}$?

We will address these questions by amalgamating the automorphism groups of $\mathcal{P}$ and $\mathcal{Q}$. However, we cannot yet answer a harder question: does the construction work when around each ridge we attempt to alternate just k copies each of $\mathcal{P}$ and $\mathcal{Q}$ before definitely closing up, with $k < \infty$?

We begin by extracting from [1, Ch. I, §7.3] some properties of groups amalgamated along subgroups, as they apply to our construction. Suppose therefore that we have specified base flags for $\mathcal{P}$ and $\mathcal{Q}$ and hence also corresponding lists of n standard involutory generators for $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. Then $\mathcal{P}$ and $\mathcal{Q}$ will have isomorphic facets precisely when there is an isomorphism from the facet subgroup of $\Gamma(\mathcal{P})$ to that of $\Gamma(\mathcal{Q})$ which pairs in order the first $n - 1$ generators of $\Gamma(\mathcal{P})$ with those of $\Gamma(\mathcal{Q})$. With these data understood we can unambiguously amalgamate $\Gamma(\mathcal{P})$ with $\Gamma(\mathcal{Q})$ along $\Gamma(\mathcal{K})$, giving the group

$$\Pi := \Pi(\mathcal{P}, \mathcal{Q}) := \Gamma(\mathcal{P}) *_{\Gamma(\mathcal{K})} \Gamma(\mathcal{Q}) \, .$$

But an important feature of the construction is that $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ embed into Π [1, Ch. I, §7.3, Prop. 4]. Therefore we are justified at the outset in simply assuming that
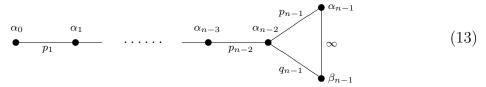
$$
\begin{aligned}
\Gamma(\mathcal{P}) &= \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle, \\
\Gamma(\mathcal{Q}) &= \langle \alpha_0, \dots, \alpha_{n-2}, \beta_{n-1} \rangle, \\
\Gamma(\mathcal{K}) &= \langle \alpha_0, \dots, \alpha_{n-2} \rangle
\end{aligned}
$$

are all subgroups of $\Pi = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$.

One can prove that a presentation of Π is obtained by amalgamating defining relations for $\Gamma(\mathcal{P})$ (on its generators $\alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}$) with defining relations for $\Gamma(\mathcal{Q})$ (on $\alpha_0, \dots, \alpha_{n-2}, \beta_{n-1}$). We can therefore represent this arrangement of groups schematically in the diagram

$$(13)$$

which evidently has the structure displayed earlier in (5).

The group Π is usually not a Coxeter group; but it will be a quotient of some Coxeter group with a diagram like this. We make no assumptions about the branch labels $p_1, \dots, p_{n-2}, p_{n-1}, q_{n-1}$. Such a label could even be '2', indicating no branch at all. However, it is a property of free products with amalgamation that $\alpha_{n-1}\beta_{n-1}$ has infinite period, which explains the label of the right-hand branch. In other words, $\langle \alpha_{n-1}, \beta_{n-1} \rangle$ is an infinite dihedral group.

For more significant calculations we must employ a standard factorization in amalgamated products. Let $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ be transversals to $\Gamma(\mathcal{K})$ in $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$, respectively, with $1 \in T_{\mathcal{P}} \cap T_{\mathcal{Q}}$. Then every $\mu \in \Pi$ has a unique *reduced decomposition* of length $m \geqslant 0$, say

$$
\mu = \kappa \tau_1 \cdots \tau_m \, . \tag{14}
$$

Here $\kappa \in \Gamma(\mathcal{K})$; and if $m \geqslant 1$, then all $\tau_i \neq 1$ and τ_i, τ_{i+1} belong to different transversals $T_{\mathcal{P}}, T_{\mathcal{Q}}$ for $1 \leqslant i < n$ [1, Ch. I, §7.3, Prop. 5]. We call κ and $\tau_1, \dots, \tau_m$, respectively, the *leading element* and the *transversal elements* for the reduced decomposition of μ.

In fact, we shall select our transversals in a special way.

Lemma 5.1. *There are transversals $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ such that for $0 \leqslant j \leqslant n-1$, $T_{\mathcal{P}}$ contains a transversal $T_{\mathcal{P},j}$ for $\langle \alpha_j, \dots, \alpha_{n-2} \rangle$ in $\langle \alpha_j, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle$; and $T_{\mathcal{Q}}$ contains a transversal $T_{\mathcal{Q},j}$ for $\langle \alpha_j, \dots, \alpha_{n-2} \rangle$ in $\langle \alpha_j, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$. Moreover,*

$$
\{1, \alpha_{n-1}\} = T_{\mathcal{P},n-1} \subseteq T_{\mathcal{P},n-2} \subseteq \dots \subseteq T_{\mathcal{P},1} \subseteq T_{\mathcal{P},0} = T_{\mathcal{P}}
$$

and

$$
\{1, \beta_{n-1}\} = T_{\mathcal{Q},n-1} \subseteq T_{\mathcal{Q},n-2} \subseteq \dots \subseteq T_{\mathcal{Q},1} \subseteq T_{\mathcal{Q},0} = T_{\mathcal{Q}}.
$$

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Proof. We proceed by induction on j, as j runs from $n-1$ down to 0. When $j = n-1$ we simply pick $T_{\mathcal{P},n-1} = \{1, \alpha_{n-1}\}$ as transversal for $\langle 1 \rangle$ in $\langle \alpha_{n-1} \rangle$. For the inductive step, suppose that $\tau_1, \tau_2 \in T_{\mathcal{P},j+1}$ (in $\langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle$) satisfy

$$\tau_1 \equiv \tau_2 \bmod \langle \alpha_j, \ldots, \alpha_{n-2} \rangle .$$

Then

$$\tau_1^{-1}\tau_2 \in \langle \alpha_j, \ldots, \alpha_{n-2} \rangle \cap \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle = \langle \alpha_{j+1}, \ldots, \alpha_{n-2} \rangle,$$

by the intersection condition (2) for string C-groups. Thus $\tau_1 = \tau_2$, and we may extend $T_{\mathcal{P},j+1}$ to $T_{\mathcal{P},j}$ for $j = n-2, \ldots, 0$. $\qquad\square$

Lemma 5.2. *For $0 \leqslant j \leqslant n-1$,*

$$\langle \alpha_j, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle \simeq \langle \alpha_j, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle *_{\langle \alpha_j, \ldots, \alpha_{n-2} \rangle} \langle \alpha_j, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle . \quad (15)$$

In particular, $\langle \alpha_{n-1}, \beta_{n-1} \rangle$ is the infinite dihedral group; and $\langle \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is the Coxeter group with diagram

$$(16)$$

Proof. For convenience, let $A = \langle \alpha_j, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle$, $B = \langle \alpha_j, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$, $C = \langle \alpha_j, \ldots, \alpha_{n-2} \rangle$, all subgroups of Π. We must show that the left side of (15) is isomorphic to $A *_C B$. Since the inclusions $A \hookrightarrow \Pi$, $B \hookrightarrow \Pi$ agree on C, there is a natural map $\varphi : A *_C B \to \Pi$ whose image is $\langle \alpha_j, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$. (We abuse notation a little here.)

Suppose that $\mu \in \ker(\varphi)$ has the reduced decomposition $\mu = \kappa \tau_1 \cdots \tau_m$, where now $\kappa \in C$ and the τ_j's belong alternately to transversals for C in A or B. By Lemma 5.1, these transversals transfer under φ to subsets of the transversals $T_{\mathcal{P}}, T_{\mathcal{Q}}$ in Π. Applying φ we therefore get $1 = \kappa \tau_1 \cdots \tau_m$ in Π, where uniqueness of the reduced decomposition gives $\kappa = 1$ and $m = 0$. Thus φ is injective. (Compare [1, Ch. I, Exercise §7 - 28].) The special cases when $j = n-2, n-1$ follow at once from the standard presentation of an amalgamated product. $\qquad\square$

The two previous lemmas have immediate consequences for the structure of transversals. For $-1 \leqslant j \leqslant n-2$, let $\mathcal{P}_j$, $\mathcal{Q}_j$, and $\mathcal{K}_j$ respectively, denote the co-faces of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{K}$, at their basic j-face. Then by Lemma 5.2 the subgroup

$$\Pi_j^+ := \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$$

of Π is isomorphic to the amalgamated product

$$\begin{aligned}
\Pi(\mathcal{P}_j, \mathcal{Q}_j) &= \Gamma(\mathcal{P}_j) *_{\Gamma(\mathcal{K}_j)} \Gamma(\mathcal{Q}_j) \\
&= \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \alpha_{n-1} \rangle *_{\langle \alpha_{j+1}, \ldots, \alpha_{n-2} \rangle} \langle \alpha_{j+1}, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle.
\end{aligned}$$

It follows from the inductive setup of our transversals that, apart from a shift in subscripts, the transversals $T_{\mathcal{P}_j}$ and $T_{\mathcal{Q}_j}$ associated with $\mathcal{P}_j$ and $\mathcal{Q}_j$ relative to the amalgamated product $\Pi(\mathcal{P}_j, \mathcal{Q}_j)$ as in Lemma 5.1 are just the subsets $T_{\mathcal{P},j+1}$ of $T_{\mathcal{P}}$ and $T_{\mathcal{Q},j+1}$ of $T_{\mathcal{Q}}$. We now have an inductive argument in place to establish the following lemma.

Lemma 5.3. *Let $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ be transversals inductively built as in Lemma 5.1, let $-1 \leqslant j \leqslant n - 2$, and let $\mu = \kappa \tau_1 \dots \tau_m$ be the reduced decomposition of an element μ in Π_j^+. Then $\kappa \in \langle \alpha_{j+1}, \dots, \alpha_{n-2} \rangle$ and $\tau_i \in T_{\mathcal{P},j+1} \cup T_{\mathcal{Q},j+1}$ for $i = 1, \dots, m$.*

Proof. The subgroup Π_j^+ of Π is an amalgamated product isomorphic to $\Pi(\mathcal{P}_j, \mathcal{Q}_j)$, with unique reduced decompositions of its elements μ relative to the respective transversals $T_{\mathcal{P},j+1} = T_{\mathcal{P}_j}$ and $T_{\mathcal{Q},j+1} = T_{\mathcal{Q}_j}$. Since the latter are subsets of $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$, respectively, the lemma follows from the uniqueness of the representations of the reduced decompositions in $\Pi(\mathcal{P}_j, \mathcal{Q}_j)$ and in Π. $\qquad\square$

The previous lemmas enable us to establish the crucial intersection property for Π, required for the construction of the desired semiregular polytope.

Lemma 5.4. Π *satisfies the intersection condition* (6).

Proof. We proceed inductively. By construction, $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ already are C-groups embedded in Π. In view of Lemma 5.2 we also may assume that Π_0 is a C-group.

According to Lemma 4.12 we must verify three intersection conditions, namely $\Gamma(\mathcal{P}) \cap \Gamma(\mathcal{Q}) = \Gamma(\mathcal{K})$ and both $\Pi_i^+ \cap \Gamma(\mathcal{P}) = \langle \alpha_{i+1}, \dots, \alpha_{n-1} \rangle$ and $\Pi_i^+ \cap \Gamma(\mathcal{Q}) = \langle \alpha_{i+1}, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$ for $i = 0, \dots, n - 2$. To this end, suppose $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$ are transversals chosen as in Lemma 5.1.

First consider the subgroup $\Gamma(\mathcal{P}) \cap \Gamma(\mathcal{Q}) \, (= \Pi_n^1 \cap \Pi_n^2)$ of Π. Each element in $\Gamma(\mathcal{P})$ (resp. $\Gamma(\mathcal{Q})$) has a reduced decomposition with at most one (non-trivial) transversal element from $T_{\mathcal{P}}$ and $T_{\mathcal{Q}}$, respectively. By the uniqueness of reduced decompositions in Π, an element in $\Gamma(\mathcal{P}) \cap \Gamma(\mathcal{Q})$ cannot involve any transversal elements at all and must lie $\Gamma(\mathcal{K})$. Thus $\Gamma(\mathcal{P}) \cap \Gamma(\mathcal{Q}) = \Gamma(\mathcal{K})$.

Now suppose $\mu \in \Pi_i^+ \cap \Gamma(\mathcal{P})$ for some $i = 0, \dots, n - 2$. By Lemma 5.3, if $\mu = \kappa \tau_1 \dots \tau_m$ is the reduced decomposition in Π_i^+ (and hence in Π), then $\kappa \in \langle \alpha_{i+1}, \dots, \alpha_{n-2} \rangle$ and $\tau_j \in T_{\mathcal{P},i+1} \cup T_{\mathcal{Q},i+1}$ for $j = 1, \dots, m$. On the other hand, $\mu \in \Gamma(\mathcal{P})$, so necessarily $m \leqslant 1$, and also $\tau_1 \in T_{\mathcal{P},i+1}$ if $m = 1$. In either case $\mu \in \langle \alpha_{i+1}, \dots, \alpha_{n-2}, \alpha_{n-1} \rangle$.

If $\mu \in \Pi_i^+ \cap \Gamma(\mathcal{Q})$ for some $i = 0, \dots, n - 1$, we can similarly conclude that $\mu \in \langle \alpha_{i+1}, \dots, \alpha_{n-2}, \beta_{n-1} \rangle$, as before with $m \leqslant 1$ but now with $\tau_1 \in T_{\mathcal{Q},i+1}$ if $m = 1$. $\qquad\square$

Now we can apply Theorem 4.7 and Proposition 4.8 to the group Π.

Theorem 5.5. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes with isomorphic facets $\mathcal{K}$. Then the group $\Pi = \Gamma(\mathcal{P}) *_{\Gamma(\mathcal{K})} \Gamma(\mathcal{Q})$ is a group of automorphisms for a semiregular $(n + 1)$-polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ whose facets are copies of $\mathcal{P}$ and $\mathcal{Q}$ appearing alternately around each ridge, each a copy of $\mathcal{K}$. Each section $\mathcal{U}_{\mathcal{P},\mathcal{Q}}/\mathcal{R}$ determined by an $(n-2)$-face $\mathcal{R}$ is an apeirogon*

$\{\infty\}$. *The polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$ is regular if and only if $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. In this case $\Gamma(\mathcal{U}_{\mathcal{P},\mathcal{Q}}) = \Pi \rtimes C_2$; otherwise $\Gamma(\mathcal{U}_{\mathcal{P},\mathcal{Q}}) = \Pi$.*

Proof. It remains to determine the structure of the full automorphism group of $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$. If $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic polytopes, then Π admits an involutory group automorphism that pairs up respective generators of $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ and hence corresponds to the symmetry of the underlying diagram for Π in its horizontal axis. Note here that Π admits a presentation that is symmetric with respect to $\mathcal{P}$ and $\mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{Q}$ are not isomorphic, then clearly such an automorphism cannot exist. Now Proposition 4.8 applies and completes the proof. $\qquad\square$

In the semiregular polytope $\mathcal{U}_{\mathcal{P},\mathcal{Q}}$, an infinite number of copies of the given n-polytopes $\mathcal{P}$ and $\mathcal{Q}$ appear alternately as facets around each $(n-2)$-face $\mathcal{R}$, determining a section $\mathcal{U}_{\mathcal{P},\mathcal{Q}}/\mathcal{R}$ of rank 2 isomorphic to an apeirogon $\{\infty\}$. Ideally, for each integer $k \geqslant 2$, we would also like to construct a similar such semiregular polytope, now with k copies each of $\mathcal{P}$ and $\mathcal{Q}$ appearing alternately around each $(n-2)$-face. At the group level, this would involve analysis of the quotient $\Pi^k = \Pi^k(\mathcal{P},\mathcal{Q})$ of the universal group $\Pi = \Pi(\mathcal{P},\mathcal{Q})$ obtained by adding the extra relation

$$(\alpha_{n-1}\beta_{n-1})^k = 1 \tag{17}$$

to the defining relations of Π, that is, by factoring out the normal subgroup N^k generated by the conjugates of $(\alpha_{n-1}\beta_{n-1})^k$ in Π. Then three tasks would have to be accomplished, with the first being the most challenging: verification of the intersection condition for Π_k; proof that the groups $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ really are embedded in Π^k; and proof that $\alpha_{n-1}\beta_{n-1}$ really has order k in the quotient Π^k. Solutions for these tasks seem to require rather complicated arguments concerning reduced decompositions of certain elements of the amalgamated product Π.

6 Polytopes from reflection groups over finite fields

In this section we briefly sketch a construction of semiregular polytopes based on modular reduction techniques applied to certain reflection groups. More details are described in a forthcoming paper [26]. Reflection groups over finite fields and their related regular polytopes have been studied in [23, 24, 25].

Suppose that $\Gamma = \langle \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$ is an abstract Coxeter group represented, up to branch labels, by a tail-triangle Coxeter diagram as in (5). Let $p_{i,j}$ and $q_{i,n-1}$ denote the orders of $\alpha_i\alpha_j$ and $\alpha_i\beta_{n-1}$ in Γ, so in particular $p_{i,i} = 1$, $p_{j,i} = p_{i,j} =: p_{i+1}$ if $j = i+1$, and $p_{i,j} = 2$ otherwise, and similarly $q_{n-1,n-1} = k$, $q_{n-1,n-2} = q_{n-2,n-1} =: q_{n-1}$, and $q_{i,n-1} = 2$ otherwise. (We will not require notation for the order of β_{n-1}^2.) Let V be real $(n+1)$-space, with basis $\{a_0, \dots, a_{n-1}, b_{n-1}\}$ and symmetric bilinear form $x \cdot y$ defined by

$$a_i \cdot a_j := -2\cos\tfrac{\pi}{p_{i,j}}, \qquad a_i \cdot b_{n-1} := -2\cos\tfrac{\pi}{q_{i,n-1}}, \qquad b_{n-1} \cdot b_{n-1} := 2. \tag{18}$$

Let $R : \Gamma \to G$ be the standard (faithful) representation of Γ in V, where

$$G = \langle r_0, \dots, r_{n-1}, s_{n-1} \rangle$$

is the isometric reflection group generated by the reflections with *roots* $a_0, \dots, a_{n-1}, b_{n-1}$ (see [15, §5.3–5.4]); thus,

$$\begin{aligned}
r_i(x) &= x - (x \cdot a_i)\, a_i \quad (i = 0, \dots, n-1), \\
s_{n-1}(x) &= x - (x \cdot b_{n-1})\, b_{n-1}.
\end{aligned}$$

Then, with respect to the basis $\{a_0, \dots, a_{n-1}, b_{n-1}\}$ of V, the reflections $r_0, \dots, r_{n-1}, s_{n-1}$ are represented by matrices in the general linear group $GL_n(\mathbb{D})$ over the ring of integers $\mathbb{D}$ in an algebraic number field determined by Γ. More explicitly, if ξ is a primitive $2m$th root of unity, where m denotes the lowest common multiple of all finite $p_{i,j}$ and $q_{i,n-1}$, then $\mathbb{D} = \mathbb{Z}[\xi]$, the ring of integers in $\mathbb{Q}(\xi)$. Hence we may view G as a subgroup of $GL_n(\mathbb{D})$ and *reduce G mod p*, for any prime p (see [23] and [9, ch. XII])).

The modular reduction technique is most easily described for Coxeter groups G (or Γ) which are *crystallographic*, meaning that G leaves some lattice in V invariant. There is a simple combinatorial characterization of crystallographic Coxeter groups, which in the present context takes the following form (see [20]): G is crystallographic if and only if $p_{i,j}, q_{i,n-1} = 2$, 3, 4, 6 or ∞ for all i, j and the triangular circuit of (5) (assuming $k \geqslant 3$) contains no, or two, branches labelled 4 and no, or two, branches labelled 6. If G is crystallographic, then there is a *basic system* $\{c_0, \dots, c_{n-1}, d_{n-1}\}$, with $c_i := t_i a_i$ and $d_{n-1} = t'_{n-1} b_{n-1}$ for certain $t_i > 0$ and $t'_{n-1} > 0$, such that

$$l_{i,j} := -t_i^{-1}(a_i \cdot a_j)\, t_j \in \mathbb{Z} \quad (0 \leqslant i, j \leqslant n-1),$$

$$m_{i,n-1} := -t_i^{-1}(a_i \cdot b_{n-1})\, t'_{n-1} \in \mathbb{Z} \quad (0 \leqslant i \leqslant n-1),$$

and

$$m_{n-1,j} := -t_j(a_j \cdot b_{n-1})\, t'^{-1}_{n-1} \in \mathbb{Z} \quad (0 \leqslant j \leqslant n-1).$$

Here, $l_{i,i} = -2$ for all i, and $l_{i,j} = 0$ if $p_{i,j} = 2$; similarly, $m_{n-1,n-1} = -2$, and $m_{i,n-1} = 0$ if $q_{i,n-1} = 2$. Then, for the rescaled roots, the generating reflections of G are represented by integral matrices and are given by

$$\begin{aligned}
r_i(c_j) &= c_j + l_{i,j} c_i, \\
r_i(d_{n-1}) &= d_{n-1} + m_{i,n-1} c_i, \\
s_{n-1}(c_j) &= c_j + m_{n-1,j} d_{n-1}, \\
s_{n-1}(d_{n-1}) &= -d_{n-1}.
\end{aligned} \tag{19}$$

The corresponding *root lattice* $(\oplus_j \mathbb{Z} c_j) \oplus \mathbb{Z} d_{n-1}$ is G-invariant. Thus we may take $\mathbb{D} = \mathbb{Z}$ and reduce G modulo any positive integer. The most interesting case occurs when

the reduction is modulo a prime p, allowing $p = 2$. A crystallographic Coxeter group G generally has many basic systems, and the modular reduction may depend on the particular system chosen.

We illustrate our method for the crystallographic Coxeter group G with diagram

$$(20)$$

Now $n = 3$ and $\{c_0, c_1, c_2, d_2\}$ is a basic system for real 4-space V. As the diagram in (20) has no branches labelled 6, a change in the underlying basic system has little effect on the reduction modulo an odd prime p; in fact, passing to a different basic system merely results in conjugation inside $GL_4(\mathbb{Z}_p)$. Here we may assume that the base vectors c_0, c_1, c_2, d_2 have squared length $1, 1, 2, 4$, respectively (see [23, §4]).

When p is odd, the reduced group G^p is generated by four isometric reflections in the finite orthogonal geometry determined on $V^p := \mathbb{Z}_p^4$ by the reduction of the corresponding (integral) bilinear form modulo p. However, V^p is non-singular only when $p > 3$, since its discriminant is $-6 \bmod p$. Thus the cases $p = 2$ and $p = 3$ require special treatment.

Now, by Lemma 4.12 and the subsequent remarks, G^p is a tail-triangle C-group if and only if the 3-generator subgroups $\langle r_0, r_1, r_2 \rangle^p$, $\langle r_0, r_1, s_2 \rangle^p$ and $\langle r_1, r_2, s_2 \rangle^p$ are C-groups satisfying

$$
\begin{aligned}
\langle r_0, r_1, r_2 \rangle^p \cap \langle r_0, r_1, s_2 \rangle^p &= \langle r_0, r_1 \rangle^p, \\
\langle r_1, r_2, s_2 \rangle^p \cap \langle r_0, r_1, r_2 \rangle^p &= \langle r_1, r_2 \rangle^p, \\
\langle r_1, r_2, s_2 \rangle^p \cap \langle r_0, r_1, s_2 \rangle^p &= \langle r_1, s_2 \rangle^p.
\end{aligned}
\tag{21}
$$

When p is odd, each of the three 3-dimensional subspaces of V^p spanned by $\{c_0, c_1, c_2\}$, $\{c_0, c_1, d_2\}$ or $\{c_1, c_2, d_2\}$, respectively, is left invariant by its respective 3-generator subgroup, and is non-singular, with discriminant $\frac{1}{2}$, -1 or $-4 \bmod p$. Moreover, each of the three 2-dimensional subspaces of V^p spanned by $\{c_0, c_1\}$, $\{c_1, c_2\}$ or $\{c_1, d_2\}$, respectively, is invariant under the respective dihedral subgroup $\langle r_0, r_1 \rangle^p$, $\langle r_1, r_2 \rangle^p$ or $\langle r_1, s_2 \rangle^p$, and has discriminant $\frac{3}{4}$, 1 or $0 \bmod p$; here the third subspace is singular (for all p), but the first and second subspaces are non-singular, except for the first when $p = 3$. Now, without elaborating in detail, it turns out that the above 3-generator subgroups of G^p really are C-groups, and that the three intersection conditions in (21) can be verified using methods similar to those described in [23, 24] (when $p \geqslant 5$); more precisely, the first two intersections conditions require an analog of [23, Theorem 4.2], and the third an analog of [24, Corollary 3.2]. The case $p = 3$ can be verified by hand, or using GAP [10].

Now Theorem 4.7 and Proposition 4.8 tell us that G^p is the automorphism group of a semiregular 4-polytope $\mathcal{S}$ whose two kinds of facets $\mathcal{P}$ and $\mathcal{Q}$ are regular polyhedra determined by the subgroups $\langle r_0, r_1, r_2 \rangle^p$ and $\langle r_0, r_1, s_2 \rangle^p$. In particular, $\mathcal{P}$ is the octahedron $\{3, 4\}$; and $\mathcal{Q}$ is the regular map of type $\{3, p\}$ of [23, §5.7], with automorphism group $G^p = O_1(3, p, 0)$ if $p \geqslant 5$, and $G^p \simeq \mathbb{S}_4$ if $p = 3$. (Recall that $O_1(3, p, 0)$ denotes

the subgroup generated by the reflections of spinor norm 1 in the full orthogonal group $O(3, p, 0)$ of a non-singular orthogonal space $\mathbb{Z}_p^3$. Note that $O_1(3, p, 0) \simeq PSL_2(\mathbb{Z}_p) \rtimes C_2$ and $O(3, p, 0) \simeq PGL_2(\mathbb{Z}_p) \rtimes C_2$.) When $p = 3$, 5 or 7, respectively, $\mathcal{Q}$ is the tetrahedron $\{3, 3\}$, icosahedron $\{3, 5\}$ or Klein map $\{3, 7\}_8$. Each edge of $\mathcal{S}$ lies in four facets, namely two copies of $\mathcal{P}$ and two copies of $\mathcal{Q}$, occurring alternately. The automorphism group G^p of $\mathcal{S}$ is given by

$$
\Gamma(\mathcal{S}) = \begin{cases} O_1(4, p, 1), & \text{if } p \equiv 1, 7 \bmod 24, \\ O(4, p, 1), & \text{if } p \equiv 5, 11 \bmod 24, \\ O_1(4, p, -1), & \text{if } p \equiv 17, 23 \bmod 24, \\ O(4, p, -1), & \text{if } p \equiv 13, 19 \bmod 24, \end{cases} \tag{22}
$$

(Recall that $O_1(4, p, \varepsilon)$, with $\varepsilon = \pm 1$, denotes the subgroup generated by the reflections of spinor norm 1, in the full orthogonal group $O(4, p, \varepsilon)$ of a non-singular orthogonal space $\mathbb{Z}_p^4$. Here $\varepsilon = 1$ if the Witt index is 2, and $\varepsilon = -1$ if the Witt index is 1.)

For $p = 3$ the space V^3 is singular with 1-dimensional radical. The group G^3 has order 1296 and consists of all isometries which fix the radical pointwise, so that $G^3 \simeq \mathbb{Z}_3^3 \rtimes \langle r_0, r_1, r_2 \rangle^3$. Thus we can think of G^3 as a crystallographic group with finite invariant lattice $\mathbb{Z}_3^3$ and octahedral point group $\langle r_0, r_1, r_2 \rangle^3 \simeq O(3, 3, 0) \simeq B_3$. Referring to [2], it is now easy to see why G^3 is isomorphic to the full automorphism group of the Gray graph (see also [22]).

We now consider the special prime $p = 2$. Of the several essentially distinct basic systems admitted by G, only that in which the rescaled base vectors have squared lengths $1, 1, 2, 4$ actually yields a tail-triangle C-group. We find that G^2 has order 96 and, a little unexpectedly, that $r_1 s_2$ has period 4. The semiregular polytope $\mathcal{S} = \mathcal{S}(G^2)$ has two sets of 4 hemioctahedral facets distributed among a meager 3 vertices. Each vertex-figure is a toroid $\{4, 4\}_{(2,0)}$. Since G^2 does admit the automorphism mentioned in Proposition 4.8(a), $\mathcal{S}$ is in fact regular; and since the facets are combinatorially flat, $\mathcal{S}$ must coincide with the (flat) universal regular polytope $\{\{3, 4\}_3, \{4, 4\}_{(2,0)}\}$ (see [19, 4E5]) denoted $\{3, 4, 4\} * 192a$ in the Census [12]. It is clear from our comment in Example 4.11 that G^2 is isomorphic to the automorphism group $\Gamma(\mathcal{T}_{hh}) = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$ for the tomotope. This isomorphism is induced by mapping (r_0, r_1, r_2, s_2) to $(\rho_0, \rho_1, \rho_3, \rho_0 \rho_2)$, so the geometrical connection between $\mathcal{S}$ and $\mathcal{T}_{hh}$ is a little obscure.

The diagram in (20) can be viewed in three ways as a tail-triangle diagram for the same underlying group G (or G^p). The corresponding intersection conditions (21) involve the same 3-generator subgroups in each case. Thus G^p is a tail-triangle C-group in three different ways, and hence can give rise to three mutually non-isomorphic semiregular 4-polytopes. We have already discussed the semiregular 4-polytope $\mathcal{S}$ associated with the original diagram.

Now when the diagram is taken in the form

$$\tag{23}$$

the corresponding semiregular 4-polytope $\mathcal{S}'$ again has two kinds of facets $\mathcal{P}$ and $\mathcal{Q}$, occurring alternately around every edge of $\mathcal{S}'$. Now $\mathcal{P}$ is the 3-cube $\{4,3\}$; and $\mathcal{Q}$ is the regular map of type $\{4,p\}$ of [23, §5.9], with group $G^p = O_1(3,p,0)$ if $p \equiv \pm 1$ mod 8, and $G^p = O(3,p,0)$ otherwise. When $p \geqslant 5$ the automorphism group of $\mathcal{S}'$ is again the group in (22). However, when $p = 3$ the 4-polytope $\mathcal{S}'$ is regular and is isomorphic to the regular 3-toroid $\{4,3,4\}_{(3,3,0)}$ (see [19, 6D]).

Finally, from

$$\tag{24}$$

we obtain a semiregular 4-polytope $\mathcal{S}''$ whose facets $\mathcal{P}$ of type $\{p,4\}$ are the duals of those of type $\{4,p\}$ of $\mathcal{S}'$, and whose facets $\mathcal{Q}$ of type $\{p,3\}$ are the duals of those of type $\{3,p\}$ of $\mathcal{S}$. When $p = 3$, we can view $\mathcal{S}''$ as a semiregular tessellation of the 3-torus by 54 tetrahedra and 27 octahedra. This 3-torus can be obtained by identifying opposite faces of the parallelepiped spanned by vectors $(3,3,0), (3,0,3)$ and $(0,3,3)$ (see Figure 1). When $p \geqslant 5$, the automorphism group of $\mathcal{S}''$ is again the group in (22).

The finite semiregular polytopes $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$ derived from the diagrams for G^p in (20), (23) and (24) are quotients of the infinite semiregular polytopes associated with the infinite Coxeter groups with these diagrams as Coxeter diagrams. In fact, the obvious group epimorphisms mapping generators to generators determine coverings between the infinite polytopes and the corresponding finite polytopes (see [19, 2D]).

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