The heat equation for the Dirichlet fractional Laplacian with Hardy's potentials: properties of minimal solutions and blow-up

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Abstract

Local and global properties of minimal solutions for the heat equation generated by the Dirichlet fractional Laplacian negatively perturbed by Hardy's potentials on open subsets of $\mathbb{R}^d$ are analyzed. As a byproduct we obtain instantaneous blow-up of nonnegative solutions in the supercritical case.

Key words: fractional Laplacian, heat equation, Dirichlet form.
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1 Introduction

In this paper, we discuss mainly two questions: 1. Local and global properties in space variable of nonnegative solutions of the heat equation related to Dirichlet fractional Laplacian on open subsets negatively perturbed by potentials of the type $\frac{c}{|x|^\alpha}$, $c > 0$ and
2. Relying on the results obtained in 1. we shall prove complete instantaneous blow-up of nonnegative solutions for the same equation provided c is bigger than some critical value c^*.
To be more concrete, let $0 < \alpha < \min(2, d)$ and Ω be an open subset $\Omega \subset \mathbb{R}^d$ containing zero. We designate by $L_0^\Omega := (-\Delta)^{\frac{\alpha}{2}}|_\Omega$ the fractional Laplacian with zero Dirichlet condition on Ω^c (as explained in the next section). We consider the associated perturbed heat equation

$$\begin{cases} -\dfrac{\partial u}{\partial t} = L_0^\Omega u - \dfrac{c}{|x|^\alpha} u, & \text{in } (0, T) \times \Omega, \\ u(t, \cdot) = 0, \ on \quad \Omega^c, \ \forall \, 0 < t < T \leq \infty \\ \quad u(0, x) = u_0(x), & \text{a.e. in } \Omega, \end{cases} \tag{1.1}$$

where $c > 0$ and u_0 is a nonnegative Borel measurable square integrable function on Ω. The meaning of a solution for the equation (1.1) will be explained in the next section.

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Regarding the first addressed question, in the paper [BK], the authors established existence of nonnegative exponentially bounded solutions on bounded Lipschitz domains provided

$$0 < c \leq c^* := \frac{2^\alpha \Gamma^2(\frac{d+\alpha}{4})}{\Gamma^2(\frac{d-\alpha}{4})}. \tag{1.2}$$

They also proved that for $c > c^*$ complete instantaneous blow up takes place, provided Ω is a bounded Lipschitz domain.

Concerning properties of solutions only partial information are available in the literature. Precisely in [BRB13, Corollary 5.1] the authors proved that for bounded $C^{1,1}$ domains then under some additional condition one has the following asymptotic behavior of nonnegative solutions $u(t, x)$ for large time,

$$u(t, x) \sim c_t |x|^{-\beta(c)} |y|^{-\beta(c)} \delta^{\alpha/2}(x) \delta^{\alpha/2}(y), \ \ a.e. \tag{1.3}$$

where $0 < \beta(c) \leq \frac{d-\alpha}{2}$ and δ is the distance function to the complement of the domain.

However, as long as we know, the second question is still open: It is not clear whether for $c > c^*$ and Ω unbounded any nonnegative solution blows up immediately and completely. In these notes we shall solve definitively both problems: Sharp local estimates with respect to the spatial variable, up to the boundary, of a special nonnegative solution (the minimal solution) of the heat equation will be established in the subcritical leading thereby to global sharp L^p regularity property. We also prove complete instantaneous blow-up in the supercritical case for arbitrary domains, regardless boundedness and regularity of the boundary.

Our strategy is described as follows: At first stage we show that in the subcritical case the underlying semigroups have heat kernels. Then we shall establish sharp estimates of the heat kernels near zero of the considered semigroups on bounded sets, which in turns will lead to sharp pointwise estimate of the minimal solution near zero of (1.1). The latter result are then exploited to prove the above mentioned properties and to enable us to extend the L^2-semigroups to semigroups on some weighted L^1-space, determining therefore the optimal class of initial data. The main ingredients at this stage are a transformation procedure by harmonic functions that will transform the forms related to the considered semigroups into Dirichlet forms together with the use of the celebrated improved Hardy–Sobolev inequality.

Then the precise description of the pointwise behavior of the heat kernel on bounded sets will deserve among others to establish blow up on open sets.

The inspiring point for us were the papers [VZ00, BG84, CM99] where the problem was addressed and solved for the Dirichlet Laplacian (i.e. $\alpha = 2$). We shall record many resemblances between our results and those found in the latter cited papers though the substantial difference between the Laplacian and the fractional Laplacian.

2   Backgrounds

From now on we fix an open subset $\Omega \subset \mathbb{R}^d$ containing zero and a real number α such that $0 < \alpha < \min(2, d)$.

The Lebesgue spaces $L^2(\mathbb{R}^d, dx)$, resp. $L^2(\Omega, dx)$ will be denoted by L^2, resp. $L^2(\Omega)$ and their respective norms will be denoted by $\|\cdot\|_{L^2}$, resp. $\|\cdot\|_{L^2(\Omega)}$. We shall write $\int \cdots$ as a shorthand for $\int_{\mathbb{R}^d} \cdots$.

The letters C, C', c_t, κ_t will denote generic nonnegative finite constants which may vary in value from line to line.

Consider the bilinear symmetric form $\mathcal{E}$ defined in L^2 by

$$
\begin{aligned}
\mathcal{E}(f,g) &= \frac{1}{2}\mathcal{A}(d,\alpha)\int\int \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\,dxdy, \\
D(\mathcal{E}) &= W^{\alpha/2,2}(\mathbb{R}^d) := \{f \in L^2 : \mathcal{E}[f] := \mathcal{E}(f,f) < \infty\},
\end{aligned}
\tag{2.1}
$$

where

$$
\mathcal{A}(d,\alpha) = \frac{\alpha\Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}.
\tag{2.2}
$$

Using Fourier transform $\hat{f}(\xi) = (2\pi)^{-d/2}\int e^{-ix\cdot\xi}f(x)\,dx$, a straightforward computation yields the following identity (see [FLS08, Lemma 3.1])

$$
\int |\xi|^{\alpha}|\hat{f}(\xi)|^2\,d\xi = \mathcal{E}[f], \ \forall f \in W^{\alpha/2,2}(\mathbb{R}^d).
\tag{2.3}
$$

It is well known that $\mathcal{E}$ is a Dirichlet form, i.e.: it is densely defined bilinear symmetric and closed form moreover it holds,

$$
\forall f \in W^{\alpha/2,2}(\mathbb{R}^d) \Rightarrow f_{0,1} := (0 \vee f) \wedge 1 \in W^{\alpha/2,2}(\mathbb{R}^d) \text{ and } \mathcal{E}[f_{0,1}] \leq \mathcal{E}[f],
\tag{2.4}
$$

Furthermore $\mathcal{E}$ is regular: $C_c(\mathbb{R}^d) \cap W^{\alpha/2,2}(\mathbb{R}^d)$ is dense in both spaces $C_c(\mathbb{R}^d)$ and $W^{\alpha/2,2}(\mathbb{R}^d)$.

The form $\mathcal{E}$ is related (via Kato representation theorem) to the selfadjoint operator, commonly named the fractional Laplacian on $\mathbb{R}^d$, and which we shall denote by $L_0 := (-\Delta)^{\alpha/2}$. We note that the domain of L_0 is the fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^d)$. We quote that the following Hardy's inequality holds true

$$
\int \frac{f^2(x)}{|x|^{\alpha}}\,dx \leq \frac{1}{c^*}\mathcal{E}[f], \ \forall f \in W^{\alpha/2,2}(\mathbb{R}^d).
\tag{2.5}
$$

Furthermore $1/c^*$ is the best constant in the latter inequality.

It is also known that $\mathcal{E}$ induces a set-function called 'capacity'. We shall say that a property holds quasi-everywhere (q.e. for short if it holds true up to a set having zero capacity).

For aspects related to Dirichlet forms we refer the reader to [FOT11].

Set $L_0^{\Omega} := (-\Delta)^{\alpha/2}|_{\Omega}$, the operator which Dirichlet form in $L^2(\overline{\Omega}, dx)$ is given by

$$
\begin{aligned}
D(\mathcal{E}_{\Omega}) &= W_0^{\alpha/2,2}(\Omega) := \{f \in W^{\alpha/2,2}(\mathbb{R}^d) : f = 0 \quad q.e.\ on\ \Omega^c\} \\
\mathcal{E}_{\Omega}(f,g) &= \mathcal{E}(f,g) \\
&= \frac{1}{2}\mathcal{A}(d,\alpha)\int_{\Omega}\int_{\Omega} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\,dx\,dy + \int_{\Omega} f(x)g(x)\kappa_{\Omega}(x)\,dx,
\end{aligned}
$$

3

where

$$\kappa_\Omega(x) := \mathcal{A}(d,\alpha) \int_{\Omega^c} \frac{1}{|x-y|^{d+\alpha}} \, dy. \tag{2.6}$$

For every $t \geq 0$ we designate by $e^{-tL_0^\Omega}$ the operator semigroup related to L_0^Ω. In the case $\Omega = \mathbb{R}^d$ we omit the superscript Ω in the notations.

It is a known fact (see [BBK$^+$09]) that $e^{-tL_0^\Omega}$, $t > 0$ has a kernel (the heat kernel) $p_t^{L_0^\Omega}(x,y)$ which is symmetric jointly continuous and $p_t^{L_0^\Omega}(x,y) > 0$, $\forall\, x,y \in \Omega$.

Let us introduce the notion of solution for problem (1.1).

Definition 2.1. Let $V \in L^1_{loc}(\Omega)$ be nonnegative, $u_0 \in L^2(\Omega)$ be nonnegative as well and $0 < T \leq \infty$. We say that a Borel measurable function $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ is a solution of the heat equation

$$\begin{cases} -\dfrac{\partial u}{\partial t} = L_0^\Omega u - Vu, & \text{in } (0,T) \times \Omega, \\ u(t,\cdot) = 0, & \text{on } \Omega^c, \ \forall\, 0 < t < T \leq \infty \\ \quad u(0,\cdot) = u_0, & \text{for } \in \Omega, \end{cases} \tag{2.7}$$

if

1. $u \in \mathcal{L}^2_{loc}\big([0,T), L^2_{loc}(\Omega)\big)$, where $\mathcal{L}^2$ is the Lebesgue space of square integrable functions.

2. $u \in L^1_{loc}\big((0,T) \times \Omega, dt \otimes V\,dx\big)$.

3. For every $t > 0$, $u(t,\cdot) = 0$, a.e. on Ω^c.

4. For every $0 \leq t < T$ and every Borel function $\phi : [0,T) \times \mathbb{R}^d$ such that $supp\,\phi \subset [0,T) \times \Omega$, ϕ, $\frac{\partial \phi}{\partial t} \in L^2((0,T) \times \Omega)$, $\phi(t,\cdot) \in D(L_0)$ and

$$\int_0^t \int_\Omega u(s,x) L_0 \psi(s,x) \, ds \, dx < \infty$$

    the following identity holds true

$$\int \big((u\phi)(t,x) - u_0(x)\phi(0,x)\big) \, dx \ + \ \int_0^t \int u(s,x)(-\phi_s(s,x) + L_0^\Omega \phi(s,x)) \, dx \, ds$$

$$= \int_0^t \int u(s,x)\phi(s,x) V(x) \, dx \, ds. \tag{2.8}$$

For every $c > 0$ we denote by V_c the Hardy potential

$$V_c(x) = \frac{c}{|x|^\alpha}, \ x \neq 0.$$

In [BK] it is proved that for bounded Ω and for $0 < c \leq c^*$ equation (1.1) has a nonnegative solution, whereas for $c > c^*$ and Ω a bounded Lipschitz domain then no nonnegative

solutions occur.

In the next section we shall be concerned with properties of a special nonnegative solution which is called *minimal solution* or *semigroup solution* in the subcritical, i.e. $0 < c < c^*$ and in the critical cases, i.e. $c = c^*$. The connotation minimal solution comes from the following observation (proved in [BK] for bounded domains and in Lemma 4.1 for general domains and in [KLVW15] in a different context): If u_k is the semigroup solution for the heat equation with potential $V_c \wedge k$, $k \in \mathbb{N}$ and if u is any nonnegative solution of (1.1) then $u_\infty := \lim_{k \to \infty} u_k$ is a nonnegative solution of (1.1) and $u_\infty \leq u$ *a.e.*.

We shall name u_∞ the minimal nonnegative solution and shall denote it by u.

Let $0 < c < c^*$. We denote by $\mathcal{E}_\Omega^{V_c}$ the quadratic form defined by

$$D(\mathcal{E}_\Omega^{V_c}) = W_0^{\alpha/2,2}(\Omega), \ \mathcal{E}_\Omega^{V_c}[f] = \mathcal{E}_\Omega[f] - \int_\Omega f^2(x) V_c(x) \, dx. \tag{2.9}$$

Whereas for $c = c^*$, we set

$$\dot{\mathcal{E}}_\Omega^{V_{c^*}} : \ D(\dot{\mathcal{E}}_\Omega^{V_{c^*}}) = W_0^{\alpha/2,2}(\Omega), \ \dot{\mathcal{E}}_\Omega^{V_{c^*}}[f] = \mathcal{E}_\Omega[f] - \int_\Omega f^2(x) V_{c^*}(x) \, dx. \tag{2.10}$$

As the closability of $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$ in $L^2(\Omega)$ is not obvious we shall perform a method that enables us to prove in a unified manner the closedness of $\mathcal{E}_\Omega^{V_c}$ as well as the closability of $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$ in $L^2(\Omega)$.

To that end we recall some known facts concerning harmonic functions of $L_0 - \frac{c}{|x|^\alpha}$.

We know from [BRB13, Lemma 2.2] that for every $0 < c \leq c^*$ there is a unique $\beta = \beta(c) \in (0, \frac{d-\alpha}{2}]$ such that $w_c(x) := |x|^{-\beta(c)}$, $x \neq 0$ solves the equation

$$(-\Delta)^{\alpha/2} w - c|x|^{-\alpha} w = 0 \text{ in the sense of distributions.} \tag{2.11}$$

That is

$$< \hat{w}, |\xi|^\alpha \hat{\varphi} > -c < |x|^{-\alpha} w, \varphi >= 0 \ \forall \varphi \in \mathcal{S}. \tag{2.12}$$

Furthermore for $\beta_* := \frac{d-\alpha}{2}$, we have $c = c^*$, i.e., $w_{c^*}(x) = |x|^{-\frac{d-\alpha}{2}}$, $x \neq 0$.

Next we fix definitively $c \in (0, c^*]$.

For $0 < c < c_*$ let Q^c be the w_c-transform of $\mathcal{E}_\Omega^{V_c}$, and for $c = c_*$ let Q^{c_*} be the w_{c^*}-transform of $\dot{\mathcal{E}}_\Omega^{V_c}$ i.e., the quadratic forms defined in $L^2(\Omega, w_c^2 dx)$ and in $L^2(\Omega, w_{c^*}^2 dx)$ respectively by:

$$D(Q^c) := \{f : \ w_c f \in W_0^{\alpha/2,2}(\Omega)\} \subset L^2(\Omega, w_c^2 dx), \ Q^c[f] = \mathcal{E}_\Omega^{V_c}[w_c f], \ \forall f \in D(Q^c).$$

$$D(Q^{c_*}) := \{f : \ w_{c^*} f \in W_0^{\alpha/2,2}(\Omega)\} \subset L^2(\Omega, w_{c^*}^2 dx), \ Q^{c_*}[f] = \dot{\mathcal{E}}_\Omega^{V_c}[w_{c^*} f], \ \forall f \in D(Q^c).$$

Lemma 2.1. *For $0 < c < c^*$ the form Q^c is a regular Dirichlet form whereas for $c = c^*$ the form Q^{c^*} is closable and its closure is a regular Dirichlet form as well. It follows in particular that $\mathcal{E}_\Omega^{V_c}$ is closed and $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$ is closable in $L^2(\Omega)$.*

In both cases, it holds

$$Q^c[f] = \frac{\mathcal{A}(d,\alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} w_c(x) w_c(y) \, dx dy, \ \forall wf \in W_0^{\alpha/2,2}(\Omega). \tag{2.13}$$

Proof. The proof of formula (2.13) follows the lines of the proof of [BRB13, Lemma 3.1], where bounded Ω is considered so we omit it.

To prove regularity it suffices to prove that $C_c^\infty(\Omega) \subset D(Q^c)$. The latter claim is in turns equivalent to the following two conditions (see [FOT11, Example 1.2.1]): for every compact set K and every open set Ω_1 with $K \subset \Omega_1 \subset \Omega$ one should have

$$\int_{K \times K} |x - y|^{2-d-\alpha} w_c(x) w_c(y) \, dx \, dy < \infty, \quad \int_K \int_{\Omega \setminus \Omega_1} |x - y|^{-d-\alpha} w_c(x) w_c(y) \, dx \, dy < \infty.$$

The first part of the latter conditions was already proved for bounded sets in [BRB13, Lemma 3.1]. Let us prove the finiteness of the second integral.

Case1: $0 \in K$. Then $0 \notin \Omega \setminus \Omega_1$. Thus $\sup_{y \in \Omega \setminus \Omega_1} w_c(y) < \infty$. On the other hand we have

$$\int_{\Omega \setminus \Omega_1} |x - y|^{-d-\alpha} \, dy \leq \int_{\{|x-y|>\epsilon\}} |x - y|^{-d-\alpha} \, dy \leq C < \infty. \tag{2.14}$$

Hence the second integral is finite.

Case2: $0 \in \Omega_1 \setminus K$. Then $\sup_{x \in K} w_c(x) < \infty, \sup_{y \in \Omega \setminus \Omega_1} w_c(y) < \infty$ and once again the second integral is finite.

Case3: $0 \in \Omega \setminus \Omega_1$. Then $\sup_{x \in K} w_c(x) < \infty$. Thus if we choose an open ball B_ϵ centered at 0 such that $B \subset \Omega \setminus \Omega_1$ we obtain

$$\begin{aligned}
\int_K \int_{\Omega \setminus \Omega_1} |x - y|^{-d-\alpha} w_c(x) w_c(y) \, dx \, dy &= \int_K \int_B |x - y|^{-d-\alpha} w_c(x) w_c(y) \, dx \, dy \\
&+ \int_K \int_{\Omega \setminus (\Omega_1 \cup B)} |x - y|^{-d-\alpha} w_c(x) w_c(y) \, dx \, dy \\
&\leq C_1 + C_2 \int_K \int_{B^c} |x - y|^{-d-\alpha} \, dx \, dy < \infty
\end{aligned} \tag{2.15}$$

We turn our attention now to prove the rest of the lemma.

Let $0 < c < c^*$. Utilizing Hardy's inequality we obtain

$$(1 - \frac{c}{c^*}) \mathcal{E}_\Omega[f] \leq \mathcal{E}_\Omega^{V_c}[f] \leq \mathcal{E}_\Omega[f], \ \forall f \in W_0^{\alpha/2,2}(\Omega), \tag{2.16}$$

from which the closedness of $\mathcal{E}_\Omega^{V_c}$ follows, as well as the closedness of Q^c. On the other hand it is obvious that the normal contraction acts on $D(Q^c)$ and hence Q^c is a Dirichlet form.

For the critical case formula (2.13) indicates that Q^{c^*} is Markovian and closable, by means of Fubini theorem. Thus, according to [FOT11, Theorem 3.1.1] its closure is a Dirichlet form. $\square$

Remark 2.1. The form $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$ is not closed. Indeed if it were the case, then for every ball B centered in 0 and $B \subset \Omega$, the form $\dot{\mathcal{E}}_B^{V_{c^*}}$ would be closed as well. However, it was proved in [BRB13, Remark 4.1] that the ground state of $\dot{\mathcal{E}}_B^{V_{c^*}}$ is not in the space $W_0^{\alpha/2,2}(B)$ leading to a contradiction.

Henceforth, we denote by $\mathcal{E}_\Omega^{V_{c^*}}$ the closure of $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$, by $L_{V_c}^\Omega$ the selfadjoint operator associated to $\mathcal{E}_\Omega^{V_c}$ and by $e^{-tL_{V_c}^\Omega}$, $t \geq 0$ the related semigroup.

We designate by L^{w_c} the operator associated to Q^c in the weighted Lebesgue space $L^2(\Omega, w_c^2 dx)$ and $T_t^{w_c}$, $t \geq 0$ its semigroup. Then

$$L^{w_c} = w_c^{-1} L_{V_c}^\Omega w_c \text{ and } T_t^{w_c} = w_c^{-1} e^{-tL_{V_c}^\Omega} w_c, \ t \geq 0. \tag{2.17}$$

The next proposition explains why are minimal solutions also semigroup solutions.

Proposition 2.1. *For every $0 < c \leq c^*$, the minimal solution is given by $u(t) := e^{-tL_{V_c}^\Omega} u_0$, $t \geq 0$. Thus $u(t) \in D(L_{V_c}^\Omega)$, $t > 0$, $u \in C([0,\infty, L^2(\Omega)) \cap C^1((0,\infty, L^2(\Omega))$ furthermore it fulfills Duhamel's formula*

$$u(t,x) = e^{-tL_0^\Omega} u_0(x) + \int_0^t \int_\Omega p_{t-s}^{L_0^\Omega}(x,y) u(s,y) V(y) \, dy \, ds, \ \forall t > 0, \ a.e.x \in \Omega. \tag{2.18}$$

Proof. Let $(h_k)_k$ be the sequence of closed quadratic forms in $L^2(\Omega)$ defined by

$$h_k := \mathcal{E}_\Omega - V_c \wedge k,$$

and $(H_k)_k$ be the related selfadjoint operators. Then $(h_k)_k$ is uniformly lower semibounded and $h_k \downarrow \mathcal{E}_\Omega^{V_c}$ in the subcritical case, whereas $h_k \downarrow \dot{\mathcal{E}}_\Omega^{V_{c^*}}$ in the critical case. As both $\mathcal{E}_\Omega^{V_c}$, $\dot{\mathcal{E}}_\Omega^{V_{c^*}}$ are closable, we conclude by [Kat95, Theorem 3.11] that (H_k) converges in the strong resolvent sense to $L_{V_c}^\Omega$ for every $0 < c \leq c^*$. Hence e^{-tH_k} converges strongly to $e^{-tL_{V_c}^\Omega}$ and then the monotone sequence $u_k := e^{-tH_k} u_0$ converges to $e^{-tL_{V_c}^\Omega} u_0$ which is nothing else but the minimal solution.

The remaining claims of the proposition follow from the standard theory of semigroups. $\qquad\square$

As one interest is properties of minimal solutions and since these are given in term of semigroups one should analyze these semigroups. Here is a first result in this direction.

Proposition 2.2. *For every $t > 0$ the semigroup $e^{-tL_\Omega^{V_c}}$, $t > 0$ has a measurable nonnegative symmetric absolutely continuous kernel, $p_t^{L_{V_c}^\Omega}$, in the sense that for every $v \in L^2(\Omega)$ it holds,*

$$e^{-tL_\Omega^{V_c}} v = \int_\Omega p_t^{L_{V_c}^\Omega}(\cdot, y) v(y) \, dy, \ a.e. \ x, y \in \Omega, \ \forall t > 0. \tag{2.19}$$

We shall call $p_t^{L_{V_c}^\Omega}$ the heat kernel of $e^{-tL_\Omega^{V_c}}$.

Proof. Owing to the known fact that $e^{-tL_0^\Omega}$, $t > 0$ has a nonnegative heat kernel together with the fact that $V_c \wedge k$ is bounded we deduce that e^{-tH_k} has a nonnegative heat kernel as well, which we denote by $P_{t,k}$. As the $u_k(t) = e^{-tH_k}$ are monotone increasing we achieve that the sequence $(P_{t,k})_k$ is monotone increasing as well. Set

$$p_t^{L_{V_c}^\Omega}(x,y) := \lim_{k \to \infty} P_{t,k}(x,y), \ \forall t > 0, \ a.e. \ x, y \in \Omega. \tag{2.20}$$

Then $p_t^{L_{V_c}^\Omega}$ has all the first properties mentioned in the proposition.

Let $u_0 \in L^2(\Omega)$ be nonnegative. Then by monotone convergence theorem, together with the latter proposition we get

$$
\begin{aligned}
e^{-tL_\Omega^{V_c}}u_0 &= \lim_{k\to\infty} u_k(t) = \lim_{k\to\infty} e^{-tH_k}u_0 = \lim_{k\to\infty}\int_\Omega P_{t,k}(\cdot,y)u_0(y)\,dy \\
&= \int_\Omega p_t^{L_{V_c}^\Omega}(\cdot,y)u_0(y)\,dy, \ a.e. \ x,y \in \Omega, \ \forall\, t > 0.
\end{aligned} \tag{2.21}
$$

For an arbitrary $v \in L^2(\Omega)$ formula (2.19) follows from the last step by decomposing v into its positive and negative parts.

$\square$

3 Heat kernel estimates, local and global behavior of the minimal solution in space variable

Along this section we assume that Ω is bounded.

The study of behavior for solutions of evolution equations is often a delicate problem. To overcome the difficulties we shall make use of the pseudo-ground state transformation for forms $\mathcal{E}_\Omega^{V_c}$ performed in Lemma 2.1 together with an improved Sobolev inequality. This transformation has the considerable effect to mutate forms $\mathcal{E}_\Omega^{V_c}$ to Dirichlet forms and to mutate $e^{-tL_{V_c}^\Omega}$ to Markovian ultracontractive semigroup on some weighted Lebesgue space. The analysis of the transformed operators will then lead us to get satisfactory results concerning estimating their kernel and hence the properties of minimal solutions. As a first step we proceed to prove that Sobolev inequality holds for the w_c-transform of the form $\mathcal{E}_\Omega^{V_c}$. As a byproduct we obtain that the semigroup of the transformed from is ultracontractive and then very interesting estimates for the heat kernel are derived.

Theorem 3.1. *1. Let $0 < c < c^*$ and $p = \frac{d}{d-\alpha}$. Then the following Sobolev inequality holds true*

$$
\| f^2 \|_{L^p(w_c^2 dx)} \leq A Q^c[f], \ \forall\, f \in D(Q^c). \tag{3.1}
$$

2. For $c = c^*$ let $1 < p < \frac{d}{d-\alpha}$. Then the following Sobolev inequality holds true

$$
\| f^2 \|_{L^p(w_{c^*}^2 dx)} \leq A Q^{c^*}[f], \ \forall\, f \in D(Q^{c^*}). \tag{3.2}
$$

3. For every $t > 0$, the operator $T_t^{w_c}$ is ultracontractive.

4. For every $0 < c < c^*$, there is a finite constant $C > 0$ such that

$$
0 < p_t^{L_{V_c}^\Omega}(x,y) \leq \frac{C}{t^{\frac{d}{\alpha}}} w_c(x) w_c(y), \ a.e. \ on \ \Omega \times \Omega, \ \forall\, t > 0. \tag{3.3}
$$

5. For $c = c^*$, there is a finite constant $C > 0$ such that

$$0 < p_t^{L_{V_{c^*}}^\Omega}(x, y) \le \frac{C}{t^{\frac{p}{p-1}}} w_{c^*}(x) w_{c^*}(y), \ a.e. \ on \ \Omega \times \Omega, \ \forall \, t > 0. \tag{3.4}$$

Proof. 1) and 2): Let $0 < c < c^*$. From Hardy's inequality we derive

$$(1 - \frac{c}{c^*}) \mathcal{E}_\Omega[f] \le \mathcal{E}_\Omega^{V_c}[f], \ \forall \, f \in W_0^{\alpha/2,2}(\Omega). \tag{3.5}$$

Now we use the known fact that $W_0^{\alpha/2,2}(\Omega)$ embeds continuously into $L^{\frac{2d}{d-\alpha}}$, to obtain the following Sobolev's inequality

$$\left(\int_\Omega |f|^{\frac{2d}{d-\alpha}} \, dx\right)^{\frac{d-\alpha}{d}} \le C \mathcal{E}_\Omega^{V_c}[f], \ \forall \, f \in W_0^{\alpha/2,2}(\Omega). \tag{3.6}$$

An application of Hölder's inequality together with Lemma 2.1 and the fact that Ω is bounded, yield then inequality (3.1).

Towards proving Sobolev's inequality in the critical case one uses the improved Hardy–Sobolev inequality, due to Frank–Lieb–Seiringer [Theorem 2.3]: For every $1 \le p < \frac{d}{d-\alpha}$ there is a constant $S_{d,\alpha}(\Omega)$ such that

$$\left(\int |f|^{2p} \, dx\right)^{1/p} \le S_{d,\alpha}(\Omega) \left(\mathcal{E}_\Omega[f] - c^* \int_\Omega \frac{f^2(x)}{|x|^\alpha} \, dx\right), \ \forall \, f \in W_0^{\alpha/2,2}(\Omega), \tag{3.7}$$

and the rest of the proof runs as before.

3): As Q^c is a Dirichlet form, by the standard theory of Markovain semigroups, it is known (see [Dav89, p.75]) that Sobolev inequality implies ultracontractivity of $T_t^{w_c}$ together with the bound

$$\|T_t^{w_c}\|_{L^2(\Omega, w_c^2 dx), L^\infty(\Omega)} \le \frac{c}{t^{d/\alpha}}, \ t > 0. \tag{3.8}$$

Now ultracontractivity in turns implies that the semigroup $e^{-tL^{w_c}}$ has a nonnegative symmetric (heat) kernel, which we denote by q_t and the latter estimate yields in turns by [Dav89, p.59]) that q_t fulfills the upper bound

$$0 \le q_t(x, y) \le \frac{c}{t^{d/\alpha}}, \ a.e., \ \forall \, t > 0. \tag{3.9}$$

On the other hand we have $q_t(x, y) = \frac{p_t^{L_{V_c}^\Omega}(x,y)}{w_c(x) w_c(y)}$, *a.e.*, yielding the upper bounds (3.3) and (3.4).

The proof of 4. is similar to the latter one so we omit it. □

We turn our attention at this stage to give a lower bound for the heat kernel.

Theorem 3.2. *For every* $0 < c \le c^*$, *every compact subset* $K \subset \Omega$ *and every* $t > 0$, *there is a finite constant* $\kappa_t = \kappa_t(K) > 0$ *such that*

$$p_t^{L_{V_c}^\Omega}(x, y) \ge \kappa_t w_c(x) w_c(y), \ a.e. \ on \ K \times K, \ \forall \, t > 0. \tag{3.10}$$

9

Proof. Let us first recall that we have already proved that $p_t^{L_{V_c}^{\Omega}} > 0$, $a.e. \forall t > 0$. This observation together with the relationship between $p_t^{L_{V_c}^{\Omega}}$ and q_t yield $q_t > 0$, $a.e.$, $\forall t > 0$. From the upper bounds (3.3)-(3.4), we infer that $T_t^{w_c}$ is a Hilbert–Schmidt operator and then for almost every z we have $q_t(\cdot, z) \in L^2(w_c^2 dx)$. Thus we write

$$q_t(\cdot, z) = e^{-\frac{t}{2} L^{w_c}} q_{t/2}(\cdot, z), \tag{3.11}$$

to conclude that $q_t(\cdot, z) \in D(Q^c)$. Since every element from the domain of a Dirichlet form has a quasi-continuous representative, we may and shall assume that $q_t(\cdot, z)$ is quasi-continuous and then $q_t(\cdot, z) > 0$ q.e.. Owing to [BBA12, Lemma 2.2] we obtain that for every compact $K \subset \Omega$, every $s > 0$ there is a constant $C_{K,s}(z) > 0$ such that

$$q_s(x, z) > C_{K,s}(z), \text{ for q.e. } x \in K. \tag{3.12}$$

By the quasi-continuity of $q_t(z, \cdot)$ we obtain similarly

$$q_s(z, y) > C'_{K,s}(z) > 0, \text{ for q.e. } y \in K. \tag{3.13}$$

Both lower bounds hold *a.e.* as well. Hence for *a.e.* $x, y \in K$ we have

$$q_t(x, y) = \int_{\Omega} q_{t/2}(x, z) q_{t/2}(z, y) w_c^2(z)\, dz \geq \kappa_t := \int_K C_{K,t/2}(z) C'_{K,t/2}(z) w_c^2(z)\, dz > 0. \tag{3.14}$$

Finally having in mind $q_t(x, y) = \frac{p_t^{L_{V_c}^{\Omega}}(x,y)}{w_c(x) w_c(y)}$, *a.e.*, we obtain

$$p_t^{L_{V_c}^{\Omega}}(x, y) \geq \kappa_t w_c(x) w_c(y), \ \forall t > 0, a.e. \ x, y \in K. \tag{3.15}$$

$\square$

Remark 3.1. Along the lines of the latter proof we have demonstrated that $p_t^{L_{V_c}^{\Omega}}$ is quasi-continuous in each variable x, y. On the other hand we know from the potential theory of Dirichlet forms that a property which holds true a.e. for a quasi-continuous function it should hold q.e. as well. Thus the lower bound (3.10) is satisfied q.e. Thus we achieve the on-diagonal lower bound

$$p_t^{L_{V_c}^{\Omega}}(x, x) \geq \kappa_t w_c^2(x), \ q.e. \text{ on } K \ \forall t > 0. \tag{3.16}$$

We are now in position to describe the exact spatial behavior of the minimal solution of equation (1.1), especially near 0.

Theorem 3.3. *1. For every $t > 0$ there is a constant $c_t > 0$ such that,*

$$u(t, x) \leq c_t w_c(x), \ a.e. \text{ on } \Omega. \tag{3.17}$$

It follows in particular that $u(t)$ is bounded away from zero.

2. *For every $t > 0$, there are finite constants c_t, $c'_t > 0$ such that*

$$c'_t w_c(x) \leq u(t, x) \leq c_t w_c(x), \ a.e. \ near \ 0. \qquad (3.18)$$

Hence $u(t)$ has a standing singularity at 0.

Proof. The upper bound (3.17) follows from Theorem 3.1-4). Let us now prove the lower bound.

Let K be a compact subset of Ω containing 0 such that Lebesgue measure of the set $\{x \in K \colon u_0(x) > 0\}$ is nonnegative.

Let κ_t be as in (3.10), then

$$u(t, x) = \int_\Omega p_t^{L_{V_c}^\Omega}(x, y) u_0(y) \, dy \geq \int_K p_t^{L_{V_c}^\Omega}(x, y) u_0(y) \, dy \geq \kappa_t w_c(x) \int_K w_c(y) u_0(y) \, dy$$
$$\geq c'_t w_c(x), \ a.e. \ on \ K, \qquad (3.19)$$

with $c'_t > 0$, which was to be proved. $\qquad \square$

The local sharp estimate (3.18) leads us to a sharp global regularity property of the solution, expressing thereby the smoothing effect of the semigroup $e^{-tL_{V_c}^\Omega}$.

Proposition 3.1.   *1. The solution $u(t)$ lies in the space $L^p(\Omega)$, $p \geq 1$ if and only if $1 \leq p < \frac{d}{\beta}$.*

   2. The semigroup $e^{-tL_{V_c}^\Omega}$ maps continuously $L^2(\Omega)$ into $L^p(\Omega)$ for every $2 \leq p < \frac{d}{\beta}$.

   3. The operator $e^{-tL_{V_c}^\Omega} \colon L^q(\Omega) \to L^p(\Omega)$ is smoothing for every $\frac{d}{d-\beta} < q < p < \frac{d}{\beta}$.

   4. The operator $L_{V_c}^\Omega$ has compact resolvent. Set $(\varphi_k^{L_{V_c}})_k$ its eigenfunctions. Then $(\varphi_k^{L_{V_c}})_k \subset L^p(\Omega)$ for every $p < \frac{d}{\beta}$.

Proof. The first assertion is a straightforward consequence of Theorem (3.3).

2): Let $u_0 \in L^2(\Omega)$ and p as described in the assertion. Thanks to the upper bounds (3.3)-(3.4) a straightforward computation leads to

$$\int_\Omega e^{-tL_{V_c}^\Omega} |u_0(x)|^p \, dx \leq c_t (\int_\Omega w_c |u_0| \, dx)^p \int_\Omega w_c^p \, dx \leq C (\int_\Omega u_0^2 \, dx)^{p/2}. \qquad (3.20)$$

3): Follows from Riesz-Thorin interpolation theorem.

4): We have already observed that $e^{-tL_{V_c}^\Omega}$ is a Hilbert–Schmidt operator and hence $L_{V_c}^\Omega$ has compact resolvent. The claim about eigenfunctions follows from assertion 2. $\qquad \square$

The already established upper estimate for the heat kernel enables one to extend the semigroup to a larger class of initial data.

Theorem 3.4.   *1. The semigroup $e^{-tL_{V_c}^\Omega}$, $t > 0$ extends to a bounded linear semigroup from $L^1(\Omega, w_c dx)$ into $L^2(\Omega)$.*

2. *The semigroup $e^{-tL_{V_c}^{\Omega}}$, $t > 0$ extends to a bounded linear semigroup from $L^p(\Omega, w_c dx)$ into $L^p(\Omega)$ for every $1 \le p < \infty$.*

3. *The semigroup $e^{-tL_{V_c}^{\Omega}}$, $t > 0$ extends to a bounded linear semigroup from $L^p(\Omega, w_c dx)$ into $L^p(\Omega, w_c dx)$ for every $1 \le p < d/3$.*

Proof. Having estimate (3.3) in hands, a straightforward computation yields

$$\int_{\Omega} (e^{-tL_{V_c}} u_0)^2 \, dx \le c_t \int_{\Omega} w_c^2 \, dx \cdot \left(\int_{\Omega} |u_0| w_c \, dy \right)^2, \quad \forall t > 0, \tag{3.21}$$

Similarly, using Hölder's inequality we achieve

$$|e^{-tL_{V_c}} u_0(x)|^p \le \int_{\Omega} p_t(x, y) \, dy \int_{\Omega} p_t(x, y) |u_0|^p \, dx \le c_t w_c^2(x) \int_{\Omega} w_c(y) \, dy \int |u_0|^p w_c \, dx \tag{3.22}$$

Hence

$$\int_{\Omega} |e^{-tL_{V_c}} u_0(x)|^p \, dx \le c_t \int_{\Omega} w_c^2(x) \, dx \int_{\Omega} w_c(y) \, dy \int |u_0|^p w_c \, dx. \tag{3.23}$$

Assertion 3. can be proved in a same way. $\qquad \square$

4 Blow-up of nonnegative solutions on open sets in the supercritical case

In this section we shall make use of the lower bound for the heat kernel as well as for nonnegative solutions in the critical case on bounded open sets, which we established in the last section, to show that for $c > c^*$ any nonnegative solution of the heat equation (1.1) on arbitrary open sets containing zero blows up completely and instantaneously. This result accomplishes the corresponding one for bounded sets with Lipschitz boundary so that to get a full picture concerning existence and nonexistence of nonnegative solutions for Dirichlet fractional Laplacian with Hardy potentials.

However, the idea of the proof deviates from the one developed in [BK]. Whereas for bounded domains with Lipschitz the main tool towards proving blowup relies, among others, on the boundary behavior of the ground state of L_0^{Ω} (which disappears in general for unbounded domains), our actual proof relies on the sofar established lower bounds for $p_t^{L_{V_{c^*}}^{\Omega}}$ and for nonnegative solutions for balls.

Henceforth we fix an open unbounded set $\Omega \subset \mathbb{R}^d$ containing zero and $c > 0$.

Let $V \in L^1(\Omega, dx)$ be a nonnegative potential. We set $W_k := V \wedge k$ and (P_k) the heat equation corresponding to the Dirichlet fractional Laplacian perturbed by $-W_k$ instead of $-V$:

$$(P_k) \colon \begin{cases} -\dfrac{\partial u}{\partial t} = L_0^{\Omega} u - W_k u, & \text{in } (0, T) \times \Omega, \\ u(t, \cdot) = 0, & \text{on } \Omega^c, \ \forall 0 < t < T \le \infty \\ u(0, x) = u_0(x), & \text{for } a.e. \ x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

12

Denote by L_k the selfadjoint operator associated to the closed quadratic form $\mathcal{E}_\Omega - W_k$ and $u_k(t) := e^{-tL_k}u_0$, $t \geq 0$ the nonnegative semigroup solution of problem (P_k). Then u_k satisfies Duhamel's formula:

$$u_k(t,x) \;=\; e^{-tL_0^\Omega}u_0(x) + \int_0^t \int_\Omega p_{t-s}^{L_0^\Omega}(x,y)u_k(s,x)V_k(y)\,dy\,ds, \;\; \forall\, t > 0, \qquad (4.2)$$

Let us list the properties of the sequence (u_k) and establish existence of the minimal solution.

Lemma 4.1.   *i) The sequence (u_k) is increasing.*

*   ii) If u is any nonnegative solution of problem (2.7) solution then $u_k \leq u$, $\forall\, k$. Moreover $u_\infty := \lim_{k\to\infty} u_k$ is a nonnegative solution of problem (2.7) as well.*

Though the proof runs as the one corresponding to the case of bounded domains (see [BK]), we shall reproduce it for the convenience of the reader.

Proof. i) By Duhamel's formula, one has

$$u_{k+1}(t) - u_k(t) \;=\; e^{-tL_{k+1}^\Omega}u_0 - e^{-tL_k^\Omega}u_0 = \int_0^t e^{-(t-s)L_k^\Omega}e^{-sL_{k+1}^\Omega}(u_0 W_{k+1} - u_0 W_k)(s)\,ds$$

$$\geq \;\; 0. \qquad (4.3)$$

ii) Let u be as stated in the lemma, $0 < t < T$ be fixed and $\phi \in C_c^\infty\big([0,t)\times\Omega\big)$ be positive. From the definition of a solution we infer

$$\int_0^t \int (u_k(s) - u(s))(-\phi_s(s) + L_0^\Omega\phi(s) - W_k\phi(s))\,ds\,dx \;=\; \int_0^t \int u\phi(W_k - V)\,ds\,dx$$

$$\leq \;\; 0. \qquad (4.4)$$

Let $\psi \in C_c^\infty\big((0,t)\times\Omega\big)$ be nonnegative and consider the parabolic problem: find a positive test function ϕ solving the equation

$$-\frac{\partial\phi}{\partial s} = -L_0^\Omega\phi + V_k\phi + \psi \text{ in } (0,t)\times\Omega, \;\; \phi(t,\cdot) = 0. \qquad (4.5)$$

Then the latter problem has a positive solution which is given by (see [Kat95, Theorem 1.27, p.493])

$$\phi(s) = \int_0^{t-s} e^{-(t-s-\xi)(L_0^\Omega - V_k)}\psi(t - \xi)\,d\xi, \;\; 0 \leq s \leq t, \;\; \phi(s) = 0, \;\; \forall\, s > t, \qquad (4.6)$$

Plugging into equation (4.4) yields

$$\int_0^t \int_\Omega (u_k - u)\psi\,ds\,dx \leq 0, \;\; \forall\, 0 \leq \psi \in C_c^\infty\big((0,t)\times\Omega\big). \qquad (4.7)$$

As t is arbitrary we obtain $u_k \leq u$.

Let us prove that u_∞ is a nonnegative solution.

By the first step of (ii) we have $0 < u_\infty \leq u, a.e.$ and therefore

$$u_\infty \in \mathcal{L}_{loc}^2\big([0,T), L_{loc}^2(\Omega)\big) \cap L_{loc}^1\big([0,T) \times \Omega, dt \otimes V\,dx\big).$$

Being solution of the heat equation (P_k), the u_k's satisfy: for every $0 \leq t < T$, every $\phi \in C_c^\infty\big([0,T) \times \Omega\big)$ such that $\int_0^t \int u_\infty |L_0^\Omega \phi|\, ds\, dx < \infty$,

$$\int_\Omega \big((u_k \phi)(t,x) - u_0(x)\phi(0,x)\big)\, dx + \int_0^t \int_\Omega u_k(s,x)\big(-\phi_s(s,x) + L_0^\Omega \phi(s,x)\big)\, dx\, ds$$
$$= \int_0^t \int_\Omega u_k(s,x)\phi(s,x)W_k(x)\, dx\, ds. \quad (4.8)$$

By dominated convergence theorem we conclude that u_∞ satisfies equation (2.8) as well, which ends the proof. $\qquad\square$

We have sofar collected enough material to announce the main theorem of this section.

Theorem 4.1. *Assume that $c > c*$. Then the heat equation (1.1) has no nonnegative solutions.*

Proof. Assume that a nonnegative solution u exists. Relying on Lemma 4.1, we may and shall suppose that $u = u_\infty$. Thus u satisfies Duhamel's formula as well. Put $c' = c - c^* > 0$, then

$$u(t,x) = e^{-tL_{V_{c^*}}^\Omega} u_0(x) + c' \int_0^t \int_\Omega p_{t-s}^{L_{V_{c^*}}^\Omega}(x,y)u(s,y)|y|^{-\alpha}\, ds\, dy. \quad (4.9)$$

Let B be an open ball centered at 0 such that $B \subset \Omega$ and $u_0 \not\equiv 0$ on B. Owing to the fact that $p_t^{L_{V_{c^*}}^\Omega} \geq p_t^{L_{V_{c^*}}^B}$, the latter identity together with the lower bound from (3.10) for $p_t^{L_{V_{c^*}}^B}$ lead to

$$u(t,x) \geq e^{-tL_{V_{c^*}}^B} u_0(x) \geq e^{-tL_{V_{c^*}}^B} u_0(x) \geq c_t w_{c^*}, \quad a.e. \text{ on } B' := \frac{1}{2}B. \quad (4.10)$$

Using formula (4.9), once again we obtain the following lower bound near 0

$$\begin{aligned} u(t,x) &\geq c' \int_0^t c_s \int_B p_{t-s}^{L_{V_{c^*}}^B}(x,y)w_{c^*}(y)|y|^{-\alpha}\, ds\, dy \\ &\geq c' w_{c^*}(x) \int_0^t c_s' \int_{B'} w_{c^*}^2(y)|y|^{-\alpha}\, ds\, dy. \quad (4.11) \end{aligned}$$

However, we have

$$\int_{B'} w_{c^*}^2(y)|y|^{-\alpha}\, dy = \infty, \quad (4.12)$$

and the solution blows up, which finishes the proof. $\qquad\square$

Remark 4.1. Finally we emphasize that our method still works if one considers potentials of the form $V = 1_B V_c + V'$ where B is an open ball around zero and $V' \in L^\infty(\Omega)$.

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