# Superfluidity, Sound Velocity and Quasi Condensation in the 2D BCS-BEC Crossover

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We study finite-temperature properties of a two-dimensional superfluid made of ultracold alkalimetal atoms in the BCS-BEC crossover. We investigate the region below the critical temperature  $T_{BKT}$  of the Berezinskii-Kosterlitz-Thouless phase transition, where there is quasi-condensation, by analyzing the effects of phase and amplitude fluctuations of the order parameter. In particular, we calculate the superfluid fraction, the sound velocity and the quasi-condensate fraction as a function of the temperature and of the binding energy of fermionic pairs.

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### I. INTRODUCTION

Nowadays the manipulation of the binding energy through external magnetic fields (Feshbach-resonance technique) enables experimentalists to evolve clouds of two-component fermionic atoms from the weakly coupled BCS-like behavior of Cooper pairs to the strongly coupled Bose-Einstein condensation (BEC) of molecules [1]. This transition is characterized by a crossover in which the s-wave scattering length  $a_s$  of the inter-atomic potential diverges as it changes sign [2, 3]. Recently, a considerable theoretical effort [4–11] has been expended on studying the condensate fraction of such a tunable superfluid, also in the two-dimensional (2D) case at zero temperature within a mean-field approach [12].

Quantum and thermal fluctuations play a relevant role in any generic 2D superfluid system [13–19]. In the last years a beyond-mean-field formalism which takes into account fluctuations of the order parameter has been developed for 2D Fermi superfluids [20–25]. The recent experimental observation [26] of a pairing pseudogap in a 2D Fermi gas has strongly renewed the interest on this subject.

In this paper we use this formalism to study the superfluid density and the sound velocity of the 2D Fermi superfluid as a function of the temperature and of the binding energy of fermionic pairs. The rest of the paper is organized as follows: the finite-temperature pathintegral formulation of the problem is discussed in Section III; the mean-field approach to the 2D BCS-BEC crossover is reported in Section III. The effect of fluctuations of the phase of the order parameter is analyzed in Section IV, where the superfluid fraction is evaluated as a function of the temperature for different values of the binding energy. In Section V we consider the effect of amplitude fluctuations of the order parameter in the determination of the sound velocity of the uniform superfluid system in the crossover: at zero temperature we compare the quite different results obtained with and without amplitude fluctuations (in 2D but also in 3D). In Section VI we calculate the quasi-condensate fraction of fermionic atoms in the region below the Berezinskii-Kosterlitz-Thouless critical temperature, where there is algebraic long-range order of the two-body density matrix.

### II. FORMALISM FOR FERMIONS IN TWO SPATIAL DIMENSIONS

We consider a two-dimensional Fermi gas of ultracold and dilute two-spin-component neutral atoms. We adopt the path integral formalism, where the atomic fermions are described by the complex Grassmann fields  $\psi_s(\mathbf{r},\tau)$ ,  $\bar{\psi}_s(\mathbf{r},\tau)$  with spin  $s=(\uparrow,\downarrow)$  [16, 17]. The partition function  $\mathcal Z$  of the uniform system at temperature T, in a two-dimensional volume  $L^2$ , and with chemical potential  $\mu$  can be written as

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp\left\{-\frac{1}{\hbar} S\right\}, \qquad (1)$$

where

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2 \mathbf{r} \, \mathscr{L}$$
 (2)

is the Euclidean action functional and  ${\mathscr L}$  is the Euclidean Lagrangian density, given by

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \,\bar{\psi}_\uparrow \,\bar{\psi}_\downarrow \,\psi_\downarrow \,\psi_\uparrow \qquad (3)$$

where g is the strength of the s-wave inter-atomic coupling (g < 0 in the BCS regime) [16, 17]. Summation over the repeated index s in the Lagrangian is meant and  $\beta \equiv 1/(k_BT)$  with  $k_B$  Boltzmann's constant. It is important to stress that we want to determine the relevant physical quantities of the system at fixed density

 $n=N/L^2$ , with N the total number of fermions, and not at fixed chemical potential  $\mu$ . For this reason we shall introduce the so-called number equation which enables one to express the chemical potential  $\mu$  in terms of the density n. The inclusion of phase fluctuations in the number equation strongly modifies the functional dependence of  $\mu$  on n.

Through the usual Hubbard-Stratonovich transformation [16, 17] the Lagrangian density  $\mathcal{L}$ , quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field  $\Delta(\mathbf{r}, \tau)$  so that:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\Delta, \bar{\Delta}] \, \exp\left\{-\frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar}\right\},\,\,(4)$$

where

$$S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2 \mathbf{r} \, \mathscr{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) \tag{5}$$

and the (exact) effective Euclidean Lagrangian density  $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$  reads

$$\mathcal{L}_e = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g} .$$
(6)

Due to to the Mermin-Wagner-Hohenberg-Coleman theorem [13–15] in a 2D uniform system no off-diagonal longrange order (ODLRO) may exist at any finite temperature T and this means that the critical temperature  $T_c$  for true condensation is  $T_c = 0$ . Nevertheless, below a finite temperature which is usually identified with the Berezinskii-Kosterlitz-Thouless critical temperature  $T_{BKT}$  there is quasi condensation, characterized in our fermionic system by algebraic long-range order (ALRO) of the two-body density matrix, where phase fluctuations of  $\Delta(\mathbf{r}, \tau)$  have an algebraic decay [16–18].

In this paper we want to investigate the effect of fluctuations of the gap field  $\Delta(\mathbf{r},t)$  around its mean-field value  $\Delta_0$  which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r},\tau) = (\Delta_0 + \sigma(\mathbf{r},\tau)) e^{i\theta(\mathbf{r},\tau)}, \qquad (7)$$

where  $\theta(\mathbf{r}, \tau)$  is the phase of the gap field (it describes the Goldstone field of the U(1) symmetry) and  $\sigma(\mathbf{r}, \tau)$ describes amplitude fluctuations. The adopted polar representation for  $\Delta(\mathbf{r}, t)$  automatically satisfies Goldstone's theorem [16–18].

### III. REVIEW OF MEAN-FIELD RESULTS

By neglecting both phase and amplitude fluctuations, i.e. by setting  $\theta(\mathbf{r},t) = 0$  and  $\sigma(\mathbf{r},\tau) = 0$ , and integrating over the fermionic fields one gets immediately the mean-field partition function

$$\mathcal{Z}_{mf} = \exp\left\{-\frac{S_{mf}}{\hbar}\right\} = \exp\left\{-\beta \Omega_{mf}\right\}, \qquad (8)$$

where

$$\frac{S_{mf}}{\hbar} = -Tr[\ln(G_0^{-1})] - \beta L^2 \frac{\Delta_0^2}{g}$$

$$= -\sum_{\mathbf{k}} \left[ 2\ln(2\cosh(\beta E_k/2)) - \beta \left(\frac{\hbar^2 k^2}{2m} - \mu\right) \right]$$

$$- \beta L^2 \frac{\Delta_0^2}{g}, \qquad (9)$$

with

$$G_0^{-1} = \begin{pmatrix} \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu & \Delta_0 \\ \Delta_0 & \hbar \partial_\tau + \frac{\hbar^2}{2m} \nabla^2 + \mu \end{pmatrix}$$
(10)

the inverse mean-field Green function, and

$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2}$$
 (11)

the energy of the fermionic elementary excitations. The constant, uniform and real gap parameter  $\Delta_0$  can be obtained by minimizing  $\Omega_{mf}$ :

$$\frac{\partial \Omega_{mf}(\Delta_0)}{\partial \Delta_0} = 0 \tag{12}$$

which gives the familiar gap equation

$$-\frac{1}{g} = \frac{1}{L^2} \sum_{k} \frac{\tanh(\beta E_k/2)}{2E_k} \,. \tag{13}$$

The integral on the right hand side of this equation is formally divergent. Nevertheless this divergence is easily removed. Contrary to the 3D case, in 2D a bound-state energy  $\epsilon_B$  exists for any value of the attractive interaction strength g between atoms. By expressing the bare interaction strength g in terms of the physical binding energy  $\epsilon_B$  through [19, 21, 22, 24]

$$-\frac{1}{g} = \frac{1}{L^2} \sum_{\mathbf{k}} \frac{1}{2\frac{\hbar^2 k^2}{2m} + \epsilon_B} \,. \tag{14}$$

we obtain the regularized gap equation

$$\sum_{\mathbf{k}} \left( \frac{\tanh(\beta E_k/2)}{2E_k} - \frac{1}{2\frac{\hbar^2 k^2}{2m} + \epsilon_B} \right) = 0.$$
 (15)

It is important to observe that the binding energy  $\epsilon_B$  can be written as  $\epsilon_B \simeq 2/(ma_{2D})$ , where  $a_{2D}$  is the 2D s-wave scattering length, such that  $a_{2D} \simeq a_z \exp(az/a_{3D})$  with  $a_{3D}$  the 3D scattering length and  $a_z$  the characteristic length of the strong transverse confinement which makes the system 2D [28]. ¿From Eq. (15) one obtains the energy gap  $\Delta_0$  as a function of T,  $\mu$ , and  $\epsilon_B$ , i.e.  $\Delta_0 = \Delta_0(T,\mu,\epsilon_B)$ . The total number N of fermions is obtained from the familiar thermodynamic relation

$$N = -\left(\frac{\partial \Omega_{mf}}{\partial \mu}\right)_{L^2,T} \,, \tag{16}$$

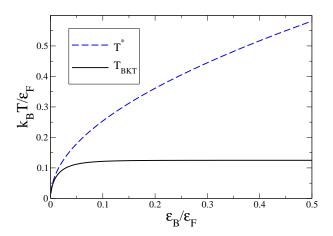


FIG. 1: (Color online). Relevant temperatures of the Fermi gas as a function of the scaled binding energy  $\epsilon_B/\epsilon_F$ , with  $\epsilon_F$  the Fermi energy. Dashed line: temperature  $T^*$  above which the quasi-condensate  $\langle |\Delta(\mathbf{r},\tau)| \rangle$  is zero; solid line: Berezinskii-Kosterlitz-Thouless critical temperature  $T_{BKT}$ . Note that only at T=0 the condensate  $\langle \Delta(\mathbf{r},\tau) \rangle$  is finite.

which gives the number equation

$$N = \sum_{\mathbf{k}} \left( 1 - \frac{\hbar^2 k^2 / 2m - \mu}{E_k} \tanh\left(\beta E_k / 2\right) \right) \tag{17}$$

which must be solved together with (15) to determine the behavior of  $\Delta_0$  and  $\mu$  as a function of the temperature T and of the binding energy  $\epsilon_B$  at fixed number density  $n = N/L^2$ . At zero temperature (T = 0) one easily finds the exact solutions of Eqs. (15) and (17) as

$$\mu = \epsilon_F - \frac{1}{2}\epsilon_B \quad \text{at } T = 0 , \qquad (18)$$

$$\Delta_0 = \sqrt{2\epsilon_F \epsilon_B} \quad \text{at } T = 0.$$
(19)

We identify the temperature  $T^*$  as the temperature at which the mean-field energy gap  $\Delta_0$  becomes zero [19, 21, 22]. Thus:  $\langle |\Delta(\mathbf{r},\tau;T^*)| \rangle = \Delta_0(T^*) = 0$ . Setting  $\Delta_0 = 0$  in Eqs. (15) and (17), in the continuum limit  $\sum_{\mathbf{k}} \to L^2 \int d^2\mathbf{k}/(2\pi)^2$  and after some manipulations one obtains the equations determining  $T^*$  as a function of n (through the 2D Fermi energy  $\epsilon_F = (\hbar/m)\pi n$ ) and the binding energy  $\epsilon_B$ :

$$\mu(T^*) = k_B T^* \ln \left( e^{\epsilon_F/(k_B T^*)} - 1 \right), (20)$$

$$\epsilon_B = k_B T^* \frac{\pi}{\gamma} \exp\left(-\int_0^{\mu(T^*)/(2k_B T^*)} \frac{\tanh(u)}{u} du\right), (21)$$

where  $\gamma=1.781$  (see also [21]). The dashed curve shown in Fig. 1 reports the scaled temperature  $k_B T^*/\epsilon_F$  as a function of the scaled binding energy  $\epsilon_B/\epsilon_F$ . Here we limit our plot to small values of  $\epsilon_B/\epsilon_F$  since for  $\epsilon_B/\epsilon_F\gtrsim 1$  and  $k_B T/\epsilon_F\gtrsim 2/3$  beyond mean-field corrections to the number equation (17), not considered above, become relevant [21, 25] for the determination of  $T^*$  vs  $\epsilon_B$  at fixed density n.

Experimentally, the BCS-BEC crossover is induced by changing the binding energy  $\epsilon_B$  with the technique of Feshbach resonances. As shown in Ref. [12], the condensate fraction of Cooper pairs at T=0 is extremely small in the BCS region, where  $\epsilon_B/\epsilon_F \ll 1$ , while it goes to one (all molecules are in the Bose-Einstein condensate) in the BEC region, where  $\epsilon_B/\epsilon_F \gg 1$ . According to Ref. [12], for the range of scaled binding energies considered in Fig. 1 the condensate fraction at zero temperature increases from nearly 0% to about 55% (see also Section VI).

### IV. PHASE FLUCTUATIONS AND SUPERFLUID FRACTION

We now consider the effect of phase fluctuations, i.e. in Eq. (7) we allow  $\theta(\mathbf{r},t) \neq 0$ , but keep  $\sigma(\mathbf{r},\tau) = 0$ . To extract the contribution of the fluctuations we perform a gauge transformation, defining a new fermionic "neutral" field

$$\chi_s(\mathbf{r},\tau) = e^{i\theta(\mathbf{r},\tau)/2} \psi_s(\mathbf{r},\tau) . \tag{22}$$

In this way the Lagrangian density (6) becomes

$$\mathcal{L}_{e} = \bar{\chi}_{s} \left[ \hbar \partial_{\tau} - \frac{\hbar^{2}}{2m} \nabla^{2} - \mu \right] \chi_{s} + i \frac{\hbar^{2}}{2m} \bar{\chi}_{s} \nabla \theta \cdot \nabla \chi_{s}$$

$$+ \bar{\chi}_{s} \chi_{s} \left[ -i \frac{\hbar}{2} \partial_{\tau} \theta - i \frac{\hbar^{2}}{4m} \nabla^{2} \theta + \frac{\hbar^{2}}{8m} (\nabla \theta)^{2} \right]$$

$$+ \Delta_{0} \chi_{\downarrow} \chi_{\uparrow} + \Delta_{0} \bar{\chi}_{\uparrow} \bar{\chi}_{\downarrow} - \frac{\Delta_{0}^{2}}{q} .$$

$$(23)$$

After functional integration over the new fermionic fields the partition function reads [16, 17]

$$Z = \int \mathcal{D}[\theta] \exp\left\{-\frac{\tilde{S}_e(\theta)}{\hbar}\right\}$$
 (24)

where

$$\frac{\tilde{S}_e(\theta)}{\hbar} = -Tr[\ln(G_0^{-1} + \Sigma_\theta)] - \beta L^2 \frac{\Delta_0^2}{g}$$
 (25)

with  $G_0^{-1}$  given by Eq. (10) and  $\Sigma_{\theta}$  given by

$$\Sigma_{\theta} = \hat{I} \left( i \frac{\hbar^2}{4m} \nabla^2 \theta + i \frac{\hbar^2}{2m} \nabla \theta \cdot \nabla \right)$$

$$- \hat{\tau}_3 \left( i \frac{\hbar}{2} \partial_{\tau} \theta - \frac{\hbar^2}{8m} (\nabla \theta)^2 \right) .$$
 (26)

Here  $\hat{I}$  is the 2 × 2 identity matrix and  $\hat{\tau}_3$  is the third Pauli matrix.

At the second order in a gradient expansion [21–24] of  $\Sigma_{\theta}$  the partition function eventually can be written as

$$\mathcal{Z} = \exp\left\{-\frac{S_{mf}}{\hbar}\right\} \int \mathcal{D}[\theta] \exp\left\{-\frac{S_{\theta}}{\hbar}\right\}, \quad (27)$$

where  $S_{mf}$  is given by Eq. (9), while the action functional  $S_{\theta}$  of the phase is given by [21–24]

$$S_{\theta} = \int_{0}^{\hbar\beta} d\tau \int_{L^{2}} d^{2}\mathbf{r} \left\{ \frac{J}{2} (\nabla \theta)^{2} + \frac{K_{\theta\theta}}{2} (\partial_{\tau}\theta)^{2} \right\} , \quad (28)$$

where

$$J = \frac{\hbar^2}{4mL^2} \sum_{\mathbf{k}} \left[ 1 - \frac{\frac{\hbar^2 k^2}{2m} - \mu}{E_k} X_T(E_k) - \frac{\hbar^2 k^2}{2m} X_T'(E_k) \right], \tag{29}$$

is the stiffness,

$$K_{\theta\theta} = \frac{\hbar^2}{4L^2} \sum_{\mathbf{k}} \left[ \frac{\Delta_0^2}{E_k^3} X_T(E_k) + \frac{(\frac{\hbar^2 k^2}{2m} - \mu)^2}{E_k^2} X_T'(E_k) \right].$$
(30)

is the phase susceptibility, and  $X_T(E_k) = \tanh{(\beta E_k/2)}$ . Notice that J and  $K_{\theta\theta}$  are non trivial functions of T,  $\mu$  and  $\Delta_0(T, \mu, \epsilon_B)$ , and from Eqs. (15) and(17) one gets  $\Delta_0$  and  $\mu$  as a function of T,  $\epsilon_B$  and n.

The action functional (28) has the form of a 2D quantum XY model [16–18], where the Goldstone field  $\theta(\mathbf{r},\tau)$  is defined in principle as an angular variable. However, it is well known [16–18] that, in addition to the characteristic temperature  $T^*$  below which quantized vortices develop, there is another relevant temperature in our system: the temperature  $T_{BKT}$  of the Berezinskii-Kosterlitz-Thouless superfluid-normal phase transition, characterized by the binding of quantized vortices below  $T_{BKT}$ . The contribution of vortices below  $T_{BKT}$  then becomes irrelevant at large distance scales and the field  $\theta$  loses its angular character, thus justifying a Gaussian treatment at small energy-momentum. This critical temperature  $T_{BKT}$  can be estimated by solving self-consistently [21, 22, 24]:

$$k_B T_{BKT} = \frac{\pi}{2} J(T_{BKT}) ,$$
 (31)

where J(T) is defined by Eq. (29) with  $\mu$  and  $\Delta_0$  given by the solutions of the gap and number equations Eqs. (15) and(17). Following the approach adopted by various authors [20–23], we use the lowest-order mean-field functions  $\Delta_0$  and  $\mu$  and plug them into the new (higher-order) effective action. Strictly speaking, instead of Eq. (17) one should use a modified number equation, where  $\Omega_{mf}$  is substituted by  $\Omega_{mf} + \Omega_{flu}$  with  $\Omega_{flu}$  taking into account fluctuations [24, 25]. However, at zero temperature  $\Omega_{flu}$  reduces to the zero-point energy of a bosonic gas with excitations  $c_s\hbar k$ , and on the basis of dimensional regularization [27] one can set  $\Omega_{flu}=0$ .

The solid curve of Fig. 1 shows  $k_B T_{BKT}$  in units of the 2D Fermi energy  $\epsilon_F$  as a function of the scaled binding energy  $\epsilon_B/\epsilon_F$ . The curve approaches very quickly its asymptotic value [21, 22]

$$k_B T_{BKT} = \frac{1}{8} \epsilon_F \ . \tag{32}$$

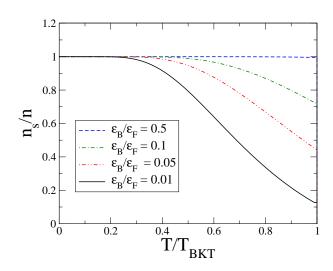


FIG. 2: (Color online). Superfluid fraction  $n_s/n$  as a function of the scaled temperature  $T/T_{BKT}$  for different values of the scaled binding energy  $\epsilon_B/\epsilon_F$ , where  $\epsilon_F = (\hbar^2/m)\pi n$  is the Fermi energy.

The domain between two curves shown in Fig. 2 is the socalled pseudo-gap region [21–25] where vortices proliferate and a more careful treatment of  $\theta$  as angular variable is needed, leading in particular to a gap for the Goldstone field.

Since  $\mathbf{v}_s = (\hbar/m)\nabla\theta$  is the superfluid velocity, the term  $(J/2)(\nabla\theta)^2$  may be identified with the superfluid kinetic energy density  $(1/2)n_s v_s^2$ , where

$$n_s = \frac{4m}{\hbar^2} J \tag{33}$$

is the superfluid number density. The renormalization group theory [18] dictates that for a 2D uniform system above  $T_{BKT}$  the phase stiffness J, and consequently also superfluid density  $n_s$ , is strictly zero. This result implies a jump to zero of the superfluid density at  $T_{BKT}$  [16, 18]. In Fig. 2 we report the superfluid fraction  $n_s/n$  as a function of the scaled temperature  $T/T_{BKT}$  for different values of the scaled binding energy  $\epsilon_B/\epsilon_F$ . The figure clearly shows that the superfluid fraction  $n_s/n$  is equal to one at very low temperatures and decreases monotonically by increasing the temperature T. Moreover, for very small values of the scaled binding energy  $\epsilon_B/\epsilon_F$ the superfluid fraction  $n_s/n$  is quite small at  $T = T_{BKT}$ while for larger values of the scaled binding energy  $\epsilon_B/\epsilon_F$ the superfluid fraction  $n_s/n$  remains close to one up to  $T = T_{BKT}$ . Notice that  $\epsilon_B/\epsilon_F = 0.5$  still corresponds to a positive zero-temperature chemical potential  $\mu(0)$ , i.e. to a system in the BCS regime.

### V. PHASE AND AMPLITUDE FLUCTUATIONS AND SOUND VELOCITY

Any superfluid system admits a density wave, the socalled first sound, where the velocities of superfluid and normal components are in-phase [16, 17]. The velocity of the Goldstone mode is nothing else than the first sound velocity of the superfluid [16, 17] and it is given by

$$c_s = \sqrt{\frac{J}{K}} \,, \tag{34}$$

where J is the stiffness and K is the susceptibility. Within the phase-only approach of the previous section we have  $K = K_{\theta\theta}$ , and using Eqs. (29) and (30) at zero temperature one immediately finds

$$J = \frac{\epsilon_F}{4\pi} \,, \tag{35}$$

and

$$K_{\theta\theta} = \frac{m}{4\pi} \frac{\epsilon_F}{\epsilon_F + \frac{1}{2}\epsilon_B} , \qquad (36)$$

and consequently, using Eq. (34) with  $K = K_{\theta\theta}$ , we obtain

$$c_s = \frac{v_F}{\sqrt{2}} \sqrt{1 + \frac{1}{2} \frac{\epsilon_B}{\epsilon_F}}$$
 at  $T = 0$  (phase-only), (37)

where  $v_F = \sqrt{2\epsilon_F/m}$  is the Fermi velocity and  $\epsilon_F = (\hbar^2/m)\pi n$  is the Fermi energy. We stress that this result is obtained by completely neglecting amplitude fluctuations  $\sigma(\mathbf{r}, \tau)$  of the order parameter  $\Delta(\mathbf{r}, \tau)$ .

Recently Schakel [31] has analyzed the 3D BCS-BEC crossover at zero temperature considering both phase  $\theta(\mathbf{r},\tau)$  and amplitude  $\sigma(\mathbf{r},\tau)$  fluctuations in  $\Delta(\mathbf{r},\tau)$ . Following the procedure of Schakel [31], in our zero-temperature 2D system after integration over  $\sigma(\mathbf{r},\tau)$  we obtain the action functional  $S_{\theta}$  of Eq. (28) with the stiffness J still given by Eq. (29) but with a new K instead of  $K_{\theta\theta}$ . In particular, the new susceptibility K is given by

$$K = \frac{K_{\theta\theta}K_{\sigma\sigma} - K_{\sigma\theta}^2}{K_{\sigma\sigma}} \,, \tag{38}$$

which is a non trivial combination of the phase-only susceptibility  $K_{\theta\theta}$  given by Eq. (30), the amplitude-only susceptibility  $K_{\sigma\sigma}$  and the amplitude-phase susceptibility  $K_{\sigma\theta}$ . Note that only when amplitude and phase fluctuations are decoupled, i.e. when  $K_{\sigma\theta} \simeq 0$  one obtains  $K \simeq K_{\theta\theta}$ .

At zero temperature, we get (see also [31]) the following formulas

$$K_{\theta\theta} = -\frac{\hbar^2}{4L^2} \left( \frac{\partial^2 \Omega_{mf}}{\partial \mu^2} \right)_{L^2 T = 0} , \qquad (39)$$

$$K_{\sigma\sigma} = -\frac{\hbar^2}{4L^2} \left( \frac{\partial^2 \Omega_{mf}}{\partial \Delta_0^2} \right)_{L^2, T=0} , \qquad (40)$$

$$K_{\sigma\theta} = \frac{\hbar^2}{4L^2} \left( \frac{\partial^2 \Omega_{mf}}{\partial \Delta_0 \partial \mu} \right)_{L^2, T=0} . \tag{41}$$

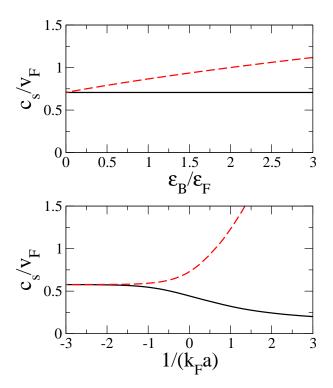


FIG. 3: (Color online). Sound velocity  $c_s$  at zero temperature (T=0) taking into account only phase fluctuations (dashed lines) or both phase and amplitude fluctuations (solid lines) of the order parameter. Upper panel: 2D scaled sound velocity  $c_s/v_F$  as a function of the scaled binding energy  $\epsilon_B/\epsilon_F$  of the 2D Fermi superfluid. Lower panel: 3D scaled sound velocity  $c_s/v_F$  as a function of the scaled inverse interaction strength  $1/(k_F a)$  of the 3D Fermi superfluid with scattering length a. Here  $\epsilon_F = \hbar^2 k_F^2/(2m)$  is the Fermi energy and  $v_F = \sqrt{2\epsilon_F/2}$  the Fermi velocity.

By using these formulas for our 2D superfluid system we easily find that  $K_{\theta\theta}$  is indeed given by Eq. (36), while  $K_{\sigma\sigma}$  and  $K_{\sigma\theta}$  are

$$K_{\sigma\sigma} = -\frac{m}{8\pi\epsilon_B} \frac{\Delta_0^2}{\epsilon_F + \frac{1}{2}\epsilon_B} , \qquad (42)$$

$$K_{\sigma\theta} = \frac{m}{8\pi} \frac{\Delta_0}{\epsilon_F + \frac{1}{2}\epsilon_B} \,. \tag{43}$$

It follows that the sound velocity of the 2D superfluid system reads  $\,$ 

$$c_s = \frac{v_F}{\sqrt{2}}$$
 at  $T = 0$  (phase and amplitude), (44)

which is exactly the 2D result obtained some years ago by Marini, Pistolesi and Strinati [20]. Taking into account both phase and amplitude fluctuations of the order parameter (Gaussian fluctuations), at zero temperature the 2D sound velocity  $c_s$  does not depend on the binding energy  $\epsilon_B$  of pairs.

Thus, as reported in the upper panel of Fig. 3, taking into account only phase fluctuations of the order parameter leads to a quite different behaviour of the zero

temperature speed of sound in the 2D Fermi superfluid from that obtained by considering both phase and amplitude fluctuations. While the latter does not depend on  $\epsilon_B$ , the former increases with it and diverges in the deep BEC regime. A similar behaviour is obtained in 3D for the dependence of the speed of sound on the the scaled inverse interaction strength  $1/(k_F a)$  which we report for completeness in the lower panel of Fig. 3. Also this panel shows that only in the deep BCS regime, where  $1/(k_F a) \ll -1$ , the two approaches give the same results  $c_s \simeq v_F/\sqrt{3}$  while, again, the phase-only sound velocity diverges in the BEC regime.

We now show that the Gaussian (phase plus amplitude) result, Eq. (44), can be re-derived by using simple thermodynamic relations [32] and it can also be easily extended at finite temperature. In fact, according to Landau [29] and Kalatnikov [30] the first sound velocity  $c_s$  is given by

$$m c_s^2 = \left(\frac{\partial P}{\partial n}\right)_{L^2,\bar{S}} ,$$
 (45)

where P is the pressure and  $\bar{S} = S/N$  is the entropy per particle of the superfluid. Moreover, at zero temperature it holds the following equality

$$\left(\frac{\partial P}{\partial n}\right)_{L^2,0} = n \left(\frac{\partial \mu}{\partial n}\right)_{L^2} . \tag{46}$$

Using Eq. (18) we immediately obtain Eq. (44).

At finite temperature we can determine the sound velocity  $c_s$  using the elegant formula of thermodynamics

$$m c_s^2 \simeq n \left(\frac{\partial \mu}{\partial n}\right)_{L^2, T}$$
 (47)

Numerically we find that  $c_s$  remains close to  $1/\sqrt{2}$  for any temperature T (up to  $T_{BKT}$ ) and for any value of the scaled binding energy  $\epsilon_B/\epsilon_F$ . This is in full agreement with experiments with 3D superfluids like <sup>4</sup>He liquid and unitary Fermi gas the sound velocity  $c_s$  does not depend significantly on the temperature T.

## VI. ALGEBRAIC LONG-RANGE ORDER AND QUASI-CONDENSATE FRACTION

As previously discussed, according to the Mermin-Wagner-Hohenberg-Coleman theorem [13–15], in a 2D uniform quantum system of interacting identical particles one can find true condensation, i.e off-diagonal-long-range-order (ODLRO), only at zero temperature (T=0). Instead, the system can have quasi condensation, i.e. algebraic-long-range-order (ALRO), below a critical finite temperature that is usually identified with the Berezinskii-Kosterlitz-Thouless temperature  $T_{BKT}$  [16, 18]. In the case of our 2D Fermi system the two-body density matrix

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \langle \bar{\psi}_{\uparrow}(\mathbf{r}_1, 0) \, \bar{\psi}_{\downarrow}(\mathbf{r}_2, 0) \, \psi_{\downarrow}(\mathbf{r}_3, 0) \, \psi_{\uparrow}(\mathbf{r}_4, 0) \rangle$$
(48)

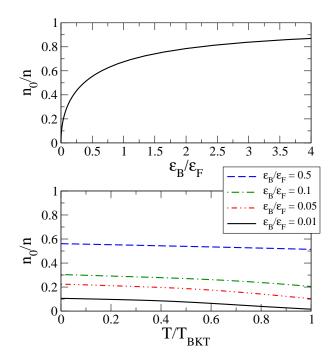


FIG. 4: (Color online). Upper panel: zero-temperature condensate fraction  $n_0/n$  vs scaled binding energy  $\epsilon_B/\epsilon_F$ . Lower panel: quasi-condensate fraction  $n_0/n$  as a fuction of the scaled temperature  $T/T_{BKT}$  for different values of the scaled binding energy  $\epsilon_B/\epsilon_F$ .

shows ODLRO at T=0 [12] and ALRO for  $0 < T < T_{BKT}$ . In particular, by using Eq. (22) and introducing the center-of-mass positions of the two Cooper pairs, given by  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  and  $\mathbf{R}' = (\mathbf{r}_3 + \mathbf{r}_4)/2$ , and their relative distances  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  and  $\mathbf{r}' = \mathbf{r}_4 - \mathbf{r}_3$ , for  $|\mathbf{R} - \mathbf{R}'| \to \infty$  we can write

$$\rho_{2}(\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}, \mathbf{r}_{4}) \simeq F^{*}(\mathbf{r}) F(\mathbf{r}') \langle e^{i(\theta(\mathbf{R}, 0) - \theta(\mathbf{R}', 0))} \rangle$$

$$\simeq F^{*}(\mathbf{r}) F(\mathbf{r}') e^{-\frac{1}{2} \langle (\theta(\mathbf{R}, 0) - \theta(\mathbf{R}', 0))^{2} \rangle}$$

$$\simeq F^{*}(\mathbf{r}) F(\mathbf{r}') \left( \frac{R_{0}}{|\mathbf{R} - \mathbf{R}'|} \right)^{\frac{k_{B}T}{8\pi J}}$$
(49)

where  $R_0 = 2c_s/(k_BT)$  is the coherence length scale of phase fluctuations [18, 23] and

$$F(\mathbf{r}') = \langle \chi_{\downarrow}(\mathbf{r}_{3}, 0) \chi_{\uparrow}(\mathbf{r}_{4}, 0) \rangle$$
$$= \frac{1}{L^{2}} \sum_{\mathbf{k}} \frac{\Delta_{0}}{2E_{k}} \tanh(\beta E_{k}/2) e^{i\mathbf{k} \cdot \mathbf{r}'} \qquad (50)$$

is the mean-field wavefunction of the Cooper pair [4–6], such that

$$n_0 = 2 \int d^2 \mathbf{r}' |F(\mathbf{r}')|^2 = \frac{\Delta_0^2}{2L^2} \sum_{\mathbf{k}} \frac{\tanh^2(\beta E_k/2)}{E_k^2}$$
 (51)

is the quasi-condensate density of atoms in the 2D superfluid

At T=0 Eq (49) displays ODLRO, i.e there is no algebraic decay of the off-diagonal part of the two-body

density matrix, and  $n_0$  is the true condensate density of the system (see also [12]). In the upper panel of Fig. 4 we plot the zero-temperature condensate fraction  $n_0/n$ as a function of the scaled binding energy  $\epsilon_B/\epsilon_F$ . At finite temperature Eq (49) displays ALRO, i.e. there is algebraic decay of the off-diagonal part of the two-body density matrix, and  $n_0$  is the quasi-condensate density of the system (see [18] for the bosonic case). In the lower panel of Fig. 4 we plot the quasi-condensate fraction  $n_0/n$  as a fuction of the scaled temperature  $T/T_{BKT}$ for different values of the scaled binding energy  $\epsilon_B/\epsilon_F$ . The figure clearly shows that for large values of the scaled binding energy  $\epsilon_B/\epsilon_F$  the quasi-condensate fraction  $n_0/n$  is practically independent on the temperature up to the Berezinskii-Kosterlitz-Thouless critical temperature  $T_{BKT}$ .

### VII. CONCLUSIONS

By using the path integral formalism and the thermodynamics of superfluids we have calculated the superfluid density, the sound velocity, and the quasi-condensate density of a 2D superfluid made of ultracold alkali-metal atoms in the BCS-BEC crossover. We have considered both phase and amplitude fluctuations of the order parameter showing that amplitude fluctuations are necessary to recover within the path integral formalism the sound velocity one gets alternatively from the mean-field

equation of state by using familiar thermodynamics relationships. Our results are obtained below the critical temperature  $T_{BKT}$  of the Berezinskii-Kosterlitz-Thouless phase transition, where there is quasi-condensation and the Goldstone field of phase fluctuations is still massless. Notice that the crucial role of phase fluctuations on the Berezinskii-Kosterlitz-Thouless transition has been very recently investigated with the attractive Hubbard model by Erez and Meir [33]. We believe that a reliable description of the pseudo-gap region above  $T_{BKT}$  [26], where the Goldstone field of phase fluctuations becomes gapped with exponential decay of correlations, requires a more sophisticated self-consistent approach to the phase fluctuations [34]. We are currently working on this issue.

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