Evolution equations of the second and third order with Lie-Bäcklund symmetries

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Preface

Chapter 4 of my book [1] is dedicated to Lie-Bäcklund group analysis of various types of differential equations. The Russian edition of this book had an *Addendum* (pages 262-266) containing a summary of new results (obtained by the end of 1982) on classification of evolution equations of the second and third order possessing Lie-Bäcklund symmetries. The *Addendum* was not included in the English translation of the book by technical reasons. I present here the missing translation. In order to make the text self-contained, I have added in the translation the equations from the main body of the book used in the *Addendum*.

The mth-order evolution equation with one spatial variable x is written

$$u_t = F(x, u, u_1, ..., u_m), \quad m \ge 2,$$
 (E)

where u_s , s = 1, ..., m, is the partial derivative of order s of u with respect to x. The expression $F_m = \partial F(x, u, u_1, ..., u_m)/\partial u_m$ is called the *separant* of Eq. (E).

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Translation from N.H. Ibragimov, Transformation groups applied to mathematical physics, Nauka, Moscow, 1983, Addendum, pp. 262-266.

Ι

The problem on classification of evolution equations having Lie-Bäcklund symmetries has been solved in [1], Section 20, for the semi-linear second-order equations of the form

$$u_t = \varphi(u)u_2 + \psi(u, u_1)$$

and for the third-order equations with a constant separant having the form

$$u_t = u_3 + \varphi(u, u_1).$$

This problem has been recently studied in more general cases.

S.I. Svinolupov and V.V. Sokolov [2] considered the second-order evolution equations of the form

$$u_t = F(u, u_1, u_2)$$

and extended the analysis of necessary conditions for existence of the recursion operator (see Eqs. (20.8), (20.9), (20.9') in [1]) by deriving three additional necessary conditions. They obtained the following forms of the function F:

$$\frac{u_2}{u_1^2} - \frac{a''}{a'} + bu_1, \quad \frac{u_2}{u_1^2} + \frac{1}{u_1} + bu_1 + c,$$

$$\frac{u_2}{(u_1+1)^2} - \frac{b'-k^2}{b+k} \frac{1}{u_1+1} + \frac{b^2-b'}{b+k} (u_1+1) + 2\frac{b'+kb}{b+k},$$

$$\frac{u_2}{(u_1+1)^2} + \frac{a''}{a'} \frac{1}{u_1+1} + \left(\frac{a''}{a'} + ka\right) u_1 - \frac{a''}{a'},$$

$$\frac{u_2+a'u_1}{(u_1+1)^2} + \frac{aa''}{a'(u_1+a)} - \left(\frac{a''}{a'} - \frac{a'}{a^2} + \frac{k}{a^2}\right) u_1,$$

where k = const., a, b, c are arbitrary functions of u and a(u) is the density of the conservation law of the corresponding equation. These functions describe all possibilities of the cases (20.9) and (20.9') from [1]. An additional analysis is necessary for the case (20.8).

The condition (19.37) from [1] yields that the third-order equation with the constant separant F_3 possessing a non-trivial Lie-Bäcklund algebra has the form

$$u_t = u_3 + a(u, u_1)u_2^2 + b(u, u_1)u_2 + c(u, u_1).$$
(1)

The equations (1) with a = b = 0 are classified in [1], Section 20.2. Furthermore, the following two integrable equations of the form (1) were known aside equations reducible to the case a = b = 0 by a simple transformation. The equation [3]

$$u_t = u_3 - \frac{3}{2} \frac{u_1}{u_1^2 + \alpha} u_2^2 - \frac{3}{2} \frac{\alpha' u_1}{u_1^2 + \alpha} u_2 - \frac{3}{8} \alpha'^2 \frac{u_1}{u_1^2 + \alpha} + \frac{1}{2} \alpha'' u_1, \tag{2}$$

where $\alpha = \sum_{i=1}^{4} k_i (u+k)^i$ is an arbitrary fourth degree polynomial in u with real coefficients, and the equation [4]

$$u_t = u_3 - \frac{3}{2} \frac{u_2^2}{u_1} - \frac{3}{2} \wp(u) u_1^3 + \frac{k}{u_1}, \quad k \neq 0,$$
(3)

where k = const., $\wp(u)$ is the Weierstrass elliptic function,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \equiv 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

$$e_1 + e_2 + e_3 = 0.$$

Upon enumerating [5] all possible equations (1) with a nontrivial algebra (assuming that elements of the algebra are independent of t, x), it became clear that Eqs. (2), (3) are exceptional, and the remaining equations (1) are reducible to the KdV equation or to a linear equation by means of rather simple transformations. Recently, S.V. Khabirov (see II) and independently S.I. Svinolupov, V.V. Sokolov (see III), have found transformations of the form

$$v = \Phi(u, u_1, \dots, u_n), \tag{4}$$

connecting (2) with the KdV equation, and proved that there are no such transformations for Eq. (3), except for degenerate cases. Thus, classification of third-order equations with the constant separant possessing a nontrivial Lie-Bäcklund algebra results in the linear equation, the KdV equation and the *Krichever-Novikov equation* (3). This solves the problem of classification of semi-linear equations

$$u_t = a(u)u_3 + \varphi(u, u_1, u_2)$$

as well. Indeed, if $a' \neq 0$ one can assume that $a(u) = u^3$ (using the substitution $a(u) = v^3$), and such equations with a nontrivial algebra are reduced to the case of the constant separant by the substitution (20.44) from [1].

For the equation of the form (3) with k = 0 (in this case one can take an arbitrary function of u instead of $\wp(u)$) one can readily obtain the recursion operator, because a substitution $w = \phi(u)$ reduces it to Eq. (20.43) from [1]:

$$w_t = w_3 - \frac{3}{2}w_1^{-1}w_2^2.$$

The latter equation has the recursion operator

$$L = D^2 - 2\frac{w_2}{w_1}D + w_1D^{-1} \cdot \left(\frac{w_3}{w_1^2} - \frac{w_2^2}{w_1^3}\right)D$$

which is obtained by the formulae (19.46), (19.50'), and (20.42) from [1].

II

(S.V. Khabirov). Equivalence transformations of the form (4) are studied for equations

$$u_t = u_3 + f(u, u_1, u_2), (5)$$

$$v_t = v_3 + h(v, v_1, v_2) \tag{6}$$

without assuming that the equations (5) and (6) admit a nontrivial algebra. All equations are considered up to pointwise changes of variables, so that transformations (4) of the order $n \ge 1$ are discussed.

It turns out that existence of equivalence transformations imposes strict restrictions on functions f and h. For example, if the KdV equation $(h = vv_1)$ is taken as (6), it appears that (5) has the form (1) and the transformation (4) has the order $n \leq 3$. Furthermore, one can enumerate the equations (1) reducible to the KdV equation by transformations of the first, second, and third order and to find the transformations (4) themselves. In particular, (2) is connected with the KdV equation

$$v_t = v_3 + vv_1 \tag{7}$$

by the third-order transformation

$$v = pu_3 + qu_2^2 + ru_2 + s,$$

where

$$p = 3\frac{z}{u_1}, \quad q = -\frac{3}{2\alpha}(1 - z^2)(1 + 2z),$$

$$r = \frac{6(1 - z)}{u + k} + \alpha' q,$$

$$s = \frac{\alpha''}{2} + 6\frac{\alpha}{(u + k)^2} + 3z\left(\frac{\alpha''}{2} - \frac{\alpha}{u + k}\right) + \frac{\alpha'^2}{4}q,$$

$$z = \pm \frac{u_1}{\sqrt{u_1^2 + \alpha}}.$$

The following equations are connected with (7) by second-order transformations (every equation is followed by the corresponding transformation):

$$u_{t} = u_{3} - \frac{3}{4} \frac{u_{2}^{2}}{u_{1}} - \frac{1}{3} u_{1}^{2} - \frac{2}{3} k u_{1}^{3/2}, \qquad v = \frac{u_{2}}{\sqrt{u_{1}}} - \frac{2}{3} u_{1} + k \sqrt{u_{1}};$$

$$u_{t} = u_{3} - \frac{1}{18} u_{1}^{3} + \frac{1}{2} k u_{1}^{2}, \qquad v = u_{2} - \frac{1}{6} u_{1}^{2} + k u_{1};$$

$$u_{t} = u_{3} + 3 \frac{a'}{a} u_{1} u_{2} + \left(\frac{a''}{a} - \frac{a^{2}}{18}\right) u_{1}^{3}, \qquad v = a u_{2} + \left(a' - \frac{a^{2}}{6}\right) u_{1}^{2}.$$

Here k is an arbitrary constant and a = a(u) is an arbitrary function.

The Krichever-Novikov equation (3) is not connected with any equation (6) by non-point transformations of the form (4) in the general case. However, there are exceptions when it can be reduced to the KdV equation. Namely, Eq. (3), written here with k = 6, is connected with (7) by the transformation

$$v = 3\left(\frac{u_3}{u_1} - \frac{3}{2}\frac{u_2^2}{u_1^2} + 4\varepsilon\frac{u_2}{u_1^2} - \frac{3}{2}\wp u_1^2 - \frac{2}{u_1^2}\right), \quad \varepsilon = \pm 1,$$
 (8)

if $\wp = \text{const}$, and by the transformation

$$v = -3\left(\frac{u_3}{u_1} - \frac{1}{2}\frac{u_2^2}{u_1^2} + \varepsilon(u)u_2 + \varepsilon'(u)u_1^2 + \frac{3}{2}\wp(u)u_1^2 + \frac{2}{u_1^2}\right)$$
(9)

if

$$\wp = \frac{1}{u^2},$$
 then $\varepsilon = \frac{2}{u},$ (9a)

or

$$\wp = \frac{\alpha^2}{4} \left(-\frac{2}{3} + \tan^2 \frac{au}{2} \right), \quad \text{then} \quad \varepsilon = \alpha \tan \frac{au}{2},$$
 (9b)

or

$$\wp = \frac{\alpha^2}{4} \left(\frac{2}{3} + \tan^2 \frac{au}{2} \right), \quad \text{then } \varepsilon = \alpha \tanh \frac{au}{2},$$
 (9c)

where $\alpha = \text{const.}$

The Lie-Bäcklund algebra for Eq. (3) with the arbitrary Weierstrass function is nontrivial and contains the following element of the fifth order:

$$u_{5} - 5\frac{u_{2}u_{4}}{u_{1}} - \frac{5}{2}\frac{u_{3}^{2}}{u_{1}} + \left(\frac{25}{2}\frac{u_{2}^{2}}{u_{1}^{2}} - \frac{5}{2}\frac{k}{u_{1}^{2}} - \frac{15}{2}\wp u_{1}^{2}\right)u_{3} - \frac{45}{8}\frac{u_{2}^{4}}{u_{1}^{3}} + \frac{25}{4}k\frac{u_{2}^{2}}{u_{1}^{3}} + \frac{15}{4}k\wp u_{1}u_{2}^{2} - \frac{15}{2}\wp' u_{1}^{3}u_{2} - \frac{3}{2}\wp'' u_{1}^{5} + \frac{27}{8}\wp^{2}u_{1}^{5} - \frac{5}{8}\frac{k^{2}}{u_{1}^{3}} + \frac{5}{4}k\wp u_{1}.$$

III

(S.I. Svinolupov, V.V. Sokolov). Eq. (2) can be connected with Eq. (20.32) from [1] by the transformation (4) of the first order. To this end, one should first reduce (2) to the form

$$u_t = u_3 - \frac{3}{2} \frac{u_1}{u_1^2 + 1} u_2^2 - \frac{3}{2} \wp(u) (u_1^3 + u_1)$$
 (2')

by a pointwise substitution and then perform the transformation

$$v = 2 \ln \left(u_1 + \sqrt{u_1^2 + 1} \right) + \ln \psi(u)$$

with the function $\psi(u)$ defined by the equation

$$A\psi^{2} + \left(\frac{3}{2}\wp(u) + C\right)\psi + B = 0.$$

If the coefficients A, B, C are expressed via the irrational invariants of the function $\wp(u)$ by the formulae

$$AB = \frac{9}{64}(e_1^2 - 4e_2e_3), \quad C = \frac{3}{4}e_1,$$

the above transformation reduces (2') to

$$v_t = v_3 - \frac{1}{8}v_1^3 + (Ae^v + Be^{-v} + C)v_1.$$

Then, one can construct a third-order transformation connecting (2') with (7) according to Lemma 20.2.2 from [1].

The following chain of transformations is suggested for Krichever-Novikov equation (3). It allows one to investigate the equation in question from various viewpoints. The substitution $v = \wp\left(\frac{u}{2}\right)$ maps (3) to the form

$$v_t = v_3 - \frac{3v_2^2}{2v_1} + \frac{av^3 + bv + c}{v_1}$$
(3')

with the constants a, b, c. This equation is equivalent to (7) when the cubic polynomial $av^3 + bv + c$ has multiple zeroes. In the general case, it is reduced by the transformation

$$w = -3\frac{v_3}{v_1} + \frac{3}{2}\frac{v_2^2}{v_1^2} - \frac{av^3 + bv + c}{v_1^2}$$

to the system

$$v_t = -2v_3 - wv_1, \quad w_t = w_3 + ww_1 - 12av_1$$

with the known (L, A)-pair [6].

IV

The equivalence problems discussed above can be considered in a more general framework by replacing the transformations (4) solved for v with a more general transformation between the differentiable variables u, v given by the differential equation

$$\Phi(u, u_1, \dots, u_n; v, v_1, \dots, v_n) = 0.$$
(10)

Let $[\Phi]$ be a differential manifold in the space of variables $(u, v; u_1, v_1; \ldots)$ given by Eq. (10). The evolutionary equations

$$u_t = F(u, u_1, \dots, u_m), \tag{11}$$

$$v_t = H(v, v_1, \dots, v_m) \tag{12}$$

will be considered as the Lie-Bäcklund equations (see Remark 17.1.1 in [1]) determining the group G with the canonical generator

$$X = F \frac{\partial}{\partial u} + H \frac{\partial}{\partial v} + \dots$$

The equations (11) and (12) are equivalent if there is a manifold $[\Phi]$ invariant with respect to the group G. The equivalence transformation between the equations (11) and (12) is given by the differential equation (10). Thus, the problem of equivalence of evolutionary equations is reduced to investigation of the invariance test

$$X\Phi|_{[\Phi]} = 0.$$

The similar generalization of the transformation (19.44) from [1],

$$y = \varphi(x, u, u_1, ..., u_n), \quad v = \Phi(x, u, u_1, ..., u_n),$$

including a change of the variable x, leads to the general $B\ddot{a}cklund\ transformation$ for evolutionary equations.

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