Asymptotic analysis for superfocusing of the electric field in between two nearly touching metallic spheres*

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Abstract

We consider the enhancement of electric field in the presence of two perfectly conducting spheres. When the two spheres get closer, the electric field have a much larger magnitude compared to the external field in the small gap region between the two spheres. The enhanced field can be arbitrary large with the generic blow-up rate $|\epsilon \ln \epsilon|^{-1}$ in three dimensional space, where ϵ is the distance between the spheres. In this paper we derive rigorously an asymptotic formula of the electric field consisting of elementary functions. The asymptotic formula explicitly characterizes superfocusing of the electric field in terms of the spheres radii, the distance between the spheres, and the external field. We illustrate our results with numerical calculations.

AMS subject classifications. 35J25; 78M35

Key words. Conductivity equation; Gradient blow-up; Bispherical coordinates

1 Introduction

Two nearly touching metallic spheres cause the enhancement of the electric field. In an external electric field of long wavelength compared to the size of spheres, the presence of nearly touching metallic spheres induces a very large electric field confined in the narrow gap region between the spheres. Since the field is concentrated in a small region compared to the wavelength of external field, this effect is often called the superfocusing. The superfocusing in nearly touching metallic spheres has attracted considerable attention due to its application to various imaging modalities such as the surface-enhanced raman spectroscopy (SERS) and the single molecule detection [26, 29].

In this paper we formulate and analyze the superfocusing of the electric field in between two nearly touching metallic spheres. When a extremely low-frequency eternal field is applied, metallic objects behave like perfect conductors according to the Drude model for metals and it is valid to consider the quasi-static approximation, *i.e.* the Laplace's equation for the electric potential. We assume that B_1 and B_2 are two perfectly conducting spheres embedded in \mathbb{R}^3 , which is occupied by the homogeneous material of the conductivity 1. Then we consider the electric potential u which satisfies the following conductivity equation:

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_1 \cup B_2}, \\
u = \text{constant} & \text{on } \partial B_j, j = 1, 2, \\
\int_{\partial B_j} \partial_{\nu} u \, d\sigma = 0, & j = 1, 2, \\
u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-2}) & \text{as } |\mathbf{x}| \to \infty,
\end{cases} \tag{1.1}$$

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where H is the external electric potential which is given by an entire harmonic function. Here and throughout ν and $\partial_{\nu}u$ respectively denote the outward unit normal vector to ∂B_j and the outward normal derivative of u on ∂B_j (j=1,2). The main goal of this paper is to understand rigorously the superfocusing of ∇u by deriving the asymptotic formula for ∇u as the distance ϵ between two spheres tends to 0.

The problem of the electrostatic interaction between two perfectly conducting spheres dates back to at least 1890s when Maxwell calculated the electric potential energy of two charged conducting spheres [22]. There are two classical methods to derive the exact solution for the electric potential. The first is the method of separation of variables in the bispherical coordinates [11, 12, 30] and the second is the method of image charges [30]. Both methods express the electric potential as an infinite series which converges fast when the two spheres are well-separated. However, in the case that the gap between the spheres are small, the solution series converges very slowly and it causes the difficulty in computing the solution accurately. Actually, the magnitude of the electric field may blow up to infinity as the distance ϵ tends to zero [16]. The asymptotic behavior of the electric field in between two closely located conductors has been studied extensively in relation with the computation of the effective conductivity in composite materials [9, 16, 23, 27]. In [23], McPhedran and colleagues considered two nearly touching cylinders of highly conducting materials. There, they approximated discrete image charges by an continuous charge distribution. And, based on this approximation, they derived asymptotics for the multipole coefficients of the electric potential and computed the effective conductivity of the composite material which consists of densely packed arrays of highly conducting cylinders. For three dimensional case, Poladian obtained similar result for highly conducting spheres in [27, 28]. It is worth to mention that this method was extended to the two-dimensional linear elasticity [24].

Lately, it has been intensively studied the singular behavior of the electric field, which is the gradient of the solution u to (1.1) in this paper. It was shown that $|\nabla u|$ is bounded independently of ϵ when the conductivities of embedded inclusions are finite and strictly positive [18, 19]. However, if the conductivities of the inclusions degenerate to ∞ (perfectly conducting), then the gradient may blow up as ϵ tends to 0. The generic rate of the gradient blow-up is $|\epsilon \ln \epsilon|^{-1}$ in three dimensions [7, 8, 15, 20], while it is $\epsilon^{-1/2}$ in two dimensions [3, 4, 5, 6, 7, 8, 10, 16, 31, 32]. The insulating case has the same blow-up rate as the perfectly conducting case in two dimensions. The gradient may or may not blow up depending on the given entire harmonic function H. In two dimensions, it was shown in [4] that the gradient may blow up only when the linear term of H is nonzero.

Let us fix some notations to state the related results in details. Since the Laplace's equation is invariant under rotation and shifting, we can denote B_1 and B_2 as

$$B_1 = B(\mathbf{c}_1, r_1), \quad B_2 = B(\mathbf{c}_2, r_2), \quad \mathbf{c}_j = (0, 0, c_j), \ j = 1, 2,$$

where

$$c_1 = \frac{r_2^2 - r_1^2 - (r_1 + r_2 + \epsilon)^2}{2(r_1 + r_2 + \epsilon)}$$
 and $c_2 = c_1 + r_1 + r_2 + \epsilon$.

Here $B(\mathbf{c}, r)$ means the ball centered at \mathbf{c} with radius r. The radii r_1 and r_2 can be different from each other. We let $\mathbf{p}_1 \in B_1$ and $\mathbf{p}_2 \in B_2$ be, respectively, the fixed points of combined reflections $R_1 \circ R_2$ and $R_2 \circ R_1$, where R_j is the reflection w.r.t. ∂B_j , *i.e.*,

$$R_j(\mathbf{x}) = \frac{r_j^2(\mathbf{x} - \mathbf{c}_j)}{|\mathbf{x} - \mathbf{c}_j|^2} + \mathbf{c}_j, \ j = 1, 2.$$

It can be easily shown that

$$\mathbf{p}_1 = (0, 0, -a_{\epsilon}) \quad \text{and} \quad \mathbf{p}_2 = (0, 0, a_{\epsilon}),$$
 (1.2)

where

$$a_{\epsilon} = \frac{\sqrt{\epsilon}\sqrt{(2r_1 + \epsilon)(2r_2 + \epsilon)(2r_1 + 2r_2 + \epsilon)}}{2(r_1 + r_2 + \epsilon)} = \sqrt{\frac{2r_1r_2}{r_1 + r_2}}\sqrt{\epsilon} + O(\epsilon\sqrt{\epsilon}). \tag{1.3}$$

The blow-up behavior of the electric field can be characterized by the solution to the following equation:

$$\begin{cases}
\Delta h = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B_1 \cup B_2}, \\
h = \text{constant} & \text{on } \partial B_j, \ j = 1, 2, \\
\int_{\partial B_j} \partial_{\nu} h \ ds = (-1)^{j+1} & \text{for } j = 1, 2, \\
h(\mathbf{x}) = O(|\mathbf{x}|^{-2}) & \text{as } |\mathbf{x}| \to \infty.
\end{cases} \tag{1.4}$$

We call h the singular function to (1.1). It was derived in [14] that the solution u to (1.1) can be decomposed into the singular and regular parts:

$$u(\mathbf{x}) = C_H^{\epsilon} h(\mathbf{x}) + H(\mathbf{x}) + r(\mathbf{x}) \quad \text{with } C_H^{\epsilon} = \frac{u|_{\partial B_1} - u|_{\partial B_2}}{h|_{\partial B_1} - h|_{\partial B_2}}, \tag{1.5}$$

where $\|\nabla r\|_{\infty}$ is bounded independently of ϵ . Throughout this paper, the symbol $\|\cdot\|_{\infty}$ denotes $\|\cdot\|_{L^{\infty}(\mathbb{R}^3\setminus\overline{(B_1\cup B_2)})}$. Once h is obtained, one can consequently compute the asymptotic of $\nabla u(\mathbf{x})$ by differentiating the right-hand side in Eq. (1.5) and that of C_H^{ϵ} by applying the following relation obtained in [31, 32]:

$$u|_{\partial B_1} - u|_{\partial B_2} = \int_{\partial B_1 \cup \partial B_2} H \partial_{\nu} h \ d\sigma. \tag{1.6}$$

If B_1 and B_2 are either disks in two dimensional space or balls in three dimensional space, then the singular function h is a potential function generated by two (for disks) or a sequence (for balls) of point charges [20]. It is worth to mention that the decomposition (1.5) and (1.6) holds for general shaped inclusions. In [2, 13], it was obtained the gradient blow-up term of u in terms of the solution to (1.4) corresponding to the disks osculating to B_j 's when B_j 's are of convex shape.

For spherical perfect conductors in \mathbb{R}^3 , h has been expressed as the electric potential generated by a sequence of point charges located at multiply reflected points with respect to the two spheres and, based on this expansion, upper and lower bounds of $|\nabla u|$ were obtained [20]. It was further investigated in [15] to derive an asymptotic formula for the same radius $r_1 = r_2 = r$: for $\mathbf{x} =$ (x_1, x_2, x_3) outside the two spheres, the solution u satisfies

$$\nabla u(\mathbf{x}) = \frac{C_H^{\epsilon}}{\pi |\ln \epsilon| (\epsilon + rx_1^2 + rx_2^2)} (\mathbf{e}_3 + \eta(\mathbf{x})) + \nabla g \quad \text{if } |(x_1, x_2)| \le \frac{r}{|\ln \epsilon|^2}, \tag{1.7}$$

where $\mathbf{e}_3 = (0, 0, 1)$, $\|\nabla g\|_{\infty}$ is bounded regardless of ϵ and $|\eta(\mathbf{x})| = O(|\ln \epsilon|^{-1})$, and the concentration factor C_H^{ϵ} satisfies

$$C_H^{\epsilon} = 2\pi \sum_{n=1}^{\infty} \frac{r}{n} \left(H\left(0, 0, \frac{r}{n}\right) - H\left(0, 0, -\frac{r}{n}\right) \right) + O(\sqrt{\epsilon} |\ln \epsilon|). \tag{1.8}$$

While the equation (1.7) provides an asymptotic of ∇u , the blow-up phenomenon of the electric field requires further investigation in view of the fact that, firstly, the unidentified function $\eta(\mathbf{x})$ and the remainder term in (1.8) can also cause the blow-up and, secondly, the valid region for (1.7) degenerates to a point as ϵ tends to 0. It is worth to remark that the formula (1.7) is slightly modified from that in [15] to be valid for any positive number r, not just 1.

In this paper, we derive an asymptotic of ∇u which completely characterizes the blow-up of the electric field due to the presence of the two nearly touching metallic spheres accepting different radii. The main results are as follows:

(i) We show that the remainder term in Eq. (1.8) (and in the modified equation which is valid for spheres of different radii) is actually $O(\epsilon | \ln \epsilon |)$. As an immediate consequence, we can replace C_H^{ϵ} by its limit as ϵ tends to zero, say C_H , in the decomposition (1.5) of u and in the asymptotic of ∇u . Furthermore, we calculate the series summation of index n in Eq. (1.8)

and completely rewrite it as the summation in terms of homogeneous polynomial order of the external field H. This reformulated expression of C_H gives directly the necessary and sufficient condition for the gradient blow-up occurrence in terms of the two spheres radii, the distance ϵ between two spheres, and the external field H, see Theorem 2.1 and Corollary 2.3.

- (ii) We provide the asymptotic formula of ∇u which is valid in the whole exterior region of the two spheres. The blow-up term is expressed explicitly in terms of elementary functions with coefficients depending on the spheres radii, the distance between the spheres, and the external field, see Theorem 2.2.
- (iii) We identify the location, size and shape of the region where the gradient blow-up occurs. The dimension of the blow-up region turns out to be of order $|\ln \epsilon|^{-1/2}$ which shrinks to a point as ϵ tends to zero. Moreover, this order $|\ln \epsilon|^{-1/2}$ is shown to be optimal. In other words, we prove and characterize the occurrence of superfocusing of the electric field, see Theorem 2.4.

The main ingredients in this paper are the bispherical coordinate system and the Euler-Maclaurin formula. In bispherical coordinates, the exact solution h of the problem (1.4) can be obtained by the method of separation of variables. By some manipulations, such as changing the order of summations, on the exact series solution, we obtain a new series of Riemann sum type. We then apply the Euler-Maclaurin formula to approximate this new series by an integral. Based on the integral expression, we investigate the blow-up feature of the electric field. This approach originally comes from the previous work by the authors [21]. There, it was derived the asymptotics of u when two circular cylinders with finite conductivities in \mathbb{R}^2 are closely located. It is worth to mention that the asymptotic of the potential difference between two spheres has been derived when an uniform external field is applied [17].

From our analysis, it turns out that the potential u can be approximated by the integral of a piecewise continuous image charge distribution. This charge distribution is of similar form to that obtained in [27] up to a multiplicative constant. There, the image charge distribution was assumed to be continuous motivated by the physical intuition and was determined to satisfy several functional equations derived from boundary conditions on the spheres and additional physical assumptions. In this paper, however, the continuous image charge distribution is derived rigorously without any physical assumptions. We emphasize that, thanks to the analysis with mathematical rigor, we are able not only to approximate the image charge distribution but also to go further to extract the blow-up term of the electric field.

The paper is organized as follows. In section 2, we state the main results. Section 3 is to review the definition and properties of the bispherical coordinate system. We provide two different series expansion for h by the bispherical coordinates and estimate the concentration factor C_H^{ϵ} in section 4. The asymptotic formula for ∇u is derived in section 5. Section 6 provides two applications of the Euler-Maclaurin formula, and we illustrate the main results with numerical calculation in section 7. The conclusion is provided in section 8.

2 Main Results

In this section we fix some notations and state the main results.

We first denote

$$\tilde{r} = \frac{r_1 r_2}{r_1 + r_2}, \quad \tilde{r}_j = \frac{r_j}{r_1 + r_2}, \ j = 1, 2,$$
(2.1)

and

$$\begin{cases}
\mu_{\epsilon} = \frac{1}{2\pi\tilde{r}} \left[|\ln \epsilon| + \ln \tilde{r} + \ln 2 - 2 \frac{\psi_{0}(\tilde{r}_{1})\psi_{0}(\tilde{r}_{2}) - \gamma^{2}}{\psi_{0}(\tilde{r}_{1}) + \psi_{0}(\tilde{r}_{2}) + 2\gamma} \right]^{-1}, \\
\mu_{j} = \frac{\psi_{0}(\tilde{r}_{j}) + \gamma}{\psi_{0}(\tilde{r}_{1}) + \psi_{0}(\tilde{r}_{2}) + 2\gamma}, \quad j = 1, 2,
\end{cases}$$
(2.2)

where ψ_0 is the digamma function and γ is the Euler's constant. We then define for $k \in \mathbb{N}$ that

$$Q_k(r_1, r_2) := 4\pi \tilde{r}^{k+1} \left[\left(\mu_1 + (-1)^{k+1} \mu_2 \right) \zeta(k+1) + \frac{\mu_1 \psi_k(\tilde{r}_2) + (-1)^{k+1} \mu_2 \psi_k(\tilde{r}_1)}{k!} \right], \qquad (2.3)$$

where ψ_k is the polygamma function of order k and ζ is the Riemann zeta function, i.e., $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ for $z \in \mathbb{C}$, $\Re(z) > 1$. Here the symbol $\Re(\cdot)$ means the real part of complex number. The polygamma function is defined as the derivatives of the Gamma function Γ , i.e.,

$$\psi_k(z) = \frac{d^{k+1}}{dz^{k+1}} \ln \Gamma(z), \quad z \in \mathbb{C}, \ k = 0, 1, \dots$$

For $z \neq 0, -1, -2, \ldots$, the polygamma function can be expressed as (see [1])

$$\psi_k(z) = \begin{cases} (-1)^{k+1} k! \sum_{m=0}^{\infty} (m+z)^{-k-1} & \text{for } k \ge 1\\ -\gamma + \sum_{k=0}^{\infty} \frac{z-1}{(k+1)(k+z)} & \text{for } k = 0. \end{cases}$$
(2.4)

The first main result is the limiting behavior of C_H^{ϵ} . We give the proof of Theorem 2.1 in section 4.3.

Theorem 2.1. (a) The concentration factor C_H^{ϵ} satisfies

$$C_H^{\epsilon} = C_H + O(\epsilon |\ln \epsilon|)$$
 as ϵ tends to zero, (2.5)

where

$$C_{H} = 4\pi\mu_{1} \sum_{m=0}^{\infty} \left[\frac{\tilde{r}}{m+1} H\left(0,0,\frac{\tilde{r}}{m+1}\right) - \frac{\tilde{r}}{m+\tilde{r}_{2}} H\left(0,0,\frac{-\tilde{r}}{m+\tilde{r}_{2}}\right) \right] + 4\pi\mu_{2} \sum_{m=0}^{\infty} \left[\frac{\tilde{r}}{m+\tilde{r}_{1}} H\left(0,0,\frac{\tilde{r}}{m+\tilde{r}_{1}}\right) - \frac{\tilde{r}}{m+1} H\left(0,0,\frac{-\tilde{r}}{m+1}\right) \right].$$
 (2.6)

(b) The constant C_H can be rewritten as follows:

$$C_H = \sum_{k=1}^{\infty} b_{H,k} \mathcal{Q}_k(r_1, r_2),$$
 (2.7)

where $Q_k(r_1, r_2)$'s are the constants defined in (2.3) and $b_{H,k}$'s are the Taylor coefficients of $g(t) := H(t\mathbf{e}_3) - H(\mathbf{0})$, i.e., $H(t\mathbf{e}_3) = H(\mathbf{0}) + b_{H,1}t + b_{H,2}t^2 + \cdots$, $t \in \mathbb{R}$.

Note that we have $\tilde{r} = r/2$ and $\mu_1 = \mu_2 = \tilde{r}_1 = \tilde{r}_2 = 1/2$ if $r_1 = r_2 = r$. Hence Eq. (2.6) coincides the series term in (1.8) for the case of two spheres of the same radius.

The second main result is the asymptotic formula for ∇u , which shows the blow-up term explicitly in terms of elementary functions. We give the proof of Theorem 2.2 in section 5.4.

Theorem 2.2. The solution u to (1.1) admits the following decomposition in $\mathbb{R}^3 \setminus (\overline{B_1 \cup B_2})$:

$$\nabla u(\mathbf{x}) = C_H \psi(\mathbf{x}) \left(\frac{\mathbf{x} - \mathbf{p}_1}{|\mathbf{x} - \mathbf{p}_1|^2} - \frac{\mathbf{x} - \mathbf{p}_2}{|\mathbf{x} - \mathbf{p}_2|^2} \right) + \nabla H(\mathbf{x}) + r(\mathbf{x}),$$

where

$$\psi(\mathbf{x}) = \frac{\mu_{\epsilon}\tilde{r}}{2a_{\epsilon}} \left(\frac{\mu_{1}r_{1}}{|\mathbf{x} - \mathbf{c}_{1}|} + \frac{\mu_{2}\tilde{r}}{|\mathbf{x} - R_{1}(\mathbf{c}_{2})|} + \frac{\mu_{2}r_{2}}{|\mathbf{x} - \mathbf{c}_{2}|} + \frac{\mu_{1}\tilde{r}}{|\mathbf{x} - R_{2}(\mathbf{c}_{1})|} \right), \tag{2.8}$$

and $||r||_{\infty}$ is bounded regardless of ϵ .

Corollary 2.3. Let u be the solution to (1.1). Then

 $|\nabla u|$ blows up as ϵ tends to zero if and only if $C_H \neq 0$.

In particular, we have the followings:

- (a) If g = 0, then $|\nabla u|$ does not blow up.
- (b) If $q(t) = t^{2m-1}$ for some $m \in \mathbb{N}$, then $|\nabla u|$ blows up.
- (c) If $r_1 = r_2 = r$ and g is a polynomial of even degree, then $|\nabla u|$ does not blow up.

Proof. Since C_H is independent of ϵ , Theorem 2.2 asserts the equivalent condition for the gradient blow-up for u. Hence, we have (a). From (2.4), we have $\psi_0(\tilde{r}_j) + \gamma < 0$ for j = 1, 2. Hence μ_1 and μ_2 are positive. We can similarly show that $\mathcal{Q}_{2m-1} > 0$ for all $m \in \mathbb{N}$, so that it follows (b). If we assume $r_1 = r_2 = r$, then $\mathcal{Q}_{2m}(r,r) = 0$. This proves (c).

For example, if $H(\mathbf{x}) = b_2 p_2(\mathbf{x}) + b_3 p_3(\mathbf{x}) + b_4 p_4(\mathbf{x})$ with $p_2(\mathbf{x}) = x_3^2 - x_1^2$, $p_3(\mathbf{x}) = x_3^3 - 3x_3x_1^2$, and $p_4(\mathbf{x}) = x_3^4 - 6x_1^2x_3^2 + x_1^4$, then the corresponding electric field blows up if and only if $C_H = b_2 \mathcal{Q}_2 + b_3 \mathcal{Q}_3 + b_4 \mathcal{Q}_4 \neq 0$. Table 1 shows $\mathcal{Q}_k(r_1, r_2)$ for $r_1 = 1$ and various r_2 values.

k	1	2	3	4	5	6
$r_2 = 1.0$	20.6709	0	13.6009	0	12.7843	0
$r_2 = 0.7$	13.8369	-1.7996	6.7967	-3.0177	5.3858	-3.6830
$r_2 = 0.3$	3.9472	-1.1121	1.4317	-1.2751	1.3062	-1.2938
$r_2 = 0.1$	0.5497	-0.1795	0.1851	-0.1828	0.1829	-0.1829

Table 1: values of the coefficients $Q_k(r_1, r_2)$ when $r_1 = 1$

2.1 Superfocusing of the electric field

Let us denote $\theta_{\epsilon} = \sqrt{\epsilon |\ln \epsilon|}$ and

$$\Omega_{\epsilon}^* = \left\{ \mathbf{x} \in \mathbb{R}^3 : \left(|(x_1, x_2)| - d_{\epsilon}^* \right)^2 + x_3^2 < \left(r_{\epsilon}^* \right)^2 \right\} \quad \text{with } d_{\epsilon}^* = \frac{a_{\epsilon}}{\sin \theta_{\epsilon}}, \ r_{\epsilon}^* = a_{\epsilon} \cot \theta_{\epsilon}, \tag{2.9}$$

which is the rotation of the shaded region in Fig. 2.1 about the x_3 -axis and can be written as $\Omega_{\epsilon}^* = \{\theta_{\epsilon} < \theta \leq \pi\}$ by the bispherical coordinates explained in the next section (see (3.3)). For small ϵ , we have

$$d_{\epsilon}^*, \ r_{\epsilon}^* \approx \sqrt{2\tilde{r}} \ |\ln \epsilon|^{-\frac{1}{2}},$$

so that the width, length and height of Ω_{ϵ}^* is of order $|\ln \epsilon|^{-1/2}$. This implies the convergence of the region Ω_{ϵ}^* to the touching point (0,0,0) as ϵ goes to 0.

We show in the following theorem that the gradient blow-up occurs only in Ω_{ϵ}^* . In other words, we have superfocusing of the electric field confined in the narrow gap region between the two nearly touching metallic spheres. Moreover, the superfocusing region Ω_{ϵ}^* is optimal in the sense that the order of its size cannot be smaller than $|\ln \epsilon|^{-1/2}$.

Theorem 2.4. To highlight the dependence on ϵ , let us denote the solution to (1.1) by u_{ϵ} . Then we have the followings.

(a) The gradient blow-up occurs only in the region Ω_{ϵ}^* . More precisely, there exists a constant C independent of ϵ satisfying

$$|\nabla u_{\epsilon}(\mathbf{x})| \leq C \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_1 \cup B_2 \cup \Omega_{\epsilon}^*}.$$

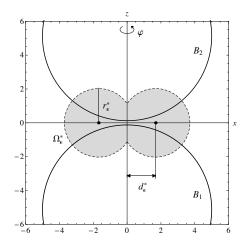


Figure 2.1: Superfocusing of the electric field: the gradient blow-up occurs only in the region Ω_{ϵ}^* (shaded in the figure) whose dimension is of order $|\ln \epsilon|^{-1/2}$.

(b) The decaying order of θ_{ϵ} is optimal in the following sense: if $C_H \neq 0$ and $\tilde{\theta}_{\epsilon}$ satisfies $\lim_{\epsilon \to 0} \tilde{\theta}_{\epsilon} = 0$ and $\lim_{\epsilon \to 0} (\tilde{\theta}_{\epsilon}/\theta_{\epsilon}) = \infty$, then we have

$$\inf_{x \in \tilde{\Omega}_{\epsilon} \setminus \overline{B_1 \cup B_2}} |\nabla u_{\epsilon}(x)| \to \infty \quad as \ \epsilon \to 0,$$

where $\tilde{\Omega}_{\epsilon}$ is defined as (2.9) with $\tilde{\theta}_{\epsilon}$ in the place of θ_{ϵ} .

We will prove the theorem in section 5.4.

3 Bispherical coordinate system

Let us introduce the bispherical coordinate system $(\xi, \theta, \varphi) \in \mathbb{R} \times [0, \pi] \times [0, 2\pi)$ with poles located at \mathbf{p}_1 and \mathbf{p}_2 . Each $\mathbf{x} = (x_1, x_2, x_3)$ in the Cartesian coordinate system of \mathbb{R}^3 corresponds to (ξ, θ, φ) through

$$e^{\xi - i\theta} = \frac{z + a_{\epsilon}}{z - a_{\epsilon}}$$
 with $z = x_3 + i |(x_1, x_2)|$ (3.1)

with φ the angle of rotation about the x_3 -axis. One can rewrite the Cartesian coordinates in terms of the bispherical coordinates as

$$x_1 = a_\epsilon \frac{\sin \theta \cos \varphi}{\cosh \xi - \cos \theta}, \quad x_2 = a_\epsilon \frac{\sin \theta \sin \varphi}{\cosh \xi - \cos \theta}, \quad x_3 = a_\epsilon \frac{\sinh \xi}{\cosh \xi - \cos \theta}.$$

It can be easily shown that the coordinate surfaces $\{\xi = c\}$ and $\{\theta = c\}$ for a nonzero c are respectively the zero level set of

$$f^{\xi}(x_1, x_2, x_3) = (x_3 - a_{\epsilon} \coth c)^2 + |(x_1, x_2)|^2 - \left(\frac{a_{\epsilon}}{\sinh c}\right)^2, \tag{3.2}$$

$$f^{\theta}(x_1, x_2, x_3) = \left(|(x_1, x_2)| - a_{\epsilon} \cot c \right)^2 + x_3^2 - \left(\frac{a_{\epsilon}}{\sin c} \right)^2.$$
 (3.3)

We illustrate the coordinate surfaces of the bispherical coordinate in Fig. 3.1.

Note that \mathbf{p}_1 and \mathbf{p}_2 are contained in $\{x_2 = 0\}$ and they are again fixed points of the combined reflections w.r.t. the two circles $\partial B_j \cap \{x_2 = 0\}$, j = 1, 2. Remind the circle of Apollonius in two dimensions: for disk $B_r(\mathbf{c})$ and $\mathbf{y} \notin \overline{B_r(\mathbf{c})}$, the circle $\partial B_r(\mathbf{c})$ is the locus of \mathbf{x} satisfying

$$\frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - R(\mathbf{y})|} = \frac{|\mathbf{y} - \mathbf{c}|}{r},$$

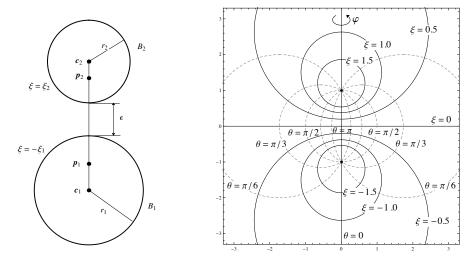


Figure 3.1: A pair of spherical perfect conductors (the left) and the coordinate level curves for the bipherical coordinate system with $a_{\epsilon} = 1$ (the right).

where R is the reflection w.r.t. the circle $\partial B_r(\mathbf{c})$. Applying this property to $\partial B_j \cap \{x_2 = 0\}$, we have that $|\mathbf{x} - \mathbf{p}_1|/|\mathbf{x} - \mathbf{p}_2|$ is constant on $\partial B_j \cap \{x_2 = 0\}$, j = 1, 2. From (3.1) and the fact

$$\left| \frac{x_3 + i|x_1| + a_{\epsilon}}{x_3 + i|x_1| - a_{\epsilon}} \right| = \frac{|\mathbf{x} - \mathbf{p}_1|}{|\mathbf{x} - \mathbf{p}_2|},$$

 ξ is constant on $\partial B_j \cap \{x_2 = 0\}$, and hence so does on ∂B_j . Furthermore, applying Eq. (3.2), we obtain for j = 1, 2 that

$$\partial B_j = \left\{ \xi = (-1)^j \xi_j \right\} \quad \text{and} \quad \mathbf{c}_j = (-1)^j a_\epsilon \coth \xi_j \mathbf{e}_3$$
 (3.4)

with two positive constants ξ_1 and ξ_2 given by

$$\xi_j = \sinh^{-1}\left(a_{\epsilon}r_i^{-1}\right) = \alpha_{\epsilon}r_i^{-1} + O(\epsilon\sqrt{\epsilon}). \tag{3.5}$$

Eq. (1.3) and Eq. (3.5) imply

$$\frac{a_{\epsilon}}{\xi_1 + \xi_2} = \tilde{r} + O(\epsilon) \tag{3.6}$$

and

$$\begin{cases}
-\xi_1 \le \xi \le \xi_2 & \text{in } \mathbb{R}^3 \setminus \overline{(B_1 \cup B_2)} \\
\xi \in (-\infty, -\xi_1) & \text{in } B_1 \\
\xi \in (\xi_2, \infty) & \text{in } B_2.
\end{cases}$$
(3.7)

Let us consider the multiple reflections of the spheres centers \mathbf{c}_1 and \mathbf{c}_2 w.r.t. the two spheres. From the definition of the reflection R_1 and (3.4), it follows

$$\mathbf{e}_{3} \cdot R_{1}(\mathbf{c}_{2}) = -a_{\epsilon} \coth \xi_{1} + \frac{r_{1}^{2}}{|\mathbf{c}_{1} - \mathbf{c}_{2}|}$$

$$= -a_{\epsilon} \frac{\cosh \xi_{1}}{\sinh \xi_{1}} + a_{\epsilon} \frac{1}{\sinh^{2} \xi_{1}} \frac{1}{\frac{\cosh \xi_{1}}{\sinh \xi_{1}} + \frac{\cosh \xi_{2}}{\sinh \xi_{2}}}$$

$$= -a_{\epsilon} \coth(\xi_{1} + \xi_{2}).$$

By the same way for $m = 0, 1, \ldots$, we have

$$\begin{cases}
(R_1 \circ R_2)^m(\mathbf{c}_1) = -\mathbf{p}_m^{\xi_1}, & (R_2 \circ R_1)^m(\mathbf{c}_2) = \mathbf{p}_m^{\xi_2}, \\
(R_2 \circ R_1)^m \circ R_2(\mathbf{c}_1) = -(R_1 \circ R_2)^m \circ R_1(\mathbf{c}_2) = \mathbf{p}_m^{\xi_1 + \xi_2},
\end{cases}$$
(3.8)

where

$$\mathbf{p}_{m}^{c} = a_{\epsilon} \coth(m(\xi_{1} + \xi_{2}) + c)\mathbf{e}_{3} \quad \text{for } c = \xi_{1}, \xi_{2}, \xi_{1} + \xi_{2}.$$
 (3.9)

3.1 Scale factors and harmonic functions

The bispherical coordinate system (ξ, θ, φ) is an orthogonal coordinate system. We denote its orthogonal coordinate directions as $\{\hat{\mathbf{e}}_{\xi}, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\varphi}\}$, *i.e.*,

$$\hat{\mathbf{e}}_{\xi} = \frac{\partial \mathbf{x}/\partial \xi}{|\partial \mathbf{x}/\partial \xi|}, \qquad \hat{\mathbf{e}}_{\theta} = \frac{\partial \mathbf{x}/\partial \theta}{|\partial \mathbf{x}/\partial \theta|}, \qquad \hat{\mathbf{e}}_{\varphi} = \frac{\partial \mathbf{x}/\partial \varphi}{|\partial \mathbf{x}/\partial \varphi|}.$$
 (3.10)

The scale factors for the bispherical coordinates are

$$\sigma_{\xi} = \sigma_{\theta} = \frac{a_{\epsilon}}{\cosh \xi - \cos \theta} \quad \text{and} \quad \sigma_{\varphi} = \frac{a_{\epsilon} \sin \theta}{\cosh \xi - \cos \theta},$$
 (3.11)

so that the gradient for scalar valued function g can be written as

$$\nabla g = \frac{1}{\sigma_{\xi}} \frac{\partial g}{\partial \xi} \hat{\mathbf{e}}_{\xi} + \frac{1}{\sigma_{\theta}} \frac{\partial g}{\partial \theta} \hat{\mathbf{e}}_{\theta} + \frac{1}{\sigma_{\varphi}} \frac{\partial g}{\partial \varphi} \hat{\mathbf{e}}_{\varphi}. \tag{3.12}$$

Here and in the remaining of the paper, the symbol ∇ denotes the gradient in the Cartesian coordinates. It can be also shown

$$\hat{\mathbf{e}}_{\xi}(\mathbf{x}) = \sigma_{\xi}(\mathbf{x})\mathbf{N}(\mathbf{x}) \quad \text{with } \mathbf{N}(\mathbf{x}) = \left(\frac{\mathbf{x} - \mathbf{p}_{1}}{|\mathbf{x} - \mathbf{p}_{1}|^{2}} - \frac{\mathbf{x} - \mathbf{p}_{2}}{|\mathbf{x} - \mathbf{p}_{2}|^{2}}\right).$$
 (3.13)

As one can see in Page 111 of [25], any harmonic function f has a general R-separation

$$f(\xi, \theta, \varphi) = \sqrt{\cosh \xi - \cos \theta} \sum_{n=1}^{+\infty} \sum_{m=0}^{n} \left[D_n^m e^{(n+\frac{1}{2})|\xi|} + E_n^m e^{-(n+\frac{1}{2})|\xi|} \right]$$

$$\times P_n^m(\cos \theta) \left[F_n^m \cos(m\varphi) + G_n^m \sin(m\varphi) \right], \tag{3.14}$$

where P_n^m 's are the Legendre associated functions and D_n^m , E_n^m , F_n^m , and G_n^m are constants. It is well known that the generating function for the Legendre polynomials $P_n(x)$'s, which are $P_n^0(x)$'s, is given by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{for } |x| \le 1, |t| < 1,$$

and they form an orthogonal basis of $L^2[0,1]$. From the equation above, the constant function 1 can be expressed as

$$1 = \sqrt{2}\sqrt{\cosh\xi - \cos\theta} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\xi|} P_n(\cos\theta).$$
 (3.15)

We also have the following two identities for $\xi \in \mathbb{R}$ [25]:

$$\int_{-1}^{1} \frac{P_n(s)}{(\cosh \xi - s)^{\frac{1}{2}}} ds = \frac{2\sqrt{2}}{2n+1} e^{-(n+\frac{1}{2})|\xi|},$$
(3.16)

$$\int_{-1}^{1} \frac{P_n(s)}{(\cosh \xi - s)^{\frac{3}{2}}} ds = \frac{2\sqrt{2}}{\sinh |\xi|} e^{-(n + \frac{1}{2})|\xi|}.$$
 (3.17)

4 The singular function h

In this section, we give the two series expansions for the solution h to (1.4) by the bispherical coordinates and give the proof of Theorem 2.1.

4.1 Solution by separation of variables

We set

$$C_j = \frac{(-1)^j}{8\pi a_{\epsilon}} \frac{U(\xi_j) - U(0)}{U(\xi_1)U(\xi_2) - U^2(0)}, \quad j = 1, 2,$$
(4.1)

where

$$U(c) = \sum_{n=0}^{\infty} \frac{e^{(2n+1)c}}{e^{(2n+1)(\xi_1 + \xi_2)} - 1} \quad \text{for } 0 < c < \xi_1 + \xi_2.$$
 (4.2)

In the following lemma we express h in the form of (3.14) with the coefficients defined using C_j 's, where C_j 's are actually potential values of h on ∂B_j 's. Note that h is independent of φ due to its symmetry under the rotation about x_3 -axis. We omit the variable φ in h for notational simplicity.

Lemma 4.1. The solution h to (1.4) can be represented as

$$h(\xi, \theta) = \sqrt{2}\sqrt{\cosh \xi - \cos \theta} \sum_{n=0}^{\infty} \left(A_n e^{(n+\frac{1}{2})\xi} + B_n e^{-(n+\frac{1}{2})\xi} \right) P_n(\cos \theta), \quad -\xi_1 \le \xi \le \xi_2, \quad (4.3)$$

where

$$A_n = \frac{C_2 e^{(2n+1)\xi_1} - C_1}{e^{(2n+1)(\xi_1 + \xi_2)} - 1} \quad and \quad B_n = \frac{C_1 e^{(2n+1)\xi_2} - C_2}{e^{(2n+1)(\xi_1 + \xi_2)} - 1}.$$

Moreover, h satisfies

$$h|_{\partial B_i} = C_j, \quad j = 1, 2.$$
 (4.4)

Proof. Let us denote the right-hand side of (4.3) as \tilde{h} . In the following, we prove \tilde{h} satisfies all the constraints in (1.4).

One can easily show that all the terms in the series expansion of \tilde{h} are harmonic, see (3.14), and they are exponentially decay (uniformly for $\xi \in [-\xi_1, \xi_2]$) in n. Hence, \tilde{h} is harmonic. From (3.15), \tilde{h} is constant on ∂B_1 and ∂B_2 . More precisely,

$$\tilde{h}(-\xi_1, \theta) = C_1$$
 and $\tilde{h}(\xi_2, \theta) = C_2$.

It can be easily shown that the outward unit normal vector ν to ∂B_i (= $\{\xi = (-1)^i \xi_i\}$) is

$$\nu = (-1)^{j+1} \hat{\mathbf{e}}_{\xi} \tag{4.5}$$

and a sufficiently smooth function v satisfies

$$\int_{\partial B_j} \partial_{\nu} v \ d\sigma = (-1)^{j+1} \int_0^{2\pi} \int_0^{\pi} \left(\frac{\partial v}{\partial \xi} \bigg|_{\xi = (-1)^j \xi_j} \sigma_{\varphi}(\xi_j, \theta, \varphi) \right) d\theta d\varphi. \tag{4.6}$$

In particular, using (3.16), we derive for j = 1, 2 that

$$\sqrt{2} \int_{\partial B_j} \partial_{\nu} \left(\sqrt{\cosh \xi - \cos \theta} \ e^{(n + \frac{1}{2})\xi} P_n(\cos \theta) \right) d\sigma = -8\pi a_{\epsilon} \delta_{2j},$$

$$\sqrt{2} \int_{\partial B_j} \partial_{\nu} \left(\sqrt{\cosh \xi - \cos \theta} \ e^{-(n + \frac{1}{2})\xi} P_n(\cos \theta) \right) d\sigma = -8\pi a_{\epsilon} \delta_{1j},$$

where $\delta_{i,j}$ is 1 if i=j and zero otherwise. Hence we have

$$\int_{\partial B_j} \partial_\nu \tilde{h} \ d\sigma = -8\pi a_\epsilon \left(\sum A_n\right) \delta_{2j} - 8\pi a_\epsilon \left(\sum B_n\right) \delta_{1j} = (-1)^{j+1} \quad \text{for } j = 1, 2.$$

Now it only remains to show the decay property at infinity for \tilde{h} . In fact, it is enough to show that $\tilde{h}(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$ because the total flux on $\partial B_1 \cup \partial B_2$ is zero. Note that the radial distance $|\mathbf{x}|$ satisfies

$$|\mathbf{x}| = a_{\epsilon} \sqrt{\frac{\cosh \xi + \cos \theta}{\cosh \xi - \cos \theta}},\tag{4.7}$$

so that $|\mathbf{x}| \to \infty$ if and only if $(\xi, \theta) \to (0, 0)$. Hence we only need to show

$$\limsup_{(\xi,\theta)\to(0,0)} \frac{|\tilde{h}(\xi,\theta)|}{\sqrt{\cosh\xi - \cos\theta}} \le C,$$

for some constant C independent of ξ and θ . Owing to

$$\frac{|\tilde{h}(\xi,\theta)|}{\sqrt{\cosh\xi - \cos\theta}} \le \sum_{m=0}^{\infty} \left(|A_n| e^{(n+\frac{1}{2})\xi_2} + |B_n| e^{(n+\frac{1}{2})\xi_1} \right) < \infty \quad \text{for all } -\xi_1 \le \xi \le \xi_2,$$

the decay condition follows. This completes the proof.

The following asymptotic of U has been derived by J. Lekner in [17].

Lemma 4.2. ([17]) For small $\epsilon > 0$, the function U defined in (4.2) satisfies

$$U(\xi_j) = \frac{1}{2(\xi_1 + \xi_2)} \left[\ln \left(\frac{2}{\xi_1 + \xi_2} \right) - \psi_0 \left(1 - \frac{\xi_j}{\xi_1 + \xi_2} \right) \right] + O(\sqrt{\epsilon}),$$

and

$$U(0) = \frac{1}{2(\xi_1 + \xi_2)} \left[\ln \left(\frac{2}{\xi_1 + \xi_2} \right) + \gamma \right] + O(\sqrt{\epsilon}).$$

Corollary 4.3. We have

$$C_j = (-1)^j \mu_{\epsilon} \mu_j + O(\epsilon), \quad j = 1, 2,$$
 (4.8)

with μ_{ϵ} , μ_{i} 's given in (2.2).

4.2 Expansion by Potentials of point charges

The fundamental solution Γ to the Laplacian in three dimensions is given by

$$\Gamma(\mathbf{x}) = -\frac{1}{4\pi |\mathbf{x}|}.$$

We can rewrite Γ by the bispherical coordinates as a fraction of w_{θ} which is defined as

$$w_{\theta}(\xi) := \sqrt{\cosh \xi - \cos \theta}. \tag{4.9}$$

Lemma 4.4. Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ be a point with the bispherical coordinate (ξ, θ, φ) . Then for $\xi \in \mathbb{R}$ we have

$$\frac{w_{\theta}(\xi)}{w_{\theta}(\xi - 2\xi_0)} = -\frac{4\pi a_{\epsilon}}{\sinh|\xi_0|} \Gamma(\mathbf{x} - \mathbf{x}_0) \quad \text{with } \mathbf{x}_0 = a_{\epsilon} \coth \xi_0 \mathbf{e}_3.$$

Proof. We have from (3.1) that

$$|x_3 + i|(x_2, x_3)| = z = \frac{2a_{\epsilon}}{e^{\xi - i\theta} - 1} + a_{\epsilon}.$$

Note that

$$\coth \xi_0 = \frac{\sinh 2\xi_0}{\cosh 2\xi_0 - 1} = \frac{2}{e^{2\xi_0} - 1} + 1 \quad \text{and} \quad |e^{\xi - i\theta} - 1| = \sqrt{2e^{\xi}(\cosh \xi - \cos \theta)}.$$

Hence it follows

$$|\mathbf{x} - \mathbf{x}_0| = \left| (x_1, x_2, x_3 - a_\epsilon \coth \xi_0) \right| = \left| x_3 + i |(x_2, x_2)| - a_\epsilon \coth(\xi_0) \right|$$

$$= 2a_\epsilon \left| \frac{1}{e^{\xi - i\theta} - 1} - \frac{1}{e^{2\xi_0} - 1} \right| = 2a_\epsilon \left| \frac{e^{2\xi_0} (e^{\xi - 2\xi_0 - i\theta} - 1)}{(e^{2\xi_0} - 1)(e^{\xi - i\theta} - 1)} \right| = \frac{a_\epsilon}{\sinh |\xi_0|} \frac{w_\theta(\xi - 2\xi_0)}{w_\theta(\xi)}.$$

Thus we prove the lemma.

Let us denote S for $(\xi, \theta, s) \in \mathbb{R} \times [0, \pi] \times \mathbb{R}$ as

$$S(\xi, \theta; s) := \sum_{m=0}^{\infty} \frac{w_{\theta}(\xi)}{w_{\theta}(\xi - 2m(\xi_1 + \xi_2) - s)}.$$
 (4.10)

Because w_{θ} is an even function for ξ and

$$\frac{w_{\theta}(\xi)}{w_{\theta}(\xi \pm 2\xi_0)} = -\frac{4\pi a_{\epsilon}}{\sinh|\xi_0|} \Gamma(\mathbf{x} \pm \mathbf{x}_0), \quad \mathbf{x}_0 = a_{\epsilon} \coth \xi_0 \mathbf{e}_3,$$

it follows

$$S(\pm \xi, \theta; 2c) = -\sum_{m=0}^{\infty} \frac{4\pi a_{\epsilon}}{\sinh |\xi_m^c|} \Gamma\left(\mathbf{x} \mp a_{\epsilon} \coth(\xi_m^c) \mathbf{e}_3\right), \quad \xi_m^c = m(\xi_1 + \xi_2) + c.$$
 (4.11)

We can express the solution h to (1.4) as a linear combination of S.

Lemma 4.5. We have

$$h(\mathbf{x}) = C_2 S(\xi, \theta; 2\xi_2) - C_1 S(\xi, \theta; 2(\xi_1 + \xi_2)) + C_1 S(-\xi, \theta; 2\xi_1) - C_2 S(-\xi, \theta; 2(\xi_1 + \xi_2)).$$

Proof. Since $\xi_1, \xi_2 > 0$, we have

$$\frac{1}{e^{(2n+1)(\xi_1+\xi_2)}-1} = \sum_{m=0}^{\infty} e^{-(m+1)(2n+1)(\xi_1+\xi_2)} = \sum_{m=0}^{\infty} e^{-(n+\frac{1}{2})[2m(\xi_1+\xi_2)+2(\xi_1+\xi_2)]}.$$

Applying the above identity and interchanging the order of summation which is possible due to the absolute convergence of the series, Eq. (4.3) becomes

$$h(\mathbf{x}) = \sqrt{2}\sqrt{\cosh\xi - \cos\theta}$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[C_2 e^{-(n+\frac{1}{2})\xi_{m,1}} - C_1 e^{-(n+\frac{1}{2})\xi_{m,2}} + C_1 e^{-(n+\frac{1}{2})\xi_{m,3}} - C_2 e^{-(n+\frac{1}{2})\xi_{m,4}} \right] P_n(\cos\theta)$$

with

$$\begin{split} \xi_{m,1} &= -\xi + 2m(\xi_1 + \xi_2) + 2\xi_2, \quad \xi_{m,2} = -\xi + 2m(\xi_1 + \xi_2) + 2(\xi_1 + \xi_2), \\ \xi_{m,3} &= \xi + 2m(\xi_1 + \xi_2) + 2\xi_1, \quad \xi_{m,4} = \xi + 2m(\xi_1 + \xi_2) + 2(\xi_1 + \xi_2). \end{split}$$

Thanks to (3.15), we obtain

$$\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\xi_{m,j}} P_n(\cos \theta) = \frac{1}{\sqrt{2}} \frac{1}{w_{\theta}(\xi_{m,j})}, \quad j = 1, \dots, 4.$$

This completes the proof.

Corollary 4.6. For c is either ξ_1, ξ_2 or $\xi_1 + \xi_2$, we denote

$$\xi_m^c = m(\xi_1 + \xi_2) + c, \quad q_m^c = \frac{4\pi a_\epsilon}{\sinh \xi_m^c}, \quad \mathbf{p}_m^c = a_\epsilon \coth \xi_m^c \mathbf{e}_3.$$
 (4.12)

Then we can expand h and C_H^{ϵ} as

$$h(\mathbf{x}) = C_1 \sum_{m=0}^{\infty} \left[q_m^{\xi_1 + \xi_2} \Gamma(\mathbf{x} - \mathbf{p}_m^{\xi_1 + \xi_2}) - q_m^{\xi_1} \Gamma(\mathbf{x} + \mathbf{p}_m^{\xi_1}) \right]$$

$$- C_2 \sum_{m=0}^{\infty} \left[q_m^{\xi_2} \Gamma(\mathbf{x} - \mathbf{p}_m^{\xi_2}) - q_m^{\xi_1 + \xi_2} \Gamma(\mathbf{x} + \mathbf{p}_m^{\xi_1 + \xi_2}) \right]$$
(4.13)

and

$$C_{H}^{\epsilon} = \frac{C_{1}}{C_{1} - C_{2}} \sum_{m=0}^{\infty} \left[q_{m}^{\xi_{1} + \xi_{2}} H \left(\mathbf{p}_{m}^{\xi_{1} + \xi_{2}} \right) - q_{m}^{\xi_{1}} H \left(-\mathbf{p}_{m}^{\xi_{1}} \right) \right] - \frac{C_{2}}{C_{1} - C_{2}} \sum_{m=0}^{\infty} \left[q_{m}^{\xi_{2}} H \left(\mathbf{p}_{m}^{\xi_{2}} \right) - q_{m}^{\xi_{1} + \xi_{2}} H \left(-\mathbf{p}_{m}^{\xi_{1} + \xi_{2}} \right) \right].$$

$$(4.14)$$

Proof. From (4.11) and Lemma 4.5, we prove (4.13). Remind that (4.4) implies

$$h|_{\partial B_1} - h|_{\partial B_2} = C_1 - C_2.$$

One can easily show (4.14) by computing $(u|_{\partial B_1} - u|_{\partial B_2})$ from (1.6) and (4.13).

Note that \mathbf{p}_m^c 's are multiply reflected points of \mathbf{c}_1 and \mathbf{c}_2 with respect to the two spheres, see (3.8). Corollary 4.6 has the same formality as Lemma 4.1 in [20], where a recursively defined series was used in the place of q_m^c . In this paper, we are able to have formulas much simpler than those in [20] thanks to adopting the bispherical coordinate system.

4.3 Proof of Theorem 2.1

Lemma 4.7. Let \mathbf{p}_m^c , q_m^c , ξ_m^c be given as in Corollary 4.6 and H be an entire harmonic function. Then there is a constant C independent of ϵ satisfying

$$\left| \sum_{m=0}^{\infty} q_m^c H(\mathbf{p}_m^c) - \sum_{m=0}^{\infty} \frac{4\pi a_{\epsilon}}{\xi_m^c} H(0, 0, \frac{a_{\epsilon}}{\xi_m^c}) \right| \le C\epsilon |\ln \epsilon|.$$

Proof. Let c be fixed to be either ξ_1 , ξ_2 or $\xi_1 + \xi_2$. We can assume $H(\mathbf{0}) = 0$ since the constant term of H does not change the gradient of the potential function. Then H can be written as

$$H(\mathbf{x}) = \sum_{k=1}^{\infty} H_k(\mathbf{x}),$$

where H_k 's are homogeneous polynomials in \mathbf{x} of degree k. Especially, we have

$$H(x_k \mathbf{e}_3) = \sum_{k=1}^{\infty} H_k(x_3 \mathbf{e}_3) = \sum_{k=1}^{\infty} b_{H,k} x_3^k \quad \text{with } b_{H,k} = \frac{1}{k!} \frac{\partial^k H}{\partial x_3^k}(\mathbf{0}).$$

From (4.12), one obtains

$$q_m^c H_k(\mathbf{p}_m^c) = 4\pi b_{H,k} a_{\epsilon}^{k+1} \frac{\cosh^k (s_0 m + c)}{\sinh^{k+1} (s_0 m + c)}, \quad s_0 = \xi_1 + \xi_2.$$

We denote

$$G_k = 4\pi b_{H,k} a_{\epsilon}^{k+1} \sum_{m=0}^{\infty} \frac{1}{(s_0 m + c)^{k+1}},$$

$$f_k(x) = \frac{\cosh^k x}{\sinh^{k+1} x} - \frac{1}{x^{k+1}}, \qquad x > 0, \ k \ge 1.$$
(4.15)

Then it follows

$$R_k := \sum_{m=0}^{\infty} q_m^c H_k(\mathbf{p}_m^c) - G_k = \frac{b_{H,k} 4\pi a_{\epsilon}^{k+1}}{s_0} \sum_{m=0}^{\infty} f_k(s_0 m + c) s_0.$$

Thanks to Lemma 6.1 in section 6, one obtains that there is a constant C independent of ϵ and k such that

$$|R_k| \le C\epsilon |\ln \epsilon| |b_{H,k}| 2^k$$
 for all $k \ge 1$.

Note that the series $\sum_{k=1}^{\infty} |b_{H,k}| 2^k$ converges since H is an entire function, so we have

$$\left| \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} q_m^c H_k(\mathbf{p}_m^c) - \sum_{k=1}^{\infty} G_k \right| = \left| \sum_{k=1}^{\infty} R_k \right| \le \tilde{C}\epsilon |\ln \epsilon| \tag{4.16}$$

for a constant \tilde{C} independent of ϵ and k. The first equality in (4.16) holds because of the absolute convergence of two series in the leftmost side. Because of the same reason, we also have

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} q_m^c H_k(\mathbf{p}_m^c) = \sum_{m=0}^{\infty} q_m^c H(\mathbf{p}_m^c),$$

$$\sum_{k=1}^{\infty} G_k = 4\pi \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{H,k} \frac{a_{\epsilon}^{k+1}}{(s_0 m + c)^{k+1}} = \sum_{m=0}^{\infty} \frac{4\pi a_{\epsilon}}{\xi_m^c} H(0, 0, \frac{a_{\epsilon}}{\xi_m^c}).$$

Therefore we prove the theorem thanks to (4.16).

Proof of Theorem 2.1 Thanks to (1.3) and (2.1), one can easily show

$$\frac{a_{\epsilon}}{\xi_m^c} = \begin{cases} \frac{\tilde{r}}{m+1} + O\left(\frac{\epsilon}{m+1}\right) & \text{for } c = \xi_1 + \xi_2\\ \frac{\tilde{r}}{m+\tilde{r}/r_j} + O\left(\frac{\epsilon}{m+1}\right) & \text{for } c = \xi_j \ (j = 1, 2). \end{cases}$$

Applying the mean-value property, we have

$$H\left(0,0,\frac{a_{\epsilon}}{\xi_{m}^{c}}\right) = \begin{cases} H\left(0,0,\frac{\tilde{r}}{m+1}\right) + \frac{1}{m+1}O(\epsilon) & \text{for } c = \xi_{1} + \xi_{2} \\ H\left(0,0,\frac{\tilde{r}}{m+\tilde{r}/r_{j}}\right) + \frac{1}{m+1}O(\epsilon) & \text{for } c = \xi_{j} \ (j=1,2), \end{cases}$$

Using Lemma 4.7, we have

$$\begin{split} &\sum_{m=0}^{\infty} q_m^{\xi_1+\xi_2} H\left(\pm \mathbf{p}_m^{\xi_1+\xi_2}\right) = \sum_{m=0}^{\infty} \frac{4\pi \tilde{r}}{m+1} H\Big(0,0,\pm \frac{\tilde{r}}{m+1}\Big) + O(\epsilon|\ln\epsilon|), \\ &\sum_{m=0}^{\infty} q_m^{\xi_j} H\left(\pm \mathbf{p}_m^{\xi_j}\right) = \sum_{m=0}^{\infty} \frac{4\pi \tilde{r}}{m+\tilde{r}/r_j} H\Big(0,0,\pm \frac{\tilde{r}}{m+\tilde{r}/r_j}\Big) + O(\epsilon|\ln\epsilon|), \quad j=1,2. \end{split}$$

Thanks to (4.14), (4.8) and the fact $\mu_1 + \mu_2 = 1$, we prove (2.5). We can assume $H(\mathbf{0}) = 0$ as in the proof of Lemma 4.7, i.e.,

$$H(0,0,x_3) = \sum_{k=1}^{\infty} b_{H,k} x_3^k.$$

To make the notation simple, let us denote

$$z_m = \frac{\tilde{r}}{m+1}$$
 and $z_{m,j} = \frac{\tilde{r}}{m+\tilde{r}/r_j}$,

then (2.6) becomes

$$C_{H} = 4\pi\mu_{1} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{H,k} \left(z_{m}^{k+1} + (-z_{m,1})^{k+1} \right) + 4\pi\mu_{2} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{H,k} \left(z_{m,2}^{k+1} + (-z_{m})^{k+1} \right)$$

$$= 4\pi \sum_{k=1}^{\infty} b_{H,k} \sum_{m=0}^{\infty} \left[\mu_{1} z_{m}^{k+1} + \mu_{2} (-z_{m})^{k+1} + \mu_{1} (-z_{m,1})^{k+1} + \mu_{2} z_{m,2}^{k+1} \right]. \tag{4.17}$$

We can exchange the order of summation in the above equation because of the absolute convergence of the series. From (2.4), we immediately see that

$$\sum_{m=0}^{\infty} z_m^{k+1} = \tilde{r}^{k+1} \zeta(k+1), \qquad \sum_{m=0}^{\infty} z_{m,j}^{k+1} = \tilde{r}^{k+1} \frac{(-1)^{k+1}}{k!} \psi_k(\tilde{r}/r_j). \tag{4.18}$$

Thus, from (4.17) and (4.18), we prove (b).

5 Asymptotics of u and ∇u

In this section, we first approximate the series S in (4.10) by an integral defined in terms of bispherical coordinates, and then derive asymptotics for h, u, and ∇u in the Cartesian coordinates.

5.1 Approximation of h and ∇h by integrals in bispherical coordinates

Let us denote with w_{θ} in (4.9) that

$$\begin{cases}
g(\xi,\theta;s) = \frac{w_{\theta}(\xi)}{w_{\theta}(\xi-s)}, \\
g_1(\xi,\theta;s) = -\frac{\sinh(\xi-s)}{2a_{\epsilon}} (g(\xi,\theta;s))^3, \\
g_2(\xi,\theta;s) = \frac{\sinh\xi}{2a_{\epsilon}} g(\xi,\theta;s).
\end{cases} (5.1)$$

Then we define

$$\mathcal{I}(\xi,\theta;c) := \frac{1}{2(\xi_1 + \xi_2)} \int_0^\infty g(\xi,\theta;t+c) dt, \tag{5.2}$$

$$\mathcal{I}_j(\xi,\theta;c) := \frac{1}{2(\xi_1 + \xi_2)} \int_0^\infty g_j(\xi,\theta;t+c) dt, \quad j = 1, 2.$$

It can be easily shown that

$$\mathcal{I}_{j}(\xi,\theta;c) = \begin{cases}
\frac{1}{2a_{\epsilon}(\xi_{1} + \xi_{2})} \frac{w_{\theta}^{3}(\xi)}{w_{\theta}(\xi - c)} & \text{for } j = 1 \\
\frac{\sinh \xi}{2a_{\epsilon}} \mathcal{I}(\xi,\theta;c) & \text{for } j = 2.
\end{cases}$$
(5.3)

Straightforward computations give

$$\hat{\mathbf{e}}_{\theta} \cdot \nabla \left(g(\xi, \theta; s) \right) = \frac{1}{\sigma_{\theta}(\xi, \theta)} \frac{\partial}{\partial \theta} \left(g(\xi, \theta; s) \right) = \frac{\sin \theta}{2a_{\epsilon}} \frac{w_{\theta}(\xi) \left(w_{\theta}^{2}(\xi - s) - w_{\theta}^{2}(\xi) \right)}{w_{\theta}^{3}(\xi - s)},$$

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla \left(g(\xi, \theta; s) \right) = \frac{1}{\sigma_{\varepsilon}(\xi, \theta)} \frac{\partial}{\partial \xi} \left(g(\xi, \theta; s) \right) = g_{1}(\xi, \theta; s) + g_{2}(\xi, \theta; s). \tag{5.4}$$

Hence the directional derivatives of \mathcal{I} at (ξ, θ, φ) becomes

$$\left(\hat{\mathbf{e}}_{\theta} \cdot \nabla \mathcal{I}\right)(\xi, \theta; c) = \frac{\sin \theta w_{\theta}(\xi)}{4a_{\epsilon}(\xi_1 + \xi_2)} \int_0^{\infty} \frac{w_{\theta}^2(\xi - t - c) - w_{\theta}^2(\xi)}{w_{\theta}^3(\xi - t - c)} dt, \tag{5.5}$$

$$(\hat{\mathbf{e}}_{\xi} \cdot \nabla \mathcal{I})(\xi, \theta; c) = \mathcal{I}_1(\xi, \theta; s) + \mathcal{I}_2(\xi, \theta; s). \tag{5.6}$$

We also have

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla \left(\mathcal{I}(-\xi, \theta; s) \right) = \frac{1}{\sigma_{\xi}} \frac{\partial}{\partial \xi} \left(\mathcal{I}(-\xi, \theta; s) \right) = -\mathcal{I}_{1}(-\xi, \theta; s) - \mathcal{I}_{2}(-\xi, \theta; s). \tag{5.7}$$

Lemma 5.1. There is a constant C independent of ϵ such that

$$\left| (S - \mathcal{I}) \right| (\xi, \theta; c) \le C, \tag{5.8}$$

$$\left|\hat{\mathbf{e}}_{\xi} \cdot \nabla \left(S - \mathcal{I}\right)\right| (\xi, \theta; c) \le C$$
 (5.9)

for all $\theta \in [0, \pi]$ and $(\xi, c) \in [-\xi_1, \xi_2] \times \mathbb{R}$ satisfying $|\xi| \leq (c - \xi) \leq 3(\xi_1 + \xi_2)$.

Proof. Note that

$$S(\xi, \theta; c) = \sum_{n=0}^{\infty} g(\xi, \theta; 2s_0 m + c), \quad s_0 = \xi_1 + \xi_2.$$

By taking the directional derivative to the above and using (5.4), we obtain

$$(\hat{\mathbf{e}}_{\xi} \cdot \nabla S)(\xi, \theta; c) = \sum_{m=0}^{\infty} g_1(\xi, \theta; 2s_0 m + c) + \sum_{m=0}^{\infty} g_2(\xi, \theta; 2s_0 m + c).$$
 (5.10)

Since $g_2(\xi, \theta; t+c)$ is positive and decreasing in $t \ge 0$ owing to $(c-\xi) \ge |\xi|$, we derive

$$\left| \sum_{m=0}^{\infty} g_2(\xi, \theta; 2s_0 m + c) - \mathcal{I}_2(\xi, \theta, c) \right| \le g_2(\xi, \theta; c) \le 1.$$
 (5.11)

One can derive (5.8) by the same way.

To deal with the summation of $g_1(\xi, \theta; t + c)$ values in (5.10), which is not monotone in t, we now apply the Euler-Maclaurin summation formula, see section 6. From Lemma 6.2, we estimate

$$\left| \sum_{m=0}^{\infty} g_1(\xi, \theta; 2s_0 m + c) - \mathcal{I}_1(\xi, \theta, c) \right|$$

$$\leq C \left(\left| g_1(\xi, \theta; c) \right| + s_0 \left| \frac{\partial}{\partial t} \left[g_1(\xi, \theta; t + c) \right] \right|_{t=0} \right| + s_0 \int_0^{\infty} \left| \frac{\partial^2}{\partial t^2} \left[g_1(\xi, \theta; t + c) \right] \right| dt \right) \leq C$$
 (5.12)

for some C independent of ϵ and (ξ, θ) . From (5.6), (5.10), (5.11), and (5.12), we prove the lemma.

Lemma 5.2. There is a constant C independent of ϵ such that

$$\left| \hat{\mathbf{e}}_{\theta} \cdot \left(\nabla \mathcal{I}(\xi, \theta; 2(\xi_1 + \xi_2)) - \nabla \mathcal{I}(-\xi, \theta; 2\xi_1) \right) \right| \leq C,
\left| \hat{\mathbf{e}}_{\theta} \cdot \left(\nabla \mathcal{I}(-\xi, \theta; 2(\xi_1 + \xi_2)) - \nabla \mathcal{I}(\xi, \theta; 2\xi_2) \right) \right| \leq C \quad \text{in } \mathbb{R}^3 \setminus \overline{(B_1 \cup B_2)}.$$

Proof. Applying (5.5) and the mean value property, we have

$$J := \left| \hat{\mathbf{e}}_{\theta} \cdot \left(\nabla \mathcal{I}(\xi, \theta; 2(\xi_1 + \xi_2)) - \nabla \mathcal{I}(-\xi, \theta; 2\xi_1) \right) \right|$$

$$= \left| \frac{\sin \theta w_{\theta}(\xi)}{4a_{\epsilon}(\xi_1 + \xi_2)} \int_{2\xi_1 + \xi}^{2(\xi_1 + \xi_2) - \xi} \frac{w_{\theta}^2(t) - w_{\theta}^2(\xi)}{w_{\theta}^3(t)} dt \right|$$

$$\leq C \left| \frac{\sin \theta w_{\theta}(\xi)}{a_{\epsilon}} \frac{w_{\theta}^2(\xi_0) - w_{\theta}^2(\xi)}{w_{\theta}^3(\xi_0)} \right|$$

for some $\xi_0 \in (2\xi_1 + \xi, 2(\xi_1 + \xi_2) - \xi)$ and a constant C independent of ϵ and (ξ, θ) . By applying the mean value property again, we have

$$w_{\theta}(\xi_0) - w_{\theta}(\xi) = (\xi_0 - \xi) \frac{\sinh \xi_*}{2w_{\theta}(\xi_*)}$$
 for some $\xi_* \in (\xi, \xi_0)$.

Note that $|\xi| \leq |\xi_0|$ and, hence,

$$w_{\theta}(\xi) \leq w_{\theta}(\xi_0).$$

Therefore, we conclude

$$J \le C \frac{|\sin \theta| |\xi_0 - \xi| \sinh \xi_*}{a_\epsilon w_\theta(\xi_*)} \left(\frac{1}{w_\theta(\xi_0)} + \frac{w_\theta(\xi)}{w_\theta(\xi_0)^2} \right) \le C \frac{|\sin \theta| \sinh \xi_*}{w_\theta(\xi_0) w_\theta(\xi_*)} \le C.$$

Similarly, we can prove the second uniform boundedness.

5.2 Asymptotics of h and u in the bispherical coordinates

Since h is a linear combination of S, see Lemma 4.5, a direct consequence of the previous lemmas is the asymptotics of h and ∇h in terms of integrals. We fix some notations for the sake of notational simplicity before deriving the asymptotics: let us denote

$$\tilde{\mu}_j = \mu_\epsilon \mu_j, \ j = 1, 2,$$

and

$$h_{s}(\mathbf{x}) = -\tilde{\mu}_{1} \Big(\mathcal{I}(-\xi, \theta; 2\xi_{1}) - \mathcal{I}(\xi, \theta; 2\xi_{1} + 2\xi_{2}) \Big) + \tilde{\mu}_{2} \Big(\mathcal{I}(\xi, \theta; 2\xi_{2}) - \mathcal{I}(-\xi, \theta; 2\xi_{1} + 2\xi_{2}) \Big),$$

$$q_{h}(\mathbf{x}) = \tilde{\mu}_{1} \Big(\mathcal{I}_{1}(-\xi, \theta; 2\xi_{1}) + \mathcal{I}_{1}(\xi, \theta; 2\xi_{1} + 2\xi_{2}) \Big) + \tilde{\mu}_{2} \Big(\mathcal{I}_{1}(\xi, \theta; 2\xi_{2}) + \mathcal{I}_{1}(-\xi, \theta; 2\xi_{1} + 2\xi_{2}) \Big). \tag{5.13}$$

Proposition 5.3. The solution h to (1.4) satisfies

$$h(\mathbf{x}) = h_s(\mathbf{x}) + b(\mathbf{x}),\tag{5.14}$$

$$\nabla h(\mathbf{x}) = q_h(\mathbf{x})\hat{\mathbf{e}}_{\xi}(\mathbf{x}) + r(\mathbf{x}), \tag{5.15}$$

where $||b||_{\infty}$, $||\nabla b||_{\infty}$ and $||r||_{\infty}$ are bounded independently of ϵ . Moreover, we have

$$C_1|\epsilon \ln \epsilon|^{-1} \le ||q_h||_{\infty} \le C_2|\epsilon \ln \epsilon|^{-1},\tag{5.16}$$

for some positive constants C_1 and C_2 independent of ϵ .

Proof. Firstly, we prove that $\|\nabla b\|_{\infty} = \|\nabla (h - h_s)\|_{\infty}$ is uniformly bounded regardless of $\epsilon > 0$. From Lemma 4.4, h_s is harmonic and has the decay property at infinity. Hence it is enough to derive the uniform boundedness of $|\nabla (h - h_s)|$ in $\mathbf{x} \in \partial B_1 \cup \partial B_2$ and $\epsilon > 0$. For simplicity, we consider only for ∂B_1 . Since h is constant on ∂B_1 , we have

$$\left|\nabla(h - h_s)\right| \le \left|\hat{\mathbf{e}}_{\xi} \cdot \nabla(h - h_s)\right| + \left|\hat{\mathbf{e}}_{\theta} \cdot \nabla h_s\right| \quad \text{on } \partial B_1. \tag{5.17}$$

In the following we show that $\|\hat{\mathbf{e}}_{\xi} \cdot \nabla(h - h_s)\|_{\infty}$ and $\|\hat{\mathbf{e}}_{\theta} \cdot \nabla h_s\|_{\infty}$ are uniformly bounded in ϵ . Note that the directional derivatives of h and h_s are combinations of those of \mathcal{I} . More precisely speaking, because of $\hat{\mathbf{e}}_{\xi} \cdot \nabla \left(S(-\xi, \theta; s)\right) = -\hat{\mathbf{e}}_{\xi} \cdot \left(\nabla S(-\xi, \theta; s)\right)$ and $\hat{\mathbf{e}}_{\xi} \cdot \nabla \left(\mathcal{I}(-\xi, \theta; s)\right) = -\hat{\mathbf{e}}_{\xi} \cdot \left(\nabla \mathcal{I}(-\xi, \theta; s)\right)$, one can rewrite $\hat{\mathbf{e}}_{\xi} \cdot \nabla h$ and $\hat{\mathbf{e}}_{\xi} \cdot \nabla h_s$ as

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla h = -C_{1} \hat{\mathbf{e}}_{\xi} \cdot \left(\nabla S \left(-\xi, \theta; 2\xi_{1} \right) + \nabla S \left(\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right)
+ C_{2} \hat{\mathbf{e}}_{\xi} \cdot \left(\nabla S \left(\xi, \theta; 2\xi_{2} \right) + \nabla S \left(-\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right),$$

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla h_{s} = \tilde{\mu}_{1} \hat{\mathbf{e}}_{\xi} \cdot \left(\nabla \mathcal{I} \left(-\xi, \theta; 2\xi_{1} \right) + \nabla \mathcal{I} \left(\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right)
+ \tilde{\mu}_{2} \hat{\mathbf{e}}_{\xi} \cdot \left(\nabla \mathcal{I} \left(\xi, \theta; 2\xi_{2} \right) + \nabla \mathcal{I} \left(-\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right).$$
(5.18)

Similarly to (5.19), we use the fact $\hat{\mathbf{e}}_{\theta} \cdot \nabla (\mathcal{I}(-\xi, \theta; s)) = \hat{\mathbf{e}}_{\theta} \cdot (\nabla \mathcal{I}(-\xi, \theta; s))$ to rewrite $\hat{\mathbf{e}}_{\theta} \cdot \nabla h_s$ as

$$\hat{\mathbf{e}}_{\theta} \cdot \nabla h_{s} = -\tilde{\mu}_{1} \hat{\mathbf{e}}_{\theta} \cdot \left(\nabla \mathcal{I} \left(-\xi, \theta; 2\xi_{1} \right) - \nabla \mathcal{I} \left(\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right) + \tilde{\mu}_{2} \hat{\mathbf{e}}_{\theta} \cdot \left(\nabla \mathcal{I} \left(\xi, \theta; 2\xi_{2} \right) - \nabla \mathcal{I} \left(-\xi, \theta; 2\xi_{1} + 2\xi_{2} \right) \right).$$
 (5.20)

Suppose that $(\tilde{\xi}, c)$ is one of $(-\xi, 2\xi_1), (\xi, 2\xi_2), (\pm \xi, 2\xi_1 + 2\xi_2)$ with $-\xi_1 \leq \xi \leq \xi_2$. Then we have

$$|\tilde{\xi}| \le (c - \tilde{\xi}) \le 3(\xi_1 + \xi_2).$$

Since $|\tilde{\xi} - c| \ge |\tilde{\xi}|$, we have $0 < g(\tilde{\xi}, \theta, c) \le 1$. Using this and the definition of \mathcal{I} and \mathcal{I}_j 's, we can easily show

$$0 < \mathcal{I}_1(\tilde{\xi}, \theta; c) \le \frac{C}{\epsilon}, \quad 0 < \mathcal{I}(\tilde{\xi}, \theta; c), \mathcal{I}_2(\tilde{\xi}, \theta; c) \le \frac{C}{\sqrt{\epsilon}} \quad \text{in } \mathbb{R}^3 \setminus (B_1 \cup B_2).$$
 (5.21)

Here and in the remaining of the proof, C indicates a positive constant independent of ϵ and (ξ, θ) . Thanks to (5.6), one obtains

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla \mathcal{I}(\tilde{\xi}, \theta; c) = O(\epsilon^{-1}). \tag{5.22}$$

We also have from Lemma 5.1 that

$$\left| \hat{\mathbf{e}}_{\xi} \cdot \nabla(S - \mathcal{I}) \right| (\tilde{\xi}, \theta; c) \le C. \tag{5.23}$$

Note that

$$\tilde{\mu}_1, \tilde{\mu}_2 = O(|\ln \epsilon|^{-1})$$
 and $\tilde{\mu}_j = (-1)^j C_j + O(\epsilon)$.

Using these facts, (5.22) and (5.23), we get

$$\begin{aligned} & \left| \hat{\mathbf{e}}_{\xi} \cdot \left(C_{j} \nabla S - (-1)^{j} \mu_{j} \nabla \mathcal{I} \right) \right| (\tilde{\xi}, \theta; c) \\ & \leq \left| C_{j} \hat{\mathbf{e}}_{\xi} \cdot \nabla \left(S - \mathcal{I} \right) \right| (\tilde{\xi}, \theta; c) + \left| \left(C_{j} - (-1)^{j} \tilde{\mu}_{j} \right) \hat{\mathbf{e}}_{\xi} \cdot \nabla \mathcal{I} \right| (\tilde{\xi}, \theta; c) \\ & < C. \end{aligned}$$

Hence we obtain from (5.18) and (5.19) that

$$\|\hat{\mathbf{e}}_{\xi} \cdot \nabla(h - h_s)\|_{\infty} \le C. \tag{5.24}$$

The θ -directional derivative of h_s satisfies

$$\left\| \hat{\mathbf{e}}_{\theta} \cdot \nabla h_s \right\|_{\infty} \le C \tag{5.25}$$

due to (5.20) and Lemma 5.2. Thanks to (5.17), (5.24) and (5.25), we derive that that $\|\nabla(h-h_s)\|_{\infty}$ is uniformly bounded independently of ϵ . This shows that $\|\nabla b\|_{\infty} \leq C$ by the discussion at the beginning of the proof. In fact, due to (5.25), we have shown a slightly stronger result as follows:

$$\nabla h(\mathbf{x}) = (\hat{\mathbf{e}}_{\xi} \cdot \nabla h_s)(\mathbf{x})\hat{\mathbf{e}}_{\xi}(\mathbf{x}) + \tilde{r}(\mathbf{x}), \quad \|\tilde{r}\|_{\infty} \le C.$$
 (5.26)

Now we prove (5.15). From (5.6) and the definition of \mathcal{I}_j 's, the ξ -directional derivative of h_s satisfies

$$\hat{\mathbf{e}}_{\xi} \cdot \nabla h_s(\mathbf{x}) = q_h(\mathbf{x}) + v(\mathbf{x}), \tag{5.27}$$

where q_h is defined as in (5.13) and

$$v(\mathbf{x}) = \frac{\sinh \xi}{2a_{\epsilon}} h_s(\mathbf{x}).$$

We need to show that $||v||_{\infty}$ is bounded regardless of $\epsilon > 0$. To do that let us consider the remainder term b in (5.14). From (5.8) and Lemma 4.5, one can easily prove

$$||b||_{\infty} = ||h - h_s||_{\infty} \le C \tag{5.28}$$

similarly to the proof of (5.24). Remind that h has the decaying property and $h|_{\partial B_1 \cup \partial B_2} = O(|\ln \epsilon|^{-1})$. Hence, $||h||_{\infty}$ is bounded independently of ϵ and so does for $||h_s||_{\infty}$ thanks to (5.28). So we have

$$||v||_{\infty} = \left\| \frac{\sinh \xi}{2a_s} h_s \right\|_{\infty} \le C.$$

Hence, we obtain (5.15) using (5.26) and (5.27).

We note from the definition of q_h that $q_h(\mathbf{x}_0)$ for $\mathbf{x}_0 \in \partial B_1$ of which bispherical coordinates are $(-\xi_1, \pi, 0)$ satisfies

$$q_h(\mathbf{x}_0) \geq \tilde{\mu}_1 \mathcal{I}_1(\xi_1, \pi; 2\xi_1) = \frac{\tilde{\mu}_1}{2a_{\epsilon}(\xi_1 + \xi_2)} \frac{(\cosh \xi_1 - \cos \pi)^{\frac{3}{2}}}{(\cosh \xi_1 - \cos \pi)^{\frac{1}{2}}} \geq \frac{C}{\epsilon |\ln \epsilon|}.$$

This proves the lower bound in (5.16), and the upper bound follows from (5.21). Hence we finish the proof.

We now have the asymptotics of u and ∇u thanks to (1.5), (2.5) and Proposition 5.3 as follows.

Proposition 5.4. The solution u to (1.1) satisfies

$$u(\mathbf{x}) = C_H h_s(\mathbf{x}) + H(\mathbf{x}) + b(\mathbf{x}),$$
$$\nabla u(\mathbf{x}) = C_H q_h(\mathbf{x}) \hat{\mathbf{e}}_{\xi}(\mathbf{x}) + \nabla H(\mathbf{x}) + r(\mathbf{x}),$$

where $||b||_{\infty}$, $||\nabla b||_{\infty}$ and $||r||_{\infty}$ are bounded independently of ϵ .

5.3 Asymptotics of h and u in the Cartesian coordinates

With the notations defined in (2.1) and (2.2), we define two density functions ρ_j , j = 1, 2, as

$$\rho_1(0,0,c) = \frac{\tilde{r}\mu_{\epsilon}}{\sqrt{c^2 - a_{\epsilon}^2}} \Big(\mu_1 \mathbb{1}_{[\mathbf{c}_1,\mathbf{p}_1]} + \mu_2 \mathbb{1}_{[R_1(\mathbf{c}_2),\mathbf{p}_1]} \Big) (0,0,c), \tag{5.29}$$

$$\rho_2(0,0,c) = \frac{\tilde{r}\mu_{\epsilon}}{\sqrt{c^2 - a_{\epsilon}^2}} \Big(\mu_2 \mathbb{1}_{[\mathbf{p}_2,\mathbf{c}_2]} + \mu_1 \mathbb{1}_{[\mathbf{p}_2,R_2(\mathbf{c}_1)]} \Big) (0,0,c), \tag{5.30}$$

where the symbol $[\mathbf{x}_1, \mathbf{x}_2]$ means the line segment connecting two points \mathbf{x}_1 and \mathbf{x}_2 , and $\mathbb{1}_{[\mathbf{x}_1, \mathbf{x}_2]}$ is the indicator function of $[\mathbf{x}_1, \mathbf{x}_2]$. From Proposition 5.4, we derive the following corollary which tells that the solution h to (1.4) can be expressed as the integral with the integrand ρ_1 and ρ_2 .

Corollary 5.5. We have

$$h(\mathbf{x}) = -\int_{[\mathbf{c}_1, \mathbf{p}_1]} \frac{\rho_1(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c} + \int_{[\mathbf{p}_2, \mathbf{c}_2]} \frac{\rho_2(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c} + r(\mathbf{x}) \quad in \ \mathbb{R}^3 \setminus \overline{B_1 \cup B_2},$$

where $\|\nabla r\|_{\infty}$ is bounded regardless of ϵ .

Proof. Let us express the function h_s in the Cartesian coordinates. Applying Lemma 4.4 and letting $c = a_{\epsilon} \coth(t + \xi_1)$, one computes

$$\mathcal{I}(-\xi,\theta;2\xi_1) = \frac{1}{\xi_1 + \xi_2} \int_0^\infty \frac{w_\theta(\xi)}{w_\theta(\xi + 2t + 2\xi_1)} dt
= \frac{a_\epsilon}{\xi_1 + \xi_2} \int_0^\infty \frac{1}{\sinh(t + \xi_1)} \frac{1}{|\mathbf{x} + a_\epsilon \coth(t + \xi_1)\mathbf{e}_3|} dt
= \frac{a_\epsilon}{\xi_1 + \xi_2} \int_{a_\epsilon \coth \xi_1}^{a_\epsilon} \frac{-1}{\sqrt{c^2 - a_\epsilon^2}} \frac{1}{|\mathbf{x} + c\mathbf{e}_3|} dc
= \int_{[\mathbf{c}_1, \mathbf{p}_1]} f(\mathbf{r}) d\mathbf{r},$$

where

$$f(\mathbf{r}) = \frac{a_{\epsilon}}{\xi_1 + \xi_2} \frac{1}{\sqrt{|\mathbf{r}|^2 - a_{\epsilon}^2}} \frac{1}{|\mathbf{x} - \mathbf{r}|}.$$

Similarly, one can easily obtain

$$\mathcal{I}(\xi, \theta; 2\xi_2) = \int_{[\mathbf{p}_2, \mathbf{c}_2]} f(\mathbf{r}) d\mathbf{r},$$

$$\mathcal{I}(-\xi, \theta; 2(\xi_1 + \xi_2)) = \int_{[R_1(\mathbf{c}_2), \mathbf{p}_1]} f(\mathbf{r}) d\mathbf{r},$$

$$\mathcal{I}(\xi, \theta; 2(\xi_1 + \xi_2)) = \int_{[\mathbf{p}_2, R_2(\mathbf{c}_1)]} f(\mathbf{r}) d\mathbf{r}.$$

Hence we have

$$h_s(\mathbf{x}) = \frac{a_{\epsilon}}{\tilde{r}(\xi_1 + \xi_2)} \tilde{h}_s(\mathbf{x}), \tag{5.31}$$

where

$$\tilde{h}_s(\mathbf{x}) = -\int_{[\mathbf{c}_1,\mathbf{p}_1]} \frac{\rho_1(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c} + \int_{[\mathbf{p}_2,\mathbf{c}_2]} \frac{\rho_2(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c}.$$

Thanks to Proposition 5.3 and the fact $\frac{a_{\epsilon}}{\tilde{r}(\xi_1+\xi_2)}=1+O(\epsilon)$, we prove the corollary.

The following corollary is the direct consequence of (1.5), Theorem 2.1 and Corollary 5.5.

Corollary 5.6. The solution u to (1.1) satisfies

$$u(\mathbf{x}) = C_H \left(-\int_{[\mathbf{c}_1, \mathbf{p}_1]} \frac{\rho_1(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c} + \int_{[\mathbf{p}_2, \mathbf{c}_2]} \frac{\rho_2(\mathbf{c})}{|\mathbf{x} - \mathbf{c}|} d\mathbf{c} \right) + H(\mathbf{x}) + b(\mathbf{x}),$$

where $\|\nabla b\|_{\infty}$ is bounded independently of ϵ .

Near the fixed points \mathbf{p}_1 and \mathbf{p}_2 , the density functions ρ_1 and ρ_2 are of similar form to that obtained in [27] as mentioned in the introduction. It is worth to emphasize that, in this paper, we derived the continuous image charge distribution by rigorous asymptotic analysis without any physical assumptions. Moreover, it turns out that each of density functions ρ_1 and ρ_2 has an discontinuity, and the coefficients in the density functions are explicitly calculated.

5.4 Proof of Theorem 2.2

Proof of Theorem 2.2 From (3.4), (3.5), (3.13) and Lemma 4.4, we have

$$\mathcal{I}_{1}(-\xi,\theta;2\xi_{1})\hat{\mathbf{e}}_{\xi} = \frac{1}{2a_{\epsilon}(\xi_{1}+\xi_{2})} \frac{w_{\theta}^{3}(\xi)}{w_{\theta}(\xi+2\xi_{1})} \hat{\mathbf{e}}_{\xi} = \frac{\cosh\xi - \cos\theta}{2a_{\epsilon}(\xi_{1}+\xi_{2})} \frac{w_{\theta}(\xi)}{w_{\theta}(\xi+2\xi_{1})} \frac{a_{\epsilon}}{\cosh\xi - \cos\theta} \mathbf{N}(\mathbf{x})$$

$$= \frac{1}{2(\xi_{1}+\xi_{2})} \frac{a_{\epsilon}}{\sinh\xi_{1}} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x}-\mathbf{c}_{1}|}$$

$$= \frac{r_{1}}{2(\xi_{1}+\xi_{2})} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x}-\mathbf{c}_{1}|}$$

and, by the same way,

$$\mathcal{I}_1(\xi, \theta; 2\xi_2)\hat{\mathbf{e}}_{\xi} = \frac{r_2}{2(\xi_1 + \xi_2)} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x} - \mathbf{c}_2|}.$$

Similarly, we compute

$$\mathcal{I}_{1}(-\xi,\theta;2\xi_{1}+2\xi_{2})\hat{\mathbf{e}}_{\xi} = \frac{1}{2(\xi_{1}+\xi_{2})} \frac{a_{\epsilon}}{\sinh(\xi_{1}+\xi_{2})} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x}-R_{1}(\mathbf{c}_{2})|}$$

$$= \frac{\tilde{r}}{2(\xi_{1}+\xi_{2})} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x}-R_{1}(\mathbf{c}_{2})|} \Big(1+O(\epsilon)\Big),$$

$$\mathcal{I}_{1}(+\xi,\theta;2\xi_{1}+2\xi_{2})\hat{\mathbf{e}}_{\xi} = \frac{\tilde{r}}{2(\xi_{1}+\xi_{2})} \frac{\mathbf{N}(\mathbf{x})}{|\mathbf{x}-R_{2}(\mathbf{c}_{1})|} \Big(1+O(\epsilon)\Big).$$

Therefore we have

$$q_h(\mathbf{x})\hat{\mathbf{e}}_{\xi}(\mathbf{x}) = \frac{a_{\epsilon}}{\tilde{r}(\xi_1 + \xi_2)}\psi(\mathbf{x})\mathbf{N}(\mathbf{x}) + O(1).$$

Note that $\frac{\tilde{r}(\xi_1+\xi_2)}{a_{\epsilon}}=1+O(\epsilon)$. From Proposition 5.4 and (5.13), we prove Theorem 2.2.

Proof of Theorem 2.4 To prove (a), it is enough to show, in view of Proposition 5.3, that there is a constant C independent of ϵ such that

$$|q_h(\xi,\theta)| \le C$$
 for $|\theta| \le \sqrt{\epsilon |\ln \epsilon|}$, $-\xi_1 \le \xi \le \xi_2$.

We estimate only one term $\mu_1 \mathcal{I}_1(-\xi, \theta; 2\xi_1)$ in h_1 ; the other three terms can be estimated in a similar way. The bispherical coordinates (ξ, θ, φ) of $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{(B_1 \cup B_2 \cup \Omega_{\epsilon}^*)}$ satisfies $|\theta| \leq \sqrt{\epsilon |\ln \epsilon|}$ and $-\xi_1 \leq \xi \leq \xi_2$, so that it follows

$$|\xi + 2\xi_1| \ge |\xi|,$$

$$w_{\theta}^2(\xi) = \cosh \xi - \cos \theta = 1 + O(\epsilon) - (1 + O(\epsilon|\ln \epsilon|)) = O(\epsilon|\ln \epsilon|).$$

We compute

$$\left|\mu_{\epsilon}\mu_{1}\mathcal{I}_{1}(-\xi,\theta;2\xi_{1})\right| = \frac{\mu_{\epsilon}\mu_{1}}{2a_{\epsilon}(\xi_{1}+\xi_{2})} \frac{w_{\theta}(\xi)}{w_{\theta}(\xi+2\xi_{1})} w_{\theta}^{2}(\xi) \leq C \frac{\mu_{\epsilon}\mu_{1}}{a_{\epsilon}(\xi_{1}+\xi_{2})} \epsilon \left|\ln \epsilon\right| \leq C,$$

where C is a constant independent of ϵ . This proves (a).

We prove (b) by showing that $|q_h(\xi, \tilde{\theta}_{\epsilon})| \to \infty$ as ϵ tends to zero. Again, we consider only $\mu_1 \mathcal{I}_1(-\xi, \tilde{\theta}_{\epsilon}; 2\xi_1)$. Let us denote $w_{\theta}(\xi) = w(\xi, \theta)$ for notational sake. Because of $-\xi_1 \le \xi \le \xi_2$, we have

$$w^{2}(\xi, \tilde{\theta}_{\epsilon}) = \cosh \xi - \cos \tilde{\theta}_{\epsilon} = \frac{\tilde{\theta}_{\epsilon}^{2}}{2} + O(\tilde{\theta}_{\epsilon}^{4})$$

and

$$\frac{w(\xi,\tilde{\theta}_{\epsilon})}{w(\xi+2\xi_{1},\tilde{\theta}_{\epsilon})} = \left(\frac{\cosh\xi-\cos\tilde{\theta}_{\epsilon}}{\cosh(\xi+2\xi_{1})-\cos\tilde{\theta}_{\epsilon}}\right)^{\frac{1}{2}} = \left(\frac{\tilde{\theta}_{\epsilon}^{2}+O(\tilde{\theta}_{\epsilon}^{4})}{\tilde{\theta}_{\epsilon}^{2}+O(\tilde{\theta}_{\epsilon}^{4})}\right)^{\frac{1}{2}} \longrightarrow 1 \quad \text{as } \epsilon \to 0.$$

Hence, we have

$$\left|\mu_{\epsilon}\mu_{1}\mathcal{I}_{1}(-\xi,\tilde{\theta}_{\epsilon};2\xi_{1})\right| = \left|\frac{\mu_{\epsilon}\mu_{1}}{2a_{\epsilon}(\xi_{1}+\xi_{2})}\frac{w^{3}(\xi,\tilde{\theta}_{\epsilon})}{w(\xi+2\xi_{1},\tilde{\theta}_{\epsilon})}\right| \geq C\frac{\tilde{\theta}_{\epsilon}^{2}}{\epsilon|\ln\epsilon|} \longrightarrow \infty \quad \text{as } \epsilon \to 0.$$

Here, we can choose C independent of ϵ and ξ satisfying $-\xi_1 \leq \xi \leq \xi_2$. This proves (b).

6 The Euler-Maclaurin formula and its two applications

The following is a special case of the Euler-Maclaurin summation formula: for $f \in C^2[x_0, \infty)$ satisfying $f, f', f'' \in L^1(x_0, \infty)$, we have for any s > 0 that

$$\sum_{k=0}^{\infty} f(x_0 + k\tilde{s})\tilde{s} = \int_{x_0}^{\infty} f(x)dx + \frac{s}{2}f(x_0) - \frac{s^2}{12}f'(x_0) + R,$$

where the remainder term satisfies

$$|R| \le \frac{s^2}{12} \int_{-\infty}^{\infty} |f''(x)| dx.$$

The followings are the applications of the Euler-Maclaurin formula, and they are essentially used to prove the main theorems in this paper.

Lemma 6.1. Fix s_0 , c such that $0 < s_0$, $c \le (\xi_1 + \xi_2)$ with ξ_1, ξ_2 given by (3.5) and set

$$f_k(x) = \frac{\coth^k x}{\sinh x} - \frac{1}{x^{k+1}}$$
 for $x > 0, \ k = 1, 2, \dots$ (6.1)

Then there is a constant C independent of ϵ and k such that

$$\left| s_0 \sum_{m=0}^{\infty} f_k(s_0 m + c) \right| \le C 2^k \epsilon^{\frac{-k+2}{2}} |\ln \epsilon|$$

for small enough $\epsilon > 0$.

Proof. Let k be a fixed positive integer. One can easily check

$$\begin{split} f_k'(x) &= \frac{k+1}{x^{k+2}} - \frac{\coth^{k+1}x}{\sinh x} - \frac{k\coth^k x}{\sinh^2 x \cosh x}, \\ f_k''(x) &= -\frac{(k+2)(k+1)}{x^{k+3}} + \frac{\coth^{k+2}x}{\sinh x} + \frac{(\coth^k x)\left[(4k+1)\cosh^2 x + k(k-1)\right]}{\sinh^3 x \cosh^2 x}. \end{split}$$

Applying the Euler-Maclaurin summation formula, we have

$$\left| \sum_{m=0}^{\infty} f_k(c+s_0 m) s_0 \right| \le \left| \int_c^{\infty} f_k(x) dx \right| + \frac{s_0}{2} |f_k(c)| + \frac{s_0^2}{12} |f_k'(c)| + \frac{s_0^2}{12} \int_c^{\infty} |f_k''(t)| dt.$$

In the following, we estimate the four terms in the right-hand side in the equation above. We first define a function v_k as

$$v_k(x) = \frac{1}{x^2} - \frac{\tanh^k x}{x^{k+2}}$$
 for $x > 0$.

Note that v_k is positive and monotonically decreasing and

$$\lim_{x \to 0^+} v_k(x) = \frac{k}{3}.$$
 (6.2)

To estimate $|f_k(c)|$ and $|\int_c^\infty f_k(x)dx|$, we decompose f_k for $k \in \mathbb{N}$ as

$$f_k(t) = \frac{1}{t^{k-1}} \frac{t^k}{\tanh^k t} (v_k(t) + p_0(t))$$
 with $p_0(t) = \frac{1}{t \sinh t} - \frac{1}{t^2}$.

Note that $p_0(t)$ is bounded on $\{t > 0\}$ and $(c^k/\tanh^k c)$ is bounded by a constant C. Here and in the remaining of the proof, C indicates a constant independent of ϵ and k. Hence we have

$$|f_k(c)| \le \frac{C}{c^{k-1}} \left(|p_0(c)| + |v_k(c)| \right) \le Ck\epsilon^{\frac{-k+1}{2}}.$$
 (6.3)

Let us now estimate $\left| \int_{c}^{\infty} f_{k}(x) dx \right|$. For k = 1, we have

$$\left| \int_{c}^{\infty} f_1(x) dx \right| \leq \int_{0}^{c} |f_1(x)| dx \leq C\sqrt{\epsilon}$$

thanks to the boundedness of $f_1(x)$ on $\{t>0\}$ and $\int_0^\infty f_1(x)dx=0$. For $k\geq 2$, we have

$$\left| \int_{c}^{\infty} f_{k}(x) dx \right| \leq \int_{c}^{1} |f_{k}(x)| dx + \int_{1}^{\infty} |f_{k}(x)| dx$$

$$\leq \int_{c}^{1} \frac{1}{x^{k-1}} \left(\frac{x}{\tanh x} \right)^{k} (|p_{0}(x)| + |v_{k}(x)|) dx + C \int_{1}^{\infty} \frac{1}{\sinh x} dx + \frac{1}{k}$$

$$\leq Ck \left(\frac{1}{\tanh 1} \right)^{k} \int_{c}^{1} x^{-k+1} dx + C \leq C2^{k} e^{\frac{-k+2}{2}} |\ln \epsilon|.$$

Hence it follows

$$\left| \int_{c}^{\infty} f_{k}(x) dx \right| \leq C 2^{k} \epsilon^{\frac{-k+2}{2}} |\ln \epsilon| \quad \text{for } k \in \mathbb{N}.$$
 (6.4)

Similar to the decomposition of f_k , we have such decompositions of f'_k and f''_k :

$$f'_k(t) = \frac{1}{t^k} \frac{t^k}{\tanh^k t} \left(-(k+1)v_k(t) + p_1(t) + kp_2(t) \right),$$

$$f''_k(t) = \frac{1}{t^{k+1}} \frac{t^k}{\tanh^k t} \left((k^2 + 3k + 2)v_k(t) + p_3(t) + (4k+1)p_4(t) + k(k-1)p_5(t) \right)$$

with bounded functions

$$p_1(t) = \frac{1}{t^2} - \frac{\coth t}{\sinh t}, \quad p_2(t) = \frac{1}{t^2} - \frac{1}{\sinh^2 t \cosh t},$$

$$p_3(t) = \frac{t \coth^2 t}{\sinh t} - \frac{1}{t^2}, \quad p_4(t) = \frac{t}{\sinh^3 t} - \frac{1}{t^2}, \quad p_5(t) = \frac{t}{\cosh^2 t \sinh^3 t} - \frac{1}{t^2}.$$

Using (6.2), we derive

$$|f_k'(c)| \le C \frac{1}{c^k} \left((k+1)|v_k(c)| + |p_1(c)| + k|p_2(c)| \right) \le Ck^2 \epsilon^{-\frac{k}{2}}$$
(6.5)

and

$$\int_{c}^{\infty} |f_{k}''(t)| dt = \int_{c}^{1} |f_{k}''(t)| dt + \int_{1}^{\infty} |f_{k}''(t)| dt
\leq C \int_{c}^{1} \frac{1}{t^{k+1}} \frac{k^{3}}{\tanh^{k} 1} dt + C \int_{1}^{\infty} \left(\frac{k^{2} + 3k + 2}{t^{k+3}} + \frac{1}{\sinh t} + \frac{k^{2} + 3k + 1}{\sinh^{3} t} \right) dt
\leq C \left(2^{k} \epsilon^{-\frac{k}{2}} + k^{2} \right) \leq C 2^{k} \epsilon^{-\frac{k}{2}}.$$
(6.6)

From (6.3), (6.4), (6.5), and (6.6), the conclusion follows.

Lemma 6.2. Let c be a constant satisfying $|\xi| \le (c - \xi) \le 3(\xi_1 + \xi_2)$ for all $-\xi_1 \le \xi \le \xi_2$ and g_1 be the function given by (5.1), i.e.,

$$g_1(\xi, \theta; c) = -\frac{\sinh(\xi - c)}{2a_{\epsilon}} \left(\frac{w_{\theta}(\xi)}{w_{\theta}(\xi - s)}\right)^3.$$

Then there exists a constant C independent of ξ , θ , ϵ and c such that

$$\left| \sum_{m=0}^{\infty} g_1(\xi, \theta; 2s_0 m + c) - \frac{1}{2s_0} \int_0^{\infty} g_1(\xi, \theta; t + c) dt \right| \le C \quad \text{with } s_0 = \xi_1 + \xi_2,$$

for all $-\xi_1 \leq \xi \leq \xi_2$ and $0 \leq \theta \leq \pi$.

Proof. We use the Euler-Maclaurin summation formula to have

$$\begin{split} &\left| \sum_{m=0}^{\infty} g_1(\xi, \theta; s_0 m + c) - \frac{1}{s_0} \int_0^{\infty} g_1(\xi, \theta; t + c) dt \right| \\ &\leq \left| g_1(\xi, \theta; c) \right| + s_0 \left| \frac{\partial}{\partial t} \left[g_1(\xi, \theta; t + c) \right] \right|_{t=0} + s_0 \int_0^{\infty} \left| \frac{\partial^2}{\partial t^2} \left[g_1(\xi, \theta; t + c) \right] \right| dt. \end{split}$$

Since $|\xi - c| \ge |\xi|$ for all $-\xi_1 \le \xi \le \xi_2$, we have

$$\left| g(\xi, \theta; c) \right| \le 1. \tag{6.7}$$

Moreover, we have $\frac{\sinh(\xi-c)}{2a_{\epsilon}} = O(1)$, so that it follows

$$\left| g_1(\xi, \theta; c) \right| \le M,\tag{6.8}$$

for a constant M independent of ϵ .

In the follows, we show that there is a positive constant C independent of ϵ, ξ, θ satisfying

$$\left| \frac{\partial}{\partial t} \left[g_1(\xi, \theta; t + c) \right] \right|_{t=0} , \int_0^\infty \left| \frac{\partial^2}{\partial t^2} \left[g_1(\xi, \theta; t + c) \right] \right| dt \le \frac{C}{\sqrt{\epsilon}}.$$

Remind that w_{θ} is the function given by

$$w_{\theta}(\xi) = \sqrt{\cosh \xi - \cos \theta}$$

A straightforward but tedious computation shows

$$\frac{\partial}{\partial t} \left[g_1(\xi, \theta; t + c) \right] \Big|_{t=0} = \frac{w_{\theta}^3(\xi)}{a_{\epsilon}} \frac{3 - \cosh^2(c - \xi) - 2\cos\theta \cosh(c - \xi)}{4\left(\cosh(c - \xi) - \cos\theta\right)^{5/2}},$$

$$\frac{\partial^2}{\partial t^2} \left[g_1(\xi, \theta; t + c) \right] = -\frac{w_{\theta}^3(\xi)}{a_{\epsilon}} \frac{\sinh \tilde{x} \left(4\cos^2\theta + 10\cos\theta \cosh \tilde{x} + \cosh^2\tilde{x} - 15\right)}{8\left(\cosh\tilde{x} - \cos\theta\right)^{7/2}}$$

with $\tilde{x} = t + c - \xi$.

Thanks to (6.7), there exists a constant C independent of ϵ such that

$$\left| \frac{\partial}{\partial t} \left[g_1(\xi, \theta; t + c) \right] \right|_{t=0} \le \frac{C}{a_{\epsilon}} \left(\left| \frac{3(1 - \cosh^2(c - \xi))}{\cosh(c - \xi) - \cos \theta} \right| + 2(\cosh(c - \xi) - \cos \theta) \right) \le \frac{C}{\sqrt{\epsilon}}. \tag{6.9}$$

For the inequality of the second derivative of g_1 , let us consider separately the cases $0 \le t \le 1$ and t > 1. For $0 \le t \le 1$, we have

$$\left| \frac{\partial^{2}}{\partial t^{2}} \left[g_{1}(\xi, \theta; t + c) \right] \right| \\
= \frac{w_{\theta}(\xi)^{3}}{a_{\epsilon} (\cosh \tilde{x} - \cos \theta)^{5/2}} \left| \frac{\sinh \tilde{x} (4\cos^{2}\theta + 10\cos\theta \cosh \tilde{x} + \cosh^{2}\tilde{x} - 15)}{8(\cosh \tilde{x} - \cos \theta)} \right| \\
\leq \frac{w_{\theta}(\xi)^{3} \sinh \tilde{x}}{a_{\epsilon} (\cosh \tilde{x} - \cos \theta)^{5/2}} \left(\left| 15 \frac{\cosh^{2}\tilde{x} - 1}{\cosh \tilde{x} - \cos \theta} \right| + \left| 4 \frac{\cosh^{2}\tilde{x} - \cos \theta}{\cosh \tilde{x} - \cos \theta} \right| + 10 \cosh \tilde{x} \right) \\
\leq \frac{C}{\sqrt{\epsilon}} \frac{w_{\theta}(\xi)^{3} \sinh \tilde{x}}{(\cosh \tilde{x} - \cos \theta)^{5/2}}.$$

Hence it follows

$$\int_{0}^{1} \left| \frac{\partial^{2}}{\partial t^{2}} \left[g_{1}(\xi, \theta; t + c) \right] \right| dt \leq \frac{C}{\sqrt{\epsilon}} w_{\theta}(\xi)^{3} \int_{c - \xi}^{1 + c - \xi} \frac{\sinh t}{(\cosh t - \cos \theta)^{5/2}} dt \\
\leq \frac{C}{\sqrt{\epsilon}} \left(\left| \frac{w_{\theta}(\xi)^{3}}{w_{\theta}(c - \xi)^{3}} \right| + \left| \frac{w_{\theta}(\xi)^{3}}{w_{\theta}(1 + c - \xi)^{3}} \right| \right) \leq \frac{C}{\sqrt{\epsilon}}.$$
(6.10)

For t > 1, $\left| \frac{\partial^2}{\partial t^2} \left[g_1(\xi, \theta; t + c) \right] \right|$ is uniformly bounded by a exponentially decreasing function of t, so that we have

$$\int_{1}^{\infty} \left| \frac{\partial^{2}}{\partial t^{2}} \left[g_{1}(\xi, \theta; t + c) \right] \right| dt \leq C,$$

for a constant of C independent of ϵ, ξ, θ and c. Using this equation and (6.8), (6.9), and (6.10) as well, we prove the lemma.

7 Numerical Illustration

In this section we illustrate the main results with some examples. More precisely, we plot the graphs of ∇h , $\nabla (u - H)$ and their blow-up terms on ∂B_1 . We consider the asymptotic behavior of ∇h and $\nabla (u - H)$ only on ∂B_1 because they have the similar behavior on ∂B_2 . To have precise values of ∇h and $\nabla (u - H)$, we use their exact solution formulas derived in section 7.1. On the other hand, gradient blow-up terms are simple elementary functions which are easy to compute as explained in section 7.2.

7.1 Exact solution

The tangential component of ∇h is zero on ∂B_1 due to the second condition in (1.4) and the normal component of ∇h has the following exact solution:

$$\partial_{\nu} h \Big|_{\partial B_{1}}(\theta) = \frac{\cosh \xi_{1} - \cos \theta}{a_{\epsilon}} \frac{\partial}{\partial \xi} h(\xi, \theta) \Big|_{\xi = -\xi_{1}}$$

$$= \frac{\sqrt{2} (\cosh \xi_{1} - \cos \theta)^{3/2}}{a_{\epsilon}} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \left(A_{n} e^{-(n + \frac{1}{2})\xi_{1}} - B_{n} e^{(n + \frac{1}{2})\xi_{1}} \right) P_{n}(\cos \theta)$$

$$+ \frac{\sinh(-\xi_{1}) (\cosh \xi_{1} - \cos \theta)^{1/2}}{\sqrt{2} a_{\epsilon}} \sum_{n=0}^{\infty} \left(A_{n} e^{-(n + \frac{1}{2})\xi_{1}} + B_{n} e^{(n + \frac{1}{2})\xi_{1}} \right) P_{n}(\cos \theta). \tag{7.1}$$

The normal derivative of (u-H) for given entire harmonic function H has a series representation similar to (7.1). Especially for the uniform external field, say $H(\mathbf{x}) = E_0 x_3$, the exact solution for u can be found in many literatures, for example [17]. To state the solution explicitly, we define

$$T(c) = \sum_{n=0}^{\infty} \frac{(2n+1)(e^{(2n+1)c}+1)}{e^{(2n+1)(\xi_1+\xi_2)}-1} \quad \text{for} \quad c > 0,$$

and

$$V_1 = -a_{\epsilon} \frac{T_2 U_1 - T_1 U_{12}}{U_1 U_2 - U_{12}^2}, \quad V_2 = a_{\epsilon} \frac{T_1 U_2 - T_2 U_{12}}{U_1 U_2 - U_{12}^2}$$

with $T_j = T(\xi_j)$, j = 1, 2. Then the solution u to (1.1) is represented as follows:

$$(u - H)(\xi, \theta) = \sqrt{2}E_0 \sqrt{\cosh \xi - \cos \theta} \sum_{n=0}^{\infty} \left(C_n e^{(n + \frac{1}{2})\xi} + D_n e^{-(n + \frac{1}{2})\xi} \right) P_n(\cos \theta),$$

where

$$C_n = \frac{e^{(2n+1)\xi_1}V_2 - V_1 - Ea_{\epsilon}(2n+1)(e^{(2n+1)\xi_1} + 1)}{e^{(2n+1)(\xi_1 + \xi_2)} - 1},$$

$$D_n = \frac{e^{(2n+1)\xi_2}V_1 - V_2 + Ea_{\epsilon}(2n+1)(e^{(2n+1)\xi_2} + 1)}{e^{(2n+1)(\xi_1 + \xi_2)} - 1}.$$

Hence, we have similarly to (7.1) that

$$\partial_{\nu}(u - H)\Big|_{\partial B_{1}}(\theta) = E_{0} \frac{\sqrt{2}(\cosh \xi_{1} - \cos \theta)^{3/2}}{a_{\epsilon}} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \left(C_{n}e^{-(n + \frac{1}{2})\xi_{1}} - D_{n}e^{(n + \frac{1}{2})\xi_{1}}\right) P_{n}(\cos \theta)$$

$$+ E_{0} \frac{\sinh(-\xi_{1})(\cosh \xi_{1} - \cos \theta)^{1/2}}{\sqrt{2}a_{\epsilon}} \sum_{n=0}^{\infty} \left(C_{n}e^{-(n + \frac{1}{2})\xi_{1}} + D_{n}e^{(n + \frac{1}{2})\xi_{1}}\right) P_{n}(\cos \theta). \tag{7.2}$$

7.2 Gradient blow-up terms

The singular function h satisfies from (5.15) and (4.5) that

$$\nabla h \big|_{\partial B_1} = \text{bounded term} + q_{\partial B_1}(\mathbf{x}) \nu(\mathbf{x}),$$

 $q_{\partial B_1}(\theta) = q_h(-\xi_1, \theta) \quad \text{for } \theta \in [0, \pi].$

From (5.3) and the definition of q_h in (5.13), we can easily derive

$$q_{\partial B_1}(\theta) = \frac{\tilde{\mu}_1}{2a_{\epsilon}(\xi_1 + \xi_2)} \left(w_{\theta}^2(\xi_1) + \frac{w_{\theta}^3(\xi_1)}{w_{\theta}(3\xi_1 + 2\xi_2)} \right) + \frac{\tilde{\mu}_2}{2a_{\epsilon}(\xi_1 + \xi_2)} \left(\frac{2w_{\theta}^3(\xi_1)}{w_{\theta}(\xi_1 + 2\xi_2)} \right). \tag{7.3}$$

Recalling $w_{\theta}(\xi) = \sqrt{\cosh \xi - \cos \theta}$, we see that the function $q_{\partial B_1}(\theta)$ consists of elementary functions which can be easily computed numerically. Similarly, the solution u to (1.1) for a given entire harmonic function H satisfies from Proposition 5.4 that

$$\nabla (u - H)|_{\partial B_1} = \text{bounded term} + C_H q_{\partial B_1}(\mathbf{x}) \nu(\mathbf{x}).$$

When an uniform field $H(\mathbf{x}) = E_0 x_3$ is applied, C_H becomes from Theorem 2.1(b) as follows:

$$C_H = E_0 Q_1(r_1, r_2) = 4\pi \tilde{r}^2 E_0 \left[(\mu_1 + \mu_2)(\pi^2/6) + \mu_1 \psi_1(\tilde{r}_2) + \mu_2 \psi_1(\tilde{r}_1) \right].$$

7.3 Examples

Data Acquisition We numerically compute $\partial_{\nu}h$ and $\partial_{\nu}(u-H)$ based on the exact solution (7.1) and (7.2). It is worth to mention the difficulty in the numerical computation of (7.1) and (7.2). Since the term $e^{-2n(\xi_1+\xi_2)}$ decays very slowly for small ϵ , the cost in numerical computation becomes very high. For instance, in Example 1, we evaluate the summation for $n \leq 5 \cdot 10^3$ to compute within a relative tolerance 10^{-5} when $\epsilon = 5 \cdot 10^{-5}$. On the other hand, the gradient blow-terms $q_{\partial B_1}$ and $C_H q_{\partial B_1}$ are consists of simple elementary functions, see (7.3). Hence, the computing cost is extremely low. For all examples, the radii of the two sphere are $r_1 = 3$ and $r_2 = 2$.

Example 1. In Fig. 7.1, we compare $\partial_{\nu}h|_{\partial B_1}$ and its blow-up term $q_{\partial B_1}$ when ϵ takes the values 0.5, 0.05, 0.00005 from left to right columns. We plot $\partial_{\nu}h|_{\partial B_1}$ (dashed graph) and $q_{\partial B_1}$ (solid graph) in the first row and the difference between them in the second row. Note that while the range of y-axis in the first row becomes huge for small ϵ , that in the second row is fixed. It means that the magnitudes of both $\partial_{\nu}h|_{\partial B_1}$ and $q_{\partial B_1}$ increase as ϵ decreases, but the difference between them decreases. Hence the blow-up term $q_{\partial B_1}$ represents $\partial_{\nu}h|_{\partial B_1}$ better when ϵ is smaller.

Example 2. In Table 2, we provide the values of $\partial_{\nu}(u-H)|_{\partial B_1}$ and its blow-up term $C_H q_{\partial B_1}$ for various ϵ and various bispherical coordinates values θ when an uniform external field $H(\mathbf{x}) = x_3$ is applied. The difference between $\partial_{\nu}(u-H)|_{\partial B_1}$ and its blow-up terms is of almost constant magnitude while the value of $\partial_{\nu}(u-H)|_{\partial B_1}$ is huge near $\theta = \pi$ for small ϵ .

8 Conclusion

In this paper we provided an asymptotic analysis for the superfocusing of the electric field due to the presence of two nearly touching perfectly conducting spheres. We expressed explicitly and completely the blow-up term of the electric field with the rigorous proof. The main ideas of this paper come from, firstly, the solution by separation of variables in the bispherical coordinates and, secondly, the idea to approximate the series solution by an integral function using the Euler-Maclaurin formula and, thirdly, the recent decomposition method to separate the blow-up term and the regular term in the electric field. The derived asymptotic formula is valid in the whole exterior region of the two spheres, and it explicitly characterizes superfocusing of the electric field.

θ (unit in π)	ϵ	$\partial_{\nu}(u-H)$	$C_H q_{\partial B_1}$	ϵ	$\partial_{\nu}(u-H)$	$C_H q_{\partial B_1}$
0	1	2.118	0.103	0.00005	1.4	0.3
0.15		2.380	0.195		1279.4	1278.8
0.30		2.754	0.505		4838.5	4837.8
0.45		3.200	1.031		9901.5	9900.8
0.60		3.807	1.668		15365.8	15364.1
0.75		4.420	2.251		20037.4	20036.6
0.90		4.827	2.621		22900.7	22900.0
1.00		4.911	2.700		23475.2	23474.5
0	0.5	1.875	0.089	0.000005	1.4	0.02
0.15	0.5	$\frac{1.873}{2.298}$	0.089 0.278	0.000003	10896	10896
0.30		2.877	0.963		41211	41211
0.45		3.938	2.131		84337	84336
0.60		5.360	3.511		130871	130871
0.75		6.688	4.746		170671	170670
0.90		7.529	5.519		195060	195060
1.00		7.700	5.675		199954	199953
0	0.05	1.586	0.059	5×10^{-7}	1.4	0.02
0.15		3.346	2.096		94900	94900
0.30		10.339	9.117		358914	358914
0.45		20.811	19.469		734490	734490
0.60		32.142	30.718		1.13976×10^{6}	1.13976×10^{6}
0.75		41.838	40.358		1.48637×10^{6}	1.48637×10^6
0.90		47.781	46.270		1.69878×10^{6}	1.69878×10^{6}
1.00		48.973	47.456		1.74140×10^{6}	1.74139×10^{6}
0	0.005	1 490	0.042	5×10^{-8}	1.4	0.02
	0.005	1.489	0.043	3 × 10	$1.4 \\ 840495$	0.02
0.15		19.800	18.855		3.17876×10^6	$840495 \\ 3.17876 \times 10^6$
0.30		74.331	73.323			
0.45		151.926	150.880		6.50509×10^6	6.50509×10^6
0.60		235.658	234.579		1.00944×10^7 1.31642×10^7	1.00944×10^7
0.75		307.272	306.166			1.31642×10^{7}
0.90		351.158	350.035		1.50454×10^{7}	1.50454×10^{7}
1.00		359.962	358.836		1.54228×10^7	1.54228×10^7
0	0.0005	1.44	0.03	5×10^{-9}	1.4	0.02
0.15		155.01	154.22		7.54254×10^6	7.54254×10^6
0.30		588.88	585.07		2.85260×10^{7}	2.85260×10^{7}
0.45		1198.81	1197.98		5.83762×10^{7}	5.83762×10^7
0.60		1860.20	1859.35		9.05863×10^{7}	9.05863×10^{7}
0.75		2425.87	2425.00		1.18135×10^{8}	1.18135×10^{8}
0.90		2772.52	2771.63		1.35017×10^{8}	1.35017×10^{8}
1.00		2842.07	2841.18		1.38404×10^{8}	1.38404×10^{8}

Table 2: comparison between the exact normal derivative $\partial_{\nu}(u-H)|_{\partial B_1}$ and its blow-up term $C_H q_{\partial B_1}$

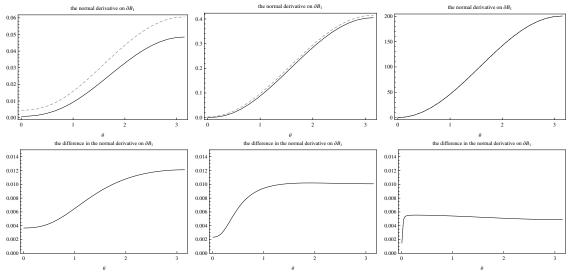


Figure 7.1: The graphs of $\partial_{\nu}h|_{\partial B_1}$ (dashed), its blow-up term $q_{\partial B_1}$ (solid) in the first row, and their difference $|\partial_{\nu}h|_{\partial B_1} - q_{\partial B_1}|$ in the second row. The distance ϵ is 0.5, 0.05, 0.00005 from left to right columns.

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