Fair Domination in Graphs

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Abstract

A fair dominating set in a graph G (or FD-set) is a dominating set S such that all vertices not in S are dominated by the same number of vertices from S; that is, every two vertices not in S have the same number of neighbors in S. The fair domination number, $\mathrm{fd}(G)$, of G is the minimum cardinality of a FD-set. We present various results on the fair domination number of a graph. In particular, we show that if G is a connected graph of order $n \geq 3$ with no isolated vertex, then $\mathrm{fd}(G) \leq n-2$, and we construct an infinite family of connected graphs achieving equality in this bound. We show that if G is a maximal outerplanar graph, then $\mathrm{fd}(G) < 17n/19$. If T is a tree of order $n \geq 2$, then we prove that $\mathrm{fd}(T) \leq n/2$ with equality if and only if T is the corona of a tree.

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1 Introduction

In this paper, we continue the study of domination in graphs. Domination in graphs is now well studied in Graph Theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [8, 9]. For notation and Graph Theory terminology we in general follow [8]. Specifically, let G = (V, E) be a graph with vertex set V of order n = |V| and edge set E of size m = |E|, and let v be a vertex in V. The open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$, while the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$.

Let G = (V, E) be a graph. A dominating set in G is a set D of vertices of G such that every vertex $v \in V$ is either in D or adjacent to a vertex of D. A vertex in D is said

to dominate a vertex outside D if they are adjacent in G. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$ -set.

Let G be a graph that is not the empty graph. For $k \geq 1$ an integer, a k-fair dominating set, abbreviated kFD-set, in G is a dominating set D such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. We note that the set D = V is a kFD-set since vacuously every vertex in $V \setminus D = \emptyset$ satisfies the desired property. The k-fair domination number of G, denoted by $\mathrm{fd}_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $\mathrm{fd}_k(G)$ is called a $\mathrm{fd}_k(G)$ -set. With this definition in mind, we point to a related problem on the so called (k, τ) -regular sets discussed in [2] (and all the references given there). Some reminiscent of this approach appears more explicitly in Proposition 7.

A fair dominating set, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. Thus a dominating set D is a FD-set in G if D = V or if $D \neq V$ and all vertices not in D are dominated by the same number of vertices from D; that is, $|N(u) \cap D| = |N(v) \cap D| > 0$ for every two vertices $u, v \in V \setminus D$. We remark that if $G \neq \overline{K}_n$, then G contains a vertex v that is not isolated in G and the set $V \setminus \{v\}$ is a FD-set in G. Hence every graph that is not empty has a FD-set of cardinality strictly less than its order. The fair domination number, denoted by $\mathrm{fd}(G)$, of a graph G that is not the empty graph is the minimum cardinality of a FD-set in G. By convention, if $G = \overline{K}_n$, we define $\mathrm{fd}(G) = n$. Hence if G is not the empty graph, then $\mathrm{fd}(G) = \min\{\mathrm{fd}_k(G)\}$, where the minimum is taken over all integers k where $1 \leq k \leq |V| - 1$. A FD-set of G of cardinality $\mathrm{fd}(G)$ is called a $\mathrm{fd}(G)$ -set. Every FD-set in a graph G is a dominating set in G. Hence we have the following observation.

Observation 1 Let G be a graph of order n. Then the following holds.

- (a) $\gamma(G) \leq \operatorname{fd}(G)$.
- (b) $fd(G) \leq n$, with equality if and only if $G = \overline{K}_n$.

We show later (see Corollary 9) that the result in Observation 1(b) can be improved as follows: if G is a graph of order n, then $\operatorname{fd}(G) \leq n-2$, unless $G = \overline{K}_n$, in which case $\operatorname{fd}(G) = n$, or G contains precisely one edge, in which case $\operatorname{fd}(G) = n-1$.

For example, consider the Petersen graph $G = G_{10}$ shown in Figure 1 which has domination number $\gamma(G) = 3$. Let V = V(G). The only possible $\gamma(G)$ -sets are the open neighborhoods D = N(v), where $v \in V$, but these are not FD-sets since if $u \in V \setminus D$, then either $u \neq v$, in which case $|N(u) \cap D| = 1$, or u = v, in which case $|N(u) \cap D| = 3$. Thus, by Observation 1(a), $\operatorname{fd}(G) > \gamma(G) = 3$. However the closed neighborhood N[v] of any vertex $v \in V$ forms a FD-set in G, and so $\operatorname{fd}(G) \leq |N[v]| = 4$. Consequently, $\operatorname{fd}(G) = 4$.

An out-regular set, abbreviated OR-set, of a graph G that is not the empty graph is a set Q of vertices such that $|N(u) \cap (V \setminus Q)| = |N(v) \cap (V \setminus Q)| > 0$ for every two vertices u and v in Q. We remark that if $G \neq \overline{K}_n$, then G contains a vertex v that is not isolated in G and the set $\{v\}$ is an OR-set in G. Hence every graph that is not empty has an OR-set. The out-regular number of a non-empty graph G, denoted by $\xi_{\rm or}(G)$, is the maximum cardinality of an OR-set. By convention, if $G = \overline{K}_n$, we define $\xi_{\rm or}(G) = 0$. An OR-set of

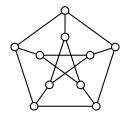


Figure 1: The Petersen graph G_{10} with $fd(G_{10}) = 4$.

G of cardinality $\xi_{\text{or}}(G)$ is called an $\xi_{\text{or}}(G)$ -set. Since every graph that is not empty has an OR-set, we have the following observation.

Observation 2 Let G be a graph of order n. Then, $\xi_{or}(G) \geq 0$, with equality if and only if $G = \overline{K}_n$.

If S is a packing in G (and so, the vertices in S are pairwise at distance at least 3 apart in G), then in order to dominate the vertices in S every dominating set in G must contain at least one vertex in N[v] for each $v \in S$, and so $\gamma(G) \geq |S|$. A perfect dominating set, abbreviated PD-set, in G is a dominating set S that is a packing in G. Thus if S is a PD-set in G, then every vertex not in S is dominated by a unique vertex in S, and so S is a 1FD-set, implying that $\mathrm{fd}(G) \leq \mathrm{fd}_1(G) \leq \gamma(G)$. Consequently, by Observation 1, we have the following observation.

Observation 3 If a graph G has a PD-set, then $\gamma(G) = \mathrm{fd}_1(G) = \mathrm{fd}(G)$.

1.1 Notation

We denote the degree of v in G by $d_G(v)$, or simply by d(v) if the graph G is clear from context. Let $\delta(G)$, $\Delta(G)$ and $\overline{d}(G)$ denote, respectively, the minimum degree, the maximum degree and the average degree in G. For a set $S \subseteq V$, we denote the number of vertices of S adjacent to v in G by $d_S(v)$. In particular, $d_V(v) = d_G(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by G[S]. We denote by $\operatorname{span}(G)$, the span of G, the number of distinct values in the degree sequence of G and by $\operatorname{rep}(G)$, the repetition number of G, the maximum multiplicity in the list of vertex degrees. The parameter $\alpha(G)$ denotes the (vertex) independence number of G, while $\chi(G)$ denotes the chromatic number of G.

Further, we denote the complete graph on n vertices by K_n and the empty graph on n vertices by \overline{K}_n . Moreover, P_n , C_n and $K_{m,n}$ denote, respectively, the path on n vertices, the cycle on n vertices and the complete bipartite graph with one partite set of cardinality m and the other of cardinality n.

2 Preliminary Results and Observations

It is a simple exercise to determine the fair domination number of certain well-studied families of graphs. Recall that for $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$, while for $m \geq n \geq 2$, $\gamma(K_{m,n}) = 2$.

Observation 4 For $m, n \ge 1$, if $G \in \{P_n, K_n, \overline{K}_n, K_{m,n}\}$, then $\operatorname{fd}(G) = \gamma(G)$. Further for $n \ge 3$, $\operatorname{fd}(C_n) = \gamma(C_n)$ unless $n \equiv 2 \pmod{3}$ and $n \ge 5$ in which case $\operatorname{fd}(C_n) = \gamma(C_n) + 1$.

We next establish a relationship between the fair domination number and the out-regular number of a graph.

Proposition 5 For every graph G of order $n \ge 2$, $fd(G) + \xi_{or}(G) = n$.

Proof. If $G = \overline{K}_n$, then $\operatorname{fd}(G) = n$ and, by convention, $\xi_{\operatorname{or}}(G) = 0$. Hence we may assume that $G \neq \overline{K}_n$, for otherwise the desired result holds. Let D be a $\operatorname{fd}(G)$ -set. By Observation 1(b), $\operatorname{fd}(G) < n$. Let $Q = V \setminus D$. Then, Q is an OR-set in G, and so $\xi_{\operatorname{or}}(G) \geq |Q| = n - \operatorname{fd}(G)$, or, equivalently, $\operatorname{fd}(G) + \xi_{\operatorname{or}}(G) \geq n$. Conversely, let Q be an $\xi_{\operatorname{or}}(G)$ -set. By Observation 2, $\xi_{\operatorname{or}}(G) > 0$. By definition, $\xi_{\operatorname{or}}(G) < n$. Let $D = V \setminus Q$. Then, D is a FD-set, and so $\operatorname{fd}(G) \leq |D| = n - \xi_{\operatorname{or}}(G)$, or, equivalently, $\operatorname{fd}(G) + \xi_{\operatorname{or}}(G) \leq n$. Consequently, $\operatorname{fd}(G) + \xi_{\operatorname{or}}(G) = n$. \square

Theorem 6 Let G be a connected graph on $n \geq 2$ vertices. Then the following holds.

- (a) If \overline{G} is connected, then $fd(G) = fd(\overline{G})$.
- (b) If \overline{G} has $q \geq 2$ components, then $fd(G) \leq n/q \leq n/2$.
- **Proof.** (a) Suppose that \overline{G} is connected. Let D be a $\operatorname{fd}(G)$ -set. Then every vertex $v \in V \setminus D$ is adjacent to precisely k vertices in D for some integer k, $1 \leq k \leq |D|$. If k = |D|, then in \overline{G} there are no edges between D and $V \setminus D$, contradicting the assumption that \overline{G} is connected. Hence, k < |D|. But then in \overline{G} every vertex in $V \setminus D$ is adjacent to precisely |D| k > 0 vertices in D, and so D is a FD-set in \overline{G} . Thus, $\operatorname{fd}(\overline{G}) \leq |D| = \operatorname{fd}(G)$. Reversing the roles of G and \overline{G} , we have that $\operatorname{fd}(G) \leq \operatorname{fd}(\overline{G})$. Consequently, $\operatorname{fd}(G) = \operatorname{fd}(\overline{G})$.
- (b) Suppose that \overline{G} is not connected and has q components. Clearly, the smallest component in \overline{G} has cardinality at most n/q. Let F be the smallest component in \overline{G} and let D = V(F). Then in G every vertex in $V \setminus D$ is adjacent to all vertices in D, and so D is a FD-set in G. Thus, $\operatorname{fd}(G) \leq |D| \leq n/q \leq n/2$. \square

Next we consider the fair domination number of the line graph, L(G), of a graph G.

Proposition 7 Let G be a graph of size m_G and let L(G) denote the line graph of G. If H is a spanning r-regular subgraph of G, where r > 0 and where H is not necessarily induced,

of size m_H , then

$$fd(L(G)) \le m_H = \left(\frac{m_H}{m_G}\right) |V(L(G))|.$$

Proof. Let D be the set of vertices in the line graph L(G) of G corresponding to the edges in H, and let $v \in V(L(G)) \setminus D$. Thus the vertex v corresponds to an edge $e \in E(G) \setminus E(H)$, and so both ends of e are incident with precisely r edges of H. Hence in L(G), the vertex v is adjacent to exactly 2r vertices in D. Thus in L(G), we have that $|N(v) \cap D| = 2r$ for every vertex $v \in V(L(G)) \setminus D$, implying that D is a FD-set in L(G), and so $\mathrm{fd}(L(G)) \leq |D| = m_H$. \square

We remark that examples of graphs that possess spanning r-regular subgraphs, where r > 0 and where H is not necessarily induced, are abundant. For example, regular graphs of even degree have a 2-factor as do Hamiltonian graphs. Several interesting families of graphs possess a 1-factor (or perfect matching), including regular bipartite graphs and connected claw-free graphs of even order. For further results about regular spanning graphs see for example [5, 6].

3 Results

3.1 Upper Bounds

We first establish upper bounds on the fair domination of a graph in terms of its order. By Observation 3(a), $fd(G) \leq n$ with equality if $G = \overline{K}_n$. However this bound can be improved slightly if we restrict our attention to graphs without isolated vertices.

Theorem 8 If G is a graph of order $n \geq 3$ with $\delta(G) \geq 1$, then $\operatorname{fd}(G) \leq n-2$, and this bound is sharp.

Proof. We proceed by induction on $n \geq 3$. If $n \in \{3,4\}$, it is a simple case to check that if G is a graph of order n with no isolated vertex, then $\operatorname{fd}(G) \leq n-2$. This establishes the base cases. Let $n \geq 5$ and assume that every graph G' of order n', where $3 \leq n' < n$, with no isolated vertex satisfies $\operatorname{fd}(G') \leq n' - 2$. Let G = (V, E) be a graph of order n with no isolated vertex. Every graph on at least two vertices has two vertices of the same degree. Let u and v be two vertices in G with the same degree. Then, $d_G(u) = d_G(v) = k$ for some k, where $1 \leq k \leq n-1$. If u and v are not adjacent in G or if u and v are adjacent in G and g and g are adjacent in g and g a

We show next that the upper bound is sharp. For this purpose we construct an infinite family of graphs G of order $n \geq 3$ with $\delta(G) \geq 1$ satisfying $\mathrm{fd}(G) = n - 2$. We consider two cases in turn, depending on the parity of n.

Claim I. There exists an infinite family of graphs H of even order n_H satisfying $fd(H) = n_H - 2$.

Proof. For $n \geq 3$ define the graph $H = H_n$ on $n_H = 2n$ vertices as follows: Let $V(H) = X \cup Y$, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, and where x_i is adjacent to y_j if and only if $i \geq j$. Further Y is an independent set and, for i, j > 1, x_i is adjacent to x_j . Thus, $d_H(x_1) = 1$ and $d_H(x_i) = i + n - 2$ for $1 < i \leq n$, while $d_H(y_i) = n - i + 1$ for $1 \leq i \leq n$. We note that $d_H(x_1) = d_H(y_n) = 1$ and $d_H(y_1) = d_H(x_2) = n$ and the degrees of all other vertices in H are distinct. We show that $f(H) = 2n - 2 = n_H - 2$. Let S be an arbitrary FD-set of H. We show that $|S| \geq n_H - 2$. We consider three cases in turn.

Case I.1. $x_1 \notin S$. Then evidently $y_1 \in S$ and S is a 1FD-set of H_n . If $x_i \notin S$ for some $i \geq 2$, then $N(x_i) \cap S = \{y_1\}$ and thus $y_2 \notin S$. Since y_2 has to be dominated by S, there is some $l \geq 2$, $l \neq i$, such that $x_l \in S$, implying that $y_1, x_l \in N(x_i) \cap S$, a contradiction. Hence, $X \setminus \{x_1\} \subseteq S$. Moreover, $y_j \in S$ for any 1 < j < n, since $|N(y_j) \cap S| \geq |N(y_j) \cap (X \setminus \{x_1\})| = n - j + 1 \geq 2$ and S is a 1FD-set. Thus $V \setminus \{x_1, y_n\} \subseteq S$ and $|S| \geq n_H - 2$, so we are done.

Case I.2. $x_1 \in S$ and $y_n \notin S$. Then evidently $x_n \in S$ and S is a 1FD-set of H_n . Since $x_1, x_n \in N(y_1) \cap S$, it follows that $y_1 \in S$. Hence, as $x_n, y_1 \in N(x_i) \cap S$ for any 1 < i < n, we have that $X \subseteq S$. This implies also that $y_j \in S$ for any 1 < j < n and thus $V \setminus \{y_n\} \subseteq S$ and therefore $|S| \ge n_H - 1$.

Case I.3. $x_1, y_n \in S$. We divide this case in three parts.

- (i) Suppose that $y_1 \notin S$. Then $y_j \in S$ for every 1 < j < n, otherwise $|N(y_1) \cap S| > |N(y_j) \cap S|$ since $N(y_j) \subset N(y_1)$ and $x_1 \in N(y_1) \setminus N(y_j)$. As $N(x_i) \subsetneq N(x_{i+1})$ for all 1 < i < n-1, it follows that there is at most one index 1 < i < n such that $x_i \notin S$, implying that $|S| \geq n_H 2$.
- (ii) Suppose that $y_1 \in S$ and $x_n \notin S$. Then as $N(x_i) \subset N(x_n)$ and $y_n \in S \cap (N(x_n) \setminus N(x_i))$ for each 1 < i < n, it follows that $X \setminus \{x_n\} \subset S$. Hence, as every y_j has a different number of neighbors in $X \setminus \{x_n\}$, $y_j \in S$ for all 1 < j < n. Hence $|S| \ge n_H 1$.
- (iii) Suppose that $y_1 \in S$ and $x_m \in S$. If $x_i \notin S$ for some 1 < i < m, then $y_j \in S$ for every 1 < j < m, since $N(y_j) \subset N(x_i)$ and $y_1 \in S \cap (N(x_i) \setminus N(y_j))$. Then $X \setminus \{x_i\} \subset S$ and thus $|S| \ge n_H 1$ and we are done. Thus assume that $X \subset S$. Then $|Y \cap S| \ge n 1$, otherwise two different vertices from Y would not be fairly dominated by S. Hence again $|S| \ge n_H 1$.

In all three cases, we have that $|S| \ge n_H - 2$. Since S is an arbitrary FD-set of H, it follows that $\mathrm{fd}(H) \ge n_H - 2$. However as shown earlier, $\mathrm{fd}(H) \le n_H - 2$. Consequently, $\mathrm{fd}(H) = n_H - 2$. This completes the proof of Claim I. (\square)

Claim II. There exists an infinite family of graphs F of odd order n_F satisfying $fd(F) = n_F - 2$.

Proof. For $n \geq 3$ define the graph $F = F_n$ on $n_F = 2n + 1$ vertices as follows: Let F be obtained from the graph H_n of even order $n_H = 2n$ defined in Claim I by adding a new vertex x_{n+1} and joining it to every vertex in $X \setminus \{x_1\}$. Let $X_F = X \cup \{x_{n+1}\}$. For the proof, consider an arbitrary FD-set of F and show that $|S| \geq n_F - 2$. The proof to show that for an arbitrary FD-set S of F we have $|S| \geq n_F - 2$ is similar to the proof presented in Claim I and is therefore omitted. (\Box)

By Claim I and Claim II, there is an infinite family of graphs G of order $n \geq 6$ with $\delta(G) \geq 1$ satisfying $\mathrm{fd}(G) = n - 2$, irrespective of whether n is even or odd. \square

Note that, by Claims I and II, for each integer $n \geq 6$, there is a connected graph G on n vertices satisfying fd(G) = n - 2. If $n \in \{3, 4, 5\}$, we can simply take $G = C_n$. Hence for all $n \geq 3$, there exists a connected graph G on n vertices satisfying fd(G) = n - 2.

Let G be a graph of order n with at least two edges. Let G^* be the subgraph obtained from G by deleting all isolated vertices in G, if any. Then, $\delta(G^*) \geq 1$ and G^* has order $n^* \geq 3$. Applying Theorem 8 to G^* , we have that $\mathrm{fd}(G^*) \leq n^* - 2$. Since every $\mathrm{fd}(G^*)$ -set can be extended to a FD-set in G by adding to it the set of isolated vertices in G, we have that $\mathrm{fd}(G) \leq n - 2$. Hence as a consequence of Theorem 8, we have the following result.

Corollary 9 If G is a graph of order n and size at least 2, then $fd(G) \leq n-2$.

We next present some upper bounds on the fair domination number in terms of its order, chromatic number and average, maximum and minimum degrees. For this purpose, we first recall the Caro-Wei Theorem (see [3, 14]).

Caro-Wei Theorem. For every graph G of order n,

$$\alpha(G) \ge \sum_{v \in V(G)} \frac{1}{1 + d_G(v)} \ge \frac{n}{\overline{d}(G) + 1}.$$

For our purposes, we also need the following useful lower bound on the repetition number of a graph, established by Caro and West in [4].

Caro-West Lemma. If G is a graph of order n, then $rep(G) \ge n/(2\overline{d}(G) - 2\delta(G) + 1)$.

Proposition 10 Let G be a graph of order n. Then the following holds.

- (a) If $n \geq 2$ and $\delta(G) \geq 1$, then $\operatorname{fd}(G) \leq n n/((\overline{d}(G) + 1)\Delta(G))$.
- (b) $fd(G) \le n n/(2\overline{d}(G) 2\delta(G) + 1)\chi(G)$.
- (c) For $r \geq 2$, if G is an r-regular graph, then $fd(G) \leq rn/(r+1)$.

- **Proof.** (a) Let B be a maximum independent set in G. Since G has no isolated vertex, the set $V \setminus B$ is a dominating set in G. We now consider the degrees of the vertices of B. There are at most $\operatorname{span}(G)$ possible distinct values for these degrees, and so, by the Pigeonhole Principle, at least one value, say q, appears at least $\alpha(G)/\operatorname{span}(G)$ times. Let Q be the set of all vertices in B with degree q. Then, $Q \subseteq B$ and |Q| = q. Let $D = V \setminus Q$. Then, D is a FD-set, and so $\operatorname{fd}(G) \leq n |Q| \leq n \alpha(G)/\operatorname{span}(G)$. The desired result now follows from the observation that $\operatorname{span}(G) \leq \Delta(G)$ and from the Caro-Wei Theorem.
- (b) Let $\operatorname{rep}(G) = m$ and suppose that $X = \{v_1, \ldots, v_m\}$ is a set of vertices with the same degree in G. Let H = G[X] be the subgraph induced by the set X. Clearly, $\chi(H) \leq \chi(G)$, and so $\alpha(H) \geq m/\chi(H) \geq m/\chi(G) \geq n/(2\overline{d}(G) 2\delta(G) + 1)\chi(G)$ by the Caro-West Lemma. Let Q be a maximum independent set in H, and so $|Q| = \alpha(H)$. Let $D = V(G) \setminus Q$. Then, D is a FD-set in G, and so $\operatorname{fd}(G) \leq n |Q| \leq n \alpha(H)$, and the desired result follows.
- (c) Since G is an r-regular graph, we note that $\overline{d}(G) = \delta(G)$ and $\chi(G) \leq r + 1$ and the result follows from Part (b). \square

Note that, in case that G is a regular graph, the complement of a fair dominating set of G is an induced regular subgraph of G. At this point, it is worth mentioning the famous Erdős-Fajtlowicz-Staton problem (see [1]) about the largest induced regular subgraph of a graph G. In the case when G is regular, the complement of such a subgraph is a fair dominating set of G.

Proposition 11 For $r \ge 1$, if G is an r-regular graph on n vertices, then $\operatorname{fd}(G) \le n - c \log n$ for some c > 0.

Proof. Let \mathcal{G}_n denote the family of all graphs of order n. By Ramsey's theory, for all graphs $G \in \mathcal{G}_n$ we have $\max\{\alpha(G), \alpha(\overline{G})\} \geq c \log n$ for some constant c. For $r \geq 1$, let G be an r-regular graph in \mathcal{G}_n . Then, G contains either an independent set or a clique of order at least $c \log n$ for some constant c. Let X be the vertex set of such an independent set or clique in G. Then, $|X| \geq c \log n$ and the subgraph G[X] induced by X is s-regular for some s. If X is an independent set in G, then s = 0, while if X is a clique, then $s = |X| - 1 \leq r$. Further if |X| - 1 = r, then we note that K_{r+1} is a component of G. On the one hand, if s < r, then for every vertex $v \in X$, we have $|N(v) \cap (V \setminus X)| = r - s > 0$ and hence $V \setminus X$ is a FD-set of G, and so $\mathrm{fd}(G) \leq n - |X| \leq n - c \log n$. On the other hand, if s = r, then we choose a vertex $v \in X$ and define $D = (V \setminus X) \cup \{v\}$. Then the set D is a FD-set of G, and so $\mathrm{fd}(G) \leq n - |X| + 1 \leq n - c \log n + 1 = n - c^* \log n$ for some constant c^* . \square

Proposition 12 If G is a connected graph on $n \ge 6$ vertices satisfying fd(G) = n - 2, then $2 \le rep(G) \le 4$.

Proof. A well-known elementary exercise states that every graph of order at least two has two vertices with the same degree, and so $\operatorname{rep}(G) \geq 2$. Hence it suffices for us to prove that $\operatorname{rep}(G) \leq 4$. Assume for the sake of contradiction, that $\operatorname{rep}(G) \geq 5$. By Ramsey Theory, $r(K_3, K_3) = 6$. Further if G is a graph on five vertices such that neither G nor

its complement \overline{G} contains a copy of K_3 , then $G = C_5$. Thus since $\operatorname{rep}(G) \geq 5$ there are either three vertices of the same degree in G that induce an independent set or clique in G or there are five vertices of the same degree in G that induce a C_5 . Let X be the vertex set of such an independent set or clique in G of cardinality 3, if it exists; otherwise let X be the vertex set of such an induced 5-cycle in G. Then, G[X] is s-regular with s = 2 or s = 0. Since G is connected and the vertices in X have all the same degree in G, they also must have the same number of neighbors in $V \setminus X$ (which is non-empty since $n \geq 6$), implying that $\operatorname{fd}(G) \leq n - 3$ or $\operatorname{fd}(G) \leq n - 5$ in case case $G[X] = C_5$. In all cases, we have $\operatorname{fd}(G) < n - 2$, a contradiction. Therefore, $\operatorname{rep}(G) \leq 4$. \square

We remark that the restriction on the order $n \geq 6$ in the statement of Proposition 12 is necessary since $G = C_5$ has order n = 5 and satisfies $\operatorname{rep}(G) = 5$ and $\operatorname{fd}(G) = 3 = n - 2$. Both Proposition 11 and Proposition 12 give further evidence that a connected graph of large order achieving the upper bound in Theorem 8 is highly non-regular. Further, Proposition 11 gives a better bound than Proposition $\operatorname{10}(a)$ when $r \geq \sqrt{n/\log n}$ and a better bound than Proposition $\operatorname{10}(b)$ when $\chi(G) \geq n/\log n$.

3.2 Trees

In this section, we focus our attention on trees. We shall need the following notation. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. The set of support vertices of a tree T is denoted by S_T , while the set of leaves by L_T . A neighbor of a vertex v that is a leaf we call a *leaf-neighbor* of v. A *strong support vertex* is a vertex adjacent to at least two leaves. The *corona* of a graph H, denoted by cor(H), is the graph of order 2|V(H)| obtained from H by attaching a leaf to each vertex of H. We note that every vertex of cor(H) is a leaf or is a support vertex with exactly one leaf-neighbor.

If we restrict our attention to trees, then the bound in Theorem 8 can be improved significantly. For this purpose, we recall that a classical result of Ore [12] established that if G is a graph of order n with no isolated vertex, then $\gamma(G) \leq n/2$. Further, Payan and Xuong [13] showed that the only connected graphs achieving equality in this bound are the 4-cycle C_4 and the corona cor(H) for a connected graph H. We next establish an upper bound on the fair domination number of a tree and characterize the extremal trees. We begin with the following two observations.

Observation 13 Every 1FD-set in a graph contains all its strong support vertices.

Proof. Let G be a graph and let D be a 1FD-set in G. Let v be an arbitrary strong support vertex in G. If $v \notin D$, then in order to dominate the leaf-neighbors of v, every leaf-neighbor of v belongs to D. Since v has at least two leaf-neighbors, this implies that $|N(v) \cap D| \geq 2$, a contradiction. Hence, $v \in D$. \square

Observation 14 If T is the corona of a tree and T has order n, then fd(T) = n/2. Further, V(T) can be partitioned into two fd(T)-sets.

Proof. The result is trivial for n=2. Hence we may assume that T is the corona of a tree and $n \geq 4$. Then, $\gamma(T) = n/2$ and we note that $|S_T| = |L_T| = n/2$. Further both sets S_T and L_T form a 1FD-set of T, and so $\operatorname{fd}(T) \leq \operatorname{fd}_1(T) \leq n/2$. Since every FD-set of T is a dominating set, we have that $n/2 = \gamma(T) \leq \operatorname{fd}(T) \leq n/2$. Consequently, we must have equality throughout this inequality chain. In particular, $\operatorname{fd}(T) = n/2$ and both S_T and L_T are $\operatorname{fd}(T)$ -sets. \square

We are now in a position to prove the following result. In the proof, we will deal with what we call 3-end-paths, which are paths xyz in a tree T such that x is a leaf, $N(y) = \{x, z\}$ and $d_T(z) \geq 2$. We will call z the base vertex of the 3-end-path. Note that, since every tree has at least two leaves, the corona of a tree on at least three vertices has at least two 3-end-paths sharing at most their base vertices.

Theorem 15 If T is a tree of order $n \ge 2$, then $\mathrm{fd}_1(T) \le n/2$ with equality if and only if T is the corona of a tree.

Proof. By Observation 14 if T is the corona of a tree, then fd(T) = n/2 and both S_T and L_T are 1FD-sets. We will prove the statement by induction on n. If $2 \le n \le 7$, this follows directly by checking all possible trees. This establishes the case cases. For the inductive hypothesis, let $n \ge 8$ and assume that every tree T' of order n', where $2 \le n' < n$, satisfies $fd_1(T) \le n'/2$, with equality only if T' is the corona of a tree. Let T be a tree of order n. If T is a star $K_{1,n}$, then the central vertex of T is a 1FD-set, implying that fd(T) = 1 < n/2 and we are done. Hence we may assume that T is not a star. Therefore, T contains a vertex w all of whose neighbors except for one, say y, are leaves. Let t be the number of leaf-neighbors of w, and so t = d(w) - 1. We distinguish the following cases.

Case 1. Suppose that t=1. Then, d(w)=2. Let z be the leaf-neighbor of w. Then, ywz is a 3-end-path in T with y as its base vertex. Let $T^*=T-\{w,z\}$ have order n^* , and so $n^*=n-2$. Applying the inductive hypothesis to T^* , $\mathrm{fd}_1(T^*) \leq n^*/2 = n/2 - 1$, with equality only if T^* is the corona of a tree. Let D^* be a $\mathrm{fd}_1(T^*)$ -set. Suppose first that T^* is not the corona of a tree. Then, $|D^*| < n/2 - 1$. Moreover, T is not the corona of a tree. If $y \in D^*$, then let $D = D^* \cup \{w\}$. If $y \notin D^*$, then let $D = D^* \cup \{z\}$. In both cases D is a 1FD-set of T, and so $\mathrm{fd}_1(T) \leq |D| < (n/2 - 1) + 1 = n/2$, and we are done. Hence we may assume that T^* is the corona of a tree. Then, y is either a leaf or a support vertex of T^* .

Suppose that y is a leaf of T^* . Since T^* has at least five vertices, it contains at least two 3-end-paths that are vertex disjoint or that have at most their base vertices in common. Therefore, T^* contains a 3-end-path, say abc where c is the base vertex of the path, that has no vertex from N[y]. We now consider the tree $T^{**} = T - \{a, b\}$. In T^{**} we note that the vertex y has degree 2 and has no leaf neighbor. Hence, T^{**} is not the corona of a tree. Applying the inductive hypothesis to T^{**} , $\operatorname{fd}_1(T^{**}) < n/2 - 1$. As above, every $\operatorname{fd}_1(T^{**})$ can be extended to a 1FD-set of T by adding to it either w or z, implying that $\operatorname{fd}_1(T) < n/2$ and we are done. Hence we may assume that y is a support vertex of T^* . Then, T is also a corona of a tree and hence, by Observation 14, $\operatorname{fd}(T) = n/2$ and we are done.

Case 2. Suppose that $t \geq 2$. Then evidently T is not the corona of a tree. Let $x_1, x_2, \ldots x_t$ be the t neighbor leaves of w and let $T^* = T - \{x_1, x_2, \ldots, x_t\}$. Suppose first that T^* is not the corona of a tree. Applying the induction hypothesis to T^* , we have that $\operatorname{fd}(T^*) < (n-t)/2$. Let D^* be a $\operatorname{fd}_1(T^*)$ -set. If $y \in D^*$, then let $D = D^* \cup \{w\}$. If $y \notin D^*$, then $w \in D^*$ and let $D = D^*$. In both cases, D is a 1FD-set of T, and so $\operatorname{fd}(T) \leq |D| \leq |D^*| + 1 < (n-t)/2 + 1 \leq n/2$, and we are done. Hence we may assume that T^* is the corona of a tree. By Observation 14, $\operatorname{fd}(T^*) = (n-t)/2$ and L_{T^*} is a $\operatorname{fd}_1(T^*)$ -set. Since $w \in L_{T^*}$, it follows that L_{T^*} is also a 1FD-set of T, implying that $\operatorname{fd}(T) \leq (n-t)/2 < n/2$, and we are done. \square

As an immediate consequence of Observation 14 and Theorem 15, we have the following result.

Corollary 16 If T is a tree of order $n \geq 2$, then $fd(T) \leq n/2$ with equality if and only if T is the corona of a tree.

Recall that the k-domination number $\gamma_k(G)$ of a graph G is the cardinality of a minimum k-dominating set, i.e., a set D of vertices such that every vertex outside D has at least k neighbors in D. In [7], Fink and Jacobson show that $\gamma_2(T) \geq \lceil (n+1)/2 \rceil$ holds for every tree T on n vertices. This fact and Theorem 15 allow us to prove that any minimum FD-set in a tree is a 1FD-set.

Observation 17 In a tree, every minimum FD-set is a 1FD-set.

Proof. Let T be a tree and let D be a $\operatorname{fd}(T)$ -set. Then, D is a kFD-set for some $k \geq 1$. Suppose that $k \geq 2$. Then, $|N(x) \cap D| = k \geq 2$ for all $x \in V \setminus D$, implying that D is a 2-dominating set of T and thus $|D| \geq \gamma_2(T)$. Hence by the result of Fink and Jacobson, we have that $\operatorname{fd}(T) = |D| \geq \gamma_2(T) \geq \lceil (n+1)/2 \rceil > n/2$, contradicting Theorem 15. Hence, k = 1, and so D is a 1FD-set. \square

The set of non-leaves in a tree is a FD-set in the tree, implying the following observation.

Observation 18 If T is a tree on $n \geq 3$ vertices with ℓ leaves, then $\operatorname{fd}(T) \leq n - \ell$.

We remark that if a tree has more leaves than internal vertices, then the upper bound on the fair domination number of a tree given by Observation 18 is better than the upper bound of Theorem 15. The next theorem characterizes the trees where the set of non-leaves is not a minimum FD-set. If H is a subtree of a tree T such that H is the corona of a tree, $H \neq T$ and $N(x) \cap S_H = \emptyset$ for every vertex $x \in V \setminus V(H)$, then we call H a special corona-subtree of T.

Theorem 19 Let T = (V, E) be a tree on $n \ge 3$ vertices with ℓ leaves. Then the following assertions are equivalent:

- (i) $fd(T) < n \ell$.
- (ii) For every fd(T)-set D, there are two vertices in $V \setminus D$ that are adjacent.
- (iii) The tree T contains a special corona-subtree.

Proof. We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Let D be an arbitrary fd(T)-set. By Observation 17, the set D is a 1FD-set in T.

- $(i) \Rightarrow (ii)$: Suppose that $\operatorname{fd}(T) < n \ell$. Then, $D \neq V \setminus L_T$ and thus either $D \subset V \setminus L_T$ (but $D \neq V \setminus L_T$) or $L_T \cap D \neq \emptyset$. Suppose $D \subset V \setminus L_T$ and let $x \in V \setminus (D \cup L_T)$. Since $d_T(x) \geq 2$ and $|N(x) \cap D| = 1$, there is a vertex $y \in N(x) \setminus D$ and (ii) holds. Hence we may assume that $L_T \cap D \neq \emptyset$. Let $z \in L_T \cap D$ and let $N(z) = \{y\}$. If $y \in D$, then $D \setminus \{z\}$ is a 1FD-set in T, contradicting the minimality of D. Hence, $y \notin D$, implying that $N(y) \cap D = \{z\}$. Since $n \geq 3$, there is a vertex $x \in N(y) \setminus D$ and, once again, (ii) holds.
- $(ii) \Rightarrow (iii)$: Suppose that for every $\mathrm{fd}(T)$ -set S there are two vertices in $V \setminus S$ that are adjacent. Let x and y be two adjacent vertices in $V \setminus D$. We note that $x, y \in V \setminus L_T$. Let $x' \in N(x) \cap D$ and $y' \in N(y) \cap D$, and consider the subtree $T' = T[\{x, x', y, y'\}]$ of T. Then, $L_{T'} = V(T') \cap D = \{x, y\}$ and $S_{L'} = V(H) \setminus L(T') = \{x, y\}$. Among all subtrees of T containing x and y with the property that the leaves of the subtree are vertices of D, let H be one of maximum order. Thus, $\{x, y\} \subset V(H)$, $L_H = V(H) \cap D$ and H is such a tree of maximum order. Let $S = V(H) \setminus L_H$.

We show that H is the corona of a tree. It suffices to show that $|N(u) \cap L_H| = 1$ for all $u \in S$. Let $u \in S$ and suppose first that $N(u) \cap L_H = \emptyset$. Since $L_H = V(H) \cap D$, there is a vertex $y \in V \setminus V(H)$ such that $y \in D$ and $y \in N(u)$. But then adding the vertex y and the edge uy to the tree H produces a tree H' such that $\{x,y\} \subset V(H')$ and $L_{H'} = V(H') \cap D$, contradicting the maximality of H. Hence, $|N(u) \cap L_H| \ge 1$ for all $u \in S$. If $|N(u) \cap L_H| > 1$ for some $u \in S$, then $|N(u) \cap D| > 1$, contradicting the fact that D is a 1FD-set. Hence, $|N(u) \cap L_H| = 1$ for all $u \in S$, implying that H is the corona of a tree and $S_H = S$. If T is the corona of a tree, then by Observation 14, the set S_T is a fd(T)-set that does not satisfy (ii), a contradiction. Hence, T is not the corona of a tree. Thus, $H \neq T$.

Let $z \in V \setminus V(H)$ and suppose that there is a vertex $v \in N(z) \cap S$. Let w be the vertex of D that is adjacent to z in T. Since T is cycle-free, $w \in V \setminus V(H)$. But then adding the vertex z and the edge wz to the tree H produces a tree H' such that $\{x,y\} \subset V(H')$ and $L_{H'} = V(H') \cap D$, contradicting the maximality of H. Hence, $N(z) \cap S = \emptyset$. This is true for all vertices $z \in V \setminus V(H)$. Consequently, H is a special corona-subtree of T, and so (iii) holds.

 $(iii) \Rightarrow (i)$: Suppose that the tree T contains a special corona-subtree H. Let $D = (V \setminus (L_T \cup S_H)) \cup (L_H \cap L_T)$. Let $v \in V \setminus D = S_H \cup (L_T \setminus V(H))$. If $v \in S_H$, then since H is a special corona-subtree of T and $L_H \subseteq D$, we have that $|N_T(v) \cap D| = |N_T(v) \cap L_H| = 1$. If $v \in L_T \setminus V(H)$, then since the neighbor of v in T belongs to D, we once again have that $|N_T(v) \cap D| = 1$. Hence, D is a 1FD-set of T, and so $\mathrm{fd}(T) \leq |D| = |V \setminus L_T| - |S_H| + |L_H \cap L_T|$. Since H is a special corona-subtree of T, at least one leaf of T is not a leaf of H. Therefore, $|L_H \cap L_T| < |L_H| = |S_H|$, implying that $\mathrm{fd}(T) < |V \setminus L_T| = n - \ell$, and so (i) holds. \square

As an immediate consequence of Theorem19, we obtain the following characterization of the trees whose set of non-leaves is a minimum FD-set.

Corollary 20 Let T = (V, E) be a tree on $n \ge 3$ vertices with ℓ leaves. Then the following assertions are equivalent:

- (i) $fd(T) = n \ell$.
- (ii) There is a fd(T)-set D such that $V \setminus D$ is an independent set.
- (iii) The tree T contains no special corona-subtree.

3.3 Maximal Outerplanar Graphs

A maximal outerplanar graph, abbreviated MOP, is a triangulation of the polygon. It is well-known that every bounded face of a MOP is a triangle. Further a MOP on n vertices is 3-colorable, 2-degenerate (i.e., every subgraph contains a vertex with degree at most two), has exactly 2n-3 edges and the neighborhood of every vertex of the graph induces a path (see [10, 11]). In particular it follows that every MOP G on n vertices has $\delta(G) = 2$ and that the vertices of degree 2 are independent when $n \geq 4$. Note also that $\overline{d}(G) = 4 - 6/n$ and hence from Proposition 10(b) it follows that $\mathrm{fd}(G) \leq 14n/15$. With more effort, this bound can be improved to 17n/19, as we will show in the next theorem. But first, we need to prove the following lemma.

Lemma 21 If G is a MOP on $n \ge 3$ vertices, then the vertices of degree 3 induce a bipartite graph.

Proof. We proceed by induction on n. If n=3 or n=4, then the theorem holds trivially. This establishes the base cases. Let $n \geq 5$ and assume that in every MOP of order n', where $3 \le n' < n$, the vertices of degree 3 induce a bipartite graph. Let G be a MOP on n vertices. Since G is maximal outerplanar, there is a vertex x of degree 2 whose deletion results in a graph G^* on n-1 vertices which is again a MOP. Let u and v be the neighbors of x in G. Then, u and v are adjacent. Further since the vertices of degree 2 in G are independent, we have that $d_G(u) \geq 3$ and $d_G(v) \geq 3$. If $d_G(u) = 3$ and $d_G(v) = 3$, then u and v would be two adjacent vertices of degree 2 in the MOP G^* of order at least 4, which is not allowed. Hence, renaming u and v if necessary, we may assume that $d_G(u) \geq 4$ and $d_G(v) \geq 3$. Let B and B^* be the sets of vertices of degree 3 in G and G^* , respectively. By the induction hypothesis, the set B^* induces a bipartite graph. If $d_G(v) \geq 4$, then $B = B^* \setminus \{u, v\}$ and B induces also a bipartite graph in G, as desired. Hence we may assume that $d_G(v) = 3$. If $d_G(u) \geq 5$, then $u, v \notin B^*$ and let $B = B^* \cup \{v\}$. If $d_G(u) = 4$, then let $B = (B^* \setminus \{u\}) \cup \{v\}$. In both cases, the vertex v is adjacent to at most one vertex of degree 3 in G, say z, and so v can be added to a partite set of $G^*[B^*]$ that does not contain z, showing then that G[B]is bipartite. \square

We are now in a position to present the following upper bound on the fair domination number of a MOP. **Theorem 22** If G is a MOP on $n \ge 3$ vertices, then fd(G) < 17n/19.

Proof. For n=3, then $\operatorname{fd}(G)=1<17n/19$. Hence we may assume that $n\geq 4$. For i=2,3,4,5, let V_i denote the set of vertices of degree i in G and let $|V_i|=n_i$. Moreover let t be the number of vertices of degree at least 6 in G. Then, $n_2+n_3+n_4+n_5+t=n$ and $2n_2+3n_3+4n_4+5n_5+6t\leq 2m=2(2n-3)=4n-6$, counting first vertices and then edges. If $n_2\leq 2n/19$, $n_3\leq 4n/19$, $n_4\leq 6n/19$ and $n_5\leq 6n/19$, then

$$4n-6 \geq 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6t$$

$$= 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6(n - (n_2 + n_3 + n_4 + n_5))$$

$$= 6n - 4n_2 - 3n_3 - 2n_4 - n_5$$

$$\geq 6n - \frac{8}{19}n - \frac{12}{19}n - \frac{6}{19}n$$

$$= 4n,$$

a contradiction. Hence, $n_2 > 2n/19$, $n_3 > 4n/19$, $n_4 > 6n/19$ or $n_5 > 6n/19$. Suppose first that $n_2 > 2n/19$. Then, as $n \ge 4$, the set V_2 is independent and is therefore an outregular set, and so $\xi_{\rm or}(G) \ge n_2 > 2n/19$. Hence by Proposition 5, we have that ${\rm fd}(G) = n - \xi_{\rm or}(G) < 17n/19$. Suppose $n_3 > 4n/19$. Let V_3' be a maximum independent subset of V_3 . By Lemma 21, the graph $G[V_3]$ is a bipartite graph, and so $|V_3'| \ge |V_3|/2 > 2n/19$. Since V_3' is an out-regular set, we have that $\xi_{\rm or}(G) \ge |V_3'| > 2n/19$. Hence by Proposition 5, we have that ${\rm fd}(G) < 17n/19$. Suppose $n_4 > 6n/19$. Let V_4' be a maximum independent subset of V_4 . Since G is 3-colorable, we have that $|V_4'| \ge |V_4|/3 > 2n/19$. The set V_4' is an out-regular set, and so $\xi_{\rm or}(G) \ge |V_4'| > 2n/19$, implying by Proposition 5 that ${\rm fd}(G) < 17n/19$. Analogously if $n_5 > 6n/19$, then ${\rm fd}(G) < 17n/19$. In all four cases, we have ${\rm fd}(G) < 17n/19$, as desired. \square

3.4 Nordhaus-Gaddum-Type Bounds

In this section we consider Nordhaus-Gaddum-type bounds for the fair domination number of a graph.

Theorem 23 Let G be a graph on n vertices. Then the following holds.

- (a) If $n \ge 5$, then $3 \le \operatorname{fd}(G) + \operatorname{fd}(\overline{G}) \le 2n 4$ and both bounds are sharp.
- (b) If $n \geq 4$, then $2 \leq \operatorname{fd}(G) \cdot \operatorname{fd}(\overline{G}) \leq (n-2)^2$ and both bounds are sharp.

Proof. We will first prove both upper bounds. Without loss of generality, we may assume that G is connected, and so, by Theorem 8, $\operatorname{fd}(G) \leq n-2$. If \overline{G} has size $m(\overline{G}) \geq 2$, then by Corollary 9, $\operatorname{fd}(\overline{G}) \leq n-2$, and so $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) \leq 2n-4$ and $\operatorname{fd}(G) \cdot \operatorname{fd}(\overline{G}) \leq (n-2)^2$, as desired. If $m(\overline{G}) = 1$, then $\overline{G} = \overline{K}_{n-2} \cup K_2$ and $G = K_n - e$ for some edge e of K_n . Hence, $\operatorname{fd}(G) = 1$ and $\operatorname{fd}(\overline{G}) = n-1$, implying $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) = n \leq 2n-4$ and $\operatorname{fd}(G) \cdot \operatorname{fd}(\overline{G}) = n-1 \leq (n-2)^2$ when $n \geq 4$. Finally, if $m(\overline{G}) = 0$, then $\overline{G} = \overline{K}_n$ and $G = K_n$. In this case we have $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) = 1 + n \leq 2n-4$, for $n \geq 5$, and

 $fd(G) \cdot fd(\overline{G}) = n \le (n-2)^2$, for $n \ge 4$. In all cases, $fd(G) + fd(\overline{G}) \le 2n-4$, for $n \ge 5$, and $fd(G) fd(\overline{G}) \le (n-2)^2$, for $n \ge 4$.

That the upper bounds are sharp may be seen as follows. For $n \geq 6$ even, let $G = H_{n/2}$, while for $n \geq 7$ odd, let $G = F_{(n-1)/2}$ where the graphs H_n and F_n are defined as in the proof of Theorem 8. Then, G has order n and, as shown in the proof of Theorem 8, $\operatorname{fd}(G) = n-2$. Since the graph \overline{G} is connected, by Theorem 6(a), we have that $\operatorname{fd}(\overline{G}) = n-2$, and so $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) = 2n-4$ and $\operatorname{fd}(G)\operatorname{fd}(\overline{G}) = (n-2)^2$. For n=4,5, we simply take $G = C_n$, which satisfies $\operatorname{fd}(G) = \operatorname{fd}(\overline{G}) = n-2$.

For the lower bounds, we may assume, without loss of generality, that $\operatorname{fd}(G) \leq \operatorname{fd}(\overline{G})$. If $\operatorname{fd}(G) \geq 2$, then $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) \geq 3$ and $\operatorname{fd}(G) \cdot \operatorname{fd}(\overline{G}) \geq 2$. Hence we may assume that $\operatorname{fd}(G) = 1$. But then G has a vertex of degree n-1, implying that \overline{G} is not connected. Hence, $\operatorname{fd}(\overline{G}) \geq 2$ and we obtain $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) \geq 3$ and $\operatorname{fd}(G) \cdot \operatorname{fd}(\overline{G}) \geq 2$. This establishes the desired lower bounds. That the lower bounds are sharp may be seen by taking $G = K_{1,n-1}$. \square

We remark that if $G = K_4$, then G has order n = 4 and $\operatorname{fd}(G) + \operatorname{fd}(\overline{G}) = 1 + 4 = 5 > 2n - 4$. Hence the constraint on the order $n \geq 5$ in the statement of Theorem 23(a) cannot be relaxed. Moreover neither can the condition $n \geq 4$ in Theorem 23(b) be relaxed since if $G = K_3$, then G has order n = 3 and we have $\operatorname{fd}(G) = 1$ and $\operatorname{fd}(\overline{G}) = 3$, implying that $\operatorname{fd}(G)\operatorname{fd}(\overline{G}) = 3 > (n-2)^2 = 1$.

3.5 Unions of Graphs

In this section we investigate the fair domination of disjoint unions of graphs. We shall prove.

Theorem 24 Let G_1, \ldots, G_k be $k \geq 1$ graphs. Let H be the disjoint union $\bigcup_{i=1}^k G_i$ of G_1, \ldots, G_k and let H have order n. Then the following holds. Then,

$$fd(H) - \sum_{i=1}^{k} fd(G_i) \le \frac{1}{k}(k-1)(n-k),$$

and this bound is sharp for k = 1, 2.

Proof. For k=1 the result is trivial since both sides of the inequality are zero. Hence we may assume that $k \geq 2$. Let $n_i = |V(G_i)|$ for $1 \leq i \leq k$. Renaming the graphs G_1, \ldots, G_k , if necessary, we may assume that $n_1 \leq n_2 \leq \cdots \leq n_k$, and so $n_k \geq n/k$ and $n - n_k \leq (k-1)n/k$. Let D_k be a fd (G_k) -set. Then, $D_k \cup (V(H) \setminus V(G_k))$ is a FD-set of

H and thus

$$fd(H) - \sum_{i=1}^{k} fd(G_i) \leq |D_k| + n - n_k - \sum_{i=1}^{k} fd(G_i)$$

$$= n - n_k - \sum_{i=1}^{k-1} fd(G_i)$$

$$\leq n - n_k - (k-1)$$

$$\leq \frac{1}{k}(k-1)(n-k).$$

This establishes the desired upper bound.

We show next that the bound is sharp for k=2. For $n \geq 3$, let $G_1 = K_{1,2n-1}$ and let $G_2 = K_{2,n-1,n-1}$. Let $H = G_1 \cup G_2$, and so H has order 4n.

We note that there are only two values of ℓ for which G_1 has an ℓ FD set D different from $V(G_1)$. The first value of ℓ is $\ell = 1$ when D consists of the central vertex of G_1 and any set of t leaves, where $0 \le t \le 2(n-1)$. The second value of ℓ is $\ell = 2n-1$ when D consists of all leaves in G_1 . Hence, $\ell \in \{1, 2n-1\}$. In particular, we note that $\mathrm{fd}(G_1) = 1$.

Next we consider the graph G_2 . Since G_2 has no dominating vertex adjacent to every other vertex, we note that G_2 has no ℓ FD-set for $\ell \in \{1, 2n-1\}$. In particular, $\mathrm{fd}(G_2) \geq 2$. However the partite set of cardinality 2 is a FD-set in G_2 , and so $\mathrm{fd}(G_2) \leq 2$. Consequently, $\mathrm{fd}(G_2) = 2$.

From our earlier observations, a FD-set of H is formed by either taking a FD-set in G_1 and all vertices in G_2 or by taking a FD-set in G_2 and all vertices in G_1 . Therefore, $fd(H) \leq \min\{fd(G_1) + |V(G_2)|, fd(G_2) + |V(G_1)|\} = 2n + 1$. Thus,

$$fd(H) - \sum_{i=1}^{k} fd(G_i) = 2(n-1) = \frac{1}{2}(4n-2) = \frac{1}{k}(k-1)(|V(H)| - k),$$

realizing the upper bound for k=2. \square

We remark that the fair domination number is highly sensitive with respect to edge deletion or edge addition. For $n \geq 3$, let $G_1 = K_{1,2n-1}$ and let $G_2 = K_{2,n-1,n-1}$. Let v_1 denote the central vertex of G_1 and let $\{u_2, v_2\}$ be the partite set of G_2 of cardinality 2. Let $H = G_1 \cup G_2$, and so H has order 4n, and let $G = H + u_2v_2$. As shown in the proof of Theorem 24, we have that fd(H) = 2n + 1. However the set $\{v_1, v_2\}$ is a FD-set of G since every vertex in $V(G) \setminus \{v_1, v_2\}$ is adjacent to exactly one of v_1 and v_2 . Hence, fd(G) = 2. Therefore we have the following observation.

Proposition 25 There exists infinitely many graph G such that $fd(G + e) - fd(G) \ge |V(G)|/2 - 1$ for some edge $e \in E(\overline{G})$.

4 Closing Remarks and Open Questions

This paper, in which we introduce the notion of fair domination, suggests many possible direction of further research. We close with the following list of open problems that we have yet to settle.

Problem 1 Find a polynomial time algorithm to compute fd(T) for trees T.

Problem 2 Improve upon the bounds of Proposition 10. In particular, find best possible upper bounds for MOP's, maximum planar graphs and regular graphs.

Problem 3 Find fd(G) for other families of graphs G, in particular the grid $P_n \times P_m$ and the torus $C_n \times C_m$.

Problem 4 *Is it true that Theorem 24 is sharp also for* $k \geq 3$?

Problem 5 In view of Proposition 25, find the exact value of $\max |fd(G+e) - fd(G)|$ over all graphs G on n vertices.

Problem 6 Characterize the graphs G on n vertices for which fd(G) = n - 2.

Problem 7 What can be said about the fair domination number fd(G) of the random graph G(n,p)?

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