Reconstructing the potential for the 1D Schrödinger equation from boundary measurements

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Abstract. We consider the inverse problem of the determining the potential in the dynamical Schrödinger equation on the interval by the measurement on the whole boundary. Provided that source is *generic* using the Boundary Control method we recover the spectrum of the problem from the observation at either left or right end points. Using the specificity of the one-dimensional situation we recover the spectral function, reducing the problem to the classical one which could be treated by known methods. We adapt the algorithm to the situation when only the finite number of eigenvalues are known and provide the result on the convergence of the method.

1. Introduction

We consider the problem of determining the potential for a one dimensional Schrödinger equation from two boundary measurements. More precisely, given $q \in L^1(0,1)$ (without loss of generality, we assume that q is real-valued) and $a \in H_0^1(0,1)$, we consider the following initial boundary value problem:

$$\begin{cases} iu_t(x,t) - u_{xx}(x,t) + q(x)u(x,t) = 0 & t > 0, \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = a(x) & 0 < x < 1. \end{cases} \tag{1.1}$$

Assuming that the initial data a is unknown, the inverse problem we are interested in is to determine the potential q from the trace of the derivative of the solution u to (1.1) on the boundary:

$$\{r_0(t), r_1(t)\} := \{u_x(0,t), u_x(1,t)\}, \qquad \forall \, t \in (0,T),$$

where $T > 0$ is a fixed constant. Once the potential has been determined, we can use the method of iterative observers recently proposed in [16] to recover the initial data.

The multidimensional inverse problems for the Schrödinger equation by one measurement were considered in [4, 5, 8, 14]. Using techniques based on Carleman estimates, the authors established global uniqueness and stability results for different geometrical conditions on the domain. We also point out the approach developed by Boumenir and Tuan for the heat equation in [9, 10, 11]. Using observation at the one end of the interval for $t \in (0, \infty)$, the authors were able to recover the spectrum of the string. Next, choosing another boundary condition, they solved the inverse problem from the two recovered spectra by the classical Levitan-Gasymov method [12, 13].

In this paper we offer a different approach. More precisely, let us introduce the unbounded operator $\mathcal{A}$ on $L^2(0,1)$ defined by

$$\mathcal{A}\phi := -\phi'' + q\phi, \qquad \forall \, \phi \in \mathcal{D}(\mathcal{A}) := H^2(0,1) \cap H_0^1(0,1). \tag{1.2}$$

Using the Boundary Control Method (denoted BCM in the rest of the paper, see [1, 6]), we first show that the eigenvalues of $\mathcal{A}$ can be recovered from the data r_0 (or r_1). To achieve this, we derive a generalized eigenvalue problem involving an integral operator (see (2.5)), whose resolution leads to the recovery of the eigenvalues of $\mathcal{A}$. Then, using the peculiarity of the one dimensional case, we recover the spectral function associated with $\mathcal{A}$, reducing the initial inverse problem to the more "classical" one of recovering an unknown potential from spectral data. We can then use either Gelfand-Levitan, Krein or Boundary Control Methods (see [2]).

Remark 1.1. *The method of solving the inverse problem presented in this paper could be applied to the case of wave and parabolic equations on the interval as well.*

The paper is organized as follows : in Section 2, we detail the different steps of our algorithm. In particular, we describe how to recover the spectral data of $\mathcal{A}$ from the boundary data and point out two methods that can be used to recover the potential. In Section 3, we show how to adapt our algorithm to the more realistic situation where not all but only a finite number of eigenvalues of $\mathcal{A}$ are known.

2. Inverse problem, application of the BCM

2.1. From boundary data to spectral data. It is well known that the selfadjoint operator $\mathcal{A}$ defined by (1.2) admits a family of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$, associated to a sequence of eigenvalues $\lambda_k \to +\infty$, forming a orthonormal basis of $L^2(0,1)$. Using the method of moments, we can represent the solution of (1.1) in the form

$$u(x,t) = \sum_1^{\infty} a_k e^{i\lambda_k t} \phi_k(x), \tag{2.1}$$

where a_k are (unknown) Fourier coefficients:

$$a_k = (a, \phi_k)_{L^2(0,1;dx)}. \tag{2.2}$$

Definition 2.1. *We call the initial data $a \in H_0^1(0,1)$ generic if $a_k \neq 0$ for all $k \geq 1$.*

For the boundary data r_0, r_1, we have the representation

$$\{r_0(t), r_1(t)\} = \left\{ \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \phi_k'(0), \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \phi_k'(1) \right\}. \tag{2.3}$$

In particular, one can note that $r_0, r_1 \in L^2(0,T)$. Throughout the paper, we always assume that the source a is generic. The reason for this requirement can be observed from (2.3): if we assume that $a_n = 0$ for some n, then from the representation (2.3) it is easy to see that the response function does not contain any information about the triplet $\{\lambda_n, \phi_n'(0), \phi_n'(1)\}$, and, consequently the potential which would be possible to recover from $\{r_0(t), r_1(t)\}$ would not be unique.

Using the method from [1] we can recover the spectrum λ_k and products $a_k \phi_k'(0)$ and $a_k \phi_k'(1)$ by the following procedure. We construct the operator $C_0^T : L^2(0,T) \longrightarrow L^2(0,T)$ by the rule:

$$(C_0^T)f(t) = \int_0^T r_0(2T - t - \tau) f(\tau) \, d\tau, \qquad 0 \leqslant t \leqslant T. \tag{2.4}$$

and consider the following generalized eigenvalue problem : Find $(\mu, f) \in \mathbb{C} \times L^2(0, T)$, $f \neq 0$, such that

$$\int_0^T \dot{r}_0(2T - t - \tau)f(\tau) = \mu \int_0^T r_0(2T - t - \tau)f(\tau) \, d\tau, \qquad 0 \leqslant t \leqslant T. \qquad (2.5)$$

Then, one can prove that the above problem admits a countable set of solutions (μ_n, f_n), $n \geqslant 1$. Moreover, the eigenvalues μ_n are precisely given by the values $i\lambda_n$, where λ_n are the eigenvalues of $\mathcal{A}$ and the family $\{f_n(t)\}_{n=1}^\infty$ is biorthogonal to $\{e^{i\lambda_k(T-t)}\}_{n=1}^\infty$.

Next, we consider the equation of the form (2.5) with $r_0(t)$ replaced by its complex conjugate $\overline{r_0(t)}$. This equation yields the sequence $\{-i\lambda_k, g_k(t)\}_{k=1}^\infty$, $(-i\lambda_n = \overline{\mu}_n)$. Let us normalize functions f_k, g_k by the rule:

$$\delta_{nk} = \left(C_0^T f_n, g_k\right) = \int_0^T \int_0^T r_0(2T - t - \tau)f_n(\tau)\overline{g_k(t)} \, d\tau \, dt \qquad (2.6)$$

and introduce constants γ_k, β_k defined by:

$$\gamma_k = \int_0^T r_0(T - \tau)f_k(\tau) \, d\tau, \qquad (2.7)$$

$$\beta_k = \int_0^T r_0(T - \tau)\overline{g_k(\tau)} \, d\tau. \qquad (2.8)$$

Then the product $a_k\phi_x'(0)$ could be evaluated by

$$a_k\phi_k'(0) = \gamma_k\beta_k. \qquad (2.9)$$

Similarly we can introduce the integral operator C_1^T associated with the response at the right endpoint $r_1(t)$, and repeat the procedure described above to find quantities $a_k\phi_k'(1)$.

Summing up, using the method from [1] we are able to recover the eigenvalues λ_k of $\mathcal{A}$ and the products $\phi_k'(0)a_k$ and $\phi_k'(1)a_k$. As a result, we can say that we recovered the spectral data consisting of

$$\mathcal{D} := \left\{\lambda_k, \frac{\phi_k'(1)}{\phi_k'(0)}\right\}_{k=1}^\infty. \qquad (2.10)$$

Precisely this data was used in [15], where the authors shown the uniqueness for the inverse spectral problem and provided the method of recovering the potential. Instead of doing this, we recover the spectral function associated to $\mathcal{A}$ and thus reduce the inverse source problem to the classical one of determining an unknown potential from spectral data.

Given $\lambda \in \mathbb{C}$, we introduce the solution $y(\cdot, \lambda)$ of the following Cauchy problem on $(0, 1)$:

$$-y''(x, \lambda) + q(x)y(x, \lambda) = \lambda y(x, \lambda), \qquad 0 < x < 1, \qquad (2.11)$$

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 1. \qquad (2.12)$$

Then the eigenvalues of the Dirichlet problem of $\mathcal{A}$ are exactly the zeroes of the function $y(1, \lambda)$, while a family of normalized corresponding eigenfunctions is given by $\phi_k(x) = \dfrac{y(x, \lambda_k)}{\|y(\cdot, \lambda_k)\|}$. Thus we can rewrite the second components in $\mathcal{D}$ in the following way:

$$\frac{\phi_k'(1)}{\phi_k'(0)} = \frac{y'(1, \lambda_k)}{y'(0, \lambda_k)} = y'(1, \lambda_k) =: A_k. \qquad (2.13)$$

Let us denote by dot the derivative with respect to λ. We use the following fact (see [15]), which is also valid for $q \in L^1(0, 1)$:

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Lemma 2.2. *If λ_n is an eigenvalue of $\mathcal{A}$, then*

$$\|y(\cdot, \lambda_n)\|_{L^2}^2 = y'(1, \lambda_n)\dot{y}(1, \lambda_n) \tag{2.14}$$

The lemma below which can be found in [**15**] for the case $q \in L^2(0,1)$ holds true for $q \in L^1(0,1)$ as well. The meaning of it is that the function $y(1, \lambda)$ is completely determined by its zeroes, which are precisely the eigenvalues of $\mathcal{A}$.

Lemma 2.3. *For $q \in L^1(0,1)$ the following representations hold*

$$y(1, \lambda) = \prod_{k \geqslant 1} \frac{\lambda_k - \lambda}{k^2 \pi^2}$$

$$\dot{y}(1, \lambda_n) = -\frac{1}{n^2 \pi^2} \prod_{k \geqslant 1, k \neq n} \frac{\lambda_k - \lambda_n}{k^2 \pi^2}.$$

Lemma 2.2 and 2.3 imply that the data $\mathcal{D}$ we have recovered (see (2.10)) allow us to evaluate the norm

$$\|y(\cdot, \lambda_n)\|_{L^2}^2 = A_n B_n =: \alpha_n^2, \tag{2.15}$$

where we denoted

$$B_n := -\frac{1}{n^2 \pi^2} \prod_{k \geqslant 1, k \neq n} \frac{\lambda_k - \lambda_n}{k^2 \pi^2}, \tag{2.16}$$

2.2. Reconstructing the potential from spectral data. The set of pairs $\{\lambda_k, \|y(\cdot, \lambda_k)\|_{L^2}^2\}_{k=1}^\infty$ is a "classical" spectral data. The potential can thus be recovered by Gelfand-Levitan, Krein method or the BCM (see [**2**]). Below we outline two possible methods of recovering the potential.

We introduce the spectral function associated with $\mathcal{A}$:

$$\rho(\lambda) = \begin{cases} -\sum\limits_{\lambda \leqslant \lambda_k \leqslant 0} \frac{1}{\alpha_k^2} & \lambda \leqslant 0, \\ \sum\limits_{0 < \lambda_k \leqslant \lambda} \frac{1}{\alpha_k^2} & \lambda > 0, \end{cases} \tag{2.17}$$

which is a monotone increasing function having jumps at the points of the Dirichlet spectra. The regularized spectral function is introduced by

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \rho_0(\lambda) & \lambda \geqslant 0, \\ \rho(\lambda) & \lambda < 0, \end{cases} \qquad \rho_0(\lambda) = \sum\limits_{0 < \lambda_k^0 \leqslant \lambda} \frac{1}{(\alpha_k^0)^2} \quad \lambda > 0, \tag{2.18}$$

where ρ_0 is the spectral function associated with the operator $\mathcal{A}$ when $q \equiv 0$. In the definition above eigenvalues and norming coefficients are $\lambda_k^0 = \pi^2 k^2$, $(\alpha_k^0)^2 = \frac{1}{2\pi^2 k^2}$ and the solution to (2.11)-(2.12) for $q \equiv 0$ is $y_0(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}}$.

Let us fix $\tau \in (0,1]$ and introduce the kernel $c^\tau(t,s)$ by the rule (see also [**2**]):

$$c^\tau(t,s) = \int_{-\infty}^{\infty} \frac{\sin \sqrt{\lambda}(\tau - t) \sin \sqrt{\lambda}(\tau - s)}{\lambda} \, d\sigma(\lambda), \quad s, t \in [0, \tau], \tag{2.19}$$

Then so-called connecting operator (see [**6**, **2**]) $C^\tau : L^2(0, \tau) \mapsto L^2(0, \tau)$ is defined by the formula

$$(C^\tau f)(t) = (I + \mathcal{C}^\tau)f(t) = f(t) + \int_0^\tau c^\tau(t,s)f(s) \, ds, \ 0 < t < \tau, \tag{2.20}$$

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Using the BCM leads to the following: for fixed $\tau \in (0,1)$ one solves the equation

$$(C^\tau f^\tau)(t) = \tau - t, \quad 0 < t < \tau, \tag{2.21}$$

Setting

$$\mu(\tau) := f^\tau(+0), \tag{2.22}$$

and then varying $\tau \in (0,1)$, the potential at the point τ is recovered by

$$q(\tau) = \frac{\mu''(\tau)}{\mu(\tau)}. \tag{2.23}$$

We can also make use of the Gelfand-Levitan theory. According to this approach, for $\tau \equiv 1$, the kernel c^τ satisfies the following integral equation with unknown V:

$$V(y,t) + c^\tau(y,t) + \int_y^\tau c^\tau(t,s)V(y,s)\,ds = 0, \quad 0 < y < t < 1. \tag{2.24}$$

Solving the equation (2.24) for all $y \in (0,1)$ we can recover the potential using

$$q(y) = 2\frac{d}{dy}V(\tau - y, \tau - y). \tag{2.25}$$

Once the potential has been found, we can recover the eigenfunctions ϕ_k, the traces $\phi'_k(0)$ and using (2.9), the Fourier coefficients a_k, $k = 1, \ldots, \infty$. Thus, the initial state can be recovered via its Fourier series. We can also use the method of observers (see [16]).

2.3. The algorithm.

1) Take $r_0(t) := u_x(0,t)$ and solve the generalized spectral problem (2.5). Denote the solution by $\{\mu_n, f_n(t)\}_{n=1}^\infty$ and note the connection with the spectra of $\mathcal{A}$: $\lambda_n = -i\mu_n$.
2) Take the function $\overline{r_0(t)}$ and repeat step one. This yields the sequence $\{\overline{\mu_n}, g_n(t)\}_{n=1}^\infty$.
3) Define the operator C_0^T by (2.4) and normalize f_n, g_n according to equation (2.6): $(C_0^T f_n, g_n) = 1$.
4) Find the quantities $a_n \phi'_n(0)$ by (2.7), (2.8) and (2.9).
5) Repeat steps 1)–4) for the function $r_1(t) := u_x(1,t)$ to find $a_k \phi'_k(1)$.
6) Define the spectral data $\mathcal{D}$ by (2.10) and find the norming coefficients α_k by using (2.15) and (2.13), (2.16).
7) Introduce the spectral function $\rho(\lambda)$, the regularized spectral function $\sigma(\lambda)$ and the kernel c^τ respectively defined by (2.17), (2.18) and (2.19).
8) Solve the inverse problem by either BCM using (2.21), (2.22), (2.23) or Gelfand-Levitan method using equations (2.24), (2.25).
9) Use the method of iterative observers described in [16] or Fourier series to recover the initial data.

Our approach yields the following uniqueness result for the inverse problem for 1-d Schrödinger equation:

Theorem 2.4. *Let the source $a \in H_0^1(0,1)$ be generic and T be an arbitrary positive number. Then the potential $q \in L^1(0,1)$ and the initial data are uniquely determined by the observation $\{u_x(0,t), u_x(1,t)\}$ for $t \in (0,T)$.*

The method could be applied to the inverse problem for the wave and parabolic equations with the potential on the interval as well. The details of the recovering the spectrum λ_k and the quantities $a_k \phi'_k(0)$, $a_k \phi'_k(1)$ could be found in [1]. The following important remark, connected with the types of the controllability of the corresponding systems, should be taken into the consideration:

Remark 2.5. *The time T of the observation could be arbitrary small for the case of Schrödinger and parabolic equations and is doubled the length of an interval ($T = 2$ in our case) for the wave equation with the potential.*

For the details see [**1**].

3. Stability of the scheme : the case of truncated spectral data

In view of applications, in this section we consider the case where only a finite number of eigenvalues of $\mathcal{A}$ are available. More precisely, let us assume that we recovered the exact values of the first N eigenvalues λ_n and traces A_n, for $n = 1, \ldots, N$. Then we can introduce the approximate normalizing coefficients $\widetilde{\alpha}_{n,N}$ by the rule

$$\widetilde{\alpha}_{n,N} = A_n \widetilde{B}_{n,N}, \quad \text{where} \quad \widetilde{B}_{n,N} := -\frac{1}{n^2 \pi^2} \prod_{k \geqslant 1, k \neq n}^{N} \frac{\lambda_k - \lambda_n}{k^2 \pi^2} \tag{3.1}$$

Then we can estimate

$$|\alpha_n - \widetilde{\alpha}_{n,N}| \leqslant |A_n||\widetilde{B}_{n,N}| \left| 1 - \prod_{k \geqslant N+1}^{\infty} \frac{\lambda_k - \lambda_n}{k^2 \pi^2} \right| \tag{3.2}$$

Since the infinite product (2.16) converges, the right hand side of the above inequality is small enough as N is getting bigger, provided n is fixed. But the following remark should be taken into the account. Let us remind the following asymptotic formulas for the eigenvalues and norming coefficients:

$$\lambda_k = \pi^2 k^2 + c + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \tag{3.3}$$

$$\alpha_k^2 = \frac{1}{2\pi^2 k^2} + O\left(\frac{1}{k^4}\right), \quad k \to \infty, \tag{3.4}$$

where $c = \displaystyle\int_0^1 q(s)\, ds$. Then the product in the right hand side of (3.2) can be rewritten as

$$\prod_{k \geqslant N+1}^{\infty} \left(1 - \frac{n^2 + O\left(\frac{1}{n^2}\right)}{k^2} + O\left(\frac{1}{k^4}\right) \right) \tag{3.5}$$

So we see that if n is close to N, then the terms $\frac{n^2 + O\left(\frac{1}{n^2}\right)}{k^2}$ are close to one, and consequently the first factors in (3.5) are small. This implies that the product in (3.2) is not close to one. This simple observation yields that we can guarantee the good estimate in (3.2) only if N is much greater than n, so that the product in (3.5) is close to one.

We need to find out how many eigenvalues (we call their number by N) we need to recover in order to n less than N approximate normalizing coefficients be recovered with a good accuracy. In other words, we need to find out the relationship between N and n such that the product in (3.2) or what is equivalent (3.5) are close to one. Let us notice that if $0 < x < \theta < 1$ then $|\ln(1 - x)| < \frac{|\ln(1-\theta)|}{\theta} x$. Using this observation, we can estimate for

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$\frac{n^2}{(N+1)^2} \leqslant \theta$:

$$\left| \ln \prod_{k \geqslant N+1}^{\infty} \left(1 - \frac{n^2}{k^2}\right) \right| \leqslant \sum_{k \geqslant N+1}^{\infty} \left| \ln \left(1 - \frac{n^2}{k^2}\right) \right|$$

$$\leqslant \sum_{k \geqslant N+1}^{\infty} \frac{|\ln(1-\theta)|}{\theta} \frac{n^2}{k^2} = n^2 \frac{|\ln(1-\theta)|}{\theta} \left(\frac{\pi^2}{6} - \sum_{k=1}^{N} \frac{1}{k^2}\right) \tag{3.6}$$

If we choose N and n such that right hand side of (3.6) is less than some $\varepsilon > 0$, then

$$e^{-\varepsilon} < \prod_{k \geqslant N+1}^{\infty} \left(1 - \frac{n^2}{k^2}\right) < 1. \tag{3.7}$$

Consequently, for such N and n we have (see (3.2)):

$$|\alpha_n - \widetilde{\alpha}_{n,N}| \leqslant |A_n||\widetilde{B}_{n,N}| \left|1 - e^{-\varepsilon}\right|. \tag{3.8}$$

Notice that since $A_n = 1 + o(1)$ and $\alpha_n = \frac{1}{\sqrt{2\pi n}} + o(1)$ as $n \to \infty$, we have that $|A_n||\widetilde{B}_{n,N}|$ is bounded by some positive $C < 4$. So, taking ε small enough, and picking N and n such that the right hand side of (3.6) be less that ε, we arrive at (3.8). Using formula

$$\frac{\pi^2}{6} = \sum_{k=1}^{N} \frac{1}{N} - \frac{1}{2N^2} + O\left(\frac{1}{N^3}\right)$$

from [**17**] we summarize all our observations in the lemma:

Lemma 3.1. *Let $0 < \varepsilon < 1$. If n and N satisfy*

$$\frac{n^2}{N} \leqslant \frac{\varepsilon}{2\ln 2}, \tag{3.9}$$

then there exists an absolute constant $C > 0$ such that

$$|\alpha_k - \widetilde{\alpha}_{k,N}| \leqslant C \left|1 - e^{-\varepsilon}\right|, \forall k = 1, \ldots, n \tag{3.10}$$

Let us recall the following important representations (see [**2**]) for the response function and for the kernel c^τ (in our case $\tau = 1$):

Lemma 3.2. *Assume that $q \in L^1(0,1)$. Then the following representation for the response function r,*

$$r(t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \, d\sigma(\lambda), \tag{3.11}$$

holds for almost all $t \in (0, 2\tau)$.

The kernel $c^\tau(s,t)$ admits the following representation:

$$c^\tau(s,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(\tau - t)\sin\sqrt{\lambda}(\tau - s)}{\lambda} \, d\sigma(\lambda), \quad s,t \in [0,\tau], \tag{3.12}$$

where the integral in the right-hand side of (2.19) converges uniformly on $[0,\tau] \times [0,\tau]$.

$$c^\tau(t,s) = p(2\tau - t - s) - p(t - s) \tag{3.13}$$

and $p(t)$ is defined by

$$p(t) := \frac{1}{2} \int_{0}^{|t|} r(s) \, ds. \tag{3.14}$$

The following useful formula follows directly from representations (3.11), (2.19):

$$c^\tau(t,t) = \frac{1}{2} \int_0^{2(\tau-t)} r(\tau)\,d\tau \tag{3.15}$$

If the exact values of the first n eigenvalues λ_k and normalizing factors α_k^2, $k = 1, \dots, n$, were known, then one could construct the "restricted" response functions and kernels defined by

$$r_n(t) = \sum_{k=1}^n \left[\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}} \frac{\operatorname{sign}\lambda_k}{\alpha_k^2} - \frac{\sin\sqrt{\lambda_k^0}t}{\sqrt{\lambda_k^0}} \frac{1}{(\alpha_k^0)^2} \right],$$

$$c_n^\tau(t,s) = \sum_{k=1}^n \frac{\sin\sqrt{\lambda_k}(T-t)\sin\sqrt{\lambda_k}(T-s)}{\lambda_k} \frac{\operatorname{sign}\lambda_k}{\alpha_k^2}$$

$$- \frac{\sin\sqrt{\lambda_k^0}(T-t)\sin\sqrt{\lambda_k^0}(T-s)}{\lambda_k^0} \frac{1}{(\alpha_k^0)^2}.$$

then every $r_n \in C^\infty(0,1)$ and Lemma 3.2 yields

$$r_n(t) \to r(t), \quad \text{for almost all } t \in (0,2), \tag{3.16}$$

$$c_n^\tau(t,s) \to c^\tau(t,s), \quad \text{uniformly on } (0,\tau)^2, \tag{3.17}$$

$$c_n^\tau(t,t) \to c^\tau(t,t), \quad \text{uniformly on the diagonal.} \tag{3.18}$$

As we only have access to approximate values of the normalizing factors in practice, we can only compute the approximate restricted kernel by

$$\widetilde{c}_{n,N}^\tau(t,s) = \sum_{k=1}^n \frac{\sin\sqrt{\lambda_k}(T-t)\sin\sqrt{\lambda_k}(T-s)}{\lambda_k} \frac{\operatorname{sign}\lambda_k}{\widetilde{\alpha}_{k,N}^2}$$

$$- \frac{\sin\sqrt{\lambda_k^0}(T-t)\sin\sqrt{\lambda_k^0}(T-s)}{\lambda_k^0} \frac{1}{(\alpha_k^0)^2}. \tag{3.19}$$

Then we can estimate the difference

$$\|c_n^\tau - \widetilde{c}_{n,N}^\tau\|_\infty \leqslant \sum_{k=1}^n \frac{|\widetilde{\alpha}_{k,N}^2 - \alpha_k^2|}{\lambda_k \widetilde{\alpha}_{k,N}^2 \alpha_k^2} = \sum_{k=1}^n \frac{|\widetilde{\alpha}_k + \alpha_k||\widetilde{\alpha}_k - \alpha_k|}{\lambda_k \widetilde{\alpha}_{k,N}^2 \alpha_k^2} \tag{3.20}$$

Using (3.10) and asymptotical expansions for the eigenvalues and norming coefficients (3.3) (3.4), we deduce from (3.20) that (below, C denotes an absolute constant that might change from line to line):

$$\|c_n^\tau - \widetilde{c}_{n,N}^\tau\|_\infty \leqslant C \sum_{k=1}^n \frac{|\widetilde{\alpha}_{k,N}^2 - \alpha_k^2|}{\lambda_k \widetilde{\alpha}_{k,N}^2 \alpha_k^2} \leqslant C \sum_{k=1}^n k|\widetilde{\alpha}_{k,N} - \alpha_k|.$$

Having remembered estimate (3.10), we finally get

$$\|c_n^\tau - \widetilde{c}_{n,N}^\tau\|_\infty \leqslant C \frac{n(n+1)}{2} \left| 1 - e^{-\varepsilon} \right|. \tag{3.21}$$

Let us fix some $\delta > 0$ and choose $n \in \mathbb{N}$ such that $\|c_n^\tau - c^\tau\|_\infty \leqslant \dfrac{\delta}{2}$ and consider the difference

$$\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty \leqslant \|\widetilde{c}_{n,N}^\tau - c_n^\tau\|_\infty + \|c_n^\tau - c^\tau\|_\infty \tag{3.22}$$

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Using (3.21) we obtain the existence of a constant $C^* > 0$ such that

$$\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty \leqslant C^* n^2 \left|1 - e^{-\varepsilon}\right| + \frac{\delta}{2}. \tag{3.23}$$

Then by choosing ε in (3.23) we achieve

$$\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty \leqslant \delta. \tag{3.24}$$

We summarize the above observations in the following Proposition, which details the main of our approximation procedure.

Proposition 3.3. *Let $\delta > 0$ be fixed. Let n be chosen that (this is possible thanks to (3.17))*

$$\|c_n^\tau - c^\tau\|_\infty \leqslant \frac{\delta}{2}.$$

Next, take $\varepsilon > 0$ such that

$$n^2 \left|1 - e^{-\varepsilon}\right| \leqslant \frac{\delta}{2},$$

Finally, choose N such that

$$\frac{n^2}{N} \leqslant \frac{\varepsilon}{2\ln 2}.$$

Then, there exists an absolute constant $C > 0$ such that the following estimate holds true:

$$\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty \leqslant C\delta,$$

where the approximate restricted kernel $\widetilde{c}_{n,N}^\tau$ and the approximate normalizing coefficients $\widetilde{\alpha}_{k,N}$, $k = 1, \ldots, n$ are respectively defined by (3.19) and (3.1).

In particular, $\widetilde{c}_{n,N}^\tau$ converges uniformly to c^τ when n tends to infinity and N is chosen as above.

Along with the equation (2.24) we consider the equation for the approximate restricted kernel $\widetilde{c}_{n,N}^\tau$:

$$\widetilde{V}_{n,N}(y,t) + \widetilde{c}_{n,N}^\tau(y,t) + \int_y^\tau \widetilde{c}_{n,N}^\tau(t,s)\widetilde{V}_{n,N}(y,s)\,ds = 0, \quad 0 < y < t < \tau. \tag{3.25}$$

Picking $\delta > 0$ we can use Proposition 3.3 to find n and N such that $\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty \leqslant \delta$. From now on, we always assume that n and N are chosen according to Proposition 3.3. Note that in particular, we have $N \to +\infty$ as $n \to +\infty$.

We introduce the operator $\widetilde{\mathcal{C}}_\tau^{n,N}$ which are given by an integral part of (2.20) with the kernel c^τ substituted by $\widetilde{c}_{n,N}^\tau$. The closeness of c^τ and $\widetilde{c}_{n,N}^\tau$ implies the operator $I + \widetilde{C}_\tau^{n,N}$ to be invertible along with $I + C_\tau$. The latter in turn implies the existence of the potential $\widetilde{q}_{n,N}$ that produces the response function

$$\widetilde{r}_{n,N}(t) = \sum_{k=1}^n \left[\frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}} \frac{\operatorname{sign}\lambda_k}{\widetilde{\alpha}_{k,N}^2} - \frac{\sin\sqrt{\lambda_k^0}\,t}{\sqrt{\lambda_k^0}} \frac{1}{(\alpha_k^0)^2} \right], \tag{3.26}$$

and the unique solvability of (3.25) (see [**2**], [**7**], [**3**]).

We denote $M := \|(I + C_\tau)^{-1}\|$. The invertibility of $I + C_\tau$ and $I + \widetilde{C}_\tau^{n,N}$ implies the norms of the solutions $\|V(y,\cdot)\|_{L^2}$ and $\|\widetilde{V}_{n,N}(y,\cdot)\|_{L^2}$ to be bounded.

Let us write down the difference (2.24) and (3.25) in the form

$$\left[V(y,t) - \widetilde{V}_{n,N}(y,t)\right] + \int_y^\tau c^\tau(t,s)\left[V(y,s) - \widetilde{V}_{n,N}(y,s)\right] ds \quad (3.27)$$

$$= \widetilde{c}_{n,N}^\tau(y,t) - c^\tau(y,t) + \int_y^\tau \left[c^\tau(t,s) - \widetilde{c}_{n,N}^\tau(t,s)\right] \widetilde{V}_{n,N}(y,s)\, ds, \quad 0 < y < t < \tau.$$

The above equality and the invertibility of $I + C^\tau$ implies the estimate:

$$\|V(y,\cdot) - \widetilde{V}_{n,N}(y,\cdot)\|_{L^2(y,\tau)} \leqslant M\left(1 + \max_{0\leqslant y\leqslant \tau} \|\widetilde{V}_{n,N}(y,\cdot)\|_{L^2}\right) \|c^\tau - \widetilde{c}_{n,N}^\tau\|_\infty. \quad (3.28)$$

To estimate the L^∞ norm, we write down (3.27) in the form

$$\left[V(y,t) - \widetilde{V}_{n,N}(y,t)\right] = \int_y^\tau c^\tau(t,s)\left[\widetilde{V}_{n,N}(y,s) - V(y,s)\right] ds \quad (3.29)$$

$$+\widetilde{c}_{n,N}^\tau(y,t) - c^\tau(y,t) + \int_y^\tau \left[c^\tau(t,s) - \widetilde{c}_{n,N}^\tau(t,s)\right] V_{n,N}(y,s)\, ds, \quad 0 < y < t < \tau.$$

The latter leads to the following inequality:

$$\|V(y,t) - \widetilde{V}_{n,N}(y,t)\|_\infty \leqslant \|c^\tau\|_\infty \max_{0\leqslant y\leqslant \tau} \int_y^\tau \left|\widetilde{V}_{n,N}(y,s) - V(y,s)\right| ds \quad (3.30)$$

$$+\|\widetilde{c}_{n,N}^\tau - c^\tau\|_\infty + \|c^\tau - \widetilde{c}_{n,N}^\tau\|_\infty \max_{0\leqslant y\leqslant \tau} \int_y^\tau |\widetilde{V}_{n,N}(y,s)|\, ds.$$

Using (3.28) and (3.30) we finally get

$$\|V - \widetilde{V}_{n,N}\|_\infty \leqslant \|c^\tau - \widetilde{c}_{n,N}^\tau\|_\infty (M\|c^\tau\|_\infty + 1)\left(1 + \max_{0\leqslant y\leqslant \tau} \|\widetilde{V}_{n,N}(y,\cdot)\|_{L^2}\right)$$

The last equality in particular implies the uniform convergence of $\widetilde{V}_{n,N}(y,y)$ to $V(y,y)$. Having remembered (2.25), we conclude that the potentials $q_{n,N}$ and q corresponding respectively to $\widetilde{V}_{n,N}$ and V satisfy

$$\int_0^t \widetilde{q}_{n,N}(s)\, ds \rightrightarrows_{n\to\infty} \int_0^t q(s)\, ds, \quad \text{uniformly in } t. \quad (3.31)$$

The latter in turn, implies that

$$\widetilde{q}_{n,N} \longrightarrow q, \quad \text{in } H^{-1}(0,1). \quad (3.32)$$

In fact, (3.31) yields more than that: we have

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \widetilde{q}_{n,N}(s)\, ds \rightrightarrows_{n\to\infty} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} q(s)\, ds, \quad \text{uniformly in } t, \varepsilon. \quad (3.33)$$

We remark that (3.33) is still not enough to guarantee the convergence $\widetilde{q}_{n,N}$ to q almost everywhere on $(0,1)$.

On the other hand let us restrict (2.24) to the diagonal $y = t$:

$$V(y,y) + c^T(y,y) + \int_y^T c^T(y,s)V(y,s)\, ds = 0, \quad 0 < y < 1, \quad (3.34)$$

and recalling (2.25), (3.15), we see that the best possible result one can expect is a convergence $\widetilde{q}_{n,N}$ to q almost everywhere on $(0,1)$.

Remark 3.4. *The stability of the scheme crucially depends on the type of the convergence $r_n \to r$. For now we know (see [2]) that the convergence is pointwise almost everywhere on the interval. The significant progress in the proving of the stability could be achieved by the improving of this result.*

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