# A MULTIPLICITY RESULT FOR A FRACTIONAL KIRCHHOFF EQUATION IN $\mathbb{R}^N$ WITH A GENERAL NONLINEARITY

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ABSTRACT. In this paper we deal with the following fractional Kirchhoff equation

$$\left(p + q(1-s) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy\right) (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N,$$

where  $s \in (0,1)$ ,  $N \ge 2$ , p > 0, q is a small positive parameter and  $g : \mathbb{R} \to \mathbb{R}$  is an odd function satisfying Berestycki-Lions type assumptions. By using minimax arguments, we establish a multiplicity result for the above equation, provided that q is sufficiently small.

#### 1. Introduction

In this paper we study the multiplicity of weak solutions to the following nonlinear fractional Kirchhoff equation

$$\left(p + q(1-s) \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy\right) (-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N$$
 (1.1)

where  $s \in (0,1)$ ,  $N \ge 2$ , p > 0, q is a small positive parameter and g is a nonlinearity which satisfies suitable assumptions. The operator  $(-\Delta)^s$  is the fractional Laplacian which may be defined for a function u belonging to the Schwartz space  $\mathcal{S}(\mathbb{R}^N)$  of rapidly decaying functions as

$$(-\Delta)^{s}u(x) = C_{N,s} P.V. \int_{\mathbb{R}^{N}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^{N}.$$

The symbol P.V. stands for the Cauchy principal value and  $C_{N,s}$  is a normalizing constant; see [15] for more details.

When  $s \to 1^-$  in (1.1), from Theorem 2 (and Corollary 2) in [12], we can see that (1.1) becomes the following Kirchhoff equation

$$-\left(p+q\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx\right)\Delta u=g(u) \text{ in } \mathbb{R}^N,$$
(1.2)

which has been extensively studied in the last decade.

The equation (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(p + q \int_{\Omega} |\nabla u(x)|^2 dx\right) \Delta u = g(x, u)$$

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with  $\Omega \subset \mathbb{R}^N$  bounded domain, which was proposed by Kirchhoff in 1883 [21] as an extension of the classical D'Alembert's wave equation

$$\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 dx\right) u_{xx}^2 = g(x, u)$$
 (1.3)

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material,  $\rho$  is the mass density and  $P_0$  is the initial tension. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [11] and Pohozaev [31]. However, equation (1.2) received much attention only after the paper by Lions [23], where a functional analysis approach was proposed to attack it. For more recent results concerning Kirchhoff-type equations we refer to [1, 2, 5, 7, 16, 22, 25, 30]. On the other hand, a great attention has been recently focused on the study of nonlinear fractional Kirchhoff problem. In [18], Fiscella and Valdinoci proposed an interesting interpretation of Kirchhoff's equation in the fractional setting, by proving the existence of nonnegative solutions for a critical Kirchhoff type problem in a bounded domain of  $\mathbb{R}^N$ . Subsequently, in [4] the authors investigated the existence and the asymptotic behavior of nonnegative solutions for a class of stationary Kirchhoff problems driven by a fractional integro-differential operator and involving a critical nonlinearity. Pucci and Saldi in [32] established the existence and multiplicity of nontrivial nonnegative entire solutions for a Kirchhoff type eigenvalue problem in  $\mathbb{R}^N$  involving a critical nonlinearity and the fractional Laplacian. More recently, in [17] has been proved the existence of infinitely many weak solutions for a Cauchy problem for a fractional Kirchhoff-type equation by using the genus theory of Krasnosel'skii; see also [6, 26, 27, 28, 29, 33] for related problems.

Motivated by the above papers, in this work we aim to study the multiplicity of weak solutions to the fractional Kirchhoff equation (1.1) with q small parameter and q is a general subcritical nonlinearity. More precisely, we suppose that  $g: \mathbb{R} \to \mathbb{R}$  satisfies Berestycki-Lions type assumptions [9, 10], that is:

 $(g_1)$   $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  and odd;

$$(g_2) -\infty < \liminf_{t \to 0^+} \frac{g(t)}{t} \le \limsup_{t \to 0^+} \frac{g(t)}{t} = -m < 0;$$

$$(g_3) \lim_{t \to \pm \infty} \frac{|g(t)|}{|t|^{2_s^* - 1}} = 0, \text{ where } 2_s^* = \frac{2N}{N - 2s};$$

$$(g_3)$$
  $\lim_{t\to\pm\infty} \frac{|g(t)|}{|t|^{2_s^*-1}} = 0$ , where  $2_s^* = \frac{2N}{N-2s}$ ;

$$(g_4)$$
 there exists  $\zeta > 0$  such that  $G(\zeta) := \int_0^{\zeta} g(t) dt > 0$ .

Let us recall that when q = 0 and p = 1 in (1.1), in [3, 14] has been established the existence and multiplicity of radially symmetric solutions to the fractional scalar field problem

$$(-\Delta)^s u = g(u) \text{ in } \mathbb{R}^N. \tag{1.4}$$

Now, we aim to study a generalization of (1.4), and we look for weak solutions to (1.1) with q > 0 sufficiently small.

Our main result is the following:

**Theorem 1.1.** Let us suppose that  $(g_1)$ ,  $(g_2)$ ,  $(g_3)$  and  $(g_4)$  are satisfied. Then, for any  $h \in \mathbb{N}$  there exists q(h) > 0 such that for any 0 < q < q(h) equation (1.1) admits at least h couples of solutions in  $H^s(\mathbb{R}^N)$  with radial symmetry.

A common approach to deal with nonlinear problems involving the fractional Laplacian, has been proposed by Caffarelli and Silvestre in [13]. It consists to realize  $(-\Delta)^s$  as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem on the upper half-space  $\mathbb{R}^{N+1}_+$ . More precisely, for  $u \in H^s(\mathbb{R}^N)$  one considers the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(x,0) = u(x) & \text{on } \partial \mathbb{R}^{N+1}_+ \end{cases}$$

from where the fractional Laplacian is obtained as

$$(-\Delta)^s u(x) = -\kappa_s \lim_{y \to 0^+} y^{1-2s} v_y(x, y),$$

where  $\kappa_s$  is a suitable constant and the equality holds in distributional sense.

In this paper we investigate the problem (1.1) directly in  $H^s(\mathbb{R}^N)$  in order to adapt the techniques developed in the classical case s=1.

More precisely, we follow the ideas in [8], and by combining the Mountain Pass approach introduced in [19] with the truncation argument of [20], we prove the multiplicity result above stated.

The paper is organized as follows: in Sec. 2 some notations and preliminaries are given, including lemmas that are required to obtain our main Theorem; in Sec. 3 we establish an abstract critical point result and finally in Sec. 4 we provide the proof of Theorem 1.1.

### 2. Preliminaries

For any  $s \in (0,1)$  we define the fractional Sobolev spaces

$$H^{s}(\mathbb{R}^{N}) = \left\{ u \in L^{2}(\mathbb{R}^{N}) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^{2}(\mathbb{R}^{2N}) \right\}$$

endowed with the natural norm

$$||u||_{H^s(\mathbb{R}^N)} = \sqrt{[u]^2_{H^s(\mathbb{R}^N)} + ||u||^2_{L^2(\mathbb{R}^N)}}$$

where the so-called Gagliardo seminorm of u is given by

$$[u]_{H^s(\mathbb{R}^N)}^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.$$

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

**Theorem 2.1.** [15] Let  $s \in (0,1)$  and N > 2s. Then  $H^s(\mathbb{R}^N)$  is continuously embedded in  $L^q(\mathbb{R}^N)$  for any  $q \in [2, 2_s^*]$  and compactly in  $L^q_{loc}(\mathbb{R}^N)$  for any  $q \in [2, 2_s^*]$ .

Let us introduce

$$H^s_{\mathrm{rad}}(\mathbb{R}^N) = \left\{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \right\}$$

the space of radial functions in  $H^s(\mathbb{R}^N)$ . For this space it holds the following compactness result due to Lions [24]:

**Theorem 2.2.** [24] Let  $s \in (0,1)$  and  $N \geq 2$ . Then  $H^s_{rad}(\mathbb{R}^N)$  is compactly embedded in  $L^q(\mathbb{R}^N)$  for any  $q \in (2,2_s^*)$ .

Finally, we recall the following fundamental compactness results:

**Lemma 2.1.** [8, 9] Let P and  $Q: \mathbb{R} \to \mathbb{R}$  be a continuous functions satisfying

$$\lim_{t \to +\infty} \frac{P(t)}{Q(t)} = 0,$$

 $\{v_j\}_{j\in\mathbb{N}},\ v\ and\ w\ be\ measurable\ functions\ from\ \mathbb{R}^N\ to\ \mathbb{R},\ with\ w\ bounded,\ such\ that$ 

$$\sup_{j \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(v_j(x))w| \, dx < +\infty,$$
$$P(v_j(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.$$

Then  $||(P(v_j) - v)w||_{L^1(\mathcal{B})} \to 0$ , for any bounded Borel set  $\mathcal{B}$ . Moreover, if we have also

$$\lim_{t \to 0} \frac{P(t)}{Q(t)} = 0,$$

and

$$\lim_{|x| \to \infty} \sup_{j \in \mathbb{N}} |v_j(x)| = 0,$$

then  $||(P(v_j) - v)w||_{L^1(\mathbb{R}^N)} \to 0$ 

**Lemma 2.2.** [14] Let  $(X, \|\cdot\|)$  be a Banach space such that X is embedded respectively continuously and compactly into  $L^q(\mathbb{R}^N)$  for  $q \in [q_1, q_2]$  and  $q \in (q_1, q_2)$ , where  $q_1, q_2 \in (0, \infty)$ . Assume that  $\{v_j\}_{j \in \mathbb{N}} \subset X$ ,  $v : \mathbb{R}^N \to \mathbb{R}$  is a measurable function and  $P \in \mathcal{C}(\mathbb{R}, \mathbb{R})$  is such that

(i) 
$$\lim_{|t|\to 0} \frac{P(t)}{|t|^{q_1}} = 0,$$

(ii) 
$$\lim_{|t| \to \infty} \frac{P(t)}{|t|^{q_2}} = 0,$$

$$(iii) \sup_{j \in \mathbb{N}} \|v_j\| < \infty,$$

(iv) 
$$\lim_{j\to\infty} P(v_j(x)) = v(x)$$
 for a.e.  $x \in \mathbb{R}^N$ .

Then, up to a subsequence, we have

$$\lim_{j \to \infty} ||P(v_j) - v||_{L^1(\mathbb{R}^N)} = 0.$$

#### 3. A CRITICAL POINT RESULT

In this section we provide an abstract multiplicity result which allows us to prove Theorem 1.1. Let us introduce the following functional defined for  $u \in H^s(\mathbb{R}^N)$ 

$$\mathcal{F}_{q}(u) = \frac{1}{2} [u]_{H^{s}(\mathbb{R}^{N})}^{2} + q \,\mathcal{R}(u) - \int_{\mathbb{R}^{N}} G(u) \, dx, \tag{3.1}$$

where q > 0 is a small parameter and  $\mathcal{R}: H^s(\mathbb{R}^N) \to \mathbb{R}$ . We suppose that

$$\mathcal{R} = \sum_{i=1}^{k} \mathcal{R}_i$$

and, for each i = 1, ..., k the functional  $\mathcal{R}_i$  satisfies

 $(\mathcal{R}_1)$   $\mathcal{R}_i \in \mathcal{C}^1(H^s(\mathbb{R}^N), \mathbb{R})$  is nonnegative and even;

 $(\mathcal{R}_2)$  there exists  $\delta_i > 0$  such that  $\langle \mathcal{R}'_i(u), u \rangle \leq C \|u\|_{H^s(\mathbb{R}^N)}^{\delta_i}$  for any  $u \in H^s(\mathbb{R}^N)$ ;

 $(\mathcal{R}_3)$  if  $\{u_j\}_{j\in\mathbb{N}}\subset H^s(\mathbb{R}^N)$  is weakly convergent to  $u\in H^s(\mathbb{R}^N)$ , then

$$\limsup_{j \to \infty} \langle \mathcal{R}'_i(u_j), u - u_j \rangle \le 0;$$

 $(\mathcal{R}_4)$  there exist  $\alpha_i, \beta_i \geq 0$  such that if  $u \in H^s(\mathbb{R}^N)$ , t > 0 and  $u_t = u\left(\frac{\cdot}{t}\right)$ , then

$$\mathcal{R}_i(u_t) = t^{\alpha_i} \, \mathcal{R}_i(t^{\beta_i} u);$$

 $(\mathcal{R}_5)$   $\mathcal{R}_i$  is invariant under the action of the N-dimensional orthogonal group, i.e.  $\mathcal{R}_i(u(g\cdot)) = \mathcal{R}_i(u(\cdot))$  for every  $g \in \mathcal{O}(N)$ .

Let us observe that for any  $u \in H^s(\mathbb{R}^N)$ ,  $\mathcal{R}_i(u) - \mathcal{R}_i(0) = \int_0^1 \frac{d}{dt} \mathcal{R}_i(tu) dt$ , so by the assumption  $(\mathcal{R}_2)$  we have

$$\mathcal{R}_i(u) \le C_1 + C_2 \|u\|_{H^s(\mathbb{R}^N)}^{\delta_i}.$$
 (3.2)

The main result of this section is the following

**Theorem 3.1.** Let us suppose  $(g_1) - (g_4)$  and  $(\mathcal{R}_1) - (\mathcal{R}_5)$ . Then, for any  $h \in \mathbb{N}$  there exists q(h) > 0 such that for any 0 < q < q(h) the functional  $\mathcal{F}_q$  admits at least h couples of critical points in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ .

Let us define, for any  $t \geq 0$ ,

$$g_1(t) := (g(t) + mt)^+$$
  
 $g_2(t) := g_1(t) - g(t),$ 

and we extend them as odd functions for  $t \leq 0$ . Observing that

$$\lim_{t \to 0} \frac{g_1(t)}{t} = 0,\tag{3.3}$$

$$\lim_{t \to \infty} \frac{g_1(t)}{t^{2_s^* - 1}} = 0,\tag{3.4}$$

$$g_2(t) \ge mt \quad \forall t \ge 0,$$
 (3.5)

we deduce that for any  $\varepsilon > 0$  there exists  $C_{\varepsilon} > 0$  such that

$$g_1(t) \le C_{\varepsilon} t^{2_s^* - 1} + \varepsilon g_2(t) \quad \forall t \ge 0. \tag{3.6}$$

Setting

$$G_i(t) := \int_0^t g_i(\tau) d\tau \quad i = 1, 2,$$

by (3.5) immediately follows that

$$G_2(t) \ge \frac{m}{2}t^2 \quad \forall t \in \mathbb{R},$$
 (3.7)

and, by (3.6) we can see that for any  $\varepsilon > 0$  there exists  $C_{\varepsilon} > 0$  such that

$$G_1(t) \le C_{\varepsilon} |t|^{2_s^*} + \varepsilon G_2(t) \quad \forall t \in \mathbb{R}.$$
 (3.8)

In view of  $(\mathcal{R}_5)$ , all functionals that we will consider along the paper are invariant under rotations, so, from now on, we will directly define our functionals in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ . Following [20], let  $\chi \in \mathcal{C}^{\infty}([0,+\infty),\mathbb{R})$  be a cut-off function such that

$$\begin{cases} \chi(t) = 1 & \text{for } t \in [0, 1] \\ 0 \le \chi(t) \le 1 & \text{for } t \in (1, 2) \\ \chi(t) = 0 & \text{for } t \in [2, +\infty) \\ \|\chi'\|_{\infty} \le 2, \end{cases}$$

and we set

$$\xi_{\Lambda}(u) = \chi\left(\frac{\|u\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}}\right).$$

Then we introduce the truncated functional  $\mathcal{F}_q^{\Lambda}: H^s_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$  defined as

$$\mathcal{F}_q^{\Lambda}(u) = \frac{1}{2} [u]_{H^s(\mathbb{R}^N)}^2 + q \, \xi_{\Lambda}(u) \, \mathcal{R}(u) - \int_{\mathbb{R}^N} G(u) \, dx.$$

Clearly, a critical point u of  $\mathcal{F}_q^{\Lambda}$  with  $||u||_{H^s(\mathbb{R}^N)} \leq \Lambda$  is a critical point of  $\mathcal{F}_q$ . Our first aim is to prove that the truncated functional  $\mathcal{F}_q^{\Lambda}$  has a symmetric mountain pass geometry:

**Lemma 3.1.** There exist  $r_0 > 0$  and  $\rho_0 > 0$  such that

$$\mathcal{F}_q^{\Lambda}(u) \ge 0, \quad \text{for } \|u\|_{H^s(\mathbb{R}^N)} \le r_0$$
  
$$\mathcal{F}_q^{\Lambda}(u) \ge \rho_0, \quad \text{for } \|u\|_{H^s(\mathbb{R}^N)} = r_0.$$
 (3.9)

Moreover, for any  $n \in \mathbb{N}$  there exists an odd continuous map

$$\gamma_n: \mathbb{S}^{n-1} \to H^s_{\mathrm{rad}}(\mathbb{R}^N)$$

such that

$$\mathcal{F}_q^{\Lambda}(\gamma_n(\sigma)) < 0 \quad \text{for all } \sigma \in \mathbb{S}^{n-1},$$
 (3.10)

where

$$\mathbb{S}^{n-1} = \{ \sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n : |\sigma| = 1 \}.$$

*Proof.* Taking  $\varepsilon = \frac{1}{2}$  in (3.8), and by using (3.7), the positivity of  $\mathcal{R}$ , and Theorem 2.1, we have

$$\mathcal{F}_{q}^{\Lambda}(u) = \frac{1}{2} [u]_{H^{s}(\mathbb{R}^{N})}^{2} + \int_{\mathbb{R}^{N}} G_{2}(u) \, dx + q \, \xi_{\Lambda}(u) \, \mathcal{R}(u) - \int_{\mathbb{R}^{N}} G_{1}(u) \, dx$$

$$\geq \frac{1}{2} [u]_{H^{s}(\mathbb{R}^{N})}^{2} + \frac{m}{4} ||u||_{L^{2}(\mathbb{R}^{N})}^{2} - C_{\frac{1}{2}} ||u||_{L^{2_{s}^{*}}(\mathbb{R}^{N})}^{2_{s}^{*}}$$

$$\geq \min \left\{ \frac{1}{2}, \frac{m}{4} \right\} ||u||_{H^{s}(\mathbb{R}^{N})}^{2} - C_{\frac{1}{2}} C^{*} ||u||_{H^{s}(\mathbb{R}^{N})}^{2_{s}^{*}}$$

from which easily follows (3.9).

Proceeding similarly to Theorem 10 in [10], for any  $n \in \mathbb{N}$ , there exists an odd continuous map  $\pi_n : \mathbb{S}^{n-1} \to H^s_{\mathrm{rad}}(\mathbb{R}^N)$  such that

$$0 \notin \pi_n(\mathbb{S}^{n-1}),$$
 
$$\int_{\mathbb{R}^N} G(\pi_n(\sigma)) \, dx \ge 1 \text{ for all } \sigma \in \mathbb{S}^{n-1}.$$

Let us define

$$\psi_n^t(\sigma) = \pi_n(\sigma) \left(\frac{\cdot}{t}\right) \text{ with } t \ge 1.$$

Then, for t sufficiently large, we get

$$\mathcal{F}_{q}^{\Lambda}(\psi_{n}^{t}(\sigma)) = \frac{t^{N-2s}}{2} [\pi_{n}(\sigma)]_{H^{s}(\mathbb{R}^{N})}^{2} + q \chi \left( \frac{t^{N-2s} [\pi_{n}(\sigma)]_{H^{s}(\mathbb{R}^{N})}^{2} + t^{N} \|\pi_{n}(\sigma)\|_{L^{2}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \mathcal{R}(\psi_{n}^{t}(\sigma)) - t^{N} \int_{\mathbb{R}^{N}} G(\pi_{n}(\sigma)) dx$$

$$\leq t^{N-2s} \left\{ \frac{[\pi_{n}(\sigma)]_{H^{s}(\mathbb{R}^{N})}^{2} - t^{2s}}{2} \right\} < 0.$$

Therefore, we can choose  $\bar{t}$  such that  $\mathcal{F}_q^{\Lambda}(\psi_n^{\bar{t}}(\sigma)) < 0$  for all  $\sigma \in \mathbb{S}^{n-1}$ , and by setting  $\gamma_n(\sigma)(x) := \psi_n^{\bar{t}}(\sigma)(x)$ , we can see that  $\gamma_n$  satisfies the required properties.

Now we define the minimax value of  $\mathcal{F}_q^{\Lambda}$  by using the maps  $\gamma_n:\partial\mathcal{D}_n\to H^s_{\mathrm{rad}}(\mathbb{R}^N)$  obtained in Lemma 3.1. For any  $n\in\mathbb{N}$ , let

$$b_n = b_n(q, \Lambda) = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in \mathcal{D}_n} \mathcal{F}_q^{\Lambda}(\gamma(\sigma)),$$

where  $\mathcal{D}_n = \{ \sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n : |\sigma| \leq 1 \}$  and

$$\Gamma_n = \left\{ \gamma \in \mathcal{C}(\mathcal{D}_n, H^s_{\mathrm{rad}}(\mathbb{R}^N)) : \begin{array}{l} \gamma(-\sigma) = -\gamma(\sigma) & \text{ for all } \sigma \in \mathcal{D}_n \\ \gamma(\sigma) = \gamma_n(\sigma) & \text{ for all } \sigma \in \partial \mathcal{D}_n \end{array} \right\}.$$

Let us introduce the following modified functionals

$$\begin{split} \widetilde{\mathcal{F}_q}(\theta, u) &= \mathcal{F}_q(u(\cdot/e^\theta)) \\ \widetilde{\mathcal{F}_q^{\Lambda}}(\theta, u) &= \mathcal{F}_q^{\Lambda}(u(\cdot/e^\theta)) \end{split}$$

for  $(\theta, u) \in \mathbb{R} \times H^s_{\mathrm{rad}}(\mathbb{R}^N)$ . We set

$$\widetilde{\mathcal{F}}'_{q}(\theta, u) = \frac{\partial}{\partial u} \widetilde{\mathcal{F}}_{q}(\theta, u),$$

$$(\widetilde{\mathcal{F}}_{q}^{\Lambda})'(\theta, u) = \frac{\partial}{\partial u} \widetilde{\mathcal{F}}_{q}^{\Lambda}(\theta, u),$$

$$\widetilde{b}_{n} = \widetilde{b}_{n}(q, \Lambda) = \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}_{n}} \max_{\sigma \in \mathcal{D}_{n}} \widetilde{\mathcal{F}}_{q}^{\Lambda}(\widetilde{\gamma}(\sigma)),$$

where

$$\widetilde{\Gamma}_n = \left\{ \widetilde{\gamma} \in \mathcal{C}(\mathcal{D}_n, \mathbb{R} \times H^s_{\mathrm{rad}}(\mathbb{R}^N)) : \begin{array}{l} \widetilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} \\ (\theta(-\sigma), \eta(-\sigma)) = (\theta(\sigma), -\eta(\sigma)) & \text{for all } \sigma \in \mathcal{D}_n \\ (\theta(\sigma), \eta(\sigma)) = (0, \gamma_n(\sigma)) & \text{for all } \sigma \in \partial \mathcal{D}_n \end{array} \right\}.$$

By the assumption  $(\mathcal{R}_4)$  we get

$$\widetilde{\mathcal{F}}_q(\theta, u) = \frac{e^{(N-2s)\theta}}{2} [u]_{H^s(\mathbb{R}^N)}^2 + q \sum_{i=1}^k e^{\alpha_i \theta} \mathcal{R}_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u) \, dx,$$

and

$$\widetilde{\mathcal{F}_q^{\Lambda}}(\theta, u) = \frac{e^{(N-2s)\theta}}{2} [u]_{H^s(\mathbb{R}^N)}^2 + q\chi \left( \frac{e^{(N-2s)\theta} [u]_{H^s(\mathbb{R}^N)}^2 + e^{N\theta} ||u||_{L^2(\mathbb{R}^N)}^2}{\Lambda^2} \right) \sum_{i=1}^k e^{\alpha_i \theta} \mathcal{R}_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u) dx.$$

Proceeding as in [3, 19, 34], we can see that the following results hold.

## Lemma 3.2. We have

- (1) there exists  $\bar{b} > 0$  such that  $b_n \geq \bar{b}$  for any  $n \in \mathbb{N}$ ,
- (2)  $b_n \to +\infty$ ,
- (3)  $b_n = \tilde{b}_n$  for any  $n \in \mathbb{N}$ .

**Lemma 3.3.** For any  $n \in \mathbb{N}$  there exists a sequence  $\{(\theta_j, u_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times H^s_{\mathrm{rad}}(\mathbb{R}^N)$  such that

- (i)  $\theta_j \to 0$ ,
- (ii)  $\mathcal{F}_{\underline{q}}^{\Lambda}(\theta_j, u_j) \to b_n$ ,
- (iii)  $(\widetilde{\mathcal{F}_q^{\Lambda}})'(\theta_j, u_j) \to 0$  strongly in  $(H_{\text{rad}}^s(\mathbb{R}^N))^{-1}$ ,
- (iv)  $\frac{\partial}{\partial \theta} \widetilde{\mathcal{F}_q^{\Lambda}}(\theta_j, u_j) \to 0.$

Our goal is to prove that, for a suitable choice of  $\Lambda$  and q, the sequence  $\{(\theta_j, u_j)\}_{j \in \mathbb{N}}$  given by Lemma 3.3 is a bounded Palais-Smale sequence for  $\mathcal{F}_q$ . We begin proving the boundedness of  $\{u_j\}_{j \in \mathbb{N}}$  in  $H^s(\mathbb{R}^N)$ .

**Proposition 3.1.** Let  $n \in \mathbb{N}$  and  $\Lambda_n > 0$  sufficiently large. There exists  $q_n$ , depending on  $\Lambda_n$ , such that for any  $0 < q < q_n$ , if  $\{(\theta_j, u_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times H^s_{\mathrm{rad}}(\mathbb{R}^N)$  is the sequence given in Lemma 3.3, then, up to a subsequence,  $\|u_j\|_{H^s(\mathbb{R}^N)} \leq \Lambda_n$ , for any  $j \in \mathbb{N}$ .

Proof. Taking into account Lemma 3.2 and Lemma 3.3 we have

$$N\widetilde{\mathcal{F}_{q}^{\Lambda}}(\theta_{j}, u_{j}) - \frac{\partial}{\partial \theta}\widetilde{\mathcal{F}_{q}^{\Lambda}}(\theta_{j}, u_{j}) = Nb_{n} + o_{j}(1),$$

which can be written as

$$se^{(N-2s)\theta_j}[u_j]_{H^s(\mathbb{R}^N)}^2$$

$$= q \chi \left( \frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \sum_{i=1}^{k} (\alpha_{i} - N) \mathcal{R}_{i}(u_{j}(\cdot/e^{\theta_{j}}))$$

$$+ q \chi \left( \frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \sum_{i=1}^{k} e^{\alpha_{i}\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}}u_{j}), \beta_{i}e^{\beta_{i}\theta_{j}}u_{j} \rangle$$

$$+ q \chi' \left( \frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \frac{(N - 2s)e^{(N - 2s)\theta_{j}} [u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} + Ne^{N\theta_{j}} \|u_{j}\|_{L^{2}(\mathbb{R}^{N})}^{2} \mathcal{R}(u_{j}(\cdot/e^{\theta_{j}}))$$

$$+ Nb_{n} + o_{j}(1)$$

$$=: \mathcal{I}_{i} + \mathcal{I}\mathcal{I}_{j} + \mathcal{I}\mathcal{I}\mathcal{I}_{j} + Nb_{n} + o_{j}(1). \tag{3.11}$$

By the definition of  $b_n$ , if  $\gamma \in \Gamma_n$ , we deduce that

$$b_{n} \leq \max_{\sigma \in \mathcal{D}_{n}} \mathcal{F}_{q}^{\Lambda}(\gamma(\sigma))$$

$$\leq \max_{\sigma \in \mathcal{D}_{n}} \left\{ \frac{1}{2} [\gamma(\sigma)]_{H^{s}(\mathbb{R}^{N})}^{2} - \int_{\mathbb{R}^{N}} G(\gamma(\sigma)) dx \right\} + \max_{\sigma \in \mathcal{D}_{n}} \left( q \, \xi_{\Lambda}(\gamma(\sigma)) \, \mathcal{R}(\gamma(\sigma)) \right)$$

$$=: A_{1} + A_{2}(\Lambda). \tag{3.12}$$

Now, if  $\|\gamma(\sigma)\|_{H^s(\mathbb{R}^N)}^2 \ge 2\Lambda^2$  then  $A_2(\Lambda) = 0$ , otherwise, by (3.2), we can find  $\delta > 0$  such that

$$A_2(\Lambda) \le q \left( C_1 + C_2 \| \gamma(\sigma) \|_{H^s(\mathbb{R}^N)}^{\delta} \right) \le q \left( C_1 + C_2' \Lambda^{\delta} \right).$$

In addition we have the following estimates:

$$|\mathcal{I}_j| \le q \left( C_3 + C_4 \Lambda^{\delta} \right), \tag{3.13}$$

$$|\mathcal{I}\mathcal{I}_j| \le C_5 \, q \, \Lambda^\delta, \tag{3.14}$$

$$|\mathcal{I}\mathcal{I}\mathcal{I}_j| \le q \left( C_6 + C_7 \Lambda^{\delta} \right).$$
 (3.15)

Putting together (3.12), (3.13), (3.14) and (3.15), from (3.11) we obtain

$$[u_j]_{H^s(\mathbb{R}^N)}^2 \le C' + q \left( C_8 + C_9 \Lambda^{\delta} \right).$$
 (3.16)

On the other hand, by (iv) of Lemma 3.3 and (3.8), we deduce that

$$\frac{(N-2s)e^{(N-2s)\theta_{j}}}{2}[u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} + q\chi\left(\frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}}\right) \sum_{i=1}^{k} \alpha_{i} \mathcal{R}_{i}(u_{j}(\cdot/e^{\theta_{j}}))$$

$$+ q\chi\left(\frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}}\right) \sum_{i=1}^{k} e^{\alpha_{i}\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}}u_{j}), \beta_{i}e^{\beta_{i}\theta_{j}}u_{j}\rangle$$

$$+ q\chi'\left(\frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}}\right) \frac{(N-2s)e^{(N-2s)\theta_{j}}[u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} + Ne^{N\theta_{j}}\|u_{j}\|_{L^{2}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \mathcal{R}(u_{j}(\cdot/e^{\theta_{j}}))$$

$$+ Ne^{N\theta_{j}} \int_{\mathbb{R}^{N}} G_{2}(u_{j}) dx$$

$$= Ne^{N\theta_{j}} \int_{\mathbb{R}^{N}} G_{1}(u_{j}) dx + o_{j}(1)$$

$$\leq Ne^{N\theta_{j}} \left(C_{\varepsilon} \int_{\mathbb{R}^{N}} |u_{j}|^{2_{s}^{*}} dx + \varepsilon \int_{\mathbb{R}^{N}} G_{2}(u_{j}) dx\right) + o_{j}(1).$$
(3.17)

Then, by using (3.7), (3.14), (3.15), (3.16), (3.17) and Theorem 2.1, we can infer

$$\frac{Ne^{N\theta_{j}}m(1-\varepsilon)}{2} \int_{\mathbb{R}^{N}} u_{j}^{2} dx 
\leq (1-\varepsilon)Ne^{N\theta_{j}} \int_{\mathbb{R}^{N}} G_{2}(u_{j}) dx 
\leq Ne^{N\theta_{j}} C_{\varepsilon} \int_{\mathbb{R}^{N}} |u_{j}|^{2_{s}^{*}} dx - q \chi \left( \frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \sum_{i=1}^{k} e^{\alpha_{i}\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}}u_{j}), \beta_{i}e^{\beta_{i}\theta_{j}}u_{j} \rangle 
- q \chi' \left( \frac{\|u_{j}(\cdot/e^{\theta_{j}})\|_{H^{s}(\mathbb{R}^{N})}^{2}}{\Lambda^{2}} \right) \frac{(N-2s)e^{(N-2s)\theta_{j}}[u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} + Ne^{N\theta_{j}} \|u_{j}\|_{L^{2}(\mathbb{R}^{N})}^{2} \mathcal{R}(u_{j}(\cdot/e^{\theta_{j}})) + o_{j}(1) 
\leq C_{10} \left( [u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} \right)^{\frac{2_{s}^{*}}{2}} + q \left( C_{11} + C_{12}\Lambda^{\delta} \right) + o_{j}(1) 
\leq C_{10} \left( C' + q \left( C_{8} + C_{9}\Lambda^{\delta} \right) \right)^{\frac{2_{s}^{*}}{2}} + q \left( C_{11} + C_{12}\Lambda^{\delta} \right) + o_{j}(1).$$
(3.18)

Now, we argue by contradiction. If we suppose that there exists no subsequence  $\{u_j\}_{j\in\mathbb{N}}$  which is uniformly bounded by  $\Lambda$  in the  $H^s$ -norm, we can find  $j_0\in\mathbb{N}$  such that

$$||u_j||_{H^s(\mathbb{R}^N)} > \Lambda \text{ for all } j \ge j_0.$$
(3.19)

Without any loss of generality, we can assume that (3.19) is true for any  $u_j$ . As a consequence, by using (3.16), (3.18) and (3.19), we can deduce that

$$\Lambda^2 < \|u_j\|_{H^s(\mathbb{R}^N)}^2 \le C_{13} + C_{14} q \Lambda^{\frac{2s}{2}\delta}$$

which is impossible for  $\Lambda$  large and q small enough. Indeed, to see this, we can observe that it is possible to find  $\Lambda_0$  such that  $\Lambda_0^2 > C_{13} + 1$  and  $q_0 = q_0(\Lambda_0)$  such that  $C_{14}q \Lambda^{\frac{2_s^*}{2}\delta} < 1$ , for any  $q < q_0$ , and this gives a contradiction.

At this point, we prove the following compactness result:

**Lemma 3.4.** Let  $n \in \mathbb{N}$ ,  $\Lambda_n, q_n > 0$  as in Proposition 3.1 and  $\{(\theta_j, u_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times H^s_{\mathrm{rad}}(\mathbb{R}^N)$  be the sequence given in Lemma 3.3. Then  $\{u_j\}_{j \in \mathbb{N}}$  admits a subsequence which converges in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$  to a nontrivial critical point of  $\mathcal{F}_q$  at the level  $b_n$ .

*Proof.* By Proposition 3.1, we know that  $\{u_j\}_{j\in\mathbb{N}}$  is bounded, so, by using Theorem 2.2, we can suppose, up to a subsequence, that there exists  $u\in H^s_{\mathrm{rad}}(\mathbb{R}^N)$  such that

$$u_j \to u$$
 weakly in  $H^s_{\text{rad}}(\mathbb{R}^N)$ ,  
 $u_j \to u$  in  $L^p(\mathbb{R}^N)$ ,  $2 , (3.20)
 $u_j \to u$  a.e. in  $\mathbb{R}^N$ .$ 

By the weak lower semicontinuity we know that

$$[u]_{H^s(\mathbb{R}^N)}^2 \le \liminf_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2. \tag{3.21}$$

Recalling that  $||u_j||_{H^s(\mathbb{R}^N)} \leq \Lambda_n$  for any  $j \in \mathbb{N}$ , we can see that, for every  $v \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ ,

$$\langle \widetilde{\mathcal{F}}'_{q}(\theta_{j}, u_{j}), v \rangle = \langle (\widetilde{\mathcal{F}}_{q}^{\Lambda_{n}})'(\theta_{j}, u_{j}), v \rangle$$

$$= e^{(N-2s)\theta_{j}} \iint_{\mathbb{R}^{2N}} \frac{u_{j}(x) - u_{j}(y)}{|x - y|^{N+2s}} (v(x) - v(y)) \, dx dy$$

$$+ q \sum_{i=1}^{k} e^{(\alpha_{i} + \beta_{i})\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}} u_{j}), v \rangle$$

$$+ e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{2}(u_{j}) v \, dx - e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{1}(u_{j}) v \, dx. \tag{3.22}$$

Taking into account (3.22) and (iii) of Lemma 3.3 we have

$$o_{j}(1) = \langle \widetilde{\mathcal{F}}'_{q}(\theta_{j}, u_{j}), u \rangle - \langle \widetilde{\mathcal{F}}'_{q}(\theta_{j}, u_{j}), u_{j} \rangle$$

$$= e^{(N-2s)\theta_{j}} \iint_{\mathbb{R}^{2N}} \frac{u_{j}(x) - u_{j}(y)}{|x - y|^{N+2s}} [(u(x) - u(y)) - (u_{j}(x) - u_{j}(y))] dxdy$$

$$+ q \sum_{i=1}^{k} e^{(\alpha_{i} + \beta_{i})\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}}u_{j}), u - u_{j} \rangle$$

$$+ e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{2}(u_{j})(u - u_{j}) dx - e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{1}(u_{j})(u - u_{j}) dx.$$
(3.23)

Now, by applying the first part of Lemma 2.1 for  $P(t) = g_i(t)$ , i = 1, 2,  $Q(t) = |t|^{2_s^* - 1}$ ,  $v_j = u_j$ ,  $v = g_i(u)$ , i = 1, 2 and  $w \in C_0^{\infty}(\mathbb{R}^N)$ , by  $(g_3)$ , (3.4) and (3.20) we can see,

as  $j \to \infty$ 

$$\int_{\mathbb{R}^N} g_i(u_j) w \, dx \to \int_{\mathbb{R}^N} g_i(u) w \, dx \quad i = 1, 2,$$

so we obtain

$$\int_{\mathbb{R}^N} g_i(u_j) u \, dx \to \int_{\mathbb{R}^N} g_i(u) u \, dx \quad i = 1, 2.$$
(3.24)

Taking  $X = H^s(\mathbb{R}^N)$ ,  $q_1 = 2$ ,  $q_2 = 2_s^*$ ,  $v_j = u_j$ ,  $v = g_1(u)u$  and  $P(t) = g_1(t)t$  in Lemma 2.2, by (3.3), (3.4) and (3.20) we deduce

$$\int_{\mathbb{R}^N} g_1(u_j) u_j \, dx \to \int_{\mathbb{R}^N} g_1(u) u \, dx. \tag{3.25}$$

On the other hand, (3.20) and Fatou's Lemma yield

$$\int_{\mathbb{R}^N} g_2(u)u \, dx \le \liminf_{j \to \infty} \int_{\mathbb{R}^N} g_2(u_j)u_j \, dx. \tag{3.26}$$

Putting together (3.23), (3.24), (3.25), (3.26), and by using  $(\mathcal{R}_3)$  we get

$$\lim \sup_{j \to \infty} [u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2} = \lim \sup_{j \to \infty} e^{(N-2s)\theta_{j}} [u_{j}]_{H^{s}(\mathbb{R}^{N})}^{2}$$

$$= \lim \sup_{j \to \infty} \left[ e^{(N-2s)\theta_{j}} \iint_{\mathbb{R}^{2N}} \frac{u_{j}(x) - u_{j}(y)}{|x - y|^{N+2s}} (u(x) - u(y)) \, dx dy \right]$$

$$+ q \sum_{i=1}^{k} e^{(\alpha_{i} + \beta_{i})\theta_{j}} \langle \mathcal{R}'_{i}(e^{\beta_{i}\theta_{j}}u_{j}), u - u_{j} \rangle$$

$$+ e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{2}(u_{j})(u - u_{j}) \, dx - e^{N\theta_{j}} \int_{\mathbb{R}^{N}} g_{1}(u_{j})(u - u_{j}) \, dx \right]$$

$$\leq [u]_{H^{s}(\mathbb{R}^{N})}^{2}. \tag{3.27}$$

Therefore (3.21) and (3.27) give

$$\lim_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2 = [u]_{H^s(\mathbb{R}^N)}^2, \tag{3.28}$$

which, in view of (3.23), yields

$$\lim_{j \to \infty} \int_{\mathbb{R}^N} g_2(u_j) u_j \, dx = \int_{\mathbb{R}^N} g_2(u) u \, dx. \tag{3.29}$$

Since  $g_2(t)t = mt^2 + h(t)$ , with h a positive and continuous function, by Fatou's Lemma follows that

$$\int_{\mathbb{R}^N} h(u) \, dx \le \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx \tag{3.30}$$

$$\int_{\mathbb{R}^N} u^2 \, dx \le \liminf_{j \to \infty} \int_{\mathbb{R}^N} u_j^2 \, dx. \tag{3.31}$$

By using (3.29) and (3.30) we can see that

$$\begin{split} \limsup_{j \to \infty} \int_{\mathbb{R}^N} m u_j^2 \, dx &= \limsup_{j \to \infty} \int_{\mathbb{R}^N} (g_2(u_j) u_j - h(u_j)) \, dx \\ &= \int_{\mathbb{R}^N} g_2(u) u \, dx + \limsup_{j \to \infty} \left( -\int_{\mathbb{R}^N} h(u_j) \, dx \right) \\ &= \int_{\mathbb{R}^N} (m u^2 + h(u)) \, dx - \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx \\ &= \int_{\mathbb{R}^N} m u^2 \, dx + \int_{\mathbb{R}^N} h(u) \, dx - \liminf_{j \to \infty} \int_{\mathbb{R}^N} h(u_j) \, dx \\ &\leq \int_{\mathbb{R}^N} m u^2 \, dx \end{split}$$

which, together with (3.31), implies that  $u_j \to u$  strongly in  $L^2(\mathbb{R}^N)$ . Then, we have proved that  $u_j \to u$  strongly in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ . Since  $b_n > 0$ , u is a nontrivial critical point of  $\mathcal{F}_q$  at the level  $b_n$ .

Now, we are ready to prove the main result of this Section:

Proof of Theorem 3.1. Let  $h \ge 1$ . Since  $b_n \to +\infty$  (see (2) of Lemma 3.2), up to a subsequence, we can consider  $b_1 < b_2 < \cdots < b_h$ . Then, in view of Lemma 3.4, we define  $q(h) = q_h > 0$  and we get the thesis.

#### 4. Proof of Theorem 1.1

In this Section we give the proof of Theorem 1.1. Let us introduce the following functional

$$\mathcal{F}_q(u) = \frac{1}{2} \left( p + \frac{q}{2} (1 - s) [u]_{H^s(\mathbb{R}^N)}^2 \right) [u]_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} G(u) \, dx.$$

In view of Theorem 3.1, it is enough to verify that

$$\mathcal{R}(u) = \frac{1-s}{4} [u]_{H^s(\mathbb{R}^N)}^4 \tag{4.1}$$

satisfies the assumptions  $(\mathcal{R}_1)$ - $(\mathcal{R}_5)$ .

Clearly  $\mathcal{R}$  is an even and nonnegative  $\mathcal{C}^1$ -functional in  $H^s(\mathbb{R}^N)$ . Since  $[u]_{H^s(\mathbb{R}^N)}^2 \leq ||u||_{H^s(\mathbb{R}^N)}^2$ , we can see that the assumptions  $(\mathcal{R}_1)$  and  $(\mathcal{R}_2)$  are satisfied.

Regarding  $(\mathcal{R}_3)$ , suppose that  $u_j \rightharpoonup u$  weakly in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$  and  $[u_j]^2_{H^s(\mathbb{R}^N)} \to \ell \geq 0$ . If  $\ell = 0$ , then we have finished. Let us assume  $\ell > 0$ . From the weak lower semicontinuity, we have

$$[u]_{H^s(\mathbb{R}^N)}^2 \le \liminf_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2. \tag{4.2}$$

By using the following properties of lim inf and lim sup for sequences of real numbers

$$\limsup_{j \to \infty} a_j b_j = a \limsup_{j \to \infty} b_j \text{ if } a_j \to a > 0,$$

$$\lim_{j \to \infty} \sup (a_j + b_j) = a + \limsup_{j \to \infty} b_j \text{ if } a_j \to a,$$

$$\lim_{j \to \infty} \sup (-a_j) = - \liminf_{j \to \infty} a_j,$$

and by applying (4.2), we obtain

$$\lim \sup_{j \to \infty} \langle \mathcal{R}'(u_j), u - u_j \rangle =$$

$$= (1 - s) \lim \sup_{j \to \infty} \left( [u_j]_{H^s(\mathbb{R}^N)}^2 \int_{\mathbb{R}^{2N}} \frac{(u_j(x) - u_j(y))}{|x - y|^{N + 2s}} [(u(x) - u(y)) - (u_j(x) - u_j(y))] dx dy \right)$$

$$= (1 - s)\ell \lim \sup_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{(u_j(x) - u_j(y))}{|x - y|^{N + 2s}} [(u(x) - u(y)) - (u_j(x) - u_j(y))] dx dy$$

$$= (1 - s)\ell \left( \lim_{j \to \infty} \int_{\mathbb{R}^{2N}} \frac{(u_j(x) - u_j(y))}{|x - y|^{N + 2s}} (u(x) - u(y)) dx dy - \lim_{j \to \infty} \inf_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2 \right)$$

$$= (1 - s)\ell \left( [u]_{H^s(\mathbb{R}^N)}^2 - \lim_{j \to \infty} \inf_{j \to \infty} [u_j]_{H^s(\mathbb{R}^N)}^2 \right) \le 0,$$

which gives  $(\mathcal{R}_3)$ .

Now, recalling the definition of  $u_t$  and by using (4.1), it follows that  $(\mathcal{R}_4)$  is verified because of

$$\mathcal{R}(u_t) = \frac{1-s}{4} \left( \iint_{\mathbb{R}^{2N}} \frac{|u(\frac{x}{t}) - u(\frac{y}{t})|^2}{|x - y|^{N+2s}} dx dy \right)^2$$

$$= \frac{(1-s)t^{2(N-2s)}}{4} \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^4$$

$$= t^{2(N-2s)} \mathcal{R}(u).$$

Finally, we prove the condition  $(\mathcal{R}_5)$ . By using a change of variable, we can see that, for any  $g \in \mathcal{O}(N)$ 

$$\mathcal{R}(u(g \cdot)) = \frac{1-s}{4} [u(g \cdot)]_{H^s(\mathbb{R}^N)}^4 = \frac{1-s}{4} [u]_{H^s(\mathbb{R}^N)}^4 = \mathcal{R}(u).$$

Then, by applying Theorem 3.1, we can infer that for any  $h \in \mathbb{N}$ , there exists q(h) > 0 such that for any 0 < q < q(h) the functional  $\mathcal{F}_q$  admits at least h couples of critical points in  $H^s(\mathbb{R}^N)$  with radial symmetry. This means that (1.1) admits at least h couples of weak solutions in  $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ .

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