A specialization property of index

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Abstract

In [Kol13] Kollár defined i-th index of a proper scheme over a field. In this note we study how index behaves under specialization, in any characteristic.

1 Introduction

In [Kol13], Kollár generalized the classical definition of index of a variety. For a proper scheme X defined over a field k (of any characteristic), he defines $\operatorname{elw}_i(X)$ to be the ideal in \mathbb{Z} generated by $\chi(\mathcal{F})$ as \mathcal{F} ranges over all coherent sheaves on X supported in a subscheme of dimension at most i. One can easily notice,

$$\operatorname{elw}_0(X) \subset \operatorname{elw}_1(X) \subset ... \subset \operatorname{elw}_{\dim X}(X).$$

Among other important properties, it is known that $\operatorname{elw}_i(-)$ is birational invariant. In this note we ask whether this property is locally constructible i.e., given a flat family $\pi: X \to B$ of proper k-schemes and any closed irreducible subset $Y \subset B$, does there exist a non-empty open subset U of Y on which the function $u \mapsto \operatorname{elw}_i(\pi^{-1}(u))$ is constant? We observe that this is not the case in general (see Example 3.1). However, we prove that the function $\operatorname{elw}_i(-)$ is invariant under specialization. In particular, we prove:

Theorem 1.1. Let R be a discrete valuation ring, $\pi: X_R \to \operatorname{Spec}(R)$ be a flat, projective morphism and X_R is an integral scheme. Suppose that the dimension of the fibers is equal to n. Denote by K (resp. k) the fraction field (resp. residue field) of R. Then, for any $0 \le i \le n$, $\operatorname{elw}_i(X_K) = \operatorname{elw}_i(X_k)$.

We note that there is no assumption on the characteristic of K or k in this article.

We now discuss the strategy of the proof. The inequality $\operatorname{elw}_i(X_K) \geq \operatorname{elw}_i(X_k)$ is a direct consequence of [Kol13, Proposition 4]. To prove the reverse inequality we introduce a concept of semistable elw-indices. The key observation is that this coincides with Kollár's definition of the elw-index (see Proposition 2.4). We then check that any

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semistable sheaf on the special fiber can be lifted to the generic fiber (see Proposition 3.3). This gives the reverse inequality, hence proves the theorem.

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2 Semi-stable Esnault-Levine-Wittenberg index

Recall, the definition of the i-th index as given in [Kol13]:

Definition 2.1. Let Y be a projective scheme over a field k. For $0 \le i \le \dim Y$, the i-th Esnault-Levine-Wittenberg index, denoted by $elw_i(Y)$ is defined as the ideal of \mathbb{Z} generated by $\chi(\mathcal{F})$ as where \mathcal{F} runs through all coherent sheaves on Y of dimension $\dim \mathcal{F} := \dim \operatorname{Supp} \mathcal{F} \le i$.

We now define a new semi-stable version of the above index:

Definition 2.2. Let Y be as before. For $0 \le i \le \dim Y$, the *semi-stable Esnault-Levine-Wittenberg index*, denoted by $\operatorname{elw}_i^{ss}(Y)$ is defined as the ideal of $\mathbb Z$ generated by $\chi(\mathcal F)$ as $\mathcal F$ runs through all coherent semi-stable sheaves over Y of dimension $\dim \mathcal F \le i$.

Abuse of Notations 2.3. We will sometimes abuse notation $elw_i(Y)$ to mean the generator of the ideal. This usage will be clear from the context.

Proposition 2.4. Let Y be a projective scheme of pure dimension n. Then, for all $0 \le i \le \dim Y$, $\operatorname{elw}_i^{ss}(Y) = \operatorname{elw}_i(Y)$.

Proof. Clearly, $\operatorname{elw}_m^{ss}(Y) \subset \operatorname{elw}_m(Y)$ for all $0 \leq m \leq n$. We now prove the converse. Let \mathcal{F} be a coherent sheaf of dimension m. There exists an unique torsion filtration:

$$0 \subset T_0(\mathcal{F}) \subset ... \subset T_m(\mathcal{F}) = \mathcal{F}$$

where $T_i(\mathcal{F})$ is the maximal subsheaf of \mathcal{F} of dimension at most equal to i. Note by definition, $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is zero or pure of dimension i. If $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is non-zero then, by [HL10, Theorem 1.3.4], there exists an unique Harder-Narasimahan filtration

$$0 = E_{-1} \subsetneq E_0 \subsetneq E_1 \subsetneq \dots \subsetneq E_l = T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$$

of the pure sheaf $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ such that E_j/E_{j-1} is semistable of dimension i, for every j=1,...,l. As Euler characteristic is additive,

$$\chi(T_i(\mathcal{F})/T_{i-1}(\mathcal{F})) = \sum_j \chi(E_j/E_{j-1})$$
 and $\chi(\mathcal{F}) = \sum_i \chi(T_i(\mathcal{F})/T_{i-1}(\mathcal{F})).$

Since $i \leq m$, this implies $\operatorname{elw}_m(Y) \subset \operatorname{elw}_m^{ss}(Y)$. Therefore, $\operatorname{elw}_m^{ss}(Y) = \operatorname{elw}_m(Y)$, for all $0 \leq m \leq n$. This proves the proposition.

3 Specialization properties of the index function

We first observe (Example 3.1) that the Esnault-Levine-Wittenberg index does not vary in an algebraic way in flat families of projective varieties (in particular, not semi-continuous). But $elw_i(-)$ is invariant under specialization (Theorem 3.4).

Example 3.1. Denote by $Y := \mathbb{A}^1_{\mathbb{R}}$ and X the hypersurface in $\mathbb{P}^2 \times Y$ defined by the polynomial $X^2 + Y^2 = t$, where t is the coordinate corresponding to \mathbb{A}^1 . Denote by $\pi : X \to Y$ the natural projection map. Observe that π is a flat morphism.

Denote by $I_{\pi}^{i}: Y \to \mathbb{Z}$ the function that associates to any $y \in Y$, the number $\operatorname{elw}_{i}(X_{y})$. We prove that I_{π}^{0} is not semi-continuous. Observe that for t > 0, there always exists rational point on the fiber X_{t} , hence $\operatorname{elw}_{0}(X_{t}) = 1$. For $t \leq 0$, there does not exist a rational point but the resulting curve X_{t} contains a \mathbb{C} -point. But \mathbb{C} is an extension of degree 2 over \mathbb{R} . Hence, $\operatorname{elw}_{0}(X_{t}) = 2$ for $t \leq 0$. This shows that I_{π}^{0} is not semi-continuous (in the Zariski topology).

Notation 3.2. Let R be a discrete valuation ring, $\pi: X_R \to \operatorname{Spec}(R)$ be a flat, projective morphism and X_R is an integral scheme. Suppose that the dimension of the fibers is equal to n. Denote by K (resp. k) the fraction field (resp. residue field) of R. Denote by X_k the special fiber and by X_K the generic fiber. There is no assumption on the characteristic of k and K.

Proposition 3.3. Suppose \mathcal{F}_k is a semistable coherent sheaf on X_k . Then, there exists a semistable coherent sheaf \mathcal{F}_R on X_R such that $\mathcal{F}_R \otimes_R k \cong \mathcal{F}_k$.

Proof. As \mathcal{F}_k is semistable there exists a free \mathcal{O}_{X_k} -module \mathcal{H}_k such that \mathcal{F}_k is a quotient of \mathcal{H}_k . Let the resulting short exact sequence be of the form:

$$0 \to \mathcal{G}_k \to \mathcal{H}_k \to \mathcal{F}_k \to 0.$$

As \mathcal{H}_k is a free \mathcal{O}_{X_k} -module, there exists an unique free \mathcal{O}_{X_R} -module \mathcal{H}_R such that $\mathcal{H}_R \otimes_R k$. Denote by \mathcal{G}_R the fiber product $\mathcal{H}_R \times_{\mathcal{H}_k} \mathcal{G}_k$ i.e., the set of sections of \mathcal{H}_R , which when restricted to \mathcal{H}_k gives us \mathcal{G}_k . Denote by \mathcal{F}_R the quotient of \mathcal{H}_R by \mathcal{G}_R . As $\mathcal{G}_R \otimes_R k \cong \mathcal{G}_k$, the universal property of cokernel implies $\mathcal{F}_R \otimes_R k \cong \mathcal{F}_k$. Since \mathcal{G}_R is a subsheaf of a free sheaf it is torsion free and as R is principal ideal domain, \mathcal{G}_R is R-flat (see [Har77, Example III.9.1.3]). Using [HL10, Lemma 2.1.4], we have \mathcal{F}_R is R-flat. By the open nature of semi-stability [HL10, Proposition 2.3.1], this implies \mathcal{F}_R is semi-stable. This proves the proposition.

We finally prove the main theorem of this article.

Theorem 3.4. The following is true: for any $0 \le i \le n$, $\operatorname{elw}_i(X_K) = \operatorname{elw}_i(X_k)$.

Proof. By [Kol13, Proposition 4]

$$\operatorname{elw}_i(X_K) = (\chi(\mathcal{O}_Z)|Z \subset X, \text{ of dimension } \leq i).$$

Given, any Z on X_K , the scheme-theorectic closure \bar{Z} is flat over $\operatorname{Spec}(R)$ (see [Har77, Proposition III.9.8]). As Euler characteristic is invariant in flat families, $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_{\bar{Z}_k})$, where \bar{Z}_k is the special fiber for the flat morphism from \bar{Z} to $\operatorname{Spec}(R)$. Hence, for $0 \leq i < n$, $\operatorname{elw}_i(X_K) \geq \operatorname{elw}_i(X_k)$.

Using Proposition 3.3, we conclude that $\operatorname{elw}_i^{ss}(X_K) \leq \operatorname{elw}_i^{ss}(X_k)$, for all $0 \leq i \leq n$. By Proposition 2.4, we get

$$\operatorname{elw}_i(X_K) = \operatorname{elw}_i^{ss}(X_K) \le \operatorname{elw}_i^{ss}(X_k) = \operatorname{elw}_i(X_k).$$

This implies $elw_i(X_K) = elw_i(X_k)$.

This completes the proof of the theorem.

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