# KIMURA-FINITENESS OF QUADRIC FIBRATIONS OVER SMOOTH CURVES

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ABSTRACT. In this short note, making use of the recent theory of noncommutative mixed motives, we prove that the Voevodsky's mixed motive of a quadric fibration over a smooth curve is Kimura-finite.

#### 1. Introduction

Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a  $\mathbb{Q}$ -linear, idempotent complete, symmetric monoidal category. Given a partition  $\lambda$  of an integer  $n \geq 1$ , consider the corresponding irreducible  $\mathbb{Q}$ -linear representation  $V_{\lambda}$  of the symmetric group  $\mathfrak{S}_n$  and the associated idempotent  $e_{\lambda} \in \mathbb{Q}[\mathfrak{S}_n]$ . Under these notations, the Schur-functor  $S_{\lambda} \colon \mathcal{C} \to \mathcal{C}$  sends an object a to the direct summand of  $a^{\otimes n}$  determined by  $e_{\lambda}$ . In the particular case of the partition  $\lambda = (1, \ldots, 1)$ , resp.  $\lambda = (n)$ , the associated Schur-functor  $\wedge^n := S_{(1,\ldots,1)}$ , resp. Sym<sup>n</sup> :=  $S_{(n)}$ , is called the  $n^{\text{th}}$  wedge product, resp. the  $n^{\text{th}}$  symmetric product. Following Kimura [9], an object  $a \in \mathcal{C}$  is called even-dimensional, resp. odd-dimensional, if  $\wedge^n(a)$ , resp. Sym<sup>n</sup>(a) = 0, for some  $n \gg 0$ . The biggest integer  $\lim_{n \to \infty} (a)$ , resp. kim<sub>-</sub>(a), for which  $\wedge^{\text{kim}_+(a)} \neq 0$ , resp. Sym<sup>kim<sub>-</sub>(a)</sup>(a)  $\neq 0$ , is called the even, resp. odd, Kimura-dimension of a. An object  $a \in \mathcal{C}$  is called Kimura-finite if  $a \simeq a_+ \oplus a_-$ , with  $a_+$  even-dimensional and  $a_-$  odd-dimensional. The integer  $\lim_{n \to \infty} (a) = \lim_{n \to \infty} (a) + \lim_{n \to \infty} (a) = \lim_{n \to \infty} (a) + \lim_{n \to \infty} (a) + \lim_{n \to \infty} (a) = \lim_{n \to \infty} (a) + \lim_{n \to \infty} (a)$ 

Voevodsky introduced in [18] an important triangulated category of geometric mixed motives  $\mathrm{DM_{gm}}(k)_{\mathbb{Q}}$  (over a perfect base field k). By construction, this category is  $\mathbb{Q}$ -linear, idempotent complete, rigid symmetric monoidal, and comes equipped with a symmetric monoidal functor  $M(-)_{\mathbb{Q}} \colon \mathrm{Sm}(k) \to \mathrm{DM_{gm}}(k)_{\mathbb{Q}}$ , defined on smooth k-schemes. An important open problem is the classification of all the Kimura-finite mixed motives and the computation of the corresponding Kimura-dimensions. On the negative side, O'Sullivan constructed a certain smooth surface S whose mixed motive  $M(S)_{\mathbb{Q}}$  is not Kimura-finite; consult [12, §5.1] for details. On the positive side, Guletskii [6] and Mazza [12] proved, independently, that the mixed motive  $M(C)_{\mathbb{Q}}$  of every smooth curve C is Kimura-finite.

The following result bootstraps Kimura-finiteness from smooth curves to families of quadrics over smooth curves:

**Theorem 1.1.** Let k be a field, C a smooth k-curve, and  $q: Q \to C$  a flat quadric fibration of relative dimension d-2. Assume that Q is smooth and that q has only simple degenerations, i.e. that all the fibers of q have  $corank \leq 1$ .

(i) When d is even, the mixed motive  $M(Q)_{\mathbb{O}}$  is Kimura-finite. Moreover, we have

$$kim(M(Q)_{\mathbb{Q}}) = kim(M(\widetilde{C})_{\mathbb{Q}}) + (d-2)kim(M(C)_{\mathbb{Q}}),$$

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<sup>&</sup>lt;sup>1</sup>Among other consequences, Kimura-finiteness implies rationality of the motivic zeta function.

- where  $D \hookrightarrow C$  stands for the finite set of critical values of q and  $\widetilde{C}$  for the discriminant double cover of C (ramified over D).
- (ii) When d is odd, k is algebraically closed, and  $1/2 \in k$ , the mixed motive  $M(Q)_{\mathbb{Q}}$  is Kimura-finite. Moreover, we have the following equality:

$$kim(M(Q)_{\mathbb{O}}) = \#D + (d-1)kim(M(C)_{\mathbb{O}}).$$

To the best of the authors' knowledge, Theorem 1.1 is new in the literature. It not only provides new (families of) examples of Kimura-finite mixed motives but also computes the corresponding Kimura dimensions.

Remark 1.2. In the particular case where k is algebraically closed and Q, C are moreover projective, Vial proved in [17, Cor. 4.4] that the Chow motive  $\mathfrak{h}(Q)_{\mathbb{Q}}$  is Kimura-finite. Since the category of Chow motives embeds fully-faithfully into  $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$  (see [18, §4]), we then obtain in this particular case an alternative "geometric" proof of the Kimura-finiteness of  $M(Q)_{\mathbb{Q}}$ . Moreover, when  $k=\mathbb{C}$  and d is odd, Bouali refined Vial's work by showing that  $\mathfrak{h}(Q)_{\mathbb{Q}}$  is isomorphic to  $\mathbb{Q}(-\frac{d-1}{2})^{\oplus \#D} \oplus \bigoplus_{i=0}^{d-2} \mathfrak{h}(C)_{\mathbb{Q}}(-i)$ ; see [4, Rk. 1.10(i)]. In this particular case, this leads to an alternative "geometric" computation of the Kimura-dimension of  $M(Q)_{\mathbb{Q}}$ .

#### 2. Preliminaries

In what follows, k denotes a base field.

**Dg categories.** For a survey on dg categories consult Keller's ICM talk [8]. In what follows, we write  $\operatorname{dgcat}(k)$  for the category of (small) dg categories and dg functors. Every (dg) k-algebra gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks since the category of perfect complexes  $\operatorname{perf}(X)$  of every k-scheme X (or, more generally, algebraic stack  $\mathcal{X}$ ) admits a canonical dg enhancement  $\operatorname{perf}_{\operatorname{dg}}(X)$ ; see [8, §4.6][11].

Noncommutative mixed motives. For a book, resp. survey, on noncommutative motives consult [13], resp. [14]. Recall from [13, §8.5.1] the construction of Kontsevich's triangulated category of noncommutative mixed motives  $\mathrm{NMot}(k)$ ; denoted by  $\mathrm{NMot}^{\mathbb{A}^1}_{\mathrm{loc}}(k)$  in *loc. cit.* By construction, this category is idempotent complete, closed symmetric monoidal, and comes equipped with a symmetric monoidal functor  $U: \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$ .

**Root stacks.** Let X be a k-scheme,  $\mathcal{L}$  a line bundle on X,  $\sigma \in \Gamma(X, \mathcal{L})$  a global section, and r > 0 an integer. In what follows, we write  $D \hookrightarrow X$  for the zero locus of  $\sigma$ . Recall from [5, Def. 2.2.1] (see also [1, Appendix B]) that the associated *root stack* is defined as the following fiber-product of algebraic stacks

$$\sqrt[r]{(\mathcal{L}, \sigma)/X} \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] 
\downarrow \qquad \qquad \downarrow \theta_r 
X \xrightarrow{(\mathcal{L}, \sigma)} [\mathbb{A}^1/\mathbb{G}_m],$$

where  $\theta_r$  stands for the morphism induced by the  $r^{\text{th}}$  power maps on  $\mathbb{A}^1$  and  $\mathbb{G}_m$ .

**Proposition 2.1.** We have an isomorphism  $U(\sqrt[r]{(\mathcal{L},\sigma)/X}) \simeq U(D)^{\oplus (r-1)} \oplus U(X)$  whenever X and D are k-smooth.

*Proof.* By construction, the root stack comes equipped with a forgetful morphism  $f \colon \sqrt[r]{(\mathcal{L}, \sigma)/X} \to X$ . As proved by Ishii-Ueda in [7, Thm. 1.6], the pull-back functor  $f^*$  is fully-faithful. Moreover, we have a semi-orthogonal decomposition

$$\operatorname{perf}(\mathcal{X}) = \langle \operatorname{perf}(D)_{r-1}, \dots, \operatorname{perf}(D)_1, f^*(\operatorname{perf}(X)) \rangle,$$

where all the categories  $perf(D)_i$  are equivalent (via a Fourier-Mukai type functor) to perf(D). Consequently, the proof follows from the fact that the functor U sends semi-orthogonal decomposition to direct sums (see [13, §8.4.1 and §8.4.5]).

Orbit categories. Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be an  $\mathbb{Q}$ -linear symmetric monoidal additive category and  $\mathcal{O} \in \mathcal{C}$  a  $\otimes$ -invertible object. The *orbit category*  $\mathcal{C}/_{-\otimes\mathcal{O}}$  has the same objects as  $\mathcal{C}$  and morphisms  $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a,b) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a,b \otimes \mathcal{O}^{\otimes n})$ . Given objects a,b,c and morphisms  $f = \{f_n\}_{n \in \mathbb{Z}}$  and  $g = \{g_n\}_{n \in \mathbb{Z}}$ , the  $i^{\mathrm{th}}$ -component of  $g \circ f$  is defined as  $\sum_n (g_{i-n} \otimes \mathcal{O}^{\otimes n}) \circ f_n$ . The canonical functor  $\pi \colon \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$ , given by  $a \mapsto a$  and  $f \mapsto f = \{f_n\}_{n \in \mathbb{Z}}$ , where  $f_0 = f$  and  $f_n = 0$  if  $n \neq 0$ , is endowed with an isomorphism  $\pi \circ (-\otimes \mathcal{O}) \Rightarrow \pi$  and is 2-universal among all such functors. Finally, the category  $\mathcal{C}/_{-\otimes\mathcal{O}}$  is  $\mathbb{Q}$ -linear, additive, and inherits from  $\mathcal{C}$  a symmetric monoidal structure making  $\pi$  symmetric monoidal.

### 3. Proof of Theorem 1.1

Following Kuznetsov [10, §3] (see also Auel-Bernardara-Bolognesi [3, §1.2]), let E be a vector bundle of rank d on C,  $p \colon \mathbb{P}(E) \to C$  the projectivization of E on C,  $\mathcal{O}_{\mathbb{P}(E)}(1)$  the Grothendieck line bundle on  $\mathbb{P}(E)$ ,  $\mathcal{L}$  a line bundle on C, and finally  $\rho \in \Gamma(C, S^2(E^{\vee}) \otimes \mathcal{L}^{\vee}) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \mathcal{L}^{\vee})$  a global section. Given this data,  $Q \subset \mathbb{P}(E)$  is defined as the zero locus of  $\rho$  on  $\mathbb{P}(E)$  and  $q \colon Q \to C$  as the restriction of p to Q; the relative dimension of q is equal to d-2. Consider also the discriminant global section  $\mathrm{disc}(q) \in \Gamma(C, \det(E^{\vee})^{\otimes 2} \otimes (\mathcal{L}^{\vee})^{\otimes d})$  and the associated zero locus  $D \hookrightarrow C$ . Note that D agrees with the finite set of critical values of q. Recall from [10, §3.5](see also [3, §1.6]) that, when d is even we have a discriminant double cover  $\widetilde{C}$  of C ramified over D. Moreover, since by hypothesis q has only simple degenerations,  $\widetilde{C}$  is k-smooth. Under the above notations, we have the following computation:

**Proposition 3.1.** Let  $q: Q \to C$  be a flat quadric fibration as above.

- (i) When d is even, we have an isomorphism  $U(Q)_{\mathbb{Z}[1/2]} \simeq U(\widetilde{C})_{\mathbb{Z}[1/2]} \oplus U(C)_{\mathbb{Z}[1/2]}^{\oplus (d-2)}$ .
- (ii) When d is odd, k is algebraically closed, and  $1/2 \in k$ , we have an isomorphism  $U(Q) \simeq U(D) \oplus U(C)^{\oplus (d-1)}$ .

*Proof.* Recall from [10, §3] (see also [3, §1.5]) the construction of the sheaf  $C_0$  of even parts of the Clifford algebra associated to q. As proved in [10, Thm. 4.2] (see also [3, Thm. 2.2.1]), we have a semi-orthogonal decomposition

$$\operatorname{perf}(Q) = \langle \operatorname{perf}(C; \mathcal{C}_0), \operatorname{perf}(C)_1, \dots, \operatorname{perf}(C)_{d-2} \rangle,$$

where  $\operatorname{perf}(C; \mathcal{C}_0)$  stands for the category of perfect  $\mathcal{C}_0$ -modules and  $\operatorname{perf}(C)_i := q^*(\operatorname{perf}(C)) \otimes \mathcal{O}_{Q/C}(i)$ . Note that all the categories  $\operatorname{perf}(C)_i$  are equivalent (via a Fourier-Mukai type functor) to  $\operatorname{perf}(C)$ . Since the functor U sends semi-orthogonal decompositions to direct sums, we then obtain a direct sum decomposition

(3.2) 
$$U(Q) \simeq U(\operatorname{perf}^{\operatorname{dg}}(C; \mathcal{C}_0)) \oplus U(C)^{\oplus (d-2)},$$

where  $\operatorname{perf}^{\operatorname{dg}}(C;\mathcal{C}_0)$  stands for the dg enhancement of  $\operatorname{perf}(C;\mathcal{C}_0)$  induced from  $\operatorname{perf}_{\operatorname{dg}}(Q)$ . As explained in [10, Prop. 4.9] (see also [3, §2.2]), the inclusion of categories  $\operatorname{perf}(C;\mathcal{C}_0) \hookrightarrow \operatorname{perf}(Q)$  is of Fourier-Mukai type. Therefore, the associated kernel leads to a Fourier-Mukai Morita equivalence between  $\operatorname{perf}^{\operatorname{dg}}(C;\mathcal{C}_0)$  and  $\operatorname{perf}_{\operatorname{dg}}(C;\mathcal{C}_0)$ . Consequently, we can replace the dg category  $\operatorname{perf}^{\operatorname{dg}}(C,\mathcal{C}_0)$  by  $\operatorname{perf}_{\operatorname{dg}}(C;\mathcal{C}_0)$  in the above decomposition (3.2).

Item (i). As explained in [10, §3.5] (see also [3, §1.6]), the category  $\operatorname{perf}(C; \mathcal{C}_0)$  is equivalent (via a Fourier-Mukai type functor) to  $\operatorname{perf}(\widetilde{C}; \mathcal{B}_0)$ , where  $\mathcal{B}_0$  is a certain sheaf of Azumaya algebras over  $\widetilde{C}$  of rank  $2^{(d/2)-1}$ . Therefore, the associated kernel leads to a Fourier-Mukai equivalence between  $\operatorname{perf}_{\operatorname{dg}}(C; \mathcal{C}_0)$  and  $\operatorname{perf}_{\operatorname{dg}}(\widetilde{C}; \mathcal{B}_0)$ . As proved in [16, Thm. 2.1], since  $\mathcal{B}_0$  is a sheaf of Azumaya algebras of rank  $2^{(d/2)-1}$ , the noncommutative mixed motive  $U(\operatorname{perf}_{\operatorname{dg}}(\widetilde{C}; \mathcal{B}_0))_{\mathbb{Z}[1/2]}$  is canonically isomorphic to  $U(\widetilde{C})_{\mathbb{Z}[1/2]}$ . Consequently, the  $\mathbb{Z}[1/2]$ -linearization of the right-hand side of (3.2) reduces to  $U(\widetilde{C})_{\mathbb{Z}[1/2]} \oplus U(C)_{\mathbb{Z}[1/2]}^{\oplus (d-2)}$ .

Item (ii). As explained in [10, Cor. 3.16] (see also [3, §1.7]), since by assumption k is algebraically closed and  $1/2 \in k$ , the category  $\operatorname{perf}(C; \mathcal{C}_0)$  is equivalent (via a Fourier-Mukai type functor) to  $\operatorname{perf}_{\mathrm{dg}}(\mathcal{X})$ . This implies that the dg category  $\operatorname{perf}_{\mathrm{dg}}(C; \mathcal{C}_0)$  is Morita equivalent to  $\operatorname{perf}_{\mathrm{dg}}(\mathcal{X})$ . Consequently, since C and D are k-smooth, we conclude from the above Proposition 2.1 that the right-hand side of (3.2) reduces to  $U(D) \oplus U(C)^{\oplus (d-1)}$ .

Item (i). As proved in [15, Thm. 2.8], there exists a  $\mathbb{Q}$ -linear, fully-faithful, symmetric monoidal functor  $\Phi$  making the following diagram commute

$$(3.3) \qquad \operatorname{Sm}(k) \xrightarrow{X \mapsto \operatorname{perf}_{\operatorname{dg}}(X)} \operatorname{dgcat}(k)$$

$$M(-)_{\mathbb{Q}} \downarrow \qquad \qquad \downarrow U(-)_{\mathbb{Q}}$$

$$\operatorname{DM}_{\operatorname{gm}}(k)_{\mathbb{Q}} \qquad \operatorname{NMot}(k)_{\mathbb{Q}}$$

$$\downarrow \operatorname{Hom}(-,U(k)_{\mathbb{Q}})$$

$$\operatorname{DM}_{\operatorname{gm}}(k)_{\mathbb{Q}}/_{-\otimes \mathbb{Q}(1)[2]} \xrightarrow{\Phi} \operatorname{NMot}(k)_{\mathbb{Q}},$$

where  $\underline{\text{Hom}}(-,-)$  stands for the internal Hom of the closed symmetric monoidal structure and  $\mathbb{Q}(1)[2]$  for the Tate object. Since the functor  $\pi$ , resp.  $\Phi$ , is additive, resp. fully-faithful and additive, we hence conclude from the combination of Proposition 3.1 with the above commutative diagram (3.3) that

(3.4) 
$$\pi(M(Q)_{\mathbb{Q}}) \simeq \pi(M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-2)}).$$

By definition of the orbit category, there exist then morphisms

$$\mathbf{f} = \{f_n\}_{n \in \mathbb{Z}} \in \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(M(Q)_{\mathbb{Q}}, (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-1)})(n)[2n])$$

$$\mathbf{g} = \{g_n\}_{n \in \mathbb{Z}} \in \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-1)}, M(Q)_{\mathbb{Q}}(n)[2n])$$

verifying the equalities  $g \circ f = id = f \circ g$ ; in order to simplify the exposition, we write -(n)[2n] instead of  $-\otimes \mathbb{Q}(1)[2]^{\otimes n}$ . Moreover, only finitely many of these morphisms are non-zero. Let us choose an integer  $N \gg 0$  such that  $f_n = g_n = 0$ 

for every |n| > N. The sets  $\{f_n \mid -N \leq n \leq N\}$  and  $\{g_{-n}(n) \mid -N \leq n \leq N\}$  give then rise to the following morphisms between mixed motives:

$$\alpha \colon M(Q)_{\mathbb{Q}} \longrightarrow \bigoplus_{n=-N}^{N} (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-1)})(n)[2n]$$

$$\beta \colon \bigoplus_{n=-N}^{N} (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-1)})(n)[2n] \longrightarrow M(Q)_{\mathbb{Q}}.$$

The composition  $\beta \circ \alpha$  agrees with the 0<sup>th</sup> component of  $g \circ f = id$ , *i.e.* with the identity of  $M(Q)_{\mathbb{Q}}$ . Consequently,  $M(Q)_{\mathbb{Q}}$  is a direct summand of the direct sum  $\bigoplus_{n=-N}^N (M(\widetilde{C})_{\mathbb{Q}} \oplus M(C)_{\mathbb{Q}}^{\oplus (d-1)})(n)[2n]$ . Using the fact that  $M(\widetilde{C})_{\mathbb{Q}}$  and  $M(C)_{\mathbb{Q}}$  are both Kimura-finite, that  $\wedge^2(\mathbb{Q}(1)[2]) = 0$ , and that Kimura-finiteness is stable under direct sums, direct summands, and tensor products, we hence conclude that the mixed motive  $M(Q)_{\mathbb{Q}}$  is also Kimura-finite. This finishes the proof of the first claim. Let us now prove the second claim.

Let X be a smooth k-scheme whose mixed motive  $M(X)_{\mathbb{Q}}$  is Kimura-finite. Note that since the functor  $\pi$  is symmetric monoidal and additive, the object  $\pi(M(X)_{\mathbb{Q}})$  of the orbit category  $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)[2]}$  is also Kimura-finite. As explained in [2, §3], we have the following equality

$$\mathrm{kim}(M(X)_{\mathbb{Q}}) = \chi(M(X)_{\mathbb{Q},+}) - \chi(M(X)_{\mathbb{Q},-})\,,$$

where  $\chi$  stands for the Euler characteristic computed in the rigid symmetric monoidal category  $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ . Therefore, since the functor  $\pi$  is moreover faithful, we observe that  $\mathrm{kim}(M(X)_{\mathbb{Q}}) = \mathrm{kim}(\pi(M(X)_{\mathbb{Q}}))$ . This leads to the following equalities:

$$(3.5) \qquad \qquad \lim(M(?)_{\mathbb{Q}}) = \lim(\pi(M(?)_{\mathbb{Q}})) \qquad ? \in \{Q, \widetilde{C}, C\}.$$

The Kimura-dimension of a direct sum of Kimura-finite objects is equal to the sum of the Kimura-dimension of each one of the objects. Hence, using the above computation (3.4) and the fact that the functor  $\pi$  is additive, we conclude that

$$(3.6) \qquad \lim(\pi(M(Q)_{\mathbb{Q}})) = \lim(\pi(M(\widetilde{C})_{\mathbb{Q}})) + (d-1)\lim(\pi(M(C)_{\mathbb{Q}})).$$

The proof of the second claim follows now from the above equalities (3.5)-(3.6).

**Item (ii).** The proof is similar to the one of item (i): simply replace C by D, (d-1) by (d-2), and use the fact that  $\lim(M(D)_{\mathbb{Q}}) = \#D$ .

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