Some stability problem to the Navier-Stokes equations in the periodic case

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Abstract. The Navier-Stokes motions in a box with periodic boundary conditions are considered. First the existence of global regular two-dimensional solutions is proved. The solutions are such that continuous with respect to time norms are controlled by the same constant for all $t \in \mathbb{R}_+$. Assuming that the initial velocity and the external force are sufficiently close to the initial velocity and the external force of the two-dimensional solutions we prove existence of global three-dimensional regular solutions which remain close to the two-dimensional solutions for all time. In this way we mean stability of two-dimensional solutions.

MSC 2010: 35Q30, 76D05, 76N10, 35B35, 76D03

Key words: incompressible Navier-Stokes equations, stability of twodimensional solutions, global regular solutions, periodic boundary conditions

1. Introduction

The aim of this paper is to prove stability of two-dimensional periodic solutions in the set of three-dimensional periodic solutions to the Navier-Stokes equation. We consider the three-dimensional fluid motions in the box $\Omega = [0, L]^3$, L > 0, described by

$$v_t + v \cdot \nabla v - \nu \Delta v + \nabla p = f \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$\text{div } v = 0 \qquad \qquad \text{in } \Omega \times \mathbb{R}_+,$$

$$v|_{t=0} = v(0) \qquad \qquad \text{in } \Omega,$$

where $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $x = (x_1, x_2, x_3)$ with $x_i \in (0, L)$, i = 1, 2, 3, is a given Cartesian system of coordinates, $p = p(x,t) \in \mathbb{R}$ is the pressure and $f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field. Finally, $\nu > 0$ is the constant viscosity coefficient and the dot denotes the scalar product in \mathbb{R}^3 .

Two-dimensional solutions to (1.1) are such that $v = v_s = (v_{s1}(x_1, x_2, t), v_{s2}(x_1, x_2, t), 0) \in \mathbb{R}^2$, $p = p_s(x_1, x_2, t) \in \mathbb{R}$, $f = f_s = (f_{s1}(x_1, x_2, t), f_{s2}(x_1, x_2, t), 0) \in \mathbb{R}^2$ and satisfy the problem

$$v_{st} + v_s \cdot \nabla v_s - \nu \Delta v_s + \nabla_{p_s} = f_s \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

$$\text{div } v_s = 0 \qquad \qquad \text{in} \quad \Omega \times \mathbb{R}_+,$$

$$v_s|_{t=0} = v_s(0) \qquad \qquad \text{in} \quad \Omega.$$

To show stability, we introduce the quantities

$$(1.3) u = v - v_s, \quad q = p - p_s$$

which are solutions to the problem

$$\begin{aligned} u_t + u \cdot \nabla u - \nu \Delta u + \nabla q &= -v_s \cdot \nabla u - u \cdot \nabla v_s + g & \text{in } \Omega \times \mathbb{R}_+, \\ (1.4) & \text{div } u &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u|_{t=0} &= u(0) & \text{in } \Omega, \end{aligned}$$

with $g = f - f_s$. Therefore, to show stability of solutions to (1.2) we need to prove smallness of quantities (1.3) in some norms for all $t \in \mathbb{R}_+$. For this purpose we apply the energy method. For this we need the Poincaré inequality. Since it does not hold for solutions to problems (1.2) and (1.4) we introduce the quantities

(1.5)
$$\bar{v}_s = v_s - \oint_{\Omega} v_s dx, \quad \bar{p}_s = p_s - \oint_{\Omega} p_s dx, \quad \bar{f}_s = f_s - \oint_{\Omega} f_s dx,$$

$$\bar{u} = u - \oint_{\Omega} u dx, \quad \bar{q} = q - \oint_{\Omega} q dx, \quad \bar{g} = g - \oint_{\Omega} g dx,$$

where

$$\int_{\Omega} \omega dx = \frac{1}{|\Omega|} \int_{\Omega} \omega dx \quad \text{and} \quad |\Omega| = L^3.$$

Applying the mean operator to $(1.2)_1$ and $(1.4)_1$ and using the periodic boundary conditions we have

$$\frac{d}{dt} \oint_{\Omega} v_s dx = \oint_{\Omega} f_s dx$$

and

(1.7)
$$\frac{d}{dt} \oint_{\Omega} u dx = \oint_{\Omega} g dx.$$

In view of (1.6), (1.7) and that any space derivative of the mean vanishes we see that for quantities (1.5) problems (1.2) and (1.4) take the forms

and

$$\begin{aligned}
\bar{u}_t + u \cdot \nabla \bar{u} - \nu \Delta \bar{u} + \nabla \bar{q} &= -v_s \cdot \nabla \bar{u} - u \cdot \nabla \bar{v}_s + \bar{g} & \text{in } \Omega \times \mathbb{R}_+, \\
(1.9) & \operatorname{div} \bar{u} &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\
\bar{u}|_{t=0} &= \bar{u}(0) & \text{in } \Omega.
\end{aligned}$$

Now, we formulate the main results of this paper (for any notation see Section 2). From Lemmas 3.1–3.3 we have

Theorem 1. (two-dimensional solutions) Assume that $\bar{f}_s \in L_2(kT, (k+1)T; L_{\sigma}(\Omega)), k \in \mathbb{N}_0, \sigma > 3$ and $\bar{v}_s(0) \in B^1_{\sigma,2}(\Omega)$. Then there exists a solution to problem (1.8) such that $\bar{v}_s \in W^{2,1}_{\sigma,2}(\Omega \times (kT, (k+1)T)), \nabla \bar{p}_s \in L_2(kT, (k+1)T; L_{\sigma}(\Omega)), k \in \mathbb{N}_0$ and the estimates hold

$$(1.10) \|\bar{v}_s\|_{W^{2,1}_{\sigma,2}(\Omega\times(0,T))} \le c(A^2 + \|\bar{f}_s\|_{L_2(0,T;L_{\sigma}(\Omega))} + \|v_s(0)\|_{B^1_{\sigma,2}(\Omega)})$$

and

(1.11)

$$\|\bar{v}_s\|_{W^{2,1}_{\sigma,2}(\Omega\times(kT,(k+1)T))} \le c \left(\frac{1}{\delta}A + A^2 + \|\bar{f}_s\|_{L_2((k-1)T,(k+1)T;L_{\sigma}(\Omega))}\right)$$

where $k \in \mathbb{N}$, $\delta \in (1/2, 1)$, $A = A_5 = \left(1 + \frac{c_{s_1}}{1 - \exp(-\nu c_{s_1} T)}\right) A_1^2 + \|v_{sx}(0)\|_{L_2}^2$. Moreover, $A_1^2 = \frac{1}{\nu c_{s_1}} \sum_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{f}_s(t)\|_{L_2}^2 dt$, c_{s_1} is the constant from the Poincaré inequality (2.3).

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Lemma 4.2 implies

Theorem 2. (stability) Let the assumptions of Theorem 1 hold. Let $\gamma \in (0, \gamma_*]$, where $\nu c_4 - \frac{c_5}{\nu^3} \gamma_*^2 \geq \frac{c_*}{2}$, $c_* < \nu c_4$ where $c_4(c_1)$ and c_1 is the constant from the Poincaré inequality (2.4). Assume that $\bar{g} \in C(\mathbb{R}_+; L_2)$, $\bar{u}(0) \in H^1$. Assume that (1.13)

 $\|\bar{u}(0)\|_{H^1}^2 \le \gamma,$

$$G^{2}(t) \equiv \frac{c_{5}}{\nu} \left[\|\bar{v}_{sx}\|_{L_{3}}^{2} \left| \int_{0}^{t} \int_{\Omega} g(x, t') dx dt' + \int_{\Omega} u(0) dx \right|^{2} + \|\bar{g}\|_{L_{2}}^{2} \right] \leq c_{*} \frac{\gamma}{4},$$

Let T > 0 be given and $k \in \mathbb{N}_0$. Assume that

$$\frac{c_5}{\nu} \int_{kT}^{(k+1)T} \|\bar{v}_{sx}\|_{L_3}^2 dt \le \frac{c_*}{4} T, \qquad \int_{kT}^{(k+1)T} G^2(t) dt \le \alpha \gamma,$$

$$\alpha \exp\left(\frac{c_*}{4} T\right) \exp\left(-\frac{c_*}{4} T\right) \le 1$$

where constant c_5 appears in (4.16). Then

(1.14)
$$\|\bar{u}(t)\|_{H^1}^2 \le \gamma \text{ for } t \in \mathbb{R}_+.$$

Finally by the regularity theory to the Navier-Stokes equations we have

Theorem 3. Let the assumptions of Theorems 1 and 2 hold. Then there exists a solution to problem (1.1) such that $v = v_s + u \in W_2^{2,1}(\Omega \times (kT, (k+1)T)), \nabla p = \nabla(p_s + q) \in L_2(kT, (k+1)T; L_2(\Omega)), k \in \mathbb{N}_0$, where v_s , p_s , u are determined by Theorems 1 and 2, respectively.

The first results connected with the stability of global regular solutions to the nonstationary Navier-Stokes equations were proved by Beirao da Veiga and Secchi [1], followed by Ponce, Racke, Sideris and Titi [2]. Paper [1] is concerned with the stability in L_p -norm of a strong three-dimensional solution of the Navier-Stokes system with zero external force in the whole space. In [2], assuming that the external force is zero and a three-dimensional initial function is close to a two-dimensional one in $H^1(\mathbb{R}^3)$, the authors showed the existence of a global strong solution in \mathbb{R}^3 which remains close to a two-dimensional strong solution for all times. In [3] Mucha obtained a similar result under weaker assumptions about the smallness of the initial velocity perturbation.

In the class of weak Leray-Hopf solutions the first stability result was obtained by Gallagher [4]. She proved the stability of two-dimensional

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solutions of the Navier-Stokes equations with periodic boundary conditions under three-dimensional perturbations both in L_2 and $H^{\frac{1}{2}}$ norms.

The stability of nontrivial periodic regular solutions to the Navier-Stokes equations was studied by Iftimie [5] and by Mucha [6]. The paper [6] is devoted to the case when the external force is a potential belonging to $L_{r,loc}(\mathbb{T}^3 \times [0,\infty))$ and when the initial data belongs to the space $W_r^{2-2/r}(\mathbb{T}^3) \cap L_2(\mathbb{T}^3)$, where $r \geq 2$ and \mathbb{T} is a torus. Under the assumption that there exists a global solution with data of regularity mentioned above and that small perturbations of data have the same regularity as above, the author proves that perturbations of the velocity and the gradient of the pressure remain small in the spaces $W^{2,1}_r(\mathbb{T}^3 \times (k,k+1))$ and $L_r(\mathbb{T}^3 \times (k, k+1)), k \in \mathbb{N}$, respectively. Paper [5] contains results concerning the stability of two-dimensional regular solutions to the Navier-Stokes system in a three-dimensional torus but here the initial data in the threedimensional problem belongs to an anisotropic space of functions having different regularity in the first two directions than in the third direction, and the external force vanishes. Moreover, Mucha [7] studies the stability of regular solutions to the nonstationary Navier-Stokes system in \mathbb{R}^3 assuming that they tend in $W_r^{2,1}$ spaces $(r \ge 2)$ to constant flows.

The papers of Auscher, Dubois and Tchamitchian [8] and of Gallagher, Iftimie and Planchon [9] concern the stability of global regular solutions to the Navier-Stokes equations in the whole space \mathbb{R}^3 with zero external force. These authors assume that the norms of the considered solutions decay as $t \to \infty$.

It is worth mentioning the paper of Zhou [10], who proved the asymptotic stability of weak solutions u with the property: $u \in L_2(0, \infty, BMO)$ to the Navier-Stokes equations in \mathbb{R}^n , $n \geq 3$, with force vanishing as $t \to \infty$.

An interesting result was obtained by Karch and Pilarczyk [11], who concentrate on the stability of Landau solutions to the Navier-Stokes system in \mathbb{R}^3 . Assuming that the external force is a singular distribution they prove the asymptotic stability of the solution under any L_2 -perturbation.

Paper [12] of Chemin and Gallagher is devoted to the stability of some unique global solution with large data in a very weak sense.

Finally, the stability of Leray-Hopf weak solutions has recently been examined by Bardos et al. [13], where equations with vanishing external force are considered. That paper concerns the following three cases: two-dimensional flows in infinite cylinders under three-dimensional perturbations which are periodic in the vertical direction; helical flows in circular cylinders under general three-dimensional perturbations; and axisymmetric flows under general three-dimensional perturbations. The theorem

concerning the first case extends a result obtained by Gallagher [4] for purely periodic boundary conditions.

Most of the papers discussed above concern to the case with zero external force [1, 2, 3, 5, 8, 9, 12, 13], or with force which decays as $t \to \infty$ ([10]). Exceptions are [6, 7, 11], where very special external forces, which are singular distributions in [11] or potentials in [6, 7], are considered. However, the case of potential forces is easily reduced to the case of zero external forces.

The aim of our paper is to prove the stability result for a large class of external forces f_s which do not produce solutions decaying as $t \to \infty$.

It is essential that our stability results are obtained together with the existence of a global strong three-dimensional solution close to a twodimensional one.

The paper is divided into two main parts. In the first we prove existence of global strong two-dimensional solutions not vanishing as $t \to \infty$ because the external force does not vanish either. To prove existence of such solutions we use the step by step method. For this purpose we have to show that the data in the time interval $[kT, (k+1)T], k \in \mathbb{N}$, do not increase with k. We do not need any restrictions on the time step T.

In the second part we prove existence of three-dimensional solutions that remain close to two-dimensional solutions. For this we need the initial velocity and the external force to be sufficiently close in appropriate norms to the initial velocity and the external force of the two-dimensional problems.

The proofs of this paper are based on the energy method, which strongly simplifies thanks to the periodic boundary conditions. The proofs of global existence which follow from the step by step technique are possible thanks to the natural decay property of the Navier-Stokes equations. This is mainly used in the first part of the paper (Section 3). To prove stability (Section 4) we use smallness of data $(v(0) - v_s(0)), (f - f_s)$ and a contradiction argument applied to the nonlinear ordinary differential inequality (4.20).

We restrict ourselves to proving estimates only, because existence follows easily by the Faedo-Galerkin method.

The paper is a serious generalization of [14] because proofs are simpler, there is imposed less restrictions on data and there is no relation between T, ν and f_s which in [14] implies some smallness for two-dimensional solutions.

The paper is organized as follows. In Section 2 we introduce notation and give some auxiliary results. Section 3 is devoted to the existence of a two-dimensional solution. It also contains some useful estimates of the

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solution. In Section 4 we prove the existence of a global strong solution to problem (1.1) close to the two-dimensional solution for all time.

2. Notation and auxiliary results

By $L_p(\Omega)$, $p \in [1, \infty]$, we denote the Lebesgue space of integrable functions and by $H^s(\Omega)$, $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Sobolev space of functions with the finite norm

$$||u||_{H^s} \equiv ||u||_{H^s(\Omega)} = \sum_{|\alpha| \le s} \left(\int_{\Omega} |D_x^{\alpha} u|^2 dx \right)^{1/2},$$

where $D_x^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_i \in \mathbb{N}_0$, i = 1, 2, 3.

Lemma 2.1. Assume that $\int_{\Omega} f_s(t)dx$, $\int_{\Omega} g(t)dx$ are locally integrable on \mathbb{R}_+ and $\int_{\Omega} v_s(0)dx$, $\int_{\Omega} u(0)dx$ are finite. Then for all $t \in \mathbb{R}_+$,

(2.1)
$$\oint_{\Omega} v_s(t)dx = \int_{0}^{t} \oint_{\Omega} f_s(t')dxdt' + \oint_{\Omega} v_s(0)dx,$$

(2.2)
$$\oint_{\Omega} u(t)dx = \int_{0}^{t} \oint_{\Omega} g(t')dxdt' + \oint_{\Omega} u(0)dx.$$

Proof. Applying the mean operator to (1.2) and (1.4), integrating by parts and using the periodic boundary conditions, we get (2.1) and (2.2) after integration with respect to time, respectively. This concludes the proof.

Lemma 2.2. By the Poincaré inequality holds

$$(2.3) c_{s1} \|\bar{v}_s\|_{H^1}^2 \le \|\nabla \bar{v}_s\|_{L_2}^2$$

and

$$(2.4) c_1 \|\bar{u}\|_{H^1}^2 \le \|\nabla \bar{u}\|_{L_2}^2,$$

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where c_{s1} , c_1 are positive constants.

Let as introduce the anisotropic Lebesgue and Sobolev spaces with the mixed norms, $L_{p_1,p_2}(\Omega \times (0,T))$ and $W_{p_1,p_2}^{2,1}(\Omega \times (0,T))$, $p_1,p_2 \in (1,\infty)$, with the following norms

$$\|u\|_{L_{p_2}(0,T;L_{p_1}(\Omega))} \equiv \|u\|_{L_{p_1,p_2}(\Omega\times(0,T))} = \left(\int\limits_0^T \left(\int\limits_\Omega |u|^{p_1}dx\right)^{p_2/p_1}dt\right)^{1/p_2},$$

$$||u||_{W_{p_1,p_2}^{2,1}(\Omega\times(0,T))} = ||D_x^2 u||_{L_{p_1,p_2}(\Omega\times(0,T))} + ||\partial_t u||_{L_{p_1,p_2}(\Omega\times(0,T))} + ||u||_{L_{p_1,p_2}(\Omega\times(0,T))}.$$

We introduce the Besov space $B_{p,q}^s(\Omega)$ (see [[15], Ch. 7, Sect. 7.32]) by

$$B_{p,q}^{s}(\Omega) = (L_{p}(\Omega), W_{p}^{m}(\Omega))_{s/m,q,J}.$$

In [16, Ch. 4, Sect. 18] the Besov spaces are introduced more explicitly. Let us consider the Stokes system

(2.5)
$$\begin{aligned} \omega_t - \nu \Delta \omega + \nabla q &= f & \text{in } \Omega \times (0, T), \\ \operatorname{div} \omega &= 0 & \text{in } \Omega \times (0, T), \\ \omega|_{t=0} &= \omega(0) & \text{in } \Omega. \end{aligned}$$

Lemma 2.3. Let $f \in L_{p_2}(0, T; L_{p_1}(\Omega)), p_1, p_2 \in (1, \infty),$ $\omega(0) \in B_{p_1, p_2}^{2-2/p_2}(\Omega).$ Then there exists a solution to problem (2.5) such that $\omega \in W_{p_1, p_2}^{2,1}(\Omega \times (0, T)), \nabla q \in L_{p_2}(0, T; L_{p_1}(\Omega))$ and

$$(2.6) \qquad \begin{aligned} \|\omega\|_{W^{2,1}_{p_1,p_2}(\Omega\times(0,T))} + \|\nabla q\|_{L_{p_2}(0,T;L_{p_1}(\Omega))} &\leq c(\|f\|_{L_{p_2}(0,T;L_{p_1}(\Omega))}) \\ + \|\omega(0)\|_{B^{2-2/p_2}_{p_1,p_2}(\Omega)}). \end{aligned}$$

Proof. To prove the lemma we use the idea of regularizer from [17, Sect. 3], where all estimates are made in the Hölder spaces. Performing the estimates in the Sobolev spaces with the mixed norm (see [18–21]) we prove the lemma.

From [22] we have

Lemma 2.4.

(i) Let $u \in W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1, \infty)$. Then $u(x,t_0) = u(x,t)|_{t=t_0}$ for $t_0 \in [0,T]$ belongs to $B^{s-2/p_0}_{p,p_0}(\Omega)$ and

$$||u(\cdot,t_0)||_{B_{p,p_0}^{s-2/p_0}(\Omega)} \le c||u||_{W_{p,p_0}^{s,s/2}(\Omega^T)},$$

where constant c does not depend on u.

(ii) For a given $\tilde{u} \in B_{p,p_0}^{s-2/p_0}(\Omega)$, $s \in \mathbb{R}_+$, $s > 2/p_0$, $p, p_0 \in (1, \infty)$, there exists a function $u \in W_{p,p_0}^{s,s/2}(\Omega^T)$ such that $u|_{t=t_0} = \tilde{u}$ for $t_0 \in [0,T]$ and

$$||u||_{W^{s,s/2}_{p,p_0}(\Omega^T)} \le c||\tilde{u}||_{B^{s-2/p_0}_{p,p_0}(\Omega)},$$

where constant c does not depend on u.

3. Two-dimensional solutions

First we have

Lemma 3.1. Let T > 0 be given. Assume that

1.
$$A_1^2 \equiv \frac{1}{\nu c_{s1}} \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} ||\bar{f}_s(t)||_{L_2}^2 dt < \infty,$$

2.
$$A_2^2 \equiv \frac{A_1^2}{1 - e^{-\nu c_{s1}T}} + \|\bar{v}_s(0)\|_{L_2}^2 < \infty,$$

where c_{s1} is introduced in (2.3). Then

and

(3.2)
$$\|\bar{v}_s(t)\|_{L_2}^2 + \nu c_{s1} \int_{kT}^t \|\bar{v}_s(t')\|_{H^1}^2 dt' \le A_1^2 + A_2^2 \equiv A_3^2,$$

where $t \in (kT, (k+1)T]$.

Proof. Multiplying $(1.8)_1$ by \bar{v}_s , integrating over Ω , using the periodic boundary conditions, the Poincaré inequality (2.3) and applying the Young inequality to the r.h.s. yield

(3.3)
$$\frac{d}{dt} \|\bar{v}_s\|_{L_2}^2 + \nu c_{s1} \|\bar{v}_s\|_{H^1}^2 \le \frac{1}{\nu c_{s1}} \|\bar{f}_s\|_{L_2}^2.$$

Continuing, we obtain

$$\frac{d}{dt}(\|\bar{v}_s\|_{L_2}^2 e^{\nu c_{s1}t}) \le \frac{1}{\nu c_{s1}} \|\bar{f}_s\|_{L_2}^2 e^{c_{s1}t}.$$

Integrating with respect to time from kT to $t \in (kT, (k+1)T]$ implies

$$\|\bar{v}_s(t)\|_{L_2}^2 \le \frac{1}{\nu c_{s1}} \int_{kT}^t \|\bar{f}_s(t')\|_{L_2}^2 dt' + e^{-\nu c_{s1}(t-kT)} \|\bar{v}_s(kT)\|_{L_2}^2.$$

Setting t = (k+1)T we get

$$\|\bar{v}_s((k+1)T)\|_{L_2}^2 \le \frac{1}{\nu c_{s1}} \int_{kT}^{(k+1)T} \|\bar{f}_s(t)\|_{L_2}^2 dt + e^{-\nu c_{s1}T} \|\bar{v}_s(kT)\|_{L_2}^2.$$

By iteration we have

$$\|\bar{v}_s(kT)\|_{L_2}^2 \le \frac{A_1^2}{1 - e^{-\nu c_{s1}T}} + e^{-\nu c_{s1}kT} \|v_s(0)\|_{L_2}^2 \le A_2^2.$$

Hence (3.1) is proved. Integrating (3.3) with respect to time from t = kT to $t \in (kT, (k+1)T]$ and employing (3.1), we obtain (3.2). This concludes the proof.

Next we obtain estimate for the second derivatives

Lemma 3.2. Let assuptions of Lemma 3.1 hold. Let $\bar{v}_s(0) \in H^1(\Omega)$. Then

(3.4)
$$\|\bar{v}_{sx}(kT)\|_{L_2}^2 \le \frac{c_{s1}A_1^2}{1 - e^{-\nu c_{s1}T}} + \|\bar{v}_{sx}(0)\|_{L_2}^2 \equiv A_4^2$$

and

(3.5)
$$\|\bar{v}_{sx}(t)\|_{L_2}^2 + \nu c_{s1} \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt' \le A_1^2 + A_4^2 \equiv A_5^2,$$

where $t \in (kT, (k+1)T]$.

Proof. Multiplying $(1.8)_1$ by $-\Delta \bar{v}_s$, integrating over Ω and using that \bar{v}_s is divergence free yields

(3.6)
$$-\int_{\Omega} \bar{v}_{st} \cdot \Delta \bar{v}_{s} dx + \nu \int_{\Omega} |\Delta \bar{v}_{s}|^{2} = \int_{\Omega} v_{s} \cdot \nabla \bar{v}_{s} \cdot \Delta \bar{v}_{s} dx - \int_{\Omega} \bar{f}_{s} \cdot \Delta \bar{v}_{s} dx.$$

Integrating by parts the first term on the l.h.s. equals

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}|\nabla\bar{v}_s|^2dx.$$

To examine the first term on the r.h.s. of (3.6) we use the formula

$$\Delta \bar{v}_s = \begin{pmatrix} -(\operatorname{rot}\bar{v}_s)_{,x_2} \\ (\operatorname{rot}\bar{v}_s)_{,x_1} \end{pmatrix}$$

where $\operatorname{rot} \bar{v}_s = \bar{v}_{s2,x_1} - \bar{v}_{s1,x_2}$. Then

$$\int_{\Omega} v_s \cdot \nabla \bar{v}_s \cdot \Delta \bar{v}_s dx = \int_{\Omega} (v_s \cdot \nabla \bar{v}_{s2} \operatorname{rot} \bar{v}_{s,x_1} - v_s \cdot \nabla \bar{v}_{s1} \operatorname{rot} \bar{v}_{s,x_2}) dx \equiv I.$$

Performing integration by parts yields

$$I = -\int_{\Omega} (v_s \cdot \nabla \bar{v}_{s2,x_1} - v_s \cdot \nabla \bar{v}_{s1,x_2}) \operatorname{rot} \bar{v}_s dx$$
$$-\int_{\Omega} (\bar{v}_{s,x_1} \cdot \nabla \bar{v}_{s2} - \bar{v}_{s,x_2} \cdot \nabla \bar{v}_{s1}) \operatorname{rot} \bar{v}_s dx \equiv I_1 + I_2,$$

where

$$I_1 = -\int\limits_{\Omega} v_s \cdot \nabla \operatorname{rot} \bar{v}_s \operatorname{rot} \bar{v}_s dx = 0$$

and

$$I_{2} = -\int_{\Omega} (\bar{v}_{s1,x_{1}} \bar{v}_{s2,x_{1}} + \bar{v}_{s2,x_{1}} \bar{v}_{s2,x_{2}} - \bar{v}_{s1,x_{2}} \bar{v}_{s1,x_{1}} - \bar{v}_{s2,x_{2}} \bar{v}_{s1,x_{2}}) \operatorname{rot} \bar{v}_{s} dx$$

$$= -\int_{\Omega} \operatorname{div} \bar{v}_{s} |\operatorname{rot} \bar{v}_{s}|^{2} dx = 0.$$

In view of the above considerations and the Hölder and Young inequalites applied to the last term on the r.h.s. of (3.6), we obtain from (3.6) the relation

(3.7)
$$\frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + \nu \|\Delta \bar{v}_s\|_{L_2}^2 \le \frac{1}{\nu} \|\bar{f}_s\|_{L_2}^2.$$

Applying the Poincaré inequality (see (2.3)) yields

$$\frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + \nu c_{s1} \|\bar{v}_{sx}\|_{L_2}^2 \le \frac{1}{\nu} \|\bar{f}_s\|_{L_2}^2$$

Hence

$$\frac{d}{dt}(\|\bar{v}_{sx}\|_{L_2}^2 e^{\nu c_{s1}t}) \le \frac{1}{\nu} \|\bar{f}_s\|_{L_2}^2 e^{\nu c_{s1}t}$$

Integrating with respect to time from kT to (k+1)T implies

$$\|\bar{v}_{sx}((k+1)T)\|_{L_2}^2 \le \frac{1}{\nu} \int_{kT}^{(k+1)T} \|\bar{f}_s\|_{L_2}^2 dt + e^{-\nu c_{s1}T} \|v_{sx}(kT)\|_{L_2}^2.$$

Then iteration implies

$$\|\bar{v}_{sx}(kT)\|_{L_2}^2 \le \frac{c_{s1}A_1^2}{1 - e^{-\nu c_{s1}T}} + e^{-\nu c_{s1}T} \|\bar{v}_{sx}(0)\|_{L_2}^2 \le A_4^2$$

Hence (3.4) is proved. Integrating (3.7) with respect to time from kT to $t \in (kT, (k+1)T]$ and using Lemma 2.3 yields (3.5). This concludes the proof.

To show stability of the two-dimensional solutions we need higher regularity of these solutions than the one proved in Lemma 3.2. We need such regularity that $v_s \in C(\mathbb{R}_+; W^1_{\sigma}(\Omega)), \sigma > 3$. Moreover, we want to show that

(3.8)
$$||v_s(t)||_{W^1_{\sigma}(\Omega)} \le c,$$

where c is a constant independent of time.

Finally, we do not want to apply the energy type method for higher derivatives (see [14]), because it implied stronger restrictions on the external force. Instead, we are going to apply the increasing regularity technique. This is possible because in view of Lemma 3.2 the term $\bar{v}_s \cdot \nabla \bar{v}_s \in L_2(\mathbb{R}_+; L_{\sigma}(\Omega))$ with $\sigma \in (1, \infty)$. As it will be seen in Section 4 we will need to show only that $\bar{v}_s \in C(\mathbb{R}_+; W^1_{\sigma}(\Omega))$ with $\sigma > 3$ (see (4.17)).

Lemma 3.3. Assume that $v_s(0) \in B^1_{\sigma,2}(\Omega)$, $\bar{f}_s \in L_2(kT,(k+1)T;L_{\sigma}(\Omega))$, $k \in \mathbb{N}_0$, $\sigma > 3$. Then $v_s \in C(\mathbb{R}_+;W^1_{\sigma}(\Omega))$, $\sigma > 3$ and (3.8) holds.

Proof. Since $v_s \cdot \nabla \bar{v}_s \in L_2(kT, (k+1)T; L_{\sigma}(\Omega)), \quad \sigma \in (1, \infty),$ $\bar{f}_s \in L_2(kT, (k+1)T; L_{\sigma}(\Omega)), \quad \bar{v}_s(0) \in B^1_{\sigma,2}(\Omega)$ the theory from [17–21], and Lemma 3.2 imply the existence of solutions to (1.8) such that $v_s \in W^{2,1}_{\sigma,2}(\Omega \times \mathbb{R}_+)$ and the estimate holds

$$(3.9) \|\bar{v}_s\|_{W^{2,1}_{\sigma^2}(\Omega\times(0,T))} \le c(A_5^2 + \|\bar{f}_s\|_{L_2(0,T;L_{\sigma}(\Omega))} + \|v_s(0)\|_{B^1_{\sigma,2}(\Omega)}).$$

However, we do not know how the constant c depends on time. Therefore, we are not able to claim that (3.8) holds. Hence, we have to prove (3.8) step by step in time. Let us consider interval (kT, (k+1)T). Let $\zeta = \zeta(t)$ be a smooth cut-off function such that $\zeta(t) = 0$ for $t \in [kT, kT + \delta/2]$ and $\zeta(t) = 1$ for $t \geq kT + \delta$. Introducing the new functions

$$\tilde{v}_s = \bar{v}_s \zeta, \quad \tilde{p}_s = \bar{p}_s \zeta, \quad \tilde{f}_s = \bar{f}_s \zeta, \quad \dot{\zeta} = \zeta_{,t}$$

we see that $(\tilde{v}_s, \tilde{p}_s)$ is a solution to the problem (3.10)

$$\tilde{v}_{st} - \nu \Delta \tilde{v}_s + \nabla \tilde{p}_s = \bar{v}_s \dot{\zeta} - v_s \cdot \nabla \tilde{v}_s + \tilde{f}_s$$
 in $\Omega \times (kT, (k+1)T)$,
 $\text{div } \tilde{v}_s = 0$ in $\Omega \times (kT, (k+1)T)$,
 $\tilde{v}_s|_{t=kT} = 0$.

In view of Lemma 3.2 and [17-21] we have the existence of solutions to (3.10) such that

$$\tilde{v}_s \in W_{\sigma,2}^{2,1}(kT+\delta,(k+1)T;\Omega), \quad \nabla \tilde{p}_s \in L_{\sigma,2}(kT+\delta,(k+1)T;\Omega)$$

and the estimate holds

$$(3.11) \|\tilde{v}_s\|_{W^{2,1}_{\sigma,2}(kT+\delta,(k+1)T;\Omega)} \le c \left(\frac{1}{\delta}A_5 + A_5^2 + \|\tilde{f}_s\|_{L_2(kT+\delta/2,(k+1)T;\Omega)}\right),$$

where c might depend on T but it does not depend on k. Hence by imbedding for $\sigma > 3$ estimate (3.11) implies (3.8).

To get (3.8) we need only estimate for the interval $(kT, kT + \delta)$, $k \in \mathbb{N}$, because for k = 0 we have (3.9). From (3.11) for k replaced by k - 1 we obtain the estimate for

$$\|\tilde{v}_s\|_{W^{2,1}_{\sigma,2}((k-1)T+\delta,kT;\Omega)} \le c \left(\frac{1}{\delta} A_5 + A_5^2 + \|\tilde{f}_s\|_{L_2((k-1)^+\frac{\delta}{2},kT;\Omega)}\right),$$

so by the trace theorem (see Lemma 2.3) we derive

(3.12)
$$\|\tilde{v}_s(kT)\|_{B^1_{\sigma,2}(\Omega)} \le c \|\bar{v}_s\|_{W^{2,1}_{\sigma,2}((k-1)T+\delta,kT,\Omega)}.$$

Hence, repeating the considerations leading to (3.9) for time interval $(kT, kT + \delta)$ we obtain that $\tilde{v}_s \in W^{2,1}_{\sigma,2}(\Omega \times (kT, kT + \delta))$ and the estimate

(3.13)
$$\|\tilde{v}_s\|_{W^{2,1}_{\sigma,2}(\Omega\times(kT,kT+\delta))} \le c(A_5^2 + \|\bar{f}_s\|_{L_{\sigma,2}(\Omega\times(kT,kT+\delta))} + \|\bar{v}_s(kT)\|_{B^1_{\sigma,2}(\Omega)}).$$

Hence (3.8) holds for all $t \in \mathbb{R}_+$ and Lemma 3.3 is proved.

4. Stability

In this Section we examine problem (1.4). First we derive a global estimate for L_2 norm of u. We show how appears restriction from Assumption 2 of Lemma 4.1 (it is much more restrictive in [14]). Fortunately, we do not need Lemma 4.1 to prove stability. Hence we have

Lemma 4.1. Let the assumptions of Lemmas 3.1, 3.2 hold. Let

1.
$$B_1^2 = \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \left(\frac{\nu c_1}{2c_3}\right| \int_0^t \int_{\Omega} g(t') dx dt' + \int_{\Omega} u(0) dx \Big|^2 + \frac{2c_3}{\nu c_1} \|\bar{g}(t)\|_{L_{6/5}}^2 dt$$
, where c_1 follows from Poincaré inequality (2.4) and c_3 from imbedding (4.3).
2. $\frac{-\nu c_1}{2} T + \frac{4c_3}{\nu c_1} A_3^2 \leq 0$.

2.
$$\frac{-\nu c_1}{2}T + \frac{4c_3}{\nu c_1}A_3^2 \leq 0$$
.

3.
$$B_2^2 = \exp\left(\frac{4c_3}{\nu c_1}A_5^2\right)B_1^2$$
.

(4.1)
$$\|\bar{u}(kT)\|_{L_2}^2 \le \frac{B_2^2}{1 - \exp(-\nu c_1 T/2)} + \|\bar{u}(0)\|_{L_2}^2 = B_3^2,$$

$$\|\bar{u}(t)\|_{L_2}^2 \le B_2^2 + B_3^2 \equiv B_4^2,$$

for $t \in [kT, (k+1)T]$ and any $k \in \mathbb{N}_0$.

Proof. Multiplying $(1.9)_1$ by \bar{u} and integrating over Ω gives

$$(4.2) \qquad \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L_{2}}^{2} + \nu c_{1} \|\bar{u}_{x}\|_{L_{2}}^{2} \leq \left| \int_{\Omega} u \cdot \nabla \bar{v}_{s} \cdot \bar{u} dx \right| + \left| \int_{\Omega} \bar{g} \cdot \bar{u} dx \right|$$

$$\leq \left| \int_{\Omega} \bar{u} \cdot \nabla \bar{v}_{s} \cdot \bar{u} dx \right| + \left| \int_{\Omega} u dx \cdot \int_{\Omega} \nabla \bar{v}_{s} \cdot \bar{u} dx \right| + \left| \int_{\Omega} \bar{g} \cdot \bar{u} dx \right|.$$

Employing the estimates

$$\begin{split} &\left| \int\limits_{\Omega} \bar{u} \cdot \nabla \bar{v}_s \cdot \bar{u} dx \right| \leq \frac{\varepsilon_1}{2} \|\bar{u}\|_{L_6}^2 + \frac{1}{2\varepsilon_1} \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}\|_{L_2}^2, \\ &\left| \int\limits_{\Omega} u dx \cdot \int\limits_{\Omega} \nabla \bar{v}_s \cdot \bar{u} dx \right| = \frac{1}{2\varepsilon_2} \|\nabla \bar{v}_s\|_{L_2}^2 \|\bar{u}\|_{L_2}^2 + \frac{\varepsilon_2}{2} \left| \int\limits_{\Omega} u dx \right|^2, \\ &\left| \int\limits_{\omega} \bar{g} \cdot \bar{u} dx \right| \leq \frac{\varepsilon_3}{2} \|\bar{u}\|_{L_6}^2 + \frac{1}{2\varepsilon_3} \|\bar{g}\|_{L_{6/5}}^2, \end{split}$$

(2.4), the imbedding

$$\|\bar{u}\|_{L_6}^2 \le c_3 \|\bar{u}\|_{H^1}^2$$

and that $\frac{\varepsilon_i}{2}$ $c_3 \leq \frac{c_1}{4}$, i = 1, 3, we obtain from (4.2) the inequality

$$(4.4) \frac{\frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L_{2}}^{2} + \frac{\nu c_{1}}{2} \|\bar{u}\|_{H^{1}}^{2} \leq \frac{c_{3}}{\nu c_{1}} (\|\bar{v}_{sx}\|_{L_{3}}^{2} + \|\bar{v}_{sx}\|_{L_{2}}^{2}) \|\bar{u}\|_{L_{2}}^{2}}{+ \frac{\nu c_{1}}{4c_{3}} \left| \int u dx \right|^{2} + \frac{c_{3}}{\nu c_{1}} \|\bar{g}\|_{L_{6/5}}^{2},}$$

where $\varepsilon_2 = \frac{\nu c_1}{2c_3}$ is set. Employing (2.2) in (4.4) yields

$$\frac{d}{dt} \|\bar{u}\|_{L_{2}}^{2} + \nu c_{1} \|\bar{u}\|_{H^{1}}^{2} \leq \frac{2c_{3}}{\nu c_{1}} (\|\bar{v}_{sx}\|_{L_{3}}^{2} + \|\bar{v}_{sx}\|_{L_{2}}^{2}) \|\bar{u}\|_{L_{2}}^{2}
+ \frac{\nu c_{1}}{2c_{3}} \left| \int_{0}^{t} \int_{\Omega} g(t') dx dt' + \int_{\Omega} u(0) dx \right|^{2} + \frac{2c_{3}}{\nu c_{1}} \|\bar{g}\|_{L_{6/5}}^{2}.$$

Considering (4.5) for $t \in (kT, (k+1)T)$ we have

$$\frac{d}{dt} \left[\|\bar{u}(t)\|_{L_{2}}^{2} \exp\left(\nu c_{1} t - \frac{2c_{3}}{\nu c_{1}} \int_{kT}^{t} (\|\bar{v}_{sx}(t')\|_{L_{3}}^{2} + \|\bar{v}_{sx}(t')\|_{L_{2}}^{2}) dt' \right) \right]
(4.6) \leq \left(\frac{\nu c_{1}}{2c_{3}} \Big| \int_{0}^{t} \int_{\Omega} g(t') dx dt' + \int_{\Omega} u(0) dx \Big|^{2} + \frac{2c_{3}}{\nu c_{1}} \|\bar{g}(t)\|_{L_{6/5}}^{2} \right) \cdot \exp\left(\nu c_{1} t - \frac{2c_{3}}{\nu c_{1}} \int_{kT}^{t} (\|\bar{v}_{sx}(t')\|_{L_{3}}^{2} + \|\bar{v}_{sx}(t')\|_{L_{2}}^{2}) dt' \right).$$

Integrating (4.6) with respect to time from t = kT to $t \in (kT, (k+1)T]$ implies

$$\|\bar{u}(t)\|_{L_{2}}^{2} \leq \exp\left[\frac{2c_{3}}{\nu c_{1}} \int_{kT}^{t} (\|v_{sx}(t')\|_{L_{3}}^{2} + \|v_{sx}(t')\|_{L_{2}}^{2}) dt'\right].$$

$$(4.7) \qquad \int_{kT}^{t} \left(\frac{\nu c_{1}}{2c_{3}} \Big| \int_{0}^{t'} \int_{\Omega} g(t'') dx dt'' + \int_{\Omega} u(0) dx \Big|^{2} + \frac{2c_{3}}{\nu c_{1}} \|\bar{g}(t')\|_{L_{6/5}}^{2}\right) dt'$$

$$+ \|\bar{u}(kT)\|_{L_{2}}^{2} \exp\left[-\nu c_{1}(t-kT) + \frac{2c_{3}}{\nu c_{1}} \int_{kT}^{t} (\|\bar{v}_{sx}(t')\|_{L_{3}}^{2}\right]$$

$$+ \|\bar{v}_{sx}(t')\|_{L_{2}}^{2} dt'\right].$$

Setting t = (k+1)T and using (3.5), inequality (4.7) yields

$$\|\bar{u}((k+1)T)\|_{L_2}^2$$

$$(4.8) \qquad \leq \exp\left(\frac{4c_3}{\nu c_1}A_3^2\right) \int_{kT}^{(k+1)T} \left[\frac{\nu c_1}{2c_3}\right] \int_{0}^{t} \int_{\Omega} g(t')dxdt' + \int_{\Omega} u(0)dx \Big|^2 + \frac{2c_3}{c_1} \|\bar{g}(t)\|_{L_{6/5}}^2 dt + \|\bar{u}(kT)\|_{L_2}^2 \exp\left(-\nu c_1 T + \frac{4c_3}{\nu c_1} A_3^2\right).$$

In view of assumptions 1–3 of the lemma we have

(4.9)
$$\|\bar{u}((k+1)T)\|_{L_2}^2 \le B_2^2 + \exp\left(\frac{-\nu c_1}{2}T\right) \|\bar{u}(kT)\|_{L_2}^2.$$

Iteration implies

$$(4.10) \|\bar{u}(kT)\|_{L_2}^2 \le \frac{B_2^2}{1 - \exp(-\nu c_1 T/2)} + \exp\left(\frac{-\nu c_1}{2}kT\right) \|\bar{u}(0)\|_{L_2}^2.$$

Hence $(4.1)_1$ is proved. Employing assumptions of the lemma and $(4.1)_1$ in (4.7) gives $(4.1)_2$. This concludes the proof.

Remark 4.2. Assumption 2 of Lemma 4.1 has the explicit form

(4.11)
$$\frac{2 - \exp(-\nu c_{s1}T)}{c_{s1}\nu(1 - \exp(-\nu c_{s1}T))} \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{f}_s(t)\|_{L_2}^2 dt + \|\bar{v}_{sx}(0)\|_{L_2}^2 \le \frac{\nu^2 c_1^2}{8c_3} T.$$

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Assuming that $\|\bar{v}_{sx}(0)\|_{L_2}$ is given we see that (4.11)holds for $T > \frac{8c_3}{\nu^2c_1^2}\|\bar{v}_{sx}(0)\|_{L_2}^2$. For such large T we have a strong restriction on $\sup_{k\in\mathbb{N}_0}\int_{kT}^{(k+1)T}\|\bar{f}_s(t)\|_{L_2}^2dt$. Physically, it means that the energy introduced to the considered region must not to be too large comparing with the dissipation.

Finally, we show that 3d solutions to (1.1) remain close to 2d solutions to (1.2) for all time if their initial data and the external forces are sufficiently close. In this proof we omit the heavy restriction (4.11).

Lemma 4.3. Let $\bar{v}_s \in C(\mathbb{R}_+; W_3^1)$, $\bar{g} \in C(\mathbb{R}_+; L_2)$, $\bar{u}(0) \in H^1$. Let $\gamma \in (0, \gamma_*]$, where $\nu c_4 - \frac{c_5}{\nu^3} \gamma_*^2 \geq \frac{c_*}{2}$, $c_* < \nu c_4$ and c_4 , c_5 are introduced in (4.16). Assume that

$$\|\bar{u}(0)\|_{H^{1}}^{2} \leq \gamma$$

$$(4.12) \qquad G^{2}(t) = \frac{c_{5}}{\nu} \left[\|\bar{v}_{sx}\|_{L_{3}}^{2} \middle| \int_{0}^{t} \oint_{\Omega} g(x, t') dx dt' + \oint_{\Omega} u(0) dx \middle|^{2} \right]$$

$$+ \|\bar{g}\|_{L_{2}}^{2} \leq c_{*} \frac{\gamma}{4}.$$

Let T > 0 be given and $k \in \mathbb{N}_0$. assume that

$$\begin{split} &\frac{c_5}{\nu} \int\limits_{kT}^{(k+1)T} \|\bar{v}_{sx}\|_{L_3}^2 dt \leq \frac{c_*}{4}T, \qquad \int\limits_{kT}^{(k+1)T} G^2(t) dt \leq \alpha \gamma, \\ &\alpha \exp\left(\frac{c_*}{4}T\right) + \exp\left(-\frac{c_*}{4}T\right) \leq 1. \end{split}$$

Then

(4.13)
$$\|\bar{u}(t)\|_{H^1}^2 \le \gamma \text{ for } t \in \mathbb{R}_+.$$

Proof. Differentiating $(1.9)_1$ with respect to x, multiplying the result by \bar{u}_x , integrating over Ω and employing the periodic boundary conditions yield

$$(4.14) \frac{\frac{1}{2} \frac{d}{dt} \|\bar{u}_x\|_{L_2}^2 + \nu \|\bar{u}_{xx}\|_{L_2}^2 \leq \|\bar{u}_x\|_{L_3}^3 + \left| \int_{\Omega} \bar{v}_{sx} \cdot \nabla u \cdot \bar{u}_x dx \right| + \left| \int_{\Omega} \bar{u}_x \cdot \nabla \bar{v}_s \cdot \bar{u}_x dx \right| + \left| \int_{\Omega} u \cdot \nabla \bar{v}_s \cdot \bar{u}_{xx} dx \right| + \left| \int_{\Omega} \bar{g} \cdot \bar{u}_{xx} dx \right|.$$

Adding (4.2) and (4.14), applying the Hölder, the Young and the Poincaré inequalities, we derive

$$\frac{d}{dt} \|\bar{u}\|_{H^{1}}^{2} + \nu c \|\bar{u}\|_{H^{2}}^{2} \leq c(\|\bar{u}_{x}\|_{L_{3}}^{3} + \frac{1}{\nu} \|\bar{v}_{sx}\|_{L_{3}}^{2} \|\bar{u}_{x}\|_{L_{2}}^{2} + \frac{1}{\nu} \|u\|_{L_{6}}^{2} \|\bar{v}_{sx}\|_{L_{3}}^{2} + \frac{1}{\nu} \|\bar{g}\|_{L_{2}}^{2}).$$

Using $||u||_{L_6}^2 \leq c(||\bar{u}||_{L_6}^2 + |\int_{\Omega} u dx|^2)$ and $||\bar{u}||_{L_6} \leq c||\bar{u}||_{H^1} \leq c||\bar{u}_x||_{L_2}$, which holds in view of the Poincaré inequality, we get

$$\frac{d}{dt} \|\bar{u}\|_{H^{1}}^{2} + \nu c \|\bar{u}\|_{H^{2}}^{2}$$

$$\leq c \left[\|\bar{u}_{x}\|_{L_{3}}^{3} + \frac{1}{\nu} \|\bar{v}_{sx}\|_{L_{3}}^{2} \left(\|\bar{u}_{x}\|_{L_{2}}^{2} + \left| \oint_{\Omega} u dx \right|^{2} \right) + \frac{1}{\nu} \|\bar{g}\|_{L_{2}}^{2} \right].$$

In view of (2.2) and the interpolation inequality (see [16, Ch. 3, Sect. 15])

$$\|\bar{u}_x\|_{L_3} \le c \|\bar{u}_{xx}\|_{L_2}^{1/2} \|\bar{u}_x\|_{L_2}^{1/2}$$

(which holds without the lower order term because $\int_{\Omega} \bar{u}_x dx = 0$), we obtain from (4.15) the inequality

$$\frac{d}{dt} \|\bar{u}\|_{H^{1}}^{2} + \nu c_{4} \|\bar{u}\|_{H^{2}}^{2} \leq \frac{c_{5}}{\nu^{3}} \|\bar{u}_{x}\|_{L_{2}}^{6} + \frac{c_{5}}{\nu} \|\bar{v}_{sx}\|_{L_{3}}^{2} \|\bar{u}_{x}\|_{L_{2}}^{2}
+ \frac{c_{5}}{\nu} \|\bar{v}_{sx}\|_{L_{3}}^{2} \left| \int_{0}^{t} \int_{\Omega} g(x, t') dx dt' + \int_{\Omega} u(0) dx \right|^{2} + \frac{c_{5}}{\nu} \|\bar{g}\|_{L_{2}}^{2}.$$

To prove the lemma we need to know that the r.h.s. of (4.16) is bounded. We consider inequality (4.16) in the time interval $(kT,(k+1)T), k \in \mathbb{N}_0$. Assume that we have proved that $u(kT) \in H^1(\Omega)$ and $\|u(kT)\|_{H^1}^2 \leq \gamma$, where γ is sufficiently small. Using that $g \in L_2(\Omega \times (kT,(k+1)T))$ is sufficiently small we have existence of solutions to problem (1.9) in $W_2^{2,1}(\Omega \times (kT,(k+1)T))$ because the other terms on the r.h.s. of (1.9) also belong to $L_2(\Omega \times (kT,(k+1)T))$ in view of imbeddings and assumption that $v_s \in W_2^{2,1}(\Omega \times (kT,(k+1)T))$. The last assertion holds in view of the assumptions of Lemma 3.2 and the restriction that v_s is a two-dimensional solution to the Navier-Stokes equations. However, to have the r.h.s. of (4.16) bounded we need that $v_s \in L_\infty(kT,(k+1)T;W_{3+}^1(\Omega))$, where $3^+ > 3$ but close to 3. This follows from Lemma 3.3, where it is proved that $v_s \in W_{\sigma,2}^{2,1}(\Omega \times (kT,(k+1)T))$ for any σ if data are sufficiently smooth.

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In view of the above remarks we can introduce the quantities

(4.17)
$$G^{2}(t) = \frac{c_{5}}{\nu} \left(\|\bar{v}_{sx}\|_{L_{3}}^{2} \Big| \int_{0}^{t} \int_{\Omega} g(x, t') dx dt' + \int_{\Omega} u(0) dx \Big|^{2} + \frac{1}{2} \|\bar{g}\|_{L_{2}}^{2} \right), \quad A^{2}(t) = \frac{c_{5}}{\nu} \|\bar{v}_{sx}\|_{L_{3}}^{2},$$

$$X(t) = \|\bar{u}(t)\|_{H^{1}}, \quad Y(t) = \|\bar{u}(t)\|_{H^{2}}.$$

Then (4.17) takes the form

$$\frac{d}{dt}X^2 + \nu c_4 Y^2 \le \frac{c_5}{\nu^3} X^4 X^2 + A^2 X^2 + G^2.$$

Since $X \leq Y$ we have

(4.18)
$$\frac{d}{dt}X^2 \le -X^2 \left(\nu c_4 - \frac{c_5}{\nu^3}X^4\right) + A^2 X^2 + G^2$$

Let $\gamma \in (0, \gamma_*]$, where γ_* is so small that

(4.19)
$$\nu c_4 - \frac{c_5}{\nu^3} \gamma_*^2 \ge c_*/2, \quad c_* < \nu c_4.$$

Since the coefficients of equation (4.18) depend on the two-dimensional solution determined step by step in time we consider (4.18) in the interval $[kT, (k+1)T], k \in \mathbb{N}_0$, with the assumptions

$$X^2(kT) \le \gamma$$
, $G^2(t) \le c_*\gamma/4$ for all $t \in [kT, (k+1)T]$.

Let us introduce the quantity

$$Z^{2}(t) = \exp\left(-\int_{kT}^{t} A^{2}(t')dt'\right)X^{2}(t), \quad t \in [kT, (k+1)T].$$

Then (4.18) takes the form

(4.20)
$$\frac{d}{dt}Z^2 \le -\left(\nu c_4 - \frac{c_5}{\nu^3}X^4\right)Z^2 + \bar{G}^2,$$

where $\bar{G}^2 = G^2 \exp \left(-\int_{kT}^t A^2(t')dt'\right)$. Suppose that

$$t_* = \inf\{t \in (kT, (k+1)T] : X^2(t) > \gamma\}$$

$$= \inf\left\{t \in (kT, (k+1)T] : Z^2(t) > \gamma \exp\left(-\int_{kT}^t A^2(t')dt'\right)\right\} > kT.$$

By (4.19) for $t \in (0, t_*]$ inequality (4.20) takes the form

(4.21)
$$\frac{d}{dt}Z^2 \le -\frac{c_*}{2}Z^2 + \bar{G}^2(t).$$

Clearly, we have

$$Z^{2}(t_{*}) = \gamma \exp\left(-\int_{kT}^{t_{*}} A^{2}(t')dt'\right) \quad \text{and}$$

$$Z^{2}(t) > \gamma \exp\left(-\int_{kT}^{t_{*}} A^{2}(t')dt'\right)$$
for $t > t_{*}$.

Then (4.21) yields

$$\frac{d}{dt}Z^2|_{t=t_*} \le c_* \left(-\frac{\gamma}{2} + \frac{\gamma}{4}\right) \exp\left(-\int_{kT}^{t_*} A^2(t')dt'\right) < 0$$

contradicting with (4.22). Therefore

(4.23)
$$Z^{2}(t) < \gamma \exp\left(-\int_{kT}^{t_{*}} A^{2}(t')dt'\right) \text{ for } t > t_{*}.$$

Then definition of $Z^2(t)$ implies

$$X^{2}(t) \leq \gamma \exp\left(\int_{t_{*}}^{t} A^{2}(t')dt'\right) \text{ for } t > t_{*}.$$

For sufficiently small γ inequality (4.18) takes the form

(4.24)
$$\frac{d}{dt}X^2 + \frac{c_*}{2}X^2 \le A^2X^2 + G^2.$$

Integrating (4.24) with respect to time from t = kT to t = (k+1)T gives

(4.25)
$$X^{2}((k+1)T) \leq \exp\left(\int_{kT}^{(k+1)T} A^{2}(f)dt\right) \int_{kT}^{(k+1)T} G^{2}(t)dt + \exp\left(-\frac{c_{*}}{2}T + \int_{kT}^{(k+1)T} A^{2}(t)dt\right) X^{2}(kT).$$

In view of the assumptions

(4.26)
$$\frac{c_*}{4}T \ge \int_{kT}^{(k+1)T} A^2(t)dt, \quad \int_{kT}^{(k+1)T} G^2(t)dt \le \alpha \gamma,$$

where α is so small and T so targe that

(4.27)
$$\alpha \exp\left(\int_{kT}^{(k+1)T} A^2(t)dt\right) + \exp\left(-\frac{c_*}{4}T\right) \le 1,$$

we have that $X^2((k+1)T) < \gamma$. Then by the induction we prove the lemma.

Acknowledgements

The author thanks to professor Y.Shibata for very important comments concerning the proof of Lemma 4.3.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement n° 319012 and from the Funds for International Co-operation under Polish Ministry of Science and Higher Education grant agreement n° 2853/7.PR/2013/2.

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