SURFACES EXPANDING BY NON-CONCAVE CURVATURE FUNCTIONS

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ABSTRACT. In this paper, we first investigate the flow of convex surfaces in the space form $\mathbb{R}^3(\kappa)$ ($\kappa = 0, 1, -1$) expanding by $F^{-\alpha}$, where F is a smooth, symmetric, increasing and homogeneous of degree one function of the principal curvatures of the surfaces and the power $\alpha \in (0, 1]$ for $\kappa = 0, -1$ and $\alpha = 1$ for $\kappa = 1$. By deriving that the pinching ratio of the flow surface M_t is no greater than that of the initial surface M_0, we prove the long time existence and the convergence of the flow. No concavity assumption of F is required. We also show that for the flow in $\mathbb{H}^3$ with $\alpha \in (0, 1)$, the limit shape may not be necessarily round after rescaling.

1. INTRODUCTION

Let $\mathbb{R}^3(\kappa)$ ($\kappa = 0, 1, -1$) be a real space form, i.e., when $\kappa = 0$, $\mathbb{R}^3(0) = \mathbb{R}^3$, when $\kappa = 1$, $\mathbb{R}^3(1) = \mathbb{S}^3$, and when $\kappa = -1$, $\mathbb{R}^3(-1) = \mathbb{H}^3$. Given a compact smooth immersion $X_0 : M \to \mathbb{R}^3(\kappa)$, we consider the smooth family of immersions $X(x, t) : M \times [0, T) \to \mathbb{R}^3(\kappa)$ solving the evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} X(x, t) = F^{-\alpha}(x, t)\nu(x, t), \\ X(\cdot, 0) = X_0(\cdot), \end{cases} \tag{1.1}$$

where $F(x, t) = F(\lambda_1(x, t), \lambda_2(x, t))$ is a smooth symmetric function of the principal curvatures of the surfaces and ν is the outer unit normal of $M_t = X_t(M)$. Throughout this paper, we assume that F satisfies the following conditions:

Assumption 1.1. *Let $\Gamma_+ = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_i > 0, i = 1, 2\}$ be the positive quadrant in $\mathbb{R}^2$. Assume that*

(i) *F is smooth, symmetric and positive on Γ_+.*
(ii) *F is strictly increasing in each argument, i.e., $\partial F/\partial \lambda_i > 0$ on Γ_+, $\forall\ i = 1, 2$.*
(iii) *F is homogeneous of degree 1, i.e., $F(k\lambda) = kF(\lambda)$ for any $k > 0$ and $\lambda = (\lambda_1, \lambda_2) \in \Gamma_+$.*
(iv) *F is normalized such that $F(1, 1) = 2$.*

We refer the reader to [6] for examples of F satisfying Assumption 1.1.

For strictly convex surfaces in $\mathbb{R}^3(\kappa)$, (1.1) is a parabolic equation and has a smooth solution on a maximal time interval $[0, T)$ ($T \leq \infty$) for any F satisfying Assumption 1.1 (cf. [16]). In this paper, we will first study the long time behavior of flow (1.1).

Theorem 1.2. *For any smooth, closed strictly convex surface M_0 in $\mathbb{R}^3$, there exists a smooth solution of flow (1.1) with F satisfying Assumption 1.1 and $\alpha \in (0,1]$. The solution exists for all time $t \in [0,\infty)$, M_t converges to infinity as $t \to \infty$ and the properly rescaled surfaces converge exponentially in C^∞-topology to the unit sphere $\mathbb{S}^2$.*

Theorem 1.3. *For any smooth, closed strictly convex surface M_0 in $\mathbb{H}^3$, there exists a smooth solution of flow (1.1) with F satisfying Assumption 1.1 and $\alpha \in (0,1]$. The solution exists for all time $t \in [0,\infty)$, and each surface M_t can be written as a graph of a function $u(t,\theta)$ over $\mathbb{S}^2$. The principal curvatures $\lambda_i(i = 1,2)$ of M_t satisfy the following decay estimate*

$$|\lambda_i - 1| \leq Ce^{-2^{(1-\alpha)} \cdot t}, \ i = 1, 2, \ as \ t \to \infty, \tag{1.2}$$

where $C = C(\alpha, M_0)$ is a positive constant depending only on α and M_0. Moreover, the defining function u of M_t satisfies the following asymptotic behavior

$$u(t,\theta) = \frac{t}{2^\alpha} + f(\theta) + o(1), \quad as \quad t \to \infty, \tag{1.3}$$

where $f(\theta)$ is a smooth function on $\mathbb{S}^2$.

Theorem 1.4. *For any smooth, closed strictly convex surface M_0 in $\mathbb{S}^3$, there exists a smooth solution of flow (1.1) with F satisfying Assumption 1.1 and $\alpha = 1$. The solution exists for finite time $t \leq T$ with $T < \infty$, M_t expands to the equator as $t \to T$ and the properly rescaled surfaces converge exponentially in C^∞-topology to the unit sphere $\mathbb{S}^2$.*

The expanding curvature flow for convex hypersurfaces driven by powers of a symmetric, increasing, homogeneous of degree one function of the principal curvatures has been studied by many authors. For $\alpha \in (0,1]$, Urbas [35] proved that the flow (1.1) in $\mathbb{R}^{n+1}$ exists for all time and converges to a round sphere after suitable rescaling, provided that either (i) the speed function F is inverse concave and its dual function F_* vanishes on the boundary of Γ_+, or (ii) F is concave and inverse concave. For power $\alpha > 1$, Gerhardt [12] proved the convergence of the flow (1.1) in $\mathbb{R}^{n+1}$ if F is concave and F vanishes on the boundary of Γ_+ (see earlier results by Schnürer [33] and Li [21]); Kröner and Scheuer [17] considered the case that F is concave and initial hypersurface M_0 satisfies a suitable curvature pinching. If the speed function F is concave and F vanishes on the boundary of Γ_+, Urbas [34] ($\alpha = 1$) and Gerhardt [8, 12] ($\alpha \in (0,1]$) proved the long time existence and convergence of the flow (1.1) in $\mathbb{R}^{n+1}$, provided that the initial hypersurface is star-shaped (not necessarily convex) and admissible. The hyperbolic version was studied by Gerhardt [11], Scheuer [30, 29] and the third author [36]. Flow (1.1) for convex hypersurfaces in sphere was considered by Gerhardt [13], Makowski-Scheuer [23] and also by the third author [36]. Inverse curvature flows have been also studied in other Riemannian manifolds: in [31, 32, 37] the ambient manifolds belong to a class of warped products that includes the space forms, while in [27, 28] the ambient manifolds are non-compact rank one symmetric spaces and a different notion of mass is used to discuss the roundness of the limit of the rescaled metric.

In the previously mentioned papers, the concavity or the inverse concavity of F plays an important role in deriving the curvature estimate and in applying the second derivative Hölder estimates of Krylov [19]. Our Theorems 1.2 – 1.4 say that the condition on the second derivatives of the speed function is not necessary in the two-dimensional case. The

proof is inspired by Andrews' work [5] on contracting curvature flow in $\mathbb{R}^3$. One ingredient of the proof is the pinching ratio estimate

$$\lambda_2 \ \leq \ C\lambda_1$$

along the flow (1.1) without any concavity assumption on the speed function F. This follows from applying the maximum principle to the evolution equation of

$$G = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}, \tag{1.4}$$

which yields that the supremum of G over M_t is monotone decreasing in time along the flow (1.1) in $\mathbb{R}^3$ and $\mathbb{H}^3$ with $\alpha \in (0,1]$, and in $\mathbb{S}^3$ with $\alpha = 1$. The reason that we can do this without condition on the second derivative of F is that we can write down the gradient term completely in two-dimensional case and show that it has a favourable sign at the critical point of G by using the condition $\nabla G = 0$ at the critical point.

The curvature pinching estimate together with the bound on the speed F will imply the bound on the principal curvatures as well as the preserving of the convexity of the evolving surfaces M_t. Since the speed function F does not satisfy any second derivative condition, we use the second derivative Hölder estimate derived by Andrews [3] in two-dimensional case and the standard parabolic Schauder estimate to derive the higher regularity of the flow. Then the long time existence of the flow (1.1) follows.

To prove the convergence of the flow, we need some extra work. In the Euclidean case, by using the curvature pinching estimate, we will refine the upper bound on F and show that the properly rescaled F is uniformly bounded from above. This together with the lower bound on the rescaled F (obtained by Gerhardt [12]) implies the uniform two sides positive bounds on the rescaled principal curvatures. The Alexandrov reflection argument implies that the rescaled surfaces converge to a round sphere continuously. The uniform estimates on the rescaled principal curvatures and Andrews' [3] second derivative Hölder estimate and the Schauder estimate can be applied to derive the smooth convergence of the rescaled flow. Finally, we can prove the exponential convergence of the rescaled flow. The method is to consider the quantity G on the rescaled flow and show that the supremum of G satisfies an exponential decay. The exponential decay of the curvature and the embedding of the flow surfaces then follow from a similar argument to that in [1]. Note that this exponential convergence of the flow (1.1) in Euclidean space was not considered in Gerhardt's paper [12].

For the convergence of the flow in the hyperbolic space, the evolution equation of the quantity G together with the contribution from the negative curvature of the hyperbolic case yields that the pinching ratio not only is controlled by its initial value but also decays to one exponentially. Thus in order to show that both principal curvatures λ_1, λ_2 converge to one as time goes to infinity, it suffices to show that the speed function F converges to 2 as time goes to infinity. For this, we adapt an argument used by Scheuer [30]. Our pinching estimate will also be used in the proof. The remaining proof of Theorem 1.3 follows from the similar argument as in [30].

Finally, for the flow (1.1) in the sphere, applying the argument in [23] together with the pinching estimate and the regularity estimate of Andrews [3], we can first prove that the evolving surface M_t expands to the equator in $C^{1,\gamma}$, $0 < \gamma < 1$ as $t \to T < \infty$. To show the smooth convergence we employ the dual flow method introduced by Gerhardt [13]. McCoy

[24] proved that the dual contracting curvature flow of (1.1) contracts convex surface to a point in finite time and properly rescaled surfaces converge to a unit sphere smoothly. This could be used to show that the rescaled solution of the inverse curvature flow (1.1) converges to unit sphere continuously. By deriving the higher regularity estimate, we can eventually show that the rescaled solution converges to the unit sphere smoothly.

In Theorem 1.3, the convergence of the flow states that the principal curvatures converge to 1 exponentially and the shifted defining function of the flow surface converges to a smooth function $f(\theta)$ over the sphere S^2 as time goes to infinity. In general, the function $f(\theta)$ may not be a constant function nor a first eigenfunction of the Laplacian operator on S^2. This means that the conformal metric $e^{2f}g_{S^2}$ may not be a round metric on the sphere. This property for the inverse curvature flow in hyperbolic space cannot be improved as we prove in the following theorem.

Theorem 1.5. *There exists a smooth star-shaped and mean convex surface M_0 in $\mathbb{H}^3$ such that along flow (1.1) with $\alpha \in (0,1), F = H$, as $t \to \infty$, the rescaled metric $e^{-2^{(1-\alpha)} \cdot t} g_t$ converges to a metric on $\mathbb{S}^2$ which is not the round metric, where g_t is the induced metric on M_t.*

An analogous counterexample for the inverse mean curvature flow (i.e., flow (1.1) with $\alpha = 1, F = H$) in $\mathbb{H}^3$ was constructed by Hung-Wang [18]. The proof of our Theorem 1.5 is inspired by their argument. Consider the following quantity for any smooth surface M in $\mathbb{H}^3$

$$Q(M) = -|M| \int_M |\mathring{A}|^2 d\mu, \tag{1.5}$$

where $\mathring{A}$ is the trace-less part of the second fundamental form of M. Let $\tilde{M}_s$ be a family of surfaces in $\mathbb{H}^3$ that are radial graphs of the functions $u(s,\theta) = cs + f(\theta) + o(1)$ over the sphere S^2 and g_{ij} be the induced metric on $\tilde{M}_s$. Hung-Wang proved that the limit of the rescaled metric $e^{-2cs}g_{ij}$ as $s \to \infty$ is a round metric if and only if $\lim_{s\to\infty} Q(\tilde{M}_s) = 0$. This characterizes when the limit of rescaled metric is a round metric in terms of the value of Q.

To construct the example in Theorem 1.5, we choose a function $\bar{f}(\theta)$ on $\mathbb{S}^2$ such that

$$\int_{\mathbb{S}^2} e^{2\bar{f}} d\mu_{g_{\mathbb{S}^2}} \int_{\mathbb{S}^2} |\mathring{D}^2 e^{-\bar{f}}|^2_{g_{\mathbb{S}^2}} d\mu_{g_{\mathbb{S}^2}} = c_0 > 0,$$

where $\mathring{D}^2 e^{-\bar{f}}$ means the traceless part of the Hessian of $e^{-\bar{f}}$. Let $\tilde{M}_s$ be the family of surfaces given by the radial graph of $u(s,\theta) = s + \bar{f}(\theta)$ over $\mathbb{S}^2$ in the (r,θ) coordinates of $\mathbb{H}^3$. We know from [18] that

$$\lim_{s\to\infty} Q(\tilde{M}_s) = -c_0. \tag{1.6}$$

By choosing $s_0 > 0$ large enough and using some estimate from Neves' paper [25], we have that for any $s \geq s_0$, $\tilde{M}_s$ is mean-convex, star-shaped and is strongly pinched. We consider the solution M_t^s of the flow (7.1) starting from $\tilde{M}_s, s \geq s_0$, where t is the time parameter. We will complete the proof of Theorem 1.5 in §7 by estimating the limit

$$\lim_{t\to\infty} Q(M_t^s) \leq -c_0/4 < 0 \tag{1.7}$$

for $s \geq s_0$, where s_0 is chosen large enough. To obtain (1.7), we need the following crucial estimate

$$|\tilde{M}_s|^2 |H - 2|^2 + |\tilde{M}_s|^2 |\mathring{A}|^2 + |\tilde{M}_s|^2 |\nabla A|^{4/3} \leq Ce^{-2^{(2-\alpha)} \cdot t}, \tag{1.8}$$

where H, A and $\mathring{A}$ are respectively mean curvature, the second fundamental form and the trace-less part of the second fundamental form of the solution M_t^s of the flow (7.1) starting from $\tilde{M}_s$ for any $s \geq s_0$. The key point of the estimate (1.8) is that the constant C is independent of the parameter s so that we can obtain the desired estimate (1.7) by properly choosing s_0 large enough. The proof of (1.8) is the central part of §7 and is technical. The idea is from Neves [25] in the case $\alpha = 1$. However, for $\alpha \in (0, 1)$, the flow (1.1) is not scale invariant, several technique difficulties will arise in obtaining the sharp exponential decay estimate (1.8). Our strategy is by first obtaining a weaker decay estimate and then improving the rate step by step. See §7.1 for details.

2. NOTATIONS AND PRELIMINARIES

In this section, we give some notations and preliminary results. Throughout the paper, we adopt the Einstein summation convention of summing over repeated indices. Let $X_t = X(x, t) : M_t \to \mathbb{R}^3(\kappa)$ be a family of surfaces in a real space form $\mathbb{R}^3(\kappa)$ moving according to

$$\frac{\partial}{\partial t} X(x, t) = -\Phi(x, t) \nu(x, t), \tag{2.1}$$

where $\Phi(x, t) = \Phi(F(x, t)) = -F^{-\alpha}(x, t)$, F is a smooth, symmetric, and homogeneous of degree one function of the principal curvatures of the surfaces $M_t = X_t(M)$ and ν is the outer unit normal of M_t.

We use $g = \{g_{ij}\}$, $A = \{h_{ij}\}$ and $\mathcal{W} = \{h_j^i\}$ to denote the components of induced metric, the second fundamental form and the Weingarten map of the surfaces, respectively. Here $h_{ij} = -\bar{g}(\bar{\nabla}_{\frac{\partial X}{\partial x^i}} \frac{\partial X}{\partial x^j}, \nu)$ in local coordinates x^1, x^2, where $\bar{g}$ denotes the metric of $\mathbb{R}^3(\kappa)$ and $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric $\bar{g}$. Note that at a given point $x \in M$, we can always choose local coordinates x^1, x^2 such that $g_{ij} = \delta_{ij}$, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ and $h_j^i = \text{diag}(\lambda_1, \lambda_2)$ at x. The function $F = F(\mathcal{W}) = F(\lambda(\mathcal{W}))$ can be considered as a function of $\mathcal{W} = (h_i^j)$ or the principal curvatures $\lambda(\mathcal{W}) = (\lambda_1, \lambda_2)$. The derivatives of F with respect to λ_i and h_i^j are related in the following way (see [1, 4, 9]). If A is a diagonal and B a symmetric matrix, then

$$\dot{F}^{kl}(A) = \frac{\partial F}{\partial \lambda_k}(\lambda(A))\delta^{kl},$$

and if A has distinct eigenvalues, then the second derivative of F in direction B is given by

$$\ddot{F}^{kl,rs}(A)B_{kl}B_{rs} = \frac{\partial^2 F(\lambda(A))}{\partial\lambda_k\partial\lambda_l}B_{kk}B_{ll} + 2\sum_{k<l}\frac{\frac{\partial F}{\partial\lambda_k} - \frac{\partial F}{\partial\lambda_l}}{\lambda_k - \lambda_l}B_{kl}^2. \tag{2.2}$$

The second term makes sense as a limit if $\lambda_k = \lambda_l$. Since F is symmetric, we may assume that at each point $(x,t) \in M \times [0,T)$, the principal curvatures satisfy $\lambda_1 \le \lambda_2$.

2.1. Evolution equations.

For the surfaces moving according to (2.1), we have the following evolution equations (see [2, 10]):

$$\frac{\partial}{\partial t}\nu = \nabla\Phi, \tag{2.3}$$

$$\frac{\partial}{\partial t}g_{ij} = -2\Phi h_{ij}, \tag{2.4}$$

$$\frac{\partial}{\partial t}\Phi - \dot{\Phi}\dot{F}^{ij}\nabla_i\nabla_j\Phi = \Phi\dot{\Phi}\dot{F}^{ij}h_{ip}h_j^p + \kappa\Phi\dot{\Phi}\dot{F}^{ij}g_{ij}, \tag{2.5}$$

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} - \dot{\Phi}\dot{F}^{kl}\nabla_k\nabla_l h_{ij} =& \dot{\Phi}\ddot{F}^{kl,mn}\nabla_i h_{kl}\nabla_j h_{mn} + \ddot{\Phi}\dot{F}^{kl}\nabla_i h_{kl}\dot{F}^{mn}\nabla_j h_{mn} \\
&+ \dot{\Phi}\dot{F}^{kl}h_{ij}h_k^p h_{pl} - \dot{\Phi}Fh_i^k h_{kj} - \Phi h_i^k h_{kj} \\
&+ \kappa(\dot{\Phi}Fg_{ij} + \Phi g_{ij} - \dot{\Phi}\dot{F}^{kl}g_{kl}h_{ij}),
\end{aligned} \tag{2.6}$$

where ∇ is the Levi-Civita connection with respect to the induced metric g, $\nabla_i h_{kl}$ is the covariant derivative of the second fundamental form, and $\dot{\Phi} = d\Phi/dr$ by considering $\Phi = \Phi(r) = -r^{-\alpha}$ as a function of the real variable r. For any function ω on M, we also use $\omega_i = \nabla_i\omega$, $\omega_{ij} = \nabla_i\nabla_j\omega$ to denote the covariant derivatives of ω with respect to g_{ij}.

2.2. Graphical representation.

We recall the warped product model of the space form $\mathbb{R}^3(\kappa)$, i.e., $\mathbb{R}^3(\kappa) = I \times \mathbb{S}^2$ equipped with the warped product metric

$$\bar{g} = dr^2 + s_\kappa^2(r)g_{\mathbb{S}^2},$$

where $I = (0,\infty)$ if $\kappa = 0,-1$, and $I = (0,\pi)$ if $\kappa = 1$ with

$$s_\kappa(r) = \begin{cases} r, & \kappa = 0, \\ \sin r, & \kappa = 1, \\ \sinh r, & \kappa = -1. \end{cases}$$

Suppose that M is a star-shaped surface in $\mathbb{R}^3(\kappa)$ and can be expressed as a graph over the sphere $\mathbb{S}^2$, i.e., $M = \{(u(\theta),\theta),\ \theta \in \mathbb{S}^2\}$ for some function $u \in C^\infty(\mathbb{S}^2)$, then the induced metric on M in terms of the coordinates θ^j is given by

$$g_{ij} = u_i u_j + s_\kappa^2(u)\sigma_{ij}, \tag{2.7}$$

where $\sigma_{ij} = g_{\mathbb{S}^2}(\partial_{\theta^i},\partial_{\theta^j})$ are the components of the round metric $g_{\mathbb{S}^2}$. The second fundamental form h_{ij} satisfies

$$h_{ij}v^{-1} = -u_{ij} + s_\kappa'(u)s_\kappa(u)\sigma_{ij}, \tag{2.8}$$

where u_{ij} are the covariant derivatives of u with respect to the induced metric g_{ij}, $s_\kappa'(r)$ is the derivative of $s_\kappa(r)$ and v is defined by

$$v = \sqrt{1 + s_\kappa^{-2}(u)|Du|_{g_{\mathbb{S}^2}}^2}. \tag{2.9}$$

The unit normal vector field on the surface is given by

$$\nu = v^{-1}(\partial_r - s_\kappa^{-2}(u)u^j\partial_{\theta^j}). \tag{2.10}$$

We denote

$$\varphi(u) = \int_{u_0}^u \frac{1}{s_\kappa(r)}dr,$$

then $\varphi'(u) = s_\kappa^{-1}(u)$ and h_j^i (the components of the Weingarten map) can be expressed as

$$h_j^i = v^{-1}s_\kappa^{-1}(u)\left(-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + s_\kappa'(u)\delta_j^i\right), \tag{2.11}$$

where $\varphi^i = \sigma^{ik}\varphi_k$, $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and the covariant derivatives here are taken with respect to σ_{ij}.

If M_t is a smooth star-shaped solution of (2.1) for $t \in [0, T)$ and each flow surface is expressed as a graph $M_t = \text{graph } u(t, \theta)$ over the sphere $\mathbb{S}^2$, we can easily deduce that the defining function $u(t, \theta)$ of M_t satisfies the following scalar parabolic equation (see [10])

$$\frac{\partial}{\partial t}u(t) = -\Phi v, \tag{2.12}$$

on $[0, T) \times \mathbb{S}^2$, where v is the function defined in (2.9).

2.3. **Support function.** The support function of a star-shaped surface M in $\mathbb{R}^3(\kappa)$ is defined by

$$\chi = \bar{g}(s_\kappa(u)\partial_r, \nu) = s_\kappa(u)v^{-1}.$$

In this subsection, we derive the evolution equation of χ for the flow surfaces M_t along the flow (2.1). We choose local coordinates x^1, x^2 such that $g_{ij} = \delta_{ij}$, $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = 0$ and $h_j^i = \text{diag}(\lambda_1, \lambda_2)$ at a given point $x \in M$. It is easy to check that the vector field $s_\kappa(u)\partial_r$ is a conformal vector field in the sense that (cf. [26, page 206])

$$\bar{\nabla}_Y(s_\kappa(u)\partial_r) = s_\kappa'(u)Y \tag{2.13}$$

for any tangent vector field Y on $\mathbb{R}^3(\kappa)$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to the metric $\bar{g}$ on $\mathbb{R}^3(\kappa)$. Then using (2.13) and (2.3), we have

$$\begin{aligned}\frac{\partial}{\partial t}\chi &= \bar{g}(\bar{\nabla}_{-\Phi\nu}(s_\kappa(u)\partial_r), \nu) + \bar{g}(s_\kappa(u)\partial_r, \partial_t\nu) \\ &= -\Phi s_\kappa'(u) + \bar{g}(s_\kappa(u)\partial_r, \nabla\Phi).\end{aligned} \tag{2.14}$$

Similarly, using (2.13), we derive that

$$\nabla_i\chi = \bar{g}(s_\kappa(u)\partial_r, h_i^k\partial_{x^k}), \tag{2.15}$$

$$\nabla_j\nabla_i\chi = \bar{g}(s_\kappa(u)\partial_r, g^{kl}\nabla_l h_{ij}\partial_{x^k}) + h_{ij}s_\kappa'(u) - h_i^k h_{kj}\chi. \tag{2.16}$$

Combining the equations (2.14)-(2.16) gives that

$$\frac{\partial}{\partial t}\chi - \dot{\Phi}\dot{F}^{ij}\nabla_j\nabla_i\chi = \dot{\Phi}\dot{F}^{ij}h_i^k h_{kj}\chi - (\Phi + \dot{\Phi}F)s_\kappa'(u). \tag{2.17}$$

Note that

$$-(\Phi + \dot{\Phi}F)s_\kappa'(u) = (1 - \alpha)F^{-\alpha}s_\kappa'(u) \geq 0$$

for $\alpha \in (0, 1]$. The maximum principle applied to (2.17) then implies that the star-shaped condition $\chi > 0$ of the flow surfaces is preserved along the flow (2.1).

3. Estimate on the pinching ratio

In this section, we show that the pinching ratio of M_t, which is the supremum over the surface M_t of the ratio of largest to smallest principal curvatures at each point, is no greater than that of M_0. This is the first key step of the proof of Theorems 1.2 – 1.4. The idea is to apply the maximum principle to the evolution equation for the following quantity G.

Theorem 3.1. *Let M_t $(0 \leq t < T)$ be a family of smooth, closed strictly convex surfaces in $\mathbb{R}^3(\kappa)(\kappa = 0, 1, -1)$ flowing according to (1.1) with F satisfying Assumption 1.1. We assume that $\alpha \in (0, 1]$ in the cases $\kappa = 0, -1$, and assume that $\alpha = 1$ in the case $\kappa = 1$. Then we have that the supremum $\tilde{G}$ of the quantity $G = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}$ is non-increasing in time.*

Proof. Denote $\Phi = -F^{-\alpha}$. Then Φ is a symmetric homogeneous of degree $-\alpha$ function of the principal curvatures of M_t. By a direct computation, we have (see [2])

$$
\begin{aligned}
\frac{\partial}{\partial t} G - \dot{\Phi}^{ij} \nabla_i \nabla_j G =& (\dot{G}^{ij} \ddot{\Phi}^{kl,mn} - \dot{\Phi}^{ij} \ddot{G}^{kl,mn}) \nabla_i h_{kl} \nabla_j h_{mn} \\
& - \dot{G}^{ij} \dot{\Phi}^{kl} h_{kl} h_i^p h_{pj} + \dot{G}^{ij} \dot{\Phi}^{kl} h_{ij} h_k^p h_{pl} + \Phi \dot{G}^{ij} h_i^k h_{kj} \\
& + \kappa (\dot{G}^{ij} \dot{\Phi}^{kl} h_{kl} g_{ij} - \dot{G}^{ij} \dot{\Phi}^{kl} g_{kl} h_{ij} + \Phi \dot{G}^{ij} g_{ij}),
\end{aligned}
\tag{3.1}
$$

where $\dot{\Phi}^{ij} = \dfrac{\partial \Phi}{\partial h_{ij}}$, $\ddot{\Phi}^{kl,mn} = \dfrac{\partial^2 \Phi}{\partial h_{kl} \partial h_{mn}}$. Note that G is homogeneous of degree zero, Φ is homogeneous of degree $-\alpha$, and the Euler relation gives that $\dot{G}^{ij} h_{ij} = 0$ and $\dot{\Phi}^{ij} h_{ij} = -\alpha \Phi$, so (3.1) can be simplified by

$$
\begin{aligned}
\frac{\partial}{\partial t} G - \dot{\Phi}^{ij} \nabla_i \nabla_j G =& (\dot{G}^{ij} \ddot{\Phi}^{kl,mn} - \dot{\Phi}^{ij} \ddot{G}^{kl,mn}) \nabla_i h_{kl} \nabla_j h_{mn} \\
& + (1 + \alpha) \Phi \dot{G}^{ij} h_i^k h_{kj} + \kappa (1 - \alpha) \Phi \dot{G}^{ij} g_{ij}.
\end{aligned}
\tag{3.2}
$$

We denote the zero-order terms and the first-order terms in (3.2) by Q_0 and Q_1, respectively:

$$
\begin{aligned}
Q_0 &= (1 + \alpha) \Phi \dot{G}^{ij} h_i^k h_{kj} + \kappa (1 - \alpha) \Phi \dot{G}^{ij} g_{ij}, \\
Q_1 &= (\dot{G}^{ij} \ddot{\Phi}^{kl,mn} - \dot{\Phi}^{ij} \ddot{G}^{kl,mn}) \nabla_i h_{kl} \nabla_j h_{mn}.
\end{aligned}
$$

In the following, at a given point $x_t \in M_t$, we will choose local coordinates x^1, x^2 such that $g_{ij} = \delta_{ij}$, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ and $h_j^i = \text{diag}(\lambda_1, \lambda_2)$ at x_t. Thus we can use (2.2) to simplify Q_0 and Q_1 as follows:

$$
Q_0 = (1 + \alpha) \Phi \left(\frac{\partial G}{\partial \lambda_1} \lambda_1^2 + \frac{\partial G}{\partial \lambda_2} \lambda_2^2 \right) + \kappa (1 - \alpha) \Phi \left(\frac{\partial G}{\partial \lambda_1} + \frac{\partial G}{\partial \lambda_2} \right),
\tag{3.3}
$$

$$Q_1 = (\frac{\partial G}{\partial \lambda_1} \frac{\partial^2 \Phi}{\partial \lambda_1^2} - \frac{\partial \Phi}{\partial \lambda_1} \frac{\partial^2 G}{\partial \lambda_1^2})(\nabla_1 h_{11})^2 + (\frac{\partial G}{\partial \lambda_1} \frac{\partial^2 \Phi}{\partial \lambda_2^2} - \frac{\partial \Phi}{\partial \lambda_1} \frac{\partial^2 G}{\partial \lambda_2^2})(\nabla_1 h_{22})^2$$
$$+ (\frac{\partial G}{\partial \lambda_2} \frac{\partial^2 \Phi}{\partial \lambda_1^2} - \frac{\partial \Phi}{\partial \lambda_2} \frac{\partial^2 G}{\partial \lambda_1^2})(\nabla_2 h_{11})^2 + (\frac{\partial G}{\partial \lambda_2} \frac{\partial^2 \Phi}{\partial \lambda_2^2} - \frac{\partial \Phi}{\partial \lambda_2} \frac{\partial^2 G}{\partial \lambda_2^2})(\nabla_2 h_{22})^2$$
$$+ 2(\frac{\partial G}{\partial \lambda_1} \frac{\partial^2 \Phi}{\partial \lambda_1 \partial \lambda_2} - \frac{\partial \Phi}{\partial \lambda_1} \frac{\partial^2 G}{\partial \lambda_1 \partial \lambda_2})\nabla_1 h_{11} \nabla_1 h_{22} \qquad (3.4)$$
$$+ 2(\frac{\partial G}{\partial \lambda_2} \frac{\partial^2 \Phi}{\partial \lambda_1 \partial \lambda_2} - \frac{\partial \Phi}{\partial \lambda_2} \frac{\partial^2 G}{\partial \lambda_1 \partial \lambda_2})\nabla_2 h_{11} \nabla_2 h_{22}$$
$$+ 2\frac{\frac{\partial G}{\partial \lambda_1} \frac{\partial \Phi}{\partial \lambda_2} - \frac{\partial G}{\partial \lambda_2} \frac{\partial \Phi}{\partial \lambda_1}}{\lambda_2 - \lambda_1}(\nabla_1 h_{12})^2 + 2\frac{\frac{\partial G}{\partial \lambda_1} \frac{\partial \Phi}{\partial \lambda_2} - \frac{\partial G}{\partial \lambda_2} \frac{\partial \Phi}{\partial \lambda_1}}{\lambda_2 - \lambda_1}(\nabla_2 h_{12})^2.$$

By the definition of G (see (1.4)), we have

$$\frac{\partial G}{\partial \lambda_1} = \frac{4\lambda_2(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)^3}, \quad \frac{\partial G}{\partial \lambda_2} = \frac{4\lambda_1(\lambda_2 - \lambda_1)}{(\lambda_1 + \lambda_2)^3}. \qquad (3.5)$$

Using (3.5), we can simplify (3.3):

$$Q_0 = \frac{4\Phi G}{\lambda_1 + \lambda_2}\Big(\kappa(\alpha - 1) + (1 + \alpha)\lambda_1 \lambda_2\Big). \qquad (3.6)$$

For each time $t > 0$, at the critical point of G with $\lambda_1 \neq \lambda_2$, from (3.5), we have $\frac{\partial G}{\partial \lambda_i} \neq 0$, and the gradient condition on G, i.e., $0 = \nabla_i G = \frac{\partial G}{\partial \lambda_1}\nabla_i h_{11} + \frac{\partial G}{\partial \lambda_2}\nabla_i h_{22}$, $i = 1, 2$, leads to the following two equations:

$$\nabla_1 h_{11} = -\frac{\frac{\partial G}{\partial \lambda_2}}{\frac{\partial G}{\partial \lambda_1}}\nabla_1 h_{22} = \frac{\lambda_1}{\lambda_2}\nabla_1 h_{22}, \quad \nabla_2 h_{22} = -\frac{\frac{\partial G}{\partial \lambda_1}}{\frac{\partial G}{\partial \lambda_2}}\nabla_2 h_{11} = \frac{\lambda_2}{\lambda_1}\nabla_2 h_{11}. \qquad (3.7)$$

Note that the Codazzi equations say that $\nabla_k h_{ij}$ is totally symmetric, i.e., we have $\nabla_1 h_{12} = \nabla_2 h_{11}$ and $\nabla_2 h_{12} = \nabla_1 h_{22}$. Using (3.7) and the homogeneity of G and Φ, we arrive at

$$Q_1 = \frac{4\alpha\Phi}{(\lambda_1 + \lambda_2)^3}\Big(((1 + \alpha)\frac{\lambda_1}{\lambda_2} + (1 - \alpha))(\nabla_1 h_{22})^2 + ((1 - \alpha) + (1 + \alpha)\frac{\lambda_2}{\lambda_1})(\nabla_2 h_{11})^2\Big). \quad (3.8)$$

In view of (3.6) and (3.8), we have $Q_0 \leq 0$ and $Q_1 \leq 0$ in the cases $\alpha \in (0, 1]$ for $\kappa = 0, -1$, and $\alpha = 1$ for $\kappa = 1$. Applying the maximum principle, we conclude that the supremum of G on M_t is non-increasing in time t. □

Since the pinching ratio is given by $r = \frac{2}{1 - \sqrt{\tilde{G}}} - 1$, as a corollary of Theorem 3.1, we have

Proposition 3.2. *Let M_t ($0 \leq t < T$) be a family of smooth closed strictly convex surfaces in $\mathbb{R}^3(\kappa)(\kappa = 0, 1, -1)$ flowing according to (1.1) with F satisfying Assumption 1.1. We assume that $\alpha \in (0, 1]$ for $\kappa = 0, -1$, and $\alpha = 1$ for $\kappa = 1$. Then the pinching ratio of M_t is no greater than the pinching ratio of M_0, i.e, there exists a positive constant $\beta > 1$ which only depends on M_0 such that*

$$\lambda_2 \leq \beta\lambda_1. \qquad (3.9)$$

Remark 3.3. Since F is homogeneous of degree one, we have that $\partial F/\partial\lambda_i, i = 1, 2$ are homogeneous of degree zero, and then we have that

$$\frac{\partial F}{\partial \lambda_i}(\lambda_1, \lambda_2) = \frac{\partial F}{\partial \lambda_i}(\frac{\lambda_1}{\lambda_1 + \lambda_2}, \frac{\lambda_2}{\lambda_1 + \lambda_2}).$$

By the pinching ratio estimate in Proposition 3.2, the supremum and infimum of $\partial F/\partial\lambda_i, i = 1, 2$ are attained on the compact set $\{(a, 1-a)) : |a - \frac{1}{2}| \le \frac{\beta-1}{2(\beta+1)}\}$ and hence there exists a positive constant c depending only on β (and therefore depending only on M_0) such that

$$0 < c^{-1} \le \frac{\partial F}{\partial \lambda_i} \le c, \quad i = 1, 2, \quad \forall\, t \in [0, T). \tag{3.10}$$

Remark 3.4. We also have the following consequences of the pinching ratio estimate (3.9). As F is homogeneous of degree one, normalized with $F(1, 1) = 2$ and is strictly monotone in each argument, we have

$$\frac{2\lambda_2}{\beta} \le 2\lambda_1 = F(\lambda_1, \lambda_1) \le F(\lambda_1, \lambda_2) \le F(\lambda_2, \lambda_2) = 2\lambda_2 \le 2\beta\lambda_1,$$

which is equivalent to

$$\frac{F}{2\beta} \le \lambda_1 \le \frac{F}{2} \le \lambda_2 \le \frac{\beta F}{2}. \tag{3.11}$$

So we have

$$\dot{F}^{ij}g_{ij} = \frac{\partial F}{\partial \lambda_1} + \frac{\partial F}{\partial \lambda_2} \ge \frac{1}{\lambda_2}\left(\frac{\partial F}{\partial \lambda_1}\lambda_1 + \frac{\partial F}{\partial \lambda_2}\lambda_2\right) = \frac{F(\lambda_1, \lambda_2)}{\lambda_2} \ge \frac{2}{\beta}, \tag{3.12}$$

and

$$\dot{F}^{ij}h_i^k h_{kj} = \frac{\partial F}{\partial \lambda_i}\lambda_i^2 \ge \left(\frac{\partial F}{\partial \lambda_1}\lambda_1 + \frac{\partial F}{\partial \lambda_2}\lambda_2\right)\lambda_1 \ge \frac{\lambda_2}{\beta}F \ge \frac{1}{2\beta}F^2. \tag{3.13}$$

Similarly, we have

$$\dot{F}^{ij}g_{ij} \le 2\beta, \quad \text{and} \quad \dot{F}^{ij}h_i^k h_{kj} \le \frac{\beta}{2}F^2. \tag{3.14}$$

Remark 3.5. We further remark that the pinching ratio estimate in Proposition 3.2 cannot be improved. For $\alpha > 1$, after a similar argument to that in [5, §5], we can obtain an example of smooth, strictly convex surface in $\mathbb{R}^3$ for which the pinching ratio becomes larger for any flow (1.1) with $\alpha > 1$. Moreover, Kröner and Scheuer [17] constructed a counterexample to show that along the flow (1.1) in $\mathbb{R}^3$ with $F = H$ and $\alpha > 1$, the convexity of the initial surface will be lost. However, the fact that the pinching ratio does not improve obviously does not rule out the possibility that other curvature estimates may yield useful results, cf. the results in [12, 30, 17] for $\alpha > 1$ case.

4. FLOW IN EUCLIDEAN SPACE

In this section, we consider flow (1.1) in Euclidean space $\mathbb{R}^3$ and prove Theorem 1.2. We assume that F satisfies Assumption 1.1 and $\alpha \in (0, 1]$. Since the flow (1.1) is a parabolic equation with strictly convex initial data, by short time existence theorem we have a smooth solution on a maximal time interval $[0, T)$, $0 < T \le \infty$. It remains to study the long time behavior of the flow. In [12], the concavity of F is essentially used in order to get the curvature estimates. Here, we do not have the concavity assumption and we use

the pinching ratio estimate obtained in Section 3 to prove the curvature estimates and the convergence of the flow. We also show that the rescaled flow converges exponentially to the sphere.

The following lemma gives the evolution of spheres in $\mathbb{R}^3$ along the flow.

Lemma 4.1. *Given $x_0 \in \mathbb{R}^3, \rho_0 \in \mathbb{R}^+$. Denote*

$$\rho(t, \rho_0) = \begin{cases} \left((1-\alpha)2^{-\alpha}t + \rho_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}}, & \alpha \in (0, 1) \\ \rho_0 e^{t/2}, & \alpha = 1. \end{cases} \tag{4.1}$$

Then the spheres $\partial B_{\rho(t,\rho_0)}(x_0)$ solve (1.1) for $t \in [0, +\infty)$ with ρ_0 as the initial radius.

Proof. Since the flow (1.1) preserves the symmetry, in the sphere case, the equation (1.1) reduces to the following ODE for the radius of the spheres

$$\begin{cases} \dfrac{d}{dt}\rho(t, \rho_0) = 2^{-\alpha}\rho(t, \rho_0)^\alpha, \\ \rho(0, \rho_0) = \rho_0. \end{cases} \tag{4.2}$$

Then the lemma follows immediately by solving (4.2). $\qquad\square$

Since the initial surface M_0 is assumed to be closed and strictly convex in $\mathbb{R}^3$, we can choose a point x_0 in the domain enclosed by M_0 such that M_0 is given by the graph of a smooth function u_0 over the sphere $\mathbb{S}^2(x_0) = \partial B_1(x_0)$. Without loss of generality, we can assume that x_0 is the origin and denote $\mathbb{S}^2(x_0)$ simply by $\mathbb{S}^2$. Then $M_0 = \{(u_0(\theta), \theta), \theta \in \mathbb{S}^2\}$. Under the assumption of Theorem 1.2, the flow (1.1) preserves the star-shaped condition (see Section 2.3) and each flow surface M_t can be written as a graph of $u(t, \theta)$ over the sphere. By (2.12), the defining function u satisfies the following scalar flow equation

$$\frac{\partial u}{\partial t} = \sqrt{1 + |D \log u|_{g_{\mathbb{S}^2}}^2} \; F^{-\alpha}, \tag{4.3}$$

which is clearly parabolic.

We assume that the initial surface $M_0 = \text{graph } u_0$ satisfies

$$\rho_1 < u_0(\theta) < \rho_2, \quad \forall\, \theta \in \mathbb{S}^2,$$

where ρ_1 and ρ_2 are two positive constants. By applying the maximum principle, we have

Lemma 4.2 ([12]). *As long as the flow (1.1) exists, the defining function $u(t, \theta)$ of the solution of the flow satisfies*

$$\rho(t, \rho_1) < u(t, \theta) < \rho(t, \rho_2), \; \forall\, t \in [0, T), \; \forall\, \theta \in \mathbb{S}^2. \tag{4.4}$$

Moreover, for $\rho_1 < \bar{r} < \rho_2$, there exist two positive constants c_1, c_2 depending only on ρ_1, ρ_2 and α such that

$$0 < c_1 \leq u(t, \theta)\rho^{-1}(t, \bar{r}) \leq c_2, \quad \forall\, t \in [0, T), \; \forall\, \theta \in \mathbb{S}^2. \tag{4.5}$$

In particular, the flow is compactly contained in $\mathbb{R}^3$ for finite time t.

Moreover, by applying the maximum principle to the evolution equation of $|D \log u|_{g_{\mathbb{S}^2}}$, we have

Lemma 4.3 ([12]). *The smooth solution u of (4.3) satisfies the following C^1-estimate*

$$|D \log u(t, \theta)|_{g_{\mathbb{S}^2}} \leq |D \log u(0, \theta)|_{g_{\mathbb{S}^2}}, \quad \forall\, t \in [0, T), \; \forall\, \theta \in \mathbb{S}^2. \tag{4.6}$$

4.1. **Long time existence.**

Proposition 4.4. *Under the assumption of Theorem 1.2, the flow (1.1) exists for all time, i.e., $T = +\infty$.*

Proof. Firstly, applying the maximum principle to the evolution equation (2.5) gives the uniform upper bound on F:

$$F(x,t) \leq \max_{M_0} F, \quad \forall \ (x,t) \in M \times [0,T). \tag{4.7}$$

Moreover, Gerhardt [12] proved that there exists a positive constant c_3 depending only on α and M_0 such that for $\rho_1 < \bar{r} < \rho_2$, we have

$$\rho(t,\bar{r})F(x,t) \geq c_3 > 0, \quad \forall \ (x,t) \in M \times [0,T). \tag{4.8}$$

Then if $T < \infty$, the estimates (4.8) and (4.7) imply that F is bounded from above and below by positive constants depending on α, M_0 and T. The bounds on F and (3.11) give us the uniform upper and lower positive bounds on the principal curvatures, which combined with the C^0 estimate (4.5) and C^1-estimate (4.6) yield the C^2-estimate of u. Moreover, by (3.10) and the bounds on F, the flow (1.1) and Eq. (4.3) are uniformly parabolic. We may apply the second order derivative Hölder estimates in [3] and parabolic Schauder estimates [22] to derive uniform bounds on all higher derivatives of the principal curvatures and of $u(t)$. Then we have a smooth limit function $u_T := \lim_{t \to T} u(t)$ which defines a smooth strictly convex surface M_T. The short time existence theorem then implies that we can continue the flow beyond the time T, which contradicts the definition of T. So we conclude that $T = \infty$. □

Remark 4.5. We note that the concavity of F was not used in the proof of C^0 estimate (4.5), C^1-estimate (4.6) and (4.8) in [12], but it was used in the proof of the curvature estimate (see Lemma 4.10 in [12]) and applying the second derivative Hölder estimates of Krylov. In order to overcome the difficulties without assumption of concavity, we use the pinching ratio estimate to derive curvature estimate. In general, for the application of the second derivative Hölder estimates of Krylov [19], we need that F is concave in its arguments. In the two-dimensional case, Andrews [3] proved that the second derivative Hölder estimates also hold without any concavity assumption on F.

4.2. **Convergence.** First, we will use the curvature pinching estimate to refine estimate (4.7) on the upper bound of F.

Lemma 4.6. *Under the assumption of Theorem 1.2, there exists a positive constant c_4 depending only on α and M_0 such that for $\rho_1 < \bar{r} < \rho_2$, we have*

$$\rho(t,\bar{r})F \leq c_4, \quad \forall \ t \in [0,\infty). \tag{4.9}$$

Proof. We consider the case $\alpha \in (0,1)$ and the case $\alpha = 1$ separately.

For $\alpha \in (0,1)$, we define $\phi = \Phi + \mu\rho^\alpha(t,\bar{r}) = -F^{-\alpha} + \mu\rho^\alpha(t,\bar{r})$, where $\mu \in (0,1]$ is a small constant such that $\phi < 0$ on M_0 and

$$\mu < 2^{-\alpha}\beta^{\frac{\alpha}{\alpha-1}}.$$

We aim to show that ϕ stays negative for such μ along the flow (1.1). By (2.5) and (4.2), we have

$$
\begin{aligned}
\partial_t \phi - \dot{\Phi} \dot{F}^{ij} \nabla_i \nabla_j \phi =& \Phi \dot{\Phi} \dot{F}^{ij} h_i^k h_{kj} + \mu \alpha 2^{-\alpha} \rho^{2\alpha-1}(t, \bar{r}) \\
\leq& -\frac{\alpha}{2\beta} F^{-2\alpha+1} + \mu \alpha 2^{-\alpha} \rho^{2\alpha-1}(t, \bar{r}),
\end{aligned}
\tag{4.10}
$$

where we used the estimate (3.13). Let $t_0 > 0$ be the first time such that ϕ touches zero at some point $x_0 \in M_{t_0}$. Then at this point, we have that

$$
0 = \phi(x_0, t_0) = -F^{-\alpha}(x_0, t_0) + \mu \rho^{\alpha}(t_0, \bar{r}).
\tag{4.11}
$$

Applying the maximum principle to (4.10) and using (4.11), we derive that

$$
0 \leq \mu \alpha 2^{-\alpha} \rho(t_0, \bar{r})^{2\alpha-1} \left(1 - 2^{\alpha-1} \beta^{-1} \mu^{1-\frac{1}{\alpha}} \right) < 0,
$$

which is a contradiction. Thus ϕ stays negative for all time $t \in [0, \infty)$ and we obtain that

$$
\rho(t, \bar{r}) F \leq \mu^{-\frac{1}{\alpha}}.
$$

For $\alpha = 1$, we define $\phi = \Phi + \mu \chi$, where χ is the support function and μ is a positive constant such that $\phi \leq 0$ on M_0. Combining (2.5) and (2.17), we have

$$
\partial_t \phi - \dot{\Phi} \dot{F}^{ij} \nabla_i \nabla_j \phi = \phi \dot{\Phi} \dot{F}^{ij} h_i^k h_{kj}.
\tag{4.12}
$$

Since $\dot{\Phi} = F^{-2}$, the estimates (3.13)-(3.14) give that $(2\beta)^{-1} \leq \dot{\Phi} \dot{F}^{ij} h_i^k h_{kj} \leq \beta/2$. Then the coefficient of ϕ on the right of (4.12) is a uniformly bounded function. By applying the maximum principle to (4.12), we conclude that $\phi \leq 0$ for all M_t, $\forall\, t \in [0, \infty)$. Therefore, we have $F\chi \leq 1/\mu$. On the other hand, by the C^0, C^1 estimates (4.5), (4.6), we know that $\chi \geq c\rho$ for some positive constant c. Thus $F\rho \leq 1/(c\mu)$. $\qquad\square$

Now we complete the proof of Theorem 1.2. Note that we can always find a constant $\bar{r} \in (\rho_1, \rho_2)$ such that

$$
\lim_{t \to \infty} u(t, \theta) \rho^{-1}(t, \bar{r}) = 1.
\tag{4.13}
$$

The estimate (4.13) follows from the Alexandrov reflection argument as in [12, Lemma 3.5] by using the result of [7]. If $\alpha \in (0, 1)$, we present a simple proof for (4.13). In this case, for any $\bar{r} \in (\rho_1, \rho_2)$, (4.13) follows from (4.4) and

$$
1 = \lim_{t \to \infty} \rho(t, \rho_1) \rho^{-1}(t, \bar{r}) \leq \lim_{t \to \infty} u(t, \theta) \rho^{-1}(t, \bar{r}) \leq \lim_{t \to \infty} \rho(t, \rho_2) \rho^{-1}(t, \bar{r}) = 1,
$$

so we get that $\lim_{t \to \infty} u(t, \theta) \rho^{-1}(t, \bar{r}) = 1$, $\forall\, \bar{r} \in (\rho_1, \rho_2)$.

We now rescale the surface by

$$
\tilde{X}(x, t) = \rho(t, \bar{r})^{-1} X(x, t).
\tag{4.14}
$$

Define a new time function $\tau = \tau(t)$ by

$$
\frac{d\tau}{dt} = \rho(t, \bar{r})^{\alpha-1}
\tag{4.15}
$$

such that $\tau(0) = 0$. Then τ ranges from 0 to ∞. It's easy to check that the rescaled surface satisfies the following evolution equation

$$
\frac{\partial \tilde{X}}{\partial \tau} = \tilde{F}^{-\alpha} \nu - 2^{-\alpha} \tilde{X}.
\tag{4.16}
$$

Note that $\partial \tilde{F}/\partial \tilde{\lambda}_i$ is homogeneous of degree zero and is scaling invariant, then from (3.10) it is also bounded on the rescaled surfaces. By the pinching estimate (3.9) and the bounds (4.8), (4.9) on the rescaled speed function $\tilde{F} = \rho(t, \bar{r})F$, we have that the principal curvatures of the rescaled surfaces are also uniformly bounded from above and below by positive constants. Moreover, the uniform bounds on $\tilde{F}$ and $\partial \tilde{F}/\partial \tilde{\lambda}_i$ imply that the flow (4.16) is uniformly parabolic. By the standard argument using the Hölder estimates [3], parabolic Schauder estimates [22] and interpolation inequalities, we can conclude that the rescaled flow converges in C^∞-topology to the unit sphere $\mathbb{S}^2$.

Finally, we show that the rescaled flow converges exponentially. Since the quantity G defined in (1.4) is homogeneous of degree zero, on the rescaled surface $\tilde{M}_\tau$ we have $\tilde{G} = G$. Then

$$\frac{\partial}{\partial \tau}\tilde{G} = \rho(t, \bar{r})^{1-\alpha}\frac{\partial}{\partial t}G.$$

By the computation in §3,

$$\frac{d}{d\tau}\max_{\tilde{M}_\tau}\tilde{G} \leq (Q_0 + Q_1)\,\rho(t, \bar{r})^{1-\alpha}$$

$$\leq -4(1+\alpha)\frac{F^{-\alpha}\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\rho(t, \bar{r})^{1-\alpha}\max_{\tilde{M}_\tau}\tilde{G}$$

$$= -4(1+\alpha)\frac{\tilde{F}^{-\alpha}\tilde{\lambda}_1\tilde{\lambda}_2}{\tilde{\lambda}_1 + \tilde{\lambda}_2}\max_{\tilde{M}_\tau}\tilde{G} \;\leq\; -\delta\max_{\tilde{M}_\tau}\tilde{G}, \tag{4.17}$$

where $\delta > 0$ is a positive constant and in the last inequality we used the facts that $\tilde{\lambda}_i$ and $\tilde{F}$ are uniformly bounded from above and below by positive constants. (4.17) implies that the trace-less part of the second fundamental form of $\tilde{M}_\tau$ has the following exponential decay

$$|\mathring{\tilde{A}}|^2(x, \tau) \leq Ce^{-\delta\tau}, \quad \forall\, x \in \tilde{M}_\tau, \tag{4.18}$$

where $C = C(M_0)$. This gives by interpolation that

$$|\nabla\mathring{\tilde{A}}|^2(x, \tau) \leq Ce^{-\delta\tau}, \quad \forall\, x \in \tilde{M}_\tau, \tag{4.19}$$

as we already have uniform bounds on $\nabla^k\mathring{\tilde{A}}$, $\forall\, k \geq 0$. Note that in 2-dimensional case, we have

$$|\nabla\tilde{A}|^2 = |\nabla\mathring{\tilde{A}}|^2 + \frac{1}{2}|\nabla\tilde{H}|^2 \leq |\nabla\mathring{\tilde{A}}|^2 + \frac{2}{3}|\nabla\tilde{A}|^2,$$

where we used the inequality (cf. [15, §2])

$$|\nabla\tilde{H}|^2 \leq \frac{4}{3}|\nabla\tilde{A}|^2.$$

Therefore, we obtain

$$|\nabla\tilde{A}|^2(x, \tau) \leq 3|\nabla\mathring{\tilde{A}}|^2(x, \tau) \leq Ce^{-\delta\tau}, \quad \forall\, x \in \tilde{M}_\tau \tag{4.20}$$

and for all higher derivatives of $\tilde{A}$ by interpolation. Then the estimate on the metric and the exponential convergence of the immersions are the same as in [1]. This finishes the proof of Theorem 1.2.

5. Flow in Hyperbolic space

In this section, we consider flow (1.1) in hyperbolic space $\mathbb{H}^3$ and prove Theorem 1.3. We assume that F satisfies Assumption 1.1 and $\alpha \in (0,1]$. Since the flow (1.1) is a parabolic equation with strictly convex initial data, by short time existence theorem we have a smooth solution on a maximal time interval $[0,T)$, $0 < T \leq \infty$. It remains to study the long time behavior of the flow.

Fix a point $x_0 \in \mathbb{H}^3$. We consider geodesic polar coordinates centered at x_0. The metric on $\mathbb{H}^3$ can be expressed as

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^2}.$$

As in Euclidean case, if the initial surface is a geodesic sphere S_{ρ_0} in $\mathbb{H}^3$, then along the flow (1.1), the flow surfaces are also geodesic spheres with radius $\rho(t,\rho_0)$ solving the following ODE

$$\frac{d}{dt}\rho(t,\rho_0) = 2^{-\alpha} \tanh^\alpha \rho(t,\rho_0). \tag{5.1}$$

with $\rho(0,\rho_0) = \rho_0$. Since $2^{-\alpha} \tanh^\alpha \rho(t,\rho_0) \leq 2^{-\alpha}$, we have that (5.1) has solution for all $t \in [0,\infty)$. Moreover, it follows from $\tanh^\alpha \rho_0 \leq \tanh^\alpha \rho(t,\rho_0) \leq 1$ that

$$\rho_0 + \frac{t}{2^\alpha} \tanh^\alpha \rho_0 \leq \rho(t,\rho_0) \leq \rho_0 + \frac{t}{2^\alpha}. \tag{5.2}$$

Suppose that M_0 is a closed strictly convex surface in $\mathbb{H}^3$, then M_0 can be given by a graph of a positive function $u_0(\theta)$ over the geodesic sphere S^2 centered at some point x_0 in the enclosed domain by M_0. Under the assumption of Theorem 1.3, the flow (1.1) preserves the star-shaped condition (see Section 2.3) and each flow surface M_t can be written as a graph of a function $u(t,\theta)$ over the sphere. By (2.12), the defining function $u(t,\theta)$ satisfies the following scalar flow equation

$$\frac{\partial u}{\partial t} = vF^{-\alpha} \tag{5.3}$$

with

$$v = \sqrt{1 + |Du|^2_{g_{\mathbb{S}^2}}/\sinh^2 u}. \tag{5.4}$$

By applying the maximum principle, we have

Lemma 5.1 ([11, 30]).   (1) *The solution u of (5.3) satisfies that*

$$\rho(t,\inf u(0,\cdot)) \leq u(t,\theta) \leq \rho(t,\sup u(0,\cdot)), \quad \forall\, t \in [0,T),\ \forall\, \theta \in \mathbb{S}^2. \tag{5.5}$$

(2) *There exists a constant c depending on M_0 and α such that*

$$0 < c^{-1} \leq e^{-\frac{t}{2\alpha}} \sinh u(t,\theta) \leq c, \quad \forall\, t \in [0,T),\ \forall\, \theta \in \mathbb{S}^2,$$

and

$$\coth u(t,\theta) - 1 \leq ce^{-2^{(1-\alpha)}\cdot t}, \ \forall\, t \in [0,T),\ \forall\, \theta \in \mathbb{S}^2.$$

Lemma 5.2 ([11, 30]). *The solution u of (5.3) satisfies the following C^1-estimate*

$$v(t,\theta) \leq \ \sup v(0,\cdot), \quad \forall\, t \in [0,T),\ \forall\, \theta \in \mathbb{S}^2, \tag{5.6}$$

where v is defined in (5.4).

Remark 5.3. We note that the concavity of F was not used in the proof of C^0 estimate and C^1-estimate in [11, 30], it was used in the proof of the curvature estimate (see Lemma 4.4 in [11] and Proposition 3.11 in [30]) and applying the second derivative Hölder estimates of Krylov.

Lemma 5.4. *Under the assumption of Theorem 1.3, there exists a constant c depending only on M_0, α such that*

$$0 < c^{-1} \leq F(x,t) \leq c, \quad \forall\, (x,t) \in M \times [0,T]. \tag{5.7}$$

Proof. The proof is essentially given in [11, 30]. The only difference is that when deriving the lower bound of F, the inequality $\dot{F}^{ij} g_{ij} \geq 2$ which is due to the concavity of F, is used crucially in [11, 30]. In our case, the pinching estimate (3.9) implies a similar lower bound $\dot{F}^{ij} g_{ij} \geq 2/\beta$ given in (3.12). This is enough to derive the estimate (5.7). □

The estimate (5.7) on F and (3.10) on $\partial F/\partial \lambda_i, i = 1, 2$, give us the uniform upper and lower positive bounds on the principal curvatures λ_1, λ_2. Using the same argument as that in the proof of Proposition 4.4, we derive that

Proposition 5.5. *Under the assumption of Theorem 1.3, the flow (1.1) in $\mathbb{H}^3$ exists for all time, i.e., $T = \infty$.*

Up to now, we have proved that the curvature function F and the principal curvatures λ_1, λ_2 of M_t are bounded from above and below by uniform positive constants depending only on α and M_0 for all time $t \in [0, \infty)$.

Lemma 5.6. *Under the assumption of Theorem 1.3, the pinching ratio β of M_t tends to 1 exponentially as $t \to \infty$.*

Proof. From the proof of Theorem 3.1, we know that

$$\frac{d}{dt} \max_{M_t} G \leq - \frac{4F^{-\alpha}}{\lambda_1 + \lambda_2} \left(1 - \alpha + (1 + \alpha)\lambda_1 \lambda_2\right) \max_{M_t} G \;\leq\; -C \max_{M_t} G,$$

where we used the estimates on F and λ_1, λ_2 and therefore $C > 0$ is a positive constant depending only on α and M_0. Then

$$G \leq e^{-Ct} \max_{M_0} G \to 0, \quad \text{as } t \to \infty.$$

□

From Section 2.2, we know that for a graph $M = \text{graph}\, u$ over a geodesic sphere in $\mathbb{H}^3$, the induced metric on M is

$$g_{ij} = u_i u_j + \sinh^2 u \, \sigma_{ij}, \tag{5.8}$$

where σ_{ij} is the components of $g_{\mathbb{S}^2}$. The second fundamental form of M satisfies

$$h_{ij} v^{-1} = -u_{ij} + \sinh u \cosh u \, \sigma_{ij}, \tag{5.9}$$

where v is defined in (5.4) and the covariant derivatives u_{ij} are taken with respect to the induced metric of M. In Theorem 4.1 of [30], it was proved that there exist constants $\lambda > 0$, $c > 0$ depending on α, M_0 such that

$$v - 1 \leq c e^{-\lambda t}, \quad \forall\, t \in [0, \infty). \tag{5.10}$$

Note that the concavity of F is not required in the proof of (5.10). Recall that the support function χ of the graph $M = \text{graph } u$ in $\mathbb{H}^3$ is defined by

$$\chi = \bar{g}((\sinh u)\partial_r, \nu) = v^{-1}\sinh u.$$

In view of the estimate (5.10) and Lemma 5.1, along the flow (1.1) in $\mathbb{H}^3$, we have that

$$\sup_{M_t} |\log \chi - u + \log 2| = \sup_{M_t} |\log(2e^{-u}v^{-1}\sinh u)| \leq ce^{-\lambda t}. \tag{5.11}$$

By (2.15), the first order derivative of the support function χ satisfies $\chi_i = \bar{g}(\sinh u\, \partial_r, h_i^k \partial_k)$, so we have

$$\chi^{-1}|\chi_i| \leq \chi^{-1}(\sum_{i=1}^{2} \chi_i^2)^{\frac{1}{2}} \leq \lambda_2 \frac{(1 - \bar{g}(\partial_r, \nu)^2)^{1/2}}{\bar{g}(\partial_r, \nu)} = \lambda_2\sqrt{v^2 - 1} \leq ce^{-\lambda t} \tag{5.12}$$

on M_t, as the principal curvature λ_2 is uniformly bounded. We note that the c and λ in Eqs. (5.10), (5.11) and (5.12) are not the same, but all of them only depend on α and M_0.

Lemma 5.7. *Under the assumption of Theorem 1.3, the principal curvatures λ_1, λ_2 of M_t converge to 1 as $t \to \infty$.*

Proof. Recall that we have the following relation between F and λ_i (see (3.11)) which is due to the pinching estimate:

$$2\lambda_1 \leq F(\lambda_1, \lambda_2) \leq 2\lambda_2 \leq 2\beta\lambda_1.$$

Since the pinching ratio β of M_t tends to 1 as $t \to \infty$ by Lemma 5.6, in order to prove this lemma it suffices to prove

$$\sup_{M_t} |F(x,t) - 2| \to 0, \quad \text{as } t \to \infty. \tag{5.13}$$

(i). First, we prove

$$\liminf_{t \to \infty} \inf_{M_t} F(x,t) \geq 2. \tag{5.14}$$

The proof is similar to part (ii) of Lemma 4.3 in [30]. We define a function on M_t by

$$\omega = \log(-\Phi) - \log\chi + u + (\alpha - 1)\log 2, \tag{5.15}$$

which is slightly different from the one in part (ii) of Lemma 4.3 of [30]. In view of the estimate (5.11), in order to prove (5.14), it suffices to prove that

$$\limsup_{t \to \infty} \sup_{M_t} \omega \leq 0. \tag{5.16}$$

By using (2.5), (5.3) and (2.17), we obtain that

$$\begin{aligned}
\partial_t \omega =& \Phi^{-1}\partial_t\Phi - \chi^{-1}\partial_t\chi + \partial_t u \\
=& \dot{\Phi}\dot{F}^{ij}(\Phi^{-1}\nabla_i\nabla_j\Phi - \chi^{-1}\nabla_i\nabla_j\chi) - \dot{\Phi}\dot{F}^{ij}g_{ij} + (1-\alpha)\Phi\chi^{-1}\cosh u + vF^{-\alpha} \\
=& \dot{\Phi}\dot{F}^{ij}\omega_{ij} + \dot{\Phi}\dot{F}^{ij}\left(\omega_i\omega_j + 2\omega_i(\chi^{-1}\chi_j - u_j) - 2\chi^{-1}\chi_i u_j + u_i u_j - u_{ij}\right) \\
& - \dot{\Phi}\dot{F}^{ij}g_{ij} + (1-\alpha)\Phi\chi^{-1}\cosh u + vF^{-\alpha} \\
=& \dot{\Phi}\dot{F}^{ij}\omega_{ij} + \dot{\Phi}\dot{F}^{ij}\left(\omega_i\omega_j + 2\omega_i(\chi^{-1}\chi_j - u_j) - 2\chi^{-1}\chi_i u_j + (1+\coth u)u_i u_j\right) \\
& - (1+\coth u)\dot{\Phi}\dot{F}^{ij}g_{ij} + \dot{\Phi}\dot{F}^{ij}h_{ij}v^{-1} + (1-\alpha)\Phi\chi^{-1}\cosh u + vF^{-\alpha}, \tag{5.17}
\end{aligned}$$

where in the last equality we used the expressions (5.8) and (5.9). Note that all the covariant derivatives in the equation above are taken with respect to the induced metric on M_t. Denote $\tilde{\omega}(t) = \sup_{M_t} \omega = \omega(t, \theta_t)$, which is a Lipschitz function on $\mathbb{R}^+$. Then at the point (t, θ_t), $\dot{\Phi} \dot{F}^{ij} \omega_{ij} \leq 0$ and $\omega_i = 0$. Moreover, by using the estimates (5.10), (5.12) and noting that $|\nabla u(t)|^2_{g_t} = \sinh^{-2} u(t) v(t)^{-2} |Du(t)|^2_{g_{\mathbb{S}^2}}$, we have that

$$\dot{\Phi} \dot{F}^{ij} \left(-2\chi^{-1}\chi_i u_j + (1 + \coth u) u_i u_j \right) \to 0, \quad \text{as } t \to \infty.$$

Then from (5.17), we have that

$$\frac{d}{dt}\tilde{\omega}(t) \leq o(1) - 4\alpha\beta^{-1} F^{-\alpha-1}(t, \theta_t) + (2\alpha - 1 + v) F^{-\alpha}(t, \theta_t)$$

$$\leq o(1) + 2\alpha(F(t, \theta_t) - 2) F^{-\alpha-1}(t, \theta_t) \quad \text{as } t \to \infty, \tag{5.18}$$

where we used that $\dot{F}^{ij} g_{ij} \geq 2/\beta$ (see (3.12)), $\coth u \geq 1$, and the estimates $\beta, v \to 1$ as $t \to \infty$ which are obtained in (5.10) and Lemma 5.6.

We claim that: $\forall\, \epsilon > 0$, there exist $t_\epsilon > 0$ and $\delta_\epsilon > 0$ such that

$$A_\epsilon := \left\{ t \in [t_\epsilon, \infty) \cap D : \tilde{\omega}(t) > \epsilon \right\} \subset \left\{ t \in [t_\epsilon, \infty) \cap D : \frac{d}{dt}\tilde{\omega}(t) \leq -\delta_\epsilon \right\}, \tag{5.19}$$

where D is the set of points of differentiability of $\tilde{\omega}$.

Since we have already proved that F is bounded, there exists a constant c which depends on α and M_0 such that $2\alpha F^{-\alpha-1} \geq c(\alpha, M_0)$. For any $\epsilon > 0$, we define $\delta_\epsilon = c(1 - e^{-\frac{\epsilon}{2\alpha}}) > 0$, which depends on α, M_0 and ϵ. By the estimate (5.11) and (5.18), we can choose $t_\epsilon > 0$ sufficiently large such that

$$-\log\chi + u - \log 2 \;<\; \frac{\epsilon}{2}, \quad \frac{d}{dt}\tilde{\omega}(t) \leq \; \delta_\epsilon + 2\alpha(F(t, \theta_t) - 2) F^{-\alpha-1}(t, \theta_t), \; \forall\, t \in [t_\epsilon, \infty). \tag{5.20}$$

Then for any $t \in A_\epsilon$, $\log(-\Phi 2^\alpha)(t, \theta_t) > \epsilon/2$, which implies that $F(t, \theta_t) < 2e^{-\frac{\epsilon}{2\alpha}}$. Then in view of (5.20), we have

$$\frac{d}{dt}\tilde{\omega}(t) \leq \; \delta_\epsilon + c(2e^{-\frac{\epsilon}{2\alpha}} - 2) = -\delta_\epsilon.$$

This proves the claim (5.19). By an easy exercise of Calculus (cf. Lemma 4.2 in [30]), the estimate (5.16) follows from the claim (5.19).

(ii). Second, we prove the other direction

$$\limsup_{t\to\infty} \sup_{M_t} F(x, t) \leq 2. \tag{5.21}$$

We modify (5.15) and define a new function on M_t by

$$\phi = -\log(-\Phi) - \log\chi + u - (\alpha + 1)\log 2.$$

We aim to prove that

$$\limsup_{t\to\infty} \sup_{M_t} \phi \leq 0, \tag{5.22}$$

which is equivalent to (5.21) in view of the estimate (5.11). By a direct calculation, ϕ satisfies the following evolution equation

$$\partial_t \phi = \dot{\Phi}\dot{F}^{ij}\phi_{ij} - \dot{\Phi}\dot{F}^{ij}\left(\phi_i\phi_j + 2\phi_i(\chi^{-1}\chi_j - u_j) + 2\chi^{-2}\chi_i\chi_j - 2\chi^{-1}\chi_i u_j + (1 - \coth u)u_i u_j\right)$$

$$- 2\dot{\Phi}\dot{F}^{ij}h_i^k h_{kj} + (1 - \coth u)\dot{\Phi}\dot{F}^{ij}g_{ij} + \dot{\Phi}\dot{F}^{ij}h_{ij}v^{-1} + (1 - \alpha)\Phi\chi^{-1}\cosh u + vF^{-\alpha}.$$

Denote $\tilde{\phi}(t) = \sup_{M_t} \phi = \phi(t, \theta_t)$, which is also a Lipschitz function on $\mathbb{R}^+$. Then

$$\frac{d}{dt}\tilde{\phi}(t) \leq o(1) - \alpha\beta^{-1}F^{-\alpha+1}(t, \theta_t) + \alpha(v^{-1} + v)F^{-\alpha}(t, \theta_t)$$
$$\leq o(1) + \alpha(2 - F(t, \theta_t))F^{-\alpha}(t, \theta_t) \quad \text{as } t \to \infty,$$

where we used that $\dot{F}^{ij}h_i^k h_{kj} \geq F^2/2\beta$ (see (3.13)), $\coth u \geq 1$ and $\beta, v \to 1$ as $t \to \infty$ (see (5.10) and Lemma 5.6). Then the remaining proof follows after a similar argument to that in part (i). $\qquad\square$

Now we can follow the similar argument as that in [30] to complete the proof of Theorem 1.3. First, by the same argument as that in Theorem 4.4 of [30], we can obtain the optimal convergence rate of the principal curvatures, i.e.,

$$|h_i^j - \delta_i^j| \leq ce^{-2^{(1-\alpha)}\cdot t}, \quad \forall\, t > 0, \tag{5.23}$$

where c is a positive constant depending on α and M_0. The conclusion of Lemma 5.7 plays a crucial role in dealing with the bad terms involving the derivatives of the second fundamental form. Second, using the argument in [30, §5], we can also obtain uniform higher order derivative estimates. Finally, using the conformally flat parametrization as that in [30, §6], we obtain the convergence (1.3) of the defining function $u(t, \theta)$. We refer the reader to [30] for more details.

6. Flow in sphere

In this section, we consider the flow (1.1) in sphere $\mathbb{S}^3$ and prove Theorem 1.4. We assume that F satisfies Assumption 1.1 and $\alpha = 1$. Since M_0 is assumed to be strictly convex in $\mathbb{S}^3$, then M_0 is embedded, contained in an open hemisphere and is the boundary of a convex body in $\mathbb{S}^3$. Since the flow is a parabolic equation with strictly convex initial data, by short time existence theorem, the flow surfaces M_t exist on a maximal time interval $[0, T)$ for some $0 < T \leq \infty$.

Firstly, we show that T is finite and is characterized as the time when the velocity of the flow blows up. Recall the evolution equation (2.5) of Φ,

$$\frac{\partial}{\partial t}\Phi - \dot{\Phi}\dot{F}^{ij}\Phi_{ij} = \Phi\dot{\Phi}(\dot{F}^{ij}h_{ip}h_j^p + \dot{F}^{ij}g_{ij}). \tag{6.1}$$

As before, the pinching ratio estimate (3.9) implies that (cf. (3.12) and (3.13))

$$\dot{F}^{ij}h_{ip}h_j^p + \dot{F}^{ij}g_{ij} \geq \frac{1}{2\beta}F^2 + \frac{2}{\beta}. \tag{6.2}$$

Recall that $\Phi = -F^{-1}$ in this case, then we have

$$\frac{\partial}{\partial t}(F^{-1}) - \dot{\Phi}\dot{F}^{ij}(F^{-1})_{ij} \geq \frac{2}{\beta}F^{-3}. \tag{6.3}$$

The maximum principle then implies that F^{-1} will blow up in finite time and thus $T < \infty$.

From (6.3), we can also see that F^{-1} is bounded from below by its initial value, and therefore F is uniformly bounded above. By the pinching ratio estimate, we conclude that the principal curvatures λ_1, λ_2 are also uniformly bounded above. Then as in previous two sections, we can derive C^0, C^1 and C^2 estimates of the flow surfaces. Moreover, arguing as [23, §5-6] and using the pinching ratio estimate (3.9) and Andrews' [3] second order

derivative Hölder estimates and parabolic Schauder estimates [22], we can conclude that $\liminf_{t \to T} \inf_{M_t} F \to 0$ as $t \to T$, $M_t \to M_T$ in $C^{1,\gamma}, 0 < \gamma < 1$ as $t \to T$ and that there exists a point $x_0 \in \mathbb{S}^3$ such that the limit surface M_T is the equator $S(x_0) = \partial \mathcal{H}(x_0)$. More precisely, we have

Proposition 6.1. *Under the assumption of Theorem 1.4, the flow surfaces M_t converge to the equator $S(x_0) = \partial \mathcal{H}(x_0)$ in $C^{1,\gamma}, 0 < \gamma < 1$ as $t \to T$.*

To show the smooth convergence, we employ the dual flow method introduced by Gerhardt in [13]. Let $M_0 \subset \mathbb{S}^{n+1}$ be a closed strictly convex hypersurface given by the immersion

$$X : M \to M_0 \subset \mathbb{S}^{n+1}. \tag{6.4}$$

Then M_0 is embedded and homeomorphic to $\mathbb{S}^n$, contained in an open hemisphere and is the boundary of a convex body $\Omega_0 \subset \mathbb{S}^{n+1}$. M_0 can also be viewed as a submanifold of codimension 2 in $\mathbb{R}^{n+2}$. The Gauss formula then is

$$X_{ij} = -g_{ij}X - h_{ij}\tilde{X},$$

where X_{ij} denotes the covariant derivatives of X, g_{ij} is the induced metric, h_{ij} the second fundamental form of M_0 considered as a hypersurface of $\mathbb{S}^{n+1}$, and $\tilde{X}$ represents the exterior normal vector of M_0 in $\mathbb{S}^{n+1}$. Then the map $\tilde{X} : M \to \mathbb{S}^{n+1}$ is an embedding of a closed strictly convex hypersurface $\tilde{M}_0 = \tilde{X}(M)$, which is called the Gauss map of M_0.

Viewing $\tilde{M}_0$ as a codimension 2 submanifold in $\mathbb{R}^{n+2}$, its Gauss formula is

$$\tilde{X}_{ij} = -\tilde{g}_{ij}\tilde{X} - \tilde{h}_{ij}X,$$

where X is the embedding (6.4) which represents the exterior normal vector of $\tilde{M}_0$. The second fundamental forms of $M_0, \tilde{M}_0$ and the corresponding principal curvatures $\lambda_i, \tilde{\lambda}_i$ satisfy $h_{ij} = \tilde{h}_{ij} = \langle \tilde{X}_i, X_j \rangle$ and $\tilde{\lambda}_i = \lambda_i^{-1}$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean metric of $\mathbb{R}^{n+2}$. We have the following result from [13] which does not need the concavity of F.

Theorem 6.2 ([13]). *Let $\Phi \in C^\infty(\mathbb{R}_+)$ be strictly monotone and F be the curvature function satisfying Assumption 1.1. Assume that $M_0 \subset \mathbb{S}^{n+1}$ is a closed and strictly convex hypersurface and $\tilde{M}_0$ is its dual hypersurface. We have that the dual flows*

$$\frac{\partial}{\partial t}X = -\Phi(F)\nu \tag{6.5}$$

and

$$\frac{\partial}{\partial t}\tilde{X} = \Phi(\tilde{F}^{-1})\tilde{\nu} \tag{6.6}$$

with initial data M_0 and $\tilde{M}_0$ resp. exist on maximal time intervals $[0, T)$ resp. $[0, T^*)$, where the flow hypersurfaces are strictly convex. Moreover, $T = T^*$ and the corresponding flow hypersurfaces M_t and $\tilde{M}_t$ are polar sets of each other.

Since $\Phi(F) = -F^{-1}$, we have $\Phi(\tilde{F}^{-1}) = -\tilde{F}$. Then the flow (6.6) corresponds to the contracting curvature flow in sphere with speed $\tilde{F}$. McCoy [24] proved the convergence result for such contracting curvature flow in sphere without needing concavity of the speed function: Let $X : M \times [0, T) \to \mathbb{S}^3$ be a solution to the contracting curvature flow

$$\frac{\partial}{\partial t}X = -F\nu, \tag{6.7}$$

with speed $F = F(\lambda_1, \lambda_2)$ satisfying Assumption 1.1. Using the argument in [13, §6–7] and the $C^{2,\alpha}$ estimate without needing concavity of the speed function by Andrews [3], McCoy [24] first proved that the solution M_t of the flow (6.7) contracts any strictly convex surface to a point $p \in \mathbb{S}^3$ as $t \to T$. Let $\Theta = \Theta(t, T)$ denote the radius of the shrinking spheres along the flow (6.7) with extinction time T, i.e., $\Theta(t, T) = \arccos e^{2(t-T)}$ which satisfies

$$\frac{d}{dt}\Theta(t, T) = -2\cot\Theta(t, T), \quad \Theta(T, T) = 0.$$

Then it has been proved in [24] that the rescaled principal curvatures satisfy

$$0 < C^{-1} \leq \lambda_i \Theta \leq C, \qquad i = 1, 2 \tag{6.8}$$

for some constant $C > 0$ and the rescaled polar graph function $\tilde{u}(\cdot, \tau) = u/\Theta$ has uniform C^m regularity for all $m \geq 2$, where u is the polar graph function of the solution M_t with respect to the point p and the rescaled time parameter is $\tau = -\ln\Theta$. Note that the rescaled polar graph function $\tilde{u}$ satisfies

$$\frac{\partial}{\partial \tau}\tilde{u} = -\frac{1}{2}vF\tan e^{-\tau} + \tilde{u}, \tag{6.9}$$

where $F = F(h_i^j)$ is the function evaluated at the Weingarten matrix of the unrescaled solution $M_t = \mathrm{graph}(u)$ and $v = \sqrt{1 + \sin^{-2} u |Du|_{g_{\mathbb{S}^2}}^2}$ is given in (2.9). Moreover, $\tilde{u}(\cdot, \tau)$ converges to 1 smoothly as $\tau \to \infty$. Since the detail is not given in [24] and the convergence will be used crucially in our dual expanding curvature flow, for the readers' convenience we describe the convergence of the flow (6.7) in the following by using the rescaling process in [2, §5-6].

For any constant $A > 0$, we define $X^{(A)} : M \times [0, A^2 T) \to (\mathbb{S}^3, A^2 g_{\mathbb{S}^3})$ by $X^{(A)}(x, t) = X(x, t/A^2)$. Then the principal curvatures $\lambda_i^{(A)}$ and the unit normal vector field on $M_t^{(A)} = X^{(A)}(M, t)$ satisfy $\lambda_i^{(A)} = \lambda_i/A$ and $\nu^{(A)} = \nu/A$. Therefore, $X^{(A)}$ is also a solution to the equation (6.7) in $(\mathbb{S}^3, A^2 g_{\mathbb{S}^3})$. We consider a sequence of times $\{t_k\}$ approaching to the maximal existence time T of X, such that

$$\sup_{M \times [0, t_k)} |\mathcal{W}|(x, t) = |\mathcal{W}|(x_k, t_k)$$

for some point $x_k \in M_{t_k}$. For each k, we rescale the flow (6.7) with respect to the parameter $A = A_k = |\mathcal{W}|(x_k, t_k)$ and define a sequence of flows $X^k(x, t) = X^{(A_k)}(x, t) = X(x, t_k + t/A_k^2)$, where $t \in [0, A_k^2(T - t_k))$. Without loss of generality, we can assume that k is large enough such that t_k is sufficiently close to T. Since $\Theta(t_k, T) = \arccos e^{2(t_k - T)}$, we have $\Theta(t_k, T) \approx 2\sqrt{(T - t_k)}$ when k is sufficiently large. (6.8) implies that $0 < C^{-1} \leq A_k^2(T - t_k) \approx \lambda_i^2 \Theta^2 \leq C$ for another constant $C > 0$. Therefore, there exists a constant $\delta > 0$ such that $X^k(x, t)$ exists on the interval $t \in [0, \delta]$ for all k.

For each large k, we choose an isometry from $\mathbb{R}^3$ to $T_{x_k}\mathbb{S}^3$ so that we can identify the tangent spaces of $\mathbb{S}^3$ at each x_k. Using the exponential maps from $T_{x_k}\mathbb{S}^3$ to $\mathbb{S}^3$, we obtain a family of surfaces $\check{M}_t^k$ in $\mathbb{R}^3$ which corresponds to the family $M_t^k = X^k(M, t) \subset \mathbb{S}^3$. The regularity estimate of M_t^k gives the corresponding regularity estimate of $\check{M}_t^k$. From (6.8) we know that $uA_k \approx u/\Theta$, so we also have that $\check{M}_t^k$ are contained in a bounded region of $\mathbb{R}^3$. Since $A_k = |\mathcal{W}|(x_k, t_k)$ goes to infinity as $k \to \infty$, the metric on $\mathbb{R}^3$ induced by the exponential maps from $T_{x_k}\mathbb{S}^3$ to $(\mathbb{S}^3, A_k^2 g_{\mathbb{S}^3})$ will converge to a flat metric as $k \to \infty$.

We can find a subsequence $\{k'\}$ of $\{k\}$ such that $\check{M}_t^{k'}$ converges to a limiting family of complete surfaces $\check{M}_t^\infty$ in $\mathbb{R}^3$ as $k' \to \infty$. Since each surface $\check{M}_t^\infty$ has pinched principal curvatures, a theorem by Hamilton [14] implies that $\check{M}_t^\infty$ is compact. Moreover, $\check{M}_t^\infty$ is a smooth solution of the flow (6.7) in $\mathbb{R}^3$ for $t \in [0, \delta]$ with $\delta > 0$.

The limit solution $\check{M}_t^\infty$ must be shrinking spheres in $\mathbb{R}^3$. This follows from the evolution equation of the following quantity

$$G = \frac{(\lambda_2 - \lambda_1)^2}{(\lambda_1 + \lambda_2)^2}.$$

In [5], Andrews calculated that along the flow (6.7) in $\mathbb{R}^3$, G satisfies

$$\frac{\partial}{\partial t} G = \dot{F}^{kl} \nabla_k \nabla_l G + Q(\nabla h, \nabla h), \tag{6.10}$$

where $Q(\nabla h, \nabla h)$ are quadratic terms involving the first derivatives of the second fundamental form. At the maximum point of G where G is non-zero (otherwise, $\check{M}_t^\infty$ is a sphere and the proof is trivial), the gradient terms $Q(\nabla h, \nabla h)$ in (6.10) satisfy

$$Q(\nabla h, \nabla h) = -\frac{8F}{H^3} \left((\nabla_1 h_{22})^2 + (\nabla_2 h_{11})^2 \right) \tag{6.11}$$

and are non-positive. The strong maximum principle implies that the maximum of G is strictly decreasing along the limit solution unless G is a constant in both space and time. Since the quantity G is a scaling-invariant, the maximum of G can not be strictly decreasing on the limit solution. Hence, the value of G on the limit solution $\check{M}_t^\infty$ must be a constant, independent of space and time. In fact, this constant corresponds to the limit of $\max_{M_{t_k}} G$ as $t_k \to T$ along the unrescaled flow (6.7) in $\mathbb{S}^3$. Moreover, this constant must be zero: If this constant is not zero, then (6.11) implies that $\nabla_1 h_{22} \equiv 0$ and $\nabla_2 h_{11} \equiv 0$. The Codazzi equations imply that $\nabla_1 h_{12} \equiv 0$ and $\nabla_2 h_{12} \equiv 0$. The fact $\nabla G \equiv 0$ then implies that $\nabla_i h_{jk} \equiv 0$ for all i, j, k. Therefore, $\check{M}_t^\infty$ are shrinking spheres. This implies that there exists a subsequence of time t_k such that the maximum of G on M_{t_k} along the flow (6.7) in $\mathbb{S}^3$ converges to zero as $t_k \to T$. Since the maximum of G is monotone decreasing along the flow (6.7), we conclude that $\max_{M_t} G$ converges to zero as $t \to T$. That is, the solution M_t of the flow (6.7) converges to a round point as $t \to T$. The stronger exponentially smooth convergence of the rescaled equation (6.9) can be deduced by considering the linearization of the flow about the shrinking sphere solution.

Now we go back to the expanding curvature flow (6.5). As was shown in Proposition 6.1, the solution M_t of the flow (1.1) will expand to the equator $\partial \mathcal{H}(x_0)$ as $t \to T$. There exists a time $\bar{t} < T$ such that M_t can be written as a polar graph of a function u with respect to the geodesic polar coordinates centered at x_0 for all $t \in [\bar{t}, T)$. The dual contracting curvature flow can be written as graphs of a function u^* with respect to the geodesic polar coordinates centered at $-x_0$ for all $t \in [\bar{t}, T)$. By Lemma 9.1 of [13], the functions u, u^* satisfy the relations

$$u_{\max} = \frac{\pi}{2} - u^*_{\min}, \qquad u_{\min} = \frac{\pi}{2} - u^*_{\max}, \qquad \forall\, t \in [\bar{t}, T).$$

Since the rescaled function $u^* \Theta^{-1} \to 1$ as $t \to T$, we have $\omega = (\frac{\pi}{2} - u)\Theta^{-1} \to 1$ as $t \to T$. The C^1 estimate of ω follows from Lemma 9.3 of [13]. The dual relation and the C^2 estimate of the contracting flow give that $\lambda_i \Theta^{-1} \to 1$ as $t \to T$, which then leads to the

C^2 estimate of ω. Since ω satisfies

$$\frac{\partial}{\partial \tau}\omega = -\frac{1}{2}vF^{-1}\tan(e^{-\tau}) - \omega \tag{6.12}$$

with respect to the new time parameter $\tau = -\ln\Theta(t,T)$, it is a direct calculation to check that (6.12) is a uniformly parabolic equation of ω. Since we already have C^0, C^1 and C^2 estimates of ω, the regularity estimate of Andrews [3] gives the $C^{2,\alpha}$ estimate of ω and the Schauder theory gives the $C^{m,\alpha}$ estimate of ω for all $m \geq 2$. Therefore, applying the interpolation inequalities (cf. Lemma 6.1 of [11]) for the C^m norms of ω, we have that ω converges in C^∞ to 1 exponentially. This completes the proof of Theorem 1.4.

7. Proof of Theorem 1.5

We define the quantity for a smooth surface M in $\mathbb{H}^3$ by

$$Q(M) = -|M|\int_M |\mathring{A}|^2 d\mu,$$

where $\mathring{A}$ is the trace-less part of the second fundamental form of M. The following proposition characterizes when the limit of rescaled metric is a round metric in terms of the value of Q.

Proposition 7.1 ([18]). *Let $\tilde{M}_s$ be a family of surfaces in $\mathbb{H}^3$ that are radial graphs of the function $u(s,\theta) = cs + f(\theta) + o(1)$ and g_{ij} be the induced metric on $\tilde{M}_s$. Then the limit of the rescaled metric $e^{-2cs}g_{ij}$ as $s \to \infty$ is a round metric if and only if $\lim_{s\to\infty} Q(\tilde{M}_s) = 0$.*

Given an initial star-shaped and mean convex surface M_0 in $\mathbb{H}^3$, we consider the smooth solution M_t of the following flow with $0 < \alpha < 1$:

$$\begin{cases} \dfrac{\partial}{\partial t}X(x,t) = H^{-\alpha}(x,t)\nu(x,t), \\ X(\cdot,0) = X_0(\cdot). \end{cases} \tag{7.1}$$

Lemma 7.2. *Suppose that M_t is a smooth solution of the flow (7.1). We have that*

$$\frac{d}{dt}Q(M_t) = |M_t|\int_{M_t} H^{1-\alpha}|\mathring{A}|^2 d\mu_t - \int_{M_t} H^{1-\alpha}\int_{M_t}|\mathring{A}|^2 d\mu_t$$
$$+ \alpha|M_t|\int_{M_t} H^{-1-\alpha}|\nabla H|^2 d\mu_t. \tag{7.2}$$

Proof. Along the flow (7.1), it follows from (2.4) and (2.6) that the second fundamental form h_i^j of M_t evolves by

$$\frac{\partial}{\partial t}h_i^j = -\nabla^j\nabla_i H^{-\alpha} - H^{-\alpha}h_i^k h_k^j + H^{-\alpha}\delta_i^j$$

$$= -\nabla^j\nabla_i H^{-\alpha} - H^{-\alpha}\mathring{h}_i^k \mathring{h}_k^j - H^{1-\alpha}\mathring{h}_i^j + H^{-\alpha}(1 - \frac{H^2}{4})\delta_i^j,$$

where $\mathring{h}_i^j = h_i^j - \frac{H}{2}\delta_i^j$ is the trace-less part of the second fundamental form. Then we have that

$$\frac{\partial}{\partial t}\mathring{h}_i^j = -\nabla^j\nabla_i H^{-\alpha} - H^{-\alpha}\mathring{h}_i^k \mathring{h}_k^j - H^{1-\alpha}\mathring{h}_i^j + H^{-\alpha}(1 - \frac{H^2}{4})\delta_i^j - \frac{1}{2}\partial_t H\delta_i^j.$$

Hence, we obtain that

$$\frac{\partial}{\partial t}|\mathring{A}|^2 = -2\mathring{h}_j^i \nabla^j \nabla_i H^{-\alpha} - 2H^{1-\alpha}|\mathring{A}|^2, \tag{7.3}$$

where we used the facts $\mathring{h}_i^k \mathring{h}_k^j \mathring{h}_j^i = 0$ and $\mathring{h}_j^i \delta_i^j = 0$. Since the area form $d\mu_t$ evolves by

$$\frac{\partial}{\partial t} d\mu_t = H^{1-\alpha} d\mu_t, \tag{7.4}$$

we have

$$\begin{aligned}
\frac{d}{dt} \int_{M_t} |\mathring{A}|^2 d\mu_t &= -2 \int_{M_t} \mathring{h}_j^i \nabla^j \nabla_i H^{-\alpha} d\mu_t - \int_{M_t} H^{1-\alpha}|\mathring{A}|^2 d\mu_t \\
&= 2 \int_{M_t} \nabla^j \mathring{h}_j^i \nabla_i H^{-\alpha} d\mu_t - \int_{M_t} H^{1-\alpha}|\mathring{A}|^2 d\mu_t \\
&= -\alpha \int_{M_t} H^{-1-\alpha}|\nabla H|^2 d\mu_t - \int_{M_t} H^{1-\alpha}|\mathring{A}|^2 d\mu_t, \tag{7.5}
\end{aligned}$$

where in the last equality we used the Codazzi equation to obtain

$$\nabla^j \mathring{h}_j^i = \nabla^j h_j^i - \frac{1}{2}\nabla^i H = \frac{1}{2}\nabla^i H.$$

The evolution equation (7.2) follows by combining (7.5) and (7.4). $\qquad\square$

We now describe the procedure of constructing the example in Theorem 1.5. Choose a function $\bar{f}(\theta)$ on $\mathbb{S}^2$ such that

$$\int_{\mathbb{S}^2} e^{2\bar{f}} d\mu_{g_{\mathbb{S}^2}} \int_{\mathbb{S}^2} |\mathring{D}^2 e^{-\bar{f}}|^2_{g_{\mathbb{S}^2}} d\mu_{g_{\mathbb{S}^2}} = c_0 > 0,$$

where $\mathring{D}^2 e^{-\bar{f}}$ means the traceless part of the Hessian of $e^{-\bar{f}}$, and $g_{\mathbb{S}^2}$ means the round metric on $\mathbb{S}^2$. Recall that the metric on $\mathbb{H}^3$ can be expressed as

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^2}$$

for the (r,θ) coordinates centered at some point $x_0 \in \mathbb{H}^3$. Let $\tilde{M}_s$ be the family of surfaces given by the radial graph of $u(s,\theta) = s + \bar{f}(\theta)$ over $\mathbb{S}^2$ in the (r,θ) coordinates of $\mathbb{H}^3$. We know from [18] that

$$\lim_{s\to\infty} Q(\tilde{M}_s) = -c_0. \tag{7.6}$$

Proposition 7.3 ([25]). *There exists a constant C_0 depending only on the bound*

$$E := \sup_{\mathbb{S}^2} \sum_{i=0}^3 |\nabla^i \bar{f}|_{g_{\mathbb{S}^2}}$$

such that for all $s \geq 1$, the mean curvature H and the second fundamental form A of $\tilde{M}_s$ satisfy that

$$|\tilde{M}_s||H - 2| + |\tilde{M}_s||\mathring{A}| + |\tilde{M}_s|^3|\nabla A|^2 \leq C_0. \tag{7.7}$$

By choosing $s_0 > 0$ large enough and using the estimate (7.7), we can make sure that there exists a constant $\epsilon_0 > 0$ such that for any $s \geq s_0$, we have

$$3 \geq H \geq \epsilon_0, \quad \bar{g}(\partial_r, \nu) \geq \epsilon_0, \quad |\mathring{A}|^2 < \frac{1}{4}H^2. \tag{7.8}$$

Then $\tilde{M}_s$ is mean-convex and star-shaped for any $s \geq s_0$. Denote the in-radius and out-radius of the initial surface $\tilde{M}_s = \text{graph } u(s, \theta)(= s + \bar{f}(\theta))$ in the (r, θ) coordinates of $\mathbb{H}^3$ by $\underline{r}_0$ and $\bar{r}_0$, i.e., $\underline{r}_0 = \inf u(s, \theta)$, $\bar{r}_0 = \sup u(s, \theta)$. Clearly, $\bar{r}_0 - \underline{r}_0 = \text{osc}(\bar{f})$ is independent of s. We consider the solution M_t^s of the flow (7.1) starting from $\tilde{M}_s$, where t is the time parameter.

Proposition 7.4. *There exist constants* $s_0 = s_0(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ *and* $C = C(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ *depending only on* $\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0$, *and* E *such that for any* $s \geq s_0$, *we have*

$$|\tilde{M}_s|^2|H - 2|^2 + |\tilde{M}_s|^2|\mathring{A}|^2 \leq Ce^{-2^{(2-\alpha)} \cdot t}$$

$$|\tilde{M}_s|^3|\nabla A|^2 \leq Ce^{-3 \cdot 2^{(1-\alpha)} \cdot t} \tag{7.9}$$

on the solution M_t^s *of the flow* (7.1) *starting from* $\tilde{M}_s$.

The key point of the estimates (7.9) is that the constant C is independent of the parameter s. The proof of (7.9) is technical and will be given in the next subsection.

We now complete the proof of Theorem 1.5 using the estimate (7.9). By (7.2), we have

$$\frac{d}{dt}Q(M_t^s) = |M_t^s| \int_{M_t^s} |\mathring{A}|^2 \left(H^{1-\alpha} - \frac{1}{|M_t^s|} \int_{M_t^s} H^{1-\alpha}\right) d\mu_t$$

$$+ \alpha|M_t^s| \int_{M_t^s} H^{-1-\alpha}|\nabla H|^2 d\mu_t.$$

Since the volume element evolves by (7.4), then using the estimate (7.9) we have

$$|M_t^s| \leq \tilde{C}|\tilde{M}_s|e^{2^{(1-\alpha)} \cdot t}$$

and

$$\frac{d}{dt}Q(M_t^s) \leq \tilde{C}|\tilde{M}_s|^{-1}e^{-2^{(1-\alpha)} \cdot t}, \tag{7.10}$$

where $\tilde{C}$ is a constant independent of s. In view of (7.6), we can choose s_0 large enough such that for any $s \geq s_0$,

$$Q(\tilde{M}_s) < -\frac{c_0}{2}, \quad \tilde{C}|\tilde{M}_s|^{-1} \leq 2^{-1-\alpha}c_0.$$

Integrating (7.10) gives that

$$\lim_{t\to\infty} Q(M_t^s) \leq -\frac{c_0}{4} < 0. \tag{7.11}$$

By the convergence result of Theorem 1.2 in [30], the solution M_t^s exists for all time and is given by the radial graph of the function

$$u(t, \theta) = \frac{t}{2\alpha} + f(\theta) + o(1), \quad \text{as } t \to \infty,$$

where $f(\theta)$ is a smooth function on $\mathbb{S}^2$. Then by combining (7.11) and Proposition 7.1, we obtain that the limit of the rescaled metric $e^{-2^{(1-\alpha)} \cdot t}g_t$ is not a round metric on $\mathbb{S}^2$. Note that

$$\lim_{t\to\infty} e^{-2^{(1-\alpha)} \cdot t}g_t = \lim_{t\to\infty} e^{-2^{(1-\alpha)} \cdot t}\sinh^2 u(t, \theta)g_{\mathbb{S}^2} = \frac{1}{4}e^{2f(\theta)}g_{\mathbb{S}^2}.$$

The fact that $e^{2f(\theta)}g_{\mathbb{S}^2}$ is not a round metric implies that $e^{-f(\theta)}$ is not a linear combination of constants and first eigenfunctions of $\mathbb{S}^2$ (cf. Lemma 4 in [18]).

7.1. **Proof of Proposition 7.4.** For the simplicity of the notations, in this subsection, we write $\tilde{M}_s$ as M_0, and M_t^s as M_t. First, we have the following evolution equations for g, χ, H and A, which follow from direct calculation using (2.4), (2.6) and (2.17).

Lemma 7.5. *Along the flow (7.1), we have*

(1) *The induced metric on M_t evolves by*

$$\partial_t g_{ij} = 2H^{-\alpha} h_{ij}. \tag{7.12}$$

(2) *The support function $\chi = \bar{g}(\sinh u \partial_r, \nu)$ satisfies*

$$\partial_t \chi = \alpha H^{-\alpha-1} \Delta \chi + \alpha H^{-\alpha-1} |A|^2 \chi + (1-\alpha) H^{-\alpha} \cosh u. \tag{7.13}$$

(3) *The mean curvature H satisfies*

$$\begin{aligned} \partial_t H =& \alpha H^{-\alpha-1} \Delta H - \alpha(\alpha+1) H^{-\alpha-2} |\nabla H|^2 - |A|^2 H^{-\alpha} + 2H^{-\alpha} \\ =& -\Delta H^{-\alpha} - |A|^2 H^{-\alpha} + 2H^{-\alpha}. \end{aligned} \tag{7.14}$$

(4) *The second fundamental form $A = (h_{ij})$ satisfies*

$$\begin{aligned} \partial_t h_{ij} =& \alpha H^{-\alpha-1} \Delta h_{ij} - \alpha(\alpha+1) H^{-\alpha-2} \nabla_i H \nabla_j H + \alpha H^{-\alpha-1} |A|^2 h_{ij} \\ &+ (1-\alpha) H^{-\alpha} h_i^k h_{kj} + (1-\alpha) H^{-\alpha} g_{ij} + 2\alpha H^{-\alpha-1} h_{ij}, \end{aligned} \tag{7.15}$$

$$\begin{aligned} \partial_t |A|^2 =& \alpha H^{-\alpha-1} \Delta |A|^2 - 2\alpha H^{-\alpha-1} |\nabla A|^2 - 2\alpha(\alpha+1) H^{-\alpha-2} h_{ij} \nabla^i H \nabla^j H \\ &+ 2\alpha H^{-\alpha-1} |A|^4 - 2(1+\alpha) H^{-\alpha} h_i^k h_k^j h_j^i \\ &+ 2(1-\alpha) H^{-\alpha+1} + 4\alpha H^{-\alpha-1} |A|^2. \end{aligned} \tag{7.16}$$

Lemma 7.6. *Let $\tilde{\chi} = e^{-\frac{t}{2\alpha}} \chi$. Then for each $t \geq 0$, we have*

$$\frac{1}{2} e^{\bar{r}_0} \geq \tilde{\chi}(x,t) \geq \min \tilde{\chi}(\cdot, 0) \geq \epsilon_0 \sinh \underline{r}_0 \tag{7.17}$$

on the solution M_t of (7.1).

Proof. First, since $\alpha \in (0,1)$, we know from (7.13) that $\tilde{\chi} = e^{-\frac{t}{2\alpha}} \chi > 0$ is preserved along the flow. By using (7.13), we have that

$$\partial_t \tilde{\chi} = \alpha H^{-\alpha-1} \Delta \tilde{\chi} + \alpha H^{-\alpha-1} |A|^2 \tilde{\chi} + (1-\alpha) H^{-\alpha} e^{-\frac{t}{2\alpha}} \cosh u - \frac{1}{2\alpha} \tilde{\chi}. \tag{7.18}$$

Since

$$\cosh u \geq \sinh u \geq \chi, \quad |A|^2 \geq H^2/2, \quad \alpha \in (0,1),$$

(7.18) implies that

$$\partial_t \tilde{\chi} \geq \alpha H^{-\alpha-1} \Delta \tilde{\chi} + \tilde{\chi} \left(\frac{\alpha}{2} H^{-\alpha+1} + (1-\alpha) H^{-\alpha} - \frac{1}{2\alpha} \right). \tag{7.19}$$

We claim that the terms in the bracket of (7.19) are always nonnegative for $\forall H > 0, \forall \alpha \in (0,1)$. In fact, for each fixed $\alpha \in (0,1)$, the minimum of

$$\frac{\alpha}{2} H^{-\alpha+1} + (1-\alpha) H^{-\alpha} - \frac{1}{2\alpha}$$

is achieved at $H = 2$. Therefore,

$$\frac{\alpha}{2} H^{-\alpha+1} + (1-\alpha) H^{-\alpha} - \frac{1}{2\alpha} \geq \frac{\alpha}{2} \cdot 2^{-\alpha+1} + (1-\alpha) 2^{-\alpha} - \frac{1}{2\alpha} = 0.$$

Since $\tilde{\chi}$ is always positive, (7.19) implies that

$$\tilde{\chi}(x,t) \geq \min \tilde{\chi}(\cdot,0) \geq \epsilon_0 \sinh \underline{r}_0.$$

On the other hand, by using (5.2) and (5.5), we know that $u(t) \leq \rho(t,\bar{r}_0) \leq \bar{r}_0 + \frac{t}{2\alpha}$, so we obtain that

$$\tilde{\chi}(x,t) = e^{-\frac{t}{2\alpha}} \sinh u(t) v^{-1} \leq \frac{1}{2} e^{u(t)-\frac{t}{2\alpha}} v^{-1} \leq \frac{e^{\bar{r}_0}}{2}.$$

$\square$

Lemma 7.7. *The mean curvature H of M_t is bounded from below by a positive constant $C_1 = C_1(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$ depending only on α, ϵ_0, and $\bar{r}_0 - \underline{r}_0$.*

Proof. Let $\zeta_t(x) = \tilde{\chi}^{-1}(x,t) H^{-\alpha}(x,t)$, where $\tilde{\chi} = e^{-\frac{t}{2\alpha}} \chi$ is defined in Lemma 7.6. Combining (7.14) and (7.18) yields

$$\partial_t \zeta_t = \alpha H^{-\alpha-1} \Delta \zeta_t + 2\alpha H^{-\alpha-1} \nabla \zeta_t \cdot \nabla \ln \tilde{\chi}$$

$$+ (\alpha - 1)\zeta_t^2 e^{-\frac{t}{2\alpha}} \cosh u - 2\alpha H^{-\alpha-1}\zeta_t + \frac{1}{2\alpha}\zeta_t$$

$$\leq \alpha H^{-\alpha-1} \Delta \zeta_t + 2\alpha H^{-\alpha-1} \nabla \ln \tilde{\chi} \cdot \nabla \zeta_t + \left(-2\alpha \zeta_t^{\frac{\alpha+1}{\alpha}} \tilde{\chi}^{\frac{\alpha+1}{\alpha}} + \frac{1}{2\alpha}\right)\zeta_t,$$

where we used the assumption $0 < \alpha < 1$. Then applying the maximum principle to the differential inequality above, we conclude that

$$\zeta_t(x) \leq \max\left\{\max_{M_0} \zeta_0, 2^{-\alpha}\alpha^{-\frac{\alpha}{\alpha+1}}(\min\tilde{\chi})^{-1}\right\}$$

$$\leq (\min\tilde{\chi}(\cdot,0))^{-1} \max\left\{\epsilon_0^{-\alpha}, 2^{-\alpha}\alpha^{-\frac{\alpha}{\alpha+1}}\right\},$$

where ϵ_0 is the same constant with that in (7.8). By the definition of ζ_t and the estimate in Lemma 7.6, we have that

$$H(x,t) \geq \left(\frac{\max\tilde{\chi}(\cdot,t)}{\min\tilde{\chi}(\cdot,0)}\right)^{-\frac{1}{\alpha}} \left(\max\{\epsilon_0^{-\alpha}, 2^{-\alpha}\alpha^{-\frac{\alpha}{\alpha+1}}\}\right)^{-\frac{1}{\alpha}}$$

$$\geq \epsilon_0^{\frac{1}{\alpha}} e^{-\frac{1}{\alpha}(\bar{r}_0 - \underline{r}_0)} \left(1 - e^{-2\underline{r}_0}\right)^{\frac{1}{\alpha}} \cdot \min\{\epsilon_0, 2\alpha^{\frac{1}{\alpha+1}}\}$$

$$\geq 2^{-\frac{1}{\alpha}} \epsilon_0^{\frac{1}{\alpha}} e^{-\frac{1}{\alpha}(\bar{r}_0 - \underline{r}_0)} \cdot \min\{\epsilon_0, 2\alpha^{\frac{1}{\alpha+1}}\}, \tag{7.20}$$

since s_0 can be chosen large enough in (7.8) and then $1 - e^{-2\underline{r}_0}$ will be greater than $1/2$. Then the lemma follows by defining C_1 as the right hand side of (7.20). $\square$

In the following, we prove Proposition 7.4 by proving Lemma 7.8 and Lemma 7.9 (Note that in the proof of Lemma 7.8, we obtain that H is bounded from below and above by positive constants which do not depend on s; hence, $|H - 2|$ and $|H^2 - 4|$ have the same decay rates). We note that in the proof of Lemma 7.8 and Lemma 7.9, we always choose s_0 large enough such that $|M_0| > 1$.

Lemma 7.8. *There exist constants $s_0 = s_0(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ and $C_2 = C_2(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ such that for any $s \geq s_0$,*

$$|M_0|^2 |\mathring{A}|^2(x,t) \leq C_2 e^{-2(2-\alpha)\cdot t}, \tag{7.21}$$

$$|M_0||H^2(x,t) - 4| \leq C_2 e^{-2(1-\alpha)\cdot t} \tag{7.22}$$

on M_t along the flow (7.1).

Proof. We define $\gamma_t = |\mathring{A}|^2 H^{-2}$, then $\gamma_t = |A|^2 H^{-2} - \frac{1}{2}$. Combining (7.14) and (7.16), we have

$$
\begin{aligned}
\partial_t(|A|^2 H^{-2}) =& \alpha H^{-\alpha-1}\Delta(|A|^2 H^{-2}) + 2(\alpha+1)H^{1-\alpha}(|A|^2 H^{-2})^2 \\
&+ 4(\alpha-1)H^{-\alpha-1}(|A|^2 H^{-2}) + 2(1-\alpha)H^{-\alpha-1} \\
&- 2(\alpha+1)H^{-\alpha-2}h_i^k h_k^j h_j^i + Q,
\end{aligned}
\tag{7.23}
$$

where Q is the gradient terms

$$
\begin{aligned}
Q =& 4\alpha H^{-\alpha-4}\nabla|A|^2 \cdot \nabla H - 2\alpha H^{-\alpha-3}|\nabla A|^2 \\
&- 2\alpha(\alpha+1)H^{-\alpha-4}h_{ij}\nabla^i H\nabla^j H + 2\alpha(\alpha-2)H^{-\alpha-5}|A|^2|\nabla H|^2.
\end{aligned}
$$

As the dimension of M_t is 2, then

$$
h_i^k h_k^j h_j^i = \sum_{i=1}^{2}\lambda_i^3 = \sum_{i=1}^{2}(\mathring{\lambda}_i + \frac{H}{2})^3 = \frac{3}{2}|\mathring{A}|^2 H + \frac{H^3}{4},
\tag{7.24}
$$

where $\mathring{\lambda}_1, \mathring{\lambda}_2$ are the eigenvalues of $\mathring{A}$. Substituting $|A|^2 H^{-2} = \gamma_t + \frac{1}{2}$ and (7.24) into (7.23), we have

$$
\partial_t\gamma_t = \alpha H^{-\alpha-1}\Delta\gamma_t + 2(\alpha+1)H^{1-\alpha}\gamma_t(\gamma_t - \frac{1}{2}) + 4(\alpha-1)H^{-\alpha-1}\gamma_t + Q.
\tag{7.25}
$$

We denote by $\{\nu_1, \nu_2\}$ an orthonormal eigenbasis for $\mathring{A}$ at the critical point p of γ_t, and assume that $\mathring{A}(\nu_1, \nu_2) \geq 0$ without loss of generality. Then at p we have (cf. [25, p.208])

$$
\begin{aligned}
&\nabla|\mathring{A}|^2 = 2|\mathring{A}|^2 H^{-1}\nabla H, \ \nabla|A|^2 = 2|A|^2 H^{-1}\nabla H, \\
&|\mathring{A}|H^{-1}\nabla H = \sqrt{2}\nabla\mathring{A}(\nu_1, \nu_1) = -\sqrt{2}\nabla\mathring{A}(\nu_2, \nu_2), \\
&2|\nabla\mathring{A}(\nu_1, \nu_2)|^2 = (\gamma_t + \frac{1}{2})|\nabla H|^2 - \frac{2}{H}\mathring{h}_{ij}\nabla^i H\nabla^j H.
\end{aligned}
\tag{7.26}
$$

It follows that at p, we have

$$
\begin{aligned}
|\nabla\mathring{A}|^2 &= |\mathring{A}|^2 H^{-2}|\nabla H|^2 + 2|\nabla\mathring{A}(\nu_1, \nu_2)|^2 \\
&= (2\gamma_t + \frac{1}{2})|\nabla H|^2 - \frac{2}{H}\mathring{h}_{ij}\nabla^i H\nabla^j H
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla A|^2 &= |\nabla\mathring{A}|^2 + \frac{1}{2}|\nabla H|^2 \\
&= (2\gamma_t + 1)|\nabla H|^2 - \frac{2}{H}\mathring{h}_{ij}\nabla^i H\nabla^j H.
\end{aligned}
$$

Then at p, the gradient term Q satisfies

$$
\begin{aligned}
Q &= 4\alpha H^{-\alpha-4} \cdot 2|A|^2 H^{-1}|\nabla H|^2 - 2\alpha H^{-\alpha-3}\left((2\gamma_t+1)|\nabla H|^2 - \frac{2}{H}\mathring{h}_{ij}\nabla^i H \nabla^j H\right) \\
&\quad - 2\alpha(\alpha+1)H^{-\alpha-4}(\mathring{h}_{ij} + \frac{H}{2}g_{ij})\nabla^i H \nabla^j H + 2\alpha(\alpha-2)H^{-\alpha-5}|A|^2|\nabla H|^2 \\
&= 2\alpha^2 H^{-\alpha-3}|\nabla H|^2\gamma_t - \alpha H^{-\alpha-3}|\nabla H|^2 + 2\alpha(1-\alpha)H^{-\alpha-4}\mathring{h}_{ij}\nabla^i H \nabla^j H \qquad (7.27) \\
&\leq 2\alpha(\alpha+(1-\alpha)^2)H^{-\alpha-3}|\nabla H|^2\gamma_t - \frac{\alpha}{2}H^{-\alpha-3}|\nabla H|^2 \\
&\leq 2\alpha H^{-\alpha-3}|\nabla H|^2(\gamma_t - \frac{1}{4}),
\end{aligned}
$$

where $0 < 1 - \alpha < 1$ is used in the last inequality and the Cauchy-Schwartz inequality

$$
(1-\alpha)H^{-1}\mathring{h}_{ij}\nabla^i H \nabla^j H \leq (1-\alpha)^2 H^{-2}|\mathring{A}|^2|\nabla H|^2 + \frac{1}{4}|\nabla H|^2
$$

is used in the first inequality.

As we assumed in (7.8) that the initial surface satisfies $\gamma_0 < 1/4$, by (7.25) and (7.27) and noting that $\alpha \leq 1$, we have that $\gamma_t < 1/4$ for each $t > 0$. Moreover, using $\gamma_t < 1/4$, Lemma 7.7 and applying the maximum principle to (7.25) and (7.27) again, we have that

$$
\gamma_t \leq (\max_{M_0} \gamma_0)e^{-\delta_1 t}, \qquad (7.28)
$$

where $0 < \delta_1 = \frac{\alpha+1}{2}C_1^{1-\alpha}$ is a positive constant depending only on α, ϵ_0, and $\bar{r}_0 - \underline{r}_0$, i.e., $\gamma_t \to 0$ exponentially as $t \to \infty$ in the rate δ_1. In the following, we improve the rate of the exponential decay in (7.28) step by step.

We define another function on M_t by

$$
\phi_t = e^{2(1-\alpha)\cdot t}(H^2 - 4). \qquad (7.29)
$$

By using (7.14), we have that

$$
\begin{aligned}
\partial_t \phi_t &= \alpha H^{-\alpha-1}\Delta\phi_t - \alpha(\alpha+2)H^{-\alpha-2}\nabla\phi_t \cdot \nabla H \\
&\quad - 2H^{1-\alpha}\left(|A|^2 - 2\right)e^{2(1-\alpha)\cdot t} + 2^{1-\alpha}\phi_t \\
&\leq \alpha H^{-\alpha-1}\Delta\phi_t - \alpha(\alpha+2)H^{-\alpha-2}\nabla\phi_t \cdot \nabla H - \left(H^{1-\alpha} - 2^{1-\alpha}\right)\phi_t \qquad (7.30) \\
&\leq \alpha H^{-\alpha-1}\Delta\phi_t - \alpha(\alpha+2)H^{-\alpha-2}\nabla\phi_t \cdot \nabla H,
\end{aligned}
$$

where we used the facts that $|A|^2 \geq H^2/2$ and $H^{1-\alpha} - 2^{1-\alpha}$ has the same sign with ϕ_t. Applying the maximum principle to (7.30), we obtain that

$$
H^2 \leq 4 + (\max_{M_0}|H^2 - 4|)e^{-2(1-\alpha)\cdot t}. \qquad (7.31)
$$

Substituting (7.28) into (7.25), using the estimate (7.27) and (7.31), we have that at the critical point of γ_t,

$$
\partial_t \gamma_t \leq \alpha H^{-\alpha-1}\Delta\gamma_t - \left(2 - 2(\alpha+1)(\max_{M_0}\gamma_0)e^{-\delta_1 t} - \frac{1-\alpha}{4}(\max_{M_0}|H^2 - 4|)e^{-2(1-\alpha)\cdot t}\right)H^{1-\alpha}\gamma_t. \qquad (7.32)
$$

Then the maximum principle implies that

$$\gamma_t \leq (\max_{M_0} \gamma_0) e^{-\delta_2 t} \exp\left(\frac{2(\alpha+1)}{\delta_1}(\max_{M_0} \gamma_0)(\max H^{1-\alpha}) + 2^{\alpha-3}(1-\alpha)(\max_{M_0} |H^2 - 4|)(\max H^{1-\alpha})\right)$$

$$=: C_3(\max_{M_0} \gamma_0) e^{-\delta_2 t}, \tag{7.33}$$

where $C_3 = C_3(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$, and $\delta_2 = 2C_1^{1-\alpha} > \delta_1$ are both positive constants depending only on α, ϵ_0, and $\bar{r}_0 - \underline{r}_0$.

We now improve the lower bound of H in Lemma 7.7. From the evolution equation (7.14), we have

$$\partial_t H^{1-\alpha} = \alpha H^{-\alpha-1}\Delta H^{1-\alpha} - \alpha(1-\alpha)H^{-2\alpha-2}|\nabla H|^2$$

$$- (1-\alpha)H^{-2\alpha}|\mathring{A}|^2 - \frac{1-\alpha}{2}H^{-2\alpha}(H^2 - 4).$$

We consider the minimum point p of H on M_t. Our aim is to show that there exists a constant $C_4(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$ such that $H^{1-\alpha} \geq 2^{1-\alpha} - C_4 e^{-C_1^{(1-\alpha)} \cdot t}$ with C_1 the lower bound given in Lemma 7.7, then we can improve the estimate in (7.32). If $H(p, t) \geq 2$, then we are done. If $H(p, t) < 2$, then using the elementary inequality

$$(1+x)^a \geq 1 + ax, \quad \forall\, x > 0,\ a > 1,$$

we have

$$4 - H^2 = H^2\left(1 + H^{\alpha-1}(2^{1-\alpha} - H^{1-\alpha})\right)^{\frac{2}{1-\alpha}} - H^2$$

$$\geq \frac{2}{1-\alpha}H^{1+\alpha}(2^{1-\alpha} - H^{1-\alpha}),$$

since $2^{1-\alpha} - H^{1-\alpha} > 0$ and $\frac{2}{1-\alpha} > 1$. So at the point p, we have

$$\partial_t(2^{1-\alpha} - H^{1-\alpha}) \leq - H^{1-\alpha}(2^{1-\alpha} - H^{1-\alpha}) + (1-\alpha)H^{-2\alpha}|\mathring{A}|^2$$

$$\leq - C_1^{1-\alpha}(2^{1-\alpha} - H^{1-\alpha}) + (1-\alpha)(4 + \max_{M_0}|H^2 - 4|)C_1^{-2\alpha}C_3(\max_{M_0} \gamma_0)e^{-\delta_2 t},$$

where we used the estimates (7.31) and (7.33). Then the maximum principle implies that

$$2^{1-\alpha} - H^{1-\alpha} \leq e^{-C_1^{(1-\alpha)} \cdot t}\left(\max_{M_0}|2^{1-\alpha} - H^{1-\alpha}| + (1-\alpha)(4 + \max_{M_0}|H^2 - 4|)C_1^{-\alpha-1}C_3 \max_{M_0}\gamma_0\right)$$

$$=: C_4 e^{-C_1^{(1-\alpha)} \cdot t}, \tag{7.34}$$

where $C_4 = C_4(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$.

Applying the estimate (7.34) in (7.32), we have that at the critical point of γ_t,

$$\partial_t \gamma_t \leq \alpha H^{-\alpha-1}\Delta\gamma_t - \left(2^{2-\alpha} - C_5(e^{-\delta_1 t} + e^{-2^{(1-\alpha)} \cdot t} + e^{-C_1^{(1-\alpha)} \cdot t})\right)\gamma_t, \tag{7.35}$$

for some constant $C_5 = C_5(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$. By applying the maximum principle again, we obtain that

$$\gamma_t \leq \max_{M_0} \gamma_0 e^{-2^{(2-\alpha)} \cdot t} \exp(C_5(\frac{1}{\delta_1} + 2^{\alpha-1} + C_1^{\alpha-1}))$$

$$=: C_6(\max_{M_0} \gamma_0)e^{-2^{(2-\alpha)} \cdot t} \tag{7.36}$$

with $C_6 = C_6(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0)$. Since $|\mathring{A}|^2 = \gamma_t H^2$, the estimate (7.21) follows from (7.36), (7.31), (7.8) and the estimate (7.7).

In the following, we prove (7.22). Note that from (7.31), (7.7) and (7.8), we immediately get that

$$|M_0|(H^2(x,t) - 4) \leq |M_0|(\max_{M_0}|H^2 - 4|)e^{-2^{(1-\alpha)}\cdot t} \leq Ce^{-2^{(1-\alpha)}\cdot t}$$

with C depending only on E. In order to prove (7.22), it remains to estimate the lower bound of $|M_0|(H^2(x,t) - 4)$. We prove this by two steps.

From the definition of C_4 given in (7.34) and the estimate given in (7.7), we can choose s_0 large enough such that $0 < C_4 \leq 2^{1-\alpha} - 1$, then (7.34) yields that

$$H^{1-\alpha} \geq 2^{1-\alpha} - C_4 e^{-C_1^{(1-\alpha)}\cdot t} \geq 2^{1-\alpha} - C_4 \geq 2^{1-\alpha} - (2^{1-\alpha} - 1) = 1,$$

from which we defer that $H \geq 1$. This means that we can take $C_1 = 1$ as the lower bound of H. By taking $C_1 = 1$, choosing s_0 large enough and repeating the same procedure as that to obtain (7.34), we can get

$$2^{1-\alpha} - H^{1-\alpha} \leq e^{-t}. \tag{7.37}$$

Let $\eta_t = e^t(H^2 - 4)$, we have

$$\partial_t \eta_t = \alpha H^{-\alpha-1}\Delta\eta_t - \alpha(\alpha+2)H^{-\alpha-2}\nabla\eta_t \cdot \nabla H$$
$$+ \left(1 - H^{1-\alpha}\right)\eta_t - 2H^{1-\alpha}e^t|\mathring{A}|^2.$$

We consider the minimum point p of η_t. If $\eta_t(p) \geq 0$, i.e., $H^2 - 4 \geq 0$, then we are done. If $\eta_t(p) < 0$, then $(1 - H^{1-\alpha})\eta_t(p) \geq 0$, so at p, using the estimate (7.21) and (7.31), we have

$$\partial_t \eta_t \geq -2H^{1-\alpha}e^t|\mathring{A}|^2 \geq -C_7|M_0|^{-2}e^{-(2^{2-\alpha}-1)t},$$

with C_7 depending only on α, ϵ_0, $\bar{r}_0 - \underline{r}_0$ and E. Then applying the maximum principle, we obtain that

$$\eta_t \geq \min_{M_0}\eta_t - \frac{C_7}{2^{2-\alpha} - 1}|M_0|^{-2} \geq -\max_{M_0}|H^2 - 4| - \frac{C_7}{2^{2-\alpha} - 1}|M_0|^{-2}.$$

We can choose s_0 large enough such that $|M_0| > 1$, then from the estimates given in (7.7) and (7.8), we get

$$\eta_t \geq -C_8(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)|M_0|^{-1}. \tag{7.38}$$

Now we improve the estimate in (7.38). For the function ϕ_t defined by (7.29), we have

$$\partial_t \phi_t = \alpha H^{-\alpha-1}\Delta\phi_t - \alpha(\alpha+2)H^{-\alpha-2}\nabla\phi_t \cdot \nabla H$$
$$+ \left(2^{1-\alpha} - H^{1-\alpha}\right)\phi_t - 2H^{1-\alpha}e^{2^{(1-\alpha)}\cdot t}|\mathring{A}|^2.$$

We consider the minimum point p of ϕ_t. If $\phi_t(p) \geq 0$, i.e., $H \geq 2$, then we are done. If $\phi_t(p) < 0$, then at p, using (7.21), (7.37) and (7.38), we have

$$\partial_t \phi_t \geq \left(2^{1-\alpha} - H^{1-\alpha}\right)\phi_t - 2H^{1-\alpha}e^{2^{(1-\alpha)}\cdot t}|\mathring{A}|^2$$
$$= \left(2^{1-\alpha} - H^{1-\alpha}\right)e^{(2^{1-\alpha}-1)t}\eta_t - 2H^{1-\alpha}e^{2^{(1-\alpha)}\cdot t}|\mathring{A}|^2 \tag{7.39}$$
$$\geq -C_8 e^{-(2-2^{1-\alpha})t}|M_0|^{-1} - 2^{2-\alpha}C_2 e^{-2^{(1-\alpha)}\cdot t}|M_0|^{-2}.$$

Then applying the maximum principle, we obtain that

$$\phi_t \geq \min_{M_0} \phi_t - C|M_0|^{-1} - C|M_0|^{-2} \geq -\max_{M_0} |H^2 - 4| - C|M_0|^{-1} - C|M_0|^{-2},$$

for some C depending only on $\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0$ and E. From the estimates given in (7.7) and (7.8), we get

$$\phi_t \geq -C(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)|M_0|^{-1}. \tag{7.40}$$

Then (7.22) follows immediately.   □

Lemma 7.9. *There exist constants $s_0 = s_0(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ and $C_9 = C_9(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$ such that for any $s \geq s_0$,*

$$|M_0|^3 |\nabla A|^2(x, t) \leq C_9 e^{-3 \cdot 2^{(1-\alpha)} \cdot t} \tag{7.41}$$

on M_t along the flow (7.1).

Proof. First, recall that in 2-dimensional case, we have (cf. [15, §2])

$$|\nabla H|^2 \leq \frac{4}{3}|\nabla A|^2,$$

which implies that

$$|\nabla A|^2 \leq 3|\nabla \mathring{A}|^2, \quad \text{and} \quad |\nabla H|^2 \leq 4|\nabla \mathring{A}|^2. \tag{7.42}$$

Therefore, in order to estimate $|\nabla A|^2$, it is equivalent to estimating $|\nabla \mathring{A}|^2$.

By combining (7.12), (7.14) and (7.15), we obtain that

$$\begin{aligned}
\partial_t \mathring{h}_{ij} =& \partial_t h_{ij} - \frac{1}{2}\partial_t H g_{ij} - \frac{H}{2}\partial_t g_{ij} \\
=& \alpha H^{-\alpha-1}\Delta \mathring{h}_{ij} - \alpha(\alpha+1)H^{-\alpha-2}\nabla_i H \nabla_j H \\
&+ \frac{1}{2}\alpha(\alpha+1)H^{-\alpha-2}|\nabla H|^2 g_{ij} + \alpha H^{-\alpha-1}|\mathring{A}|^2 \mathring{h}_{ij} + \frac{\alpha+1}{2}H^{-\alpha}|\mathring{A}|^2 g_{ij} \\
&- \frac{\alpha}{2}H^{1-\alpha}\mathring{h}_{ij} + (1-\alpha)H^{-\alpha}\mathring{h}_i^k \mathring{h}_{kj} + 2\alpha H^{-\alpha-1}\mathring{h}_{ij}.
\end{aligned} \tag{7.43}$$

If Rm denotes the curvature tensor of M_t, then for any tensor S on M_t, the Ricci identity implies that

$$\Delta \nabla S = \nabla \Delta S + Rm * \nabla S + \nabla Rm * S, \tag{7.44}$$

where $*$ means the contraction of two tensors using the metric $g(t)$ of M_t. We also have the formula of commuting ∇ with ∂_t,

$$\partial_t \nabla S - \nabla \partial_t S = S * \nabla \partial_t g(t). \tag{7.45}$$

By Gauss equations, the curvature of M_t satisfies

$$\begin{aligned}
R_{ijkl} =& -(g_{ik}g_{jl} - g_{il}g_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk} \\
=& (\frac{H^2}{4} - 1)(g_{ik}g_{jl} - g_{il}g_{jk}) + \mathring{h}_{ik}\mathring{h}_{jl} - \mathring{h}_{il}\mathring{h}_{jk} \\
&+ \frac{H}{2}\left(\mathring{h}_{ik}g_{jl} + \mathring{h}_{jl}g_{ik} - \mathring{h}_{il}g_{jk} - \mathring{h}_{jk}g_{il}\right)
\end{aligned}$$

in local coordinates. Then

$$|Rm| \leq C_{10}\left(|H^2 - 4| + |\mathring{A}|^2 + H|\mathring{A}|\right) \tag{7.46}$$

and

$$|\nabla Rm| \leq C_{10}\left(H|\nabla H| + |\mathring{A}||\nabla\mathring{A}| + |\mathring{A}||\nabla H| + H|\nabla\mathring{A}|\right)$$

$$\leq 3C_{10}\left(|\mathring{A}||\nabla\mathring{A}| + H|\nabla\mathring{A}|\right) \tag{7.47}$$

for a universal constant C_{10} depending only on the dimension $n = 2$.

Note that in the proof of Lemma 7.8, we obtain that H and $|A|^2$ are bounded from below and above by positive constants which do not depend on s, using (7.12), (7.42)–(7.47), we compute and estimate the evolution equation of $|\nabla\mathring{A}|^2$.

$$\partial_t|\nabla\mathring{A}|^2 = 2g(\partial_t\nabla\mathring{A}, \nabla\mathring{A}) - 2\partial_t g_{rs}g^{ir}g^{ms}g^{jn}g^{kp}\mathring{h}_{ij,k}\mathring{h}_{mn,p} - \partial_t g_{rs}g^{kr}g^{ps}g^{im}g^{jn}\mathring{h}_{ij,k}\mathring{h}_{mn,p}$$

$$= 2g(\partial_t\nabla\mathring{A}, \nabla\mathring{A}) - 4H^{-\alpha}(\mathring{h}_{rs} + \frac{H}{2}g_{rs})g^{ir}g^{ms}g^{jn}g^{kp}\mathring{h}_{ij,k}\mathring{h}_{mn,p}$$

$$- 2H^{-\alpha}(\mathring{h}_{rs} + \frac{H}{2}g_{rs})g^{kr}g^{ps}g^{im}g^{jn}\mathring{h}_{ij,k}\mathring{h}_{mn,p}$$

$$\leq 2g(\partial_t\nabla\mathring{A}, \nabla\mathring{A}) + C|\nabla\mathring{A}|^2|\mathring{A}| - 3H^{1-\alpha}|\nabla\mathring{A}|^2$$

$$= 2g(\nabla\partial_t\mathring{A}, \nabla\mathring{A}) + \mathring{A}*\nabla(H^{-\alpha}A)*\nabla\mathring{A} + C|\nabla\mathring{A}|^2|\mathring{A}| - 3H^{1-\alpha}|\nabla\mathring{A}|^2$$

$$\leq 2\alpha H^{-\alpha-1}g(\nabla\Delta\mathring{A}, \nabla\mathring{A}) + \left(-3H^{1-\alpha} - \alpha H^{1-\alpha} + 4\alpha H^{-\alpha-1}\right)|\nabla\mathring{A}|^2$$

$$+ C\left(|\nabla^2\mathring{A}| + |\nabla^2 H|\right)|\nabla\mathring{A}|^2 + C|\nabla\mathring{A}|^4 + C|\nabla\mathring{A}|^2\left(|\mathring{A}|^3 + |\mathring{A}|^2 + |\mathring{A}|\right)$$

$$\leq \alpha H^{-\alpha-1}\Delta|\nabla\mathring{A}|^2 - 2\alpha H^{-\alpha-1}|\nabla^2\mathring{A}|^2$$

$$+ \left(-3H^{1-\alpha} - \alpha H^{1-\alpha} + 4\alpha H^{-\alpha-1}\right)|\nabla\mathring{A}|^2 + C|\nabla\mathring{A}|^4 \tag{7.48}$$

$$+ C\left(|\nabla^2\mathring{A}| + |\nabla^2 H|\right)|\nabla\mathring{A}|^2 + C|\nabla\mathring{A}|^2\left(|H^2 - 4| + |\mathring{A}|^3 + |\mathring{A}|^2 + |\mathring{A}|\right),$$

where the constant C depends only on α, ϵ_0 and $\bar{r}_0 - \underline{r}_0$. We claim that

$$|\nabla^2 H| \leq C_{11}|\nabla^2\mathring{A}| \tag{7.49}$$

for a universal constant C_{11} depending on the dimension $n = 2$. In fact, let $\{e_1, e_2\}$ be a local orthonormal frame of M_t. Then

$$|\nabla^2 H|^2 = \sum_{i,j=1}^{2}(h_{11,ij} + h_{22,ij})^2 \leq \sum_{i,j=1}^{2}\left(2h_{11,ij}^2 + 2h_{22,ij}^2\right). \tag{7.50}$$

Using the Codazzi equation $h_{kl,m} = h_{km,l}$ and the fact $\nabla^2 h_{kl} = \nabla^2\mathring{h}_{kl}$ for any pair $k \neq l$, we have

$$h_{22,1j} = h_{21,2j} = \mathring{h}_{21,2j}, \quad \forall\, j = 1, 2, \tag{7.51}$$

$$h_{11,1j} = \mathring{h}_{11,1j} + \frac{1}{2}H_{,1j} = \mathring{h}_{11,1j} + h_{22,1j} - \mathring{h}_{22,1j}$$

$$= \mathring{h}_{11,1j} + \mathring{h}_{21,2j} - \mathring{h}_{22,1j}, \quad \forall\, j = 1, 2. \tag{7.52}$$

Similarly, we have

$$h_{22,2j} = \mathring{h}_{22,2j} + \mathring{h}_{12,1j} - \mathring{h}_{11,2j}, \quad \forall\, j = 1, 2, \tag{7.53}$$

$$h_{11,2j} = \mathring{h}_{12,1j} \quad \forall\, j = 1, 2. \tag{7.54}$$

Then (7.49) follows by combining (7.50) – (7.54). In view of the estimates (7.21) and (7.22), (7.49) and applying the Cauchy-Schwartz inequality

$$2\alpha H^{-\alpha-1}|\nabla^2\mathring{A}|^2 + \frac{C^2\cdot C_{11}^2}{8\alpha}|\nabla\mathring{A}|^4 H^{\alpha+1} \geq C\cdot C_{11}|\nabla^2\mathring{A}||\nabla\mathring{A}|^2$$

to kill out the second order term $|\nabla^2\mathring{A}|^2$ in (7.48), we have

$$\partial_t|\nabla\mathring{A}|^2 \leq \alpha H^{-\alpha-1}\Delta|\nabla\mathring{A}|^2 + \left(-3\cdot 2^{1-\alpha} + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t}\right)|\nabla\mathring{A}|^2 + C|\nabla\mathring{A}|^4, \quad (7.55)$$

where the constant C depends only on $\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0$ and E.

However, we can not obtain the estimate (7.41) by applying the maximum principle to (7.55), as the coefficient of $|\nabla\mathring{A}|^4$ is a positive constant. To kill out this bad term, we will combine the following evolution equation of $|\mathring{A}|^2$, which follows from (7.43), (7.12) and the estimates (7.21), (7.22).

$$\begin{aligned}
\partial_t|\mathring{A}|^2 =&\,\alpha H^{-\alpha-1}\Delta|\mathring{A}|^2 - 2\alpha H^{-\alpha-1}|\nabla\mathring{A}|^2 - 2\alpha(\alpha+1)H^{-\alpha-2}\mathring{h}_{ij}\nabla^i H\nabla^j H\\
&+ 2\alpha H^{-\alpha-1}|\mathring{A}|^4 - (\alpha+2)H^{1-\alpha}|\mathring{A}|^2 + 4\alpha H^{-\alpha-1}|\mathring{A}|^2\\
\leq&\,\alpha H^{-\alpha-1}\Delta|\mathring{A}|^2 + (-2^{-\alpha}\cdot\alpha + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t})|\nabla\mathring{A}|^2\\
&+ (-2^{2-\alpha} + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t})|\mathring{A}|^2,
\end{aligned} \quad (7.56)$$

where the constant C depends only on $\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0$ and E. Define

$$\psi_t = \log|\nabla\mathring{A}|^2 + K|\mathring{A}|^2 \quad (7.57)$$

for a constant K to be determined later. Then

$$\begin{aligned}
\partial_t\psi_t \leq&\,\alpha H^{-\alpha-1}\Delta\psi_t + \alpha H^{-\alpha-1}|\nabla\log|\nabla\mathring{A}|^2|^2 - 3\cdot 2^{1-\alpha} + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t}\\
&+ \left(C + K(-2^{-\alpha}\alpha + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t})\right)|\nabla\mathring{A}|^2\\
&+ K(-2^{2-\alpha} + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t})|\mathring{A}|^2,
\end{aligned} \quad (7.58)$$

where the constant C depends only on $\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0$ and E. At the critical point p of ψ_t, we have

$$\nabla\log|\nabla\mathring{A}|^2 = -K\nabla|\mathring{A}|^2,$$

hence at this p, we have

$$\begin{aligned}
|\nabla\log|\nabla\mathring{A}|^2|^2 = K^2|\nabla|\mathring{A}|^2|^2 &= K^2|2|\mathring{A}|\nabla|\mathring{A}||^2\\
&= 4K^2|\mathring{A}|^2|\nabla|\mathring{A}||^2 \leq 4K^2|\mathring{A}|^2|\nabla\mathring{A}|^2\\
&\leq 4K^2 C_2|M_0|^{-2}e^{-2^{(2-\alpha)}\cdot t}|\nabla\mathring{A}|^2,
\end{aligned}$$

where we used the inequality $|\nabla|\mathring{A}||^2 \leq |\nabla\mathring{A}|^2$ in the first inequality and (7.21) in the last inequality. Therefore, at p,

$$\begin{aligned}
\partial_t\psi_t \leq&\,\alpha H^{-\alpha-1}\Delta\psi_t - 3\cdot 2^{1-\alpha} + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t}\\
&+ \left(C + K(-2^{-\alpha}\cdot\alpha + C|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t}) + 4K^2 C_2|M_0|^{-2}e^{-2^{(2-\alpha)}\cdot t}\right)|\nabla\mathring{A}|^2\\
&+ KC|M_0|^{-1}e^{-2^{(1-\alpha)}\cdot t}C_2|M_0|^{-2}e^{-2^{(2-\alpha)}\cdot t}
\end{aligned} \quad (7.59)$$

for some constant $C = C(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$. We can choose $K = C\alpha^{-1}2^{2+\alpha}$ and choose s_0 sufficiently large such that $|M_0|^{-1} < \min\{\frac{\alpha}{C \cdot 2^{1+\alpha}}, \frac{\alpha}{\sqrt{2C \cdot C_2} \cdot 2^{3+\alpha}}\}$. Then we have

$$C + K(-2^{-\alpha} \cdot \alpha + C|M_0|^{-1}e^{-2^{(1-\alpha)} \cdot t}) + 4K^2C_2|M_0|^{-2}e^{-2^{(2-\alpha)} \cdot t} \ < \ 0$$

and

$$C|M_0|^{-1} < \alpha \cdot 2^{-1-\alpha}, \ \ KC|M_0|^{-1}C_2|M_0|^{-2} < \alpha^2 \cdot 2^{-6-2\alpha}.$$

Therefore, at the critical point of ψ_t, we have

$$\partial_t \psi_t \leq \alpha H^{-\alpha-1}\Delta\psi_t - 3 \cdot 2^{1-\alpha} + \alpha \cdot 2^{-1-\alpha}e^{-2^{(1-\alpha)} \cdot t} + \alpha^2 \cdot 2^{-6-2\alpha}e^{-3 \cdot 2^{(1-\alpha)} \cdot t}. \tag{7.60}$$

Applying maximum principal to (7.60), we conclude that

$$\psi_t \leq \max_{M_0} \psi_0 - 3 \cdot 2^{1-\alpha} \cdot t + C, \tag{7.61}$$

for some constant $C = C(\alpha, \epsilon_0, \bar{r}_0 - \underline{r}_0, E)$. Then (7.41) follows immediately from (7.61) and the estimate (7.7). $\qquad\square$

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