FROBENIUS STRATIFICATION OF MODULI SPACES OF RANK 3 VECTOR BUNDLES IN CHARACTERISTIC 3, I

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ABSTRACT. Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$, $F_X : X \to X$ the absolute Frobenius morphism. Let $\mathfrak{M}_X^s(r,d)$ be the moduli space of stable vector bundles of rank r and degree d on X. We study the Frobenius stratification of $\mathfrak{M}_X^s(3,0)$ in terms of Harder-Narasimhan polygons of Frobenius pull backs of stable vector bundles and obtain the irreducibility and dimension of each non-empty Frobenius stratum in the case $(p,g) = (3,2)$.

Dedicated To The Memory of Professor Michel Raynaud.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$. Let $\mathfrak{M}_X^s(r,d)$ (resp. $\mathfrak{M}_X^{ss}(r,d)$) be the moduli space of stable (resp. semistable) vector bundles of rank r and degree d on X.

Let $\mathscr{E}$ be a vector bundle on X, and

$$\mathrm{HN}_\bullet(\mathscr{E}) : 0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{m-1} \subset \mathscr{E}_m = \mathscr{E}$$

the Harder-Narasimhan filtration of $\mathscr{E}$. Consider the points

$$(\mathrm{rk}(\mathscr{E}_i), \deg(\mathscr{E}_i))(0 \leq i \leq m)$$

in the coordinate plane of rank-degree, we connect the point $(\mathrm{rk}(\mathscr{E}_i), \deg(\mathscr{E}_i))$ to the point $(\mathrm{rk}(\mathscr{E}_{i+1}), \deg(\mathscr{E}_{i+1}))$ successively by line segment for $0 \leq i \leq m-1$. Then we get a convex polygon in the plane which we call the *Harder-Narasimhan Polygon* of $\mathscr{E}$, denoted by $\mathrm{HNP}(\mathscr{E})$.

Let (r,d) be a point in the coordinate plane of rank-degree. If $r \leq \mathrm{rk}(\mathscr{E})$, and $d \geq (\leq)d'$ for some point $(r,d') \in \mathrm{HNP}(\mathscr{E})$, then we say (r,d) *lies above* (*below*) the $\mathrm{HNP}(\mathscr{E})$. Given two convex polygons $\mathscr{P}_1$ and $\mathscr{P}_2$, if for any vertex on $\mathscr{P}_1$ lies above $\mathscr{P}_2$, then we say $\mathscr{P}_1$ lies above $\mathscr{P}_2$, denoted by $\mathscr{P}_1 \succcurlyeq \mathscr{P}_2$. There is a natural partial order structure, in the sense of $\succcurlyeq$, on the set $\{\mathrm{HNP}(\mathscr{E}) \mid \mathscr{E} \in \mathfrak{Vect}_X(r,d)\}$, where $\mathfrak{Vect}_X(r,d)$ is the category of vector bundles of rank r and degree d on X.

In general, the semistability of vector bundles is possibly destabilized under Frobenius pull back F_X^* (cf. [2], [14]). Thus, there is a natural set-theoretic map

$$S_{\mathrm{Frob}}^s : \mathfrak{M}_X^s(r,d)(k) \quad \to \quad \mathfrak{ConPgn}(r, pd)$$
$$[\mathscr{E}] \quad \mapsto \quad \mathrm{HNP}(F_X^*(\mathscr{E}))$$

where $\mathfrak{ConPgn}(r, pd)$ is the partially ordered set of all convex polygons in the coordinate plane such that their vertexes have integral coordinates, start at the origin $(0,0)$ and end at the point (r, pd). For any $\mathscr{P} \in \mathfrak{ConPgn}(r, pd)$, we denote

$$S_X(r, d, \mathscr{P}) \quad := \quad \{ [\mathscr{E}] \in \mathfrak{M}_X^s(r,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P} \}.$$
$$S_X(r, d, \mathscr{P}^+) \quad := \quad \{ [\mathscr{E}] \in \mathfrak{M}_X^s(r,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathscr{P} \}.$$

Then we have a canonical stratification of $\mathfrak{M}_X^s(r,d)$ by Harder-Narasimhan polygons of Frobenius pull backs of stable vector bundles of rank r and degree d. We call this the *Frobenius stratification*. By a theorem of S. S. Shatz [15, Theorem 3] and the geometric invariant theory construction of $\mathfrak{M}_X^s(r,d)$, we know that the Frobenius stratum $S_X(r, d, \mathscr{P}^+)$ is a closed subvariety of $\mathfrak{M}_X^s(r,d)$ for any $\mathscr{P} \in \mathfrak{ConPgn}(r, pd)$.

The main goal of the paper is to study the geometric properties of Frobenius strata, such as non-emptiness, irreducibility, connectedness, smoothness, dimension and so on. Some results are known in special cases for small values of p, g, r and d. In particular, Joshi-Ramanan-Xia-Yu [5] give a complete description of the Frobenius stratification of $\mathfrak{M}_X^s(2,d)$ for any integer d when $p = 2$ and $g \geq 2$. They obtain the irreducibility and dimension of each non-empty Frobenius stratum. Fix integers r and d with $r > 0$, we denote

$$W_X(r,d) := \{ [\mathscr{E}] \in \mathfrak{M}_X^s(r,d)(k) | \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathrm{HNP}(F_X^*(\mathscr{F})) \text{ for any } [\mathscr{F}] \in \mathfrak{M}_X^s(r,d)(k) \}.$$

Then $W_X(r,d)$ is a closed subvariety of $\mathfrak{M}_X^s(r,d)$. Joshi-Ramanan-Xia-Yu [5, Theorem 4.6.4] show that $W_X(2,d)$ is irreducible and of dimension g when $p = 2$ and $g \geq 2$. Moreover, under the assumption $p > r(r-1)(g-1)$, Joshi and Pauly [4, Theorem 6.2.1] show that the dimension of $W_X(r,0)$ is g. Other results about Frobenius stratification can be found in [6][7][8][9][13][19].

In [10] the author shows that $W_X(p,d)$ is a closed subvariety of $\mathfrak{M}_X^s(p,d)$ which is isomorphic to the Jacobian variety Jac_X of X for any integer d (cf. [10, Theorem 3.2]). In particular, $W_X(p,d)$ is an smooth irreducible projective variety of dimension g. Combining [11, Theorem 1.1] with [10, Theorem 2.5], we can obtain the geometric properties of a specific Frobenius stratum.

Theorem 1.1. (*Theorem 5.5*) *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k. Then the subset*

$$V_{rp,d} = \{ [\mathscr{E}] \in \mathfrak{M}_X^s(rp,d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_{rp,d}^{\mathrm{can}} \}$$

is a smooth irreducible closed subvariety of dimension $r^2(g-1) + 1$ in $\mathfrak{M}_X^s(rp,d)$.

In this paper, we mainly study the Frobenius stratification of $\mathfrak{M}_X^s(3,0)$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3. In this case, there are 4 possible Harder-Narasimhan polygons $\{\mathscr{P}_i\}_{1 \leq i \leq 4}$ for Frobenius pull backs of Frobenius destabilized semistable vector bundles of rank 3 and degree 0 (See Section 2). We obtain the irreducibility and dimension of each non-empty Frobenius stratum in the moduli space $\mathfrak{M}_X^s(3,0)$ when $(p,g) = (3,2)$. The structure of Frobenius stratification of $\mathfrak{M}_X^s(3,d)$ is easily deduced from $\mathfrak{M}_X^s(3,0)$, when $3|d$.

In general, it is difficult to determine the Harder-Narasimhan polygon of $F_X^*(\mathscr{E})$ for a stable bundle $\mathscr{E}$ on X. We first show that any rank 3 and degree 0 Frobenius destabilized stable vector bundle $\mathscr{E}$ with $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$ can be embedded into $F_{X*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ of degree -1, when $(p, g) = (3, 2)$ (Proposition 3.3). Then we can determine $\mathrm{HNP}(F_X^*(\mathscr{E}))$ by analysing the intersection of $F_X^*(\mathscr{E})$ with the canonical filtration of $F_X^*(F_{X*}(\mathscr{L}))$. Moreover, we show that any rank 3 and degree 0 subsheaf $\mathscr{E}$ of $F_{X*}(\mathscr{L})$ is semistable for any line bundle $\mathscr{L}$ of degree -1 (Proposition 3.4). Then we have a morphism

$$\theta : \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X)) \rightarrow \mathfrak{M}_X^{ss}(3, 0)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}],$$

where $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$ is the Quot scheme parameterizing all rank 3 and degree 0 subsheaves of $F_{X*}(\mathscr{L})$ for any line bunde $\mathscr{L} \in \mathrm{Pic}^{(-1)}(X)$. Restricting the morphism θ to the stable locus $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$ of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$, we can obtain the geometric properties of Frobenius strata of $\mathfrak{M}_X^s(3, 0)$ from the geometric properties of Frobenius strata of $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$. These techniques are generalizations of the methods which are first introduced by Joshi-Ramanan-Xia-Yu in [5].

The main result of this paper is the following Theorem.

Theorem 1.2. (*Theorem 5.2*) *Let k be an algebraically closed field of characteristic* 3, *X a smooth projective curve of genus* 2 *over k. Then*

(1) $S_X(3, 0, \mathscr{P}_1^+) \cong S_X(3, 0, \mathscr{P}_2^+)$, $S_X(3, 0, \mathscr{P}_1) \cong S_X(3, 0, \mathscr{P}_2)$, *and*

$$S_X(3, 0, \mathscr{P}_1^+) \cap S_X(3, 0, \mathscr{P}_2^+) = S_X(3, 0, \mathscr{P}_3^+).$$

(2) $S_X(3, 0, \mathscr{P}_i^+) = \overline{S_X(3, 0, \mathscr{P}_i)}$, $S_X(3, 0, \mathscr{P}_i)$ *and* $S_X(3, 0, \mathscr{P}_i^+)$ *are irreducible quasi-projective varieties for* $1 \leq i \leq 4$, *and*

$$\dim S_X(3, 0, \mathscr{P}_i^+) = \dim S_X(3, 0, \mathscr{P}_i) = \begin{cases} 5, \text{when } i = 1 \\ 5, \text{when } i = 2 \\ 4, \text{when } i = 3 \\ 2, \text{when } i = 4 \end{cases}$$

In section 2, we show that there are 4 possible Harder-Narasimhan polygons $\{\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$ for Frobenius destabilized semistable vector bundles of rank 3 and degree 0 in the case $(p, g) = (3, 2)$.

In section 3, we show that any Frobenius destabilized stable bundle $\mathscr{E}$ of rank 3 and degree 0 with $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$ can be embedded into $F_{X*}(\mathscr{L})$ for some line bundle $\mathscr{L}$ of degree -1. Moreover, we show that for any line bundle $\mathscr{L}$ of degree -1 on X, each rank 3 and degree 0 subsheaf $\mathscr{E} \subset F_{X*}(\mathscr{L})$ is semistable.

In section 4, we will study the Frobenius stratification of the Quot scheme $\mathrm{Quot}_X(3, d, \mathrm{Pic}^{(d-1)}(X))$ and obtain the smoothness, irreducibility and dimension of each non-empty stratum.

In section 5, we study the Frobenius stratification of moduli space $\mathfrak{M}_X^s(3, 0)$ when $(p, g) = (3, 2)$. We obtain the geometric properties of Frobenius strata in $\mathfrak{M}_X^s(3, 0)$ from the geometric properties of Frobenius strata in $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$ by the morphism $\theta^s : \mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X)) \rightarrow \mathfrak{M}_X^s(3, 0) : [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}]$. Moreover, we obtain the geometric properties of a special Frobenius stratum in $\mathfrak{M}_X^s(rp, d)$ for any integers p, g, r, d with $r > 0, p > 0$ and $g \geq 2$.
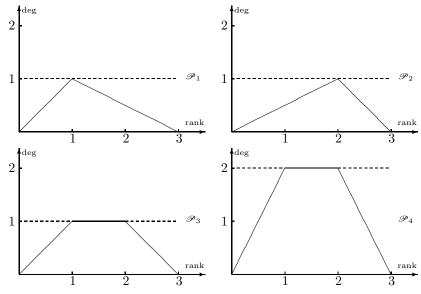
2. CLASSIFICATION OF FROBENIUS HARDER-NARASIMHAN POLYGONS

In this section, we will determine all of the possible Harder-Narasimhan polygons of $F_X^*(\mathscr{E})$ for any Frobenius destabilized semistable vector bundles $\mathscr{E} \in \mathfrak{M}_X^s(3,0)$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

Lemma 2.1 (N. I. Shepherd-Barron [16] and V. Mehta, C. Pauly [12]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k, $\mathscr{E}$ a semistable vector bundle on X. Let $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{m-1} \subset \mathscr{E}_m = F_X^*(\mathscr{E})$ be the Harder-Narasimhan filtration of $F_X^*(\mathscr{E})$. Then*

(1) *For any $1 \leq i \leq m-1$, $\mu(\mathscr{E}_i/\mathscr{E}_{i-1}) - \mu(\mathscr{E}_{i+1}/\mathscr{E}_i) \leq 2g - 2$.*
(2) *$\mu_{\max}(F_X^*(\mathscr{E})) - \mu_{\min}(F_X^*(\mathscr{E})) \leq \min\{r-1, p-1\} \cdot (2g-2)$.*

According to the Lemma 2.1, in the case $(p, g, r, d) = (3, 2, 3, 0)$, there are 4 possible Harder-Narasimhan polygons for all Frobenius destabilized semistable vector bundles of rank 3 and degree 0 when $(p, g) = (3, 2)$ as the following:

3. CONSTRUCTION OF STABLE VECTOR BUNDLES

Definition 3.1 (Joshi-Ramanan-Xia-Yu [5]). Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve over k. For any coherent sheaf $\mathscr{F}$ on X, let

$$\nabla_{\mathrm{can}} : F_X^* F_{X*}(\mathscr{F}) \to F_X^* F_{X*}(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1$$

be the canonical connection on $F_X^* F_{X*}(\mathscr{F})$. Set

$$V_1 \quad := \quad \ker(F_X^* F_{X*}(\mathscr{F}) \twoheadrightarrow \mathscr{F}),$$

$$V_{l+1} \quad := \quad \ker\{V_l \xrightarrow{\nabla_{\mathrm{can}}} F_X^* F_{X*}(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1 \to (F_X^* F_{X*}(\mathscr{F})/V_l) \otimes_{\mathscr{O}_X} \Omega_X^1\}$$

The filtration

$$\mathbb{F}_{\mathscr{F}}^{\mathrm{can}} {}_\bullet : F_X^* F_{X*}(\mathscr{F}) = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{p-1} \supset V_p = 0$$

is called the *canonical filtration* of $F_X^* F_{X*}(\mathscr{F})$.

Lemma 3.2 (Joshi-Ramanan-Xia-Yu [5] and X. Sun [17])**.** *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k, $\mathscr{E}$ a vector bundle on X. Then the canonical filtration of $F_X^* F_{X*}(\mathscr{E})$ is*

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{l+1} \subset V_l \subset \cdots \subset V_1 \subset V_0 = F_X^* F_{X*}(\mathscr{E})$$

such that

(1) *$\nabla_{\mathrm{can}}(V_{i+1}) \subset V_i \otimes_{\mathscr{O}_X} \Omega_X^1$ for $0 \le i \le p-1$.*
(2) *$V_l/V_{l+1} \xrightarrow{\nabla_{\mathrm{can}}} \mathscr{E} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes l}$ are isomorphic for $0 \le l \le p-1$.*
(3) *If $g \ge 1$, then $F_{X*}(\mathscr{E})$ is semistable whenever $\mathscr{E}$ is semistable. If $g \ge 2$, then $F_{X*}(\mathscr{E})$ is stable whenever $\mathscr{E}$ is stable.*
(4) *If $g \ge 2$ and $\mathscr{E}$ is semistable, then the canonical filtration of $F_X^* F_{X*}(\mathscr{E})$ is nothing but the Harder-Narasimhan filtration of $F_X^* F_{X*}(\mathscr{E})$.*

Proposition 3.3. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{E}$ be a rank 3 and degree 0 stable vector bundle on X and one has non-trivial homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$, where $\mathscr{L}$ is a line bundle on X with $\deg(\mathscr{L}) = -1$. Then the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection.*

Proof. By adjunction, there is a non-trivial homomorphism $\mathscr{E} \to F_{X*}(\mathscr{L})$. Denote the image by $\mathscr{G}$. By stability of $\mathscr{E}$, we have $\deg(\mathscr{G}) \ge 0$ and $\deg(\mathscr{G}) = 0$ if and only if $\mathscr{E} \cong \mathscr{G}$. On the other hand, we have $\deg(F_{X*}(\mathscr{L})) = 1$ (cf. [5, Sect. 2.9]). Moreover, by Lemma 3.2(3), the stability of $F_{X*}(\mathscr{L})$ implies that $\deg(\mathscr{G}) \le 0$. Hence $\deg(\mathscr{G}) = 0$. Thus, $\mathscr{E} \cong \mathscr{G}$, i.e. the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection. $\qquad\square$

Proposition 3.4. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{L}$ be a line bundle on X with $\deg(\mathscr{L}) = -1$, $\mathscr{E}$ a subsheaf of $F_X^*(\mathscr{L})$ with $\mathrm{rk}(\mathscr{E}) = 3$ and $\deg(\mathscr{E}) = 0$. Then $\mathscr{E}$ is a semistable vector bundle. Moreover, there exists some rank 3 and degree 0 stable subsheaf of $F_{X*}(\mathscr{L})$.*

Proof. Let $\mathscr{G} \subset \mathscr{E}$ be a subsheaf of $\mathscr{E}$ with $\mathrm{rk}(\mathscr{G}) < \mathrm{rk}(\mathscr{E}) = 3$. By [18, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$, we have

$$\mu(\mathscr{G}) - \mu(F_{X*}(\mathscr{L})) \le -\frac{p - \mathrm{rk}(\mathscr{G})}{p}(g-1) = -\frac{3 - \mathrm{rk}(\mathscr{G})}{3}.$$

It follows that

$$\mu(\mathscr{G}) \le -\frac{2 - \mathrm{rk}(\mathscr{G})}{3} \le 0.$$

Thus $\mathscr{E}$ is a semistable vector bundle.

Let x be a closed point of X. Then $F_{X*}(\mathscr{L}(-x))$ is a rank 3 and degree 0 stable subsheaf of $F_{X*}(\mathscr{L})$ by [5, Sect. 2.9] and Lemma 3.2(3). $\qquad\square$

By Proposition 3.3, Proposition 3.4 and the classification of Harder-Narasimhan polygons of Frobenius pull backs of Frobenius destabilized semistable vector bundles in the case $(p, g, r, d) = (3, 2, 3, 0)$, we have any Frobenius destabilized stable bundle $\mathscr{E} \in \mathfrak{M}_X^s(3, 0)$ with $\mathrm{HNP}(F_X^*(\mathscr{E})) \ne \mathscr{P}_1$ can be embedded into $F_X^*(\mathscr{L})$ for some line bundle $\mathscr{L}$ on X of degree -1.

4. Frobenius Stratification of Quot Schemes

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k, $F_X : X \to X$ the absolute Frobenius morphism. Let $P^t(T) \in \mathbb{Q}[T]$ be the Hilbert polynomial of $F_{X*}(\mathscr{L})$ for any line bundle $\mathscr{L}$ of degree t on X, and $P_{r,d}(T) \in \mathbb{Q}[T]$ the Hilbert polynomial of any vector bundle $\mathscr{F}$ of rank r and degree d on X. Denote

$$\Phi_{r,d}^t(T) := P^t(T) - P_{r,d}(T) \in \mathbb{Q}[T].$$

Let $\mathrm{Pic}^{(t)}(X)$ be the Picard scheme parameterizing all line bundles of degree t on X, $\mathcal{L}$ the universal line bundle on $\mathrm{Pic}^{(t)}(X) \times_k X$, and

$$\pi : \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) := \mathrm{Quot}_{(\mathrm{Id}_{\mathrm{Pic}^{(t)}(X)} \times F_X)_*(\mathcal{L})/\mathrm{Pic}^{(t)}(X) \times_k X/\mathrm{Pic}^{(t)}(X)}^{\Phi_{r,d}^t} \to \mathrm{Pic}^{(t)}(X)$$

the Quot scheme classifying the subsheaves of $(\mathrm{Id}_{\mathrm{Pic}^{(t)}(X)} \times F_X)_*(\mathcal{L})$ on $\mathrm{Pic}^{(t)}(X) \times_k X$, which are flat families of vector bundles of rank r and degree d on X parameterized by $\mathrm{Pic}^{(t)}(X)$. Then for any $\mathscr{L} \in \mathrm{Pic}^{(t)}(X)$, the fiber of π over $[\mathscr{L}]$ is the Quot scheme $\mathrm{Quot}_X(r, d, \mathscr{L}) := \mathrm{Quot}_{F_{X*}(\mathscr{L})/X/k}^{\Phi_{r,d}^t}$ which parameterizing all rank r and degree d subsheaves of $F_{X*}(\mathscr{L})$ on X.

Consider the commutative diagram of morphisms

$$
\begin{array}{ccc}
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) \times_k X & \xrightarrow{\mathrm{Id}_{\mathrm{Quot}} \times F_X} & \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) \times_k X \\
\downarrow {\scriptstyle \pi \times \mathrm{Id}_X} & & \downarrow {\scriptstyle \pi \times \mathrm{Id}_X} \\
\mathrm{Pic}^{(t)}(X) \times_k X & \xrightarrow{\mathrm{Id}_{\mathrm{Pic}^{(t)}(X)} \times F_X} & \mathrm{Pic}^{(t)}(X) \times_k X.
\end{array}
$$

and the universal subsheaf

$$\mathcal{E} \hookrightarrow (\pi \times \mathrm{Id}_X)^*(\mathrm{Id}_{\mathrm{Quot}} \times F_X)_*(\mathcal{L}) = (\mathrm{Id}_{\mathrm{Quot}} \times F_X)_*(\pi \times \mathrm{Id}_X)^*(\mathcal{L})$$

on $\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) \times_k X$. By adjunction, we have homomorphism

$$(\mathrm{Id}_{\mathrm{Quot}} \times F_X)^*(\mathcal{E}) \to (\pi \times \mathrm{Id}_X)^*(\mathcal{L})$$

on $\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) \times_k X$. Let $\mathscr{G}$ be the co-kernel of above homomorphism. Then $\mathrm{pr}_*(\mathscr{G})$ is a coherent sheaf on $\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))$, where

$$\mathrm{pr} : \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) \times_k X \to \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))$$

is the natural projection. Then the subset

$$\mathrm{Quot}_X^{\sharp}(r, d, \mathrm{Pic}^{(t)}(X)) := \{ q \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) |\, \dim_{\kappa(q)}(\mathrm{pr}_*(\mathscr{G})_q \otimes_{\mathscr{O}_q} \kappa(q)) = 0 \}$$

is an open subscheme of $\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))$. Then we have

$$\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))(k) = \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mid \mathrm{rk}(\mathscr{E}) = r, \deg(\mathscr{E}) = d, \mathscr{L} \in \mathrm{Pic}^{(t)}(X) \}$$

$$\mathrm{Quot}_X^{\sharp}(r, d, \mathrm{Pic}^{(t)}(X))(k) = \left\{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))(k) \,\middle|\, \begin{array}{l} \text{The adjoint homomorphism} \\ F_X^*(\mathscr{E}) \to \mathscr{L} \text{ is surjective.} \end{array} \right\}$$

Let $\mathscr{P} \in \mathfrak{ConPgn}(r, pd)$, we denote the subschemes of $\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))$ and $\mathrm{Quot}_X^{\sharp}(r, d, \mathrm{Pic}^{(t)}(X))$ as the following (For simplicity, we describe these Quot

schemes in the sense of closed points):

$$
\begin{aligned}
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) &:= \{\; [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P} \;\} \\
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) &:= \{\; [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathscr{P} \;\} \\
\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) &:= \{\; [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P} \;\} \\
\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) &:= \{\; [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathscr{P} \;\}.
\end{aligned}
$$

Let $\mathscr{L} \in \mathrm{Pic}^{(t)}(X)$. The fibers of the projections

$$
\begin{aligned}
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X) \\
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}) \hookrightarrow \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X) \\
\mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+) \hookrightarrow \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X)) \hookrightarrow \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X) \\
\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}) \hookrightarrow \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X)) \hookrightarrow \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X) \\
\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+) \hookrightarrow \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X)) \hookrightarrow \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X)) &\overset{\pi}{\to} \mathrm{Pic}^{(t)}(X).
\end{aligned}
$$

over the $[\mathscr{L}]$ are denoted by $\mathrm{Quot}_X(r, d, \mathscr{L})$, $\mathrm{Quot}_X(r, d, \mathscr{L}, \mathscr{P})$, $\mathrm{Quot}_X(r, d, \mathscr{L}, \mathscr{P}^+)$, $\mathrm{Quot}_X^\sharp(r, d, \mathscr{L})$, $\mathrm{Quot}_X^\sharp(r, d, \mathscr{L}, \mathscr{P})$ and $\mathrm{Quot}_X^\sharp(r, d, \mathscr{L}, \mathscr{P}^+)$ respectively. For example, the scheme $\mathrm{Quot}_X^\sharp(r, d, \mathscr{L}, \mathscr{P})$ parameterizing all rank r and degree d subsheaves $\mathscr{E} \subset F_{X*}(\mathscr{L})$ with surjective adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ such that $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}$.

In this section, we are interested in the Frobenius stratification of moduli spaces of vector bundles in the case

$$(p, g, r, d, t) = (3, 2, 3, 0, -1).$$

In this case, the scheme $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$ parameterizing all the rank 3 and degree 0 subsheaves of $F_{X*}(\mathscr{L})$ for any line bundle $\mathscr{L}$ of degree -1 on X. By Proposition 3.3 and Proposition 3.4, we know that these vector bundles are semistable. This induces a natural morphism

$$
\begin{aligned}
\theta : \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X)) &\to \mathfrak{M}_X^{ss}(3, 0) \\
[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] &\mapsto [\mathscr{E}].
\end{aligned}
$$

Now, we first analysis the structure of the $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$. Let

$$e := [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})]$$

be a closed point of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$, where $\mathscr{L} \in \mathrm{Pic}^{(-1)}(X)$. The nontrivial adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ implies that

$$\mu(F_X^*(\mathscr{E})) > \mu(\mathscr{L}) \geq \mu_{\min}(F_X^*(\mathscr{E})),$$

so $\mathscr{E}$ is a Frobenius destabilized semistable vector bundle.

Proposition 4.1. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Let $\mathscr{L}$ be a line bundle of degree -1 on X, $0 \subset E_2 \subset E_1 \subset F_X^*(F_{X*}(\mathscr{L}))$ the canonical filtration of $F_X^*(F_{X*}(\mathscr{L}))$. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, 0, \mathscr{L})(k)$. Then $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$, and*

(1) $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4$ *if and only if* $\deg(F_X^*(\mathscr{E}) \cap E_2) = 2$ *if and only if the adjoint homomorphism* $F_X^*(\mathscr{E}) \to \mathscr{L}$ *is not surjective. In this case, the Harder-Narasimhan filtration of* $F_X^*(\mathscr{E})$ *is*

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

(2) $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3$ *if and only if* $\deg(F_X^*(\mathscr{E}) \cap E_2) = 1$. *In this case,* $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(3, 0, \mathscr{L})(k)$ *and the Harder-Narasimhan filtration of* $F_X^*(\mathscr{E})$ *is*

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

(3) $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_2$ *if and only if* $\deg(F_X^*(\mathscr{E}) \cap E_2) = 0$. *In this case,* $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(3, 0, \mathscr{L})(k)$ *and the Harder-Narasimhan filtration of* $F_X^*(\mathscr{E})$ *is*

$$0 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

Proof. The canonical filtration $0 \subset E_2 \subset E_1 \subset F_X^*(F_{X*}(\mathscr{L}))$ induces the filtration

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

Then the injections $F_X^*(\mathscr{E}) \cap E_2 \hookrightarrow E_2$ and $F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1) \hookrightarrow F_X^*(F_{X*}(\mathscr{L}))/E_1$ imply that $\deg(F_X^*(\mathscr{E}) \cap E_2) \leq 3$ and $\deg(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) \leq -1$. Suppose that $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, 0, \mathscr{L})(k)$ such that $F_X^*(\mathscr{E})$ is semistable or $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_1$, then $\mu_{\min}(F_X^*(\mathscr{E})) \geq -\frac{1}{2}$. This contradicts the fact $\deg(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) \leq -1$. Hence $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$.

We first claim that

$$0 \leq \deg(F_X^*(\mathscr{E}) \cap E_2) \leq 2.$$

The fact $\deg(F_X^*(\mathscr{E}) \cap E_2) \leq 2$ is followed by [15, Theorem 2] and the classification of Harder-Narasimhan polygons of Frobenius pull backs of Frobenius destabilized semistable vector bundles in the case $(p, g, r, d) = (3, 2, 3, 0)$. On the other hand, suppose that $\deg(F_X^*(\mathscr{E}) \cap E_2) < 0$, then

$$\deg((F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2)) \geq 2.$$

Then the injection

$$(F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2) \hookrightarrow (F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) \otimes_{\mathscr{O}_X} \Omega_X^1$$

induces a contradiction, since $\deg((F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) \otimes_{\mathscr{O}_X} \Omega_X^1) \leq 1$. This completes the proof of the claim.

(1). I. If $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4$, there exists a unique maximal destabilizing subline bundle $E' \subset F_X^*(\mathscr{E})$ with $\deg(E') = 2$. Suppose that $E' \nsubseteq F_X^*(\mathscr{E}) \cap E_1$, then the composition

$$E' \hookrightarrow F_X^*(\mathscr{E}) \hookrightarrow F_X^*(F_{X*}(\mathscr{L})) \twoheadrightarrow F_X^*(F_{X*}(\mathscr{L}))/E_1 \cong \mathscr{L}$$

is non-trivial. This induces a contradiction since $\deg(E') > \deg(\mathscr{L})$. Suppose that $E' \subset F_X^*(\mathscr{E}) \cap E_1$ and $E' \nsubseteq F_X^*(\mathscr{E}) \cap E_2$, then the composition

$$E' \hookrightarrow F_X^*(\mathscr{E}) \cap E_1 \hookrightarrow E_1 \twoheadrightarrow E_1/E_2$$

is non-trivial. This induces a contradiction since $\deg(E') > \deg(E_1/E_2)$. Hence $E' \subset F_X^*(\mathscr{E}) \cap E_2$. In fact $E' = F_X^*(\mathscr{E}) \cap E_2$. Thus $\deg(F_X^*(\mathscr{E}) \cap E_2) = 2$.

II. If $\deg(F_X^*(\mathscr{E}) \cap E_2) = 2$, then the injection

$$F_X^*(\mathscr{E}) \cap E_2 \hookrightarrow ((F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2)) \otimes_{\mathscr{O}_X} \Omega_X^1$$

implies that $\deg((F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2)) \geq 0$. So $\deg(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)) \leq -2$. Hence the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is not surjective.

III. If $F_X^*(\mathscr{E}) \to \mathscr{L}$ is not surjective, then $\mu_{\min}(F_X^*(\mathscr{E})) \leq -2$. Then by [15, Theorem 2] and the classification of Harder-Narasimhan polygons of Frobenius pull backs of Frobenius destabilized semistable vector bundles in the case $(p, g, r, d) = (3, 2, 3, 0)$, we have $\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4$.

In this case, it is easy to see that the Harder-Narasimhan filtration $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

(2). I. If $\deg(F_X^*(\mathscr{E}) \cap E_2) = 1$, then the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is surjective by (1) and

$$\mu(F_X^*(\mathscr{E}) \cap E_2) > \mu((F_X^*(\mathscr{E}) \cap E_1)/(F_X^*(\mathscr{E}) \cap E_2)) > \mu(F_X^*(\mathscr{E})/(F_X^*(\mathscr{E}) \cap E_1)).$$

Hence, $\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3$ and the Harder-Narasimhan filtration of $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_2 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

II. If $\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3$, there exists a unique maximal destabilizing sub-line bundle $E' \subset F_X^*(\mathscr{E}) \cap E_1$ with $\deg(E') = 1$. Suppose that $E' \nsubseteq F_X^*(\mathscr{E}) \cap E_2$, then

$$E' \hookrightarrow F_X^*(\mathscr{E}) \cap E_1 \hookrightarrow E_1 \twoheadrightarrow E_1/E_2$$

is non-trivial. This implies $E' \cong E_1/E_2$ since E' and E_1/E_2 are line bundles with same degree. Then $E_1 = E' \oplus E''$ for some line bundle E'' of degree 3. This induces a contradiction by Corollary 4.4. Hence $E' \subseteq F_X^*(\mathscr{E}) \cap E_2$. In fact $E' = F_X^*(\mathscr{E}) \cap E_2$, so $\deg(F_X^*(\mathscr{E}) \cap E_2) = 1$.

(3). By the proof of (1) and (2), we can conclude that $\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_2$ if and only if $\deg(F_X^*(\mathscr{E}) \cap E_2) = 0$. In this case, $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \text{Quot}_X^\sharp(3, 0, \mathscr{L})(k)$ and the Harder-Narasimhan filtration $F_X^*(\mathscr{E})$ is

$$0 \subset F_X^*(\mathscr{E}) \cap E_1 \subset F_X^*(\mathscr{E}).$$

$\square$

Lemma 4.2 (A. Grothendieck, M. Raynaud). *Let k be an algebraically closed field of characteristic $p > 2$, X a smooth projective curve of genus $g \geq 2$ over k, $F : X \to X_1 := X \times_k k$ the relative Frobenius morphism over k. Let B_X^1 be the locally free sheaf of locally exact differential forms on X_1 defined by the exact sequence of locally free sheaves*

$$0 \to \mathscr{O}_{X_1} \to F_*(\mathscr{O}_X) \to B_X^1 \to 0.$$

Then the Harder-Narasimhan filtration of $F^*(B_X^1)$ is

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{l+1} \subset V_l \subset \cdots \subset V_1 = F^*(B_X^1)$$
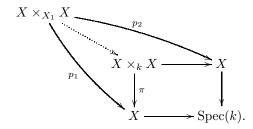
such that $V_i/V_{i+1} \cong \Omega_X^{\otimes i}$ for any $1 \leq i \leq p-1$, and $p|(g-1)$ if and only if

$$F^*(B_X^1) \cong \Omega_X^{\otimes p-1} \oplus \Omega_X^{\otimes p-2} \oplus \cdots \oplus \Omega_X^1.$$

In particular, in the case $p = 3$ and $g = 2$, we have

$$F^*(B_X^1) \ncong \Omega_X^{\otimes 2} \oplus \Omega_X^1.$$

Proof. Consider the fibred product of X and X over X_1

$$
\begin{array}{ccc}
X \times_{X_1} X & & \\
& \searrow^{p_2} & \\
& X \times_k X \longrightarrow X & \\
^{p_1}\swarrow & \downarrow^{\pi} & \downarrow \\
& X \longrightarrow \mathrm{Spec}(k).
\end{array}
$$

Let $\triangle$ be the diagonal which is defined by an invertible ideal sheaf $\mathscr{I}$ on $X \times_k X$. Then $X \times_{X_1} X$ is the $(p-2)$-th infinitesimal neighborhood of $\triangle$ in $X \times_k X$, which is defined by ideal sheaf $\mathscr{I}^{p-1}$, and we have a filtration of ideal sheaves

$$0 \subset \mathscr{I}^{p-1} \subset \mathscr{I}^{p-2} \subset \cdots \subset \mathscr{I}^2 \subset \mathscr{I} \subset \mathscr{O}_{X \times_k X}.$$

Taking direct images of above filtration under the first projection $\pi : X \times_k X \to X$, we can get the filtration

$$0 \subset \pi_*(\mathscr{I}^{p-1}) \subset \pi_*(\mathscr{I}^{p-2}) \subset \cdots \subset \pi_*(\mathscr{I}^2) \subset \pi_*(\mathscr{I}) \subset \pi_*(\mathscr{O}_{X \times_k X})$$

such that $\pi_*(\mathscr{I}^i)/\pi_*(\mathscr{I}^{i+1}) \cong \Omega_X^{\otimes i}$ for $1 \le i \le p-1$ and $\pi_*(\mathscr{I}) \cong F^*(B_X^1)$. The result of Grothendieck says: the extension of $\pi_*(\mathscr{I}^1)/\pi_*(\mathscr{I}^2)$ by $\pi_*(\mathscr{I}^2)/\pi_*(\mathscr{I}^3)$ corresponds to the element $(g-1)c$ of $H^1(X, \Omega_X^1)$, where c is the canonical base of $H^1(X, \Omega_X^1)$. Hence this extension is trivial iff p divides g-1. Then using $H^1(X, \Omega_X^i) = 0$ for $i \ge 2$, we see that, $F^*(B_X^1)$ is a direct sum of $\Omega_X^{\otimes i}$ for $1 \le i \le p-1$ iff $p|(g-1)$. $\qquad\square$

Corollary 4.3. *Let k be an algebraically closed field of characteristic $p > 2$, X a smooth projective curve of genus $g \ge 2$ over k, $F : X \to X_1 := X \times_k k$ the relative Frobenius morphism over k, and $\mathscr{L}$ a line bundle on X. Then the Harder-Narasimhan filtration of $F_X^* F_{X*}(\mathscr{L})$ is*

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{l+1} \subset V_l \subset \cdots \subset V_1 \subset V_0 = F^* F_*(\mathscr{L})$$

such that $V_i/V_{i+1} \cong \Omega_X^{\otimes i} \otimes \mathscr{L}$ for any $0 \le i \le p-1$, and $p|(g-1)$ if and only if

$$F_X^* F_{X*}(\mathscr{L}) \cong (\Omega_X^{\otimes p-1} \otimes \mathscr{L}) \oplus (\Omega_X^{\otimes p-2} \otimes \mathscr{L}) \oplus \cdots \oplus (\Omega_X^1 \otimes \mathscr{L}) \oplus \mathscr{L}.$$

In particular, in the case $p = 3$ and $g = 2$, we have

$$F_X^* F_{X*}(\mathscr{L}) \not\cong (\Omega_X^{\otimes 2} \otimes \mathscr{L}) \oplus (\Omega_X^1 \otimes \mathscr{L}) \oplus \mathscr{L}.$$

Proof. Applying Frobenius pull back to the exact sequence of locally free sheaves

$$0 \to \mathscr{O}_{X_1} \to F_*(\mathscr{O}_X) \to B_X^1 \to 0,$$

we have

$$0 \to \mathscr{O}_X \cong F^*(\mathscr{O}_{X_1}) \to F^* F_*(\mathscr{O}_X) \to F^*(B_X^1) \to 0.$$

On the other hand, it is easy to see that the composition $\mathscr{O}_X \cong F^*(\mathscr{O}_{X_1}) \to F^* F_*(\mathscr{O}_X) \to \mathscr{O}_X$ is an isomorphism. Therefore $F^* F_*(\mathscr{O}_X) = V_1 \oplus \mathscr{O}_X$ and $V_1 \cong F^*(B_X^1)$. Thus by Lemma 4.2 we have $p|(g-1)$ if and only if

$$F_X^* F_{X*}(\mathscr{O}_X) \cong \Omega_X^{\otimes p-1} \oplus \Omega_X^{\otimes p-2} \oplus \cdots \oplus \Omega_X^1 \oplus \mathscr{O}_X.$$

As $F^* F_*(\mathscr{L}) \cong F^* F_*(\mathscr{O}_X) \otimes \mathscr{L}$, so we have $p|(g-1)$ if and only if

$$F_X^* F_{X*}(\mathscr{L}) \cong (\Omega_X^{\otimes p-1} \otimes \mathscr{L}) \oplus (\Omega_X^{\otimes p-2} \otimes \mathscr{L}) \oplus \cdots \oplus (\Omega_X^1 \otimes \mathscr{L}) \oplus \mathscr{L}.$$

$\square$

Proposition 4.4. *Let k be an algebraically closed field of characteristic* 3, *X a smooth projective curve of genus* 2 *over k. Then*

$$\mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X),\mathscr{P}_i^+) = \overline{\mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X),\mathscr{P}_i)},$$

and $\mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X),\mathscr{P}_i^+)$ *are smooth irreducible projective varieties for* $2 \leq i \leq 4$,

$$\dim \mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X),\mathscr{P}_i^+) = \dim \mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X),\mathscr{P}_i) = \begin{cases} 5, \text{when } i=2 \\ 4, \text{when } i=3 \\ 3, \text{when } i=4 \end{cases}$$

Proof. By [3], there is a morphism

$$\begin{aligned} \Pi : \mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X)) &\rightarrow X \times \mathrm{Pic}^{(-1)}(X) \\ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] &\mapsto (\mathrm{Supp}(F_{X*}(\mathscr{L})/\mathscr{E}),\mathscr{L}). \end{aligned}$$

For any point $x \in X$ and any $\mathscr{L} \in \mathrm{Pic}^{(-1)}(X)$, we denote the fiber of Π over $(x,[\mathscr{L}])$ by $\mathrm{Quot}_X(3,0,\mathscr{L},x)$. Then there is a one to one correspondence between the set of closed points $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})]$ of $\mathrm{Quot}_X(3,0,\mathscr{L},x)$ and the set of $\mathscr{O}_x$-submodules $V \subset F_{X*}(\mathscr{L})_x$ such that $F_{X*}(\mathscr{L})_x/V \cong k$. The latter has a natural structure of algebraic variety which is isomorphic to projective space $\mathbb{P}_k^2$. Hence $\mathrm{Quot}_X(3,0,\mathrm{Pic}^{(-1)}(X))$ is a smooth irreducible projective variety of dimension 5. Without loss of generality, we can assume that $\mathscr{O}_x \cong k[[t^3]]$, then $F_{X*}(\mathscr{L})_x \cong k[[t]]$ endows with $k[[t^3]]$-module structure induced by injection $k[[t^3]] \hookrightarrow k[[t]]$ and

$$F_X^*(F_{X*}(\mathscr{L}))_x \cong k[[t]] \otimes_{k[[t^3]]} k[[t]].$$

Suppose that the $\mathscr{O}_x$-submodule $\mathscr{E}_x$ of $F_{X*}(\mathscr{L})_x$ corresponds to the $k[[t^3]]$-submodule $V_{\mathscr{E}}$ of $k[[t]]$, then the $\mathscr{O}_x$-submodule $F_X^*(\mathscr{E})_x$ of $F_X^*(F_{X*}(\mathscr{L}))_x$ corresponds to the $k[[t]]$-submodule $V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ of $k[[t]] \otimes_{k[[t^3]]} k[[t]]$.

Consider the decomposition of $k[[t]] = k[[t^3]] \oplus k[[t^3]] \cdot t \oplus k[[t^3]] \cdot t^2$ as $k[[t^3]]$-module. Then the $k[[t^3]]$-submodule $V_{\mathscr{E}} \subset k[[t]]$ with $k[[t]]/V_{\mathscr{E}} \cong k$ implies that

$$k[[t^3]] \cdot t^3 \oplus k[[t^3]] \cdot t^4 \oplus k[[t^3]] \cdot t^5 \subset V_{\mathscr{E}}.$$

Now, we investigate the intersection of $F_X^*(\mathscr{E})$ with the canonical filtration

$$0 \subset E_2 \subset E_1 \subset F_X^*(F_{X*}(\mathscr{L})).$$

Locally, the stalk E_{1x} has a basis $\{t \otimes 1 - 1 \otimes t, (t \otimes 1 - 1 \otimes t)^2\}$ and E_{2x} has a basis $\{(t \otimes 1 - 1 \otimes t)^2\}$ as $k[[t]]$-submodules of $F_X^*(F_{X*}(\mathscr{L}))_x \cong k[[t]] \otimes_{k[[t^3]]} k[[t]]$ by [17, Lemma 3.2]. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3,0,\mathscr{L},x)(k)$, we claim that

(a) $(t \otimes 1 - 1 \otimes t)^2 \notin V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$
(b) $(t \otimes 1 - 1 \otimes t)^2 t \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ if and only if $\{t,t^2\} \subset V_{\mathscr{E}}$.
(c) $(t \otimes 1 - 1 \otimes t)^2 t^2 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ if and only if $t^2 \in V_{\mathscr{E}}$.
(d) $(t \otimes 1 - 1 \otimes t)^2 t^3 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$.

Suppose that $(t \otimes 1 - 1 \otimes t)^2 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$, then we have

$$V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]] = (t \otimes 1 - 1 \otimes t)^2 k[[t]].$$

It follows that $F_X^*(\mathscr{E}) \cap E_2 = E_2$, so $\deg(F_X^*(\mathscr{E}) \cap E_2) = 3$. This contradicts to Proposition 4.1.

Since
$$(t \otimes 1 - 1 \otimes t)^2 t \quad = \quad t^2 \otimes t - 2t \otimes t^2 + 1 \otimes t^3$$
$$= \quad t^2 \otimes t - 2t \otimes t^2 + t^3 \otimes 1$$

and $\{t^3\} \subset V_{\mathscr{E}}$ by $k[[t^3]] \cdot t^3 \oplus k[[t^3]] \cdot t^4 \oplus k[[t^3]] \cdot t^5 \subset V_{\mathscr{E}}$, then we have
$(t \otimes 1 - 1 \otimes t)^2 t \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ if and only if $t^2 \otimes t - 2t \otimes t^2 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$,
which is equivalent to $\{t, t^2\} \subset V_{\mathscr{E}}$.

Since
$$(t \otimes 1 - 1 \otimes t)^2 t^2 \quad = \quad t^2 \otimes t^2 - 2t \otimes t^3 + 1 \otimes t^4$$
$$= \quad t^2 \otimes t^2 - 2t^4 \otimes 1 + t^3 \otimes t$$

and $\{t^3, t^4\} \subset V_{\mathscr{E}}$ by $k[[t^3]] \cdot t^3 \oplus k[[t^3]] \cdot t^4 \oplus k[[t^3]] \cdot t^5 \subset V_{\mathscr{E}}$, then we have
$\quad (t \otimes 1 - 1 \otimes t)^2 t^2 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$ if and only if $t^2 \otimes t^2 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$,
which is equivalent to $x^2 \in V_{\mathscr{E}}$.

Since
$$(t \otimes 1 - 1 \otimes t)^2 t^3 \quad = \quad t^2 \otimes t^3 - 2t \otimes t^4 + 1 \otimes t^5$$
$$= \quad t^5 \otimes 1 - 2t^4 \otimes t + t^3 \otimes t^2$$

and $\{t^3, t^4, t^5\} \subset V_{\mathscr{E}}$ by $k[[t^3]] \cdot t^3 \oplus k[[t^3]] \cdot t^4 \oplus k[[t^3]] \cdot t^5 \subset V_{\mathscr{E}}$. It follows that
$(t \otimes 1 - 1 \otimes t)^2 t^3 \in V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]$.

In summary, by above claim, we have
$$1 \le \dim E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x}) \le 3.$$

Precisely,

$$\dim E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x}) = \begin{cases} 1 & \text{if ond only if } \{t, t^2\} \subset V_{\mathscr{E}} \\ 2 & \text{if ond only if } t \notin V_{\mathscr{E}} \text{ and } t^2 \in V_{\mathscr{E}} \\ 3 & \text{if ond only if } t^2 \notin V_{\mathscr{E}} \end{cases}$$

Consider the exact sequence of $\mathscr{O}_X$-modules
$$0 \to F_X^*(\mathscr{E}) \cap E_2 \to E_2 \to E_2/(F_X^*(\mathscr{E}) \cap E_2) \to 0.$$

Notice that $E_2/(F_X^*(\mathscr{E}) \cap E_2) = E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x})$. Therefore, by Proposition 4.1, we have

$$\begin{aligned}
\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_2 \quad &\Leftrightarrow \quad \deg(F_X^*(\mathscr{E}) \cap E_2) = 0 \\
&\Leftrightarrow \quad \deg(E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x})) = 3 \\
&\Leftrightarrow \quad t^2 \notin V_{\mathscr{E}}.
\end{aligned}$$

$$\begin{aligned}
\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_3 \quad &\Leftrightarrow \quad \deg(F_X^*(\mathscr{E}) \cap E_2) = 1 \\
&\Leftrightarrow \quad \deg(E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x})) = 2 \\
&\Leftrightarrow \quad t \notin V_{\mathscr{E}} \text{ and } t^2 \in V_{\mathscr{E}}.
\end{aligned}$$

$$\begin{aligned}
\text{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_4 \quad &\Leftrightarrow \quad \deg(F_X^*(\mathscr{E}) \cap E_2) = 2 \\
&\Leftrightarrow \quad \deg(E_{2x}/((V_{\mathscr{E}} \otimes_{k[[t^3]]} k[[t]]) \cap E_{2x})) = 1 \\
&\Leftrightarrow \quad \{t, t^2\} \subset V_{\mathscr{E}}.
\end{aligned}$$

For $i = 2, 3, 4$, let $\mathrm{Quot}_X(3, d, \mathscr{L}_x, \mathscr{P}_i^+(d))$ be the closed subschemes of $\mathrm{Quot}_X(3, d, \mathscr{L}_x)$ consisting of closed points

$$\mathrm{Quot}_X(3, 0, \mathscr{L}, x, \mathscr{P}_i^+)(k) = \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, 0, \mathscr{L}, x)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \succcurlyeq \mathscr{P}_i \}$$

Then

$$\mathrm{Quot}_X(3, 0, \mathscr{L}, x, \mathscr{P}_2^+) \cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k\} \cong \mathbb{P}_k^2$$
$$\mathrm{Quot}_X(3, 0, \mathscr{L}, x, \mathscr{P}_3^+) \cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k, t^2 \in V_{\mathscr{E}}\} \cong \mathbb{P}_k^1$$
$$\mathrm{Quot}_X(3, 0, \mathscr{L}, x, \mathscr{P}_4^+) \cong \{V \mid k[[t^3]]\text{-submodule } V \subset k[[t]], k[[t]]/V \cong k, \{t, t^2\} \subset V_{\mathscr{E}}\} \cong \{p\}.$$

So $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+)$ are smooth irreducible projective varieties for $2 \leq i \leq 4$, and

$$\dim \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+) = \begin{cases} 5, \text{ when } i = 2 \\ 4, \text{ when } i = 3 \\ 3, \text{ when } i = 4 \end{cases}$$

Since $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i)$ is an open subvariety of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+)$ for any $2 \leq i \leq 4$, we have

$$\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+) = \overline{\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i)},$$
$$\dim \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+) = \dim \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i).$$

$$\square$$

5. Frobenius Stratification of Moduli Spaces

We first introduce some notations which will be used in this section. Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k. Let r, d and t be integers with $r > 0$, and $\mathscr{P} \in \mathfrak{ConPgn}(r, pd)$. Then we define the subschemes of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$ and $\mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})$ in the sense of closed points as the following:

$$\mathrm{Quot}_X^s(r, d, \mathrm{Pic}^{(t)}(X))(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathscr{E} \text{ is stable } \}$$
$$\mathrm{Quot}_X^s(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) \mid \mathscr{E} \text{ is stable } \}.$$
$$\mathrm{Quot}_X^s(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) \mid \mathscr{E} \text{ is stable } \}$$
$$\mathrm{Quot}_X^{s,\sharp}(r, d, \mathrm{Pic}^{(t)}(X))(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X))(k) \mid \mathscr{E} \text{ is stable } \}$$
$$\mathrm{Quot}_X^{s,\sharp}(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P})(k) \mid \mathscr{E} \text{ is stable } \}.$$
$$\mathrm{Quot}_X^{s,\sharp}(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) := \{ [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^\sharp(r, d, \mathrm{Pic}^{(t)}(X), \mathscr{P}^+)(k) \mid \mathscr{E} \text{ is stable } \}$$

We now study the geometric properties of Frobenius strata in the Frobenius stratification of moduli space $\mathfrak{M}_X^s(3, 0)$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

In the case $(p, g, r, d) = (3, 2, 3, 0)$, by Proposition 3.4, there is a proper morphism

$$\theta : \mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X)) \rightarrow \mathfrak{M}_X^{ss}(3, 0)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \mapsto [\mathscr{E}].$$

By restricting the morphism θ to the stable locus $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$ of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X))$, then we have a morphism from $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$ to $\mathfrak{M}_X^s(3, 0)$, denoted by θ^s. Hence, θ^s is a proper morphism.

Proposition 5.1. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Then the image of the morphism*

$$\theta^s : \mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X)) \ \rightarrow \ \mathfrak{M}_X^s(3, 0)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \ \mapsto \ [\mathscr{E}]$$

is the subset

$$\{ \ [\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k) \ | \ \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\} \ \}.$$

Moreover, the restriction $\theta^s|_{\mathrm{Quot}_X^{s,\sharp}(3,0,\mathrm{Pic}^{(-1)}(X))}$ is an injective morphism and the image of $\theta^s|_{\mathrm{Quot}_X^{s,\sharp}(3,0,\mathrm{Pic}^{(-1)}(X))}$ is the subset

$$\{ \ [\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k) \ | \ \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3\} \ \}.$$

Proof. Let $[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X(3, 0, \mathscr{L})(k)$, then $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$ by Proposition 4.1. It follows that the image of θ lies in the following subset

$$\{ \ [\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k) \ | \ \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\} \ \}.$$

On the other hand, let $[\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k)$ such that $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\}$. Then $F_X^*(\mathscr{E})$ has a quotient line bundle $\mathscr{L}'$ of $\deg(\mathscr{L}') \leq -1$. Embedding $\mathscr{L}'$ into some line bundle $\mathscr{L}$ of $\deg(\mathscr{L}) = -1$, then we have the non-trivial homomorphism

$$F_X^*(\mathscr{E}) \twoheadrightarrow \mathscr{L}' \hookrightarrow \mathscr{L}.$$

Then the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection by Proposition 3.3. Hence, the image of θ is

$$\{ \ [\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k) \ | \ \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4\} \ \}.$$

Now, we will prove $\theta|_{\mathrm{Quot}_X^{s,\sharp}(3,0,\mathrm{Pic}^{(-1)}(X))}$ is an injective morphism. Let

$$e_i := [\mathscr{E}_i \hookrightarrow F_{X*}(\mathscr{L}_i)] \in \mathrm{Quot}_X^{s,\sharp}(3, 0, \mathrm{Pic}^{(-1)}(X))(k),$$

where $\mathscr{L}_i \in \mathrm{Pic}^{(-1)}(X)$, $i = 1, 2$. Suppose that $\theta(e_1) = \theta(e_2) \in \mathfrak{M}_X^s(3, 0)(k)$, i.e. $\mathscr{E}_1 \cong \mathscr{E}_2$. Since $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3\}$, we have $\mu_{\min}(F_X^*(\mathscr{E}_i)) = -1$. So the surjection $F_X^*(\mathscr{E}_i) \rightarrow \mathscr{L}_i$ implies that $\mathscr{L}_i$ is the quotient line bundle of $F_X^*(\mathscr{E}_i)$ with minimal slope in the Harder-Narasimhan filtration of $F_X^*(\mathscr{E}_i)$ for $i = 1, 2$. By the uniqueness of Harder-Narasimhan filtration, there exists an isomorphism $\psi : \mathscr{L}_1 \rightarrow \mathscr{L}_2$ making the following diagram

$$
\begin{array}{ccccc}
F_X^*(\mathscr{E}_1) & \longrightarrow & \mathscr{L}_1 & \longrightarrow & 0 \\
\phi \downarrow \cong & & \psi \downarrow \cong & & \\
F_X^*(\mathscr{E}_2) & \longrightarrow & \mathscr{L}_2 & \longrightarrow & 0
\end{array}
$$

commutative, where the isomorphism ϕ is induced from an isomorphism $\mathscr{E}_1 \xrightarrow{\cong} \mathscr{E}_2$. By adjunction, we have the commutative diagram

$$
\begin{array}{ccccc}
0 \longrightarrow & \mathscr{E}_1 & \longrightarrow & F_{X*}(\mathscr{L}_1) \\
 & \downarrow \cong & F_{X*}(\psi) \downarrow \cong & \\
0 \longrightarrow & \mathscr{E}_2 & \longrightarrow & F_{X*}(\mathscr{L}_2)
\end{array}
$$

where the vertical homomorphism is the isomorphism

$$F_{X*}(\psi) : F_{X*}(\mathscr{L}_1) \xrightarrow{\cong} F_{X*}(\mathscr{L}_2).$$

This implies $\mathscr{E}_1 = \mathscr{E}_2$ as subsheaves of $F_{X*}(\mathscr{L})$, where

$$[\mathscr{L}] = [\mathscr{L}_1] = [\mathscr{L}_2] \in \mathrm{Pic}^{(-1)}(X)(k).$$

Thus, e_1 and e_2 are the some point in the $\mathrm{Quot}_X^{s,\sharp}(3, 0, \mathrm{Pic}^{(-1)}(X))$. Hence the morphism $\theta|_{\mathrm{Quot}_X^{s,\sharp}(3,0,\mathrm{Pic}^{(-1)}(X))}$ is injective.

Let $[\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k)$ and $\mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3\}$. Then $F_X^*(\mathscr{E})$ has a quotient line bundle $\mathscr{L}$ of $\deg(\mathscr{L}) = -1$. Then by Proposition 3.3, the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injective homomorphism. Therefore

$$e := [\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \in \mathrm{Quot}_X^{s,\sharp}(3, 0, \mathrm{Pic}^{(-1)}(X))(k)$$

and $\theta(e) = [\mathscr{E}]$. Hence the image of $\theta|_{\mathrm{Quot}_X^{s,\sharp}(3,0,\mathrm{Pic}^{(-1)}(X))}$ is the subset

$$\{\ [\mathscr{E}] \in \mathfrak{M}_X^s(3, 0)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) \in \{\mathscr{P}_2, \mathscr{P}_3\}\ \}.$$

$\square$

We can obtain the geometric properties of Frobenius strata in the Frobenius stratification of moduli space $\mathfrak{M}_X^s(3, 0)$ from the geometric properties of Frobenius strata in $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X))$, where X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

Theorem 5.2. *Let k be an algebraically closed field of characteristic 3, X a smooth projective curve of genus 2 over k. Then*

(1) $S_X(3, 0, \mathscr{P}_1^+) \cong S_X(3, 0, \mathscr{P}_2^+)$, $S_X(3, 0, \mathscr{P}_1) \cong S_X(3, 0, \mathscr{P}_2)$, *and*

$$S_X(3, 0, \mathscr{P}_1^+) \cap S_X(3, 0, \mathscr{P}_2^+) = S_X(3, 0, \mathscr{P}_3^+).$$

(2) $S_X(3, 0, \mathscr{P}_i^+) = \overline{S_X(3, 0, \mathscr{P}_i)}$, $S_X(3, 0, \mathscr{P}_i)$ *and* $S_X(3, 0, \mathscr{P}_i^+)$ *are irreducible quasi-projective varieties for* $1 \leq i \leq 4$, *and*

$$\dim S_X(3, 0, \mathscr{P}_i^+) = \dim S_X(3, 0, \mathscr{P}_i) = \begin{cases} 5, \text{when } i = 1 \\ 5, \text{when } i = 2 \\ 4, \text{when } i = 3 \\ 2, \text{when } i = 4 \end{cases}$$

Proof. (1). The morphism

$$\iota : \mathfrak{M}_X^s(3, 0) \to \mathfrak{M}_X^s(3, 0)$$

$$[\mathscr{E}] \mapsto [\mathscr{E}^\vee]$$

induces an involution of $\mathfrak{M}_X^s(3, 0)$, which maps $S_X(3, 0, \mathscr{P}_1^+)$ (resp. $S_X(3, 0, \mathscr{P}_1)$) onto $S_X(3, 0, \mathscr{P}_2^+)$ (resp. $S_X(3, 0, \mathscr{P}_2)$). Hence we have

$$S_X(3, 0, \mathscr{P}_1^+) \cong S_X(3, 0, \mathscr{P}_2^+),$$

$$S_X(3, 0, \mathscr{P}_1) \cong S_X(3, 0, \mathscr{P}_2).$$

The fact $S_X(3, 0, \mathscr{P}_1^+) \cap S_X(3, 0, \mathscr{P}_2^+) = S_X(3, 0, \mathscr{P}_3^+)$ is followed by the classification of Harder-Narasimhan polygons of Frobenius pull backs of Frobenius destabilized stable vector bundles in the case $(p, g, r, d) = (3, 2, 3, 0)$.

(2). By Proposition 3.4 and the openness of stability of vector bundles, we have $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+)$ are non-empty open subvarieties of $\mathrm{Quot}_X(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+)$ for $2 \leq i \leq 4$. Hence, $\mathrm{Quot}_X^s(3, 0, \mathrm{Pic}^{(-1)}(X), \mathscr{P}_i^+)$ are smooth quasi-projective varieties for $2 \leq i \leq 4$.

The morphism

$$\theta^s : \operatorname{Quot}_X^s(3,0,\operatorname{Pic}^{(-1)}(X)) \quad \to \quad \mathfrak{M}_X^s(3,0)$$
$$[\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})] \quad \mapsto \quad [\mathscr{E}]$$

maps $\operatorname{Quot}_X^s(3,0,\operatorname{Pic}^{(-1)}(X),\mathscr{P}_i^+)$ onto $S_X(3,0,\mathscr{P}_i^+)$ for $2 \le i \le 4$. Then by Proposition 4.4 and properness of θ^s, we have $S_X(3,0,\mathscr{P}_i^+)$ are irreducible quasi-projective varieties for $2 \le i \le 4$. Since $S_X(3,0,\mathscr{P}_i)$ is an open subvariety of $S_X(3,0,\mathscr{P}_i^+)$, we have

$$S_X(3,0,\mathscr{P}_i^+) = \overline{S_X(3,0,\mathscr{P}_i)}$$

for any $2 \le i \le 4$. Moreover, by Proposition 5.1, the injection $\theta^s|_{\operatorname{Quot}_X^{s,\sharp}(3,0,\operatorname{Pic}^{(-1)}(X))}$ maps $\operatorname{Quot}_X^s(3,0,\operatorname{Pic}^{(-1)}(X),\mathscr{P}_i)$ onto $S_X(3,0,\mathscr{P}_i)$ for $i = 2,3$. Then by Proposition 4.4, we have

$$\dim S_X(3,0,\mathscr{P}_i^+) = \dim S_X(3,0,\mathscr{P}_i) = \dim \operatorname{Quot}_X^s(3,0,\operatorname{Pic}^{(-1)}(X),\mathscr{P}_i) = \begin{cases} 5 & i = 2 \\ 4 & i = 3 \end{cases}$$

As $S_X(3,0,\mathscr{P}_1^+) \cong S_X(3,0,\mathscr{P}_2^+)$, so $S_X(3,d,\mathscr{P}_1^+) = \overline{S_X(3,0,\mathscr{P}_1^+)}$ is an irreducible quasi-projective variety and $\dim S_X(3,d,\mathscr{P}_1^+) = \dim S_X(3,d,\mathscr{P}_1) = 5$.

Now we study the properties of stratum $S_X(3,0,\mathscr{P}_4)$. Any $[\mathscr{E}] \in S_X(3,0,\mathscr{P}_4)$ has the form $F_{X*}(\mathscr{L}')$ for some line bundle $\mathscr{L}'$ of $\deg(\mathscr{L}') = -2$ by [10, Lemma 3.1] (or Lemma 5.4). Moreover, by Lemma 5.3, the morphism

$$P_{\operatorname{Frob}}^s : \mathfrak{M}_X^s(1,-2) \quad \to \quad \mathfrak{M}_X^s(3,0)$$
$$[\mathscr{L}'] \quad \mapsto \quad [F_{X*}(\mathscr{L}')]$$

is a closed immersion and the image of $P_{\operatorname{Frob}}^s$ is just the $S_X(3,0,\mathscr{P}_4)$. Thus $S_X(3,0,\mathscr{P}_4) = S_X(3,0,\mathscr{P}_4^+)$ is isomorphic to Jacobian variety Jac_X of X which is a smooth irreducible projective variety of dimension 2. $\qquad\square$

Lemma 5.3 (L. Li [10]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \ge 2$ over k. Then the well-defined set-theoretic map*

$$P_{\operatorname{Frob}}^s : \mathfrak{M}_X^s(r,d) \quad \to \quad \mathfrak{M}_X^s(rp, d + r(p-1)(g-1))$$
$$[\mathscr{E}] \quad \mapsto \quad [F_{X*}(\mathscr{E})]$$

is a closed immersion.

For any integer d with $3|d$, we can construct an isomorphism of

$$\mathfrak{M}_X^s(3,d) \cong \mathfrak{M}_X^s(3,0)$$

by tensoring any given line bundle of degree $\frac{d}{3}$. Then the Frobenius stratification of $\mathfrak{M}_X^s(3,d)$ is easily deduced from $\mathfrak{M}_X^s(3,0)$, when X is a smooth projective curve of genus 2 over an algebraically closed field k of characteristic 3.

Now, we will study the geometric properties of a specific Frobenius stratum in the moduli spaces of stable vector bundles of higher rank. We first generalize [10, Lemma 3.1] to the higher rank case.

Fix a quadruple (p,g,r,d), we construct a convex polygon $\mathscr{P}_{rp,d}^{\operatorname{can}} \in \mathfrak{ConPgn}(r,pd)$ as follows:

$$\mathscr{P}_{rp,d}^{\operatorname{can}} : \text{ with vertexes } \left(i \cdot r, d \cdot i + r \cdot i(p-i)(g-1)\right) \text{ for } 0 \le i \le p.$$

It is easy to check that the difference between the slopes of two successive line segments is $2g - 2$.

Lemma 5.4 (C. Liu, M. Zhou [11]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k. Let $\mathscr{E}$ be a stable vector bundle of rank rp on X. Then the following statements are equivalent:*

- *(i) $\mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_{rp,d}^{\mathrm{can}}$.*
- *(ii) $\mu_{\max}(F_X^*(\mathscr{E})) - \mu_{\min}(F_X^*(\mathscr{E})) = (p-1)(2g-2)$.*
- *(iii) There exists a stable vector bundle $\mathscr{F}$ such that $\mathscr{E} = F_{X*}(\mathscr{F})$.*

Theorem 5.5. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k. Then the subset*

$$V_{rp,d} = \{[\mathscr{E}] \in \mathfrak{M}_X^s(rp, d)(k) \mid \mathrm{HNP}(F_X^*(\mathscr{E})) = \mathscr{P}_{rp,d}^{\mathrm{can}}\}$$

is a smooth irreducible closed subvariety of dimension $r^2(g-1) + 1$ in $\mathfrak{M}_X^s(rp, d)$.

Proof. By Lemma 5.4, we know that $V_{rp,d}$ is precisely the image of the morphism $P_{\mathrm{Frob}}^s : \mathfrak{M}_X^s(r, d - r(p-1)(g-1)) \to \mathfrak{M}_X^s(rp, d)$. Then this Theorem is followed by Lemma 5.3. $\square$

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