ON WELL POSEDNESS FOR THE INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION

CARLOS M. GUZMÁN

ABSTRACT. The purpose of this paper is to study well-posedness of the initial value problem (IVP) for the inhomogeneous nonlinear Schrödinger equation (INLS)

$$iu_t + \Delta u + \lambda |x|^{-b} |u|^{\alpha} u = 0,$$

where $\lambda = \pm 1$ and $\alpha, b > 0$.

We obtain local and global results for initial data in $H^s(\mathbb{R}^N)$, with $0 \le s \le 1$. To this end, we use the contraction mapping principle based on the Strichartz estimates related to the linear problem.

1. Introduction

In this work, we study the initial value problem (IVP), also called the Cauchy problem, for the inhomogenous nonlinear Schrödinger equation (INLS)

$$\begin{cases}
i\partial_t u + \Delta u + \lambda |x|^{-b} |u|^{\alpha} u = 0, & t \in \mathbb{R}, \ x \in \mathbb{R}^N, \\
u(0, x) = u_0(x),
\end{cases}$$
(1.1)

where u=u(t,x) is a complex-valued function in space-time $\mathbb{R}\times\mathbb{R}^N$, $\lambda=\pm 1$ and $\alpha,b>0$. The equation is called "focusing INLS" when $\lambda=+1$ and "defocusing INLS" when $\lambda=-1$.

In the end of the last century, it was suggested that stable high power propagation can be achieved in a plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel, see Gill [14] and Liu-Tripathi [23]. In this case, the beam propagation can be modeled by the inhomogeneous nonlinear Schrödinger equation in the following form:

$$i\partial_t u + \Delta u + K(x)|u|^\alpha u = 0.$$

This model has been investigated by several authors, see, for instance, Merle [24] and Raphaël-Szeftel [25], for $k_1 < K(x) < k_2$ with $k_1, k_2 > 0$, and Fibich-Wang [11], for $K(\epsilon|x|)$ with ϵ small and $K \in C^4(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$. However, in these works K(x) is bounded which is not verified in our case.

Our main goal here is to establish local and global results for the Cauchy problem (1.1) in $H^s(\mathbb{R}^N)$, with $0 \le s \le 1$ applying Kato's method. Indeed, we construct a closed subspace of $C([-T,T];H^s(\mathbb{R}^N))$ such that the operator defined by

$$G(u)(t) = U(t)u_0 + i\lambda \int_0^t U(t - t')|x|^{-b}|u(t')|^{\alpha}u(t')dt',$$
 (1.2)

where U(t) denotes the solution to the linear problem $i\partial_t u + \Delta u = 0$, with initial data u_0 , is stable and contractive in this space. Thus by the contraction mapping principle we obtain a unique fixed point. The fundamental tool to prove these

results are the classic Strichartz estimates satisfied by the solution of the linear Schrödinger equation.

Notice that if u(t,x) is solution of (1.1) so is $u_{\delta}(t,x) = \delta^{\frac{2-b}{\alpha}} u(\delta^2 t, \delta x)$, with initial data $u_{0,\delta}(x)$ for all $\delta > 0$. Computing the homogeneous Sobolev norm we get

$$||u_{0,\delta}||_{\dot{H}^s} = \delta^{s - \frac{N}{2} + \frac{2-b}{\alpha}} ||u_0||_{\dot{H}^s}.$$

Hence, the scale-invariant Sobolev norm is $H^{s_c}(\mathbb{R}^N)$ with $s_c = \frac{N}{2} - \frac{2-b}{\alpha}$ (critical Sobolev index). Note that, if $s_c = 0$ (alternatively $\alpha = \frac{4-2b}{N}$) the problem is known as the mass-critical or L^2 -critical; if $s_c = 1$ (alternatively $\alpha = \frac{4-2b}{N-2}$) it is called energy-critical or \dot{H}^1 -critical, finally the problem is known as mass-supercritical and energy-subcritical if $0 < s_c < 1$. On the other hand, the inhomogeneous nonlinear Schrödinger equation has the following conserved quantities:

$$Mass \equiv M[u(t)] = \int_{\mathbb{R}^N} |u(t,x)|^2 dx = M[u_0]$$
 (1.3)

and

$$Energy \equiv E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^{N}} |\nabla u(t,x)|^{2} dx - \frac{\lambda}{\alpha+2} \left\| |x|^{-b} |u|^{\alpha+2} \right\|_{L_{x}^{1}} = E[u_{0}].$$

The well-posedness theory for the INLS equation (1.1) was studied for many authors in recent years. Let us briefly recall the best results available in the literature. Cazenave [2] studied the well-posedness in $H^1(\mathbb{R}^N)$ using an abstract theory. To do this, he analyzed (1.1) in the sense of distributions, that is, $i\partial_t u + \Delta u + |x|^{-b}|u|^{\alpha}u = 0$ in $H^{-1}(\mathbb{R}^N)$ for almost all $t \in I$. Therefore, using some results of Functional Analysis and Semigroups of Linear Operators, he proved that it is appropriate to seek solutions of (1.1) satisfying

$$u \in C([0,T); H^1(\mathbb{R}^N)) \cap C^1([0,T); H^{-1}(\mathbb{R}^N))$$
 for some $T > 0$.

It was also proved that for the defocusing case $(\lambda = -1)$ any local solution of the IVP (1.1) with $u_0 \in H^1(\mathbb{R}^N)$ extends globally in time.

Other authors like Genoud-Stuart [13] (see also references therein) also studied this problem for the focusing case ($\lambda=1$). Using the abstract theory developed by Cazenave [2], they showed that the IVP (1.1) is locally well-posed in $H^1(\mathbb{R}^N)$ if $0 < \alpha < 2^*$, where

$$2^* := \begin{cases} \frac{4-2b}{N-2} & N \ge 3, \\ \infty & N = 1, 2. \end{cases}$$
 (1.4)

Recently, using some sharp Gagliardo-Nirenberg inequalities, Genoud [12] and Farah [10] extended for the focusing INLS equation (1.1) some global well-posedness results obtained, respectively, by Weinstein [27] for the L^2 -critical NLS equation and by Holmer-Roudenko [18] for the L^2 -supercritical and H^1 -subcritical case. These authors proved that the solution u of the Cauchy problem (1.1) is globally defined in $H^1(\mathbb{R}^N)$ quantifying the smallness condition in the initial data.

However, the abstract theory developed by Cazenave and later used by Genoud-Stuart [13] to show well-posedness for (1.1), does not give sufficient tools to study other interesting questions, for instance, scattering and blow up investigated by Kenig-Merle [21], Holmer-Roudenko-Duyckaerts [9] and others, for the NLS equation. To study these problems, the authors rely on the Strichartz estimates for NLS equation and the classical fixed point argument combining with the concentration-compactness and rigidity technique.

Inspired by these papers and working toward the proof of scattering and blow up for the INLS equation, we show the well-posedness for the IVP (1.1) using the classic Strichartz estimates and the contraction mapping principle.

Applying this technique in the case b = 0 (classical nonlinear Schrödinger equation (NLS)), the IVP (1.1) has been extensively studied over the three decades. The L²-theory was obtained by Y. Tsutsumi [26] in the case $0 < \alpha < \frac{4}{N}$. The H^1 -subcritical case was studied by Ginibre-Velo [15]-[16] and Kato [19] (these papers also consider nonlinearities much more general than a pure power). Later, Cazenave-Weissler [4] treated the L^2 -critical case and the H^1 -critical case.

We summarize the well known well-posedness theory for the NLS equation in the following theorem (we refer, for instance, to Linares-Ponce [22] for a proof of these results).

Theorem 1.1. Consider the Cauchy problem for the NLS equation ((1.1) with b=0). Then, the following statements hold

- (1) If $0 < \alpha < \frac{4}{N}$, then the IVP (1.1) is locally and globally well posed in $L^2(\mathbb{R}^N)$. Moreover if $\alpha = \frac{4}{N}$, it is globally well posed in $L^2(\mathbb{R}^N)$ for small
- (2) The IVP (1.1) with b=0 is locally well posed in $H^1(\mathbb{R}^N)$ if $0 < \alpha \le \frac{4}{N-2}$ for $N \ge 3$ or $0 < \alpha < +\infty$, for N=1,2. Also, it is globally well-posed in $H^1(\mathbb{R}^N)$ if
 - (i) $\lambda < 0$,

 - $\begin{array}{ll} \text{(ii)} \ \, \lambda > 0 \ \, and \, \, 0 < \alpha < \frac{4}{N}, \\ \text{(iii)} \ \, \lambda > 0, \, \frac{4}{N} < \alpha < \frac{4}{N-2} \, \, and \, \, small \, \, initial \, \, data, \\ \text{(iv)} \ \, \lambda > 0, \, \, \alpha = \frac{4}{N-2} \, \, and \, \, small \, \, initial \, \, data. \\ \end{array}$

In addition, Cazenave-Weissler [5] and recently Cazenave-Fang-Han [3] showed that the IVP for the NLS is locally well posed in $H^s(\mathbb{R}^N)$ if $0 < \alpha \le \frac{4}{N-2s}$ and $0 < s < \frac{4}{N-2s}$ $\frac{N}{2}$, moreover the local solution extends globally in time for small initial data.

Our main interest in this paper is to prove similar results for the INLS equation. To this end, we divide in two parts.

The first part is devoted to study the local theory of the IVP (1.1). We start considering the local well-posedness in $L^2(\mathbb{R}^N)$.

Theorem 1.2. Let $0 < \alpha < \frac{4-2b}{N}$ and $0 < b < \min\{2, N\}$, then for all $u_0 \in L^2(\mathbb{R}^N)$ there exist $T = T(\|u_0\|_{L^2}, N, \alpha) > 0$ and a unique solution u of (1.1) satisfying

$$u \in C\left([-T,T];L^2(\mathbb{R}^N)\right) \cap L^q\left([-T,T];L^r(\mathbb{R}^N)\right),$$

for any (q,r) L^2 -admissible. Moreover, the continuous dependence upon the initial data holds.

It is worth to mention that the last theorem is an extension of a result by Tsutsumi [26] (which asserts local well-posedness for the NLS equation, (1.1) with b=0, when $0<\alpha<\frac{4}{N}$) to the INLS model.

Next, we treat the local well-posedness in $H^s(\mathbb{R}^N)$ for $0 < s \le 1$. Before stating the theorem, we define the following numbers

$$\widetilde{2} := \begin{cases} \frac{N}{3} & N = 1, 2, 3, \\ 2 & N \ge 4 \end{cases} \quad \text{and} \quad \alpha_s := \begin{cases} \frac{4-2b}{N-2s} & s < \frac{N}{2}, \\ +\infty & s = \frac{N}{2}. \end{cases}$$

$$(1.5)$$

Theorem 1.3. Assume $0 < \alpha < \alpha_s$, $0 < b < \widetilde{2}$ and $\max\{0, s_c\} < s \le \min\{\frac{N}{2}, 1\}$. If $u_0 \in H^s(\mathbb{R}^N)$ then there exist $T = T(\|u_0\|_{H^s}, N, \alpha) > 0$ and a unique solution u of (1.1) with

$$u \in C\left([-T,T]; H^s(\mathbb{R}^N)\right) \cap L^q\left([-T,T]; H^{s,r}(\mathbb{R}^N)\right)$$

for any (q,r) L^2 -admissible. Moreover, the continuous dependence upon the initial data holds.

Remark 1.4. Observe that $\alpha < \frac{4-2b}{N-2s}$ is equivalent to $s_c < s$. On the other hand, if $0 < \alpha < \frac{4-2b}{N}$ then $s_c < 0$, for this reason we add the restriction $s > \max\{0, s_c\}$ in the above statement.

As an immediate consequence of the Theorem 1.3, we have that the IVP (1.1) is locally well-posed in $H^1(\mathbb{R}^N)$.

Corollary 1.5. Assume $N \geq 2$, $0 < \alpha < 2^*$ and $0 < b < \widetilde{2}$. If $u_0 \in H^1(\mathbb{R}^N)$ then the initial value problem (1.1) is locally well-posed and

$$u \in C\left([-T,T];H^1(\mathbb{R}^N)\right) \cap L^q\left([-T,T];H^{1,r}(\mathbb{R}^N)\right),$$

for any (q,r) L^2 -admissible.

Remark 1.6. One important difference of the previous results and its its counterpart for the NLS model (see Theorem 1.1-(2)) is that we do not treat the critical case here, i.e. $\alpha = \frac{4-2b}{N-2s}$ with $0 \le s \le 1$ and $N \ge 3$. It is still an open problem.

In the second part, we consider the global well-posedness of the Cauchy problem (1.1). We begin with a global result in $L^2(\mathbb{R}^N)$ which is an immediate consequence of Theorem 1.2.

Theorem 1.7. If $0 < \alpha < \frac{4-2b}{N}$ and $0 < b < \min\{2, N\}$, then for all $u_0 \in L^2(\mathbb{R}^N)$ the local solution u of the IVP (1.1) extends globally with

$$u \in C\left(\mathbb{R}; L^2(\mathbb{R}^N)\right) \cap L^q\left(\mathbb{R}; L^r(\mathbb{R}^N)\right),$$

for any (q,r) L^2 -admissible.

In the sequel we establish a small data global theory for the INLS model (1.1).

Theorem 1.8. Let $\frac{4-2b}{N} < \alpha < \alpha_s$ with $0 < b < \widetilde{2}$, $s_c < s \le \min\{\frac{N}{2}, 1\}$ and $u_0 \in H^s(\mathbb{R}^N)$. If $\|u_0\|_{H^s} \le A$ then there exists $\delta = \delta(A)$ such that if $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$, then the solution of (1.1) is globally defined. Moreover,

$$||u||_{S(\dot{H}^{s_c})} \le 2||U(t)u_0||_{S(\dot{H}^{s_c})}$$

and

$$||u||_{S(L^2)} + ||D^s u||_{S(L^2)} \le 2c||u_0||_{H^s}.$$

Remark 1.9. Note that in the last result we don't need the condition $s > \max\{0, s_c\}$ as in Theorem 1.3, since $\alpha > \frac{4-2b}{N}$ implies $s_c > 0$.

Remark 1.10. Also note that by the Strichartz estimates (2.10), the condition $||U(t)u_0||_{S(\dot{H}^{s_c})} < \delta$ is automatically satisfied if $||u_0||_{\dot{H}^{s_c}} \leq \frac{\delta}{c}$.

A similar small data global theory for the NLS model can be found in Cazenave-Weissler [6], Holmer-Roudenko [18] and Guevara [17]. A consequence of the Theorem 1.8 is the following global well-posed result in $H^1(\mathbb{R}^N)$.

Corollary 1.11. Let $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2^*$ with $0 < b < \widetilde{2}$ and $u_0 \in H^1(\mathbb{R}^N)$. Assume $\|u_0\|_{H^1} \leq A$ then there exists $\delta = \delta(A) > 0$ such that if $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$, then there exists a unique global solution u of (1.1) such that

$$||u||_{S(\dot{H}^{s_c})} \le 2||U(t)u_0||_{S(\dot{H}^{s_c})}$$

and

$$||u||_{S(L^2)} + ||\nabla u||_{S(L^2)} \le 2c||u_0||_{H^1}.$$

The rest of the paper is organized as follows. In section 2, we introduce some notations and give a review of the Strichartz estimates. In section 3, we prove the local well-posedness results: Theorems 1.2 and 1.3. Finally, in Section 4, we prove the results concerning the global theory: Theorems 1.7 and 1.8.

2. Notation and preliminares

Let us start this section by introducing the notation used throughout the paper. We use c to denote various constants that may vary line by line. Let a set $A \subset \mathbb{R}^N$, $A^C = \mathbb{R}^N \setminus A$ denotes the complement of A. Given $x, y \in \mathbb{R}^N$, x.y denotes the inner product of x and y on \mathbb{R}^N .

Let $q, r \geq 1, T > 0$ and $s \in \mathbb{R}$, the mixed norms in the spaces $L^q_{[0,T]}L^r_x$ and $L^q_{[0,T]}H^s_x$ of f(x,t) are defined, respectively, as

$$||f||_{L_{0,T}^q L_x^r} = \left(\int_0^T ||f(t,.)||_{L_x^r}^q dt \right)^{\frac{1}{q}}$$

and

$$||f||_{L_{0,T}^q H_x^s} = \left(\int_0^T ||f(t,.)||_{H_x^s}^q dt\right)^{\frac{1}{q}}$$

with the usual modifications when I = [0, T] and we restrict the x-integration to a subset $A \subset \mathbb{R}^N$ then the mixed norm will be denoted by $||f||_{L^q_I L^r_x(A)}$. Moreover, when f(t,x) is defined for every time $t \in \mathbb{R}$ we shall consider the notations $||f||_{L^q_I L^r_x}$ and $||f||_{L^q_I L^r_x}$.

For $s \in \mathbb{R}$, J^s and D^s denote the Bessel and the Riesz potentials of order s, given via Fourier transform by the formulas

$$\widehat{J^sf} = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f} \quad \text{and} \quad \widehat{D^sf} = |\xi|^s \widehat{f},$$

where the Fourier transform of f(x) is given by

$$\widehat{f}(y) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} f(x) dx.$$

On the other hand, we define the norm of the Sobolev spaces $H^{s,r}(\mathbb{R}^N)$ and $\dot{H}^{s,r}(\mathbb{R}^N)$, respectively, by

$$||f||_{H^{s,r}} := ||J^s f||_{L^r}$$
 and $||f||_{\dot{H}^{s,r}} := ||D^s f||_{L^r}$.

If r=2 we denote $H^{s,2}$ simply by H^s .

Next, we recall some Strichartz type estimates associated to the linear Schrödinger propagator.

 $^{{}^{1}||}f||_{L_{0,T}^{\infty}} = \sup_{t \in [0,T]} |f(t)|.$

Strichartz type estimates. We say the pair (q, r) is L^2 -admissible or simply admissible par if they satisfy the condition²

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r},$$

where

$$\begin{cases} 2 \leq r \leq \frac{2N}{N-2} & \text{if } N \geq 3, \\ 2 \leq r < +\infty & \text{if } N = 2, \\ 2 \leq r \leq +\infty & \text{if } N = 1. \end{cases}$$
 (2.1)

We also called the pair \dot{H}^s -admissible if

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s, (2.2)$$

where

$$\begin{cases}
\frac{2N}{N-2s} \le r \le \left(\frac{2N}{N-2}\right)^{-} & \text{if } N \ge 3, \\
\frac{2}{1-s} \le r \le \left(\left(\frac{2}{1-s}\right)^{+}\right)' & \text{if } N = 2, \\
\frac{2}{1-2s} \le r \le +\infty & \text{if } N = 1.
\end{cases} \tag{2.3}$$

Here, a^- is a fixed number slightly smaller than a $(a^- = a - \varepsilon \text{ with } \varepsilon > 0 \text{ small enough})$ and, in a similar way, we define a^+ . Moreover $(a^+)'$ is the number such that

$$\frac{1}{a} = \frac{1}{(a^+)'} + \frac{1}{a^+},\tag{2.4}$$

that is $(a^+)' := \frac{a^+ \cdot a}{a^+ - a}$ with a^+ . Finally we say that (q, r) is \dot{H}^{-s} -admissible if

$$\frac{2}{a} = \frac{N}{2} - \frac{N}{r} + s,$$

where

$$\begin{cases}
\left(\frac{2N}{N-2s}\right)^{+} \leq r \leq \left(\frac{2N}{N-2}\right)^{-} & \text{if } N \geq 3, \\
\left(\frac{2}{1-s}\right)^{+} \leq r \leq \left(\left(\frac{2}{1+s}\right)^{+}\right)' & \text{if } N = 2, \\
\left(\frac{2}{1-2s}\right)^{+} \leq r \leq +\infty & \text{if } N = 1.
\end{cases} \tag{2.5}$$

Given $s \in \mathbb{R}$, let $\mathcal{A}_s = \{(q,r); (q,r) \text{ is } \dot{H}^s - \text{admissible}\}$ and (q',r') is such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$ for $(q,r) \in \mathcal{A}_s$. We define the following Strichartz norm

$$||u||_{S(\dot{H}^s)} = \sup_{(q,r)\in\mathcal{A}_s} ||u||_{L_t^q L_x^r}$$

and the dual Strichartz norm

$$||u||_{S'(\dot{H}^{-s})} = \inf_{(q,r)\in\mathcal{A}_{-s}} ||u||_{L_t^{q'}L_x^{r'}}.$$

Note that, if s=0 then \mathcal{A}_0 is the set of all L^2 -admissible pairs. Moreover, if s=0, $S(\dot{H}^0)=S(L^2)$ and $S'(\dot{H}^0)=S'(L^2)$. We just write $S(\dot{H}^s)$ or $S'(\dot{H}^{-s})$ if the mixed norm is evaluated over $\mathbb{R}\times\mathbb{R}^N$. To indicate a restriction to a time interval $I\subset (-\infty,\infty)$ and a subset A of \mathbb{R}^N , we will consider the notations $S(\dot{H}^s(A);I)$ and $S'(\dot{H}^{-s}(A);I)$.

We now list (without proving) some estimates that will be useful in our work.

 $^{^2}$ We included in the above definition the improvement, due to M. Keel and T. Tao [20], to the limiting case for Strichartzs inequalities.

Lemma 2.1. (Sobolev embedding) Let $s \in (0, +\infty)$ and $1 \le p < +\infty$.

(i) If $s \in (0, \frac{N}{p})$ then $H^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ where $s = \frac{N}{p} - \frac{N}{r}$. Moreover,

$$||f||_{L^r} \le c||D^s f||_{L^p}. \tag{2.6}$$

(ii) If $s = \frac{N}{2}$ then $H^s(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ for all $r \in [2, +\infty)$. Furthermore,

$$||f||_{L^r} \le c||f||_{H^s}. \tag{2.7}$$

Proof. See Bergh-Löfström [1, Theorem 6.5.1] (see also Linares-Ponce [22, Theorem 3.3] and Demenguel-Demenguel [8, Proposition 4.18]). \Box

Remark 2.2. Using (i), with p=2, we have that $H^s(\mathbb{R}^N)$, with $s\in(0,\frac{N}{2})$, is continuously embedded in $L^r(\mathbb{R}^N)$ and

$$||f||_{L^r} \le c||f||_{H^s},\tag{2.8}$$

where $r \in [2, \frac{2N}{N-2s}]$.

Lemma 2.3. (Fractional product rule) Let $s \in (0,1]$ and $1 < r, r_1, r_2, p_1, p_2 < +\infty$ are such that $\frac{1}{r} = \frac{1}{r_i} + \frac{1}{p_i}$ for i = 1, 2. Then,

$$||D^s(fg)||_{L^r} \le c||f||_{L^{r_1}} ||D^s g||_{L^{p_1}} + c||D^s f||_{L^{r_2}} ||g||_{L^{p_2}}.$$

Proof. See Christ-Weinstein [7, Proposition 3.3].

Lemma 2.4. (Fractional chain rule) Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$, and $1 < r, r_1, r_2 < +\infty$ are such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then,

$$||D^s G(u)||_{L^r} \le c||G'(u)||_{L^{r_1}} ||D^s u||_{L^{r_2}}.$$

Proof. See Christ-Weinstein [7, Proposition 3.1].

The main tool to show the local and global well-posedness are the well-known Strichartz estimates. See for instance Linares-Ponce [22] and Kato [19] (see also Holmer-Roudenko [18] and Guevara [17]).

Lemma 2.5. The following statements hold.

(i) (Linear estimates).

$$||U(t)f||_{S(L^2)} \le c||f||_{L^2}, \tag{2.9}$$

$$||U(t)f||_{S(\dot{H}^s)} \le c||f||_{\dot{H}^s}. \tag{2.10}$$

(ii) (Inhomogeneous estimates).

$$\left\| \int_{\mathbb{R}} U(t-t')g(.,t')dt' \right\|_{S(L^2)} + \left\| \int_0^t U(t-t')g(.,t')dt' \right\|_{S(L^2)} \le c\|g\|_{S'(L^2)}, (2.11)$$

$$\left\| \int_0^t U(t - t')g(., t')dt' \right\|_{S(\dot{H}^s)} \le c \|g\|_{S'(\dot{H}^{-s})}. \tag{2.12}$$

The relations (2.11) and (2.12) will be very useful to perform estimates on the nonlinearity $|x|^{-b}|u|^{\alpha}u$.

We end this section with three important remarks.

Remark 2.6. Let $F(x,u) = |x|^{-b}|z|^{\alpha}z$, where $f(z) = |z|^{\alpha}z$. The complex derivative of f is

$$f_z(z) = \frac{\alpha+2}{2}|z|^{\alpha}$$
 and $f_{\bar{z}}(u) = \frac{\alpha}{2}|z|^{\alpha-2}z^2$.

For $z, w \in \mathbb{C}$, we have

$$f(z) - f(w) = \int_0^1 \left[f_z(w + t(z - w))(z - w) + f_{\bar{z}}(w + t(z - w))\overline{(z - w)} \right] dt.$$

Thus,

$$|F(x,z) - F(x,w)| \lesssim |x|^{-b} (|z|^{\alpha} + |w|^{\alpha}) |z - w|.$$
 (2.13)

Remark 2.7. Let $B = B(0,1) = \{x \in \mathbb{R}^N; |x| \le 1\}$ and b > 0. If $x \in B^C$ then $|x|^{-b} < 1$ and so

$$|||x|^{-b}f||_{L_x^r} \le ||f||_{L_x^r(B^C)} + |||x|^{-b}f||_{L_x^r(B)}.$$

The next remark provides a condition for the integrability of $|x|^{-b}$ on B and B^C .

Remark 2.8. We notice that if $\frac{N}{\gamma} - b > 0$ then $|||x|^{-b}||_{L^{\gamma}(B)} < +\infty$, indeed

$$\int_{B} |x|^{-\gamma b} dx = c \int_{0}^{1} r^{-\gamma b} r^{N-1} dr = c_{1} r^{N-\gamma b} \Big|_{0}^{1} < +\infty \text{ if } N - \gamma b > 0.$$

Similarly, we have that $||x|^{-b}||_{L^{\gamma}(B^{C})}$ is finite if $\frac{N}{\gamma} - b < 0$.

3. Local well-posedness

In this section we prove the local well-posedness results. The theorems follows from a contraction mapping argument based on the Strichartz estimates. First, we show the local well-posedness in $L^2(\mathbb{R}^N)$ (Theorem 1.2) and then in $H^s(\mathbb{R}^N)$ for $0 < s \le 1$ (Theorem 1.3) as well as Corollary 1.5.

3.1. L^2 -Theory. We begin with the following lemma. It provides an estimate for the INLS model nonlinearity in the Strichartz spaces.

Lemma 3.1. Let $0 < \alpha < \frac{4-2b}{N}$ and $0 < b < \min\{2, N\}$. Then,

$$||x|^{-b}|u|^{\alpha}v||_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})||u||_{S(L^2;I)}^{\alpha}||v||_{S(L^2;I)}, \tag{3.1}$$

where I = [0, T] and $c, \theta_1, \theta_2 > 0$.

Proof. By Remark 2.7, we have

$$\begin{aligned} \||x|^{-b}|u|^{\alpha}v\|_{S'(L^2;I)} &\leq \||u|^{\alpha}v\|_{S'(L^2(B^C);I)} + \||x|^{-b}|u|^{\alpha}v\|_{S'(L^2(B);I)} \\ &\equiv A_1 + A_2. \end{aligned}$$

Note that in the norm A_1 we don't have any singularity, so we know that

$$A_1 \le cT^{\theta_1} \|u\|_{S(L^2;I)}^{\alpha} \|v\|_{S(L^2;I)}, \tag{3.2}$$

where $\theta_1 > 0$. See Kato [19, Theorem 0] (also see Linares-Ponce [22, Theorem 5.2 and Corollary 5.1]).

On the other hand, we need to find an admissible pair to estimate A_2 . In fact, using the Hölder inequality twice we obtain

$$\begin{split} A_2 & \leq & \left\| |x|^{-b} |u|^{\alpha} v \right\|_{L_I^{q'} L_x^{r'}(B)} \leq \left\| \||x|^{-b} \|_{L^{\gamma}(B)} \|u\|_{L_x^{\alpha r_1}}^{\alpha \alpha r_1} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\ & \leq & \left\| |x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_1}} \|u\|_{L_I^{\alpha q_2} L_x^{\alpha r_1}}^{\alpha r_1} \|v\|_{L_I^q L_x^r} \\ & \leq & T^{\frac{1}{q_1}} \||x|^{-b} \|_{L^{\gamma}(B)} \|u\|_{L_I^q L_x^r}^{\alpha} \|v\|_{L_I^q L_x^r}, \end{split}$$

if (q,r) L^2 -admissible and

$$\begin{cases}
\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\
\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\
q = \alpha q_2, \quad r = \alpha r_1.
\end{cases}$$
(3.3)

In order to have $||x|^{-b}||_{L^{\gamma}(B)} < +\infty$ we need $\frac{N}{\gamma} > b$, by Remark 2.8. Hence, in view of (3.3) (q, r) must satisfy

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{N(\alpha+2)}{r} > b \\
\frac{1}{q_1} = 1 - \frac{\alpha+2}{q}.
\end{cases}$$
(3.4)

From the first equation in (3.4) we have $N-b-\frac{N(\alpha+2)}{r}>0$, which is equivalent to

$$\alpha < \frac{r(N-b) - 2N}{N},\tag{3.5}$$

for $r > \frac{2N}{N-b}$. By hypothesis $\alpha < \frac{4-2b}{N}$, then setting r such that

$$\frac{r(N-b)-2N}{N} = \frac{4-2b}{N},$$

we get³ $r = \frac{4-2b+2N}{N-b}$ satisfying (3.5). Consequently, since (q,r) is L^2 -admissible we obtain $q = \frac{4-2b+2N}{N}$. Next, applying the second equation in (3.4) we deduce

$$\frac{1}{q_1} = \frac{4 - 2b - \alpha N}{4 - 2b + 2N},$$

which is positive by the hypothesis $\alpha < \frac{4-2b}{N}$. Thus,

$$A_2 \le cT^{\theta_2} \|u\|_{S(L^2;I)}^{\alpha} \|v\|_{S(L^2;I)},$$

where $\theta_2 = \frac{1}{q_1}$. Therefore, combining (3.2) and the last inequality we prove (3.1).

Our goal now is to show Theorem 1.2.

Proof of Theorem 1.2. We define

$$X = C\left([-T, T]; L^{2}(\mathbb{R}^{N})\right) \bigcap L^{q}\left([-T, T]; L^{r}(\mathbb{R}^{N})\right),$$

for any (q, r) L^2 -admissible, and

$$B(a,T) = \{ u \in X : ||u||_{S(L^2:[-T,T])} \le a \},\$$

³Since $0 < b < \min\{N,2\}$ the denominator of r is positive and $r > \frac{2N}{N-b}$. Moreover, by a simple computations we have $2 \le r \le \frac{2N}{N-2}$ if $N \ge 3$, and $2 \le r < +\infty$ if N = 1, 2, that is r satisfies (2.1). Therefore, the pair (q,r) above defined is L^2 -admissible.

where a and T are positive constants to be determined later. We follow the standard fixed point argument to prove this result. It means that for appropriate values of a, T we shall show that G defined in (1.2) defines a contraction map on B(a, T).

Without loss of generality we consider only the case t > 0. Applying Strichartz inequalities (2.9) and (2.11), we have

$$||G(u)||_{S(L^2;I)} \le c||u_0||_{L^2} + c|||x|^{-b}|u|^{\alpha+1}||_{S'(L^2;I)}, \tag{3.6}$$

where I = [0, T]. Moreover, Lemma 3.1 yields

$$||G(u)||_{S(L^2;I)} \leq c||u_0||_{L^2} + c(T^{\theta_1} + T^{\theta_2})||u||_{S(L^2;I)}^{\alpha+1}$$

$$\leq c||u_0||_{L^2} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1},$$

provided $u \in B(a,T)$. Hence,

$$||G(u)||_{S(L^2;[-T,T])} \le c||u_0||_{L^2} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1}.$$

Next, choosing $a = 2c||u_0||_{L^2}$ and T > 0 such that

$$ca^{\alpha}(T^{\theta_1} + T^{\theta_2}) < \frac{1}{4},\tag{3.7}$$

we conclude $G(u) \in B(a,T)$.

Now we prove that G is a contraction. Again using Strichartz inequality (2.11) and (2.13), we deduce

$$\begin{split} \|G(u) - G(v)\|_{S(L^2;I)} & \leq c \, \big\| |x|^{-b} (|u|^\alpha u - |v|^\alpha v) \big\|_{S'(L^2;I)} \\ & \leq c \, \big\| |x|^{-b} |u|^\alpha |u - v| \big\|_{S'(L^2;I)} \\ & + c \, \big\| |x|^{-b} |v|^\alpha |u - v| \big\|_{S'(L^2;I)} \\ & \leq c (T^{\theta_1} + T^{\theta_2}) \|u\|_{S(L^2;I)}^\alpha \|u - v\|_{S(L^2;I)} \\ & + c (T^{\theta_1} + T^{\theta_2}) \|v\|_{S(L^2;I)}^\alpha \|u - v\|_{S(L^2;I)}, \end{split}$$

where I = [0, T]. That is,

$$||G(u) - G(v)||_{S(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2}) \left(||u||_{S(L^2;I)}^{\alpha} + ||v||_{S(L^2;I)}^{\alpha} \right) ||u - v||_{S(L^2;I)}$$

$$\le 2c(T^{\theta_1} + T^{\theta_2})a^{\alpha} ||u - v||_{S(L^2;I)},$$

provided $u, v \in B(a, T)$. Therefore, the inequality (3.7) implies that

$$\begin{split} \|G(u) - G(v)\|_{S(L^2; [-T, T])} & \leq & 2c(T^{\theta_1} + T^{\theta_2})a^{\alpha} \|u - v\|_{S(L^2; [-T, T])} \\ & < & \frac{1}{2} \|u - v\|_{S(L^2; [-T, T])}, \end{split}$$

i.e., G is a contraction on S(a,T).

The proof of the continuous dependence is similar to the one given above and it will be omitted. $\hfill\Box$

3.2. H^s -Theory. The aim of this subsection is to prove the local well-posedness in $H^s(\mathbb{R}^N)$ with $0 < s \le 1$ (Theorem 1.3) as well as Corollary 1.5. Before doing that we establish useful estimates for the nonlinearity $|x|^{-b}|u|^{\alpha}u$. First, we consider the nonlinearity in the space $S'(L^2)$ and in the sequel in the space $D^{-s}S'(L^2)$, that is, we estimate the norm $||x|^{-b}|u|^{\alpha}u||_{S'(L^2 \setminus I)}$ and $||D^s(|x|^{-b}|u|^{\alpha}u)||_{S'(L^2 \setminus I)}$.

We start this subsection with the following remarks.

Remark 3.2. Since we will use the Sobolev embedding (Lemma 2.1), we divide our study in three cases: $N \geq 3$ and $s < \frac{N}{2}$; N = 1, 2 and $s < \frac{N}{2}$; N = 1, 2 and $s = \frac{N}{2}$. (see respectively Lemmas 3.4, 3.5 and 3.6 bellow).

Remark 3.3. Another interesting remark is the following claim

$$D^{s}(|x|^{-b}) = C_{N,b}|x|^{-b-s}. (3.8)$$

Indeed, we use the facts $\widehat{D^s f} = |\xi|^s \widehat{f}$ and $\widehat{(|x|^{-\beta})} = \frac{C_{N,\beta}}{|\xi|^{N-\beta}}$ for $\beta \in (0,N)$. Let $f(x) = |x|^{-b}$, we have

$$\widehat{D^s(|x|^{-b})} = |\xi|^s \widehat{(|x|^{-b})} = |\xi|^s \frac{C_{N,\beta}}{|\xi|^{N-b}} = \frac{C_{N,\beta}}{|\xi|^{N-(b+s)}}.$$

Since $0 < b < \widetilde{2}$ and $0 < s \le \min\{\frac{N}{2}, 1\}$ then 0 < b + s < N, so taking $\beta = s + b$, we get

$$D^{s}(|x|^{-b}) = \left(\frac{C_{N,\beta}}{|y|^{N-(b+s)}}\right)^{\vee} = C_{N,\beta}|x|^{-b-s}.$$

Lemma 3.4. Let $N \geq 3$ and $0 < b < \widetilde{2}$. If $s < \frac{N}{2}$ and $0 < \alpha < \frac{4-2b}{N-2s}$ then the following statements hold

(i)
$$||x|^{-b}|u|^{\alpha}v||_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})||D^s u||_{S(L^2;I)}^{\alpha}||v||_{S(L^2;I)}$$

(ii)
$$\|D^s(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})\|D^s u\|_{S(L^2;I)}^{\alpha+1}$$
,

where I = [0, T] and $c, \theta_1, \theta_2 > 0$.

Proof. (i) We divide the estimate in B and B^C , indeed

$$\begin{aligned} |||x|^{-b}|u|^{\alpha}v||_{S'(L^2;I)} &\leq |||x|^{-b}|u|^{\alpha}v||_{S'(L^2(B^C);I)} + |||x|^{-b}|u|^{\alpha}v||_{S'(L^2(B);I)} \\ &\equiv B_1 + B_2. \end{aligned}$$

First, we consider B_1 . Let (q_0, r_0) L^2 -admissible given by

$$q_0 = \frac{4(\alpha + 2)}{\alpha(N - 2s)}$$
 and $r_0 = \frac{N(\alpha + 2)}{N + \alpha s}$. (3.9)

If $s<\frac{N}{2}$ then $s<\frac{N}{r_0}$ and so using the Sobolev inequality (2.6) and the Hölder inequality twice, we get

$$B_{1} \leq \||x|^{-b}|u|^{\alpha}v\|_{L_{I}^{q'_{0}}L_{x}^{r'_{0}}(B^{C})} \leq \|\||x|^{-b}\|_{L^{\gamma}(B^{C})}\|u\|_{L_{x}^{\alpha r_{1}}}^{\alpha}\|v\|_{L_{x}^{r_{0}}}\|_{L_{I}^{q'_{0}}}$$

$$\leq \||x|^{-b}\|_{L^{\gamma}(B^{C})}\|\|D^{s}u\|_{L_{x}^{r_{0}}}^{\alpha}\|v\|_{L_{x}^{r_{0}}}\|_{L_{I}^{q'_{0}}}$$

$$\leq \||x|^{-b}\|_{L^{\gamma}(B^{C})}T^{\frac{1}{q_{1}}}\|D^{s}u\|_{L_{I}^{\alpha q_{2}}L_{x}^{r_{0}}}^{\alpha}\|v\|_{L_{I}^{q_{0}}L_{x}^{r_{0}}}$$

$$= \||x|^{-b}\|_{L^{\gamma}(B^{C})}T^{\frac{1}{q_{1}}}\|D^{s}u\|_{L_{I}^{\alpha q_{1}}L_{x}^{r_{0}}}^{\alpha}\|v\|_{L_{I}^{q_{0}}L_{x}^{r_{0}}}, \tag{3.10}$$

where

$$\begin{cases}
\frac{1}{r'_0} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_0} \\
\frac{1}{q'_0} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_0} \\
q_0 = \alpha q_2, \quad s = \frac{N}{r_0} - \frac{N}{\alpha r_1}.
\end{cases}$$
(3.11)

⁴It is not difficult to check that q_0 and r_0 satisfy the conditions of admissible pair, see (2.1).

In view of Remark 2.8 in order to show that the first norm in the right hand side of (3.10) is bounded we need $\frac{N}{\gamma} - b < 0$. Indeed, (3.11) is equivalent to

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{2N}{r_0} - \frac{N\alpha}{r_0} + \alpha s \\
\frac{1}{q_1} = 1 - \frac{\alpha + 2}{q_0},
\end{cases}$$
(3.12)

which implies, by (3.9)

$$\frac{N}{\gamma} = 0$$
 and $\frac{1}{q_1} = \frac{4 - \alpha(N - 2s)}{4}$. (3.13)

So $\frac{N}{\gamma} - b < 0$ and $\frac{1}{q_1} > 0$, by our hypothesis $\alpha < \frac{4-2b}{N-2s}$. Therefore, setting $\theta_1 = \frac{1}{q_1}$ we deduce

$$B_1 \le cT^{\theta_1} \|D^s u\|_{S(L^2;I)}^{\alpha} \|v\|_{S(L^2;I)}. \tag{3.14}$$

We now estimate B_2 . To do this, we use similar arguments as the ones in the estimation of A_2 in Lemma 3.1. It follows from Hölder's inequality twice and Sobolev embedding (2.6) that

$$\begin{split} B_2 & \leq & \left\| |x|^{-b} |u|^{\alpha} v \right\|_{L_I^{q'} L_x^{r'}(B)} \leq \left\| \||x|^{-b} \|_{L^{\gamma}(B)} \|u\|_{L_x^{\alpha r_1}}^{\alpha} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\ & \leq & \left\| \||x|^{-b} \|_{L^{\gamma}(B)} \|D^s u\|_{L_x^r}^{\alpha} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\ & \leq & \left\| |x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_1}} \|D^s u\|_{L_I^{\alpha q_2} L_x^r}^{\alpha} \|v\|_{L_I^q L_x^r} \\ & = & \left\| |x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_1}} \|D^s u\|_{L_I^q L_x^r}^{\alpha} \|v\|_{L_I^q L_x^r} \end{split}$$

if (q,r) L^2 -admissible and the following system is satisfied

$$\begin{cases}
\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\
s = \frac{N}{r} - \frac{N}{\alpha r_1}, \quad s < \frac{N}{r} \\
\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\
q = \alpha q_2.
\end{cases} (3.15)$$

Similarly as in Lemma 3.1 we need to check that $\frac{N}{\gamma} > b$ (so that $||x|^{-b}||_{L^{\gamma}(B)}$ is finite) and $\frac{1}{q_1} > 0$ for a certain choice of (q, r) L^2 -admissible pair. From (3.15) this is equivalent to

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{2N}{r} - \frac{N\alpha}{r} + \alpha s > b \\
\frac{1}{q_1} = 1 - \frac{\alpha + 2}{q} > 0.
\end{cases}$$
(3.16)

The first equation in (3.16) implies that $\alpha < \frac{(N-b)r-2N}{N-rs}$ (assuming $s < \frac{N}{r}$), then let us choose r such that

$$\frac{(N-b)r-2N}{N-rs} = \frac{4-2b}{N-2s}$$

since, by our hypothesis $\alpha < \frac{4-2b}{N-2s}$. Therefore r and q are given by⁵

$$r = \frac{2N[N - b + 2(1 - s)]}{N(N - 2s) + 4s - bN} \text{ and } q = \frac{2[N - b + 2(1 - s)]}{N - 2s},$$
 (3.17)

⁵It is easy to see that r > 2 if, and only if, $s < \frac{N}{2}$ and $r < \frac{2N}{N-2}$ if, and only if, b < 2. Therefore the pair (q, r) given in (3.17) is L^2 -admissible.

where we have used that (q,r) is a L^2 -admissible pair to compute the value of q. Note that $s < \frac{N}{r}$ if, and only if, b + 2s - N < 0. Since $s \le 1$, $b < \widetilde{2}$ (see (1.5)) and $N \ge 3$ it is easy to see that $s < \frac{N}{r}$ holds. In addition, from the second equation of (3.16) and (3.17) we also have

$$\frac{1}{q_1} = \frac{4 - 2b - \alpha(N - 2s)}{2(N - b + 2 - 2s)} > 0,$$
(3.18)

since $\alpha < \frac{4-2b}{N-2s}$. Hence,

$$B_2 \le cT^{\theta_2} \|D^s u\|_{S(L^2:I)}^{\alpha} \|v\|_{S(L^2:I)}, \tag{3.19}$$

where θ_2 is given by (3.18). Finally, collecting the inequalities (3.14) and (3.19) we obtain (i).

(ii) Observe that

$$||D^s(|x|^{-b}|u|^{\alpha}u)||_{S'(L^2 \cdot I)} \le C_1 + C_2,$$

where

$$C_1 = \|D^s(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2(B^C);I)}$$
 and $C_2 = \|D^s(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2(B);I)}$.

To estimate C_1 we use the same admissible pair (q_0, r_0) used to estimate the term B_1 in item (i). Indeed, let

$$C_{11}(t) = \left\| D^s(|x|^{-b}|u|^{\alpha}u) \right\|_{L_x^{r_0'}(B^C)}$$

then Lemma 2.3 (fractional product rule), Lemma 2.4 (fractional chain rule) and Remark 3.3 yield

$$C_{11}(t) \leq ||x|^{-b}||_{L^{\gamma}(B^{C})} ||D^{s}(|u|^{\alpha}u)||_{L^{\beta}_{x}} + ||D^{s}(|x|^{-b})||_{L^{d}(B^{C})} ||u||_{L^{(\alpha+1)e}_{x}}^{\alpha+1}$$

$$\leq ||x|^{-b}||_{L^{\gamma}(B^{C})} ||u||_{\alpha r_{1}}^{\alpha} ||D^{s}u||_{L^{r_{0}}_{x}}^{\epsilon+1} + ||x|^{-b-s}||_{L^{d}(B^{C})} ||D^{s}u||_{L^{r_{0}}_{x}}^{\alpha+1}$$

$$\leq ||x|^{-b}||_{L^{\gamma}(B^{C})} ||D^{s}u||_{L^{r_{0}}_{x}}^{\alpha+1} + ||x|^{-b-s}||_{L^{d}(B^{C})} ||D^{s}u||_{L^{r_{0}}_{x}}^{\alpha+1}, \qquad (3.20)$$

where we also have used the Sobolev inequality (2.6) and (3.8). Moreover, we have the following relations

$$\begin{cases} \frac{1}{r'_0} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\ \frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_0} \\ s = \frac{N}{r_0} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r_0} \\ s = \frac{N}{r_0} - \frac{N}{(\alpha + 1)e} \end{cases}$$

which implies that

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{2N}{r_0} - \frac{\alpha N}{r_0} + \alpha s \\
\frac{N}{d} = N - \frac{2N}{r_0} - \frac{\alpha N}{r_0} + \alpha s + s.
\end{cases}$$
(3.21)

Note that, in view of (3.9) we have $\frac{N}{\gamma} - b < 0$ and $\frac{N}{d} - b - s < 0$. These relations imply that $|||x|^{-b}||_{L^{\gamma}(B^{C})}$ and $|||x|^{-b-s}||_{L^{d}(B^{C})}$ are bounded quantities (see Remark 2.8). Therefore, it follows from (3.20) that

$$C_{11}(t) \le c \|D^s u\|_{L_{\infty}^{r_0}}^{\alpha+1}$$

On the other hand, using $\frac{1}{q_0'} = \frac{1}{q_1} + \frac{\alpha+1}{q_0}$ and applying the Hölder inequality in the time variable we conclude

$$||C_{11}||_{L_{r}^{q'_{0}}} \le cT^{\frac{1}{q_{1}}} ||D^{s}u||_{L_{I}^{q_{0}}L_{x}^{r_{0}}}^{\alpha+1},$$

where $\frac{1}{q_1}$ is given in (3.13). The estimate of C_1 is finished since $C_1 \leq \|C_{11}\|_{L^{q_1'}}$.

We now consider C_2 . Let $C_{22}(t) = \|D^s(|x|^{-b}|u|^{\alpha}u)\|_{L_x^{r'}(B)}$, we have $C_2 \le \|C_{22}\|_{L_x^{q'}}$. Using the same arguments as in the estimate of C_{11} we obtain

$$C_{22}(t) \le ||x|^{-b}||_{L^{\gamma}(B)} ||D^{s}u||_{L^{r}_{r}}^{\alpha+1} + ||x|^{-b-s}||_{L^{d}(B)} ||D^{s}u||_{L^{r}_{r}}^{\alpha+1}, \tag{3.22}$$

if (3.21) is satisfied replacing r_0 by r (to be determined later), that is

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{2N}{r} - \frac{\alpha N}{r} + \alpha s \\
\frac{N}{d} = N - \frac{2N}{r} - \frac{\alpha N}{r} + \alpha s + s.
\end{cases}$$
(3.23)

In order to have that $||x|^{-b}||_{L^{\gamma}(B)}$ and $||x|^{-b-s}||_{L^{d}(B)}$ are bounded, we need $\frac{N}{\gamma} > b$ and $\frac{N}{d} > b + s$, respectively, by Remark 2.8. Therefore, since the first equation in (3.23) is the same as the first one in (3.16), we choose r as in (3.17). So we get $\frac{N}{\gamma} > b$, which also implies that $\frac{N}{d} - s > b$. Finally, (3.22) and the Hölder inequality in the time variable yield

$$C_{2} \leq cT^{\frac{1}{q_{1}}} \|D^{s}u\|_{L_{I}^{(\alpha+1)q_{2}}L_{x}^{r}}^{\alpha+1}$$

$$= cT^{\frac{1}{q_{1}}} \|D^{s}u\|_{L_{I}^{q_{L}r}}^{\alpha+1}, \qquad (3.24)$$

where

$$\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2} \qquad q = (\alpha + 1)q_2. \tag{3.25}$$

Notice that (3.25) is exactly to the second equation in (3.16), thus $\frac{1}{q_1} > 0$ (see the relation (3.18)). This completes the proof of Lemma 3.4.

One important remark is that Lemma 3.4 only holds for $N \geq 3$, since the admissible par (q,r) defined in (3.17) doesn't satisfy the condition $s < \frac{N}{r}$, for N = 1, 2. In the next lemma we study these cases.

Lemma 3.5. Let N = 1, 2 and $0 < b < \widetilde{2}$. If $s < \frac{N}{2}$ and $0 < \alpha < \frac{4-2b}{N-2s}$ then

(i)
$$||x|^{-b}|u|^{\alpha}v||_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})||D^s u||_{S(L^2;I)}^{\alpha}||v||_{S(L^2;I)}$$

(ii)
$$\|D^s(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})\|D^s u\|_{S(L^2;I)}^{\alpha+1}$$
,

where I = [0, T] and $c, \theta_1, \theta_2 > 0$.

Proof. (i) As before, we divide the estimate in B and B^C . The estimate on B^C is the same as the term B_1 in Lemma 3.4-(i), since (q_0, r_0) given in (3.9) is L^2 -admissible for $s < \frac{N}{2}$ in all dimensions. Thus we only consider the estimate on B

Indeed, set the L^2 -admissible pair $(\bar{q}, \bar{r}) = (\frac{8}{2N-s}, \frac{4N}{s})$. We deduce from the Hölder inequality twice and Sobolev embedding (2.6)

$$\begin{split} \big\| |x|^{-b} |u|^{\alpha} v \big\|_{L_{I}^{\overline{q}'} L_{x}^{\overline{r}'}(B)} & \leq & \Big\| \||x|^{-b} \|_{L^{\gamma}(B)} \|u\|_{L_{x}^{\alpha r_{1}}}^{\alpha} \|v\|_{L_{x}^{r}} \Big\|_{L_{I}^{q'}} \\ & \leq & \||x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_{1}}} \|D^{s} u\|_{L_{I}^{\alpha q_{2}} L_{x}^{r}}^{\alpha} \|v\|_{L_{I}^{q} L_{x}^{r}} \\ & = & \||x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_{1}}} \|D^{s} u\|_{L_{I}^{q} L_{x}^{r}}^{\alpha} \|v\|_{L_{I}^{q} L_{x}^{r}} \end{split}$$

if (q, r) is L^2 -admissible and the following system is satisfied

$$\begin{cases}
\frac{1}{\bar{r}'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\
s = \frac{N}{r} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r} \\
\frac{1}{\bar{q}'} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\
q = \alpha q_2.
\end{cases} (3.26)$$

Using the values of \bar{q} and \bar{r} given above, the previous system is equivalent to

$$\begin{cases} \frac{N}{\gamma} = \frac{4(N-b)-s}{4} - \frac{N}{r} - \frac{\alpha(N-sr)}{r} + b\\ \frac{1}{q_1} = \frac{8-2N-s}{8} - \frac{\alpha+1}{q}. \end{cases}$$
(3.27)

From the first equation in (3.27) if $\alpha < \frac{r(4(N-b)-s)-4N}{N-sr}$ then $\frac{N}{\gamma} > b$, and so $|x|^{-b} \in L^{\gamma}(B)$. Now, in view of the hypothesis $\alpha < \frac{4-2b}{N-2s}$ we set r such that

$$\frac{r(4(N-b)-s)-4N}{4(N-sr)} = \frac{4-2b}{N-2s},$$

that is^6

$$r = \frac{4N(N - 2s + 4 - 2b)}{4s(4 - 2b) + (N - 2s)(4N - 4b - s)}.$$
(3.28)

Note that, in order to satisfy the second equation in the system (3.26) we need to verify $s < \frac{N}{r}$. A simple calculation shows that it is true if, and only if, 4b + 5s < 4Nand this is true since $b < \frac{N}{3}$ and $s < \frac{N}{2}$. On the other hand, since we are looking for a pair (q,r) L^2 -admissible one has

$$q = \frac{8(N - 2s + 4 - 2b)}{(8 - 2N + s)(N - 2s)}. (3.29)$$

Finally, from (3.29) the second equation in (3.27) is given by

$$\frac{1}{q_1} = \left(\frac{8 - 2N + s}{8}\right) \left(\frac{4 - 2b - \alpha(N - 2s)}{N - 2s + 4 - 2b}\right). \tag{3.30}$$

which is positive, since $\alpha < \frac{4-2b}{N-2s}$, $s < \frac{N}{2}$ and N = 1, 2. (ii) Similarly as in item (i) we only consider the estimate on B. Let

$$D_2(t) = \||x|^{-b}|u|^{\alpha}u\|_{L^{\bar{r}'}(B)}.$$

⁶We claim that r satisfies (2.1). In fact, obviously $r < +\infty$. Moreover $r \geq 2$ if, and only if, $8-2N+s \ge 0$ and this is true since s > 0 and N = 1, 2.

We use analogous arguments as the ones in the estimate of C_2 in Lemma 3.4-(ii). Lemmas 2.3-2.4, the Hölder inequality, the Sobolev embedding (2.6) and Remark 3.3 imply

$$\begin{split} D_{2}(t) \leq & \||x|^{-b}\|_{L^{\gamma}(B)} \|D^{s}(|u|^{\alpha}u)\|_{L^{\beta}_{x}} + \|D^{s}(|x|^{-b})\|_{L^{d}(B)} \|u\|_{L^{(\alpha+1)e}_{x}}^{\alpha+1} \\ \leq & \||x|^{-b}\|_{L^{\gamma}(B)} \|u\|_{\alpha r_{1}}^{\alpha} \|D^{s}u\|_{L^{r}_{x}} + \||x|^{-b-s}\|_{L^{d}(B)} \|D^{s}u\|_{L^{r}_{x}}^{\alpha+1} \\ \leq & \||x|^{-b}\|_{L^{\gamma}(B)} \|D^{s}u\|_{L^{r}_{x}}^{\alpha+1} + \||x|^{-b-s}\|_{L^{d}(B)} \|D^{s}u\|_{L^{r}_{x}}^{\alpha+1}, \end{split}$$
(3.31)

where

$$\begin{cases} \frac{1}{\bar{r}'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\ \frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r} \\ s = \frac{N}{r} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r} \\ s = \frac{N}{r} - \frac{N}{(\alpha + 1)e}, \end{cases}$$

which is equivalent to

$$\begin{cases}
\frac{N}{\gamma} = N - \frac{N}{\bar{r}} - \frac{(\alpha+1)N}{r} + \alpha s \\
\frac{N}{d} = N - \frac{N}{\bar{r}} - \frac{(\alpha+1)N}{r} + \alpha s + s.
\end{cases}$$
(3.32)

Hence, setting again $(\bar{q}, \bar{r}) = (\frac{8}{2N-s}, \frac{4N}{s})$ the first equation in (3.32) the same as the first one in (3.27). Therefore choosing r as in (3.28) we have $\frac{N}{\gamma} > b$, which also implies $\frac{N}{d} > b + s$. Therefore, it follows from Remark 2.8 and (3.31) that

$$D_2(t) \le c \|D^s u\|_{L_x^r}^{\alpha+1}.$$

Since, $\frac{1}{q'} = \frac{1}{q_1} + \frac{\alpha+1}{q}$ (recall that q is given in (3.29)) and applying the Hölder inequality in the time variable we conclude

$$||D_2||_{L_T^{\overline{q}'}} \le cT^{\frac{1}{q_1}} ||D^s u||_{L_T^q L_x^r}^{\alpha+1},$$

where
$$\frac{1}{a_1} > 0$$
 (see (3.30)).

We finish the estimates for the nonlinearity considering the case $s = \frac{N}{2}$. Note that this case can only occur if N = 1, 2, since here we are interested in local (and global) results in $H^s(\mathbb{R}^N)$ for $\max\{0, s_c\} < s \le \min\{\frac{N}{2}, 1\}$.

Lemma 3.6. Let N = 1, 2 and $0 < b < \frac{N}{3}$. If $s = \frac{N}{2}$ and $0 < \alpha < +\infty$ then

- (i) $||x|^{-b}|u|^{\alpha}v||_{S'(L^2;I)} \le cT^{\theta_1}||u||_{L_I^{\infty}H_x^s}^{\alpha}||v||_{L_I^{\infty}L_x^2}$
- (ii) $\|D^s(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2;I)} \le cT^{\theta_1}\|u\|_{L_x^{\infty}H_x^s}^{\alpha+1}$,

where I = [0, T] and $c, \theta_1 > 0$.

Proof. (i) To this end we start defining the following numbers

$$r = \frac{N(\alpha + 2)}{N - 2b}$$
 and $q = \frac{4(\alpha + 2)}{N\alpha + 4b}$, (3.33)

it is easy to check that (q, r) is L^2 -admissible.

We divide the estimate in B and B^C . We first consider the estimate on B. From Hölder's inequality

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{r'}(B)} \le |||x|^{-b}||_{L^{\gamma}(B)}||u||_{L_{x}^{\alpha r_{1}}}^{\alpha}||v||_{L_{x}^{2}}, \tag{3.34}$$

where

$$\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{2}.\tag{3.35}$$

In view of Remark 2.8 to show that $|x|^{-b} \in L^{\gamma}(B)$, we need $\frac{N}{\gamma} - b > 0$. So, the relations (3.33) and (3.35) yield

$$\frac{N}{\gamma} - b = \frac{\alpha(N - 2b)}{2(\alpha + 2)} - \frac{N}{r_1}.$$
(3.36)

If we choose $\alpha r_1 \in \left(\frac{2N(\alpha+2)}{N-2b}, +\infty\right)$ then the right hand side of (3.36) is positive. Therefore,

$$|||x|^{-b}|u|^{\alpha}v||_{L_{r}^{r'}(B)} \le c||u||_{L_{x}^{\alpha r_{1}}}^{\alpha}||v||_{L_{x}^{2}}.$$

On the other hand, since $\frac{2N(\alpha+2)}{N-2b} > 2$ we can apply the Sobolev embedding (2.7) to obtain

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{r'}(B)} \le c||u||_{H^{s}}^{\alpha}||v||_{L_{x}^{2}}.$$
(3.37)

Next, we consider the estimate on $\mathcal{B}^{\mathcal{C}}$. Using the same argument as in the first case we get

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L^{r'}_x(B^C)} \leq \||x|^{-b}\|_{L^{\gamma}(B^C)} \|u\|_{L^{\alpha r_1}_x}^{\alpha r_1} \|v\|_{L^2_x},$$

where the relations (3.35) and (3.36) hold. Thus, choosing $\alpha r_1 \in \left(2, \frac{2N(\alpha+2)}{N-2b}\right)$ we have that $\frac{N}{\gamma} - b < 0$, which implies $|x|^{-b} \in L^{\gamma}(B^C)$, by Remark 2.8. Therefore, again the Sobolev embedding (2.7) leads to

$$|||x|^{-b}|u|^{\alpha}v||_{L_x^{r'}(B^C)} \le c||u||_{H_x^s}^{\alpha}||v||_{L_x^2}.$$

Finally, it follows from the Hölder inequality in time variable, (3.37) and the last inequality that

$$|||x|^{-b}|u|^{\alpha}v||_{L_{t}^{q'}L_{x}^{r'}} \le cT^{\theta_{1}}||u||_{L_{L}^{\infty}H^{s}}^{\alpha}||v||_{L_{L}^{\infty}L_{x}^{2}}, \tag{3.38}$$

where $\theta_1 = \frac{1}{q'} > 0$, by (3.33).

(ii) Similarly as in the proof of item (i) we begin setting

$$r = \frac{N(\alpha + 2)}{N - b - s}$$
 and $q = \frac{4(\alpha + 2)}{\alpha N + 2b + 2s}$. (3.39)

Observe that, since $s=\frac{N}{2}$ and $0 < b < \frac{N}{3}$ the denominator of r is a positive number. Furthermore it is easy to verify that (q,r) is L^2 -admissible.

First, we consider the estimate on B. Lemma 2.4 together with the Hölder inequality and (3.8) imply

$$E_{1}(t) \leq \||x|^{-b}\|_{L^{\gamma}(B)} \|D^{s}(|u|^{\alpha}u)\|_{L^{\beta}_{x}} + \|D^{s}(|x|^{-b})\|_{L^{d}(B)} \|u\|_{L^{(\alpha+1)e}_{x}}^{\alpha+1}$$

$$\leq \||x|^{-b}\|_{L^{\gamma}(B)} \|u\|_{L^{\alpha r_{1}}_{x}}^{\alpha r_{1}} \|D^{s}u\|_{L^{2}_{x}} + \||x|^{-b-s}\|_{L^{d}(B)} \|u\|_{L^{(\alpha+1)e}_{x}}^{\alpha+1},$$

where $E_1(t) = \|D^s(|x|^{-b}|u|^{\alpha}u)\|_{L_x^{r'}(B)}$ and

$$\left\{ \begin{array}{ll} \frac{1}{r'} = & \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\ \frac{1}{\beta} = & \frac{1}{r_1} + \frac{1}{2}, \end{array} \right.$$

which implies

$$\begin{cases}
\frac{N}{\gamma} = \frac{N}{2} - \frac{N}{r} - \frac{N}{r_1} \\
\frac{N}{d} = N - \frac{N}{r} - \frac{N}{e}.
\end{cases}$$
(3.40)

Now, we claim that $||x|^{-b}||_{L^{\gamma}(B)}$ and $||x|^{-b-s}||_{L^{d}(B)}$ are bounded quantities for a suitable choice of r_1 and e. Indeed, using the value of r in (3.39), (3.40) and the fact that $s = \frac{N}{2}$ we get

$$\begin{cases}
\frac{N}{\gamma} - b &= \frac{(\alpha+1)(N-2b)}{2(\alpha+2)} - \frac{N}{r_1} \\
\frac{N}{d} - b - s &= \frac{(\alpha+1)(N-2b)}{2(\alpha+2)} - \frac{N}{e}.
\end{cases}$$
(3.41)

By Remark 2.8, if $r_1, e > \frac{2N(\alpha+2)}{(\alpha+1)(N-2b)}$ then the right hand side of both equations in (3.41) are positive, so $|x|^{-b} \in L^{\gamma}(B)$ and $|x|^{-b-s} \in L^{d}(B)$. Hence

$$E_1(t) \le c \|u\|_{L_x^{\alpha r_1}}^{\alpha} \|D^s u\|_{L_x^2} + c \|u\|_{L^{(\alpha+1)e}}^{\alpha+1}.$$

Choosing r_1 and e as before, it is easy to see that $\alpha r_1 > 2$ and $(\alpha + 1)e > 2$, thus we can use the Sobolev inequality (2.7)

$$E_{1}(t) \leq c \|u\|_{H_{x}^{s}}^{\alpha} \|D^{s}u\|_{L_{x}^{2}} + c \|u\|_{H_{x}^{s}}^{\alpha+1}$$

$$\leq c \|u\|_{H_{x}^{s}}^{\alpha+1}.$$
(3.42)

To complete the proof, we need to consider the estimate on B^C . By the same arguments as before we have

$$E_2(t) \le \||x|^{-b}\|_{L^{\gamma}(B^C)} \|u\|_{L^{\alpha r_1}_x}^{\alpha} \|D^s u\|_{L^2_x} + \||x|^{-b-s}\|_{L^d(B^C)} \|u\|_{L^{(\alpha+1)e}_x}^{\alpha+1},$$

where $E_2(t) = \|D^s(|x|^{-b}|u|^{\alpha}u)\|_{L_x^{r'}(B^C)}$ and (3.41) holds. Similarly as in item (i), since $\frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)}$, $\frac{2N(\alpha+2)}{N-b-s} > 2$, we can choose r_1 and e such that

$$\alpha r_1 \in \left(2, \frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)}\right)$$
 and $(\alpha+1)e \in \left(2, \frac{2N(\alpha+2)}{N-2b}\right)$,

and so we obtain from (3.41) that $\frac{N}{\gamma} - b < 0$ and $\frac{N}{d} - b - s < 0$. In other words, $|||x|^{-b}||_{L^{\gamma}(B^{C})}$ and $|||x|^{-b-s}||_{L^{d}(B^{C})}$ are bounded quantities for these choices of r_1 and e (see Remark 2.8). In addition, by the Sobolev inequality (2.7) we conclude

$$E_2(t) \le c \|u\|_{H_x^s}^{\alpha+1}.$$

Finally, (3.42) and the last inequality lead to

$$\| \|D^s(|x|^{-b}|u|^\alpha u) \|_{L_I^{q'}L_x^{r'}} \le c T^{\frac{1}{q'}} \|u\|_{L_I^\infty H_x^s}^{\alpha+1},$$
 where $\frac{1}{q'} > 0$ by (3.39). \Box

We now have all tools to prove the main result of this section, Theorem 1.3.

$$\frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)} > \frac{2N\alpha}{N-2b} > \frac{2(4-2b)}{N-2b} > 2.$$

⁷Increasing the value of r_1 if necessary.

⁸Notice that, since N=1,2 and by hypothesis $\alpha>\frac{4-2b}{N}$ we have

Proof of Theorem 1.3. We define

$$X = C\left([-T, T]; H^s(\mathbb{R}^N)\right) \bigcap L^q\left([-T, T]; H^{s,r}(\mathbb{R}^N)\right),\,$$

for any (q, r) L^2 -admissible, and

$$||u||_T = ||u||_{S(L^2;[-T,T])} + ||D^s u||_{S(L^2;[-T,T])}.$$

We shall show that $G = G_{u_0}$ defined in (1.2) is a contraction on the complete metric space

$$S(a,T) = \{ u \in X : ||u||_T \le a \}$$

with the metric

$$d_T(u, v) = ||u - v||_{S(L^2:[-T,T])},$$

for a suitable choice of a and T.

First, we claim that S(a,T) with the metric d_T is a complete metric space. Indeed, the proof follows similar arguments as in [2] (see Theorem 1.2.5 and the proof of Theorem 4.4.1 page 94). Since $S(a,T) \subset X$ and X is a complete space, it suffices to show that S(a,T), with the metric d_T , is closed in X. Let $u_n \in S(a,T)$ such that $d_T(u_n,u) \to 0$ as $n \to +\infty$, we want to show that $u \in S(a,T)$. If $u_n \in C([-T,T]; H^s(\mathbb{R}^N))$ (see the definition of S(a,T)) we have, for almost all $t \in [-T,T]$, $u_n(t)$ bounded in $H^s(\mathbb{R}^N)$ and so (since $H^s(\mathbb{R}^N)$) is reflexive)

$$u_n(t) \rightharpoonup v(t) \text{ in } H^s(\mathbb{R}^N) \text{ and } \|v(t)\|_{H^s} \le \liminf_{n \to +\infty} \|u_n\|_{H^s} \le a.$$
 (3.43)

On the other hand, the hypothesis $d_T(u_n, u) \to 0$ implies that $u_n \to u$ in $L_I^q L_x^r$ for all (q, r) L^2 -admissible. Since $(\infty, 2)$ is L^2 -admissible we get $u_n(t) \to u(t)$ in L^2 , for almost all $t \in [-T, T]$. Therefore, by uniqueness of the limit we deduce that u(t) = v(t). Also, we have from (3.43)

$$||u(t)||_{H^s} \le a.$$

That is, $u \in C([-T,T]; H^s(\mathbb{R}^N))$. From similar arguments, if $u_n \in L^q(I; H^{s,r}(\mathbb{R}^N))$ we obtain $u \in S(a,I)$. This completes the proof of the claim.

Returning the proof of the theorem, it follows from the Strichartz inequalities (2.9) and (2.11) that

$$||G(u)||_{S(L^2:[-T,T])} \le c||u_0||_{L^2} + c||F||_{S'(L^2:[-T,T])}$$
(3.44)

and

$$||D^{s}G(u)||_{S(L^{2};[-T,T])} \le c||D^{s}u_{0}||_{L^{2}} + c||D^{s}F||_{S'(L^{2};[-T,T])}, \tag{3.45}$$

where $F(x,u) = |x|^{-b}|u|^{\alpha}u$. Similarly as in the proof of Theorem 1.2, without loss of generality we consider only the case t > 0. So, we deduce using Lemmas 3.4-3.5-3.6 and (3.2)

$$||F||_{S'(L^2:I)} \le c(T^{\theta_1} + T^{\theta_2})||u||_I^{\alpha+1}$$

and

$$||D^s F||_{S'(L^2;I)} \le c(T^{\theta_1} + T^{\theta_2})||u||_I^{\alpha+1}.$$

where I = [0, T] and $\theta_1, \theta_2 > 0$. Hence, if $u \in S(a, T)$ then

$$||G(u)||_T \le c||u_0||_{H^s} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1}.$$

Now, choosing $a = 2c||u_0||_{H^s}$ and T > 0 such that

$$ca^{\alpha}(T^{\theta_1} + T^{\theta_2}) < \frac{1}{4},\tag{3.46}$$

we obtain $G(u) \in S(a,T)$. Such calculations establish that G is well defined on S(a,T).

To prove that G is a contraction we use (2.13) and an analogous argument as before

$$d_T(G(u), G(v)) \leq c \|F(x, u) - F(x, v)\|_{S'(L^2; [-T, T])}$$

$$\leq c(T^{\theta_1} + T^{\theta_2}) (\|u\|_T^{\alpha} + \|v\|_T^{\alpha}) d_T(u, v),$$

and so, taking $u, v \in S(a, T)$ we get

$$d_T(G(u), G(v)) \le c(T^{\theta_1} + T^{\theta_2})a^{\alpha}d_T(u, v).$$

Therefore, from (3.46), G is a contraction on S(a,T) and by the Contraction Mapping Theorem we have a unique fixed point $u \in S(a,T)$ of G.

We finish this section noting that Corollary 1.5 follows directly from Theorem 1.3. It is worth to mention that Corollary 1.5 only holds for $N \geq 2$ since we assume $s \leq \min\{\frac{N}{2}, 1\}$ in Theorem 1.3.

4. Global Well-Posedness

This section is devoted to study the global well-posedness of the Cauchy problem (1.1). Similarly as the local theory we use the fixed point theorem to prove our small data results in $H^s(\mathbb{R}^N)$. We start with a global result in $L^2(\mathbb{R}^N)$, which does not require any smallness assumption.

- 4.1. L^2 -Theory. The global well-posedness result in $L^2(\mathbb{R}^N)$ (see Theorem 1.7) is an immediate consequence of Theorem 1.2. Indeed, using (3.7) we have that $T(\|u_0\|_{L^2}) = \frac{C}{\|u_0\|_{L^2}^d}$ for some C, d > 0, then the conservation law (1.3) allows us to reapply Theorem 1.2 as many times as we wish preserving the length of the time interval to get a global solution.
- 4.2. H^s -Theory. In this subsection, we turn our attention to proof the Theorem 1.8. Again the heart of the proof is to establish good estimates on the nonlinearity $F(x,u) = |x|^{-b}|u|^{\alpha}u$. First, we estimate the norm $||F(x,u)||_{S'(\dot{H}^{-s_c})}$ (see Lemma 4.1 below), next we estimate $||F(x,u)||_{S'(L^2)}$ (see Lemma 4.2) and finally we consider the norm $||D^sF(x,u)||_{S'(L^2)}$ (see Lemmas 4.3, 4.5 and 4.7).

We begin defining the following numbers (depending only on N, α and b)

$$\widehat{q} = \frac{4\alpha(\alpha + 2 - \theta)}{\alpha(N\alpha + 2b) - \theta(N\alpha - 4 + 2b)} \quad \widehat{r} = \frac{N\alpha(\alpha + 2 - \theta)}{\alpha(N - b) - \theta(2 - b)} \tag{4.1}$$

and

$$\widetilde{a} = \frac{2\alpha(\alpha + 2 - \theta)}{\alpha[N(\alpha + 1 - \theta) - 2 + 2b] - (4 - 2b)(1 - \theta)} \quad \widehat{a} = \frac{2\alpha(\alpha + 2 - \theta)}{4 - 2b - (N - 2)\alpha}, \quad (4.2)$$

where $\theta > 0$ sufficiently small⁹. It is easy to see that $(\widehat{q}, \widehat{r})$ L^2 -admissible, $(\widehat{a}, \widehat{r})$ \dot{H}^{s_c} -admissible $(\widehat{a}, \widehat{r})$ \dot{H}^{-s_c} -admissible. Moreover, we observe that

$$\frac{1}{\widehat{a}} + \frac{1}{\widetilde{a}} = \frac{2}{\widehat{q}}.\tag{4.3}$$

Using the same notation of the previous section, we set B=B(0,1) and we recall that $|x|^{-b}\in L^{\gamma}(B)$ if $\frac{N}{\gamma}>b$. Similarly, we have that $|x|^{-b}\in L^{\gamma}(B^C)$ if $\frac{N}{\gamma}< b$ (see Remark 2.8).

Our first result reads follows.

Lemma 4.1. Let $\frac{4-2b}{N} < \alpha < \alpha_s$ and $0 < b < \widetilde{2}$. If $s_c < s \le \min\{\frac{N}{2}, 1\}$ then the following statement holds

$$|||x|^{-b}|u|^{\alpha}v||_{S'(\dot{H}^{-s_c})} \le c||u||_{L^{\infty}_{+}H^{s}_{x}}^{\theta}||u||_{S(\dot{H}^{s_c})}^{\alpha-\theta}||v||_{S(\dot{H}^{s_c})}, \tag{4.4}$$

where c > 0 and $\theta \in (0, \alpha)$ is a sufficiently small number.

Proof. The proof follows from similar arguments as the ones in the previous lemmas. We study the estimates in B and B^C separately.

We first consider the set B. From the Hölder inequality we deduce

$$\begin{aligned} \||x|^{-b}|u|^{\alpha}v\|_{L_{x}^{\widehat{r}'}(B)} &\leq \||x|^{-b}\|_{L^{\gamma}(B)}\|u\|_{L_{x}^{\theta r_{1}}}^{\theta - r_{1}}\|u\|_{L_{x}^{(\alpha - \theta)r_{2}}}^{\alpha - \theta}\|v\|_{L_{x}^{\widehat{r}}} \\ &= \||x|^{-b}\|_{L^{\gamma}(B)}\|u\|_{L_{x}^{\theta r_{1}}}^{\theta - r_{1}}\|u\|_{L_{x}^{\widehat{r}}}^{\alpha - \theta}\|v\|_{L_{x}^{\widehat{r}}}, \end{aligned}$$
(4.5)

where

$$\frac{1}{\hat{r}'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\hat{r}} \text{ and } \hat{r} = (\alpha - \theta)r_2.$$
 (4.6)

Now, we make use of the Sobolev embedding (Lemma 2.1), so we consider two cases: $s = \frac{N}{2}$ and $s < \frac{N}{2}$.

Case $s = \frac{N}{2}$. Since $s \leq \min\{\frac{N}{2}, 1\}$, we only have to consider the cases where (N, s) is equal to $(1, \frac{1}{2})$ or (2, 1). In order to have the norm $||x|^{-b}||_{L^{\gamma}(B)}$ bounded we need $\frac{N}{\gamma} > b$. In fact, observe that (4.6) implies

$$\frac{N}{\gamma} = N - \frac{N(\alpha + 2 - \theta)}{\widehat{r}} - \frac{N}{r_1},$$

and from (4.1) it follows that

$$\frac{N}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{N}{r_1}.\tag{4.7}$$

Since $\alpha > \frac{4-2b}{N}$ then $\frac{N\alpha}{2-b} > 2$, therefore choosing

$$\theta r_1 \in \left(\frac{N\alpha}{2-b}, +\infty\right),$$
 (4.8)

⁹First note that, since $\theta>0$ is sufficiently small, we have that the denominators of $\widehat{q},\widehat{r},\widehat{a}$ and \widetilde{a} are all positive numbers. Moreover, it is easy to see that \widehat{r} satisfies (2.3). In fact \widehat{a} can be rewritten as $\widehat{a}=\frac{\alpha+2-\theta}{1-s_c}$ and since $\theta<\alpha$ we have $\widehat{a}>\frac{2}{1-s_c}$, which implies that $\widehat{r}<\frac{2N}{N-2}$, for $N\geq 3$. We also note that $\widehat{r}\leq ((\frac{2}{1-s_c})^+)'$, for N=2. Indeed, the last inequality is equivalent to $\varepsilon\widehat{r}<(\frac{2}{1-s_c})^+(\frac{2}{1-s_c})$ (recall (2.4)) and this is true since $\varepsilon>0$ is a small enough number. For N=1, we see that $\widehat{r}<\infty$. Finally, we have $\widehat{r}>\frac{2N}{N-s_c}=\frac{N\alpha}{2-b}$. Indeed, this is equivalent to $(\alpha+2-\theta)(2-b)>\alpha(N-b)-\theta(2-b)\Leftrightarrow (\alpha+2)(2-b)>\alpha(N-b)\Leftrightarrow \alpha<\frac{4-2b}{N-2}$. So since $\alpha<\frac{4-2b}{N-2s}$ and $s\leq 1$ (hypothesis) we have that $\alpha<\frac{4-2b}{N-2}$ holds, consequently $\widehat{r}>\frac{2N}{N-s_c}$. ¹⁰Recall that s_c is the critical Sobolev index given by $s_c=\frac{N}{2}-\frac{2-b}{\alpha}$.

we get $\frac{N}{\gamma} > b$. Hence, inequality (4.5) and the Sobolev embedding (2.7) yield

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{\hat{r}'}(B)} \le c||u||_{H_{x}^{s}}^{\theta}||u||_{L_{x}^{\hat{r}}}^{\alpha-\theta}||v||_{L_{x}^{\hat{r}}}.$$
(4.9)

Case $s < \frac{N}{2}$. Our goal here is to also obtain the inequality (4.9). Indeed we already have the relation (4.7), then the only change is the choice of θr_1 since we can not apply the Sobolev embedding (2.7) when $s < \frac{N}{2}$. In this case we set

$$\theta r_1 = \frac{2N}{N - 2s},\tag{4.10}$$

SO

$$\frac{N}{\gamma} - b = \theta(s - s_c) > 0,$$

that is, the quantity $||x|^{-b}||_{L^{\gamma}(B)}$ is finite. Therefore by the Sobolev embedding (2.8) we obtain the desired inequality (4.9).

Next, we consider the set B^C . We claim that

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{\widehat{r}'}(B^{C})} \le c||u||_{H_{x}^{s}}^{\theta}||u||_{L_{x}^{\widehat{r}}}^{\alpha-\theta}||v||_{L_{x}^{\widehat{r}}}.$$
(4.11)

Indeed, Arguing in the same way as before we deduce

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{\hat{r}'}(B^{C})} \leq |||x|^{-b}||_{L^{\gamma}(B^{C})}||u||_{L_{x}^{\theta_{T_{1}}}}^{\theta_{T_{1}}}||u||_{L_{x}^{\hat{r}}}^{\alpha-\theta}||v||_{L_{x}^{\hat{r}}},$$

where the relation (4.7) holds. We first show that $|||x|^{-b}||_{L^{\gamma}(B^C)}$ is finite for a suitable of r_1 . Here we also consider two cases: $s = \frac{N}{2}$ and $s < \frac{N}{2}$. In the first case, we choose r_1 such that

$$\theta r_1 \in \left(2, \frac{N\alpha}{2-b}\right) \tag{4.12}$$

then, from (4.7), $\frac{N}{\gamma} - b < 0$, so $|x|^{-b} \in L^{\gamma}(B^C)$. Thus, by the Sobolev inequality (2.7) and using the last inequality we deduce (4.11). Now if $s < \frac{N}{2}$, choosing again θr_1 as (4.12) one has $\frac{N}{\gamma} - b < 0$. In addition, since $\alpha < \frac{4-2b}{N-2s}$ we obtain $\frac{N\alpha}{2-b} < \frac{2N}{N-2s}$, therefore the Sobolev inequality (2.8) implies (4.11). This completes the proof of the claim.

Now, inequalities (4.9) and (4.11) yield

$$|||x|^{-b}|u|^{\alpha}v||_{L_{x}^{\widehat{r}'}} \le c||u||_{H_{x}^{s}}^{\theta}||u||_{L_{x}^{\widehat{r}}}^{\alpha-\theta}||v||_{L_{x}^{\widehat{r}}}$$

$$\tag{4.13}$$

and the Hölder inequality in the time variable leads to

$$\begin{split} \big\| |x|^{-b} |u|^\alpha v \big\|_{L_t^{\widehat{a}'} L_x^{\widehat{r}'}} & \leq & c \|u\|_{L_t^{\infty} H_x^s}^{\theta} \|u\|_{L_t^{(\alpha - \theta)a_1} L_x^{\widehat{r}}}^{\alpha - \theta} \|v\|_{L_t^{\widehat{a}} L_x^{\widehat{r}}}, \\ & = & c \|u\|_{L_t^{\infty} H_x^s}^{\theta} \|u\|_{L_t^{\widehat{a}} L_x^{\widehat{r}}}^{\alpha - \theta} \|v\|_{L_t^{\widehat{a}} L_x^{\widehat{r}}}, \end{split}$$

where

$$\frac{1}{\widetilde{a}'} = \frac{\alpha - \theta}{\widehat{a}} + \frac{1}{\widehat{a}}.$$

Since \widehat{a} and \widetilde{a} defined in (4.2) satisfy the last relation we conclude the proof of (4.4).¹¹

Lemma 4.2. Let
$$\frac{4-2b}{N} < \alpha < \alpha_s$$
 and $0 < b < \widetilde{2}$. If $s_c < s \le \min\{\frac{N}{2}, 1\}$ then
$$|||x|^{-b}|u|^{\alpha}v||_{S'(L^2)} \le c||u||_{L_t^{\infty}H_x^s}^{\theta}||u||_{S(\dot{H}^{s_c})}^{\alpha-\theta}||v||_{S(L^2)},$$
 (4.14)

where c > 0 and $\theta \in (0, \alpha)$ is a sufficiently small number.

¹¹Recall that $(\widehat{a},\widehat{r})$ is \dot{H}^{s_c} -admissible and $(\widetilde{a},\widehat{r})$ is \dot{H}^{-s_c} -admissible.

Proof. By the previous lemma we already have (4.13), then applying Hölder's inequality in the time variable we obtain

$$|||x|^{-b}|u|^{\alpha}v||_{L_{t}^{\widehat{q}'}L_{x}^{\widehat{p}'}} \le c||u||_{L_{t}^{\infty}H_{x}^{s}}^{\theta}||u||_{L_{t}^{\widehat{q}}L_{x}^{\widehat{p}}}^{\alpha-\theta}||v||_{L_{t}^{\widehat{q}}L_{x}^{\widehat{p}}}, \tag{4.15}$$

since

$$\frac{1}{\widehat{a}'} = \frac{\alpha - \theta}{\widehat{a}} + \frac{1}{\widehat{a}} \tag{4.16}$$

by (4.1) and (4.2). The proof is finished in view of $(\widehat{q}, \widehat{r})$ be L^2 -admissible.

We now estimate $||D^s(|x|^{-b}|u|^{\alpha}u)||_{S'(L^2)}$. To this end we divide our study in three cases: $N \ge 4$, N = 3 and N = 1, 2.

Lemma 4.3. Let $N \geq 4$, $0 < b < \widetilde{2}$ and $\frac{4-2b}{N} < \alpha < \alpha_s$. If $s_c < s \leq 1$ then the following statement holds

$$\left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{S'(L^{2})} \le c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha - \theta} \|D^{s} u\|_{S(L^{2})}, \tag{4.17}$$

where c > 0 and $\theta \in (0, \alpha)$ is a sufficiently small number.

Proof. First note that we always have $s < \frac{N}{2}$ in this lemma, since we are assuming $N \ge 4$ and $s_c < s \le 1$. Here, we also divide the estimate in B and B^C separately. We begin estimating on B. The fractional product rule (Lemma 2.3) yields

$$||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{L^{\frac{\alpha}{p'}}(B)} \le N_{1}(t,B) + N_{2}(t,B),$$
 (4.18)

where

$$N_1(t,B) = \left\| |x|^{-b} \right\|_{L^{\gamma}(B)} \left\| D^s(|u|^{\alpha}u) \right\|_{L^{\beta}_x} \qquad N_2(t,B) = \left\| D^s(|x|^{-b}) \right\|_{L^d(B)} \left\| |u|^{\alpha}u \right\|_{L^{\alpha}_x} = \left\| D^s(|x|^{-b}) \right\|_{L^{\alpha}(B)} = \left\| D^s(|$$

and

$$\frac{1}{\hat{r}'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e}.$$
 (4.19)

It follows from the fractional chain rule (Lemma 2.4) and Hölder's inequality that

$$N_{1}(t,B) \leq \||x|^{-b}\|_{L^{\gamma}(B)} \|u\|_{L^{\theta r_{1}}_{x}}^{\theta} \|u\|_{L^{(\alpha-\theta)r_{2}}_{x}}^{\alpha-\theta} \|D^{s}u\|_{L^{\hat{r}}_{x}}$$

$$= \||x|^{-b}\|_{L^{\gamma}(B)} \|u\|_{L^{\theta r_{1}}_{x}}^{\theta} \|u\|_{L^{\hat{r}}_{x}}^{\alpha-\theta} \|D^{s}u\|_{L^{\hat{r}}_{x}}, \tag{4.20}$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\hat{r}}$$
 and $\hat{r} = (\alpha - \theta)r_2$. (4.21)

Notice that the right hand side of (4.20) is the same as the right hand side of (4.5), with $v = D^s u$, so combining (4.19) and (4.21) we also have (4.6). Thus, arguing in the same way as in Lemma 4.1 we obtain (recall that (4.9) also holds when $s < \frac{N}{2}$)

$$N_1(t,B) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^{\widehat{r}}}^{\alpha-\theta} \|D^s u\|_{L_x^{\widehat{r}}}.$$
 (4.22)

On the other hand, we deduce from (3.8), Hölder's inequality and the Sobolev emdebbing (2.6)

$$N_{2}(t,B) \leq ||x|^{-b-s}||_{L^{d}(B)}||u||_{L^{\theta_{T_{1}}}}^{\theta_{T_{1}}}||u||_{L^{\alpha-\theta_{T_{2}}}}^{\alpha-\theta}||u||_{L^{r_{3}}}$$

$$= ||x|^{-b-s}||_{L^{d}(B)}||u||_{L^{\theta_{T_{1}}}}^{\theta_{T_{1}}}||u||_{L^{\hat{r}_{x}}}^{\alpha-\theta}||D^{s}u||_{L^{\hat{r}_{x}}}, \qquad (4.23)$$

where

$$\begin{cases}
\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} & \widehat{r} = (\alpha - \theta)r_2 \\
s = \frac{N}{\widehat{r}} - \frac{N}{r_3} & \text{with} \quad s < \frac{N}{\widehat{r}},
\end{cases} (4.24)$$

which implies using (4.19) that

$$\frac{N}{d} - s = N - \frac{N(\alpha + 2 - \theta)}{\widehat{r}} - \frac{N}{r_1}$$

and so, by (4.1)

$$\frac{N}{d} - b - s = \frac{\theta(2-b)}{\alpha} - \frac{N}{r_1}.$$
 (4.25)

Observe that the right hand side of (4.25) is the same as the right hand side of (4.7). Hence, choosing θr_1 as in (4.10) (recall that $s < \frac{N}{2}$) we have $\frac{N}{d} - b - s > 0$, so the quantity $||x|^{-b-s}||_{L^d(B)}$ is bounded, by Remark 2.8. Now, the Sobolev embedding (2.8) and (4.23) imply that

$$N_2(t,B) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^{\widehat{r}}}^{\alpha-\theta} \|D^s u\|_{L_x^{\widehat{r}}}.$$

Therefore, the last inequality together with (4.22) lead to

$$||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{L_{r}^{\widehat{r}'}(B)} \le c||u||_{H_{x}^{s}}^{\theta}||u||_{L_{x}^{\widehat{r}}}^{\alpha-\theta}||D^{s}u||_{L_{x}^{\widehat{r}}}.$$
(4.26)

Thus applying Hölder's inequality in the time variable and recalling (4.16),

$$\begin{split} \left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{t}^{\widehat{q}'} L_{x}^{\widehat{r}'}(B)} & \leq c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{L_{t}^{\widehat{q}} L_{x}^{\widehat{r}}}^{\alpha - \theta} \|D^{s} u\|_{L_{t}^{\widehat{q}} L_{x}^{\widehat{r}}} \\ & \leq c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{s}c)}^{\alpha - \theta} \|D^{s} u\|_{S(L^{2})}. \end{split} \tag{4.27}$$

Next we consider the norm $\|D^s(|x|^{-b}|u|^{\alpha}u)\|_{L_x^{\tilde{r}'}(B^C)}$. Similarly as before, replacing B by B^C , we also get (4.20)-(4.21) and consequently by the proof of Lemma 4.1 we have the inequality (4.22), that is

$$N_1(t, B^C) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^{\widehat{r}}}^{\alpha - \theta} \|D^s u\|_{L_x^{\widehat{r}}}.$$

We also have (replacing B by B^C)

$$N_2(t, B^C) \le |||x|^{-b-s}||_{L^d(B^C)} ||u||_{L^{\theta_{r_1}}_x}^{\theta_{r_1}} ||u||_{L^{\hat{\tau}}_x}^{\alpha - \theta} ||D^s u||_{L^{\hat{\tau}}_x},$$

where the relation (4.25) holds, thus setting $\theta r_1 = 2$ we deduce

$$\frac{N}{d} - b - s = -\theta s_c < 0,$$

which implies that $|x|^{-b-s} \in L^d(B^C)$, by Remark 2.8. Then the Sobolev embedding (2.8) yields

$$N_2(t, B^C) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^{\widehat{r}}}^{\alpha - \theta} \|D^s u\|_{L_x^{\widehat{r}}}.$$

Therefore,

$$\begin{aligned} \left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{x}^{\hat{r}'}(B^{C})} & \leq & N_{1}(t,B) + N_{2}(t,B^{C}) \\ & \leq & \|u\|_{H_{x}^{s}}^{\theta} \|u\|_{L_{x}^{\hat{r}}}^{\alpha - \theta} \|D^{s} u\|_{L_{x}^{\hat{r}}}. \end{aligned}$$

Finally, using Hölder's inequality in the time variable, the last inequality (recalling (4.16)) and the relation (4.27) we get the estimate (4.17).

Remark 4.4. Notice that Lemma 4.3 doesn't hold in dimension three for every $\alpha < \alpha_s$ (recall (1.5)). In fact, the condition $s < \frac{N}{r}$ (used in (4.24)) is only true for $N \ge 4$. In the next lemma we consider the case N = 3.

Before stating the lemma, we define the following numbers

$$k = \frac{4\alpha(\alpha + 1 - \theta)}{4 - 2b - \alpha} \qquad p = \frac{6\alpha(\alpha + 1 - \theta)}{(4 - 2b)(\alpha - \theta) + \alpha} \tag{4.28}$$

and

$$l = \frac{4\alpha(\alpha + 1 - \theta)}{\alpha(3\alpha - 2 + 2b) - \theta(3\alpha - 4 + 2b)},$$
(4.29)

where $\theta \in (0, \alpha)$. It is not difficult to verify that (l, p) is L^2 -admissible and (k, p) is \dot{H}^{s_c} -admissible¹².

We also define

$$m = \frac{4D}{D - \varepsilon} \qquad n = \frac{6D}{2D + \varepsilon} \tag{4.30}$$

and

$$a^* = \frac{4\theta}{2 + \varepsilon - D} \quad r^* = \frac{6\alpha\theta}{(4 - 2b)\theta - (2 + \varepsilon - D)\alpha},\tag{4.31}$$

where $D = \alpha - \theta + \mu$ with $\mu \in (b, 1)$ and ε is a sufficiently small number such that $\varepsilon < \mu - b$. Note that 2 < n < 3 (n satisfies the condition (2.1) for N=3) and (m,n) is L^2 -admissible. Moreover, choosing $\theta = F\alpha$ with $H^{13} F = \frac{2-\varepsilon + \mu - 2b}{4-2b}$ we claim that (a^*, r^*) is \dot{H}^{s_c} -admissible. We first show that the denominators of a^* and r^* are positive numbers. Indeed

$$2+\varepsilon-D=2+\varepsilon-\mu+F\alpha-\alpha=2+\varepsilon-\mu-\alpha(1-F)=2+\varepsilon-\mu-\alpha\left(\frac{2+\varepsilon-\mu}{4-2b}\right),$$

so by our hypothesis $\alpha < \frac{4-2b}{3-2s}$ and since $s \leq 1$ we deduce $2 + \varepsilon - D > 0$. We also have (using the value of F and the fact that $D > \mu$)

$$(4-2b)\theta - (2+\varepsilon-D)\alpha = \alpha \left((4-2b)F - 2 - \varepsilon + D \right) > \left(2(\mu-b) - 2\varepsilon \right),$$

which is positive setting $\varepsilon < \mu - b$.

Next, we show that r^* satisfies the condition (2.3), with N=3. Note that r^* can be rewritten as $r^*=\frac{6\alpha F}{2(\mu-b-\varepsilon)+\alpha(1-F)}$. Hence, $r^*<6$ is equivalent to

$$\alpha F < 2(\mu - b - \varepsilon) + \alpha(1 - F) \Leftrightarrow \alpha < \frac{2(\mu - b - \varepsilon)}{2F - 1} = 4 - 2b,$$

which is true since $\alpha < \frac{4-2b}{3-2s}$ and $s \le 1$. In addition, $r^* > \frac{6}{3-2s} = \frac{3\alpha}{2-b}$ is equivalent to

$$(4-2b)F > 2(\mu-b-\varepsilon) + \alpha(1-F) \iff \alpha < 4-2b.$$

Finally, it is easy to see that (a^*, r^*) satisfy the condition (2.2).

Lemma 4.5. Let N = 3, $\frac{4-2b}{3} < \alpha < \frac{4-2b}{3-2s}$ and 0 < b < 1. If $s_c < s \le 1$ then there exists $\mu \in (b,1)$ such that

$$\|D^{s}(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^{2})} \leq c\|u\|_{L^{\infty}_{t}H^{s}_{x}}^{\theta}\|u\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta}(\|D^{s}u\|_{S(L^{2})} + \|u\|_{S(L^{2})})$$

$$+c\|u\|_{L^{\infty}_{t}H^{s}_{x}}^{1-\mu}\|u\|_{S(\dot{H}^{s_{c}})}^{\theta}\|D^{s}u\|_{S(L^{2})}^{\alpha-\theta+\mu}, \tag{4.32}$$

where c>0, $\theta=\alpha F$ with $F=\frac{2-\varepsilon+\mu-2b}{4-2h}$ and $\varepsilon>0$ is a sufficiently small number.

 $^{^{12}\}text{We see that }\frac{3\alpha}{2-b}=\frac{6}{3-2s_c}< p<6, \text{ i.e., }p \text{ satisfies the condition (2.3) (and therefore (2.1), since }\frac{6}{3-2s_c}>2) \text{ for }N=3.$ $^{13}\text{It is easy to see that }F\in\left(\frac{1}{2},1\right) \text{ if }\varepsilon<\mu-b.$ Therefore, since $\theta=F\alpha,$ we have $\theta<\alpha.$

Proof. Observe that

$$\left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{S'(L^{2})}\leq \left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{S'(L^{2}(B))}+\left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{S'(L^{2}(B^{C}))}.$$

Let $A \subset \mathbb{R}^N$ that can be B or B^C . Since (2,6) is L^2 -admissible in 3D we have

$$||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{S'(L^{2}(A))} \leq ||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{L^{2'}_{+}L^{6'}_{-}(A)}$$

As before, applying the fractional product rule (Lemma 2.3) we have

$$\|D^{s}(|x|^{-b}|u|^{\alpha}u)\|_{L_{x}^{\theta'}(A)} \le M_{1}(t,A) + M_{2}(t,A),$$
 (4.33)

where

$$M_1(t,A) = \left\| |x|^{-b} \right\|_{L^{\gamma}(A)} \|D^s(|u|^{\alpha}u)\|_{L^{\beta}_x}, \quad M_2(t,A) = \left\| D^s(|x|^{-b}) \right\|_{L^d(A)} \||u|^{\alpha}u\|_{L^{e}_x}$$
 and

$$\frac{1}{6'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e}.$$
 (4.34)

Estimating $M_1(t,A)$. It follows by the fractional chain rule (Lemma 2.4) and Hölder's inequality that

$$M_{1}(t,A) \leq ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L^{\alpha-\theta_{r_{2}}}}^{\alpha-\theta} ||D^{s}u||_{L^{p}_{x}}$$

$$= ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L^{p}_{x}}^{\alpha-\theta} ||D^{s}u||_{L^{p}_{x}}, \qquad (4.35)$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{p}$$
 and $p = (\alpha - \theta)r_2$. (4.36)

Combining (4.34) and (4.36) we obtain

$$\frac{3}{\gamma} = \frac{5}{2} - \frac{3}{r_1} - \frac{3(\alpha + 1 - \theta)}{p},$$

which implies, by (4.28)

$$\frac{3}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{3}{r_1}.$$
 (4.37)

In to order to show that $||x|^{-b}||_{L^{\gamma}(A)}$ is finite we need to verify that $\frac{3}{\gamma} - b > 0$ if A = B and $\frac{3}{\gamma} - b < 0$ if $A = B^C$, by Remark 2.8. Indeed if $\theta r_1 = \frac{6}{3-2s}$, by (4.37), we have

$$\frac{3}{\gamma} - b = \theta(s - s_c) > 0$$

and if $\theta r_1 = 2$ then

$$\frac{3}{\gamma} - b = -\theta s_c < 0.$$

Therefore, the inequality (4.35) and the Sobolev embedding (2.8) yield

$$M_1(t,A) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^p}^{\alpha-\theta} \|D^s u\|_{L_x^p}. \tag{4.38}$$

We now estimate $M_2(t, A)$. Let $A = B^C$, applying the Hölder inequality and (3.8) we have

$$M_{2}(t, B^{C}) \leq ||x|^{-b-s}||_{L^{d}(B^{C})} ||u||_{L_{x}^{\theta r_{1}}}^{\theta} ||u||_{L_{x}^{(\alpha-\theta)r_{2}}}^{\alpha-\theta} ||u||_{L_{x}^{p}}^{\theta}$$

$$\leq ||x|^{-b-s}||_{L^{d}(B^{C})} ||u||_{L_{x}^{\theta r_{1}}}^{\theta} ||u||_{L_{x}^{p}}^{\alpha-\theta} ||u||_{L_{x}^{p}}^{\theta},$$

where

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{p}$$
 and $p = (\alpha - \theta)r_2$.

The relation (4.34) and the last relation imply

$$\frac{3}{d} = \frac{5}{2} - \frac{3}{r_1} - \frac{3(\alpha + 1 - \theta)}{p}.$$

In view of (4.28) we deduce

$$\frac{3}{d} - b = \frac{\theta(2-b)}{\alpha} - \frac{3}{r_1}.$$

Setting $\theta r_1 = 2$ we have $\frac{3}{d} - b = -\theta s_c$, so $\frac{3}{d} - b - s = -\theta s_c - s < 0$, i.e., $|x|^{-b-s} \in L^d(B^C)$. So, by the Sobolev inequality (2.8)

$$M_2(t, B^C) \le c \|u\|_{H^s_x}^{\theta} \|u\|_{L^p_x}^{\alpha - \theta} \|u\|_{L^p_x}.$$
 (4.39)

We also deduce from the Hölder inequality, the Sobolev embedding 14 (2.6) and (3.8)

$$\begin{array}{lcl} M_2(t,B) & \leq & \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x}\|u\|^{\mu}_{L^{\mu r_3}_x}\|u\|^{1-\mu}_{L^{(1-\mu)r_4}_x} \\ & \leq & \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|D^su\|^{\alpha-\theta}_{L^n_x}\|D^su\|^{\mu}_{L^n_x}\|u\|^{1-\mu}_{L^{(1-\mu)r_4}_x} \\ & = & \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L^{r_s}_x}\|D^su\|^{\alpha-\theta+\mu}_{L^n_x}\|u\|^{1-\mu}_{L^{(1-\mu)r_4}_x}, \end{array}$$

if the following system is satisfied

$$\begin{cases} \frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \\ s = \frac{3}{n} - \frac{3}{(\alpha - \theta)r_2} \qquad s = \frac{3}{n} - \frac{3}{\mu r_3} \\ r^* = \theta r_1. \end{cases}$$

It follows from (4.34) and the previous system that

$$\frac{3}{d} = \frac{5}{2} + sD - \frac{3\theta}{r^*} - \frac{3D}{n} - \frac{3}{r_4},\tag{4.40}$$

which implies by (4.30) and (4.31)

$$\frac{3}{d} = \frac{7}{2} + sD - \frac{(2-b)\theta}{\alpha} - \frac{3D}{2} - \frac{3}{r_4},\tag{4.41}$$

where $D = \alpha - \theta + \mu$. In view of Remark 2.8 to show that $||x|^{-b-s}||_{L^d(B)}$ is bounded we need $\frac{3}{d} - b - s > 0$. In fact, choosing $(1 - \mu)r_4 = \frac{6}{3-2s}$ we have

$$\frac{3}{d} - b - s = 2 - b - \frac{3\alpha}{2} + \frac{3\theta}{2} + s(\alpha - \theta) - \frac{(2 - b)\theta}{\alpha}$$

$$= -\alpha \left(\frac{3}{2} - \frac{2 - b}{\alpha}\right) + \theta \left(\frac{3}{2} - \frac{2 - b}{\alpha}\right) + s(\alpha - \theta)$$

$$= (s - s_c)(\alpha - \theta),$$

which is positive since $s > s_c$. So $|x|^{-b-s} \in L^d(B)$ and

$$M_2(t,B) \le c \|u\|_{H_s^s}^{1-\mu} \|u\|_{L_x^{r*}}^{\theta} \|D^s u\|_{L_x^n}^{\alpha-\theta+\mu}. \tag{4.42}$$

where we have used the Sobolev embedding (2.8).

Therefore, combining (4.33), (4.38) with $A = B^{C}$ and (4.39) we obtain

$$\left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{L_{x}^{\theta'}(B^{C})}\leq c\|u\|_{H_{x}^{s}}^{\theta}\|u\|_{L_{x}^{p}}^{\alpha-\theta}\|D^{s}u\|_{L_{x}^{p}}+c\|u\|_{H_{x}^{s}}^{\theta}\|u\|_{L_{x}^{p}}^{\alpha-\theta}\|u\|_{L_{x}^{p}}$$

¹⁴We can use the Sobolev embedding (2.6) since $s \le 1 < \frac{3}{n}$.

Moreover by (4.38) with A = B and (4.42) we have

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{\theta'}_x(B)} \leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L^p_x}\|u\|_{L^p_x} + c\|u\|^{1-\mu}_{H^s_x}\|u\|^\theta_{L^{r*}_x}\|D^s u\|^{\alpha-\theta+\mu}_{L^n_x}.$$

Finally, since

$$\frac{1}{2'} = \frac{\alpha - \theta}{k} + \frac{1}{l}$$

and

$$\frac{1}{2'} = \frac{\theta}{a^*} + \frac{\alpha - \theta + \mu}{m},$$

we can use Hölder's inequality in the time variable in the last two inequalities to conclude

$$\left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{t}^{2'} L_{x}^{\theta'}(B^{C})} \leq c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{L_{t}^{k} L_{x}^{p}}^{\alpha - \theta} \left(\|D^{s} u\|_{L_{t}^{l} L_{x}^{p}} + \|u\|_{L_{t}^{l} L_{x}^{p}} \right)$$

and

$$\begin{split} \left\| D^s \left(|x|^{-b} |u|^\alpha u \right) \right\|_{L^{2'}_t L^{\theta'}_x(B)} & \leq & c \|u\|^\theta_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{L^k_t L^p_x} \|D^s u\|_{L^l_t L^p_x} \\ & + c \|u\|^{1-\mu}_{L^\infty_t H^s_x} \|u\|^\theta_{L^{q^*}_t L^{r^*}_x} \|D^s u\|^{\alpha-\theta+\mu}_{L^m_t L^n_x}, \end{split}$$

The proof is completed recalling that (m, n) and (l, p) are L^2 -admissible as well as (k, p) and (a^*, r^*) are \dot{H}^{s_c} -admissible.

Remark 4.6. It is worth to mention that in the previous lemma $\theta > 0$ is given by $\theta = F\alpha$ and since F < 1, we only have that $\theta < \alpha$ and it might be not true that θ is close to 0.

Before proving our global well-posedness result, we finish estimating the norm $\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)}$ in the dimensions N=1,2.

Lemma 4.7. Let N = 1, 2 and $\frac{4-2b}{N} < \alpha < \alpha_s$ with $0 < b < \widetilde{2}$. If $s_c < s \le \min\{\frac{N}{2}, 1\}$ then

$$||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{S'(L^{2})} \leq c||u||_{L_{t}^{\infty}H_{x}^{s}}^{\theta}||u||_{S(\dot{H}^{s_{c}})}^{\alpha-\theta}||D^{s}u||_{S(L^{2})} + c||u||_{L_{t}^{\infty}H_{x}^{s}}^{1+\theta}||u||_{S(\dot{H}^{s_{c}})}^{\alpha-\theta},$$

$$(4.43)$$

where c > 0 and $\theta \in (0, \alpha)$ is a sufficiently small number.

Proof. The proof follows from analogous arguments as the ones used in the previous lemmas. Let $A \subset \mathbb{R}^N$ that can be B or B^C and (q, r) any L^2 -admissible pair. By the fractional product rule (Lemma 2.3) we get

$$||D^{s}(|x|^{-b}|u|^{\alpha}u)||_{L_{x}^{r'}(A)} \le P_{1}(t,A) + P_{2}(t,A),$$
 (4.44)

where

$$P_1(t,A) = \||x|^{-b}\|_{L^{\gamma}(A)} \|D^s(|u|^{\alpha}u)\|_{L^{\beta}_x}, \quad P_2(t,A) = \|D^s(|x|^{-b})\|_{L^d(A)} \||u|^{\alpha}u\|_{L^{e}_x}$$

$$(4.45)$$

and

$$\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e}.$$
 (4.46)

To estimate $P_1(t,A)$ and $P_2(t,A)$, we consider three cases: N=1 and $s<\frac{1}{2}$; N=2 and s<1; N=1,2 and $s=\frac{N}{2}$.

Case N=1 and $s<\frac{1}{2}$. We define the following numbers

$$k^* = \frac{4\alpha(\alpha + 1 - \theta)}{(4 - 2b)(\alpha - \theta + 1) - \alpha} \qquad l^* = \frac{4(\alpha + 1 - \theta)}{\alpha - \theta} \qquad p^* = 2(\alpha + 1 - \theta) \quad (4.47)$$

$$q_0 = \frac{2\alpha}{\alpha b + \theta(2-b)}, \quad \text{and} \quad r_0 = \frac{2\alpha}{\alpha(1-2b) - \theta(4-2b)}.$$
 (4.48)

It is straightforward to verify that, if $\theta > 0$ is a small enough number, the assumption $0 < b < \frac{1}{3}$ implies that the denominators of q_0 , r_0 , k^* and l^* are all positive numbers. Furthermore, (q_0, r_0) , (l^*, p^*) are L^2 -admissible l^{15} and (k^*, p^*) is \dot{H}^{s_c} -admissible.

First, we estimate $P_1(t, A)$ with $r = r_0$. The fractional chain rule (Lemma 2.4) and Hölder's inequality yield

$$P_{1}(t,A) \leq ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L_{x}^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L_{x}^{(\alpha-\theta)_{r_{2}}}}^{\alpha-\theta} ||D^{s}u||_{L_{x}^{p^{*}}}$$

$$= ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L_{x}^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L_{x}^{p^{*}}}^{\alpha-\theta} ||D^{s}u||_{L_{x}^{p^{*}}}, \qquad (4.49)$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{p^*} \quad \text{and} \quad p^* = (\alpha - \theta)r_2.$$
 (4.50)

This implies

$$\frac{1}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{1}{r_1},$$
 (4.51)

where we have used (4.46), (4.50), (4.47) and (4.48). Now, if A=B and setting $\theta r_1 = \frac{2}{1-2s}$ we get $\frac{1}{\gamma} - b = \theta(s-s_c) > 0$, furthermore, taking $A=B^C$ and choosing $\theta r_1 = 2$ one has $\frac{1}{\gamma} - b = -\theta s_c < 0$. Hence, from the Sobolev embedding ¹⁶ (2.8) and Remark 2.8

$$P_1(t,A) \le c \|u\|_{H_x^s}^{\theta} \|u\|_{L_x^{p^*}}^{\alpha-\theta} \|D^s u\|_{L_x^{p^*}}. \tag{4.52}$$

We now consider $P_2(t, A)$ with $r = r_0$. It follows from (4.45) and (3.8) that

$$P_2(t,A) \leq ||x|^{-b-s}||_{L^d(A)}||u||_{L_x^{(\theta+1)e}}^{\theta+1}||u||_{L_x^{\infty}}^{\alpha-\theta}$$

$$(4.53)$$

and by (4.46)

$$\frac{1}{d} - b = \frac{1}{2} + \frac{\theta(2-b)}{\alpha} - \frac{1}{e}.$$
 (4.54)

We claim that $||x|^{-b-s}||_{L^d(A)}$ is a finite quantity for a suitable choice of e. If A=B we choose $(\theta+1)e=\frac{2}{1-2s}$, and if $A=B^C$ we set $(\theta+1)e=2$. We obtain in the first case

$$\frac{1}{d} - b - s = \theta(s - s_c) > 0,$$

and in the second case

$$\frac{1}{\gamma} - b - s = -\theta s_c < 0.$$

So, the Sobolev embedding (2.8), Remark 2.8 and (4.53) yield

$$P_2(t,A) \le c \|u\|_{H_x^s}^{\theta+1} \|u\|_{L_x^{\infty}}^{\alpha-\theta}.$$

¹⁵Note that, $r_0 > 2$ (see (2.1) for N = 1). Moreover, since $0 < b < \frac{1}{3}$ we have $p^* \ge \frac{2}{1 - 2s_c} = 1$ $\begin{array}{c} \frac{\alpha}{2-b} \text{ (see (2.2) for } N=1). \\ \text{ } \\$

Therefore, relations (4.44), (4.52) and the last inequality with A=B and $A = B^C$ imply that

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{r_0'}_x(B)} \leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L^{p^*}_x}\|D^s u\|_{L^{p^*}_x} + c\|u\|^{\theta+1}_{H^s_x}\|u\|^{\alpha-\theta}_{L^\infty_x}$$

and

$$\left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{L_{x}^{r_{0}'}(B^{C})}\leq c\|u\|_{H_{x}^{s}}^{\theta}\|u\|_{L_{x}^{p^{*}}}^{\alpha-\theta}\|D^{s}u\|_{L_{x}^{p^{*}}}+c\|u\|_{H_{x}^{s}}^{\theta+1}\|u\|_{L_{x}^{\infty}}^{\alpha-\theta}.$$

Finally since

$$\frac{1}{q_0'} = \frac{\alpha - \theta}{k^*} + \frac{1}{l^*}$$

we apply the Hölder inequality in the time variable to get (recalling (l^*, p^*) is L^2 admissible and (k^*, p^*) is H^{s_c} -admissible)

$$\begin{split} \left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{t}^{q'_{0}} L_{x}^{r'_{0}}} & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{L_{t}^{k^{*}} L_{x}^{p^{*}}}^{\theta} \|D^{s} u\|_{L_{t}^{l^{*}} L_{x}^{p^{*}}}^{\theta} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta+1} \|u\|_{L_{t}^{(\alpha-\theta)q'_{0}} L_{x}^{\infty}}^{\alpha-\theta} \\ & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta} \|D^{s} u\|_{S(L^{2})}^{\theta} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta+1} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta}. \end{split}$$

where we have used the fact that $(\alpha - \theta)q'_0 = \frac{4}{1-2s_c}$, by (4.48), and $(\frac{4}{1-2s_c}, \infty)$ is \dot{H}^{s_c} -admissible.

Case N=2 and s<1. We consider the following numbers

$$\widetilde{q} = \frac{2\alpha}{\alpha[b + 2\varepsilon(\alpha - \theta)] + \theta(2 - b)} \qquad \widetilde{r} = \frac{2\alpha}{\alpha[1 - b - 2\varepsilon(\alpha - \theta)] - \theta(2 - b)}, \quad (4.55)$$

$$l_0 = \frac{2(\alpha + 1 - \theta)}{(\alpha - \theta)(1 - 2\varepsilon)} \qquad p_0 = \frac{2(\alpha + 1 - \theta)}{1 + 2\varepsilon(\alpha - \theta)}$$
(4.56)

and

$$k_0 = \frac{2\alpha(\alpha + 1 - \theta)}{\alpha[1 - b - 2\varepsilon(\alpha - \theta)] + (2 - b)(1 - \theta)}$$

$$(4.57)$$

Note that $(\widetilde{q}, \widetilde{r})$, (l_0, p_0) are L^2 -admissible L^2 and (k_0, p_0) is \dot{H}^{s_c} -admissible L^3 .

Estimating $P_1(t, A)$ (recall (4.45)-(4.46)) with $r = \tilde{r}$. The fractional chain rule (Lemma 2.4) and Hölder's inequality lead to

$$P_{1}(t,A) \leq ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}_{x}}^{\theta_{r_{1}}} ||u||_{L^{\alpha-\theta_{r_{2}}}_{x}}^{\alpha-\theta} ||D^{s}u||_{L^{p_{0}}_{x}}$$

$$= ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}_{x}}^{\theta_{r_{1}}} ||u||_{L^{p_{0}}_{x}}^{\alpha-\theta} ||D^{s}u||_{L^{p_{0}}_{x}}, \qquad (4.58)$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{p_0}$$
 and $p_0 = (\alpha - \theta)r_2$, (4.59)

¹⁷The hypothesis $0 < b < \frac{N}{3}$ with N = 2 guarantee that the denominators of \tilde{q} , \tilde{r} , k_0 , l_0 and p_0 are all positive numbers. Moreover, $\tilde{r} > 2$ is equivalent to $\alpha(b + 2\varepsilon(\alpha - \theta)) > -\theta(2 - b)$ which is true, therefore \tilde{r} satisfies (2.1) for N = 2.

18We claim that $\frac{2\alpha}{2-b} = \frac{2}{1-s_c} \le p_0 \le ((\frac{2}{1-s_c})^+)'$. Indeed, the first inequality is equivalent to $\alpha(1-b) + (1-\theta)(2-b) \ge 2\varepsilon\alpha(\alpha-\theta)$ which is true since $\varepsilon > 0$ is a small enough number. On the other hand, the later inequality holds choose $\varepsilon < (\frac{2}{2})^+ (\frac{2}{2})^+$ (recall (2.4)) can be provided

the other hand, the later inequality holds since $\varepsilon p_0 \leq (\frac{2}{1-s_c})^+(\frac{2}{1-s_c})$ (recall (2.4)) can be verified for $\varepsilon > 0$ small enough.

so by the relations (4.46), (4.59), (4.56) and (4.55) one has

$$\frac{2}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{2}{r_1}. (4.60)$$

As in the previous case, if A = B we set $\theta r_1 = \frac{2}{1-s}$ and then $\frac{2}{\gamma} - b > 0$. On the other hand, if $A = B^C$, we set $\theta r_1 = 2$ and then $\frac{2}{\gamma} - b < 0$. Hence, the Sobolev embedding (2.8) and Remark 2.8 yield

$$P_1(t,A) \le c \|u\|_{H^s}^{\theta} \|u\|_{L^{p_0}}^{\alpha-\theta} \|D^s u\|_{L^{p_0}}. \tag{4.61}$$

Next we estimate $P_2(t, A)$ with with $r = \tilde{r}$. An application of the Hölder inequality together with (4.45) and (3.8) imply

$$\begin{array}{lcl} P_2(t,A) & \leq & \||x|^{-b-s}\|_{L^d(A)} \|u\|_{L^{(\theta+1)r_1}_x}^{\theta+1} \|u\|_{L^{(\alpha-\theta)r_2}_x}^{\alpha-\theta} \\ & \leq & \||x|^{-b-s}\|_{L^d(A)} \|u\|_{L^{(\theta+1)r_1}_x}^{\theta+1} \|u\|_{L^{\frac{1}{s}}}^{\alpha-\theta}, \end{array}$$

where

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2}, \quad (\alpha - \theta)r_2 = \frac{1}{\varepsilon}.$$
(4.62)

We deduce from (4.62) and (4.46)

$$\frac{2}{d} = 2 - \frac{2}{\tilde{r}} - \frac{1}{r_1} - 2\varepsilon(\alpha - \theta)
= 1 + b + \frac{\theta(2 - b)}{\alpha} - \frac{2}{r_1},$$
(4.63)

where we have used (4.55). In addition, if A = B and $(\theta + 1)r_1 = \frac{2}{1-s}$ we get

$$\frac{2}{d} - b - s = \theta(s - s_c) > 0,$$

likewise if $A = B^C$ and $(\theta + 1)r_1 = 2$, we have

$$\frac{2}{d} - b - s = -\theta s_c - s < 0.$$

Thus

$$P_2(t,A) \le c \|u\|_{H_x^s}^{\theta+1} \|u\|_{L_x^{\frac{1}{\varepsilon}}}^{\alpha-\theta},$$

where we have used the Sobolev inequality (2.8) and and Remark 2.8.

Hence, by the relations (4.44), (4.61) and the last inequality

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{\widetilde{r}}_x} \leq c\|u\|_{H^s_x}^{\theta}\|u\|_{L^{p_0}_x}^{\alpha-\theta}\|D^s u\|_{L^{p_0}_x} + c\|u\|_{H^s_x}^{\theta+1}\|u\|_{L^{\frac{s}{2}}_x}^{\alpha-\theta}.$$

Finally, from (4.55) and (4.57)

$$\frac{1}{\widetilde{q}'} = \frac{\alpha - \theta}{k_0} + \frac{1}{l_0},$$

so applying the Hölder inequality in the time variable we deduce

$$\begin{split} \left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{t}^{\widetilde{q}'} L_{x}^{\widetilde{r}'}} & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{L_{t}^{k_{0}} L_{x}^{p_{0}}}^{\alpha - \theta} \|D^{s} u\|_{L_{t}^{l_{0}} L_{x}^{p_{0}}}^{p_{0}} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta + 1} \|u\|_{L_{t}^{(\alpha - \theta)\widetilde{q}'} L_{x}^{\frac{1}{\varepsilon}}}^{\theta - \theta} \\ & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha - \theta} \|D^{s} u\|_{S(L^{2})}^{s} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta + 1} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha - \theta}. \end{split}$$

where we have used the fact that $(\alpha - \theta)\tilde{q}' = \frac{2\alpha}{2-b-2\varepsilon\alpha}$ and $(\frac{2\alpha}{2-b-2\varepsilon\alpha}, \frac{1}{\varepsilon})$ is \dot{H}^{s_c} -admissible.

Case N=1,2 and $s=\frac{N}{2}$. As before, we start defining the following numbers

$$\bar{a} = \frac{2(\alpha + 1 - \theta)}{2 - s_c} \quad \bar{q} = \frac{2(\alpha + 1 - \theta)}{2 + s_c(\alpha - \theta)}$$
 (4.64)

$$\bar{r} = \frac{2N(\alpha + 1 - \theta)}{N(\alpha + 1 - \theta) - 2s_c(\alpha - \theta) - 4} \tag{4.65}$$

and

$$\bar{k} = \frac{2(\alpha + 1 - \theta)^2}{2(\alpha - \theta)(1 - s_c) - s_c} \quad \bar{l} = \frac{2(\alpha + 1 - \theta)^2}{2(\alpha - \theta)(1 - s_c) + s_c\left((\alpha + 1 - \theta)^2 - 1\right)} \quad (4.66)$$

$$\bar{p} = \frac{2N(\alpha + 1 - \theta)^2}{(N - 2s_c)(\alpha + 1 - \theta)^2 - 4(\alpha - \theta)(1 - s_c) + 2s_c}.$$
(4.67)

It is not difficult to check that (\bar{q}, \bar{r}) and (\bar{l}, \bar{p}) L^2 -admissible and (\bar{a}, \bar{r}) , (\bar{k}, \bar{p}) \dot{H}^{s_c} -admissible.¹⁹

First, we estimate $P_1(t, A)$ with $r = \bar{r}$. The fractional chain rule (Lemma 2.4) and Hölder's inequality lead to

$$P_{1}(t,A) \leq ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L^{(\alpha-\theta)r_{2}}}^{\alpha-\theta} ||D^{s}u||_{L^{\bar{p}}}$$

$$= ||x|^{-b}||_{L^{\gamma}(A)} ||u||_{L^{\theta_{r_{1}}}}^{\theta_{r_{1}}} ||u||_{L^{\bar{p}}}^{\alpha-\theta} ||D^{s}u||_{L^{\bar{p}}}, \tag{4.68}$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\bar{p}} \quad \text{and} \quad \bar{p} = (\alpha - \theta)r_2,$$
(4.69)

and so combining (4.46), (4.69) (4.65) and (4.67) we obtain

$$\frac{N}{\gamma} - b = N - b - \frac{N}{r_1} - \frac{N}{\bar{r}} - \frac{N(\alpha + 1 - \theta)}{\bar{p}}$$

$$= N - b - \frac{N}{r_1} - \left(\frac{(\alpha + 1 - \theta)(N - 2s_c) + N - 2(2 - s_c)}{2}\right)$$

$$= \frac{\theta(2 - b)}{\alpha} - \frac{N}{r_1}.$$
(4.70)

In order to have that the first norm in the right hand side of (4.68) is finite, we need to verify $\frac{N}{\gamma} - b > 0$ if A = B and $\frac{N}{\gamma} - b < 0$ if $A = B^C$ for suitable choices of r_1 . To this end, we set r_1 such that

$$\theta r_1 > \frac{N\alpha}{(2-b)}$$
 (when $A = B$) and $2 < \theta r_1 < \frac{N\alpha}{(2-b)}$ (when $A = B^C$) (4.71)

$$\alpha (2(4-2b) - N - 2\alpha(N-2)) + (4-2b) \ge 2\theta(4-2b-\alpha(N-2)),$$

this is true since θ small enough, N = 1, 2 and $b < \frac{N}{3}$.

 $^{^{19}{\}rm It}$ is easy to see that the denominators of \bar{a} and \bar{q} are positive numbers (since $s_c<1$ and $\alpha>\theta)$. Furthermore, the denominators of \bar{r},\bar{k},\bar{l} and \bar{p} are also positive numbers for $\theta>0$ sufficiently small and $b<\frac{N}{3}$. We also have $\bar{r},\bar{p}\geq\frac{2N}{N-2s_c}=\frac{N\alpha}{2-b}.$ Indeed $\bar{r}=\frac{2N(\alpha+1-\theta)}{N-2b-\theta(N-2s_c)}\geq\frac{N\alpha}{2-b}\Leftrightarrow\alpha(4-N)+(1-\theta)(4-2b)>-\theta\alpha(N-2s_c)$ which is true since N=1,2 and $\theta<1.$ Moreover, $\bar{p}\geq\frac{N\alpha}{2-b}$ is equivalent to $2(\alpha-\theta)(4-2b-\alpha(N-2))\geq N\alpha-(4-2b)$ so

Hence, the Sobolev embedding (2.7) and (4.68) yield

$$P_1(t,A) \leq c \|u\|_{H^s_x}^{\theta} \|u\|_{L^{\bar{p}}_x}^{\alpha-\theta} \|D^s u\|_{L^{\bar{p}}_x}. \tag{4.72}$$

We now consider $P_2(t, A)$ with $r = \bar{r}$. By the Hölder inequality and (4.45)

$$P_{2}(t,A) \leq ||x|^{-b-s}||_{L^{d}(A)}||u||_{L_{x}^{(\theta+1)r_{1}}}^{\theta+1}||u||_{L_{x}^{\alpha-\theta}}^{\alpha-\theta}$$

$$= ||x|^{-b-s}||_{L^{d}(A)}||u||_{L_{x}^{(\theta+1)r_{1}}}^{\theta+1}||u||_{L_{x}^{\alpha}}^{\alpha-\theta}, \tag{4.73}$$

where

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} \quad \text{and} \quad \bar{r} = (\alpha - \theta)r_2.$$
(4.74)

The relations (4.46) and (4.74) as well as \bar{r} defined in (4.65), yield (recall $s = \frac{N}{2}$)

$$\frac{N}{d} - b - s = N - b - s - \frac{N}{r_1} - \frac{N(\alpha + 1 - \theta)}{\bar{r}}$$

$$= \frac{N}{2} + (2 - b) - \frac{N}{r_1} - \frac{N(\alpha + 1 - \theta)}{2} + s_c(\alpha - \theta)$$

$$= \frac{\theta(2 - b)}{\alpha} - \frac{N}{r_1}.$$
(4.75)

We see that the right hand side of (4.75) is equal to the right hand side of (4.70), so choosing r_1 as in (4.71) and again applying the Sobolev inequality (2.7), we conclude

$$P_2(t,A) \le c \|u\|_{H_x^s}^{\theta+1} \|u\|_{L_x^{\bar{r}}}^{\alpha-\theta}.$$

The inequalities (4.44), (4.72) and the last inequality imply that

$$\left\|D^{s}\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{L_{x}^{\bar{r}'}} \leq c\|u\|_{H_{x}^{s}}^{\theta}\|u\|_{L_{x}^{\bar{r}}}^{\alpha-\theta}\|D^{s}u\|_{L_{x}^{\bar{p}}} + c\|u\|_{H_{x}^{s}}^{\theta+1}\|u\|_{L_{x}^{\bar{r}}}^{\alpha-\theta}.$$

Since

$$\frac{1}{\bar{q}'} = \frac{\alpha - \theta}{\bar{k}} + \frac{1}{\bar{l}}$$

we can apply the Hölder inequality in the time variable to deduce

$$\begin{split} \left\| D^{s} \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L_{t}^{\bar{q}'} L_{x}^{\bar{r}'}} & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{L_{t}^{\bar{k}} L_{x}^{\bar{p}}}^{\bar{p}} \|D^{s} u\|_{L_{t}^{\bar{l}} L_{x}^{\bar{p}}} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta+1} \|u\|_{L_{t}^{(\alpha-\theta)\bar{q}'} L_{x}^{\bar{r}}}^{\alpha-\theta} \\ & \leq & c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{sc})}^{\alpha-\theta} \|D^{s} u\|_{S(L^{2})} \\ & + c \|u\|_{L_{t}^{\infty} H_{x}^{s}}^{\theta+1} \|u\|_{L_{t}^{\bar{t}} L_{r}^{\bar{r}}}^{\alpha-\theta}, \end{split}$$

where in the last equality we have used the fact that $\bar{a} = (\alpha - \theta)\bar{q}'$. This completes the proof since $(\bar{a}, \bar{r}) \dot{H}^{s_c}$ -admissible.

The next result follows directly from Lemmas 4.3, 4.5 and 4.7.

Corollary 4.8. Assume $\frac{4-2b}{N} < \alpha < \alpha_s$ and $0 < b < \widetilde{2}$. If $s_c < s \leq \min\{\frac{N}{2}, 1\}$ then following statement hold:

$$||D^{s}F||_{S'(L^{2})} \leq c||u||_{L_{t}^{\infty}H_{x}^{s}}^{\theta}||u||_{S(\dot{H}^{s_{c}})}^{\alpha-\theta} (||D^{s}u||_{S(L^{2})} + ||u||_{S(L^{2})} + ||u||_{L_{t}^{\infty}H_{x}^{s}})$$

$$+ c||u||_{L_{t}^{\infty}H_{x}^{s}}^{1-\mu}||u||_{S(\dot{H}^{s_{c}})}^{\theta}||D^{s}u||_{S(L^{2})}^{\alpha-\theta+\mu},$$

where $F(x, u) = |x|^{-b}|u|^{\alpha}u$.

Now, we have all the tools to prove the Theorem 1.8. Similarly as in the local theory, we use the contraction mapping principle.

Proof of Theorem 1.8. First, we define

$$B = \{u: \|u\|_{S(\dot{H}^{s_c})} \le 2\|U(t)u_0\|_{S(\dot{H}^{s_c})} \text{ and } \|u\|_{S(L^2)} + \|D^s u\|_{S(L^2)} \le 2c\|u_0\|_{H^s}\}.$$

We prove that $G = G_{u_0}$ defined in (1.2) is a contraction on B equipped with the metric

$$d(u,v) = ||u-v||_{S(L^2)} + ||u-v||_{S(\dot{H}^{s_c})}.$$

Indeed, we deduce by the Strichartz inequalities (2.9), (2.10), (2.11) and (2.12)

$$||G(u)||_{S(\dot{H}^{s_c})} \le ||U(t)u_0||_{S(\dot{H}^{s_c})} + c||F||_{S'(\dot{H}^{-s_c})}$$

$$\tag{4.76}$$

$$||G(u)||_{S(L^2)} \le c||u_0||_{L^2} + c||F||_{S'(L^2)} \tag{4.77}$$

and

$$||D^{s}G(u)||_{S(L^{2})} \le c||D^{s}u_{0}||_{L^{2}} + c||D^{s}F||_{S'(L^{2})}, \tag{4.78}$$

where $F(x,u) = |x|^{-b}|u|^{\alpha}u$. On the other hand, it follows from Lemmas 4.1 and 4.2 together with Corollary 4.8 that

$$||F||_{S'(\dot{H}^{-s_c})} \leq c||u||_{L_t^{\infty}H_x^s}^{\theta}||u||_{S(\dot{H}^{s_c})}^{\alpha-\theta}||u||_{S(\dot{H}^{s_c})}$$

$$||F||_{S'(L^2)} \leq c||u||_{L_t^{\infty}H_x^s}^{\theta}||u||_{S(\dot{H}^{s_c})}^{\alpha-\theta}||u||_{S(L^2)}$$

and

$$\begin{split} \|D^{s}F\|_{S'(L^{2})} & \leq c\|u\|_{L^{\infty}_{t}H^{s}_{x}}^{\theta}\|u\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta}\left(\|D^{s}u\|_{S(L^{2})} + \|u\|_{S(L^{2})} + \|u\|_{L^{\infty}_{t}H^{s}_{x}}\right) \\ & + c\|u\|_{L^{\infty}_{t}H^{s}_{x}}^{1-\mu}\|u\|_{S(\dot{H}^{s_{c}})}^{\theta}\|D^{s}u\|_{S(L^{2})}^{\alpha-\theta+\mu}. \end{split}$$

Combining (4.76)-(4.78) and the last inequalities, we get for $u \in B$

$$||G(u)||_{S(\dot{H}^{s_c})} \leq ||U(t)u_0||_{S(\dot{H}^{s_c})} + c||u||_{L_t^{\infty}H_x^s}^{\theta} ||u||_{S(\dot{H}^{s_c})}^{\alpha-\theta} ||u||_{S(\dot{H}^{s_c})}$$

$$\leq ||U(t)u_0||_{S(\dot{H}^{s_c})} + 2^{\alpha+1}c^{\theta+1}||u_0||_{H^s}^{\theta} ||U(t)u_0||_{S(\dot{H}^{s_c})}^{\alpha-\theta+1}.$$

In addition, setting $X = ||D^s u||_{S(L^2)} + ||u||_{S(L^2)} + ||u||_{L^\infty_t H^s_x}$

$$\begin{split} \|G(u)\|_{S(L^{2})} + \|D^{s}G(u)\|_{S(L^{2})} & \leq c\|u_{0}\|_{H^{s}} + c\|u\|_{L_{t}^{\infty}H_{x}^{s}}^{\theta} \|u\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta} X \\ & + c\|u\|_{L_{t}^{\infty}H_{x}^{s}}^{1-\mu} \|u\|_{S(\dot{H}^{s_{c}})}^{\theta} \|D^{s}u\|_{S(L^{2})}^{\alpha-\theta+\mu} \\ & \leq c\|u_{0}\|_{H^{s}} + 2^{\alpha+2}c^{\theta+2}\|u_{0}\|_{H^{s}}^{\theta+1}\|U(t)u_{0}\|_{S(\dot{H}^{s_{c}})}^{\alpha-\theta} \\ & + 2^{\alpha+1}c^{\alpha-\theta+2}\|u_{0}\|_{H^{s}}^{\alpha-\theta+1}\|U(t)u_{0}\|_{S(\dot{H}^{s_{c}})}^{\theta}, \end{split}$$

where we have used the fact that $X \leq 2^2 c \|u_0\|_{H^s}$ since $u \in B$. Now if $\|U(t)u_0\|_{L^\infty} \leq \delta$ with

Now if $||U(t)u_0||_{S(\dot{H}^{s_c})} < \delta$ with

$$\delta \leq \min \left\{ \sqrt[\alpha-\theta]{\frac{1}{2c^{\theta+1}2^{\alpha+1}A^{\theta}}}, \sqrt[\alpha-\theta]{\frac{1}{4c^{\theta+1}2^{\alpha+2}A^{\theta}}}, \sqrt[\theta]{\frac{1}{4c^{\alpha-\theta+1}2^{\alpha+1}A^{\alpha-\theta}}} \right\}, \quad (4.79)$$

where A > 0 is a number such that $||u_0||_{H^s} \leq A$, we get

$$||G(u)||_{S(\dot{H}^{s_c})} \le 2||U(t)u_0||_{S(\dot{H}^{s_c})}$$

and

$$||G(u)||_{S(L^2)} + ||D^sG(u)||_{S(L^2)} \le 2c||u_0||_{H^s},$$

that is $G(u) \in B$.

To complete the proof we show that G is a contraction on B. From (2.13) and repeating the above computations one has

$$\begin{split} \|G(u) - G(v)\|_{S(\dot{H}^{s_c})} &\leq c \|F(x, u) - F(x, v)\|_{S(\dot{H}^{-s_c})} \\ &\leq c \||x|^{-b} |u|^{\alpha} |u - v|\|_{S(\dot{H}^{-s_c})} + \||x|^{-b} |v|^{\alpha} |u - v|\|_{S(\dot{H}^{-s_c})} \\ &\leq c \|u\|_{L_t^{\infty} H_x^s}^{\theta} \|u\|_{S(\dot{H}^{s_c})}^{\alpha - \theta} \|u - v\|_{S(\dot{H}^{s_c})} \\ &+ c \|v\|_{L_t^{\infty} H_x^s}^{\theta} \|v\|_{S(\dot{H}^{s_c})}^{\alpha - \theta} \|u - v\|_{S(\dot{H}^{s_c})} \end{split}$$

which implies, taking $u, v \in B$

$$\begin{split} \|G(u) - G(v)\|_{S(\dot{H}^{s_c})} & \leq 2c(2c)^{\theta} \|u_0\|_{H^s}^{\theta} 2^{\alpha - \theta} \|U(t)u_0\|_{S(\dot{H}^{s_c})}^{\alpha - \theta} \|u - v\|_{S(\dot{H}^{s_c})} \\ & = 2^{\alpha + 1} c^{\theta + 1} \|u_0\|_{H^s}^{\theta} \|U(t)u_0\|_{S(\dot{H}^{s_c})}^{\alpha - \theta} \|u - v\|_{S(\dot{H}^{s_c})}. \end{split}$$

By similar arguments we also obtain

$$||G(u) - G(v)||_{S(L^2)} \le 2^{\alpha + 1} c^{\theta + 1} ||u_0||_{H^s}^{\theta} ||U(t)u_0||_{S(\dot{H}^{s_c})}^{\alpha - \theta} ||u - v||_{S(L^2)}.$$

Finally, from the two last inequalities and (4.79)

$$d(G(u), G(v)) \le 2^{\alpha+1} c^{\theta+1} \|u_0\|_{H^s}^{\theta} \|U(t)u_0\|_{S(\dot{H}^{s_c})}^{\alpha-\theta} d(u, v) \le \frac{1}{2} d(u, v),$$

i.e., G is a contraction.

Therefore, by the Banach Fixed Point Theorem, G has a unique fixed point $u \in B$, which is a global solution of (1.1).

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CARLOS M. GUZMÁN

Department of Mathematics, University Federal of Minas Gerais, BRAZIL $E\text{-}mail\ address:\ \mathtt{carlos.guz.j@gmail.com}$