Rationality of capped descendent vertex in K-theory

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Abstract

In this paper we analyze the fundamental solution of the *quantum difference equation* (qde) for the moduli space of instantons on two-dimensional projective space. The qde is a K-theoretic generalization of the quantum differential equation in quantum cohomology. As in the quantum cohomology case, the fundamental solution of qde provides the capping operator in K-theory (the rubber part of the capped vertex). We study the dependence of the capping operator on the equivariant parameters a_i of the torus acting on the instanton moduli space by changing the framing. We prove that the capping operator factorizes at $a_i \to 0$. The rationality of the K-theoretic 1-leg capped descendent vertex follows from factorization of the capping operator as a simple corollary.

Contents

1 Introduction

1.1 Summary

The rationality of partition functions in the presence of descendents is an important and long standing conjecture in enumerative geometry of 3-folds. The deepest insight into this conjecture was achieved in the theory of stable pairs where the conjecture was proved for nonsingular toric 3-folds and local curve 3-folds [10, 9] (the theory of stable pairs is conjecturally equivalent to Gromov-Witten theory and cohomological Donaldson-Thomas theory of 3-folds [2]). The central result here is the rationality of the cohomological capped 1-leg descendent vertex, see Theorem 3 in [10], which was obtained by detailed analysis of the poles of the descendent vertex. The rationality

of the partition functions for local curve 3-folds and nonsingular toric 3-folds can be derived from this result by applying standard arguments such as degeneration and geometric reduction of 3-leg descendent vertex to the case of 1-leg.

In this paper we prove the rationality of the capped K-theoretic 1-leg descendent vertex - Theorem 2. This function is defined in the theory of stable quasimaps to the Hilbert scheme of points on the complex plane. The enumerative geometry of stable quasimaps developed in [5] is a K-theoretic generalization of quantum cohomology. We should stress here that our proof of rationality is completely different (and, we believe, simpler) than one in [10]. Our main idea is to analyze the capped K-theory vertex in broader context: instead of restricting ourself to quasimaps to the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{C}^2)$ we consider the quasimaps to the moduli of instantons $\mathscr{M}(n,r)$ of arbitrary rank r and topological charge n. The Hilbert schemes are covered by a special case $r = 1$ when $\mathscr{M}(n,1) = \mathrm{Hilb}^n(\mathbb{C}^2)$. For a general r the coefficients of power series for both the bare vertex and the capping operator are rational functions of additional equivariant parameters $a_1 \cdots a_r$ corresponding to the action of framing torus $\mathsf{A} = (\mathbb{C}^*)^r$ on $\mathscr{M}(n,r)$. We analyze the asymptotic behaviour of the capping operator and vertex in the limit $a_i \to 0$ and find that these functions factorize as stated in Theorem 3 and Theorem 4. In Section 1.6 we show that the rationality of the capped descendent vertex follows in an elementary way from these two factorization theorems.

The main ingredient of our approach is the *quantum difference equation* (qde), see equation (7). As explained in [8], in K-theory this equation plays the role similar to one of the quantum differential equation in quantum cohomology of instanton moduli space $\mathscr{M}(n,r)$. The capping operator in K-theory is given by the fundamental solution of qde. We note here, that in the special case $r = 1$, the cohomological limit of the operator $\mathbf{M}_{\mathcal{O}(1)}(z)$ in qde (7) coincides with the operator of quantum multiplication by the first Chern class of tautological bundle in the quantum cohomology of $\mathrm{Hilb}^n(\mathbb{C}^2)$ given by formula (6) in [6]. Therefore, in this limit the qde turns to the quantum differential equation for $\mathrm{Hilb}^n(\mathbb{C}^2)$ described in [7]. As a consequence, rationality of the descendent vertex in the quantum cohomology follows from our main Theorem 2 through the cohomological limit.

The paper is organized as follows. First we recall some necessary facts about $\mathscr{M}(n,r)$ and various tori acting on this moduli space. Following [5] we define the bare and capped vertex functions in Section 1.3. We then

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formulate the factorization Theorems 3 and 4 and prove the main Theorem 2 as a corollary.

In Section 2 we outline the theory of the quantum toroidal algebra $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$. Using the results of [8] we then describe the action of this algebra on the equivariant K-theory of the instanton moduli spaces. In Section 3 we give a universal formula for qde for instanton moduli space of arbitrary rank r in terms of Heisenberg subalgebras of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$. We then prove the factorization Theorem 3 for the capping operator. In Section 4 we give an explicit formula for the bare 1-leg K-theoretic vertex. We use it to prove a factorization theorem for the vertex.

1.2 Instanton moduli

Let $\mathscr{M}(n, r)$ be the moduli space of framed rank r torsion-free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with fixed second Chern class $c_2(\mathcal{F}) = n$. A framing of a sheaf $\mathcal{F}$ is a choice of an isomorphism:

$$\phi : \mathcal{F}|_{L_\infty} \to \mathcal{O}_{L_\infty}^{\oplus r} \tag{1}$$

where L_∞ is the line at infinity of $\mathbb{C}^2 \subset \mathbb{P}^2$. This moduli space is usually referred to as rank r instanton moduli space. Note that in the special case $r = 1$ this moduli space is isomorphic to the Hilbert scheme of n-points on the complex plane

$$\mathscr{M}(n, 1) = \mathrm{Hilb}^n(\mathbb{C}^2).$$

Let $\mathsf{A} \simeq (\mathbb{C}^\times)^r$ be the framing torus acting on $\mathscr{M}(n, r)$ by scaling the i-th summand in isomorphism (1) with a character which we denote by a_i. This torus acts on the instanton moduli space preserving the symplectic form. Let us denote by $\mathsf{T} = \mathsf{A} \times (\mathbb{C}^\times)^2$ where the second factor acts on $\mathbb{C}^2 \subset \mathbb{P}^2$ by scaling the coordinates on the plane with characters which we denote by t_1 and t_2. This induces an action of T on $\mathscr{M}(n, r)$. The action of this torus scales the symplectic form with a character $\hbar = t_1 t_2$.

Let $\mathsf{C} \subset \mathsf{A}$ be a one-dimensional subtorus acting on the $r = r_1 + r_2$-dimensional framing space with the character $r_1 + ar_2$. In this situation we say that subtorus C splits the framing r to $r_1 + ar_2$. We note that the components of C-fixed point set are of the form:

$$\mathscr{M}(n, r)^{\mathsf{C}} = \coprod_{n_1 + n_2 = n} \mathscr{M}(n_1, r_1) \times \mathscr{M}(n_2, r_2). \tag{2}$$

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If we set $\mathscr{M}(r) = \coprod_{n=0}^{\infty} \mathscr{M}(n,r)$ then

$$\mathscr{M}(r)^{\mathsf{C}} = \mathscr{M}(r_1) \times \mathscr{M}(r_2). \tag{3}$$
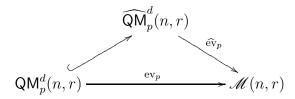
1.3   Vertex functions

Let $\mathsf{QM}^d(n,r)$ be the moduli space of stable quasimaps from $\mathbb{P}^1$ to $\mathscr{M}(n,r)$ [1]. Let us consider an action of a one-dimensional torus on $\mathbb{P}$ which comes from scaling the standard coordinate on $\mathbb{P}$. The fixed point set of this action consist of two points $\{p_1, p_2\} = \{0, \infty\} \subset \mathbb{P}$. We denote the character of $T_{p_1}\mathbb{P}$ by q and the torus by $\mathbb{C}^*_q$. This action induces an action of $\mathbb{C}^*_q$ on $\mathsf{QM}^d(n,r)$. We denote the total torus acting on $\mathsf{QM}^d(n,r)$ by $\mathsf{G} = \mathsf{T} \times \mathbb{C}^*_q$.

For a point $p \in \mathbb{P}$ let $\mathsf{QM}^d_p(n,r) \subset \mathsf{QM}^d(n,r)$ be the open subset of quasimaps non-singular at p. This subset comes together with the evaluation map:

$$\mathrm{ev}_p : \mathsf{QM}^d_p(n,r) \longrightarrow \mathscr{M}(n,r)$$

sending a quasimap to its value at p.

The moduli space of relative quasimaps $\widehat{\mathsf{QM}}^d_p(n,r)$ is a resolution of the map ev meaning that we have a commutative diagram:

$$
\begin{array}{ccc}
& \widehat{\mathsf{QM}}^d_p(n,r) & \\
& \nearrow \qquad \searrow^{\widehat{\mathrm{ev}}_p} & \\
\mathsf{QM}^d_p(n,r) & \xrightarrow{\quad \mathrm{ev}_p \quad} & \mathscr{M}(n,r)
\end{array}
$$

with a *proper* evaluation map $\widehat{\mathrm{ev}}_p$. The explicit construction of the moduli space of relative quasimaps is given in Section 6.4 of [5].

Let $L = \mathbb{C}^n$ and $\tau \in K_{GL(L)} = \Lambda[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ be a symmetric Laurent polynomial. Every such polynomial corresponds to a virtual representation $\tau(L)$ of $GL(L)$ which is a tensorial polynomial in L. For example, elementary symmetric function $\tau = e_k(x_i)$ corresponds to $\tau(L) = \Lambda^k L$. The *bare vertex* with a descendent τ is defined by the following formal power series:

$$V^{(\tau)}(z) = \sum_{d=0}^{\infty} \mathrm{ev}_{p_2,*}\Big(\mathsf{QM}^d_{p_2}(n,r), \widehat{\mathscr{O}}_{\mathrm{vir}} \otimes \tau(\mathscr{V}|_{p_1})\Big) z^d \in K_{\mathsf{G}}\Big(\mathscr{M}(n,r)\Big)_{loc} \otimes \mathbb{Q}[[z]] \tag{4}$$

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[1] In this paper we follow the terminology and notations of [5]. A good introduction to the stable quasimaps is Section 4.3 of [5].

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Here $\mathscr{V}$ is the rank n degree d bundle on $\mathbb{P}^1$ defining the quasimap and $\widehat{\mathscr{O}}_{\text{vir}}$ is the virtual structure of $\mathsf{QM}^d_{p_2}(n,r)$. The pushforward $\text{ev}_{p_2,*}$ is not proper; however, the fixed locus of G is (in fact, the fixed locus is a set of finitely many isolated points in this case). Therefore, the pushforward is well defined in localized K-theory.

The *capped vertex* with a descendent τ is a similar object defined for quasimap moduli space relative to p_2:

$$\hat{V}^{(\tau)}(z) = \sum_{d=0}^{\infty} \widehat{\text{ev}}_{p_2,*}\left(\widehat{\mathsf{QM}}^d_{p_2}(n,r), \widehat{\mathscr{O}}_{\text{vir}} \otimes \tau(\mathscr{V}|_{p_1})\right) z^d \in K_{\mathsf{G}}\left(\mathscr{M}(n,r)\right) \otimes \mathbb{Q}[[z]] \ (5)$$

Now, $\widehat{\text{ev}}_{p_2}$ is proper and the result lives in non-localized K-theory.

By definition, the degree zero quasimaps $\widehat{\mathsf{QM}}^0_{p_2}(n,r) = \mathscr{M}(n,r)$ correspond to the constant maps from $\mathbb{P}$ to the instanton moduli space. In this case we have $\mathscr{V}|_{p_1} = \mathcal{V}$ where $\mathcal{V}$ is a tautological bundle on $\mathscr{M}(n,r)$. By definition of the capped vertex :

$$\hat{V}^{(\tau)}(z) = \tau(\mathcal{V})\mathcal{K}^{1/2} + O(z)$$

where $\mathcal{K}$ is the canonical bundle on $\mathscr{M}(n,r)$.

As an element of non-localized K-theory the capped vertex is a simpler object. For example at large r all quantum corrections vanish:

Theorem 1. *(Theorem 7.5.23 in [5]) For every τ there exist $r \gg 0$ such that $\hat{V}^{(\tau)}(z) = \tau(\mathcal{V})\mathcal{K}^{1/2}$.*

In fact, numerical computations shows that one can give the precise bound for r in this theorem. Assume that the descendent is given by a Schur polynomial $\tau = s_\lambda(x_1, \cdots, x_n)$ for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$.

Conjecture 1. *For every $\tau = s_\lambda(x_1, \cdots, x_n)$ the capped vertex is classical $\hat{V}^{(\tau)}(z) = \tau(\mathcal{V})\mathcal{K}^{1/2}$ if and only if $r > \lambda_1$.*

In particular, this conjecture implies that for the Hilbert schemes of points on a plane $\text{Hilb}^n(\mathbb{C}^2)$ corresponding to $r = 1$ the capped vertex is classical only in the absence of descendents, i.e, for $\tau = 1$. In general, all terms in the power series (5) are non-vanishing. However, our main theorem says that this power series is in fact a rational function:

Theorem 2. *The power series $\hat{V}^{(\tau)}(z)$ is the Taylor expansion in z of a rational function in $\mathbb{Q}(u_1, \cdots, u_r, t_1, t_2, q, z)$.*

1.4 Capping operator

In computing the capped vertex one can separate the contributions of the $\mathsf{QM}^d_{p_2}(n,r)$ (which corresponds to the bare vertex) and the "rubber" part of the relative moduli space, see Section 7 of [5]:

$$\hat{V}^{(\tau)}(z) = \Psi(z) V^{(\tau)}(z). \tag{6}$$

Here

$$\Psi(z) = K_{\mathsf{G}}\Big(\mathscr{M}(n,r)\Big)^{\otimes 2}_{loc} \otimes \mathbb{Q}[[z]]$$

is the so called capping operator corresponding to the contribution of the rubber part. The matrix $\Psi(z)$ can be computed explicitly as a matrix of the *fundamental solution of the quantum difference equation*:

$$\Psi(z)\mathcal{O}(1) = \mathbf{M}_{\mathcal{O}(1)}(z)\Psi(z) \tag{7}$$

where $\mathcal{O}(1)$ is an operator of multiplication by the corresponding line bundle in $K_{\mathsf{T}}(\mathscr{M}(n,r))$. The operator $\mathbf{M}_{\mathcal{O}(1)}(z)$ acts in $K_{\mathsf{T}}(\mathscr{M}(n,r))$ and has rational in z matrix coefficients. It is constructed explicitly in Section 3.1.

1.5 Factorization theorems

Assume that the torus C splits the framing r to $r_1 + ar_2$, so that the set of C-fixed points is given by (3). By definition, the coefficients of power series $\Psi(z)$ and $V^{(\tau)}(z)$ are given by classes of localized K-theory and thus are rational functions of a. We are interested in $a \to 0$ limits of these power series. In Section 2.4 we describe an action of the quantum Heisenberg algebra $\mathfrak{h}$ on $K_{\mathsf{T}}\Big(\mathscr{M}(r)\Big)$. Let α_k, $k \in \mathbb{Z} \setminus \{0\}$ and K be the standard generators of $\mathfrak{h}$. In Section 3.5 we prove the following result:

Theorem 3. *If $\Psi^{(r)}(z)$ is the solution of (7) for $\mathscr{M}(r)$ then:*

$$\lim_{a \to 0} \Psi^{(r)}(z) = Y^{(r_1),(r_2)}(z)\, \Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}}) \tag{8}$$

where $Y^{(r_1),(r_2)}(z)$ is the evaluation of the following universal element

$$Y(z) = \exp\Big(-\sum_{k=1}^{\infty} \frac{(\hbar - \hbar^{-1})K^k \otimes K^{-k}}{1 - z^{-k}K^k \otimes K^{-k}} \alpha_{-k} \otimes \alpha_k\Big) \in \mathfrak{h}^{\otimes 2}(z) \tag{9}$$

in the $\mathfrak{h}^{\otimes 2}$ - representation $K_{\mathsf{T}}\Big(\mathscr{M}(r_1)\Big) \otimes K_{\mathsf{T}}\Big(\mathscr{M}(r_2)\Big)$.

Note that $\alpha_{-k} \otimes \alpha_k$ act on the corresponding K-theory as locally nilpotent operators. Thus, the coefficients of $Y^{(r_1),(r_2)}(z)$ are rational functions of z for all r_1 and r_2. In Section 4 we also prove a similar result for the bare vertex:

Theorem 4. *For a descendent* $\tau \in \Lambda[x_1, \cdots, x_n]$ *we have[2]:*

$$\lim_{a \to 0} V^{(r),(\tau)}(z) = V^{(r_1),(\tau)}(z \hbar^{\frac{r_2}{2}}) \otimes V^{(r_2),(1)}(z \hbar^{-\frac{r_1}{2}} q^{-r_1}) \tag{10}$$

Here, as well as in Theorem 3, the additional superscript (r) corresponds to the ranks of the instanton moduli.

1.6   Proof of the main Theorem 2

The proof of Theorem 2 is now elementary. Indeed, for arbitrary r_1 and τ and in Theorem 4 let $\hat{V}^{(r_1),(\tau)}(z)$ be the corresponding capped vertex. We need to check that $\hat{V}^{(r_1),(\tau)}(z)$ is a rational function of z. First, by Theorem 1 we can find r large enough for the corresponding capped descendent vertex to be classical $\hat{V}^{(r),(\tau)}(z) = \tau(\mathcal{V})\mathcal{K}^{1/2}$, i.e. independent of z.

We have:
$$\tau(\mathcal{V})\mathcal{K}^{1/2} = \Psi^{(r)}(z) V^{(r),(\tau)}(z).$$

The bundle $\mathcal{K}$ does not depend on a and by our choice of τ we have $\lim_{a \to 1} \tau(\mathcal{V}) = \tau(\mathcal{V}) \otimes 1$. Thus, by the factorization theorems we obtain:

$$(\tau(\mathcal{V}) \otimes 1)\mathcal{K}^{1/2} =$$

$$Y^{(r_1),(r_2)}(z)\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}})V^{(r_1),(\tau)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})V^{(r_2),(1)}(z\hbar^{-\frac{r_1}{2}}q^{-r_1})$$

Now, the first factor on the right side gives the capped vertex with shifted parameter $\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}})V^{(r_1),(\tau)}(z\hbar^{\frac{r_2}{2}}) = \hat{V}^{(r_1),(\tau)}(z\hbar^{\frac{r_2}{2}})$. The operator $Y^{(r_1),(r_2)}(z)$ from Theorem 3 is explicitly invertible, therefore:

$$\hat{V}^{(r_1),(\tau)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})V^{(r_2),(1)}(z\hbar^{-\frac{r_1}{2}}q^{-r_1}) = Y^{(r_1),(r_2)}(z)^{-1}(\tau(\mathcal{V})\otimes 1)\mathcal{K}^{1/2}$$

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[2]In full generality the descendent τ can be a symmetric Laurent polynomial $\tau \in K_{GL(n)}(\cdot) = \Lambda[x_1^{\pm}, \cdots, x_n^{\pm}]$, but it will be clear in the proof of this proposition that cases with inverse powers of x_i are treated in the same way when $a \to \infty$. The result remains the same if we change the roles of r_1 and r_2.

The matrix $Y^{(r_1),(r_2)}(z)$ has rational coefficients, so $Y^{(r_1),(r_2)}(z)^{-1}$ has. Thus, in the right side we have a vector whose components are rational functions of z. The first component of this vector is $\hat{V}^{(r_1),(\tau)}(z\hbar^{\frac{r_2}{2}})$ and thus is also a rational function. Of course, the property to be rational does not depend on the shift, so $\hat{V}^{(r_1),(\tau)}(z)$ is a rational function of z. $\square$

1.7 Acknowledgements

2 Quantum toroidal algebra $\mathfrak{gl}_1$

2.1 Generators and relations

Let us set $\mathbf{Z} = \mathbb{Z}^2$, $\mathbf{Z}^* = \mathbf{Z} \setminus \{(0,0)\}$ and:

$$\mathbf{Z}^+ = \{(i,j) \in \mathbf{Z}; i > 0 \ \text{ or } \ i = 0, \ \ j > 0\}, \ \ \mathbf{Z}^- = -\mathbf{Z}^+$$

Set

$$n_k = \frac{(t_1^{\frac{k}{2}} - t_1^{-\frac{k}{2}})(t_2^{\frac{k}{2}} - t_2^{-\frac{k}{2}})(\hbar^{-\frac{k}{2}} - \hbar^{\frac{k}{2}})}{k}$$

and for vector $\mathbf{a} = (a_1, a_2) \in \mathbf{Z}$ denote by $\deg(\mathbf{a})$ the greatest common divisor of a_1 and a_2. We set $\epsilon_{\mathbf{a}} = \pm 1$ for $\mathbf{a} \in \mathbf{Z}^\pm$. For a pair non-collinear vectors we set $\epsilon_{\mathbf{a},\mathbf{b}} = \text{sign}(\det(\mathbf{a},\mathbf{b}))$.

The "toroidal" algebra $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ is an associative algebra with 1 generated by elements $e_{\mathbf{a}}$ and $K_{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{Z}$, subject to the following relations [12]:

- elements $K_{\mathbf{a}}$ are central and

$$K_0 = 1, \quad K_{\mathbf{a}} K_{\mathbf{b}} = K_{\mathbf{a}+\mathbf{b}}$$

- if $\mathbf{a}$, $\mathbf{b}$ are two collinear vectors then:

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] = \delta_{\mathbf{a}+\mathbf{b}} \frac{K_{\mathbf{a}}^{-1} - K_{\mathbf{a}}}{n_{\deg(\mathbf{a})}} \tag{11}$$

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- if $\mathbf{a}$ and $\mathbf{b}$ are such that $\deg(\mathbf{a}) = 1$ and the triangle $\{(0,0), \mathbf{a}, \mathbf{b}\}$ has no interior lattice points then

$$[e_{\mathbf{a}}, e_{\mathbf{b}}] = \epsilon_{\mathbf{a},\mathbf{b}} K_{\alpha(\mathbf{a},\mathbf{b})} \frac{\Psi_{\mathbf{a}+\mathbf{b}}}{n_1}$$

where

$$\alpha(\mathbf{a}, \mathbf{b}) = \begin{cases} \epsilon_{\mathbf{a}}(\epsilon_{\mathbf{a}}\mathbf{a} + \epsilon_{\mathbf{b}}\mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}))/2 & \text{if } \epsilon_{\mathbf{a},\mathbf{b}} = 1 \\ \epsilon_{\mathbf{b}}(\epsilon_{\mathbf{b}}\mathbf{b} + \epsilon_{\mathbf{b}}\mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a}+\mathbf{b}))/2 & \text{if } \epsilon_{\mathbf{a},\mathbf{b}} = -1 \end{cases}$$

and elements $\Psi_{\mathbf{a}}$ are defined by:

$$\sum_{k=0}^{\infty} \Psi_{k\mathbf{a}} z^k = \exp\left(\sum_{m=1}^{\infty} n_m e_{m\mathbf{a}} z^m\right)$$

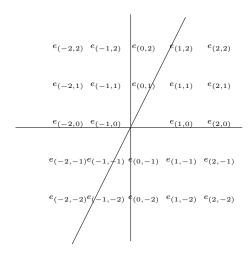for $\mathbf{a} \in \mathbf{Z}$ such that $\deg(\mathbf{a}) = 1$.

Figure 1: The line with slope $w = 2$ corresponds to Heisenberg subalgebra generated by $\alpha_k^2 = e_{k,2k}$ for $k \in \mathbb{Z} \setminus \{0\}$.

2.2 Slope Heisenberg subalgebras

For $w \in \mathbb{Q} \cup \{\infty\}$ we denote by $d(w)$ and $n(w)$ the denominator and numerator of rational number. We set $d(\infty) = 0$, $n(\infty) = 1$ and $n(0) = 0$, $d(0) = 1$. Let us set:
$$\alpha_k^w = e_{(d(w)k, n(w)k)}, \quad k \in \mathbb{Z} \setminus \{0\}$$
From (11) we see that for fixed $w \in \mathbb{Q} \cup \{\infty\}$ these elements generate a Heisenberg with the following relations:

$$[\alpha_{-k}^w, \alpha_k^w] = \frac{K_{(1,0)}^{kd(w)} - K_{(1,0)}^{-kd(w)}}{n_k}$$

We will informally refer to this algebra as "Heisenberg subalgebra with a slope w" and denote it $\mathfrak{h}_w \subset U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$. It is convenient to visualize the algebra $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ as in the Figure 1. Heisenberg subalgebras of $\mathfrak{h}_w$ correspond to lines with slope w in this picture.

The Heisenberg subalgebra with slope $w = 0$ will play distinguished role in this paper. In this special case we will often omit the slope superscript: $\mathfrak{h} = \mathfrak{h}_0$, $\alpha_k = \alpha_k^0$ and $K = K_{(1,0)}$.

2.3 Hopf structures

The algebra $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ carried different inequivalent Hopf structures. We will use the Hopf structure with zero slope defined by a coproduct $\Delta : U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1) \to U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1) \otimes U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$, which has the following explicit form on Heisenberg subalgebra with slope zero:

$$\Delta(\alpha_{-k}) = \alpha_{-k} \otimes 1 + K^{-k} \otimes \alpha_{-k}$$

$$\Delta(\alpha_k) = \alpha_k \otimes K^k + 1 \otimes \alpha_k \tag{12}$$

$$\Delta(K) = K \otimes K$$

where $k > 0$. The algebra $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ is a triangular Hopf algebra, which means there exist an element $\mathscr{R} \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1) \otimes U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ (the universal R-matrix) which enjoys the following properties. In $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 3}$ it satisfies the quantum Yang-Baxter equation:

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}$$

where indexes show in which component of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 3}$ the corresponding R-matrix acts. In addition we have:

$$1 \otimes \Delta(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}, \quad \Delta \otimes 1(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$$

and:

$$\mathscr{R}\Delta(g) = \Delta^{op}(g)\mathscr{R}, \quad \forall g \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1).$$

where Δ^{op} is the opposite coproduct. Explicitly, the universal R-matrix is given by the following formula (Khoroshkin-Tolstoy factorization formula)

$$\mathscr{R} = \overleftarrow{\prod_{w \in \mathbb{Q}\cup\{\infty\}}} \exp\left(\sum_{k=0}^{\infty} n_k\, \alpha_{-k}^w \otimes \alpha_k^w\right) \tag{13}$$

the order of factors in this product is given explicitly as [3] :

$$\mathscr{R} = \overrightarrow{\prod_{\substack{w\in\mathbb{Q}\\w<0}}} R_w^-\, R_\infty \overleftarrow{\prod_{\substack{w\in\mathbb{Q}\\w\geq 0}}} R_w^+ = \overleftarrow{\prod_{\substack{w\in\mathbb{Q}\\w\geq 0}}} \left(R_w^-\right)^{-1} \left(R_\infty\right)^{-1} \overrightarrow{\prod_{\substack{w\in\mathbb{Q}\\w<0}}} \left(R_w^+\right)^{-1}$$

where the order is the standard order on the set of rational numbers, and:

$$R_w^\pm = \prod_{k=0}^{\infty} \exp(n_k\, \alpha_{\pm k}^w \otimes \alpha_{\mp k}^w) = \exp\left(\sum_{k=0}^{\infty} n_k\, \alpha_{\pm k}^w \otimes \alpha_{\mp k}^w\right) \tag{14}$$

2.4   Fock space representations

Set $\mathbf{R} = \mathbb{Q}(t_1^{1/2}, t_2^{1/2}, a)$. Let $\mathcal{F}(a) = \mathbf{R}[p_1, p_2, \cdots]$ be the set of polynomials in infinitely many variables which we consider as $\mathbf{R}$ vector space. Let ν be a partition and $P_\nu \in \mathcal{F}(a)$ be the corresponding Macdonald polynomials in Haiman's normalization [1][4]. Recall that P_ν is a basis of the vector space $\mathcal{F}(a)$.

Let us define a homomorphism $\mathrm{ev}_a : U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}\left(\mathcal{F}(a)\right)$ defined explicitly in the basis of Macdonald polynomials by:

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[3]These two infinite products give the same element in the completion of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 2}$. There is, however, an important difference - evaluated in the tensor product of two Fock representations $\mathcal{F}(a_1) \otimes \mathcal{F}(a_2)$ the fist product converges in the topology of power series in a_1/a_2 while the second as power series in a_2/a_1. In this paper we are dealing with limits $a_2/a_1 \to 0$ of various operators and thus, use the second infinite product.

[4]The parameters t, q of Macdonald polynomials are related to t_1 and t_2 by $q = t_1^{1/2}$, $t = t_2^{1/2}$

- On central elements:

$$\mathrm{ev}_a(K_{(1,0)})\, P_\nu = t_1^{-\frac{1}{2}} t_2^{-\frac{1}{2}} P_\nu, \quad \mathrm{ev}_a(K_{(0,1)})\, P_\nu = P_\nu \tag{15}$$

- On Heisenberg subalgebra with slope 0:

$$\mathrm{ev}_a(e_{(m,0)}) P_\nu \mapsto \begin{cases} \dfrac{1}{(t_1^{m/2} - t_1^{-m/2})(t_2^{m/2} - t_2^{-m/2})}\, p_{-m} P_\nu & m < 0 \\[2em] -m \dfrac{\partial P_\nu}{\partial p_m} & m > 0 \end{cases} \tag{16}$$

- On Heisenberg subalgebra with slope ∞:

$$\mathrm{ev}_a(e_{(0,m)})\, P_\nu = a^{-m}\mathrm{sign}(k) \left(\frac{1}{1 - t_1^m} \sum_{i=1}^{\infty} t_1^{m(\nu_i - 1)} t_2^{m(i-1)} \right) P_\nu \tag{17}$$

Note that $e_{0,n}$, $e_{n,0}$, $K_{(0,1)}$ and $K_{(1,0)}$ generate $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ thus the last set of formulas define a homomorphism ev_u.

The representation ev_a is called *Fock representation* of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ evaluated at a. The parameter a is called the evaluation parameter of Fock representation. Let us note here that by (17) the generators act in $\mathcal{F}(a)$ as monomials in the evaluation parameter a:

$$\mathrm{ev}_a(\alpha_k^w) \sim a^{-kn(w)} \tag{18}$$

2.5 Tensor product of Fock representations

Let us denote

$$\mathcal{F}^{(r)} = \mathcal{F}(a_1) \otimes \cdots \otimes \mathcal{F}(a_r) \tag{19}$$

and define representation of quantum toroidal algebra $\mathrm{ev}^{(r)} : U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}\!\left(\mathcal{F}^{(r)}\right)$ by:

$$\mathrm{ev}^{(r)}(\alpha) = \mathrm{ev}_{a_1} \otimes \cdots \otimes \mathrm{ev}_{a_r}\!\left(\Delta^{(r)}(\alpha)\right), \quad \alpha \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$$

where Δ is a coproduct from Section 2.3. We set:

$$\mathscr{R}^{(r_1),(r_2)} = \mathrm{ev}^{(r_1)} \otimes \mathrm{ev}^{(r_2)}(\mathscr{R})$$

As shown in [8], in the tensor product of Fock spaces, the infinite product (13) converges in the topology of power series in a_i and $\mathscr{R}^{(r_1),(r_2)}$ is a well defined element of $\mathrm{End}\left(\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)}\right)$ whose matrix coefficients are rational functions of evaluation parameters a_i. Let $\mathrm{ev}_a^{(r)} = \mathrm{ev}_{a_1 a} \otimes \cdots \otimes \mathrm{ev}_{a_r a}$ be a shifted evaluation. Denote $\mathscr{R}^{(r_1),(r_2)}(a) = \mathrm{ev}^{(r_1)} \otimes \mathrm{ev}_a^{(r_2)}(\mathscr{R})$ and $R_w^{\pm,(r_1),(r_2)}(a) = \mathrm{ev}^{(r_1)} \otimes \mathrm{ev}_a^{(r_2)}(R_w^{\pm})$.

Proposition 1. *(Section 2 of [8])*

- *The operator $R_0^{-,(r_1),(r_2)} \overset{def}{=} R_0^{-,(r_1),(r_2)}(a)$ does not depend on a.*

- *$R_w^{-,(r_1),(r_2)}(0) = 1$ for $w > 0$ and $R_w^{+,(r_1),(r_2)}(0) = 1$ for $w < 0$*

- *$R_\infty^{(r_1),(r_2)}(0) = \hbar^{-\Omega}$ where $\hbar^\Omega$ acts on $\mathcal{F}_{(n_1)}^{(r_1)} \otimes \mathcal{F}_{(n_2)}^{(r_2)}$ by multiplication on $\hbar^{(n_1 r_2 + n_2 r_1)/2}$.*

- *In particular, we have: $\mathscr{R}^{(r_1),(r_2)}(0) = (R_0^{-,(r_1),(r_2)})^{-1} \hbar^\Omega$.*

The universal R-matrix $\mathscr{R}$ satisfies the quantum Yang-Baxter equation. Thus, its limit $\mathsf{R} = \hbar^{-\Omega} R_0^-$ is also a solution of this equation. It is not difficult to see that R is a universal R-matrix for $\mathfrak{h}$.

2.6   Triangular and block-diagonal operators

Let us introduce some convenient notations and terminology here. The Fock space is equipped with the grading:

$$\mathcal{F}(a) = \bigoplus_{n=0}^{\infty} \mathcal{F}_{(n)}(a)$$

corresponding to the degree function $\deg(p_k) = k$. This induces the grading on the tensor products:

$$\mathcal{F}^{(r)} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{(n)}^{(r)}$$

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where $\mathcal{F}^{(r)}_{(n)}$ denotes the subspace spanned by elements of degree n. Throughout this paper we use the following terminology: an operator $A : \mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)} \to \mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)}$ is *lower-triangular* (upper-triangular) if $A = \bigoplus_{k=0}^{\infty} A_{(k)}$ (respectively $A = \bigoplus_{k=0}^{\infty} A_{(-k)}$) where $A_{(k)} : \mathcal{F}^{(r_1)}_{(n_1)} \otimes \mathcal{F}^{(r_2)}_{(n_2)} \to \mathcal{F}^{(r_1)}_{(n_1+k)} \otimes \mathcal{F}^{(r_2)}_{n_1-k}$. We say that the operator A is *strictly* lower or upper-triangular if in addition $A_{(0)} = 1$. We say the operator is block-diagonal if $A_{(k)} = 0$ for $k \neq 0$. For example, the operators R_w^- and R_w^+ defined by (14) are strictly lower and upper-triangular respectively. Finally, for a formal variable z we denote a block-diagonal operator $z_{(i)}^d$ acting on a $\mathcal{F}^{(r_1)}_{(n_1)} \otimes \mathcal{F}^{(r_2)}_{(n_2)}$ as multiplication by z^{n_i}.

2.7   Geometric realization of Fock representations

In the special case $r = 1$ the instanton moduli space coincides with the Hilbert scheme of n points on the complex plane $\mathscr{M}(n, 1) = \mathrm{Hilb}^n(\mathbb{C}^2)$. As a vector space the equivariant K-theory of the Hilbert scheme of points is isomorphic to the Fock space:

$$\bigoplus_{n=0}^{\infty} K_{\mathsf{T}}(\mathrm{Hilb}^n(\mathbb{C}^2)) = \mathcal{F}(a) \tag{20}$$

where T is a torus from Section 1.2. The summands here correspond to the grading from the previous section $K_{\mathsf{T}}(\mathrm{Hilb}^n(\mathbb{C}^2)) = \mathcal{F}_{(n)}(a)$. The parameters t_1, t_2 and a of the Fock space are identified with the corresponding equivariant parameters of T.

The fixed point set $\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$ is discrete. Its elements are labeled by partitions ν with $|\nu| = n$. The structure sheaves of the fixed points $\mathcal{O}_\nu$ form a basis of the localized K-theory. The polynomials representing the elements of this basis under isomorphism (20) are the Macdonald polynomials P_ν in Haiman normalization [1].

In [12] the geometric action of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ on the equivariant K-theory of $\mathrm{Hilb}^n(\mathbb{C}^2)$ was constructed. As a representation, this action coincides with the Fock representation described in Section 2.4. In particular, formulas (15)-(17) describe this geometric action explicitly in the basis of the fixed points.

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2.8 Coproducts and stable envelopes

In [4, 3] the geometric action of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ on $K_{\mathsf{T}}(\mathscr{M}(r))$ was constructed. As a representation $K_{\mathsf{T}}(\mathscr{M}(r))$ is isomorphic to a product of r Fock spaces $\mathcal{F}^{(r)}$. The evaluation parameters a_i of this representation are identified with the equivariant parameters of framing torus A. An alternative construction of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ action on equivariant K-theory was given in [8] using FRT formalism [11]. The key ingredient of [8] is the notion of the stable map, which is a geometric description of the coproduct. Here we recall basic facts about the stable map, the details can be found in Section 2 of [8].

Assume that a one-dimensional subtorus $\mathsf{C} \subset \mathsf{A}$ splits the framing as r to $r_1 + ar_2$ such that $\mathscr{M}(r)^{\mathsf{C}} = \mathscr{M}(r_1) \times \mathscr{M}(r_2)$. In this case we have a canonical isomorphism of the corresponding K-theories, called *stable envelope*[5]:

$$K_{\mathsf{T}}(\mathscr{M}(r)^{\mathbb{C}^*}) = K_{\mathsf{T}}(\mathscr{M}(r_1)) \otimes K_{\mathsf{T}}(\mathscr{M}(r_1)) \xrightarrow{\text{Stab}} K_{\mathsf{T}}(\mathscr{M}(r))$$

The stable map Stab and the antipode Δ make the following diagram commutative:

$$
\begin{array}{ccc}
K_{\mathsf{T}}(\mathscr{M}(r_1)) \otimes K_{\mathsf{T}}(\mathscr{M}(r_1)) & \xrightarrow{\text{Stab}} & K_{\mathsf{T}}(\mathscr{M}(r)) \\
\downarrow{\scriptstyle \Delta(\alpha)} & & \downarrow{\scriptstyle \alpha} \\
K_{\mathsf{T}}(\mathscr{M}(r_1)) \otimes K_{\mathsf{T}}(\mathscr{M}(r_1)) & \xrightarrow{\text{Stab}} & K_{\mathsf{T}}(\mathscr{M}(r))
\end{array}
$$

for every element $\alpha \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$.

Let us consider a chain of splittings by whole A, such that all factors $r_i = 1$ in the end:

$$r \to a_1 r_1 + a_2 r_2 \to a_1 r_1 + a_2 r_2 + a_3 r_3 \to \cdots \to a_1 + \cdots + a_r \qquad (21)$$

This gives a canonical isomorphism of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$-modules:

$$K_{\mathsf{T}}(\mathscr{M}(1)) \otimes \cdots \otimes K_{\mathsf{T}}(\mathscr{M}(1)) = \mathcal{F}^{(r)} \xrightarrow{\Phi^{(r)}} K_{\mathsf{T}}(\mathscr{M}(r)) \qquad (22)$$

---

[5]To be precise, the definition of stable envelope map requires a choice of a chamber in $\mathfrak{C} \subset Lie(\mathsf{C})$ and an alcove $\nabla \subset Pic(\mathscr{M}(n,r)) \otimes \mathbb{Q}$. In this paper the chamber $\mathfrak{C}$ corresponds to $a \to 0$ and ∇ unique alcove lying in the opposite of the ample cone whose closure contains $0 \in H^2(\mathscr{M}(n,r), \mathbb{R})$. This is the same choice as in Theorem 3 in [8]. The other choices of alcove ∇ correspond to a freedom in the choice of the Hopf structure, i.e., the antipode Δ for $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$.

where $\mathcal{F}^{(r)}$ is defined by (19), and $\Phi^{(r)}$ is a composition of the stable envelopes corresponding to splittings. By construction of stable envelope, $\Phi^{(r)}$ does not depend on the order of splittings in (21) and thus it is well defined. This property reflects the coassociativity of coproduct.

3 Quantum difference equations

3.1 The quantum difference operator

First let us consider an element $B(z) \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)[[z,q]]$ given explicitly by:

$$B(z) = \overleftarrow{\prod_{\substack{w \in \mathbb{Q} \\ -1 \le w < 0}}} : \exp\Big(\sum_{k=0}^{\infty} \frac{n_k\, \hbar^{-krd(w)/2}}{1 - z^{-kd(w)} q^{kn(w)} \hbar^{-krd(w)/2}} \, \alpha_{-k}^w \alpha_k^w \Big): .$$

The symbol :: here denotes the normal ordered exponent - all annihilation operators α_k^w with $k > 0$ are moved to the right. We identify $K_\mathsf{T}(\mathscr{M}(r))$ with representation $\mathcal{F}^{(r)}$ of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ through isomorphism $\Phi^{(r)}$. Define the following operator acting on $K_\mathsf{T}(\mathscr{M}(r))$:

$$\mathbf{M}_{\mathcal{O}(1)}^{(r)}(z) = \mathcal{O}(1)\, \mathrm{ev}_{a_1} \otimes \cdots \otimes \mathrm{ev}_{a_1}\Big(\Delta^{\otimes r}(B(z)) \Big) \tag{23}$$

where $\mathcal{O}(1)$ is the operator of multiplication by the corresponding line bundle in K-theory. Note that the coefficients of normally ordered exponentials act in $\mathcal{F}^{(r)}$ as a locally nilpotent operators. This means that the operators $\mathbf{M}_{\mathcal{O}(1)}^{(r)}(z)$ has rational matrix coefficients for all r. The qde for the rank r instanton moduli space is given by the following equation [8]:

$$\Psi^{(r)}(zq)\mathcal{O}(1) = \mathbf{M}^{(r)}(z)\Psi^{(r)}(z). \tag{24}$$

The capping operator in quantum K-theory is a fundamental solution of qde, i.e., corresponds to the following boundary condition[6]:

$$\Psi^{(r)}(0) = 1 \tag{25}$$

which geometrically means that the classical limit $z = 0$ of the capped descendent vertex coincides with the corresponding K-theory class.

---

[6]Here $1 \in K_\mathsf{G}(\mathscr{M}(n,r))$ stands for the class of the structure sheaf $\mathcal{O}_{\mathscr{M}(n,r)}$.

3.2 Cocycle identity

Out goal to to compare the capping operator $\Psi^{(r)}(z)$ for $\mathcal{M}(r)$ with the (shifted) capping operator for the C-fixed point set given by $\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})$. By definition, the first operator is the fundamental solution of (24) while the second solves

$$\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}q) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}}q)\mathcal{O}(1) =$$

$$\mathbf{M}_{\mathcal{O}(1)}^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \mathbf{M}_{\mathcal{O}(1)}^{(r_2)}(z\hbar^{-\frac{r_1}{2}})\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})$$

We conclude that the operator:

$$\mathbf{J}^{(r_1),(r_2)}(z) = \Psi^{(r)}(z)\left(\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})\right)^{-1} \tag{26}$$

is the solution of the following difference equation:

$$\mathbf{J}^{(r_1),(r_2)}(zq)\mathbf{M}_{\mathcal{O}(1)}^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \mathbf{M}_{\mathcal{O}(1)}^{(r_2)}(z\hbar^{-\frac{r_1}{2}}) = \mathbf{M}_{\mathcal{O}(1)}^{(r)}(z)\mathbf{J}^{(r_1),(r_2)}(z) \tag{27}$$

Proposition 2. *The operator $\boldsymbol{J}^{(r_1),(r_2)}(z)$ satisfies the dynamical cocycle identity:*

$$\boldsymbol{J}^{(r_1+r_2),(r_3)}(z)\boldsymbol{J}^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}}) = \boldsymbol{J}^{(r_1),(r_2+r_3)}(z)\boldsymbol{J}^{(r_2),(r_3)}(z\hbar^{-\frac{r_1}{2}}) \tag{28}$$

Proof. Consider the chain of splittings:

$$\mathcal{M}(r_1 + r_2 + r_3) \to \mathcal{M}(r_1 + r_2) \times \mathcal{M}(r_3) \to \mathcal{M}(r_1) \times \mathcal{M}(r_2) \times \mathcal{M}(r_3)$$

or, alternatively:

$$\mathcal{M}(r_1 + r_2 + r_3) \to \mathcal{M}(r_1) \times \mathcal{M}(r_2 + r_3) \to \mathcal{M}(r_1) \times \mathcal{M}(r_2) \times \mathcal{M}(r_3)$$

The result does not depend on the choice of the order and the proposition follows from definition of $\mathbf{J}^{(r_1),(r_2)}(z)$. $\square$

By definition, $\Psi^{(r)}(z)$ is a power series in z whose coefficients are rational functions of a. As we discussed in [8] the limit $\lim\limits_{a \to 0} \Psi^{(r)}(z)$ exists and is lower-triangular. The power series $\Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})$ is independent of a. We conclude:

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Corollary 1. *The operator*

$$Y^{(r_1),(r_2)}(z) = \lim_{a \to 0} \boldsymbol{J}^{(r_1),(r_2)}(z) \tag{29}$$

is a lower-triangular operator satisfying the dynamical cocycle identity (28).

The difference operators $\mathbf{M}^{(r)}_{\mathcal{O}(1)}(z)$ are given by evaluation of a universal elements from $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)[[z,q]]$ in the Fock representation. It means that the solution $\boldsymbol{J}^{(r_1),(r_2)}(z)$ and $Y^{(r_1),(r_2)}(z)$ are also given by evaluation of some universal elements. In particular $Y^{(r_1),(r_2)}(z) = \mathrm{ev}^{(r_1)} \otimes \mathrm{ev}^{(r_2)}(Y(z))$ for some universal $Y(z)$. In the following sections we find $Y(z)$ explicitly.

Finally, let us note here that by definition (26) we have $\boldsymbol{J}^{(r),(0)}(z) = \boldsymbol{J}^{(0),(r)}(z) = 1$ and thus:

$$Y^{(r),(0)}(z) = Y^{(0),(r)}(z) = 1, \quad \forall r. \tag{30}$$

3.3 Shift operators

The global dependence of the capping operator $\Psi^{(r)}(z)$ on the equivariant parameter a is governed by certain q-difference equation dual to qde, the so called quantum Knizhnik-Zamolodchikov equation, [8] (Section 4.2):

$$\Psi^{(r)}(z, aq)\mathsf{E}(z, a) = \mathsf{S}(z, a)\Psi^{(r)}(z, a) \tag{31}$$

Moreover, under identification (22) the operators are explicitly expressible through the universal R-matrices from Section 2.5:

$$\mathsf{S}(z, a) = z^d_{(1)}\mathscr{R}^{(r_1),(r_2)}(a), \quad \mathsf{E}(z, a) = z^d_{(1)}\left(R^{(r_1),(r_2)}_\infty(a)\right).$$

Proposition 3. *The operator $\Psi^{(r)}(z, 0)$ is a lower-triangular solution of wall Knizhnik-Zamolodchikov (wKZ) equation:*

$$z^d_{(1)}(R^{-,(r_1),(r_2)}_0)^{-1}\hbar^\Omega\Psi^{(r)}(z, 0) = \Psi^{(r)}(z, 0)\hbar^\Omega z^d_{(1)} \tag{32}$$

Proof. The existence of a limit $\Psi^{(r)}(z, 0)$ was shown in [8]. The proof follows from Proposition 1. $\square$

Note that the operator $\hbar^\Omega z^d_{(1)}$ commutes with any block diagonal-operator, such as for example $\Psi^{(r_1)}(z\hbar^{r_2/2}) \otimes \Psi^{(r_2)}(z\hbar^{-r_1/2})$. Thus, we conclude:

Corollary 2. *The operator $Y^{(r_1),(r_2)}(z)$ defined by (29) is a lower-triangular solution of wKZ equation:*

$$z^d_{(1)}(R^{-,(r_1),(r_2)}_0)^{-1}\hbar^\Omega Y^{(r_1),(r_2)}(z) = Y^{(r_1),(r_2)}(z)\hbar^\Omega z^d_{(1)} \tag{33}$$

3.4   Uniqueness of solutions of wKZ equation

Lemma 1. *For a given block diagonal operator $D^{(r_1),(r_2)}$ there exists unique lower triangular solution $J^{(r_1),(r_2)}$ of wKZ:*

$$z_{(1)}^d (R_0^{-,(r_1),(r_2)})^{-1} \hbar^\Omega J^{(r_1),(r_2)} = J^{(r_1),(r_2)} \hbar^\Omega z_{(1)}^d \tag{34}$$

and diagonal part $J_{(0)}^{(r_1),(r_2)} = D^{(r_1),(r_2)}$.

Proof. Write the last equation in the form:

$$Ad_{\hbar^{-\Omega} z_{(1)}^{-d}}(J^{(r_1),(r_2)}) = \tilde{R} J^{(r_1),(r_2)}$$

with $\tilde{R} = \hbar^{-\Omega}(R_0^{-,(r_1),(r_2)})^{-1}\hbar^\Omega$. The operator $\tilde{R}$ is strictly lower triangular, thus, taking the n-th component of this equation we obtain:

$$Ad_{\hbar^\Omega z_1^{-d}}(J_{(n)}^{(r_1),(r_2)}) = J_{(n)}^{(r_1),(r_2)} + \cdots$$

where dots stand for terms $J_{(k)}^{(r_1),(r_2)}$ with $k < n$. As the operator $1 - Ad_{\hbar^\Omega z_1^{-d}}$ is invertible for general z we can solve this linear equation for $J_{(n)}^{(r_1),(r_2)}$ recursively through the lower terms $J_{(k)}^{(r_1),(r_2)}$ with $k < n$. By induction, all terms are thus expressed through the lowest term $J_{(0)}^{(r_1),(r_2)} = D^{(r_1),(r_2)}$. $\qquad\square$

Corollary 3. *The solution of wKZ with $J_{(0)}^{(r_1),(r_2)} = D^{(r_1),(r_2)}$ is given by $J^{(r_1),(r_2)} = E^{(r_1),(r_2)} D^{(r_1),(r_2)}$ where $E^{(r_1),(r_2)}$ is unique strictly lower triangular solution.*

Proof. Indeed, by above Proposition there exist unique strictly lower-triangular solution $E^{(r_1),(r_2)}$. The operator $\hbar^\Omega z_{(1)}^d$ commutes with any block diagonal operator $D^{(r_1),(r_2)}$ and thus $J^{(r_1),(r_2)} = E^{(r_1),(r_2)} D^{(r_1),(r_2)}$ is the solution of wKZ with $J_{(0)}^{(r_1),(r_2)} = D^{(r_1),(r_2)}$. $\qquad\square$

Finally, we give explicit universal formula for the strictly lower triangular solution E.

Proposition 4. *There exist the universal element $E(z) \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 2}[[z]]$ given explicitly by:*

$$E(z) = \exp\Big(\sum_{k=1}^\infty \frac{n_k}{1 - z^{-k}K^{-k} \otimes K^k} K^{-k}\alpha_{-k} \otimes K^k\alpha_k\Big) \tag{35}$$

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such that $E = ev^{(r_1)} \otimes ev^{(r_2)}(E(z))$ is the strictly lower-triangular solution of wKZ equation in $\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)}$.

Proof. To show that $E(z)$ is a universal solution of wKZ it is enough to check that it solves the following equation in $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 2}(z)$:

$$z_{(1)}^{-d}E(z)z_{(1)}^{d} = (R_0^-)^{-1}\hbar^\Omega E(z)\hbar^{-\Omega} \tag{36}$$

The operator R_0^- is given by (14). Note, that the operators $z_{(1)}^{d}$ and $\hbar^\Omega$ are only defined in the representation $\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)}$. However, the operators $Ad_{z_{(1)}^{d}}$ and $Ad_{\hbar^\Omega}$ are well defined in $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 2}$:

$$z_{(1)}^{-d}\alpha_{-k} \otimes \alpha_k z_{(1)}^{d} = z^{-k}\alpha_{-k} \otimes \alpha_k,$$

and Lemma 2 below gives:

$$\hbar^\Omega \alpha_{-k} \otimes \alpha_k \hbar^{-\Omega} = K^k \alpha_{-k} \otimes K^{-k}\alpha_k$$

From (14) we see that (36) is equivalent to the set of equations:

$$\frac{n_k K^{-k} \otimes K^k z^{-k}}{1 - z^{-k}K^{-k} \otimes K^k} = -n_k + \frac{n_k}{1 - z^{-k}K^{-k} \otimes K^k}$$

which are identities. $\qquad\square$

Lemma 2. *For all r_1, r_2 the identity:*

$$\hbar^\Omega \alpha_{-k} \otimes \alpha_k \hbar^{-\Omega} = K^{-k}\alpha_{-k} \otimes K^k \alpha_k$$

holds in $\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)}$.

Proof. Note that $\alpha_{-k} \otimes \alpha_k : \mathcal{F}_{(n_1)}^{(r_1)} \otimes \mathcal{F}_{(n_2)}^{(r_2)} \to \mathcal{F}_{(n_1+k)}^{(r_1)} \otimes \mathcal{F}_{(n_2-k)}^{(r_2)}$. Thus, from the explicit action of $\hbar^\Omega$ on the fock space described in Proposition 1 we obtain:

$$\hbar^\Omega \alpha_{-k} \otimes \alpha_k \hbar^{-\Omega} = \hbar^{k(r_2-r_1)/2}\alpha_{-k} \otimes \alpha_k$$

The central element K acts on $\mathcal{F}^{(r)}$ by multiplication on a scalar $\hbar^{-r/2}$. The lemma is proven. $\qquad\square$

Corollary 4. *For all r_1 and r_2 we have:*

$$Y^{(r_1),(r_2)}(z) = ev^{(r_1)} \otimes ev^{(r_2)}(E(z)D(z))$$

where $D(z)$ universal operator such that $D^{(r_1),(r_2)}(z)$ is block diagonal in every representation.

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3.5 Universal form of $Y(z)$

Our final step is to prove that $D = 1$ which means:

Proposition 5. *For all r_1 and r_2 we have:*

$$Y^{(r_1),(r_2)}(z) = ev^{(r_1)} \otimes ev^{(r_2)}(E(z))$$

Proof. By Corollary 4 we have:

$$Y^{(r_1),(r_2)}(z) = E^{(r_1),(r_2)}(z)D^{(r_1),(r_2)}$$

for some block diagonal matrix $D^{(r_1),(r_2)}$ and we need to show that $D^{(r_1),(r_2)} = 1$. By Corollary 1 we have:

$$Y^{(r_1+r_2),(r_3)}(z)Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}}) = Y^{(r_1),(r_2+r_3)}(z)Y^{(r_2),(r_3)}(z\hbar^{-\frac{r_1}{2}}) \qquad (37)$$

Let us consider left and right wKZ operators (they are coproducts of the wKZ operators in the first and second components respectively):

$$A_L(J) = \mathsf{R}_{23}\mathsf{R}_{13}z_{(3)}^d J z_{(3)}^{-d}\hbar^{\Omega_{13}+\Omega_{23}}$$

$$A_R(J) = \mathsf{R}_{12}\mathsf{R}_{13}z_{(1)}^{-d} J z_{(1)}^d\hbar^{\Omega_{12}+\Omega_{13}}$$

where R as in Section 2.5. It is obvious that the left side of (37) satisfies the equation $A_L(J) = J$. Similarly, the right side satisfies $A_R(J) = J$. Therefore we must have:

$$A_R(Y^{(r_1+r_2),(r_3)}(z)Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})) = Y^{(r_1+r_2),(r_3)}(z)Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})$$

Taking degree zero part of this equality in the third component we obtain:

$$\mathsf{R}_{12}^{-}\hbar^{-\Omega_{13}}z_{(1)}^{-d}D^{(r_1+r_2),(r_3)}Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})z_{(1)}^d\hbar^{\Omega_{12}+\Omega_{13}}$$

$$= D^{(r_1+r_2),(r_3)}Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})$$

The block-diagonal operator $D^{(r_1+r_2),(r_3)}$ commutes with $\hbar^{-\Omega_{13}}$ and $z_{(1)}^{-d}$ so we can rewrite the last equation in the form:

$$\mathsf{R}_{12}D^{(r_1+r_2),(r_3)}\hbar^{-\Omega_{13}}z_{(1)}^{-d}Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})z_{(1)}^d\hbar^{\Omega_{12}+\Omega_{13}}$$

$$= D^{(r_1+r_2),(r_3)}Y^{(r_1),(r_2)}(z\hbar^{\frac{r_3}{2}})$$

By Lemma 3 this is equivalent to the equality:

$$\mathsf{R}_{12}D^{(r_1+r_2),(r_3)}\big(z\hbar^{\frac{r_3}{2}}\big)_{(1)}^{-d}Y^{(r_1),(r_2)}\big(z\hbar^{\frac{r_3}{2}}\big)\big(z\hbar^{\frac{r_3}{2}}\big)_{(1)}^{d}\hbar^{-\Omega_{12}}$$

$$= D^{(r_1+r_2),(r_3)}Y^{(r_1),(r_2)}\big(z\hbar^{\frac{r_3}{2}}\big)$$

By definition the operator $Y^{r_1,r_2}(z)$ solves wKZ equation (34) which gives:

$$\mathsf{R}_{12}D^{(r_1+r_2),(r_3)}(\mathsf{R}_{12})^{-1} = D^{(r_1+r_2),(r_3)} \tag{38}$$

The same argument for A_L gives similar restriction for the second component of D:

$$\mathsf{R}_{23}D^{(r_1),(r_2+r_3)}(\mathsf{R}_{23})^{-1} = D^{(r_1),(r_2+r_3)} \tag{39}$$

The universal element has the form: $D(z) = \sum_{i,j} f_{i,j}(z)b_i\otimes b_j \in U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)^{\otimes 2}[[z]]$ in a basis of degree zero elements b_i. The first equation (38) gives $\Delta^{op}(b_i) = \Delta(b_i)$. The only elements of $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}_1)$ with this properties are $b_i = K^i$. Similarly, the second (39) equation gives $b_j = K^j$. We conclude that the universal operator has the following form: $D(z) = \sum_{i,j} f_{i,j}(z)K^i \otimes K^j$. The diagonal part of (30) gives $D^{(r_1),(0)}(z) = D^{(0),(r_2)}(z) = 1$ for all r_1 and r_2. which means that $f_{0,0}(z) = 1$ and $f_{i,j}(z) = 0$ of $i \neq 0$ or $j \neq 0$. We conclude that $D = 1$. $\qquad\square$

To finish the proof we need the following simple lemma.

Lemma 3. *The following identity holds in* $\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)} \otimes \mathcal{F}^{(r_3)}$

$$\hbar^{-\Omega_{13}}(\alpha_{-k} \otimes \alpha_k \otimes 1)\hbar^{\Omega_{13}} = (\hbar^{r_3/2})_{(1)}^{-d}(\alpha_{-k} \otimes \alpha_k \otimes 1)(\hbar^{r_3/2})_{(1)}^{d} \tag{40}$$

Proof. From explicit action of Heisenberg algebra we know that the operator $\alpha_{-k} \otimes \alpha_k \otimes 1$ maps:

$$\mathcal{F}^{(r_1)} \otimes \mathcal{F}^{(r_2)} \otimes \mathcal{F}^{(r_3)} : \mathcal{F}^{(r_1)}_{(n_1)} \otimes \mathcal{F}^{(r_2)}_{(n_2)} \otimes \mathcal{F}^{(r_3)}_{(n_3)} \to \mathcal{F}^{(r_1)}_{(n_1+k)} \otimes \mathcal{F}^{(r_2)}_{(n_2-k)} \otimes \mathcal{F}^{(r_3)}_{(n_3)}$$

Thus, from the definition of $\hbar^\Omega$ given in Proposition 1 we obtain:

$$\hbar^{-\Omega_{13}}(\alpha_{-k} \otimes \alpha_k \otimes 1)\hbar^{\Omega_{13}} = \hbar^{-kr_3/2}(\alpha_{-k} \otimes \alpha_k \otimes 1)$$

Similarly, by the definition of $z_{(i)}^d$ from Section 2.6 we obtain the same result

$$(\hbar^{r_3/2})_{(1)}^{-d}(\alpha_{-k} \otimes \alpha_k \otimes 1)(\hbar^{r_3/2})_{(1)}^{d} = \hbar^{-kr_3/2}(\alpha_{-k} \otimes \alpha_k \otimes 1)$$

$$\square$$

3.6 Proof of Theorem 3

The theorem 3 in now proved. Indeed, by definition of $\mathbf{J}^{(r_1),(r_2)}$ (26) and definition of $Y^{(r_1),(r_2)}(z)$ (9) we have

$$\lim_{a \to 0} \Psi^{(r)}(z) = Y^{(r_1),(r_2)}(z) \Psi^{(r_1)}(z\hbar^{\frac{r_2}{2}}) \otimes \Psi^{(r_2)}(z\hbar^{-\frac{r_1}{2}})$$

By Proposition 5 the operator $Y^{(r_1),(r_2)}(z)$ is an evaluation of the universal element (35), which is the claim of Theorem 3. $\square$

4 Bare vertex

4.1 Localization to the fixed points

In this section we consider the descendent bare vertex defined by (4). The push-forward in (4) can be computed explicitly by the equivariant localization to the locus of the fixed point set of G-action on the moduli space of nonsingular quasimaps $\mathsf{QM}^d_{p_2}(n, r)$. This provides a formula for the bare vertex as a power series in z with explicit coefficients. Here we sketch its derivation and use it to prove the factorization Theorem 4.

First, recall that the fixed set $\mathsf{QM}^d_{p_2}(n, r)^{\mathsf{G}}$ consist of finitely many isolated points $\mathbf{p}$ which parameterize the following data:

$$\mathsf{QM}^d_{p_2}(n, r)^{\mathsf{G}} = \{\mathbf{p} = (\overrightarrow{\lambda}, \overrightarrow{d}) : |\overrightarrow{\lambda}| = n, \;\; |\overrightarrow{d}| = d\}$$

where $\overrightarrow{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ is an r-tuple of Young diagrams with total number of $|\overrightarrow{\lambda}| = |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$ boxes. The degree data $\overrightarrow{d} = \{d_\square \geq 0 : \square \in \overrightarrow{\lambda}\}$ assigns a non-negative integer to each box of $\overrightarrow{\lambda}$ such that $|\overrightarrow{d}| = \sum_{\square \in \overrightarrow{\lambda}} d_\square = d$. The degree $\overrightarrow{d}$ is assumed to be stable, which means that the data $d_\square$ for $\square \in \lambda^{(i)}$ for $i = 1, \cdots, r$ define r *plane partitions*. The equivariant localization formula thus gives the following power series for the bare descendent vertex:

$$V^{(\tau)}(z) = \sum_{d=0}^{\infty} \sum_{\mathbf{p} \in \mathsf{QM}^d_{p_2}(n,r)^{\mathsf{G}}} a(\mathbf{p}) z^d \in K_{\mathsf{G}}(\mathscr{M}(n, r))_{loc} \otimes \mathbb{Q}[[z]] \qquad (41)$$

where $a(\mathbf{p})$ is a contribution of a fixed point $\mathbf{p}$.

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The fixed set $\mathscr{M}(n,r)^{\mathsf{T}}$ consists of isolated points labeled by r-tuples of Young diagrams $\overrightarrow{\lambda}$ with $|\overrightarrow{\lambda}| = n$. In fact, this is a special case of the fixed points on the moduli space of quasimaps (the degree zero or constant quasimaps) $\mathscr{M}(n,r) = \mathsf{QM}^0_{p_2}(n,r)$. The classes of the fixed points $[\overrightarrow{\lambda}] \in K_{\mathsf{G}}(\mathscr{M}(n,r))_{loc}$ form a basis in the localized K-theory. Assume that in this basis the bare vertex has the expansion $V^{(\tau)}(z) = \sum\limits_{|\overrightarrow{\lambda}|=n} V^{(\tau)}_{\overrightarrow{\lambda}}(z)[\overrightarrow{\lambda}]$. In (41) only the fixed points $\mathbf{p} = (\overrightarrow{\nu}, \overrightarrow{d})$ with $\overrightarrow{\nu} = \overrightarrow{\lambda}$ contribute to the coefficient $V^{(\tau)}_{\overrightarrow{\lambda}}(z)$. Therefore we have:

$$V^{(\tau)}_{\overrightarrow{\lambda}}(z) = \sum_{\substack{d_\square \geq 0 \\ \square \in \overrightarrow{\lambda}}} a_{\overrightarrow{\lambda}}((\overrightarrow{\lambda}, \overrightarrow{d}))z^{|\overrightarrow{d}|}$$

where the sum runs over the stable degrees $\overrightarrow{d}$. Before we give an explicit formula for $a_{\overrightarrow{\lambda}}((\overrightarrow{\lambda}, \overrightarrow{d}))$ let us introduce the following auxiliary functions: we set $n(\square) = k$ if $\square \in \lambda^{(k)}$ and denote by $x(\square)$ and $y(\square)$ the standard coordinates of a box in a partition. Define:

$$\varphi_{\overrightarrow{\lambda}}(\square) = a_{n(\square)}t_1^{x(\square)}t_2^{y(\square)}$$

and

$$\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d}) = \sum_{\square \in \overrightarrow{\lambda}} \varphi_{\overrightarrow{\lambda}}(\square) q^{d_\square}, \quad \mathcal{W} = a_1 + \cdots + a_r. \tag{42}$$

For each fixed point we define a Laurent polynomial in equavariant parameters $S(\overrightarrow{\lambda}, \overrightarrow{d}) \in K_{\mathsf{G}}(pt)$ by:

$$S(\overrightarrow{\lambda}, \overrightarrow{d}) = \mathcal{W}^* \mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d}) + \mathcal{W} \mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})^* t_1^{-1} t_2^{-1}$$
$$- \mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})^* \mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})(1 - t_1^{-1})(1 - t_2^{-1}). \tag{43}$$

The symbol $*$ corresponds to taking dual in K-theory and acts by inversion of all weights $* : w \mapsto w^{-1}$.

4.2 Virtual tangent space

The G-character of the virtual tangent space $T^{vir}_{(\overrightarrow{\lambda},dd)}\mathsf{QM}^d_{p_2}(n,r)$ is given explicitly by [5]:

$$T^{vir}_{(\overrightarrow{\lambda},\overrightarrow{d})}\mathsf{QM}^d_{p_2}(n,r) = S(\overrightarrow{\lambda},\overrightarrow{0}) + \frac{S(\overrightarrow{\lambda},\overrightarrow{d}) - S(\overrightarrow{\lambda},\overrightarrow{0})}{q-1} \tag{44}$$

where $\overrightarrow{0} = \{d_\square = 0 : \forall \square \in \overrightarrow{\lambda}\}$. The stability of $\overrightarrow{d}$ implies that $T^{vir}_{(\overrightarrow{\lambda},\overrightarrow{d})}\mathsf{QM}^d_{p_2}(n,r)$ is a Laurent polynomial without a constant term and with both positive and negative coefficients (negative coefficients appear because it is a *virtual* tangent space).

As we noted above $\mathscr{M}(n,r) = \mathsf{QM}^0_{p_2}(n,r)$ and thus the T-character of the tangent space to the instanton moduli space at a point $\overrightarrow{\lambda} \in \mathscr{M}(n,r)^\mathsf{T}$ is given by

$$T_{\overrightarrow{\lambda}}\mathscr{M}(n,r) = S(\overrightarrow{\lambda},\overrightarrow{0})$$

This character is a Laurent polynomial with positive coefficients and no constant term.

4.3 Bare descendent vertex explicitly

Let $\tau \in K_{GL(n)} = \Lambda[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$ a symmetric polynomial representing the descendent. We define the evaluation of such polynomial at fixed point $(\overrightarrow{\lambda},\overrightarrow{d})$ by $\tau(\overrightarrow{\lambda},\overrightarrow{d}) = \tau(x_\square = \varphi_{\overrightarrow{\lambda}}(\square)q^{d(\square)})$ (recall that the r-tuple of Young diagrams $\overrightarrow{\lambda}$ corresponding to a fixed point $(\overrightarrow{\lambda},\overrightarrow{d})$ has exactly $|\overrightarrow{\lambda}| = n$ boxes so we may think that the variables x_i are labeled by boxes in $\overrightarrow{\lambda}$).

The localization in the equivariant K-theory gives the coefficients in (41)[7]:

$$V^{(\tau)}_{\overrightarrow{\lambda}}(z) = \sum_{\substack{d_\square \geq 0 \\ \square \in \overrightarrow{\lambda}}} \hat{a}\left(T^{vir}_{(\overrightarrow{\lambda},\overrightarrow{d})}\mathsf{QM}^d_{p_2}(n,r)\right)\tau(\overrightarrow{\lambda},\overrightarrow{d})(-1)^{|d|r}q^{-\frac{r|d|}{2}}z^{|d|} \tag{45}$$

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[7]In this formula $q^{-\frac{r|d|}{2}}$ is the contribution of polarization to virtual structure sheaf - see (6.1.9) in [5]. In fact, as we discussed in [8] the capping operator is given by the fundamental solution of qde in which we should substitute $z \to (-1)^r z$. In this paper, however, we prefer to move this sign to the vertex, in which case all final formulas look simpler. This corresponds to the appearance of factor the $(-1)^{|d|r}$ in (45).

where following Section 3.4.40 of [5] we use a "roof" function defined on Laurent polynomials without a constant term by:

$$\hat{a}(x+y) = \hat{a}(x)\hat{a}(y), \quad \hat{a}(x) = \frac{1}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}.$$

4.4 Proof of the Theorem 4

The coefficients of the power series for the bare descendent vertex (45) are rational functions of the equivariant parameters $a_1, \cdots, a_r$ corresponding to framing torus A. The action of subtorus $\mathsf{C} \subset \mathsf{A}$ corresponds to the substitution of framing characters

$$(a_1, \cdots, a_{r_1}, a_{r_1+1} \cdots, a_r) \to (a_1, \cdots, a_{r_1}, aa_{r_1+1} \cdots, aa_r). \tag{46}$$

We are interested in the limit $\lim_{a \to 0} V^{(r),(\tau)}(z)$ under this substitution. Let $\overrightarrow{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(r_1)}, \lambda^{(r_1+1)} \cdots, \lambda^{(r)})$ be a fixed point on $\mathscr{M}(n,r)$. Denote by $\overrightarrow{\lambda}_1 = (\lambda^{(1)}, \cdots, \lambda^{(r_1)})$ and $\overrightarrow{\lambda}_2 = (\lambda^{(r_1+1)} \cdots, \lambda^{(r)})$ the corresponding classes on $\mathscr{M}(n,r_1)$ and $\mathscr{M}(n,r_2)$. The theorem 4 is equivalent to the following factorization of the coefficients of the bare vertex:

$$\lim_{a \to 0} V^{(r),(\tau)}_{\overrightarrow{\lambda}}(z) = V^{(r_1),(\tau)}_{\overrightarrow{\lambda}_1}(z\hbar^{\frac{r_2}{2}})V^{(r_2),(1)}_{\overrightarrow{\lambda}_2}(z\hbar^{-\frac{r_1}{2}}q^{-r_1}) \tag{47}$$

To prove it, first note that after substitution (46) for the functions (42) we obtain:

$$\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d}) = \mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1) + a\mathcal{V}(\overrightarrow{\lambda}_2, \overrightarrow{d}_2), \quad \mathcal{W} = \mathcal{W}_1 + a\mathcal{W}_2$$

where $\mathcal{W}_1 = a_1 + \cdots a_{r_1}$ and $\mathcal{W}_2 = a_{r_1+1} + \cdots + a_r$. Let us consider the contribution to the vertex of the term $\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})^*$ in (43). The contribution of a weight $w \in \mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})^*$ to $\hat{a}(T^{vir}_{(\overrightarrow{\lambda},\overrightarrow{d})}\mathsf{QM}^d_{p_2}(n,r))$ has the following form:

$$\hat{a}\left(-w(1-t_1^{-1})(1-t_2^{-1})\right) = \frac{(w^{\frac{1}{2}} - w^{-\frac{1}{2}})(w^{\frac{1}{2}}t_1^{-\frac{1}{2}}t_2^{-\frac{1}{2}} - w^{-\frac{1}{2}}t_1^{\frac{1}{2}}t_2^{\frac{1}{2}})}{(w^{\frac{1}{2}}t_1^{-\frac{1}{2}} - w^{-\frac{1}{2}}t_1^{\frac{1}{2}})(w^{\frac{1}{2}}t_2^{-\frac{1}{2}} - w^{-\frac{1}{2}}t_2^{\frac{1}{2}})}$$

If $w \in \mathcal{V}(\overrightarrow{\lambda}_i, \overrightarrow{d}_i)\mathcal{V}(\overrightarrow{\lambda}_j, \overrightarrow{d}_j)^*$ with $i \neq j$ then $w \sim a^{\pm 1}$ and in the limit $a \to 0$ the last expression is equal to 1. Thus, in the limit $a \to 0$ the

contribution of the term $\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})^*$ to $T^{vir}_{\overrightarrow{\lambda}, \overrightarrow{d}}\mathsf{QM}^d_{p_2}(n, r)$ splits to a sum of contributions of $\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)^*$ and $\mathcal{V}(\overrightarrow{\lambda}_2, \overrightarrow{d}_2)\mathcal{V}(\overrightarrow{\lambda}_2, \overrightarrow{d}_2)^*$. Thus, the contribution to the vertex, given by roof function $\hat{a}$ factors to a product of corresponding contributions.

Second, we analyse terms $\mathcal{V}(\overrightarrow{\lambda}_i, \overrightarrow{d}_i)\mathcal{W}_j$ with $i \neq j$. We assume $i = 1$ and $j = 2$. The corresponding contribution to $S(\overrightarrow{\lambda}, \overrightarrow{d})$ has the form:

$$\mathcal{W}_2^* \frac{\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1) - \mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{0})}{q - 1} + \mathcal{W}_2 \frac{(\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)^* - \mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{0})^*)}{(q - 1)t_1 t_2} =$$

$$\sum_{j=1}^{r_2} \sum_{\square \in \overrightarrow{\lambda}_1} \left(\frac{\varphi_{\overrightarrow{\lambda}}(\square)}{aa_j} \sum_{i=0}^{d(\square)-1} q^i - \frac{aa_j}{\varphi_{\overrightarrow{\lambda}}(\square)t_1 t_2 q} \sum_{i=0}^{d(\square)-1} q^{-i} \right)$$

Thus the corresponding contribution to the vertex gives:

$$\prod_{\square \in \overrightarrow{\lambda}_1} \prod_{j=1}^{r_2} \prod_{i=0}^{d(\square)-1} \frac{\left(\frac{aa_j}{\varphi_{\overrightarrow{\lambda}_1}(\square)q^i t_1 t_2 q} \right)^{1/2} - \left(\frac{aa_j}{\varphi_{\overrightarrow{\lambda}_1}(\square)q^i t_1 t_2 q} \right)^{-1/2}}{\left(\frac{\varphi_{\overrightarrow{\lambda}_1}(\square)q^i}{aa_j} \right)^{1/2} - \left(\frac{\varphi_{\overrightarrow{\lambda}_1}(\square)q^i}{aa_j} \right)^{-1/2}}$$

$$\xrightarrow{a \to 0} \prod_{\square \in \overrightarrow{\lambda}_1} \prod_{j=1}^{r_2} \prod_{i=0}^{d(\square)-1} \left(-t_1^{1/2} t_2^{1/2} q^{1/2} \right) = (-(\hbar q)^{1/2})^{r_2|\overrightarrow{d}_1|}$$

Same calculation for $i = 2, j = 1$ gives $(-(\hbar q)^{1/2})^{-r_1|\overrightarrow{d}_2|}$. The terms $\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)\mathcal{W}_1$ and $\mathcal{V}(\overrightarrow{\lambda}_2, \overrightarrow{d}_2)\mathcal{W}_2$ do not depend on a. We conclude, that the contribution of $\mathcal{V}(\overrightarrow{\lambda}, \overrightarrow{d})\mathcal{W}$ to the vertex in the limit $a \to 0$ factors to the contributions of $\mathcal{V}(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)\mathcal{W}_1$ and $\mathcal{V}(\overrightarrow{\lambda}_2, \overrightarrow{d}_2)\mathcal{W}_2$ shifted by some powers of $-(\hbar q)^{1/2}$. Finally, we have:

$$\lim_{a \to 0} \varphi_{\overrightarrow{\lambda}}(\square) = \begin{cases} \varphi_{\overrightarrow{\lambda}}(\square) & \text{if } \square \in \overrightarrow{\lambda}_1 \\ 0 & \text{if } \square \in \overrightarrow{\lambda}_2 \end{cases}$$

which for a symmetric polynomial τ means $\lim\limits_{a \to 0} \tau(\overrightarrow{\lambda}, \overrightarrow{d}) = \tau(\overrightarrow{\lambda}_1, \overrightarrow{d}_1)$ Over-

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all, we obtain:

$$\lim_{a \to 0} V^{(r),(\tau)}_{\vec{\lambda}}(z) =$$

$$\sum_{\vec{d}_1, \vec{d}_2} \hat{a}\left(T^{vir}_{(\vec{\lambda}_1, \vec{d}_1)} \mathsf{QM}^{d_1}_{p2}(n_1, r_1)\right) \hat{a}\left(T^{vir}_{(\vec{\lambda}_2, \vec{d}_2)} \mathsf{QM}^{d_2}_{p2}(n_2, r_2)\right)$$

$$\times (-(\hbar q)^{\frac{1}{2}})^{r_2|\vec{d}_1|}(-(\hbar q)^{\frac{1}{2}})^{-r_1|\vec{d}_2|} z^{|\vec{d}_1|+|\vec{d}_2|} \tau(\vec{\lambda}_1, \vec{d}_1)(-1)^{|d|r} q^{-\frac{|d|r}{2}}$$

$$= V^{(r_1),(\tau)}_{\vec{\lambda}_1}\left(z\hbar^{\frac{r_2}{2}}\right) V^{(r_2),(1)}_{\vec{\lambda}_2}\left(z\hbar^{-\frac{r_1}{2}} q^{-r_1}\right)$$

□

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