New Bounds on the Minimum Density of a Vertex Identifying Code for the Infinite Hexagonal Grid *

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Abstract

For a graph, G, and a vertex $v \in V(G)$, let $N[v]$ be the set of vertices adjacent to and including v. A set $D \subseteq V(G)$ is a vertex identifying code if for any two distinct vertices $v_1, v_2 \in V(G)$, the vertex sets $N[v_1] \cap D$ and $N[v_2] \cap D$ are distinct and non-empty. We consider the minimum density of a vertex identifying code for the infinite hexagonal grid. In 2000, Cohen et al. constructed two codes with a density of $\frac{3}{7} \approx 0.428571$, and this remains the best known upper bound. Until now, the best known lower bound was $\frac{12}{29} \approx 0.413793$ and was proved by Cranston and Yu in 2009. We present three new codes with a density of $\frac{3}{7}$, and we improve the lower bound to $\frac{5}{12} \approx 0.416667$.

1 Introduction

The study of vertex identifying codes is motivated by the desire to detect failures efficiently in a multiprocessor network. Such a network can be modelled as an undirected graph, G, where $V(G)$ represents the set of processors and $E(G)$ represents the set of connections among processors. Suppose we place detectors on a subset of these processors. These detectors monitor all processors within a neighborhood of radius r and send a signal to a central controller when a failure occurs. We assume that no two failures occur simultaneously. A signal from a detector, d, indicates that a processor in the r-neighborhood of d has failed but provides no further information. Now, any given processor, p, might be in the r-neighborhood of several detectors, d_1, d_2, d_3... Then, when p fails, the central controller receives signals from d_1, d_2, d_3... Let us call $\{d_1, d_2, d_3, ...\}$ the trace of p in G. If each processor has a unique and non-empty trace, then the central controller can determine which processor failed simply by noting the detectors from which signals were received. In this case, we call the subset of processors on which detectors were placed an identifying code.

Vertex identifying codes were first introduced in 1998 by Karpovsky, Chakrabarty and Levitin [5]. The processors of the preceding paragraph become the vertices of a graph, and the processors on which detectors have been placed become the vertex subset called a vertex identifying code. In the example above, we considered detectors which monitor a neighborhood of radius r. In this paper, we concern ourselves with the case in which $r = 1$.

Let $N_i(v)$ be the set of vertices at distance-i from a vertex, v, and let $N[v] = N_1(v) \cup \{v\}$.

Definition 1.1. Consider a graph, G. A set $D \subseteq V(G)$ is a **vertex identifying code** if

(i) For all $v \in V(G)$, $N[v] \cap D \neq \emptyset$

(ii) For all $v_1, v_2 \in V(G)$ where $v_1 \neq v_2$, $N[v_1] \cap D \neq N[v_2] \cap D$

From Definition 1.1, we see that some graphs do not admit vertex identifying codes. In particular, if $N[v_1] = N[v_2]$ for some distinct $v_1, v_2 \in V(G)$ then G does not admit a vertex identifying code because $N[v_1] \cap D = N[v_2] \cap D$ for any $D \subseteq V(G)$. On the other hand, if $N[v_1] \neq N[v_2]$ for all distinct $v_1, v_2 \in V(G)$ then G admits a vertex identifying code because $V(G)$ is such a code.

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Of particular interest are vertex identifying codes of minimal cardinality. When dealing with infinite graphs, we consider instead the *density* of a vertex identifying code, i.e., the ratio of the number of vertices in the code to the total number of vertices. Let G be an infinite graph, and let $D \subseteq V(G)$ be a vertex identifying code for G. Then, for some $v \in V(G)$, the set of vertices in D within distance-k of v is given by $\bigcup_{i=0}^{k} N_i(v) \cap D$. Let $\sigma(D,G)$ be the density of D in G. Then,

$$\sigma(D,G) = \limsup_{k\to\infty} \frac{\left|\bigcup_{i=0}^{k} N_i(v) \cap D\right|}{\left|\bigcup_{i=0}^{k} N_i(v)\right|} \tag{1.1}$$

Let $\sigma_0(G)$ be the minimum density of a vertex identifying code for G; that is,

$$\sigma_0(G) = \min_{D}\{\sigma(D,G)\} \tag{1.2}$$

Karpovsky et al. [5] considered the minimum density of vertex identifying codes for the infinite triangular (G_T), square (G_S) and hexagonal (G_H) grids. They showed $\sigma_0(G_T) = 1/4$. In 1999, Cohen et al. [2] proved $\sigma_0(G_S) \leq 7/20$, and, in 2005, Ben-Haim and Litsyn [1] completed the proof by showing $\sigma_0(G_S) \geq 7/20$.

We concern ourselves in this paper with $\sigma_0(G_H)$. In 1998, Karpovsky et al. [5] showed $\sigma_0(G_H) \geq 2/5 = 0.4$. In 2000, Cohen et al. [3] improved this result to $\sigma_0(G_H) \geq 16/39 \approx 0.410256$ and constructed two codes with a density of $3/7 \approx 0.428571$ implying $\sigma_0(G_H) \leq 3/7$. In 2009, Cranston and Yu [4] proved $\sigma_0(G_H) \geq 12/29 \approx 0.413793$. For other results on identifying codes for the hexagonal grid, see [6, 7].

In this paper, we present three new codes with a density of $3/7$ and prove $\sigma_0(G_H) \geq 5/12 \approx 0.416667$. In conclusion, it is now known that $5/12 \leq \sigma_0(G_H) \leq 3/7$.

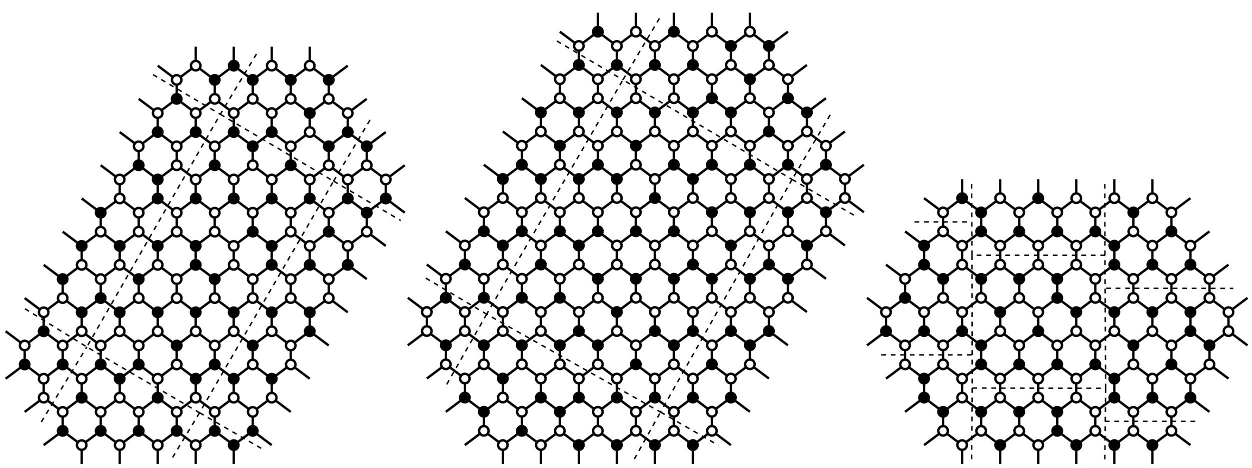


Figure 1.1: Three new codes with a density of 3/7. The solid vertices are in the code.

Suppose β is an upper bound on $\sigma_0(G_H)$. To prove this, we need only show the existence of a code, D, with $\sigma(D,G_H) \leq \beta$. When constructing such codes, we usually look for tiling patterns. Since the pattern repeats ad infinitum, the density of one tile is the density of the whole graph. Figure 1.1 shows three new codes for the infinite hexagonal grid with a density of 3/7.

Theorem 1.2. *The minimum density of a vertex identifying code for the infinite hexagonal grid is greater than or equal to* 5/12*.*

To prove Theorem 1.2, we employ the discharging method. Let D be an arbitrary vertex identifying code for G_H. We assign 1 "charge" to each vertex in D which we then redistribute so that every vertex in G_H retains at least 5/12 charge. The charge is redistributed in accordance with a set of "Discharging Rules". Since D was chosen arbitrarily, we then conclude that 5/12 is a lower bound on $\sigma_0(G_H)$.

As the proof of Theorem 1.2 is rather lengthy, we include a sketch of the proof in Section 2. In Section 3, we introduce several properties of vertex identifying codes for G_H which we will reference throughout the paper. Section 4 is devoted to terminology and notations; the vast majority of relevant notions are defined here. In Section 5, we state several lemmas concerning the structure of vertex identifying codes for G_H. However, we defer the proofs of these lemmas to Section 7. The main result of this paper, Theorem 1.2, is proved in Section 6.

For the rest of the paper, if not explicitly stated, D is to be interpreted as a vertex identifying code for the infinite hexagonal grid.

2 Sketch of the Proof

As mentioned in the introduction, our proof of Theorem 1.2 makes use of the discharging method. We assign 1 charge to each vertex in D and then redistribute this charge so that each vertex in G_H retains at least 5/12. To design the proper discharging rules, we start with the following (Rule 1 in Section 6):

If a vertex, v, is not in D and has k neighbors in D, then v receives $\frac{5}{12k}$ from each of these neighbors.

We can easily verify that Rule 1 suffices to allow each vertex in $G_H \setminus D$ to retain 5/12 charge (Claim 6.1). As a result, the remaining discharging rules are concerned exclusively with vertices in D. Now, any vertex, v, in D with a neighbor in $G_H \setminus D$ loses charge by Rule 1. We show in Section 6 that only one type of vertex loses too much by Rule 1; we call such a vertex a *poor* 1*-cluster* (Definition 4.1). Consequently, we must find charge to send to poor 1-clusters from nearby vertices. We find that it is helpful to consider a cluster (Definition 3.1) as a single entity. Thus we first need to determine the surplus charge each cluster may have after Rule 1.

We observe that some 1-clusters may have surplus charge and that their surplus differs according to the neighbors they may have; for this reason we define *non-poor* 1*-clusters* (Definition 4.12) and *one-third vertices* (Definition 4.13). In Lemmas 5.2-5.4, we determine how many poor 1-clusters can lie in the neighborhood of a non-poor 1-cluster, and then in Rules 2, 3d and 3e, we design the appropriate discharging rules to distribute the surplus charge. In Claim 6.4, we show that non-poor 1-clusters ultimately retain a charge of at least 5/12.

For 3^+-clusters, the situation is more complicated. We first see a difference of surplus charge according to the distribution of vertices at distance-2 from a given 3^+-cluster; for this reason we define *open/closed* k*-clusters* (Definition 4.3), *crowded/uncrowded* k*-clusters* (Definition 4.4) and the P-function (Definition 4.5). These definitions allow us to distinguish among 3^+-clusters with varying amounts of surplus charge. We will see in Section 6 that for very large k, a k-cluster can always afford to send charge to all nearby poor 1-clusters. Consequently, we are mostly concerned with k-clusters with $3 \leq k \leq 6$. In Lemmas 5.6-5.16, we determine the number of poor 1-clusters that can lie in the neighborhood of a given k-cluster. Discharging Rules 3a-3c are designed in accordance with these lemmas to send charge from 3^+-clusters to poor 1-clusters lying in a distance-2 or distance-3 neighborhood.

Now, some poor 1-clusters do not lie in a neighborhood that receives charge by Rule 3. We call these *very poor* 1*-clusters* (Definition 4.14), and we distinguish between two orientations: symmetric and asymmetric (Definition 4.15). In Lemmas 5.17, 5.20 and 5.24 we scan the neighborhood of a very poor 1-cluster for clusters with charge available for redistribution after Rule 3. Crucially, we find in Lemma 5.20 that if there is no other way to squeeze charge for a given very poor 1-cluster from a single nearby cluster, there must be *type*-1 *paired* 3*-clusters* or *type*-2 *paired* 3*-clusters* (Definition 4.9) in the neighborhood. These are structures which tend to form in the extended neighborhood of an asymmetric very poor 1-cluster and which always have extra charge after Rule 3. In order to reserve this extra charge for very poor 1-clusters, several discharging rules make exceptions for type-1 and type-2 paired 3-clusters. That this creates no new deficiency of charge is proved in Section 6. We prove some properties of type-1 and type-2 paired 3-clusters in Lemmas 5.25 and 5.26. Discharging Rules 4-7 are designed in accordance with the above-mentioned lemmas to send charge to very poor 1-clusters.

On an additional note, the structure of type-1 and type-2 paired 3-clusters is very specific, and this forces us to introduce some very specific notions (for example, Definitions 4.5 and 4.6). This is done so that our analysis can penetrate to the properties of individual vertices. As a result, the proofing process is somewhat tedious though more or less straightforward.

3 General Structural Properties

Definition 3.1. A component of the subgraph induced by D is called a **cluster**. A cluster containing k vertices is called a **k-cluster**; a cluster containing k or more vertices is called a **k$^+$-cluster**. Let D_k be the set of all vertices in k-clusters; and let $\mathcal{K}_k$ be the set of all k-clusters. Let $d_C(v)$ be the degree of a vertex, v, in a 3^+-cluster, C; and let $\Delta(C) = \max\{d_C(v) : v \in C\}$.

Proposition 3.2. *There exist no* 2*-clusters.*

Proof. Suppose by contradiction that there exists a 2-cluster, C, and let $V(C) = \{v, w\}$. Then, $N[v] \cap D = \{v, w\}$ and $N[w] \cap D = \{v, w\}$. Now, if $N[v] \cap D = N[w] \cap D$, then $v = w$ (Definition 1.1), which is a contradiction. □

Corollary 3.3. *If a vertex, v, is not in a 3^+-cluster, then either v is not in D or v is a* 1*-cluster.*

Proposition 3.4. *If a vertex not in D has* 2 *adjacent vertices not in D, then the remaining adjacent vertex is in a 3^+-cluster.*

Proof. Consider a vertex, v, such that that $N_1(v) = \{a, b, c\}$ and let $a, b, v \notin D$. Suppose by contradiction that $c \notin D_{3+}$. Then, $c \notin D$ or $c \in D_1$ (Corollary 3.3). If $c \notin D$, then $N[v] \cap D = \emptyset$, which is a contradiction (Definition 1.1). If $c \in D_1$, then $N[v] \cap D = N[c] \cap D = \{c\}$; therefore, $c = v$ (Definition 1.1), which is a contradiction. □

Proposition 3.5. *Each of the vertices adjacent to a* 1*-cluster, v, has at least one adjacent vertex in $D \setminus \{v\}$.*

Proof. Let $v \in D_1$, and let u be an adjacent vertex. Suppose by contradiction that u has no adjacent vertices in $D \setminus \{v\}$. Then, $v \in D_{3+}$ (Proposition 3.4), which is a contradiction. □

Proposition 3.6. *Each leaf of a 3^+-cluster, C, has at least one distance-*2 *vertex in $D \setminus C$.*

Proof. Let v be a leaf of a 3^+-cluster, C. Then, exactly 2 of the vertices adjacent to v are not in D; let u and w be these vertices. Suppose by contradiction that v has no distance-2 vertices in $D \setminus C$. Then, $N[u] \cap D = N[w] \cap D = \{v\}$; therefore, $u = w$ (Definition 1.1), which is a contradiction. □

4 Terminology and Notations

We introduce the following convention which we will use throughout the paper. Let G be a graph, and suppose $D \subseteq V(G)$ is a vertex identifying code for G. In the figures, we use a solid vertex to denote that a vertex is in D, and we use a hollow vertex to denote that a vertex is not in D. The status of all other vertices is undetermined. In Figure 4.1, for instance, $u \in D$ and $v \notin D$, while the status of w is undetermined.

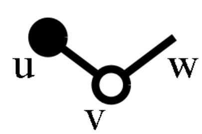


Figure 4.1

Definition 4.1. A 1-cluster with exactly 3 distance-2 vertices in D is called a **poor 1-cluster**. Let D_1^p be the set of all poor 1-clusters.

Corollary 4.2. *Each of the neighbors of a poor* 1*-cluster, v, has exactly one neighbor in $D \setminus \{v\}$.*

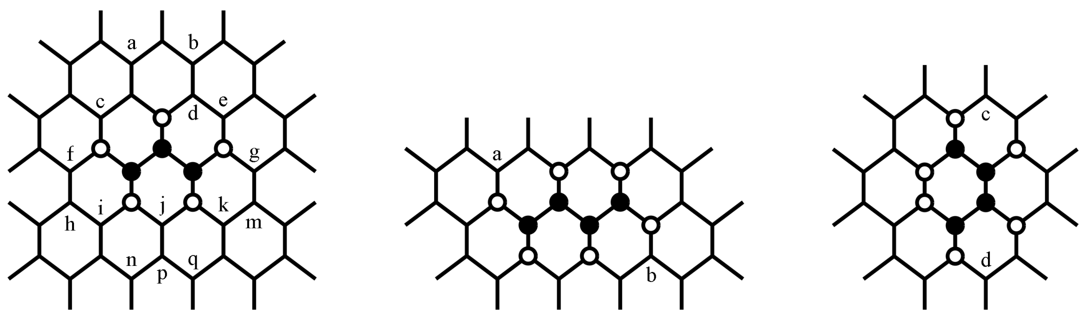
Proof. This follows from Proposition 3.5 and Definition 4.1. □

Definition 4.3. For $k \geq 3$, let C be a k-cluster with $\Delta(C) = 2$. If none of the non-leaf vertices of C has a distance-2 vertex in $D \setminus C$, then C is an **open k-cluster**. If at least one of the non-leaf vertices of C has a distance-2 vertex, v, in $D \setminus C$, then C is a **closed k-cluster** and v **closes** C. Let D_k^o be the set of all vertices in open k-clusters and D_k^c the set of all vertices in closed k-clusters; let $\mathcal{K}_k^o$ be the set of all open k-clusters and $\mathcal{K}_k^c$ the set of all closed k-clusters.

Definition 4.4. If an open k-cluster, C, has exactly 2 distance-2 vertices in D, both of which are poor 1-clusters, then C is **uncrowded**. Otherwise, C is **crowded**.

Definition 4.5. For a given cluster, C, let $P(C) = \sum_{v \in C} |N_2(v) \cap D \setminus C|$.

Definition 4.6. Let C be the 3-cluster shown in Figure 4.2a. Vertices a and b are in the **head positions** of C; c and e are in the **shoulder positions**; f and g are in the **arm positions**; h and m are in the **hand positions**; i and k are in the **foot positions**; j is in the **tail position**; and n and q are in the **fin positions**. If q is not in D, then b, d, e, g, k and m are on the **finless side** of C. If d is in D, then b, e, g, k, m and q are on the **closed side** of C.

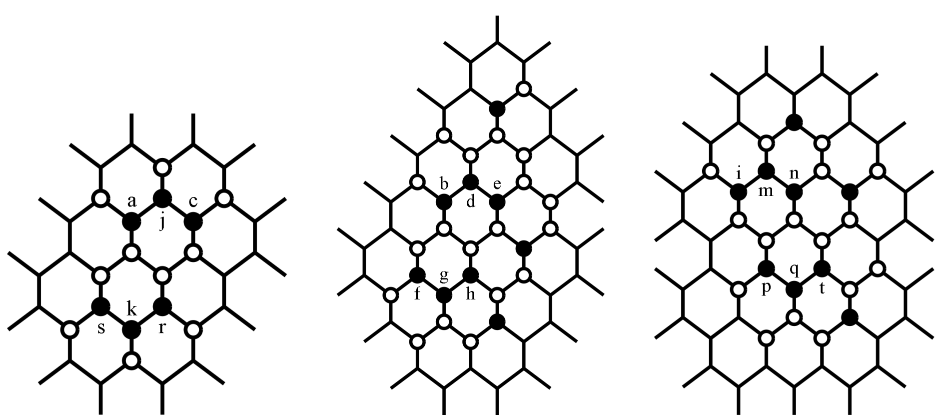


(a) 3-Cluster (b) Linear 4-Cluster (c) Curved 4-Cluster

Figure 4.2

Definition 4.7. Let C be a 4-cluster with $\Delta(C) = 2$. If the leaves of C do not lie on the same 6-cycle, then C is a **linear** 4**-cluster**. Otherwise, C is a **curved** 4**-cluster**. Let C_1 be the linear 4-cluster shown in Figure 4.2b. Vertices a and b are in the **one-turn positions** of C_1. Let C_2 be the curved 4-cluster shown in Figure 4.2c. Vertices c and d are in the **backwards positions** of C_2.

Definition 4.8. A vertex, v, is **distance-k from a cluster**, C, if k is the minimum distance from v to any of the vertices of C. If $k \leq \ell$, then v is **within distance-ℓ** of C. If a vertex, v, is within distance-3 of a cluster, C, then v is **nearby** C.



(a) Paired 3-Clusters (b) Type-1 Paired 3-Clusters (c) Type-2 Paired 3-Clusters

Figure 4.3

Definition 4.9. Let C_1 be the 3-cluster described by a, j and c in Figure 4.3a, and let C_2 be the 3-cluster described by s, k and r. Then, C_1 and C_2 are **paired 3-clusters**. Let C_3 be the 3-cluster described by b, d and e in Figure 4.3b, and let C_4 be the 3-cluster described by f, g and h. Then, C_3 and C_4 are **type-1 paired**, and C_3 is **type-1 paired on top**. Let C_5 be the 3-cluster described by i, m and n in Figure 4.3c, and let C_6 be the 3-cluster described by p, q and t. Then, C_5 and C_6 are **type-2 paired**.

Corollary 4.10. *If a* 3*-cluster, C, is type-*1 *paired, then C is not type-*2 *paired, and vice versa.*

Definition 4.11. A poor 1-cluster, v, is **stealable** if v is distance-3 from a 4^+-cluster and distance-2 from a 3^+-cluster, C, such that if C is an open 3-cluster, then

(i) v is not in a shoulder position;

(ii) if v is in an arm position, then C is neither type-1 nor type-2 paired.

Definition 4.12. A 1-cluster that is not poor is called a **non-poor 1-cluster**. Let D_1^{np} be the set of all non-poor 1-clusters. If 3 non-poor 1-clusters, u, v and w, are adjacent to the same one-third vertex, then u, v and w are referred to as a **group of non-poor 1-clusters**. A vertex, v, is **distance-k from a group of non-poor 1-clusters**, H, if v is distance-k from any of the 1-clusters in H.

Definition 4.13. If a vertex, v, is not in D and has 3 neighbors in D, then v is called a **one-third vertex**.

Definition 4.14. If a poor 1-cluster, v, is neither distance-2 from a 3^+-cluster or a non-poor 1-cluster nor distance-3 from a closed 3-cluster or 4^+-cluster, then v is called a **very poor 1-cluster**. Let D_1^{vp} be the set of all very poor 1-clusters.

(a) Symmetric Orientation

(b) Asymmetric Orientation

Figure 4.4

Definition 4.15. If a very poor 1-cluster, v, has 3 distance-2 vertices in D which are all distance-4 from each other, then v is in a **symmetric orientation** (see Figure 4.4a). A very poor 1-cluster which is not in a symmetric orientation is in an **asymmetric orientation** (see Figure 4.4b). The vertex u in Figure 4.4b is in the **u-position** of v, the vertex w is in the **w-position** of v, and the vertex x is in the **x-position** of v.

5 Structural Lemmas

In this section, we state several lemmas concerning the structure of a vertex identifying code for the infinite hexagonal grid. As the primary purpose of these lemmas is to abridge the proof of Theorem 1.2, we defer the proofs of all lemmas to page 17. Additionally, we defer the proofs of Proposition 5.1 and Corollaries 5.22 and 5.23.

Proposition 5.1. *If a poor* 1*-cluster, v, is distance-*2 *from exactly one of the* 1*-clusters in a group of non-poor* 1*-clusters, then v is distance-*2 *from an open* 3*-cluster or within distance-*3 *of a closed* 3*-cluster or* 4^+*-cluster.*

Lemma 5.2. *Consider a group of non-poor* 1*-clusters, H. There exist at most* 2 *poor* 1*-clusters which are distance-*2 *from H and neither distance-*2 *from an open* 3*-cluster nor within distance-*3 *of a closed* 3*-cluster or* 4^+*-cluster.*

Lemma 5.3. *Let v be a one-third vertex, and let v have exactly 2 adjacent 1-clusters, c and d. Each of c and d has at most one distance-2 poor 1-cluster that is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4^+-cluster.*

Lemma 5.4. *If a one-third vertex has exactly one adjacent 1-cluster, d, then d has at most 2 distance-2 poor 1-clusters.*

Lemma 5.5. *A 3-cluster has at most one finless side.*

Lemma 5.6. *Let C_1 be a closed 3-cluster with $P(C_1) = 3$.*

(i) C_1 has at most 8 nearby poor 1-clusters. If C_1 has 8 such clusters, at least one of the poor 1-clusters at distance-3, v, is distance-2 from another 3^+-cluster, C_2, such that

- *(a) if C_2 is an open 3-cluster, then v is not in a shoulder position;*
- *(b) if C_2 is an open 3-cluster and v is in an arm position, then C_2 is not type-1 paired; if C_2 is type-2 paired, then C_1 is type-2 paired with C_2.*

(ii) If neither the shoulder positions nor the tail position are in D, then C_1 has at most 5 nearby poor 1-clusters.

Lemma 5.7. *Let C be a closed 3-cluster with $P(C) = 4$. If C is adjacent to a one-third vertex, then C has at most 8 nearby poor 1-clusters. Furthermore, if an arm position or a foot position of C is a poor 1-cluster, then C has at most 7 nearby poor 1-clusters. If 2 arm or foot positions are poor 1-clusters, then C has at most 6 nearby poor 1-clusters.*

Lemma 5.8. *Let C_1 be a linear open 4-cluster with $P(C_1) = 2$.*

(i) C_1 has at most 8 nearby poor 1-clusters. Furthermore, if C_1 has 8 such clusters, then at least 2 of the distance-3 poor 1-clusters are stealable; if C_1 has 7 such clusters, then at least one is stealable.

(ii) If one one-turn position is not in D, then C_1 has at most 6 nearby poor 1-clusters. If C_1 has exactly 6 such clusters, then at least one of the distance-3 poor 1-clusters is stealable.

(iii) If neither one-turn position is in D, then C_1 has at most 4 nearby poor 1-clusters.

Lemma 5.9. *Let C be a linear 4-cluster with $P(C) = 3$.*

(i) If C is adjacent to no one-third vertices, then C has at most 9 nearby poor 1-clusters.

(ii) If C is adjacent a one-third vertex, then C has at most 6 nearby poor 1-clusters.

Lemma 5.10. *Let C_1 be a curved open 4-cluster with $P(C_1) = 2$.*

(i) C_1 has at most 8 nearby poor 1-clusters. Furthermore, if C_1 has 8 such clusters, then at least 2 of the distance-3 poor 1-clusters are stealable; if C_1 has 7 such clusters, then at least one is stealable.

(ii) If one backwards position is not in D, then C_1 has at most 6 nearby poor 1-clusters. If C_1 has exactly 6 such 1-clusters, then at least one of the distance-3 poor 1-clusters is stealable.

(iii) If neither backwards position is in D, then C_1 has at most 2 nearby poor 1-clusters.

Lemma 5.11. *Let C be a curved 4-cluster with $P(C) = 3$.*

(i) If C is adjacent to no one-third vertices, then C has at most 11 nearby poor 1-clusters. Furthermore, if C has k backwards positions not in D, then C has at most $11 - k$ nearby poor 1-clusters.

(ii) If C is adjacent to a one-third vertex, then C has at most 6 nearby poor 1-clusters.

Lemma 5.12. *An open* 5*-cluster, C, has at most* 9 *nearby poor* 1*-clusters. If C has exactly* 9 *such* 1*-clusters, then at least one of the distance-*3 *poor* 1*-clusters is stealable.*

Lemma 5.13. *If a* 4*-cluster, C, has one degree-*3 *vertex and $P(C) = 3$, then C has at most* 8 *nearby poor* 1*-clusters.*

Lemma 5.14. *If a* 5*-cluster, C, has one degree-*3 *vertex, then C has at most* 12 *nearby poor* 1*-clusters.*

Lemma 5.15. *An open* 6*-cluster, C, with $\Delta(C) = 2$ has at most* 10 *nearby poor* 1*-clusters.*

Lemma 5.16 was proved by Cranston and Yu [4, p. 14] in 2009. We state it here without proof.

Lemma 5.16. *For $k \geq 3$, a k-cluster has at most $k + 8$ nearby clusters.*

Lemma 5.17. *Let v be a very poor* 1*-cluster in a symmetric orientation, and let a, b and c be the vertices in D at distance-*2 *from v. There exist* 3 *open* 3*-clusters, C_1, C_2 and C_3, such that v is in a head position of all* 3 *and exactly one of a, b and c is in a shoulder position of each of C_1, C_2 and C_3. Furthermore, if each of C_1, C_2 and C_3 is uncrowded, then each of a, b and c is distance-*3 *from a closed* 3*-cluster or 4^+-cluster.*

Corollaries 5.18 and 5.19 follow directly from the proof of Lemma 5.17.

Corollary 5.18. *None of the open* 3*-clusters of which a very poor* 1*-cluster in a symmetric orientation is in a head position is type-*1 *paired on top.*

Corollary 5.19. *Each of the vertices in D at distance-*2 *from a very poor* 1*-cluster in a symmetric orientation is distance-*2 *from no other very poor* 1*-cluster.*

Lemma 5.20. *Let v be a very poor* 1*-cluster, and let u, w and x be in the u-position, w-position and x-position, respectively, of v. There exists an open* 3*-cluster, C_0, such that v and x are in the head positions and w is in a shoulder position of C_0; and one of the following holds:*

(i) C_0 is crowded.

(ii) There exists a closed 3*-cluster or 4^+-cluster at distance-*3 *from w.*

(iii) There exists an open 3*-cluster, C, such that the tail position of C is in D and u, v or w is in the hand position on the finless side of C.*

*(iv) There exists a leaf, ℓ, of a 4^+-cluster, C, at distance-*2 *from u such that u is the only vertex in $D \setminus C$ at distance-*2 *from ℓ. If C is a linear* 4*-cluster, then C has at most one one-turn position. If C is a curved* 4*-cluster, then C has at most one backwards position in D.*

(v) There exists a leaf, ℓ, of a closed 3*-cluster, C, at distance-*2 *from u such that u is the only vertex in $D \setminus C$ at distance-*2 *from ℓ and u is in a foot or arm position. Either C is type-*2 *paired and u is in the arm position on the closed side of C, or C has at most* 6 *nearby poor* 1*-clusters.*

(vi) There exists an open 3*-cluster, C, such that v or w is in a hand position of C and the hand and arm positions on the opposite side of C are both in D.*

(vii) There exists an open 3*-cluster, C, such that u is in a foot position and C is type-*1 *paired on top.*

Corollary 5.21 follows directly from the proof of Lemma 5.20.

Corollary 5.21. *Let v be a very poor* 1*-cluster in an asymmetric orientation, and let w be in the w-position of v. The open* 3*-cluster of which v is in a head position and w is in a shoulder position is not type-*1 *paired on top.*

Corollary 5.22. *Let u and w be vertices in the u-position and w-position, respectively, of a very poor* 1*-cluster, v. Neither u nor w is distance-*2 *from any very poor* 1*-cluster other than v.*

Corollary 5.23. *Consider a very poor 1-cluster, v, in an asymmetric orientation, and let x be in the x-position of v. If x is a very poor 1-cluster, then x is in an asymmetric orientation and v is in the x-position of x.*

Lemma 5.24. *If a very poor 1-cluster is in a head position of an open 3-cluster, C, then both shoulder positions of C are in D.*

Lemma 5.25. *If a poor 1-cluster, v, is in a shoulder or arm position of an open 3-cluster that is type-1 paired on top, then v is nearby another 3^+-cluster, C_2, such that if C_2 is an open 3-cluster then v is distance-2 from C_2 but not in an arm position and C_2 is not type-1 paired on top.*

Lemma 5.26. *Let C_1 be a closed 3-cluster that is type-2 paired with the open 3-cluster, C_2. If C_1 has 7 nearby poor 1-clusters, then the arm position, n, of C_2 is in D and the hand position, k, on the same side is not in D. Furthermore, if n is a poor 1-cluster, then n is nearby a third 3^+-cluster, C_3, such that if C_3 is an open 3-cluster then n is distance-2 from C_3 but not in an arm position and C_3 is not type-1 paired on top.*

6 Proof of Theorem 1.2

We employ the discharging method. Suppose each vertex in D has 1 charge. We redistribute this charge so that each vertex in G_H has at least $\frac{5}{12}$ charge. Below are the discharging rules:

1. If a vertex, v, is not in D and has k neighbors in D, then v receives $\frac{5}{12k}$ from each of these neighbors.
2. Let $v_{\frac{1}{3}}$ be a one-third vertex, and let a, b and c be the vertices adjacent to $v_{\frac{1}{3}}$.
   (a) If a and b are 1-clusters and c is in a 3^+-cluster, C, then each of a and b receives $\frac{1}{72}$ from C.
   (b) If a is a 1-cluster and b and c are in 3^+-clusters, C_1 and C_2, then a receives $\frac{1}{36}$ from each of C_1 and C_2. If $C_1 = C_2$, then a receives $2 \cdot \frac{1}{36} = \frac{1}{18}$ from C_1.
3. Let v be a poor 1-cluster.
   (a) If v is distance-2 from a closed 3-cluster or 4^+-cluster, C, then v receives $\frac{1}{24}$ from C.
   (b) If v is distance-2 from an open 3-cluster, C, and has not received charge by previous rules, then v receives $\frac{1}{24}$ from C unless (a) C is type-1 paired on top and v is in a shoulder or arm position, or (b) C is type-2 paired and v is in an arm position.
   (c) If v is distance-3 from a closed 3-cluster or 4^+-cluster, C, and has not received charge by previous rules, then v receives $\frac{1}{24}$ from C unless C is a closed 3-cluster and v is in the arm position of an open 3-cluster that is type-2 paired with C.
   (d) Let h be a non-poor 1-cluster that is not in a group of non-poor 1-clusters. If v is distance-2 from h and has not received charge by previous rules, then v receives $\frac{1}{24}$ from h.
   (e) If v is distance-2 from a group of non-poor 1-clusters, H, and has not received charge by previous rules, then v receives $\frac{1}{24}$ from H.
4. If a closed 3-cluster, C_1, and an open 3-cluster, C_2, are type-2 paired and the arm position of C_2 is in D and the hand position on the same side is not in D, then C_1 receives $\frac{1}{24}$ from C_2.
5. If a very poor 1-cluster, v, is in a head position of a crowded open 3-cluster, C_0, then v receives $\frac{1}{24}$ from C_0.
6. Let v be a very poor 1-cluster in a symmetric orientation, and let a be a distance-2 poor 1-cluster. If a is in a shoulder position of an open 3-cluster and distance-3 from a closed 3-cluster or 4^+-cluster, C, then a receives $\frac{1}{24}$ from C in addition to any charge received by previous rules and v receives $\frac{1}{24}$ from a.

7. Let v be a very poor 1-cluster in an asymmetric orientation, and let u, and w be poor 1-clusters in the u-position and w-position, respectively, of v. The following applies only if v does not receive charge by Discharging Rule 5.

   (a) If w is distance-3 from a closed 3-cluster or 4^+-cluster, C, then w receives $\frac{1}{24}$ from C in addition to any charge received by previous rules and v receives $\frac{1}{24}$ from w.

   (b) Let C be an open 3-cluster, and let the tail position of C be in D. If u, v or w is in the hand position on the finless side of C, then u, v or w, respectively, receives $\frac{1}{24}$ from C in addition to any charge received by previous rules. If u or w receives this charge, then v receives $\frac{1}{24}$ from u or w, respectively.

   (c) If u distance-2 from a leaf, ℓ, of a type-2 paired closed 3-cluster, a closed 3-cluster with at most 6 nearby poor 1-clusters or a 4^+-cluster, C, such that u is not in a shoulder or tail position, a one-turn position or a backwards position of a closed 3-cluster, a linear 4-cluster or a curved 4-cluster, respectively, and u is the only vertex in $D \setminus C$ at distance-2 from ℓ, then u receives $\frac{1}{24}$ from C in addition to any charge received by previous rules and v receives $\frac{1}{24}$ from u.

   (d) Let C be an open 3-cluster, and let the hand and arm positions on one side of C be in D. If v or w is in the hand position on the other side of C, then v or w, respectively, receives $\frac{1}{24}$ from C in addition to any charge received by previous rules. If w receives this charge, then v receives $\frac{1}{24}$ from w.

   (e) Let C be an open 3-cluster that is type-1 paired on top. If u is in the foot position of C, then u receives $\frac{1}{24}$ from C in addition to any charge received by previous rules and v receives $\frac{1}{24}$ from u.

Now we verify that the above discharging rules allow each vertex in G_H to retain at least $\frac{5}{12}$ charge. For a given vertex, v, let $f(v)$ be the final charge of v and let $f_n(v)$ be the charge of v after Discharging Rule n. And for a given k-cluster, C, where $k \geq 3$, let $f(C)$ be the final charge of C and let $f_n(C)$ be the charge of C after Discharging Rule n; note that $f(C) \geq \frac{5k}{12}$ immediately implies that each vertex in C can retain at least $\frac{5}{12}$ charge.

If $v \in V(G_H)$, then $v \notin D$ or $v \in D_1$ or $v \in D_3$ or $v \in D_{4^+}$. We consider vertices not in D in Claim 6.1. We partition D_1 such that $D_1 = (D_1^p \setminus D_1^{vp}) \cup D_1^{np} \cup D_1^{vp}$, and we consider each case separately in Claims 6.3, 6.4 and 6.5, respectively. Rather than considering individual vertices in D_3, we consider $\mathcal{K}_3$. We partition $\mathcal{K}_3$ such that $\mathcal{K}_3 = \mathcal{K}_3^o \cup \mathcal{K}_3^c$, and we consider each case separately in Claims 6.7 and 6.11, respectively. We defer our discussion of 4^+-clusters until after Claim 6.11.

Claim 6.1. *If a vertex, v, is not in D, then $f(v) = \frac{5}{12}$.*

Proof. Let $v \notin D$, and suppose v has k neighbors in D. Then, by Discharging Rule 1, v receives $\frac{5}{12k}$ from each of these neighbors. That is, $f(v) = f_1(v) = k \cdot \frac{5}{12k} = \frac{5}{12}$. ☐

Proposition 6.2. *Any poor 1-cluster at distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or 4^+-cluster receives charge by Discharging Rules 3a-3c.*

Proof. Let $v \in D_1^p$ such that v is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or 4^+-cluster. Then, by Rules 3a-3c, v receives $\frac{1}{24}$ from a nearby 3^+-cluster except, potentially, in 2 cases. In the first case, v is in a shoulder or arm position of an open 3-cluster which is type-1 paired on top. But, by Lemma 5.25, v is nearby another 3^+-cluster, C_1, such that if C_1 is an open 3-cluster then v is distance-2 from C_1 but not in an arm position and C_1 is not type-1 paired on top; therefore, v receives $\frac{1}{24}$ from C_1 by Discharging Rules 3a-3c. In the second case, v is in an arm position of an open 3-cluster which is type-2 paired with a closed 3-cluster. But, by Lemma 5.26, v is nearby a third 3^+-cluster, C_2, such that if C_2 is an open 3-cluster then v is distance-2 from C_2 but not in an arm position and C_2 is not type-1 paired on top; therefore, v receives $\frac{1}{24}$ from C_2 by Discharging Rules 3a-3c. ☐

Claim 6.3. *If a poor 1-cluster, v, is not very poor, then $f(v) = \frac{5}{12}$.*

Proof. Let $v \in D_1^p \setminus D_1^{vp}$. Then, v must send charge to all 3 neighbors, each of which has exactly 2 neighbors in D. That is, $f_1(v) = 1 - 3 \cdot \frac{5}{12 \cdot 2} = \frac{9}{24}$. But v is not very poor; therefore, v is distance-2 from a 3^+-cluster or non-poor 1-cluster or distance-3 from a closed 3-cluster or 4^+-cluster. If v is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or 4^+-cluster, then v receives $\frac{1}{24}$ by Rules 3a-3c (Proposition 6.2); if not, then v receives charge from a distance-2 non-poor 1-cluster by Rules 3d-3e. Thus, we have shown that v will receive $\frac{1}{24}$ from a nearby cluster. Therefore, $f_3(v) = f_1(v) + \frac{1}{24} = \frac{9}{24} + \frac{1}{24} = \frac{5}{12}$. If v is distance-2 from a very poor 1-cluster, w, then v may need to receive charge from a nearby cluster and send charge to w by Rules 6-7. If w is in a symmetric orientation, then v is not distance-2 from any very poor 1-cluster other than w (Corollary 5.19). If w is in an asymmetric orientation and v is in the u-position or w-position of w, then v is not distance-2 from any very poor 1-cluster other than w (Corollary 5.22). Thus, if Rules 6-7 require v to receive and send charge, then v must only send charge to one very poor 1-cluster. Then, if Rule 6 is applicable, v receives $\frac{1}{24}$ and sends $\frac{1}{24}$. The same is true of Rules 7a-7e. Therefore, Rules 6-7 have no effect on the final charge of v. Therefore, $f(v) = f_7(v) = f_3(v) = \frac{5}{12}$. ☐

Claim 6.4. *For every non-poor 1-cluster, v, $f(v) \geq \frac{5}{12}$.*

Proof. Let $v \in D_1^{np}$. Then, v must send charge to all 3 of its neighbors, at least one of which has 3 neighbors in D. Therefore, $f_1(v) \geq 1 - \left(2 \cdot \frac{5}{12 \cdot 2} + \frac{5}{12 \cdot 3}\right) = \frac{4}{9}$.

Suppose v is in a group of non-poor 1-clusters, H. Then, $f_1(H) \geq 3 \cdot f_1(v) = \frac{4}{3}$. By Discharging Rule 3e, H may need to send charge to distance-2 poor 1-clusters that do not receive charge by Rules 3a-3d; therefore, H must send charge to a distance-2 poor 1-cluster, u, only if u is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4^+-cluster (Proposition 6.2). Then, H must send charge to at most 2 distance-2 poor 1-clusters (Lemma 5.2). Therefore, $f(H) = f_3(H) \geq f_1(H) - 2 \cdot \frac{1}{24} \geq \frac{4}{3} - \frac{1}{12} = \frac{15}{12} = 3 \cdot \frac{5}{12}$. Therefore, $f(v) \geq \frac{5}{12}$.

Now, suppose v shares a one-third vertex with a non-poor 1-cluster, w, and a 3^+-cluster, C. By Discharging Rule 2a, each of v and w receives $\frac{1}{72}$ from C. And, by Discharging Rule 3d, each of v and w may need to send charge to distance-2 poor 1-clusters that do not receive charge by Rules 3a-3c; therefore, v and w send charge to a poor 1-cluster, u, only if u is neither distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4^+-cluster (Proposition 6.2). Then, each of v and w has at most one distance-2 poor 1-cluster that does not receive charge by Rules 3a-3c (Lemma 5.3). Therefore, $f(v) = f_3(v) \geq f_2(v) - \frac{1}{24} = \left(f_1(v) + \frac{1}{72}\right) - \frac{1}{24} \geq \left(\frac{4}{9} + \frac{1}{72}\right) - \frac{1}{24} = \frac{5}{12}$

Now, suppose v shares a one-third vertex with 3^+-clusters only. Then, by Rule 2b, v receives $\frac{1}{18}$ from these 3^+-clusters. By Rule 3d, v may need to send charge to distance-2 poor 1-clusters. However, v has at most 2 distance-2 poor 1-clusters (Lemma 5.4). Therefore, $f(v) = f_3(v) \geq f_2(v) - 2 \cdot \frac{1}{24} = \left(f_1(v) + \frac{1}{18}\right) - \frac{1}{12} \geq \left(\frac{4}{9} + \frac{1}{18}\right) - \frac{1}{12} = \frac{5}{12}$. ☐

Claim 6.5. *For every very poor 1-cluster, v, $f(v) \geq \frac{5}{12}$.*

Proof. Let $v \in D_1^{vp}$. We saw above that $f_1(v) = \frac{9}{24}$.

If v is in a symmetric orientation, then v is in a head position of 3 open 3-clusters, C_1, C_2 and C_3 (Lemma 5.17). If any of C_1, C_2 and C_3 is crowded, say C_1, then v receives $\frac{1}{24}$ from C_1. Then, $f_5(v) = f_1(v) + \frac{1}{24} = \frac{5}{12}$. Let a, b and c be the vertices in D at distance-2 from v. Since $v \in D_1^{vp}$, we must have $a, b, c \in D_1^p$. However, exactly one of a, b and c is in a shoulder position of each of C_1, C_2 and C_3 (Lemma 5.17); therefore $a, b, c \notin D_1^{vp}$. Then, Rules 6-7 do not require v to send any charge; therefore, $f(v) \geq f_5(v) \geq \frac{5}{12}$. Now, if each of C_1, C_2 and C_3 is uncrowded, then each of a, b and c is distance-3 from a closed 3-cluster or 4^+-cluster (Lemma 5.17). Discharging Rule 5 is not applicable, but, by Discharging Rule 6, each of a, b and c receives $\frac{1}{24}$ from a distance-3 closed 3-cluster or 4^+-cluster and sends $\frac{1}{24}$ to v. Rule 7 is not applicable; therefore $f(v) = f_6(v) = f_1(v) + 3 \cdot \frac{1}{24} = \frac{9}{24} + \frac{1}{8} = \frac{1}{2} > \frac{5}{12}$.

Now, suppose v is in an asymmetric orientation. Let u, w and x be in the u-position, w-position and x-position, respectively. Now, $u, w \notin D_1^{vp}$ (Corollary 5.22); and if $x \in D_1^{vp}$, then v is in the x-position of x (Corollary 5.23). Therefore, none of the Discharging Rules requires v to send charge. Now, by Lemma 5.20, v and x are in the head positions of an open 3-cluster, C_0, and one of the following holds:

- C_0 is crowded. In this case, v receives $\frac{1}{24}$ from C_0 by Rule 5.
- There exists a closed 3-cluster or 4^+-cluster at distance-3 from w. In this case, v receives $\frac{1}{24}$ from w by Rule 7a.

- There exists an open 3-cluster, C, such that the tail position of C is in D and u, v or w is in the hand position on the finless side of C. In this case, v receives $\frac{1}{24}$ from u, w or C by Rule 7b.
- There exists a leaf, ℓ, of a 4^+-cluster, C, at distance-2 from u such that u is not in a one-turn position or a backwards position of a linear 4-cluster or a curved 4-cluster, respectively, and u is the only vertex in $D \setminus C$ at distance-2 from ℓ. In this case, v receives $\frac{1}{24}$ from u by Rule 7c.
- There exists a leaf, ℓ, of a closed 3-cluster, C, at distance-2 from u such that u is in a foot or arm position and u is the only vertex in $D \setminus C$ at distance-2 from ℓ. Furthermore, either C is type-2 paired or C has at most 6 nearby poor 1-clusters. In this case, v receives $\frac{1}{24}$ from u by Rule 7c.
- There exists an open 3-cluster, C, such that v or w is in a hand position of C and the hand and arm positions on the other side of C are in D. In this case, v receives $\frac{1}{24}$ from C or w by Rule 7d.
- There exists an open 3-cluster, C, such that u is in a foot position and C is type-1 paired on top. In this case, v receives $\frac{1}{24}$ from u by Rule 7e.

Therefore, v receives at least $\frac{1}{24}$ from a nearby cluster by Discharging Rules 5 and 7. Therefore, $f(v) = f_7(v) \geq f_1(v) + \frac{1}{24} = \frac{9}{24} + \frac{1}{24} = \frac{5}{12}$. □

Proposition 6.6. *If an open* 3*-cluster, C, is neither type*-1 *paired nor type*-2 *paired and* $f_2(C) \geq \frac{34+P(C)}{24}$, *then* $f(C) \geq 3 \cdot \frac{5}{12}$.

Proof. Let C be an open 3-cluster that is neither type-1 nor type-2 paired. Note that Rule 3b requires C to send at most $P(C) \cdot \frac{1}{24}$, Rule 5 requires C to send at most $\frac{2}{24}$ and Rules 7b and 7d each require C to send at most $\frac{1}{24}$ (Lemma 5.5); therefore, C sends at most $\frac{P(C)+4}{24}$ by Rules 3-7. Therefore, if $f_2(C) \geq \frac{34+P(C)}{24}$, then $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. □

Claim 6.7. *For every open* 3*-cluster,* C, $f(C) \geq 3 \cdot \frac{5}{12}$.

Proof. Consider an open 3-cluster, C. Then, by Discharging Rule 1, the middle vertex of C must send $\frac{5}{12}$. Each of the leaf vertices has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6); therefore, each of the leaf vertices must send at most $\frac{5}{12 \cdot 2} + \frac{5}{12} = \frac{15}{24}$ by Rule 1. Thus, $f_1(C) \geq 3 - \frac{5}{12} - 2 \cdot \frac{15}{24} = \frac{4}{3}$.

First, suppose C is type-1 paired on top (see Figure 4.3b). Now, the shoulder position of C is in D or the arm position is in D (Proposition 3.6). Suppose exactly one is in D, and let v be this vertex. Then, C has exactly 2 distance-2 vertices in D; therefore, $f_1(C) = \frac{4}{3}$ and Rule 2 does not apply. By Rule 3b, C does not send charge to v even if $v \in D_1^p$, but C may need to send charge to the poor 1-cluster, u, in the foot position. Therefore, $f_3(C) \geq f_1(C) - \frac{1}{24} = \frac{31}{24}$. Now, C is not type-2 paired (Corollary 4.10); therefore, Rule 4 does not apply. Since at least one of the shoulder positions of C is not in D, Rule 5 does not apply (Lemma 5.24). Since the tail position of a paired 3-cluster is not in D, Rule 7b does not apply. At least one of the hand positions of C is not in D; therefore, Rule 7d does not apply. By Rule 7e, C may need to send $\frac{1}{24}$ to u; therefore, $f(C) = f_7(C) \geq f_3(C) - \frac{1}{24} \geq \frac{31}{24} - \frac{1}{24} = 3 \cdot \frac{5}{12}$.

Now, suppose C is type-1 paired on top and both the shoulder position and the arm position of C are in D. Then, $f_1(C) = 3 - 3 \cdot \frac{5}{12} - \frac{5}{12 \cdot 2} - \frac{5}{12 \cdot 3} = \frac{101}{72}$. Let v and w be the shoulder and arm positions of C, respectively. If v and w are 1-clusters, then C sends $\frac{1}{72}$ to both by Discharging Rule 2a. If exactly one of v and w is a 1-cluster, say v, then C sends $\frac{1}{36}$ to v by Rule 2b. In both cases, C sends a total of $\frac{1}{36}$; thus, $f_2(v) \geq f_1(v) - \frac{1}{36} = \frac{33}{24}$. By Rule 3b, C may need to send $\frac{1}{24}$ to the poor 1-cluster, u, in the foot position; therefore, $f_3 \geq \frac{32}{24}$. Again, Rules 4-7d do not apply. By Rule 7e, C may need to send $\frac{1}{24}$ to u; therefore, $f(C) = f_7(C) \geq f_3(C) - \frac{1}{24} \geq \frac{32}{24} - \frac{1}{24} = \frac{31}{24} > 3 \cdot \frac{5}{12}$.

Suppose C is type-2 paired (see Figure 4.3c). One of the shoulder positions of C is in D. On the other side of C, the shoulder position is in D or the arm position is in D (Proposition 3.6). Let v and w be the shoulder and arm positions of C, respectively. Then, $w \in D_1^{np}$ or $w \in D_1^p$ or $w \in D_{3+}$ or $w \notin D$. If $w \in D_1^{np}$ and $v \notin D$, then $f_1(C) = \frac{32}{24}$. Now, C is not type-1 paired (Corollary 4.10); therefore, C may need to send charge to the shoulder position on the side opposite w by Rule 3b. Then, $f_3(C) \geq \frac{31}{24}$. By Rule 4, C sends $\frac{1}{24}$ to the closed 3-cluster with which C is type-2 paired. Then, $f_4(C) \geq \frac{30}{24}$. Rule 5 does not apply (Lemma 5.24). And Rules 7b, 7c and 7e do not apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \in D_1^{np}$ and $v \in D$, then $f_1(C) = \frac{101}{72}$. By Rule 2, C may need to send $\frac{1}{36}$ to distance-2 non-poor 1-clusters. Then, $f_2(C) \geq \frac{33}{24}$.

By Rule 3b, C sends at most $\frac{1}{24}$. By Rule 4, C sends $\frac{1}{24}$. Since $v \in D_1^{np} \cup D_{3+}$, at least one of the head positions of C is not a very poor 1-cluster; therefore, C sends at most $\frac{1}{24}$ by Rule 5. Rules 6-7 do not apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. Now, if $w \in D_1^p$, then $v \notin D$ (Corollary 4.2) and $f_1(C) = \frac{32}{24}$. Then, C sends at most $\frac{1}{24}$ by Rule 3b, and C sends $\frac{1}{24}$ by Rule 4. Rules 5-7 do not apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \in D_{3+}$ and $v \notin D$, then $f_1(C) = \frac{32}{24}$. By Rule 3b, C sends at most $\frac{1}{24}$. If the hand position adjacent to w is in D, then C does not send charge by Rule 4 but C may need to send charge by Rule 7d. If the hand position adjacent to w is not in D, then C must send charge by Rule 4 but not by Rule 7d. Therefore, C sends charge by at most one of Rules 4 and 7d. No other rules apply. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \in D_{3+}$ and $v \in D$, then $f_1(C) = \frac{101}{72}$. By Rule 2, C sends at most $\frac{1}{36}$ to v. Then, $f_2(C) \geq \frac{33}{24}$. By Rule 3b, C sends at most $\frac{1}{24}$. Therefore, $f_3(C) \geq \frac{32}{24}$. Now, $v \in D_1^{np} \cup D_{3+}$ and v is distance-2 from a head position of C; therefore, at most one of the head positions of C is a very poor 1-cluster. Then, C sends at most $\frac{1}{24}$ by Rule 5. Again, C sends charge by at most one of Rules 4 and 7d. Therefore, $f(C) \geq \frac{30}{24} = 3 \cdot \frac{5}{12}$. If $w \notin D$, then $v \in D$ (Proposition 3.6). Then, $f_1(C) = \frac{32}{24}$ and both shoulder positions of C are in D. If both are poor 1-clusters, then C sends at most $\frac{2}{24}$ by Rule 3b and sends no charge by other rules. Therefore, $f(C) \geq 3 \cdot \frac{5}{12}$. If one is not a poor 1-cluster, then at most one of the head positions of C is a very poor 1-cluster. Therefore, C sends at most $\frac{1}{24}$ by Rule 3b and at most $\frac{1}{24}$ by Rule 5, and C sends no charge by other rules. Therefore, $f(C) \geq \frac{30}{24}$. If neither is a poor 1-cluster, then C does not send charge by any rule; therefore, $f(C) = \frac{32}{24} > 3 \cdot \frac{5}{12}$.

Suppose C is neither type-1 paired nor type-2 paired and $P(C) = 2$. Then, 4 and 7e do not apply and $f_1(C) = \frac{32}{24}$. Since $P(C) = 2$, there exists no one-third vertex adjacent to C; therefore, Rule 2 does not apply. Now, C may need to send charge by Rules 3b, 5, 7b, and 7d. If C must send charge by Rule 5, then both shoulder positions are in D (Lemma 5.24); therefore, the arm positions and the tail position of C are not in D and, hence, Rules 7b and 7d do not apply. If C must send charge by Rule 7b, then the tail position of C is in D; therefore, the arm and shoulder positions of C are not in D and, hence, Rules 5 and 7d do not apply. If C must send charge by Rule 7d, then an arm position of C is in D. Since each leaf must have at least one distance-2 vertex in D (Proposition 3.6) and $P(C) = 2$, the tail position of C is not in D and at least one of the shoulder positions is not in D; therefore, Rules 5 and 7b do not apply. Thus, we have shown that C sends charge by at most one of Rules 5, 7b and 7d. First, suppose C sends by none of Rules 5, 7b and 7d. Then, C is uncrowded and the tail position of C is not in D; therefore, C has exactly 2 distance-2 poor 1-clusters. Then, C sends $\frac{2}{24}$ by Rule 3b and no charge by any other rule; therefore, $f(C) = \frac{30}{24}$. Suppose C sends charge by Rule 5. Then, both shoulder positions of C are in D (Lemma 5.24), and C is crowded; therefore, one of the shoulder positions is not a poor 1-cluster. But then one of the head positions of C is distance-2 from a non-poor 1-cluster or 3^+-cluster; therefore, there exists only one very poor 1-cluster in a head position of C. Then, C sends $\frac{1}{24}$ by 3b and $\frac{1}{24}$ by Rule 5; therefore, $f(C) = \frac{30}{24}$. Now suppose C sends charge by Rule 7b. Then, the tail position of C is in D. Since $P(C) = 2$, the tail position is the only vertex in D at distance-2 from C; therefore, there exists at most one distance-2 poor 1-cluster. Then, $f_3(C) \geq \frac{31}{24}$ and Rule 7b requires C to send at most $\frac{1}{24}$; therefore, $f(C) \geq \frac{30}{24}$. Finally, suppose C sends charge by Rule 7d. Then, the hand and arm positions on one side of C are in D; therefore, at least one of the vertices in D at distance-2 from C is not a poor 1-cluster. Then, $f_3(C) \geq \frac{31}{24}$ and, by Rule 7d, C sends at most $\frac{1}{24}$; therefore, $f(C) \geq \frac{30}{24}$.

Suppose C is neither type-1 paired nor type-2 paired and $P(C) = 3$. If C is adjacent to no one-third vertices, then $f_1(C) = 3 - 3 \cdot \frac{5}{24} - 2 \cdot \frac{5}{12} = \frac{37}{24}$; therefore, $f(C) \geq \frac{30}{24}$ (Proposition 6.6). If C is adjacent to a one-third vertex, then $f_1(C) = \frac{101}{72}$ and $f_2(C) \geq \frac{99}{72} = \frac{33}{24}$. Since C is adjacent to a one-third vertex and $P(C) = 3$, there exists at most one distance-2 poor 1-cluster; therefore, $f_3(C) \geq \frac{32}{24}$. First, suppose the tail position of C is in D. Then, a foot position is also in D and no other distance-2 vertices are in D; therefore, Rules 5 and 7d do not apply. By Rule 7b, C sends at most $\frac{1}{24}$; therefore, $f(C) \geq \frac{31}{24}$. Now, suppose the tail position of C is not in D; therefore, Rule 7b does not apply. If C does not send charge by Rule 5, then C sends at most by Rule 7d and $f(C) \geq \frac{31}{24}$. If C sends charge by Rule 5, then both shoulder positions of C are in D (Lemma 5.24). However, C is adjacent to a one-third vertex; therefore, one of the shoulder positions is not a poor 1-cluster. Then, at most one of the head positions is a very poor 1-cluster. Therefore, C sends at most $\frac{1}{24}$ by Rule 5 and at most $\frac{1}{24}$ by Rule 7d. Therefore, $f(C) \geq \frac{30}{24}$.

Suppose C is neither type-1 paired nor type-2 paired and $P(C) = 4$. First, suppose C is adjacent to no one-third vertices. Then, $f_1(C) = \frac{42}{24}$; therefore, $f(C) \geq \frac{30}{24}$ (Proposition 6.6). Now, suppose C is adjacent to exactly one one-third vertex. Then, $f_1(C) = \frac{116}{72}$. By Rule 2, C sends at most $\frac{1}{36}$; therefore,

$f_2(C) = \frac{114}{72} = \frac{38}{24}$. Therefore, $f(C) \geq \frac{30}{24}$ (Proposition 6.6). Now, suppose C is adjacent to exactly 2 one-third vertices. Then, $f_1(C) = \frac{53}{36}$. By Rule 2, C sends at most $2 \cdot \frac{1}{36}$; therefore, $f_2(C) \geq \frac{51}{36} = \frac{34}{24}$. Since C is adjacent to 2 one-third vertices and $P(C) = 4$, there exist no distance-2 poor 1-clusters; therefore, $f_3(C) \geq f_2(C) \geq \frac{34}{24}$. In total, Rules 5-7 require C to send at most $\frac{4}{24}$; therefore, $f(C) \geq \frac{30}{24}$.

Suppose C is neither type-1 paired nor type-2 paired and $P(C) \geq 5$. Then, $f_2(C) \geq \frac{39}{24}$. Now, an open 3-cluster has at most 4 distance-2 poor 1-clusters. Therefore, C sends at most $\frac{4}{24}$ by Rule 3b. By Rules 5-7, C sends at most $\frac{4}{24}$. Therefore, $f(C) \geq \frac{31}{24}$. □

Proposition 6.8. *A closed 3-cluster or 4^+-cluster sends charge to a distance-3 poor 1-cluster by at most one of Rules 3c, 6 and 7a.*

Proof. Let C_1 be closed 3-cluster or 4^+-cluster. If a poor 1-cluster, v, is distance-2 from a very poor 1-cluster in a symmetric orientation, then v is distance-2 from exactly one very poor 1-cluster. If v is in the u-position or w-position of a very poor 1-cluster in an asymmetric orientation, then v is distance-2 from exactly one very poor 1-cluster. Therefore, C_1 sends charge to a poor 1-cluster by at most one of Rules 6 and 7a. If C_1 sends charge to a poor 1-cluster, a, by Rule 6, then a is distance-2 from a very poor 1-cluster in a symmetric orientation and in a shoulder position of an open 3-cluster, C_2 (Lemma 5.17). Now, C_2 is not type-1 paired on top (Corollary 5.18); therefore, a receives charge from C_2 by Rule 3b and not from C_1 by Rule 3c. Therefore, C_1 sends charge to a by at most one of Rules 3c and 6. If C_1 sends charge to a poor 1-cluster, w, by Rule 7a, then w is in the w-position of a very poor 1-cluster in an asymmetric orientation; therefore, w is in the shoulder position of an open 3-cluster, C_0 (Lemma 5.20). Now, C_0 is not type-1 paired on top (Corollary 5.21); therefore, w receives charge from C_0 by Rule 3b and not from C_1 by Rule 3c. Therefore, C_1 sends charge to w by at most one of Rules 3c and 7a. □

Proposition 6.9. *If C is a closed 3-cluster and $f_2(C) \geq \frac{43}{24}$, then $f(C) \geq 3 \cdot \frac{5}{12}$.*

Proof. By Rules 3c, 6 and 7a, C sends at most $\frac{1}{24}$ to each distance-3 poor 1-cluster (Proposition 6.8). By Rule 3a, C sends at most $\frac{1}{24}$ to each distance-2 poor 1-cluster. Now, C has at most 11 nearby poor 1-clusters (Lemma 5.16); therefore, by Rules 3-7a, C sends at most $\frac{11}{24}$. By Rule 7c, C sends at most $\frac{2}{24}$ to distance-2 poor 1-clusters. Therefore, C sends at most $\frac{13}{24}$ by Rules 3-7. □

Corollary 6.10. *A closed 3-cluster sends at most $\frac{11}{24}$ by Rules 3-7a and at most $\frac{2}{24}$ by Rule 7c.*

Claim 6.11. *For every closed 3-cluster, C, $f(C) \geq 3 \cdot \frac{5}{12}$.*

Proof. Consider a closed 3-cluster, C_1, and let $P(C_1) = 3$. Then, $f_1(C_1) = \frac{37}{24}$. By Lemma 5.6, C_1 has at most 8 nearby poor 1-clusters; however, if C_1 has 8 such clusters, at least one of the poor 1-clusters at distance-3, v, is distance-2 from another 3^+-cluster, C_2, such that

(a) if C_2 is an open 3-cluster, then v is not in a shoulder position;

(b) if C_2 is an open 3-cluster and v is in an arm position, then C_2 is not type-1 paired; if C_2 is type-2 paired, then C_1 is type-2 paired with C_2.

Therefore, v receives charge from C_2 by Rules 3a-3b and not from C_1 by Rule 3c; additionally, C_1 does not send charge to v by Rules 6 and 7a. Then, C_1 sends at most $\frac{7}{24}$ by Rules 3, 6 and 7a (Proposition 6.8). If C_1 sends charge by Rule 7c, then C_1 has at most 6 nearby poor 1-clusters or C_1 is type-2 paired. A poor 1-cluster, u, receives charge by Rule 7c only if u is the only vertex in $D \setminus C_1$ at distance-2 from a leaf of C_1 and u is not in a shoulder or tail position; therefore, C_1 sends at most $\frac{2}{24}$ by Rule 7c. If C_1 sends $\frac{2}{24}$ by Rule 7c, then the shoulder positions and the tail position of C_1 are not in D; therefore, C_1 has at most 5 nearby poor 1-clusters (Lemma 5.6). Then, C sends at most $\frac{5}{24}$ by Rules 3, 6 and 7a (Proposition 6.8) and $\frac{2}{24}$ by Rule 7c, and $f(C_1) \geq \frac{30}{24}$. If C_1 sends $\frac{1}{24}$ by Rule 7c and C_1 has at most 6 nearby poor 1-clusters, then C_1 sends at most $\frac{6}{24}$ by Rules 3, 6 and 7a (Proposition 6.8) and $\frac{1}{24}$ by Rule 7c, and $f(C_1) \geq \frac{30}{24}$. If C_1 sends $\frac{1}{24}$ by Rule 7c and C_1 is type-2 paired with the open 3-cluster, C_2, then the argument is identical to the previous case unless C_1 has 7 nearby poor 1-clusters. In this case, the arm position of C_2 is in D and the hand position on the same side is not in D (Lemma 5.26); therefore, C_1 receives $\frac{1}{24}$ from C_2 by Rule 4. Then, C_1 sends at most $\frac{7}{24}$ by Rules 3, 6 and 7a (Proposition 6.8) and $\frac{1}{24}$ by Rule 7c; however, C_1 receives $\frac{1}{24}$ by Rule 4 and, therefore, $f(C_1) \geq \frac{30}{24}$.

Consider a closed 3-cluster, C, and let $P(C) = 4$. Then, either C is adjacent to no one-third vertices or C is adjacent to exactly one one-third vertex. In the former case, $f_1(C) = \frac{42}{24}$; and since C is not adjacent to any one-third vertices, $f_2(C) = \frac{42}{24}$. Now, C sends at most $\frac{11}{24}$ by Rules 3-7a and at most $\frac{2}{24}$ by Rule 7c (Corollary 6.10). Since C is adjacent to no one-third vertices, C is closed by a single vertex; therefore, since $P(C) = 4$, one of the leaves of C is distance-2 from 2 vertices in $D \setminus C$; therefore, C sends at most $\frac{1}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{30}{24}$. Now suppose C is adjacent to a one-third vertex. Then, $f_2(C) \geq \frac{38}{24}$. Now, C has at most 8 nearby poor 1-clusters (Lemma 5.7); therefore, C sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 6.8). Therefore, if C sends no charge by Rule 7c, then $f(C) \geq \frac{30}{24}$. If C sends $\frac{1}{24}$ by Rule 7c, then an arm position or foot position of C is a poor 1-cluster. But then C has at most 7 nearby poor 1-clusters (Lemma 5.7) and $f_{7a} \geq \frac{31}{24}$; therefore, $f(C) \geq \frac{30}{24}$. If C sends $\frac{2}{24}$ by Rule 7c, then 2 arm or foot positions are poor 1-clusters. But then C has at most 6 nearby poor 1-clusters (Lemma 5.7) and $f_{7a} \geq \frac{32}{24}$; therefore, $f(C) \geq \frac{30}{24}$.

Let $P(C) = 5$. Then, either C is adjacent to no one-third vertices, one one-third vertex or 2 one-third vertices. In the first case, $f_2(C) = \frac{47}{24}$; therefore, $f(C) \geq \frac{30}{24}$ (Proposition 6.9). In the second case, $f_2(C) \geq \frac{43}{24}$; therefore, $f(C) \geq \frac{30}{24}$ (Proposition 6.9). In the last case, $f_2(C) \geq \frac{39}{24}$. Now, C has at most 11 nearby clusters (Lemma 5.16). Since C is adjacent to 2 one-third vertices, at least 3 of these clusters are not poor 1-clusters; additionally, at least one of the leaves of C is distance-2 from more than one vertex in $D \setminus C$. Therefore, C sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 6.8) and at most $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{30}{24}$.

Let $P(C) \geq 6$. If $f_2(C) \geq \frac{43}{24}$, then $f(C) \geq \frac{30}{24}$ (Proposition 6.9). If $f_2(C) < \frac{43}{24}$, then C is adjacent to at least 3 one-third vertices. Therefore, at least 5 of the clusters nearby C are not poor 1-clusters. Therefore, C has at most 6 nearby poor 1-clusters (Lemma 5.16). Then, C sends at most $\frac{6}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$. Therefore, C sends at most $\frac{8}{24}$. But if $P(C) \geq 6$, then $f_2(C) \geq \frac{40}{24}$; therefore, $f(C) \geq \frac{32}{24}$. $\square$

Now we begin our discussion of 4^+-clusters. For $k \geq 4$, let C be a k-cluster, and let v be a vertex in C. Then, $d_C(v) \in \{1, 2, 3\}$. Let

$$\alpha_i = |\{v \in C : d_C(v) = i\}|$$

Now, C has at most $k + 8$ nearby poor 1-clusters (Lemma 5.16); therefore, C sends at most $\frac{k+8}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{1}{24}$ for each leaf of C. Now, the number of leaves of C is α_1; therefore, C sends at most $\frac{1}{24}\alpha_1$ by Rule 7c. Rules 1 and 2 are the only others by which C may need to send charge; therefore, $f(C) \geq f_2(C) - \frac{1}{24}[(k + 8) + \alpha_1]$. Now, $f_2(C)$ is minimal when $P(C) = 2$; that is, $f_2(C) \geq k - (\frac{5}{12} + \frac{5}{24})\alpha_1 - \frac{5}{12}\alpha_2$. Let $F(C) = f(C) - \frac{5}{12}k$. Then,

$$F(C) \geq \left[k - \left(\frac{5}{12} + \frac{5}{24}\right)\alpha_1 - \frac{5}{12}\alpha_2\right] - \frac{1}{24}\left[(k+8) + \alpha_1\right] - \frac{5}{12}k$$

Now, $k = \alpha_1 + \alpha_2 + \alpha_3$. Then, substituting and simplifying,

$$F(C) \geq \frac{1}{24}\left(-3\alpha_1 + 3\alpha_2 + 13\alpha_3 - 8\right)$$

Now, $\Delta(C) = 3$; therefore, $\alpha_1 \leq \alpha_3 + 2$. Then,

$$F(C) \geq \frac{1}{24}\left[-3\left(\alpha_3 + 2\right) + 3\alpha_2 + 13\alpha_3 - 8\right] = \frac{1}{24}\left(3\alpha_2 + 10\alpha_3 - 14\right) \tag{6.1}$$

Now, $F(C) < 0$ if, and only if, $f(C) < \frac{5}{12}k$. Let $A = \{(0,0), (1,0), (2,0), (3,0), (4,0), (0,1), (1,1)\}$. If $F(C) < 0$, then $(\alpha_2, \alpha_3) \in A$. That is, for all $(\alpha_2, \alpha_3) \notin A$, Equation 6.1 implies $f(C) \geq \frac{5}{12}k$. Therefore, we have only left to consider the cases in which $(\alpha_2, \alpha_3) \in A$.

If $(\alpha_2, \alpha_3) \in \{(0,0), (1,0)\}$, then $C \in \mathcal{K}_1 \cup \mathcal{K}_3$. But we assumed $C \in \mathcal{K}_{4^+}$; therefore, we need not consider this case. If $(\alpha_2, \alpha_3) \in \{(2,0), (0,1)\}$, then $C \in \mathcal{K}_4$; we consider this case in Claim 6.12. If $(\alpha_2, \alpha_3) \in \{(3,0), (1,1)\}$, then $C \in \mathcal{K}_5$; we consider this case in Claim 6.13. Finally, if $(\alpha_2, \alpha_3) = (4,0)$, then $C \in \{L \in \mathcal{K}_6 : \Delta(L) = 2\}$; we consider this case in Claim 6.14.

Claim 6.12. *For every 4-cluster, C, $f(C) \geq 4 \cdot \frac{5}{12}$.*

Proof. First, consider a linear 4-cluster, C, and let $P(C) = 2$. Then, $f_1(C) = \frac{46}{24}$. Rule 2 does not apply; therefore, $f_2(C) = \frac{46}{24}$. First suppose C sends no charge by Rule 7c. Then, C has at most 8 nearby poor 1-clusters; however, if C has k nearby poor 1-clusters, where $k > 6$, then $k - 6$ of the distance-3 poor 1-clusters are stealable (Lemma 5.8) – that is, $k - 6$ of the nearby poor 1-clusters will receive charge from other 3^+-clusters by Rules 3a-3b and not from C by Rules 3c, 6 or 7a. Therefore, C sends charge to at most 6 nearby poor 1-clusters. Then, C sends at most $\frac{6}{24}$ (Proposition 6.8) and, therefore, $f(C) \geq \frac{40}{24} = 4 \cdot \frac{5}{12}$. Now suppose C sends $\frac{1}{24}$ by Rule 7c. Then, one one-turn position is not in D; therefore, C has at most 6 nearby poor 1-clusters and if C has exactly 6 such clusters then at least one of the distance-3 poor 1-clusters is stealable (Lemma 5.8) – that is, at least one of the distance-3 poor 1-clusters will receive charge by Rules 3a-3b and not from C by Rules 3c, 6 or 7a. Therefore, C sends charge to at most 5 nearby poor 1-clusters by Rules 3, 6 and 7a. Then, C sends at most $\frac{5}{24}$ by Rules 3, 6 and 7a (Proposition 6.8) and $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. Finally, suppose C sends $\frac{2}{24}$ by Rule 7c. Then, neither one-turn position is in D; therefore, C has at most 4 nearby poor 1-clusters. Then, C sends at most $\frac{4}{24}$ by Rules 3, 6 and 7a (Proposition 6.8) and $\frac{2}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$.

Let $P(C) = 3$. First, suppose C is adjacent to no one-third vertices. Then, $f_2(C) = \frac{51}{24}$. Now, C has at most 9 nearby poor 1-clusters (Lemma 5.9); therefore, C sends at most $\frac{9}{24}$ by Rules 3-7a (Proposition 6.8). And C sends at most $\frac{2}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. Now, suppose C is adjacent to a one-third vertex, $v_{\frac{1}{3}}$. Then, $f_2(C) \geq \frac{47}{24}$. Now, C has at most 6 nearby poor 1-clusters (Lemma 5.9); therefore, C sends at most $\frac{6}{24}$ by Rules 3-7a (Proposition 6.8). Since $P(C) = 3$, one of the leaves of C must be adjacent to $v_{\frac{1}{3}}$; therefore, one of the leaves of C has more than one distance-2 vertex in $D \setminus C$. Then, C sends at most $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$.

Now, consider a curved 4-cluster, C, and let $P(C) = 2$. Then, $f_1(C) = \frac{46}{24}$. Rule 2 does not apply; therefore, $f_2(C) = \frac{46}{24}$. First, suppose C sends no charge by Rule 7c. Then, C has at most 8 nearby poor 1-clusters; however, if C has k such clusters, where $k > 6$, then at least $k - 6$ of the distance-3 poor 1-clusters are stealable (Lemma 5.10) – that is, C sends charge to at most 6 nearby poor 1-clusters by Rules 3-7a. Then, C sends no charge by Rule 7c and at most $\frac{6}{24}$ by Rules 3-7a (Proposition 6.8); therefore, $f(C) \geq \frac{40}{24}$. Now, suppose C sends $\frac{1}{24}$ by Rule 7c. Then, one backwards position of C is not in D; therefore, C has at most 6 nearby poor 1-clusters, and if C has 6 such clusters then at least one is stealable (Lemma 5.10) – that is, C sends at most $\frac{5}{24}$ by Rules 3-7a (Proposition 6.8). Therefore, $f(C) \geq \frac{40}{24}$. Finally, suppose C sends $\frac{2}{24}$ by Rule 7c. Then, neither backwards position of C is in D; therefore, C has at most 2 nearby poor 1-clusters (Lemma 5.10). Then, C sends at most $\frac{2}{24}$ by Rules 3-7a (Proposition 6.8); therefore, $f(C) \geq \frac{42}{24}$.

Let $P(C) = 3$. First, suppose C is adjacent to no one-third vertices. Then, $f_2(C) = \frac{51}{24}$. Now, C has at most 11 nearby poor 1-clusters (Lemma 5.11); therefore, if C sends no charge by Rule 7c, then $f(C) \geq \frac{40}{24}$ (Proposition 6.8). If C sends $\frac{1}{24}$ by Rule 7c, then one backwards position of C is not in D; therefore, C has at most 10 nearby poor 1-clusters (Lemma 5.11). Then, C sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 6.8) and $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. If C sends $\frac{2}{24}$, then both backwards positions are not in D; therefore, C has at most 9 nearby poor 1-clusters (Lemma 5.11). Then, C sends at most $\frac{9}{24}$ by Rules 3-7a (Proposition 6.8); therefore, $f(C) \geq \frac{40}{24}$. Now, suppose C is adjacent to a one-third vertex, $v_{\frac{1}{3}}$. Then, $f_2(C) \geq \frac{47}{24}$. Since $P(C) = 3$ and each leaf of C has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6), $v_{\frac{1}{3}}$ is adjacent to one of the leaves of C; therefore, C sends at most $\frac{1}{24}$ by Rule 7c. Now, C has at most 6 nearby poor 1-clusters (Lemma 5.11). Therefore, C sends at most $\frac{6}{24}$ by Rules 3-7a (Proposition 6.8) and at most $\frac{1}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$.

Consider a linear or curved 4-cluster, C, and let $P(C) \geq 4$. First, suppose C is adjacent to no one-third vertices. Then, $f_2(C) \geq \frac{56}{24}$. Now, C has at most 12 nearby poor 1-clusters (Lemma 5.16); therefore, C sends at most $\frac{12}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$; therefore, $f(C) \geq \frac{42}{24}$. Now, suppose C is adjacent to exactly one one-third vertex. Then, $f_2(C) \geq \frac{52}{24}$. Now, C has at most 12 nearby clusters (Lemma 5.16). However, since C is adjacent to a one-third vertex, at least 2 of these clusters are not poor 1-clusters; therefore, C has at most 10 nearby poor 1-clusters. Then, C sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 6.8) and at most $\frac{2}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. Finally, suppose C is adjacent to 2 one-third vertices. Now, if $P(C) = 4$, then each leaf is adjacent to a one-third vertex and $f_2(C) \geq \frac{48}{24}$. Then, at least 4 of the 12 possible nearby clusters are not poor 1-clusters; therefore, C sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 6.8). Since both leaves have more than one distance-2 vertex in $D \setminus C$, no charge is sent by Rule 7c; therefore, $f(C) \geq \frac{40}{24}$. If $P(C) \geq 5$ and C is adjacent to 3 one-third vertices, then $f_2(C) \geq \frac{49}{24}$

and at least 5 of the 12 possible nearby clusters are not poor 1-clusters; therefore, C sends at most $\frac{7}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$; therefore, $f(C) \geq \frac{40}{24}$. If $P(C) \geq 5$ and C is adjacent to exactly 2 one-third vertices, then $f_2(C) \geq \frac{53}{24}$. Now, at least 3 of the 12 possible nearby clusters are not poor 1-clusters; therefore, C sends at most $\frac{9}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$; therefore, $f(C) \geq \frac{42}{24}$.

Consider a 4-cluster, C, and let C have a degree-3 vertex. First, suppose $P(C) = 3$. Then, $f_2(C) = \frac{51}{24}$. Now, C has at most 8 nearby poor 1-clusters; therefore, C sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 6.8). Since C has 3 leaves, C sends at most $\frac{3}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{40}{24}$. Now, suppose $P(C) \geq 4$. If C is adjacent to a one-third vertex, $v_{\frac{1}{3}}$, then $f_2(C) \geq \frac{52}{24}$ and at least 2 of the 12 possible nearby clusters (Lemma 5.16) are not poor 1-clusters; therefore, C sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 6.8). From the structure of C, we see that $v_{\frac{1}{3}}$ must be adjacent to a leaf of C; therefore, at least one of the leaves of C has more than one distance-2 vertex in $D \setminus C$. Therefore, C sends at most $\frac{2}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{40}{24}$. If C is adjacent to no one-third vertices, then $f_2(C) \geq \frac{56}{24}$. Now, C has at most 12 nearby poor 1-clusters (Lemma 5.16), and C sends at most $\frac{3}{24}$ by Rule 7c; therefore, $f(C) \geq \frac{41}{24}$ (Proposition 6.8). □

Claim 6.13. *For every* 5*-cluster, C, $f(C) \geq 5 \cdot \frac{5}{12}$.*

Proof. Consider a 5-cluster, C with $\Delta(C) = 2$. If $C \in \mathcal{K}_5^c$, then $f_2(C) \geq \frac{65}{24}$. Now, C has at most 13 nearby poor 1-clusters (Lemma 5.16); therefore, C sends at most $\frac{13}{24}$ by Rules 3-7a (Proposition 6.8). Since C has exactly 2 leaves, C sends at most $\frac{2}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{50}{24} = 5 \cdot \frac{5}{12}$. If $C \in \mathcal{K}_5^o$, then $f_2(C) \geq \frac{60}{24}$. Now, C has at most 9 nearby poor 1-clusters; furthermore, if C has exactly 9 such clusters, then at least one is stealable (Lemma 5.12) – that is, C sends at most $\frac{8}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{50}{24}$.

Now, let C have a degree-3 vertex. Then, $f_2(C) \geq \frac{65}{24}$. Now, C has at most 12 nearby poor 1-clusters (Lemma 5.14); therefore, C sends at most $\frac{12}{24}$ by Rules 3-7a (Proposition 6.8). Since C has 3 leaves, C sends at most $\frac{3}{24}$ by Rule 7c. Therefore, $f(C) \geq \frac{50}{24}$. □

Claim 6.14. *For every* 6*-cluster, C, with $\Delta(C) = 2$, $f(C) \geq 6 \cdot \frac{5}{12}$.*

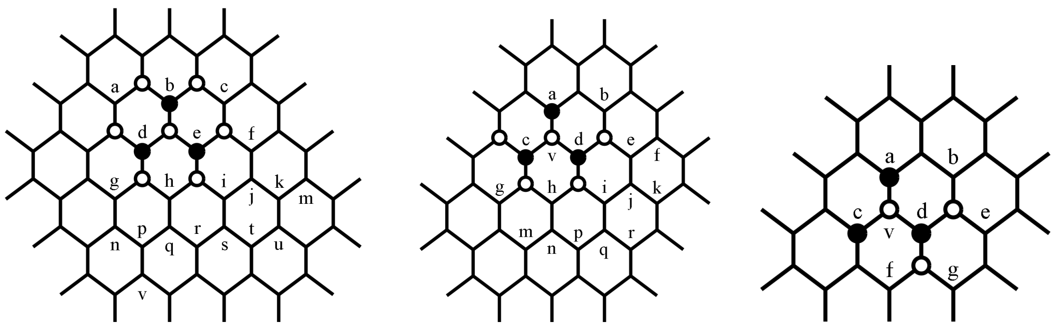
Proof. Consider a 6-cluster, C with $\Delta(C) = 2$. Then, C has exactly 2 leaves. If $C \in \mathcal{K}_6^c$, then $f_2(C) \geq \frac{79}{24}$. Now, C has at most 14 nearby poor 1-clusters; therefore, C sends at most $\frac{14}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{63}{24} > 6 \cdot \frac{5}{12}$. If $C \in \mathcal{K}_6^o$, then $f_2(C) \geq \frac{74}{24}$. Now, C has at most 10 nearby poor 1-clusters (Lemma 5.15); therefore, C sends at most $\frac{10}{24}$ by Rules 3-7a (Proposition 6.8). By Rule 7c, C sends at most $\frac{2}{24}$. Therefore, $f(C) \geq \frac{62}{24} > 6 \cdot \frac{5}{12}$. □

7 Deferred Proofs

Proof of Proposition 5.1. Let H be the group of non-poor 1-clusters described by b, d and e in Figure 7.1a. We choose $i \in D_1^p$. Then, i is distance-2 from e and not distance-2 from the other 1-clusters in H. By symmetry, this is the general case. Since $i \in D_1^p$, we have $j, s \notin D$ and, by Corollary 4.2, $h \notin D$. Therefore, $q \in D_{3+}$ (Proposition 3.4) and $g \in D$ (Proposition 3.5). Let C be the 3^+-cluster at q. If $r \in D$, then $r \in C$; therefore, i is distance-2 from a 3^+-cluster. If $r \notin D$, then $p \in C$. If $n \in D$, then $g, n, p, q \in C$ and C is a 4^+-cluster; therefore, i is distance-3 from a 4^+-cluster. If $n \notin D$, then $v \in D$ and $p, q, v \in C$. Now, g closes C. Therefore, i is distance-3 from a closed 3^+-cluster. □

Proof of Lemma 5.2. Let H be the group of non-poor 1-clusters described by b, d and e in Figure 7.1a. Now, if a poor 1-cluster, w, is distance-2 from exactly one of b, d and e, then w is distance-2 from an open 3-cluster or within distance-3 of a closed 3-cluster or 4^+-cluster (Proposition 5.1). Thus, we need only consider poor 1-clusters which are distance-2 from 2 of the 1-clusters in H. There are 3 possibilities: a, c and h. Suppose by contradiction that each of a, c and h is a poor 1-cluster that is not distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4^+-cluster. Since $h \in D_1^p$, we have $p \in D$ or $r \in D$ but not both (Corollary 4.2). By symmetry, we choose $r \in D$. Now, by hypothesis, h is not distance-2 from any 3^+-cluster; therefore, $r \in D_1$ (Corollary 3.3). So we have $s \notin D$. Since $h \in D_1^p$ and $e \in D$, we also have $i \notin D$ (Corollary 4.2). Therefore, $j \in D_{3+}$ (Proposition 3.4). Also, since $i, s \notin D$ and $r \in D_1$, we have $t \in D$ (Proposition 3.5). Now, $e \in D$ and, by hypothesis, $c \in D_1^p$; therefore, $f \notin D$ (Corollary 4.2). Let C be the

3^+-cluster at j. Since $f, i \notin D$, we have $k \in C$. Now, if $u \in D$, then $j, k, t, u \in C$; therefore, c and h are distance-3 from a 4^+-cluster, which is a contradiction. If $u \notin D$, then $m \in C$ and t closes C; therefore, c and h are distance-3 from a closed 3^+-cluster, which is a contradiction. □



(a) Proposition 5.1 and Lemma 5.2 (b) Lemma 5.3 (c) Lemma 5.4

Figure 7.1

Proof of Lemma 5.3. Let v be the one-third vertex shown in Figure 7.1b, and let c and d be 1-clusters. Then, by hypothesis, $a \in D \setminus D_1$; therefore, $a \in D_{3+}$ (Corollary 3.3). Suppose by contradiction that one of c and d has at least 2 distance-2 poor 1-clusters that are not distance-2 from an open 3-cluster nor within distance-3 of a closed 3-cluster or 4^+-cluster. By symmetry, we consider d. Now, there are 4 candidates for distance-2 poor 1-clusters: b, e, h and i. However, b is distance-2 from the 3^+-cluster at a, so we need not consider b.

If $h \in D_1^p$, then $n \notin D$ and $i \notin D$ (Corollary 4.2). So we have $e \in D_1^p$; therefore, $f, j \notin D$ and $k \in D$ (Proposition 3.5). Since $i, j \notin D$, we have $q \in D_{3+}$ (Proposition 3.4). Let C be the 3^+-cluster at q. If $p \in D$, then $p \in C$ and h is distance-2 from a 3^+-cluster, which is a contradiction. If $p \notin D$, then $r \in C$; therefore, either $k \in C$ and e is distance-2 from a 4^+-cluster, or k closes C and both e and h are distance-3 from a closed 3^+-cluster, which is a contradiction.

If $h \notin D_1^p$, then we have $e, i \in D_1^p$. Therefore, $f, j, q \notin D$ and, by Corollary 4.2, $h \notin D$. Then $n \in D_{3+}$ (Proposition 3.4) and $g \in D$ (Proposition 3.5). Let C be the 3^+-cluster at n. If $p \in D$, then $p \in C$ and i is distance-2 from a 3^+-cluster, which is contradiction. If $p \notin D$, then $m \in C$. Then, either $g \in C$ and i is distance-3 from a 4^+-cluster, or g closes C and i is distance-3 from a closed 3^+-cluster, which is a contradiction. □

Proof of Lemma 5.4. Let v be the one-third vertex shown in Figure 7.1c, and let d be a 1-cluster. Then, by hypothesis, $a, c \in D \setminus D_1$; therefore, $a, c \in D_{3+}$ (Corollary 3.3). Now, d has 6 distance-2 vertices: a, b, c, e, f and g. However, $a, c \in D_{3+}$. If $b \in D_1^p$, then $e \notin D$ (Corollary 4.2); and vice versa. If $f \in D_1^p$, then $g \notin D$ (Corollary 4.2); and vice versa. Therefore, at most 2 of b, e, f and g are poor 1-clusters. □

Proof of Lemma 5.5. Let C be the 3-cluster shown in Figure 4.2a. Suppose by contradiction that C has 2 finless sides; then, $n, p \notin D$ (Definition 4.6). If $j \notin D$, then $p \in D_{3+}$ (Proposition 3.4). But none of the vertices adjacent to p is in D; therefore, $p \in D_1$, which is a contradiction. If $j \in D$ and $p \in D$, then $j, p \in D_2$, which is a contradiction (Proposition 3.2). If $j \in D$ and $p \notin D$, then $j \in D_{3+}$ (Proposition 3.4). But none of the vertices adjacent to j is in D; therefore, $j \in D_1$, which is a contradiction. □

Proof of Lemma 5.6. Let C_1 be the 3-cluster shown in Figure 7.2. If $C_1 \in \mathcal{K}_3^c$, then the non-leaf vertex of C_1 has at least one distance-2 vertex in $D \setminus C$ (Definition 4.3); by symmetry, we choose $f \in D$. Now, $P(C_1) = 3$ and each leaf of C has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6); therefore, $e \notin D$ and

$$|\{d, j, p, q\} \cap D| = |\{g, k, r, q\} \cap D| = 1$$

There are 11 candidates for nearby poor 1-clusters: a, c/d, f, h, i/j, k/m, n, p/t, q/v, r/x and s.

First we consider the cases for which $q \notin D$. To begin, we show that there are at most 9 nearby poor 1-clusters. Now, $v \in D_{3+}$ (Proposition 3.4); therefore, there are at most 10 nearby poor 1-clusters. If $p \in D$, then $n \notin D_1^p$ and there are at most 9 nearby poor 1-clusters. If $p \notin D$ and $t \notin D$, then there are at most 9 nearby poor 1-clusters. If $r \in D$, then $s \notin D_1^p$ and there are at most 9 nearby poor 1-clusters. If $r \notin D$ and $x \notin D$, then there are at most 9 nearby poor 1-clusters. So we consider $p, r \notin D$ and $t, x \in D$. Let C_v be the 3^+-cluster at v. At least one of u and w is in C_v; therefore, at least one of t and x is not a 1-cluster. Therefore, there are at most 9 nearby poor 1-clusters.

Now we consider the cases for which at least one of d and g is in D. If $g \in D$, then $f, h \notin D_1^p$. Therefore, there are at most 7 nearby poor 1-clusters. So now we consider $d \in D$ and $g \notin D$. Either $k \in D$ or $r \in D$ but not both. If $k \in D$, then $s \notin D_1^p$ and there are at most 8 nearby poor 1-clusters. Now, if $k \notin D_1^p$, then there are at most 7 nearby poor 1-clusters. If $k \in D_1^p$, then $s \notin D$. Since $r \notin D$, we have $x \in D_{3+}$ (Proposition 3.4). If $t \notin D_1^p$, then C_1 has at most 7 nearby poor 1-clusters. If $t \in D_1^p$, then t is distance-2 from C_v. If $C_v \in \mathcal{K}_3^o$, then $v, w, x \in C_v$ and t is in an arm position. If C_v is paired, then it is type-2 paired with C_1. Therefore, the lemma holds with $k \in D$. If $r \in D$, then $s \notin D_1^p$ and there are at most 8 nearby poor 1-clusters. If $r \notin D_1^p$, then C_1 has at most 7 nearby poor 1-clusters. If $r \in D_1^p$, then $s \notin D$. Since $k \notin D$, we have $m \in D_{3+}$ (Proposition 3.4); therefore, C_1 has at most 7 nearby poor 1-clusters and the lemma holds.

Now we consider the cases for which neither shoulder position is in D. Since $d \notin D$, we have $a, c \in D_{3+}$. Therefore, C_1 has at most 7 nearby poor 1-clusters. Either $j \in D$ or $p \in D$; in both cases, $n \notin D_1^p$. Therefore, C_1 has at most 6 nearby poor 1-clusters. Either $k \in D$ or $r \in D$; in both cases, $s \notin D_1^p$. Therefore, C_1 has at most 5 nearby poor 1-clusters and the lemma holds.

Now we consider the case for which $q \in D$. If $q \in D$, then $d, g, j, k, p, r \notin D$. Then, $a, c \in D_{3+}$ (Proposition 3.4). Therefore, C_1 has at most 9 nearby poor 1-clusters. If $q \notin D_1^p$, then C_1 has at most 8 nearby poor 1-clusters. If $q \in D_1^p$, then either $u \in D$ or $w \in D$ (Proposition 3.5). If $u \in D$, then $t \notin D_1^p$ and there are at most 8 nearby poor 1-clusters; if $w \in D$, then $x \notin D_1^p$ and there are at most 8 nearby poor 1-clusters. Therefore, C_1 has at most 8 nearby poor 1-clusters. Now, if $i \notin D_1^p$, then C_1 has at most 7 nearby poor 1-clusters. If $i \in D_1^p$, then i is distance-2 from the 3^+-cluster at c, C_c; since $a \in D$, we have $C_c \notin \mathcal{K}_3^o$. $\square$

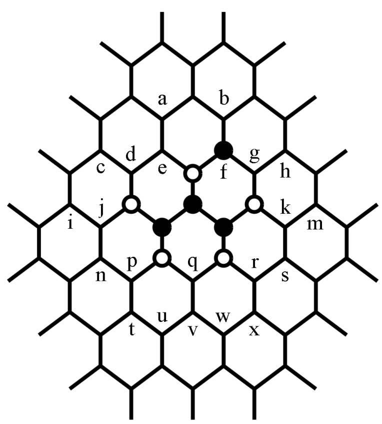


Figure 7.2: Lemma 5.6 and Lemma 5.7

Proof of Lemma 5.7. Let C be the closed 3-cluster shown in Figure 7.2. By symmetry, we choose $f \in D$. There are 5 possible one-third vertices adjacent to C, and there are 11 candidates for nearby poor 1-clusters: a/e, c/d, f, h, i/j, k/m, n, p/t, q/v, r/x and s.

Suppose $e \in D$. Then, $a, e, f \notin D_1^p$; therefore, C has at most 9 nearby poor 1-clusters. Since $P(C) = 4$, each leaf has exactly one distance-2 vertex in $D \setminus C$. If $d, g \in D$, then $c, d, h \notin D_1^p$; therefore, C has at most 7 nearby poor 1-clusters. If $q \in D$, then either $q \in D_1^p$ or $q \notin D_1^p$. If $q \notin D_1^p$, then C has at most 8 nearby poor 1-clusters. If $q \in D_1^p$, then $p, r \notin D$ (Corollary 4.2). If $t \notin D_1^p$ or $x \notin D_1^p$, then C has at most 8 nearby poor 1-clusters. So assume $t, x \in D_1^p$. Since $q \in D_1^p$, we have $u \in D$ or $w \in D$ (Proposition 3.5); therefore, at least one of t and x is not a poor 1-cluster. Therefore, C has at most 8 nearby poor 1-clusters. If $q \notin D$, then $v \in D_{3^+}$ (Proposition 3.4); therefore, C has at most 8 nearby poor 1-clusters. If $j \in D$ or $p \in D$, then $n \notin D_1^p$; and if $k \in D$ or $r \in D$, then $s \notin D_1^p$. Therefore, if one foot or arm position is a poor 1-cluster, then C has at most 7 nearby poor 1-clusters; and if 2 foot or arm positions are poor 1-clusters, then C has at most 6 nearby poor 1-clusters.

Suppose $d, j \in D$. Then $c, d, i, j, n \notin D_1^p$. Therefore, C has at most 8 nearby poor 1-clusters. If $k \in D$ or $r \in D$, then $s \notin D_1^p$; in this case, C has at most 7 nearby poor 1-clusters.

The argument is nearly identical to the one above for the cases in which $g, k \in D$, $p, q \in D$ and $q, r \in D$. □

Proof of Lemma 5.8. Let C_1 be the linear open 4-cluster shown in Figure 7.3a. Both leaves of C_1 must have at least one distance-2 vertex in D (Proposition 3.6), and, by hypothesis, $P(C_1) = 2$; therefore, $f, g, r, s \notin D$ and

$$|\{e, k, q\} \cap D| = |\{h, m, t\} \cap D|$$

By Proposition 3.4, we have $b, w \in D_{3^+}$. Let C_b and C_w be the 3^+-clusters at b and w, respectively. There are 10 candidates for nearby poor 1-clusters: a, c/h, d/e, i, j/k, m/n, p, q/v, t/u and x. Now, at least one of a and c is adjacent to or in C_b; therefore, at least one of a and c is not a poor 1-cluster. A similar argument may be made for v and x. Therefore, there are at most 8 nearby poor 1-clusters.

If both one-turn positions are in D, then we have $e, t \in D$. If C_1 has 8 nearby poor 1-clusters, then at most one of a and c is not a poor 1-cluster. If $a \notin D_1^p$, then c is distance-2 from C_b. If $C_b \in \mathcal{K}_3$, then c is either in a foot position or arm position. If c is in an arm position, then $a \in C_b$ and C_b is not paired. If $c \notin D_1^p$, then a similar argument may be made for a. And a symmetric argument may be made for v and x and C_w. Therefore, at least 2 of the poor 1-clusters at distance-3 are stealable. If C_1 has exactly 7 nearby poor 1-clusters, then at least one of a, c, v and x is a poor 1-cluster. Then, the above argument suffices; therefore, at least one of the distance-3 poor 1-clusters is stealable.

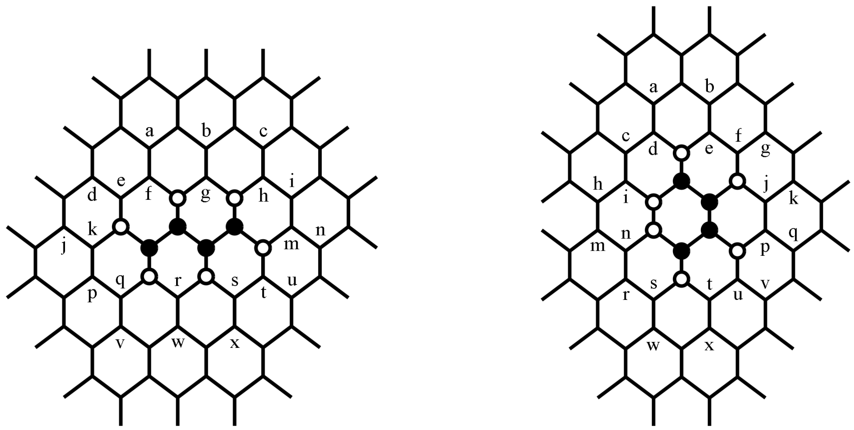
If exactly one one-turn position is in D, then $e \notin D$ or $t \notin D$. By symmetry we choose $e \notin D$. Then, $d \in D_{3^+}$ (Proposition 3.4). Therefore, there are at most 7 nearby poor 1-clusters. By hypothesis, at least one of k and q is in D. In both cases, $p \notin D_1^p$. Therefore, there are at most 6 nearby poor 1-clusters. Now, $a \in D_{3^+}$ (Proposition 3.4). Thus, if C_1 has exactly 6 nearby poor 1-clusters, then $c \in D_1^p$. Then c is distance-2 from C_b and $a \in C_b$. If $C_b \in \mathcal{K}_3$, then c is in an arm position and C_b is not paired. Therefore, c is stealable.

If neither one-turn position is in D, then $e, t \notin D$. Therefore, $a, d, u, x \in D_{3^+}$ (Proposition 3.4). Therefore, C_1 has at most 6 nearby poor 1-clusters. By hypothesis, at least one of k and q and at least one of h and m is in D. If $k \in D$ or $q \in D$, then $p \notin D_1^p$; and if $h \in D$ or $m \in D$, then $i \notin D_1^p$. Therefore, C_1 has at most 4 nearby poor 1-clusters. □

Proof of Lemma 5.9. Let C be the linear 4-cluster shown in Figure 7.3a. First, suppose C is adjacent to no one-third vertices. Now, either $g \in D$ or $g \notin D$. If $g \in D$, then one of the leaves of C and one of the middle vertices has a distance-2 vertex in $D \setminus C$. Now, each leaf has at least one distance-2 vertex in $D \setminus C$ (Propostion 3.6) and, by hypothesis, $P(C) = 3$; therefore, $f, h, m, r, s, t \notin D$ and $|\{e, k, q\} \cap D| = 1$. Then, $u, w, x \in D_{3^+}$ (Proposition 3.4) and there are at most 9 nearby poor 1-clusters: a, c, d/e, g, i, j/k, n, p and q/v. Now, suppose $g \notin D$. If $r \in D$, then this case can be reduced, by symmetry, to the above case. So assume $r \notin D$. Then, $b, w \in D_{3^+}$ (Propostion 3.4). There are 10 candidates for nearby poor 1-clusters: a/f, c/h, d/e, i, j/k, m/n, p, q/v, s/x and t/u. Suppose by contradiction that there exist 10 nearby poor 1-clusters. Then, $i, p \in D_1^p$. Therefore, $h, m \notin D$ and $n, v \in D$ (Proposition 3.5). And we must have $n, v \in D_1^p$; otherwise, C has fewer than 10 nearby poor 1-clusters. Since $g, h, m \notin D$, we have $t \in D$ (Proposition 3.6). And, as above, we must have $t \in D_1^p$; therefore, $x \in D$ (Proposition 3.5). And, again, we must have $x \in D_1^p$. Therefore, $v, x \in D_1^p$. But $w \in D_{3^+}$ and $r \notin D$; therefore, at least one of v and x is not a poor 1-cluster, which is a contradiction.

Now, suppose C is adjacent to a one-third vertex, $v_{\frac{1}{3}}$. Since $P(C) = 3$ and each leaf must have at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6), $v_{\frac{1}{3}}$ must be adjacent to e and k or m and t. By symmetry, we choose $e, k \in D$. Then, $|\{h, m, t\} \cap D| = 1$ and $f, g, q, r, s \notin D$; therefore, $b, w \in D_{3^+}$ (Proposition 3.4) and there are at most 7 nearby poor 1-clusters: a, c/h, m/n, p, t/u, v and x. Suppose by contradiction that

C has 7 nearby poor 1-clusters. Then, $v, x \in D_1^p$. But $w \in D_{3+}$ and $r \notin D$; therefore, at least one of v and x is not a poor 1-cluster, which is a contradiction. □



(a) Lemma 5.8 and Lemma 5.9

(b) Lemma 5.10 and Lemma 5.11

Figure 7.3

Proof of Lemma 5.10. Let C_1 be the curved open 4-cluster shown in Figure 7.3b. Both leaves of C_1 must have at least one distance-2 vertex in D (Proposition 3.6), and, by hypothesis, $P(C_1) = 2$; therefore, $f, j, p, u \notin D$ and

$$|\{d, e, i\} \cap D| = |\{n, s, t\} \cap D| = 1$$

By symmetry, there are only 6 cases to consider: $e, t \in D$; $e, s \in D$; $e, n \in D$; $d, s \in D$; $d, n \in D$; and $i, n \in D$. Note that $k, q \in D_{3+}$ in every case (Proposition 3.4). First, we consider the cases with backwards positions.

$e, t \in D$: There are 9 candidates for nearby poor 1-clusters: a, c, e, g, h, r, s, v and w. We could have chosen m instead of h, but the proof would be symmetric so we consider only h as a candidate. Now, at most one of h and r is a poor 1-cluster; therefore, C_1 has at most 8 nearby poor 1-clusters. When C_1 has exactly 8 such 1-clusters, all of the candidates other than h and r are poor 1-clusters. Therefore, $g, v \in D_1^p$ and q and k are in the same 4^+-cluster, C_2. Then we have g and v at distance-2 from C_2, where C_2 is not an open 3-cluster. When C_1 has exactly 7 nearby poor 1-clusters, at most one of g and v is no longer a poor 1-cluster. Therefore, at least one of g and v is distance-2 from a 3^+-cluster. If $g \in D_1^p$, then $k \in D_{3+} \setminus D_3^o$ and g is distance-2 from k; a symmetric argument can be made for v and q. Therefore, at least one of g and v is distance-2 from a 3^+-cluster, C_2, where C_2 is not an open 3-cluster.

$e, s \in D$: Since $t \notin D$, we have $k, q, v, x \in D_{3+}$. There are 6 candidates for nearby poor 1-clusters: a, c, e, g, s and h/m. However, h and s cannot both be poor 1-clusters; and m and c cannot both be poor 1-clusters. Therefore, there are at most 5 nearby poor 1-clusters.

$e, n \in D$: Again, $k, q, v, x \in D_{3+}$. There are 7 candidates for nearby poor 1-clusters: a, c, e, g, h, n and w. However, at most one of n and w is a poor 1-cluster. Therefore, there are at most 6 nearby poor 1-clusters. If C_1 has exactly 6 such 1-clusters, then $g \in D_1^p$. Then, g is distance-2 from the 3^+-cluster at k, C_k, and C_k is not an open 3-cluster.

$d, s \in D$: Since $e, t \notin D$, we have $b, g, k, q, v, x \in D_{3+}$. There are 3 candidates for nearby poor 1-clusters: d, h and s. We could have chosen m instead of h but the proof would be symmetric. It cannot be the case that both h and s are poor 1-clusters. Therefore, there are at most 2 nearby poor 1-clusters.

$d, n \in D$: Again, $b, g, k, q, v, x \in D_{3+}$. There are 4 candidates for nearby poor 1-clusters: d, h, n and w. However, at most one of d and h is a poor 1-cluster; likewise for n and w. Therefore, there are at most 2 nearby poor 1-clusters.

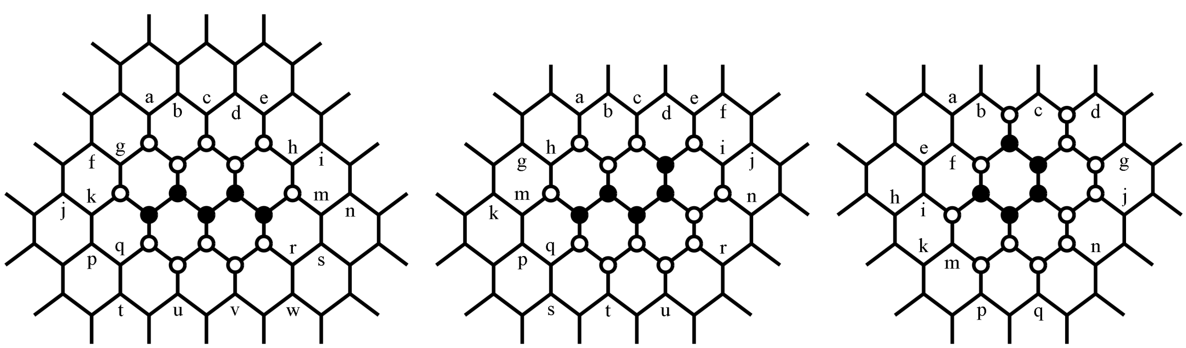
$i, n \in D$: Once again, $b, g, k, q, v, x \in D_{3+}$. There are 4 candidates for nearby poor 1-clusters: a, i, n and w. However, at most one of a and i is a poor 1-cluster; likewise for n and w. Therefore, there are at most 2 nearby poor 1-clusters. □

Proof of Lemma 5.11. Let C be the curved 4-cluster shown in Figure 7.3b. First, suppose C is adjacent to no one-third vertices and both backwards positions are in D; that is, $e, t \in D$. Then, either $j \in D$ or $j \notin D$. Now, $P(C) = 3$; therefore, if $j \in D$ then $d, f, i, n, p, s, u \notin D$. Therefore, C has at most 11 nearby poor 1-clusters: a, b, c, e, g, h/m, j, q, r, t and v. Now, consider the case in which $j \notin D$. If $p \in D$, then this case can be reduced by symmetry to the previous case. So we assume $p \notin D$. Then, $k, q \in D_{3+}$ (Proposition 3.4), and C has at most 10 nearby poor 1-clusters: a/d, c, e, g, h/i, m/n, r, s/w, t and u/v.

Now, suppose C is adjacent to no one-third vertices and one backwards position is not in D. By symmetry, we choose $e \notin D$. Again, either $j \in D$ or $j \notin D$. First, assume $j \in D$. Since $P(C) = 3$ and each leaf has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6), we must have $f \notin D$; then, $b, g \in D_{3+}$ (Proposition 3.4), and C has at most 10 nearby poor 1-clusters: a/d, c, h/i, j, m/n, p/q, r, s/w, t/x and u/v. Now, assume $j \notin D$. If $p \in D$, then this case can be reduced by symmetry to the previous case. So we assume $p \notin D$; then, $k, q \in D_{3+}$ (Proposition 3.4), and C has at most 10 nearby poor 1-clusters: a/d, b, c, f/g, h/i, m/n, r, s/w, t/x and u/v.

Now, suppose C is adjacent to no one-third vertices and both backwards positions are not in D; that is, $e, t \notin D$. First, assume $j \in D$. Since $P(C) = 3$ and each leaf has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6), we must have $f, p, u \notin D$; then, $b, g, v, x \in D_{3+}$ (Proposition 3.4), and C has at most 8 nearby poor 1-clusters: a/d, c, h/i, j, m/n, q, r and s/w. Now, assume $j \notin D$. If $p \in D$, then this case can be reduced by symmetry to the previous case. So we assume $p \notin D$. Then, $k, q \in D_{3+}$ (Proposition 3.4). There are 10 candidates for nearby poor 1-clusters: a/d, b, c, f/g, h/i, m/n, r, s/w, u/v and x. Each leaf of C has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6). Therefore, $d \in D$ or $i \in D$; in both cases, $c \notin D_1^p$. Therefore, C has at most 9 nearby poor 1-clusters.

Finally, suppose C is adjacent to a one-third vertex, $v_{\frac{1}{3}}$. Since $P(C) = 3$ and each leaf must have at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6), $v_{\frac{1}{3}}$ must be adjacent to a leaf of C. By symmetry, we choose $d, e \in D$. Then, for the same reasons, we must have $f, i, j, p, u \notin D$. Then, $k, q \in D_{3+}$ (Proposition 3.4), and C has at most 6 nearby poor 1-clusters: g, h/m, n/r, s/w, t/x and v. □



(a) Case 1 (b) Case 2 (c) Case 3

Figure 7.4: Lemma 5.12

Proof of Lemma 5.12. By symmetry, there are only 3 cases to consider.

Let C_1 be the open 5-cluster shown in Figure 7.4a. By Proposition 3.4, we have $c, u, v \in D_{3+}$. Then the candidates for nearby poor 1-clusters are a, e, f/g, h/i, j/k, m/n, p/q, r/s, t and w. There are 10 candidates in total. However, $c \in D_{3+}$. Let C_c be the 3^+-cluster at c. Now, $b \in C_c$ or $d \in C_c$ or both. If both, then $a, e \notin D_1^p$ and there are at most 8 nearby poor 1-clusters. So consider the case in which only one of b and d is in D. By symmetry, we choose $d \in D$; therefore, $e \notin D_1^p$ and there are at most 9 nearby poor 1-clusters. However, a is distance-2 from C_c. Now, if $e \notin D$, then $h, i \in D_{3+}$ (Proposition 3.4) and there are at most 8 nearby poor 1-clusters. Therefore, if $C_c \in \mathcal{K}_3^o$ and C_1 has 9 nearby poor 1-clusters, then $c, d, e \in C_c$. Then, a is not in a shoulder position and C_c is not paired.

Let C_2 be the open 5-cluster shown in Figure 7.4b. By Proposition 3.4, we have $r, t, u \in D_{3^+}$. There are 9 candidates for nearby poor 1-clusters: a/b, c/d, e/f, g/h, i/j, k/m, n, p/q and s. However, s is distance-2 from the 3^+-cluster at t; let C_t be this 3^+-cluster. If $C_t \in \mathcal{K}_3^o$, then $u \in C_t$; furthermore, s is in an arm position but C_t is not paired.

Let C_3 be the open 5-cluster shown in Figure 7.4c. By Proposition 3.4, we have $g, j, n, q \in D_{3^+}$. There are 7 candidates for nearby poor 1-clusters: a/b, c, d, e/f, h/i, k/m and p. $\square$

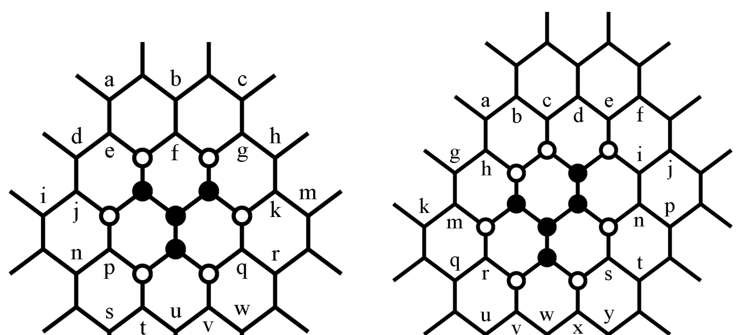
Proof of Lemma 5.13. Let C be the 4-cluster shown in Figure 7.5a. Then C has one degree-3 vertex. Each of the 3 leaves of C has at least one distance-2 vertex in $D \setminus C$ (Proposition 3.6) and, by hypothesis, $P(C) = 3$; therefore

$$|\{e, f, j, p\} \cap D| = |\{f, g, k, q\} \cap D| = |\{p, q, t, v\} \cap D| = 1$$

By symmetry, we must consider only 2 cases: $f \in D$ and $g \in D$. There are 12 candidates for nearby poor 1-clusters: a/e, b/f, c/g, d, h, i/j, k/m, n/p, q/r, s/t, u and v/w.

Now, if $f \in D$, then $e, g, j, k, p, q \notin D$. Therefore, $n, r \in D_{3^+}$ (Proposition 3.4). Thus, C has at most 10 nearby poor 1-clusters. Then, either $t \in D$ or $v \in D$; in both cases, $u \notin D_1^p$. Therefore, C has at most 9 nearby poor 1-clusters. By symmetry, we choose $v \in D$ and $t \notin D$. If $v \notin D_1^p$, then the lemma holds; so we assume $v \in D_1^p$. Then, $u \notin D$. But we also have $t \notin D$; therefore, $s \in D_{3^+}$ (Proposition 3.4). Therefore, C has at most 8 nearby poor 1-clusters.

If $g \in D$, then $f, k, q \notin D$. Therefore, $b, r \in D_{3^+}$ and $h \notin D_1^p$. Therefore, C has at most 9 nearby poor 1-clusters. If $g \notin D_1^p$, then the lemma holds; so we assume $g \in D_1^p$. Then, $h \notin D$. But we also have $k \notin D$; therefore, $m \in D_{3^+}$ (Proposition 3.4). Therefore, C has at most 8 nearby poor 1-clusters. $\square$



(a) Lemma 5.13 (b) Lemma 5.14

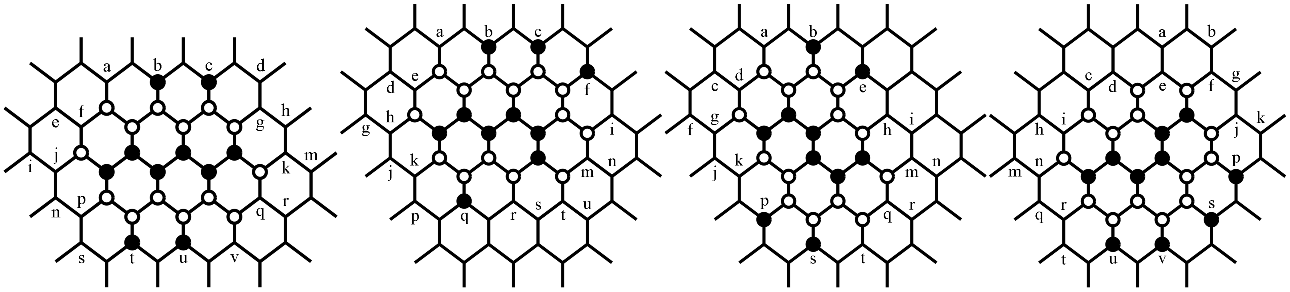
Figure 7.5

Proof of Lemma 5.14. Let C be the 5-cluster shown in Figure 7.5b. Then C has one degree-3 vertex. There are 13 candidates for nearby poor 1-clusters: a/h, b/c, d, e/f, g, i/j, k/m, n/p, q/r, s/t, u/v, w and x/y.

Now, if $c \in D$, then $d \notin D_1^p$. Therefore, C has at most 12 nearby poor 1-clusters.

Now we consider the case for which $c \notin D$. If $b \notin D_1^p$, then C has at most 12 nearby poor 1-clusters and the lemma holds. So we assume $b \in D_1^p$. Then we have $a \notin D_1^p$. If $h \notin D_1^p$, then C has at most 12 nearby poor 1-clusters and the lemma holds. So we assume $h \in D_1^p$. Then $g \notin D_1^p$; therefore, C has at most 12 nearby poor 1-clusters. $\square$

Proof of Lemma 5.15. By symmetry there are only 4 cases to consider. Let C_1 be the open 6-cluster shown in Figure 7.6a. Then, $b, c, t, u \in D_{3^+}$ (Proposition 3.4). There are 10 candidates for nearby poor 1-clusters: a, d/g, e/f, h, i/j, k/m, n, p/s, q/r and v. Therefore, C_1 has at most 10 nearby poor 1-clusters. Let C_2

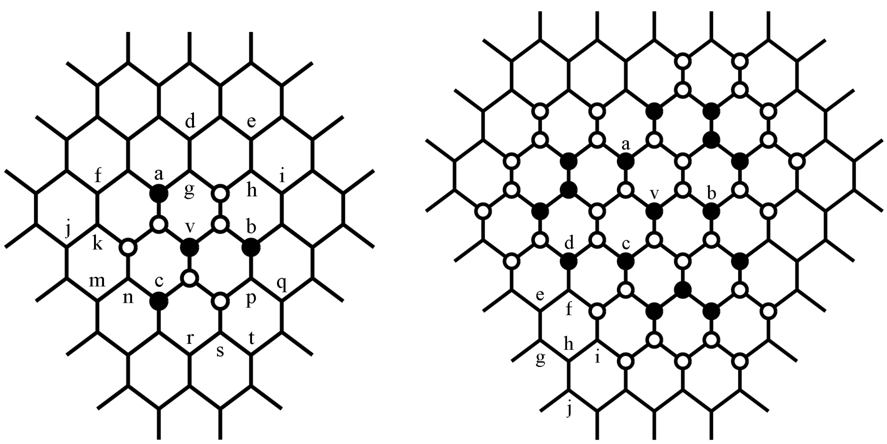


(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

Figure 7.6: Lemma 5.15

be the open 6-cluster shown in Figure 7.6b. Then, $b, c, f, q \in D_{3^+}$. Therefore, C_2 has at most 9 nearby poor 1-clusters: a, d/e, g/h, i, j/k, m/n, p, r/s and t/u. Let C_3 be the open 6-cluster shown in Figure 7.6c. Then, $b, e, p, s \in D_{3^+}$. Therefore, C_3 has at most 8 nearby poor 1-clusters: a, c/d, f/g, h/i, j/k, m/n, q/r and t. Let C_4 be the open 6-cluster shown in Figure 7.6d. Then, $p, s, u, v \in D_{3^+}$. Therefore, C_4 has at most 9 nearby poor 1-clusters: a/e, b/f, c/d, g, h/i, j/k, m/n, q/r and t. $\square$

Proof of Lemma 5.17. Let v, a, b and c be as shown in Figure 7.7a. Now, $v \in D_1^{vp}$ and a, b and c are distance-2 from v; therefore, $a, b, c \in D_1^p$. Therefore, $g, n, p \notin D$ and $d, m, q \in D$ (Proposition 3.5). Then $h, k, s \in D_{3^+}$ (Proposition 3.4). Let C_h be the 3^+-cluster at h. Now, C_h is distance-3 from v; therefore, $C_h \in \mathcal{K}_3^o$. Since $b, d \in D$, we must have $e, h, i \in C_h$. Symmetric arguments may be made to show $f, j, k \in D_{3^+}^o$ and $r, s, t \in D_{3^+}^o$. Therefore, v is in a head position of 3 open 3-clusters and exactly one of a, b and c is in a shoulder position of each of these open 3-clusters.

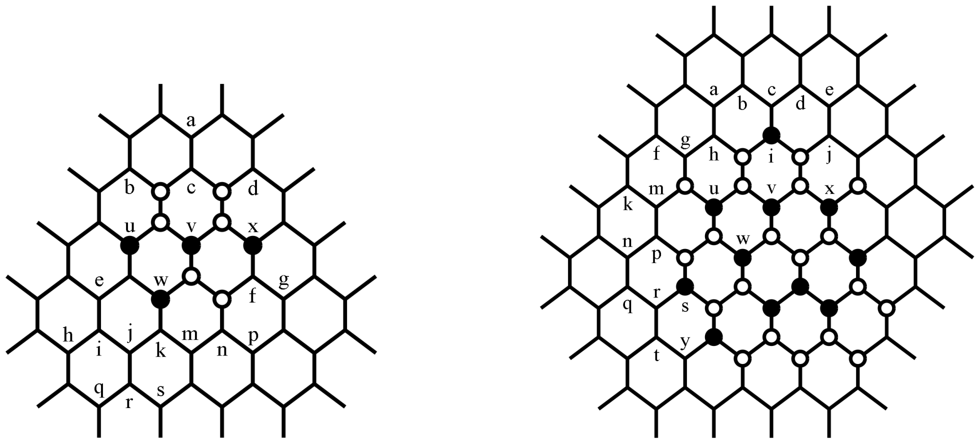


(a) (b)

Figure 7.7: Lemma 5.17

Now suppose each of the open 3-clusters at distance-3 from v is uncrowded as in Figure 7.7b. All of the vertices have been relabelled except v, a, b and c. Now, the graph is rotationally symmetric about v, so we need only consider one of a, b and c. We choose c. Now, d is in a shoulder position of an open 3-cluster, C, which is distance-3 from v. By hypothesis, C is uncrowded; thus, $d \in D_1^p$. Therefore, $f \notin D$ and $e \in D$ (Proposition 3.5). Then we have $i \in D_{3^+}$. Let C_i be the 3^+-cluster at i. Since 2 of the 3 neighbors of i are not in D, we also have $h \in C_i$. If $g \in D$, then $e, g, h, i \in C_i$. If $g \notin D$, then $h, i, j \in C_i$ and e closes C_i. In both cases, C_i is a closed 3-cluster or 4^+-cluster at distance-3 from c. $\square$

Proof of Lemma 5.20. Let u, v, w and x be as shown in Figure 7.8a. Now, by hypothesis, $v \in D_1^{vp}$; therefore, $u, w, x \in D_1^p$. Then, we have $f \notin D$; therefore, $n \in D_{3^+}$. Since $x \in D_1^p$, we have $g \in D$ (Proposition 3.5). Let C_0 be the 3^+-cluster at n. Now, v is distance-3 from C, so $C \in \mathcal{K}_3^o$. The only possibility is to have $m, n, p \in C$. Therefore, v and x are in the head positions of an open 3-cluster, and w is in a shoulder position.



(a) Lemma 5.20

(b) Lemma 5.20, where (i) and (ii) are not satisfied.

Figure 7.8: Lemma 5.20

If C_0 is crowded, then (i) is satisfied and the lemma holds. So assume C_0 is uncrowded. Then, $j, k, s \notin D$, and since $u \in D_1^p$ we have $e \notin D$ (Corollary 4.2). If $i \notin D$, then $h, r \in D_{3^+}$. Let C_r be the 3^+-cluster at r. If $h \in C_r$, then w is distance-3 from a 4^+-cluster. If $h \notin C_r$, then h closes C_r and w is distance-3 from a closed 3^+-cluster. In both cases, (ii) is satisfied. So assume w is not distance-3 from a closed 3-cluster or 4^+-cluster. Then, we have $i \in D$. Now, if $r \notin D$, then $i, q \in D_{3^+}$. Let C_i be the 3^+-cluster at i. If $q \in C_i$, then w is distance-3 from a 4^+-cluster; and if $q \notin C_i$, then q closes C_i and w is distance-3 from a closed 3^+-cluster. But we assumed that (ii) is not satisfied; therefore, $r \in D$.

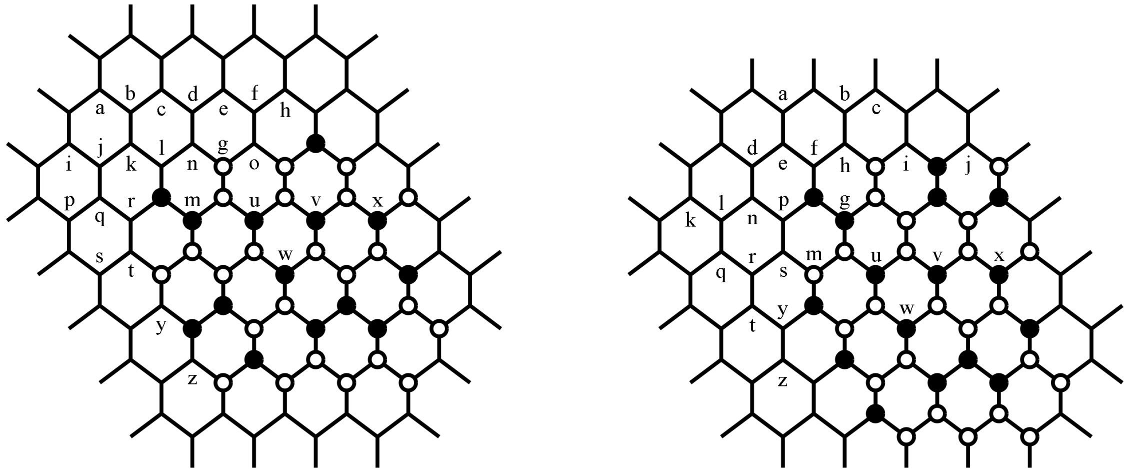
If $c \notin D$, then $a, b, d \in D_{3^+}$. Let C_b be the 3^+-cluster at b. If b is a leaf of C_b, then either $u \in C_b$ and $C_b \in \mathcal{K}_{4^+}$, or u closes C_b and $C_b \in \mathcal{K}_{3^+}^c$, or $a \in C_b$ and $C_b \in \mathcal{K}_{4^+}$, or a closes C_b and $C_b \in \mathcal{K}_{3^+}^c$. But v is very poor and distance-3 from b, so we must have $C_b \in \mathcal{K}_3^o$. This is only possible if b is the middle vertex of C_b. Let C_a be the 3^+-cluster at a, and let C_d be the 3^+-cluster at d. Now, $C_b \in \mathcal{K}_3^o$, so a is a leaf of C_a and either $d \in C_a$ or d closes C_a. In the first case, v is distance-3 from a 4^+-cluster . But, by hypothesis, v is not distance-3 from a 4^+-cluster. Therefore, d closes C_a. But then d is a leaf of C_d and x closes C_d; therefore, v is distance-3 from a closed 3^+-cluster. But, by hypothesis, v is not distance-3 from a closed 3^+-cluster. Therefore, $c \in D$.

Figure 7.8b shows the surrounding vertices of v when neither (i) nor (ii) is satisfied. All the vertices except u, v, w and x have been relabelled. Now, $u \in D_1^p$; therefore, exactly one of g and m is in D (Corollary 4.2).

First, we consider the case for which $m \in D$ and $g \notin D$. If $p \in D$, then $m, p \in D_{3^+}$. Let $C_{m,p}$ be the 3^+-cluster at m and p. If $k, m, p \in C_{m,p}$, then u closes $C_{m,p}$. But p is distance-3 from w and, by assumption, w is not distance-3 from a closed 3^+-cluster. If $m, n, p \in C_{m,p}$, then s closes $C_{m,p}$. But, again, by assumption, w is not distance-3 from a closed 3^+-cluster. Therefore, $p \notin D$. Then, by Proposition 3.4, we have $s \in D_{3^+}$. Let C_s be the 3^+-cluster at s. Since 2 of the 3 neighbors of s are not in D, we have $r \in C_s$. Now, C_s is distance-3 from w; therefore, by assumption, $C_s \in \mathcal{K}_3^o$. Therefore, either $q, r, s \in C_s$ or $r, s, t \in C_s$. In both cases we have $n, y \notin D$. Then, by Proposition 3.4, we have $m \in D_{3^+}$. Let C_m be the 3^+-cluster at m. Since 2 of the 3 neighbors of m are not in D, we must have $k \in C_m$.

Figure 7.9a shows the surrounding vertices of v when neither (i) nor (ii) is satisfied and $m \in D$. All of the vertices except g, m, u, v, w and x have been relabelled.

Now, if $z \in D$, then $z \in D_{3^+}$. Let C_z be the 3^+-cluster at z. By assumption, (ii) is not satisfied, so



(a) Lemma 5.20, where (i) and (ii) are not satisfied and $m \in D$.

(b) Lemma 5.20, where (i) and (ii) are not satisfied and $g \in D$.

Figure 7.9: Lemma 5.20

$C_z \in \mathcal{K}_3^o$. Then w is in the hand position of an open 3-cluster, C_z, such that the tail position is in D and w is on the finless side. Therefore, (iii) is satisfied.

Now assume (iii) is not satisfied. Then $z \notin D$ and $y \in D_{3^+}$. Let C_y be the 3^+-cluster at y. Since w is distance-3 from C_y, we have $C_y \in \mathcal{K}_3^o$. Let C_m be the 3^+-cluster at m. Then, m is a leaf of C_m and u is the only vertex in $D \setminus C_m$ at distance-2 from m. If C_m is a linear 4-cluster, then either $q, r \in C_m$ or $l, k \in C_m$; in both cases, the one-turn position at distance-2 from m is not in D. If C_m is a curved 4-cluster, then either $r, t \in C_m$ or $l, n \in C_m$; in both cases, the backwards position at distance-2 from m is not in D. Therefore, if $C_m \in \mathcal{K}_{4^+}$, then (iv) is satisfied.

Now assume (iv) is not satisfied. Then, $C_m \in \mathcal{K}_3$. Either $l \in C_m$ or $r \in C_m$; in both cases, u is in a foot or arm position. First, we consider the case in which $r \in C_m$. Then, $l, q, t \notin D$. If $n \notin D$, then $d, o \in D_{3^+}$ (Proposition 3.4). Since 2 of the 3 neighbors of o are not in D, we also have $f \in D_{3^+}$. Let $C_{f,o}$ be the 3^+-cluster at f and o. If $e \in D$, then $d, e, f, o \in C_{f,o}$ and $C_{f,o} \in \mathcal{K}_{4^+}$. If $e \notin D$, then $h \in C_{f,o}$ and d closes $C_{f,o}$. But, by hypothesis, v is not within distance-3 of a closed 3-cluster or 4^+-cluster. Therefore, $n \in D$ and $C_m \in \mathcal{K}_3^c$. Then, C_m is type-2 paired with C_y and u is in the arm position on the closed side of C_m. Then, (v) is satisfied. Therefore, with $r \in D$, the lemma holds.

Now assume (v) is not satisfied. So we have $l \in C_m$. Recall $C_m \in \mathcal{K}_3$; therefore, $k, n, r \notin D$. Then, $f, o \in D_{3^+}$ (Proposition 3.4). Again, let $C_{f,o}$ be the 3^+-cluster at f and o. Now, $C_{f,o}$ is distance-3 from v, so we must have $C_{f,o} \in \mathcal{K}_3^o$. If $h \in C_{f,o}$, then the tail position of $C_{f,o}$ is in D and u is in the hand position on the finless side. But, by assumption, (iii) is not satisfied. Therefore, $e \in C_{f,o}$ and $d, h \notin D$. If $C_{f,o} \in \mathcal{K}_3^c$, then $C_{f,o}$ has at most 6 neary poor 1-clusters: $a/b, c, i/j, p/q, s/t$ and u. But, by assumption, (v) is not satisfied. So we have $C_{f,o} \in \mathcal{K}_3^o$. Then, $q, t \notin D$; therefore, $s \in D_{3^+}$. Let C_s be the 3^+-cluster at s. If C_s occupies the arm position of C_y, then w is in the hand position of an open 3-cluster satisfying (vi).

Now assume (vi) is not satisfied. Then, then arm position of C_y is not in D. Therefore, u is in the foot position of an open 3-cluster which is type-1 paired on top. Therefore, (vii) is satisfied, and the lemma holds.

Now we return to Figure 7.8b and consider the case for which $g \in D$ and $m \notin D$.

If $h \in D$, then $g, h \in D_{3^+}$ and either $a \in D_{3^+}$ or $f \in D_{3^+}$. In both cases, v is distance-3 from a closed 3^+-cluster. Therefore, $h \notin D$. Then, by Proposition 3.4, we have $i \in D_{3^+}$. Let C_i be the 3^+-cluster at i. Since 2 of the 3 neighbors of i are not in D, we have $c \in C_i$ and i is a leaf of C_i. Therefore, $j \in D$ (Proposition 3.6). Now, C_i is distance-3 from v; therefore, $C_i \in \mathcal{K}_3^o$. Either $b \in C_i$ or $d \in C_i$. In both cases, we have $a, e \notin D$. Then, $g \in D_{3^+}$ (Proposition 3.4). Let C_g be the 3^+-cluster at g. Since 2 of the 3 neighbors of g are not in D, we have $f \in C_g$. If $p \notin D$, then $n, s \in D_{3^+}$ (Proposition 3.4). Let C_s be the 3^+-cluster at s. Since 2 of the 3 neighbors of s are not in D, we have $r \in C_s$. If $q \in D$, then $n, q, r, s \in C_s$ and $C_s \in \mathcal{K}_{4^+}$.

If $q \notin D$, then $r, s, t \in C_s$ and n closes C_s. But s is distance-3 from w and we assumed (ii) is not satisfied. Therefore, $p \in D$.

Figure 7.9b shows the surrounding vertices of v when neither (i) nor (ii) is satisfied and $g \in D$. All of the vertices except g, m, u, v, w and x have been relabelled.

If $j \in D$, then $j \in D_{3^+}$. Let C_j be the 3^+-cluster at j. Since, C_j is distance-3 from v, we must have $C_j \in \mathcal{K}_3^o$. Then, the tail position of C_j is in D and v is in the hand position on the finless side. Therefore, (iii) is satisfied.

Now assume (iii) is not satisfied. Then we have $i \in D_3^o$ and $j \notin D$. Let C_i be the open 3-cluster at i. Now, u is distance-2 from the 3^+-cluster at g; let C_g be this 3^+-cluster. Then, g is a leaf of C_g and u is the only distance-2 vertex of g in $D \setminus C_g$. If C_g is a linear 4-cluster, then either $n, p \in C_g$ or $e, f \in C_g$; in both cases, the one-turn position at distance-2 from g is not in D. If C_g is a curved 4-cluster then either $f, h \in C_g$ or $p, s \in C_g$; in both cases, the backwards position at distance-2 from g is not in D. Therefore, if $C_g \in \mathcal{K}_{4^+}$, then (iv) is satisfied.

Now assume (iv) is not satisfied. Then, $C_g \in \mathcal{K}_3$. Either $f \in C_g$ or $p \in C_g$; in both cases, u is in a foot or arm position. First we consider the case in which $f \in C_g$. Then, $e, h, p \notin D$. If $s \notin D$, then $r, y \in D_{3^+}$ (Proposition 3.4). Let C_y be the 3^+-cluster at y. If $t \in D$, then $r, t \in C_y$ and $C_y \in \mathcal{K}_{4^+}$. If $t \notin D$, then $z \in C_y$ and r closes C_y. In both cases, w is distance-3 from a closed 3-cluster or 4^+-cluster. But we assumed that (ii) is not satisfied. Therefore, $s \in D$. Then, u is in an arm position on the closed side of C_g , and C_g is type-2 paired with C_i. Then, (v) is satisfied. Therefore, with $f \in D$, the lemma holds.

Now assume (v) is not satisfied. Then $f \notin D$. Therefore, $p \in C_g$ and $f, n, s \notin D$. Then, $y \in D_{3^+}$ (Proposition 3.4). Let C_y be the 3^+-cluster at y. Now, u is distance-3 from C_y and we assumed (ii) is not satisfied. Therefore, $C_y \in \mathcal{K}_3^o$. If $z \in C_y$, then the tail position of C_y is in D and u is in the hand position on the finless side of C_y. But we assumed that (iii) is not satisfied. Therefore, $t \in C_y$. Now, if C_g is a closed 3-cluster, then there are at most 6 nearby poor 1-clusters: $b/h, a/e, d, k/l, q$ and u. But we assumed (v) is not satisfied. Therefore, $C_g \in \mathcal{K}_3^o$. Then, $h \notin D$. By Proposition 3.4, we have $b \in D_{3^+}$. If $c \in D$, then v is in the hand position of C_i and the hand and arm positions on the other side of C_i are both in D. Therefore, (vi) is satisfied.

Now assume (vi) is not satisfied. Then we have $c \notin D$. Therefore, u is in the foot position of an open 3-cluster which is type-1 paired on top. Therefore, (vii) is satisfied and the lemma holds. ☐

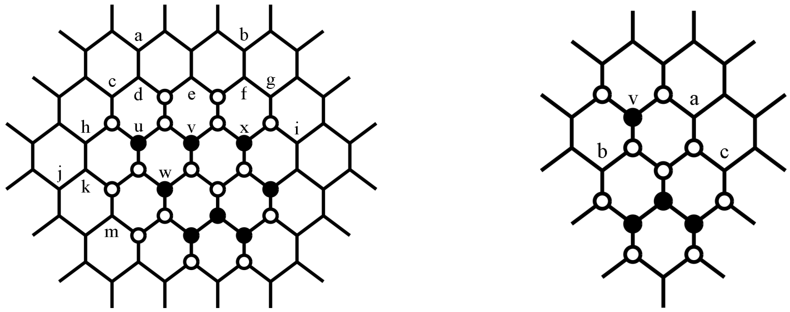
Proof of Corollary 5.22. Let x be in the x-position of v. Now, $v \in D_1^{vp}$; therefore, $u, w, x \in D_1^p$. Additionally, v is in a head position of an open 3-cluster, C_0, and w is in a shoulder position (Lemma 5.20). Let u, v, w, x and C_0 be as shown in Figure 7.10a.

Suppose by contradiction that u is distance-2 from a very poor 1-cluster other than v. There are 2 possibilities: $c \in D_1^{vp}$ or $h \in D_1^{vp}$. If $c \in D_1^{vp}$, then $d \notin D$ and $a \in D$ (Proposition 3.5). Then, $e \in D_{3^+}$ (Proposition 3.4). Let C_e be the 3^+-cluster at e. Either $a \in C_e$ or a closes C_e; in both cases c is within distance-3 of a closed 3-cluster or 4^+-cluster. Therefore, $c \notin D_1^{vp}$. If $h \in D_1^{vp}$, then $k \notin D$ and $j \in D$ (Proposition 3.5). Then, $m \in D_{3^+}$ (Proposition 3.4). Let C_m be the 3^+-cluster at m. Either $j \in C_m$ or j closes C_m; in both cases h is within distance-3 of a closed 3-cluster or 4^+-cluster. Therefore, $h \notin D_1^{vp}$.

Now, suppose by contradiction that w is distance-2 from a very poor 1-cluster other than v. The only possibility is u. If $u \in D_1^{vp}$, then $c \in D_1^p$ or $h \in D_1^p$. But we already saw that $c \in D_1^p$ or $h \in D_1^p$ implies $e \in D_3^c \cup D_{4^+}$ or $m \in D_3^c \cup D_{4^+}$. Since u is distance-3 from both e and m, we have $u \notin D_1^{vp}$. ☐

Proof of Corollary 5.23. Let v and x be as shown in Figure 7.10a. If $x \in D_1^{vp}$, then $g \in D_1^p$ or $i \in D_1^p$. If $g \in D_1^p$, then $f \notin D$ and $b \in D$ (Proposition 3.5). Therefore, $e \in D_{3^+}$ (Proposition 3.4). Let C_e be the 3^+-cluster at e. Either $b \in C_e$ or b closes C_e; in both cases, x is distance-3 from a closed 3-cluster or 4^+-cluster. But, by hypothesis, $v \in D_1^{vp}$. Therefore, $i \in D_1^p$. Therefore, x is in an asymmetric orientation and v is in the x-position of x. ☐

Proof of Lemma 5.24. Let C be the open 3-cluster shown in Figure 7.10b, and let v be a very poor 1-cluster. Then, v is in a head position of C. Since $v \in D_1$, we have $b \in D$ (Proposition 3.5). Suppose by contradiction that $c \notin D$. Then, $a \in D_{3^+}$ (Proposition 3.4). Then $v \in D_1^{vp}$ and v is distance-2 from a 3^+-cluster, which is a contradiction. Therefore, $c \in D$. ☐



(a) Corollary 5.22 and Corollary 5.23

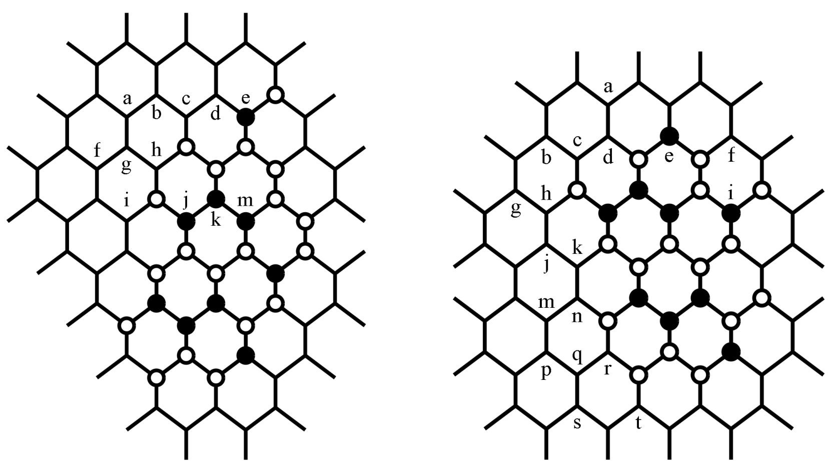
(b) Lemma 5.24

Figure 7.10

Proof of Lemma 5.25. Let C_1 be the open 3-cluster described by j, k and m in Figure 7.11a. Then C_1 is type-1 paired on top. Now, $e \in D_{3^+}$ (Proposition 3.4). Let C_e be the 3^+-cluster at e. Since 2 of the 3 neighbors of e are not in D, we have $d \in C_e$. If C_1 has a poor 1-cluster in a shoulder or arm position, then either $h \in D_1^p$ or $i \in D_1^p$.

First suppose $h \in D_1^p$. Then $c \in D$ (Proposition 3.5). Therefore, $c \in C_e$ and h is distance-2 from C_e. If $C_e \in \mathcal{K}_3^o$, then h is in a foot position and C_e is not paired.

Now suppose $i \in D_1^p$. Then $h \notin D$ (Corollary 4.2). Therefore, $c, g \in D_{3^+}$ (Proposition 3.4). Let C_g be the 3^+-cluster at g. If $f \in D$, then i is distance-2 from C_g. If $f \notin D$, then either $a, b, c, g \in C_g$ or c closes C_g; in both cases, i is distance-3 from a closed 3-cluster or 4^+-cluster. Thus, if $C_g \in \mathcal{K}_3^o$, then $a, f, g \in C_g$. Then, c and i are in the shoulder positions of C_g and, hence, C_g is not type-1 paired on top. □



(a) Lemma 5.25

(b) Lemma 5.26

Figure 7.11

Proof of Lemma 5.26. Let C_1 be the type-2 paired closed 3-cluster shown in Figure 7.11b, and let C_2 be the type-2 paired open 3-cluster. Suppose by contrapositive that $n \notin D$ or $k \in D$. First, we deal with case in which $n \notin D$. Then C_1 has 7 candidates for nearby poor 1-clusters: a/d, b/c, e, f, g/h, i and j/k. It suffices to eliminate one of these candidates. If $k \notin D$, then $j \in D_{3^+}$ (Proposition 3.4) and the lemma holds. If $k \notin D_1^p$, then the lemma holds; so assume $k \in D_1^p$. Then, $j \notin D$. If $h \notin D$, then $g \in D_{3^+}$ (Proposition 3.4) and the lemma holds. If $h \notin D_1^p$, then the lemma holds; so assume $h \in D_1^p$. Then, $c \notin D$ (Corollary 4.2). If

$b \notin D_1^p$, then the lemma holds; so assume $b \in D_1^p$. Then, $d \in D$ (Proposition 3.5). But then e is adjacent to a one-third vertex; therefore, $e \notin D_1^p$. Therefore, C_1 does not have 7 nearby poor 1-clusters. Now, we deal with the case in which $k \in D$. Then, C_1 has 7 candidates for nearby poor 1-clusters: a/d, b/c, e, f, g/h, i and k. It suffices to eliminate one of these candidates. If $k \notin D_1^p$, then the lemma holds; so assume $k \in D_1^p$. Then, $j \notin D$. If $h \notin D$, then $g \in D_{3+}$ (Proposition 3.4) and the lemma holds; so assume $h \in D$. If $h \notin D_1^p$, then the lemma holds; so assume $h \in D_1^p$. Then, $c \notin D$ (Corollary 4.2). If $b \notin D_1^p$, then the lemma holds; so assume $b \in D_1^p$. Then, $d \in D$ (Proposition 3.5). But then e is adjacent to a one-third vertex; therefore, $e \notin D_1^p$. Therefore, C_1 does not have 7 nearby poor 1-clusters.

If $n \in D_1^p$, then $m \notin D$ and $r \notin D$ (Corollary 4.2). Then, $q, t \in D_{3+}$ (Proposition 3.4). Let C_q be the 3^+-cluster at q. If $C_q \in \mathcal{K}_3^o$, then $p, q, s \in C_q$. However, n and t are in shoulder positions; therefore, C_q is not type-1 paired on top. If $C_q \notin \mathcal{K}_3^o$, then n closes C_q or t closes C_q or $t \in C_q$; in each case, n is nearby a closed 3-cluster or 4^+-cluster. □

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