SUSY N-supergroups and their real forms

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Abstract

We study SUSY N-supergroups, N=1,2, their classification and explicit realization, together with their real forms. In the end, we give the supergroup of SUSY preserving automorphism of $\mathbb{C}^{1|1}$ and we identify it with a subsupergroup of the SUSY preserving automorphisms of $\mathbb{P}^{1|1}$.

1 Introduction

The papers [5] and [6] carry out a thorough study of the real compact supergroups $S^{1|1}$ and $S^{1|2}$, called *supercircles*, in odd dimension 1 and 2, and their theory of representation, together with the Peter-Weyl theorem. These supercircles are realized as real forms of $(\mathbb{C}^{1|1})^{\times}$ and $(\mathbb{C}^{1|2})^{\times}$ respectively, and in the case of $S^{1|1}$, we have a precise relation between the real structures and real forms of $(\mathbb{C}^{1|1})^{\times}$ and the SUSY preserving automorphism of the SUSY 1-curve $(\mathbb{C}^{1|1})^{\times}$. In this paper we want to study the SUSY N-curves, which also admit a supergroup structure leaving invariant their SUSY structure, namely the SUSY N-supergroups. SUSY N-curves have been the object of study of several papers. After the foundational work by Manin [21], in [16, 22] Rabin et al. study families of super elliptic curves over non trivial odd bases, which are SUSY 1-curves, whose reduced part is an elliptic curve. These families of supercurves however do not admit a natural supergroup structure leaving invariant their SUSY structure, hence they are not SUSY N-supergroups according to our terminology (see also Remark 2.6). On the other hand, studying the real forms and the supergroup structure of SUSY

curves gives the opportunity of exploiting the representation theory for physical applications. We also point out that we are not merely studying SUSY-N curves that are also supergroups, but our requirement that the supergroup structure preserves the SUSY N-structure is quite restrictive and reduces drastically the possibilities for such supercurves, yet provides a useful local model for generic ones.

Our paper is organized as follows.

In Sections 2, 3 we give the definition of SUSY N-supergroups and we classify them. We also give an interpretation of the supergroup SL(1|1) as the SUSY 2-curve incidence supermanifold of the SUSY 1-curve $(\mathbb{C}^{1|1})^{\times}$ and its dual. In Section 4 we study of real forms of SUSY N-supergroups of type 1 and classify them. Finally in Section 5 we compute the supergroup of SUSY preserving automorphism of $\mathbb{C}^{1|1}$.

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2 The SUSY N-supergroups

We want to study SUSY N-curves which also have a supergroup structure, which preserves the SUSY N-structure. In the following we shall use the notation and terminology as in [20] Ch. 2. X is a SUSY 1-curve if X is a 1|1 complex supermanifold and there is a 0|1 distribution \mathcal{D} such that the Frobenius map $\mathcal{D} \otimes \mathcal{D} \to TX/\mathcal{D}$ given by $Y_1 \otimes Y_2 \mapsto [Y_1, Y_2] \pmod{\mathcal{D}}$ is an isomorphism. X is a SUSY 2-curve if it is a 1|2 complex supermanifold and there are two 0|1 distributions \mathcal{D}_i such that $[\mathcal{D}_i, \mathcal{D}_i] \subset \mathcal{D}_i$ and the Frobenius map $\mathcal{D}_1 \otimes \mathcal{D}_2 \to TX/[\mathcal{D}_1, \mathcal{D}_2]$ is an isomorphism. In [20] Ch. 2, Manin provides local models for such distributions:

$$\mathcal{D}=\zeta\partial_z+\partial_\zeta$$
 on X a SUSY 1-curve $\mathcal{D}_1=\zeta_2\partial_z+\partial_{\zeta_1}$ $\mathcal{D}_2=\zeta_1\partial_z+\partial_{\zeta_2}$ on X a SUSY 2-curve

Definition 2.1. Let X be a SUSY N-curve, with 0|1 distribution(s) \mathcal{D}_i , where i = 1 for N = 1 and i = 1, 2 for N = 2. We say that X is a SUSY N-supergroup if X is a supergroup and the distribution(s) \mathcal{D}_i are left invariant.

If X and Y are SUSY N-supergroups, we say that $f: X \longrightarrow Y$ is a morphism of SUSY N-supergroups if f is a supergroup morphism and $f_*(\mathcal{D}_i) = \mathcal{D}_i$, that is f preserves the distributions or exchanges them.

We can immediately compute the Lie superalgebra of SUSY N-supergroups.

Proposition 2.2. Let X be a SUSY N-supergroup. Then:

$$\operatorname{Lie}(X) = \langle Z, C \rangle, \quad [Z, Z] = 0, \quad \text{for } N = 1$$

$$Lie(X) = \langle Z_1, Z_2, C \rangle, \quad [Z_1, Z_2] = C, \quad for \quad N = 2$$

where Z, Z_i are suitably chosen odd elements and we assume to be zero all the brackets we do not write.

Proof. For N=1 see [5]. Let N=2, \mathcal{D}_i the left invariant distributions on the SUSY 2-supergroup X. Let $Z_i \in \operatorname{Lie}(X)$ be a left invariant (odd) generator of \mathcal{D}_i . We have $\operatorname{Lie}(X) = \langle Z_1, Z_2, [Z_1, Z_2] \rangle$. Notice that in general $[\mathcal{D}_i, \mathcal{D}_i] \neq 0$, however since $Z_i \in \operatorname{Lie}(X)$ and the bracket must preserve the parity, we have $[Z_i, Z_i] = 0$. Let $C := [Z_1, Z_2]$. The Jacobi identity gives immediately that C is central.

Since SUSY N-supergroups are in particular supergroups, we shall use freely the formalism of Super Harish Chandra Pairs (SHCP), see [3] Ch. 7 for more details.

Proposition 2.3. Let X, X' be SUSY N-supergroups. \widetilde{X} , \widetilde{X}' their underlying reduced groups. Then $X \cong X'$ if and only if $\widetilde{X} \cong \widetilde{X}'$.

Proof. Suppose $f:X\to X'$ is an isomorphism. Then \widetilde{X} and \widetilde{X}' are isomorphic.

Conversely, suppose $|f|: \widetilde{X} \to \widetilde{X}'$ is an isomorphism. $|f|: \widetilde{X} \to \widetilde{X}'$ lifts to the universal covers, giving an isomorphism $\widetilde{f}: \mathbb{C} \to \mathbb{C}$ which fixes the origin. By a standard result from one-variable complex analysis, \widetilde{f} is multiplication by a nonzero scalar λ . If N=1, define the super Lie algebra morphism $\varphi: \mathfrak{g} \to \mathfrak{g}$ by $C \mapsto \lambda C', Z \mapsto \sqrt{\lambda} Z'$. Clearly $\varphi|_{\mathfrak{g}_0} = d|f|$ so $F:=(|f|,\varphi)$ is an isomorphism of SHCP, hence $X\cong X'$ as supergroups.

By construction $dF(\mathcal{D}_e) = \mathcal{D}'_e$ at the identity e. However, as \mathcal{D} , \mathcal{D}' are left-invariant and F is a supergroup isomorphism, we have $dF(\mathcal{D}_p) = \mathcal{D}'_p$ at every point p of X, hence F is an isomorphism of SUSY N-supergroups.

If N=2, define the super Lie algebra morphism $\varphi:\mathfrak{g}\to\mathfrak{g}$ by $C\mapsto\lambda C',\ Z_1\mapsto\sqrt{\lambda}Z_1',\ Z_2\mapsto\sqrt{\lambda}Z_2'$. Again, $\varphi|_{\mathfrak{g}_0}=d|f|$ so $F:=(|f|,\varphi)$ is an isomorphism of SHCP. This shows that $X\cong X'$ as supergroups, however, reasoning as above, we obtain our result.

Corollary 2.4. Let X be a SUSY N-supergroup. Then, $X = \mathbb{C}^{1|N}$ or $X \cong \mathbb{C}^{1|N}/G$, where G is either a rank 1 free abelian subgroup of $\mathbb{C}^{1|N}$, or a lattice in $\mathbb{C}^{1|N}$. Furthermore, X inherits its SUSY N-structure from its universal cover $\mathbb{C}^{1|N}$.

Proof. If X is simply connected, then $X=\mathbb{C}^{1|N}$. Assume X is not simply connected. The universal cover of X is readily seen to be $\mathbb{C}^{1|N}$. The kernel of the covering morphism $\pi:\mathbb{C}^{1|N}\to X$ is a 0|0-subsupergroup G of $\mathbb{C}^{1|N}$ and, by a classical argument, is either a rank 1 free abelian subgroup of $\mathbb{C}^{1|N}$, or a lattice in $\mathbb{C}^{1|N}$. So $X\cong\mathbb{C}^{1|N}/G$.

Definition 2.5. We say that X is a SUSY N-supergroup of $type\ 1$ (resp. $type\ 2$) if $X = \mathbb{C}^{1|N}/G$, where G is a rank 1 free abelian subgroup (resp. a lattice) in $\mathbb{C}^{1|N}$.

We end this section with a remark relating our treatment with [16, 22].

Remark 2.6. In [16, 22], Rabin et al. consider families $X \to \mathcal{B}$ of SUSY 1-elliptic curves over a base superspace $\mathcal{B} = (pt, \Lambda)$, where Λ is a non trivial Grassmann algebra. In this setting, families of SUSY 1-curves do not admit any natural structure of supergroup *over* \mathcal{B} .

This is consistent with our treatment, because we work in the absolute setting, i.e. taking $\mathcal{B} = \{pt\}$, so it is possible to endow a SUSY 1-elliptic curve with the supergroup structure inherited from its universal cover $\mathbb{C}^{1|1}$ (Prop. 2.4).

3 SUSY N-supergroups of type 1

We want to classify the SUSY N-supergroups of type 1 and relate them to Manin's approach to SUSY curves. We shall use interchangeably the formalisms of functor of points and also of super Harish-Chandra pairs (SHCP).

Proposition 3.1. Up to isomorphism, for N = 1, 2 fixed, we have only two SUSY N-supergroups of type 1.

1. For N=1 they are $(\mathbb{C}^{1|1})^{\times}$ and $\mathbb{C}^{1|1}$ with group law respectively:

$$(w,\eta) \cdot (w,\eta') = (ww' + \eta\eta', w\eta' + \eta w')$$

$$(z,\zeta) \cdot (z',\zeta') = (z + z' + \zeta\zeta', \zeta + \zeta')$$
(1)

2. For N=2 they are $(\mathbb{C}^{1|2})^{\times}$ and $\mathbb{C}^{1|2}$ with group laws:

$$(v, \xi, \eta) \cdot (v', \xi', \eta') = (vv' + \eta \xi', v\xi' + \xi v' + \xi v^{-1} \eta \xi', \eta v' + v\eta' + \eta \xi' v'^{-1} \eta')$$

$$(z, \zeta, \chi) \cdot (z', \zeta', \chi') = (z + z' + \zeta \chi', \zeta + \zeta', \chi + \chi')$$

Proof. For N=1 the statements are contained in [5], provided that one verifies left invariance, which is a straightforward check. Let N=2, \mathcal{D}_i the left invariant distributions on the SUSY 2-supergroup X. Let $D_i \in \operatorname{Lie}(X)$ be a left invariant (odd) generator of \mathcal{D}_i . By 2.2 we have $\operatorname{Lie}(X) = \langle D_1, D_2, [D_1, D_2] \rangle$. The given group laws correspond to the Lie superalgebra we have obtained respectively for $G_0 = \mathbb{C}^\times$ and $G_0 = \mathbb{C}$. For example let us compute the Lie superalgebra structure for $(\mathbb{C}^{1|2})^\times$ (the case of $\mathbb{C}^{1|2}$ is similar). The tangent space is $\mathbb{C}^{1|2}$ and let e_1 , \mathcal{E}_1 , \mathcal{E}_2 be the canonical basis. The corresponding left invariant vector fields in the global coordinates (v, ξ, η) are:

$$D_{1} = (d\ell_{(u,\mu,\nu)})_{(1,0,0)} \mathcal{E}_{1} = v\partial_{\eta}$$

$$D_{2} = (d\ell_{(u,\mu,\nu)})_{(1,0,0)} \mathcal{E}_{2} = -\eta\partial_{v} + (v + \xi v^{-1}\eta)\partial_{\xi}$$

$$E = (d\ell_{(u,\mu,\nu)})_{(1,0,0)} e_{1} = v\partial_{v} + \xi\partial_{\xi} + \eta\partial_{\eta}$$

As one can readily check $[D_1, D_2] = -2E$ (so we set C = -2E) and $[D_i, D_i] = 0$.

Remark 3.2. We can interpret the multiplicative and additive SUSY 2-supergroups using matrix supergroups. $(\mathbb{C}^{1|2})^{\times}$ is SL(1|1). In functor of

points notation:

$$(\mathbb{C}^{1|2})^{\times}(T) = \mathrm{SL}(1|1)(T) = \left\{ \begin{pmatrix} u & \xi \\ \eta & v \end{pmatrix} \mid v^{-1}(u - \xi v^{-1}\eta) = 1 \right\}$$

 $\mathbb{C}^{1|2}$ is the subgroup of SL(2|1) given in the functor of points notation by:

$$\mathbb{C}^{1|2}(T) = \left\{ \begin{pmatrix} 1 & z & \zeta \\ 0 & 1 & 0 \\ 0 & \chi & 1 \end{pmatrix} \right\}$$

For example, let us check the second of these statements, the first one being the same. We can verify the claim reasoning in terms of the functor of points. So we have to compute:

$$\begin{pmatrix} 1 & z & \zeta \\ 0 & 1 & 0 \\ 0 & \chi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z' & \zeta' \\ 0 & 1 & 0 \\ 0 & \chi' & 1 \end{pmatrix} = \begin{pmatrix} 1 & z' + z + \zeta \chi' & \zeta + \zeta' \\ 0 & 1 & 0 \\ 0 & \chi + \chi' & 1 \end{pmatrix}$$

which is precisely the multiplication as in (2). This approach can be helpful in calculations. We leave to the reader the straightforward checks regarding the group law of the first statement.

We now want to interpret some of the discussion in [20] Sec. 6 in the framework of SUSY supergroups. Let X be a SUSY 1-curve and \widehat{X} its dual. The T-points of \widehat{X} are the 0|1 subvarieties of X(T). Let Δ be the superdiagonal subscheme of $X \times \widehat{X}$. It is locally defined by the incidence relation:

$$z - z' - \zeta'\zeta = 0 \tag{3}$$

 (z,ζ') and (z',ζ') local coordinates of X and \widehat{X} (see Def. 6.2 in [20]). Δ is a SUSY 2-curve, with distributions \mathcal{D}_1 , \mathcal{D}_2 and we have the commutative diagram:

$$\Delta \qquad (4)$$

$$X = \Delta/\mathcal{D}_1 \qquad \longrightarrow \Delta/\mathcal{D}_2 = \hat{X}$$

where Δ/\mathcal{D}_i means the superspace whose reduced space is $|\Delta|$, and whose structure sheaf is the subsheaf of \mathcal{O}_{Δ} consisting of sections which are invariant under \mathcal{D}_i . Specifying a SUSY-1 structure on X gives an isomorphism $X \cong \widehat{X}$.

On $(\mathbb{C}^{1|2})^{\times}$ we have the global SUSY 2-structure:

$$D_1 = \partial_{\zeta_1} + \zeta_2 \partial_z \quad D_2 = \partial_{\zeta_2} + \zeta_1 \partial_z \tag{5}$$

Clearly $D_1^2 = D_2^2 = 0$ and $[D_1, D_2] = 2\partial_z$. In Remark 3.2, we have viewed $(\mathbb{C}^{1|2})^{\times}$ as the supergroup $\mathrm{SL}(1|1)$. The condition on the berezinian is the (global) incidence relation (3) and allow us to identify $\mathrm{SL}(1|1)$ with Δ for $X = (\mathbb{C}^{1|1})^{\times}$ and $\mathrm{GL}(1|1)$ with $X \times \widehat{X}$. Notice that in our special case, namely for $X = \widehat{X} = (\mathbb{C}^{1|1})^{\times}$, we have that $X, \widehat{X} \subset \mathrm{SL}(1|1)$ as the subsupergroups:

$$X(T) = \left\{ \begin{pmatrix} x & \xi \\ \xi & x \end{pmatrix} \right\}, \qquad \widehat{X}(T) = \left\{ \begin{pmatrix} y & \eta \\ -\eta & y \end{pmatrix} \right\}$$

These inclusions correspond to the Lie superalgebra inclusions:

$$\langle C, U = E + F \rangle$$
, $\langle C, V = E - F \rangle \subset \langle C, E, F \rangle = \text{sl}(1|1)$

where C, E, F are the usual generators for sl(1|1), namely:

$$[C, E] = [C, F] = [E, E] = [F, F] = 0,$$
 $[E, F] = C$ (6)

4 Real forms of SUSY supergroups

We want to study the real forms of SUSY N-supergroups of type 1. In [5] and [6] we proved that, up to isomorphism, there is one real form of the SUSY 1-supergroup $(\mathbb{C}^{1|1})^{\times}$ and the corresponding involution is the composition of complex conjugation and the SUSY preserving automorphisms P_{\pm} . We wish to prove a similar result for the SUSY 2-supergroups.

Definition 4.1. Let X be a SUSY 2-curve with distributions \mathcal{D}_i . We say that an automorphism $\phi: X \longrightarrow X$ is SUSY preserving if $\phi_*(\mathcal{D}_i) = \mathcal{D}_j$, that is, if ϕ preserves individually each distribution or exchanges them. If X is a SUSY 2-supergroup we additionally require ϕ to be a supergroup automorphism.

Notice that in a SUSY 2 curve the roles of \mathcal{D}_1 and \mathcal{D}_2 are interchangeable; this forces us to give such a definition of SUSY preserving automorphism.

We start our discussion by observing that, up to isomorphism, there is only one real form of the Lie superalgebra sl(1|1) with compact even part.

In fact, assume $\mathfrak{g}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iC, U, V\}$ is such real form, with central even element iC (see (6)). If U = aE + bF, V = cE + dF, there is no loss of generality in assuming a = 1 because $E \mapsto a^{-1}E$, $F \mapsto aF$, $C \mapsto C$ is a Lie superalgebra automorphism of $\operatorname{sl}(1|1)$. Assume furtherly $[U, U] \neq 0$ (when both [U, U] = [V, V] = 0 we leave to the reader the easy check of what the real form is). Then we have that b = i, up to a constant, that we absorbe in C. An easy calculation shows that V = iE - F, hence we have proven the following proposition.

Proposition 4.2. Up to isomorphism, there is a unique real form of sl(1|1) with compact even part, namely

$$\mathrm{su}(1|1) = \mathrm{span}_{\mathbb{R}}\{iC, U = E + iF, V = iE - F\} \subset \mathrm{sl}(1|1).$$

We can then state the theorem giving all real forms of SUSY 2-supergroups with compact support.

Theorem 4.3. Up to isomorphism, there exists a unique real form of the SUSY 2-supergroup SL(1|1) and it is obtained with an involution $\sigma = c \circ \phi$ where ϕ is a SUSY preserving automorphism and c is a complex conjugation. Explicitly:

$$\sigma \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} \overline{d}^{-1} & -i\overline{a}^{-2}\overline{\gamma} \\ -i\overline{a}^{-2}\overline{\beta} & \overline{a}^{-1} \end{pmatrix}$$

Proof. We first check that the given σ is of the prescribed type, namely that

$$\phi \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} d^{-1} & -ia^{-2}\gamma \\ -ia^{-2}\beta & a^{-1} \end{pmatrix}$$

is a SUSY preserving automorphism. The fact that ϕ is a supergroup automorphism is a simple check, that can be verified using the functor of point formalism, namely one sees that:

$$\phi\left(\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \cdot \begin{pmatrix} a' & \beta' \\ \gamma' & d' \end{pmatrix}\right) = \begin{pmatrix} d^{-1} & -ia^{-2}\gamma \\ -ia^{-2}\beta & a^{-1} \end{pmatrix} \begin{pmatrix} d'^{-1} & -ia'^{-2}\gamma \\ -ia'^{-2}\beta & a'^{-1} \end{pmatrix}$$

Since ϕ is a supergroup morphism, $d\phi$ preserves left invariant vector fields (see [3] Ch. 7), hence it preserves the SUSY structure.

As for uniqueness, by Prop. 4.2 we know there exists a unique real form su(1|1) of sl(1|1) = Lie(SL(1|1)) and one readily checks su(1|1) = Lie(SU(1|1)). By the equivalence of categories in SHCP theory we obtain the result.

5 The SUSY preserving automorphisms of $\mathbb{C}^{1|1}$

Let our notation and terminology be as in [5], [6].

On $\mathbb{C}^{1|1}$ we have the globally defined SUSY structure given by the vector field:

$$D = \partial_{\zeta} + \zeta \partial_z$$

where (z,ζ) are the global coordinates. This structure is unique up to isomorphism (see [13] Sec. 4). We want to determine the supergroup of automorphism of $\mathbb{C}^{1|1}$ preserving such SUSY structure. We will denote it with $\operatorname{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1})$. In the work [13] we have provided the \mathbb{C} -points of such supergroup; they are obtained by looking at the transformations leaving invariant the 1-form:

$$s = dz - \zeta d\zeta$$

and are given by the endomorphisms

$$F(z,\zeta) = (az + b, \sqrt{a}\zeta)$$

We can identify the \mathbb{C} -points of the supergroup of SUSY preserving automorphism with the matrix group

$$\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C}) = \left\{ \begin{pmatrix} c & d & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \subset \operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{P}^{1|1})(\mathbb{C}) \quad (7)$$

This is a subgroup of the \mathbb{C} -points of the SUSY-preserving automorphisms of the SUSY 1-curve $\mathbb{P}^{1|1}$, namely those fixing the point at infinity (see Sec. 5 [13] and Sec. 5). In such identification $a=c^2$ and b=dc. Notice that, though $\mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C})$ is a matrix group, it is not obvious that also $\mathrm{Aut}_{\mathrm{SUSY}}(\mathbb{C}^{1|1})$ should be, since we are looking at the SUSY preserving automorphism of $\mathbb{C}^{1|1}$ as supermanifold morphisms. Neverthless we will show that this is the case.

An automorphism $F: \mathbb{C}^{1|1} \longrightarrow \mathbb{C}^{1|1}$ induces an automorphism $F^*: \mathcal{O}(\mathbb{C}^{1|1}) \longrightarrow \mathcal{O}(\mathbb{C}^{1|1})$ of the superalgebra of global sections. F is SUSY preserving if and only if

$$F^* \circ D = kD \circ F^* \tag{8}$$

where D is now interpreted as a derivation of $\mathcal{O}(\mathbb{C}^{1|1})$ and k is a suitable constant. We first consider the infinitesimal picture and compute

 $\operatorname{Lie}(\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1}))$. By (7), $\operatorname{Lie}(\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1}))_0$ is 2 dimensional, and as one can readily check, it is spanned by the two even vector fields:

$$U_1 = 2z\partial_z + \zeta\partial_\zeta, \quad U_2 = \partial_z$$

We hence only need to compute $Lie(Aut_{SUSY}(\mathbb{C}^{1|1}))_1$.

Proposition 5.1. Lie(Aut_{SUSY}($\mathbb{C}^{1|1}$)) is the Lie subsuperalgebra of the vector fields on $\mathbb{C}^{1|1}$ spanned by

$$U_1 = 2z\partial_z + \zeta\partial_\zeta, \quad U_2 = \partial_z, \quad V = \zeta\partial_z - \partial_\zeta.$$

with brackets:

$$[V, V] = 2U_2, \quad [U_2, U_1] = -2U_1, \quad [U_2, V] = -V, \quad [U_1, V] = 0$$

Proof. Consider $I + \theta \chi$, for $\chi \in \text{Lie}(\text{Aut}_{\text{SUSY}}(\mathbb{C}^{1|1}))_1$. The condition (8) gives immediately that the odd derivation χ^* of $\mathcal{O}(\mathbb{C}^{1|1})$ induced by χ must satisfy $[\chi^*, D] = 0$. A small calculation gives then the result.

In the Super Harish-Chandra pair (SHCP) formalism, we can immediately write the supergroup of SUSY preserving automorphism:

$$\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1}) = (\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C}), \operatorname{Lie}(\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1}))$$

The next proposition identifies such supergroup with a natural subsupergroup of $\operatorname{Aut}_{SUSY}(\mathbb{P}^{1|1}) = \operatorname{SpO}(2|1)$ (Ref. [14]) using the more geometric functor of points approach.

Proposition 5.2. Aut_{SUSY}($\mathbb{C}^{1|1}$) is the stabilizer subsupergroup in SpO(2|1) of the point at infinity:

$$\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1})(T) = \left\{ \begin{pmatrix} c & d & \gamma \\ 0 & c^{-1} & 0 \\ 0 & c^{-1}\gamma & 1 \end{pmatrix} \mid c, d \in \mathcal{O}(T)_0, \ \gamma \in \mathcal{O}(T)_1 \right\}$$

 $T \in (\text{smflds})_{\mathbb{R}}$.

Proof. The first statement is an immediate consequence of Proposition 5.1. As for the second one, consider the subgroup G of $SpO(2|1) = Aut_{SUSY}(\mathbb{P}^{1|1})$ that fixes the point at infinity. Its functor of points is given by:

$$G(T) = \left\{ \begin{pmatrix} c & d & \gamma \\ 0 & c^{-1} & 0 \\ 0 & c^{-1}\gamma & 1 \end{pmatrix} \mid c, d \in \mathcal{O}(T)_0, \ \gamma \in \mathcal{O}(T)_1 \right\}$$

G is representable and its SHCP coincides with $\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1})$, because $\operatorname{Lie}(G)$ = $\operatorname{Lie}(\operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1}), |G| = \operatorname{Aut}_{\operatorname{SUSY}}(\mathbb{C}^{1|1})(\mathbb{C})$ and we have the compatibility conditions.

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