An Analogue of Hilton-Milner Theorem for Set Partitions

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Abstract

Let $\mathcal{B}(n)$ denote the collection of all set partitions of [n]. Suppose $\mathcal{A} \subseteq \mathcal{B}(n)$ is a non-trivial t-intersecting family of set partitions i.e. any two members of \mathcal{A} have at least t blocks in common, but there is no fixed t blocks of size one which belong to all of them. It is proved that for sufficiently large n depending on t,

$$|\mathcal{A}| \le B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} + t$$

where B_n is the *n*-th Bell number and \tilde{B}_n is the number of set partitions of [n] without blocks of size one. Moreover, equality holds if and only if A is equivalent to

$$\{P \in \mathcal{B}(n) : \{1\}, \{2\}, \dots, \{t\}, \{i\} \in P \text{ for some } i \neq 1, 2, \dots, t, n\} \cup \{Q(i, n) : 1 \leq i \leq t\}$$

where $Q(i,n) = \{\{i,n\}\} \cup \{\{j\} : j \in [n] \setminus \{i,n\}\}$. This is an analogue of the Hilton-Milner theorem for set partitions.

KEYWORDS: intersecting family, Hilton-Milner, Erdős-Ko-Rado, set partitions

1 Introduction

1.1 Finite sets

Let $[n] = \{1, \dots, n\}$ and $\binom{[n]}{k}$ denote the family of all k-subsets of [n].

One of the most beautiful result in extremal combinatorics is the Erdős-Ko-Rado theorem ([5], [6], [18]) which asserts that if a family $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting (i.e. $|A \cap B| \ge t$ for any $A, B \in \mathcal{A}$) and n > 2k - t, then $|\mathcal{A}| \le \binom{n-t}{k-t}$ for $n \ge (k - t + 1)(t + 1)$.

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Theorem 1.1 (Erdős, Ko, and Rado [5], Frankl [6], Wilson [18]). Suppose $A \subseteq {n \choose k}$ is t-intersecting and n > 2k - t. Then for $n \ge (k - t + 1)(t + 1)$,

$$|\mathcal{A}| \le \binom{n-t}{k-t}.$$

Moreover, if n > (k - t + 1)(t + 1), equality holds if and only if $A = \{A \in {[n] \choose k} : T \subseteq A\}$ for some t-set T.

For a family \mathcal{A} of k-subsets, \mathcal{A} is said to be *trivially t*-intersecting if there exists a t-set $T = \{x_1, \ldots, x_t\}$ such that all members of \mathcal{A} contains T. The Erdős-Ko-Rado theorem implies that a t-intersecting family of maximum size must be trivially t-intersecting when n is sufficiently large in terms of k and t.

Hilton and Milner [9] proved a strengthening of the Erdős-Ko-Rado theorem for t = 1 by determining the maximum size of a non-trivial 1-intersecting family. A short and elegant proof was later given by Frankl and Füredi [7] using the shifting technique.

Theorem 1.2 (Hilton-Milner). Let $A \subseteq {[n] \choose k}$ be a non-trivial 1-intersecting family with $k \geq 4$ and n > 2k. Then

$$|\mathcal{A}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality holds if and only if

$$\mathcal{A} = \{X \in \binom{[n]}{k} : x \in X, X \cap Y \neq \emptyset\} \cup \{Y\}$$

for some k-subset $Y \in {[n] \choose k}$ and $x \in X \setminus Y$.

1.2 Permutations and set partitions

The main result of this paper is motivated by recent investigations of the Erdős-Ko-Rado type of problems for permutations and set partitions.

The study of intersecting families of permutations was initiated by Deza and Frankl [2] in the context of coding theory. Let $\operatorname{Sym}(n)$ denote the set of all permutations of [n]. A family $\mathcal{A} \subseteq \operatorname{Sym}(n)$ is t-intersecting if $|\{x:g(x)=h(x)\}| \geq t$ for any $g,h \in \mathcal{A}$.

Recently, Ellis, Friedgut and Pilpel [4] showed that for sufficiently large n depending on t, a t-intersecting family \mathcal{A} of permutations has size at most (n-t)!, with equality if and only if \mathcal{A} is a coset of the stabilizer of t points, thus settling an old conjecture of Deza and Frankl in the affirmative. The proof uses spectral methods and representations of the symmetric group. Subsequently, building on the representation theoretic approach, Ellis [3] proved an analogue of the Hilton-Milner theorem for t-intersecting families of permutations. The readers may also refer to [1, 8, 10, 12, 14, 15, 16, 17] for some recent results on the Erdős-Ko-Rado type of problems.

On the other hand, recall that a set partition of [n] is a collection of pairwise disjoint nonempty subsets (called *blocks*) of [n] whose union is [n]. Let $\mathcal{B}(n)$ denote the family of all set partitions of [n]. It is well-known that the size of $\mathcal{B}(n)$ is the n-th Bell number, denoted by B_n . A block of size one is also known as a *singleton*. We denote the number of all set partitions of [n] which are singleton-free (i.e. without any singleton) by \tilde{B}_n .

A family $A \subseteq \mathcal{B}(n)$ is said to be *t-intersecting* if any two of its members have at least t blocks in common. It is *trivially* t-intersecting if it consists of set partitions containing t fixed singletons. Note that if it is trivially t-intersecting of maximum size then it consists of all set partitions containing the t fixed singletons.

Motivated by the Erdős-Ko-Rado theorem, Ku and Renshaw [13, Theorem 1.7 and Theorem 1.8] proved the following analogue of the Erdős-Ko-Rado theorem for set partitions.

Theorem 1.3 (Ku-Renshaw). Suppose $A \subseteq \mathcal{B}(n)$ is a t-intersecting family. Then for $n \geq n_0(t)$,

$$|\mathcal{A}| \leq B_{n-t},$$

with equality if and only if A is a trivially t-intersecting family of maximum size.

In view of the Hilton-Milner theorem, the aim of this paper is to determine the size and the structure of non-trivial t-intersecting families of set partitions. The analogy to set systems and permutations suggests that almost all members of such a family should share t common singletons. The following is an example of a large non-trivial t-intersecting family.

Let $a_1, a_2, \ldots, a_t, b \in [n]$, and all the a_i 's and b are distinct. Let $Q(a_i, b) \in \mathcal{B}(n)$ be the set partition containing $\{a_i, b\}$ and $\{j\}$ for all $j \neq a_i, b$. Set $\mathcal{Q} = \{Q(a_1, b)\} \cup \cdots \cup \{Q(a_t, b)\}$. A Hilton-Milner type family is given by

$$\mathcal{H}(a_1, \dots, a_t, b) = \{ P \in \mathcal{B}(n) : \{a_1\}, \dots, \{a_t\}, \{c\} \in P \text{ for some } c \neq a_1, \dots, a_t, b\} \cup \mathcal{Q}.$$

It is easily verified that

$$|\mathcal{H}(a_1,\ldots,a_t,b)| = B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} + t.$$

Indeed, the number of set partitions having $\{a_1\}$, $\{a_2\}$, ..., $\{a_t\}$ as the only singletons is \tilde{B}_{n-t} , and the number of set partitions having $\{a_1\}$, $\{a_2\}$, ..., $\{a_t\}$ and $\{b\}$ as the only singletons is \tilde{B}_{n-t-1} .

Using an analogue of the shifting operation for set partitions (called the *splitting* operation) first introduced by Ku and Renshaw in [13], we prove the following analogue of the Hilton-Milner theorem.

Theorem 1.4. Suppose A is a non-trivial t-intersecting family of set partitions of [n]. Then, for $n \geq n_0(t)$,

$$|\mathcal{A}| \le B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} + t,$$

with equality if and only if $A = \mathcal{H}(a_1, \ldots, a_t, b)$ for some $a_1, \ldots, a_t, b \in [n]$.

2 Splitting operation

In this section, we summarize some important results regarding the splitting operation for intersecting family of set partitions. We refer the reader to [13] for proofs which are omitted here.

Let $i, j \in [n]$, $i \neq j$, and $P \in \mathcal{B}(n)$. Denote by $P_{[i]}$ the block of P which contains i. We define the (i, j)-split of P to be the following set partition:

$$s_{ij}(P) = \begin{cases} P \setminus \{P_{[i]}\} \cup \{\{i\}, P_{[i]} \setminus \{i\}\} & \text{if } j \in P_{[i]}, \\ P & \text{otherwise.} \end{cases}$$

For a family $A \subseteq \mathcal{B}(n)$, let $s_{ij}(A) = \{s_{ij}(P) : P \in A\}$. Any family A of set partitions can be decomposed with respect to given $i, j \in [n]$ as follows:

$$\mathcal{A} = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},$$

where $A_{ij} = \{P \in \mathcal{A} : s_{ij}(P) \notin \mathcal{A}\}$. Define the (i, j)-splitting of \mathcal{A} to be the family

$$S_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup s_{ij}(\mathcal{A}_{ij}).$$

Let I(n,t) denote the set of all t-intersecting families of set partitions of [n]. Surprisingly, it turns out that for any $A \in I(n,t)$, splitting operations preserve the size and the intersecting property.

Proposition 2.1 ([13], Proposition 3.2). Let $A \in I(n,t)$. Then $S_{ij}(A) \in I(n,t)$ and $|S_{ij}(A)| = |A|$.

A family \mathcal{A} of set partitions is *compressed* if for any $i, j \in [n]$, $i \neq j$, we have $S_{ij}(\mathcal{A}) = \mathcal{A}$. For a set partition P, let $\sigma(P) = \{x : \{x\} \in P\}$ denote the union of its singletons (block of size 1). For a family \mathcal{A} of set partitions, let $\sigma(\mathcal{A}) = \{\sigma(P) : P \in \mathcal{A}\}$. Note that $\sigma(\mathcal{A})$ is a family of subsets of [n].

Proposition 2.2 ([13], Proposition 3.3). Given a family $A \in I(n,t)$, by repeatedly applying the splitting operations, we eventually obtain a compressed family $A^* \in I(n,t)$ with $|A^*| = |A|$.

For a compressed family \mathcal{A} , its intersecting property can be transferred to $\sigma(\mathcal{A})$, thus allowing us to access the structure of \mathcal{A} via the structure of $\sigma(\mathcal{A})$.

Proposition 2.3 ([13], Proposition 3.4). If $A \in I(n,t)$ is compressed, then $\sigma(A)$ is a t-intersecting family of subsets of [n].

Lemma 2.4. Suppose $A \in I(n,t)$ and $S_{ij}(A) = \mathcal{H}(a_1,\ldots,a_t,b)$. If

(a)
$$P_e = \{\{a_1\}, \dots, \{a_t\}, \{e\}, [n] \setminus \{a_1, \dots, a_t, e\}\} \in \mathcal{A},$$
 for all $e \in [n] \setminus \{a_1, \dots, a_t, b\}$, and

(b) $Q(a_l, b) \in \mathcal{A} \text{ for all } 1 \leq l \leq t,$

then $\mathcal{A} = \mathcal{H}(a_1, \ldots, a_t, b)$.

Proof. Suppose there is a $P \in S_{ij}(\mathcal{A}) \setminus \mathcal{A}$. Then $P = s_{ij}(T)$ for some $T \in \mathcal{A}$ and $T \notin S_{ij}(\mathcal{A})$.

Case 1. Suppose $i \neq a_1, \ldots, a_t$. If $i \neq b$, then P consists of $\{a_1\}, \ldots, \{a_t\}, \{i\}$ and B, where B is a set partition of $[n] \setminus \{a_1, \ldots, a_t, i\}$. Suppose B does not contain any singleton. If $j \neq a_1, \ldots, a_t$, then the only singletons in T are $\{a_1\}, \ldots, \{a_t\}$. If $j = a_{l_1}$ for some $1 \leq l_1 \leq t$, then the only singletons in T are $\{a_1\}, \ldots, \{a_{l_1-1}\}, \{a_{l_1+1}\}, \ldots, \{a_t\}$. In all cases, T has no singletons other than $\{a_1\}, \ldots, \{a_t\}$. Therefore $|T \cap Q(a_1, b)| \leq t - 1$, contradicting the fact that A is t-intersecting. Similarly if B contains the singleton $\{b\}$ or $\{j\}$ only, then $|T \cap Q(a_1, b)| \leq t - 1$, a contradiction. Hence B contains a singleton $\{e\}$ for some $e \neq b, j$. This means that T contains the singletons $\{a_1\}, \ldots, \{a_t\}, \{e\}$, and so $T \in \mathcal{H}(a_1, \ldots, a_t, b)$, contradicting the fact that $T \notin S_{ij}(A)$.

If i = b, then P consists of $\{a_1\}, \ldots, \{a_t\}, \{b\}, \{e_1\}$ and B, where $e_1 \in [n] \setminus \{a_1, \ldots, a_t, b\}$ and B is a set partition of $[n] \setminus \{a_1, \ldots, a_t, b, e_1\}$. If $j \neq a_1, \ldots, a_t$, then T contains the singletons $\{a_1\}, \ldots$,

 $\{a_t\}$, $\{e_1\}$, a contradiction. Suppose $j=a_{l_0}$ for some $1 \leq l_0 \leq t$. Suppose B contains a block of size at least 2. Let e_0 be an element in this block. Note that $|P_{e_0} \cap T| = t - 1$, a contradiction. So we may assume that B consists of singletons, but then $T = Q(a_{l_0}, b)$, a contradiction.

Case 2. Suppose $i = a_{l_0}$ for some $1 \le l_0 \le t$. Then P consists of $\{a_1\}, \ldots, \{a_t\}, \{e\}$ and B, where $e \in [n] \setminus \{a_1, \ldots, a_t, b\}$ and B is a set partition of $[n] \setminus \{a_1, \ldots, a_t, e\}$. Suppose B contains a block of size at least 2. Let e_0 be an element in this block. We may assume $e_0 \ne b$. Note that $|P_{e_0} \cap T| = t - 1$, a contradiction. So we may assume that B consists of singletons, but then $T = Q(a_{l_0}, b)$, a contradiction.

Hence
$$S_{ij}(\mathcal{A}) = \mathcal{A}$$
.

The following proposition says that a Hilton-Milner type family is preserved when 'undoing' the splitting operations.

Proposition 2.5. Suppose $n \geq t+3$, $A \in I(n,t)$ and $S_{ij}(A) = \mathcal{H}(a_1,\ldots,a_t,b)$. Then $A = \mathcal{H}(a_1,\ldots,a_t,b)$.

Proof. It is sufficient to show that conditions (a) and (b) of Lemma 2.4 hold.

Let $e \in [n] \setminus \{a_1, \ldots, a_t, b\}$ and

$$P_e = \{\{a_1\}, \dots, \{a_t\}, \{e\}, [n] \setminus \{a_1, \dots, a_t, e\}\}.$$

Note that $P_e \in S_{ij}(\mathcal{A})$ and $|[n] \setminus \{a_1, \ldots, a_t, e\}| \geq 2$.

Case 1. Suppose $i, j \neq a_1, \ldots, a_t$. Assume that $e \neq i$. If $P_e \notin \mathcal{A}$, then $P_e = s_{ij}(T_e)$ for some $T_e \in \mathcal{A}$, a contradiction, for i cannot be contained in a block of size greater than 1 after the splitting operation. So $P_e \in \mathcal{A}$ for all $e \in [n] \setminus \{a_1, \ldots, a_t, b, i\}$.

Suppose $j \neq b$. Now if $Q(a_l, b) \notin \mathcal{A}$ for some $1 \leq l \leq t$, then

$$W_1 = \{\{a_l, b\}\} \cup \{\{i, j\}\} \cup \{\{q\} : q \in [n] \setminus \{a_l, b, i, j\}\} \in \mathcal{A},$$

and $|W_1 \cap P_j| = t - 1$, contradicting the fact that \mathcal{A} is t-intersecting. So $Q(a_l, b) \in \mathcal{A}$ for all $1 \leq l \leq t$. It remains to show that $P_i \in \mathcal{A}$ when $i \neq b$. If $P_i \notin \mathcal{A}$, then

$$W_2 = \{\{a_1\}, \dots, \{a_t\}, [n] \setminus \{a_1, \dots, a_t\}\} \in \mathcal{A},$$

a contradiction, for $|W_2 \cap Q(a_1, b)| = t - 1$.

Suppose j = b. Then $i \neq b$. If $P_i \notin \mathcal{A}$, then $W_2 \in \mathcal{A}$. If $Q(a_l, b) \notin \mathcal{A}$ for some $1 \leq l \leq t$, then

$$W_3 = \{\{a_l, b, i\}\} \cup \{\{q\} : q \in [n] \setminus \{a_l, b, i\}\} \in \mathcal{A},$$

and $|W_2 \cap W_3| = t - 1$, a contradiction. If $Q(a_l, b) \in \mathcal{A}$ for some $1 \leq l \leq t$, then $|W_2 \cap Q(a_l, b)| = t - 1$, again a contradiction. Hence $P_i \in \mathcal{A}$. Since $|W_3 \cap P_i| = t - 1$, we conclude that $Q(a_l, b) \in \mathcal{A}$ for all $1 \leq l \leq t$.

Case 2. Suppose $i = a_{l_0}$ and $j = a_{l_1}$ for some $1 \le l_0, l_1 \le t$. Without loss of generality assume that $l_0 = 1$ and $l_1 = 2$. Note that $Q(a_1, b) \in \mathcal{A}$. Now if $P_e \notin \mathcal{A}$, then

$$W_1 = \{\{a_1, a_2\}, \{a_3\}, \dots, \{a_t\}, \{e\}, [n] \setminus \{a_1, \dots, a_t, e\}\} \in \mathcal{A},$$

a contradiction, for $|Q(a_1,b) \cap W_1| = t-1$. So $P_e \in \mathcal{A}$ for all $e \in [n] \setminus \{a_1,\ldots,a_t,b\}$.

Now if $Q(a_l, b) \notin \mathcal{A}$ for some $3 \leq l \leq t$, then

$$W_2 = \{\{a_l, b\}\} \cup \{\{a_1, a_2\}\} \cup \{\{q\} : q \in [n] \setminus \{a_1, a_2, a_l, b\}\} \in \mathcal{A},$$

a contradiction, as $|P_e \cap W_2| = t - 2$. Next if $Q(a_2, b) \notin \mathcal{A}$, then

$$W_3 = \{\{a_1, a_2, b\}\} \cup \{\{q\} : q \in [n] \setminus \{a_1, a_2, b\}\} \in \mathcal{A},$$

again a contradiction, as $|P_e \cap W_3| = t - 1$. Thus $Q(a_l, b) \in \mathcal{A}$ for all $1 \leq l \leq t$.

Case 3. Suppose $j = a_{l_0}$ for some $1 \leq l_0 \leq t$. Without loss of generality assume that $l_0 = 1$. As in Case 1, $P_e \in \mathcal{A}$ for all $e \in [n] \setminus \{a_1, \ldots, a_t, b, i\}$. By Case 2, we may assume that $i \neq a_1, \ldots, a_t$.

Suppose i = b. Then $Q(a_l, b) \in \mathcal{A}$ for all $1 \le l \le t$, and we are done.

Suppose $i \neq b$. If $Q(a_1, b) \notin \mathcal{A}$, then

$$W = \{\{a_1, b, i\}\} \cup \{\{q\} : q \in [n] \setminus \{a_1, b, i\}\} \in \mathcal{A}.$$

Since $P_i \in S_{ia_1}(\mathcal{A})$, and $|P_i \cap W| = t - 1$, we must have

$$R' = \{\{a_1, i\}, \{a_2\}, \dots, \{a_t\}, [n] \setminus \{a_1, \dots, a_t, i\}\} \in \mathcal{A},$$

but then $|R' \cap W| = t - 1$, a contradiction. Hence $Q(a_1, b) \in \mathcal{A}$. Now $|Q(a_1, b) \cap R'| = t - 1$ implies that $P_i \in \mathcal{A}$. Next if $Q(a_l, b) \notin \mathcal{A}$ for some $2 \leq l \leq t$, then

$$W_2 = \{\{a_l, b\}\} \cup \{\{a_1, i\}\} \cup \{\{q\} : q \in [n] \setminus \{a_1, i, a_l, b\}\} \in \mathcal{A},$$

a contradiction, as $|P_i \cap W_2| = t - 2$. Thus $Q(a_l, b) \in \mathcal{A}$ for all $1 \leq l \leq t$.

Case 4. Suppose $i = a_{l_0}$ for some $1 \le l_0 \le t$. Without loss of generality assume that $l_0 = 1$. By Case 2, we may assume that $j \ne a_1, \ldots, a_t$.

Suppose j = b. Note that $Q(a_1, b) \in \mathcal{A}$. Let $e_0, e_1 \in [n] \setminus \{a_1, \dots, a_t, b\}$, $e_0 \neq e_1$. If both P_{e_0} and P_{e_1} are not contained in \mathcal{A} , then $W_0, W_1 \in \mathcal{A}$, where

$$W_0 = \{\{a_2\}, \dots, \{a_t\}, \{e_0\}, [n] \setminus \{a_2, \dots, a_t, e_0\}\},$$

$$W_1 = \{\{a_2\}, \dots, \{a_t\}, \{e_1\}, [n] \setminus \{a_2, \dots, a_t, e_1\}\}.$$

We have obtained a contradiction, as $|W_0 \cap W_1| = t - 1$. So we may assume $P_{e_0} \in \mathcal{A}$. If $Q(a_l, b) \notin \mathcal{A}$ for some $2 \leq l \leq t$, then

$$W_2 = \{\{a_l, b, a_1\}\} \cup \{\{q\} : q \in [n] \setminus \{a_l, b, a_1\}\} \in \mathcal{A},$$

and $|W_2 \cap P_{e_0}| = t - 1$, a contradiction. Thus $Q(a_l, b) \in \mathcal{A}$ for all $1 \le l \le t$. Since $|Q(a_2, b) \cap W_1| = t - 1$, we conclude that $P_{e_1} \in \mathcal{A}$. In fact, $P_e \in \mathcal{A}$ for all $e \in [n] \setminus \{a_1, \ldots, a_t, b\}$.

Suppose $j \neq b$. Note that $Q(a_1, b) \in \mathcal{A}$. If $P_j \notin \mathcal{A}$, then

$$W_3 = \{\{a_1, j\}, \{a_2\}, \dots, \{a_t\}, [n] \setminus \{a_1, \dots, a_t, j\}\} \in \mathcal{A}.$$

This contradicts that \mathcal{A} is t-intersecting as $|Q(a_1,b) \cap W_3| = t-1$. Thus $P_j \in \mathcal{A}$. Now if $Q(a_l,b) \notin \mathcal{A}$ for some $2 \leq l \leq t$, then

$$W_4 = \{\{a_l, b\}\} \cup \{\{a_1, j\}\} \cup \{\{q\} : q \in [n] \setminus \{a_1, j, a_l, b\}\} \in \mathcal{A}.$$

This contradicts that \mathcal{A} is t-intersecting as $|P_j \cap W_4| = t - 2$. Thus $Q(a_l, b) \in \mathcal{A}$ for all $1 \leq l \leq t$. Finally if $P_e \notin \mathcal{A}$ and $e \neq j$, then

$$W_5 = \{\{a_2\}, \dots, \{a_t\}, \{e\}, [n] \setminus \{a_1, a_2, \dots, a_t, e\}\} \in \mathcal{A},$$

a contradiction, for $|Q(a_2,b) \cap W_5| = t-1$. Hence $P_e \in \mathcal{A}$ for all $e \in [n] \setminus \{a_1,\ldots,a_t,b\}$.

3 Proof of main result

The following identities for B_n and \tilde{B}_n are straightforward.

Lemma 3.1. Let $n \geq 2$. Then

$$B_n = \sum_{k=0}^n \binom{n}{k} \tilde{B}_{n-k}, \tag{1}$$

$$\tilde{B}_n = \sum_{k=1}^{n-1} {n-1 \choose k} \tilde{B}_{n-1-k}, \tag{2}$$

with the conventions $B_0 = \tilde{B}_0 = 1$.

Note in passing that $\tilde{B}_1 = 0$. By (1) and (2),

$$B_n = \tilde{B}_n + \tilde{B}_{n+1} = B_{n-1} - \tilde{B}_{n-1} + \tilde{B}_{n+1} \le B_{n-1} + \tilde{B}_{n+1}.$$
(3)

Since $\lim_{n\to\infty} B_n/B_{n-1} = \infty$ (see [11, Corollary 2.7]), we deduce that

$$\lim_{n \to \infty} \tilde{B}_{n+1}/B_{n-1} = \infty. \tag{4}$$

Next note that $(B_{n-1} - \tilde{B}_{n-1} - \tilde{B}_{n-2})/B_{n-2} = (\tilde{B}_n - \tilde{B}_{n-2})/B_{n-2} \ge \tilde{B}_n/B_{n-2} - 1$. So $\lim_{n \to \infty} (B_{n-1} - \tilde{B}_{n-1} - \tilde{B}_{n-2})/B_{n-2} = \infty$ and Lemma 3.2 follows. Lemma 3.3 follows by noting that $B_{n-r+1} \le B_{n-t-3}$.

Lemma 3.2. Let c be a fixed positive integer. Then, for $n \geq n_0(t)$,

$$cB_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}.$$

Lemma 3.3. If $t + 4 \le r \le n - 2$ and $n \ge n_0(t)$, then

$$tB_{n-r+1} < \tilde{B}_{n-t-1}.$$

Lemma 3.4. *For* $n \ge n_0(t)$,

$$\tilde{B}_{n-t-1} > \sum_{k=\left|\frac{n}{t+1}+t-1\right|+1}^{n} {n \choose k} \tilde{B}_{n-k}.$$

Proof. By (2),

$$\sum_{k=\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+1}^{n} \binom{n}{k} \tilde{B}_{n-k} \leq \tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+1} \sum_{k=\left\lfloor n/(t+1)+t-1\right\rfloor+1}^{n} \binom{n}{k}$$

$$\leq 2^{n} \tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+1}.$$

So it is sufficient to show that $\tilde{B}_{n-t-1}/\tilde{B}_{n-\lfloor \frac{n}{t+1}+t-1\rfloor+1} > 2^n$.

Again by (2), for any fixed r, $\tilde{B}_m/\tilde{B}_{m-2} > r$ for sufficiently large m. Therefore

$$\frac{\tilde{B}_{n-t-1}}{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+1}} \ge \begin{pmatrix} \frac{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+2u-1}}{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+2u-3}} \end{pmatrix} \cdots \begin{pmatrix} \frac{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+5}}{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+3}} \end{pmatrix} \begin{pmatrix} \frac{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+3}}{\tilde{B}_{n-\left\lfloor\frac{n}{t+1}+t-1\right\rfloor+1}} \\ > r^{u-1}, \end{pmatrix}$$

where $u = \lfloor \frac{1}{2}(\lfloor \frac{n}{t+1} + t - 1 \rfloor - t - 2)\rfloor$. Clearly $u - 1 \geq \frac{n}{4(t+1)}$. So if we choose $r = 2^{4(t+1)}$, then for sufficiently large n, the lemma follows.

Lemma 3.5. Let \mathcal{A} be a non-trivial t-intersecting family of set partitions of [n] of maximum size. Suppose for all $i, j \in [n]$ such that $S_{ij}(\mathcal{A}) \neq \mathcal{A}$, $S_{ij}(\mathcal{A})$ is trivially t-intersecting. If $S_{ab}(\mathcal{A}) \neq \mathcal{A}$ for some $a, b \in [n]$, then for $n \geq n_0(t)$, we have

$$\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2,$$

and either (5) or (6) holds:

$$\mathcal{A}_1 \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{b\} \in C\},$$

$$\emptyset \neq \mathcal{A}_2 \subseteq \{C \in \mathcal{B}(n) : \{a, b\} \in C\},$$

$$\mathcal{A} \subseteq \{C \in \mathcal{B}(n) : \{y_3\}, \dots, \{y_t\} \in C\},$$

$$(5)$$

for some fixed $y_3, \ldots, y_t \in [n] \setminus \{a, b\}$, or

$$\mathcal{A}_{1} \subseteq \{C \in \mathcal{B}(n) : \{a\} \in C\},$$

$$\emptyset \neq \mathcal{A}_{2} \subseteq \{C \in \mathcal{B}(n) : \{a,b\} \in C\},$$

$$\mathcal{A} \subseteq \{C \in \mathcal{B}(n) : \{x_{2}\}, \dots, \{x_{t}\} \in C\},$$

$$(6)$$

for some fixed $x_2, ..., x_t \in [n] \setminus \{a, b\}$. Here, $y_3, ..., y_t$ only exist if $t \geq 3$ and $x_2, ..., x_t$ only exist if $t \geq 2$.

Proof. By assumption, $S_{ab}(A)$ is trivially t-intersecting. This means that either

- (a) $\{a\}, \{b\}, \{y_3\}, \dots, \{y_t\} \in P \text{ for all } P \in S_{ab}(\mathcal{A}), \text{ or } \{y_t\} \in P$
- (b) $\{a\}, \{x_2\}, \dots, \{x_t\} \in P \text{ for all } P \in S_{ab}(A), \text{ or } A \in A$
- (c) $\{b\}, \{x_2\}, \dots, \{x_t\} \in P \text{ for all } P \in S_{ab}(A).$

Suppose (a) holds. Since A is non-trivially t-intersecting, we conclude that (5) holds.

Suppose (b) or (c) holds. Since \mathcal{A} is non-trivially t-intersecting, there is a $P_0 \in \mathcal{A}$ such that $s_{ab}(P_0) \notin \mathcal{A}$ and $P_0 = \{\{a,b\} \cup X_1\} \cup \{\{x_l\} : 2 \leq l \leq t\} \cup B_1 \text{ where } X_1 \subseteq [n] \setminus \{a,b,x_2,\ldots,x_t\} \text{ and } B_1 \text{ is a set partition of } [n] \setminus (\{a,b,x_2,\ldots,x_t\} \cup X_1) \text{ (we allow } X_1 = \emptyset).$

Suppose there is a $Q \in \mathcal{A}$ such that $Q = \{\{a,b\} \cup X_2\} \cup \{\{x_l\} : 2 \leq l \leq t\} \cup B_2$ where $X_2 \subseteq [n] \setminus \{a,b,x_2,\ldots,x_t\}$ and B_2 is a set partition of $[n] \setminus (\{a,b,x_2,\ldots,x_t\} \cup X_2)$. Suppose $X_2 \not\subseteq X_1$. Let $d \in X_2 \setminus X_1$. If $s_{ad}(Q), s_{bd}(Q) \in \mathcal{A}$, then $s_{ab}(s_{bd}(Q)) = s_{bd}(Q)$ and $s_{ab}(s_{ad}(Q)) = s_{ad}(Q)$. But this contradicts (b) and (c), as $\{a\}$ is not a block in $s_{bd}(Q)$ and $\{b\}$ is not a block in $s_{ad}(Q)$. So we may assume $s_{bd}(Q) \notin \mathcal{A}$. Since $s_{bd}(P_0) = P_0$, we see that $S_{bd}(\mathcal{A})$ is non-trivially t-intersecting and $S_{bd}(\mathcal{A}) \neq \mathcal{A}$, a contradiction. So we may assume $X_2 \subseteq X_1$. If there is a $c \in X_1 \setminus X_2$, then $s_{ac}(P_0) = s_{ab}(P_0)$, $s_{ac}(Q) = Q$, and thus $S_{ac}(\mathcal{A})$ is non-trivially t-intersecting and $S_{ac}(\mathcal{A}) \neq \mathcal{A}$, a contradiction. Therefore we may assume that

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

where

$$\mathcal{A}_1 \subseteq \{C \in \mathcal{B}(n) : \{a\} \in C\},$$

$$\emptyset \neq \mathcal{A}_2 \subseteq \{C \in \mathcal{B}(n) : \{a,b\} \cup X_1 \in C\},$$

$$\mathcal{A} \subseteq \{C \in \mathcal{B}(n) : \{x_2\}, \dots, \{x_t\} \in C\}.$$

Suppose $X_1 \neq \emptyset$. This implies that (b) holds. Note that $s_{ba}(P_0) \notin \mathcal{A}$, for otherwise $s_{ab}(s_{ba}(P_0)) = s_{ba}(P_0) \in S_{ab}(\mathcal{A})$ and it does not contain the singleton $\{a\}$. Now $S_{ba}(\mathcal{A}) \neq \mathcal{A}$ implies that $S_{ba}(\mathcal{A})$ is trivially t-intersecting (by assumption). Furthermore every element in $S_{ba}(\mathcal{A})$ contains the singleton $\{b\}$. Since $S_{ba}(\mathcal{A}_1) = \mathcal{A}_1$, we must have $\mathcal{A}_1 \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{b\}, \{x_2\}, \dots, \{x_t\} \in C\}$. Therefore $|\mathcal{A}_1| \leq B_{n-t-1}$, $|\mathcal{A}_2| \leq B_{n-t-1}$ and $|\mathcal{A}| \leq 2B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$ (Lemma 3.2), a contradiction, as \mathcal{A} is a non-trivial t-intersecting family of maximum size.

Theorem 3.6. Let \mathcal{A} be a non-trivial t-intersecting family of set partitions of [n] of maximum size. If \mathcal{A} is not compressed, then for $n \geq n_0(t)$, there exist $k, l \in [n]$ such that $S_{kl}(\mathcal{A}) \neq \mathcal{A}$ and $S_{kl}(\mathcal{A})$ is non-trivially t-intersecting.

Proof. Assume, for a contradiction, that for all $i, j \in [n]$ such that $S_{ij}(\mathcal{A}) \neq \mathcal{A}$, $S_{ij}(\mathcal{A})$ is trivially t-intersecting.

Since \mathcal{A} is not compressed, there exist $a, b \in [n]$ with $S_{ab}(\mathcal{A}) \neq \mathcal{A}$. By Lemma 3.5, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, and either (5) or (6) holds. Note that in either case $S_{aj}(\mathcal{A}) = \mathcal{A}$ and $S_{ja}(\mathcal{A}) = \mathcal{A}$ for all $j \in [n] \setminus \{a, b\}$. Note also that $S_{ab}(\mathcal{A}) = S_{ba}(\mathcal{A})$.

We have two cases.

Case 1. Suppose (5) holds. Then $S_{bj}(\mathcal{A}) = \mathcal{A}$ and $S_{jb}(\mathcal{A}) = \mathcal{A}$ for all $j \in [n] \setminus \{a, b\}$. Suppose there exist $k, l \in [n]$ with $S_{kl}(\mathcal{A}) \neq \mathcal{A}$ and $k, l \neq a, b$. Again by Lemma 3.5, $\mathcal{A} = \mathcal{A}_3 \cup \mathcal{A}_4$ where $\mathcal{A}_3 \subseteq \{C \in \mathcal{B}(n) : \{k\} \in C\}$ and $\mathcal{A}_4 \subseteq \{C \in \mathcal{B}(n) : \{k, l\} \in C\}$. Therefore

$$\mathcal{A} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4,$$

where

$$\mathcal{D}_{1} \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{b\}, \{k\} \in C\},\$$

$$\mathcal{D}_{2} \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{b\}, \{k, l\} \in C\},\$$

$$\mathcal{D}_{3} \subseteq \{C \in \mathcal{B}(n) : \{a, b\}, \{k\} \in C\},\$$

$$\mathcal{D}_{4} \subseteq \{C \in \mathcal{B}(n) : \{a, b\}, \{k, l\} \in C\}.$$

Now $k \neq y_3, \ldots, y_t$, for $S_{kl}(\mathcal{A}) \neq \mathcal{A}$. Therefore $|\mathcal{A}| \leq 4B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$ (Lemma 3.2), a contradiction, as \mathcal{A} is a maximum size non-trivial t-intersecting family.

So we may assume that $S_{ij}(\mathcal{A}) = \mathcal{A}$ for all $i, j \in [n]$ with $(i, j) \neq (a, b), (b, a)$. We first show that the interesting property of \mathcal{A} can be partially transferred to the family $\sigma(\mathcal{A})$ of sets which are union of singletons. In particular, we show the following *cross-intersecting* property:

$$|\sigma(P) \cap \sigma(R) \cap [n] \setminus \{a, b, y_3, \dots, y_t\}| \ge 2, \quad \forall P \in \mathcal{A}_1, R \in \mathcal{A}_2. \tag{7}$$

Assume for a contradiction that there exist $P \in \mathcal{A}_1$ and $R \in \mathcal{A}_2$ such that

$$|\sigma(P) \cap \sigma(R) \cap [n] \setminus \{a, b, y_3, \dots, y_t\}| \le 1.$$

Since P contains $\{a\}, \{b\}, \{y_3\}, \ldots, \{y_t\}$ and R contains $\{a, b\}, \{y_3\}, \ldots, \{y_t\}$, we conclude that P and R must have at least one block of size at least 2 in common. Suppose there are $s \geq 1$ such common blocks of P and R, say C_1, \ldots, C_s , which are disjoint from $\sigma(P) \cup \sigma(R) \cup \{a, b, y_3, \ldots, y_t\}$. Fix two distinct points w_i , z_i from each block C_i . Then, since $S_{ij}(A) = A$ for all $i, j \in [n]$ with $(i, j) \neq (a, b), (b, a)$, we have

$$R^* = s_{w_s, z_s}(\cdots(s_{w_1, z_1}(R))\cdots) \in \mathcal{A}.$$

However, $|P \cap R^*| \leq t - 1$, contradicting the t-intersecting property of A. This proves (7).

Note that $\{a\}, \{b\}, \{y_3\}, \ldots, \{y_t\} \in P$ for all $P \in S_{ab}(\mathcal{A})$. This implies that $s_{ab}(P) \notin \mathcal{A}$ for all $P \in \mathcal{A}_2$. Furthermore if $P \in \mathcal{A}_2$, then by (7), $|\sigma(P) \cap [n] \setminus \{a, b, y_3, \ldots, y_t\}| \geq 2$. Therefore $|\sigma(s_{ab}(P))| \geq t + 2$ for all $P \in \mathcal{A}_2$. Similarly, $|\sigma(P)| \geq t + 2$ for all $P \in \mathcal{A}_1$. Therefore $|\mathcal{A}| \leq B_{n-t} - \tilde{B}_{n-t} - (n-t)\tilde{B}_{n-t-1} < B_{n-t} - \tilde{B}_{n-t-1}$, a contradiction, as \mathcal{A} is a maximum size non-trivial t-intersecting family.

Case 2. Suppose (6) holds. Suppose there exist $k, l \in [n]$ with $S_{kl}(\mathcal{A}) \neq \mathcal{A}, k \neq a$, and $(k, l) \neq (b, a)$. Again by Lemma 3.5, we deduce that

$$\mathcal{A} = \mathcal{D}_5 \cup \mathcal{D}_6 \cup \mathcal{A}_2$$

where

$$\mathcal{D}_5 \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{k\} \in C\},$$

$$\mathcal{D}_6 \subseteq \{C \in \mathcal{B}(n) : \{a\}, \{k, l\} \in C\}.$$

Now $k \neq x_2, \ldots, x_t$, for $S_{kl}(\mathcal{A}) \neq \mathcal{A}$. Therefore $|\mathcal{A}| \leq 3B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$ (Lemma 3.2), a contradiction.

So we may assume that $S_{ij}(A) = A$ for all $i, j \in [n]$ with $(i, j) \neq (a, b), (b, a)$. As in the proof of (7), we can show that the following cross-intersecting property holds:

$$|\sigma(P) \cap \sigma(R) \cap [n] \setminus \{a, b, x_2, \dots, x_t\}| \ge 1, \quad \forall P \in \mathcal{A}_1, R \in \mathcal{A}_2. \tag{8}$$

Suppose A_2 contains a P_1 with $|\sigma(P_1)| = t$. Let $\sigma(P_1) = \{x_2, \dots, x_t, y\}$. Note that $y \in [n] \setminus \{a, b, x_2, \dots, x_t\}$. By (8), every element in A_1 contains the singletons $\{a\}, \{x_2\}, \dots, \{x_t\}$, and $\{y\}$. Therefore $|A_1| \leq B_{n-t-1}$, $|A_2| \leq B_{n-t-1}$ and $|A| \leq 2B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$ (Lemma 3.2), a contradiction. So we may assume that A_2 does not contain any P with $|\sigma(P)| = t$.

Note that by (8), $|\sigma(P)| \ge t + 1$ for all $P \in \mathcal{A}_1$. So there are two subcases to be considered.

Subcase 2.1. Suppose \mathcal{A}_1 contains a P with $|\sigma(P)| = t + 1$. Let P_1, \ldots, P_r be the only elements in $\sigma(\mathcal{A})$ with $|\sigma(P_i)| = t + 1$. Let $\sigma(P_i) = \{a, x_2, \ldots, x_t, z_i\}$. Note that by (8), $z_i \neq b$. Furthermore $r \leq n - t - 1$. If r = n - t - 1, then by (8), $\mathcal{A}_2 = \{Q(a,b)\}$. If t = 1, then we conclude that $\mathcal{A} = \mathcal{H}(a,b)$, as \mathcal{A} is a non-trivial 1-intersecting family of maximum size, but this contradicts that \mathcal{A} is not compressed. If t > 1, then $Q(x_2,b) \notin \mathcal{A}$ since $Q(x_2,b)$ is not of the form given in (6). But $\mathcal{A} \cup \{Q(x_2,b)\}$ is t-intersecting, contradicting the fact that \mathcal{A} is a non-trivial t-intersecting family of maximum size. Similarly, $r \neq n - t - 2$. So $r \leq n - t - 3$.

Note that if $P \in \mathcal{A}_1$, then $\sigma(P) \neq \{a, x_2, \dots, x_t\}$ and $\sigma(P) \neq \{a, x_2, \dots, x_t, v\}$ for $v \in [n] \setminus \{a, b, x_2, \dots, x_t, z_1, \dots, z_r\}$. So

$$|\mathcal{A}_1| \le B_{n-t} - \tilde{B}_{n-t} - (n-t-r-1)\tilde{B}_{n-t-1}.$$

Now if $P \in \mathcal{A}_2$, then by (8), $\sigma(P) \supseteq \{x_2, \dots, x_t, z_1, \dots, z_r\}$. So $|\mathcal{A}_2| \leq B_{n-1-t-r}$. Assume for the moment that $r \geq 2$. Then $|\mathcal{A}_2| \leq B_{n-t-3} < \tilde{B}_{n-t-1}$ (by (4)), and

$$|\mathcal{A}| \leq B_{n-t} - \tilde{B}_{n-t} - (n-t-r-2)\tilde{B}_{n-t-1}$$

$$\leq B_{n-t} - \tilde{B}_{n-t} - (n-t-2)\tilde{B}_{n-t-1} + (n-t-3)\tilde{B}_{n-t-1}$$

$$= B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1},$$

a contradiction.

Suppose r = 1. Then by (8), every element in A_2 contains the singletons $\{x_2\}, \ldots, \{x_t\}, \{z_1\}$. Since A_2 does not contain any P with $|\sigma(P)| = t$, we have $|A_2| \leq B_{n-t-2} - \tilde{B}_{n-t-2} = \tilde{B}_{n-t-1}$ (by (3)), and

$$|\mathcal{A}| \le B_{n-t} - \tilde{B}_{n-t} - (n-t-2)\tilde{B}_{n-t-1} + \tilde{B}_{n-t-1}$$

 $< B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1},$

a contradiction.

Subcase 2.2. Suppose $|\sigma(P)| \ge t + 2$ for all $P \in \mathcal{A}_1$. Then

$$|\mathcal{A}_1| \le B_{n-t} - \tilde{B}_{n-t} - (n-t)\tilde{B}_{n-t-1}.$$

By (8), every $P \in \mathcal{A}_2$ must contain a singleton distinct from $\{a\}, \{b\}, \{x_2\}, \ldots, \{x_t\}$. Since \mathcal{A}_2 does not contain any P with $|\sigma(P)| = t$, we have $|\mathcal{A}_2| \leq (n-t-1)(B_{n-t-2} - \tilde{B}_{n-t-2}) = (n-t-1)\tilde{B}_{n-t-1}$

(by (3)), and

$$|\mathcal{A}| \le B_{n-t} - \tilde{B}_{n-t} - (n-t)\tilde{B}_{n-t-1} + (n-t-1)\tilde{B}_{n-t-1}$$

= $B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$,

a contradiction. This completes the proof of the theorem.

Theorem 3.7. Let A be a non-trivial t-intersecting family of set partitions of [n] of maximum size. Suppose $\sigma(A)$ is a non-trivial t-intersecting family of subsets of [n]. Then, for $n \geq n_0(t)$,

$$|\mathcal{A}| = B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} + t.$$

Moreover, $A = \mathcal{H}(a_1, \ldots, a_t, b)$ for some $a_1, \ldots, a_t, b \in [n]$.

Proof. For $k \ge t+1$, let $\mathcal{F}_k = \sigma(\mathcal{A}) \cap {[n] \choose k}$. Since $\sigma(\mathcal{A})$ is t-intersecting, by applying the Erdős-Ko-Rado theorem to \mathcal{F}_k for each $k \le \lfloor \frac{n}{t+1} + t - 1 \rfloor$, we have

$$|\mathcal{A}| \le \sum_{k=t+1}^{\lfloor \frac{n}{t+1} + t - 1 \rfloor} {n-t \choose k-t} \tilde{B}_{n-k} + \sum_{k=\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1}^{n} {n \choose k} \tilde{B}_{n-k}. \tag{9}$$

We consider the following cases.

Case 1. $\mathcal{F}_{t+1} = \emptyset$.

Then the sum in (9) starts from k = t + 2, and by (1) and Lemma 3.4:

$$|\mathcal{A}| \leq \sum_{k=t+2}^{\lfloor \frac{n}{t+1} + t - 1 \rfloor} \binom{n-t}{k-t} \tilde{B}_{n-k} + \sum_{k=\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1}^{n} \binom{n}{k} \tilde{B}_{n-k}$$

$$< \sum_{k=t+2}^{n} \binom{n-t}{k-t} \tilde{B}_{n-k} + \tilde{B}_{n-t-1}$$

$$= B_{n-t} - \binom{n-t}{0} \tilde{B}_{n-t} - \binom{n-t}{1} \tilde{B}_{n-t-1} + \tilde{B}_{n-t-1}$$

$$= B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} - (n-t-2) \tilde{B}_{n-t-1}$$

$$< B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}$$

for sufficiently large n. This contradicts the maximality of A.

Case 2. $\mathcal{F}_{t+1} \neq \emptyset$.

Subcase 2.1. $|\cap_{F \in \mathcal{F}_{t+1}} F| < t$.

Then there exist three sets $F_1, F_2, F_3 \in \mathcal{F}_{t+1}$ such that $F_1 \cap F_2 \nsubseteq F_3$. Note that F_3 must contain the symmetric difference $F_1 \Delta F_2$, and since $|F_3 \cap F_i| \ge t$ for $i = 1, 2, F_3$ must take the form $(F_1 \cup F_2) \setminus \{x\}$ for some $x \in F_1 \cap F_2$. Indeed, all sets in \mathcal{F}_{t+1} other than F_1 and F_2 must also have this form.

Let

$$A_0 = \{1, 2, \dots, t, t+1\},$$

$$A_1 = \{1, 2, \dots, t, t+1, t+2\} \setminus \{1\},$$

$$A_2 = \{1, 2, \dots, t, t+1, t+2\} \setminus \{2\},$$

$$\vdots$$

$$A_{t+1} = \{1, 2, \dots, t, t+1, t+2\} \setminus \{t+1\}.$$

Without loss of generality, we may assume that $A_0, A_1, A_2 \in \mathcal{F}_{t+1}$ and $\mathcal{F}_{t+1} \subseteq \{A_0, A_1, A_2, \dots, A_{t+1}\}$. In view of the t-intersecting property of $\sigma(A)$, if $P \in A$ and $i \notin \sigma(P)$ for some $1 \le i \le t+1$, then $A_i \subseteq \sigma(P)$, for $A_0, A_1, A_2 \in \mathcal{F}_{t+1}$. Hence for any $P \in A$, $A_i \subseteq \sigma(P)$ for some $0 \le i \le t+1$. Now for sufficiently large n (Lemma 3.2),

$$|\mathcal{A}| \le (t+2)B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1},$$

contradicting the maximality of A.

Subcase 2.2. $|\cap_{F \in \mathcal{F}_{t+1}} F| = t$.

Without loss of generality, there exists $r \geq t + 1$ such that

$$\mathcal{F}_{t+1} = \{\{1, 2, \dots, t, i\} : t+1 \le i \le r\}$$

for some $r \in \{t+1,\ldots,n\}$. Notice that $r \leq n-1$; otherwise, all the set partitions in \mathcal{A} will contain $\{1\}, \{2\}, \ldots, \{t\}$, contradicting the non-triviality of $\sigma(\mathcal{A})$.

Let $P \in \mathcal{A}$. Then either $\{1, 2, ..., t\} \subseteq \sigma(P)$, or there is a $j \in \{1, 2, ..., t\}$ with $j \notin \sigma(P)$ and $(\{1, 2, ..., t\} \setminus \{j\}) \cup \{t + 1, ..., r\} \subseteq \sigma(P)$ (since $\sigma(P)$ must intersect every element in \mathcal{F}_{t+1}). In the former, we cannot have $\sigma(P) = \{1, 2, ..., t\}$ or $\sigma(P) = \{1, 2, ..., t, x\}$ for all $x \in [n] \setminus \{1, 2, ..., t, t + 1, ..., r\}$; in the later, $(\{1, 2, ..., t\} \setminus \{j\}) \cup \{t + 1, ..., r\} \subseteq \sigma(P)$ where j can take at most t values. So if $t + 4 \le r \le n - 2$, then

$$|\mathcal{A}| \leq B_{n-t} - \tilde{B}_{n-t} - \binom{n-r}{1} \tilde{B}_{n-t-1} + tB_{n-r+1}$$

$$< B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} \quad \text{(Lemma 3.3)}.$$

Suppose $t+2 \le r \le t+3$. Assume $\{1,2,\ldots,t\} \subseteq \sigma(P)$. The number of $P \in \mathcal{A}$ with $\{1,2,\ldots,t,i\} \subseteq \sigma(P)$ $(t+1 \le i \le r)$ is at most $3B_{n-t-1}$. The number of $P \in \mathcal{A}$ with $\{1,2,\ldots,t,i\} \nsubseteq \sigma(P)$ for all $i=t+1,t+2,\ldots,r$, is at most $\sum_{k=2}^{n-r} {n-r \choose k} \tilde{B}_{n-r-k} < B_{n-r} < B_{n-t-1}$. Therefore for sufficiently large n (Lemma 3.2),

$$|\mathcal{A}| \le 3B_{n-t-1} + B_{n-t-1} + tB_{n-r+1} \le (t+4)B_{n-t-1} < B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1}.$$

Suppose r = t + 1 i.e. $\mathcal{F}_{t+1} = \{\{1, 2, \dots, t, t + 1\}\}$. As in Case 1, for sufficiently large n

$$|\mathcal{A}| \leq \tilde{B}_{n-t-1} + \sum_{k=t+2}^{\lfloor \frac{n}{t+1} + t - 1 \rfloor} {n-t \choose k-t} \tilde{B}_{n-k} + \sum_{k=\lfloor \frac{n}{t+1} + t - 1 \rfloor + 1}^{n} {n \choose k} \tilde{B}_{n-k}$$

$$< B_{n-t} - \tilde{B}_{n-t} - \tilde{B}_{n-t-1} - (n-t-3)\tilde{B}_{n-t-1}$$

$$< B_{n-1} - \tilde{B}_{n-1} - \tilde{B}_{n-2}.$$

Hence, r = n - 1 and $\mathcal{A} = \mathcal{H}(1, 2, \dots, t, n)$.

Proof of Theorem 1.4.

Let \mathcal{A} be a non-trivial t-intersecting family of maximum size. Repeatedly apply the splitting operations until we obtain a family \mathcal{A}^* such that \mathcal{A}^* is compressed (Proposition 2.2). Note that by Theorem 3.6, we may choose the splitting operations so that \mathcal{A}^* is non-trivially t-intersecting. Therefore $\sigma(\mathcal{A}^*)$ is non-trivially t-intersecting (for $\sigma(\mathcal{A}^*)$ is t-intersecting by Proposition 2.3), and the result follows from Theorem 3.7 and Proposition 2.5.

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