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Turán's problem and Ramsey numbers for trees

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Abstract

Let $T_n^1=(V,E_1)$ and $T_n^2=(V,E_2)$ be the trees on n vertices with $V=\{v_0,v_1,\ldots,v_{n-1}\}$, $E_1=\{v_0v_1,\ldots,v_0v_{n-3},v_{n-4}v_{n-2},v_{n-3}v_{n-1}\}$, and $E_2=\{v_0v_1,\ldots,v_0v_{n-3},v_{n-3}v_{n-2},v_{n-3}v_{n-1}\}$. In this paper, for $p\geq n\geq 5$ we obtain explicit formulas for $\operatorname{ex}(p;T_n^1)$ and $\operatorname{ex}(p;T_n^2)$, where $\operatorname{ex}(p;L)$ denotes the maximal number of edges in a graph of order p not containing L as a subgraph. Let $r(G_1,G_2)$ be the Ramsey number of the two graphs G_1 and G_2 . In this paper we also obtain some explicit formulas for $r(T_m,T_n^i)$, where $i\in\{1,2\}$ and T_m is a tree on m vertices with $\Delta(T_m)\leq m-3$.

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1. Introduction

In this paper, all graphs are simple graphs. For a graph G = (V(G), E(G)) let e(G) = |E(G)| be the number of edges in G and let $\Delta(G)$ be the maximal degree of G. For a forbidden graph L, let $\operatorname{ex}(p;L)$ denote the maximal number of edges in a graph of order p not containing any copies of L. The corresponding Turán problem is to evaluate $\operatorname{ex}(p;L)$. For a graph G of order p, if G does not contain any copies of L and $e(G) = \operatorname{ex}(p;L)$, we say that G is an extremal graph. In this paper we also use $\operatorname{Ex}(p;L)$ to denote the set of extremal graphs of order p not containing L as a subgraph.

Let \mathbb{N} be the set of positive integers. Let $p,n\in\mathbb{N}$ with $p\geq n\geq 2$. For a given tree T_n on n vertices, it is difficult to determine the value of $\operatorname{ex}(p;T_n)$. The famous Erdős-Sós conjecture asserts that $\operatorname{ex}(p;T_n)\leq \frac{(n-2)p}{2}$. For the progress on the Erdős-Sós conjecture, see for example [8, 11]. Write p=k(n-1)+r, where $k\in\mathbb{N}$ and $r\in\{0,1,\ldots,n-2\}$. Let P_n be the path on n vertices. In [4] Faudree and Schelp showed that

(1.1)
$$\operatorname{ex}(p; P_n) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n-2$. For $n \geq 4$ let $T^*_n = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In [10] we determine $\exp(p; K_{1,n-1}), \exp(p; T'_n)$ and $\exp(p; T^*_n)$. For i = 1, 2 let $T^i_n = (V, E_i)$ be the tree on n vertices with

$$V = \{v_0, v_1, \dots, v_{n-1}\},$$

$$E_1 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\},$$

$$E_2 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-3}v_{n-1}\}.$$

In this paper, for $p \ge n \ge 5$ we obtain explicit formulas for $ex(p; T_n^1)$ and $ex(p; T_n^2)$ (see Theorems 2.1 and 3.1).

For a graph G, as usual \overline{G} denotes the complement of G. Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer p such that, for every graph G with p vertices, either G contains a copy of G_1 or else \overline{G} contains a copy of G_2 .

Let $n \in \mathbb{N}$, $n \geq 6$ and let T_n be a tree on n vertices. As mentioned in [7], recently Zhao proved the following conjecture of Burr and Erdős [2]: $r(T_n, T_n) \leq 2n - 2$. Let $m, n \in \mathbb{N}$. In 1973 Burr and Roberts [3] showed that for $m, n \geq 3$,

(1.2)
$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases}$$

In 1995, Guo and Volkmann [5] proved that for $n > m \ge 4$,

(1.3)
$$r(K_{1,m-1}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1), \\ m+n-4 & \text{if } 2 \nmid m(n-1). \end{cases}$$

Recently the first author evaluated the Ramsey number $r(T_m, T_n^*)$ for $T_m \in \{P_m, K_{1,m-1}, T_m', T_m^*\}$. In particular, he proved that (see [9]) for $n > m \ge 7$,

(1.4)
$$r(K_{1,m-1}, T_n^*) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3. \end{cases}$$

Suppose $m, n \in \mathbb{N}$ and $i, j \in \{1, 2\}$. In this paper, using the formula for $\operatorname{ex}(p; T_n^i)$ and the method in [9] we evaluate $r(T_m, T_n^i)$ for $T_m \in \{K_{1,m-1}, T_m', T_m^*, T_m^j\}$. In particular, we have the following typical results:

$$\begin{split} &r(T_n^i,T_n^j)=2n-6-(1-(-1)^n)/2,\ r(P_n,T_n^j)=2n-7\quad\text{for}\quad n\geq 17,\\ &r(T_n^i,T_n')=r(T_n^i,T_n^*)=2n-5\quad\text{for}\quad n\geq 8,\\ &r(K_{1,m-1},T_n^i)=m+n-4\quad\text{for}\quad n>m\geq 7\quad\text{and}\quad 2\mid mn,\\ &r(T_m^i,T_n^j)=m+n-5\quad\text{for}\quad m\geq 7,\ n\geq (m-3)^2+3\quad\text{and}\quad m-1\nmid n-4, \end{split}$$

and for $n > m \ge 16$,

$$r(T'_m, T^i_n) = \begin{cases} m + n - 4 & \text{if } m - 1 \mid n - 4, \\ m + n - 6 & \text{if } n = m + 1 \equiv 1 \pmod{2}, \\ m + n - 5 & \text{otherwise.} \end{cases}$$

In addition to the notation introduced above, throughout the paper we also use the following symbols: [x] is the greatest integer not exceeding x, d(v) is the degree of the vertex v in a graph, $\Gamma(v)$ is the set of vertices adjacent to the vertex v, d(u,v) is the distance between the two vertices u and v in a graph, K_n is the complete graph on n vertices, $G[V_0]$ is the subgraph of G induced by vertices in the set V_0 (we write $G[v_1, \ldots, v_m]$ instead of $G[\{v_1, \ldots, v_m\}]$), $G - V_0$ is the subgraph of G obtained by deleting vertices in V_0 and all edges incident to them, and finally $e(V_1V_1')$ is the number of edges with one endpoint in V_1 and another endpoint in V_1' .

2. Evaluation of $ex(p; T_n^1)$

Lemma 2.1. Let $p, n \in \mathbb{N}$ with $p \ge n - 1 \ge 1$. Then $ex(p; K_{1,n-1}) = [\frac{(n-2)p}{2}]$.

This is a known result. See for example [10, Theorem 2.1].

Lemma 2.2. Let $p, n \in \mathbb{N}$, $p \ge n \ge 7$ and $G \in Ex(p; T_n^1)$. Suppose that G is connected. Then $\Delta(G) = n - 4$ and $e(G) = \left[\frac{(n-4)p}{2}\right]$.

Proof. Since a graph not containing $K_{1,n-3}$ as a subgraph implies that the graph does not contain T_n^1 as a subgraph, by Lemma 2.1 we have

(2.1)
$$e(G) = \exp(p; T_n^1) \ge \exp(p; K_{1,n-3}) = \left\lceil \frac{(n-4)p}{2} \right\rceil.$$

If $\Delta(G) \leq n-5$, using Euler's theorem we see that $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-5)p}{2}$. This together with (2.1) yields $\frac{(n-4)p-1}{2} \leq \left[\frac{(n-4)p}{2}\right] \leq e(G) \leq \frac{(n-5)p}{2}$. This is impossible. Hence $\Delta(G) \geq n-4$. Now we show that $\Delta(G) = n-4$.

Suppose $q \ge n$ and q = k(n-1) + r with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then clearly $kK_{n-1} \cup K_r$ does not contain any copies of T_n^1 and so $\exp(q; T_n^1) \ge e(kK_{n-1} \cup K_r)$. For q = n we see that $e(kK_{n-1} \cup K_r) = e(K_{n-1} \cup K_1) = \frac{(n-1)(n-2)}{2} > 2n-1$. For $q \ge n+1$ we see that $(n-6)q \ge (n-6)(n+1) > (\frac{n-1}{2})^2 - 2$ and so $e(kK_{n-1} \cup K_r) = \frac{k(n-1)(n-2)}{2} + \frac{r(r-1)}{2} = \frac{(n-2)q-r(n-1-r)}{2} \ge \frac{(n-2)q-(\frac{n-1}{2})^2}{2} > 2q-1$. Hence

(2.2)
$$\operatorname{ex}(q; T_n^1) \ge e(kK_{n-1} \cup K_r) > 2q - 1 \text{ for } q \ge n.$$

Suppose $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \ldots, v_m\}$. If m = p - 1, as G does not contain T_n^1 as a subgraph, we see that $G[v_1, \ldots, v_m]$ does not contain $2K_2$ as a subgraph and hence $e(G[v_1, \ldots, v_m]) \leq m - 1$. Therefore

(2.3)
$$e(G) = d(v_0) + e(G[v_1, \dots, v_m]) \le m + m - 1 = 2p - 3.$$

By (2.2), we have $e(G) = \exp(p; T_n^1) > 2p-1$ and we get a contradiction. Hence m < p-1. Suppose that u_1, \ldots, u_t are all vertices in G such that $d(u_1, v_0) = \cdots = d(u_t, v_0) = 2$. Then $t \geq 1$. Assume $u_1v_1 \in E(G)$ with no loss of generality. If m = p-2, then $V(G) = \{v_0, v_1, \ldots, v_m, u_1\}$ and $v_iv_j \notin E(G)$ for $1 \leq i < j \leq m$. If $i \leq i < j \leq m$. If $i \leq i < j \leq m$ some $i \in \{2, 3, \ldots, m\}$, then $i \leq i \leq i < j \leq m$. Hence $i \leq i \leq i < j \leq m$ and $i \leq i \leq i < j \leq m$. Hence $i \leq i \leq i < j \leq m$ and $i \leq i \leq i < j \leq m$.

By the above, m < p-2. We first assume $m \ge n-2$. As G does not contain any copies of T_n^1 , we see that $\{v_2, \ldots, v_m\}$ is an independent set, $u_i v_j \notin E(G)$ for any $i \in \{2, 3, \ldots, t\}$ and $j \in \{2, 3, \ldots, m\}$, and $u_i v_1 \in E(G)$ for any $i = 1, 2, \ldots, t$. Set $V_1 = \{v_0, v_2, v_3, \ldots, v_m\}$. Then

 $e(G[V_1]) = m - 1$. If u_1 is adjacent to at least two vertices in $\{v_2, v_3, \ldots, v_m\}$, then $v_1 v_j \notin E(G)$ for any $j = 2, 3, \ldots, m$. If v_1 is adjacent to at least two vertices in $\{v_2, v_3, \ldots, v_m\}$, then $u_1 v_j \notin E(G)$ for any $j = 2, 3, \ldots, m$. Hence there are at most m edges with one endpoint in V_1 and another endpoint in $G - V_1$. Therefore,

$$(2.4) e(G) \le e(G[V_1]) + m + e(G - V_1) = 2m - 1 + e(G - V_1).$$

For $m \in \{n-2, n-1\}$ let $G_1 = K_m$. Then clearly $e(G_1) = \frac{m(m-1)}{2} > 2m-1$. For $m = k(n-1) + r \ge n$ with $k \in \mathbb{N}$ and $0 \le r \le n-2$ let $G_1 = kK_{n-1} \cup K_r$. Then G_1 does not contain any copies of T_n^1 and $e(G_1) > 2m-1$ by (2.2). Thus, by (2.4) we have $e(G) \le 2m-1+e(G-V_1) < e(G_1 \cup (G-V_1))$ for $m \ge n-2$. This contradicts the fact $G \in \operatorname{Ex}(p; T_n^1)$.

Suppose m=n-3 and $d(v_1)=n-3$. Then $v_1v_s \not\in E(G)$ for some $s\in\{2,3,\ldots,n-3\}$. We claim that $V(G)=\{v_0,v_1,\ldots,v_m,u_1,\ldots,u_t\}$. Otherwise, there exists $w\in V(G)$ such that $d(v_0,w)=3$. As $d(v_1)=n-3$, we see that the subgraph induced by $\{v_1,v_s,w\}\cup\Gamma(v_1)$ contains a copy of T_n^1 . This contradicts the assumption $G\in \operatorname{Ex}(p;T_n^1)$. Hence the claim is true and so |V(G)|=p=n-2+t. Since $p\geq n$ we have $t\geq 2$. For $i=1,2,\ldots,t$ and $j=2,3,\ldots,n-3$ we have $u_iv_j\not\in E(G),\,u_iv_1\in E(G)$ and so $t+1\leq d(v_1)=n-3$. Therefore $1\leq 1\leq 1\leq n-4$ and hence

$$e(G) = e(G[v_0, v_2, v_3, \dots, v_{n-3}]) + d(v_1) + e(G[u_1, \dots, u_t])$$

$$\leq {\binom{n-3}{2}} + n - 3 + {\binom{t}{2}} = {\binom{n-2}{2}} + {\binom{t}{2}}.$$

Clearly $K_{n-1} \cup K_{t-1}$ does not contain T_n^1 and

$$e(K_{n-1} \cup K_{t-1}) = \binom{n-1}{2} + \binom{t-1}{2} = \binom{n-2}{2} + \binom{t}{2} + n - 1 - t > e(G).$$

This contradicts the assumption $G \in \text{Ex}(n-2+t; T_n^1)$.

Now suppose m = n - 3 and $d(v_1) \le n - 4$. If t = 1, setting $V_2 = \{v_0, v_1, \dots, v_{n-3}, u_1\}$ we see that

$$e(G) = e(G[v_0, v_2, v_3, \dots, v_{n-3}]) + d(v_1) + d(u_1) - 1 + e(G - V_2)$$

$$\leq {n-3 \choose 2} + n - 4 + n - 4 + e(G - V_2)$$

$$= \frac{n^2 - 3n - 4}{2} + e(G - V_2) < e(K_{n-1} \cup (G - V_2)).$$

This contradicts the assumption $G \in \operatorname{Ex}(p; T_n^1)$. Hence $t \geq 2$. For $i = 1, 2, \dots, t$ and $j = 2, 3, \dots, n-3$ we see that $u_i v_j \notin E(G)$ and $u_i v_1 \in E(G)$. Let $V_3 = \{v_0, v_1, \dots, v_{n-3}\}$. Then

$$e(G) = d(v_1) + e(G[v_0, v_2, v_3, \dots, v_{n-3}]) + e(G - V_3)$$

$$\leq n - 4 + \binom{n-3}{2} + e(G - V_3) = \frac{n^2 - 5n + 4}{2} + e(G - V_3)$$

$$< e(K_{n-2} \cup (G - V_3)).$$

Since G is an extremal graph, we get a contradiction.

Summarizing all the above we obtain $\Delta(G) = n-4$ and so $e(G) = \sum_{v \in V(G)} d(v) \le \frac{(n-4)p}{2}$. This together with (2.1) yields $e(G) = \left[\frac{(n-4)p}{2}\right]$, which completes the proof.

Lemma 2.3. Let $n, n_1, n_2 \in \mathbb{N}$ with $n_1 < n - 1$ and $n_2 < n - 1$. Then

$$\binom{n_1}{2} + \binom{n_2}{2} < \min\left\{ \binom{n_1 + n_2}{2}, \binom{n-1}{2} + \binom{n_1 + n_2 - n + 1}{2} \right\}.$$

Proof. It is clear that

$$\binom{n_1}{2} + \binom{n_2}{2} = \frac{(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1n_2}{2} < \binom{n_1 + n_2}{2}$$

and

$${\binom{n-1}{2} + \binom{n_1 + n_2 - n + 1}{2} - \binom{n_1}{2} - \binom{n_2}{2}}$$

$$= \frac{(n-1)(n-2) + (n_1 + n_2 - n + 1)(n_1 + n_2 - n)}{2} - \frac{(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1n_2}{2}$$

$$= (n-1 - n_1)(n-1 - n_2) > 0.$$

Thus the lemma is proved.

Lemma 2.4. Suppose that $p \in \mathbb{N}$, $p \geq 6$, and G is a connected graph of order p that does not contain any copies of T_6^1 . Then $e(G) \leq 2p - 3$.

Proof. Clearly $\Delta(T_6^1)=3$. Suppose $v_0\in V(G),\ d(v_0)=\Delta(G)=m$ and $\Gamma(v_0)=\{v_1,\ldots,v_m\}$. If $\Delta(G)=m\leq 3$, using Euler's theorem we see that $e(G)\leq \frac{3p}{2}\leq 2p-3$. From now on we assume $\Delta(G)=m\geq 4$. If $d(v)\leq 2$ for all $v\in V(G)-\{v_0\}$, then

$$e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \le \frac{1}{2} (m + 2(p-1)) \le \frac{3(p-1)}{2} < 2p - 3.$$

So the result is true. Now we assume $d(v) \ge 3$ for some $v \in V(G) - \{v_0\}$. We may choose a vertex $u_0 \in V(G)$ so that $u_0 \ne v_0$, $d(u_0) \ge 3$ and $d(u_0, v_0)$ is as small as possible.

We first assume $d(u_0,v_0)=1$ and $u_0=v_1$ with no loss of generality. That is, $d(v_1)\geq 3$. Suppose $\Gamma(v_1)\subset\{v_0,v_1,\ldots,v_m\}$. Since $d(v_1)\geq 3$ and G does not contain any copies of T_6^1 , we see that $V(G)=\{v_0,\ldots,v_m\},\ m=p-1\geq 5$ and $G[v_1,\ldots,v_m]$ does not contain any copies of $2K_2$. Thus $e(G)\leq d(v_0)+m-1=2m-1\leq 2(m+1)-3=2p-3$. Now assume $\Gamma(v_1)-\{v_0,v_1,\ldots,v_m\}=\{w_1,\ldots,w_t\}$. Since $d(v_0)=m\geq 5,\ d(v_1)\geq 3$ and G does not contain any copies of T_6^1 , we see that $V(G)=\{v_0,v_1,\ldots,v_m,w_1,\ldots,w_t\}$ and $\{v_2,\ldots,v_m\}$ is an independent set. For $t\geq 2$, we have $e(G[w_1,\ldots,w_t])\leq 1$ and $v_iw_j\notin E(G)$ for any $i\in\{2,3,\ldots,m\}$ and $j\in\{1,2,\ldots,t\}$. Thus $e(G)\leq d(v_0)+d(v_1)-1+1\leq 2m<2(m+1+t)-3=2p-3$. Now assume t=1. Then $v_1v_i\in E(G)$ for some $i\in\{2,3,\ldots,m\}$ and $v_jw_1\notin E(G)$ for $j\in\{2,3,\ldots,m\}-\{i\}$. Hence $e(G)\leq d(v_0)+d(v_1)-1+1\leq 2m<2(m+2)-3=2p-3$.

Next we assume $d(u_0, v_0) = 2$. Then $\{v_1, \ldots, v_m\}$ is an independent set. If $\Gamma(u_0) \subseteq \{v_1, \ldots, v_m\}$, then $V(G) = \{v_0, \ldots, v_m, u_0\}$ and so $e(G) = d(v_0) + d(u_0) \le m + m < 2(m+2) - 3 = 2p - 3$. If $\Gamma(u_0) - \{v_2, \ldots, v_m\} = \{v_1, w_1, \ldots, w_t\}$, we see that $V(G) = \{v_0, v_1, \ldots, v_m, u_0, w_1, \ldots, w_t\}$ and so $e(G) = d(v_0) + d(u_0) + e(G[w_1, \ldots, w_t]) \le m + m + 1 < 2(m + 2 + t) - 3 = 2p - 3$.

Finally we assume $d(u_0, v_0) \geq 3$. Suppose that $v_0v_1u_1u_2\cdots u_ku_0$ is the shortest path in G between v_0 and u_0 , and $\Gamma(u_0) = \{w_1, \ldots, w_t, u_k\}$. Since G is connected and G does not contain any copies of T_6^1 , it is easily seen that $V(G) = \{v_0, v_1, \ldots, v_m, u_1, \ldots, u_k, u_0, w_1, \ldots, w_t\}$,

 $d(v_2) = \cdots = d(v_m) = 1, d(v_1) = d(u_1) = \cdots = d(u_k) = 2 \text{ and } e(G[w_1, \dots, w_t]) \le 1.$ Clearly G is a tree or a graph obtained by adding an edge to a tree. Hence $e(G) \leq p < 2p - 3$. Summarizing all the above proves the lemma.

Theorem 2.1. Suppose $p, n \in \mathbb{N}$, $p \ge n - 1 \ge 4$ and p = k(n - 1) + r, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then

$$\begin{split} ex(p;T_n^1) &= \max \left\{ \left[\frac{(n-2)p}{2} \right] - (n-1+r), \frac{(n-2)p - r(n-1-r)}{2} \right\} \\ &= \begin{cases} \left[\frac{(n-2)p}{2} \right] - (n-1+r) & \textit{if } n \geq 16 \textit{ and } 3 \leq r \leq n-6 \textit{ or if} \\ & 13 \leq n \leq 15 \textit{ and } 4 \leq r \leq n-7, \\ \frac{(n-2)p - r(n-1-r)}{2} & \textit{otherwise.} \end{cases} \end{split}$$

Proof. Clearly $\operatorname{ex}(n-1;T_n^1)=e(K_{n-1})=\frac{(n-2)(n-1)}{2}$. Thus the result is true for p=n-1. From now on we assume $p\geq n$. Since $T_5^1\cong P_5$, by (1.1) we obtain the result in the case n=5. Now we assume $n\geq 6$. Suppose $G\in \mathrm{Ex}(p;T_n^1)$ and G_1,\ldots,G_t are all components of G with $|V(G_i)| = p_i$ and $p_1 \le p_2 \le \cdots \le p_t$. Then clearly $G_i \in \operatorname{Ex}(p_i; T_n^1)$ for $i = 1, 2, \dots, t$.

We first consider the case n=6. If $p_i \leq 5$, then clearly $G_i \cong K_{p_i}$ and $e(G_i) = \binom{p_i}{2}$. If $p_i \geq 6$ and $p_i = 5k_i + r_i$ with $k_i \in \mathbb{N}$ and $0 \leq r_i \leq 4$, from Lemma 2.4 we have $e(G_i) \leq 2p_i - 3 \leq 2p_i - \frac{r_i(5-r_i)}{2} = e(k_iK_5 \cup K_{r_i})$. Since $k_iK_5 \cup K_{r_i}$ does not contain any copies of T_6^1 and $G_i \in \text{Ex}(p_i; T_6^1)$, we see that $e(G_i) \ge e(k_i K_5 \cup K_{r_i})$ and so $e(G_i) = e(k_i K_5 \cup K_{r_i})$. Therefore, there is a graph $G' \in \text{Ex}(p; T_6^1)$ such that $G' = a_1 K_1 \cup a_2 K_2 \cup a_3 K_3 \cup a_4 K_4 \cup a_5 K_5$, where a_1, \ldots, a_5 are nonnegative integers. If $a_1 + a_2 + a_3 + a_4 \le 1$, then $\operatorname{ex}(p; T_6^1) = e(G') =$ $e(a_5K_5 \cup K_r) = k\binom{5}{2} + \binom{r}{2}$. If $a_1 + a_2 + a_3 + a_4 > 1$, then $2a_1 + 3a_2 + 3a_3 + 2a_4 > 3 \ge \frac{r(5-r)}{2}$ and so

$$e(a_1K_1 \cup a_2K_2 \cup a_3K_3 \cup a_4K_4)$$

$$= a_2 + 3a_3 + 6a_4 < 2(a_1 + 2a_2 + 3a_3 + 4a_4) - \frac{r(5-r)}{2} = (k-a_5)\binom{5}{2} + \binom{r}{2}.$$

Thus, $\exp(p; T_6^1) = e(G') = e(a_1K_1 \cup a_2K_2 \cup a_3K_3 \cup a_4K_4) + e(a_5K_5) < k\binom{5}{2} + \binom{r}{2}$. Since $kK_5 \cup K_r$ does not contain any copies of T_6^1 , we get a contradiction. Thus $\exp(p; T_6^1) = e(kK_5 \cup K_r) = k\binom{5}{2} + \binom{r}{2} = 2p - \frac{r(5-r)}{2}$. This proves the result for n = 6. From now on we assume $n \geq 7$. If t = 1, then G is connected. Thus, by Lemma 2.2 we

have

(2.5)
$$e(G) = \left\lceil \frac{(n-4)p}{2} \right\rceil \quad \text{for} \quad t = 1.$$

Now we assume $t \geq 2$. We claim that $p_i \geq n-1$ for $i \geq 2$. Otherwise, $p_1 \leq p_2 < n-1$ and so $G_1 \cup G_2 \cong K_{p_1} \cup K_{p_2}$. If $p_1 + p_2 < n$, by Lemma 2.3 we have $e(G_1 \cup G_2) =$ $e(K_{p_1} \cup K_{p_2}) = \binom{p_1}{2} + \binom{p_2}{2} < \binom{p_1 + p_2}{2} = e(K_{p_1 + p_2})$. Since $K_{p_1 + p_2}$ does not contain T_n^1 and $G_1 \cup G_2 \in \text{Ex}(p_1 + p_2; T_n^1)$ we get a contradiction. Hence $p_1 + p_2 \ge n$. Using Lemma 2.3 again we see that

$$e(G_1 \cup G_2) = e(K_{p_1} \cup K_{p_2}) = \binom{p_1}{2} + \binom{p_2}{2}$$

$$< \binom{n-1}{2} + \binom{p_1 + p_2 - n + 1}{2} = e(K_{n-1} \cup K_{p_1 + p_2 - n + 1}).$$

Since $p_1 \leq p_2 < n-1$, we have $p_1 + p_2 - n + 1 < n-1$. Therefore $K_{n-1} \cup K_{p_1 + p_2 - n + 1}$ does not contain T_n^1 . As $G_1 \cup G_2$ is an extremal graph without T_n^1 , we also get a contradiction. Thus, the claim is true.

Next we claim that $p_i \leq n-1$ for all $i=1,2,\ldots,t-1$. If $p_{t-1} \geq n$, by Lemma 2.2 we have

$$e(G_{t-1} \cup G_t) = e(G_{t-1}) + e(G_t) = \left\lceil \frac{(n-4)p_{t-1}}{2} \right\rceil + \left\lceil \frac{(n-4)p_t}{2} \right\rceil \le \left\lceil \frac{(n-4)(p_{t-1}+p_t)}{2} \right\rceil.$$

Let $H \in \text{Ex}(p_{t-1} + p_t - n + 1; K_{1,n-3})$. As $p_{t-1} + p_t - n + 1 \ge p_t + 1 \ge n + 1$, we have $e(H) = \left[\frac{(n-4)(p_{t-1} + p_t - n + 1)}{2}\right]$ by Lemma 2.1. Clearly $K_{n-1} \cup H$ does not contain any copies of T_n^1 and

$$e(K_{n-1} \cup H) = e(K_{n-1}) + e(H) = \binom{n-1}{2} + \left[\frac{(n-4)(p_{t-1} + p_t - n + 1)}{2} \right]$$
$$= \left[\frac{(n-4)(p_{t-1} + p_t)}{2} \right] + n - 1 > e(G_{t-1} \cup G_t).$$

Since $G_{t-1} \cup G_t \in \operatorname{Ex}(p_{t-1} + p_t; T_n^1)$, we get a contradiction. Hence $p_1 \leq p_2 \leq \cdots \leq p_{t-1} \leq n-1$. Combining this with the previous assertion that $p_t \geq \cdots \geq p_2 \geq n-1$ we obtain

$$(2.6) p_1 \le n - 1, p_2 = \dots = p_{t-1} = n - 1 and p_t \ge n - 1.$$

As G is an extremal graph, we must have

(2.7)
$$G_1 \cong K_{p_1}, \quad G_2 \cong K_{n-1}, \quad \dots, \quad G_{t-1} \cong K_{n-1}.$$

If $p_t = n - 1$, then $G_t \cong K_{n-1}$. By (2.7), $G \cong K_{p_1} \cup (t-1)K_{n-1} \cong kK_{n-1} \cup K_r$. Thus,

(2.8)
$$e(G) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}$$
 for $t \ge 2$ and $p_t = n-1$.

Now we assume $p_t \ge n$. By Lemma 2.2, $e(G_t) = \left[\frac{(n-4)p_t}{2}\right]$. Since $p_1 \le n-1$, we have $G_1 \cong K_{p_1}$ and so $e(G_1) = e(K_{p_1}) = {p_1 \choose 2}$. Let $H_1 \in \operatorname{Ex}(p_1 + p_t; K_{1,n-3})$. Then H_1 does not contain T_n^1 as a subgraph. By Lemma 2.1, for $p_1 \le n-4$ we have

$$e(H_1) = \left[\frac{(n-4)(p_1+p_t)}{2}\right] \ge \left[\frac{(n-4)p_t}{2}\right] + \left[\frac{(n-4)p_1}{2}\right]$$

$$\ge \left[\frac{(n-4)p_t}{2}\right] + \frac{(n-4)(p_1-1)}{2} + 1$$

$$> \left[\frac{(n-4)p_t}{2}\right] + \frac{p_1(p_1-1)}{2} = e(G_1 \cup G_t).$$

This contradicts $G_1 \cup G_t \in \text{Ex}(p_1 + p_t; T_n^1)$. Hence $n - 3 \le p_1 \le n - 1$. For $p_1 \in \{n - 3, n - 2\}$ and $p_t \ge n$, we have $p_1(p_1 - (n - 3)) \le 2n - 4$ and so

$$e(G_1 \cup G_t) = e(G_1) + e(G_t) = \binom{p_1}{2} + \left[\frac{(n-4)p_t}{2}\right]$$

$$\leq \frac{p_1(p_1-1) + (n-4)p_t}{2} = \frac{p_1(p_1-(n-3)) + (n-4)(p_1+p_t)}{2}$$

$$\leq \frac{2n-4+(n-4)(p_1+p_t)}{2} = \binom{n-1}{2} + \frac{(n-4)(p_1+p_t-n+1)-2}{2}$$
$$< \binom{n-1}{2} + \left[\frac{(n-4)(p_1+p_t-n+1)}{2}\right].$$

Let $H_2 \in \text{Ex}(p_1 + p_t - n + 1; K_{1,n-3})$. Then $K_{n-1} \cup H_2$ does not contain any copies of T_n^1 . Since $p_1 + p_t - n + 1 \ge p_1 + 1 \ge n - 2$, applying Lemma 2.1 we have $e(H_2) = \left[\frac{(n-4)(p_1 + p_t - n + 1)}{2}\right]$. Thus, we have $e(K_{n-1} \cup H_2) = \binom{n-1}{2} + \left[\frac{(n-4)(p_1 + p_t - n + 1)}{2}\right] > e(G_1 \cup G_t)$. This contradicts $G_1 \cup G_t \in \text{Ex}(p_1 + p_t; T_n^1)$.

By the above, for $t \ge 2$ and $p_t \ge n$ we have $p_1 = p_2 = \cdots = p_{t-1} = n-1$. If $p_t \ge 2n-2$, setting $H_3 \in \text{Ex}(p_t - (n-1); K_{1,n-3})$ and then applying Lemmas 2.1 and 2.2 we find that

$$e(G_t) = \left\lceil \frac{(n-4)p_t}{2} \right\rceil < \binom{n-1}{2} + \left\lceil \frac{(n-4)(p_t - (n-1))}{2} \right\rceil = e(K_{n-1} \cup H_3).$$

This contradicts the fact $G_t \in \operatorname{Ex}(p_t; T_n^1)$. Hence $n \leq p_t < 2n-2$ and so $r \geq 1$. Note that p = k(n-1) + r = (k-1)(n-1) + n - 1 + r and $n \leq n-1+r < 2n-2$. Hence t = k, $p_t = n-1+r$ and therefore

(2.9)
$$e(G) = e((k-1)K_{n-1}) + e(G_t) = (k-1)\binom{n-1}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right] = \left[\frac{(n-2)p}{2}\right] - (n-1+r) \text{ for } t \ge 2 \text{ and } p_t \ge n.$$

Since $G \in \text{Ex}(p; T_n^1)$, by comparing (2.5), (2.8) and (2.9) we get

$$e(G) = \max \left\{ \left[\frac{(n-4)p}{2} \right], \frac{(n-2)p - r(n-1-r)}{2}, \left[\frac{(n-2)p}{2} \right] - (n-1+r) \right\}.$$

Observe that $p = k(n-1) + r \ge n - 1 + r$. We see that $\left[\frac{(n-4)p}{2}\right] = \left[\frac{(n-2)p}{2}\right] - p \le \left[\frac{(n-2)p}{2}\right] - (n-1+r)$ and therefore

(2.10)
$$\exp(p; T_n^1) = e(G) = \max \left\{ \frac{(n-2)p - r(n-1-r)}{2}, \left[\frac{(n-2)p}{2} \right] - (n-1+r) \right\}$$

$$= \frac{(n-2)p - r(n-1-r)}{2} + \max \left\{ 0, \left[\frac{r(n-3-r) - 2(n-1)}{2} \right] \right\}.$$

For $7 \le n \le 12$ we have $r(n-3-r)-2(n-1) \le \frac{(n-3)^2}{4}-2(n-1) = \frac{(n-7)^2-32}{4} < 0$. For $r \in \{0,1,2,n-5,n-4,n-3,n-2\}$ we see that r(n-3-r)-2(n-1) < 0. Suppose $n \ge 13$ and $3 \le r \le n-6$. For $4 \le r \le n-7$ we have $|r-\frac{n-3}{2}| \le \frac{n-11}{2}$ and so

$$r(n-3-r) - 2(n-1) = \frac{n^2 - 14n + 17}{4} - \left(r - \frac{n-3}{2}\right)^2$$
$$\ge \frac{n^2 - 14n + 17}{4} - \left(\frac{n-11}{2}\right)^2 = 2n - 26 \ge 0.$$

For $r \in \{3, n-6\}$ we have r(n-3-r) - 2(n-1) = 3(n-6) - 2(n-1) = n-16. Now combining the above with (2.10) we deduce the result.

Corollary 2.1. Suppose $p, n \in \mathbb{N}$, $p \ge n \ge 5$ and $n - 1 \nmid p$. Then $\frac{(n-2)p}{2} - \frac{(n-1)^2}{8} \le ex(p; T_n^1) \le \frac{(n-2)(p-1)}{2}$.

Proof. Suppose p = k(n-1) + r with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then $r \ge 1$. Clearly $\frac{(n-1)^2}{4} \ge r(n-1-r) = (\frac{n-1}{2})^2 - (\frac{n-1}{2}-r)^2 \ge (\frac{n-1}{2})^2 - (\frac{n-1}{2}-1)^2 = n-2$ and $n-1+r > \frac{n-2}{2}$. Thus, from Theorem 2.1 we deduce that $\exp(p; T_n^1) \le \frac{(n-2)p-(n-2)}{2}$ and $\exp(p; T_n^1) \ge \frac{(n-2)p-r(n-1-r)}{2} \ge \frac{(n-2)p-(n-1)^2/4}{2}$. This proves the corollary.

3. Evaluation of $ex(p; T_n^2)$

Lemma 3.1. Let $p, n \in \mathbb{N}$, $p \ge n \ge 7$ and $G \in Ex(p; T_n^2)$. Suppose that G is connected. Then $\Delta(G) \le n-3$. Moreover, for p < 2n-2 we have $\Delta(G) \le n-4$.

Proof. Since a graph does not contain $K_{1,n-3}$ implies that the graph does not contain T_n^2 , by Lemma 2.1 we have

(3.1)
$$e(G) = \exp(p; T_n^2) \ge \exp(p; K_{1,n-3}) = \left\lceil \frac{(n-4)p}{2} \right\rceil.$$

Suppose that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = m$ and $\Gamma(v_0) = \{v_1, \dots, v_m\}$. If $V(G) = \{v_0, v_1, \dots, v_m\}$, then $m = p - 1 \ge n - 1$. Since G does not contain T_n^2 , we see that $G[v_1, \dots, v_m]$ does not contain $K_{1,2}$ and hence $e(G[v_1, \dots, v_m]) \le \frac{m}{2}$. Therefore $e(G) = d(v_0) + e(G[v_1, \dots, v_m]) \le m + \frac{m}{2} = \frac{3(p-1)}{2} \le \frac{(n-4)p-3}{2} < [\frac{(n-4)p}{2}]$. This contradicts (3.1). Thus p > m + 1. Suppose that u_1, \dots, u_t are all vertices such that $d(u_1, v_0) = \dots = d(u_t, v_0) = 2$. Then $t \ge 1$. We may assume without loss of generality that v_1, \dots, v_s are all vertices in $\Gamma(v_0)$ adjacent to some vertex in the set $\{u_1, \dots, u_t\}$. Then $1 \le s \le m$. Let $V_1 = \{v_0, v_1, \dots, v_m\}, V_1' = V(G) - V_1$ and let $e(V_1V_1')$ be the number of edges with one endpoint in V_1 and another endpoint in V_1' . Since G does not contain T_n^2 , for $m \ge n - 3$ each $v_i(1 \le i \le s)$ has one and only one adjacent vertex in the set $\{u_1, \dots, u_t\}$. Thus, for $m \ge n - 3$ we must have $e(V_1V_1') = s \ge t$.

If $m \geq n-1$, since G does not contain T_n^2 as a subgraph, we see that $d(v_i) \leq 2$ for $i=1,\ldots,m$ and so $e(G[V_1])=d(v_0)+e(G[v_{s+1},\ldots,v_m])\leq m+\frac{m-s}{2}$. Hence

$$e(G) = e(G[V_1]) + e(V_1V_1') + e(G - V_1)$$

$$\leq \frac{3m - s}{2} + s + e(G - V_1) \leq 2m + e(G - V_1).$$

Suppose m+1=k(n-1)+r with $k \in \mathbb{N}$ and $0 \le r \le n-2$. Set $G_1=kK_{n-1} \cup K_r$. Since $m+1 \ge n$, by (2.2) we have $e(G_1) > 2(m+1)-1 > 2m$. Thus, $e(G_1 \cup (G-V_1)) = e(G_1) + e(G-V_1) > 2m + e(G-V_1) \ge e(G)$. As G_1 does not contain any copies of T_n^2 and G is an extremal graph, we get a contradiction. Hence $\Delta(G) = m \le n-2$.

Suppose m = n - 2. As G does not contain T_n^2 as a subgraph, we see that $d(v_1) = \cdots =$

$$d(v_s) = 2 \text{ and so } e(G[V_1]) \le n - 2 + \binom{n-2-s}{2}. \text{ Since } 1 \le s \le m = n - 2 \le 2n - 8, \text{ we have}$$

$$e(G) = e(G[V_1]) + e(V_1V_1') + e(G - V_1)$$

$$\le \binom{n-2-s}{2} + n - 2 + s + e(G - V_1)$$

$$= \frac{(n-2)(n-1) - s(2n-7-s)}{2} + e(G - V_1)$$

$$< \binom{n-1}{2} + e(G - V_1) = e(K_{n-1} \cup (G - V_1)).$$

This is impossible since G is an extremal graph.

By the above, $\Delta(G) \leq n-3$. We first assume $\Delta(G) = n-3$. We claim that $d(v_i) \leq n-4$ for $i=1,2,\ldots,s$. If $i \in \{1,2,\ldots,s\}$ and $d(v_i) = n-3$, let u_j be the unique adjacent vertex of v_i in $\{u_1,\ldots,u_t\}$ and let $V_2 = \{v_0,v_1,\ldots,v_{n-3},u_j\}$. Then there is at most one vertex adjacent to u_j in $G-V_2$. Hence $e(G-V_1) \leq 1 + e(G-V_2)$. Since each v_r $(1 \leq r \leq s)$ is adjacent to one and only one vertex in $\{u_1,\ldots,u_t\}$ and $\Delta(G[V_1]) \leq n-3$, we see that

$$e(G[V_1]) = \frac{1}{2} \sum_{r=0}^{n-3} d_{G[V_1]}(v_r) \le \frac{s(n-4) + (n-2-s)(n-3)}{2} = \frac{(n-2)(n-3) - s}{2}.$$

Note that $s \leq \Delta(G) = n - 3$. From the above we deduce that

$$e(G) = e(G[V_1]) + e(V_1V_1') + e(G - V_1) = e(G[V_1]) + s + e(G - V_1)$$

$$\leq e(G[V_1]) + s + 1 + e(G - V_2) \leq \frac{(n-2)(n-3) - s}{2} + s + 1 + e(G - V_2)$$

$$= \frac{(n-2)(n-3) + s + 2}{2} + e(G - V_2) \leq \frac{(n-2)(n-3) + n - 1}{2} + e(G - V_2)$$

$$\leq \frac{(n-1)(n-2)}{2} + e(G - V_2) = e(K_{n-1} \cup (G - V_2)).$$

Since $K_{n-1} \cup (G-V_2)$ does not contain T_n^2 and G is an extremal graph, we get a contradiction. Hence the claim is true. Thus, for $\Delta(G) = n-3$ we have $d_{G[V_1]}(v_i) \leq n-5$ for $i = 1, 2, \ldots, s$ and so

$$(3.2) \quad e(G[V_1]) = \frac{1}{2} \sum_{i=0}^{n-3} d_{G[V_1]}(v_i) \le \frac{s(n-5) + (n-2-s)(n-3)}{2} = \frac{(n-2)(n-3)}{2} - s.$$

Now we assume p < 2n-2 and p = n-1+r. Then $1 \le r < n-1$. By the above, $\Delta(G) \le n-3$. Assume $\Delta(G) = n-3$. Then $|V(G-V_1)| = p - (n-2) = r+1 < n$, $\Delta(G-V_1) \le n-3$ and so $e(G-V_1) \le \min\{\binom{r+1}{2}, \frac{(r+1)(n-3)}{2}\}$. Since $e(G[V_1]) \le \frac{(n-2)(n-3)}{2} - s$ by (3.2), we deduce that

$$e(G) = e(G[V_1]) + e(V_1V_1') + e(G - V_1)$$

$$\leq \frac{(n-2)(n-3)}{2} - s + s + \min\left\{\frac{r(r+1)}{2}, \frac{(r+1)(n-3)}{2}\right\}$$

$$= \begin{cases} \frac{(n-2)(n-3)}{2} + \binom{r+1}{2} & \text{if } r \leq n-3 \\ \frac{(n-2)(n-3)}{2} + \frac{(n-3)(n-1)}{2} & \text{if } r = n-2 \end{cases}$$

$$< \binom{n-1}{2} + \binom{r}{2} = e(K_{n-1} \cup K_r).$$

This is impossible since G is an extremal graph. Thus, $\Delta(G) \leq n-4$ for p < 2n-2. Now the proof is complete.

Lemma 3.2. Let $p, n \in \mathbb{N}$, $p \ge n \ge 7$ and $G \in Ex(p; T_n^2)$. Suppose that G is connected. Then p < 2n - 2.

Proof. By Lemma 3.1, we have $\Delta(G) \leq n-3$ and so $e(G) \leq \frac{(n-3)p}{2}$. Assume that p = k(n-1) + r with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Let $G_1 \in \operatorname{Ex}(n-1+r; K_{1,n-3})$. Then $e(G_1) = [\frac{(n-4)(n-1+r)}{2}]$ by Lemma 2.1. Hence, if $(k-2)(n-1) - r \geq 2$, then

$$e((k-1)K_{n-1} \cup G_1) = (k-1)\binom{n-1}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right]$$

$$= \frac{(p-r-(n-1))(n-2)}{2} + \left[\frac{(n-4)(n-1+r)}{2}\right]$$

$$= \left[\frac{(n-3)p}{2} + \frac{p-2r-2(n-1)}{2}\right]$$

$$= \left[\frac{(n-3)p}{2} + \frac{(k-2)(n-1)-r}{2}\right] > \left[\frac{(n-3)p}{2}\right] \ge e(G).$$

This is impossible since $(k-1)K_{n-1} \cup G_1$ does not contain T_n^2 as a subgraph and $G \in \text{Ex}(p;T_n^2)$. Thus $(k-2)(n-1)-r \leq 1$. If k=3, then r=n-2 and p=3(n-1)+n-2=4n-5 and so

$$e(G) \le \left[\frac{(n-3)p}{2}\right] \le \frac{(n-3)(4n-5)}{2} = \frac{4n^2 - 17n + 15}{2}$$
$$< \frac{4n^2 - 14n + 12}{2} = 3\binom{n-1}{2} + \binom{n-2}{2} = e(3K_{n-1} \cup K_{n-2}).$$

Since $3K_{n-1} \cup K_{n-2}$ does not contain T_n^2 and $G \in \text{Ex}(p; T_n^2)$, we get a contradiction. Thus $k \leq 2$.

For p = 2(n-1) + r with $r \in \{0, 1, 2, n-4, n-3, n-2\}$ we see that r(n-2-r) < 2n-2 and so $e(2K_{n-1} \cup K_r) = \frac{2(n-1)(n-2) + r(r-1)}{2} > \frac{(n-3)(2n-2+r)}{2} \ge e(G)$. This contradicts the assumption $G \in \text{Ex}(p; T_n^2)$. Now suppose p = 2(n-1) + r with $3 \le r \le n-5$. If $\Delta(G) \le n-4$, then $e(G) \le \frac{(n-4)p}{2}$. From previous argument we have

$$e(K_{n-1} \cup G_1) = {\binom{n-1}{2}} + \left[\frac{(n-4)(n-1+r)}{2}\right] = \left[\frac{(n-3)p-r}{2}\right]$$
$$= \left[\frac{(n-4)p}{2}\right] + n-1 > \frac{(n-4)p}{2} \ge e(G).$$

Since $K_{n-1} \cup G_1$ does not contain T_n^2 as a subgraph and $G \in \operatorname{Ex}(p; T_n^2)$, we get a contradiction. Hence $\Delta(G) = n - 3$. Suppose $v_0 \in V(G)$, $d(v_0) = n - 3$, $\Gamma(v_0) = \{v_1, \dots, v_{n-3}\}$, $V_1 = \{v_0, v_1, \dots, v_{n-3}\}$ and $V_1' = V(G) - V_1$. Suppose also that there are exactly s vertices in $\Gamma(v_0)$ adjacent to some vertex in V_1' . Then $1 \leq s \leq n - 3$. By (3.2), $e(G[V_1]) \leq \frac{(n-2)(n-3)}{2} - s$. As G does not contain any copies of T_n^2 , we see that $e(V_1V_1') = s$. Since $|V(G - V_1)| = |V_1'| = p - (n-2) = n + r$ and $G - V_1$ does not contain any copies of T_n^2 we see that $e(G - V_1) \leq \operatorname{ex}(n+r; T_n^2)$.

We claim that

$$\exp(n+r;T_n^2) \le \max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2)+r(r+1)}{2}\right\}$$

for $3 \le r \le n-5$. Let $G' \in \operatorname{Ex}(n+r;T_n^2)$. If G' is connected, using Lemma 3.1 we have $\Delta(G') \le n-4$ and so $e(G') \le \frac{(n-4)(n+r)}{2}$. Now suppose that G' is not connected. If $n_1, n_2 \in \{1, 2, \dots, n-2\}$, from Lemma 2.3 we have $e(K_{n_1} \cup K_{n_2}) < e(K_{n_1+n_2})$ for $n_1 + n_2 < n_2$ and $e(K_{n_1} \cup K_{n_2}) < e(K_{n-1} \cup K_{n_1+n_2-(n-1)})$ for $n_1 + n_2 \ge n$. Thus, $G' = G'_1 \cup G'_2$, where G'_1 and G'_2 are components of G' with $|V(G'_1)| = p'_1 < n - 1$ and $|V(G'_2)| = p'_2 \ge n - 1$. For $p'_2 \ge n$ we have $p'_1 \le r \le n - 3$ and so $e(G'_1) = \frac{p'_1(p'_1 - 1)}{2} \le \frac{(n - 4)p'_1}{2}$. For $p'_2 \ge n$ we also have $\Delta(G'_2) \leq n-4$ and so $e(G'_2) \leq \frac{(n-4)p'_2}{2}$ by Lemma 3.1. Hence for $p'_2 \geq n$ we find that $e(G') = e(G'_1) + e(G'_2) \leq \frac{(n-4)p'_1}{2} + \frac{(n-4)p'_2}{2} = \frac{(n-4)(n+r)}{2}$. Now assume $p'_2 = n-1$. Then $p'_1 = r + 1$ and

$$e(G') = e(K_{n-1} \cup K_{r+1}) = \frac{(n-1)(n-2) + r(r+1)}{2}.$$

Hence the claim is true and so

$$e(G - V_1) \le \exp(n + r; T_n^2) \le \max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2) + r(r+1)}{2}\right\}.$$

Thus,

$$e(G) = e(G[V_1]) + e(V_1V_1') + e(G - V_1)$$

$$\leq \frac{(n-2)(n-3)}{2} - s + s + \max\left\{\frac{(n-4)(n+r)}{2}, \frac{(n-1)(n-2) + r(r+1)}{2}\right\}$$

$$= \binom{n-1}{2} + \max\left\{\frac{(n-4)(n-1+r) - n}{2}, \frac{(n-1)(n-2) + r(r-1)}{2} - (n-2-r)\right\}$$

$$< \binom{n-1}{2} + \max\left\{\left[\frac{(n-4)(n-1+r)}{2}\right], \frac{(n-1)(n-2) + r(r-1)}{2}\right\}$$

$$= \max\left\{e(K_{n-1} \cup G_1), e(2K_{n-1} \cup K_r)\right\}.$$

This is impossible since G is an extremal graph.

By the above we must have k = 1 and so p = k(n - 1) + r < 2n - 2 as asserted.

Lemma 3.3. Let $p, n \in \mathbb{N}$, $p \geq n \geq 7$ and $G \in Ex(p; T_n^2)$. Suppose that G is connected. Then $\Delta(G) = n - 4$ and $e(G) = [\frac{(n-4)p}{2}]$.

Proof. By (3.1), $e(G) \geq \left[\frac{(n-4)p}{2}\right]$. If $\Delta(G) \leq n-5$, using Euler's theorem we see that $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-5)p}{2}$. Hence $\frac{(n-4)p-1}{2} \leq \left[\frac{(n-4)p}{2}\right] \leq e(G) \leq \frac{(n-5)p}{2}$. This is impossible. Thus $\Delta(G) \geq n-4$. By Lemmas 3.1 and 3.2, $\Delta(G) \leq n-4$. Therefore $\Delta(G) = n-4$ and so $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{(n-4)p}{2}$. Recall that $e(G) \geq \left[\frac{(n-4)p}{2}\right]$. Then $e(G) = \left[\frac{(n-4)p}{2}\right]$ as asserted.

Lemma 3.4. Let p and k be nonnegative integers, p = 5k + r and $r \in \{0, 1, 2, 3, 4\}$.

Suppose that G is a graph of order p without T_6^2 . Then $e(G) \leq 2p - \frac{r(5-r)}{2}$.

Proof. Clearly $\Delta(T_6^2) = 3$. We prove the lemma by induction on p. For $p \leq 5$ we have $e(G) \leq \frac{p(p-1)}{2} = 2p - \frac{r(5-r)}{2}$. Now suppose that $p \geq 6$ and the lemma is true for all graphs of order $p_0 < p$ without T_6^2 . If $\Delta(G) \leq 3$, then $e(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \leq \frac{3p}{2} \leq 2p - 3 \leq 2p - 3$ $2p - \frac{r(5-r)}{2}$

Suppose $\Delta(G) = m \ge 4$, $v_0 \in V(G)$, $d(v_0) = m$, $\Gamma(v_0) = \{v_1, \dots, v_m\}$, $V_1 = \{v_0, v_1, \dots, v_m\}$ and $V_1' = V(G) - V_1$. If $G[V_1]$ is a component of G, then $e(G[V_1]) = e(K_5) = 10$ for m = 4,

and $e(G[V_1]) \le m + \frac{m}{2} = \frac{3m}{2}$ for $m \ge 5$ since $d(v_i) \le 2$ for i = 1, 2, ..., m. By the inductive hypothesis, $e(G[V_1']) \le 2(p - m - 1) - \frac{r_1(5 - r_1)}{2}$, where $r_1 \in \{0, 1, 2, 3, 4\}$ is given by $p - m - 1 \equiv r_1 \pmod{5}$. Thus, for m = 4 we have $e(G) = e(G[V_1]) + e(G[V_1']) \le 10 + 2(p - 5) - \frac{r(5 - r)}{2} = 2p - \frac{r(5 - r)}{2}$, and for $m \ge 5$ we have $e(G) = e(G[V_1]) + e(G[V_1']) \le \frac{3m}{2} + 2(p - m - 1) - \frac{r_1(5 - r_1)}{2} \le 2p - 2 - \frac{m}{2} \le 2p - 3 \le 2p - \frac{r(5 - r)}{2}$.

From now on we assume that $G[V_1]$ is not a component of G and $m = \Delta(G) \geq 4$. Hence there is a vertex u_1 such that $d(u_1, v_0) = 2$ and $u_1v_1 \in E(G)$ with no loss of generality. Then $v_1v_i \notin E(G)$ for $i = 2, 3, \ldots, m$. For m = 4 we see that $e(G[V_1]) + e(V_1V_1') \leq 4 + 4 = 8$. For $m \geq 5$ we see that $d(v_i) \leq 2$ for $i = 1, 2, \ldots, m$ and so $e(G[V_1]) + e(V_1V_1') \leq \sum_{i=1}^m d(v_i) \leq 2m$. Hence, for $m \geq 4$ we have $e(G) = e(G[V_1]) + e(V_1V_1') + e(G[V_1']) \leq 2m + e(G[V_1'])$. By the inductive hypothesis, $e(G[V_1']) \leq 2(p-m-1) - \frac{r_1(5-r_1)}{2}$, where $r_1 \in \{0,1,2,3,4\}$ is given by $p-m-1 \equiv r_1 \pmod{5}$. Thus, $e(G) \leq 2m+2(p-m-1) - \frac{r_1(5-r_1)}{2} = 2p-2 - \frac{r_1(5-r_1)}{2}$. For $r_1 \geq 1$ we have $e(G) \leq 2p-2-2 < 2p - \frac{r(5-r)}{2}$. For $r_1 = 0$ and r = 0, 1, 4 we have $e(G) \leq 2p-2 \leq 2p - \frac{r(5-r)}{2}$. Therefore, we only need to consider the case $p \equiv m+1 \equiv 2, 3 \pmod{5}$. Now assume $p \equiv m+1 \equiv 2, 3 \pmod{5}$ and $\Gamma(u_1) - \{v_1, \ldots, v_m\} = \{w_1, \ldots, w_t\}$. As $m \geq 4$ we have $m \geq 6$. Set $V_2 = \{v_0, v_1, \ldots, v_m, u_1\}$ and $V_2' = V(G) - V_2$. Since $d(v_i) \leq 2$ for $i = 1, 2, \ldots, m$, we see that

$$e(G) = e(G[V_2]) + e(V_2V_2') + e(G[V_2']) \le \sum_{i=1}^{m} d(v_i) + t + e(G[V_2']) \le 2m + t + e(G[V_2']).$$

Note that $p-m-2 \equiv 4 \pmod 5$ and $e(G[V_2']) \le 2(p-m-2) - \frac{4(5-4)}{2}$ by the inductive hypothesis. We then have $e(G) \le 2m + t + 2(p-m-2) - 2 = 2p + t - 6$. For $t \le 3$ we get $e(G) \le 2p + t - 6 \le 2p - 3 = 2p - \frac{r(5-r)}{2}$. For $t \ge 4$ set $V_3 = \{v_0, v_1, \dots, v_m, u_1, w_1, \dots, w_t\}$ and $V_3' = V(G) - V_3$. Since $d(v_i) \le 2$ for $i = 1, 2, \dots, m$ and $d(w_j) \le 2$ for $j = 1, 2, \dots, t$, using the inductive hypothesis we see that

$$e(G) = e(G[V_3]) + e(V_3V_3') + e(G[V_3']) \le \sum_{i=1}^m d(v_i) + \sum_{j=1}^t d(w_j) + e(G[V_3'])$$

$$\le 2m + 2t + e(G[V_3']) \le 2m + 2t + 2(p - m - 2 - t) = 2p - 4$$

$$< 2p - \frac{r(5 - r)}{2}.$$

By the above, the lemma has been proved by induction.

Theorem 3.1. Let $p, n \in \mathbb{N}$, $p \ge n - 1 \ge 4$ and p = k(n - 1) + r, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n - 2\}$. Then

$$\begin{split} ex(p;T_n^2) &= \max \left\{ \left[\frac{(n-2)p}{2} \right] - (n-1+r), \frac{(n-2)p - r(n-1-r)}{2} \right\} \\ &= \begin{cases} \left[\frac{(n-2)p}{2} \right] - (n-1+r) & \text{if } n \geq 16 \text{ and } 3 \leq r \leq n-6 \text{ or if} \\ & 13 \leq n \leq 15 \text{ and } 4 \leq r \leq n-7, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases} \end{split}$$

Proof. Clearly $\operatorname{ex}(n-1;T_n^2)=e(K_{n-1})=\frac{(n-2)(n-1)}{2}$. Thus the result is true for p=n-1. Now we assume $p\geq n$. Since $T_5^2\cong T_5'$, taking n=5 in [10, Theorem 3.1] we obtain the result

in the case n = 5. For n = 6 we see that $\exp(p; T_6^2) \ge e(kK_5 \cup K_r) = 10k + \frac{r(r-1)}{2} = 2p - \frac{r(5-r)}{2}$. This together with Lemma 3.4 gives the result in this case. Applying Lemmas 3.3, 2.3 and replacing T_n^1 with T_n^2 in the proof of Theorem 2.1 we deduce the result for $n \geq 7$.

Corollary 3.1. Suppose $p, n \in \mathbb{N}, p \geq n \geq 5$ and $n-1 \nmid p$. Then $\frac{(n-2)p}{2} - \frac{(n-1)^2}{8} \leq n$ $ex(p; T_n^2) \le \frac{(n-2)(p-1)}{2}$

4. The Ramsey number $r(T_n^i, T_n)$

Lemma 4.1 ([9, Lemma 2.1]). Let G_1 and G_2 be two graphs. Suppose $p \in \mathbb{N}$, $p \geq$ $max\{|V(G_1)|, |V(G_2)|\}$ and $ex(p; G_1) + ex(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \le p$.

Proof. Let G be a graph of order p. If $e(G) \leq ex(p; G_1)$ and $e(\overline{G}) \leq ex(p; G_2)$, then $\exp(p;G_1) + \exp(p;G_2) \ge e(G) + e(\overline{G}) = \binom{p}{2}$. This contradicts the assumption. Hence, either $e(G) > \exp(p; G_1)$ or $e(\overline{G}) > \exp(p; G_2)$. Therefore, G contains a copy of G_1 or \overline{G} contains a copy of G_2 . This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved.

Lemma 4.2 ([9, Lemma 2.3]). Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then

- (i) $r(G_1, G_2) \ge d_1 + d_2 (1 (-1)^{(d_1 1)(d_2 1)})/2$.
- (ii) Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \le m$. Then $r(G_1, G_2) \ge d_1$ $2d_2 - 1 \ge d_1 + d_2.$
- (iii) If G_1 is a connected graph of order m, $d_1 \neq m-1$ and $d_2 > m$, then $r(G_1, G_2) \geq m$ $d_1 + d_2$.

Theorem 4.1. Let $n \in \mathbb{N}$ and $i, j \in \{1, 2\}$.

- (i) If n is odd with $n \ge 17$, then $r(T_n^i, T_n^j) = 2n 7$.
- (ii) If n is even with $n \ge 12$, then $r(T_n^i, T_n^j) = 2n 6$.

Proof. Suppose $n \geq 12$. Since $\Delta(T_n^i) = \Delta(T_n^j) = n - 3$, from Lemma 4.2 we know that $r(T_n^i, T_n^j) \ge 2n - 7$ for odd n, and $r(T_n^i, T_n^j) \ge 2n - 6$ for even n. If n is odd with $n \ge 17$, using Theorems 2.1 and 3.1 (with k = 1 and r = n - 6) we see that

$$ex(2n-7;T_n^i) = \frac{(n-2)(2n-7)-1}{2} - (2n-7) < \frac{(n-4)(2n-7)}{2} = \frac{1}{2} {2n-7 \choose 2}$$

and so $\exp(2n-7;T_n^i) + \exp(2n-7;T_n^j) < {2n-7 \choose 2}$. Thus, by Lemma 4.1 we have $r(T_n^i,T_n^j) \le r(2n-7)$ 2n-7. Hence (i) is true. From Theorems 2.1 and 3.1 (with k=1 and r=n-5) we see that for $n \geq 12$,

$$ex(2n-6;T_n^i) = \frac{(n-2)(2n-6) - 4(n-5)}{2} = n^2 - 7n + 16$$

$$< n^2 - \frac{13}{2}n + \frac{21}{2} = \frac{1}{2} {2n-6 \choose 2}$$

and so $\exp(2n-6;T_n^i)+\exp(2n-6;T_n^j)<\binom{2n-6}{2}$. Thus, by Lemma 4.1 we have $r(T_n^i,T_n^j)\leq r(2n-6)$ 2n-6. Hence $r(T_n^i, T_n^j) = 2n-6$ for even n, proving (ii).

Lemma 4.3. Let $n \in \mathbb{N}$, $n \geq 5$ and $i \in \{1,2\}$. Let G_n be a connected graph of order n

such that $ex(2n-5; G_n) < n^2 - 5n + 4$. Then $r(T_n^i, G_n) \le 2n - 5$. Proof. By Theorems 2.1 and 3.1, $ex(2n-5; T_n^i) = \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2 - 6n + 11$. Thus,

$$ex(2n-5;G_n) + ex(2n-5;T_n^i) < n^2 - 5n + 4 + n^2 - 6n + 11 = {2n-5 \choose 2}.$$

Appealing to Lemma 4.1 we obtain $r(T_n^i, G_n) \leq 2n - 5$.

Lemma 4.4 ([10, Theorem 3.1]). Let $p, n \in \mathbb{N}$ with $p \ge n \ge 5$. Let $r \in \{0, 1, \dots, n-2\}$ be given by $p \equiv r \pmod{n-1}$. Then

$$ex(p; T'_n) = \begin{cases} \left[\frac{(n-2)(p-1) - r - 1}{2} \right] & \text{if } n \ge 7 \text{ and } 2 \le r \le n - 4, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Theorem 4.2. Let $n \in \mathbb{N}$, $n \geq 8$ and $i \in \{1, 2\}$. Then $r(T_n^i, T_n^i) = r(T_n^i, T_n^*) = 2n - 5$. Proof. Let $T_n \in \{T_n^i, T_n^*\}$. As $2K_{n-3}$ does not contain any copies of T_n^i and $\overline{2K_{n-3}} = K_{n-3,n-3}$ does not contain any copies of T_n , we see that $r(T_n^i, T_n) \geq 1 + 2(n-3) = 2n - 5$. Taking p = 2n - 5 and r = n - 4 in Lemma 4.4 we find that

$$ex(2n-5;T'_n) = \left[\frac{(n-2)(2n-6) - (n-4) - 1}{2}\right] \le n^2 - \frac{11}{2}n + \frac{15}{2} < n^2 - 5n + 4.$$

By [10, Theorem 4.1],

$$ex(2n-5;T_n^*) = \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11 < n^2 - 5n + 4.$$

Thus, applying Lemma 4.3 we obtain $r(T_n^i, T_n) \leq 2n - 5$. Hence $r(T_n^i, T_n) = 2n - 5$ as asserted.

Remark 4.1 Let $n \in \mathbb{N}$, $n \geq 5$ and $i \in \{1, 2\}$. From [5, Theorem 3.1(ii)] we know that $r(K_{1,n-1}, T_n^i) = 2n - 3$.

Theorem 4.3. Let $n \in \mathbb{N}$ and $i \in \{1,2\}$. Then $r(P_n, T_n^i) = 2n - 7$ for $n \ge 17$, $r(P_{n-1}, T_n^i) = 2n - 7$ for $n \ge 13$, $r(P_{n-2}, T_n^i) = 2n - 7$ for $n \ge 11$ and $r(P_{n-3}, T_n^i) = 2n - 7$ for $n \ge 8$.

Proof. Suppose $n \ge 8$ and $s \in \{0, 1, 2, 3\}$. From Lemma 4.2(ii) we have $r(P_{n-s}, T_n^i) \ge 2(n-3) - 1 = 2n - 7$. By (1.1),

$$\exp(2n-7;P_{n-s}) = \begin{cases} \frac{(n-2)(2n-7)-5(n-6)}{2} = \frac{(n-4)(2n-7)+16-n}{2} & \text{if } s = 0, \\ \frac{(n-3)(2n-7)-3(n-5)}{2} = \frac{(n-4)(2n-7)+8-n}{2} & \text{if } s = 1, \\ \frac{(n-4)(2n-7)-(n-4)}{2} & \text{if } s = 2, \\ \frac{(n-5)(2n-7)-(n-5)}{2} = \frac{(n-4)(2n-7)+12-3n}{2} & \text{if } s = 3. \end{cases}$$

By Theorems 2.1 and 3.1,

$$ex(2n-7;T_n^i) = \begin{cases} \left[\frac{(n-4)(2n-7)}{2}\right] & \text{if } n \ge 16, \\ \frac{(n-2)(2n-7)-5(n-6)}{2} = \frac{(n-4)(2n-7)+16-n}{2} & \text{if } n < 16. \end{cases}$$

For $n \ge 17, 13, 11$ or 8 according as s = 0, 1, 2 or 3, from the above we find $\exp(2n - 7; P_{n-s}) + \exp(2n - 7; T_n^i) < \binom{2n-7}{2}$ and so $r(P_{n-s}, T_n^i) \le 2n - 7$ by Lemma 4.1. This completes the proof.

5. The Ramsey number $r(T_m^i, T_n)$ for m < n

Proposition 5.1 (Burr[1]). Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $m-1 \mid n-2$. Let T_m be a tree on m vertices. Then $r(T_m, K_{1,n-1}) = m + n - 2$.

Proposition 5.2 (Guo and Volkmann [5, Theorem 3.1]). Let $m, n \in \mathbb{N}, m \geq 3$ and n = k(m-1) + b with $k \in \mathbb{N}$ and $b \in \{0, 1, \dots, m-2\} \setminus \{2\}$. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. Then $r(T_m, K_{1,n-1}) \leq m+n-3$. Moreover, if $k \geq m-b$, then $r(T_m, K_{1,n-1}) = m+n-3$.

Lemma 5.1 ([6, Theorem 8.3, pp.11-12]). Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a-1)(b-1)$, then there are two nonnegative integers x and y such that n = ax + by.

Theorem 5.1. Let $m, n \in \mathbb{N}$, $n > m \geq 5$, $m-1 \nmid n-2$ and $i \in \{1,2\}$. Then $r(T_m^i, K_{1,n-1}) = m+n-3$ or m+n-4. Moreover, if $n \geq (m-3)^2+1$ or m+n-4=(m-1)x+(m-2)y for some nonnegative integers x and y, then $r(T_m, K_{1,n-1}) = m+n-3$ for any tree $T_m \neq K_{1,m-1}$ of order m.

Proof. Let $T_m \neq K_{1,m-1}$ be a tree on m vertices. From Proposition 5.2 we know that $r(T_m, K_{1,n-1}) \leq m+n-3$. By Lemma 4.2(iii), $r(T_m^i, K_{1,n-1}) \geq m-3+n-1$. Thus, $r(T_m^i, K_{1,n-1}) = m+n-3$ or m+n-4. If $n \geq (m-3)^2+1$, then $m+n-4 \geq (m-2)(m-3)$ and so m+n-4=(m-1)x+(m-2)y for some nonnegative integers x and y by Lemma 5.1. If m+n-4=(m-1)x+(m-2)y for $x,y \in \{0,1,2,\ldots\}$, setting $G=xK_{m-1}\cup yK_{m-2}$ we see that G does not contain any copies of T_m and \overline{G} does not contain any copies of $K_{1,n-1}$. Thus $r(T_m,K_{1,n-1})\geq 1+|V(G)|=m+n-3$. Now putting all the above together we obtain the theorem.

Theorem 5.2. Let $m, n \in \mathbb{N}$, $n > m \geq 6$, $m - 1 \mid n - 3$ and $i \in \{1, 2\}$. Then $r(T_m^i, T_n') = m + n - 3$.

Proof. By Theorems 2.1 and 3.1, $ex(m+n-3; T_m^i) = \frac{(m-2)(m+n-3)-(m-2)}{2} < \frac{(m-2)(m+n-3)}{2}$. Thus applying [9, Theorem 5.1] we obtain the conclusion.

Theorem 5.3. Suppose $i \in \{1,2\}$, $m, n \in \mathbb{N}$, $n > m \ge 7$ and $m-1 \nmid (n-3)$. Then $m+n-5 \le r(T_m^i, T_n') \le m+n-4$ and $m+n-6 \le r(T_m^i, T_n^*) \le m+n-4$. Moreover, if n = k(m-1) + b = q(m-2) + a, $k, q \in \mathbb{N}$, $a \in \{0, 1, ..., m-3\}$, $b \in \{0, 1, ..., m-2\}$ and one of the following conditions holds:

- (1) $b \in \{1, 2, 4\},\$
- (2) b = 0 and $k \ge 3$,
- (3) $n > (m-3)^2 + 2$,
- (4) $n \ge m^2 1 b(m-2)$,
- (5) $a \ge 3$ and $n \ge (a-4)(m-1)+4$,

then $r(T_m^i, T_n^*) = r(T_m^i, T_n') = m + n - 4$.

Proof. By Lemma 4.2 we have $r(T_m^i, T_n') \ge m-3+n-2$ and $r(T_m^i, T_n^*) \ge m-3+n-3$. Since $m-1 \nmid n-3$, we have $m-1 \nmid m+n-4$. From Corollaries 2.1 and 3.1 we find $\operatorname{ex}(m+n-4;T_m^i) \le \frac{(m-2)(m+n-5)}{2}$. Hence, by [9, Lemma 5.2] we have $r(T_m^i, T_n') \le m+n-4$, and by [9, Lemma 4.2] we have $r(T_m^i, T_n^*) \le m+n-4$. Now applying [9, Theorems 4.4 and 5.4] we deduce the remaining assertion.

6. The Ramsey number $r(G_m, T_n^j)$ for m < n

Theorem 6.1. Let $m, n \in \mathbb{N}$, $m \geq 5$, $n \geq 8$, n > m and $j \in \{1, 2\}$. Then $r(K_{1,m-1}, T_n^j) = m + n - 4$ or m + n - 5. Moreover, if $2 \mid mn$, then $r(K_{1,m-1}, T_n^j) = m + n - 4$.

Proof. From Lemma 4.2 we deduce that $r(K_{1,m-1},T_n^j) \geq m-1+n-3-(1-(-1)^{(m-2)(n-4)})/2 = m+n-4-(1-(-1)^{mn})/2$. So, it suffices to prove that $r(K_{1,m-1},T_n^j) \leq m+n-4$. By Lemma 2.1, $\exp(m+n-4;K_{1,m-1}) = \left[\frac{(m-2)(m+n-4)}{2}\right]$. By Theorems 2.1 and 3.1, we have

$$ex(m+n-4;T_n^j) = \left[\frac{(n-4)(m+n-4)}{2}\right] \text{ or } \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2}.$$

Since
$$\left[\frac{(m-2)(m+n-4)}{2}\right] + \left[\frac{(n-4)(m+n-4)}{2}\right] \le \frac{(m+n-6)(m+n-4)}{2} < {m+n-4 \choose 2}$$
 and

$$\frac{(m-2)(m+n-4)}{2} + \frac{(n-2)(m+n-4) - (m-3)(n-m+2)}{2}$$

$$= \frac{(m+n-4)(m+n-5) - (m-4)(n-m-\frac{2}{m-4})}{2} < {m+n-4 \choose 2},$$

we see that $ex(m+n-4;K_{1,m-1})+ex(m+n-4;T_n^j)<\binom{m+n-4}{2}$ and so $r(K_{1,m-1},T_n^j)\leq m+n-4$ by Lemma 4.1. This completes the proof.

Theorem 6.2. Let $m, n \in \mathbb{N}$, $m \ge 4$, $n \ge 7$, $m - 1 \mid n - 4$ and $j \in \{1, 2\}$.

- (i) If G_m is a connected graph of order m with $ex(m+n-4;G_m) \leq \frac{(m-2)(m+n-5)}{2}$, then $r(G_m,T_n^j)=m+n-4$.
- (ii) $r(T'_m, T^j_n) = r(T^1_m, T^j_n) = r(T^2_m, T^j_n) = m + n 4$ for $m \ge 5$, $r(T^*_m, T^j_n) = m + n 4$ for $m \ge 6$, and $r(P_m, T^j_n) = m + n 4$.

Proof. Set t = (n-4)/(m-1). Suppose that G_m is a connected graph of order m with $\exp(m+n-4;G_m) \leq \frac{(m-2)(m+n-5)}{2}$. Then clearly $\Delta(\overline{(t+1)K_{m-1}}) = t(m-1) = n-4$. Thus, $(t+1)K_{m-1}$ does not contain any copies of G_m and $\overline{(t+1)K_{m-1}}$ does not contain any copies of T_n^j . Hence $r(G_m, T_n^j) \geq 1 + (t+1)(m-1) = m+n-4$. By Theorems 2.1 and 3.1,

$$ex(m+n-4;T_n^j) = \left[\frac{(n-4)(m+n-4)}{2}\right] \text{ or } \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2}.$$

If $ex(m+n-4; T_n^j) = \left[\frac{(n-4)(m+n-4)}{2}\right]$, then

$$ex(m+n-4;G_m) + ex(m+n-4;T_n^j)$$

$$\leq \frac{(m-2)(m+n-5) + (n-4)(m+n-4)}{2} < {m+n-4 \choose 2}.$$

If
$$ex(m+n-4;T_n^j) = \frac{(n-2)(m+n-4)-(m-3)(n-m+2)}{2}$$
, then

$$\exp(m+n-4; G_m) + \exp(m+n-4; T_n^j)
\leq \frac{(m-2)(m+n-5) + (n-2)(m+n-4) - (m-3)(n-m+2)}{2}
= {m+n-4 \choose 2} - \frac{(m-4)(n-m+1)}{2} < {m+n-4 \choose 2}.$$

Therefore, by Lemma 4.1 we always have $r(G_m, T_n^j) \leq m + n - 4$ and hence $r(G_m, T_n^j) = m + n - 4$. This proves (i).

Now consider (ii). Note that $m+n-4\equiv 1 \pmod{m-1}$. By (1.1), we have $\exp(m+n-4;P_m)=\frac{(m-2)(m+n-5)}{2}$. By Lemma 4.4, $\exp(m+n-4;T_m')=\frac{(m-2)(m+n-5)}{2}$ for $m\geq 5$. By [10, Theorem 4.2], $\exp(m+n-4;T_m')=\frac{(m-2)(m+n-5)}{2}$ for $m\geq 6$. By Theorems 2.1 and 3.1, $\exp(m+n-4;T_m')=\frac{(m-2)(m+n-5)}{2}$ for $i\in\{1,2\}$ and $m\geq 5$. Thus from (i) and the above we deduce (ii). The proof is complete.

Lemma 6.1. Let $j \in \{1, 2\}$, $m, n \in \mathbb{N}$, $m \ge 7$ and $m-1 \nmid n-4$. Assume $n = m+1 \ge 12$ or $n \ge max \{m+2, 19-m\}$.

(i) If G_m is a connected graph of order m with $ex(m+n-5;G_m) \leq \frac{(m-2)(m+n-6)}{2}$, then $r(G_m,T_n^j) \leq m+n-5$.

(ii) For $T_m \in \{P_m, T'_m, T^*_m, T^1_m, T^2_m\}$ we have $r(T_m, T^j_n) \le m + n - 5$. Proof. Since m + n - 5 = n - 1 + m - 4, by Theorems 2.1 and 3.1 we have

$$ex(m+n-5;T_n^j) = \left[\frac{(n-4)(m+n-5)}{2}\right]$$
or
$$\frac{(n-2)(m+n-5) - (m-4)(n-1-(m-4))}{2}$$

If n=m+1, then (m-4)(n-3-(m-4))=2(n-5). If $n\geq m+2$, then $3\leq m-4\leq n-6$ and so $(m-4)(n-3-(m-4))=(\frac{n-3}{2})^2-(m-4-\frac{n-3}{2})^2\geq (\frac{n-3}{2})^2-(n-6-\frac{n-3}{2})^2=3(n-6)$. Thus,

$$\frac{(n-4)(m+n-5)+m-2}{2} - \frac{(n-2)(m+n-5)-(m-4)(n-1-(m-4))}{2}$$

$$= \frac{(m-4)(n-3-(m-4))-2n+m}{2}$$

$$\geq \begin{cases} \frac{2(n-5)-2n+m}{2} = \frac{m-10}{2} > 0 & \text{if } n=m+1 \geq 12, \\ \frac{3(n-6)-2n+m}{2} = \frac{n-10+m-8}{2} > 0 & \text{if } n \geq \max \{m+2,19-m\}. \end{cases}$$

Therefore, from the above we deduce that

(6.1)
$$\operatorname{ex}(m+n-5;T_n^j) < \frac{(n-4)(m+n-5)+m-2}{2}.$$

Hence, if G_m is a connected graph of order m with $\exp(m+n-5;G_m) \leq \frac{(m-2)(m+n-6)}{2}$, then

$$ex(m+n-5; G_m) + ex(m+n-5; T_n^j) < \frac{(m-2)(m+n-6)}{2} + \frac{(n-4)(m+n-5) + m-2}{2} = {m+n-5 \choose 2}.$$

Applying Lemma 4.1 we obtain (i).

Now we consider (ii). Since $m-1 \nmid (m+n-5)$, by Corollaries 2.1 and 3.1 we have $\exp(m+n-5;T_m^i) \leq \frac{(m-2)(m+n-6)}{2}$ for $i \in \{1,2\}$. By (1.1), $\exp(m+n-5;P_m) \leq \frac{(m-2)(m+n-6)}{2}$. By Lemma 4.4, $\exp(m+n-5;T_m') \leq \frac{(m-2)(m+n-6)}{2}$. By [10, Theorems 4.1-4.5], $\exp(m+n-5;T_m') \leq \frac{(m-2)(m+n-6)}{2}$. Thus, from the above and (i) we deduce (ii). This proves the lemma.

Theorem 6.3. Let $m \in \mathbb{N}$ and $j \in \{1, 2\}$.

(i) We have

$$r(T'_m, T^j_{m+1}) = \begin{cases} 2m - 4 & \text{if } 2 \nmid m \text{ and } m \ge 9, \\ 2m - 5 & \text{if } 2 \mid m \text{ and } m \ge 16. \end{cases}$$

(ii) If $n \in \mathbb{N}$, $m \ge 7$, $n \ge max\{m+2, 19-m\}$ and $m-1 \nmid n-4$, then $r(T'_m, T^j_n) = m+n-5$. Proof. We first assume $2 \nmid m$ and $m \geq 9$. By Lemma 4.2(i), we have $r(T'_m, T^{\jmath}_{m+1}) \geq$ m-2+m-2=2m-4. By Lemma 4.4, $\exp(2m-4;T_m')=\frac{(m-2)(2m-4)-2(m-3)}{2}=m^2-5m+7$. By Theorems 2.1 and 3.1, $\exp(2m-4;T_{m+1}^j)=\frac{(m-1)(2m-4)-4(m-4)}{2}=m^2-5m+10$. Thus,

$$ex(2m-4;T'_m) + ex(2m-4;T^j_{m+1})$$

$$= m^2 - 5m + 7 + m^2 - 5m + 10 = 2m^2 - 10m + 17 < 2m^2 - 9m + 10 = {2m-4 \choose 2}.$$

Hence, by Lemma 4.1 we obtain $r(T'_m, T^j_{m+1}) \leq 2m - 4$ and so $r(T'_m, T^j_{m+1}) = 2m - 4$.

Now we assume $2 \mid m$ and $m \ge 16$. By Lemma 4.2(i), $r(T'_m, T^j_{m+1}) \ge m - 2 + m - 2 - 1 = 2m - 5$. By Lemma 4.4, $\exp(2m - 5; T'_m) = \left[\frac{(m-2)(2m-6)-(m-3)}{2}\right] = \frac{2m^2-11m+14}{2}$. By Theorems 2.1 and 3.1, $\exp(2m - 5; T^j_{m+1}) = \left[\frac{(m-1)(2m-5)}{2}\right] - (2m - 5) = \frac{2m^2-11m+14}{2}$. Thus,

$$ex(2m-5;T'_m) + ex(2m-5;T^j_{m+1}) = 2m^2 - 11m + 14 < 2m^2 - 11m + 15 = \binom{2m-5}{2}.$$

Hence, by Lemma 4.1 we obtain $r(T'_m, T^j_{m+1}) \leq 2m-5$ and so $r(T'_m, T^j_{m+1}) = 2m-5$. This proves (i).

Now we consider (ii). Suppose $n \in \mathbb{N}$, $m \ge 7$ and $n \ge \max \{m+2, 19-m\}$. By Lemma 6.1(ii), $r(T'_m, T_n^j) \le m + n - 5$. By Lemma 4.2, we have $r(T'_m, T_n^j) \ge m - 2 + n - 3$. Thus, $r(T'_m, T_n^j) = m + n - 5$. This proves (ii). The proof is complete.

Theorem 6.4. Let $j \in \{1, 2\}, m, n \in \mathbb{N}, m \ge 7 \text{ and } m-1 \nmid n-4.$ Suppose $n = m+1 \ge 12$ or $n \ge max \ \{m+2, 19-m\}$. Assume that $G_m \in \{P_m, T_m^*, T_m^1, T_m^2\}$ or G_m is a connected graph of order m such that $ex(m+n-5; G_m) \le \frac{(m-2)(m+n-6)}{2}$. If $n \ge (m-3)^2 + 3$ or m+n-6=(m-1)x+(m-2)y for some nonnegative integers x and y, then $r(G_m,T_n^{\jmath})=$ m + n - 5.

Proof. If $n \ge (m-3)^2 + 3$, then $m+n-6 \ge (m-2)(m-3)$ and so m+n-6 =(m-1)x + (m-2)y for some $x, y \in \{0, 1, 2, ...\}$ by Lemma 5.1. Now suppose m + n - 6 =(m-1)x + (m-2)y, where $x, y \in \{0, 1, 2, ...\}$. Set $G = xK_{m-1} \cup yK_{m-2}$. Then $\Delta(\overline{G}) \le n-4$. Thus, G does not contain any copies of G_m and \overline{G} does not contain any copies of T_n^j . Hence $r(G_m, T_n^{\mathfrak{I}}) \geq 1 + |V(G)| = m + n - 5$. On the other hand, by Lemma 6.1 we have $r(G_m, T_n^{\mathfrak{I}}) \leq m+n-5$. Thus $r(G_m, T_n^{\mathfrak{I}}) = m+n-5$. This proves the theorem.

Corollary 6.1. Let $m, n \in \mathbb{N}$, $m \ge 7$, $m-1 \mid n-b, b \in \{2, 3, 5\}$, $n \ge max\{m+2, 19-m\}$ and $j \in \{1,2\}$. Assume that $G_m \in \{P_m, T_m^*, T_m^1, T_m^2\}$ or G_m is a connected graph of order m with $ex(m+n-5; G_m) \leq \frac{(m-2)(m+n-6)}{2}$. Then $r(G_m, T_n^j) = m+n-5$. Proof. Set k = (n-b)/(m-1). Then $k \in \mathbb{N}$. For b=2 we have $k \geq 2$. Since

$$m+n-6 = \begin{cases} (k-2)(m-1) + 3(m-2) & \text{if } b = 2, \\ (k-1)(m-1) + 2(m-2) & \text{if } b = 3, \\ (k+1)(m-1) & \text{if } b = 5, \end{cases}$$

the result follows from Theorem 6.4.

Theorem 6.5. Let $m \in \mathbb{N}$, $m \ge 12$ and $i, j \in \{1, 2\}$. Then

$$r(T_m^i, T_{m+1}^j) = r(T_m^*, T_{m+1}^j) = 2m - 5.$$

Proof. Let $T_m \in \{T_m^i, T_m^*\}$. By Theorems 2.1, 3.1 and [10, Theorem 4.1],

$$ex(2m-5;T_m) = \frac{(m-2)(2m-5)-3(m-4)}{2},$$

$$ex(2m-5;T_{m+1}^j) = \frac{(m-1)(2m-5)-5(m-5)}{2} \text{ or } \left[\frac{(m-3)(2m-5)}{2}\right].$$

Since $\frac{(m-2)(2m-5)-3(m-4)}{2} + \frac{(m-3)(2m-5)}{2} = \frac{(2m-5)(2m-6)+7-m}{2} < {2m-5 \choose 2}$ and

$$\frac{(m-2)(2m-5) - 3(m-4)}{2} + \frac{(m-1)(2m-5) - 5(m-5)}{2}$$
$$= 2m^2 - 12m + 26 < 2m^2 - 11m + 15 = {2m-5 \choose 2},$$

we see that $\exp(2m-5;T_m) + \exp(2m-5;T_{m+1}^j) < \binom{2m-5}{2}$. Hence, applying Lemma 4.1 we deduce that $r(T_m,T_{m+1}^j) \leq 2m-5$. Since $\Delta(T_m) = m-3$ and $\Delta(T_{m+1}^j) = m-2$, by Lemma 4.2(i) we have $r(T_m,T_{m+1}^j) \geq m-3+m-2 = 2m-5$. Hence $r(T_m,T_{m+1}^j) = 2m-5$. This proves the theorem.

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