A Method for Image Reduction Based on a Generalization of Ordered Weighted Averaging Functions*

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Abstract

In this paper we propose a special type of aggregation function which generalizes the notion of Ordered Weighted Averaging Function - OWA. The resulting functions are called **Dynamic Ordered Weighted Averaging Functions** — **DY-OWAs**. This generalization will be developed in such way that the weight vectors are variables depending on the input vector. Particularly, this operators generalize the aggregation functions: *Minimum*, *Maximum*, *Arithmetic Mean*, *Median* etc, which are extensively used in image processing. In this field of research two problems are considered: The determination of methods to reduce images and the

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construction of techniques which provide noise reduction. The operators described here are able to be used in both cases. In terms of image reduction we apply the methodology provided in [1]. We use the noise reduction operators obtained here to treat the images obtained in the first part of the paper, thus obtaining images with better quality.

Keywords: Aggregation functions, *OWA* functions, Image reduction, Noise reduction.

1 Introduction

Image processing has great applicability in several areas. In medicine, for example, they can be applied to: Identify tumors [2]; support techniques in advancing dental treatments [3], etc. Such images are not always obtained with a suitable quality, and to detect the desired information, various methods have been developed in order to eliminate most of the noise contained in these images.

Another problem addressed in image processing is the decrease of resolution, usually aiming the reduction of memory consumption required for its storage [4].

There are several techniques for image reduction in the literature, more recently Paternain *et. al.* [1] constructed reduction operators using weighted averaging aggregation functions. The proposed method consists of: (1) To reduce a given image by using a reduction operator; (2) To build a new image from the reduced one, and (3) To analyze the quality of the last image by using the measures *PSNR* and *MSIM* [4].

In this work we introduce a class of aggregation functions called: **Dynamic Ordered Weighted Averaging Function** - (**DYOWA**). They generalize the *OWA* function introduced by Yager [5] and in particular the operators: *Arithmetic Mean*, *Median*, *Maximum*, *Minimum* and *cOWA*. We provide a range of their properties such as: idempotence, symmetry and homogeneity as well two methods ¹: (1) for image reduction

¹These methods were implemented by using Java 1.8.0_31 software in a 64 bits MS Windows machine.

and (2) for noise treatment.

This paper is structured in the following way: SECTION 2 provides some basics of Aggregation Functions Theory. SECTION 3 introduces Dynamic Ordered Weighted Averaging functions, shows some examples and properties, and introduces a particular DYOWA function, called **H**, which will be fundamental for this work. In SECTION 4 we provide an application of DYOWA's to image reduction and in SECTION 5, we show that DYOWA functions are able to treat images with noise, aiming to improve the reduction method used in SECTION 4. Finally, section SECTION 6 gives the final remarks of this work.

2 Aggregation Functions

Aggregation functions are important mathematical tools for applications in several fields: Information fuzzy [6]; Decision making [9, 8, 7, 11, 10]; Image processing [1, 2, 12] and Engineering [13]. In this section we introduce them together with examples and properties. We also present a special family of aggregation functions called *Ordered Weighted Averaging* - **OWA** and show some of its features.

2.1 Definition and Examples

Aggregation functions associate each entry \mathbf{x} with n arguments in the closed interval [0,1] an output value also in the interval [0,1]; formally we have:

Definition 1. An n-ary aggregation function is a mapping $f:[0,1]^n \to [0,1]$, which associates each n-dimensional vector \mathbf{x} to a single value $f(\mathbf{x})$ in the interval [0,1] such that:

1.
$$f(0,...,0) = 0$$
 and $f(1,...,1) = 1$;

2. If
$$\mathbf{x} \leq \mathbf{y}$$
, i.e., $x_i \leq y_i$, for all $i = 1, 2, ..., n$, then $f(\mathbf{x}) \leq f(\mathbf{y})$.

Example 1.

- (a) Arithmetic Mean: $Arith(\mathbf{x}) = \frac{1}{n}(x_1 + x_2... + x_n)$
- (b) *Minimum*: $Min(\mathbf{x}) = min\{x_1, x_2, ..., x_n\};$
- (c) *Maximum*: $Max(\mathbf{x}) = max\{x_1, x_2, ..., x_n\};$
- (d) Harmonic mean: $f_n(\mathbf{x}) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}};$

From now on we will use the short term "aggregation" instead of "n-ary aggregation function".

Aggregations can be divided into four distinct classes: *Averaging, Conjunctive, Disjunctive* and *Mixed*. Since this paper focus on averaging aggregations, we will define only this class. A wider approach in aggregation can be found in [15, 14, 16, 17].

Definition 2. An aggregation is called **Averaging**, if for all $\mathbf{x} \in [0,1]^n$ we have:

$$Min(\mathbf{x}) < f(\mathbf{x}) < Max(\mathbf{x})$$

Example 2. The Arithmetic Mean, the Maximum and the Minimum are averaging aggregation functions.

2.2 Special Types Aggregation Functions

An aggregation function f:

- (1) is **Idempotent** if, and only if, f(x,...,x) = x for all $x \in [0,1]$.
- (2) is **Homogeneous** of order k if, and only if, for all $\lambda \in [0,1]$ and $\mathbf{x} \in [0,1]^n$, $f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^k f(x_1, x_2, ..., x_n)$. When f is homogeneous of order 1 we simply say that f is homogeneous.

- (3) is **Shift-invariant** if, and only if, $f(x_1+r, x_2+r, ..., x_n+r) = f(x_1, x_2, ..., x_n)+r$, for all $r \in [-1, 1]$, $\mathbf{x} \in [0, 1]^n$ such that $(x_1+r, x_2+r, ..., x_n+r) \in [0, 1]^n$ and $f(x_1, x_2, ..., x_n) + r \in [0, 1]$.
- (4) is **Monotonic** if, and only if, $\mathbf{x} \leq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$.
- (5) is **Strictly Monotone** if, and only if, $f(\mathbf{x}) < f(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$, i.e. $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$.
- (6) has a **Neutral Element** $e \in [0,1]$, if for all $t \in [0,1]$ at any coordinate input vector \mathbf{x} , it has to be:

$$f(e, ..., e, t, e, ..., e) = t$$
, and

(7) f is **Symmetric** if, and only if, its value is not changed under the permutations of the coordinates of x, i.e, we have:

$$f(x_1, x_2, ..., x_n) = f(x_{p_{(1)}}, x_{p_{(2)}}, \cdots, x_{p_{(n)}})$$

For all x and any permutation $P:\{1,2,...,n\} \rightarrow \{1,2,...,n\}$.

(8) An **Absorbing Element** (*Annihilator*) of an aggregation function f, is an element $a \in [0, 1]$ such that:

$$f(x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = a$$

- (9) A **Zero Divisor** of an aggregation function is an element $a \in]0,1[$, such that, there is some vector \mathbf{x} with $x_j > 0$, for all $1 \leq j \leq n$, and $f(x_1,...,x_{j-1},a,x_{j+1},...,x_n) = 0$.
- (10) A **One Divisor** of an aggregation function f is an element $a \in [0, 1]$ such that,

there is some vector $\mathbf x$ with $x_j < 1$, for all $1 \le j \le n$, and $f(x_1,...,x_{j-1},a,x_{j+1},...,x_n) = 1$.

(11) If $N:[0,1] \to [0,1]$ is a strong negation² and $f:[0,1]^n \to [0,1]$ is an aggregation function, then the **dual aggregation function** of f is:

$$f^{d}(x_{1}, x_{2}, ..., x_{n}) = N(f(N(x_{1}), N(x_{2}), ..., N(x_{n})))$$

which is also an aggregation function.

Example 3.

- (i) The functions: Arith, Min and Max are examples of idempotent, homogeneous, shift-invariant, monotonic and symmetric functions.
- (ii) Min and Max have 0 and 1 elements as annihilator, respectively, but Arith does not have annihiladors.
- (iii) Min, Max and Arith do not have zero divisors and one divisors.
- (iv) The dual of Max with respect to negation N(x) = 1 x is the Min function.

2.3 Ordered Weighted Averaging Function - OWA

In the field of aggregation functions there is a very important subclass in which the elements are parametric; they are called: *Ordered Weighted Averaging* or simply **OWA** [5].

An OWA is an aggregation function which associates weights to all components x_i of an input vector \mathbf{x} . To achieve that observe the following definition.

²A **strong negation** is an antitonic function $N:[0,1]\to [0,1]$ such that $N(N(\alpha))=\alpha$ for all $\alpha\in[0,1]$.

Definition 3. Let be an input vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0, 1]^n$ and a vector of weights $\mathbf{w} = (w_1, \dots, w_n)$, such that $\sum_{i=1}^n w_i = 1$. Assuming the permutation:

$$Sort(\mathbf{x}) = (x_{p(1)}, x_{p(2)}, \dots, x_{p(n)})$$

such that $x_{p(i)} \ge x_{p(i+1)}$, i.e., $x_{p(1)} \ge x_{p(2)} \ge \ldots \ge x_{p(n)}$, the Ordered Weighted Averaging (OWA) Function with respect to \mathbf{w} , is the function $OWA_{\mathbf{w}} : [0,1]^n \to [0,1]$ such that:

$$OWA_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i \cdot x_{p(i)}$$

In what follows we remove ${\bf w}$ from $OWA_{\bf w}({\bf x})$. The main properties of such functions are:

- (a) For any vector of weights w, the function $OWA(\mathbf{x})$ is idempotent and monotonic. Moreover, $OWA(\mathbf{x})$ is strictly increasing if all weights w are positive;
- (b) The dual of a $OWA_{\mathbf{w}}$ is denoted by $(OWA)^d$, with the vector of weights dually ordered, i.e. $(OWA_{\mathbf{w}})^d = OWA_{\mathbf{w}^d}$, where $\mathbf{w}^d = (w_{p(n)}, w_{p(n-1)}, ..., w_{p(1)})$.
- (c) OWA are continuous, symmetric and shift-invariant functions;
- (d) They do not have neutral or absorption elements, except in the special case of functions OWA of Max and Min.

2.3.1 Examples of special functions *OWA*

- 1. If all weight vector components are equal to $\frac{1}{n}$, then $OWA(\mathbf{x}) = Arith((\mathbf{x}))$.
- 2. If $\mathbf{w} = (1, 0, 0, ..., 0)$, then $OWA(\mathbf{x}) = Max(\mathbf{x})$.
- 3. If $\mathbf{w} = (0, 0, 0, ..., 1)$, then $OWA(\mathbf{x}) = Min(\mathbf{x})$.
- 4. if $w_i=0$, for all i, with the exception of a k-th member, i.e, $w_k=1$, then this OWA is called **static** and $OWA_{\mathbf{w}}(x)=x_k$

5. Given a vector \mathbf{x} and its ordered permutation $Sort(\mathbf{x}) = (x_{(1)}, \dots, x_{(n)})$, the *Median* function

$$Med(\mathbf{x}) = \begin{cases} \frac{1}{2}(x_{(k)} + x_{(k+1)}), & \text{if } n = 2k\\ x_{(k+1)}, & \text{if } n = 2k+1 \end{cases}$$

is an OWA function in which the vector of weights is defined by:

- If n is odd, then $w_i = 0$ for all $i \neq \lceil \frac{n}{2} \rceil$ and $w_{\lceil n/2 \rceil} = 1$.
- If n is even, then $w_i=0$ for all $i\neq \lceil\frac{n+1}{2}\rceil$ and $i\neq \lfloor\frac{n+1}{2}\rfloor$, and $w_{\lceil n/2\rceil}=w_{\lfloor n/2\rfloor}=\frac{1}{2}$.

Example 4. The n-dimensional cOWA function [18] is the OWA operator, with weighted vector defined by:

- If n is even, then $w_j = \frac{2(2j-1)}{n^2}$, for $1 \le j \le \frac{n}{2}$, and $w_{n/2+i} = w_{n/2-i+1}$, for $1 \le i \le \frac{n}{2}$.
- If n is odd, then $w_j = \frac{2(2j-1)}{n^2}$, for $1 \le j \le \frac{n-1}{2}$, $w_{n/2+i} = w_{n/2-i+1}$, for $1 \le i \le \frac{n}{2}$, and $w_{(n+1)/2} = 1 2\sum_{j=1}^{(n-1)/2} w_i$.

The *OWA* functions are defined in terms of a predetermined vector of weights. In the next section we propose the generalization of the concept of *OWA* in order to relax the vector of weights. To achieve that we replace the vector of weights by a family of functions. The resulting functions are called **Dynamic Ordered Weighted Avegaring Functions** or in short: **DYOWA**s.

3 Dynamic Ordered Weighted Avegaring Functions -

DYOWA

Before defining the notion of *DYOWA* functions, we need to establish the notion of *weight-function*.

Definition 4. A finite family of functions $\Gamma = \{f_i : [0,1]^n \to [0,1] \mid 1 \le i \le n\}$ such that:

$$\sum_{i=1}^{n} f_i(\mathbf{x}) = 1.$$

is called family of weight-function (FWF).

A Dynamic Ordered Weighted Averaging Function or simply DYOWA associated to a FWF Γ is a function of the form:

$$DYOWA_{\Gamma}(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i$$

Below we show some examples of DYOWA operators with their respective weight-functions.

Example 5. Let be $\Gamma = \{f_i(\mathbf{x}) = \frac{1}{n} \mid 1 \le i \le n\}$. The DYOWA operator associated to Γ , DYOWA $_{\Gamma}(\mathbf{x})$, is $Arith(\mathbf{x})$.

Example 6. The function Minimum can be obtained from $\Gamma = \{f_i \mid 1 \leq i \leq n\}$, where $f_1(\mathbf{x}) = f_2(\mathbf{x}) = \dots = f_{n-1}(\mathbf{x}) = 0$ and $f_n(\mathbf{x}) = 1$, for all $\mathbf{x} \in [0,1]^n$.

Example 7. Similarly, the function Maximum is also of type DYOWA with Γ dually defined.

Example 8. For any vector of weights $\mathbf{w} = (w_1, w_2, ..., w_n)$, A function $OWA_{\mathbf{w}}(\mathbf{x})$ is a DYOWA in which the weight-functions are given by: $f_i(\mathbf{x}) = w_{p(i)}$, where $p: \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\}$ is the permutation, such that p(i) = j with $x_i = x_{(j)}$. For example: If $\mathbf{w} = (0.3, 0.4, 0.3)$, then for $\mathbf{x} = (0.1, 1.0, 0.9)$ we have $x_1 = x_{(3)}$, $x_2 = x_{(1)}$ and $x_3 = x_{(2)}$. Thus, $f_1(\mathbf{x}) = 0.3$, $f_2(\mathbf{x}) = 0.3$, $f_3(\mathbf{x}) = 0.4$, and $DYOWA(\mathbf{x}) = 0.3 \cdot 0.1 + 0.3 \cdot 1.0 + 0.4 \cdot 0.9 = 0.69$

Remark 1. Example 8 shows that the functions OWA, introduced by Yager, are special cases of DYOWA functions. There are, however, some DYOWA functions which are not OWA.

Example 9. Let $\Gamma = \{\sin(x) \cdot y, 1 - \sin(x) \cdot y\}$. The respective DYOWA function is $DYOWA(x, y) = (\sin(x) \cdot y) \cdot x + (1 - \sin(x) \cdot y) \cdot y$, which is not an OWA function.

3.1 Properties of *DYOWA* **Functions**

The next theorem characterizes the DYOWA functions which are also aggregations.

Theorem 1. Let $\Gamma = \{f_1, \dots, f_n\}$ be a FWF. A DYOWA $_{\Gamma}$ is an aggregation function if, and only if, it is monotonic.

Proof. Obviously, if $DYOWA_{\Gamma}$ is an aggregation, then it is monotonic function. Conversely, if $DYOWA_{\Gamma}$ is monotonic, then for it to become an aggregation, enough to show that

$$DYOWA_{\Gamma}(0,...,0) = 0$$
 e $DYOWA_{\Gamma}(1,...,1) = 1$,

this follows from the definition of DYOWA.

Corollary 1. A DYOWA is an aggregation function if, and only if, it is an a aggregation of type averaging.

Proof. For all $\mathbf{x} = (x_1, ..., x_n)$ have to

$$Min(\mathbf{x}) \le x_i \le Max(\mathbf{x}), \ \forall i = 1, 2, ..., n.$$

So,

$$\sum_{i=1}^{n} f_i(\mathbf{x}) \cdot Min(\mathbf{x}) \le \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i \le \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot Max(\mathbf{x}),$$

but as $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, it follows that

$$Min(\mathbf{x}) \le \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i \le Max(\mathbf{x})$$

Corollary 2. All functions of the type DYOWA presented in examples 4, 5, 6, 7 and 8 are averaging aggregation functions.

Proof. Just see that those functions are monotonic.

Proposition 1. For every Γ , $DYOWA_{\Gamma}$ is idempotent.

Proof. If $\mathbf{x} = (x, ..., x)$ with $t \in [0, 1]$, then:

$$DYOWA_{\Gamma}(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \cdot x = x \cdot \sum_{i=1}^{n} f_i(\mathbf{x}) = x$$

This property is important because it tells us that every DYOWA is idempotent, regardless it is an aggregation or not.

Proposition 2. If Γ is invariant under translations, i.e, $f_i(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) = f_i(x_1, x_2, ..., x_n)$ for any $\mathbf{x} \in [0, 1]^n$, for $i \in \{1, 2, \cdots, n\}$ and $\lambda \in [-1, 1]$, then $DYOWA_{\Gamma}$ is shift-invariant.

Proof. Let $\mathbf{x} = (x_1, ..., x_n) \in [0, 1]^n$ and $\lambda \in [-1, 1]$ such that $(x_1 + \lambda, x_2 + \lambda, ..., x_n + \lambda) \in [0, 1]^n$. then,

$$DYOWA_{\Gamma}(x_1 + \lambda, ..., x_n + \lambda) =$$

$$= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot (x_i + \lambda)$$

$$= \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot x_i$$

$$+ \sum_{i=1}^{n} f_i(x_1 + \lambda, ..., x_n + \lambda) \cdot \lambda$$

$$= \sum_{i=1}^{n} f_i(x_1, ..., x_n) \cdot x_i + \lambda$$

$$= DYOWA_{\Gamma}(x_1, ..., x_n) + \lambda$$

Proposition 3. If Γ is homogeneous of order k (i.e., if f_i is homogeneous of order k, for each $f_i \in \Gamma$), then $DYOWA_{\Gamma}(\mathbf{x})$ is homogeneous of order k+1.

Proof. Of course that, if $\lambda=0$, then $DYOWA_{\Gamma}(\lambda x_1,...,\lambda x_n)=\lambda f(x_1,...,x_n)$. Now, to $\lambda\neq 0$ we have:

$$DYOWA_{\Gamma}(\lambda x_{1},...,\lambda x_{n}) = \sum_{i=1}^{n} f_{i}(\lambda x_{1},...,\lambda x_{n}) \cdot \lambda x_{i}$$
$$= \lambda \cdot \sum_{i=1}^{n} \lambda^{k} f_{i}(x_{1},...,x_{n}) x_{i}$$
$$= \lambda^{k+1} \cdot DYOWA_{\Gamma}(x_{1},...,x_{n})$$

Example 10. Let Γ be defined by

$$f_i(x_1, ..., x_n) = \begin{cases} \frac{1}{n}, & \text{if } x_1 = ... = x_n = 0\\ \frac{x_i}{\sum\limits_{j=1}^n x_j}, & \text{otherwise} \end{cases}$$

Then,

$$DYOWA_{\Gamma}(\mathbf{x}) = \begin{cases} 0, & \text{if } x_1, ..., x_n = 0\\ \sum\limits_{i=1}^n x_i^2 \\ \sum\limits_{i=1}^n x_i, & \text{otherwise} \end{cases}$$

This $DYOWA_{\Gamma}$ is idempotent, homogeneous and shift-invariant. However, $DYOWA_{\Gamma}$ is not monotonic, since $DYOWA_{\Gamma}(0.5, 0.2, 0.1) = 0.375$ and $DYOWA_{\Gamma}(0.5, 0.22, 0.2) = 0.368$.

The next definition provides a special FWF, which will be used to build a DYOWA whose properties are very important for this paper.

Definition 5. *Consider the family* Γ *of functions*

$$f_i(\mathbf{x}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{x} = (x, ..., x) \\ \frac{1}{n-1} \left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |x_j - Med(\mathbf{x})|} \right), & \text{otherwise} \end{cases}$$

Then, Γ is a FWF, i.e. $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$, for all $\mathbf{x} \in [0,1]^n$. Let \mathbf{H} be the associated DYOWA. The computation of \mathbf{H} can be performed using the following expressions:

$$\mathbf{H}(\mathbf{x}) = \begin{cases} x, & \text{if } \mathbf{x} = (x, ..., x) \\ \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \frac{x_i | x_i - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} | x_j - Med(\mathbf{x})|} \right), & \text{otherwise} \end{cases}$$

Example 11. Let be n = 3. So, for $\mathbf{x} = (0.1, 0.3, 0)$ we have

$$f_1(\mathbf{x}) = 0.5, \ f_2(\mathbf{x}) = 0.167, \ f_3(\mathbf{x}) = 0.333 \ and \ \mathbf{H}(\mathbf{x}) = 0.08.$$

Proposition 4. The weight-functions defined in Definition 5 are such that: $f_i(x_1 + \lambda, ..., x_n + \lambda) = f_i(x_1, x_2, ..., x_n)$ and $f_i(\lambda x_1, ..., \lambda x_n) = f_i(x_1, ..., x_n)$, for any $1 \le i \le n$.

Proof. Writing $\mathbf{x}' = (x_1 + \lambda, ..., x_n + \lambda)$, then $f(x_1 + \lambda, ..., x_n + \lambda) = (f_1(\mathbf{x}'), ..., f_n(\mathbf{x}'))$. Clearly, $Med(\mathbf{x}') = Med(\mathbf{x}) + \lambda$. Thus, for $\mathbf{x} \neq (x, ..., x)$ we have:

$$f_{i}(\mathbf{x}') = \frac{1}{n-1} \left(1 - \frac{|x_{i} + \lambda - Med(\mathbf{x}')|}{\sum\limits_{j=1}^{n} |x_{j} + \lambda - Med(\mathbf{x}')|} \right)$$

$$= \frac{1}{n-1} \left(1 - \frac{|x_{i} + \lambda - (Med(\mathbf{x}) + \lambda)|}{\sum\limits_{j=1}^{n} |x_{j} + \lambda - (Med(\mathbf{x}) + \lambda)|} \right)$$

$$= \frac{1}{n-1} \left(1 - \frac{|x_{i} - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |x_{j} - Med(\mathbf{x})|} \right)$$

$$= f_{i}(\mathbf{x}).$$

Therefore, $f(\mathbf{x}') = (f_1(\mathbf{x}'), ..., f_n(\mathbf{x}')) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))$. The case in which $\mathbf{x} = (x, ..., x)$ is immediate.

To check the second property, make $\mathbf{x}'' = (\lambda x_1, ..., \lambda x_n)$, note that $Med(\mathbf{x}'') = \lambda med(\mathbf{x})$ and for $\mathbf{x} \neq (x, ..., x)$

$$f_{i}(\mathbf{x}'') = \frac{1}{n-1} \left(1 - \frac{|\lambda x_{i} - Med(\lambda \mathbf{x})|}{\sum\limits_{j=1}^{n} |\lambda x_{j} - Med(\lambda \mathbf{x})|} \right)$$

$$= \frac{1}{n-1} \left(1 - \frac{|\lambda x_{i} - \lambda Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |\lambda x_{j} - \lambda Med(\mathbf{x})|} \right)$$

$$= \frac{1}{n-1} \left(1 - \frac{|\lambda| \cdot |x_{i} - Med(\mathbf{x})|}{|\lambda| \cdot \sum\limits_{j=1}^{n} |x_{j} - Med(\mathbf{x})|} \right)$$

$$= \frac{1}{n-1} \left(1 - \frac{|x_{i} - Med(\mathbf{x})|}{\sum\limits_{j=1}^{n} |x_{j} - Med(\mathbf{x})|} \right)$$

$$= f_{i}(\mathbf{x})$$

Therefore, $f(\mathbf{x}'') = (f_1(\mathbf{x}''), ..., f_n(\mathbf{x}'')) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x})) = f(\mathbf{x})$. The case in which $\mathbf{x} = (x, ..., x)$ is also immediately

Corollary 3. H is shift-invariant and homogeneous.

Proof. Straightforward for propositions 2 and 3.

The function \mathbf{H} is of great importance to this work, since this function, as well as some DYOWA's already mentioned will provide us tools able: (1) To reduce the size of images and (2) To deal with noise reduction.

Now, we present some other properties of function **H**.

3.2 Properties of H

In addition to idempotency, homogeneity and shift-invariance ${\bf H}$ has the following proprerties.

Proposition 5. H has no neutral element.

Proof. Suppose **H** has a neutral element e, find the vector of weight for $\mathbf{x} = (e, ..., e, x, e, ..., e)$. Note that if $n \geq 3$, then $Med(\mathbf{x}) = e$ and therefore,

$$f_i(\mathbf{x}) = \frac{1}{n-1} \left(1 - \frac{|x_i - Med(\mathbf{x})|}{\sum\limits_{j=1}^n |x_j - Med(\mathbf{x})|} \right)$$
$$= \frac{1}{n-1} \left(1 - \frac{|x_i - e|}{\sum\limits_{j=1}^n |x_j - e|} \right)$$
$$= \frac{1}{n-1} \left(1 - \frac{|x_i - e|}{|x - e|} \right)$$

therefore,

$$f_i(\mathbf{x}) = \begin{cases} \frac{1}{n-1}, & \text{if } x_i = e \\ 0, & \text{if } x_i = x \end{cases}, \text{ to } n \ge 3$$

i.e.,

$$f(\mathbf{x}) = \left(\frac{1}{n-1}, ..., \frac{1}{n-1}, 0, \frac{1}{n-1}, ..., \frac{1}{n-1}\right)$$

and

$$\mathbf{H}(\mathbf{x}) = (n-1) \cdot \frac{e}{n-1} = e$$

But since e is a neutral element of \mathbf{H} , $\mathbf{H}(\mathbf{x})=x$. Absurd, since we can always take $x\neq e$.

For n=2, we have $Med(\mathbf{x})=\frac{x+e}{2}$, where $\mathbf{x}=(x,e)$ or $\mathbf{x}=(e,x)$. In both cases it is not difficult to show that $f(\mathbf{x})=(0.5,0.5)$ and $\mathbf{H}(\mathbf{x})=\frac{x+e}{2}$. Thus, taking $x\neq e$, again we have $\mathbf{H}(x,e)\neq x$.

Proposition 6. H has no absorbing elements.

Proof. To n=2, we have $\mathbf{H}(\mathbf{x})=\frac{x_1+x_2}{2}$, which has no absorbing elements. Now

for $n \geq 3$ we have to $\mathbf{x} = (a,0,...,0)$ with $Med(\mathbf{x}) = 0$ therefore,

$$f_1(\mathbf{x}) = \frac{1}{n-1} \left(1 - \frac{a}{a} \right) = 0 \text{ and } f_i = \frac{1}{n-1}, \forall i = 2, ..., n.$$

therefore,

$$\mathbf{H}(a,0,...,0) = 0 \cdot a + \frac{1}{n-1} \cdot 0 + ... + \frac{1}{n-1} \cdot 0 = a \Rightarrow a = 0,$$

but to $\mathbf{x}=(a,1,...,1)$ we have to $Med(\mathbf{x})=1.$ Furthermore,

$$f_1(\mathbf{x}) = \frac{1}{n-1} \left(1 - \frac{1-a}{1} - a \right) = 0$$

and

$$f_i = \frac{1}{n-1}$$
 para $i = 2, 3, ..., n$.

therefore,

$$\mathbf{H}(a, 1, ..., 1) = 0 \cdot a + \frac{1}{n-1} \cdot 1 + ... + \frac{1}{n-1} \cdot 1 = a \Rightarrow a = 1.$$

With this we prove that **H** does note have annihiladors.

Proposition 7. H has no zero divisors.

Proof. Let $a \in]0,1[$ and consider $\mathbf{x}=(a,x_2,...,x_n)\in]0,1]^n$. In order to have $\mathbf{H}(\mathbf{x})=\sum_{i=1}^n f_i(\mathbf{x})\cdot x_i=0$ we have $f_i(\mathbf{x})\cdot x_i=0$ for all i=1,2,...,n. But as $a\neq 0$ and we can always take $x_2,x_3,...,x_n$ also different from zero, then for each i=1,2,...,n there remains only the possibility of terms:

$$f_i(\mathbf{x}) = 0$$
 para $i = 1, 2, ..., n$.

This is absurd, for $f_i(\mathbf{x}) \in [0,1]$ e $\sum_{i=1}^n f_i(\mathbf{x}) = 1$. like this, **H** has no zero divisors. \square

Proposition 8. H does not have one divisors

Proof. Just to see that $a \in]0,1[$, we have to $\mathbf{H}(a,0,...,0) = f_1(\mathbf{x}).a \le a < 1.$

Proposition 9. H is symmetric.

Proof. Let $P: \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}$ be a permutation. So we can easily see that $Med(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) = Med(x_1, x_2, ..., x_n)$ for all $\mathbf{x} = (x_1, x_2, ..., x_n) \in [0, 1]^n$. We also have to $\sum_{i=1}^n |x_{P(i)} - Med(x_{P(1)}, x_{P(2)}, ..., x_{P(n)})| = \sum_{i=1}^n |x_i - Med(\mathbf{x})|$. Thus, it suffices to consider the case where $(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) \neq (x, x, ..., x)$. But $(x_{P(1)}, x_{P(2)}, ..., x_{P(n)}) \neq (x, x, ..., x)$ we have to:

$$\begin{split} &\mathbf{H}\left(x_{P(1)},x_{P(2)},...,x_{P(n)}\right) = \\ &= \frac{1}{n-1}\sum_{i=1}^{n} \left(x_{P(i)} - \frac{x_{P(i)}|x_{P(i)} - Med(x_{P(1)},...,x_{P(n)})|}{\sum\limits_{j=1}^{n}|x_{P(i)} - Med(x_{P(1)},...,x_{P(n)})|}\right) \\ &= \frac{\sum\limits_{i=1}^{n}x_{P(i)}}{n-1} - \frac{1}{n-1} \cdot \sum\limits_{i=1}^{n} \frac{x_{P(i)}|x_{P(i)} - Med(x_{1},...,x_{n})|}{\sum\limits_{j=1}^{n}|x_{P(i)} - Med(x_{1},...,x_{n})|} \\ &= \frac{\sum\limits_{i=1}^{n}x_{i}}{n-1} - \frac{1}{n-1} \cdot \sum\limits_{i=1}^{n} \frac{x_{P(i)}|x_{P(i)} - Med(x_{1},...,x_{n})|}{\sum\limits_{j=1}^{n}|x_{i} - Med(x_{1},...,x_{n})|} \\ &= \frac{\sum\limits_{i=1}^{n}x_{i}}{n-1} - \frac{1}{n-1} \cdot \sum\limits_{i=1}^{n} \frac{x_{i}|x_{i} - Med(x_{1},...,x_{n})|}{\sum\limits_{j=1}^{n}|x_{i} - Med(x_{1},...,x_{n})|} \\ &= \mathbf{H}(x_{1},...,x_{n}). \end{split}$$

Therefore, **H** satisfies the following properties:

- Idempotence
- Homogeneity
- Shift-invariance

- Symmetry.
- H has no neutral element
- H has no absorbing elements
- H has no zero divisors
- H does not have one divisors

Remark 2. Unfortunately we do not prove here the monotonicity of **H**, due to its complexity, but we suspect that it is true. This demonstration will be relegated to a future work.

The next two sections show the suitability of *DYOWA*. They will provide applications for image and noise reduction.

4 DYOWA's as images reduction tools

In this part of our work we use the functions *DYOWA* studied in Examples 4, 5, 6, 7 and 8, and definition 5 to build image reduction operators, the resulting images will be compared with the reduced image obtained from the operator function **H**.

An image is a matrix $m \times n$, M = A(i,j), where each A(i,j) represents a pixel. In essence, a reduction operator reduces a given image $m \times n$ to another $m' \times n'$, such that m' < m and n' < n. For example,

$$\begin{bmatrix} 0.1 & 0.2 & 0 & 0.5 \\ 0.3 & 0.3 & 0.2 & 0.8 \\ 1 & 0.5 & 0.6 & 0.4 \\ 0 & 0.3 & 0.5 & 0.7 \end{bmatrix} \longmapsto \begin{bmatrix} 0.1 & 0 \\ 1 & 0.6 \end{bmatrix}$$

In Grayscale images the value of pixels belong to the set [0, 255], which can be

normalized by dividing them by 255, so that we can think of pixels as values in the range [0,1].



Figure 1: Example of image in Grayscale.

There are several possible ways to reduce a given image, as shown in the following example:

Example 12. The image

$$M = \begin{bmatrix} 0.8 & 0.7 & 0.2 & 1 & 0.5 & 0.5 \\ 0.6 & 0.2 & 0.3 & 0.1 & 1 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0.9 & 1 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{bmatrix},$$

can be reduced to another 2×3 by partitioning M in blocks 2×2 :

$$\overline{M} = \begin{bmatrix} \begin{bmatrix} 0.8 & 0.7 \\ 0.6 & 0.2 \end{bmatrix} & \begin{bmatrix} 0.2 & 1 \\ 0.3 & 0.1 \end{bmatrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0.1 & 0.2 \end{bmatrix} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.4 \end{bmatrix} & \begin{bmatrix} 0.9 & 1 \\ 0.5 & 0.6 \end{bmatrix} \end{bmatrix},$$

and applying to each block, for example, the function f(x,y,z,w)=Max(x,y,z,w): We obtain, the image:

$$M_* = \left[\begin{array}{ccc} 0.8 & 1 & 1 \\ 0.2 & 0.6 & 1 \end{array} \right]$$

Applying g(x, y, z, w) = Min(x, y, z, w) we would obtain:

$$M_{**} = \left[\begin{array}{ccc} 0.2 & 0.1 & 0 \\ 0 & 0.2 & 0.5 \end{array} \right]$$

In fact, if we apply any other function, we get a new image (usually different from the previous one), but what is the best?

One possible answer to this question involves a method called **magnification** or **extension** (see [19, 20, 21]), which is a method which magnifies the reduced image to another with the same size of the original one. The magnified image is then compared with the original input image.

Example 13. From M_* and M_{**} , we get images 4×6 , M' and M'', simply cloning each pixel,

$$\left[\begin{array}{c} x \end{array}\right] \longmapsto \left[\begin{array}{cc} x & x \\ x & x \end{array}\right]$$

We obtain, new images

and

$$M_2 = \left[\begin{array}{cccccccc} 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0.2 & 0.5 & 0.5 \end{array} \right]$$

Since M_1 e M_2 have the same size as the original image M, we can now measure what is the best reduction. This can be done by comparing the initial image M with each of the resulting images, M_1 and M_2 . But, how do we compare?

One of the possibilities to compare the images M_1 and M_2 with the original image M is to use the mensure PSNR [4], calculated as follows:

$$PSNR(I, K) = 10 \cdot log_{10} \left(\frac{MAX_I^2}{MSE(I, K)} \right),$$

where I = I(i,j) and K = K(i,j) are two images, $MSE(I,K) = \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} [I(i,j) - K(i,j)]^n$ and MAX_I is the maximum possible pixel value of pixel. Observe that the closer the image the smaller the value of MSE and the larger the value of PSNR 3 .

In what follows, we use DYOWA operators: $\mathbf{H}, cOWA, Median$ and Arith to reduce size of images in grayscale. We apply the following method:

Method 1

1. Reduce the input images using the H, cOWA, Arithmetic Mean and Median;

³In particular, if the input image are equal, then the MSE value is zero and the PSNR will be infinity.

- 2. Magnify the reduced image to the size of the original image using the method described in example 13;
- 3. Compare the last image with the original one using the measure PSNR.

Remark 3. This general method can be applied to any kind of image. In this work we applied it to the 10 images in grayscale of size 512×512 (Figure 2) ⁴.

In the tables I and II we present the PSNR values between the input images and the output provided by Method 1. Table 1 provides results for operators using blocks 2×2 and Table II for blocks 4×4 .

	Н	cOWA	Arith	Median
Img 01	29.63	29.66	29,71	29.50
Img 02	33.15	33.14	33.18	33.09
Img 03	29.52	29.53	29.57	29.44
Img 04	31.54	31.54	31.61	31.46
Img 05	27.87	27.88	27.91	27.80
Img 06	40.78	40.78	40.79	40.78
Img 07	27.40	27.42	27.47	27.30
Img 08	26.56	26.57	26.61	26.47
Img 09	28.84	28.85	28.89	28.73
Img 10	24.43	24.45	24.53	24.27
Avg	29.97	29.98	30.03	29.88

Table 1: PSNR values after a reduction using the DYOWAs operators using blocks 2×2

According to PSNR, Arith provided the higher quality image. However, the reduction operators generated by \mathbf{H} and cOWA provide us quite similar images to those given by Arith.

Observe that although the Method 1 is very simple, it introduces noise in the resulting image. In what follows we show that the operator **H** is suitable to filter images with noise. This is done by using **H** to define the weights which are used in the process of convolution. This new process will, then, be used to provide a better comparison in the Method 1.

 $^{^4}$ In this paper we made two reductions: using 2×2 blocks and 4×4 blocks.

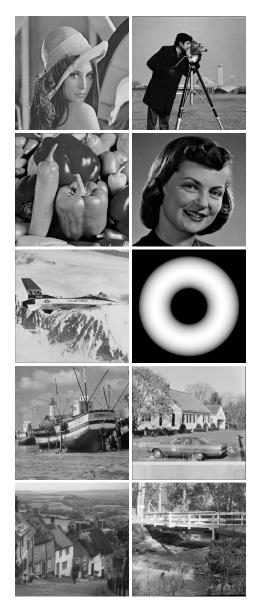


Figure 2: Original images

5 DYOWA's as Tools of Noise Reduction

In this section we show that the DYOWA operators studied in section III can be used to deal with images containing noise.

	H	cOWA	Arith	Median
Img 01	26.34	26.26	26.36	26.70
Img 02	23.64	23.60	23.65	22.78
Img 03	25.55	25.46	25.56	24.84
Img 04	27.53	27.45	27.54	26.86
Img 05	24.14	24.06	24.14	23.28
Img 06	34.39	34.34	34.41	33.83
Img 07	23.98	23.88	23.99	23.19
Img 08	23.07	22.97	23.07	22.18
Img 09	25.78	25.69	25.79	25.05
Img 10	21.71	21.61	21.71	20.62
Avg	25.61	25.53	25.62	24.83

Table 2: PSNR values after a reduction using the DYOWAs operators using blocks 4×4 .

The methodology employed here consists to analyze the previous images with Gaussian noise $\sigma=10\%$ and 15%; apply a filter built upon the operators \mathbf{H} , cOWA and Arith based on convolution method (See [4]), and compare the resulting images with the original one using PSNR.

	H	cOWA	Arith	No Tratament
Img 01	30.96	30.56	30.96	23.83
Img 02	28.16	27.78	27.36	24.36
Img 03	31.33	30.99	31.08	24.23
Img 04	32.33	32.09	32.20	24.48
Img 05	30.39	30.09	30.10	24.06
Img 06	31.66	31.69	31.38	25.73
Img 07	28.97	28.65	28.80	23.93
Img 08	28.51	28.28	28.25	24.02
Img 09	30.03	29.73	30.02	23.91
Img 10	25.97	25.84	25.81	23.76
Avg	29.83	29.57	29.60	24.23

Table 3: PSNR values between the output image with original one, in which $\sigma = 10\%$

Tables III and IV demonstrate the power of **H** on images with noise. All listed operators improved significantly the quality of the image with noise. However, **H** exceeded all other analyzed.

Figure 4 shows an example of a image with Gaussian noise $\sigma=15\%$ and the

	Н	cOWA	Arith	No Tratament
Img 01	29.88	29.40	29.97	21.19
Img 02	27.33	27.04	26.74	21.48
Img 03	29.92	29.55	29.79	21.32
Img 04	30.23	30.06	30.08	21.51
Img 05	29.38	28.96	29.27	21.30
Img 06	27.95	28.03	27.56	22.36
Img 07	28.23	27.84	28.14	21.26
Img 08	27.87	27.57	27.70	21.28
Img 09	29.17	28.76	29.22	21.25
Img 10	25.55	25.39	25.44	21.41
Avg	28.55	28.26	28.39	21.44

Table 4: PSNR values between the output image with original one, in which $\sigma=15\%$.

Figure 5 the output image after applying the filter of convolution using ${\bf H}$.

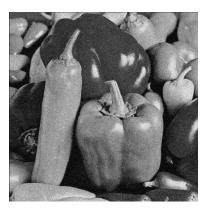


Figure 3: Image 03 with Gaussian noise $\sigma=15\%$.

The reader can see in tables III and IV that ${\bf H}$ proved to be an excellent operator for noise reduction.

In what follows, we modify the Method 1 in order to provide a better magnified image to be compare with the original one.

Method 1'

1. Reduce the input images using the \mathbf{H} , cOWA, $Arithmetic\ Mean$ and Median;



Figure 4: Figure 4 after being treated with H by convolution.

- 2. (A) Magnify the reduced image to the size of the original image using the method described in Example 13, and (B) Use the convolution filter, using **H**, on the last image;
- 3. Compare the last image with the original one using the measure PSNR.

Tables V and VI show the obtained results:

	Н	cOWA	Arith	Median
Img 01	30.60	30.55	30.62	30.50
Img 02	28.68	28.67	28.70	28.68
Img 03	30.89	30.83	30.85	30.84
Img 04	32.74	32.71	32.71	32.71
Img 05	29.22	29.18	29.16	29.19
Img 06	43.88	43.80	43.93	43.90
Img 07	28.05	28.01	28.03	27.95
Img 08	27.39	27.37	27.37	27.36
Img 09	29.36	29.35	29.40	29.30
Img 10	24.94	24.92	24.95	24.83
Avg	30.57	30.54	30.57	30.53

Table 5: PSNR between the original image and the magnified image from the image reduced by blocks 2×2 .

Since the output of convolution using \mathbf{H} is closer to the original input image, the tables V and VI show that the process of reduction using \mathbf{H} is more efficient.

	H	cOWA	Arith	Median
Img 01	27.44	27.41	27.41	27.01
Img 02	23.91	23.88	23.91	23.24
Img 03	26.86	26.85	26.85	26.28
Img 04	28.87	28.86	28.85	28.39
Img 05	25.15	25.15	25.12	24.64
Img 06	28.13	28.05	28.13	27.04
Img 07	24.68	24.63	24.68	24.13
Img 08	23.78	23.76	23.78	23.14
Img 09	26.45	26.40	26.45	25.92
Img 10	22.27	22.21	22.26	21.49
Avg	25.75	25.72	25.74	25.12

Table 6: PSNR between the original image and the magnified image from the image reduced by blocks 4×4

6 Final Remarks

In this paper we propose a generalized form of Ordered Weighted Averaging function, called **Dynamic Ordered Weighted Averaging** function or simply **DYOWA**. This functions are defined by weights, which are obtained dynamically from of each input vector $\mathbf{x} \in [0,1]^n$. We demonstrate, among other results, that OWA functions are instances of DYOWAs, and, hence, functions like: Arithmetic Mean, Median, Maximum, Minimum and cOWA are also examples of DYOWA.

In the second part of this work we present a particular DYOWA, called of \mathbf{H} , and show that it is idempotent, symmetric, homogeneous, shift-invariant, and moreover, it has no zero divisors and one divisors, and also does not have neutral elements. Since aggregation functions which satisfy these properties are extensively used in image processing, we tested its usefulness to: (1) reduce the size of images and (2) deal with noise in images.

In terms of image reduction, Method 1 showed a weakness, since it adds noise during the process of magnification. However, the treatment of noise with function **H** improved the magnification step providing an evidence that the function **H** is more efficient to perform the image reduction process.



Figure 5: Magnification of image 06 reduce by bloks 2×2 using the operator H by Method 1



Figure 6: Magnification of image 06 reduce by bloks 2×2 using the operator H by Method 1'



Figure 7: Image 06



Figure 8: Magnification of image 01 reduce by bloks 4×4 using the operator H by Method 1

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Figure 9: Magnification of image 01 reduce by bloks 4×4 using the operator H by Method 1'



Figure 10: Image 01

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