UNIFIED FORMALISM FOR HIGHER-ORDER NON-AUTONOMOUS DYNAMICAL SYSTEMS

Pedro Daniel Prieto-Martínez* Narciso Román-Roy[†]

Departamento de Matemática Aplicada IV. Edificio C-3, Campus Norte UPC C/ Jordi Girona 1. 08034 Barcelona. Spain

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Abstract

This work is devoted to giving a geometric framework for describing higher-order non-autonomous mechanical systems. The starting point is to extend the Lagrangian-Hamiltonian unified formalism of Skinner and Rusk for these kinds of systems, generalizing previous developments for higher-order autonomous mechanical systems and first-order non-autonomous mechanical systems. Then, we use this unified formulation to derive the standard Lagrangian and Hamiltonian formalisms, including the Legendre-Ostrogradsky map and the Euler-Lagrange and the Hamilton equations, both for regular and singular systems. As applications of our model, two examples of regular and singular physical systems are studied.

Key words: Higher-order non-autonomous systems, Lagrangian and Hamiltonian formalisms, Symplectic and presymplectic manifolds.

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^{*}e-mail: peredaniel@ma4.upc.edu

 $^{^{\}dagger}$ e-mail: nrr@ma4.upc.edu

Contents

1	Inti	Introduction	
2	Geo	ometric structures of higher-order jet bundles over $\mathbb R$	4
	2.1	Higher-order jet bundles over $\mathbb R$	4
	2.2	Total time derivative	4
	2.3	Higher-order semisprays. Holonomic sections	6
3	Ski	nner-Rusk unified formalism	7
	3.1	Unified phase space. Geometric and dynamical structures	7
	3.2	Dynamical equations	11
		3.2.1 Dynamical equations for sections	11
		3.2.2 Dynamical equations for vector fields	14
4	Lagrangian formalism		19
	4.1	General setting	19
	4.2	Dynamical equations for sections	20
	4.3	Dynamical equations for vector fields	21
5	Hamiltonian formalism		24
	5.1	General setting	24
	5.2	Hyperregular and regular systems. Dynamical equations for sections and vector fields	27
	5.3	Singular (almost-regular) Lagrangians. Dynamical equations for sections and vector fields	30
6	Examples		31
	6.1	The shape of a deformed elastic cylindrical beam with fixed ends $\dots \dots$.	31
	6.2	The second-order relativistic particle subjected to a potential	35
7	Cor	nclusions and outlook	39
${f A}$	Αp	particular situation: trivial bundles	40

1 Introduction

Higher-order dynamical systems play a relevant role in certain branches of theoretical physics, applied mathematics and numerical analysis. In particular, they appear in theoretical physics, in the mathematical description of relativistic particles with spin, string theories, Hilbert's Lagrangian for gravitation, Podolsky's generalization of electromagnetism and others [3, 6, 7, 8, 27, 32, 34, 37, 38, 40, 47, as well as in some problems of fluid mechanics and classical physics (see, for instance, the example in Section 6.1 taken from [9, 25]), and in numerical models arising from the discretization of first-order dynamical systems that preserve their inherent geometric structures [21]. In these kinds of systems, the dynamics have explicit dependence on accelerations or higher-order derivatives of the generalized coordinates of position.

In recent years, much works has been devoted to the development of geometric formalisms for higher-order mechanics and field theory (see, for instance, [1, 2, 11, 12, 16, 22, 26, 28, 29, 31, 33, 44). These formulations use higher-order tangent and jet bundles as the main tool. In particular, in a recent paper [39] a new geometric formulation has been proposed, which is an extension to higher-order autonomous mechanical systems of the formalism proposed by R. Skinner and R. Rusk in his seminal paper [45]. This formulation compresses the Lagrangian and Hamiltonian formalisms into a single one, originally developed for first-order autonomous mechanical systems and later generalized to non-autonomous systems [5, 14], control systems [4], and first-order classical field theories (see [41] and references therein). Nevertheless, to our knowledge, there is neither a complete geometrical description of the Lagrangian and Hamiltonian formalisms (partial studies on this subject can be found in [15, 20, 23, 19, 30]), nor of the Skinner-Rusk unified formalism for non-autonomous higher-order mechanical systems.

The aim of this work is to fill this gap. In order to do this, we first develop the Lagrangian-Hamiltonian unified formalism of Skinner-Rusk for higher-order non-autonomous mechanical systems, studying in particular how this formulation enables us to obtain the generalized Legendre-Ostrogradsky map connecting the Lagrangian and Hamiltonian formalisms, as well as the Euler-Lagrange and the Hamilton equations of motion. Thus, starting from this unified framework, we obtain a geometric description for the Lagrangian and Hamiltonian formalisms for higher-order non-autonomous mechanical systems. This study is conducted both for regular and singular dynamical systems. Our analysis is performed by using higher-order jet bundles, since we wish this work to serve as a model to develop an unambiguous framework for higher-order classical field theories that complete previous approaches in this way [10, 46].

The paper is organized as follows: in Section 2, we review the geometric structures needed to develop the formalism, such as the higher-order jet bundles, the total derivatives and higherorder semisprays. Section 3 is devoted to the geometric formulation of the Skinner-Rusk unified formalism for higher-order non-autonomous mechanical systems, including the description of the dynamical equations using sections and vector fields. In Sections 4 and 5, we recover the standard Lagrangian and Hamiltonian formalisms, presenting a complete description of both for regular and singular systems. Finally, in Section 6, two examples are analyzed; the first is a regular system which models the shape of a deformed elastic cylindrical beam with fixed ends and has applications in Statics and other branches of classical physics [9, 25]; the second is a modification of a singular system describing a relativistic particle [36, 35, 7, 34, 39], which in our case is subjected to a generic time-dependent potential. The paper concludes in Section 7 with a summary of results and future research, and an appendix in Section A where the particular situation of higher-order trivial bundles is briefly analyzed.

All the manifolds are real, second countable and C^{∞} . The maps and the structures are assumed to be C^{∞} . Sum over repeated indices is understood.

2.1 Higher-order jet bundles over \mathbb{R}

(See [22, 43] for details).

Let $E \xrightarrow{\pi} \mathbb{R}$ be a bundle (dim E = n+1), and let $\eta \in \Omega^1(\mathbb{R})$ be the canonical volume form in \mathbb{R} . If $k \in \mathbb{N}$, the *kth* order jet bundle of the projection π , $J^k\pi$, is the ((k+1)n+1)-dimensional manifold of the *k*-jets of sections $\phi \in \Gamma(\pi)$. A point in $J^k\pi$ is denoted by $j^k\phi$, where $\phi \in \Gamma(\pi)$ is any representative of the equivalence class. We have the following natural projections: if $r \leq k$,

$$\pi_r^k : \quad J^k \pi \quad \longrightarrow \quad J^r \pi \quad , \quad \pi^k : \quad J^k \pi \quad \longrightarrow \quad E \\
j^k \phi \quad \longmapsto \quad j^r \phi \quad , \qquad \qquad j^k \phi \quad \longmapsto \quad \phi \, ,$$

Notice that $\pi_0^k = \pi^k$, where $J^0\pi$ is canonically identified with E, and $\pi_k^k = \operatorname{Id}_{J^k\pi}$. Furthermore, we denote $\bar{\pi}^k = \pi \circ \pi^k \colon J^k\pi \to \mathbb{R}$.

Local coordinates in $J^k\pi$ are constructed as follows: let t be the global coordinate in \mathbb{R} such that $\eta = \mathrm{d}t$, and (t, q^A) , $(1 \leq A \leq n)$, local coordinates in E adapted to the bundle structure. Let $\phi \in \Gamma(\pi)$ such that $\phi = (t, \phi^A)$. Then, local coordinates in $J^k\pi$ are $(t, q^A, q_1^A, \ldots, q_k^A)$, with

$$q^A = \phi^A$$
 , $q_i^A = \frac{\mathrm{d}^i \phi^A}{\mathrm{d}t^i}$.

Usually we write q_0^A instead of q^A , and so the local coordinates in $J^k\pi$ are written $(t, q_0^A, q_1^A, \dots, q_k^A)$.

Using these coordinates, the local expression of the natural projections are

$$\pi_r^k(t, q_0^A, q_1^A, \dots, q_k^A) = (t, q_0^A, q_1^A, \dots, q_r^A)$$
, $\pi^k(t, q_0^A, q_1^A, \dots, q_k^A) = (t, q_0^A)$.

If $\phi \in \Gamma(\pi)$ is a section of π , we denote by $j^k \phi$ the canonical lifting of ϕ to $J^k \pi$, that is, the map $j^k \phi \colon \mathbb{R} \to J^k \pi$, which is a section of the projection $\bar{\pi}^k$.

Remark: We use the same notation for points of $J^k\pi$ and liftings of sections to $J^k\pi$, since giving a point in $J^k\pi$ is equivalent to giving the lifting to $J^k\pi$ of a section of π (see [43] for details).

2.2 Total time derivative

(See [43] for details).

Definition 1 Let $E \xrightarrow{\pi} \mathbb{R}$ be a bundle, $t_o \in \mathbb{R}$, $\phi \in \Gamma(\pi)$ and $u \in T_{t_o}\mathbb{R}$. The kth holonomic lift of u by ϕ is defined as $((j^k\phi)_*(u), j_{t_o}^{k+1}\phi) \in (\pi_k^{k+1})^*TJ^k\pi$, where $j_{t_o}^{k+1}\phi \equiv (j^{k+1}\phi)(t_o)$.

In local coordinates, if u is given by $u = u_o \frac{\partial}{\partial t}\Big|_{t_o}$, the kth holonomic lift of u is given by

$$(j^k \phi)_* u = u_o \left(\frac{\partial}{\partial t} \Big|_{j_{t_o}^k \phi} + \sum_{i=0}^k q_{i+1}^A (j_{t_o}^{k+1} \phi) \frac{\partial}{\partial q_i^A} \Big|_{j_{t_o}^k \phi} \right). \tag{1}$$

The vector space $(\pi_k^{k+1})^*(\mathrm{T}J^k\pi)_{j_{t_0}^{k+1}\pi}$ has a canonical splitting as a direct sum, as follows:

$$(\pi_k^{k+1})^* (\mathrm{T}J^k \pi)_{j_{t_o}^{k+1} \pi} = (\pi_k^{k+1})^* (V(\bar{\pi}^k))_{j_{t_o}^{k+1} \phi} \oplus (j^k \phi)_* \mathrm{T}_{t_o} \mathbb{R} ,$$

where $(j^k\phi)_*T_{t_o}\mathbb{R}$ denotes the set of kth holonomic lifts of tangent vectors in $T_{t_o}\mathbb{R}$ by ϕ . As a consequence, the vector bundle $(\pi_k^{k+1})^* T J^k \pi \xrightarrow{(\pi_k^{k+1})^* \tau_{J^k \pi}} J^k \pi$ may be written as the direct sum of two subbundles:

$$(\pi_k^{k+1})^*V(\bar{\pi}^k)\oplus H(\pi_k^{k+1})\xrightarrow{(\pi_k^{k+1})^*\tau_{J^k\pi}} J^k\pi\ ,$$

where $H(\pi_k^{k+1})$ is the union of the fibres $(j^k \phi)_* T_t \mathbb{R}$, for $t \in \mathbb{R}$.

Now, if $\mathfrak{X}(\pi_k^{k+1})$ denotes the module of vector fields along the projection π_k^{k+1} , the submodule corresponding to sections of $(\pi_k^{k+1})^* \tau_{J^k \pi} \Big|_{(\pi_k^{k+1})^* V(\bar{\pi}^k)}$ is denoted by $\mathfrak{X}^v(\pi_k^{k+1})$, and the submodule corresponding to sections of $(\pi_k^{k+1})^* \tau_{J^k \pi} \Big|_{H(\pi_k^{k+1})}$ is denoted by $\mathfrak{X}^h(\pi_k^{k+1})$. The splitting for the bundles given above induces the following canonical splitting for the module $\mathfrak{X}(\pi_k^{k+1})$:

$$\mathfrak{X}(\pi_k^{k+1}) = \mathfrak{X}^v(\pi_k^{k+1}) \oplus \mathfrak{X}^h(\pi_k^{k+1})$$
.

An element of the submodule $\mathfrak{X}^h(\pi_k^{k+1})$ is called a total derivative.

Definition 2 Given a vector field $X \in \mathfrak{X}(\mathbb{R})$, a section $\phi \in \Gamma(\pi)$ and a point $t_o \in \mathbb{R}$, the kth holonomic lift of X by ϕ , $X^k \equiv j^k X \in \mathfrak{X}^h(\pi_k^{k+1})$, is defined as

$$X_{j_{t_o}^{k+1}\phi}^k = (j^k\phi)_* X_{t_o}.$$

Hence, every vector field $X \in \mathfrak{X}(\mathbb{R})$ defines a total derivative given by its holonomic lift.

Alternatively, we have the following characterization of X^k as a derivation: for every $f \in$ $C^{\infty}(J^k\pi)$ we have

$$(d_{X^k}f)(j_{t_o}^{k+1}\phi) = d_X(f \circ j^k\phi)(t_o),$$

where d_{X^k} is the derivation associated to X^k and d_X is the derivation corresponding to X.

In local coordinates, if $X \in \mathfrak{X}(\mathbb{R})$ is given by $X = X_o \frac{\partial}{\partial t}$, then, bearing in mind the local expression of the kth holonomic lift for tangent vectors (1), the kth holonomic lift of X is

$$X^{k} = X_{o} \left(\frac{\partial}{\partial t} + \sum_{i=0}^{k} q_{i+1}^{A} \frac{\partial}{\partial q_{i}^{A}} \right).$$

Finally, the total time derivative is the kth holonomic lift of the coordinate vector field $\partial/\partial t \in \mathfrak{X}(\mathbb{R})$, which is denoted by $d_T \in \mathfrak{X}(\pi_k^{k+1})$, and whose local expression is

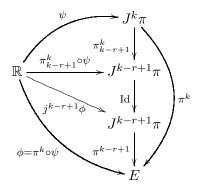
$$d_T = \frac{\partial}{\partial t} + \sum_{i=0}^k q_{i+1}^A \frac{\partial}{\partial q_i^A}.$$
 (2)

Remark: The usual notation for the total time derivative is d/dt, as seen in [43], while the notation d_T is usually reserved for the same operator in the autonomous case. Nevertheless, in this paper we use the same notation for both operators, and the one that is considered will be understood from the context.

2.3 Higher-order semisprays. Holonomic sections

Now we generalize the concept of semispray introduced in [22] to the time-dependent case.

Definition 3 A section $\psi \in \Gamma(\bar{\pi}^k)$ is holonomic of type $r, 1 \leqslant r \leqslant k$, if $j^{k-r+1}\phi = \pi_{k-r+1}^k \circ \psi$, where $\phi = \pi^k \circ \psi \in \Gamma(\pi)$; that is, the section ψ is the lifting of a section of π up to $J^{k-r+1}\pi$.



In particular, a section ψ is holonomic of type 1 if, with $\phi = \pi^k \circ \psi$, then $j^k \phi = \psi$; that is, ψ is the canonical k-jet lifting of a section $\phi \in \Gamma(\pi)$. Throughout this paper, sections that are holonomic of type 1 are simply called holonomic.

Definition 4 A vector field $X \in \mathfrak{X}(J^k\pi)$ is a semispray of type $r, 1 \leqslant r \leqslant k$, if every integral section ψ of X is holonomic of type r.

The local expression of a holonomic section of type $r, \psi \in \Gamma(J^k \pi)$, is

$$\psi(t) = (t, q_0^A, q_1^A, \dots, q_{k-r+1}^A, \psi_{k-r+2}^A, \dots, \psi_k^A).$$

Thus, the local expression of a semispray of type r is

$$X = f \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + \dots + q_{k-r+1}^A \frac{\partial}{\partial q_{k-r}^A} + X_{k-r+1}^A \frac{\partial}{\partial q_{k-r+1}^A} + \dots + X_k^A \frac{\partial}{\partial q_k^A}.$$

From the local expression, it is clear that every holonomic section of type r is also holonomic of type s, for $s \ge r$. The same remark is true for semisprays.

We observe that, from the definition, semisprays of type 1 in $J^k\pi$ are the analogue to the holonomic vector fields in first-order mechanics; that is, they are the vector fields whose integral sections (curves) are the canonical liftings to $J^k\pi$ of sections (curves) on the basis. Their local expressions are

$$X = f \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + \dots + q_k^A \frac{\partial}{\partial q_{k-1}^A} + X_k^A \frac{\partial}{\partial q_k^A}.$$

If $X \in \mathfrak{X}(J^k\pi)$ is a semispray of type r, a section $\phi \in \Gamma(\pi)$ is said to be a path or solution of X if $j^k \phi$ is an integral curve of X; that is, $\widetilde{j^k \phi} = X \circ j^k \phi$, where $\widetilde{j^k \phi}$ denotes the canonical lifting of $j^k \phi$ from $J^k \pi$ to $T(J^k \pi)$. Then, in coordinates, ϕ verifies the following system of differential equations of order k+1:

$$\frac{\mathrm{d}^{k-r+2}\phi^A}{\mathrm{d}t^{k-r+2}} = X_{k-r+1}^A \left(\phi, \frac{\mathrm{d}\phi}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^k\phi}{\mathrm{d}t^k} \right), \quad \dots, \frac{\mathrm{d}^{k+1}\phi^A}{\mathrm{d}t^{k+1}} = X_k^A \left(\phi, \frac{\mathrm{d}\phi}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^k\phi}{\mathrm{d}t^k} \right) \; .$$

3 Skinner-Rusk unified formalism

Unified phase space. Geometric and dynamical structures 3.1

Consider the configuration bundle $\pi \colon E \to \mathbb{R}$, where E is an (n+1)-dimensional smooth manifold. Let $\mathcal{L} \in \Omega^1(J^k\pi)$ be a kth order Lagrangian density, that is, a $\bar{\pi}^k$ -semibasic 1-form. Thus, we can write \mathcal{L} as $\mathcal{L} = L \cdot (\bar{\pi}^k)^* \eta = L dt$, where $L \in C^{\infty}(J^k \pi)$ is the kth-order Lagrangian function.

According to [5, 24, 39], we consider the following bundles:

$$W = J^{2k-1}\pi \times_{J^{k-1}\pi} T^*(J^{k-1}\pi)$$
 ; $W_r = J^{2k-1}\pi \times_{J^{k-1}\pi} J^{k-1}\pi^*$,

(the fiber product of the above bundles), where $J^{k-1}\pi^* = T^*(J^{k-1}\pi)/(\bar{\pi}^{k-1})^*T^*\mathbb{R}$. The bundles W and W_r are called the higher-order extended jet-momentum bundle and the higher-order restricted jet-momentum bundle, respectively.

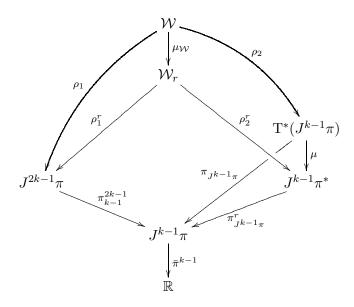
Comment: The reason for taking these bundles is in order to recover the Lagrangian and Hamiltonian formalisms from this unified framework, and as we see in Sections 4 and 5, those formalisms take place in the bundles $J^{2k-1}\pi$ and $J^{k-1}\pi^*$.

These bundles are endowed with the canonical projections

$$\rho_1 \colon \mathcal{W} \to J^{2k-1}\pi \quad ; \quad \rho_2 \colon \mathcal{W} \to \mathrm{T}^*(J^{k-1}\pi) \quad ; \quad \rho_{J^{k-1}\pi} \colon \mathcal{W} \to J^{k-1}\pi \quad ; \quad \rho_{\mathbb{R}} \colon \mathcal{W} \to \mathbb{R}$$

$$\rho_1^r \colon \mathcal{W}_r \to J^{2k-1}\pi \quad ; \quad \rho_2^r \colon \mathcal{W}_r \to J^{k-1}\pi^* \quad ; \quad \rho_{J^{k-1}\pi}^r \colon \mathcal{W}_r \to J^{k-1}\pi \quad ; \quad \rho_{\mathbb{R}}^r \colon \mathcal{W}_r \to \mathbb{R}$$

In addition, the natural quotient map $\mu \colon \mathrm{T}^*(J^{k-1}\pi) \to J^{k-1}\pi^*$ induces a natural projection (that is, a surjective submersion) $\mu_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}_r$. Thus, we have the following diagram



where $\pi_{J^{k-1}\pi} \colon \mathrm{T}^*(J^{k-1}\pi) \to J^{k-1}\pi$ is the canonical submersion and $\pi^r_{J^{k-1}\pi} \colon J^{k-1}\pi^* \to J^{k-1}\pi$ is the map satisfying $\pi_{J^{k-1}\pi} = \pi^r_{J^{k-1}\pi} \circ \mu$.

If $(U; t, q_0^A)$ is a local chart of coordinates in E, we denote by $((\pi^{2k-1})^*(U); t, q_0^A, \dots, q_{2k-1}^A)$ and $((\pi_{J^{k-1}\pi} \circ \pi^{k-1})^*(U); t, q_0^A, \dots, q_{k-1}^A, p, p_A^0, \dots, p_A^{k-1})$ the induced local charts in $J^{2k-1}\pi$ and $T^*(J^{k-1}\pi)$, respectively. Thus $(t, q_0^A, \dots, q_{k-1}^A, p_A^0, \dots, p_A^{k-1})$ are the natural coordinates in $J^{k-1}\pi^*$, and the coordinates in \mathcal{W} and \mathcal{W}_r are $(t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A, p, p_A^0, \dots, p_A^{k-1})$ and $(t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A, p_A^0, \dots, p_A^{k-1})$, respectively. Note that dim $\mathcal{W} = 3kn + 2$ and $\dim(\mathcal{W}_r) = 3kn + 1$.

The bundle W is endowed with some canonical geometric structures. The first one is:

Definition 5 Let $\Theta_{k-1} \in \Omega^1(\mathrm{T}^*(J^{k-1}\pi))$ be the tautological 1-form, and $\Omega_{k-1} = -\mathrm{d}\Theta_{k-1} \in \Omega^2(\mathrm{T}^*(J^{k-1}\pi))$ the canonical symplectic 2-form on $\mathrm{T}^*(J^{k-1}\pi)$. We define the higher-order unified canonical forms as

$$\Theta = \rho_2^* \Theta_{k-1} \in \Omega^1(\mathcal{W}) \quad ; \quad \Omega = \rho_2^* \Omega_{k-1} \in \Omega^2(\mathcal{W}). \tag{3}$$

Bearing in mind that the local expressions for the canonical forms on $T^*(J^{k-1}\pi)$ are

$$\Theta_{k-1} = p_A^i dq_i^A + p dt \quad ; \quad \Omega_{k-1} = dq_i^A \wedge dp_A^i - dp \wedge dt \,, \tag{4}$$

the above forms can be written locally as

$$\Theta = \rho_2^*(p_A^i dq_i^A + p dt) = p_A^i dq_i^A + p dt \quad ; \quad \Omega = \rho_2^*(dq_i^A \wedge dp_A^i - dp \wedge dt) = dq_i^A \wedge dp_A^i - dp \wedge dt \quad (5)$$

Notice that from the local expressions (5) we have

$$\ker \Omega = \left\langle \frac{\partial}{\partial q_k^A}, \dots, \frac{\partial}{\partial q_{2k-1}^A} \right\rangle = \mathfrak{X}^{V(\rho_2)}(\mathcal{W}) . \tag{6}$$

Thus, Ω is a presymplectic form in \mathcal{W} .

The second canonical structure in \mathcal{W} is the following:

Definition 6 The higher-order coupling 1-form in W is the $\rho_{\mathbb{R}}$ -semibasic 1-form $\hat{\mathcal{C}} \in \Omega^1(W)$ defined as follows: for every $w = (\bar{y}, \alpha_q) \in W$ (that is, $\alpha_q \in T_q^*(J^{k-1}\pi)$, where $q = \pi_{k-1}^{2k-1}(\bar{y})$ is the projection of \bar{y} to $J^{k-1}\pi$) and $u \in T_wW$, then

$$\langle \hat{\mathcal{C}}(w) \mid u \rangle = \langle \alpha_q \mid (T_w(j^{k-1}\phi \circ \rho_{\mathbb{R}}))(u) \rangle, \tag{7}$$

where $\phi \in \Gamma(\pi)$ is any representative of \bar{y} (that is, $j^{2k-1}\phi = \bar{y}$).

 $\hat{\mathcal{C}}$ being a $\rho_{\mathbb{R}}$ -semibasic form, there exists $\hat{C} \in C^{\infty}(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}\rho_{\mathbb{R}}^*\eta = \hat{C}dt$. An easy computation in coordinates gives the following local expression for the coupling 1-form:

$$\hat{\mathcal{C}} = (p + p_A^i q_{i+1}^A) dt. \tag{8}$$

We denote $\hat{\mathcal{L}} = (\pi_k^{2k-1} \circ \rho_1)^* \mathcal{L} \in \Omega^1(\mathcal{W})$. As the Lagrangian density is a $\bar{\pi}^k$ -semibasic form, we have that $\hat{\mathcal{L}}$ is a $\rho_{\mathbb{R}}$ -semibasic 1-form, and thus we can write $\hat{\mathcal{L}} = \hat{L}\rho_{\mathbb{R}}^* \eta = \hat{L}dt$, where $\hat{L} = (\pi_k^{2k-1} \circ \rho_1)^* L \in C^{\infty}(\mathcal{W})$ is the pull-back of the Lagrangian function associated with \mathcal{L} . Then, we define a *Hamiltonian submanifold*

$$\mathcal{W}_o = \left\{ w \in \mathcal{W} \colon \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w) \right\} \stackrel{j_o}{\hookrightarrow} \mathcal{W}.$$

 $\hat{\mathcal{C}}$ and $\hat{\mathcal{L}}$ being $\rho_{\mathbb{R}}$ -semibasic 1-forms, the submanifold \mathcal{W}_o is defined by the constraint $\hat{C} - \hat{L} = 0$. In natural coordinates, bearing in mind the local expression (8) of $\hat{\mathcal{C}}$, the constraint function is

$$\hat{C} - \hat{L} = p + p_A^i q_{i+1}^A - \hat{L} = 0.$$

We have the following natural projections in \mathcal{W}_o :

$$\rho_{\mathbb{R}}^{o} \colon \mathcal{W}_{o} \to \mathbb{R} \quad ; \quad \rho_{1}^{o} \colon \mathcal{W}_{o} \to J^{2k-1}\pi \quad ; \quad \rho_{2}^{o} \colon \mathcal{W}_{o} \to \mathrm{T}^{*}(J^{k-1}\pi)$$
$$\hat{\rho}_{2}^{o} = \mu \circ \rho_{2}^{o} \colon \mathcal{W}_{o} \to J^{k-1}\pi^{*} \quad ; \quad \rho_{J^{k-1}\pi}^{o} \colon \mathcal{W}_{o} \to J^{k-1}\pi .$$

Local coordinates in \mathcal{W}_o are $(t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A, p_A^0, \dots, p_A^{k-1})$, and the local expressions of the above maps are

$$\begin{split} &\rho_1^o(t,q_i^A,q_j^A,p_A^i) = (t,q_i^A,q_j^A) \quad ; \quad \rho_2^o(t,q_i^A,q_j^A,p_A^i) = (t,q_i^A,\hat{L} - p_A^iq_{i+1}^A,p_A^i) \\ &\hat{\rho}_2^o(t,q_i^A,q_j^A,p_A^i) = (t,q_i^A,p_A^i) \quad ; \quad j_o(t,q_i^A,q_j^A,p_A^i) = (t,q_i^A,q_j^A,\hat{L} - p_A^iq_{i+1}^A,p_A^i) \,. \end{split}$$

Proposition 1 The submanifold $W_o \hookrightarrow W$ is 1-codimensional, μ_W -transverse and diffeomorphic to \mathcal{W}_r .

(Proof) \mathcal{W}_o is obviously 1-codimensional, since it is defined by one constraint function.

To see that \mathcal{W}_o is diffeomorphic to \mathcal{W}_r , we show that the smooth map $\mu_{\mathcal{W}} \circ j_o \colon \mathcal{W}_o \to \mathcal{W}_r$ is one-to-one. First, for every $(\bar{y}, \alpha) \in \mathcal{W}_o$, we have $L(\pi_k^{2k-1}(\bar{y})) = \hat{L}(\bar{y}, \alpha) = \hat{C}(\bar{y}, \alpha)$, and

$$(\mu_{\mathcal{W}} \circ j_o)(\bar{y}, \alpha) = \mu_{\mathcal{W}}(\bar{y}, \alpha) = (\bar{y}, \mu(\alpha)) = (\bar{y}, [\alpha]).$$

First, $\mu_{\mathcal{W}} \circ j_o$ is injective; in fact, let $(\bar{y}_1, \alpha_1), (\bar{y}_2, \alpha_2) \in \mathcal{W}_o$, then we wish to prove that

$$(\mu_{\mathcal{W}} \circ j_o)(\bar{y}_1, \alpha_1) = (\mu_{\mathcal{W}} \circ j_o)(\bar{y}_2, \alpha_2) \Leftrightarrow (\bar{y}_1, \alpha_1) = (\bar{y}_2, \alpha_2) \Leftrightarrow \bar{y}_1 = \bar{y}_2 \text{ and } \alpha_1 = \alpha_2.$$

Now, using the previous expression for $(\mu_{\mathcal{W}} \circ j_{o})(\bar{y}, \alpha)$, we have

$$(\mu_{\mathcal{W}} \circ j_o)(\bar{y}_1, \alpha_1) = (\mu_{\mathcal{W}} \circ j_o)(\bar{y}_2, \alpha_2) \Leftrightarrow (\bar{y}_1, [\alpha_1]) = (\bar{y}_2, [\alpha_2]) \Leftrightarrow \bar{y}_1 = \bar{y}_2 \text{ and } [\alpha_1] = [\alpha_2].$$

Hence, by definition of W_o , we have $L(\pi_k^{2k-1}(\bar{y}_1)) = L(\pi_k^{2k-1}(\bar{y}_2)) = \hat{C}(\bar{y}_1, \alpha_1) = \hat{C}(\bar{y}_2, \alpha_2)$. Locally, from the third equality we obtain

$$p(\alpha_1) + p_A^i(\alpha_1)q_{i+1}^A(\bar{y}_1) = p(\alpha_2) + p_A^i(\alpha_2)q_{i+1}^A(\bar{y}_2),$$

but $[\alpha_1] = [\alpha_2] \implies p_A^i(\alpha_1) = p_A^i([\alpha_1]) = p_A^i([\alpha_2]) = p_A^i(\alpha_2)$. Then $p(\alpha_1) = p(\alpha_2)$, and $\alpha_1 = \alpha_2$. Furthermore, $\mu_{\mathcal{W}} \circ j_o$ is surjective. In fact, given $(\bar{y}, [\alpha]) \in \mathcal{W}_r$, we wish to find $(\bar{y}, \beta) \in j_o(\mathcal{W}_o)$ such that $[\beta] = [\alpha]$. It suffices to take $[\beta]$ such that, in local coordinates of \mathcal{W} ,

$$p_A^i(\beta) = p_A^i([\beta])$$
 , $p(\beta) = L(\pi_k^{2k-1}(\bar{y})) - p_A^i([\alpha])q_{i+1}^A(\bar{y})$.

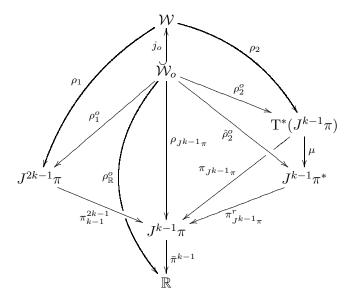
This β exists as a consequence of the definition of W_o . Now, since $\mu_W \circ j_o$ is a one-to-one submersion, then, by equality on the dimensions of W_o and W_r , it is a one-to-one local diffeomorphism, and thus a global diffeomorphism.

Finally, in order to prove that W_o is μ_W -transversal, it is necessary to check if $L(Y)(\xi) \equiv$ $Y(\xi) \neq 0$, for every $Y \in \ker \mu_{\mathcal{W}_*}$ and every constraint function ξ defining \mathcal{W}_o . Since \mathcal{W}_o is defined by the constraint function $\hat{C} - \hat{L} = 0$ and $\ker \mu_{\mathcal{W}_*} = \{\partial/\partial p\}$, we have

$$\frac{\partial}{\partial p}(\hat{C} - \hat{L}) = \frac{\partial}{\partial p}(p + p_A^i q_{i+1}^A - \hat{L}) = 1,$$

then W_o is $\mu_{\mathcal{W}}$ -transversal.

As a consequence of this last result, in the following we consider the diagram:



As a consequence of Proposition 1, the submanifold W_o induces a section $\hat{h} \in \Gamma(\mu_W)$, that is, a map $\hat{h} \colon W_r \to W$. This section is specified by giving the local Hamiltonian function

$$\hat{H} = -\hat{L} + p_A^i q_{i+1}^A \,, \tag{9}$$

that is, $\hat{h}(t, q_i^A, q_j^A, p_A^i) = (t, q_i^A, q_j^A, -\hat{H}, p_A^i)$. The section \hat{h} is called a Hamiltonian section of μ_W , or a Hamiltonian μ_W -section.

Next, we can define the forms

$$\Theta_o = j_o^* \Theta = (\rho_2^o)^* \Theta_{k-1} \in \Omega^1(\mathcal{W}_o) \quad ; \quad \Omega_o = j_o^* \Omega = (\rho_2^o)^* \Omega_{k-1} \in \Omega^2(\mathcal{W}_o) \quad ,$$

with local expressions

$$\Theta_o = p_A^i dq_i^A + (\hat{L} - p_A^i q_{i+1}^A) dt \quad ; \quad \Omega_o = dq_i^A \wedge dp_A^i + d(p_A^i q_{i+1}^A - \hat{L}) \wedge dt \quad , \tag{10}$$

and we have the presymplectic Hamiltonian systems (W_o, Ω_o) and (W_r, Ω_r) , with $\Omega_r = \hat{h}^*(\Omega)$.

Finally, it is necessary to introduce the following concepts:

Definition 7 A section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ is holonomic of type r in W_o , $1 \leqslant r \leqslant 2k-1$, if the section $\rho_1^o \circ \psi_o \in \Gamma(\bar{\pi}^{2k-1})$ is holonomic of type r in $J^{2k-1}\pi$.

Definition 8 A vector field $X_o \in \mathfrak{X}(W_o)$ is said to be a semispray of type r in W_o if every integral section ψ_o of X_o is holonomic of type r in W_o .

The local expression of a semispray of type r in W_o is

$$X_o = f \frac{\partial}{\partial t} + \sum_{i=0}^{2k-1-r} q_{i+1}^A \frac{\partial}{\partial q_i^A} + \sum_{i=2k-r}^{2k-1} X_i^A \frac{\partial}{\partial q_i^A} + \sum_{i=0}^{k-1} G_A^i \frac{\partial}{\partial p_A^i} ,$$

and, in particular, for a semispray of type 1 in W_o we have

$$X_o = f \frac{\partial}{\partial t} + \sum_{i=0}^{2k-2} q_{i+1}^A \frac{\partial}{\partial q_i^A} + X_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A} + \sum_{i=0}^{k-1} G_A^i \frac{\partial}{\partial p_A^i} .$$

3.2 Dynamical equations

The dynamical equations for non-autonomous dynamical systems in general can be geometrically written in several equivalent ways, using sections (curves) which are the dynamical trajectories, or vector fields whose integral curves are the dynamical trajectories. In this section we explore both of these ways, and prove their equivalence.

3.2.1 Dynamical equations for sections

The Lagrangian-Hamiltonian problem for sections associated with the system (W_o, Ω_o) consists in finding sections $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ (that is, curves $\psi_o : \mathbb{R} \to W_o$) characterized by the condition

$$\psi_o^* i(Y)\Omega_o = 0$$
, for every $Y \in \mathfrak{X}(\mathcal{W}_o)$. (11)

In natural coordinates, let $Y \in \mathfrak{X}(\mathcal{W}_o)$ be a generic vector field given by

$$Y = f \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} f_i^A \frac{\partial}{\partial q_i^A} + \sum_{j=k}^{2k-1} F_j^A \frac{\partial}{\partial q_j^A} + \sum_{i=0}^{k-1} G_A^i \frac{\partial}{\partial p_A^i},$$
 (12)

bearing in mind the coordinate expression (10) of Ω_o , the contraction $i(X)\Omega_o$ is

$$i(Y)\Omega_{o} = f\left(\frac{\partial \hat{L}}{\partial q_{r}^{A}} dq_{r}^{A} - q_{i+1}^{A} dp_{A}^{i} - p_{A}^{i} dq_{i+1}^{A}\right) + f_{0}^{A} \left(dp_{A}^{0} - \frac{\partial \hat{L}}{\partial q_{0}^{A}} dt\right)$$

$$+ f_{i}^{A} \left(dp_{A}^{i} - \frac{\partial \hat{L}}{\partial q_{i}^{A}} dt + p_{A}^{i-1} dt\right) + F_{k}^{A} \left(p_{A}^{k-1} - \frac{\partial \hat{L}}{\partial q_{k}^{A}}\right) dt + G_{A}^{i} \left(q_{i+1}^{A} dt - dq_{i}^{A}\right).$$

Thus, taking the pull-back by the section $\psi_o = (t, q_i^A(t), q_i^A(t), p_A^i(t))$, we obtain

$$\psi_o^* i(Y)\Omega_o = f\left(\frac{\partial \hat{L}}{\partial q_r^A} \dot{q}_r^A - q_{i+1}^A \dot{p}_A^i - p_A^i \dot{q}_{i+1}^A\right) dt + f_0^A \left(\dot{p}_A^0 - \frac{\partial \hat{L}}{\partial q_0^A}\right) dt + f_i^A \left(\dot{p}_A^i - \frac{\partial \hat{L}}{\partial q_i^A} + p_A^{i-1}\right) dt + F_k^A \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A}\right) dt + G_A^i \left(q_{i+1}^A - \dot{q}_i^A\right) dt.$$

Finally, requiring this last expression to vanish and bearing in mind that the equation must hold for every vector field $Y \in \mathfrak{X}(\mathcal{W}_o)$ (that is, it must hold for every function $f, f_i^A, F_j^A, G_A^i \in C^{\infty}(\mathcal{W}_o)$) we obtain the following system of equations

$$\frac{\partial \hat{L}}{\partial q_r^A} \dot{q}_r^A - q_{i+1}^A \dot{p}_A^i - p_A^i \dot{q}_{i+1}^A = 0$$
 (13)

$$\dot{p}_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \tag{14}$$

$$\dot{p}_A^i = \frac{\partial \hat{L}}{\partial q^A} - p_A^{i-1} \tag{15}$$

$$p_A^{k-1} = \frac{\partial \hat{L}}{\partial q_h^A} \tag{16}$$

$$\dot{q}_i^A = q_{i+1}^A \ . \tag{17}$$

It is easy to check that equation (13) is redundant, since it is a consequence of the others. Equations (14), (15) and (17) are differential equations whose solutions are the functions defining

the section ψ_o . In fact, equations (14), (15) give the higher-order Euler-Lagrange equations, as we see at the end of this Section and in Section 4. In addition, observe that equations (16) do not involve any derivative of ψ_o : they are pointwise algebraic conditions. These equations arise from the $\hat{\rho}_2^o$ -vertical part of the vector fields Y. Moreover, we have the following result:

Lemma 1 If $Y \in \mathfrak{X}^{V(\hat{\rho}_2^o)}(\mathcal{W}_o)$, then $i(Y)\Omega_o$ is $\rho_{\mathbb{R}}^o$ -semibasic.

(Proof) A direct calculation in coordinates leads to this result. Bearing in mind that a local basis for the $\hat{\rho}_{2}^{o}$ -vertical vector fields is given by (6) and the local expression (10) of Ω_{o} , we have

$$i\left(\frac{\partial}{\partial q_j^A}\right)\Omega_o = \begin{cases} \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A}\right) dt, & \text{for } j = k; \\ 0 = 0 \cdot dt, & \text{for } j = k+1, \dots, 2k-1. \end{cases}$$

Thus, in both cases we obtain a $\rho_{\mathbb{R}}^o$ -semibasic form.

As a consequence of this result, we can define the submanifold

$$\mathcal{W}_c = \left\{ w \in \mathcal{W}_o \colon (i(Y)\Omega_o)(w) = 0 \text{ for every } Y \in \mathfrak{X}^{V(\hat{\rho}_2^o)}(\mathcal{W}_o) \right\} \stackrel{j_1}{\hookrightarrow} \mathcal{W}_o,$$

where every section ψ_o solution of equation (11) must take values. It is called the first constraint submanifold of the Hamiltonian presymplectic system (W_o, Ω_o) .

Locally, W_c is defined in W_o by the constraints $p_A^{k-1} - \partial \hat{L}/\partial q_k^A = 0$, as we have seen in (16) and in the proof of the previous Lemma. In combination with equations (15), we have:

Proposition 2 W_c contains a submanifold $W_1 \hookrightarrow W_c$ which can be identified as the graph of a map $\mathcal{FL}: J^{2k-1}\pi \to J^{k-1}\pi^*$ defined locally by

$$\mathcal{F}\mathcal{L}^*t = t \quad , \quad \mathcal{F}\mathcal{L}^*q_r^A = q_r^A \quad , \quad \mathcal{F}\mathcal{L}^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{r+i}^A}\right) \, .$$

(Proof) As W_c is defined locally by the constraints $p_A^{k-1} - \partial \hat{L}/\partial q_k^A = 0$, it suffices to prove that these constraints give rise to the functions defining the map given above, and thus to the submanifold W_1 . We do this in coordinates.

Taking into account that $d_T(p_A^i) = \dot{p}_A^i$ along sections, the constraint function defining \mathcal{W}_c , in combination with equations (15) give rise to the following constraint functions

$$\begin{split} p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A} &= 0 \\ p_A^{k-2} - \left(\frac{\partial \hat{L}}{\partial q_{k-1}^A} - d_T(p_A^{k-1})\right) &= p_A^{k-2} - \sum_{i=0}^1 (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{k-1+i}^A}\right) = 0 \\ & \vdots \\ p_A^1 - \left(\frac{\partial \hat{L}}{\partial q_2^A} - d_T(p_A^2)\right) &= p_A^1 - \sum_{i=0}^{k-2} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{2+i}^A}\right) = 0 \\ p_A^0 - \left(\frac{\partial \hat{L}}{\partial q_1^A} - d_T(p_A^1)\right) &= p_A^0 - \sum_{i=0}^{k-1} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{1+i}^A}\right) = 0 \ , \end{split}$$

Therefore, these constraints define a submanifold $W_1 \hookrightarrow W_c$ and we may consider that this W_1 is the graph of a map $\mathcal{FL}: J^{2k-1}\pi \to J^{k-1}\pi^*$ given by

$$\mathcal{F}\mathcal{L}^*t = t \quad , \quad \mathcal{F}\mathcal{L}^*q_r^A = q_r^A \quad , \quad \mathcal{F}\mathcal{L}^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{r+i}^A}\right) \; .$$

Bearing in mind that the submanifold $W_o \hookrightarrow W$ is defined locally by the constraint function $p + p_A^i q_{i+1}^A - \hat{L} = 0$, and that W_1 is a submanifold of W_c , and thus a sumbanifold of W_o , from the above Proposition we can state the following result, which is a straightforward consequence of the previous result:

Corollary 1 W_1 is the graph of a map $\widetilde{\mathcal{FL}}: J^{2k-1}\pi \to T^*(J^{k-1}\pi)$ defined locally by

$$\widetilde{\mathcal{FL}}^* t = t$$
 , $\widetilde{\mathcal{FL}}^* q_r^A = q_r^A$,

$$\widetilde{\mathcal{FL}}^*p = \hat{L} - \sum_{r=1}^k q_r^A \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{r+i}^A} \right) \quad , \quad \widetilde{\mathcal{FL}}^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{r+i}^A} \right) \; .$$

Remark: The submanifold W_1 can be obtained from W_c using a constraint algorithm. Hence, W_1 acts as the initial phase space of the system.

The maps $\widetilde{\mathcal{FL}}$ and \mathcal{FL} are called the extended Legendre-Ostrogradsky map and the restricted Legendre-Ostrogradsky map associated to the Lagrangian density \mathcal{L} , respectively. A justification of this terminology is given in Section 5. Now we can give the following definition:

Definition 9 A Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is regular if the restricted Legendre-Ostrogradsky map \mathcal{FL} is a local diffeomorphism. If the map \mathcal{FL} is a global diffeomorphism, then \mathcal{L} is said to be hyperregular.

Computing in natural coordinates the local expression of the tangent map to \mathcal{FL} , the regularity condition for \mathcal{L} is equivalent to

$$\det\left(\frac{\partial^2 L}{\partial q_k^B \partial q_k^A}\right)(\bar{y}) \neq 0, \quad \text{for every } \bar{y} \in J^k \pi.$$

Equivalently, if we denote $\hat{p}_A^{r-1} = \mathcal{FL}^* p_A^{r-1}$, then the Lagrangian density \mathcal{L} is regular if, and only if, the set (t, q_i^A, \hat{p}_A^i) , $0 \leq i \leq k-1$, is a set of local coordinates in $J^{2k-1}\pi$. The local functions \hat{p}_A^i are called the *Jacobi-Ostrogradsky momentum coordinates*, and they satisfy that

$$\hat{p}_A^{r-1} = \frac{\partial L}{\partial q_r^A} - d_T(\hat{p}_A^r) \quad , \tag{18}$$

which are exactly the relations given by (15), taking into account that $d_T = d/dt$ along sections.

Notice that equations (14), (15), and (17) do not allow us to determinate the functions q_j^A , $k \leq j \leq 2k-1$, of the section ψ_o . Thus, in the general case, we need an additional condition when stating the problem, which is the holonomy condition for the section ψ_o . Therefore, the Lagrangian-Hamiltonian problem must be reformulated as follows:

The Lagrangian-Hamiltonian problem consists in finding holonomic sections $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ characterized by the equation (11).

Remarks:

• In fact, the functions q_j^A , $k \leq j \leq 2k-1$, are determined by the equations (14) and (15), bearing in mind that the section ψ_o must lie in the submanifold $\mathcal{W}_1 = \operatorname{graph}(\widetilde{\mathcal{FL}})$. It is easy to see that, by replacing the local expression of the extended Legendre-Ostrogradsky map in the equations (14) and (15), these equations lead to the Euler-Lagrange equations and to the remaining (k-1)n equations that give the full holonomy condition:

$$(\dot{q}_{j}^{B} - q_{j+1}^{B}) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} \bigg|_{\psi_{o}} - \sum_{i=k}^{j-1} (\dot{q}_{i}^{B} - q_{i+1}^{B}) (\cdots) = 0 \qquad (k \leqslant j \leqslant 2k - 2)$$

$$\frac{\partial \hat{L}}{\partial q_{0}^{A}} \bigg|_{\psi_{o}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \hat{L}}{\partial q_{1}^{A}} \bigg|_{\psi_{o}} + \frac{\mathrm{d}^{2}}{\mathrm{d}t^{2}} \frac{\partial \hat{L}}{\partial q_{2}^{A}} \bigg|_{\psi_{o}} + \cdots + (-1)^{k} \frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}} \frac{\partial \hat{L}}{\partial q_{k}^{A}} \bigg|_{\psi_{o}} = 0 ,$$

where the terms in brackets (\cdots) contain terms involving partial derivatives of the Lagrangian function and iterated total time derivatives, and the first sum (for j = k) is empty. However, observe that these equations may or may not be compatible, and a sufficient condition to ensure compatibility is the regularity of the Lagrangian density. Thus, for singular Lagrangian densities, the holonomy condition for the section ψ_0 is required.

- The requirement of the section ψ_o to be holonomic is a relevant difference from the first-order case, where the holonomy condition is deduced straightforwardly from the dynamical equations when written in local coordinates. Nevertheless, in the higher-order case, the equations allow us to recover only the holonomy of type k, as seen in (17), and the highest-order holonomy condition can only be recovered from the equations if the Lagrangian density is regular. Hence, this condition is required "ad hoc".
- The regularity of the Lagrangian density has no relevant role at first sight. However, as we have seen in the first remark, equations (14) and (15) give the higher-order Euler-Lagrange equations, which have a unique solution if the Lagrangian density is regular. For singular Lagrangians, these equations may give rise to new constraints, and a constraint algorithm should be used for finding a submanifold where the equations can be solved.

3.2.2 Dynamical equations for vector fields

The Lagrangian-Hamiltonian problem for vector fields associated with the system (W_o, Ω_o) consists in finding vector fields $X_o \in \mathfrak{X}(W_o)$ such that

$$i(X_o)\Omega_o = 0 \quad ; \quad i(X_o)(\rho_{\mathbb{R}}^o)^* \eta = 1.$$
 (19)

According to [13] and [18], we have:

Proposition 3 A solution $X_o \in \mathfrak{X}(W_o)$ to equation (19) exists only on the points of the submanifold S_c defined by

$$S_c = \left\{ w \in \mathcal{W}_o \colon (i(Z) d\hat{H})(w) = 0 \text{ for every } Z \in \ker \Omega \right\} \stackrel{j_1}{\hookrightarrow} \mathcal{W}_o.$$
 (20)

We have the following result:

Proposition 4 The submanifold $S_c \hookrightarrow W_o$ contains a submanifold $S_1 \hookrightarrow S_c$ which is the graph of the extended Legendre-Ostrogradsky map; that is, $S_1 = \operatorname{graph} \widetilde{\mathcal{FL}}$; and hence $S_1 = W_1$.

(Proof) As S_c is defined by (20), it suffices to prove that the constraints defining S_c give rise to the constraint functions defining the graph of the extended Legendre-Ostrogradsky map associated to \mathcal{L} . We do this calculation in coordinates. Taking the local expression (9) of the local Hamiltonian function $\hat{H} \in C^{\infty}(\mathcal{W}_o)$, we have

$$d\hat{H} = \sum_{i=0}^{k-1} (q_{i+1}^{A} dp_{A}^{i} + p_{A}^{i} dq_{i+1}^{A}) - \sum_{i=0}^{k} \frac{\partial \hat{L}}{\partial q_{i}^{A}} dq_{i}^{A},$$

and using the local basis of ker Ω given in (6), we obtain that the equations defining the submanifold S_c are

$$i(Z)\mathrm{d}\hat{H}=0\Longleftrightarrow p_A^{k-1}-\frac{\partial\hat{L}}{\partial q_k^A}=0\,,\ \mathrm{for\ every}\ 1\leqslant A\leqslant n\ .$$

Note that these expressions relate the momentum coordinates p_A^{k-1} with the Jacobi-Ostrogradsky functions $\hat{p}_A^{k-1} = \partial \hat{L}/\partial q_k^A$, and so we obtain the last group of equations of the restricted Legendre-Ostrogradsky map. Now, using the same argument as in the proof of Proposition 2 and the relations (18) for the momenta, we can consider that \mathcal{S}_c contains a submanifold \mathcal{S}_1 which is the graph of a map

$$\begin{array}{cccc} F \colon J^{2k-1}\pi & \longrightarrow & J^{k-1}\pi^* \\ (t,q_i^A,q_i^A) & \longmapsto & (t,q_i^A,p_A^i) \end{array}$$

which we identify with the restricted Legendre-Ostrogradsky map by making the identification $p_A^{r-1} = \hat{p}_A^{r-1}$.

Finally, taking into account that S_1 is also a submanifold of W_o , which is defined by the constraint $p + p_A^i q_{i+1}^A - \hat{L} = 0$, we have as a direct consequence that S_1 is the graph of the extended Legendre-Ostrogradsky map $\widetilde{\mathcal{FL}}$, and hence $S_1 = W_1$.

We denote by $\mathfrak{X}_{W_1}(W)$ the set of vector fields in W_o at support on W_1 . Hence, we look for vector fields $X_o \in \mathfrak{X}_{W_1}(W_o)$ which are solutions to equations (19) at support on W_1 ; that is

$$i(X_o)\Omega_o|_{\mathcal{W}_1} = 0 \quad ; \quad i(X_o)(\rho_{\mathbb{R}}^o)^*\eta|_{\mathcal{W}_1} = 1 .$$
 (21)

In natural coordinates, let $X_o \in \mathfrak{X}(W_o)$ be a generic vector field given locally by (12). Thus, from (19) we obtain the following system of (2k+1)n+2 equations

$$-f_0^A \frac{\partial \hat{L}}{\partial q_0^A} + f_i^A \left(p_A^{i-1} - \frac{\partial \hat{L}}{\partial q_i^A} \right) + F_k^A \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A} \right) + G_A^i q_{i+1}^A = 0 , \qquad (22)$$

$$f_i^A = f q_{i+1}^A \,\,, \tag{23}$$

$$G_A^0 = f \frac{\partial \hat{L}}{\partial q_0^A} \quad , \quad G_A^i = f \left(\frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} \right) = f d_T(p_A^i) \quad ,$$
 (24)

$$f = 1 (25)$$

$$f\left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A}\right) = 0 , \qquad (26)$$

where $0 \le i \le k-1$ in (23), and $1 \le i \le k-1$ in (24). By a simple calculation one can see that equation (22) is redundant, since it is a combination of the others. Therefore

$$X_o = \frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_i^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^O} + d_T(p_A^i) \frac{\partial}{\partial p_A^i}.$$
 (27)

Remark: In a more general situation, the second equation in (19) is written $i(X_o)\rho_{\mathbb{R}}^*\eta \neq 0$, that is, a $\rho_{\mathbb{R}}^o$ -transversal condition for the vector field X_o . In local coordinates, this replaces equation (25) by $f \neq 0$, thus giving the vector field

$$X_o = f \left(\frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i} \right) ,$$

where $f \in C^{\infty}(W_o)$ is any non-vanishing function. This gives a whole family of vector field solutions to the dynamical equations, and taking a particular constant value for f just fixes a specific vector field in this family. From a physical viewpoint, taking a particular value for f is just fixing the gauge.

Observe that equations (26) are just a compatibility condition for the vector field X_o , which, together with the relations (18) for the momenta, state that vector field X_o solutions to equations (19) exist only at support on the submanifold defined by the graph of the extended Legendre-Ostrogradsky map. Thus, we recover, in coordinates, the result stated in Propositions 3 and 4. Furthermore, equations (23) show that X_o is a semispray of type k in W_o .

The component functions F_j^A , $k \leq j \leq 2k-1$, are undetermined. Nevertheless, recall that X_o is a vector field that must be tangent to the submanifold \mathcal{W}_1 . Thus, it is necessary to impose that $L(X_o)\xi|_{\mathcal{W}_1} = 0$ for every constraint function ξ defining \mathcal{W}_1 . Locally, this is equivalent to imposing $X_o(\xi)|_{\mathcal{W}_1} = 0$. Hence, taking into account Prop. 4, these conditions lead to

$$\left(\frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i} \right) \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A} \right) = 0$$

$$\left(\frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i} \right) \left(p_A^{k-2} - \sum_{i=0}^1 (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{k-1+i}^A} \right) \right) = 0$$

$$\vdots$$

$$\left(\frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i} \right) \left(p_A^1 - \sum_{i=0}^{k-2} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{2+i}^A} \right) \right) = 0$$

$$\left(\frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i} \right) \left(p_A^0 - \sum_{i=0}^{k-1} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{1+i}^A} \right) \right) = 0$$

(observe that we do not need to check $L(X_o)(p-\widetilde{\mathcal{FL}}^*p)=0$, since this is the constraint defining the submanifold $\mathcal{W}_o \hookrightarrow \mathcal{W}$, and X_o is a vector field already defined in \mathcal{W}_o) and, from here, we obtain the following kn equations

$$(F_{k}^{B} - q_{k+1}^{B}) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} = 0$$

$$(F_{k+1}^{B} - q_{k+2}^{B}) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} - (F_{k}^{B} - q_{k+1}^{B}) d_{T} \left(\frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} \right) = 0$$

$$\vdots$$

$$(F_{2k-2}^{B} - q_{2k-1}^{B}) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} - \sum_{i=0}^{k-3} (F_{k+i}^{B} - q_{k+i+1}^{B}) (\cdots) = 0$$

$$(-1)^{k} (F_{2k-1}^{B} - d_{T} (q_{2k-1}^{B})) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} + \sum_{i=0}^{k} (-1)^{i} d_{T}^{i} \left(\frac{\partial \hat{L}}{\partial q_{i}^{A}} \right) - \sum_{i=0}^{k-2} (F_{k+i}^{B} - q_{k+i+1}^{B}) (\cdots) = 0$$

$$(28)$$

where the terms in brackets (\cdots) contain relations involving partial derivatives of the Lagrangian function \hat{L} and applications of the total derivative d_T , which are not written for simplicity. These equations may or may not be compatible, and a sufficient condition for compatibility is the regularity of the Lagrangian density \mathcal{L} . In particular, we have:

Proposition 5 If $\mathcal{L} \in \Omega^1(J^k\pi)$ is a regular Lagrangian density, then there exists a unique vector field $X_o \in \mathfrak{X}_{W_1}(W_o)$ which is a solution to equation (21); it is tangent to W_1 , and is a semispray of type 1 in W_o .

(Proof) As the Lagrangian density \mathcal{L} is regular, the Hessian matrix $\left(\frac{\partial^2 \hat{L}}{\partial q_k^B \partial q_k^A}\right)$ is regular at every point, and this enables us to solve the above k systems of n equations (28) determining all the functions F_i^A uniquely, as follows

$$F_{i}^{A} = q_{i+1}^{A} \quad , \quad (k \leqslant i \leqslant 2k - 2)$$

$$(-1)^{k} \left(F_{2k-1}^{B} - d_{T} \left(q_{2k-1}^{B} \right) \right) \frac{\partial^{2} \hat{L}}{\partial q_{k}^{B} \partial q_{k}^{A}} + \sum_{i=0}^{k} (-1)^{i} d_{T}^{i} \left(\frac{\partial \hat{L}}{\partial q_{i}^{A}} \right) = 0 .$$

$$(29)$$

In this way, the tangency condition holds for X_o at every point on \mathcal{W}_1 . Furthermore, the equalities (29) show that X_o is a semispray of type 1 in \mathcal{W}_o with local expression

$$X_o = \frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i}.$$
 (30)

However, if \mathcal{L} is not regular, the equations (28) may or may not be compatible, and the compatibility condition may give rise to new constraints. In the most favourable cases, there is a submanifold $W_f \hookrightarrow W_1$ (it could be $W_f = W_1$) such that there exist vector fields $X_o \in \mathfrak{X}_{W_1}(W_o)$, tangent to W_f , which are solutions to the equations

$$i(X_o)\Omega_o|_{\mathcal{W}_f} = 0 \quad , \quad i(X_o)(\rho_{\mathbb{R}}^o)^*\eta|_{\mathcal{W}_f} = 1 .$$
 (31)

Finally, the relation among the results obtained in the two last sections is as follows:

Theorem 1 The following assertions on a holonomic section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ are equivalent:

1. ψ_o is a solution to equation (11), that is,

$$\psi_o^* i(Y)\Omega_o = 0$$
, for every $Y \in \mathfrak{X}(\mathcal{W}_o)$.

2. If ψ_o is given locally by $\psi_o(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t)), 0 \le i \le k-1, k \le j \le 2k-1$, then the components of ψ_o satisfy equations (14) and (15), that is, the following system of kn differential equations

$$\dot{p}_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \quad ; \quad \dot{p}_A^i = \frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} \,. \tag{32}$$

3. ψ_o is a solution to the equation

$$i(\psi_o')(\Omega_o \circ \psi_o) = 0, \qquad (33)$$

where $\psi'_o : \mathbb{R} \to TW_o$ is the canonical lifting of ψ_o to the tangent bundle.

4. ψ_o is an integral curve of a vector field contained in a class of $\rho_{\mathbb{R}}^o$ -transverse semisprays of type 1, $\{X_o\} \subset \mathfrak{X}(\mathcal{W}_o)$, satisfying the first equation in (19), that is,

$$i(X_o)\Omega_o = 0$$
.

(Proof)

 $(1 \Leftrightarrow 2)$ As we have seen in Section 3.2.1, equation (11) gives, in natural coordinates, the equations (13), (14), (15), (16) and (17). As stated there, equation (13) is redundant, since it is a combination of the others, and from equations (16) we deduce that the section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ lies in the submanifold \mathcal{W}_1 . Hence, equation (11) is locally equivalent to equations (14), (15), and (17). However, as we assume that ψ_o is holonomic, equations (17) hold identically, and thus equation (11) is locally equivalent to equations (14) and (15), that is, to equations (32).

 $(2 \Leftrightarrow 3)$ If $\psi_o(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t))$ is the local expression of ψ_o in natural coordinates, then $\psi_o'(t) = (1, \dot{q}_i^A(t), \dot{q}_i^A(t), \dot{p}_i^A(t))$, and the inner product $i(\psi_o')(\Omega_o \circ \psi_o)$ gives, in coordinates,

$$i(\psi_o')(\Omega_o \circ \psi)_o = \left(p_A^i \dot{q}_{i+1}^A - \dot{q}_r^A \frac{\partial \hat{L}}{\partial q_r^A} + \dot{p}_A^i q_{i+1}^A\right) dt + \left(\frac{\partial \hat{L}}{\partial q_0^A} - \dot{p}_A^0\right) dq_0^A$$

$$+ \left(\frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} - \dot{p}_A^i\right) dq_i^A + \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A}\right) dq_k^A + \left(\dot{q}_i^A - q_{i+1}^A\right) dp_A^i.$$

Now, requiring this last expression to vanish, we obtain the system of (2k+1)n+1 equations

$$p_{A}^{i}\dot{q}_{i+1}^{A} - \dot{q}_{r}^{A}\frac{\partial \hat{L}}{\partial q_{r}^{A}} + \dot{p}_{A}^{i}q_{i+1}^{A} = 0 \quad ; \quad \dot{p}_{A}^{0} = \frac{\partial \hat{L}}{\partial q_{0}^{A}} \quad ; \quad \dot{p}_{A}^{i} = \frac{\partial \hat{L}}{\partial q_{i}^{A}} - p_{A}^{i-1}$$

$$p_{A}^{k-1} = \frac{\partial \hat{L}}{\partial q_{k}^{A}} \quad ; \quad \dot{q}_{i}^{A} = q_{i+1}^{A}$$

Observe that this system of equations is the same given by (13), (14), (15), (16) and (17). The same remarks given in the proof of $(1 \Leftrightarrow 2)$ apply in this case. In particular, the fifth group of kn equations $\dot{q}_i^A = q_{i+1}^A$ is identically satisfied by the section ψ_o , since we assume it to be holonomic. Thus, bearing in mind the above item, we have proved that equation (33) is locally equivalent to the kn differential equations (32).

 $(2 \Leftrightarrow 4)$ As we have seen in this Section, if a generic vector field $X_o \in \mathfrak{X}(W_o)$ is given locally by (12), then the first equation in (19) is locally equivalent to equations (22), (23), (24) and (26). As already stated, equation (22) is redundant, since it is a combination of the others; and the n equations (26) state, in coordinates, the result given in Proposition 3. In addition, since the vector fields X_o in the class are semisprays of type 1, the kn equations (23) are identically satisfied. Thus, the first equation in (19) is locally equivalent to the kn equations (24). Finally, the $\rho_{\mathbb{R}}^o$ -transverse condition for the class $\{X_o\}$ is locally equivalent to $f \neq 0$.

Now, let $\sigma \in \Gamma(\rho_{\mathbb{R}}^o)$ be an integral curve of X_o , that is, $\sigma' = X_o \circ \sigma$. If σ is given locally by $\sigma(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t))$, then $\sigma'(t) = (1, \dot{q}_i^A(t), \dot{q}_j^A(t), \dot{p}_A^i(t))$, and, taking f = 1 as a representative of the class $\{X_o\}$, the condition of σ to be an integral curve is locally equivalent to the equations

$$\dot{q}_i^A = f_i^A \circ \sigma \quad ; \quad \dot{q}_i^A = F_i^A \circ \sigma \quad ; \quad \dot{p}_A^i = G_A^i \circ \sigma .$$

Replacing these equations in (23) and (24), we obtain the following 2kn differential equations

$$\dot{q}_i^A = q_{i+1}^A \quad ; \quad \dot{p}_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \quad ; \quad \dot{p}_A^i = \frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} \, .$$

Observe that, as every vector field in the class is a semispray of type 1, the first kn equations are identically satisfied. Thus, the condition of σ to be an integral curve of a $\rho_{\mathbb{R}}^o$ -transverse semispray of type 1, $X_o \in \mathfrak{X}(\mathcal{W}_o)$, satisfying the first equation in (19) is locally equivalent to equations (32).

4 Lagrangian formalism

4.1 General setting

Now we recover the Lagrangian dynamics from the unified formalism. We do not distinguish between the regular and singular cases, since the results remain the same in either case, but a few comments on the singular case will be given. First, we have:

Proposition 6 The map $\rho_1^1 = \rho_1^o \circ j_1 \colon \mathcal{W}_1 \to J^{2k-1}\pi$ is a diffeomorphism.

(Proof) As $W_1 = \operatorname{graph} \mathcal{FL}$, we have that $J^{2k-1}\pi \simeq W_1$. Furthermore, ρ_1^1 is a surjective submersion and, by the equality between dimensions, it is also an injective immersion and hence it is a diffeomorphism.

Now, we must define the Poincaré-Cartan forms in order to establish the dynamical equations for the Lagrangian formalism. First, we have the following result:

Lemma 2 Let $\Theta_{k-1} \in \Omega^1(\mathrm{T}^*(J^{k-1}\pi))$, $\Omega_{k-1} = -\mathrm{d}\Theta_{k-1} \in \Omega^2(\mathrm{T}^*(J^{k-1}\pi))$ be the canonical forms in $\mathrm{T}^*(J^{k-1}\pi)$. We define the Poincaré-Cartan forms as $\Theta_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Theta_{k-1} \in \Omega^1(J^{2k-1}\pi)$, $\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Omega_{k-1} \in \Omega^2(J^{2k-1}\pi)$. Then $\Theta_o = (\rho_1^o)^*\Theta_{\mathcal{L}}$ and $\Omega_o = (\rho_1^o)^*\Omega_{\mathcal{L}}$.

(Proof) We have for $\Theta_{\mathcal{L}}$:

$$(\rho_1^o)^*\Theta_{\mathcal{L}} = (\rho_1^o)^* (\widetilde{\mathcal{FL}}^*\Theta_{k-1}) = (\widetilde{\mathcal{FL}} \circ \rho_1^o)^*\Theta_{k-1} = (\rho_2^o)^*\Theta_{k-1} = \Theta_o,$$

and for $\Omega_{\mathcal{L}}$:

$$(\rho_1^o)^*\Omega_{\mathcal{L}} = (\rho_1^o)^*(-d\Theta_{\mathcal{L}}) = -d(\rho_1^o)^*\Theta_{\mathcal{L}} = -d\Theta_o = \Omega_o.$$

Alternatively, according to [42] and [43] (see also [1], [26]), we can define the *Poincaré-Cartan* 1-form using the canonical structures of the higher-order jet bundles; in particular,

$$\Theta_{\mathcal{L}} = S_n^{(k)}(\mathrm{d}L) + \mathcal{L} \in \Omega^1(J^{2k-1}\pi),$$

where $S_{\eta}^{(k)}$ is the generalization to higher-order jet bundles of the operator used in the classical Hamilton-Cartan formalism for problems in the calculus of variations which involve time explicitly (see [42] and [43] for details).

Using natural coordinates, the local expression of the Poincaré-Cartan 1-form is

$$\Theta_{\mathcal{L}} = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial L}{\partial q_{r+i}^A} \right) \left(dq_{r-1}^A - q_r^A dt \right) + L dt.$$
 (34)

Remark: $\Theta_{\mathcal{L}}$ is a π_{k-1}^{2k-1} -semibasic 1-form.

From the Poincaré-Cartan 1-form, the concept of regularity for a higher-order Lagrangian density is a straightforward generalization of the well-known definition for first-order non-autonomous dynamical systems. In fact, first we define the *Poincaré-Cartan 2-form* as $\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^2(J^{2k-1}\pi)$. Then

Definition 10 A Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is regular if the Poincaré-Cartan 2-form $\Omega_{\mathcal{L}}$ has maximal rank. Elsewhere \mathcal{L} is singular.

In natural coordinates, the local expression of the 2-form $\Omega_{\mathcal{L}}$ is

$$\Omega_{\mathcal{L}} = \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^{i+1} \left(d_T^i \left(\frac{\partial^2 L}{\partial t \partial q_{r+i}^A} dt + \frac{\partial^2 L}{\partial q_j^B \partial q_{r+i}^A} dq_j^B \right) \wedge (dq_{r-1}^A - q_r^A dt) - d_T^i \left(\frac{\partial L}{\partial q_{r+i}^A} \right) dq_r^A \wedge dt \right) - \frac{\partial L}{\partial q_j^B} dq_j^B \wedge dt.$$
(35)

From this expression in local coordinates, we can see that the regularity condition for \mathcal{L} is equivalent to

$$\det\left(\frac{\partial^2 L}{\partial q_k^B \partial q_k^A}\right)(\bar{y}) \neq 0,$$

for every $\bar{y} \in J^{2k-1}\pi$. Thus, this notion of regularity is equivalent to the one given before. Geometrically, \mathcal{L} is regular if, and only if, $(\Omega_{\mathcal{L}}, (\bar{\pi}^{2k-1})^*\eta)$ is a cosymplectic structure on $J^{2k-1}\pi$, that is, $\Omega_{\mathcal{L}}$ and $(\bar{\pi}^{2k-1})^*\eta$ are both closed and $\Omega_{\mathcal{L}}^{kn} \wedge (\bar{\pi}^{2k-1})^*\eta$ is a volume form.

4.2 Dynamical equations for sections

Using the previous results, we can recover the Lagrangian sections in $J^{2k-1}\pi$ from the sections in the unified formalism.

Proposition 7 Let $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ be a holonomic section solution to equation (11). Then the section $\psi_{\mathcal{L}} = \rho_1^o \circ \psi_o \in \Gamma(\bar{\pi}^{2k-1})$ is holonomic, and is a solution to the equation

$$\psi_{\mathcal{L}}^* i(Y)\Omega_{\mathcal{L}} = 0, \quad \text{for every } Y \in \mathfrak{X}(J^{2k-1}\pi)$$
 (36)

(Proof) Since, by definition, $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ is holonomic if $\rho_1^o \circ \psi_o \in \Gamma(\bar{\pi}^{2k-1})$ is holonomic, it is obvious that $\psi_{\mathcal{L}} = \rho_1^o \circ \psi_o$ is a holonomic section.

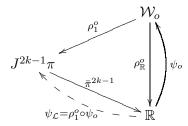
Now, recall that, since ρ_1^o is a submersion, for every $Y \in \mathfrak{X}(J^{2k-1}\pi)$ there exist some $Z \in \mathfrak{X}(W_o)$ such that $\rho_{1*}^o Z = Y$, that is, Y and Z are ρ_1^o -related. Note that this vector field is not unique, since $Z + Z_o$, with $Z_o \in \ker \rho_{1*}^o$ is also ρ_1^o -related with Y. Thus, using this particular choice of ρ_1^o -related vector fields, we have

$$\psi_{\mathcal{L}}^* i(Y) \Omega_{\mathcal{L}} = (\rho_1^o \circ \psi_o)^* i(Y) \Omega_{\mathcal{L}} = \psi_o^* ((\rho_1^o)^* i(Y) \Omega_{\mathcal{L}}) = \psi_o^* (i(Z)(\rho_1^o)^* \Omega_{\mathcal{L}}) = \psi_o^* i(Z) \Omega_o.$$

Since the equality $\psi_o^* i(Z)\Omega_o = 0$ holds for every $Z \in \mathfrak{X}(\mathcal{W}_o)$, in particular it holds for every $Z \in \mathfrak{X}(\mathcal{W}_o)$ which is ρ_1^o -related with $Y \in \mathfrak{X}(J^{2k-1}\pi)$. Hence, we obtain

$$\psi_{\mathcal{L}}^* i(Y) \Omega_{\mathcal{L}} = \psi_o^* i(Z) \Omega_o = 0.$$

The diagram for this situation is the following:



Remark: Observe that, from this result, we have no equivalence between section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ solutions to equation (11) and section $\psi_{\mathcal{L}} \in \Gamma(\bar{\pi}^{2k-1})$ solutions to equation (36), but only that every holonomic section ψ_o solution to the dynamical equations in the unified formalism can be projected to a holonomic section $\psi_{\mathcal{L}}$ solution to the Lagrangian equations. Nevertheless, recall that section ψ_o solutions to equation (11) take their values in the submanifold \mathcal{W}_1 , which is diffeomorphic to $J^{2k-1}\pi$, and thus it is possible to establish an equivalence using the diffeomorphism ρ_1^1 .

Assume $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ is given locally by $\psi_o(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t)), \ 0 \leqslant i \leqslant k-1, k \leqslant j \leqslant 2k-1$. Since ψ_o is assumed to be a holonomic section solution to equation (11), it must satisfy equations (14), (15) and (17). The last group of equations is automatically satisfied because of the holonomy condition. Now, bearing in mind that the section ψ_o takes values in the submanifold W_1 , and the characterization of W_1 given in Proposition 2, equations (14) and (15) can be ρ_1^o -projected to $J^{2k-1}\pi$, thus giving the following equations for the section $\psi_{\mathcal{L}} = \rho_1^o \circ \psi_o$:

$$\frac{\partial L}{\partial q_0^A}\Big|_{\psi_{\mathcal{L}}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial q_1^A}\Big|_{\psi_{\mathcal{L}}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial L}{\partial q_2^A}\Big|_{\psi_{\mathcal{L}}} + \ldots + (-1)^k \left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_k^A}\right|_{\psi_{\mathcal{L}}} = 0.$$

Finally, bearing in mind that $\psi_{\mathcal{L}}$ is holonomic in $J^{2k-1}\pi$, there exists a section $\phi \in \Gamma(\pi)$, whose local expression is $\phi(t) = (t, q_0^A(t))$, such that $j^{2k-1}\phi = \psi_{\mathcal{L}}$, and thus the above equations can be rewritten in the following form

$$\frac{\partial L}{\partial q_0^A}\Big|_{j^{2k-1}\phi} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial q_1^A}\Big|_{j^{2k-1}\phi} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial L}{\partial q_2^A}\Big|_{j^{2k-1}\phi} + \dots + (-1)^k \left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_k^A}\right|_{j^{2k-1}\phi} = 0.$$
(37)

Therefore, we obtain the Euler-Lagrange equations for a kth order non-autonomous system. As stated before, equation (37) may or may not be compatible, and in this last case a constraint algorithm must be used in order to obtain a submanifold $S_f \hookrightarrow J^{2k-1}\pi$ (if such submanifold exists) where the equations can be solved.

4.3 Dynamical equations for vector fields

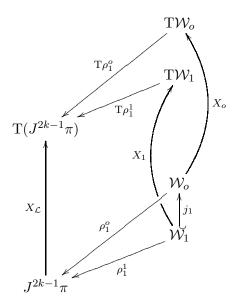
Now, using the results stated at the beginning of the Section, we can recover a vector field solution to the Lagrangian equations starting from a vector field solution to the equation in the unified formalism. First we have:

Lemma 3 Let $X_o \in \mathfrak{X}(W_o)$ be a vector field tangent to W_1 . Then there exists a unique vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ such that $X_{\mathcal{L}} \circ \rho_1^o \circ j_1 = \mathrm{T}\rho_1^o \circ X_o \circ j_1$.

(Proof) Since X_o is tangent to \mathcal{W}_1 , there exists a vector field $X_1 \in \mathfrak{X}(\mathcal{W}_1)$ such that $\mathrm{T} j_1 \circ X_1 = X_o \circ j_1$. Furthermore, as ρ_1^1 is a diffeomorphism, there is a unique vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ which is ρ_1^1 -related with X_1 ; that is, $X_{\mathcal{L}} \circ \rho_1^1 = \mathrm{T} \rho_1^1 \circ X_1$. Then

$$X_{\mathcal{L}} \circ \rho_1^o \circ j_1 = X_{\mathcal{L}} \circ \rho_1^1 = \mathrm{T} \rho_1^1 \circ X_1 = \mathrm{T} \rho_1^o \circ \mathrm{T} j_1 \circ X_1 = \mathrm{T} \rho_1^o \circ X_o \circ j_1 \,.$$

The above result states that for every $X_o \in \mathfrak{X}_{W_1}(W_o)$ there exists a vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ such that the following diagram commutes



As a consequence we obtain:

Theorem 2 Let $X_o \in \mathfrak{X}_{W_1}(W_o)$ be a vector field solution to equations (21) and tangent to W_1 (at least on the points of a submanifold $W_f \hookrightarrow W_1$). Then there exists a unique semispray of type k, $X_L \in \mathfrak{X}(J^{2k-1}\pi)$, which is a solution to the equations

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$$
 , $i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^*\eta = 1$ (38)

(at least on the points of $S_f = \rho_1^o(W_f)$). In addition, if $\mathcal{L} \in \Omega^1(J^k\pi)$ is a regular Lagrangian density, then $X_{\mathcal{L}}$ is a semispray of type 1.

Conversely, if $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ is a semispray of type k (resp., of type 1), which is a solution to equations (38) (at least on the points of a submanifold $S_f \hookrightarrow J^{2k-1}\pi$), then there exists a unique vector field $X_o \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_o)$ which is a solution to equations (21) (at least on the points of $\mathcal{W}_f = (\rho_1^1)^{-1}(S_f) \hookrightarrow \mathcal{W}_1 \hookrightarrow \mathcal{W}_o$), and it is a semispray of type k in \mathcal{W}_o (resp., of type 1).

(*Proof*) Applying Lemmas 2 and 3, we have:

$$0 = i(X_o)\Omega_o|_{\mathcal{W}_1} = i(X_o)(\rho_1^o)^*\Omega_{\mathcal{L}}|_{\mathcal{W}_1} = (\rho_1^o)^* i(X_{\mathcal{L}})\Omega_{\mathcal{L}}|_{\mathcal{W}_1} ,$$

$$1 = i(X_o)(\rho_{\mathbb{R}}^o)^*\eta|_{\mathcal{W}_1} = i(X_o)(\bar{\pi}^{2k-1} \circ \rho_1^o)^*\eta\Big|_{\mathcal{W}_1} = (\rho_1^o)^* i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^*\eta\Big|_{\mathcal{W}_1} .$$

However, as ρ_1^o is a surjective submersion, this is equivalent to

$$0 = i(X_{\mathcal{L}})\Omega_{\mathcal{L}}|_{\rho_1^o(\mathcal{W}_1)} = i(X_{\mathcal{L}})\Omega_{\mathcal{L}}|_{J^{2k-1}\pi} ,$$

$$1 = i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^* \eta \Big|_{\rho_1^o(\mathcal{W}_1)} = i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^* \eta \Big|_{J^{2k-1}\pi} ,$$

since $\rho_1^o(W_1) = J^{2k-1}\pi$ (or the submanifold $S_f \hookrightarrow J^{2k-1}\pi$). The converse is immediate, reversing this reasoning.

In order to prove that $X_{\mathcal{L}}$ is a semispray of type k, we compute its local expression in coordinates. From the local expression (27) for the vector field X_o (where the functions F_j^A are the solutions of equations (28)), and using Lemma 3, we obtain that the local expression of the vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ is

$$X_{\mathcal{L}} = \frac{\partial}{\partial t} + \sum_{i=0}^{k} q_{i+1}^{A} \frac{\partial}{\partial q_{i}^{A}} + \sum_{j=k}^{2k-1} F_{j}^{A} \frac{\partial}{\partial q_{j}^{A}},$$

which is the local expression for a semispray of type k in $J^{2k-1}\pi$.

Finally, if $\mathcal{L} \in \Omega^1(J^k\pi)$ is a regular Lagrangian density, equations (28) become (29), and hence the local expression of $X_{\mathcal{L}}$ is

$$X_{\mathcal{L}} = \frac{\partial}{\partial t} + \sum_{i=0}^{2k-2} q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A},$$

which is the local expression for a semispray of type 1 in $J^{2k-1}\pi$.

Remarks:

• It is important to point out that, if \mathcal{L} is not a regular Lagrangian density, then X_o is a semispray of type k in \mathcal{W}_o , but not necessarily a semispray of type 1. This means that $X_{\mathcal{L}}$ may be a solution to the Lagrangian equations for vector fields, but the trajectories given by its integral sections are not solutions to the dynamical system (the sections solution to the dynamical problem must be holonomic, but the integral sections of $X_{\mathcal{L}}$ are only holonomic of type k). Thus, for singular Lagrangians, this must be imposed as an additional condition. This constitutes a relevant difference from the case of first-order dynamical systems, where this condition ($X_{\mathcal{L}}$ is a semispray of type 1) is obtained straightforwardly in the unified formalism.

For singular Lagrangians, only in the most interesting cases can we assure the existence of a submanifold $W_f \hookrightarrow W_1$ and vector fields $X_o \in \mathfrak{X}_{W_1}(W_o)$ tangent to W_f which are solutions to equations (31). Then, considering the submanifold $S_f = \rho_1^1(W_f) \hookrightarrow J^{2k-1}\pi$, in the best cases we have that those semisprays of type 1 $X_{\mathcal{L}}$ exist, perhaps on another submanifold $M_f \hookrightarrow S_f$ where they are tangent, and are solutions to equations

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}}|_{M_f} = 0 \quad , \quad i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^* \eta \Big|_{M_f} = 1.$$
 (39)

• Notice that Theorem 2 states that there is a one-to-one correspondence between vector field $X_o \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_o)$ solutions to equations (21) and vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ solutions to (38), but not uniqueness. In fact, we cannot assure uniqueness of the vector field $X_{\mathcal{L}}$ unless the Lagrangian density is regular, as we can see in the following result:

Corollary 2 If the Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is regular, then there is a unique semispray of type 1, $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$, which is a solution to equations (38).

(Proof) If the Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is regular, using Proposition 5, there exists a unique semispray of type 1, $X_o \in \mathfrak{X}(\mathcal{W}_o)$, solution to equations (21) and tangent to \mathcal{W}_1 . Then, using Theorem 2, there is a unique vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$, which is a semispray of type 1 in $J^{2k-1}\pi$ and is a solution to equations (38).

In other words, uniqueness of the vector field $X_{\mathcal{L}}$ is a consequence of uniqueness of X_o .

Finally, as a consequence of Theorem 1 and the results stated in this Section, we obtain:

Theorem 3 The following assertions on a section $\phi \in \Gamma(\pi)$ are equivalent:

1. $j^{2k-1}\phi$ is a solution to equation (36), that is,

$$(j^{2k-1}\phi)^* i(Y)\Omega_{\mathcal{L}} = 0$$
, for every $Y \in \mathfrak{X}(J^{2k-1}\pi)$.

2. In natural coordinates, if $\phi = (t, q_0^A(t))$, then $j^{2k-1}\phi = (t, q_0^A(t), q_1^A(t), \dots, q_{2k-1}^A(t))$ is a solution to the kth order Euler-Lagrange equations given by (37), that is,

$$\frac{\partial L}{\partial q_0^A}\bigg|_{j^{2k-1}\phi} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial q_1^A}\bigg|_{j^{2k-1}\phi} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial L}{\partial q_2^A}\bigg|_{j^{2k-1}\phi} + \ldots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_k^A}\bigg|_{j^{2k-1}\phi} = 0.$$

3. Denoting $\psi_{\mathcal{L}} = j^{2k-1}\phi$, then $\psi_{\mathcal{L}}$ is a solution to the equation

$$i(\psi_{\mathcal{L}}')(\Omega_{\mathcal{L}} \circ \psi_{\mathcal{L}}) = 0$$
,

where $\psi'_{\mathcal{L}} \colon \mathbb{R} \to \mathrm{T}(J^{2k-1}\pi)$ is the canonical lifting of $\psi_{\mathcal{L}}$ to the tangent bundle.

4. $j^{2k-1}\phi$ is an integral curve of a vector field contained in a class of $\bar{\pi}^{2k-1}$ -transverse semisprays of type 1, $\{X_{\mathcal{L}}\}\subset \mathfrak{X}(J^{2k-1}\pi)$, satisfying the first equation in (38), that is,

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}}=0$$
.

5 Hamiltonian formalism

5.1 General setting

In order to describe the Hamiltonian formalism on the basis of the unified one, we must distinguish between the regular and non-regular cases. In fact, the only "non-regular" case we consider is the almost-regular one, so we need to define the concept of almost-regular Lagrangian density.

Before doing so, we must define the generalization of the Legendre map from the first-order time-dependent case. Since $\Theta_{\mathcal{L}} \in \Omega^1(J^{2k-1}\pi)$ is a π_{k-1}^{2k-1} -semibasic 1-form, we can give the following definition:

Definition 11 The extended Legendre-Ostrogradsky map associated with the Lagrangian density \mathcal{L} is the map $\widetilde{\mathcal{FL}}: J^{2k-1}\pi \to \mathrm{T}^*(J^{k-1}\pi)$ defined as follows: for every $u \in \mathrm{T}(J^{2k-1}\pi)$,

$$\Theta_{\mathcal{L}}(u) = \left\langle \mathrm{T} \pi_{k-1}^{2k-1}(u) \mid \widetilde{\mathcal{FL}}(\tau_{J^{2k-1}\pi}(u)) \right\rangle,$$

where $\tau_{J^{2k-1}\pi} \colon \mathrm{T}(J^{2k-1}\pi) \to J^{2k-1}\pi$ is the canonical submersion.

This map verifies that $\pi_{J^{k-1}\pi} \circ \widetilde{\mathcal{FL}} = \pi_{k-1}^{2k-1}$, where $\pi_{J^{k-1}\pi} \colon \mathrm{T}^*(J^{k-1}\pi) \to J^{k-1}\pi$ is the natural projection. Furthermore, if $\Theta_{k-1} \in \Omega^1(\mathrm{T}^*(J^{k-1}\pi))$ and $\Omega_{k-1} = -\mathrm{d}\Theta_{k-1} \in \Omega^2(\mathrm{T}^*(J^{k-1}\pi))$ are the canonical 1 and 2 forms of the cotangent bundle $\mathrm{T}^*(J^{k-1}\pi)$, we have that

$$\widetilde{\mathcal{FL}}^* \Theta_{k-1} = \Theta_{\mathcal{L}} \quad , \quad \widetilde{\mathcal{FL}}^* \Omega_{k-1} = \Omega_{\mathcal{L}} .$$

Bearing in mind the local expression (4) of the tautological 1-form on $T^*(J^{k-1}\pi)$ and the local expression (34) of $\Theta_{\mathcal{L}}$, we have that the local expression of the map $\widetilde{\mathcal{FL}}$ is:

$$\widetilde{\mathcal{FL}}^* t = t$$
 , $\widetilde{\mathcal{FL}}^* q_r^A = q_r^A$,

$$\widetilde{\mathcal{FL}}^*p = L - \sum_{r=1}^k q_r^A \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial L}{\partial q_{r+i}^A} \right) \quad , \quad \widetilde{\mathcal{FL}}^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial L}{\partial q_{r+i}^A} \right) \; ,$$

that is, this map coincides with the extended Legendre-Ostrogradsky map defined locally in Section 3.2.1, thus justifying the notation and terminology introduced therein.

Notice that dim $T^*(J^{k-1}\pi) = 2kn + 2 > 2kn + 1 = \dim J^{2k-1}\pi$. Thus, $T^*(J^{k-1}\pi)$ is not a suitable dual bundle to $J^{2k-1}\pi$ for giving a Hamiltonian description of the dynamical system. Therefore, according to, for instance, [41] and the references therein, we consider the bundle $J^{k-1}\pi^* = T^*(J^{k-1}\pi)/(\bar{\pi}^{k-1})^*T^*\mathbb{R}$, with the natural projections

$$\mu \colon \mathrm{T}^*(J^{k-1}\pi) \to J^{k-1}\pi^* \quad , \quad \pi^r_{J^{k-1}\pi} \colon J^{k-1}\pi^* \to J^{k-1}\pi \quad , \quad \bar{\tau} = \pi^r_{J^{k-1}\pi} \circ \bar{\pi}^{k-1} \colon J^{k-1}\pi^* \to \mathbb{R} \, ,$$

where $\pi^r_{J^{k-1}\pi}$ is the map satisfying $\pi_{J^{k-1}\pi} = \pi^r_{J^{k-1}\pi} \circ \mu$. Notice that dim $J^{k-1}\pi^* = 2kn + 1$.

Thus, we define the restricted Legendre-Ostrogradsky map as $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}} : J^{2k-1}\pi \to J^{k-1}\pi^*$. This map satisfies $\pi^r_{J^{k-1}\pi} \circ \mathcal{FL} = \pi^{2k-1}_{k-1}$, and has the following local expression

$$\mathcal{F}\mathcal{L}^*t = t \quad , \quad \mathcal{F}\mathcal{L}^*q_r^A = q_r^A \quad , \quad \mathcal{F}\mathcal{L}^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial L}{\partial q_{r+i}^A}\right) \, .$$

In other words, this map coincides with the restricted Legendre-Ostrogradsky map defined locally in Section 3.2.1. This justifies the notation and terminology introduced in that Section.

Proposition 8 For every $\bar{y} \in J^{2k-1}\pi$ we have that $\operatorname{rank}(\widetilde{\mathcal{FL}}(\bar{y})) = \operatorname{rank}(\mathcal{FL}(\bar{y}))$.

We do not prove this result. Following the patterns in [17], the idea is to compute in natural coordinates the local expressions of the Jacobian matrices of \mathcal{FL} and $\widetilde{\mathcal{FL}}$. Then, observe that the ranks of both maps depend on the rank of the Hessian matrix of L with respect to q_k^A at the point \bar{y} , and that the additional row in the Jacobian matrix of $\widetilde{\mathcal{FL}}$ is a linear combination of the others. See [17] for details in the first-order case.

As a consequence of Proposition 8, and taking into account the different definitions given for the regularity of the Lagrangian density, we arrive at the following result:

Proposition 9 Given a Lagrangian $\mathcal{L} \in \Omega^1(J^k\pi)$, the following statements are equivalent:

- 1. $\Omega_{\mathcal{L}}$ has maximal rank on $J^{2k-1}\pi$.
- 2. The pair $(\Omega_{\mathcal{L}}, (\bar{\pi}^{2k-1})^*\eta)$ is a cosymplectic structure on $J^{2k-1}\pi$.

- 3. $\mathcal{FL}: J^{2k-1}\pi \to J^{k-1}\pi^*$ is a local diffeomorphism.
- 4. $\widetilde{\mathcal{FL}}: J^{2k-1}\pi \to \mathrm{T}^*(J^{k-1}\pi)$ is an immersion.

(Proof) It is easy to check that all the statements are locally equivalent to

$$\det\left(\frac{\partial^2 L}{\partial q_k^B\partial q_k^A}\right)(\bar{y})\neq 0,\quad \text{for every } \bar{y}\in J^k\pi\,.$$

Now, we denote by $\widetilde{\mathcal{P}} = \operatorname{Im}(\widetilde{\mathcal{FL}}) = \widetilde{\mathcal{FL}}(J^{2k-1}\pi) \stackrel{\widetilde{\mathcal{I}}}{\hookrightarrow} \operatorname{T}^*(J^{k-1}\pi)$ the image of the extended Legendre-Ostrogradsky map; and by $\mathcal{P} = \operatorname{Im}(\mathcal{FL}) = \mathcal{FL}(J^{2k-1}\pi) \stackrel{\mathcal{I}}{\hookrightarrow} J^{k-1}\pi^*$ the image of the restricted Legendre-Ostrogradsky map. Let $\bar{\tau}_o = \bar{\tau} \circ \jmath \colon \mathcal{P} \to \mathbb{R}$ be the natural projection. We can now give the following definition:

Definition 12 A Lagrangian $\mathcal{L} \in \Omega^1(J^k\pi)$ is called an almost-regular Lagrangian density if:

- 1. \mathcal{P} is a closed submanifold of $J^{k-1}\pi^*$.
- 2. FL is a submersion onto its image.
- 3. For every $\bar{y} \in J^{2k-1}\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of $J^{2k-1}\pi$.

As a consequence of Prop. 8, we have that $\widetilde{\mathcal{P}}$ is diffeomorphic to \mathcal{P} . This diffeomorphism is just μ restricted to the image set $\widetilde{\mathcal{P}}$, and we denote it by $\widetilde{\mu}$. This enables us to state:

Lemma 4 If the Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is, at least, almost-regular, the Hamiltonian section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$ induces a Hamiltonian section $h \in \Gamma(\mu)$ defined by

$$h([\alpha]) = (\rho_2 \circ \hat{h})([(\rho_2^r)^{-1}(\jmath([\alpha]))]), \quad \text{for every } [\alpha] \in \mathcal{P}.$$
(40)

(Proof) It is clear that, given $[\alpha] \in J^{k-1}\pi^*$, the section \hat{h} maps every point $(\bar{y}, [\alpha]) \in (\rho_2^r)^{-1}([\alpha])$ into $\rho_2^{-1}[\rho_2(\hat{h}(\bar{y}, [\alpha]))]$. So we have the diagram

$$\widetilde{\mathcal{P}} \xrightarrow{\widetilde{\jmath}} T^*(J^{k-1}\pi) \xleftarrow{\rho_2} \mathcal{W}$$

$$\downarrow \widetilde{\mu} \qquad \qquad \downarrow \mu \qquad \qquad \mu_{\mathcal{W}} \qquad \downarrow \widehat{h}$$

$$\mathcal{P} \xrightarrow{\jmath} J^{k-1}\pi^* \xleftarrow{\rho_2^r} \mathcal{W}_r$$

Thus, the crucial point is the ρ_2 -projectability of the local function \hat{H} . However, since a local base for ker ρ_{2*} is given by

$$\ker \rho_{2*} = \left\langle \frac{\partial}{\partial q_k^A}, \dots, \frac{\partial}{\partial q_{2k-1}^A} \right\rangle,$$

we have that \hat{H} is ρ_2 -projectable if and only if

$$p_A^{k-1} = \frac{\partial L}{\partial q_k^A} \,.$$

This condition is fulfilled when $[\alpha] \in \mathcal{P}$, which implies that $\rho_2[\hat{h}((\rho_2^r)^{-1}([\alpha]))] \in \widetilde{\mathcal{P}}$.

Remark: In the hyperregular case, we have $\mathcal{P} = J^{k-1}\pi^*$.

Locally, this Hamiltonian μ -section is specified by the local Hamiltonian function $H \in C^{\infty}(J^{k-1}\pi^*)$, that is,

$$h(t, q_i^A, p_A^i) = (t, q_i^A, -H, p_A^i).$$

5.2 Hyperregular and regular systems. Dynamical equations for sections and vector fields

Now we analyze the case when \mathcal{L} is a regular Lagrangian density, although by simplicity we focus on the hyperregular case (the regular case is recovered from this by restriction on the corresponding open sets where \mathcal{FL} is a local diffeomorphism). This means that the phase space of the system is $J^{k-1}\pi^*$ (or the corresponding open sets).

In this case, we can give the explicit expression for the local Hamiltonian function, which is

$$H = \sum_{i=0}^{k-2} p_A^i q_{i+1}^A + p_A^{k-1} (\mathcal{F} \mathcal{L}^{-1})^* q_k^A - (\pi_k^{2k-1} \circ \mathcal{F} \mathcal{L}^{-1})^* L.$$
 (41)

The Hamiltonian section h is used to construct the Hamilton-Cartan forms in $J^{k-1}\pi^*$ by making

$$\Theta_h = h^* \Theta_{k-1} \in \Omega^1(J^{k-1} \pi^*) \quad , \quad \Omega_h = h^* \Omega_{k-1} \in \Omega^2(J^{k-1} \pi^*) ,$$

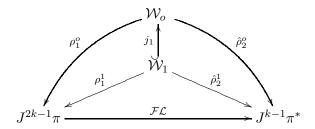
where Θ_{k-1} and Ω_{k-1} are the canonical 1 and 2 forms of the cotangent bundle $T^*(J^{k-1}\pi)$. Bearing in mind the local expression (4) of Θ_{k-1} and Ω_{k-1} , the local expression of the forms Θ_h and Ω_h is

$$\Theta_h = p_A^i dq_i^A - H dt$$
 , $\Omega_h = dq_i^A \wedge dp_A^i + dH \wedge dt$,

Notice that $\mathcal{FL}^*\Theta_h = \Theta_{\mathcal{L}}$ and $\mathcal{FL}^*\Omega_h = \Omega_{\mathcal{L}}$.

Proposition 10 If $\mathcal{L} \in \Omega^1(J^k\pi)$ is a hyperregular Lagrangian, then $\hat{\rho}_2^1 = \hat{\rho}_2^o \circ j_1 \colon \mathcal{W}_1 \to J^{k-1}\pi^*$ is a diffeomorphism.

(Proof) The following diagram is commutative



that is, we have $\hat{\rho}_2^1 = \hat{\rho}_2^o \circ j_1 = \mathcal{FL} \circ \rho_1^1$. Now, by Proposition 6, the map ρ_1^1 is a diffeomorphism. In addition, as \mathcal{L} is hyperregular, the map \mathcal{FL} is also a diffeomorphism, and thus $\hat{\rho}_2^1$ is a composition of diffeomorphisms, and hence a diffeomorphism itself.

This last result allows us to recover the Hamiltonian formalism in the same way we recovered the Lagrangian one (see Section 4), just using the diffeomorphism to define a correspondence between the solutions of both equations.

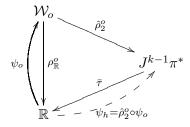
Using the previous results, we can recover the Hamiltonian sections in $J^{k-1}\pi^*$ from the sections solution to the equations in the unified formalism.

Proposition 11 Let $\mathcal{L} \in \Omega^1(J^k\pi)$ be a hyperregular Lagrangian. Let $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ be a section solution to equation (11). Then the section $\psi_h = \hat{\rho}_2^o \circ \psi_o \in \Gamma(\bar{\tau})$ is a solution to the equation

$$\psi_h^* i(Y)\Omega_h = 0, \quad \text{for every } Y \in \mathfrak{X}(J^{k-1}\pi^*)$$
 (42)

(Proof) The proof of this result is analogous to the proof given for Proposition 7.

The diagram for this situation is the following:



Remarks:

- Observe that, for the Hamiltonian sections, the condition of holonomy on the section ψ_o is not required. This is because we only need ψ_o to be a holonomic section of type k, and this condition is always fulfilled.
- As for the Lagrangian sections given by Proposition 7, this last result does not give an equivalence between sections $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$, which are solutions to equation (11), and sections $\psi_h \in \Gamma(\bar{\tau})$, which are solutions to equation (42). However, recall that sections ψ_o , which are solutions to the dynamical equations in the unified formalism, take values in \mathcal{W}_1 , and hence we are able to establish the equivalence using the diffeomorphism $\hat{\rho}_2^1$.

Let $\psi_o(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t)) \in \Gamma(\rho_{\mathbb{R}}^o)$, $0 \le i \le k-1$, $k \le j \le 2k-1$, be a solution to equation (11). Hence, ψ_o must satisfy equations (14), (15) and (17). Now, bearing in mind the local expression for the local Hamiltonian function H given in (41), we obtain the following 2kn equations for the section $\psi_h = \hat{\rho}_2^o \circ \psi_o = (t, q_i^A(t), p_A^i(t))$:

$$\dot{q}_i^A = \frac{\partial H}{\partial p_A^i} \Big|_{\psi_h} \quad ; \quad \dot{p}_A^i = -\frac{\partial H}{\partial q_i^A} \Big|_{\psi_h} .$$
(43)

So we obtain the Hamilton equations for a kth-order non-autonomous system.

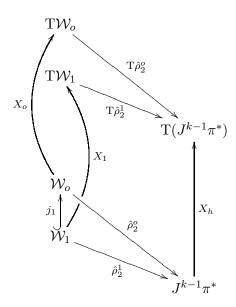
Next, we recover the Hamiltonian vector field from the vector field solution to the dynamical equations (19) in the hyperregular case. As $\hat{\rho}_2^1$ is a diffeomorphism by Proposition 10, the reasoning we follow is the same as that for the Lagrangian formalism.

Lemma 5 Let $\mathcal{L} \in \Omega^1(J^k\pi)$ be a hyperregular Lagrangian. Let $X_o \in \mathfrak{X}(\mathcal{W}_o)$ be a vector field tangent to \mathcal{W}_1 . Then there exists a unique vector field $X_h \in \mathfrak{X}(J^{k-1}\pi^*)$ such that $X_h \circ \hat{\rho}_2^o \circ j_1 = \mathrm{T}\hat{\rho}_2^o \circ X_o \circ j_1$.

(Proof) The proof of this result is similar to the proof given for Lemma 3.

This result states that, for every $X_o \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_o)$, we have a vector field $X_h \in \mathfrak{X}(J^{k-1}\pi^*)$

such that the following diagram commutes



Theorem 4 Let $\mathcal{L} \in \Omega^1(J^k\pi)$ be a hyperregular Lagrangian, and $X_o \in \mathfrak{X}_{W_1}(W_o)$ the vector field solution to equations (21) and tangent to W_1 . Then, there exists a unique vector field $X_h \in \mathfrak{X}(J^{k-1}\pi^*)$, which is a solution to the equations

$$i(X_h)\Omega_h = 0$$
 , $i(X_h)\bar{\tau}^*\eta = 1$. (44)

Conversely, if $X_h \in \mathfrak{X}(J^{k-1}\pi^*)$ is a solution to equations (44), then there exists a unique vector field $X_o \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_o)$, tangent to \mathcal{W}_1 , which is a solution to equations (21).

(*Proof*) The proof of this result is analogous to the first part of the proof given for Theorem 2, Lemma 5 now being used to obtain the vector field $X_h \in \mathfrak{X}(J^{k-1}\pi^*)$.

In local coordinates, if the vector field $X_o \in \mathfrak{X}_{W_1}(W_o)$ solution to equations (21) is given by (30), by using Lemma 5 we obtain the local expression for the vector field X_h , which is

$$X_h = \frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + \frac{\partial L}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + d_T(p_A^i) \frac{\partial}{\partial p_A^i}.$$

Finally, to close the hyperregular case, as a consequence of Theorem 1 and the results stated in this Section, we obtain the following result:

Theorem 5 The following assertions on a section $\psi_h \in \Gamma(\bar{\tau})$ are equivalent:

1. ψ_h is a solution to equation (42), that is,

$$\psi_h^* i(Y)\Omega_h = 0$$
, for every $Y \in \mathfrak{X}(J^{k-1}\pi^*)$.

2. In natural coordinates, if ψ_h is given by $\psi_h(t) = (t, q_i^A(t), p_A^i(t)), \ 0 \leqslant i \leqslant k-1$, then the components of ψ_h satisfy the kth order Hamilton equations given by (43), that is,

$$\dot{q}_i^A = \left. \frac{\partial H}{\partial p_A^i} \right|_{\psi_h} \quad ; \quad \dot{p}_A^i = -\left. \frac{\partial H}{\partial q_i^A} \right|_{\psi_h} \, .$$

3. ψ_h is a solution to the equation

$$i(\psi_h')(\Omega_h \circ \psi_h) = 0$$
,

where $\psi'_h \colon \mathbb{R} \to \mathrm{T}(J^{k-1}\pi^*)$ is the canonical lifting of ψ_h to the tangent bundle.

4. ψ_h is an integral curve of a vector field contained in a class of $\bar{\tau}$ -transverse vector fields, $\{X_h\} \subset \mathfrak{X}(J^{k-1}\pi^*)$, satisfying the first equation in (44), that is,

$$i(X_h)\Omega_h=0$$
.

5.3 Singular (almost-regular) Lagrangians. Dynamical equations for sections and vector fields

Recall that, for almost-regular Lagrangians, only in the most favourable cases can we assure the existence of some submanifold $W_f \hookrightarrow W_1$ where the dynamical equations can be solved. In this case, the solutions to the Hamiltonian formalism cannot be obtained straightforwardly from the solutions in the unified formalism, but rather by passing through the Lagrangian formalism and using the Legendre-Ostrogradsky map.

In this case, the phase space of the system is $\mathcal{P} = \operatorname{Im}(\mathcal{FL}) \hookrightarrow J^{k-1}\pi^*$. We denote by \mathcal{FL}_o : $J^{2k-1}\pi \to \mathcal{P}$ the map defined by $\mathcal{FL} = \jmath \circ \mathcal{FL}_o$. As in the hyperregular case, the Hamiltonian section h is used to construct the Hamilton-Cartan forms on \mathcal{P} as follows:

$$\Theta_h^o = h^* \Theta_{k-1} \in \Omega^1(\mathcal{P})$$
 , $\Omega_h^o = h^* \Omega_{k-1} \in \Omega^2(\mathcal{P})$.

They verify that $\mathcal{FL}_o^*\Theta_h^o = \Theta_{\mathcal{L}}$ and $\mathcal{FL}_o^*\Omega_h^o = \Omega_{\mathcal{L}}$.

Proposition 12 Let $\mathcal{L} \in \Omega^1(J^k\pi)$ be an almost-regular Lagrangian. Let $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ be a section solution to equation (11). Then, the section $\psi_h^o = \mathcal{FL}_o \circ \psi_{\mathcal{L}} \in \Gamma(\bar{\tau}_o)$ is a solution to the equation

$$(\psi_h^o)^* i(Y)\Omega_h^o = 0$$
, for every $Y \in \mathfrak{X}(\mathcal{P})$. (45)

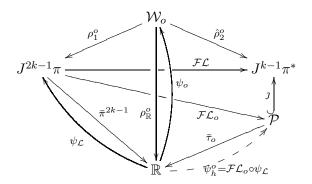
(Proof) Since the Lagrangian density is almost-regular, the map \mathcal{FL}_o is a submersion onto its image, \mathcal{P} . Hence, for every $Y \in \mathfrak{X}(\mathcal{P})$ there exist some $Z \in \mathfrak{X}(J^{2k-1}\pi)$ such that Z is \mathcal{FL}_o -related with Y, that is, $\mathcal{FL}_{o*}Z = Y$. Using this, we have

$$(\psi_h^o)^* i(Y)\Omega_h^o = (\mathcal{F}\mathcal{L}_o \circ \psi_{\mathcal{L}})^* i(Y)\Omega_h^o = \psi_{\mathcal{L}}^* (\mathcal{F}\mathcal{L}^* i(Y)\Omega_h^o) = \psi_{\mathcal{L}}^* i(Z)\mathcal{F}\mathcal{L}_o^*\Omega_h^o = \psi_{\mathcal{L}}^* i(Z)\Omega_{\mathcal{L}}.$$

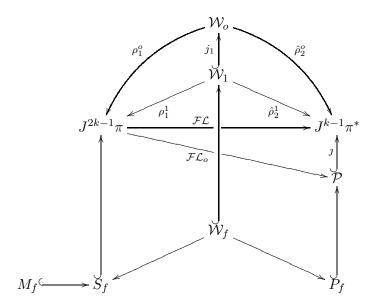
Then, using Proposition 7, we have proved

$$(\psi_h^o)^* i(Y)\Omega_h^o = \psi_{\mathcal{L}}^* i(Z)\Omega_{\mathcal{L}} = 0.$$

The diagram for this situation is the following:



Now, assume that there exists a submanifold $W_f \hookrightarrow W_1$ and vector fields $X_o \in \mathfrak{X}_{W_1}(W_o)$ tangent to W_f which are solutions to equations (31). Now consider the submanifolds $S_f = \rho_1^1(W_f) \hookrightarrow J^{2k-1}\pi$ and $P_f = \hat{\rho}_2^1(W_f) = \mathcal{FL}(S_f) \hookrightarrow \mathcal{P} \hookrightarrow J^{k-1}\pi^*$. Using Theorem 2, from the vector fields $X_o \in \mathfrak{X}_{W_1}(W_o)$ we obtain the corresponding vector fields $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$, and from these, the semisprays of type 1 (if they exist), which are perhaps defined on a submanifold $M_f \hookrightarrow S_f$, are tangent to M_f and are solutions to equations (39). So we have the diagram



Now, following analogous procedures for autonomous and non-autonomous systems [18, 29], one can prove that there are semisprays of type 1 in M_f (perhaps only on the points of another submanifold $\bar{M}_f \hookrightarrow M_f$), which are \mathcal{FL} -projectable on P_f . These vector fields $X_h^o = \mathcal{FL}_*X_{\mathcal{L}} \in \mathfrak{X}(\mathcal{P})$ are tangent to P_f and are solutions to equations

$$i(X_h^o)\Omega_h^o|_{P_f} = 0 \quad , \quad i(X_h^o)\bar{\tau}_o^*\eta|_{P_f} = 1$$
 (46)

Conversely, as \mathcal{FL}_o is a submersion, for every vector field $X_h^o \in \mathfrak{X}(\mathcal{P})$ solution to equations (46), there is a semispray of type 1, $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$, such that $\mathcal{FL}_{o*}X_{\mathcal{L}} = X_h^o$, and we can recover solutions to equations (31) using Theorem 2.

Of course, for the almost-regular case, we have a similar result to Theorem 5, on the points of the final constraint submanifold P_f .

6 Examples

6.1 The shape of a deformed elastic cylindrical beam with fixed ends

As a first example we consider a deformed elastic cylindrical beam with both ends fixed. The problem is to determinate its shape; that is, the width of every section transversal to the axis. This system has been studied on many occasions, such as [9] (Chapter 3, §3.9) and [25] (Chapter IV, §4). Strictly speaking, it is not a time-dependent mechanical system, but it can be modeled using a configuration bundle over a compact subset of \mathbb{R} , where the base coordinate represents every transversal section of the beam, thus allowing us to show an application of our formalism. For simplicity, instead of a compact subset, we take the whole real line as the base manifold.

The configuration bundle for this system is $\pi \colon E \to \mathbb{R}$, where E is a 2-dimensional smooth manifold. Let x be the global coordinate in \mathbb{R} , and $\eta \in \Omega^1(\mathbb{R})$ the volume form in \mathbb{R} with local expression $\eta = \mathrm{d}x$. Natural coordinates in E adapted to the bundle structure are (x, q_0) . Now, taking natural coordinates in the higher-order jet bundle of π , the second-order Lagrangian density for this system, $\mathcal{L} \in \Omega^1(J^2\pi)$, is locally given by

$$\mathcal{L}(x, q_0, q_1, q_2) = L \cdot (\bar{\pi}^2)^* \eta = \left(\frac{1}{2}\mu(x)q_2^2 + \rho(x)q_0\right) dx,$$

where $\mu, \rho \in C^{\infty}(J^2\pi)$ are functions that only depend on the coordinate x and represent physical parameters of the beam: ρ is the linear density and μ is a non-vanishing function involving Young's modulus of the material, the radius of curvature and the sectional moment of the cross-section considered (see [9] for a detailed description). This is a regular Lagrangian density, since the Hessian matrix of the Lagrangian function $L \in C^{\infty}(J^2\pi)$ associated with \mathcal{L} with respect to q_2 is

$$\left(\frac{\partial^2 L}{\partial q_2 \partial q_2}\right) = \mu(x) \,,$$

and this 1×1 matrix has maximum rank, since μ is a non-vanishing function.

Remark: If the beam is homogeneous, μ and ρ are constants (with $\mu \neq 0$), and thus the Lagrangian density is "autonomous", that is, it does not depend explicitly on the coordinate of the base manifold. This case is analyzed in [25].

As this is a second-order system, we consider the bundles $W = J^3\pi \times_{J^1\pi} T^*(J^1\pi)$ and $W_r = J^3\pi \times_{J^1\pi} J^1\pi^*$, with natural coordinates $(x, q_0, q_1, q_2, q_3, p, p^0, p^1)$ and $(x, q_0, q_1, q_2, q_3, p^0, p^1)$, respectively. Now, using the notation and terminology introduced throughout this article, if $\Theta_1 \in \Omega^1(T^*(J^1\pi))$ and $\Omega_1 \in \Omega^2(T^*(J^1\pi))$ are the canonical forms of $T^*(J^1\pi)$, we define the forms $\Theta = \rho_2^*\Theta_1 \in \Omega^1(W)$ and $\Omega = \rho_2^*\Omega_1 \in \Omega^2(W)$, whose local expressions are

$$\Theta = p^0 dq_0 + p^1 dq_1 + p dx \quad ; \quad \Omega = dq_0 \wedge dp^0 + dq_1 \wedge dp^1 - dp \wedge dx.$$

The coupling 1-form $\hat{\mathcal{C}} \in \Omega^1(\mathcal{W})$ has the local expression

$$\hat{\mathcal{C}} = \hat{C} \cdot \rho_{\mathbb{R}}^* \eta = (p + p^0 q_1 + p^1 q_2) \mathrm{d}x ,$$

and then we can introduce the Hamiltonian submanifold

$$\mathcal{W}_o = \left\{ w \in \mathcal{W} \colon \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w) \right\} \stackrel{j_o}{\hookrightarrow} \mathcal{W},$$

which is locally defined by the constraint function $\hat{C} - \hat{L} = 0$, whose coordinate expression is

$$\hat{C} - \hat{L} = p + p^0 q_1 + p^1 q_2 - \frac{1}{2} \mu(x) q_2^2 - \rho(x) q_0 = 0.$$

Finally, we construct the Hamiltonian $\mu_{\mathcal{W}}$ -section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$, which is specified by giving the local Hamiltonian function \hat{H} , whose local expression is

$$\hat{H}(x, q_0, q_1, q_2, q_3, p^0, p^1) = p^0 q_1 + p^1 q_2 - \frac{1}{2} \mu(x) q_2^2 - \rho(x) q_0;$$

that is, we have $\hat{h}(x, q_0, q_1, q_2, q_3, p^0, p^1) = (x, q_0, q_1, q_2, q_3, -\hat{H}, p^0, p^1)$. Using this Hamiltonian section, we define the forms $\Theta_o = j_o^* \Theta \in \Omega^1(\mathcal{W}_o)$ and $\Omega_o = j_o^* \Omega \in \Omega^2(\mathcal{W}_o)$, with local expressions

$$\Theta_o = p^0 dq_0 + p^1 dq_1 + \left(\frac{1}{2}\mu(x)q_2^2 + \rho(x)q_0 - p^0q_1 - p^1q_2\right) dx,$$

$$\Omega_o = dq_0 \wedge dp^0 + dq_1 \wedge dp^1 + \left(-\rho(x)dq_0 + p^0dq_1 + (p^1 - \mu(x)q_2)dq_2 + q_1dp^0 + q_2dp^1\right) \wedge dx.$$

In order to state the Lagrangian-Hamiltonian problem for sections in this system, let $Y \in \mathfrak{X}(\mathcal{W}_o)$ be a generic vector field locally given by

$$Y = f \frac{\partial}{\partial x} + f_0 \frac{\partial}{\partial q_0} + f_1 \frac{\partial}{\partial q_1} + F_2 \frac{\partial}{\partial q_2} + F_3 \frac{\partial}{\partial q_3} + G^0 \frac{\partial}{\partial p^0} + G^1 \frac{\partial}{\partial p^1}.$$

Now, if $\psi_o(x) = (x, q_0(x), q_1(x), q_2(x), q_3(x), p^0(x), p^1(x))$ is a holonomic section of the projection $\rho_{\mathbb{R}}^o$, equation (11) leads to the following 5 equations (the redundant equation (13) is omitted):

$$\dot{q}_0 = q_1 \quad ; \quad \dot{q}_1 = q_2 \tag{47}$$

$$\dot{p}^0 = \rho(x) \quad ; \quad \dot{p}^1 = -p^0 \tag{48}$$

$$p^1 = q_2 \mu(x) \tag{49}$$

Equations (47) give us the condition of holonomy of type 2 for the section, which are also redundant since we assume that ψ_o is holonomic. Equation (49) is a pointwise algebraic condition, from which we know that the section ψ_o must lie in a submanifold W_1 that can be identified with the graph of the extended Legendre-Ostrogradsky map, $\widetilde{\mathcal{FL}}$.

Now we compute the local expression of the map $\widetilde{\mathcal{FL}}: J^3\pi \to \mathrm{T}^*(J^1\pi)$. From Corollary 1 we know the general expression for this map, and we obtain:

$$\widetilde{\mathcal{F}\mathcal{L}}^* p^0 = -q_2 \frac{\partial \mu}{\partial x} - q_3 \mu \quad ; \quad \widetilde{\mathcal{F}\mathcal{L}}^* p^1 = q_2 \mu \quad ; \quad \widetilde{\mathcal{F}\mathcal{L}}^* p = -\frac{1}{2} \mu q_2^2 + q_1 q_2 \frac{\partial \mu}{\partial x} + q_1 q_3 \mu + q_0 \rho \,. \tag{50}$$

Therefore, the section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ is a holonomic section of the projection $\rho_{\mathbb{R}}^o$, which lies in the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}_o$ defined by the above constraint functions, and whose last components satisfy the differential equations

$$\dot{p}^0 = \rho(x)$$
 ; $\dot{p}^1 = -p^0$.

Now we state the Lagrangian-Hamiltonian problem for vector fields: we wish to find $X_o \in \mathfrak{X}(\mathcal{W}_o)$ solution to (19). If X_o is locally given by

$$X_o = f \frac{\partial}{\partial x} + f_0 \frac{\partial}{\partial q_0} + f_1 \frac{\partial}{\partial q_1} + F_2 \frac{\partial}{\partial q_2} + F_3 \frac{\partial}{\partial q_3} + G^0 \frac{\partial}{\partial p^0} + G^1 \frac{\partial}{\partial p^1},$$

then equations (19) lead to the following (again, the redundant equation (22) is omitted):

$$f_0 = f \cdot q_1 \quad ; \quad f_1 = f \cdot q_2 \tag{51}$$

$$G^0 = f \cdot \rho(x) \quad ; \quad G^1 = -f \cdot p^0$$
 (52)

$$f = 1 \tag{53}$$

$$f \cdot \left(p^1 - q_2 \mu(x)\right) = 0 \tag{54}$$

Equations (51) give us the condition of semispray of type 2 in W_o for X_o . In addition, equation (54) is an algebraic relation from which we obtain, in coordinates, the result stated in Propositions 3 and 4, that is, the vector field X_o is defined along a submanifold W_1 which we identify with the graph of the extended Legendre-Ostrogradsky map and is defined by

$$\mathcal{W}_1 = \{ w \in \mathcal{W}_o : \xi_0(w) = \xi_1(w) = 0 \}$$
,

where $\xi_r = p^r - \widetilde{\mathcal{FL}}^* p^r$, r = 1, 2. Thus, using (51), (52) and (53), X_o is given locally by

$$X_o = \frac{\partial}{\partial x} + q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + F_2 \frac{\partial}{\partial q_2} + F_3 \frac{\partial}{\partial q_3} + \rho \frac{\partial}{\partial p^0} - p^0 \frac{\partial}{\partial p^1}.$$
 (55)

Notice that the functions F_2 and F_3 in (55) are not determined until the tangency of X_o on W_1 is required. This condition is locally equivalent to checking if the following identities hold

$$L(X_o)\xi_0|_{W_1} = 0$$
 , $L(X_o)\xi_1|_{W_1} = 0$.

As we have seen in Section 3.2.2, these equations lead to the Lagrangian equations for the vector field X_o ; that is, on the points of W_o we obtain

$$L(X_o)\xi_0 = \rho + q_2 \frac{\partial^2 \mu}{\partial x \partial x} + q_3 \frac{\partial \mu}{\partial x} + F_2 \frac{\partial \mu}{\partial x} + F_3 \mu = 0$$
 (56)

$$L(X_o)\xi_1 = (q_3 - F_2)\mu = 0. (57)$$

Equation (57) gives us the condition of semispray of type 1 for the vector field X_o (recall that μ is non-vanishing), and equation (56) is the Euler-Lagrange equation for X_o . Observe that, since μ is a non-vanishing function, these equations have a unique solution for F_2 and F_3 . Hence, there is a unique vector field $X_o \in \mathfrak{X}(\mathcal{W}_o)$ solution to the equations $i(X_o)\Omega_o|_{\mathcal{W}_1} = 0$ and $i(X_o)(\rho_{\mathbb{R}}^o)^*\eta|_{\mathcal{W}_1} = 1$, which is tangent to the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}_o$, and is given locally by

$$X_o = \frac{\partial}{\partial x} + q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{1}{\mu} \left(\rho + q_2 \frac{\partial^2 \mu}{\partial x \partial x} + 2q_3 \frac{\partial \mu}{\partial x} \right) \frac{\partial}{\partial q_3} + \rho \frac{\partial}{\partial p^0} - p^0 \frac{\partial}{\partial p^1}.$$

Finally, we recover the Lagrangian and Hamiltonian solutions for sections and vector fields. For the Lagrangian solutions, by Proposition 7, from the holonomic section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ we can recover a holonomic section $\psi_{\mathcal{L}} = \rho_1^o \circ \psi_o \in \Gamma(\bar{\pi}^3)$ solution to equation (36). In particular, if $\psi_o(x) = (x, q_0(x), q_1(x), q_2(x), q_3(x), p^0(x), p^1(x))$, then $\psi_{\mathcal{L}}(x) = (x, q_0(x), q_1(x), q_2(x), q_3(x))$ is a holonomic section solution to equations (48), which, bearing in mind the local expression (50) of the extended Legendre-Ostrogradsky map, can be written locally as

$$\rho + \dot{q}_2 \frac{\partial \mu}{\partial x} + q_2 \frac{\partial^2 \mu}{\partial x \partial x} + \dot{q}_3 \mu + q_3 \frac{\partial \mu}{\partial x} = 0$$
 (58)

$$(\dot{q}_2 - q_3)\mu = 0 (59)$$

Equation (59) gives the condition for the section $\psi_{\mathcal{L}}$ to be holonomic, and it is redundant since we required this condition to be fulfilled at the beginning. Now, if $\phi(x) = (x, y(x))$ is a section of π such that $j^3 \phi = \psi_{\mathcal{L}}$, then the Euler-Lagrange equation can be written locally

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mu\ddot{y}) + \rho = 0 \ .$$

In the case of an homogeneous beam, the Euler-Lagrange equation reduces to $\mu y^{(iv)} + \rho = 0$.

For the Lagrangian vector field, from Lemma 3 and Theorem 2, we can recover, from the semispray of type 1 $X_o \in \mathfrak{X}(W_o)$ a semispray of type 1, $X_{\mathcal{L}} \in \mathfrak{X}(J^3\pi)$, which is a solution to equations (38), and is locally given by

$$X_{\mathcal{L}} = \frac{\partial}{\partial x} + q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{1}{\mu} \left(\rho + q_2 \frac{\partial^2 \mu}{\partial x \partial x} + 2q_3 \frac{\partial \mu}{\partial x} \right) \frac{\partial}{\partial q_3}.$$

Now, as \mathcal{L} is a regular Lagrangian density, for the Hamiltonian solutions we can use the results stated in Section 5.2 and recover the Hamiltonian solutions directly from the unified formalism. For the Hamiltonian sections, using Proposition 11, from a section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ fulfilling equation (11) we can recover a section $\psi_h = \rho_2^o \circ \psi_o \in \Gamma(\bar{\tau})$ solution to equation (42). In particular, if

 $\psi_o(x) = (x, q_0(x), q_1(x), q_2(x), q_3(x), p^0(x), p^1(x)), \text{ then } \psi_h(x) = (x, q_0(x), q_1(x), p^0(x), p^1(x)) \text{ is a section solution to equations (47) and (48), which can be written locally as$

$$\dot{q}_0 = \frac{\partial H}{\partial p^0}\Big|_{\psi_h} \quad ; \quad \dot{q}_1 = \frac{\partial H}{\partial p^1}\Big|_{\psi_h} \quad ; \quad \dot{p}^0 = -\frac{\partial H}{\partial q_0}\Big|_{\psi_h} \quad ; \quad \dot{p}^1 = -\frac{\partial H}{\partial q_1}\Big|_{\psi_h} .$$

where $H \in C^{\infty}(J^1\pi^*)$ is the local Hamiltonian function with local expression

$$H = p^0 q_1 + \frac{(p^1)^2}{2\mu} - \rho q_0.$$

For the Hamiltonian vector field, from Lemma 5 and Theorem 4, the vector field $X_o \in \mathfrak{X}(\mathcal{W}_o)$ gives a vector field $X_h \in \mathfrak{X}(J^1\pi^*)$ solution to equations (44), which is locally given by

$$X_h = \frac{\partial}{\partial x} + q_1 \frac{\partial}{\partial q_0} + \frac{p^1}{\mu} \frac{\partial}{\partial q_1} + \rho \frac{\partial}{\partial p^0} - p^0 \frac{\partial}{\partial p^1}.$$

6.2 The second-order relativistic particle subjected to a potential

Consider a relativistic particle whose action is proportional to its extrinsic curvature [36, 35, 7, 34, 39]. Now assume this system is subjected to the action of a generic potential depending on the time and the position of the particle, thus obtaining a time-dependent dynamical system.

The configuration bundle for this system is $E \xrightarrow{\pi} \mathbb{R}$, where E is a (n+1)-dimensional smooth manifold. Let t be the global coordinate in \mathbb{R} , and $\eta \in \Omega^1(\mathbb{R})$ the volume form in \mathbb{R} with local expression $\eta = \mathrm{d}t$. Natural coordinates in E adapted to the bundle structure are denoted by (t, q_0^i) , $1 \le i \le n$. Now, bearing in mind the natural coordinates in the higher-order jet bundle of π , the second-order Lagrangian density for this system, $\mathcal{L} \in \Omega^1(J^2\pi)$, is locally given by

$$\mathcal{L}(t, q_0^i, q_1^i, q_2^i) = \left(\frac{\alpha}{(q_1^i)^2} \left[(q_1^i)^2 (q_2^i)^2 - (q_1^i q_2^i)^2 \right]^{1/2} + V(t, q_0^i) \right) dt \equiv \left(\frac{\alpha}{(q_1^i)^2} \sqrt{g} + V(t, q_0^i) \right) dt ,$$

where α is some nonzero constant and $V \in C^{\infty}(J^2\pi)$ is a function depending only on t and q_0^i . This is a singular Lagrangian density, as we can see by computing the Hessian matrix of the Lagrangian function $L \in C^{\infty}(J^2\pi)$ associated with \mathcal{L} with respect to q_2^A , which is

$$\frac{\partial^2 L}{\partial q_2^B \partial q_2^A} = \begin{cases} \frac{\alpha}{2(q_1^i)^2 \sqrt{g^3}} \left[\left((q_1^i q_2^i)^2 - 2(q_1^i)^2 (q_2^i)^2 \right) q_1^B q_1^A \right. \\ \left. + (q_1^i)^2 (q_1^i q_2^i) (q_2^B q_1^A - q_1^B q_2^A) - (q_1^i)^2 (q_2^i)^2 q_2^B q_2^A \right] & \text{if } B \neq A \\ \frac{\alpha}{\sqrt{g^3}} \left[g - (q_2^i)^2 q_1^A q_1^A + 2(q_1^i q_2^i) q_1^A q_2^A - (q_1^i)^2 q_2^A q_2^A \right] & \text{if } B = A \end{cases},$$

and a long calculation shows that $\det \left(\frac{\partial^2 L}{\partial q_2^B \partial q_2^A} \right) = 0.$

Consider the bundles $\mathcal{W}=J^3\pi\times_{J^1\pi}\mathrm{T}^*(J^1\pi)$ and $\mathcal{W}_r=J^3\pi\times_{J^1\pi}J^1\pi^*$, with natural coordinates $(t,q_0^i,q_1^i,q_2^i,q_3^i,p,p_i^0,p_i^1)$ and $(t,q_0^i,q_1^i,q_2^i,q_3^i,p_i^0,p_i^1)$, respectively. Now, if $\Theta_1\in\Omega^1(\mathrm{T}^*(J^1\pi))$ and $\Omega_1\in\Omega^2(\mathrm{T}^*(J^1\pi))$ are the canonical forms of the cotangent bundle of $J^1\pi$, we define

$$\Theta = \rho_2^* \Theta_1 = p_i^0 \mathrm{d} q_0^i + p_i^1 \mathrm{d} q_1^i + p \mathrm{d} t \in \Omega^1(\mathcal{W}) \quad ; \quad \Omega = \rho_2^* \Omega_1 = d q_0^i \wedge d p_i^0 + d q_1^i \wedge d p_i^1 - \mathrm{d} p \wedge \mathrm{d} t \in \Omega^2(\mathcal{W}) .$$

The coupling 1-form $\hat{\mathcal{C}} \in \Omega^1(\mathcal{W})$ has the local expression $\hat{\mathcal{C}} = \hat{C} \cdot \rho_{\mathbb{R}}^* \eta = (p + p_i^0 q_1^i + p_i^1 q_2^i) dt$, and from this we can introduce the Hamiltonian submanifold $\mathcal{W}_o \stackrel{j_o}{\hookrightarrow} \mathcal{W}$, which is locally defined by the constraint function $\hat{C} - \hat{L} = 0$, whose coordinate expression is

$$\hat{C} - \hat{L} = p + p_i^0 q_1^i + p_i^1 q_2^i - \frac{\alpha}{(q_1^i)^2} \sqrt{g} - V(t, q_0^i).$$

This allows us to construct the Hamiltonian $\mu_{\mathcal{W}}$ -section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$, which is specified by giving the local Hamiltonian function \hat{H} , whose local expression is

$$\hat{H}(t,q_0^i,q_1^i,q_2^i,q_3^i,p_i^0,p_i^1) = p_i^0q_1^i + p_i^1q_2^i - \frac{\alpha}{(q_1^i)^2}\sqrt{g} - V(t,q_0^i)\,,$$

that is, we have $\hat{h}(t,q_0^i,q_1^i,q_2^i,q_3^i,p_i^0,p_i^1)=(t,q_0^i,q_1^i,q_2^i,q_3^i,-\hat{H},p_i^0,p_i^1)$. Using this Hamiltonian section, we define the forms $\Theta_o=j_o^*\Theta\in\Omega^1(\mathcal{W}_o)$ and $\Omega_o=j_o^*\Omega\in\Omega^2(\mathcal{W}_o)$, with local expressions

$$\begin{split} \Theta_o &= p_i^0 \mathrm{d} q_0^i + p_i^1 \mathrm{d} q_1^i + \left(\frac{\alpha}{(q_1^i)^2} \sqrt{g} + V(t, q_0^i) - p_i^0 q_1^i - p_i^1 q_2^i \right) \\ \Omega_o &= \mathrm{d} q_0^i \wedge \mathrm{d} p_i^0 + \mathrm{d} q_1^i \wedge \mathrm{d} p_i^1 + \left(q_1^A \mathrm{d} p_A^0 + q_2^A \mathrm{d} p_A^1 - \frac{\partial V}{\partial q_0^i} \mathrm{d} q_0^i \right. \\ &+ \left[p_A^0 + \frac{\alpha}{((q_1^i)^2)^2 \sqrt{g}} \left[\left((q_1^i)^2 (q_2^i)^2 - 2(q_1^i q_2^i)^2 \right) q_1^A + (q_1^i q_2^i) (q_1^i)^2 q_2^A \right] \right] \mathrm{d} q_1^A \\ &+ \left[p_A^1 - \frac{\alpha}{(q_1^i)^2 \sqrt{g}} \left((q_1^i)^2 q_2^A - (q_1^i q_2^i) q_1^A \right) \right] \mathrm{d} q_2^A \right) \wedge \mathrm{d} t \,. \end{split}$$

In order to state the Lagrangian-Hamiltonian problem for sections for this second-order system, let $Y \in \mathfrak{X}(\mathcal{W}_o)$ be a generic vector field locally given by

$$Y = f \frac{\partial}{\partial t} + f_0^A \frac{\partial}{\partial q_0^A} + f_1^A \frac{\partial}{\partial q_1^A} + F_2^A \frac{\partial}{\partial q_2^A} + F_3^A \frac{\partial}{\partial q_3^A} + G_A^0 \frac{\partial}{\partial p_A^0} + G_A^1 \frac{\partial}{\partial p_A^1} .$$

Now, if $\psi_o(t) = (t, q_0^i(t), q_1^i(t), q_2^i(t), q_3^i(t), p_i^0(t), p_i^1(t))$ is a holonomic section of the projection $\rho_{\mathbb{R}}^o$, equation (11) leads to the following 5n equations (the redundant equation (13) is omitted):

$$\dot{q}_0^A = q_1^A \quad , \quad \dot{q}_1^A = q_2^A \tag{60}$$

$$\dot{p}_A^0 = \frac{\partial V}{\partial q_0^A} \quad , \quad \dot{p}_A^1 = -p_A^0 - \frac{\alpha}{((q_1^i)^2)^2 \sqrt{g}} \left[\left((q_1^i)^2 (q_2^i)^2 - 2(q_1^i q_2^i)^2 \right) q_1^A + (q_1^i q_2^i)(q_1^i)^2 q_2^A \right] \tag{61}$$

$$p_A^1 = \frac{\alpha}{(q_1^i)^2 \sqrt{g}} \left((q_1^i)^2 q_2^A - (q_1^i q_2^i) q_1^A \right) \tag{62}$$

Equations (60) give the condition of holonomy of type 2 for the section ψ_o , which are also redundant since the holonomy of ψ_o is already assumed. Equations (62) are an algebraic condition, from which we conclude that the section ψ_o must lie in a submanifold W_1 that can be identified with the graph of the extended Legendre-Ostrogradsky map, $\widetilde{\mathcal{FL}}$. The expression in natural coordinates of this map $\widetilde{\mathcal{FL}}: J^3\pi \to \mathrm{T}^*(J^1\pi)$ is obtained from Corollary 1 and is

$$\begin{split} \mathcal{F}\mathcal{L}^*p_A^0 &= \frac{\alpha}{(q_1^i)^2\sqrt{g^3}} \left[\left((q_2^i)^2 g + (q_1^i)^2 (q_2^i)^2 (q_1^i q_3^i) - (q_1^i)^2 (q_1^i q_2^i) (q_2^i q_3^i) \right) q_1^A \right] \\ &+ \frac{\alpha}{(q_1^i)^2\sqrt{g^3}} \left[\left(((q_1^i)^2)^2 (q_2^i q_3^i) - (q_1^i)^2 (q_1^i q_2^i) (q_1^i q_3^i) - (q_1^i q_2^i) g \right) q_2^A - (q_1^i)^2 g q_3^A \right] \\ \mathcal{F}\mathcal{L}^*p_A^1 &= \frac{\alpha}{(q_1^i)^2\sqrt{g}} \left[(q_1^i)^2 q_2^A - (q_1^i q_2^i) q_1^A \right] \\ \mathcal{F}\mathcal{L}^*p &= \frac{\alpha}{(q_1^i)^2} \sqrt{g} + V(t, q_0^i) - \left(\frac{\alpha}{(q_1^i)^2\sqrt{g^3}} \left[((q_2^i)^2 g + (q_1^i)^2 (q_2^i)^2 (q_1^i q_3^i) - (q_1^i)^2 (q_1^i q_2^i) (q_2^i q_3^i) \right) q_1^A \right] \\ &+ \frac{\alpha}{(q_1^i)^2\sqrt{g^3}} \left[\left(((q_1^i)^2)^2 (q_2^i q_3^i) - (q_1^i)^2 (q_1^i q_2^i) (q_1^i q_3^i) - (q_1^i q_2^i) g \right) q_2^A - (q_1^i)^2 g q_3^A \right] \right) q_1^A \\ &- \frac{\alpha}{(q_1^i)^2\sqrt{g}} \left[(q_1^i)^2 q_2^A - (q_1^i q_2^i) q_1^A \right] q_2^A \end{split}$$

Hence, the section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ is holonomic and lies in the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}_o$ defined by the constraint functions given above, and its last components satisfy the 2n differential equations

$$\dot{p}_A^0 = \frac{\partial V}{\partial q_0^A} \quad , \quad \dot{p}_A^1 = -p_A^0 - \frac{\alpha}{((q_1^i)^2)^2 \sqrt{g}} \left[\left((q_1^i)^2 (q_2^i)^2 - 2(q_1^i q_2^i)^2 \right) q_1^A + (q_1^i q_2^i) (q_1^i)^2 q_2^A \right] .$$

Now we state the Lagrangian-Hamiltonian problem for vector fields, that is, we wish to find a vector field $X_o \in \mathfrak{X}(\mathcal{W}_o)$ solution to equations (19). If the vector field X_o is locally given by

$$X_o = f \frac{\partial}{\partial t} + f_0^A \frac{\partial}{\partial q_0^A} + f_1^A \frac{\partial}{\partial q_1^A} + F_2^A \frac{\partial}{\partial q_2^A} + F_3^A \frac{\partial}{\partial q_3^A} + G_A^0 \frac{\partial}{\partial p_A^0} + G_A^1 \frac{\partial}{\partial p_A^1} .$$

The equations (19) lead to the following 5n + 1 equations (the redundant equation (22) is omitted):

$$f_0^A = f \cdot q_1^A \quad ; \quad f_1^A = f \cdot q_2^A$$
 (64)

$$G_A^0 = \frac{\partial V}{\partial q_0^A} \quad ; \quad G_A^1 = -p_A^0 - \frac{\alpha}{((q_1^i)^2)^2 \sqrt{g}} \left[\left((q_1^i)^2 (q_2^i)^2 - 2(q_1^i q_2^i)^2 \right) q_1^A + (q_1^i q_2^i)(q_1^i)^2 q_2^A \right] \quad (65)$$

$$f = 1 \tag{66}$$

$$f \cdot \left(p_A^1 - \frac{\alpha}{(q_1^i)^2 \sqrt{g}} \left((q_1^i)^2 q_2^A - (q_1^i q_2^i) q_1^A \right) \right) = 0 \tag{67}$$

From equations (64) we obtain the condition of semispray of type 2 for the vector field X_o . In addition, equations (67) are algebraic relations between the coordinates in W_o which give, in coordinates, the result stated in Propositions 3 and 4, that is, the vector field X_o is defined along a submanifold W_1 which we identify with the graph of the extended Legendre-Ostrogradsky. Thus, using (64), (65) and (66), the vector field X_o is given locally by

$$X_o = \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + F_2^A \frac{\partial}{\partial q_2^A} + F_3^A \frac{\partial}{\partial q_3^A} + \frac{\partial V}{\partial q_0^A} \frac{\partial}{\partial p_A^O} + G_A^1 \frac{\partial}{\partial p_A^D} , \qquad (68)$$

where the functions G_A^1 are determined by (65). Since we wish to recover the solutions in the Lagrangian formalism from the vector field X_o , we must require it to be a semispray of type 1. This condition reduces the set of vector fields $X_o \in \mathfrak{X}(\mathcal{W}_o)$ given by (68) to the following ones

$$X_o = \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + q_3^A \frac{\partial}{\partial q_2^A} + F_3^A \frac{\partial}{\partial q_3^A} + \frac{\partial V}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + G_A^1 \frac{\partial}{\partial p_A^1} . \tag{69}$$

Notice that the functions F_3^A are not determinated until the tangency of the vector field X_o on W_1 is required. From the expression in local coordinates (63) of the map $\widetilde{\mathcal{FL}}$, we obtain the primary constraints defining the closed submanifold $\widetilde{\mathcal{P}} = \operatorname{Im}(\widetilde{\mathcal{FL}}) \hookrightarrow \operatorname{T}^*(J^1\pi)$, which are

$$\phi_1^{(0)} \equiv p_i^1 q_1^i = 0 \quad ; \quad \phi_2^{(0)} \equiv (p_i^1)^2 - \frac{\alpha^2}{(q_1^i)^2} = 0 \; ; \tag{70}$$

Let $\mathcal{FL}_o: J^3\pi \to \widetilde{\mathcal{P}}$. Then, the submanifold $\mathcal{W}_1 = \operatorname{graph} \mathcal{FL}_o = \operatorname{graph} \widetilde{\mathcal{FL}}$ is defined by

$$\mathcal{W}_1 = \left\{ w \in \mathcal{W}_o \colon \xi(w) = \xi_0^A(w) = \xi_1^A(w) = \phi_1^{(0)}(w) = \phi_2^{(1)}(w) = 0 \right\}$$

where $\xi_r^A = p_A^r - \widetilde{\mathcal{FL}}^* p_A^r$, $\xi = p - \widetilde{\mathcal{FL}}^* p$.

Next, we compute the tangency condition for the vector field $X_o \in \mathfrak{X}(\mathcal{W}_o)$, given locally by (69) on the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}_o \hookrightarrow \mathcal{W}$, by checking if the following identities hold

$$L(X)\xi_0^A\big|_{\mathcal{W}_1} = 0 \quad ; \quad L(X)\xi_1^A\big|_{\mathcal{W}_1} = 0$$
 (71)

$$L(X)\phi_1^{(0)}\Big|_{\mathcal{W}_1} = 0 \quad ; \quad L(X)\phi_2^{(0)}\Big|_{\mathcal{W}_1} = 0 .$$
 (72)

As we have seen in Section 3.2.2, equations (71) give us the Lagrangian equations for the vector field X_o . However, equations (72) do not hold, since

$$L(X_o)\phi_1^{(0)} = L(X_o)(p_i^1 q_1^i) = -p_i^0 q_1^i \quad , \quad L(X_o)\phi_2^{(0)} = L(X_o)((p_i^1)^2 - \alpha^2/(q_1^i)^2) = -2p_i^0 p_i^1 ,$$

and hence we obtain two first-generation secondary constraints

$$\phi_1^{(1)} \equiv p_i^0 q_1^i = 0 \quad , \quad \phi_2^{(1)} \equiv p_i^0 p_i^1 = 0$$
 (73)

that define a new submanifold $W_2 \hookrightarrow W_1$. Now, by checking the tangency of the vector field X_o to this new submanifold, we obtain

$$L(X_o)\phi_1^{(1)} = L(X_o)(p_i^0 q_1^i) = 0$$
 , $L(X_o)\phi_2^{(1)} = L(X_o)(p_i^0 p_i^1) = -(p_i^0)^2$,

and a second-generation secondary constraint appears,

$$\phi^{(2)} \equiv (p_i^0)^2 = 0 , \qquad (74)$$

which defines a new submanifold $W_3 \hookrightarrow W_2$. Finally, the tangency of the vector field X_o on this submanifold gives no new constraints, since

$$L(X_o)\phi^{(2)} = L(X_o)((p_i^0)^2) = 0$$
.

So we have two primary constraints (70), two first-generation secondary constraints (73), and a single second-generation secondary constraint (74). Notice that these five constraints only depend on q_1^A , p_A^0 and p_A^1 , and so they are $\hat{\rho}_2^o$ -projectable.

Notice that we still have to check (71). As we have seen in Section 3.2.2, we obtain the following equations

$$\left(F_3^B - d_T\left(q_3^B\right)\right) \frac{\partial^2 \hat{L}}{\partial q_2^B \partial q_2^A} + \frac{\partial \hat{L}}{\partial q_0^A} - d_T\left(\frac{\partial \hat{L}}{\partial q_1^A}\right) + d_T^2\left(\frac{\partial \hat{L}}{\partial q_2^A}\right) + \left(F_2^B - q_3^B\right) d_T\left(\frac{\partial^2 \hat{L}}{\partial q_2^B \partial q_2^A}\right) = 0$$
(75)

$$\left(F_2^B - q_3^B\right) \frac{\partial^2 \hat{L}}{\partial q_2^B \partial q_2^A} = 0$$
(76)

Since we have already required the vector field X_o to be a semispray of type 1 in \mathcal{W}_o , equations (76) are satisfied identically and equations (75) become

$$\left(F_3^B - d_T\left(q_3^B\right)\right) \frac{\partial^2 \hat{L}}{\partial q_2^B \partial q_2^A} + \frac{\partial \hat{L}}{\partial q_0^A} - d_T\left(\frac{\partial \hat{L}}{\partial q_1^A}\right) + d_T^2\left(\frac{\partial \hat{L}}{\partial q_2^A}\right) = 0.$$
(77)

A long calculation shows that this equation is compatible if, and only if, $\frac{\partial V}{\partial q_0^A} = 0$, for $1 \leqslant A \leqslant n$. That is, we have n first-generation secondary constraints arising from the tangency condition of X_o along W_1 , thus defining a new submanifold $W_4 \hookrightarrow W_3$ with constraint functions

$$\phi_{3,A}^{(1)} \equiv \frac{\partial V}{\partial q_0^A} = 0 \quad \text{for } 1 \leqslant A \leqslant n \,.$$

Observe that, since V is a function that depends only on t and q_0^A , these new constraints also depend only on the coordinates t and q_0^A , and thus they are $\hat{\rho}_2^o$ -projectable. From a physical viewpoint, these constraints mean that the dynamics of the particle can take place on every level set of the potential with respect to the position coordinates.

Finally, we recover the Lagrangian and Hamiltonian dynamics from the unified formalism. For the Lagrangian solutions, using Proposition 7, we know that from the holonomic section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ we can recover a holonomic section $\psi_{\mathcal{L}} = \rho_1^o \circ \psi_o \in \Gamma(\bar{\pi}^3)$ solution to equation (36). In particular, if $\psi_o(t) = (t, q_0^i(t), q_1^i(t), q_2^i(t), q_3^i(t), p_i^0(t), p_i^1(t))$, then $\psi_{\mathcal{L}}(t) = (t, q_0^i(t), q_1^i(t), q_2^i(t), q_3^i(t))$ is a holonomic section solution to equations (61). Now, bearing in mind the local expression (63) of the extendend Legendre-Ostrogradsky map, equations (61) give the last n equations of the holonomy condition for $\psi_{\mathcal{L}}$, which are identically satisfied since the holonomy condition has been already required, and the classical higher-order Euler-Lagrange equations

$$\left. \frac{\partial L}{\partial q_0^A} \right|_{\psi_{\mathcal{L}}} - \left. \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial q_1^A} \right|_{\psi_{\mathcal{L}}} + \left. \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial L}{\partial q_2^A} \right|_{\psi_{\mathcal{L}}} = 0.$$

For the Lagrangian vector field, from Lemma 3 and Theorem 2, we can recover from the semispray of type 1 $X_o \in \mathfrak{X}(W_o)$ a semispray of type 1, $X_{\mathcal{L}} \in \mathfrak{X}(J^3\pi)$, solution to equations (39) (with $M_f = \rho_1^o(W_4)$), and it is locally given by

$$X_o = \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + q_3^A \frac{\partial}{\partial q_2^A} + F_3^A \frac{\partial}{\partial q_3^A} \; ,$$

where F_3^A are the solutions of equations (77).

One can check that, if the semispray condition is not required at the beginning and we perform all this procedure with the vector field given by (68), the final result is the same. This means that, in this case, the semispray condition does not give any additional constraint.

Now, since \mathcal{L} is an almost-regular Lagrangian density, for the Hamiltonian dynamics we must use the results stated in Section 5.3 and recover the Hamiltonian solutions passing through the Lagrangian formalism. For the Hamiltonian sections, by Proposition 12, from a section $\psi_o \in \Gamma(\rho_{\mathbb{R}}^o)$ solution to equation (11), we can recover a section $\psi_h = \mathcal{F}\mathcal{L} \circ \rho_1^o \circ \psi_o \in \Gamma(\bar{\tau}_o)$ solution to the equation (45).

For the Hamiltonian vector fields, we know that there are semisprays of type $1 \ X_{\mathcal{L}} \in \mathfrak{X}(J^3\pi)$, solutions to equations (39), which are \mathcal{FL}_o -projectable on $P_4 = \hat{\rho}_2^o(\mathcal{W}_4)$, tangent to P_4 and solutions to the Hamilton equation.

7 Conclusions and outlook

The objective of this work is to develop a complete and detailed geometric framework for describing the Lagrangian and Hamiltonian formalisms of higher-order non-autonomous mechanical systems, and to give some applications of it.

Our approach to the problem consists in extending the Lagrangian-Hamiltonian unified formalism of Skinner and Rusk to this case, starting from the generalization of this formalism previously made for first-order non-autonomous dynamical systems [5] and higher-order autonomous mechanical systems [39]. This enables us to derive the Lagrangian and the Hamiltonian formalisms for these kinds of systems (in a natural way). We pay special attention to describing the equations of motion in several equivalent ways, using sections and vector fields in the bundles that constitute the phase spaces of these systems, and showing how the equivalence between the Lagrangian and the Hamiltonian formalisms is stated through the Legendre-Ostrogradsky map, which is also obtained in a natural way from the unified formalism. Our analysis is performed both for regular and singular systems.

As applications of our formalism, we study two physical examples: a regular system describing the shape of a deformed elastic cylindrical beam with fixed ends, and a singular system describing a relativistic particle subjected to a generic potential depending on time and positions.

The background geometrical tools that we use are higher-order jet bundles, in general, rather than the simpler and more usual trivial bundles (this particular case is also analyzed in the work), since our aim is for this geometric framework to serve as a guideline towards a geometric model for higher-order field theories, which is free of the ambiguities present in their standard geometrical descriptions (concerning the definition of the Poincaré-Cartan forms and the Legendre transformation). Some advances on this subject have been already obtained [10, 46], and we trust that our future work will contribute to completing them.

A A particular situation: trivial bundles

Assume that the bundle $E \xrightarrow{\pi} \mathbb{R}$ is trivial; that is, $E = \mathbb{R} \times Q$, where Q is a n-dimensional manifold. In this case, we have that $J^k \pi \cong \mathbb{R} \times \mathrm{T}^k Q$, where $\mathrm{T}^k Q$ is the kth order tangent bundle of Q (see [22] for details). The natural coordinates in this case are defined in the same way as in the general case, and are denoted by $(t, q_0^A, q_1^A, \dots, q_k^A)$. In this case, the bundles involved in the construction are

$$J^{2k-1}\pi \cong \mathbb{R} \times \mathcal{T}^{2k-1}Q \quad , \quad \mathcal{T}^*(J^{k-1}\pi) \cong \mathbb{R} \times \mathbb{R}^* \times \mathcal{T}^*(\mathcal{T}^{k-1}Q) \quad , \quad J^{k-1}\pi^* \cong \mathbb{R} \times \mathcal{T}^*(\mathcal{T}^{k-1}Q)$$

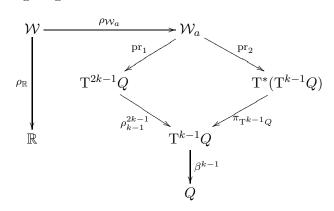
Thus, the higher-order restricted jet-momentum bundle is

$$\mathcal{W}_r = J^{2k-1}\pi \times_{J^{k-1}\pi} J^{k-1}\pi^* \cong (\mathbb{R} \times \mathrm{T}^{2k-1}Q) \times_{\mathbb{R} \times \mathrm{T}^{k-1}Q} (\mathbb{R} \times \mathrm{T}^*(\mathrm{T}^{k-1}Q))$$
$$\cong \mathbb{R} \times \mathrm{T}^{2k-1}Q \times_{\mathrm{T}^{k-1}Q} \mathrm{T}^*(\mathrm{T}^{k-1}Q) \cong \mathbb{R} \times \mathcal{W}_a,$$

where $W_a = T^{2k-1}Q \times_{T^{k-1}Q} T^*(T^{k-1}Q)$ denotes the unified phase space in the autonomous formalism. Natural coordinates in this bundle are the same as in the non-autonomous case, that is, $(t, q_0^A, q_1^A, \dots, q_{2k-1}^A, p_A^0, p_A^1, \dots, p_A^{k-1})$.

Remark: As we will see, in this particular situation the extended jet-momentum bundle is not needed. Thus, we denote W_r simply by W in this section. The differential forms Θ_r and Ω_r (or, equivalently, Θ_o and Ω_o) are also denoted by Θ and Ω , respectively.

We have the following diagram



where all the maps are the natural projections (see [39] for details). In coordinates, we have

$$\rho_{\mathbb{R}}(t, q_i^A, q_j^A, p_A^i) = t \quad , \quad \rho_{\mathcal{W}_a}(t, q_i^A, q_j^A, p_A^i) = (q_i^A, q_j^A, p_A^i)$$
$$\operatorname{pr}_1(q_i^A, q_j^A, p_A^i) = (q_i^A, q_j^A) \quad , \quad \operatorname{pr}_2(q_i^A, q_j^A, p_A^i) = (q_i^A, p_A^i)$$

Now we see how to construct the canonical structures in W, described previously, from the canonical structures in W_a . Let $\theta_a \in \Omega^1(W_a)$ and $\Omega_a \in \Omega^2(W_a)$ be the canonical forms on W_a defined as

$$\theta_a = \operatorname{pr}_2^* \theta_{k-1}$$
 , $\Omega_a = \operatorname{pr}_2^* \omega_{k-1} = -\operatorname{d}\theta_a$,

where θ_{k-1} and ω_{k-1} are the canonical 1 and 2 forms on the cotangent bundle $T^*(T^{k-1}Q)$.

As stated before, the dynamics of the system is described by a Lagrangian density $\mathcal{L} \in \Omega^1(\mathbb{R} \times \mathrm{T}^k Q)$, with associated Lagrangian function $L \in \mathrm{C}^{\infty}(\mathbb{R} \times \mathrm{T}^k Q)$. Then, if $C \in \mathrm{C}^{\infty}(\mathcal{W}_a)$ is the coupling function in the autonomous formalism [39], we can construct a globally defined Hamiltonian function in the following way:

$$H = \rho_{\mathcal{W}_a}^* C - L$$
.

Then, the forms $\Theta \in \Omega^1(\mathcal{W})$ and $\Omega \in \Omega^2(\mathcal{W})$ can be constructed as follows

$$\Theta = \rho_{\mathcal{W}_a}^* \theta_a - H \rho_{\mathbb{R}}^* \eta$$
 , $\Omega = -d\Theta = \rho_{\mathcal{W}_a}^* \Omega_a + dH \wedge \rho_{\mathbb{R}}^* \eta$.

In local coordinates, bearing in mind the local expressions of θ_{k-1} , ω_{k-1} and \mathcal{C} :

$$\theta_{k-1} = p_A^i dq_i^A$$
 , $\omega_{k-1} = dq_i^A \wedge dp_A^i$, $\mathcal{C} = p_A^i q_{i+1}^A$,

we have that the local expression for the forms Θ and Ω are

$$\Theta = p_A^i dq_i^A - (p_A^i q_{i+1}^A - L) dt \quad , \quad \Omega = dq_i^A \wedge dp_A^i + d(p_A^i q_{i+1}^A - L) \wedge dt;$$

that is, we obtain the local expressions given in (10).

The dynamical equations for sections and vector fields are now stated as in Section 3, and the local expressions are the same. There is only one difference: in Proposition 3 a connection in W is needed in order to split the presymplectic form Ω into the sum of a 2-form with the wedge product of two 1-forms (which are the differential of the local Hamiltonian function, and the volume form in \mathbb{R}). In this case, we do not need to use such a connection, since the bundles are trivial and this splitting is natural.

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