On extreme Bosonic linear channels

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Abstract

The set of all channels with fixed input and output is convex. We first give a convenient formulation of necessary and sufficient condition for a channel to be extreme point of this set in terms of complementary channel, a notion of big importance in quantum information theory. This formulation is based on the general approach to extremality of completely positive maps in an operator algebra due to Arveson. We then apply this formulation to prove the main result of this note: under certain nondegeneracy conditions, purity of the environment is necessary and sufficient for extremality of Bosonic linear (quasi-free) channel. It follows that Gaussian channel between finite-mode Bosonic systems is extreme if and only if it has minimal noise.

1 Extremality in terms of complementary channels

In what follows \mathcal{H} (possibly with indices) denotes a separable Hilbert space, $\mathfrak{T}(\mathcal{H})$ – the Banach space of trace-class operators and $\mathfrak{B}(\mathcal{H}) = \mathfrak{T}(\mathcal{H})^*$ – the algebra of all bounded operators in \mathcal{H} . Let A, B be two quantum systems with the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, which we call the input and the output systems. In this paper we call by *channel* a normal, unital, completely positive map Φ : $\mathfrak{B}(\mathcal{H}_B) \longrightarrow \mathfrak{B}(\mathcal{H}_A)$. There is a unique linear trace-preserving map $\Phi_* : \mathfrak{T}(\mathcal{H}_A) \longrightarrow \mathfrak{T}(\mathcal{H}_B)$ such that $\Phi = (\Phi_*)^*$, which maps density operators (states) in \mathcal{H}_A into density operators in \mathcal{H}_B . In physical terms we work in the Heisenberg picture, while Φ_* is the channel in the Schrödinger picture.

The set of all channels with input A and output B is convex. We first give a convenient formulation of necessary and sufficient condition for a channel to be extreme point of this set in terms of complementary channel, a notion of big importance in quantum information theory [1,2,3]. This formulation is based on the general approach to extremality of completely positive maps of a C^* -algebra due to Arveson [4] (see also Appendix). We then apply this formulation to prove the main result of this note: under certain nondegeneracy conditions, purity of the environment is necessary and sufficient for extremality of Bosonic linear (quasi-free) channel. It follows that a Gaussian channel between finite-mode Bosonic systems is extreme if and only if it has minimal noise. For channels

in one Bosonic mode this was conjectured by Ivan, Sabapathy and Simon [5] basing on consideration of finite-dimensional criterion of Choi [6]. Finding the proof for this statement was the initial motivation of the present work.

Given three quantum systems A, B, E with the spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E$ and an isometric operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$, the relations

$$\Phi[X] = V^*(X \otimes I_E)V, \qquad X \in \mathfrak{B}(\mathcal{H}_B), \tag{1}$$

$$\tilde{\Phi}[Y] = V^*(I_B \otimes Y)V, \qquad Y \in \mathfrak{B}(\mathcal{H}_E)$$
 (2)

define two channels $\Phi : \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$, $\tilde{\Phi} : \mathfrak{B}(\mathcal{H}_E) \to \mathfrak{B}(\mathcal{H}_A)$, which are called mutually *complementary*.

The Stinespring dilation theorem implies that for given a channel Φ the representation (1) and hence a complementary channel $\tilde{\Phi}$ given by (2) always exists. The representation (1) is *minimal* if the subspace

$$\mathcal{M} = \{ (X \otimes I_E) V \psi : \psi \in \mathcal{H}_A, X \in \mathfrak{B}(\mathcal{H}_B) \} \subset \mathcal{H}_B \otimes \mathcal{H}_E$$

is dense in $\mathcal{H}_B \otimes \mathcal{H}_E$. Let

$$\Phi[X] = V'^*(X \otimes I_{E'})V', \qquad X \in \mathfrak{B}(\mathcal{H}_B)$$
(3)

be another representation for Φ , then there exists an isometric operator W from \mathcal{H}_E into $\mathcal{H}_{E'}$ such that

$$(I_B \otimes W)V = V', \tag{4}$$

so that the new complementary channel is $\tilde{\Phi}'[Y'] = \tilde{\Phi}[W^*Y'W]$. In particular, if the representation (3) is also minimal, the operator W maps \mathcal{H}_E onto $\mathcal{H}_{E'}$, so that the minimal representation and the corresponding complementary channel are unique up to the unitary equivalence. In this case the complementary channel is also called minimal.

Let Y be an operator in \mathcal{H}_E such that $0 \leq Y \leq I_E$, then the relation

$$\Phi_Y[X] = V^*(X \otimes Y)V, \qquad X \in \mathfrak{B}(\mathcal{H}_B),$$

apparently defines a completely positive map satisfying $\Phi_Y \leq \Phi$. The operatoral gebraic version of the Radon-Nikodym theorem established by Arveson [4], Theorem 1.4.2, implies: assuming that the representation (1) is minimal, this relation sets one-to-one affine correspondence between the order intervals $[0, I_E]$ and $[0, \Phi]$.

Proposition 1. Channel Φ is extreme if and only if it has a complementary channel $\tilde{\Phi}'$ such that $\operatorname{Ker}\tilde{\Phi}'=0$, or, equivalently, $\operatorname{Ran}\tilde{\Phi}'_*=\mathfrak{T}(\mathcal{H}_E)$. Any such complementary channel is minimal.

This follows from Theorem 1.4.6 [4] (see Appendix) but we give here a proof for completeness. Assume that $\operatorname{Ker}\tilde{\Phi}'=0$, and let $\tilde{\Phi}$ be a complementary channel in the minimal representation (1) for Φ , so that $\tilde{\Phi}'[Y'] = \tilde{\Phi}[W^*Y'W]$ where $W^*W = I_E$. Then $\tilde{\Phi}'[I_{E'} - WW^*] = \tilde{\Phi}[W^*(I_{E'} - WW^*)W] = 0$, therefore $I_{E'} - WW^* = 0$, so that W is unitary operator onto $\mathcal{H}_{E'}$. Therefore we can assume $\tilde{\Phi}' = \tilde{\Phi}$ and $\operatorname{Ker}\tilde{\Phi} = 0$. Let $\Phi = \frac{1}{2}(\Phi_1 + \Phi_2)$, where Φ_j are channels.

Then by Arveson's Theorem, $\Phi_j = \Phi_{Y_j}$; j = 1, 2, where $0 \le Y_j \le 2I_E$. Since Φ_j are unital, $V^*(I_B \otimes Y_j)V = V^*(I_B \otimes I_E)V$, whence $V^*(I_B \otimes (I_E - Y_j))V = 0$. But this means that $\tilde{\Phi}[I_E - Y_j] = 0$, hence $I_E - Y_j = 0$. Therefore $\Phi_j = \Phi$, so Φ is extreme channel.

Conversely, let Φ be extreme channel and let us show that $\operatorname{Ker}\tilde{\Phi}=0$ for the minimal complementary channel. Let $\tilde{\Phi}[Y]=0$. Without loss of generality we can assume that $Y=Y^*$ and $\|Y\|\leq 1$. Define $\Phi_{\pm}[X]=V^*(X\otimes(I_B\pm Y))V$, for the minimal representation (1) of the channel Φ . Then Φ_{\pm} are normal completely positive maps by Arveson's Theorem and $\Phi_{\pm}[I_B]=I_A$ by the assumption. Since Φ is extreme, $\Phi=\Phi_{\pm}$, implying $V^*(X\otimes Y)V=0$ for all X,Y. It follows that $\langle \psi_2|V^*(X_2^*\otimes I_E)(I_B\otimes Y)(X_1\otimes I_E)V|\psi_1\rangle\equiv 0$, hence the bilinear form of the operator $I_B\otimes Y$ vanishes on \mathcal{M} , hence Y=0 by the minimality of the representation.

Remark. Choosing an orthonormal basis $\{e_j\}$ in \mathcal{H}_E , introduce the operators $V_j : \mathcal{H}_A$ to \mathcal{H}_B , defined by $V_j = \langle e_j | V$ i.e.

$$\langle \phi | V_i \psi \rangle = \langle \phi \otimes e_i | V \psi \rangle, \quad \phi \in \mathcal{H}_B, \psi \in \mathcal{H}_A,$$
 (5)

and let $y_{jk} = \langle e_j | Y e_k \rangle$ be the matrix of the operator Y. Then the above result amounts to the following: channel Φ is extreme if and only if it has a representation

$$\Phi[X] = \sum_{k} V_k^* X V_k, \qquad X \in \mathfrak{B}(\mathcal{H}_B), \tag{6}$$

where the system $\{V_j^*V_k\}$ is strongly independent in the sense that $\sum_{jk} y_{jk} V_j^* V_k = 0$ (strong operator convergence) for a matrix $[y_{jk}]$ of bounded operator implies $y_{jk} \equiv 0$. This result contained in [7] generalizes Choi's criterion for finite dimensional case [6]. However for our purposes the formulation in terms of the complementary channel turns out to be more convenient.

2 Extremality of Linear Bosonic Channels

In what follows we consider Bosonic system with s modes described by irreducible Weyl-Segal system

$$W(z) = \exp i (Rz), \tag{7}$$

in a Hilbert space \mathcal{H} , where

$$Rz = \sum_{j=1}^{s} (x_j q_j + y_j p_j),$$
 (8)

so that $R = [q_1, p_1, \dots, q_s, p_s]$ is the row vector of the *canonical observables* and $z = [x_1, y_1, \dots, x_s, y_s]^{\top}$ is the column vector of real parameters. We refer to [8, 9, 10, 11] for relevant definitions and results.

A real vector space Z equipped with a nondegenerate antisymmetric bilinear form Δ is called symplectic space. Particularly important case is the standard symplectic space $Z = \mathbb{R}^{2s}$ equipped with the form $z^{\top} \Delta z'$, where

$$\Delta = \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}$$
 (9)

is the matrix of commutators of the canonical observables, $[Rz,Rz'] = -iz^{\top}\Delta z'I$.

The noncommutative Fourier transform of a trace class operator τ in $\mathcal H$ is defined as

$$\phi_{\tau}(z) = \text{Tr}\tau W(z).$$

The complex function $\phi_{\tau}(z)$ is bounded and continuous on Z. If ρ is a density operator (state), ϕ_{ρ} is called its characteristic function. Then $\phi_{\rho}(0) = 1$. Operator τ is positive if and only if $\phi_{\tau}(z)$ is Δ -nonnegative definite: all the matrices with the elements

$$\phi_{\tau}(z_r - z_s) \exp \frac{i}{2} (z_r^{\top} \Delta z_s), \tag{10}$$

where z_1, \ldots, z_n is an arbitrary finite subset of Z, are nonnegative definite.

Let $(Z_A, \Delta_A), (Z_B, \Delta_B)$ be the symplectic spaces of dimensionalities $2s_A, 2s_B$, which will describe the input and the output of the channel (here Δ_A, Δ_B have the canonical form (9)) and let $W_A(z_A), W_B(z_B)$ be the Weyl operators in the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ of the corresponding Bosonic systems. Channel Φ transforming the Weyl operators according to the rule

$$\Phi[W_B(z_B)] = W(Kz_B)f(z_B),\tag{11}$$

where K is a linear map between output and input symplectic spaces and f is a complex continuous function such that f(0) = 1, is called *linear Bosonic* [12] or quasi-free [9]. Define the real skew-symmetric $2s_B \times 2s_B$ — matrix

$$\Delta_K = \Delta_B - K^{\top} \Delta_A K.$$

The map (11) is completely positive if and only if f is Δ_K -nonnegative definite: all the matrices with the elements

$$f(z_r - z_s) \exp \frac{i}{2} (z_r^{\top} \Delta_K z_s), \tag{12}$$

where z_1, \ldots, z_n is an arbitrary finite subset of Z, are nonnegative definite [9],[13]. In what follows we assume that the real skew-symmetric $2s_B \times 2s_B - matrix$ $\Delta_K = \Delta_B - K^\top \Delta_A K$ is nondegenerate, i.e.

$$\det \Delta_K \neq 0. \tag{13}$$

Under this condition there exist real nondegenerate $2s_B\times 2s_B-$ matrix K_D such that

$$K_D^{\top} \Delta_D K_D = \Delta_K, \tag{14}$$

where $\Delta_D = \Delta_B$. This is just the canonical form of the nondegenerate skew-symmetric matrix Δ_K . Comparing (12) with (10) we find that there exists a state ρ_D of the Bosonic system in the space \mathcal{H}_D corresponding to the standard symplectic space $(Z_D, \Delta_D) \simeq (Z_B, \Delta_B)$ such that

$$f(z_B) = \text{Tr}\rho_D W_D(K_D z_B) = \phi_{\rho_D}(K_D z_B)$$
(15)

i.e.

$$\Phi[W_B(z_B)] = W(Kz_B)\phi_{\rho_D}(K_D z_B). \tag{16}$$

The relation (14) implies that

$$\Delta_B = K^{\top} \Delta_A K + K_D^{\top} \Delta_D K_D. \tag{17}$$

We will make use of the unitary dilation of the channel Φ from [10]. Consider the composite Bosonic system AD = BE with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_D \simeq$ $\mathcal{H}_B \otimes \mathcal{H}_E$ corresponding to the symplectic space $Z = Z_A \oplus Z_D = Z_B \oplus Z_E$, where $(Z_E, \Delta_E) \simeq (Z_A, \Delta_A)$. Thus $[R_A R_D] = [R_B R_E]$ describe different splits of the set of canonical observables for the composite system. The channel Φ is described by the linear input-output relation (preserving the commutators as follows from (17)

$$R_B' = R_A K + R_D K_D \tag{18}$$

where the system D is in the state ρ_D (for simplicity of notations we write R_A, \ldots instead of $R_A \otimes I_D, \ldots$). It is shown that the commutator-preserving relation (18) can be complemented to the full linear canonical transformation by putting

$$R_E' = R_A L + R_D L_D, \tag{19}$$

where $(2s_A) \times (2s_A)$ - matrix L and $(2s_B) \times (2s_A)$ - matrix L_D are such that the $2(s_A + s_B) \times 2(s_A + s_B)$ - matrix

$$T = \left[\begin{array}{cc} K & L \\ K_D & L_D \end{array} \right] \tag{20}$$

is symplectic, i.e. satisfies

$$T^{\top} \left[\begin{array}{cc} \Delta_A & 0 \\ 0 & \Delta_D \end{array} \right] T = \left[\begin{array}{cc} \Delta_B & 0 \\ 0 & \Delta_E \end{array} \right]$$

and $\Delta_E = \Delta_A$.

Lemma 1. Under the condition (13), $\det L \neq 0$.

Proof. The fact that T is symplectic implies, in addition to (17),

$$0 = K^{\top} \Delta_A L + K_D^{\top} \Delta_D L_D,$$

$$\Delta_E = L^{\top} \Delta_A L + L_D^{\top} \Delta_D L_D.$$

Taking into account that $\det K_D \neq 0$, the first equation implies

$$L_D = -\left(K_D^{\top} \Delta_D\right)^{-1} K^{\top} \Delta_A L.$$

Substituting into the second equation gives $\Delta_D = L^{\top}ML$, where $M = \Delta_A + \Delta_A K (K_D \Delta_D)^{-1} \Delta_D (K_D^{\top} \Delta_D)^{-1} K^{\top} \Delta_A$. Therefore $(\det L)^2 \det M = 1$, whence $\det L \neq 0.\square$

Denote by the U_T the unitary operator in $\mathcal{H}_A \otimes \mathcal{H}_D \simeq \mathcal{H}_B \otimes \mathcal{H}_E$ implementing the symplectic transformation T so that

$$[R_B' R_E'] = U_T^* [R_B R_E] U_T = [R_A R_D] T.$$
(21)

Then we have the unitary dilation

$$\Phi[W_B(z_B)] = \operatorname{Tr}_D(I_A \otimes \rho_D) U_T^*(W_B(z_B) \otimes I_E) U_T. \tag{22}$$

The weakly complementary channel [10] is then

$$\tilde{\Phi}^w[W_E(z_E)] = \operatorname{Tr}_D(I_A \otimes \rho_D) U_T^*(I_B \otimes W_E(z_E)) U_T.$$

The equation (19) is nothing but the input-output relation for the weakly complementary channel which thus acts as

$$\tilde{\Phi}^w[W_E(z_E)] = W_A(Lz_E)\phi_{\rho_D}(L_D z_E). \tag{23}$$

In the case of pure state $\rho_D = |\psi_D\rangle\langle\psi_D|$ the relation (22) amounts to the Stinespring representation (1) for the channel Φ with the isometry $V = U_T |\psi_D\rangle$, and the relation (22) amounts to (2) implying that $\tilde{\Phi}^w = \tilde{\Phi}$.

Apparently if the channel Φ given by (16) is extreme then ρ_D is a pure state (otherwise the spectral decomposition of ρ_D would provide a nontrivial convex decomposition of Φ). In the converse direction we prove

Theorem. Assume that ρ_D is a pure state with nonvanishing characteristic function ϕ_{ρ_D} which is L^2 -differentiable to the order $2s_D$. Then the channel Φ given by (16) is extreme.

Remark. We conjecture that a similar result should hold without assumption (13) for a Bosonic linear channels on the CCR-algebra. In [9] purity of ρ_D , in the situation where K is symplectic transformation or symplectic projection (so that $\det \Delta_K = 0$), was shown to be sufficient for extremality of a quasi-free map on the CCR-algebra.

<u>Proof.</u> We shall show that the complementary channel $\tilde{\Phi}$ satisfies the condition $\overline{\text{Ran}\tilde{\Phi}_*} = \mathfrak{T}(\mathcal{H}_E)$ of the Proposition 1.

Lemma 2. Let $\Psi_{K,f}$ be a Bosonic linear channel with the same input and output space \mathcal{H} ,

$$\Psi_{K,f}[W(z)] = W(Kz)f(z),$$

where K is a nondegenerate square matrix. Then the restriction of $\Psi_{K,f}$ onto $\mathfrak{T}(\mathcal{H})$ coincides with $|\det K|^{-1} \left(\Psi_{\hat{K},\hat{f}}\right)_*$, where $\hat{K} = K^{-1}, \hat{f}(z) = f(-K^{-1}z)$.

Proof. We use the inversion formula for the noncommutative Fourier transform [8]: if τ is trace-class operator and $\phi_{\tau}(z) = \text{Tr}\tau W(z)$, then

$$\tau = \frac{1}{(2\pi)^s} \int \phi_{\tau}(z) W(-z) d^{2s} z.$$

It follows that

$$\begin{split} \Psi_{K,f}[\tau] &= \frac{1}{(2\pi)^s} \int \phi_{\tau}(z) W(-Kz) f(-z) d^{2s} z \\ &= \frac{1}{(2\pi)^s |\det K|} \int \phi_{\tau}(K^{-1}z) W(-z) f(-K^{-1}z) d^{2s} z \\ &= \frac{1}{(2\pi)^s |\det K|} \int \phi_{\tau}(\hat{K}z) W(-z) \hat{f}(z) d^{2s} z. \end{split}$$

Then

$$\operatorname{Tr}\Psi_{K,f}[\tau]W(z) = |\det K|^{-1} \phi_{\tau}(\hat{K}z)\hat{f}(z)$$

$$= |\det K|^{-1} \operatorname{Tr}\tau W(\hat{K}z)\hat{f}(z)$$

$$= |\det K|^{-1} \operatorname{Tr}\tau \Psi_{\hat{K},\hat{f}}(W(z))$$

and the Lemma is proved. \square

Lemma 3. Under the condition of Lemma 2, $\operatorname{Ran}\tilde{\Phi}_* = \mathfrak{T}(\mathcal{H}_E)$.

Proof. From Lemma 2 it follows that up to a positive factor Φ_* is itself Bosonic linear channel satisfying the condition of Lemma 2. It is sufficient to prove that arbitrary positive trace-class operator τ in \mathcal{H}_E can be approximated in the trace norm by operators of the form $\Phi_*[\tau_n], \tau_n \in \mathfrak{T}(\mathcal{H}_A)$. By the Parceval identity for the noncommutative Fourier transform, the function $\phi_{\sqrt{\tau}}(z)$ is square integrable. Denote by \mathcal{C} the class of infinitely differentiable functions with finite support. Let $\{\phi_n(z)\}\subset \mathcal{C}$ be a sequence converging to $\phi_{\sqrt{\tau}}(z)$ in $L^2(Z)$. Consider the operators

$$\sigma_n = \frac{1}{(2\pi)^s} \int \phi_n(z) W(-z) d^{2s} z.$$

We can assume that $\phi_n(-z) = \overline{\phi_n(-z)}$ so that σ_n is Hermitean. By the Parceval identity for the noncommutative Fourier transform, the Hilbert-Schmidt norms $\|\sqrt{\tau} - \sigma_n\|_2 \longrightarrow 0$. By using the inequality $\|AB\|_1 \le \|A\|_2 \|B\|_2$ we conclude $\|\tau - \sigma_n^2\|_1 \longrightarrow 0$. The function $\varphi_n(z) = \text{Tr}\sigma_n^2 W(z)$ is twisted convolution of two functions $\phi_n(z)$ and hence also belongs to \mathcal{C} . Then by change of variables

$$\sigma_n^2 = \frac{1}{(2\pi)^s} \int \varphi_n(z) W(-z) d^{2s} z$$

=
$$\frac{1}{(2\pi)^s |\det L|} \int \varphi_n(L^{-1}z) W(-L^{-1}z) d^{2s} z = \tilde{\Phi}_*[\tau_n],$$

where $\tau_n = \frac{1}{(2\pi)^s} \int \frac{\varphi_n(L^{-1}z)}{f(L^{-1}z)} W(-z) d^{2s}z$ and f(z) is given by (15), so it does not vanish and is L^2 -differentiable to the order 2s by the assumption of Theorem. Hence the function $\frac{\varphi_n(L^{-1}z)}{f(L^{-1}z)}$ is well defined, finitely supported and also

 L^2 -differentiable to the order 2s. It remains to show that τ_n is a trace class operator.

Lemma 4. If ϕ_{τ} has finite support and is L^2 -differentiable of the order 2s then $\tau \in \mathfrak{T}(\mathcal{H})$.

Proof. By using the formula (see [8], Lemma V.4.2),

$$\phi_{\tau(Rw)}(z) = \left[-\frac{1}{2} z^{\mathsf{T}} \Delta w - \nabla_w \right] \phi_{\tau}(z), \quad w, z \in \mathbb{Z},$$

we see that $\phi_{\tau(Rw)^{2s}}$ is square integrable. It follows that $\tau(Rw)^{2s}$ extends to a Hilbert-Schmidt operator, and similarly the operator $\sigma = \tau\left(2N_1+1\right)\ldots\left(2N_s+1\right)$, where $2N_j+1=(Re_j)^2+(Rh_j)^2=q_j^2+p_j^2$ for a symplectic basis $\{e_j,h_j\}_{j=1,\ldots,s}$ in Z. Here N_j is the number operator of j-th mode which is selfadjoint with the eigenvalues $n_j=0,1,\ldots$, and the operators N_1,\ldots,N_s commute. Therefore $\sigma^*\sigma$ is a positive trace class operator. From this we conclude

$$\operatorname{Tr} \sigma^* \sigma = \sum_{n_1, \dots n_s} (2n_1 + 1)^2 \dots (2n_s + 1)^2 \langle n_1, \dots n_s | \tau^* \tau | n_1, \dots n_s \rangle < \infty,$$

where $\{|n_1, \ldots n_s\rangle\}$ is the orthonormal basis of common eigenvectors of operators N_1, \ldots, N_s . By using Cauchy-Schwarz inequality and the inequality $\langle \psi | |\tau| |\psi \rangle^2 \leq \langle \psi | \tau^* \tau | \psi \rangle$ for a unit vector ψ , we have

$$(\operatorname{Tr}|\tau|)^{2} = \left(\sum_{n_{1},\dots n_{s}} \langle n_{1},\dots n_{s} | |\tau| | n_{1},\dots n_{s} \rangle \right)^{2}$$

$$\leq \sum_{n_{1},\dots n_{s}} (2n_{1}+1)^{2} \dots (2n_{s}+1)^{2} \langle n_{1},\dots n_{s} | \tau^{*}\tau | n_{1},\dots n_{s} \rangle$$

$$\cdot \sum_{n_{1},\dots n_{s}} (2n_{1}+1)^{-2} \dots (2n_{s}+1)^{-2} < \infty,$$

Thus Lemma 4 and hence Lemma 3 are proved. Applying the Proposition 1 proves the Theorem. \Box

3 The case of Gaussian channels

The density operator (state) ρ is called Gaussian, if its characteristic function $\phi_{\rho}(z) = \text{Tr}\rho W(z)$ has the form

$$\phi_{\rho}(z) = \exp\left(il^{\top}z - \frac{1}{2}z^{T}\alpha z\right),\tag{24}$$

where α is a real symmetric $(2s) \times (2s)$ -matrix, called covariance (or correlation) matrix of ρ . The necessary and sufficient condition for α to be a covariance matrix is the inequality

$$\alpha \ge \frac{i}{2}\Delta,\tag{25}$$

where both parts are considered as complex Hermitian matrices. This is equivalent to the Δ -nonnegative definiteness of $\phi_{\rho}(z)$.

Proposition 2. Gaussian state ρ is pure if and only one of the following equivalent conditions holds:

- 1. α is a minimal (in the sense of partial ordering of real symmetric matrices) solution of the inequality (25);
- 2. the symplectic eigenvalues of the matrix α are all equal to their minimal possible value $\frac{1}{2}$;
- 3. Rank $\left(\alpha \frac{i}{2}\Delta\right) = s;$
- 4. $\alpha + \frac{1}{4}\Delta\alpha^{-1}\Delta = 0$;
- 5. $\alpha = -\frac{1}{2}\Delta J$, where J is an operator of complex structure in (Z, Δ) .

This statement is a collection of results scattered in literature; its proof is essentially based on Williamson's canonical form of the matrix α , see [8, 14, 11].

Bosonic Gaussian channel $\Phi = \Phi_{K,l,\mu}$ is Bosonic linear channel with Gaussian function f(z), namely

$$\Phi_{K,l,\mu}[W_B(z_B)] = W(Kz_B) \exp\left[il^{\top} z_B - \frac{1}{2} z_B^{\top} \mu z_B\right],$$
 (26)

Here K is real $2s_A \times 2s_B$ —matrix and μ is real symmetric $2s_B \times 2s_B$ —matrix satisfying the inequality

$$\mu \ge \frac{i}{2} \left[\Delta_B - K^{\mathsf{T}} \Delta_A K \right], \tag{27}$$

which is necessary and sufficient condition for complete positivity. The channel (26) has minimal noise if μ is a minimal solution of this inequality [14, 10].

Assuming the condition (13), let K_D be a solution of (14), then $\alpha_D = \left(K_D^{\top}\right)^{-1} \mu\left(K_D\right)^{-1}$ is real symmetric $2s_B \times 2s_B$ —matrix such that $\alpha_D \geq \frac{i}{2}\Delta_D$. Further, from minimality of μ it follows that α_D is a minimal solution of the inequality $\alpha_D \geq \frac{i}{2}\Delta_D$, and as such it is the covariance matrix of a pure centered (l=0) Gaussian state $\rho_D = |\psi_D\rangle\langle\psi_D|$ of the Bosonic system in the space \mathcal{H}_D corresponding to the standard symplectic space $(Z_D, \Delta_D) \simeq (Z_B, \Delta_B)$. Applying Theorem 1 to the case of Gaussian ρ_D we obtain

Corollary. Bosonic Gaussian channel is extreme if and only if it has the minimal noise.

Lemma 2 in the case of Gaussian channels amounts to the statement:

Let $\Phi_{K,l,\mu}$ be the Gaussian channel with the same input and output space, where K is a nondegenerate square matrix. Then the restriction of $\Phi_{K,l,\mu}$ onto $\mathfrak{T}(\mathcal{H})$ coincides with $|\det K|^{-1} \left(\Phi_{\hat{K},\hat{l},\hat{\mu}}\right)_{*}$, where

$$\left(\hat{K},\hat{l},\hat{\mu}\right) = \left(K^{-1},-\left(K^{-1}\right)^{\top}l,\left(K^{-1}\right)^{\top}\mu K^{-1}\right).$$

This generalizes the duality observed for the one-mode Gaussian channels in the canonical form in [5].

4 Appendix

Let \mathfrak{B} be a C^* -algebra and Φ a completely positive map from \mathfrak{B} to $\mathfrak{B}(\mathcal{H})$, where \mathcal{H} is a separable Hilbert space. Let

$$\Phi[X] = V^*\pi(X)V, \quad X \in \mathfrak{B},$$

be a minimal Stinespring representation for Φ , where V is a bounded operator from \mathcal{H} to \mathcal{K} , another separable Hilbert space, and π is a representation of \mathfrak{B} on \mathcal{K} . Let $\mathfrak{E}=\pi(\mathfrak{B})'$ be the commutant of the algebra $\pi(\mathfrak{B})$ in \mathcal{K} . Theorem 1.4.6 in [4] says that Φ is an extreme point of the convex set of completely positive maps Ψ normalized so that $\Psi[I]=N\equiv V^*V$ if and only if the subspace $\mathcal{L}=[V\mathcal{H}]\subseteq\mathcal{K}$ is separating for \mathfrak{E} , i.e. for $Y\in\mathfrak{E}$ the relation $P_{\mathcal{L}}Y|_{\mathcal{L}}=0$ implies Y=0.

Consider the map

$$\tilde{\Phi}[Y] = V^* Y V, \quad Y \in \mathfrak{E},$$

which is a completely positive map satisfying $\tilde{\Phi}[I] = N$, which may be called complementary to Φ (this possibility of generalizing the notion of complementary map was noticed by Ruskai [15]). Then it is easy to see that the above Arveson's criterion of extremality is equivalent to $\text{Ker}\tilde{\Phi} = 0$.

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