SEMISTABILITY OF RATIONAL PRINCIPAL GL_n-BUNDLES IN POSITIVE CHARACTERISTIC

LINGGUANG LI

ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H. Let E be a rational $GL_n(k)$-bundle on X, and $\rho : GL_n(k) \to GL_m(k)$ a rational $GL_n(k)$-representation at most degree d such that ρ maps the radical $R(GL_n(k))$ of $GL_n(k)$ into the radical $R(GL_m(k))$ of $GL_m(k)$. We show that if $F_X^{N*}(E)$ is semistable for some integer $N \geq \max_{0 < r < m} C_m^r \cdot \log_p(dr)$, then the induced rational $GL_m(k)$-bundle $E(GL_m(k))$ is semistable. As an application, if $\dim X = n$, we get a sufficient condition for the semistability of Frobenius direct image $F_{X*}(\rho_*(\Omega_X^1))$, where $\rho_*(\Omega_X^1)$ is the locally free sheaf obtained from Ω_X^1 via the rational representation ρ.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic, X a smooth projective variety over k with a fixed ample divisor H. Let G and G' be reductive algebraic groups over k, $\rho : G \to G'$ a homomorphism of algebraic groups. One of the important and essential problem in the studying of G-bundles is to study the behavior of the semistability of G'-bundles under the extension of structure group. In precise, let E be a semistable rational G-bundle on X, does the induced rational G'-bundle $E(G')$ is also semistable?

Suppose that ρ maps the radical of $R(G)$ into the radical $R(G')$ of G' (Unless stated otherwise, we always require this condition for any homomorphisms of algebraic groups and all representations are rational representations in this paper), and E is a semistable rational G-bundle on X. If char$(k) = 0$, S. Ramanan and A. Ramanathan [9, Theorem 3.18] showed that the induced rational G'-bundle $E(G')$ is also semistable on X. If char$(k) = p > 0$, the induced rational G'-bundle $E(G')$ may be not semistable in general. However, S. Ramanan and A. Ramanathan [9, Theorem 3.23] proved that strong semistability of rational G-bundle E implies the strong semistability of rational G'-bundle $E(G')$. In addition, S. Ilangovan, V. B. Mehta and A. J. Parameswaran [4] showed that if $G' = GL_m(k)$ for some integer $m > 0$ and $p > \text{ht}(\rho)$, then the induced rational $GL_m(k)$-bundle $E(GL_m(k))$ is semistable. F. Coiai and Y. I. Holla [2] generalized some results of [9] and showed that given a representation $\rho : G \to GL_m(k)$, there exists a non-negative integer N, depending only on G and ρ, such that for any rational G-bundle E whose N-th Frobenius pull back $F_X^{N*}(E)$ is semistable, then the induced rational $GL_m(k)$-bundle $E(GL_m(k))$ is again semistable. S. Gurjar and V. Mehta [3] improved the result of [2] and obtain a explicit bound for N in terms of certain numerical data attached to ρ. The main ingredient of the proof in [2] and [3] is to give a uniform bound for the field of definition of the instability parabolics (Kempf' parabolic) associated to non-semistable points in related representing space.

We now briefly describe the main idea of their proof. Fix a representation $G \to GL_m(k)$. Let E be rational G-bundle on X, $E(G)$ the group scheme over X associated to E, and $E(GL_m(k))$ the induced rational $GL_m(k)$-bundle under the extension of structure group via ρ. Let $k(X)$ be the function field of X, the generic fiber $E(G)_0$ of $E(G)$ is a group scheme over $\mathrm{Spec}(k(X))$. Let P be a maximal parabolic subgroup of $GL_m(k)$, $E(GL_m(k)/P)$ the associated $GL_m(k)/P$-fiber space over X, and $E(GL_m(k)/P)_0$ the generic fiber of $E(GL_m(k)/P)$. Then there is an $E(G)_0$-action on the smooth projective variety $E(GL_m(k)/P)_0$ over $k(X)$ which is linearized by a suitable very ample line bundle. If $E(GL_m(k))$ adimts a reduction of structure group to this maximal parabolic subgroup P, then we get a rational section $\sigma : U \to E(GL_m(k)/P)$, where U is an open subscheme of X with $\mathrm{codim}_X(X - U) \geq 2$. Restricting to the generic fiber gives a $k(X)$-rational point σ_0 of $E(GL_m(k)/P)_0$. In [9], it is shown that if σ_0 is a semistable point in $E(GL_m(k)/P)_0$ for the above $E(G)_0$-action, then the rational reduction σ does not violate the semistability of rational $GL_m(k)$-bundle $E(GL_m(k))$. Also, if σ_0 is not semistable for the above $E(G)_0$-action and its instability parabolic $P(\sigma_0)$, which is defined over $\overline{k(X)}$, is actually defined over $k(X)$, then again σ does not contradict the semistability of rational $GL_m(k)$-bundle $E(GL_m(k))$. In the case of characteristic 0, by the uniqueness of instability parabolic and it is invariant under the action of Galois group, then $P(\sigma_0)$ is actually defined over $k(X)$. This proves that $E(GL_m(k))$ is a semistable rational $GL_m(k)$-bundle. However, in the case of characteristic $p > 0$, $P(\sigma_0)$ may be not defined over $k(X)$, and it is defined over a finite extension of $k(X)$. By the uniqueness of instability parabolic, the Galois descent argument implies that $P(\sigma_0)$ is actually defined over finite purely inseparable field extension $K^{p^{-N}}$ of K for some non-negative integer N.

In [2] and [3], the authors showed that there exists a uniform bound N, depending only on G and ρ, such that for all possible rational reductions to all maximal parabolic subgroups the instability parabolics of points corresponding to these rational reductions are actually defined over $K^{p^{-N}}$ via different methods. This can be shown to imply that if E is a semistable rational G-bundle such that $F_X^{N*}(E)$ is semistable, then the induced rational $GL_m(k)$-bundle $E(GL_m(k))$ is also semistable. The major differences between the methods of [2] and that of [3] lie in the approach of estimating the field extension L of K such that a given K-scheme M has a L-rational point. F. Coiai and Y. I. Holla [2] proved the existence of the uniform bound by bounding the non-separability of the group action and the non-reducedness of the stabilizers of various unstable rational points. However, the above estimation does not seen quantifiable. On the other hand, S. Gurjar and V. Mehta [3] directly estimated the field of definition of the instability parabolics which is probably weaker than the method of [2], but it is quantifiable.

The paper is organized as follows.

In section 2, we recall some definitions and results about geometric invariant theory and rational principal bundles, such as the instability 1-PS, instability parabolics of non-semistable points, etc. These results can be found in [6] and [9].

In section 3, we mainly study the rationality of the instability parabolics of non-semistable points in $GL_n(k)$-representation spaces, and apply these results to the study of semistability of rational principal bundles under the extension of structure groups via a $GL_n(k)$-representation $\rho : GL_n(k) \to GL_m(k)$.

Theorem 1.1 (Theorem 3.2). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H, $\rho : GL_n(k) \to GL_m(k)$ a $GL_n(k)$-representation over k at most degree d. Let E be a rational $GL_n(k)$-bundle on X such that $F_X^{N*}(E)$ is semistable for some integer*

$$N \geq \max_{0 < r < m} C_m^r \cdot \log_p(dr).$$

Then the induced rational $GL_m(k)$-bundle $E(GL_m(k))$ is also semistable.

In section 4, we study the semistability of truncated symmetric powers $\mathrm{T}^l(\mathscr{E})$ which is obtained from a torsion free sheaf $\mathscr{E}$ under a special group representations. The main result is to give a sufficient condition for the semistability of $\mathrm{T}^l(\mathscr{E})$.

Theorem 1.2 (Theorem 4.4). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H, integer $0 \leq l \leq (p-1) \cdot \dim X$. Let $\mathscr{E}$ be a torsion free sheaf of rank n on X such that $F_X^{N*}(\mathscr{E})$ is semistable for some integer*

$$N \geq \max_{1 \leq r \leq N(p,n,l)} C_{N(p,n,l)}^r \cdot \log_p(lr),$$

where $N(p,n,l) = \sum_{q=0}^{l(p)} (-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq}$, and $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$. Then the torsion free sheaf $\mathrm{T}^l(\mathscr{E})$ is also a semistable sheaf.

In section 5, we study the semistability of the Frobenius direct image of $\rho_*(\Omega_X^1)$ which is obtained from Ω_X^1 via the a $GL_n(k)$-representation ρ.

Theorem 1.3 (Theorem 5.4). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k of dimension n with a fixed ample divisor H such that $\deg_H(\Omega_X^1) \geq 0$, $\rho : GL_n(k) \to GL_m(k)$ a $GL_n(k)$-representation at most degree d. Suppose that $F_X^{N*}(\Omega_X^1)$ is semistable for some integer*

$$N \geq \max_{\substack{1 \leq l \leq n(p-1) \\ 0 < r < m \cdot N(p,n,l)}} C_{m \cdot N(p,n,l)}^r \log_p((d+l)r).$$

Then the Frobenius direct image $F_{X*}(\rho_*(\Omega_X^1))$ is a semistable sheaf, where $\rho_*(\Omega_X^1)$ is the locally free sheaf obtained from Ω_X^1 via the representation ρ. Moreover, in this case, $F_{X*}(\rho_*(\Omega_X^1) \otimes \mathscr{L})$ is semistable for any line bundle $\mathscr{L} \in \mathrm{Pic}(X)$.

2. Geometric Invariant Theory and Rational Principal Bundles

2.1. **Geometric Invariant Theory.** Let K be a field, K_s the separable closure of K, $\overline{K}$ the algebraically closure of K. Let G be a connected reductive algebraic group over K, Then G has a maximal torus T defined over K (See [1, Proposition 7.10]), and T splits over K_s. Let $X_*(T)$ be the group of parameter subgroups (1-PS) of T, i.e., group homomorphisms of the multiplicative group G_m into T. Let N_T be the normalizer of T in G, then Weyl group $W_T := N_T/T$ of G with respect to T acts on $X_*(T)$ by conjugation. Fix an inner product $\langle, \rangle$ on $X_*(T)$ which is invariant under the action of Weyl group W_T as well as the Galois group $\mathrm{Gal}(K_s/K)$ (See Section 4 of [6]). Then we can define *norm* $\|\lambda\|$ of 1-PS $\lambda \in X_*(T)$ as $\langle \lambda, \lambda \rangle$. Let T' be another maximal torus of G, T is conjugate to T' by an element of G, and the isomorphism $T \to T'$ is well defined up to Weyl group action on T. Therefore the inner product $\langle, \rangle$ on $X_*(T)$ determines uniquely one in $X_*(T')$. Hence the norm of any 1-PS in G is well defined.

Let V be a finite dimensional $\overline{K}$-vector space, $\rho : G(\overline{K}) \to GL_{\overline{K}}(V)$ a representation of G. A vector $0 \neq v \in V$ is *semistable* for the $G(\overline{K})$-action if $0 \notin \overline{G(\overline{K}) \cdot v}$. One knows that this is equivalent to existence of a $G(\overline{K})$-invariant element $\phi \in S^m(V)$ for some $m > 0$ such that $\phi(v) \neq 0$. For a 1-PS λ of $G(\overline{K})$, V has a decomposition $V = \bigoplus V_i$, where $V_i = \{v \in V | \lambda(t)v = t^i v\}$. Define

$$m(\lambda, v) := \min\{i | v \text{ has a non-zero component in } V_i\},$$

$$\mu(\lambda, v) := \frac{m(\lambda, v)}{\|\lambda\|}.$$

For a 1-PS λ of $G(\overline{K})$, the associated subgroup $P(\lambda)$ of $G(\overline{K})$ is defined by

$$P(\lambda) : \{g \in G(\overline{K}) | \lim_{t \to 0} \lambda(t) \cdot g \cdot \lambda^{-1}(t) \text{ exists in } G(\overline{K})\},$$

which is a parabolic subgroup of $G(\overline{K})$.

For a non-semistable vector $v \in V$, define the *instability* 1-PS of v to be the 1-PS λ_v such that $\mu(\lambda_v, v) = \sup\{\mu(\lambda, v) | \lambda \in X_*(G(\overline{K}))\}$, which is not unique.

We now recall some basic facts in geometric invariant theory, which can be found in [6] and [9].

Lemma 2.1. [6][9] *Let K be a field, G a connected reductive algebraic group over K, $\rho : G(\overline{K}) \to GL_{\overline{K}}(V)$ a representation of $G(\overline{K})$ on a $\overline{K}$-vector space V of finite dimension. Let $v \in V$ be a non-semistable point for the $G(\overline{K})$-action. Then*

(1) *The function $\lambda \mapsto \mu(\lambda, v)$ on the set $X_*(G(\overline{K}))$ attains the maximum value.*
(2) *There is a unique instability parabolic $P(v)$ of v such that for any instability 1-PS λ of v, we have $P(v) = P(\lambda)$.*
(3) *For any maximal torus $T \subseteq P(v)$, there is a unique instability 1-PS λ_T of v such that $\lambda_T \subseteq T$.*
(4) *For $g \in G(\overline{K})$, λ is the instability 1-PS of v, then $g \cdot \lambda(t) \cdot g^{-1}$ is the instability 1-PS of $g \cdot v$, $\mu(\lambda, v) = \mu(g\lambda(t)g^{-1}, g \cdot v)$ and $P(g \cdot v) = g \cdot P(v) \cdot g^{-1}$.*
(5) *For an instability 1-PS λ of v, if λ is defined over an extension field L/K, then the instability parabolic $P(v)$ of v is also defined over L/K.*

2.2. **Rational Principal G-Bundles.** Let K be a field, G a reductive algebraic group over K, X a smooth projective variety over K with a fixed ample divisor H. A (*principal*) *G-bundle* on X is a X-scheme $\pi : E \to X$ with an G-action (acts on the right) and π is G-invariant and isotrivial, i.e., locally trivial in the étale topology.

If Y is a quasi projective G-scheme over K (on the left), the associated fibre bundle $E(Y)$ over X is the quotient $E \times_K F$ under the action of G given by

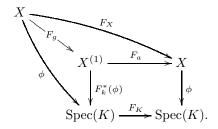$$g(e, y) = (e \cdot g, g^{-1} \cdot y), g \in G, e \in E, y \in Y.$$

Let M be a projective variety over K with a G-action, which is linearized by an ample line bundle $\mathscr{L}$ on X. Let $E(G) := E \times_{G, \text{Int}} G$ denote the group scheme over X associated to E by the action of G on itself by inner automorphisms. Then X-group scheme $E(G)$ acts naturally on the X-scheme $E(M)$ which is linearized by line bundle $E(\mathscr{L})$.

Let $x \in X$ be a point of X, $E(G)_x$, $E(M)_x$ and $E(\mathscr{L})_x$ denote the fiber of $E(G)$, $E(M)$ and $E(\mathscr{L})$ over x respectively. Then $E(G)_x$ is a group scheme over $\text{Spec}(k(x))$, and one has the action of $E(G)_x$ on $E(M)_x$ linearized by line bundle $E(\mathscr{L})_x$ which is defined over $\text{Spec}(k(x))$.

Let $P \subset G$ be a closed subgroup of G, a *reduction of structure group* of E to P is a pair $\sigma := (E_\sigma, \phi)$ with a P-bundle E_σ and an isomorphism of G-bundles $\phi : E_\sigma(G) \to E$. Note that quotient E/P is naturally isomorphic to the fiber bundle $E(G/P)$ on X and a section $\sigma : X \to E/P$ gives the P-bundle $\sigma^*(E)$ with natural isomorphism $\sigma^*(E)(G) \cong E$. This induces a bijection correspondence between sections of $E/P \to X$ and reductions of structure group of E to P. Let T_P be the tangent bundle along the fibers of the map $E/P \to X$, then $T_\sigma := \sigma^*(T_P)$ is the vector bundle on X associated to P-bundle E_σ for the natural representation of P on $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{p}$ is the Lie algebra of G and P respectively.

A *rational G-bundle* E on X is a G-bundle on a big open subscheme U of X, i.e. $\mathrm{codim}_X(X-U) \geq 2$. A *rational reduction of structure group* of a rational G-bundle E to a subgroup $P \subset G$ is a reduction σ of structure group of $E|_{U'}$ to P over a big open subscheme $U' \subseteq U$. Then the locally free sheaf T_σ on U' determine a reflexive sheaf $i_{U'*}(T_\sigma)$ where $i_{U'} : U' \to X$ is the natural open immersion, denoted by T_σ again. The rational G-bundle E is *semistable* if for any rational reduction σ of E to any parabolic subgroup P of G over any big open subscheme $U' \subseteq U$, the rational vector bundle T_σ has $\deg_H(T_\sigma) \geq 0$. If $G = GL_n(K)$, then there is an one to one correspondence between rational $GL_n(K)$-bundles between reflexive torsion free sheaves. In this case, the semistability of the rational $GL_n(K)$-bundle E is equivalent to the semistability of the torsion free sheaf $\mathscr{E}$ in the sense of Mumford-Takemoto, where $\mathscr{E}$ is the reflexive torsion free sheaf corresponds to E.

2.3. Frobenius Pull Back of Principal G-Bundles. Let K be a field of characteristic $p > 0$, $\phi : X \to \mathrm{Spec}(K)$ a scheme over k. The *absolute Frobenius morphism* $F_X : X \to X$ is induced by $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$. Consider the commutative diagram

The morphism F_a (resp. F_g) is called the *arithmetic Frobenius morphism* (resp. *geometric Frobenius morphism*) of $\phi : X \to \mathrm{Spec}(K)$.

If K is a perfect field, F_K and F_a are isomorphisms. Let G be an algebraic group over K, $\pi : E \to X$ a G-bundle over X. Pulling back by the absolute Frobenius F_X^* we get a G-bundle $F_X^*(\pi) : F_X^*(E) \to X_F$, where X_F is the scheme X endowed with the K-structure by the composition $X \overset{\phi}{\to} \mathrm{Spec}(K) \overset{F_K}{\to} \mathrm{Spec}(K)$. If K is a perfect field, we can change the K-structures of G, X_F and $F_X^*(E)$ by composing their structure morphisms with $\mathrm{Spec}(K) \overset{F_K^{-1}}{\to} \mathrm{Spec}(K)$ to get a $F_K^*(G)$-bundle $F_X^*(E) \to X$. In this case, the $F_K^*(G)$-bundle $F_X^*(E) \to X$ is the same as the bundle obtained from G-bundle E by the extension of structure group $g : G \to F_K^*(G)$.

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample line bundle H, G a reductive algebraic group over k. Let E be a rational G-bundle on X, then we can get a rational $F_K^*(G)$-bundle $F_X^*(E)$ on X. Then E is semistable when $F_X^*(E)$ is semistable. If $F_X^{m*}(E)$ is semistable for any integer $m \geq 0$, then E is called *strongly semistable*.

2.4. **Semistability of Rational G-bundles under Extension of Structure Groups.** Let k be an algebraically closed field, X a smooth projective variety over k with a fixed ample line bundle H, G a reductive algebraic group over k with a rational representation $\rho : G \to GL_m(k)$. Let E be a semistable rational G-bundle on X, we study the semistability of rational $GL_m(k)$-bundle $E(GL_m(k))$.

Let P be a maximal parabolic subgroup of $GL_m(k)$, then G acts on $GL_m(k)/P$ which is linearized by the very ample generator $\mathscr{L}$ of $\operatorname{Pic}(GL_m(k)/P)$. This gives an G-invariant embedding of $GL_m(k)/P$ inside projective space $\mathbb{P}(H^0(X, \mathscr{L}))$. Then X-group scheme $E(G)$ acts naturally on the X-scheme $E(GL_m(k)/P)$ which is linearized by line bundle $E(\mathscr{L})$.

Let $E(G)_0$, $E(GL_m(k)/P)_0$ and $E(\mathscr{L})_0$ be the fiber of $E(G)$, $E(GL_m(k)/P)$ and $E(\mathscr{L})$ over the generic point of X respectively. Then $E(G)_0$ is a group scheme over function field $\operatorname{Spec}(k(X))$, and one has the action of $E(G)_0$ on $E(GL_m(k)/P)_0$ linearized by line bundle $E(\mathscr{L})_0$ which is defined over $\operatorname{Spec}(k(X))$. Therefore there is an one to one correspondence between rational reductions of structure group of rational $GL_m(k)$-bundle $E(GL_m(k))$ to P and $k(X)$-rational points of $E(GL_m(k)/P)_0$.

Lemma 2.2. [9, Proposition 3.10, Proposition 3.13] *Let k be an algebraically closed field, X a smooth projective variety over k with an ample line bundle H. Let G be a reductive algebraic group over k, E a semistable rational G-bundle on X. Let $\sigma : U' \to E(GL_m(k)/P)$ be a rational reduction of structure group of $GL_m(k)$-bundle $E(GL_m(k))$ to a maximal parabolic subgroup P of $GL_m(k)$, where U' is a big open scheme of X. The associated $k(X)$-rational point in $E(GL_m(k)/P)_0$ is denoted by σ_0. Then*

- (1) *If σ_0 is a semistable point for the action of $E(G)_0$ on $E(GL_m(k)/P)_0$ linearized by line bundle $E(\mathscr{L})_0$, then $\deg_H T_\sigma \geq 0$.*
- (2) *If σ_0 is not a semistable point for the action of $E(G)_0$ on $E(GL_m(k)/P)_0$ linearized by line bundle $E(\mathscr{L})_0$, and its instability parabolic $P(\sigma)$ is defined over $k(X)$, then $\deg_H T_\sigma \geq 0$.*

3. The Semistability of Principal Bundles via GL_n-Representaions

Let K be an arbitrary field, $\rho : GL_n(K) \to GL_m(K)$ a representation of $GL_n(K)$ over K. Then ρ is given by an $m \times m$-matrix of regular functions

$$\frac{f_{ij}}{\det(T_{ij})^{a_{ij}}} \in K[T_{ij}, \det(T_{ij})^{-1}]_{1 \leq i,j \leq n},$$

where $a_{ij} \in \mathbb{N}$ and $f_{ij} \in K[T_{ij}]_{1 \leq i,j \leq n}$ with $(\det(T_{ij}), f_{ij}) = 1$. Denote

$$d := \max_{1 \leq i,j \leq n} \{\deg(f_{ij} + n(a - a_{ij}))\}, \text{ where } a := \max_{1 \leq i,j \leq n} \{a_{ij}\}.$$

We say ρ is a $GL_n(K)$-*representation over K of dimension m at most degree d*. Moreover, if $a_{ij} = 0$ for any $1 \leq i, j \leq n$, i.e., the above regular functions on $GL_n(K)$ lie in the subring $K[T_{ij}]_{1 \leq i,j \leq n}$, we say that ρ is a *polynomial representation* of $GL_n(K)$ over K.

In this section, we use a variant of method of S. Gurjar and V. Mehta [3] to give an explicit uniform bound for the field of definition of all instability parabolics of all non-semistable points in a given $GL_n(k)$-representation space V in terms of n, $\dim_k V$ and the maximal degree of regular functions correspond to the given

$GL_n(k)$-representation. Moreover, we use these results to the study of semistability of principal bundles under the extension of structure groups via $GL_n(k)$-representations.

Theorem 3.1. *Let K be an arbitrary field of characteristic $p > 0$, V a K-vector space of dimension m. Let $\rho : GL_n(K) \to GL_K(V)$ be a $GL_n(K)$-representation over K at most degree d. Then the instability parabolic of any unstable K-rational point in V is defined over $K^{p^{-N}}$ for some positive integer $N \geq m \cdot \log_p(d)$.*

Proof. By base change, the representation $\rho : GL_n(K) \to GL_K(V)$ over K induces a representation $\overline{\rho} : GL_n(\overline{K}) \to GL_{\overline{K}}(V_{\overline{K}})$ over $\overline{K}$ at most degree d. Fix a maximal torus T in $GL_n(\overline{K})$ which is defined over K. Denote $V_{\overline{K}} := V \otimes_K \overline{K}$, we can choose a simultaneous eigen basis $e_1, \ldots, e_m$ of $V_{\overline{K}}$ for all 1-PS of $GL_n(k)$ which lie in T.

Let $v \in V$ be a non-semistable K-rational point with respect to the $GL_n(\overline{K})$-action $\overline{\rho}$. Then there are $f_l \in K[T_{ij}]_{1 \leq i,j \leq n}$ of $\deg(f_l) \leq d$ and $a \in \mathbb{N}$, such that for any $\overline{K}$-rational point $g \in GL_n(\overline{K})$, we have

$$g \cdot v = \sum_{l=1}^m \frac{f_l(g)}{\det(g)^a} e_l.$$

Let $\lambda(t) \in X_*(GL_n(\overline{K}))$ be an instability 1-PS of v. Then there exists $\overline{K}$-rational point $g \in GL_n(\overline{K})$ such that $g \cdot \lambda(t) \cdot g^{-1} \subseteq T$. Then $g \cdot \lambda(t) \cdot g^{-1}$ is defined over K_s, the separable closure of K, and $g \cdot \lambda(t) \cdot g^{-1}$ is an instability 1-PS of $g \cdot v$ with $\mu(\lambda(t), v) = \mu(g \cdot \lambda(t) \cdot g^{-1}, g \cdot v)$.

Let $f_{l_1}, \ldots, f_{l_r}$ (resp. $f_{l_{r+1}}, \ldots, f_{l_m}$) denote the set of polynomials which vanish at g (resp. non-vanish at g). Consider the K-affine scheme

$$X := \mathrm{Spec}(K[T_{ij}]_{1 \leq i,j \leq n}/(f_{l_1}, \ldots, f_{l_r})).$$

Then g is a $\overline{K}$-rational point of $X(\overline{K}) \subseteq \mathbb{A}^{n \times n}(\overline{K})$ with

$$\det(g) \prod_{l=r+1}^m f_l(g) \neq 0.$$

Therefore, by [3, Lemma 10], there exists a finite extension field $L \subseteq \overline{K}$ of K with

$$[L : K] \leq \prod_{l=1}^r \deg(f_l) = d^r$$

such that X has a L-rational point g' in X and $\det(g') \prod_{l=r+1}^m f_l(g') \neq 0$. Thus $g' \in GL_n(L)$ and $g \cdot v$ and $g' \cdot v$ have the same set of monomials with non-zero coefficients when expanded in terms of the basis $e_1, \ldots, e_m$. Since $e_1, \ldots, e_m$ is a simultaneous basis for $g \cdot \lambda(t) \cdot g^{-1}$, so $\mu(g \cdot \lambda(t) \cdot g^{-1}, g \cdot v) = \mu(g \cdot \lambda(t) \cdot g^{-1}, g' \cdot v)$. Hence

$$\mu(g' \cdot \lambda(t) \cdot g'^{-1}, g' \cdot v) = \mu(\lambda(t), v) = \mu(g \cdot \lambda(t) \cdot g^{-1}, g \cdot v) = \mu(g \cdot \lambda(t) \cdot g^{-1}, g' \cdot v).$$

As $g' \cdot \lambda(t) \cdot g'^{-1}$ is an instability 1-PS of $g' \cdot v$, so $g \cdot \lambda(t) \cdot g^{-1}$ is also an instability 1-PS of $g' \cdot v$. It follows that $g'^{-1}(g \cdot \lambda(t) \cdot g^{-1})g'$ is an instability 1-PS of v which is defined over L. Thus, by Lemma 2.1, the instability parabolic $P(v)$ of v is defined over $L \cdot K_s$. By the Galois descent argument, any instability parabolic is defined over a purely inseparable extension of K. Suppose that for some positive integer N with $p^N \geq d^m \geq d^r$. Then $P(v)$ must be defined over $L \cap K^{p^{-N}}$. By the arbitrariness of non-semistable point $v \in V$, this theorem is followed. $\square$

Theorem 3.2. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H, $\rho : GL_n(k) \to GL_m(k)$ a $GL_n(k)$-representation over k at most degree d. Let E be a rational $GL_n(k)$-bundle on X such that $F_X^{N*}(E)$ is semistable for some integer*

$$N \geq \max_{0 < r < m} C_m^r \cdot \log_p(dr).$$

Then the induced $GL_m(k)$-bundle $E(GL_m(k))$ is also semistable.

Proof. Let V be a k-vector space of dimension m with a fixed basis $e_1, \ldots, e_n$. Then there is a natural isomorphism $GL_n(k) \cong GL_k(V)$ with respect to this basis of V. Let P be a maximal parabolic subgroup of $GL_k(V)$, then $GL_k(V)/P$ is isomorphic to the grassmannian $Grass(r, V)$ of r-dimensional subspaces of V for some integer $0 < r < \dim_k V$. This induces a $GL_n(k)$-equivalent embedding

$$GL_k(V)/P \hookrightarrow \mathbb{P}(\bigwedge^r V),$$

which can be lift to a $GL_n(k)$-representation

$$\rho^r : GL_n(k) \to GL_k(\bigwedge^r V).$$

Since $\rho : GL_n(k) \to GL_k(V)$ is a $GL_n(k)$-representation over k at most degree d, ρ^r is a $GL_n(k)$-representation over k at most degree dr. (In fact, $GL_n(k)$ acts on $GL_k(V)/P$ which is linearized by the very ample generator $\mathscr{L}$ of $\text{Pic}(GL_k(V)/P)$. This gives a $GL_n(k)$-invariant embedding of $GL_k(V)/P$ inside projective space $\mathbb{P}(H^0(X, \mathscr{L}))$, and a $GL_n(k)$-equivalent isomorphism $\bigwedge^r V \cong H^0(X, \mathscr{L})$.)

The X-group scheme $E(GL_n(k))$ acts naturally on the X-scheme $E(GL_k(V)/P)$ which is linearized by line bundle $E(\mathscr{L})$. Let $E(G)_0$, $E(GL_k(V)/P)_0$ and $E(\mathscr{L})_0$ denote the fiber of $E(G)$, $E(GL_k(V)/P)$ and $E(\mathscr{L})$ over the generic point of X respectively. Then $E(G)_0$ is a group scheme over function field $\text{Spec}(k(X))$, and one has an $E(G)_0$-action on $E(GL_k(V)/P)_0$ defined over $\text{Spec}(k(X))$, which is linearized by line bundle $E(\mathscr{L})_0$. One observes that $E(GL_n(k))_0$ and $E(GL_k(V)/P)_0$ became trivial after a finite separable field extension. Hence after change the base to $k(X)_s$, the separable closure of $k(X)$, we get canonical isomorphisms

$$E(GL_n(k))_0 \otimes_{k(X)} k(X)_s \cong GL_n(k) \otimes_k k(X)_s \cong GL_n(k(X)_s),$$

$$E(GL_k(V)/P)_0 \otimes_{k(X)} k(X)_s \cong (GL_k(V)/P) \otimes_k k(X)_s,$$

$$E(\mathscr{L})_0 \otimes_{k(X)} k(X)_s \cong \mathscr{L} \otimes_k k(X)_s,$$

and this isomorphisms being compatible with group actions. Let $V_s := V \times_k k(X)_s$. The $E(GL_n(k))_0 \otimes_{k(X)} k(X)_s$-action on $E(GL_k(V)/P)_0 \otimes_{k(X)} k(X)_s$ induces a $GL_n(k(X)_s)$-equivalent embedding

$$(GL_m(k)/P) \otimes_k k(X)_s \hookrightarrow \mathbb{P}(\bigwedge^r V) \otimes_k k(X)_s \cong \mathbb{P}(\bigwedge^r V_s),$$

which can be lift to a $GL_n(k(X)_s)$-representation

$$\rho_s^r : GL_n(k(X)_s) \to GL_{k(X)_s}(\bigwedge^r V_s).$$

In fact, ρ_s^r is just the base change of the $GL_n(k)$-representation ρ^r from k to $k(X)_s$. Hence ρ_s^r is a $GL_n(k(X)_s)$-representation over $k(X)_s$ of dimension C_m^r at most degree dr.

Let $\sigma : U \to E(GL_k(V)/P)$ be a rational reduction of structure group of E to P, where U is an open subscheme of X with $\operatorname{codim}_X(X - U) \geq 2$. It corresponds to a $k(X)$-rational point $\sigma_0 \in E(GL_k(V)/P)_0(k(X))$. Let T_σ denote the torsion free sheaf determined by the the locally free sheaf $\sigma^*(T_P)$ on U, where T_P is the tangent bundle along the fibers of the map $E(GL_k(V)/P) \to X$. By Lemma 2.2, if σ_0 is a semistable point for the action of $E(GL_n(k))_0$ on $E(GL_k(V)/P)_0$ linearized by line bundle $E(\mathscr{L})_0$, then

$$\deg_H T_\sigma \geq 0.$$

On the other hand, if σ_0 is not a semistable point for the action of $E(GL_n(k))_0$ on $E(GL_k(V)/P)_0$ linearized by line bundle $E(\mathscr{L})_0$, we would like to prove that

$$\deg_H T_\sigma \geq 0.$$

View σ_0 as an $k(X)_s$-rational point in $(GL_k(V)/P) \otimes_k k(X)_s$, and lift this point to a point in $\bigwedge^r V_s$, denote by σ_0 again. Then by Theorem 3.1, we have the instability parabolic $P(\sigma_0)$ of σ_0 is defined over $k(X)_s^{p^{-l}}$ for any integer $l \geq C_m^r \cdot \log_p(dr)$. Then $P(\sigma_0)$ is actually defined over $k(X)^{p^{-l}}$ by the uniqueness of the instability parabolic and Galois descent argument.

Pulling back by the Frobenius morphism, the action of the generic fibre

$$F_X^{l*}(E(GL_n(k)))_0 \cong F_{k(X)}^{l*}(E(GL_n(k))_0)$$

on

$$F_X^{l*}(E(GL_k(V)/P))_0 \cong F_{k(X)}^{l*}(E(GL_m(V)/P)_0)$$

is the base change by $F_{k(X)}^{l*}$ of the $E(GL_n(k))_0$-action on $(GL_k(V)/P)_0$. The Frobenius $F_{k(X)}^{l*}$ factors through an isomorphism:

$$
\begin{array}{ccc}
k(X) & \xrightarrow{F_{k(X)}^{l*}} & k(X) \\
\cong \Big\downarrow & \nearrow{\scriptstyle i_l} & \\
k(X)^{p^{-l}} & &
\end{array}
$$

where $i_l : \operatorname{Spec}(k(X)^{p^{-l}}) \to \operatorname{Spec}(k(X))$ is given by the inclusion $k(X) \subseteq k(X)^{p^{-l}}$. Therefore for this $F_{k(X)}^{l*}(E(GL_n(k))_0)$-action, the instability parabolic of the point $F_X^{l*}(\sigma)_0$ is defined over $k(X)$. Since $F_X^{l*}(E)$ is semistable, by Lemma 2.2, we have

$$\deg_H(T_{F_X^{l*}(\sigma)}) = \deg_H F_X^{l*}(T_\sigma) = p^l \cdot \deg_H(T_\sigma) \geq 0.$$

Thus $\deg_H T_\sigma \geq 0$. Therefore if $F_X^{N*}(E)$ is semistable for some integer

$$N \geq \max_{0 < r < m} C_m^r \cdot \log_p(dr),$$

then for any rational reduction σ of structure group of rational $GL_m(k)$-bundle $E(GL_m(k))$ to any maximal parabolic subgroup $P \subset GL_m(k)$, we have

$$\deg_H(T_\sigma) \geq 0.$$

Hence $E(GL_m(k))$ is a semistable rational $GL_m(k)$-bundle. This completes the proof of this theorem. $\qquad \square$

4. The Semistability of Truncated Symmetric Powers

The truncated symmetric powers was first introduced in [10] in order to study the semistability of Frobenius direct images. L. Li and F. Yu [8] have studied the instability of $T^l(\mathcal{E})$ and show that $T^l(\mathcal{E})$ is strongly semistable when $\mathcal{E}$ is strongly semistable. In this section, we would like to continue the further study of the semistability $T^l(\mathcal{E})$, and give a sufficient condition for semistability of $T^l(\mathcal{E})$.

Now, we recall the construction and properties of truncated symmetric powers of vector spaces (See [10, Section 3]).

Let K be an arbitrary field, V a n-dimensional K-vector space with standard representation of $GL_n(K)$. Let l be a positive integer, S_l the symmetric group of l elements with a natural action on $V^{\otimes l}$ by

$$(v_1 \otimes \cdots \otimes v_l) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)}$$

for any $v_i \in V$ and any $\sigma \in S_l$. Let $e_1, \cdots, e_n$ be a basis of V. For any non-negative partition $(k_1, \cdots, k_n)$ of l (i.e. $l = \sum_{i=1}^{n} k_i$, $k_i \geq 0$, $1 \leq i \leq n$), we define

$$v(k_1, \cdots, k_n) := \sum_{\sigma \in S_l} (e_1^{\otimes k_1} \otimes \cdots \otimes e_n^{\otimes k_n}) \cdot \sigma.$$

Let $T^l(V) \subset V^{\otimes l}$ be the linear subspace generated by all vectors

$$\{v(k_1, \cdots, k_n) \mid l = \sum_{i=1}^{n} k_i, \ k_i \geq 0, \ 1 \leq i \leq n\}.$$

Then $T^l(V)$ is a $GL_n(K)$-subrepresentation of $V^{\otimes l}$ with

$$N(p, n, l) := \dim_k T^l(V) = \sum_{q=0}^{l(p)} (-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq},$$

where $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$.

If $\mathrm{char}(K) = 0$ then we have $GL_n(K)$-equivalent $T^l(V) \cong \mathrm{Sym}^l(V)$ for any integer $l > 0$. On the other hand, if $\mathrm{char}(K) = p > 0$, then we have $GL_n(K)$-equivalent $T^l(V) \cong \mathrm{Sym}^l(V)$ when $0 < l < p$ and $T^l(V) = 0$ for $l > n(p-1)$.

Proposition 4.1. *For any integer $l > 0$, $T^l(V) = (V^{\otimes l})^{S_l}$.*

Proof. Fix a basis $e_1, \cdots, e_n$ of V. Let $(k_1, \cdots, k_n)$ be a non-negative partition of l, $W_{k_1, \cdots, k_n}$ the linear subspace of $V^{\otimes l}$ generated by vectors

$$\{e_{i_1} \otimes \cdots \otimes e_{i_l} \mid k_m = \sharp\{i_j | i_j = m, 1 \leq j \leq l\}, 1 \leq m \leq n\}.$$

Thus $W_{(k_1, \cdots, k_n)}$ is a S_l-invariant linear subspace of $V^{\otimes l}$.

By definition, it is obvious that $T^l(V) \subseteq (V^{\otimes l})^{S_l}$. It is easy to see that, any element in $(V^{\otimes l})^{S_l}$ can be expressed as the form

$$\alpha = \sum_{l = \sum_{i=1}^{n} k_i, k_i \geq 0} \alpha_{k_1, \cdots, k_n},$$

where $\alpha_{k_1, \cdots, k_n} \in W_{k_1, \cdots, k_n}$. Then we have $\alpha_{k_1, \cdots, k_n} \in (V^{\otimes l})^{S_l}$. In order to prove $\alpha \in T^l(V)$, it suffices to show that $\alpha_{k_1, \cdots, k_n} \in T^l(V)$. By simple observation, we have $\alpha_{k_1, \cdots, k_n} = a \cdot v(k_1, \cdots, k_n)$ for some $a \in k$. It follows that $(V^{\otimes l})^{S_l} \subseteq T^l(V)$. Hence $T^l(V) = (V^{\otimes l})^{S_l}$.   □

Proposition 4.2. *Let K be a field of characteristic $p > 0$, V a n-dimensional K-vector space. Then for any integer $0 < l \leq n(p-1)$, the $GL_K(V)$-representation $\rho^l : GL_K(V) \to GL_K(\mathrm{T}^l(V))$ is a polynomial representation at most degree l.*

Proof. Fix a basis $e_1, \cdots, e_n$ of V. Endowing the basis $\{e_{i_1} \otimes \cdots \otimes e_{i_l} \mid 1 \leq i_j \leq n\}$ of $V^{\otimes l}$ with lexicographic order. Under this basis, the $GL_K(V)$-representation $V^{\otimes l}$ is equivalent to the homomorphism of algebraic groups

$$\rho^l : GL_n(K) \to GL_{n^l}(K)$$

$$(g_{ij}) \mapsto (h_{st}) := ((g_{ij}) \otimes \cdots \otimes (g_{ij})).$$

Therefore ρ^l is given by a matrix of polynomials h_{st} in $K[T_{ij}]_{1 \leq i,j \leq n}$ of degree l for any integers $1 \leq s, t \leq n^l$. Then by the change of basis, under the basis $e_1, \cdots, e_n$,

$$\left(\text{resp. } \{v(k_1, \cdots, k_n) \mid l = \sum_{i=1}^{n} k_i, \ 0 \leq k_i < p, \ 1 \leq i \leq n\} \right)$$

of V (resp. $\mathrm{T}^l(V)$), the $GL_K(V)$-representation $\rho^l : GL_K(V) \to GL_K(\mathrm{T}^l(V))$ is also give by a matrix of polynomials f_{ij} in $K[T_{ij}]_{1 \leq i,j \leq n}$ of degree l for any integers $1 \leq i, j \leq \dim \mathrm{T}^l(V)$. Hence $\rho^l : GL_K(V) \to GL_K(\mathrm{T}^l(V))$ is a polynomial representation at most degree l. $\square$

Corollary 4.3. *Let K be a field of characteristic $p > 0$, V a n-dimensional K-vector space, l an integer with $0 < l \leq n(p-1)$. Then the instability parabolic of any unstable K-rational point in the $GL_K(V)$-representation space $\mathrm{T}^l(V)$ is defined over $K^{p^{-N}}$ for some integer $N \geq \log_p(l) \cdot \dim_K \mathrm{T}^l(V)$, i.e.,*

$$N \geq \log_p(l) \cdot \sum_{q=0}^{l(p)} (-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq},$$

where $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$.

Proof. By Proposition 4.2, we have $\rho^l : GL_K(V) \to GL_K(\mathrm{T}^l(V))$ a polynomial representation of $GL_K(V)$ at most degree l. Hence by Theorem 3.1, if an integer $N \geq (\dim \mathrm{T}^l(V)) \cdot \log_p(l)$, then the instability parabolic of any unstable K-rational point in $\mathrm{T}^l(V)$ is defined over $K^{p^{-N}}$. $\square$

Let X be a smooth variety over an algebraically closed field k of characteristic $p > 0$, $\mathscr{E}$ a locally free sheaf of rank n on X. Then the locally free sheaf $\mathrm{T}^l(\mathscr{E}) \subset \mathscr{E}^{\otimes l}$ is defined to be the sheaf of sections of the associated vector bundle of the frame bundle of $\mathscr{E}$ through the representation $\mathrm{T}^l(V)$ (See [10, Definition 3.4]).

Theorem 4.4. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H. Let $\mathscr{E}$ be a torsion free sheaf of rank n on X such that $F_X^{N*}(\mathscr{E})$ is semistable for some integer*

$$N \geq \max_{1 \leq r \leq N(p,n,l)} C_{N(p,n,l)}^r \cdot \log_p(lr),$$

where $N(p,n,l) = \sum_{q=0}^{l(p)} (-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq}$, and $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$. Then the torsion free sheaf $\mathrm{T}^l(\mathscr{E})$ is also a semistable sheaf.

Proof. Let V be a n-dimensional k-vector space. Then by Proposition 4.2, the $GL_K(V)$-representation on $\mathrm{T}^l(V)$ is a polynomial representation of dimension $N(p,n,l)$ at most degree l. Hence this Proposition followed by Theorem 3.2. $\square$

5. Semistability of Frobenius direct image of $\rho_*(\Omega_X^1)$

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective surface with a fixed ample divisor H such that Ω_X^1 is semistable with $\deg_H(\Omega_X^1) > 0$, and $\mathscr{L} \in \mathrm{Pic}(X)$. Y. Kitadai, H. Sumihiro [7, Theorem 3.1] showed that the $F_{X*}(\Omega_X^1)$ is semistable. Moreover, X. Sun [11, Theorem 4.11] generalized the [7, Theorem 3.1] and showed that $F_{X*}(\Omega_X^1)$ is stable if Ω_X^1 is stable. In this section, we would like to study the semistability of Frobenius direct image of $\rho_*(\Omega_X^1)$ on higher dimensional base space, where $\rho_*(\Omega_X^1)$ is the locally free sheaf obtained from Ω_X^1 via a $GL_n(k)$-representation ρ.

Lemma 5.1. [10, Theorem 4.8] *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k of dimension n with a fixed ample divisor H such that $\deg_H(\Omega_X^1) \geq 0$. Let $\mathscr{E}$ be a torsion free sheaf on X such that $\mathscr{E} \otimes \mathrm{T}^l(\Omega_X^1)$, $1 \leq l \leq n(p-1)$, are semistable sheaves. Then the Frobenius direct image $F_{X*}(\mathscr{E})$ is a semistable sheaf.*

Lemma 5.2. *Let K be a field, V, W_1 and W_2 are k-vector spaces. Let*

$$\rho_l : GL_K(V) \to GL_K(W_l)$$

be $GL_K(V)$-representation at most degree d_l, $l = 1, 2$. Then the $GL_K(V)$-representation

$$\rho_1 \otimes \rho_2 : GL_K(V) \to GL_K(W_1 \otimes W_2)$$

at most degree $d_1 + d_2$.

Proof. The $GL_K(V)$-representation ρ_l is given by a matrix M_l of regular functions

$$\frac{f_{ij}^{(l)}}{\det(T_{ij})^{a_l}} \in K[T_{ij}, \det(T_{ij})^{-1}]_{1 \leq i,j \leq \dim_K V},$$

where $a_l \in \mathbb{N}$ and $f_{ij}^{(l)} \in K[T_{ij}]_{1 \leq i,j \leq \dim_K V}$ with $\deg(f_{ij}^{(l)}) \leq d_l$, $l = 1, 2$. Then $GL_K(V)$-representation

$$\rho_1 \otimes \rho_2 : GL_K(V) \to GL_K(W_1 \otimes W_2)$$

is determined by the matrix $M_1 \otimes M_2$ of regular functions on $GL_K(V)$. Hence it is at most degree $d_1 + d_2$. $\square$

Proposition 5.3. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k with a fixed ample divisor H, and $\rho : GL_n(k) \to GL_m(k)$ a $GL_n(k)$-representation at most degree d. Let $\mathscr{E}$ be a torsion free sheaf of rank n on X, and $F_X^{N*}(\mathscr{E})$ is semistable for some integer*

$$N \geq \max_{0 < r < m \cdot N(p,n,l)} C_{m \cdot N(p,n,l)}^r \log_p((d+l)r).$$

where $N(p,n,l) = \sum_{q=0}^{l(p)} (-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq}$, and $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$. Then the torsion free part of $\rho_*(\mathscr{E}) \otimes \mathrm{T}^l(\mathscr{E})$ is also a semistable sheaf, where $\rho_*(\mathscr{E})$ is the torsion free sheaf obtained from $\mathscr{E}$ via the representation ρ.

Proof. Let V and W be k-vector spaces of dimension n and m with fixed bases respectively. Then there are natural isomorphisms $GL_n(k) \cong GL_k(V)$ and $GL_m(k) \cong GL_k(W)$ with respect to the given bases. The corresponding representation $\rho : GL_k(V) \to GL_k(W)$ is still denoted by ρ.

Let the torsion free sheaf $\mathscr{E}$ on X corresponds to the rational $GL_k(V)$-bundle E on X. Then the torsion free part of $\rho_*(\mathscr{E}) \otimes \mathrm{T}^l(\mathscr{E})$ corresponds to the rational $GL_k(W \otimes_k \mathrm{T}^l(V))$-bundle $E(GL_k(W \otimes_k \mathrm{T}^l(V)))$ which is obtained from E by extension of structure group to $GL_k(W \otimes_k \mathrm{T}^l(V))$ via the natural representation

$$\widetilde{\rho}^l := \rho^l \otimes \rho : GL_k(V) \to GL_k(W \otimes_k \mathrm{T}^l(V)).$$

One observes that $\widetilde{\rho}^l$ is a $GL_k(V)$-representation over k at most degree $d + l$ by Lemma 5.2. Hence, by Theorem 3.2, if $F_X^{N*}(\mathscr{E})$ is semistable for some integer

$$N \geq \max_{0 < r < m \cdot N(p,n,l)} C^r_{m \cdot N(p,n,l)} \log_p((d+l)r),$$

then $E(GL_k(W \otimes_k \mathrm{T}^l(V)))$ is a semistable rational $GL_k(W \otimes_k \mathrm{T}^l(V))$-bundle, i.e., the torsion free part of $\rho_*(\mathscr{E}) \otimes \mathrm{T}^l(\mathscr{E})$ is a semistable sheaf. $\qquad\square$

Theorem 5.4. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k of dimension n with a fixed ample divisor H such that $\deg_H(\Omega_X^1) \geq 0$, $\rho : GL_n(k) \to GL_m(k)$ a $GL_n(k)$-representation at most degree d. Suppose that $F_X^{N*}(\Omega_X^1)$ is semistable for some integer*

$$N \geq \max_{\substack{1 \leq l \leq n(p-1) \\ 0 < r < m \cdot N(p,n,l)}} C^r_{m \cdot N(p,n,l)} \log_p((d+l)r).$$

where $N(p,n,l) = \sum_{q=0}^{l(p)}(-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq}$, and $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$. Then the Frobenius direct image $F_{X*}(\rho_*(\Omega_X^1))$ is a semistable sheaf, where $\rho_*(\Omega_X^1)$ is the locally free sheaf obtained from Ω_X^1 via the representation ρ.

Proof. By Proposition 5.3, we have $\mathrm{T}^l(\Omega_X^1) \otimes \rho_*(\Omega_X^1)$ are semistable sheaves for all integers $1 \leq l \leq n(p-1)$. Then the Frobenius direct image $F_{X*}(\rho_*(\Omega_X^1))$ is a semistable sheaf by Lemma 5.1. $\qquad\square$

Corollary 5.5. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective variety over k of dimension n with a fixed ample divisor H such that $\deg_H(\Omega_X^1) \geq 0$, d and $m \in \mathbb{N}_+$. Suppose that $F_X^{N*}(\Omega_X^1)$ is semistable for some integer*

$$N \geq \max_{\substack{1 \leq l \leq n(p-1) \\ 0 < r < m \cdot N(p,n,l)}} C^r_{m \cdot N(p,n,l)} \log_p((d+l)r).$$

where $N(p,n,l) = \sum_{q=0}^{l(p)}(-1)^q \cdot C_n^q \cdot C_{n+l-q-1}^{l-pq}$, and $l(p)$ is the unique integer such that $0 \leq l - l(p) \cdot p < p$. Then

(1) *If $m = n^d$, then $F_{X*}((\Omega_X^1)^{\otimes d})$ is a semistable sheaf.*
(2) *If $m = C_{n+r-1}^{r-1}$, then $F_{X*}(\mathrm{Sym}^d(\Omega_X^1))$ is a semistable sheaf.*
(3) *If $m = C_n^d$, then $F_{X*}(\bigwedge^d(\Omega_X^1))$ is a semistable sheaf.*

Proof. It is obvious that for any integer $d > 0$, the natural $GL_k(V)$-action on $V^{\otimes d}$, $\mathrm{Sym}^d(V)$ and $\bigwedge^d(V)$ are all polynomial representation of degree d. Since $\mathrm{rk}((\Omega_X^1)^{\otimes d}) = n^d$, $\mathrm{rk}(\mathrm{Sym}^d(\Omega_X^1)) = C_{n+r-1}^{r-1}$ and $\mathrm{rk}(\bigwedge^d(\Omega_X^1)) = C_n^d$, this corollary follows from Theorem 5.4. $\qquad\square$

References

[1] A. Borel and T. A. Springer, Rationality properties of linear algebraic groups II, *Tohoku Math. J.* 20 (1968), 443-497.

[2] F. Coiai and Y. I. Holla, Extension of structure group of principal bundles in positive characteristic. *J. Reine Agnew. Math.* 595 (2006), 1-24.

[3] S. Gurjar and V. Mehta, Rationality of the instability parabolic and related results. *Transform. Groups* 20 (2015), no. 1, 99-112.

[4] S. Ilangovan, V. B. Mehta and A. J. Parameswaran, Semistability and semisimplicity in representations of low height in positive characteristic, *A tribute to C. S. Seshadri*, Perspectives in Geometry and Representation Theory, Hindustan Book Agency (2003), 271C282.

[5] K. Joshi, S. Ramanan, E. Z. Xia, J.-K. Yu, On vector bundles destabilized by Frobenius pull-back. *Compos. Math.* 142 (2006) 616-630.

[6] G. Kempf, Instability in invariant theory, *Ann. of Math.* 108 (1978), 299-316.

[7] Y. Kitadai, H. Sumihiro, Canonical filtrations and stability of direct images by Frobenius morphisms. II. *Hiroshima Math. J.* 38 (2008), no. 2, 243C261.

[8] L. Li and F. Yu, Instability of truncated symmetric powers of sheaves. *J. Algebra.* 386 (2013) 176-189.

[9] S. Ramanan and A. Ramanathan, Some remarks on the instability flag, *Tohoku Math. J.* 36 (1984), 269-291.

[10] X. Sun, Direct images of bundles under Frobenius morphisms. *Invent. Math.* 173 (2) (2008) 427-447.

[11] X. Sun, Frobenius morphism and semi-stable bundles. Algebraic geometry in East Asia-Seoul 2008, 161C182, *Adv. Stud. Pure Math.*, 60, Math. Soc. Japan, Tokyo, 2010.

School of Mathematical Sciences, Tongji University, Shanghai, P. R. China

E-mail address: LiLg@tongji.edu.cn