Upper Bounds on Sets of Orthogonal Colorings of Graphs

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Abstract

We generalize the notion of orthogonal latin squares to colorings of simple graphs. Two *n*-colorings of a graph are said to be *orthogonal* if whenever two vertices share a color in one coloring they have distinct colors in the other coloring. We show that the usual bounds on the maximum size of a certain set of orthogonal latin structures such as latin squares, row latin squares, equi-*n* squares, single diagonal latin squares, double diagonal latin squares, or sudoku squares are a special cases of bounds on orthogonal colorings of graphs.

Keywords: orthogonal latin squares, graph coloring, chromatic number

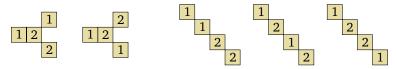
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1. Introduction

A *latin square of order* n is an $n \times n$ array filled with n symbols such that each symbol occurs exactly once in each row and column. Two latin squares of order n are said to be *orthogonal* if, when superimposed, each of the n^2 possible ordered pairs of symbols occur. A set of pairwise orthogonal latin squares is said to be a set of *mutually orthogonal latin squares* (MOLS).

We use the notation N(n) for the largest size of a set of MOLS of order n. It is well known that $N(n) \le n-1$. When n is a power of a prime one can construct a set of n-1 MOLS using finite fields, so when n is a prime power N(n) = n-1 (see [8, Ch. 2]). In this paper we introduce a generalization of the function N(n) that gives an upper bound on the cardinality of a set of pairwise orthogonal n-colorings of a graph.

Example 1. A *polyomino* is a (possibly disconnected) subset of cells from a square tiling of the plane. If we label the cells of a polyomino so that no symbol occurs more than once in any row or column, then we get a *latin polyomino*. As an example, below we exhibit five latin polyominoes in two polyomino shapes.



Note that the two latin polyominoes on the left are orthogonal in the sense that when they are superimposed each of the ordered pairs is distinct. Similarly, the three latin polyominoes on the right are mutually orthogonal. In each case we have exceeded the bound N(2) = 1 that occurs in the case where the polyomino shape is a square.

In this paper we consider latin squares as a special case of a proper *n*-coloring of a simple graph. We view orthogonality of latin squares as a special case of orthogonal colorings of a graph, a notion first studied in [4]. As a corollary to our main theorems we obtain bounds on the cardinality of orthogonal sets of several structures including equi-*n* squares, row or column latin squares, latin squares, single diagonal latin squares, double diagonal latin squares, and partial latin squares.

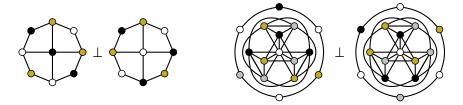
A graph, G, is a set of vertices, V(G), and a set of edges, E(G), where each edge is an unordered pair of distinct elements of V(G). The order of G is the number of vertices in V(G). Two vertices are said to be adjacent (or neighbors) if they are connected by an edge in E. We write $u \sim v$ to signify that u is adjacent to v. The degree of a vertex is the number of vertices adjacent to it. An r-clique is a (sub)graph of r pairwise adjacent vertices.

A *(proper) coloring* of G is a labeling of the vertices so that any two adjacent vertices have distinct labels (which we call *colors*). An *n-coloring* is a coloring consisting of (at most) n colors. It is usually convenient to assume these colors are the numbers $1, \ldots, n$. The *chromatic number* $\chi = \chi(G)$, of G is the minimum number, n, such that there exists an n-coloring of G. If G is a coloring of G, then we write G(v) for the color that G assigns to G.

A latin square is an n-coloring of an $n \times n$ rook's graph. An $m \times n$ rook's graph is a graph with mn vertices in m rows and n columns, where two vertices are adjacent if they share a row or a column.

Two colorings, C_1 and C_2 , of a graph, G, are said to be *orthogonal* if whenever two distinct vertices share a color in C_1 they have distinct colors in C_2 . A set of pairwise orthogonal n-colorings of G is called a set of *mutually orthogonal* n-colorings. We denote the maximum size of a set of mutually orthogonal n-colorings of the graph G by N(G, n).

Example 2. Below we display orthogonal colorings of a graph of order 9 and a graph of order 16.



A graph with a pair of orthogonal n-colorings can have at most n^2 vertices because there are at most n^2 distinct ordered pairs of labels. In fact, if any color shows up more than n times in a coloring, then the coloring has no orthogonal mate. At the other extreme we may have an overabundance of colors—if $n \ge |V(G)|$

then each vertex can be assigned a different color. Such a coloring is orthogonal to any coloring, including itself.

In the reference, [4], Caro and Yuster investigate the number of colors needed for a graph to have orthogonal colorings. The authors define the *orthogonal chromatic number of G*, $O\chi(G)$, to be the minimum number of colors in any pair of orthogonal $O\chi(G)$ -colorings of G. Similarly, the k-orthogonal chromatic number of G, $O\chi_k(G)$, is the minimum number of colors required so that there exist k mutually orthogonal $O\chi_k(G)$ -colorings of G. The authors find upper bounds on the values of $O\chi(G)$ and $O\chi_k(G)$ in terms of the parameters of G such as the maximum degree of any vertex of G or the chromatic number of G. They also show that several classes of graphs always have the lowest possible value $O\chi(G) = \left\lceil \sqrt{|V(G)|} \right\rceil$.

Here we are concerned with the function N(G, n), which is related to the function $O\chi_k(G)$ by the inequalities

$$N(G, O\chi_k(G)) \ge k$$
 and $O\chi_{N(G,n)}(G) \le n$.

By studying the function N(G, n) we obtain results that are complementary to the bounds of [4] in that they yield lower bounds for $O\chi_k(G)$. In Section 2.6 we state a few of these lower bounds for $O\chi_k(G)$ as corollaries to our main theorems.

Many authors have studied edge colorings of graphs as opposed to vertex colorings. A *proper edge coloring* of a graph is a labeling of the edges of a graph such that any two edges that share a vertex receive distinct labels. For each graph, G, the *line graph of G* is the graph with vertex set E(G), and two vertices are adjacent if and only if they share a vertex in G. Each edge coloring of G corresponds to a vertex coloring of the line graph of G in a natural way, so any question about edge coloring can be converted into a question about vertex coloring. The converse statement does not hold because most graphs are not the line graph of any graph.

Despite the fact that each edge coloring may be viewed as an example of vertex coloring, many questions about edge colorings are interesting in their own right and have been studied in the context of edge colorings. Of particular note to us is the article [2] by Archdeacon and Dinitz where they studied orthogonal edge colorings of graphs. In their article they viewed a latin square of order n as an edge coloring of the complete bipartite graph $K_{n,n}$. Results about orthogonal edge colorings can be converted to results about orthogonal vertex coloring by studying the corresponding line graph.

Remark 3. A *subgraph*, H, of a graph, G, is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It will always be the case that if H is a subgraph of G, then $N(H,n) \ge N(G,n)$ because mutually orthogonal colorings of G are also mutually orthogonal colorings of H.

Our main results are four theorems that give upper bounds for N(G, n), namely the *degree bound* of Theorem 4, the *clique bound* of Theorem 5, the *average-degree bound* of Theorem 6, and the *edge bound* of Theorem 7. The first two theorems give bounds that depend on a large number of edges locally.

Theorem 4 (degree bound). Let G be a graph of order $n^2 - m$ where $0 \le m < n - 1$, and let Δ be the maximum degree of any vertex of G. If $\Delta \ge n^2 - n$, then $N(G, n) \le 1$. Otherwise

$$N(G,n) \leq \left\lfloor \frac{n^2 - m - \Delta - 1}{n - m - 1} \right\rfloor.$$

Theorem 5 (clique bound). Let r, s, and n be integers satisfying 1 < r, $s \le n < r + s$. Let G be a graph with an s-clique, A, that is disjoint from an r-clique, B, such that each vertex in B is adjacent to at least j vertices in A. Then

$$N(G,n) \leq \left\lfloor \frac{r(s-j)}{r+s-n} \right\rfloor.$$

The next two theorems give upper bounds on N(G, n) based on the number of edges globally.

Theorem 6 (average-degree bound). Let G be a graph of order v with average vertex degree D. Then for n < v,

$$N(G,n) \leq \left\lfloor \frac{(\nu-D-1)n}{\nu-n} \right\rfloor.$$

Theorem 7 (edge bound). *Let G be a graph with v vertices and e edges, v > n. Write* v = qn + r *with* $0 \le r < n$. *Then*

$$N(G,n) \leq \left\lfloor \frac{\binom{\nu}{2} - e}{(n-r)\binom{\lfloor \frac{\nu}{n} \rfloor}{2} + r\binom{\lceil \frac{\nu}{n} \rceil}{2}} \right\rfloor.$$

If D is the average degree of the vertices of G, and e is the number of edges in G, then $D=\frac{2e}{|V(G)|}$, so the edge bound and the average-degree bound depend on the same information. In fact the edge bound will always perform as well as the average-degree bound. We have included the average-degree bound because of its simplicity and because it is sufficiently strong to give tight bounds for most latin structures.

In Section 2 we provide several corollaries to Theorems 5–7. We shall show that the best possible upper bounds on the cardinality of sets of mutually orthogonal latin structures can be derived from the four bounds we have introduced. Table 1 on page 5 summarizes many of the results of Section 2 by showing the upper bounds on the cardinality of sets of orthogonal latin structures.

In Section 2.5 we use an upper bound on $O\chi_k(G)$ from [4] to obtain a lower bound for N(G, n). In Section 2.6 we obtain lower bounds on the k-orthogonal chromatic number $O\chi_k(G)$. In Section 3 we prove each of the four bounds. In Section 4 we introduce some graph constructions that arise from sets of mutually orthogonal colorings of a graph.

Table 1: We list the upper bounds for N(G,n) (the maximum cardinality of mutually orthogonal sets of latin structures). Upper bounds that are known to be best possible are in bold.

	Degree Bound	Average- Degree Bound	Clique Bound	Edge Bound
Graph G	$\left\lfloor \frac{n^2 - m - \Delta - 1}{n - m - 1} \right\rfloor$	$\left\lfloor \frac{(\nu - D - 1)n}{\nu - n} \right\rfloor$	$\left\lfloor \frac{r(s-j)}{r+s-n} \right\rfloor$	$\frac{\binom{v}{2} - e}{(n-r)\binom{\lfloor \frac{v}{n} \rfloor}{2} + r\binom{\lceil \frac{v}{n} \rceil}{2}}$
equi- <i>n</i> square	n + 1	n+1	_	n+1
row or column latin square of order <i>n</i>	n	n	n	n
latin square of order <i>n</i>	n – 1	n – 1	n – 1	n – 1
single diagonal latin square of order <i>n</i>	n – 2	n – 2	n – 2	n – 2
double diagonal latin square of odd order, $n > 3$	n – 3	n – 2	n – 3	n-2
sudoku square of order n	n – 2	n-2	$n-\sqrt{\overline{n}}$	n – 2
$m \times n$ latin rectangle $m \le n$	_	n – 1	n – 1	n – 1

2. Upper Bounds on sets of Orthogonal Latin Structures

We first define several structures that are related to latin squares. Then we show that the four bounds give the best possible value of N(G, n) for each structure G.

2.1. Latin Squares

An equi-n square is an $n \times n$ array such that each of n symbols occurs in exactly n cells. A row (resp. column) latin square of order n is an $n \times n$ array where each of n symbols occurs in each row (resp. column). A single diagonal latin square is a latin square where each symbol on the main diagonal is distinct. A double diagonal latin square is a single diagonal latin square where each symbol along the back diagonal is also distinct.

Each of these latin structures is a balanced n-coloring of a graph with n^2 vertices. Two vertices of this underlying graph are adjacent if the structure forbids that they share a color. For instance, two vertices in the underlying graph for a single diagonal latin square are adjacent if they are in the same row or column or if they are both in the main diagonal.

Corollary 8. Let $n \ge 2$ be a positive integer.

(1) No set of mutually orthogonal equi-n squares of order n consists of more than n+1 squares.

- (2) No set of mutually orthogonal row (or column) latin squares of order n consists of more than n squares.
- (3) No set of mutually orthogonal latin squares of order n consists of more than n-1 squares.
- (4) No set of mutually orthogonal single diagonal latin squares of order n consists of more than n-2 squares.
- (5) No set of mutually orthogonal double diagonal latin squares of odd order n > 3 consists of more than n 3 squares. (See [6])

Proof. To prove each part we shall use the degree bound with m = 0.

In an equi-n square, no cells are adjacent, so $\Delta=0$. For statement (1) the degree bound is $\frac{n^2-(0)-1}{n-1}=n+1$. In a row latin square the rows form an n-clique, and so each vertex has degree

In a row latin square the rows form an n-clique, and so each vertex has degree $\Delta = n - 1$. Thus for (2) the degree bound is $\frac{n^2 - (n-1) - 1}{n-1} = n$. In a latin square each vertex is adjacent to all others in the same row and

In a latin square each vertex is adjacent to all others in the same row and column, so each vertex has degree $\Delta = 2n - 2$. Thus for (3) the degree bound is $\frac{n^2 - (2n-2) - 1}{n} = n - 1$.

In a single diagonal latin square, the upper left cell is adjacent to each vertex in the top row, the left column, or the main diagonal, so it has degree $\Delta = 3n - 3$. In statement (4) we have a degree bound of $\frac{n^2 - (3n - 3) - 1}{n^2 - (3n - 3)} = n - 2$.

statement (4) we have a degree bound of $\frac{n^2 - (3n-3)-1}{n-1} = n-2$. In a double diagonal latin square of odd order the center cell is adjacent to $\Delta = 4n-4$ cells, so we have a degree bound of $\frac{n^2 - (4n-4)-1}{n-1} = n-3$ for (5).

2.2. Gerechte Designs and Sudoku Squares

A gerechte design of order n is an $n \times n$ array that has been partitioned into n regions of size n and filled with n symbols so that each row, column, and region contains each symbol exactly once. A sudoku square of order n^2 is a gerechte design of order n^2 where the partitioned regions are the n^2 subarrays (starting from the top left corner) of size $n \times n$.

As a corollary to Theorem 5 we get a bound on the size of a set of mutually orthogonal gerechte designs. The following bound was proved in [3, Corollary 2.2].

Corollary 9. Given a partition of an $n \times n$ array into regions S_1, \ldots, S_n , each of size n, the size of a set of mutually orthogonal gerechte designs for this partition is at most n-j, where j is the maximum size of the intersection of a row or column with one of the sets S_1, \ldots, S_n with j < n.

Proof. Let A be a row (or column). If S_i intersects A in j cells (j < n), then we let $B = A \setminus S_i$. Then with r = |B|, and s = t = n the clique bound limits the number of mutually orthogonal gerechte designs of order n with partition S_1, \ldots, S_n to $\frac{|B|(n-j)}{|B|+n-n} = n-j$.

Corollary 10. The maximum size of a set of mutually orthogonal sudoku squares of order n^2 is $n^2 - n$.

Proof. A sudoku square is a gerechte design. The upper left block intersects the top row in n cells, so the clique bound on the cardinality of a set of mutually orthogonal n^2 -colorings is precisely $n^2 - n$.

When n is a power of a prime the upper bound of Corollary 10 is always obtainable (see [3] or [9] for constructions). We also note that most of the results of Corollary 8 follow as a corollary to Theorem 5.

2.3. Latin Rectangles

Let $m \le n$. An $m \times n$ latin rectangle is defined to be an $m \times n$ array based on n symbols such that no symbol occurs more than once in any row or column. Latin rectangles are a natural extension of the notion of latin squares, and they have been studied extensively [5, 8].

Two $m \times n$ latin rectangles are said to be *orthogonal* if, when superimposed, the mn ordered pairs are distinct. An $m \times n$ latin square is precisely an n-coloring of an $m \times n$ rook's graph, and orthogonality of latin rectangles corresponds precisely with mutually orthogonal n-colorings of those graphs.

It is customary to let N(m,n) be the maximum cardinality of a set of mutually orthogonal latin rectangles (MOLR) of size $m \times n$. The following bound for mutually orthogonal latin rectangles is well-known [8].

Corollary 11.
$$N(m, n) \le n - 1$$
.

Proof. In the underlying graph, G, for an $m \times n$ latin rectangle each vertex shares a row with n-1 vertices and column with m-1 vertices. Thus we may apply the average-degree bound with D=m+n-2 and v=mn. Then

$$N(G,n) \le \left| \frac{(mn - (m+n-2) - 1)n}{mn - n} \right| = n - 1.$$

For $r \le s \le t$ a *latin rectangle of type* (r, s, t) is an $r \times s$ array filled with t distinct symbols such that no symbol occurs more than once in any row or column.

Latin rectangles of type (r,s,t) occur quite naturally. Any $r \times s$ rectangle inside a latin square is a latin rectangle of type (r,s,t) for some t. A latin rectangle of type (n,n,n) is precisely a latin square, and a latin rectangle of type (r,s,s) is an $r \times s$ latin rectangle.

Let N(r,s,t) be the maximum size of a set of MOLR of type (r,s,t). Then N(r,s)=N(r,s,s) because each $r\times s$ latin rectangle is based on s symbols. The bound $N(1,s)=N(1,s,t)=\infty$ is trivial because any two $1\times s$ latin rectangles are orthogonal. Therefore, we shall assume that r>1.

First we note some obvious bounds. Clearly $N(r,s,t) \ge N(r,s)$ because a rectangle based on s symbols is also based on t symbols because $s \le t$. In fact with t and t fixed, t is a nonincreasing function of t. With t and t fixed (and t is a nonincreasing function of t because a set of mutually orthogonal t is a rectangles yields a set of mutually orthogonal t is a rectangles by deleting a row. Similarly, with t and t fixed (and t is a nondecreasing function of t because a set of t is a nondecreasing function of t because a set of t is a nonlocal probability of t in t i

The following result is a direct application of the clique bound of Theorem 5.

Corollary 12. When t < 2s and r > 1,

$$N(r,s,t) \le \frac{s^2 - s}{2s - t}.$$

Proof. The top two rows of an $r \times s$ rectangle are s-cliques, A and B. Each vertex in B shares a column with a vertex of A, so we may take r = s and j = 1. The clique bound is $\frac{s^2 - s}{2s - t}$.

Note that we get the bound $N(r,s,s) \le s-1$ of Corollary 11 as a special case of Corollary 12. Also noteworthy is the case when t=s+1 where the maximum possible size of a collection of mutually orthogonal latin rectangles of type (r,s,s+1) is s. In particular, when s is one less than a power of a prime the bound is tight because we can use a complete set of s MOLS of order s+1 to obtain s mutually orthogonal latin rectangles of type (r,s,s+1) by deleting s+1-r rows and 1 column from each square.

2.4. Orthogonal Partial Latin Squares

A partial latin square of order n is an $n \times n$ of (possibly empty) cells such that each of n distinct symbols occurs at most once in each row and column. Thus a partial latin square is a partial n-coloring of an $n \times n$ rook's graph. Two partial latin squares are said to be orthogonal if whenever two cells share a symbol in one partial latin square, then in the other partial latin square the two cells do not share the same symbol (or at least one of the two cells is empty).

Mutually orthogonal latin polyominoes were studied in [1] by Abdel-Ghaffar, motivated by an application to computer databases. In particular, Abdel-Ghaffar defined $M_n(p)$ to be the maximum number of mutually orthogonal partial latin squares of order n with the same p cells filled. Theorem 1 of [1] gives the bound

$$M_n(p) \le \left\lfloor \frac{p(p-1)}{\lfloor p/n \rfloor (2p-n-n\lfloor p/n \rfloor)} \right\rfloor - 2.$$

It can be showed that this bound for $M_n(p)$ agrees precisely with the edge bound for a partial latin square of order n with p filled cells.

2.5. Lower Bounds for N(G, n)

While upper bounds for N(G,n) are nice, it would be ideal to have some lower bounds for N(G,n). To find a lower bound for N(G,n) we look to the work of Caro and Yuster. In [4] Caro and Yuster studied the the k-orthogonal chromatic number $O\chi_k(G)$, which is the minimum number, n, such that $N(G,n) \ge k$. They obtained upper bounds for $O\chi_k(G)$ that depend on parameters of the graph such as the maximum degree of a vertex or the chromatic number $\chi(G)$. Perhaps the most useful result from [4] is Theorem 1.1, part of which we display below.

Theorem 13 (Part of Theorem 1.1 of [4]). Let G be a graph with v vertices, and with maximum degree Δ . For $k \geq 2$ the following upper bound holds:

$$O\chi_k(G) \le \min\left\{2\sqrt{k-1}\max\{\Delta,\sqrt{\nu}\},(k-1)\left\lceil\frac{\nu}{\Delta+1}\right\rceil + \Delta\right\}.$$

We can obtain a lower bound for N(G, n) by reversing engineering the bound of Theorem 13.

Corollary 14. Let G be a graph with v vertices and maximum degree Δ . If $N(G, n) \ge 2$, then

$$N(G,n) \ge \max \left\{ \left(\frac{n}{2 \max\{\Delta, \sqrt{\nu}\}} \right)^2 + 1, \frac{n-\Delta}{\lceil \frac{\nu}{\Delta+1} \rceil} + 1 \right\}.$$

The lower bound of Corollary 14 is of use only when the maximum degree Δ is smaller than n. Theorems 1.2–1.4 of [4] give good bounds on $O\chi_k(G)$ when the chromatic number is small or if the graph, G, has a special structure like a t-partite graph or a d-degenerate graph. Theorems 1.2–1.4 of [4] can also be reverse engineered to obtain lower bounds for N(G,n), though they do not apply as widely, nor are they easy to write.

2.6. Lower Bounds for Orthogonal Chromatic Numbers

In this section we note that we can solve for n in each the four bounds for N(G,n) that we have introduced, and thus find lower bounds for $O\chi_k(G)$.

The following corollary is obtained from the average-degree bound of Theorem 6.

Corollary 15. Let G be a graph with v vertices with average degree D. Then

$$O\chi_k(G) \ge \left\lceil \frac{vk}{v - D - 1 + k} \right\rceil.$$

The next corollary is obtained from the clique bound of Theorem 5.

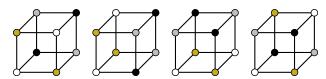
Corollary 16. Let G be a graph with an s-clique, A, that is disjoint from an r-clique, B, such that each vertex in B is adjacent to at least j vertices in A. If $O\chi_k(G) < r + s$, then we get the bound

$$O\chi_k(G) \ge \left\lceil r + s + \frac{rj - rs}{k} \right\rceil$$

We can also derive lower bounds for $O\chi_k(G)$ from the degree bound of Theorem 4 and the edge bound of Theorem 7, but the formulas are not as clean.

Example 17. Consider the cube graph, G, whose vertices and edges are the corners and edges of a cube. G has no pair of orthogonal 3 colorings. The degree bound is $N(G,3) \le 4$, while the clique bound, average-degree bound, and edge bound each yield the inequality $N(G,3) \le 2$, so none of these bounds are sharp in this

case. With 4 colors the average-degree bound and the edge bound are $N(G, 4) \le 4$. Moreover, we can find 4 mutually orthogonal 4-colorings of G.



Hence N(G, 4) = 4.

Corollary 15 yields the following lower bounds on the orthogonal chromatic numbers $O\chi(G) \ge 3$ and $O\chi_4(G) \ge O\chi_3(G) \ge 4$. The facts stated in the previous paragraph confirm that $O\chi(G) = O\chi_3(G) = O\chi_4(G) = 4$.

It is worth noting that we can obtain upper bounds on the k-orthogonal chromatic numbers of G from [4], namely $O\chi(G) \le 4$ (by Theorem 1.3 of [4]) $O\chi_3(G) \le 6$ (by Theorem 1.2 of [4]) and $O\chi_4(G) \le 9$ (by Theorem 1.1 or Theorem 1.4 of [4]).

3. Proofs of the Main Theorems

3.1. The Degree Bound

Proof of Theorem 4. Let *G* be a graph of order $n^2 - m$ where $0 \le m < n - 1$, and let u be a vertex of degree Δ . We must show

$$N(G,n) \le \left| \frac{n^2 - m - \Delta - 1}{n - m - 1} \right|.$$

Let C_1, \ldots, C_k be mutually orthogonal n-colorings of G. If $\Delta \ge n^2 - n$ then by the generalized pigeon hole principle at least one color must be applied to more than n vertices in C_1 , so no coloring of G may be orthogonal to C_1 . Thus if $\Delta \ge n^2 - n$, then $N(G,n) \le 1$.

Next assume that k > 1, so that no color may be applied to more than n vertices in any coloring C_i . Then, since no color may be applied to more than n vertices, by the generalized pigeon hole principle there must be at least n - m vertices that are assigned the color $C_i(u)$ in the coloring C_i . There are at most $n^2 - m - \Delta - 1$ vertices that can share a color with u in any coloring C_i . The n - m - 1 (or more) vertices of G that share a color with vertex u in C_i cannot share a color with u in C_j whenever $i \neq j$, so we have the inequality

$$k(n-m-1) \le n^2 - m - \Delta - 1.$$

Solving for *k* yields

$$k \le \frac{n^2 - m - \Delta - 1}{n - m - 1}.$$

3.2. The Clique Bound

Proof of Theorem 5. Let r, s, and n be integers satisfying 1 < r, $s \le n < r + s$. Let G be a graph with an s-clique, A, that is disjoint from an r-clique, B, such that each vertex in B is adjacent to at least j vertices in A. We must show that

$$N(G,n) \leq \left| \frac{r(s-j)}{r+s-n} \right|.$$

Let C_1, \ldots, C_k be mutually orthogonal n-colorings of G. We may assume without loss of generality that the vertices of A are colored with the same colors, say $1, \ldots, s$, so that each pair of colors $(1,1), (2,2), \ldots, (s,s)$ occurs in A for each pair of superimposed colorings C_i, C_j . Among the colorings C_1, \ldots, C_k , no vertex of B can have any color in the set $\{1, \ldots, s\}$ more than once, so colors from the set $\{1, \ldots, s\}$ can be applied to a vertex of B at most s-j times across all the colorings C_1, \ldots, C_k . Thus, colors from the set $\{s+1, \ldots, n\}$ will occur at least k-(s-j) times for each of the r vertices of B among the colorings C_1, \ldots, C_k . The total number of times that colors from the set $\{s+1, \ldots, n\}$ occur in B among the colorings C_1, \ldots, C_k is at least r(k-s+j).

On the other hand, at most n-s of the colors from the set $\{s+1,\ldots,n\}$ can occur in B for each coloring C_i . Thus the total number occurences of a color from the set $\{s+1,\ldots,n\}$ in B among the colorings C_1,\ldots,C_k is at most k(n-s). Therefore $r(k-s+j) \leq m(n-s)$ so that

$$k \le \frac{r(s-j)}{r+s-n}.$$

3.3. The Average-Degree Bound

Proof of Theorem 6. Let *G* be a graph of order v with average vertex degree *D*. We must show that for n < v,

$$N(G,n) \leq \left\lfloor \frac{(\nu-D-1)n}{\nu-n} \right\rfloor.$$

Let C_1,\ldots,C_k be mutually orthogonal n-colorings of G. Note that if each vertex shares a color with s vertices in each of the colorings C_1,\ldots,C_k then $ks \leq v-D-1$. Similarly if each vertex shares a color with s vertices on average across the colorings C_1,\ldots,C_k , then $ks \leq v-D-1$, so $k \leq \frac{v-D-1}{s}$. We shall find a lower bound for s and hence an upper bound for k.

Fix i and consider the coloring C_i . Let l_j be the number of times that color j is assigned to a vertex in C_i , so $l_1 + \cdots + l_n = v$. Then l_j vertices share a color with $l_j - 1$ vertices, so on average across all vertices, each vertex shares a color in C_i with s vertices on average, where

$$s = \frac{l_1(l_1-1) + l_2(l_2-1) + \dots + l_n(l_n-1)}{n\nu}.$$

Here *s* is weighted average of the values $\frac{l_1-1}{n}$, $\frac{l_2-1}{n}$,..., $\frac{l_n-1}{n}$, where the larger values are weighted more heavily, so

$$s \ge \frac{l_1 - 1}{n} + \frac{l_2 - 1}{n} + \dots + \frac{l_n - 1}{n} = \frac{v}{n} - 1.$$

Thus $\left(\frac{v}{n}-1\right)k+D \le v-1$ so

$$k \le \frac{(\nu - D - 1)n}{\nu - n}.$$

3.4. The Edge Bound

Proof of Theorem 7. Let *G* be a graph with ν vertices and e edges, $\nu > n$. We must show that if r is the remainder when ν is divided by n, then

$$N(G,n) \leq \left\lfloor \frac{\binom{v}{2} - e}{(n-r)\binom{\lfloor \frac{v}{n} \rfloor}{2} + r\binom{\lceil \frac{v}{n} \rceil}{2}} \right\rfloor.$$

Let C_1, \ldots, C_k be mutually orthogonal n-colorings of G. Each pair of vertices that share a color in C_i must have distinct colors in C_j , $j \neq i$. Once each pair of vertices has shared a color in one of the C_i 's it is not possible to find an additional n-coloring orthogonal to each of the C_i 's. Thus we need to identify the minimum number of pairs of vertices that will share a color in a given C_i . If a color is applied to m vertices, then that accounts for $\binom{m}{2}$ pairs. Therefore, if color t is applied to μ_t vertices, then

$$\binom{\mu_1}{2} + \binom{\mu_2}{2} + \dots + \binom{\mu_n}{2} \tag{1}$$

pairs of vertices are accounted for in the coloring C_i . The condition $\mu_1 + \cdots + \mu_n = \nu$ guarantees that the expression (1) is minimized when n-r of the colors appear $\lfloor \frac{\nu}{n} \rfloor$ times and the other r colors appear $\lceil \frac{\nu}{n} \rceil$ times. Therefore

$$k\left[(n-r)\binom{\left\lfloor\frac{\nu}{n}\right\rfloor}{2}+r\binom{\left\lceil\frac{\nu}{n}\right\rceil}{2}\right]+e\leq \binom{\nu}{2}.$$

Solving for *k* then yields the theorem.

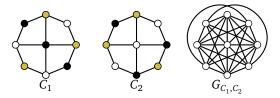
4. Constructions with Orthogonal Colorings

In this section we introduce a few graph structures that can be created from a set of orthogonal colorings.

4.1. Coloring-Extended Supergraphs

Given a graph G and a collection of colorings C_1, \ldots, C_k of G we can define a graph G_{C_1,\ldots,C_k} , which we call the *coloring-extended supergraph* to be the graph with vertex set V(G), and distinct vertices u and v are adjacent in G_{C_1,\ldots,C_k} if either $u \sim v$ in G or $C_i(u) = C_i(v)$ for some i.

Example 18. Below we display two orthogonal colorings, C_1 and C_2 , of a graph G and the supergraph G_{C_1,C_2} .

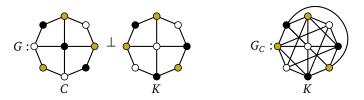


Theorem 19 (Caro and Yuster [4]). Let G be a graph with mutually orthogonal n-colorings C_1, \ldots, C_k . Then the collection, C_1, \ldots, C_k , of mutually orthogonal n-colorings of G can be extended if and only if there exists an n-coloring of the graph G_{C_1,\ldots,C_k} .

Proof. Let K be an n-coloring of G_{C_1,\dots,C_k} . Then K is also an n-coloring of G, and $(K(u),C_i(u))=(K(v),C_i(v))$ implies that K(u)=K(v) and $C_i(u)=C_i(v)$. However, the condition $C_i(u)=C_i(v)$ implies that $u \sim v$ in G_{C_1,\dots,C_k} , so the only possible way to satisfy K(u)=K(v) is if u=v. Thus K is orthogonal to each C_i .

For the converse we suppose that K is an n-coloring of G that is orthogonal to each of C_1, \ldots, C_k . If K(u) = K(v) then $u \not\sim v$ in G, and, since K is orthogonal to C_i , we must have $C_i(u) \neq C_i(v)$ for each i. Therefore K is also an n-coloring of G_{C_1, \ldots, C_k} .

Example 20. Below we display orthogonal colorings *C* and *K* of a graph *G*.



Note that K is also a coloring of the graph G_C .

In [4] Caro and Yuster made use of Theorem 19 to find an upper bound for the orthogonal chromatic number $O\chi(G)$.

Remark 21. When $n^2 \ge |G| = \nu \ge \frac{n^2}{2} + 2$ the clique bound can be applied to the graph G_{C_1} . In this case we obtain the bound

$$N(G,n) \leq \left| \frac{\left\lceil \frac{\nu}{n} \right\rceil \left\lceil \frac{\nu - \left\lceil \frac{\nu}{n} \right\rceil}{n-1} \right\rceil}{\left\lceil \frac{\nu}{n} \right\rceil + \left\lceil \frac{\nu - \left\lceil \frac{\nu}{n} \right\rceil}{n-1} \right\rceil - n} \right| + 1.$$

While this is a good bound, it does not outperform the edge bound, though it does match the edge bound for large values of ν when the initial graph G has few edges. In particular, when $\nu = n^2$ we get $N(G, n) \le n + 1$, so part (1) of Corollary 8 can also be viewed as a corollary of the clique bound.

4.2. MOLS via Mutually Orthogonal Colorings

A set of mutually orthogonal colorings of a graph can be used to create a set of MOLS under the right conditions. The existence of k-2 mutually orthogonal latin squares of order n is equivalent to the existence of k mutually orthogonal equi-n squares of order n. A set of k-2 mutually orthogonal latin squares of order n can be extended to a set of k mutually orthogonal equi-n squares by adding a square with constant rows and a square with constant columns. It is similarly possible to construct k-2 mutually orthogonal latin squares from a set of k mutually orthogonal equi-n squares. The following theorem and its proof provide a generalization of this construction using mutually orthogonal colorings of graphs.

Theorem 22. Let G be a graph of order n^2 . Suppose there exists a set of k mutually orthogonal n-colorings of G. Then there exists a set of k mutually orthogonal equi-n squares, k-1 mutually orthogonal row (or column) latin squares, k-2 MOLS of order n, and k-3 single diagonal latin squares.

Proof. We first construct a set of k-2 MOLS. Let C_1,\ldots,C_k be mutually orthogonal n-colorings of G. We can obtain a set of k-2 MOLS L_1,\ldots,L_{k-2} of order n by letting the entry in row i and column j of L_m be the color $C_m(v)$, where v is the unique vertex in G with color $C_{k-1}(v)=i$ and $C_k(v)=j$. To see that the squares L_1,\ldots,L_{k-2} are latin note that each row and column will have distinct entries because C_{k-1} and C_k are orthogonal colorings. Moreover, since C_1,\ldots,C_{k-2} are pairwise orthogonal, the squares L_1,\ldots,L_{k-2} will also be pairwise orthogonal.

To this set of MOLS we can add an equi-n square with constant columns (or constant rows) to obtain a set of k-1 row (or column) latin squares. Moreover, if we include both the square with constant rows and the square with constant columns we have a set of k mutually orthogonal equi-n squares.

To obtain a set of k-3 mutually orthogonal single diagonal latin squares we can identify a permutation, σ , that will permute the rows of L_{k-2} so that the main diagonal entries of $\sigma(L_{k-2})$ are equal. Then apply σ to each square L_1, \ldots, L_{k-3} . The resulting squares $\sigma(L_1), \ldots, \sigma(L_{k-2})$ will still be mutually orthogonal, and each of $\sigma(L_1), \ldots, \sigma(L_{k-3})$ will be a single diagonal latin square because no symbol will occur more than once down the main diagonal.

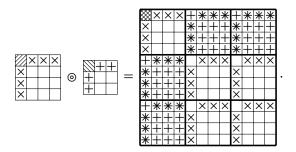
4.3. Or-Products

There are several ways to define the product of two graphs. The most commonly studied products are the tensor product graph, the cartesian product graph, the strong product graph, and the lexicographical product graph. We refer the reader to the reference [7] for definitions and results about these product graphs. Here we define yet another product graph.

Let G and H be graphs. We define the *or-product* of G and H to be the graph $G \otimes H$ that has vertex set $V(G) \times V(H)$ where vertices (u, v) and (u', v') are adjacent in $G \otimes H$ if either $u \sim u'$ in G or $v \sim v'$ in H. The or-product is commutative and associative and each of the product graphs named in the paragraph above is a subgraph of the or-product graph. (In fact $G \otimes H$ is the complement of the strong product of the complements of G and G.)

Example 23. The Kronecker product of matrices can be used to create a set of MOLS of order mn from a set of MOLS of order m together with a set of MOLS of order n. However, the $mn \times mn$ rook's graph is not the same as the graph that one obtains by taking the graph theoretic cartesian product of an $m \times m$ rook's graph and an $n \times n$ rook's graph (nor is it the same graph as any other product of rooks graphs).

Below we show a 4×4 rook's graph, G, with a vertex, \square , that is adjacent to each of the vertices of the form \boxtimes and a 3×3 rooks graph, H, with a vertex, \square , that is adjacent to each of the vertices of the form \boxplus . We have marked the vertices of the graph $G \odot H$ that are adjacent to the vertex $\boxtimes = (\square, \square)$.



The following result for graphs is a strengthening of the Kronecker product construction of mutually orthogonal latin squares.

Theorem 24. Let G and H be graphs and m and n be positive integers. If S is a subgraph of $G \odot H$, then

$$N(S, mn) \ge \min\{N(G, m), N(H, n)\}.$$

Proof. It is sufficient to show that

$$N(G \otimes H, mn) \ge \min\{N(G, m), N(H, n)\}.$$

We create mn colors by taking ordered pairs of colors where the first color is a color applied to vertices of G and the second color is a color applied to vertices

of H. We may then use a coloring, A, of G and a coloring, B, of H to obtain a coloring AB of $G \odot H$ by assigning to the vertex (u,v) the color (A(u),B(v)). To see that AB is a proper coloring of $G \odot H$ we note that if $(u,v) \neq (u',v')$ and AB(u,v) = AB(u',v') then A(u) = A(u') and B(v) = B(v'), so $u \nsim u'$ and $v \nsim v'$ which implies $(u,v) \nsim (u',v')$.

Now assume that we have mutually orthogonal colorings $A_1, ..., A_r$ of G and mutually orthogonal colorings $B_1, ..., B_r$ of H. We claim that the product colorings $A_1B_1, ..., A_rB_r$ are mutually orthogonal.

Suppose that for $i \neq j$ we have equal pairs of colors

$$(A_iB_i(u,v),A_iB_i(u,v)) = (A_iB_i(u',v'),A_iB_i(u',v')).$$

To conclude that A_iB_i is orthogonal to A_jB_j , we must show that (u,v)=(u',v'). Since $A_iB_i(u,v)=A_iB_i(u',v')$ then $A_i(u)=A_i(u')$ and $B_i(v)=B_i(v')$ while $A_jB_j(u,v)=A_jB_j(u',v')$ implies $A_j(u)=A_j(u')$ and $B_j(v)=B_j(v')$. Since A_i is orthogonal to A_j , the equality of the ordered pairs $(A_i(u),A_j(u))=(A_i(u'),A_j(u'))$ implies u=u'. Simililarly $(B_i(v),B_j(v))=(B_i(v'),B_j(v'))$ implies v=v'. Therefore the coloring A_iB_i is orthogonal to the coloring A_iB_i .

As a special case of Theorem 24 we obtain the well-known lower bound on the size of a set of MOLS that can be obtained via a Kronecker product construction [8].

Corollary 25. $N(mn) \ge \min\{N(m), N(n)\}$.

Proof. Let R_j denote the $j \times j$ rook's graph for each j. The graph, R_{mn} , is a subgraph of $R_m \odot R_n$ in a natural way, so by Theorem 24

$$N(R_{mn}, mn) \ge \min\{N(R_m, m), N(R_n, n)\}.$$

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