# Non-Isothermal Boundary in the Boltzmann Theory and Fourier Law

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#### Abstract

In the study of the heat transfer in the Boltzmann theory, the basic problem is to construct solutions to the following steady problem:

$$v \cdot \nabla_x F = \frac{1}{K_n} Q(F, F), \qquad (x, v) \in \Omega \times \mathbf{R}^3,$$
 (0.1)

$$F(x,v)|_{n(x)\cdot v<0} = \mu_{\theta} \int_{n(x)\cdot v'>0} F(x,v')(n(x)\cdot v')dv', \quad x \in \partial\Omega, \tag{0.2}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^d$ ,  $1 \leq d \leq 3$ ,  $K_n$  is the Knudsen number and  $\mu_{\theta}$  $\frac{1}{2\pi\theta^2(x)}\exp[-\frac{|v|^2}{2\theta(x)}]$  is a Maxwellian with non-constant (non-isothermal) wall temperature  $\theta(x)$ . Based on new constructive coercivity estimates for both steady and dynamic cases, for  $|\theta - \theta_0| \leq \delta \ll 1$  and any fixed value of K<sub>n</sub>, we construct a unique solution  $F_s$  to (0.1) and (0.2), continuous away from the grazing set and exponentially asymptotically stable. This solution is a genuine non equilibrium stationary solution differing from a local equilibrium Maxwellian. As an application of our results we establish the expansion  $F_s = \mu_{\theta_0} + \delta F_1 + O(\delta^2)$  and we prove that, if the Fourier law holds, the temperature contribution associated to  $F_1$  must be linear, in the slab geometry. This contradicts available numerical simulations, leading to the prediction of breakdown of the Fourier law in the kinetic regime.

### Contents

1	Introduction and notation	2
2	Background	<b>12</b>
3	$L^2$ Estimate	17
4	$L^{\infty}$ Estimate along the Stochastic Cycles	27
5	Well-posedness, Continuity and Fourier Law	43

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### 1 Introduction and notation

According to the Boltzmann equation (1.2), a rarefied gas confined in a bounded domain, in contact with a thermal reservoir modeled by (0.2) at a given constant temperature  $\theta$  (isothermal), has an equilibrium state described by the Maxwellian

constant 
$$\times e^{-|v|^2/2\theta}$$
,

and it is well known [35, 10, 18, 12, 39, 22, 38] that such an equilibrium is reached exponentially fast, at least if the initial state is close to the above Maxwellian in a suitable norm.

If the temperature  $\theta$  at the boundary is not uniform in (0.2), such a statement is not true and the existence of stationary solutions and the rate of convergence require a much more delicate analysis, because rather complex phenomena are involved. For example, suppose that the domain is just a slab between two parallel plates at fixed temperatures  $\theta_-$  and  $\theta_+$  with  $\theta_{-} < \theta_{+}$ . Then, one expects that a stationary solution is reached, where there is a steady flow of heat from the hotter plate to the colder one. The approach to the stationary solution may involve convective motions, oscillations and possibly more complicated phenomena. Even the description of the stationary solution is not obvious, and the relation between the heat flux and the temperature gradient, (e.g. the Fourier law (1.1)), is not a priori known. A first answer to such questions can be given confining the analysis to the small Knudsen number regime (i.e.,  $K_n \to 0$  in (0.1)), where the particles undergo a large number of collision per unit time and a hydrodynamic regime is established. In this case, it can be formally shown, using expansion techniques [9, 11], that the lowest order in  $K_n$  is a Maxwellian local equilibrium and the evolution is ruled by macroscopic equations, such as the Navier-Stokes equations. In particular, the heat flux vector q turns out to be proportional to the gradient of temperature, as predicted by the Fourier law

$$q = -\kappa(\theta)\nabla_x\theta\tag{1.1}$$

with the heat conductivity  $\kappa(\theta)$  depending on the interaction potential. This was first obtained by Maxwell and Boltzmann [28, 7] which relates the macroscopic heat flow to the microscopic potential of interaction between the molecules. The rigorous proof of such a statement was given in [16, 17] in the case of the slab geometry and provided that  $\theta_+ - \theta_-$  is sufficiently small (uniformly in the Knudsen number  $K_n$ ). This is a special case of a problem which has received recently a large attention in the Statistical Mechanics community, the derivation of the Fourier law from the microscopic deterministic evolutions ruled by the Newton or Schrödinger equation or from stochastic models [8, 30, 6, 1].

The aim of this paper is to analyze the thermal conduction phenomena in the kinetic regime. This problem was studied in the slab geometry, for small Knudsen numbers in [16, 17] and for large Knudsen number  $(K_n \to \infty)$  in [40]. Here we are interested in a general domain and in a regime where the Knudsen number  $K_n$  is neither small nor large. In this regime, the only construction of solutions to (0.1) and (0.2) we are aware of, was achieved in [4] in a slab for large  $\theta_+ - \theta_-$ , with  $L^1$  techniques closer in the spirit to the DiPerna-Lions renormalized solutions [13, 14] (see also [38] and references quoted therein). However, the uniqueness and

stability of such  $L^1$  solutions are unknown. To develop the quantitative analysis we have in mind, the theory of the solutions close to the equilibrium ([35, 36, 10, 22]) is better suited. For this reason we confine ourselves to temperature profiles at the boundary which do not oscillate too much. More precisely, we will assume that the temperature  $\theta(x)$  on  $\partial\Omega$  is given by  $\theta(x) = \theta_0 + \delta \vartheta(x)$  with  $\delta$  a small parameter and  $\vartheta(x)$  a prescribed bounded function on  $\partial\Omega$  such that

$$\sup_{x\in\partial\Omega}|\vartheta(x)|\leq 1.$$

This will allow us to use perturbation arguments in the neighborhood of the equilibrium at the uniform temperature  $\theta_0$ .

We shall consider the Boltzmann equation

$$\partial_t F + v \cdot \nabla F = \frac{1}{K_n} Q(F, F), \tag{1.2}$$

with F(t, x, v) the probability density that a particle of the gas at time t is in a small cell of the phase space  $\Omega \times \mathbf{R}^3$  centered at (x, v). Here  $\Omega$  is a bounded domain in  $\mathbf{R}^d$ , d = 1, 2, 3 with a smooth boundary  $\partial \Omega$ . The function F is required to be a positive function on  $\Omega \times \mathbf{R}^3$  such that  $\iint_{\Omega \times \mathbf{R}^3} F dx dv$  is fixed for any t. The right hand side of (1.2), Q(F, G), is the Boltzmann collision operator (non-symmetric)

$$Q(F,G) = \int_{\mathbf{R}^3} dv_* \int_{\mathbf{S}^2} d\omega B(v - v_*, \omega) F(v_*') G(v')$$

$$- \int_{\mathbf{R}^3} dv_* \int_{\mathbf{S}^2} d\omega B(v - v_*, \omega) F(v_*) G(v)$$

$$\equiv Q_{\text{gain}}(F, G) - Q_{\text{loss}}(F, G), \tag{1.3}$$

where  $B(v,\omega) = |v|^{\gamma} q_0 \left(\omega \cdot \frac{v}{|v|}\right)$  with  $0 \leq \gamma \leq 1$  (hard potential),  $0 \leq q_0(\omega \cdot \frac{v}{|v|}) \leq C|\omega \cdot \frac{v}{|v|}|$  (angular cutoff) is the collision cross section and  $v', v'_*$  are the incoming velocities in a binary elastic collision with outgoing velocities  $v, v_*$  and impact parameter  $\omega$ :

$$v' = v - \omega[(v - v_*) \cdot \omega], \quad v'_* = v_* + \omega[(v - v_*) \cdot \omega].$$
 (1.4)

The contact of the gas with thermal reservoirs is described by suitable boundary conditions. We confine ourselves to the simplest interesting case of the diffuse reflection (0.2), although more general boundary data could be studied [10]. On  $\partial\Omega$  (supposed to be a  $C^1$ -smooth surface with external normal n(x) well defined in each point  $x \in \partial\Omega$ ) we assume the condition:

$$F(t, x, v) = \mu_{\theta}(x, v) \int_{n(x) \cdot v' > 0} F(t, x, v') \{n(x) \cdot v'\} dv', \tag{1.5}$$

for  $x \in \partial \Omega$  and  $n(x) \cdot v < 0$ , where  $\mu_{\theta}$  is the Maxwellian at temperature  $\theta$ ,

$$\mu_{\theta}(x,v) = \frac{1}{2\pi\theta^2(x)} \exp\left[-\frac{|v|^2}{2\theta(x)}\right],\tag{1.6}$$

normalized so that

$$\int_{n(x)\cdot v>0} \mu_{\theta}(x,v) \{n(x)\cdot v\} dv = 1.$$
 (1.7)

Throughout this paper,  $\Omega$  is a connected and bounded domain in  $\mathbf{R}^d$ , for d=1,2,3 and the velocity  $\bar{v} \in \mathbf{R}^d$  and  $\hat{v} \in \mathbf{R}^{3-d}$  such that

$$v = (v_1, \dots, v_d, v_{d+1}, \dots, v_3) = (\bar{v}, \hat{v}).$$
 (1.8)

We denote the phase boundary in the phase space  $\Omega \times \mathbf{R}^3$  as  $\gamma = \partial \Omega \times \mathbf{R}^3$ , and split it into the outgoing boundary  $\gamma_+$ , the incoming boundary  $\gamma_-$ , and the grazing boundary  $\gamma_0$ :

$$\gamma_{+} = \{(x, v) \in \partial\Omega \times \mathbf{R}^{3} : n(x) \cdot v > 0\},$$
  

$$\gamma_{-} = \{(x, v) \in \partial\Omega \times \mathbf{R}^{3} : n(x) \cdot v < 0\},$$
  

$$\gamma_{0} = \{(x, v) \in \partial\Omega \times \mathbf{R}^{3} : n(x) \cdot v = 0\}.$$

The backward exit time  $t_{\mathbf{b}}(x,v)$  is defined for  $(x,v) \in \overline{\Omega} \times \mathbf{R}^3$ 

$$t_{\mathbf{b}}(x,v) = \inf\{\ t \ge 0 : x - t\bar{v} \in \partial\Omega\},\tag{1.9}$$

and  $x_{\mathbf{b}}(x,v) = x - t_{\mathbf{b}}(x,v)\bar{v} \in \partial\Omega$ . Furthermore, we define the singular grazing boundary  $\gamma_0^{\mathbf{S}}$ , a subset of  $\gamma_0$ , as:

$$\gamma_0^{\mathbf{S}} = \{(x, v) \in \gamma_0 : t_{\mathbf{b}}(x, -v) \neq 0 \text{ and } t_{\mathbf{b}}(x, v) \neq 0\},$$
 (1.10)

and the discontinuity set in  $\overline{\Omega} \times \mathbf{R}^3$ :

$$\mathfrak{D} = \gamma_0 \cup \{(x, v) \in \overline{\Omega} \times \mathbf{R}^3 : (x_{\mathbf{b}}(x, v), v) \in \gamma_0^{\mathbf{S}} \}. \tag{1.11}$$

We will use the short notation  $\mu_{\delta}$  for the Maxwellian

$$\mu_{\delta}(x,v) = \mu_{\theta_0 + \delta\vartheta(x)}(v) = \frac{1}{2\pi[\theta_0 + \delta\vartheta(x)]^2} \exp\left[-\frac{|v|^2}{2[\theta_0 + \delta\vartheta(x)]}\right]. \tag{1.12}$$

Moreover, to denote the global Maxwellian at temperature  $\theta_0$ ,  $\mu_{\theta_0}$ , we will simply use the symbol  $\mu$ :

$$\mu \equiv \mu_{\theta_0}$$
.

Since (1.7) is valid for all  $\delta$ , we have

$$\int_{n(x)\cdot v>0} \mu(v) \{n(x)\cdot v\} dv = 1.$$
 (1.13)

We denote by L the standard linearized Boltzmann operator

$$Lf = -\frac{1}{\sqrt{\mu}} [Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)] = \nu(v)f - Kf$$

$$= \nu(v)f - \int_{\mathbf{R}^3} \mathbf{k}(v, v_*) f(v_*) dv_*, \qquad (1.14)$$

with the collision frequency  $\nu(v) \equiv \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \mu(v_*) d\omega dv_* \sim \{1 + |v|\}^{\gamma}$  for  $0 \le \gamma \le 1$ . Moreover, we set

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2) \equiv \Gamma_{\text{gain}}(f_1, f_2) - \Gamma_{\text{loss}}(f_1, f_2).$$
 (1.15)

Finally, we define

$$P_{\gamma}f(x,v) = \sqrt{\mu(v)} \int_{n(x)\cdot v'>0} f(x,v') \sqrt{\mu(v')} (n(x)\cdot v') dv'.$$
 (1.16)

Thanks to (1.13),  $P_{\gamma}f$ , viewed as function on  $\{v \in \mathbf{R}^3 \mid v \cdot n(x) > 0\}$  for any fixed  $x \in \partial\Omega$ , is a  $L_v^2$ -projection with respect to the measure  $|n(x) \cdot v|$  for any boundary function f defined on  $\gamma_+$ .

We denote  $\|\cdot\|_{\infty}$  either the  $L^{\infty}(\Omega \times \mathbf{R}^3)$ -norm or the  $L^{\infty}(\Omega)$ -norm in the bulk, while  $|\cdot|_{\infty}$  is either the  $L^{\infty}(\partial\Omega \times \mathbf{R}^3)$ -norm or the  $L^{\infty}(\partial\Omega)$ -norm at the boundary. Also we adopt the Vinogradov notation:  $X \lesssim Y$  is equivalent to  $|X| \leq CY$  where C is a constant not depending on X and Y. We subscript this to denote dependence on parameters, thus  $X \lesssim_{\alpha} Y$  means  $|X| \leq C_{\alpha} Y$ . Denote  $\langle v \rangle = \sqrt{1 + |v|^2}$ . Our main results are as follows.

**Theorem 1.1** There exists  $\delta_0 > 0$  such that for  $0 < \delta < \delta_0$  in (1.12) and for all M > 0, there exists a non-negative solution  $F_s = M\mu + \sqrt{\mu}f_s \ge 0$  with  $\iint_{\Omega \times \mathbf{R}^3} f_s \sqrt{\mu} dx dv = 0$  to the steady problem (0.1) and (0.2) such that for all  $0 \le \zeta < \frac{1}{4+2\delta}$ ,  $\beta > 4$ ,

$$\|\langle v \rangle^{\beta} e^{\zeta |v|^2} f_s\|_{\infty} + |\langle v \rangle^{\beta} e^{\zeta |v|^2} f_s|_{\infty} \lesssim \delta.$$
 (1.17)

If  $M\mu + \sqrt{\mu}g_s$  with  $\iint_{\Omega \times \mathbf{R}^3} g_s \sqrt{\mu} dx dv = 0$  is another solution such that, for  $\beta > 4$ 

$$\|\langle v \rangle^{\beta} g_s\|_{\infty} + |\langle v \rangle^{\beta} g_s|_{\infty} \ll 1,$$

then  $f_s \equiv g_s$ . Furthermore, if  $\theta(x)$  is continuous on  $\partial\Omega$  then  $F_s$  is continuous away from  $\mathfrak{D}$ . In particular, if  $\Omega$  is convex then  $\mathfrak{D} = \gamma_0$ . On the other hand, if  $\Omega$  is not convex, we can construct a continuous function  $\vartheta(x)$  on  $\partial\Omega$  in (1.25) with  $|\vartheta|_{\infty} \leq 1$  such that the corresponding solution  $F_s$  in not continuous.

We stress that the solution  $F_s$  is a genuine non equilibrium steady solution. Indeed, it is not a local Maxwellian because it would not satisfy equation (0.1), nor a global Maxwellian because it would not satisfy the boundary condition (0.2).

We also remark that in addition to the wellposedness property, the continuity property in this theorem is the first step to understand higher regularity of  $F_s$  in a convex domain. The robust  $L^{\infty}$  estimates used in the proof enable us to establish the following  $\delta$ -expansion of  $F_s$ , which is crucial for deriving the necessary condition of the Fourier law (1.1). In the rest of this paper we assume the normalization M=1.

#### Theorem 1.2 Let

$$\mu_{\delta} = \mu + \delta \mu_1 + \delta^2 \mu_2 + \cdots, \qquad (1.18)$$

with  $\int \mu_i d\gamma = 0$  for all i from (1.7), and set

$$F_s = \mu + \sqrt{\mu} f_s.$$

Then there exist  $f_1, f_2, ..., f_{m-1}$  with  $\|\langle v \rangle^{\beta} e^{\zeta |v|^2} f_i\|_{\infty} \lesssim 1$ , for  $0 \leq \zeta < \frac{1}{4}, \beta > 4$ , such that the following  $\delta$ -expansion is valid

$$f_s = \delta f_1 + \delta^2 f_2 + \dots + \delta^m f_m^{\delta},$$

with  $\|\langle v \rangle^{\beta} e^{\zeta |v|^2} f_m^{\delta}\|_{\infty} \lesssim 1$  for  $0 \leq \zeta < \frac{1}{4+2\delta}, \ \beta > 4$ . In particular,  $f_1$  satisfies

$$v \cdot \nabla_{x} f_{1} + \frac{1}{K_{n}} L f_{1} = 0,$$

$$f_{1}|_{\gamma_{-}} = \sqrt{\mu(v)} \int_{n(x) \cdot v' > 0} f_{1}(x, v') \sqrt{\mu(v')} \{n(x) \cdot v'\} dv' + \frac{\mu_{1}}{\sqrt{\mu}}.$$

$$(1.19)$$

We have the following dynamical stability result:

**Theorem 1.3** For every fixed  $0 \le \zeta < \frac{1}{4+2\delta}$ ,  $\beta > 4$ , there exist  $\lambda > 0$  and  $\varepsilon_0 > 0$ , depending on  $\delta_0$ , such that if

$$\|\langle v\rangle^{\beta} e^{\zeta|v|^2} [f(0) - f_s]\|_{\infty} + |\langle v\rangle^{\beta} e^{\zeta|v|^2} [f(0) - f_s]|_{\infty} \le \varepsilon_0, \tag{1.20}$$

then there exists a unique non-negative solution  $F(t) = \mu + \sqrt{\mu} f(t) \ge 0$  to the dynamical problem (1.2) and (1.5) such that

$$\|\langle v \rangle^{\beta} e^{\zeta |v|^2} [f(t) - f_s]\|_{\infty} + |\langle v \rangle^{\beta} e^{\zeta |v|^2} [f(t) - f_s]\|_{\infty}$$
  
 
$$\lesssim e^{-\lambda t} \{ \|\langle v \rangle^{\beta} e^{\zeta |v|^2} [f(0) - f_s]\|_{\infty} + |\langle v \rangle^{\beta} e^{\zeta |v|^2} [f(0) - f_s]\|_{\infty} \}.$$

If the domain is convex,  $\theta(x)$  is continuous on  $\partial\Omega$  and moreover  $F_0(x,v)$  is continuous away from  $\gamma_0$  and satisfies the compatibility condition

$$F_0(x, v) = \mu^{\theta}(x, v) \int_{n(x) \cdot v' > 0} F_0(x, v') (n(x) \cdot v') dv',$$

then F(t,x,v) is continuous away from  $\gamma_0$ .

The asymptotic stability of  $F_s$  further justifies the physical importance of such a steady state solution. We remember that, when  $\delta > 0$ , then  $F_s$  fails to be a global Maxwellian or even a local Maxwellian, and its stability analysis marks a drastic departure from relative entropy approach (e.g. [12]). Moreover, such an asymptotic stability plays a crucial role in our proof of non-negativity of  $F_s$ .

An important consequence of Theorem 1.2 is the Corollary below which specializes the result to the case of a slab  $-\frac{1}{2} \le x \le \frac{1}{2}$  between two parallel plates kept at temperatures  $\theta(-\frac{1}{2}) = 1 - \delta$  and  $\theta(\frac{1}{2}) = 1 + \delta$ .

Corollary 1.4 Let  $x \in \Omega = [-\frac{1}{2}, \frac{1}{2}]$  and  $f_s$  be the solution to

$$v_{1}\partial_{x}f_{s} + \frac{1}{K_{n}}Lf_{s} = \frac{1}{K_{n}}\Gamma(f_{s}, f_{s}), \qquad -\frac{1}{2} < x < \frac{1}{2},$$

$$\sqrt{\mu(v)}f_{s}(-\frac{1}{2}, v) = \frac{1}{2\pi[1 - \delta]^{2}}\exp\left[-\frac{|v|^{2}}{2[1 - \delta]}\right] \int_{u_{1} < 0} u_{1}\sqrt{\mu(u)}f_{s}(-\frac{1}{2}, u)du, \qquad v_{1} > 0,$$

$$\sqrt{\mu(v)}f_{s}(\frac{1}{2}, v) = \frac{1}{2\pi[1 + \delta]^{2}}\exp\left[-\frac{|v|^{2}}{2[1 + \delta]}\right] \int_{u_{1} < 0} u_{1}\sqrt{\mu(u)}f_{s}(\frac{1}{2}, u)du, \qquad v_{1} < 0.$$

$$(1.21)$$

where  $v_1$  is the component in the direction x of the velocity v.

Then  $f_1$ , the first order in  $\delta$  correction to  $\mu$  according to the expansion of Theorem 1.2, is the unique solution to

$$v_{1}\partial_{x}f_{1} + \frac{1}{K_{n}}Lf_{1} = 0, \quad -\frac{1}{2} < x < \frac{1}{2},$$

$$f_{1}(-\frac{1}{2},v) = \sqrt{\mu(v)} \int_{u_{1}<0} f_{1}(-\frac{1}{2},u)\sqrt{\mu(u)}\{-u_{1}\}du + \frac{\mu_{1}(-\frac{1}{2},v)}{\sqrt{\mu(v)}}, \quad v_{1} > 0,$$

$$f_{1}(\frac{1}{2},v) = \sqrt{\mu(v)} \int_{u_{1}>0} f_{1}(\frac{1}{2},u)\sqrt{\mu(u)}\{u_{1}\}du + \frac{\mu_{1}(\frac{1}{2},v)}{\sqrt{\mu(v)}}, \quad v_{1} < 0.$$

$$(1.22)$$

In order to establish a criterion of validity of the Fourier law, let us remember that the temperature associated to the stationary solution  $F_s$  is given by

$$\theta_s(x) = \frac{1}{3\rho_s} \int_{\mathbf{R}^3} |v - u_s|^2 F_s(x, v) dv,$$
 (1.23)

where  $\rho_s = \int_{\mathbf{R}^3} F_x(x, v) dv$  and  $u_s(x) = \rho_s^{-1} \int_{\mathbf{R}^3} v F_s(x, v) dv$ . The heat flux associated to the distribution  $F_s$  is the vector field defined as

$$q_s(x) = \frac{1}{2} \int_{\mathbf{R}^3} (v - u_s(x))|v - u_s(x)|^2 F_s(x, v) dv.$$
 (1.24)

We state the Fourier law in the following formulation: there is a positive  $C^1$  function  $\kappa(\theta)$ , the heat conductivity, such that (1.1) is valid for  $F_s$ .

**Theorem 1.5** If the Fourier law holds for  $F_s$ , then  $\theta_1(x) \equiv \frac{1}{3} \int_{\mathbf{R}^3} |v - u_1|^2 f_1$  is a linear function over  $-\frac{1}{2} \leq x \leq \frac{1}{2}$ .

Available numerical simulations, Figure 1, [29] indicate the linearity is clearly violated for all finite Knudsen number  $K_n$  (= k in Figure 1).

We therefore predict that the general Fourier law (1.1) is invalid and inadequate in the kinetic regime with finite Knudsen number  $K_n$ . It is necessary to at least solve the linear Boltzmann equation (1.19) to capture the heat transfer in the Boltzmann theory.

Without loss of generality, we may assume  $\theta_0 = 1$  and  $K_n = 1$  throughout the rest of the paper. Therefore in the rest of this paper we assume

$$\theta(x) = 1 + \delta \theta(x). \tag{1.25}$$

The key difficulty in the study of the steady Boltzmann equation lies in the fact that the usual entropic estimate for  $\int F \ln F$ , coming from  $\partial_t F$ , is absent. The only a-priori estimate is given by the entropy production  $\int Q(F,F) \ln F$ , which is very hard to use [4]. In the context of small perturbation of  $\mu$ , only the linearized dissipation rate  $\|(\mathbf{I}-\mathbf{P})f\|_{\nu}^2$  is controlled <sup>1</sup>, and the key is to estimate the missing hydrodynamic part  $\mathbf{P}f$ , in term of  $(\mathbf{I}-\mathbf{P})f$ . This is a well-known basic question in the Boltzmann theory. Motivated by the studies of collisions in a plasma, a new nonlinear energy method in high Sobolev norms was initiated in the Boltzmann study, to estimate  $\mathbf{P}f$  ([19]) in terms of  $(\mathbf{I}-\mathbf{P})f$ . For  $F=\mu+\sqrt{\mu}f$ , remembering the definitions

<sup>&</sup>lt;sup>1</sup>We denote by  $\mathbf{P}f$  the projection of f on the null space of L.

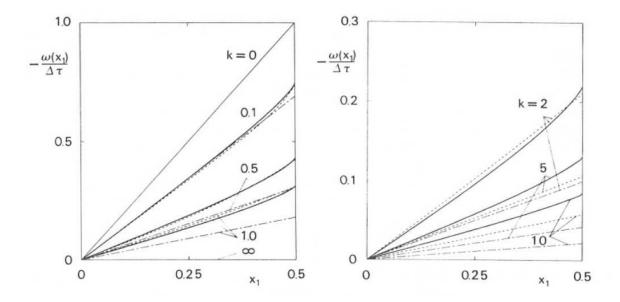


Figure 1: Nonlinearity of  $\theta_1$  in a kinetic regime

(1.14) of the linearized Boltzmann operator and (1.15) of the quadratic nonlinear collision operator, the Boltzmann equation (1.2) can be rewritten for the perturbation f as

$$\{\partial_t + v \cdot \nabla_x + L\} f = \Gamma(f, f), \tag{1.26}$$

If the operator L were positive definite, then global solutions (for small f) could be constructed "easily" for (1.26). However, L is only semi-positive

$$\langle Lf, f \rangle \gtrsim \|\{\mathbf{I} - \mathbf{P}\}f\|_{\nu}^2,$$
 (1.27)

where  $\|\cdot\|_{\nu}$  is the  $\nu$ -weighted  $L^2$  norm. The kernel (the hydrodynamic part) is given by

$$\mathbf{P}f \equiv \{a_f(t,x) + v \cdot b_f(t,x) + \frac{(|v|^2 - 3)}{2}c_f(t,x)\}\sqrt{\mu}.$$
 (1.28)

Note that we use slight different definitions of  $a_f(t,x)$  and  $c_f(t,x)$  from [19] to capture the crucial total mass constraint. The so-called 'hydrodynamic part' of f,  $\mathbf{P}f$ , is the  $L_v^2$ -projection on the kernel of L, for every given x. The novelty of such energy method is to show that L is indeed positive definite for small solutions f to the nonlinear Boltzmann equation (1.26). The key macroscopic equations for  $\xi_f = (a_f, b_f, c_f)$  connect  $\mathbf{P}f$  and  $\{\mathbf{I} - \mathbf{P}\}f$  via the Boltzmann equation as in page 621 of [19]:

$$\Delta \xi_f \sim \partial^2 \{ \mathbf{I} - \mathbf{P} \} f$$
 + higher order terms, (1.29)

where  $\partial^2$  denotes some second order differential operator in [19]. Such hidden ellipticity in  $H^k$   $(k \ge 1)$  implies that the hydrodynamic part  $\mathbf{P}f$ , missing from the lower bound of L, can be controlled via the microscopic part  $\{\mathbf{I} - \mathbf{P}\}f$ , so that L can control the full  $||f||_{\nu}^2$  from (1.27). Such nonlinear energy framework has led to resolutions to several open problems in the Boltzmann theory [19, 20, 25].

It should be noted that all of these results deal with idealized periodic domains, in which the solutions can remain smooth in  $H^k$  for  $k \ge 1$ . Of course, a gas is usually confined within a container, and its interaction with the boundary plays a crucial role both from physical and mathematical view points. Mathematically speaking, the phase boundary  $\partial\Omega \times \mathbf{R}^3$  is always characteristic but not uniformly characteristic at the grazing set  $\gamma_0 = \{(x, v) : x \in \partial\Omega, \text{ and } v \cdot n(x) = 0\}$ . In particular, many of the natural physical boundary conditions create singularities in general domains ([22, 26]), for which the high Sobolev estimates break down in the crucial elliptic estimates (1.29). Discontinuities are expected to be created at the boundary, and then propagate inside a non-convex domain. Therefore completely new tools need to be developed.

A new  $L^2-L^\infty$  framework is developed in [22] (see [21] for a short summary of the method) to resolve such a difficulty in the Boltzmann theory, which leads to resolution of asymptotic stability of Maxwellians for specular reflection in an analytic and convex domain, and for diffuse reflection (with uniform temperature!) in general domains (no convexity is needed). We remark that the non-convex domains occur naturally for non-isothermal boundary (e.g. two non flat separated boundaries). Furthermore, the solutions to the boundary problems are shown to be *continuous* if the domain  $\Omega$  is strictly convex. Different  $L^2 - L^\infty$  methods have been used in [2, 3, 5, 16, 17] in particular geometries.

The new  $L^2 - L^{\infty}$  framework introduced in [22] has two parts:

 $L^2$  Positivity: Assume the wall temperature  $\theta$  is constant in (1.5). It suffices to establish the following finite-time estimate

$$\int_0^1 \|\mathbf{P}g(s)\|_{\nu}^2 ds \lesssim \left\{ \int_0^1 \|\{\mathbf{I} - \mathbf{P}\}g(s)\|_{\nu}^2 + \text{boundary contributions} \right\}. \tag{1.30}$$

The natural attempt is to establish  $L^2$  estimate for  $\xi_f$  from the macroscopic equation (1.29). However, this is very challenging due to the fact that only f has trace in the sense of Green's identity (Lemma 2.2), neither  $\mathbf{P}f$  nor  $\{\mathbf{I} - \mathbf{P}\}f$  even make sense on the boundary of a general domain. Instead, the proof of (1.30) given in [22] is based on a delicate contradiction argument because it is difficult to estimate  $\mathbf{P}f$  via  $\{\mathbf{I} - \mathbf{P}\}f$  directly in a  $L^2$  setting in the elliptic equation (1.29), in the presence of boundary conditions. The heart of this argument, lies in an exact computation of  $\mathbf{P}f$  which leads to the contradiction. As a result, such an indirect method fails to provide a constructive estimate of (1.30) with explicit constants.

 $L^{\infty}$  Bound: The method of characteristics can bootstrap the  $L^2$  bound into a point-wise bound to close the nonlinear estimate. Let U(t) be the semigroup generated by  $v \cdot \nabla_x + L$  and G(t) the semigroup generated by  $v \cdot \nabla_x + \nu$ , with the prescribed boundary conditions. By two iterations, one can establish:

$$U(t) = G(t) + \int_0^t G(t - s_1) KG(s_1) ds_1 + \int_0^t \int_0^{s_1} G(t - s_1) KG(s_1 - s) KU(s) ds ds_1. \quad (1.31)$$

From the compactness property of K, the main contribution in (1.31) is roughly

$$\int_{0}^{t} \int_{0}^{s_{1}} \int_{v',v'' \text{bounded}} |f(s, X_{\mathbf{cl}}(s; s_{1}, X_{\mathbf{cl}}(s_{1}; t, x, v), v'), v'')| dv' dv'' ds ds_{1}.$$
 (1.32)

where  $X_{\mathbf{cl}}(s;t,x,v)$  denotes the generalized characteristics associated with specific boundary condition. A change of variable from v' to  $X_{\mathbf{cl}}(\cdot)$  would transform the v' and v''-integration

in (1.32) into x and v integral of f, which decays from the  $L^2$  theory. The key is to check if

$$\det\left\{\frac{X_{\mathbf{cl}}(s; s_1, X_{\mathbf{cl}}(s_1; t, x, v), v')}{dv'}\right\} \neq 0. \tag{1.33}$$

is valid. Without boundary,  $X_{\mathbf{cl}}(s; s_1, X(s_1; t, x, v), v')$  is simply  $x - (t - s_1)v - (s_1 - s)v'$ , and  $\det\left\{\frac{X_{\mathbf{cl}}(\cdot)}{dv'}\right\} \neq 0$  most of the time. For specular or diffuse reflections, each type of characteristic trajectories  $X_{\mathbf{cl}}$  repeatedly interact with the boundary. To justify (1.33), various delicate arguments were invented to overcome different difficulties, and analytic boundary and convexity are needed for the specular case.

Since its inception, this new  $L^2 - L^{\infty}$  approach has already led to new results in the study of relativistic Boltzmann equation [34], in hydrodynamic limits of the Boltzmann theory([23] [24] [33]), in stability in the presence of a large external field ([15] [27]).

Our current study of non-isothermal boundary ( $\theta$  is non-uniform) is naturally based on such a  $L^2 - L^{\infty}$  framework. The main new difficulty in contrast to [22], however, is that the presence of a non-constant temperature creates non-homogeneous terms in both the linear Boltzmann (steady and unsteady) as well as in the boundary condition, so that the exact computation, crucial to the  $L^2$  estimate (1.30), breaks down, and the  $L^2 - L^{\infty}$  scheme [22] collapses.

The main technical advance in this paper is the development of a direct (constructive) and robust approach to establish (1.30) in the presence of a non-uniform temperature diffuse boundary condition (0.2). Instead of using these macrosopic equations (e.g. (1.29)), whose own meaning is doubtful in a bounded domain, we resort to the basic Green's identity for the transport equation and choose proper test functions to recover ellipticity estimates for a, b, c and hence  $\mathbf{P}f$  directly. In light of the energy identity,  $\{1-P_{\gamma}\}f$  is controlled at the boundary  $\gamma_+$ , but not  $P_{\gamma}f$  in (1.16). The essence of the method is to choose a test function which can eliminate the  $P_{\gamma}f$  contribution at the boundary, and to control the a, b, c component of  $\mathbf{P}f$  respectively in the bulk at the same time. The choice of the test function for c is rather direct: we set

$$\psi_c = (|v|^2 - \beta_c)\sqrt{\mu}v \cdot \nabla_x \phi_c(x),$$

for some constant  $\beta_c$  to be determined, with  $-\Delta\phi_c=c$  and  $\phi_c=0$  on  $\partial\Omega$ . On the other hand, the test function for b is rather delicate. In fact, two different sets of test functions have to be constructed:

$$(v_i^2 - \beta_b)\sqrt{\mu}\partial_j\phi_b^j, \quad i, j = 1, ...d,$$
$$|v|^2 v_i v_j \sqrt{\mu}\partial_j\phi_b^i, \quad i \neq j,$$

with  $-\Delta \phi_b^j = b_j$  and  $\phi_b^j = 0$  on  $\partial \Omega$ . In particular, a unique constant  $\beta_c$  can be chosen to deduce the estimates for b, thanks to the special structure of the transport equation and the diffuse boundary condition. The choice of the test function for a requires special attention. It turns out that in order to eliminate the contribution  $P_{\gamma}f$ , we need to choose the test function

$$(|v|^2 - \beta_a)v \cdot \nabla_x \phi_a \sqrt{\mu},$$

with  $-\Delta \phi_a = a$  and Neumann boundary condition

$$\frac{\partial}{\partial n}\phi_a = 0 \quad \text{on } \partial\Omega.$$

This is only possible if the total mass of f is zero, i.e.,

$$\iint_{\Omega \times \mathbf{R}^3} f(x, v) \sqrt{\mu} dx dv \equiv \sqrt{2\pi} \int_{\Omega} a(x) dx = 0.$$
 (1.34)

This illustrates the importance of the mass constraint, which unfortunately is not valid for the steady problem (0.1) and (0.2). So we are forced to use a penalization procedure (see below) to deal with it. The key lemma which delivers the basic estimates is Lemma 3.4. Furthermore, in the dynamical case, also the time derivatives  $\partial_t \phi_c$ ,  $\partial_t \phi_b^j$  and  $\partial_t \phi_a$  need to be controlled in negative Sobolev spaces. This is possible due to the special structure of the Boltzmann equation as well as the diffuse boundary condition. Once again, the total mass zero condition (1.34) is essential. This is the key to prove the crucial Lemma 6.2. Even though this new unified procedure can be viewed as a 'weak version' of the macroscopic equations (1.29), the estimates we obtain via this approach are more general. For instance, the (1.29) was only valid for dimension  $\geq 2$ , but the new estimates are valid for any dimension. There seems to be a very rich structure in the linear Boltzmann equation.

To bootstrap such a  $L^2$  estimate into a  $L^{\infty}$  estimate, we define the stochastic cycles for the generalized characteristic lines interacting with the boundary:

**Definition 1.6 (Stochastic Cycles)** Fixed any point (t, x, v) with  $(x, v) \notin \gamma_0$ , let  $(t_0, x_0, \bar{v}_0)$  =  $(t, x, \bar{v})$ . For  $\bar{v}_{k+1}$  such that  $\bar{v}_{k+1} \cdot n(x_{k+1}) > 0$ , define the (k+1)-component of the back-time cycle as

$$(t_{k+1}, x_{k+1}, \bar{v}_{k+1}) = (t_k - t_{\mathbf{b}}(x_k, \bar{v}_k), x_{\mathbf{b}}(x_k, \bar{v}_k), \bar{v}_{k+1}). \tag{1.35}$$

Set

$$\begin{split} X_{\mathbf{cl}}(s;t,x,\bar{v}) &=& \sum_{k} \mathbf{1}_{[t_{k+1},t_k)}(s) \{x_k + (s-t_k)\bar{v}_k\}, \\ \bar{V}_{\mathbf{cl}}(s;t,x,\bar{v}) &=& \sum_{k} \mathbf{1}_{[t_{k+1},t_k)}(s)\bar{v}_k, \quad V_{\mathbf{cl}}(s;t,x,\bar{v}) = (\bar{V}_{\mathbf{cl}}(s;t,x,\bar{v}),\hat{v}). \end{split}$$

Define  $V_{k+1} = \{v \in \mathbf{R}^3 \mid \bar{v} \cdot n(x_{k+1}) > 0\}$ , and let the iterated integral for  $k \geq 2$  be defined as

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \dots \prod_{j=1}^{k-1} d\sigma_j \equiv \int_{\mathcal{V}_1} \dots \left\{ \int_{\mathcal{V}_{k-1}} d\sigma_{k-1} \right\} d\sigma_1, \tag{1.36}$$

where  $d\sigma_j = \mu(v)(n(x_j) \cdot \bar{v})dv$  is a probability measure.

We note that the  $\bar{v}_j$ 's (j=1,2,...) are all independent variables, and  $t_k, x_k$  depend on  $t_j, x_j, \bar{v}_j$  for  $j \leq k-1$ . However, the phase space  $\mathcal{V}_j$  implicitly depends on  $(t, x, \bar{v}, \bar{v}_1, \bar{v}_2, ... \bar{v}_{j-1})$ . Our method is to use the Vidav's two iterations argument([37]) and estimate the  $L^{\infty}$ -norm along these stochastic cycles with corresponding phase spaces  $\prod_{j=1}^{k-1} \mathcal{V}_j$ . The key is to estimate measures of various sets in  $\prod_{j=1}^{k-1} \mathcal{V}_j$  in Lemma 4.2. We designed an abstract and unified iteration (4.8), which is suitable for both steady and unsteady cases. New precise estimate (4.16) of non-homogeneous terms resulting from the non-constant temperature are obtained in Prop. 4.1. Based on a delicate change of variables in Lemma 2.3, such an estimate is crucial in the proof of formation of singularity for a non-convex domain in Theorem 1.1.

In order to keep the mass zero condition and to start iterating scheme, it is essential to introduce a penalization  $\varepsilon$  to solve the problem

$$\varepsilon f^{l+1} + v \cdot \nabla_x f^{l+1} + L f^{l+1} = \Gamma(f^l, f^l).$$

The presence of  $\varepsilon$  ensures the critical zero mass condition (1.34). It is important to note that our  $L^{\infty}$  estimate are intertwined with  $L^2$  at every step of the approximations, which ensures the preservation of the continuity for a convex domain. The continuity properties of our final solutions follows from the  $L^{\infty}$  limit at every step. Moreover, the proof of continuity away from the singular set  $\mathfrak{D}$  in (1.11) in a general domain is consequence of a delicate result for  $Q_{\text{gain}}$  in [26].

To illustrate the subtle nature of our construction, we remark that for the natural positivity-preserving scheme:

$$\varepsilon f^{l+1} + v \cdot \nabla_x f^{l+1} + \nu f^{l+1} - K f^l = \Gamma_{\text{gain}}(f^l, f^l) - \Gamma_{\text{loss}}(f^l, f^{l+1}) \qquad (1.37)$$

$$F^{l+1} = \mu_{\theta}(x, v) \int_{n(x) \cdot v' > 0} F^l \{n(x) \cdot v'\} dv',$$

we are unable to prove the convergence, due to breakdown of  $Lf^{l+1}$ , whence the mass zero constraint (1.34) fails to be satisfied. Consequently, we are unable to prove  $F_s \geq 0$  in our construction. Such a positivity is only proven via the dynamical asymptotic stability of  $F_s$ , in which the initial positivity plus the choice of a small time interval are crucial to guarantee the convergence of the analog of (1.37) in the dynamical setting.

Our estimates are robust and allow us to expand our steady state  $F_s$  in terms of  $\delta$ , the magnitude of the perturbation. This leads to the first order precise characterization of  $F_s$  by  $f_1$ , which satisfies (1.19).

It should be pointed out the our rather complete study of the non-isothermal boundary for the Boltzmann theory for  $\delta \ll 1$  forms a mathematically solid foundation for influential work in applied physics and engineering such as in [31][32], where the existence of such steady solutions to the Boltzmann equation is a starting point, but without mathematical justification. We expect our solutions as well as our new estimates would lead to many new developments along this direction.

The plan of the paper is the following. In next section we present some background material and in particular a version of the Ukai Trace Theorem and the Green identity as well as a new  $L^1$  estimate at the boundary in Lemma 2.3. Section 3 is devoted to the construction of  $L^2$  solutions to the stationary linearized problem. In particular, we prove Lemma 3.4 which provides the basic estimate of the  $L^2$  norms of  $\mathbf{P}f$  in terms of the  $L^2$  norm of  $(\mathbf{I} - \mathbf{P})f$ . In Section 4, after introducing an abstract iteration scheme suitable for proving  $L^{\infty}$  bounds, we prove, in Proposition 4.1, the existence of the solution to the linearized problem in  $L^{\infty}$  and related bounds. In Section 5 we combine the results of the previous sections to construct the stationary solution to the Boltzmann equation and discuss its regularity properties. In particular, we give the proof of the  $\delta$ -expansion and use it to establish a necessary condition for the validity of the Fourier law. In Section 6 we extend the  $L^2$  estimates to the time dependent problem. Section 7 contains the extension of the  $L^{\infty}$  estimates to the time dependent problem and the proof of the exponential asymptotic stability of the stationary solution. From this we then obtain its positivity.

# 2 Background

In this section we state basic preliminaries. First we shall clarify the notations of functional spaces and norms: we use  $\|\cdot\|_p$  for both of the  $L^p(\Omega \times \mathbf{R}^3)$  norm and the  $L^p(\Omega)$  norm, and  $(\cdot, \cdot)$ 

for the standard  $L^2(\Omega \times \mathbf{R}^3)$  inner product. Moreover we denote  $\|\cdot\|_{\nu} \equiv \|\nu^{1/2}\cdot\|_2$  and  $\|f\|_{H^k} = \|f\|_2 + \sum_{i=1}^k \|\nabla_x^i f\|_2$ . For the phase boundary integration, we define  $d\gamma = |n(x) \cdot v| dS(x) dv$  where dS(x) is the surface measure and define  $|f|_p^p = \int_{\gamma} |f(x,v)|^p d\gamma \equiv \int_{\gamma} |f(x,v)|^p$  and the corresponding space as  $L^p(\partial\Omega \times \mathbf{R}^3; d\gamma) = L^p(\partial\Omega \times \mathbf{R}^3)$ . Further  $|f|_{p,\pm} = |f\mathbf{1}_{\gamma\pm}|_p$  and  $|f|_{\infty,\pm} = |f\mathbf{1}_{\gamma\pm}|_{\infty}$ . We also use  $|f|_p^p = \int_{\partial\Omega} |f(x)|^p dS(x) \equiv \int_{\partial\Omega} |f(x)|^p$ . Denote  $f_{\pm} = f_{\gamma\pm}$ . Recall that  $x \in \overline{\Omega} \subset \mathbf{R}^d$  for d = 1, 2, 3, and  $v \in \mathbf{R}^3$ . For  $A \in \mathbf{R}^d$ , the notation  $v \cdot A$  means  $\sum_{i=1}^d v_i A_i$ .

In the next lemma, we prove that the trace of f is well-defined locally for a certain class of f:

### Lemma 2.1 (Ukai Trace Theorem) Define

$$\gamma^{\varepsilon} \equiv \{(x, v) \in \gamma : |n(x) \cdot v| \ge \varepsilon, |v| \le \frac{1}{\varepsilon}\}.$$
(2.1)

Then

$$|f\mathbf{1}_{\gamma^{\varepsilon}}|_{1} \lesssim_{\varepsilon,\Omega} ||f||_{1} + ||\{v \cdot \nabla_{x}\}f||_{1}, \qquad (steady)$$

$$\int_{s}^{t} |f\mathbf{1}_{\gamma^{\varepsilon}}(\tau)|_{1} d\tau \lesssim_{\varepsilon,\Omega} \int_{s}^{t} ||f(\tau)||_{1} d\tau + \int_{s}^{t} ||\{\partial_{t} + v \cdot \nabla_{x}\}f(\tau)||_{1} d\tau, \qquad (dynamic)$$

for all  $0 \le s \le t$ .

**Proof.** Recall the notation (1.8) and the definition of backward exit time  $t_{\mathbf{b}}(x,v) = t_{\mathbf{b}}(x,\bar{v})$  and  $x_{\mathbf{b}}(x,v) = x_{\mathbf{b}}(x,\bar{v}) = x - t_{\mathbf{b}}(x,\bar{v})\bar{v} \in \partial\Omega$  in (1.9).

In the steady case, from [10], page 247, we have the following identity:

$$\iint_{\Omega \times \mathbf{R}^3} f(x, v) dx dv = \iint_{\gamma_+} \int_{t_{\mathbf{b}}(x, v)}^0 f(x - s\bar{v}, v) ds d\gamma 
+ \iint_{\gamma_-} \int_{0}^{t_{\mathbf{b}}(x, -v)} f(x + s\bar{v}, v) ds d\gamma.$$

Assume  $||f||_1$ ,  $||\{v \cdot \nabla_x\}f||_1 < \infty$ . Then  $f(x + s\bar{v}, v)$  is an absolutely continuous function of s for fixed (x, v), so that, by the fundamental theorem of calculus

$$f(x,v) = f(x - s'\bar{v}, v) + \int_{-s'}^{0} v \cdot \nabla_{x} f(x + \tau \bar{v}, v) d\tau$$

$$\text{for } (s', x, v) \in [0, t_{\mathbf{b}}(x, v)] \times \gamma_{+},$$

$$f(x,v) = f(x + s'\bar{v}, v) + \int_{0}^{s'} v \cdot \nabla_{x} f(x + \tau \bar{v}, v) d\tau$$

$$\text{for } (s', x, v) \in [0, t_{\mathbf{b}}(x, -v)] \times \gamma_{-},$$

and therefore, for both cases,

$$|f(x,v)| \le |f(x-s'\bar{v},v)| + \int_{t_{\mathbf{b}}(x,v)}^{0} |v \cdot \nabla_x f(x-\tau \bar{v},v)| d\tau,$$
 (2.2)

$$|f(x,v)| \le |f(x+s'\bar{v},v)| + \int_0^{t_{\mathbf{b}}(x,-v)} |v \cdot \nabla_x f(x+\tau \bar{v},v)| d\tau.$$
 (2.3)

On the other hand, for  $x \in \partial\Omega$  assume that  $\partial\Omega$  is locally parameterized by  $\xi: \{y \in \mathbf{R}^d: |x-y| < \delta\} \to \mathbf{R}$  so that

$$\sup_{\substack{y \in \partial \Omega \\ |x-y| < \delta}} \frac{|(x-y) \cdot n(x)|}{|x-y|^2} \le \max_{\substack{y \in \partial \Omega \\ |x-y| < \delta}} |\nabla_x^2 \xi(y)|.$$

By the compactness of  $\Omega$  (and  $\partial\Omega$ ), uniformly in x, we have  $|(x-y)\cdot n(x)| \leq C_{\Omega}|x-y|^2$  for all  $y \in \partial\Omega$ . Taking inner product of  $x - x_{\mathbf{b}}(x, v) = t_{\mathbf{b}}(x, v)\overline{v}$  with n(x), we get

$$t_{\mathbf{b}}(x,v)|v \cdot n(x)| = |(x - x_{\mathbf{b}}(x,v)) \cdot n(x)| \le C_{\Omega}|x - x_{\mathbf{b}}(x,v)|^2 = C_{\Omega}|v|^2|t_{\mathbf{b}}(x,v)|^2.$$

Therefore we deduce

$$t_{\mathbf{b}}(x,v) \ge \frac{|n(x) \cdot v|}{C_{\Omega}|v|^2}.$$
(2.4)

If  $(x,v) \in \gamma^{\varepsilon} \cap \gamma_{+}$  or  $(x,v) \in \gamma^{\varepsilon} \cap \gamma_{-}$  then, by the definition of  $\gamma^{\varepsilon}$  and (40) in [22],

$$\min\{t_{\mathbf{b}}(x,v), \ t_{\mathbf{b}}(x,-v)\} \ge \frac{|n(x)\cdot v|}{C_{\Omega}|v|^2} \ge C_{\Omega}^{-1}\varepsilon^3.$$

First we integrate (2.2), for  $(x, v) \in \gamma^{\varepsilon} \cap \gamma_{+}$ , and then for  $s' \in [0, t_{\mathbf{b}}(x, v)]$ . Similarly integrate (2.3), for  $(x, v) \in \gamma^{\varepsilon} \cap \gamma_{-}$ , and then for  $s' \in [0, t_{\mathbf{b}}(x, -v)]$  to have

$$C_{\Omega}^{-1} \varepsilon^3 |f \mathbf{1}_{\gamma^{\varepsilon}}|_1 \le C \{ ||f||_1 + ||v \cdot \nabla_x f||_1 \}.$$

In the dynamic case, from [10] in page 247, we have the following identity:

$$\iiint_{[0,t]\times\Omega\times\mathbf{R}^{3}} f(s,x,v)dsdxdv = \int_{\gamma_{+}} \int_{\max\{0,t-t_{\mathbf{b}}(x,v)\}}^{t} f(s,x-(t-s)\bar{v},v)dsd\gamma + \int_{\gamma_{-}} \int_{0}^{\min\{t,t_{\mathbf{b}}(x,-v)\}} f(s,x+s\bar{v},v)dsd\gamma.$$
(2.5)

Assume again  $||f||_1$ ,  $||\{\partial_t + v \cdot \nabla_x\}f||_1 < \infty$ . Then  $f(s, x + s\bar{v}, v)$  is absolutely continuous of s for fixed x, v, so that, by the fundamental theorem of calculus

$$f(s, x, v) = f(s - s', x - s'\bar{v}, v) + \int_{s - s'}^{s} v \cdot \nabla_{x} f(\tau, x - (s - \tau)\bar{v}, v) d\tau$$

$$\text{for } (s, s', x, v) \in [0, t] \times [\max\{0, s - t_{\mathbf{b}}(x, v)\}, s] \times \gamma_{+},$$

$$f(s, x, v) = f(s + s', x + s'\bar{v}, v) + \int_{s}^{s + s'} v \cdot \nabla_{x} f(\tau, x + \tau\bar{v}, v) d\tau$$

$$\text{for } (s, s', x, v) \in [0, t] \times [0, \min\{t - s, t_{\mathbf{b}}(x, -v)\}] \times \gamma_{-},$$

and hence, for both cases,

$$|f(s,x,v)| \leq |f(s-s',x-s'\bar{v},v)| + \int_{s-s'}^{s} |v \cdot \nabla_x f(\tau,x-(s-\tau)\bar{v},v)| d\tau, |f(s,x,v)| \leq |f(s+s',x+s'\bar{v},v)| + \int_{s}^{s+s'} |v \cdot \nabla_x f(\tau,x+\tau\bar{v},v)| d\tau.$$

Then the rest part of proof is exactly the same as in the steady case.

The following Green's identities are important in this paper.

**Lemma 2.2** For the steady case, assume that f(x, v),  $g(x, v) \in L^2(\Omega \times \mathbf{R}^3)$ ,  $v \cdot \nabla_x f$ ,  $v \cdot \nabla_x g \in L^2(\Omega \times \mathbf{R}^3)$  and  $f_{\gamma}, g_{\gamma} \in L^2(\partial \Omega \times \mathbf{R}^3)$ . Then

$$\iint_{\Omega \times \mathbf{R}^3} \{ v \cdot \nabla_x f \} g + \{ v \cdot \nabla_x g \} f \ dv dx = \int_{\gamma} f g d\gamma. \tag{2.6}$$

For the dynamic case, assume that f(t, x, v),  $g(t, x, v) \in L^{\infty}([0, T]; L^{2}(\Omega \times \mathbf{R}^{3}))$ ,  $\partial_{t} f + v \cdot \nabla_{x} f$ ,  $\partial_{t} g + v \cdot \nabla_{x} g \in L^{2}([0, T] \times \Omega \times \mathbf{R}^{3})$  and  $f_{\gamma}, g_{\gamma} \in L^{2}([0, T] \times \partial \Omega \times \mathbf{R}^{3})$ . Then, for almost all  $t, s \in [0, T]$ ,

$$\int_{s}^{t} \iint_{\Omega \times \mathbf{R}^{3}} \{\partial_{t} f + v \cdot \nabla_{x} f\} g \, dv dx + \int_{s}^{t} \iint_{\Omega \times \mathbf{R}^{3}} \{\partial_{t} g + v \cdot \nabla_{x} g\} f \, dv dx$$

$$= \iint_{\Omega \times \mathbf{R}^{3}} f(t) g(t) dv dx - \iint_{\Omega \times \mathbf{R}^{3}} f(s) g(s) dv dx + \int_{s}^{t} \int_{\gamma} f g d\gamma d\tau. \tag{2.7}$$

**Proof.** See the proof in Chapter 9 of [10].

**Lemma 2.3** Let  $\Omega \subset \mathbf{R}^d$  and recall the notation (1.8). If

$$\bar{v} \cdot n(x_{\mathbf{b}}(x, \bar{v})) < 0,$$

then  $t_{\mathbf{b}}(x,\bar{v})$  and  $x_{\mathbf{b}}(x,\bar{v})$  are smooth functions of  $(x,\bar{v})$  so that

$$\nabla_{x} t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})}, \qquad \nabla_{\bar{v}} t_{\mathbf{b}} = -\frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})}, \qquad (2.8)$$

$$\nabla_{x} x_{\mathbf{b}} = I - \nabla_{x} t_{\mathbf{b}} \otimes v, \qquad \nabla_{\bar{v}} x_{\mathbf{b}} = -t_{\mathbf{b}} I - \nabla_{\bar{v}} t_{\mathbf{b}} \otimes v.$$

For d = 2, 3, if  $x \in \partial \Omega$ 

$$\int_{\mathbf{S}^{d-1}} |f(x - t_{\mathbf{b}}(x, \mathfrak{u})\mathfrak{u}||n(x) \cdot \mathfrak{u}| d\mathfrak{u} \lesssim \int_{\partial \Omega} |f(y)| dS(y), \tag{2.9}$$

and if  $x \in \Omega$  and  $\{y \in \mathbf{R}^d : |x - y| < \epsilon|\} \cap \partial\Omega = \emptyset$ 

$$\int_{\mathbf{S}^{d-1}} |f(x - t_{\mathbf{b}}(x, \mathfrak{u})\mathfrak{u})| d\mathfrak{u} \lesssim_{\epsilon} \int_{\partial\Omega} |f(y)| dS(y), \tag{2.10}$$

**Proof.** Assume that  $\partial\Omega$  is locally parameterized by  $\xi: \mathbf{R}^d \to \mathbf{R}$  and  $\Omega$  is locally  $\{x \in \mathbf{R}^d : \xi(x) < 0\}$  so that

$$\xi(x_{\mathbf{b}}(t, x, \bar{v})) = \xi(x - t_{\mathbf{b}}(x, \bar{v})\bar{v}) = 0.$$

By the implicit function theorem, taking  $x_i$  and  $\bar{v}_i$  derivative respectively,

$$\partial_{j}\xi - \sum_{k=1}^{d} \partial_{k}\xi \frac{\partial t_{\mathbf{b}}}{\partial x_{j}} \bar{v}_{k} = 0, \quad \text{so that} \quad \frac{\partial t_{\mathbf{b}}}{\partial x_{j}} = \frac{n_{j}(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})},$$

$$\partial_{j}\xi t_{\mathbf{b}} + \sum_{k=1}^{d} \partial_{k}\xi \frac{\partial t_{\mathbf{b}}}{\partial \bar{v}_{j}} \bar{v}_{k} = 0, \quad \text{so that} \quad \frac{\partial t_{\mathbf{b}}}{\partial \bar{v}_{j}} = -t_{\mathbf{b}} \frac{n_{j}(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})}.$$

We then have

$$\frac{\partial (x_{\mathbf{b}})_i}{\partial x_j} = \delta_{ij} - \frac{\bar{v}_i n_j(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})}, \qquad \frac{\partial (x_{\mathbf{b}})_i}{\partial \bar{v}_j} = -t_{\mathbf{b}} \left\{ \delta_{ij} - \frac{\bar{v}_i n_j(x_{\mathbf{b}})}{\bar{v} \cdot n(x_{\mathbf{b}})} \right\}.$$

Now we prove (2.9) for d=3. Without loss of generality we may assume that  $\partial_3 \xi(x_{\mathbf{b}}) \neq 0$ . Using the spherical coordinates,  $\mathfrak{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbf{S}^2$  and

$$\xi(x_1 - t_{\mathbf{b}}(x, \mathbf{u})\sin\theta\cos\phi, x_2 - t_{\mathbf{b}}(x, \mathbf{u})\sin\theta\cos\phi, x_3 - t_{\mathbf{b}}(x, \mathbf{u})\cos\theta) = 0.$$

By the implicit function theorem we compute

$$\frac{\partial}{\partial \theta} t_{\mathbf{b}}(x, \mathfrak{u}(\theta, \phi)) = -t_{\mathbf{b}}(x, \mathfrak{u}(\theta, \phi)) \frac{-\sin \theta + \partial_{1} \xi(x_{\mathbf{b}}) \cos \theta \cos \phi + \partial_{2} \xi(x_{\mathbf{b}}) \cos \theta \sin \phi}{\mathfrak{u}(\theta, \phi) \cdot n(x_{\mathbf{b}}) |\nabla_{x} \xi(x_{\mathbf{b}})|},$$

$$\frac{\partial}{\partial \phi} t_{\mathbf{b}}(x, \mathfrak{u}(\theta, \phi)) = t_{\mathbf{b}}(x, \mathfrak{u}(\theta, \phi)) \frac{\partial_{1} \xi(x_{\mathbf{b}}) \sin \theta \sin \phi - \partial_{2} \xi(x_{\mathbf{b}}) \sin \theta \cos \phi}{\mathfrak{u}(\theta, \phi) \cdot n(x_{\mathbf{b}}) |\nabla_{x} \xi(x_{\mathbf{b}})|}.$$

Since the Jacobian matrix is

$$\begin{split} \operatorname{Jac} \left\{ \frac{\partial ((x_{\mathbf{b}})_{1}, (x_{\mathbf{b}})_{2})}{\partial (\theta, \phi)} \right\} \\ &= \left( \begin{array}{c} -\frac{\partial t_{\mathbf{b}}}{\partial \theta} \sin \theta \cos \phi - t_{\mathbf{b}} \cos \theta \cos \phi & -\frac{\partial t_{\mathbf{b}}}{\partial \phi} \sin \theta \cos \phi + t_{\mathbf{b}} \sin \theta \sin \phi \\ -\frac{\partial t_{\mathbf{b}}}{\partial \theta} \sin \theta \sin \phi - t_{\mathbf{b}} \cos \theta \sin \phi & -\frac{\partial t_{\mathbf{b}}}{\partial \phi} \sin \theta \sin \phi - t_{\mathbf{b}} \sin \theta \cos \phi \end{array} \right), \end{split}$$

we have

$$\det\left(\frac{\partial((x_{\mathbf{b}})_{1},(x_{\mathbf{b}})_{2})}{\partial(\theta,\phi)}\right) \ge \frac{(t_{\mathbf{b}})^{2}\sin\theta|\partial_{3}\xi(x_{\mathbf{b}})|}{|n(x_{\mathbf{b}})\cdot\mathfrak{u}||\nabla\xi(x_{\mathbf{b}})|},\tag{2.11}$$

and hence

$$\int_{\mathfrak{u}\in\mathbf{S}^{2},\mathfrak{u}'\sim\mathfrak{u}} |f(x-t_{\mathbf{b}}(x,\mathfrak{u}')\mathfrak{u}')||n(x)\cdot\mathfrak{u}'|d\mathfrak{u}'$$

$$\leq \iint_{\substack{\theta'\in[0,\pi),\phi'\in[0,2\pi)\\ (\theta',\phi')\sim(\theta,\phi)}} |f(x-t_{\mathbf{b}}(x,\mathfrak{u}')\mathfrak{u}')||n(x)\cdot\mathfrak{u}'|\sin\theta'd\phi'd\theta', \qquad (2.12)$$

where  $\mathfrak{u}' = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta')$ . Now we apply a change of variables

$$(\theta', \phi') \mapsto ((x_{\mathbf{b}}(x, \mathbf{u}'))_1, (x_{\mathbf{b}}(x, \mathbf{u}'))_2) \equiv ((x'_{\mathbf{b}})_1, (x'_{\mathbf{b}})_2), \tag{2.13}$$

and use (2.11) to further bound (2.12) as

$$\iint_{\substack{(x_{\mathbf{b}}')_1 \sim (x_{\mathbf{b}})_1 \\ (x_{\mathbf{t}}')_2 \sim (x_{\mathbf{b}})_2}} |f(x_{\mathbf{b}}')| \frac{|n(x) \cdot \mathfrak{u}(x_{\mathbf{b}}')||n(x_{\mathbf{b}}') \cdot \mathfrak{u}(x_{\mathbf{b}}')|}{|t_{\mathbf{b}}(x, \mathfrak{u}(x_{\mathbf{b}}'))|^2} \frac{|\nabla \xi(x_{\mathbf{b}}')|}{|\partial_3 \xi(x_{\mathbf{b}}')|} d(x_{\mathbf{b}}')_1 d(x_{\mathbf{b}}')_2. \tag{2.14}$$

Notice the surface measure of  $\partial\Omega$  is  $dS = \frac{|\nabla \xi(x_{\mathbf{b}}')|}{|\partial_3 \xi(x_{\mathbf{b}}')|} d(x_{\mathbf{b}}')_1 d(x_{\mathbf{b}}')_2$  if  $\partial_3 \xi(x_{\mathbf{b}}') \neq 0$ . Denote  $\mathfrak{u}(x_{\mathbf{b}}') \equiv \frac{x - x_{\mathbf{b}}'}{|x - x_{\mathbf{b}}'|}$  and we use (2.4) as

$$t_{\mathbf{b}}(x, \mathfrak{u}(x'_{\mathbf{b}})) \gtrsim_{\Omega} |n(x) \cdot \mathfrak{u}(x'_{\mathbf{b}})|, \quad t_{\mathbf{b}}(x, \mathfrak{u}(x'_{\mathbf{b}})) \gtrsim_{\Omega} |n(x'_{\mathbf{b}}) \cdot \mathfrak{u}(x'_{\mathbf{b}})|,$$

to bound (2.14) by

$$\iint_{y \in \mathbf{S}^2, y \sim x_{\mathbf{b}}} |f(y)| dS(y).$$

For d=2, using  $\mathfrak{u}=(\cos\theta,\sin\theta)$  if  $\partial_2\xi(x_{\mathbf{b}})\neq 0$  we can compute

$$\frac{\partial}{\partial \theta} t_{\mathbf{b}}(x, \mathfrak{u}(\theta)) = t_{\mathbf{b}}(x, \mathfrak{u}(\theta)) \frac{n(x_{\mathbf{b}}) \cdot (\sin \theta, -\cos \theta)}{n(x_{\mathbf{b}}) \cdot \mathfrak{u}(\theta)},$$

and

$$\det\left(\frac{\partial(x_{\mathbf{b}})_{1}}{\partial\theta}\right) \ge \frac{t_{\mathbf{b}}(x,\mathfrak{u}(\theta))|\partial_{2}\xi(x_{\mathbf{b}})|}{|n(x_{\mathbf{b}})\cdot\mathfrak{u}(\theta)||\nabla\xi(x_{\mathbf{b}})|}.$$
(2.15)

The rest of proof of (2.9) is same (even simpler!) as the d=3 case.

For (2.10), since there is a lower bound for  $t_{\mathbf{b}}(x,\mathfrak{u}) \geq \epsilon$  for  $\{|x-y| < \epsilon\} \cap \partial\Omega = \emptyset$  and  $\mathfrak{u} \in \mathbf{S}^{d-1}$ , using (2.11) and (2.15), it is easy to prove (2.10).

### 3 $L^2$ Estimate

The main purpose of this section is to prove the following:

Proposition 3.1 Assume

$$\iint_{\Omega \times \mathbf{R}^3} g(x, v) \sqrt{\mu} \, dx dv = 0, \quad \int_{\gamma_-} r \sqrt{\mu} d\gamma = 0.$$
 (3.1)

Then there exists a unique solution to

$$v \cdot \nabla_x f + Lf = g, \quad f_- = P_\gamma f + r, \tag{3.2}$$

such that  $\iint_{\Omega \times \mathbf{R}^3} f \sqrt{\mu} \ dx dv = 0$  and

$$||f||_{\nu} + |f|_2 \lesssim ||g||_2 + |r|_2.$$

For the proof of Proposition 3.1 we need several lemmas. We start with the simple transport equation with a penalization term:

**Lemma 3.2** For any  $\varepsilon > 0$ , there exists a unique solution to

$$\varepsilon f + v \cdot \nabla_x f = g, \quad f_- = r,$$

so that

$$||f||_2 + |f|_2 \lesssim_{\varepsilon} ||g||_2 + |r|_2,$$

$$||\langle v \rangle^{\beta} e^{\zeta |v|^2} f||_{\infty} + |\langle v \rangle^{\beta} e^{\zeta |v|^2} f|_{\infty} \lesssim_{\varepsilon} ||\langle v \rangle^{\beta} e^{\zeta |v|^2} g||_{\infty} + |\langle v \rangle^{\beta} e^{\zeta |v|^2} r|_{\infty},$$

for all  $\beta \geq 0, \zeta \geq 0$ . Moreover if g and r are continuous away from the grazing set  $\gamma_0$ , then f is continuous away from  $\mathfrak{D}$ . In particular, if  $\Omega$  is convex then  $\mathfrak{D} = \gamma_0$ .

**Proof.** The existence of f and  $L^{\infty}$ -bound follow from integration along the characteristic lines of  $\frac{dx}{ds} = \bar{v} \in \mathbf{R}^d$ , and  $\frac{dv}{ds} = 0$  (Recall (1.8) for notations). More precisely, setting  $h(x,v) = \langle v \rangle^{\beta} e^{\zeta |v|^2} f(x,v)$ , the integrated form of the equation for h is:

$$\begin{split} h(x,v) &= \mathbf{1}_{t>t_{\mathbf{b}}(x,v)} \Big\{ h(x-t\bar{v},v) e^{-\varepsilon t} \\ &+ \int_0^t \langle v \rangle^\beta e^{\zeta |v|^2} g(x-(t-s)\bar{v},v) e^{-\varepsilon (t-s)} ds \Big\} \\ &+ \mathbf{1}_{t\leq t_{\mathbf{b}}(x,v)} \Big\{ \langle v \rangle^\beta e^{\zeta |v|^2} r(x-t\bar{v},v) e^{-\varepsilon t} \\ &+ \int_{t-t_{\mathbf{b}}(x,v)}^t \langle v \rangle^\beta e^{\zeta |v|^2} g(x-(t-s)\bar{v},v) e^{-\varepsilon (t-s)} ds \Big\}, \end{split}$$

where  $t_{\mathbf{b}}(x, v)$  is defined in (1.9). We prove the  $L^{\infty}$ -bound by choosing a large  $t = t(\varepsilon)$  such that

$$|h(x,v)| \lesssim \frac{1}{2} \{ \|h\|_{\infty} + |h|_{\infty} \} + \|\langle v \rangle^{\beta} e^{\zeta |v|^2} r\|_{\infty} + \varepsilon^{-1} \|\langle v \rangle^{\beta} e^{\zeta |v|^2} g\|_{\infty}.$$

In order to prove the continuity, let  $(x,v) \in \overline{\Omega} \times \mathbf{R}^3 \backslash \mathfrak{D}$ . Then by the definition of  $\mathfrak{D}$ ,  $n(x_{\mathbf{b}}(x,v)) \cdot v < 0$  and hence  $t_{\mathbf{b}}(x,v)$  is smooth by Lemma 2.3. Therefore, if g and r are continuous, f(x,v) is continuous at  $(x,v) \in \overline{\Omega} \times \mathbf{R}^3 \backslash \mathfrak{D}$ .

Suppose now that  $\Omega$  is convex. In order to show that  $\mathfrak{D}=\gamma_0$ , since  $\mathfrak{D}\supset\gamma_0$  is true for any  $\Omega$ , it suffices to show that  $\mathfrak{D}\subset\gamma_0$ . Since  $\Omega$  is convex  $t_{\mathbf{b}}(x,v)=0=t_{\mathbf{b}}(x,-v)$  for  $x\in\partial\Omega,\ v\neq0$ . Therefore  $\gamma_0^{\mathbf{S}}=\partial\Omega\times\{0\}$ . Hence, if  $(x_{\mathbf{b}}(x,v),v)\in\gamma_0^{\mathbf{S}}$  then v=0 and  $x_{\mathbf{b}}(x,0)=x\in\partial\Omega$  and  $(x,v)\in\gamma_0^{\mathbf{S}}$ .

The  $L^2$ -estimate and the uniqueness follow from Green's formula, since  $||f||_2 \lesssim ||\langle v \rangle^{\beta} e^{\zeta |v|^2} f||_{\infty}$ .

In next lemma we add to the penalized transport equation a suitably cut-offed linearized Boltzmann operator. Moreover we include a *reduced* diffuse reflection boundary condition, with the purpose of setting up a contracting map argument. We have

**Lemma 3.3** For any  $\varepsilon > 0$ , m > 0, and for any integer j > 0, there exists a unique solution to

$$\varepsilon f + v \cdot \nabla_x f + L_m f = g, \qquad f_- = \left(1 - \frac{1}{j}\right) P_\gamma f + r, \tag{3.3}$$

with  $L_m$  the linearized Boltzmann operator corresponding to the cut-offed cross section  $B_m = \min\{B, m\}$ . Moreover, uniformly in j, we have

$$||f||_{\nu} + |f|_2 \lesssim_{\varepsilon,m} ||g||_2 + |r|_2.$$

Finally the limit f as  $j \to \infty$  of the sequence  $\{f^j\}$  exists and solves uniquely

$$\varepsilon f + v \cdot \nabla_x f + L_m f = g, \qquad f_- = P_\gamma f + r.$$
 (3.4)

**Proof.** Denote  $L_m = \nu_m - K_m$ . For any j, we apply Lemma 3.2 to the following double-iteration in both j and  $\ell$ :

$$\varepsilon f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} + \nu_m f^{\ell+1} - K_m f^{\ell} = g, 
f_-^{\ell+1} = (1 - \frac{1}{j}) P_{\gamma} f^{\ell} + r,$$
(3.5)

with  $f^0 = 0$ .

Step 1. We first fix m and j and take  $\ell \to \infty$ .

From Green's identity,

$$\varepsilon \|f^{\ell+1}\|_{2}^{2} + \frac{1}{2}|f^{\ell+1}|_{2,+}^{2} + \|f^{\ell+1}\|_{\nu_{m}}^{2}$$

$$= (K_{m}f^{\ell}, f^{\ell+1}) + \frac{1}{2}|(1 - \frac{1}{i})P_{\gamma}f^{\ell} + r|_{2,-}^{2} + (f^{\ell+1}, g).$$

From  $(K_m(f^{\ell} + f^{\ell+1}), f^{\ell} + f^{\ell+1}) \le (\nu_m(f^{\ell} + f^{\ell+1}), f^{\ell} + f^{\ell+1})$ , we deduce  $(K_m f^{\ell}, f^{\ell+1}) < (\nu_m f^{\ell}, f^{\ell+1})$ .

Moreover, there is  $C_i$  such that

$$|(1 - \frac{1}{j})P_{\gamma}f^{\ell} + r|_{2,-}^{2} \le |(1 - \frac{1}{j})P_{\gamma}f^{\ell}|_{2,-}^{2} + \frac{1}{2j^{2}}|P_{\gamma}f^{\ell}|_{2,-}^{2} + C_{j}|r|_{2}^{2}.$$
(3.6)

We derive from  $|P_{\gamma}f^{\ell}|_{2,-}^2 \le |f^{\ell}|_{2,+}^2$  and  $\|\cdot\|_2 \ge \frac{1}{m}\|\cdot\|_{\nu_m}$ ,

$$\left\{\frac{\varepsilon}{m}+1\right\} \|f^{\ell+1}\|_{\nu_m}^2 + \frac{1}{2}|f^{\ell+1}|_{2,+}^2 \le \frac{1}{2} \|f^{\ell}\|_{\nu_m}^2 + \frac{1}{2} \|f^{\ell+1}\|_{\nu_m}^2 \\
+ \frac{1}{2} (1 - \frac{2}{j} + \frac{3}{2j^2})|f^{\ell}|_{2,+}^2 + \frac{1}{2} C_j |r|_2^2 + \frac{\varepsilon}{2m} \|f^{\ell+1}\|_{\nu_m}^2 + 4\varepsilon^{-1} \|g\|_2^2.$$

Since  $\frac{\varepsilon}{2m} + 1 - \frac{1}{2} > \frac{1}{2}$  and  $1 - \frac{2}{j} + \frac{3}{2j^2} < 1$ , by iteration over  $\ell$ , for some  $\eta_{\varepsilon,m,j} < 1$ ,

$$||f^{\ell+1}||_{\nu_m}^2 + |f^{\ell+1}||_{2,+}^2 \le \eta_{\varepsilon,m,j} \{ ||f^{\ell}||_{\nu_m}^2 + |f^{\ell}||_{2,+}^2 \} + C_{\varepsilon,m,j} \{ |r|_2^2 + ||g||_2^2 \}.$$

Taking the difference of  $f^{\ell+1} - f^{\ell}$  we conclude that  $f^{\ell}$  is a Cauchy sequence. We take  $\ell \to \infty$  to obtain  $f^j$  as a solution to the equation

$$\varepsilon f^{j} + v \cdot \nabla_{x} f^{j} + L_{m} f^{j} = g, \qquad f_{-}^{j} = (1 - \frac{1}{j}) P_{\gamma} f^{j} + r.$$
 (3.7)

Step 2. We take  $j \to \infty$  for  $f^j$ .

By Green's identity we obtain uniformly in j,

$$\varepsilon ||f^{j}||_{2}^{2} + (L_{m}f^{j}, f^{j}) + \frac{1}{2}|f^{j}|_{2,+}^{2} - \frac{1}{2}|P_{\gamma}f^{j} + r|_{2,-}^{2} = \int f^{j}g.$$

We rewrite for any  $\eta > 0$ 

$$\frac{1}{2}|P_{\gamma}f^{j}+r|_{2,-}^{2} = \frac{1}{2}|P_{\gamma}f^{j}|_{2,-}^{2} + \frac{1}{2}|r|_{2}^{2} + \int_{\gamma_{-}} P_{\gamma}f^{j} r d\gamma$$

$$\leq \frac{1}{2}|P_{\gamma}f^{j}|_{2,-}^{2} + C_{\eta}|r|_{2}^{2} + \eta|P_{\gamma}f^{j}|_{2,-}^{2}, \tag{3.8}$$

so that from  $\int f^j g \leq \frac{\varepsilon}{2} ||f^j||_2^2 + C_{\varepsilon} ||g||_2^2$  and from the spectral gap of  $L_m$ , we have

$$\frac{\varepsilon}{2} \|f^j\|_2^2 + \|(\mathbf{I} - \mathbf{P})f^j\|_{\nu_m}^2 + \frac{1}{2} |(1 - P_\gamma)f^j|_{2,+}^2 \le C_{\eta,\varepsilon} \{|r|_2^2 + \|g\|_2^2\} + \eta |P_\gamma f^j|_{2,-}^2. \tag{3.9}$$

But from the equation we have

$$v \cdot \nabla_x [f^j]^2 = -2\varepsilon [f^j]^2 - 2f^j L_m (\mathbf{I} - \mathbf{P}) f^j + 2f^j g.$$

Taking absolute value and integrating on  $\Omega \times \mathbf{R}^3$ , from (3.9) we have

$$||v \cdot \nabla_x (f^j)^2||_1 \leq C_{\varepsilon} \{||f^j||_2^2 + ||(\mathbf{I} - \mathbf{P})f^j||_{\nu_m}^2 + ||g||_2^2\}$$
  
$$\leq C_{\eta, \varepsilon} \{|r|_2^2 + ||g||_2^2\} + \eta C_{\varepsilon} |P_{\gamma} f^j|_{2, -}^2.$$

Hence, by Lemma 2.1, for any  $\gamma^{\varepsilon'}$  in (2.1) away from  $\gamma_0$ , we have

$$|f^{j} \mathbf{1}_{\gamma^{\varepsilon'}}|_{2}^{2} \leq C_{\varepsilon,\eta,\varepsilon'}\{|r|_{2}^{2} + ||g||_{2}^{2}\} + \eta C_{\varepsilon,\varepsilon'}|P_{\gamma}f^{j}|_{2,-}^{2}.$$
(3.10)

From (1.16) we can write  $P_{\gamma}f^{j} = z_{\gamma}(x)\sqrt{\mu}$  for a suitable function  $z_{\gamma}(x)$  and, from  $|P_{\gamma}f^{j}\mathbf{1}_{\gamma^{\varepsilon'}}|_{2} \lesssim |f^{j}\mathbf{1}_{\gamma^{\varepsilon'}}|_{2} < +\infty$ , for  $\varepsilon'$  small

$$\begin{split} |P_{\gamma}f^{j} \mathbf{1}_{\gamma^{\varepsilon'}}|_{2}^{2} &= \int_{\partial\Omega} |z_{\gamma}(x)|^{2} \int_{|n(x)\cdot v| \geq \varepsilon', |v| \leq \frac{1}{\varepsilon'}} \mu(v)|v\cdot n(x)| dv dx \\ &\geq \int_{\partial\Omega} |z_{\gamma}(x)|^{2} dx \times \frac{1}{2} \int_{\mathbf{R}^{3}} \mu(v)|v\cdot n(x)| dv \\ &= \frac{1}{2} |P_{\gamma}f^{j}|_{2}^{2}, \end{split} \tag{3.11}$$

where we used the fact

$$\int_{|n(x)\cdot v|\leq \varepsilon'} \mu(v)|n(x)\cdot v|dv \leq \int_{-\varepsilon'}^{\varepsilon'} e^{-v_{\parallel}^{2}} |v_{\parallel}|dv_{\parallel} \int_{\mathbf{R}^{2}} e^{-|v_{\perp}|^{2}/2} dv_{\perp} \leq C\varepsilon', \qquad (3.12)$$

$$\int_{|v|\geq 1/\varepsilon'} \mu(v)|n(x)\cdot v|dv \leq C\varepsilon'.$$

Therefore we conclude

$$\frac{1}{2} |P_{\gamma} f^{j}|_{2}^{2} - |(1 - P_{\gamma}) f^{j}|_{2,+}^{2} \leq |P_{\gamma} f^{j} \mathbf{1}_{\gamma^{\varepsilon'}}|_{2}^{2} - |(1 - P_{\gamma}) f^{j} \mathbf{1}_{\gamma^{\varepsilon'}}|_{2,+}^{2} \\
\lesssim |f^{j} \mathbf{1}_{\gamma^{\varepsilon'}}|_{2}^{2} \\
\leq C_{\varepsilon, \eta, \varepsilon'} \{|r|_{2}^{2} + ||g||_{2}^{2}\} + \eta C_{\varepsilon, \varepsilon'} |P_{\gamma} f^{j}|_{2,-}^{2}.$$
(3.13)

Adding  $4 \times (3.9)$  to (3.13), we obtain:

$$2\varepsilon ||f^{j}||_{2}^{2} + |(1 - P_{\gamma})f^{j}|_{2,+}^{2} + 4||(\mathbf{I} - \mathbf{P})f^{j}||_{\nu_{m}}^{2} + \frac{1}{2}|P_{\gamma}f^{j}|_{2}^{2}$$

$$\leq C_{\varepsilon,\eta}\{|r|_{2}^{2} + ||g||_{2}^{2}\} + \eta(1 + C_{\varepsilon,\varepsilon'})|P_{\gamma}f^{j}|_{2,-}^{2}.$$

Choosing  $\eta$  small and taking the weak limit  $j \to \infty$ , we complete the proof of the lemma.

Next lemma states the crucial  $L^2$  bound for  $\mathbf{P}f$ . It will provide uniform in  $\varepsilon$  estimates which allow to take the limit  $\varepsilon \to 0$  in (3.4).

**Lemma 3.4** Let f be a solution, in the sense of (3.15) below, to

$$v \cdot \nabla_x f + Lf = g, \qquad f_- = P_\gamma f + r, \tag{3.14}$$

with

$$\iint_{\Omega\times\mathbf{R}^3}f\sqrt{\mu}dxdv=\iint_{\Omega\times\mathbf{R}^3}g\sqrt{\mu}dxdv=\int_{\gamma_-}r\sqrt{\mu}d\gamma=0,$$

then we have

$$\|\mathbf{P}f\|_{\nu}^{2} \lesssim \|(\mathbf{I} - \mathbf{P})f\|_{\nu}^{2} + \|g\|_{2}^{2} + |(1 - P_{\gamma})f|_{2,+}^{2} + |r|_{2}^{2}.$$

**Proof.** The Green's identity (2.6) provides the following weak version of (3.14):

$$\int_{\gamma} \psi f d\gamma - \iint_{\Omega \times \mathbf{R}^3} v \cdot \nabla_x \psi f = -\iint_{\Omega \times \mathbf{R}^3} \psi L(\mathbf{I} - \mathbf{P}) f + \iint_{\Omega \times \mathbf{R}^3} \psi g.$$
 (3.15)

Recall that  $\mathbf{P}f = \{a+v\cdot b+c[\frac{|v|^2}{2}-\frac{3}{2}]\}\sqrt{\mu}$  on  $\Omega\times\mathbf{R}^3$ . The key of the proof is to choose suitable  $H^1$  test functions  $\psi$  to estimate a,b and c in (3.22), (3.29), (3.34), (3.42) thus concluding the proof of Proposition 3.1.

Step 1. Estimate of c

To estimate c, we first choose the test function

$$\psi = \psi_c \equiv (|v|^2 - \beta_c)\sqrt{\mu}v \cdot \nabla_x \phi_c(x), \tag{3.16}$$

where

$$-\Delta_x \phi_c(x) = c(x), \quad \phi_c|_{\partial\Omega} = 0,$$

and  $\beta_c$  is a constant to be determined. From the standard elliptic estimate, we have

$$\|\phi_c\|_{H^2} \lesssim \|c\|_2.$$

With the choice (3.16), the right hand side of (3.15) is bounded by

$$||c||_2 \{ ||(\mathbf{I} - \mathbf{P})f||_2 + ||g||_2 \}.$$
 (3.17)

We have

$$v \cdot \nabla_x \psi_c = \sum_{i,j=1}^d (|v|^2 - \beta_c) v_i \sqrt{\mu} v_j \partial_{ij} \phi_c(x),$$

so that the left hand side of (3.15) takes the form, for  $i = 1, \dots, d$ ,

$$\int_{\partial\Omega\times\mathbf{R}^{3}} (n(x)\cdot v)(|v|^{2} - \beta_{c})\sqrt{\mu} \sum_{i=1}^{d} v_{i}\partial_{i}\phi_{c}f$$

$$-\iint_{\Omega\times\mathbf{R}^{3}} (|v|^{2} - \beta_{c})\sqrt{\mu} \Big\{ \sum_{i,j=1}^{d} v_{i}v_{k}\partial_{ij}\phi_{c} \Big\} f. \tag{3.18}$$

We decompose

$$f_{\gamma} = P_{\gamma}f + \mathbf{1}_{\gamma_{+}}(1 - P_{\gamma})f + \mathbf{1}_{\gamma_{-}}r, \qquad \text{on } \gamma, \tag{3.19}$$

$$f = \left\{ a + v \cdot b + c \left[ \frac{|v|^2}{2} - \frac{3}{2} \right] \right\} \sqrt{\mu} + (\mathbf{I} - \mathbf{P}) f, \quad \text{on } \Omega \times \mathbf{R}^3.$$
 (3.20)

We shall choose  $\beta_c$  so that, for all i

$$\int (|v|^2 - \beta_c) v_i^2 \mu(v) dv = 0.$$
 (3.21)

Since  $\mu = \frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$  the desired value of  $\beta_c$  is  $\beta_c = 5$ . Because of the choice of  $\beta_c$ , there is no a contribution in the bulk and no  $P_{\gamma}f$  contribution at the boundary in (3.18).

Therefore, substituting (3.19) and (3.20) into (3.18), since the b terms and the off-diagonal c terms also vanish by oddness in v in the bulk, the left hand side of (3.15) becomes

$$\begin{split} &\sum_{i=1}^{d} \int_{\gamma} (|v|^{2} - \beta_{c}) v_{i}^{2} n_{i} \sqrt{\mu} \partial_{i} \phi_{c} [(1 - P_{\gamma}) f \mathbf{1}_{\gamma_{+}} + r \mathbf{1}_{\gamma_{-}}] \\ &- \sum_{i=1}^{d} \int_{\mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i}^{2} (\frac{|v|^{2}}{2} - \frac{3}{2}) \mu(v) dv \int_{\Omega} \partial_{ii} \phi_{c}(x) c(x) dx \\ &- \sum_{i=1}^{d} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i} \sqrt{\mu} (v \cdot \nabla_{x}) \partial_{i} \phi_{c} (\mathbf{I} - \mathbf{P}) f. \end{split}$$

For  $\beta_c = 5$ ,  $\int_{\mathbf{R}^3} (|v|^2 - \beta_c) v_i^2 (\frac{|v|^2}{2} - \frac{3}{2}) \mu(v) dv = 10\pi \sqrt{2\pi}$ . Therefore, we obtain from (3.17)

$$-10\pi\sqrt{2\pi}\int_{\Omega}\Delta_{x}\phi_{c}(x)c(x) \lesssim \|c\|_{2}\{|(1-P_{\gamma})f|_{2,+} + \|(\mathbf{I}-\mathbf{P})f\|_{2} + \|g\|_{2} + |r|_{2}\},$$

where we have used the elliptic estimate and the trace estimate:

$$|\nabla_x \phi_c|_2 \lesssim ||\phi_c||_{H^2} \lesssim ||c||_2.$$

Since  $-\Delta_x \phi_c = c$ , from (3.16) we obtain

$$||c||_2^2 \lesssim \{|(1-P_{\gamma})f|_{2,+} + ||(\mathbf{I}-\mathbf{P})f||_2 + ||g||_2 + |r|_2\}||c||_2,$$

and hence

$$||c||_{2}^{2} \lesssim |(1 - P_{\gamma})f|_{2,+}^{2} + ||(\mathbf{I} - \mathbf{P})f||_{2}^{2} + ||g||_{2}^{2} + |r|_{2}^{2}.$$
 (3.22)

Step 2. Estimate of b

We shall establish the estimate of b by estimating  $(\partial_i \partial_j \Delta^{-1} b_j) b_i$  for all  $i, j = 1, \dots, d$ , and  $(\partial_i \partial_j \Delta^{-1} b_i) b_i$  for  $i \neq j$ .

We fix i, j. To estimate  $\partial_i \partial_j \Delta^{-1} b_j b_i$  we choose as test function in (3.15)

$$\psi = \psi_b^{i,j} \equiv (v_i^2 - \beta_b) \sqrt{\mu} \partial_j \phi_b^j, \quad i, j = 1, \dots, d,$$
(3.23)

where  $\beta_b$  is a constant to be determined, and

$$-\Delta_x \phi_b^j(x) = b_j(x), \quad \phi_b^j|_{\partial\Omega} = 0. \tag{3.24}$$

From the standard elliptic estimate

$$\|\phi_b^j\|_{H^2} \lesssim \|b\|_2.$$

Hence the right hand side of (3.15) is now bounded by

$$||b||_2 \{ ||(\mathbf{I} - \mathbf{P})f||_2 + ||g||_2 \}.$$
 (3.25)

Now substitute (3.19) and (3.20) into the left hand side of (3.15). Note that  $(v_i^2 - \beta_b)\{n(x) \cdot v\}\mu$ is odd in v, therefore  $P_{\gamma}f$  contributions to (3.15) vanishes. Moreover, by (3.20), the a, c contributions to (3.15) also vanish by oddness. Therefore the left hand side of (3.15) takes the form

$$\int_{\partial\Omega\times\mathbf{R}^{3}} (n(x)\cdot v)(v_{i}^{2}-\beta_{b})\sqrt{\mu}\partial_{j}\phi_{b}^{j}f - \iint_{\Omega\times\mathbf{R}^{3}} (v_{i}^{2}-\beta_{b})\sqrt{\mu}\left\{\sum_{l} v_{l}\partial_{lj}\phi_{b}^{j}\right\}f$$

$$= \int_{\partial\Omega\times\mathbf{R}^{3}} (n(x)\cdot v)(v_{i}^{2}-\beta_{b})\sqrt{\mu}\partial_{j}\phi_{b}^{j}[(1-P_{\gamma})f+r]\mathbf{1}_{\gamma+}$$

$$-\int_{\Omega}\int_{\mathbf{R}^{3}} \sum_{l} (v_{i}^{2}-\beta_{b})v_{l}^{2}\mu\partial_{lj}\phi_{b}^{j}(x)b_{l}dvdx$$

$$-\iint_{\Omega\times\mathbf{R}^{3}} \sum_{l} (v_{i}^{2}-\beta_{b})v_{l}\sqrt{\mu}\partial_{lj}\phi_{b}^{j}(x)(\mathbf{I}-\mathbf{P})f.$$
(3.26)

Furthermore, since  $\mu(v) = \frac{1}{2\pi} \prod_{i=1}^{3} e^{-\frac{|v_i|^2}{2}}$  we can choose  $\beta_b > 0$  such that for all i,

$$\int_{\mathbf{R}^3} [(v_i)^2 - \beta_b] \mu(v) dv = \int_{\mathbf{R}} [v_1^2 - \beta_b] e^{-\frac{|v_1|^2}{2}} dv_1 = 0.$$
 (3.28)

We remark that the choice (3.28) also plays a crucial rule in the dynamical estimate (6.11). Since  $\mu(v) = \frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$ , the desired value is  $\beta_b = 1$ . Note that for such chosen  $\beta_b$ , and for  $i \neq k$ , by an explicit computation

$$\int (v_i^2 - \beta_b) v_k^2 \mu dv = \int (v_1^2 - \beta_b) v_2^2 \frac{1}{2\pi} e^{-\frac{|v_1|^2}{2}} e^{-\frac{|v_2|^2}{2}} e^{-\frac{|v_3|^2}{2}} dv 
= \int_{\mathbf{R}} (v_1^2 - \beta_b) e^{-\frac{|v_1|^2}{2}} dv_1 = 0, 
\int (v_i^2 - \beta_b) v_i^2 \mu dv = \int_{\mathbf{R}} [v_1^4 - \beta_b v_1^2] e^{-\frac{|v_1|^2}{2}} dv_1 = 2\sqrt{2\pi} \neq 0.$$

Therefore, (3.27) becomes, by (3.23),

$$-\iint_{\Omega \times \mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) v_{i}^{2} \mu dv \partial_{ij} \phi_{b}^{j}(x) b_{i} + \sum_{k(\neq i)} \underbrace{\left[ \int_{\mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) v_{k}^{2} \mu \right]}_{=0} \int_{\Omega} \partial_{kj} \phi_{b}^{j}(x) b_{k}$$

$$= 2\sqrt{2\pi} \int_{\Omega} (\partial_{i} \partial_{j} \Delta^{-1} b_{j}) b_{i}.$$

Hence we have the following estimate for all i, j, by (3.25):

$$\left| \int_{\Omega} \partial_i \partial_j \Delta^{-1} b_j b_i \right| \lesssim |(1 - P_{\gamma}) f|_{2,+}^2 + \|(\mathbf{I} - \mathbf{P}) f\|_2^2 + \|g\|_2^2 + |r|_2^2 + \varepsilon \|b\|_2^2.$$
 (3.29)

In order to estimate  $\partial_j(\partial_j\Delta^{-1}b_i)b_i$  for  $i\neq j$ , we choose as test function in (3.15)

$$\psi = |v|^2 v_i v_j \sqrt{\mu} \partial_j \phi_b^i(x), \quad i \neq j, \tag{3.30}$$

where  $\phi_b^i$  is given by (3.24). Clearly, the right hand side of (3.15) is again bounded by (3.25). We substitute again (3.19) and (3.20) into the left hand side of (3.15). The  $P_{\gamma}f$  contribution and a, c contributions vanish again due to oddness. Then the left hand side of (3.15) becomes

$$\int_{\partial\Omega\times\mathbf{R}^{3}} \{n\cdot v\} |v|^{2} v_{i} v_{j} \sqrt{\mu} \partial_{j} \phi_{b}^{i} f - \iint_{\Omega\times\mathbf{R}^{3}} |v|^{2} v_{i} v_{j} \sqrt{\mu} \{\sum_{k} v_{k} \partial_{kj} \phi_{b}^{i}\} f$$

$$= \int_{\partial\Omega\times\mathbf{R}^{3}} \{n\cdot v\} |v|^{2} v_{i} v_{j} \sqrt{\mu} \partial_{j} \phi_{b}^{i} [(1-P_{\gamma})f + r] \mathbf{1}_{\gamma_{+}} \tag{3.31}$$

$$-\iint_{\Omega \times \mathbf{R}^3} |v|^2 v_i^2 v_j^2 \mu [\partial_{ij} \phi_b^i b_j + \partial_{jj} \phi_b^i(x) b_i]$$
(3.32)

$$-\iint_{\Omega \times \mathbf{R}^3} |v|^2 v_i v_j v_k \sqrt{\mu} \partial_{kj} \phi_b^i(x) [\mathbf{I} - \mathbf{P}] f.$$
(3.33)

Note that (3.32) is evaluated as

$$7\sqrt{2\pi} \int_{\Omega} \{ (\partial_i \partial_j \Delta^{-1} b_i) b_j + (\partial_j \partial_j \Delta^{-1} b_i) b_i \}.$$

Furthermore, by (3.24),  $|\partial_j \phi_b^i|_2 \lesssim \|\phi_b^i\|_{H^2} \lesssim \|b\|_2$ , so that

$$(3.31) + (3.33) \lesssim ||b||_2 \{ |(1 - P_{\gamma})f|_{2,+} + |r|_2 + ||(\mathbf{I} - \mathbf{P})f||_2 \}.$$

Combining (3.29), we have the following estimate for  $i \neq j$ ,

$$\left| \int_{\Omega} \partial_{j} \partial_{j} \Delta^{-1} b_{i} b_{i} \right| 
\lesssim \left| \int_{\Omega} \partial_{i} \partial_{j} \Delta^{-1} b_{i} b_{j} \right| + \left| (1 - P_{\gamma}) f \right|_{2,+}^{2} + \left\| (\mathbf{I} - \mathbf{P}) f \right\|_{2}^{2} + \left\| g \right\|_{2}^{2} + \left| r \right|_{2}^{2} + \varepsilon \| b \|_{2}^{2} 
\lesssim \left| (1 - P_{\gamma}) f \right|_{2,+}^{2} + \left\| (\mathbf{I} - \mathbf{P}) f \right\|_{2}^{2} + \left\| g \right\|_{2}^{2} + \left| r \right|_{2}^{2} + \varepsilon \| b \|_{2}^{2}.$$
(3.34)

Moreover, by (3.29), for i = j = 1, 2, ..., d,

$$\left| \int_{\Omega} \partial_{j} \partial_{j} \Delta^{-1} b_{j} b_{j} \right|$$

$$\lesssim |(1 - P_{\gamma}) f|_{2,+}^{2} + ||(\mathbf{I} - \mathbf{P}) f||_{2}^{2} + ||g||_{2}^{2} + |r|_{2}^{2} + \varepsilon ||b||_{2}^{2}.$$
(3.35)

Combining (3.34) and (3.35), we sum over  $j = 1, 2, \dots, d$ , to obtain, for all  $i = 1, 2, \dots, d$ ,

$$||b_i||_2 \lesssim |(1 - P_\gamma)f|_{2,+}^2 + ||(\mathbf{I} - \mathbf{P})f||_2^2 + ||g||_2^2 + |r|_2^2.$$
 (3.36)

Step 3. Estimate of a

The estimate for a is more delicate because it requires the zero mass condition

$$\iint_{\Omega \times \mathbf{R}^3} f \sqrt{\mu} dx dv = 0 = \int_{\Omega} a dx.$$

We choose a test function

$$\psi = \psi_a \equiv (|v|^2 - \beta_a)v \cdot \nabla_x \phi_a \sqrt{\mu} = \sum_{i=1}^d (|v|^2 - \beta_a)v_i \partial_i \phi_a \sqrt{\mu}, \tag{3.37}$$

where

$$-\Delta_x \phi_a(x) = a(x), \qquad \frac{\partial}{\partial n} \phi_a|_{\partial \Omega} = 0.$$

It follows from the elliptic estimate with  $\int_{\Omega} a = 0$  that we have

$$\|\phi_a\|_{H^2} \lesssim \|a\|_2.$$

Since  $\int_{\mathbf{R}^3} (\frac{|v|^2}{2} - \frac{3}{2})(v_i)^2 \mu(v) dv \neq 0$ , we can choose  $\beta_a > 0$  so that, for all i,

$$\int_{\mathbf{R}^3} (|v|^2 - \beta_a) \left(\frac{|v|^2}{2} - \frac{3}{2}\right) (v_i)^2 \mu(v) = 0.$$
(3.38)

Since  $\mu(v) = \frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$ , the desired value is  $\beta_a = 10$ . Plugging  $\psi_a$  into (3.15) and its right hand side is again bounded by

$$||a||_2 \{ ||(\mathbf{I} - \mathbf{P})f||_2 + ||g||_2 \}.$$

By (3.19) and (3.20), since the c contribution vanishes in (3.15) due to our choice of  $\beta_a$  and the b contribution vanishes in (3.15) due to the oddness, the right hand side of (3.15) takes the form of

$$\sum_{i=1}^{d} \int_{\gamma} \{n \cdot v\}(|v|^2 - \beta_a) v_i \sqrt{\mu} \partial_i \phi_a(x) [P_{\gamma} f + (I - P_{\gamma}) f \mathbf{1}_{\gamma_+} + r \mathbf{1}_{\gamma_+}]$$
(3.39)

$$-\sum_{i,k=1}^{d} \iint_{\Omega \times \mathbf{R}^3} (|v|^2 - \beta_a) v_i v_k \partial_{ik} \phi_a(x) a(x) \mu(v)$$
(3.40)

$$-\sum_{i,k=1}^{d} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{a}) v_{i} v_{k} \partial_{ik} \phi_{a}(x) (\mathbf{I} - \mathbf{P}) f.$$
(3.41)

We make an orthogonal decomposition at the boundary,

$$v_i = (v \cdot n)n_i + (v_\perp)_i = v_n n_i + (v_\perp)_i$$
.

The contribution of  $P_{\gamma}f = z_{\gamma}(x)\sqrt{\mu}$  in (3.39) is

$$\int_{\gamma} (|v|^2 - \beta_a) v \cdot \nabla_x \phi_a(x) v_n \mu(v) z_{\gamma}(x)$$

$$= \int_{\gamma} (|v|^2 - \beta_a) v_n \frac{\partial \phi_a}{\partial n} v_n \mu(v) z_{\gamma}(x)$$

$$+ \int_{\gamma} (|v|^2 - \beta_a) v_{\perp} \cdot \nabla_x \phi_a v_n \mu(v) z_{\gamma}(x).$$

The crucial choice of (3.38) makes the first term vanish due to the Neumann boundary condition, while the second term also vanishes due to the oddness of  $(v_{\perp})_i v_n$  for all i. Therefore, (3.39) and (3.41) are bounded by

$$||a||_2 \{ ||(\mathbf{I} - \mathbf{P})f||_2 + |(1 - P_\gamma)f|_{2,+} + |r|_2 \}.$$

The second term (3.40), for  $k \neq i$  vanishes due to the oddness. Hence we only have the k = i contribution:

$$\sum_{i=1}^{d} \iint_{\Omega \times \mathbf{R}^3} (|v|^2 - \beta_a)(v_i)^2 \mu \partial_{ii} \phi_a a.$$

Using  $-\Delta_x \phi_a = a$  we obtain

$$||a||_{2}^{2} \lesssim ||(\mathbf{I} - \mathbf{P})f||_{2}^{2} + |(1 - P_{\gamma})f|_{2,+}^{2} + |r|_{2}^{2} + ||g||_{2}^{2}.$$
(3.42)

We close this section by proving Proposition 3.1.

### Proof of the Proposition 3.1.

We keep m fixed and take  $\varepsilon \to 0$  in Lemma 3.3 by using Lemma 3.4 which obviously holds even with the additional penalization term. Indeed  $\iint_{\Omega \times \mathbf{R}^3} g \sqrt{\mu} = 0 = \int_{\gamma_-} r \sqrt{\mu} d\gamma$ . Hence we have  $\varepsilon \iint_{\Omega \times \mathbf{R}^3} f^{\varepsilon} \sqrt{\mu} = 0$  and therefore, for any  $\varepsilon > 0$ ,

$$\iint_{\Omega \times \mathbf{R}^3} f^{\varepsilon} \sqrt{\mu} dx dv = 0.$$

We first obtain

$$\|\mathbf{P}f^{\varepsilon}\|_{2}^{2} \lesssim \|(\mathbf{I} - \mathbf{P})f^{\varepsilon}\|_{2}^{2} + \|(1 - P_{\gamma})f^{\varepsilon}\|_{2,+}^{2} + \|r\|_{2}^{2} + \|g\|_{2}^{2} + \varepsilon \|f^{\varepsilon}\|_{2}^{2}, \tag{3.43}$$

On the other hand from Green's identity:

$$\varepsilon \|f^{\varepsilon}\|_{2}^{2} + (L_{m}f^{\varepsilon}, f^{\varepsilon}) + \frac{1}{2}|f^{\varepsilon}|_{2,+}^{2} - \frac{1}{2}|P_{\gamma}f^{\varepsilon} + r|_{2,-}^{2} = \int f^{\varepsilon}g,$$

we deduce from the spectral gap of  $L_m$ 

$$\varepsilon \|f^{\varepsilon}\|_{2}^{2} + \|(\mathbf{I} - \mathbf{P})f^{\varepsilon}\|_{\nu_{m}}^{2} + \frac{1}{2}|(1 - P_{\gamma})f^{\varepsilon}|_{2,+}^{2} \le \eta [\|f^{\varepsilon}\|_{2}^{2} + |P_{\gamma}f^{\varepsilon}|_{2,-}^{2}] + C_{\eta}[|r|_{2}^{2} + \|g\|_{2}^{2}]. \quad (3.44)$$

From the argument of (3.11) and the trace theorem as well as the equation (3.2).

$$|P_{\gamma}f^{\varepsilon}|_{2}^{2} \lesssim ||v \cdot \nabla_{x}(f^{\varepsilon})^{2}||_{1} + ||f^{\varepsilon}||_{2}^{2} \lesssim ||(\mathbf{I} - \mathbf{P})f^{\varepsilon}||_{\nu_{m}}^{2} + ||g||_{2}^{2} + ||f^{\varepsilon}||_{2}^{2}$$

Plugging this into (3.44) with  $\eta$  small and adding a small constant  $\times$  (3.43), collecting terms and using the fact

$$\|\mathbf{P}f^{\varepsilon}\|_{2}^{2} \sim \|\mathbf{P}f^{\varepsilon}\|_{\nu_{m}}^{2},$$

we obtain the uniform in  $\varepsilon$  estimate

$$||f^{\varepsilon}||_{\nu_m}^2 + |f^{\varepsilon}|_2^2 \lesssim ||g||_2^2 + |r|_2^2.$$
 (3.45)

We thus obtain a weak solution  $f^{\varepsilon} \to f$  with the same bound (3.45). Moreover, we have

$$\varepsilon[f^{\varepsilon} - f] + v \cdot \nabla_x[f^{\varepsilon} - f] + L_m[f^{\varepsilon} - f] = \varepsilon f, \qquad [f^{\varepsilon} - f]_{-} = P_{\gamma}[f^{\varepsilon} - f].$$

We conclude from (3.45), written for the difference of  $f^{\varepsilon} - f$ , that

$$\begin{split} \|f^{\varepsilon} - f\|_{\nu_m}^2 + |[f^{\varepsilon} - f]|_2^2 &\lesssim \varepsilon \|f\|_2^2 \\ &\lesssim \varepsilon \{\|g\|_2^2 + \|r\|_2^2\} \to 0. \end{split}$$

The proposition follows as  $\varepsilon \to 0$ .

Finally from (3.45) again we can take the limit  $m \to \infty$  of  $f^m$  solution to

$$v \cdot \nabla_x f^m + L_m f^m = g,$$
  $f_-^m = P_\gamma f^m + r.$ 

By a diagonal process, there exists a weak solution f such that  $f^m \to f$  weakly in  $\|\cdot\|_{\nu_{m_0}}$ , for any fixed  $m_0$ . It thus follows from weak semi-continuity of the norm  $\|\cdot\|_{\nu_{m_0}}$  that

$$||f||_{\nu_{m_0}}^2 + |f|_2^2 \lesssim ||g||_2^2 + |r|_2^2.$$

Note that the limiting f so obtained satisfies  $\iint_{\Omega \times \mathbf{R}^3} f(x,v) \sqrt{\mu(v)} dv dx = 0$ . The proposition follows as  $m_0 \to \infty$  and uniqueness follows from Green's identity. We remark that due to lack of moments control of f we cannot show  $f^m \to f$  strongly in  $L^2$ .

## 4 $L^{\infty}$ Estimate along the Stochastic Cycles

We define a weight function scaled with parameter  $\varrho$ ,

$$w_{\varrho}(v) = w_{\varrho,\beta,\zeta}(v) \equiv (1 + \varrho^2 |v|^2)^{\frac{\beta}{2}} e^{\zeta |v|^2}.$$
 (4.1)

The main purpose of this section is to prove the following:

**Proposition 4.1** Assume (3.1). Then the solution f to the linear Boltzmann equation (3.2) satisfies

$$||w_{\varrho}f||_{\infty} + |w_{\varrho}f|_{\infty} \lesssim ||w_{\varrho}g||_{\infty} + |w_{\varrho}\langle v\rangle r|_{\infty}.$$

Moreover if g and r are continuous away from the grazing set  $\gamma_0$ , then f is continuous away from  $\mathfrak{D}$ . In particular, if  $\Omega$  is convex then  $\mathfrak{D} = \gamma_0$ .

We define

$$\tilde{w}_{\varrho}(v) \equiv \frac{1}{w_{\varrho,\beta,\zeta}(v)\sqrt{\mu(v)}} = \sqrt{2\pi} \frac{e^{(\frac{1}{4}-\zeta)|v|^2}}{(1+\varrho^2|v|^2)^{\frac{\beta}{2}}},\tag{4.2}$$

and  $\mathcal{V}(x) = \{v \in \mathbf{R}^3 : n(x) \cdot v > 0\}$  with a probability measure  $d\sigma = d\sigma(x)$  on  $\mathcal{V}(x)$  which is given by

$$d\sigma \equiv \mu(v)\{n(x) \cdot v\}dv. \tag{4.3}$$

We use below Definition 1.6 of stochastic cycles and iterated integral and remind the dependence of  $t_k$  on  $(t, x, v, v_1, v_2, \dots, v_{k-1})$ . We first show that the set of points in the phase space  $\prod_{j=1}^{k-1} \mathcal{V}_j$  not reaching t=0 after k bounces is small when k is large.

**Lemma 4.2** For  $T_0 > 0$  sufficiently large, there exist constants  $C_1, C_2 > 0$  independent of  $T_0$ , such that for  $k = C_1 T_0^{5/4}$ , and all  $(t, x, v) \in [0, T_0] \times \overline{\Omega} \times \mathbf{R}^3$ ,

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k(t, x, \bar{v}, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{k-1}) > 0\}} \prod_{j=1}^{k-1} d\sigma_j \le \left\{ \frac{1}{2} \right\}^{C_2 T_0^{5/4}}.$$
(4.4)

We also have, for  $\beta > 4$ ,

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \prod_{j=1}^{k-1} d\sigma_{j} \leq \left\{ 1 + \frac{C_{\beta,\zeta}}{\varrho^{4}} \right\}^{k-1}, \qquad (4.5)$$

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\{0 < t_{l+1}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \prod_{j=1}^{k-1} d\sigma_{j} \leq \left\{ 1 + \frac{C_{\beta,\zeta}}{\varrho^{4}} \right\},$$

for all  $l = 1, 2, \dots, k - 1$ .

**Proof.** Choosing  $0 < \mathfrak{z}$  sufficiently small, we further define non-grazing sets for  $1 \le j \le k-1$  as

$$\mathcal{V}_j^{\mathfrak{z}} = \{ v_j \in \mathcal{V}_j : \ \bar{v}_j \cdot n(x_j) \ge \mathfrak{z} \} \cap \{ v_j \in \mathcal{V}_j : \ |v_j| \le \frac{1}{\mathfrak{z}} \}.$$

Clearly, by the same argument used in (3.12),

$$\int_{\mathcal{V}_{j}\setminus\mathcal{V}_{j}^{\delta}} d\sigma_{j} \leq \int_{\bar{v}_{j}\cdot n(x_{j})\leq \mathfrak{z}} d\sigma_{j} + \int_{|v_{j}|\geq \frac{1}{\mathfrak{z}}} d\sigma_{j} \leq C\mathfrak{z}, \tag{4.6}$$

where C is independent of j. On the other hand, if  $v_j \in \mathcal{V}_j^{\mathfrak{z}}$ , then from the definition of diffusive back-time cycle (1.35), we have  $x_j - x_{j+1} = (t_j - t_{j+1})\bar{v}_j$ . By (2.4), since  $|v_j| \leq \frac{1}{\mathfrak{z}}$ , and  $\bar{v}_j \cdot n(x_j) \geq \mathfrak{z}$ ,

$$(t_j - t_{j+1}) \ge \frac{\mathfrak{z}^3}{C_{\Omega}}.$$

Therefore, if  $t_k(t, x, \bar{v}, \bar{v}_1, \bar{v}_2..., \bar{v}_{k-1}) > 0$ , then there can be at most  $\left[\frac{C_{\xi}T_0}{\delta^3}\right] + 1$  number of  $v_j \in \mathcal{V}_j^{\delta}$  for  $1 \leq j \leq k-1$ . We therefore have

$$\int_{\mathcal{V}_{1}} \dots \left\{ \int_{\mathcal{V}_{k-1}} \mathbf{1}_{\{t_{k}>0\}} d\sigma_{k-1} \right\} d\sigma_{k-2} \dots d\sigma_{1}$$

$$\leq \sum_{m=1}^{\left[\frac{C_{\xi}T_{0}}{\delta^{3}}\right]+1} \int_{\{\text{There are exactly } m \text{ of } v_{j_{i}} \in \mathcal{V}_{j_{i}}^{\delta}, \text{ and } k-1-m \text{ of } v_{j_{i}} \notin \mathcal{V}_{j_{i}}^{\delta}\}} \prod_{j=1}^{k-1} d\sigma_{j}$$

$$\leq \sum_{m=1}^{\left[\frac{C_{\xi}T_{0}}{\delta^{3}}\right]+1} \binom{k-1}{m} |\sup_{j} \int_{\mathcal{V}_{j}^{\delta}} d\sigma_{j}|^{m} \left\{ \sup_{j} \int_{\mathcal{V}_{j} \setminus \mathcal{V}_{j}^{\delta}} d\sigma_{j} \right\}^{k-m-1}.$$

Since  $d\sigma$  is a probability measure,  $\int_{\mathcal{V}_i^3} d\sigma_i \leq 1$ , and

$$\left\{ \int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{F}}} d\sigma_j \right\}^{k-m-1} \leq \left\{ \int_{\mathcal{V}_j \setminus \mathcal{V}_j^{\mathfrak{F}}} d\sigma_j \right\}^{k-2 - \left[\frac{C_{\xi} T_0}{\mathfrak{F}^3}\right]} \leq \left\{ C_{\mathfrak{F}} \right\}^{k-2 - \left[\frac{C_{\xi} T_0}{\mathfrak{F}^3}\right]}.$$

But, from  $\binom{k-1}{m} \le \{k-1\}^m \le \{k-1\}^{\left[\frac{C_{\Omega}T_0}{\delta^3}\right]+1}$ , we deduce that

$$\int \mathbf{1}_{\{t_k > 0\}} \Pi_{l=1}^{k-1} d\sigma_l \le \left[ \frac{C_{\xi} T_0}{\mathfrak{z}^3} \right] (k-1)^{\left[ \frac{C_{\xi} T_0}{\mathfrak{z}^3} \right] + 1} C_{\mathfrak{z}}^{k-2 - \left[ \frac{C_{\xi} T_0}{\mathfrak{z}^3} \right]}. \tag{4.7}$$

Now let  $k-2=N\{\left[\frac{C_{\Omega}T_{0}}{\mathfrak{z}^{3}}\right]+1\}$ , so that if  $\frac{C_{\Omega}T_{0}}{\mathfrak{z}^{3}}\geq 1$ , (4.7) can be further majorized by

$$\left\{ N \left( \frac{C_{\Omega} T_0}{\mathfrak{z}^3} + 1 \right) (C \mathfrak{z})^N \right\}^{\left[ \frac{C_{\Omega} T_0}{\mathfrak{z}^3} \right] + 1} \\
\leq \left\{ \frac{2N C_{\Omega} T_0}{\mathfrak{z}^3} (C \mathfrak{z})^N \right\}^{\left[ \frac{C_{\Omega} T_0}{\mathfrak{z}^3} \right] + 1} \\
\leq \left\{ C_{N,\Omega} T_0 \mathfrak{z}^{N-3} \right\}^{\left[ \frac{C_{\Omega} T_0}{\mathfrak{z}^3} \right] + 1}.$$

We choose  $C_{N,\Omega}T_0\mathfrak{z}^{N-3}=\frac{1}{2}$ , so that  $\mathfrak{z}=\left\{\frac{1}{2C_{N,\Omega}T_0}\right\}^{\frac{1}{N-3}}$  is small for  $T_0$  large and for N>3. Moreover,

$$\left[\frac{C_{N,\Omega}T_0}{\mathfrak{z}^3}\right] + 1 \sim C_{N,\Omega}T_0^{1+\frac{3}{N-3}},$$

and  $\frac{C_{N,\Omega}T_0}{i^3} \geq 2$ , if  $T_0$  is large so that we can close our estimate.

Finally we choose N=15. For  $T_0$  sufficiently large,  $\left[\frac{C_{N,\Omega}T_0}{\mathfrak{z}^3}\right]+1\sim CT_0^{5/4}$  and  $k=16\{\left[\frac{C_{\Omega}T_0}{\mathfrak{z}^3}\right]+1\}+2\sim CT_0^{5/4}$ , and (4.4) follows.

Next, we give the proof of the first estimate in (4.5). The left hand side of (4.5) is bounded by

$$\begin{split} & \int_{\Pi_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} \\ & \leq \int_{\Pi_{j=1}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\{t_{k} \leq 0 < t_{1}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} \\ & \leq \prod_{j=1}^{k-1} \left\{ \int_{\mathcal{V}_{j}} [1 + \tilde{w}_{\varrho}(v_{j}) \langle v_{j} \rangle] d\sigma_{j} \right\} \leq \left\{ 1 + \frac{1}{\sqrt{2\pi}} \int_{v_{1} > 0} \frac{v_{1} e^{-(\frac{1}{4} + \zeta)|v|^{2}}}{(1 + \varrho^{2}|v|^{2})^{\frac{\beta - 1}{2}}} dv \right\}^{k-1} \\ & \leq \left\{ 1 + \frac{1}{\sqrt{2\pi}} \int_{\mathfrak{u}_{1} > 0} \frac{C_{\zeta} \mathfrak{u}_{1}}{(1 + |\mathfrak{u}|^{2})^{\frac{\beta - 1}{2}}} \varrho^{-4} d\mathfrak{u} \right\}^{k-1} \\ & \leq \left\{ 1 + \frac{C_{\beta, \zeta}}{\varrho^{4}} \right\}^{k-1}, \end{split}$$

where we used the change of variables :  $\varrho v = \mathfrak{u}$  and the fact  $\beta > 4$ .

For the second estimate in (4.5) similarly we have

$$\int_{\Pi_{j=1}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\{0 < t_{l+1}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} \leq \int_{\Pi_{j=1}^{k-1} \mathcal{V}_{j}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} 
\leq \int_{\mathcal{V}_{l}} [1 + \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle] d\sigma_{l} \leq \left\{ 1 + \frac{1}{\sqrt{2\pi}} \int_{v_{1} > 0} \frac{v_{1} e^{-(\frac{1}{4} + \zeta)|v|^{2}}}{(1 + \varrho^{2}|v|^{2})^{\frac{\beta - 1}{2}}} dv \right\} 
\leq \left\{ 1 + \frac{1}{\sqrt{2\pi}} \int_{\mathfrak{u}_{1} > 0} \frac{C_{\zeta} \mathfrak{u}_{1}}{(1 + |\mathfrak{u}|^{2})^{\frac{\beta - 1}{2}}} \varrho^{-4} d\mathfrak{u} \right\} 
\leq \left\{ 1 + \frac{C_{\beta, \zeta}}{\varrho^{4}} \right\},$$

where we used the fact that  $d\sigma_j$  is a probability measure on  $\mathcal{V}_j$ .

Denote  $h = w_{\varrho} f$  and  $K_{w_{\varrho}}(\cdot) = w_{\varrho}K(\frac{1}{w_{\varrho}}\cdot)$ . We present an abstract iteration scheme which gives a unified way to study both steady and dynamic problem with diffuse boundary condition. Recall  $v = (\bar{v}, \hat{v})$  and  $v_l = (\bar{v}_l, \hat{v}_l)$  from (1.8) and define:

$$|h^{\ell+1}(t,x,v)| \leq \mathbf{1}_{t_{1}\leq 0}e^{-\nu(v)t}|h^{\ell+1}(0,x-t\bar{v},v)| +\mathbf{1}_{t_{1}\leq 0}\int_{0}^{t}e^{-\nu(v)(t-s)}|[K_{w_{\varrho}}h^{\ell}+w_{\varrho}g](s,x-(t-s)\bar{v},v)|ds +\mathbf{1}_{t_{1}>0}\int_{t_{1}}^{t}e^{-\nu(v)(t-s)}|[K_{w_{\varrho}}h^{\ell}+w_{\varrho}g](s,x-(t-s)\bar{v},v)|ds +\mathbf{1}_{t_{1}>0}e^{-\nu(v)(t-t_{1})}|w_{\varrho}r(t_{1},x_{1},v)| + \frac{e^{-\nu(v)(t-t_{1})}}{\tilde{w}_{\varrho}(v)}\int_{\prod_{i=1}^{k-1}\nu_{i}}|H|,$$

$$(4.8)$$

where |H| is bounded by

$$\sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \le 0 < t_l\}} |h^{\ell+1-l}(0, x_l - t_l \bar{v}_l, v_l)| d\Sigma_l(0)$$
(4.9)

$$+\sum_{l=1}^{k-1} \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} |[K_{w_{\varrho}} h^{\ell-l} + w_{\varrho} g](s, x_{l} - (t_{l} - s)\bar{v}_{l}, v_{l})| d\Sigma_{l}(s) ds$$

$$(4.10)$$

$$+\sum_{l=1}^{k-1} \int_{t_{l+1}}^{t_l} \mathbf{1}_{\{0 < t_l\}} |[K_{w_{\varrho}} h^{\ell-l} + w_{\varrho} \ g](s, x_l - (t_l - s)\bar{v}_l, v_l)| d\Sigma_l(s) ds$$

$$(4.11)$$

$$+\sum_{l=1}^{k-1} \mathbf{1}_{\{0 < t_l\}} d\Sigma_l^r \tag{4.12}$$

$$+\mathbf{1}_{\{0 < t_k\}} |h^{\ell+1-k}(t_k, x_k, v_{k-1})| d\Sigma_{k-1}(t_k), \tag{4.13}$$

and we define

$$d\Sigma_{l} = \{\Pi_{j=l+1}^{k-1} d\sigma_{j}\} \times \{\tilde{w}(v_{l}) d\sigma_{l}\} \times \Pi_{j=1}^{l-1} d\sigma_{j}$$

$$d\Sigma_{l}(s) = \{\Pi_{j=l+1}^{k-1} d\sigma_{j}\} \times \{e^{\nu(v_{l})(s-t_{l})} \tilde{w}(v_{l}) d\sigma_{l}\} \times \Pi_{j=1}^{l-1} \{e^{\nu(v_{j})(t_{j+1}-t_{j})} d\sigma_{j}\}$$

$$d\Sigma_{l}^{r} = \{\Pi_{j=l+1}^{k-1} d\sigma_{j}\} \times \{e^{\nu(v_{l})(t_{l+1}-t_{l})} \tilde{w}(v_{l}) w(v_{l}) r(t_{l+1}, x_{l+1}, v_{l}) d\sigma_{l}\}$$

$$\times \Pi_{i=1}^{l-1} \{e^{\nu(v_{j})(t_{j+1}-t_{j})} d\sigma_{j}\}.$$

$$(4.14)$$

**Remark 4.3** For the steady case you can regard the temporal variables t, s as parameters and read h(t, x, v) = h(x, v). We used the notation in (1.8),  $v = (v_1, \dots, v_d; v_{d+1}, \dots, v_3) = (\bar{v}; \hat{v})$  again and  $\bar{v} \in \mathbf{R}^d$  and  $\hat{v} \in \mathbf{R}^{3-d}$  where d is the spatial dimension so that  $x \in \overline{\Omega} \subset \mathbf{R}^d$ , for d = 1, 2, 3.

**Lemma 4.4** There exist  $\varrho_0 > 0$  and C > 0 such that for all  $\varrho > \varrho_0$ ,  $\beta > 4$ , and for  $k = \varrho = Ct^{\frac{5}{4}}$ ,

$$\sup_{0 \le s \le t} e^{\frac{\nu_0}{2}t} \|h^{\ell+1}(s)\|_{\infty} 
\le \frac{1}{8} \max_{0 \le l \le 2k} \sup_{0 \le s \le t} \left\{ e^{\frac{\nu_0}{2}s} \|h^{\ell-l}(s)\|_{\infty} \right\} + \max_{0 \le l \le 2k} \|h^{\ell+1-l}(0)\|_{\infty} 
+ \varrho \left[ \sup_{0 \le s \le t} \left\{ e^{\frac{\nu_0}{2}s} |w_{\varrho} r(s)|_{\infty} \right\} + \sup_{0 \le s \le t} \left\{ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho} g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \right] 
+ C \max_{1 \le l \le 2k} \int_0^t \left\| \frac{h^{\ell-l}(s)}{w_{\varrho}} \right\|_2 ds.$$
(4.15)

Furthermore for  $k = \varrho = Ct^{\frac{5}{4}}$ 

$$|h^{\ell+1}(t,x,v)| \leq e^{-\nu(v)(t-t_1)}w_{\varrho}(v)|r(t_1,x_1,v)| + \left\{\frac{C_{\beta,\rho,\zeta}}{N} + \varepsilon C_{\rho,\beta,N}\right\} \sup_{s} |\langle v\rangle^{\beta+4}r(s)|_{\infty} + C_{N,\varepsilon,\rho} \left|\sup_{s,v} |r(s,\cdot,v)|\right|_{1} + \frac{1}{N}\sup_{s,l} \|h^{l}(s)\|_{\infty} + e^{-\frac{\nu_{0}}{2}t}\sup_{l} \|h^{l}(0)\|_{\infty} + \varrho\sup_{s} \left\|\frac{w_{\varrho} g(s)}{\langle v\rangle}\right\|_{\infty} + e^{-\frac{\nu_{0}}{2}t}\sup_{l} \int_{0}^{t} \left\|\frac{h^{l}(s)}{w_{\varrho}}\right\|_{2} ds.$$

$$(4.16)$$

**Remark 4.5** The estimate (4.16) is used only in the proof of the singularity formation in Theorem 1.1.

**Proof.** We first prove (4.15) and then sketch the proof of (4.16).

We start with r-contribution in (4.8) and (4.12). Clearly the contribution in (4.8) is bounded by

$$e^{-\nu(v)[t-t_1(t,x,v)]}|w_{\varrho}(v)r(t_1,x_1,v)|.$$

Since the exponent of  $d\Sigma_l^r$  is bounded by  $e^{-\nu_0(t-t_{l+1})}$ , from (4.14) and Lemma 4.2

$$(4.12) \leq ke^{-\frac{\nu_0}{2}t} \frac{1}{\tilde{w}_{\varrho}(v)} \left\{ 1 + \frac{C_{\beta,\zeta}}{\varrho^4} \right\} \left| e^{\frac{\nu_0}{2}s} \frac{w_{\varrho}r(s)}{\langle v \rangle} \right|_{\infty}$$

$$\leq e^{-\frac{\nu_0}{2}t} \frac{2\varrho}{\tilde{w}_{\varrho}(v)} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_0}{2}s} \left| \frac{w_{\varrho}r(s)}{\langle v \rangle} \right|_{\infty} \right\},$$

$$(4.17)$$

for sufficiently large  $\varrho > 0$  and  $k = \varrho = Ct^{\frac{5}{4}}$ .

We now turn to the g- contribution in (4.8), (4.10) and (4.11). Rewrite

$$|w_{\varrho}g(x_{l}-(t_{l}-s)\bar{v}_{l},v_{l})| = \frac{|w_{\varrho}g(x_{l}-(t_{l}-s)\bar{v}_{l},v_{l})|}{\langle v_{l}\rangle} \times \langle v_{l}\rangle \leq \left\|\frac{w_{\varrho}g}{\langle v_{l}\rangle}\right\|_{\infty} \langle v_{l}\rangle.$$

Since the exponent of  $d\Sigma_l(s)$  is bounded by  $e^{\nu_0(s-t_1)}$ , the g-contributions in (4.8), (4.10) and (4.11) are bounded by

$$2\int_{0}^{t} e^{-\nu(v)(t-s)} |w_{\varrho}g(s, x - (t-s)\bar{v}, v)| ds$$

$$+ \frac{e^{-\nu_{0}(t-t_{1})}}{\tilde{w}_{\varrho}(v)} \int_{0}^{t} e^{-\nu_{0}(t_{1}-s)} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} ds$$

$$\times \sum_{l=1}^{k-1} \left\{ \int \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} + \max_{l} \int \mathbf{1}_{\{0 < t_{l+1}\}} \tilde{w}_{\varrho}(v_{l}) \langle v_{l} \rangle \Pi_{j=1}^{k-1} d\sigma_{j} \right\}.$$

$$(4.18)$$

From

$$\frac{1}{\tilde{w}_{\rho}} \lesssim_{\beta,\zeta} \varrho^{\beta},\tag{4.19}$$

the second line of (4.18) is bounded by

$$\begin{split} &\lesssim_{\beta,\zeta} \quad \varrho^{\beta} e^{-\nu_{0}(t-t_{1})} \int_{0}^{t} e^{-\nu_{0}(t_{1}-s)} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} ds \\ &\lesssim \quad \varrho^{\beta} e^{-\frac{\nu_{0}}{2}(t-t_{1})} \int_{0}^{t} e^{-\frac{\nu_{0}}{2}(t_{1}-s)-\frac{\nu_{0}}{2}t_{1}} ds \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \\ &\lesssim \quad \varrho^{\beta} e^{-\frac{\nu_{0}}{2}t} \left\{ \int_{0}^{t} e^{-\frac{\nu_{0}}{2}(t_{1}-s)} ds \right\} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \\ &\lesssim \quad \varrho^{\beta} e^{-\frac{\nu_{0}}{2}t} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\}. \end{split}$$

With our choice  $k = \varrho = Ct^{\frac{5}{4}}$  and from Lemma 4.2, the third and forth line of (4.18) are bounded by

$$(1+k)\left\{1+\frac{C_{\beta,\zeta}}{\varrho^4}\right\}^{k-1}+k\left\{1+\frac{C_{\beta,\zeta}}{\varrho^4}\right\} \leq 2(1+\varrho)\left\{1+\frac{C_{\beta,\zeta}}{\varrho^4}\right\}^{\varrho} \lesssim_{\beta,\zeta} (1+\varrho),$$

where we have chosen sufficiently large  $\varrho_0 > 0$  such that  $\varrho > \varrho_0$ ,

$$\left\{1 + \frac{C_{\beta,\zeta}}{\varrho^4}\right\}^\varrho < 2.$$
(4.20)

Therefore, the total g-contribution is bounded by

$$\lesssim_{\beta,\zeta} (1+\varrho^{\beta})e^{-\frac{\nu_0}{2}t} \sup_{0 \le s \le t} \left\{ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\}.$$

Notice that the exponent in  $d\Sigma_l(s)$  is bounded by  $e^{-\nu_0(t_1-s)}$  and from (4.19) and (4.2) and from Lemma 4.2 we get

$$\begin{split} e^{-\nu(v)t}|h^{\ell+1}(0,x-t\bar{v},v)| & \mathbf{1}_{t_1 \leq 0} \\ & + \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}_{\varrho}(v)} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \Big\{ \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq 0 < t_l\}} |h^{\ell+1-l}(0,x_l-t_l\bar{v}_l,v_l)| d\Sigma_l(0) \\ & + \mathbf{1}_{\{0 < t_k\}} |h^{\ell+1-k}(t_k,x_k,v_{k-1})| d\Sigma_{k-1}(t_k) \Big\} \\ \lesssim_{\beta,\zeta} & e^{-\nu_0 t} \|h^{\ell+1}(0)\|_{\infty} + \varrho^{\beta} e^{-\nu_0 t} \max_{1 \leq l \leq k} \|h^{\ell+1-l}(0)\|_{\infty} \int \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1} \leq 0 < t_l\}} d\Sigma_l \\ & + \varrho^{\beta} e^{-\nu_0 t} \sup_{0 \leq s \leq t} \Big\{ e^{\nu_0 s} \|h^{\ell+1-k}(s)\|_{\infty} \Big\} \int \mathbf{1}_{\{0 < t_k\}} d\Sigma_{k-1} \\ \lesssim_{\beta,\zeta} & e^{-\nu_0 t} \Big\{ \|h^{\ell+1}(0)\|_{\infty} + \varrho^{\beta} \Big\{ 1 + \frac{C_{\beta,\zeta}}{\varrho^4} \Big\}^{\varrho} \max_{1 \leq l \leq k} \|h^{\ell+1-l}(0)\|_{\infty} \\ & + \varrho^{\beta} \Big\{ \frac{1}{2} \Big\}^{C_2 C^{-1} \varrho} \sup_{0 \leq s \leq t} e^{\nu_0 s} \|h^{\ell+1-k}(s)\|_{\infty} \Big\} \\ \lesssim_{\beta,\zeta} & e^{-\frac{\nu_0}{2} t} \Big\{ \max_{0 \leq l \leq k} \|h^{\ell+1-l}(0)\|_{\infty} + \Big\{ \frac{1}{2} \Big\}^{\varrho} \sup_{0 \leq s \leq t} e^{\frac{\nu_0}{2} s} \|h^{\ell+1-k}(s)\|_{\infty} \Big\} , \end{split}$$

where we have chosen  $t = \varrho = Ct^{\frac{5}{4}}$  for sufficiently large  $\varrho > 0$  but fixed and used (4.19). Now we obtain an upper bound for (4.8), (4.9), (4.10), (4.11), (4.12), (4.13) as

$$|h^{\ell+1}(t,x,v)| \leq \mathbf{1}_{t_{1}\leq 0} \int_{0}^{t} e^{-\nu(v)(t-s)} |K_{w_{\varrho}}h^{\ell}(s,x-(t-s)\bar{v},v)| ds$$

$$+\mathbf{1}_{t_{1}>0} \int_{t_{1}}^{t} e^{-\nu(v)(t-s)} |K_{w_{\varrho}}h^{\ell}(s,x-(t-s)\bar{v},v)| ds$$

$$+\frac{e^{-\nu(v)(t-t_{1})}}{\tilde{w_{\varrho}}(v)} \times \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1}\leq 0 < t_{l}\}} |K_{w_{\varrho}}h^{\ell-l}(s,X_{\mathbf{cl}}(s),v_{l})| \right.$$

$$+ \int_{t_{l+1}}^{t_{l}} \mathbf{1}_{\{0 < t_{l+1}\}} |K_{w_{\varrho}}h^{\ell-l}(s,X_{\mathbf{cl}}(s),v_{l})| \right\} d\Sigma_{l}(s) ds$$

$$+ e^{-\frac{\nu_{0}}{2}t} A_{\ell}(t,x,v),$$

$$(4.21)$$

where  $A_{\ell}(t, x, v)$  denotes

$$A_{\ell}(t, x, v) = e^{\frac{\nu_{0}}{2}t_{1}}w_{\varrho}(v)|r(t_{1}, x_{1}, v)| + \frac{2\varrho}{\tilde{w}_{\varrho}(v)} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left| \frac{w_{\varrho} r(s)}{\langle v \rangle} \right|_{\infty} \right\}$$

$$+ (1 + \varrho^{\beta}) \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{w_{\varrho} g(s)}{\langle v \rangle} \right\|_{\infty} \right\} + \max_{0 \leq l \leq k} \|h^{\ell+1-l}(0)\|_{\infty}$$

$$+ \left\{ \frac{1}{2} \right\}^{\varrho} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \|h^{\ell+1-k}(s)\|_{\infty} \right\}.$$

$$(4.22)$$

Recall from (1.35) that the back-time cycle from  $(s, X_{cl}(s; t, x, v), v')$  denotes

$$(t'_1, x'_1, v'_1), (t'_2, x'_2, v'_2), \cdots, (t'_{l'}, x'_{l'}, v'_{l'}), \cdots$$

We now iterate (4.21) for  $\ell - l$  times to get the representation for  $h^{\ell-l}$  and then plug in  $K_{w_o}h^{\ell-l}(s, X_{\mathbf{cl}}(s), v_l)$  to obtain

$$K_{w_{\varrho}}h^{\ell-l}(s, X_{\mathbf{cl}}(s), v_{l}) \leq \int_{\mathbf{R}^{3}} \mathbf{k}_{w_{\varrho}}(v_{l}, v') |h^{\ell-l}(s, X_{\mathbf{cl}}(s), v')| dv'$$

$$\leq \iint \mathbf{1}_{t'_{1} \leq 0} \int_{0}^{s} e^{-\nu(v')(s-s_{1})}$$

$$\mathbf{k}_{w_{\varrho}}(v_{l}, v') \mathbf{k}_{w_{\varrho}}(v', v'') |h^{\ell-1-l}(s_{1}, X_{\mathbf{cl}}(s) - (s-s_{1})\overline{v}', v'')| ds_{1} dv' dv''$$

$$+ \iint \mathbf{1}_{t'_{1} > 0} \int_{t'_{1}}^{s} e^{-\nu(v')(s-s_{1})}$$

$$\mathbf{k}_{w_{\varrho}}(v_{l}, v') \mathbf{k}_{w_{\varrho}}(v', v'') |h^{\ell-1-l}(s_{1}, X_{\mathbf{cl}}(s) - (s-s_{1})\overline{v}', v'')| ds_{1} dv' dv''$$

$$+ \iint dv' dv'' \int_{\prod_{j=1}^{k-1} V'_{j}} \frac{e^{-\nu(v')(s-t'_{1})}}{\tilde{w}_{\varrho}(v')} \sum_{l'=1}^{k-1} \int_{0}^{t'_{l'}} ds_{1} \mathbf{1}_{\{t'_{l'+1} \leq 0 < t'_{l'}\}}$$

$$\mathbf{k}_{w_{\varrho}}(v_{l}, v') \mathbf{k}_{w_{\varrho}}(v'_{l'}, v'') |h^{\ell-1-l-l'}(s_{1}, x'_{l'} + (s_{1} - t'_{l'})\overline{v}'_{l'}, v'')| d\Sigma_{l'}(s_{1})$$

$$+ \iint dv' dv'' \int_{\prod_{j=1}^{k-1} V'_{j}} \frac{e^{-\nu(v')(s-t'_{1})}}{\tilde{w}_{\varrho}(v')} \sum_{l'=1}^{k-1} \int_{t'_{l'}}^{t'_{l'-1}} ds_{1} \mathbf{1}_{\{t'_{l'} > 0\}}$$

$$\mathbf{k}_{w_{\varrho}}(v_{l}, v') \mathbf{k}_{w_{\varrho}}(v'_{l'}, v'') |h^{n-1-l-l'}(s_{1}, x'_{l'} + (s_{1} - t'_{l'})\overline{v}'_{l'}, v'')| d\Sigma_{l'}(s_{1})$$

$$+ e^{-\frac{\nu_{0}}{2}s} \int_{\mathbf{R}^{3}} \mathbf{k}_{w_{\varrho}}(v_{l}, v') A_{\ell-1-l}(s, X_{\mathbf{cl}}(s), v') dv'.$$

$$(4.24)$$

The total contributions of  $A_{\ell-l-1}$ 's in (4.21) are obtained via plugging (4.23) with different l's into (4.21). Since  $\int \mathbf{k}_{w_{\varrho}}(v_{l}, v') dv' < \infty$ , the summation of all contributions of  $A_{\ell-l-1}$ 's leads to the bound

$$\lesssim_{\beta,\zeta} 2A_{\ell-1}(t) \int_{0}^{t} e^{-\nu_{0}(t-s)} e^{-\frac{\nu_{0}}{2}s} ds + \varrho^{\beta} e^{-\frac{\nu_{0}}{2}t} \max_{1 \leq l \leq k-1} A_{\ell-l-1}(t)$$

$$\times \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} + \int_{t_{l+1}}^{t_{l}} \mathbf{1}_{\{0 < t_{l}\}} \right\} e^{-\frac{\nu_{0}}{2}(t-s)} \tilde{w}_{\varrho}(v_{l}) d\Sigma_{l} ds$$

$$+ e^{-\frac{\nu_{0}}{2}t} A_{\ell}(t)$$

$$\lesssim_{\beta,\zeta} \varrho^{\beta} e^{-\frac{\nu_{0}}{2}t} \left\{ \left[ 1 + \frac{C_{\beta,\zeta}}{\varrho^{4}} \right]^{\varrho} + \left[ \frac{1}{2} \right]^{\varrho} \right\} \max_{0 \leq l \leq k} A_{\ell-l}(t)$$

$$\lesssim_{\beta,\zeta} \varrho^{\beta} e^{-\frac{\nu_{0}}{2}t} \left[ \varrho \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} |wr(s)|_{\infty} \right\} + \varrho \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{wg(s)}{\langle v \rangle} \right\|_{\infty} \right\} \right]$$

$$+ e^{-\frac{\nu_{0}}{2}t} \left\{ \max_{1 \leq l \leq 2k} \|h^{\ell+1-l}(0)\|_{\infty} + \left\{ \frac{1}{2} \right\}^{\varrho} \max_{0 \leq l \leq k} \sup_{0 < s < t} e^{\frac{\nu_{0}}{2}s} \|h^{\ell-l+1-k}(s)\|_{\infty} \right\}.$$

To estimate the  $h^{\ell-l-1}$  contribution, we first separate  $s-s_1 \leq \varepsilon$  and  $s-s_1 \geq \varepsilon$ . In the first case, we use the fact  $\int \mathbf{k}_{w_{\varrho}}(p,v)dv < +\infty$  and by (4.23) to obtain the small contribution

$$\varepsilon e^{-\frac{\nu_0}{2}s} \max_{1 \le l' \le k} \sup_{0 \le s_1 \le s} \{ e^{\frac{\nu_0}{2}s_1} \| h^{\ell - l - l'}(s_1) \|_{\infty} \}. \tag{4.26}$$

Now we treat the case of  $s - s_1 \ge \varepsilon$ . For any large  $N \gg 1$ , we can choose a number m(N) to define

$$\mathbf{k}_{m}(v'_{l'}, v'') \equiv \mathbf{1}_{|v'_{l'} - v''| \ge \frac{1}{m}, |v''| \le m} \mathbf{k}_{w_{\varrho}}(v'_{l'}, v''), \tag{4.27}$$

such that

$$\sup_{v'_{l'}} \int_{\mathbf{R}^3} |\mathbf{k}_m(v'_{l'}, v'') - \mathbf{k}_{w_{\varrho}}(v'_{l'}, v'')| dv'' \le \frac{1}{N}.$$

We split  $\mathbf{k}_w = \{\mathbf{k}_{w_\varrho}(v'_{l'}, v'') - \mathbf{k}_m(v'_{l'}, v'')\} + \mathbf{k}_m(v'_{l'}, v'')$ , and the first difference leads to a small contribution in (4.23) with  $k = \varrho$ ,

$$\frac{C_{\varrho}}{N} e^{-\frac{\nu_0}{2}s} \max_{1 \le l' \le k} \sup_{0 \le s_1 \le s} \{ e^{\frac{\nu_0}{2}s_1} \|h^{\ell - l - l'}(s_1)\|_{\infty} \}. \tag{4.28}$$

For the remaining main contribution of  $\mathbf{k}_m(v'_{l'}, v'')$ , note that

$$|\mathbf{k}_m(v'_{l'}, v'')| \lesssim C_N.$$

We may make a change of variable  $y = x'_{l'} + (s_1 - t'_{l'})\bar{v}_{l'}$  ( $x'_{l'}$  does not depend on  $v'_{l}$ ) so that  $\left|\frac{dy}{d\bar{v}'_{l}}\right| \geq \varepsilon^{d}$ , for  $s - s_1 \geq \varepsilon$  to estimate (using the notation in (1.8))

$$\int_{|v''| \leq m} \int_{\mathcal{V}'_{l'}} |h^{\ell-l-l'}(s_{1}, x'_{l'} + (s_{1} - t'_{l'}) \overline{v}'_{l}, v'')| \frac{e^{-(\frac{1}{4} + \zeta)|v'_{l'}|^{2}}}{(1 + \varrho^{2}|v'_{l'}|)^{\beta}} |n(x'_{l'}) \cdot v'_{l'}| dv'' \\
\leq \int_{\mathbf{R}^{3-d}} e^{-(\frac{1}{4} + \zeta)|\hat{v}_{l'}|^{2}} d\hat{v}_{l'} \int_{|v''| \leq m} dv'' \\
\times \int_{\mathbf{R}^{d}} h^{\ell-l-l'}(s_{1}, x'_{l'} + (s_{1} - t'_{l'}) \overline{v}'_{l}, v'') \mathbf{1}_{\{x'_{l'} + (s_{1} - t'_{l'}) \overline{v}'_{l} \in \overline{\Omega}\}} d\overline{v}'_{l'} \\
\leq \frac{1}{\varepsilon^{d}} \int_{\Omega} \int_{|v''| \leq m} |h^{\ell-l-l'}(s_{1}, y, v'')| dy dv'' \\
\lesssim_{\varepsilon, m} \left\| \frac{h^{\ell-l-l'}(s_{1})}{w_{\varrho}(v)} \right\|_{2}.$$

Hence the integrand of main contribution is bounded by  $\lesssim_{\varepsilon,m} \|\frac{h^{\ell-l-l'}(s_1)}{w_{\varrho}(v)}\|_2$ . Rearranging (4.23) and combining (4.22), (4.26) and (4.28) we have a bound for (4.23) as

$$K_{w_{\varrho}}h^{\ell+1-l}(s, X_{\mathbf{cl}}(s), v_{l})$$

$$\leq e^{-\frac{\nu_{0}}{2}s} \left[ \varepsilon + \frac{C_{K}}{N} + \left\{ \frac{1}{2} \right\}^{\varrho} \right] \max_{1 \leq l \leq 2k} \sup_{0 \leq s_{1} \leq s} \left\{ e^{\frac{\nu_{0}}{2}s_{1}} \|h^{\ell-l}(s_{1})\|_{\infty} \right\}$$

$$+ e^{-\frac{\nu_{0}}{2}s} \left\{ \varrho \sup_{0 \leq s_{1} \leq s} \left[ e^{\frac{\nu_{0}}{2}s_{1}} \left| \frac{w_{\varrho}r(s_{1})}{\langle v \rangle} \right|_{\infty} \right] + \varrho \sup_{0 \leq s_{1} \leq s} \left[ e^{\frac{\nu_{0}}{2}s_{1}} \left\| \frac{w_{\varrho}g(s_{1})}{\langle v \rangle} \right\|_{\infty} \right]$$

$$+ \max_{0 \leq l \leq 2k} \|h^{\ell+1-l}(0)\|_{\infty} \right\}$$

$$+ C_{\varepsilon,m,N} \max_{1 \leq l \leq 2k} \int_{0}^{s} \left\| \frac{h^{\ell-l}(s_{1})}{w_{\varrho}(v)} \right\|_{2} ds_{1}$$

$$\equiv e^{-\frac{\nu_{0}}{2}s} B.$$

$$(4.29)$$

By plugging back (4.25) and (4.29) into (4.21), we obtain

$$\begin{split} &|h^{\ell+1}(t,x,v)| \\ &\lesssim & B\left\{\mathbf{1}_{t_1 \leq 0} \int_0^t e^{-\nu(v)(t-s)} ds + \mathbf{1}_{t_1 > 0} \int_{t_1}^t e^{-\nu(v)(t-s)} ds\right\} \\ &+ B \frac{\int e^{-\nu_0(t-s)} ds}{\tilde{w}_{\varrho}} \sum_{l=1}^{k-1} \left\{ \int \mathbf{1}_{\{t_{l+1} \leq 0 < t_l\}} + \int \mathbf{1}_{\{0 < t_l\}} \right\} d\Sigma_l + e^{-\frac{\nu_0}{2}t} A^{\ell} \\ &\lesssim & e^{-\frac{\nu_0}{2}t} \left[\varepsilon + \frac{C_K}{N} + \left\{\frac{1}{2}\right\}^{\varrho}\right] \max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_0}{2}s} \|h^{\ell-l}(s)\|_{\infty} \right\} \\ &+ e^{-\frac{\nu_0}{2}t} \left\{ \varrho \sup_{0 \leq s \leq t} \left[ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho} r(s)}{\langle v \rangle} \right\|_{\infty} \right] + \varrho \sup_{0 \leq s \leq t} \left[ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho} g(s)}{\langle v \rangle} \right\|_{\infty} \right] \\ &+ \max_{0 \leq l \leq 2k} \|h^{\ell+1-l}(0)\|_{\infty} \right\} \\ &+ C_{\varepsilon,m,N} \max_{1 \leq l \leq 2k} \int_0^t \left\| \frac{h^{\ell-l}(s)}{w_{\varrho}(v)} \right\|_2 ds_1 \end{split}$$

We then deduce our lemma by choosing  $t = \varrho = Ct^{5/4}$  and letting  $\varrho$  large and fixed, (so are k and t), and then choosing  $\varepsilon$  sufficiently small and N sufficiently large. It is trivial to rescale to the case  $\varrho = 1$  with a different constant depending on  $\varrho$ .

Now we sketch the proof of (4.16). Since the proof is similar to (4.15) we just highlight the differences. The key is to estimate the r-contribution (4.12) differently, using its weaker  $L_x^1(L_{s,v}^{\infty})$  norm. For large N > 0 we choose m > 0 such that

$$\int_{\mathcal{V}} \mathbf{1}_{|v| \ge m} \langle v \rangle \tilde{w}_{\varrho}(v) d\sigma \le \frac{1}{N}.$$

From (4.14) and Lemma 4.2, we bound (4.12) by splitting  $1 = \mathbf{1}_{|v| \geq m} + \mathbf{1}_{|v| < m}$ :

$$\frac{e^{-\frac{\nu_0}{2}t}}{\tilde{w}_{\varrho}(v)} \int \sum_{l=1}^{k-1} \mathbf{1}_{0 < t_l} d\sigma_{k-1} \cdots d\sigma_{l+1} \\
\times e^{\frac{\nu_0}{2}t_{l+1}} \left| \frac{w_{\varrho}(v_l)r(t_{l+1})}{\langle v_l \rangle} \right| \mathbf{1}_{|v_l| \ge m} \langle v_l \rangle \tilde{w}_{\varrho}(v_l) d\sigma_l d\sigma_{l-1} \cdots d\sigma_1 \\
+ \frac{e^{-\frac{\nu_0}{2}t}}{\tilde{w}_{\varrho}(v)} \int \sum_{l=1}^{k-1} \mathbf{1}_{0 < t_l} d\sigma_{k-1} \cdots d\sigma_{l+1} e^{\frac{\nu_0}{2}t_{l+1}} w_{\varrho}(v_l) |r(t_{l+1})| \mathbf{1}_{|v_l| < m} \tilde{w}_{\varrho}(v_l) d\sigma_l d\sigma_{l-1} \cdots d\sigma_1 \\
\leq \frac{k}{N\tilde{w}_{\varrho}(v)} \left\{ 1 + \frac{C_{\beta,\zeta}}{\varrho^4} \right\} \sup_{0 \le s \le t} \left| \frac{w_{\varrho}r(s)}{\langle v \rangle} \right| \\
+ \frac{kC_m}{\tilde{w}_{\varrho}(v)} \int \int_{|v_l| \le m} |r(t_{l+1}, x_l - t_{\mathbf{b}}(x_l, v_l) \bar{v}_l, \bar{v}_l, \hat{v}) ||n(x_l) \cdot v_l| dv_l d\sigma_{l-1} \cdots d\sigma_1. \tag{4.30}$$

We have used the  $d\sigma_j$  is a probability measure for j > l. Using the notation (1.8),  $\bar{u}_l \equiv \frac{\bar{v}_l}{|\bar{v}_l|}$  and  $t_{\mathbf{b}}(x_l, v_l)\bar{v}_l = t_{\mathbf{b}}(x_l, \frac{\bar{v}_l}{|\bar{v}_l|})\frac{\bar{v}_l}{|\bar{v}_l|}$ , and we have an uppder bound:

$$\int_{|v_{l}| \leq m} |r(t_{l+1}, x_{l} - t_{\mathbf{b}}(x_{l}, v_{l})\bar{v}_{l}, v)||n(x_{l}) \cdot \bar{v}_{l}|dv_{l}$$

$$\leq \int_{|\hat{v}_{l}| \leq m} d\hat{v}_{l} \int_{|\bar{v}_{l}| \leq m} \sup_{s, v} |r(s, x_{l} - t_{\mathbf{b}}(x_{l}, v_{l})\bar{v}_{l}, v)||n(x_{l}) \cdot \bar{u}_{l}||\bar{v}_{l}|d\bar{v}_{l}$$

$$\lesssim_{m} \int_{\mathbf{S}^{d-1}} \sup_{s, v} |r(s, x_{l} - t_{\mathbf{b}}(x_{l}, v_{l})\bar{v}_{l}, v)||n(x_{l}) \cdot \bar{u}_{l}|d\bar{u}_{l}.$$

$$(4.31)$$

Since  $x_l - t_{\mathbf{b}}(x_l, v_l)\bar{v}_l \in \partial\Omega$  and  $x_l \in \partial\Omega$ , Now apply (2.9) in Lemma 2.3 to have

$$\int_{|v_l| \le m} \sup_{s,v} |r(s, x_l - t_{\mathbf{b}}(x_l, v_l)\bar{v}_l, v)| |n(x_l) \cdot \bar{u}_l| d\bar{u}_l$$

$$\lesssim_m \int_{\partial\Omega} \sup_{s,v} |r(s, y, v)| dS(y). \tag{4.32}$$

We now turn to (4.21). Now instead of (4.22), combining (4.32) and (4.30) yields

$$e^{-\frac{\nu_{0}}{2}t}A_{\ell}(t,x,v)$$

$$= e^{-\frac{\nu_{0}}{2}(t-t_{1})}w_{\varrho}(v)|r(t_{1},x_{1},v)|$$

$$+\frac{C\varrho}{\tilde{w}_{\varrho}(v)}\left\{1+\frac{C_{\beta,\zeta}}{\varrho^{4}}\right\}\left\{\frac{1}{N}\sup_{s}\left|\frac{w_{\varrho}r(s)}{\langle v\rangle}\right|_{\infty}+C_{N}\left|\sup_{s,v}|r(s,\cdot,v)|\right|_{1}\right\}$$

$$+C_{\beta,\zeta}(1+\varrho^{\beta})\sup_{s}\left\|\frac{w_{\varrho}g(s)}{\langle v\rangle}\right\|_{\infty}+e^{-\frac{\nu_{0}}{2}t}\sup_{l}\|h^{l}(0)\|_{\infty}+C_{\beta,\zeta}\left\{\frac{1}{2}\right\}^{\varrho}\sup_{s,l}\|h^{l}(s)\|_{\infty}$$

$$\equiv e^{-\frac{\nu_{0}}{2}(t-t_{1})}w_{\varrho}(v)|r(t_{1},x_{1},v)|+D. \tag{4.33}$$

We now plug (4.33) back into (4.24), the estimate for (4.23), for each  $\ell$ . Now we obtain the upper bound for the last term of (4.24) using the estimate of  $A_{\ell-1-l}(s, X_{\mathbf{cl}}(s), v')$  from (4.33) as

$$e^{-\frac{\nu_0}{2}s} \int_{\mathbf{R}^3} \mathbf{k}'_{w_\varrho} e^{\frac{\nu_0}{2}t'_1} w_\varrho(v') |r(t'_1, x'_1, v')| dv' + e^{-\frac{\nu_0}{2}s} C_K D.$$

where

$$t_1' = t_{\mathbf{b}}(X_{\mathbf{cl}}(s), \bar{v}') \tag{4.34}$$

is the exit time of the point  $(X_{\mathbf{cl}}(s), \bar{v}')$  and

$$x_1' = X_{\mathbf{cl}}(s) - t_{\mathbf{b}}(X_{\mathbf{cl}}(s), \overline{v}')\overline{v}', \quad \mathbf{k}_{w_{\varrho}}' = \mathbf{k}_{w_{\varrho}}(V_{\mathbf{cl}}(s), v'). \tag{4.35}$$

We further plug this bound back into (4.23) and then back into (4.21) to isolate *only* the contribution of  $A_{\ell-1-l}$ . Following exactly the same steps, it suffices to control

$$\mathbf{1}_{t_{1} \leq 0} \int_{0}^{t} e^{-\nu(v)(t-s)} e^{-\frac{\nu_{0}}{2}s} \int \mathbf{k}'_{w_{\varrho}} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | dv' ds \qquad (4.36)$$

$$+ \mathbf{1}_{t_{1} > 0} \int_{t_{1}}^{t} e^{-\nu(v)(t-s)} \int \mathbf{k}'_{w_{\varrho}} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | dv' ds$$

$$+ \frac{e^{-\nu(v)(t-t_{1})}}{\tilde{w}_{\varrho}(v)} \times \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \int \mathbf{k}'_{w_{\varrho}} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | + \int_{t_{l+1}}^{t_{l}} \mathbf{1}_{\{0 < t_{l+1}\}} \int \mathbf{k}'_{w_{\varrho}} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | \right\} d\Sigma_{l}(s) ds$$

$$+ e^{-\frac{\nu_{0}}{2}(t-t_{1})} w_{\varrho}(v) | r(t_{1}, x_{1}, v) | + D + C_{\beta, \zeta} \varrho^{\beta} \left\{ \left[ 1 + \frac{C_{\beta, \zeta}}{\varrho^{4}} \right]^{\varrho} + \frac{1}{2^{\varrho}} \right\} D,$$

by Lemma 4.2. Note  $\varrho^{\beta}\left\{[1+\frac{C_{\beta,\zeta}}{\varrho^4}]^{\varrho}+\frac{1}{2^{\varrho}}\right\}\lesssim 1$ . We shall use a sequence of approximations to estimate the above integrals. Let  $\mathbf{k}'_{w_{\varrho}}\mathbf{1}_{|v'-V_{\mathbf{cl}}(s)|\geq \frac{1}{m},|v'-V_{\mathbf{cl}}(s)|\leq m}=\mathbf{k}^m_{w_{\varrho}}$ . We again split for m large

$$\int |\mathbf{k}'_{w_{\varrho}} - \mathbf{k}^m_{w_{\varrho}}| \le \frac{1}{N}.$$

We therefore bound

$$\mathbf{1}_{t_{1} \leq 0} \int_{0}^{t} e^{-\nu(v)(t-s)} e^{-\frac{\nu_{0}}{2}s} \int \mathbf{k}_{w_{\varrho}}^{m} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | dv' ds$$

$$+ \mathbf{1}_{t_{1} > 0} \int_{t_{1}}^{t} e^{-\nu(v)(t-s)} \int \mathbf{k}_{w_{\varrho}}^{m} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | ds$$

$$+ \frac{e^{-\nu(v)(t-t_{1})}}{\tilde{w}_{\varrho}(v)} \times \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \int \mathbf{k}_{w_{\varrho}}^{m} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | \right.$$

$$+ \int_{t_{l+1}}^{t_{l}} \mathbf{1}_{\{0 < t_{l+1}\}} \int \mathbf{k}_{w_{\varrho}}^{m} e^{\frac{\nu_{0}}{2}t'_{1}} w_{\varrho}(v') | r(t'_{1}, x'_{1}, v') | \left. \right\} d\Sigma_{l}(s) ds$$

$$+ \frac{C_{\beta, \varrho, k, \zeta}}{N} \sup_{s} |w_{\varrho} r(s)|_{\infty}. \tag{4.37}$$

Since  $\mathbf{k}_{w_{\varrho}}^{m}$  is bounded, and  $w_{\varrho}(v') \lesssim_{\varrho} \langle v' \rangle^{\beta}$ , we further split  $|v'| \geq N$  and  $|v'| \leq N$ . Since  $\int_{|v| \geq N} \langle v \rangle^{-4} \lesssim \frac{1}{N}$ , up to a constant of  $C_{N,\varrho,k}$  the main integrals in (4.37) are bounded by:

$$\mathbf{1}_{t_{1} \leq 0} \int_{0}^{t} \int_{\Pi_{m}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| dv' ds + \mathbf{1}_{t_{1} > 0} \int_{t_{1}}^{t} \int_{\Pi_{m}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| dv' ds 
+ \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{0}^{t_{l}} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \int_{\Pi_{m}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| 
+ \int_{t_{l+1}}^{t_{l}} \mathbf{1}_{\{0 < t_{l+1}\}} \int_{\Pi_{m}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| \right\} d\Sigma_{l}(s) ds 
+ \frac{1}{N} \sup_{s,v,x} \langle v \rangle^{\beta+4} |r(s,x,v)|,$$
(4.38)

where

$$v' \in \Pi_m \text{ iff } |v' - V_{\mathbf{cl}}(s)| \ge \frac{1}{m}, \text{ and } |v' - V_{\mathbf{cl}}(s)| \le m, \text{ and } |v'| \le N.$$
 (4.39)

Lastly, for any  $\varepsilon > 0$ , we further split the time intervals of the main integrals in (4.38) to obtain:

$$\mathbf{1}_{t_{1} \leq 0} \int_{\varepsilon}^{t-\varepsilon} \int_{\Pi_{N}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| dv' ds + \mathbf{1}_{t_{1} > 0} \int_{t_{1}+\varepsilon}^{t-\varepsilon} \int_{\Pi_{N}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| dv' ds$$

$$+ \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \sum_{l=1}^{k-1} \left\{ \int_{\varepsilon}^{t_{l}-\varepsilon} \mathbf{1}_{\{t_{l+1} \leq 0 < t_{l}\}} \int_{\Pi_{N}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| \right.$$

$$+ \int_{t_{l+1}+\varepsilon}^{t_{l}-\varepsilon} \mathbf{1}_{\{0 < t_{l+1}\}} \int_{\Pi_{N}} \langle v' \rangle^{\beta} |r(t'_{1}, x'_{1}, v')| \right\} d\Sigma_{l}(s) dv' ds$$

$$+ \varepsilon C_{\varrho,k,\beta} \sup_{s,x,v} \langle v \rangle^{\beta} |r(s,x,v)|. \tag{4.40}$$

We now are ready to use change of variables to estimate the main v'-integrals in (4.40). Recall that, by the definition (4.35) of  $x'_1$  and Definition (1.6),  $X_{cl}(s)$  reaches  $\partial\Omega$  if and only if s at these  $t, t_1, t_2, ... t_{l+1}$ . From our splitting of time intervals, there exists  $c_{\varepsilon} > 0$  such that

$$\operatorname{dist}(X_{\mathbf{cl}}(s), \partial \Omega) > c_{\varepsilon} > 0,$$

in the integrals in (4.40), uniformly in  $\Pi_N$ . We now repeat the change of variables  $x_1' \to \frac{\bar{v}'}{|\bar{v}'|}$  as (4.31) and (4.32), but using (2.10) instead of (2.9). We finally bound (4.40) by

$$C_{N,\varepsilon,k,\varrho} \left| \sup_{s,v} |r(s,\cdot,v)| \right|_{1} + \varepsilon C_{\varrho,k,\beta,N} \sup_{s,x,v} \langle v \rangle^{\beta} |r(s,x,v)|. \tag{4.41}$$

Collecting and combining (4.33), (4.36), (4.37), (4.40) and (4.41), we conclude the r contribution in (4.21) is bounded by

$$e^{-\frac{\nu_0}{2}(t-t_1)}w_{\varrho}(v)|r(t_1,x_1,v)| + C_{\varrho}D + \left\{\frac{C_{\beta,\varrho,k,\zeta}}{N} + \varepsilon C_{\varrho,k,\beta,N}\right\} \sup_{s} |\langle v \rangle^{\beta+4}r(s)|_{\infty} + C_{N,\varepsilon,k,\varrho} \left|\sup_{s,v} |r(s,\cdot,v)|\right|_{1}.$$

We deduce (4.16) by recalling D defined in (4.33).

After preparing the above tools we return to the stationary problem to give the proof of Proposition 4.1.

**Proof of Proposition 4.1.** We use the exactly same approximation (3.5) to establish the proposition and follow the same steps in the proof of Proposition 3.1. We denote  $h^{\ell+1} = w_o f^{\ell+1}$ , and rewrite (3.5) as

$$\varepsilon h^{\ell+1} + v \cdot \nabla_x h^{\ell+1} + \nu h^{\ell+1} = K_{w_{\varrho}} h^{\ell} + w_{\varrho} g, 
h^{\ell+1}_{-} = \frac{1 - \frac{1}{j}}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} h^{\ell}(t, x, v') \tilde{w}_{\varrho}(v') d\sigma + w_{\varrho} r.$$
(4.42)

Step 1: We take  $\ell \to \infty$  in  $L^{\infty}$ . Upon integrating over the characteristic lines  $\frac{dx}{dt} = v$ , and  $\frac{dv}{dt} = 0$  and using the boundary condition repeatedly, we obtain (by replacing 1 with  $1 - \frac{1}{j}$  and  $\nu$  with  $\nu + \varepsilon$ ) that the abstract iteration (4.8) is valid for stationary  $h^{\ell+1}(s,x,v) = h^{\ell+1}(x,v)$ , g(s,x,v) = g(x,v) and r(s,x,v) = r(x,v). Therefore, for  $\ell \geq 2k$ , by Lemma 4.4, we get (choosing  $k = \varrho = Ct^{5/4}$  large)

$$\begin{split} \|h^{\ell+1}\|_{\infty} & \leq & \frac{1}{8} \max_{1 \leq l \leq 2k} \{ \|h^{\ell-l}\|_{\infty} \} + e^{-\frac{\nu_0}{2}t} \max_{0 \leq l \leq 2k} \|h^{\ell+1-l}\|_{\infty} \\ & + \varrho \left[ |w_{\varrho} \ r|_{\infty} + \left\| \frac{w_{\varrho} \ g}{\langle v \rangle} \right\|_{\infty} \right] + C_k \max_{1 \leq l \leq 2k} \left\| f^{\ell-l} \right\|_2. \end{split}$$

Then, absorbing  $e^{-\frac{\nu_0}{2}t}\|h^{\ell+1}\|_{\infty}$  in the left hand side, we have

$$\begin{split} \|h^{\ell+1}\|_{\infty} & \leq & \frac{1}{4} \max_{1 \leq l \leq 2k} \{ \|h^{\ell+1-l}\|_{\infty} \} + C_k \left[ |w_{\varrho} \ r|_{\infty} + \left\| \frac{w_{\varrho} \ g}{\langle v \rangle} \right\|_{\infty} \right] \\ & + C_k \max_{1 \leq l \leq 2k} \left\| f^{\ell-l} \right\|_2. \end{split}$$

Now this is valid for all  $\ell \geq 2k$ . By induction on  $\ell$ , we can iterate such bound for  $\ell+2,....\ell+2k$  to obtain

$$\begin{split} &\|h^{\ell+i}\|_{\infty} \leq \frac{1}{4} \max_{1 \leq l \leq 2k} \{\|h^{\ell+i-l}\|_{\infty}\} + C_k \left[ |w_{\ell}| r|_{\infty} + \left\| \frac{w_{\ell}|g|}{\langle v \rangle} \right\|_{\infty} + \max_{-2k \leq l \leq 2k} \left\| f^{\ell+l} \right\|_2 \right] \\ &\leq \frac{1}{4} \max_{1 \leq l \leq 2k} \{\|h^{\ell-1+i-l}\|_{\infty}\} + 2C_k \left[ |w_{\ell}| r|_{\infty} + \left\| \frac{w_{\ell}|g|}{\langle v \rangle} \right\|_{\infty} + \max_{-2k \leq l \leq 2k} \left\| f^{\ell+l} \right\|_2 \right] \\ &\vdots \\ &\leq \frac{1}{4} \max_{1 \leq l \leq 2k} \{\|h^{\ell-l}\|_{\infty}\} + (2i+1)C_k \left[ |w_{\ell}| r|_{\infty} + \left\| \frac{w_{\ell}|g|}{\langle v \rangle} \right\|_{\infty} + \max_{-2k \leq l \leq 2k} \left\| f^{\ell+l} \right\|_2 \right]. \end{split}$$

We now take a maximum over  $1 \le i \le 2k$  to get

$$\max_{1 \le l \le 2k} \|h^{\ell+1-l}\|_{\infty} \le \frac{1}{4} \max_{1 \le l \le 2k} \|h^{\ell-l}\|_{\infty} + (4k+1)C_k \left[ |w_{\ell} r|_{\infty} + \left\| \frac{w_{\ell} g}{\langle v \rangle} \right\|_{\infty} + \max_{-2k \le l \le 2k} \|f^{\ell+l}\|_2 \right].$$
(4.43)

This implies that, from Lemma 4.4 for all  $\ell \geq 2k$ ,

$$\max_{1 \le l \le 2k} \|h^{\ell+1-l}\|_{\infty} \lesssim_k \left[ \max_{1 \le l \le 2k} \|h^l\|_{\infty} + |w_{\ell}|_{\infty} + \sup_{0 \le s \le t} \left\| \frac{w_{\ell}|g|}{\langle v \rangle} \right\|_{\infty} + \max_{1 \le l \le \ell} \|f^l\|_2 \right]. \tag{4.44}$$

Now in order to control  $\max_{1 \le l \le 2k} \|h^l\|_{\infty}$ , we can use (4.8) repeatedly for  $h^{2k} \to h^{2k-1} \cdots \to h^0$  to obtain, by Lemma 4.2,

$$\max_{1 \le l \le 2k} \|h^l\|_{\infty} \lesssim_k |w_{\varrho}| r|_{\infty} + \left\| \frac{w_{\varrho}|g|}{\langle v \rangle} \right\|_{\infty}. \tag{4.45}$$

We therefore conclude that, from (4.44) and (4.45),

$$\max_{1 \le l \le 2k} \|h^{\ell+1-l}\|_{\infty} \lesssim_{k} \left[ \|h^{0}\|_{\infty} + |w_{\varrho}| r|_{\infty} + \sup_{0 \le s \le t} \left\| \frac{w_{\varrho} g}{\langle v \rangle} \right\|_{\infty} + \max_{1 \le l \le \ell} \|f^{l}\|_{2} \right].$$

But  $\max_{1 \leq l \leq \infty} \|f^l\|_2$  is bounded by step 1 in the proof of Lemma 3.3. Furthermore for  $\beta > 4$ ,  $\|\cdot\|_2$  is bounded by  $\|w_{\varrho}\cdot\|_{\infty}$  and  $|r|_2$  is bounded by  $|w_{\varrho}\langle v\rangle r|_{\infty}$ . Hence we have

$$\max_{1 \le l \le 2k} \|h^{\ell+1-l}\|_{\infty} \lesssim_k \left[ \|h^0\|_{\infty} + |w_{\varrho}| r|_{\infty} + \sup_{0 \le s \le t} \left\| \frac{w_{\varrho}|g|}{\langle v \rangle} \right\|_{\infty} \right].$$

Therefore there exists a limit (unique) solution  $h^{\ell} \to h = w_{\varrho} f \in L^{\infty}$ . Furthermore, h satisfies (4.8) with  $h^{\ell+1} \equiv h$ .

By subtracting  $h^{\ell+1} - h$  in (4.8) with r = g = 0, we obtain from Lemma 4.4, for  $h^{\ell+1} - h$ , (choosing  $k = \varrho = Ct^{5/4}$  large)

$$\begin{split} \|h^{\ell+1} - h\|_{\infty} & \leq & \frac{1}{8} \max_{0 \leq l \leq 2k} \|h^{\ell-l} - h\|_{\infty} + e^{-\frac{\nu_0}{2}k} \max_{0 \leq l \leq 2k} \|h^{\ell+1-l} - h\|_{\infty} \\ & + C_k \max_{1 \leq l \leq 2k} \|f^{\ell-l} - f\|_2 \\ & \leq & \frac{1}{4} \max_{1 \leq l \leq 2k} \|h^{\ell-l} - h\|_{\infty} + C_k \max_{1 \leq l \leq 2k} \|f^{\ell-l} - f\|_2, \end{split}$$

where  $e^{-\frac{\nu_0}{2}k}\|h^{\ell+1}-h\|_{\infty}$  is absorbed in the left hand side.

$$\leq \frac{1}{4^{\ell/2k}} \max_{1 \leq l \leq 2k} \{ \|h^l - h\|_{\infty} \} + C_k \left[ \max_{-2k \leq l \leq 2k} \|f^{\ell+l} - f\|_2 \right]$$

$$\leq \frac{C_k(|w_{\varrho}r|_{\infty} + \sup_{0 \leq s \leq t} \left\| \frac{w_{\varrho}g}{\langle v \rangle} \right\|_{\infty})}{4^{\ell/2k}} + C_k \left[ \max_{-2k < l \leq 2k} \|f^{\ell+l} - f\|_2 \right].$$

From step 1 of Lemma 3.3, we deduce  $||h^{\ell+1-l} - h||_{\infty} \to 0$  for  $\ell$  large.

Step 2: Now we let  $j \to \infty$ . We take  $f^j$  to be the solution to (3.7) and integrate along  $\frac{dx}{dt} = v, \frac{dv}{dt} = 0$  repeatedly. We establish (4.8) for  $h^{\ell} \equiv h^j$  (we may replace  $(1 - \frac{1}{j})$  by 1 and  $\varepsilon = 0$  to preserve inequality) and l = 0. Lemma 4.4 implies

$$\|h^{j}\|_{\infty} \leq \frac{1}{8} \|h^{j}\|_{\infty} + e^{-\frac{\nu_{0}}{2}k} \|h^{j}\|_{\infty} + \varrho^{1+4\beta} \left[ |w_{\varrho}r|_{\infty} + \left\| \frac{w_{\varrho}g}{\langle v \rangle} \right\|_{\infty} \right] + C(k) \|f^{j}\|_{2},$$

so that

$$||h^j||_{\infty} \le C_k \left[ |w_{\varrho}r|_{\infty} + \left\| \frac{w_{\varrho}g}{\langle v \rangle} \right\|_{\infty} \right] + C_k ||f^j||_2.$$

Since  $||f^j||_2$  is bounded, this implies that  $||h^j||_{\infty}$  is uniformly bounded and we obtain a (unique) solution  $h = wf \in L^{\infty}$ . Taking the difference, we have

$$\varepsilon[h^{j} - h] + v \cdot \nabla_{x}[h^{j} - h] + \nu[h^{j} - h] = K_{w}[h^{j} - h], 
h_{-}^{j} - h_{-} = \frac{1}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} [h^{j} - h](t, x, v') \tilde{w}_{\varrho}(v') d\sigma(v') 
- \frac{1}{j} \frac{1}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} h^{j}(t, x, v') \tilde{w}_{\varrho}(v') d\sigma(v').$$

We regard  $-\frac{1}{j}\frac{1}{\tilde{w}_{\varrho}(v)}\int_{n(x)\cdot v'>0}h(t,x,v')\tilde{w}_{\varrho}(v')(n(x)\cdot v')dv'=r$ . So Lemma 4.4 implies that

$$||h^{j} - h||_{\infty} \le \frac{1}{4} \{ ||h^{j} - h||_{\infty} \} + \frac{1}{i} |h^{j}|_{\infty} + C_{k} [||f^{j} - f||_{2}],$$

which goes to zero as j to  $\infty$ .

We obtained a  $L^{\infty}$  solution  $h^{\varepsilon} = w_{\rho} f^{\varrho}$  to (3.3). By integrating over the trajectory, (4.8) is valid for  $h^{\ell}$  replaced by  $h^{\varepsilon}$  so that from Lemma 4.4

$$||h^{\varepsilon}||_{\infty} \leq \frac{1}{8} \{||h^{\varepsilon}||_{\infty}\} + e^{-\frac{\nu_0}{2}t} ||h^{\varepsilon}||_{\infty} + k \left[|wr|_{\infty} + \left\|\frac{wg}{\langle v \rangle}\right\|_{\infty}\right] + C(k) ||f^{\varepsilon}||_{2},$$

and hence

$$||h^{\varepsilon}||_{\infty} \lesssim k \left[ |wr|_{\infty} + \left\| \frac{wg}{\langle v \rangle} \right\|_{\infty} \right] + C(k) ||f^{\varepsilon}||_{2},$$

which implies that, from Proposition 3.1, that  $||h^{\varepsilon}||_{\infty}$  is uniformly bounded and we obtain h = wf solution to the linear equation.

Now we have

$$\varepsilon h^{\varepsilon} + v \cdot \nabla_{x} [h^{\varepsilon} - h] + \nu L_{m} [h^{\varepsilon} - h] = 0,$$

$$h^{\varepsilon}_{-} - h_{-} = \frac{1}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} [h^{\varepsilon} - h](t, x, v') \tilde{w}_{\varrho}(v') d\sigma,$$

so that from Lemma 4.4

$$||h^{\varepsilon} - h||_{\infty} \lesssim_k \varepsilon ||h^{\varepsilon}||_{\infty} + ||f^{\varepsilon} - f||_2,$$

which goes to zero. For hard potential kernel, we recall that in previous section we have constructed an approximating sequence  $f^m$  to the equation

$$v \cdot \nabla f^m + L_m f^m = g, \qquad f_-^m = P_\gamma f^m + r,$$

with uniform bound in  $L^2$  and a limit  $f^m \to f$  weakly in  $\|\cdot\|_{\nu}$ , see Step 2 in the proof of Proposition 3.1. Moreover, we obtain from (4.4) that  $||w_{\rho}f^{m}||_{\infty}$  is uniformly bounded and so is  $w_{\rho}f$ . Note that

$$v \cdot \nabla_x [f^m - f] + L_m [f^m - f] = [-L + L_m]f, \quad [f^m - f]_- = P_\gamma [f^m - f].$$

It follows from Proposition 3.1 and the boundedness of  $w_{\rho}f$  that

$$||f^m - f||_{\nu} \le ||[-L + L_m]f||_2 \to 0.$$

Now, to show  $||f^m - f||_{\infty} \to 0$ , we apply (4.8) with  $g = (-L + L_m)f$ 

$$\|\{f^m - f\}\|_{\infty} \lesssim \|f^m - f\|_2 + \left\|\frac{\{L_m - L\}f}{\langle v \rangle}\right\|_{\infty} \to 0.$$

Since  $w_{\varrho}f \in L^{\infty}$ , the second term goes to zero. In the iteration scheme  $f^{\ell}$  is continuous away from  $\mathfrak{D}$  and hence f is continuous away

**Remark 4.6** Our construction fails to imply the  $F_s = \mu + \sqrt{\mu} f_s \ge 0$ . This can only be shown by the dynamical asymptotic stability discussed in Section 7.

## 5 Well-posedness, Continuity and Fourier Law

#### Proof of the Theorem 1.1.

Wellposedness. We consider the following iterative sequence

$$v \cdot \nabla_{x} f^{\ell+1} + L f^{\ell+1} = \Gamma(f^{\ell}, f^{\ell}),$$

$$f_{-}^{\ell+1} = P_{\gamma} f^{\ell+1} + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f^{\ell} \sqrt{\mu} (n(x) \cdot v) dv + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}},$$
(5.1)

with  $f^0 = 0$ .

Note  $\int \Gamma(f^{\ell}, f^{\ell}) \sqrt{\mu} = 0$  and from  $\int_{n(x) \cdot v < 0} \mu_{\delta}(n(x) \cdot v) dv = \int_{n(x) \cdot v < 0} \mu(n(x) \cdot v) dv = 1$ ,

$$\int_{\gamma_{-}} \sqrt{\mu} \left\{ \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f^{\ell} \sqrt{\mu} (n(x) \cdot v) dv + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \right\} d\gamma = 0.$$

Since  $|w_{\varrho} \frac{\mu_{\delta} - \mu}{\sqrt{\mu}}|_{\infty} \lesssim \delta$ , we apply Proposition 4.1 to get

$$||w_{\varrho}f^{\ell+1}||_{\infty} + |w_{\varrho}f^{\ell+1}|_{\infty} \lesssim \left|\left|\frac{w_{\varrho}\Gamma(f^{\ell}, f^{\ell})}{\langle v \rangle}\right|\right|_{\infty} + \delta|w_{\varrho}f^{\ell}|_{\infty,+} + \delta.$$

Since  $\left\|\frac{w_{\varrho}\Gamma(f^{\ell},f^{\ell})}{\langle v\rangle}\right\|_{\infty} \lesssim \|w_{\varrho}f^{\ell}\|_{\infty}^{2}$ , we deduce

$$||w_{\rho}f^{\ell+1}||_{\infty} + |w_{\rho}f^{\ell+1}||_{\infty} \lesssim ||w_{\rho}f^{\ell}||_{\infty}^{2} + \delta|w_{\rho}f^{\ell}||_{\infty,+} + \delta,$$

so that for  $\delta$  small,

$$||w_{\varrho}f^{\ell+1}||_{\infty} + |w_{\varrho}f^{\ell+1}|_{\infty} \lesssim \delta.$$

Upon taking differences, we have

$$[f^{\ell+1} - f^{\ell}] + v \cdot \nabla_x [f^{\ell+1} - f^{\ell}] + L[f^{\ell+1} - f^{\ell}]$$

$$= \Gamma(f^{\ell} - f^{\ell-1}, f^{\ell}) + \Gamma(f^{\ell-1}, f^{\ell} - f^{\ell-1}),$$

$$f_{-}^{\ell+1} - f_{-}^{\ell} = P_{\gamma} [f^{\ell+1} - f^{\ell}] + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} [f^{\ell} - f^{\ell-1}] (n(x) \cdot v) dv + \sqrt{\mu},$$

and by Proposition 4.1 again for  $f^{\ell+1} - f^{\ell}$ ,

$$\|w_{\varrho}[f^{\ell+1}-f^{\ell}]\|_{\infty} + |w_{\varrho}[f^{\ell+1}-f^{\ell}]|_{\infty} \lesssim \delta \big\{ \|w_{\varrho}[f^{\ell}-f^{\ell-1}]\|_{\infty} + |w_{\varrho}[f^{n}-f^{n-1}]|_{\infty} \big\}.$$

Hence  $f^{\ell}$  is Cauchy in  $L^{\infty}$  and we construct our solution by taking the limit  $f^{\ell} \to f_s$ . Uniqueness follows in the standard way. Moreover due to Theorem 2 and Theorem 3 in [26]  $f^{\ell}$  is continuous away from  $\mathfrak{D}$  and so is  $f_s$ . Moreover, if  $\Omega$  is convex, then  $\mathfrak{D} = \gamma_0$ .

Formation of singularities. Now we prove the formation of singularity. Recall that  $|\vartheta|_{\infty} \leq 1$  and the wall Maxwellian is defined as

$$\mu_{\delta}(v) = \frac{1}{2\pi[1 + \delta\vartheta(x)]^2} \exp\left[-\frac{|v|^2}{2[1 + \delta\vartheta(x)]}\right] ,$$

while the global Maxwellian is  $\mu(v) = \frac{1}{2\pi} e^{-\frac{|v|^2}{2}}$ . Then

$$\left| \mu_{\delta}(x,v) - \mu(v) - 2\left[\frac{|v|^2}{4} - 1\right]\mu(v)\delta\vartheta(x) \right|$$

$$\lesssim \delta^2 |\vartheta|_{\infty}^2 \left[1 + |v|^4\right] \exp\left[-\frac{|v|^2}{2(1+\delta|\vartheta|_{\infty})}\right].$$
(5.2)

For any non-convex domain  $\Omega \subset \mathbf{R}^d$  for d=2,3, there exists at least one  $(x_0,v) \in \gamma_0^{\mathbf{S}}$  with  $\bar{v} \neq 0$ . Suppose  $h=w_\varrho f$  satisfies (5.1) with  $f^{\ell+1}=f^\ell=f$ . We prove that h is discontinuous at  $(x_0,v)$  by contradiction argument. Assume h is continuous at  $(x_0,v)$  where the magnitude of v will be chosen later. By definition

$$x_1 \equiv x_0 - t_{\mathbf{b}}(x_0, \frac{v}{|v|}) \frac{\bar{v}}{|\bar{v}|}.$$

We choose  $\vartheta$  to be a continuous function such that

$$\vartheta(x_0) = |\vartheta|_{\infty} = 1, \quad \vartheta(x_1) = \frac{1}{2}|\vartheta|_{\infty} = \frac{1}{2},$$

$$(5.3)$$

and mainly zero elsewhere. Then by the continuity assumption we know that the quantities **I** and **II** defined below must coincide: **I** = **II**. **I** is obtained by evaluating  $h(x_0, v)$  through the boundary condition

$$\mathbf{I} = h(x_0, v) = \frac{1}{\tilde{w}_s(v)} \int_{n(x_0) \cdot v' > 0} h(x_0, v') \tilde{w}_s(v') d\sigma$$
 (5.4)

+ 
$$w_s(v) \frac{\mu_{\delta} - \mu}{\sqrt{\mu}}|_{(x_0,v)} \int_{n(x_0)\cdot v'>0} h(x_0,v')\tilde{w}(v')d\sigma$$
 (5.5)

+ 
$$w_s(v) \frac{\mu_\delta - \mu}{\sqrt{\mu}}|_{(x_0,v)}$$
. (5.6)

On the other hand, the existence of  $x_1$  allow us to evaluate  $h(x_0, v) = \mathbf{II}$  along the trajectory and has the expression (4.8) with H as in (4.9) – (4.13) with

$$r = \frac{\mu_{\delta} - \mu}{\sqrt{\mu}}, \quad h^{\ell} \equiv h, \ g = \Gamma(\frac{h}{w}, \frac{h}{w}),$$

which are independent of time. Since from (5.2),

$$\begin{aligned} |r|_2 &\lesssim \delta |\vartheta|_2, \\ \left|\sup_{v} |r(\cdot, v)|\right|_1 &\lesssim \delta |\vartheta|_1 \lesssim_{\Omega} \delta |\vartheta|_2, \\ |\langle v \rangle^{\beta+4} r|_{\infty} &\lesssim \delta |\vartheta|_{\infty} \lesssim \delta, \\ \left\|\frac{w_{\varrho} g}{\langle v \rangle}\right\|_{\infty} &\lesssim \left\|\frac{w_{\varrho} \Gamma(\frac{h}{w}, \frac{h}{w})}{\langle v \rangle}\right\|_{\infty} \lesssim \|w_{\varrho} h\|_{\infty}^2 \lesssim \delta^2 |\vartheta|_{\infty}^2 \lesssim \delta^2, \\ \|f\|_2 &\lesssim \delta^2 |\vartheta|_{\infty}^2 + \delta |\vartheta|_2, \end{aligned}$$

where we have employed Propositions 3.1 and 4.1 for the last two estimates. Due to (5.2) we can use the estimate (4.16) for II:

$$|h(x_{0}, v)| \leq e^{-\nu(v)(t-t_{1})}w_{\varrho}(v)|r(x_{1}, v)| + \left\{\frac{C_{\beta,\rho,\zeta}}{N} + \varepsilon C_{\rho,\beta,N}\right\}|\langle v\rangle^{\beta+4}r|_{\infty} + C_{N,\varepsilon,\rho}\left|\sup_{v}|r(\cdot, v)\right|_{1} + \left\{\frac{1}{N} + e^{-\frac{\nu_{0}}{2}\rho}\right\}|h||_{\infty} + \varrho\left\{\left\|\frac{w_{\varrho}}{\langle v\rangle}\right\|_{\infty}\right\} + C\int_{0}^{t}||f||_{2}ds \leq e^{-\nu(v)(t-t_{1})}w_{\varrho}(v)|r(x_{1}, v)| + \left\{\frac{C_{\beta,\rho,\zeta}}{N} + \varepsilon C_{\rho,\beta,N} + \frac{C}{N} + Ce^{-\frac{\nu_{0}}{2}\rho} + C\delta\right\}\delta + C_{\rho,\varepsilon,N}\delta|\vartheta|_{2}$$

$$\equiv e^{-\nu(v)(t-t_{1})}w_{\varrho}(v)|r(x_{1}, v)| + B.$$
(5.7)

Now we will show that, for a suitable choice of  $\vartheta$ ,  $\mathbf{I} - \mathbf{II} \ngeq 0$ , which is a contradiction so that  $h = w_{\varrho} f$  has a discontinuity at  $(x_0, v)$ .

Rewrite

$$\mathbf{I} - \mathbf{II} \geq \{(5.6) - e^{-\nu(v)(t-t_1)} w_{\varrho}(v) \frac{\mu_{\delta}(x_1, v) - \mu(v)}{\sqrt{\mu}} \} -\{|(5.4)| + |(5.5)| + B\}.$$
(5.8)

For large v, by (5.2),  $t \ge t_1, e^{-\nu(v)(t-t_1)} \le 1$ , we have

$$(5.6) - e^{-\nu(v)(t-t_1)} w_{\varrho}(v) \frac{\mu_{\delta}(x_1, v) - \mu(v)}{\sqrt{\mu}}$$

$$\geq 2w_{\varrho}(v) \left[\frac{|v|^2}{4} - 1\right] \sqrt{\mu} \delta |\vartheta|_{\infty} - w_{\varrho}(v) \left[\frac{|v|^2}{4} - 1\right] \sqrt{\mu} \delta |\vartheta|_{\infty} - C\delta^2$$

$$\geq w_{\varrho}(v) \left[\frac{|v|^2}{4} - 1\right] \sqrt{\mu} \delta |\vartheta|_{\infty} - C\delta^2. \tag{5.9}$$

Recall that  $||h||_{\infty} \lesssim \delta |\vartheta|_{\infty} \lesssim \delta$  to conclude that

$$|(5.4)|, |(5.5)| \le \left\{ \varrho^{-4} w_{\varrho}(v) \sqrt{\mu(v)} + \left(1 + \left[\frac{|v|^2}{4} - 1\right] w_{\varrho}(v) \sqrt{\mu(v)}\right) \delta |\vartheta|_{\infty} \right\} \delta |\vartheta|_{\infty}.$$

By (5.9), we can find  $|v_0| > 0$  so that  $c_{v_0} > 0$  sufficiently large so that

$$w_{\varrho}(v_0)\left[\frac{|v_0|^2}{4}-1\right]\sqrt{\mu(v_0)}-\varrho^{-4}w_{\varrho}(v_0)\sqrt{\mu(v_0)}\geq c_{v_0}>0.$$

On the other hand in (5.7) and (5.9), we choose  $\delta$  sufficiently small, and  $\rho$  sufficiently large, then N sufficiently large to get, then  $\varepsilon$  sufficiently small, such that for  $v = v_0$ ,

$$\mathbf{I} - \mathbf{II} \ge \frac{c_{v_0}}{2} \delta - C\delta |\vartheta|_2.$$

Since  $\vartheta$  is almost zero except for  $x_0$  and  $x_1$ , we can can make  $|\vartheta|_2$  arbitrarily small. In particular, there is a continuous function  $\vartheta$  such that  $\frac{c_{v_0}}{2}\delta - C\delta|\vartheta|_2 > \frac{c_{v_0}}{4}\delta > 0$ . Hence  $\mathbf{I} - \mathbf{II} > 0$  and this is a contradiction.

#### Proof of Theorem 1.2.

We now prove the  $\delta$ -expansion. Since  $\mu_{\delta}(x)$  is analytic with respect to  $\delta$  we have

$$\mu_{\delta}(x,v) = \mu + \delta\mu_1 + \delta^2\mu_2 + \cdots$$
 (5.10)

Note that  $|\mu_i(v)| \lesssim p_i(v)e^{-\frac{|v|^2}{2}}$ , where  $p_i(v)$  is some polynomial. Further we seek a formal expansion

$$f_s \sim \delta f_1 + \delta^2 f_2 + \cdots$$

Plugging this into the equation,

$$v \cdot \nabla_{x} [\delta f_{1} + \delta^{2} f_{2} + \dots] + L[\delta f_{1} + \delta^{2} f_{2} + \dots]$$

$$= \Gamma(\delta f_{1} + \delta^{2} f_{2} + \dots, \delta f_{1} + \delta^{2} f_{2} + \dots),$$

$$[\delta f_{1} + \delta^{2} f_{2} + \dots]_{-} = P_{\gamma} [\delta f_{1} + \delta^{2} f_{2} + \dots]$$

$$+ \frac{[\delta \mu_{1} + \delta^{2} \mu_{2} + \dots]}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} [\sqrt{\mu} + \delta f_{1} + \delta^{2} f_{2} + \dots] \sqrt{\mu} (n(x) \cdot v) dv.$$

We compare the coefficients of power of  $\delta$  to get an equation for  $f_i$  for  $i=1,\ldots,m-1,$  (assuming  $f_0\equiv 0$ )

$$v \cdot \nabla_{x} f_{i} + L f_{i} = \sum_{j=1}^{i-1} \Gamma(f_{j}, f_{i-j}),$$

$$f_{i}|_{\gamma_{-}} = P_{\gamma} f_{i} + \sum_{j=1}^{i-1} \frac{\mu_{j}}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f_{i-j} \sqrt{\mu} (n(x) \cdot v) dv + \frac{\mu_{i}}{\sqrt{\mu}}.$$
(5.11)

Note that, from the  $\delta$ -expansion,

$$\int_{\gamma_{\pm}} (\mu^{\delta}(x,v) - \mu(v)) |n(x) \cdot v| dv = 0, \quad \sum_{i=1}^{\infty} \delta^{i} \int_{\gamma_{\pm}} \mu_{i} |n(x) \cdot v| dv = 0.$$

Since

$$\int_{\mathbf{R}^3} \sum_{j=1}^{i-1} \Gamma(f_j, f_{i-j}) \sqrt{\mu} dv = 0, \quad \int_{n(x) \cdot v < 0} \left[ \sum_{j=1}^{i-1} \frac{\mu_j}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f_{i-j} \sqrt{\mu} + \frac{\mu_i}{\sqrt{\mu}} \right] = 0,$$

by applying Proposition 4.1 repeatedly, we can construct  $f_1, f_2, ..., f_{m-1}$  inductively so that for  $0 \le \zeta < \frac{1}{4}$ ,

$$||w_{\varrho}f_i||_{\infty} + |w_{\varrho}f_i|_{\infty} \lesssim_m 1. \tag{5.12}$$

In particular  $f_1$  satisfies (1.19). Now we define the remainder  $f_m^{\delta}$  satisfying

$$f_s = \delta f_1 + \dots + \delta^{m-1} f_{m-1} + \delta^m f_m^{\delta},$$

and we obtain the equation for the remainder  $f_m^{\delta}$ 

$$v \cdot \nabla_{x} f_{m}^{\delta} + L f_{m}^{\delta} = \delta[\Gamma(f_{1} + \delta f_{2} + \dots + \delta^{m-2} f_{m-1}, f_{m}^{\delta}) + \Gamma(f_{m}^{\delta}, f_{1} + \delta f_{2} + \dots + \delta^{m-2} f_{m-1}) + \delta^{m} \Gamma(f_{m}^{\delta}, f_{m}^{\delta}) + g^{\delta},$$

$$f_{m}^{\delta}|_{\gamma_{-}} = P_{\gamma} f_{m}^{\delta} + \frac{\mu^{\delta} - \mu}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f_{m}^{\delta} \sqrt{\mu} (n(x) \cdot v) dv + r^{\delta}.$$

where

$$g^{\delta} = \sum_{i=m}^{2m-1} \sum_{j=1}^{m-1} \delta^{i-m} \Gamma(f_j, f_{i-j}), \quad r^{\delta} = \sum_{i=1}^{m} \delta^{i-1} \frac{\mu_i}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f_{m-i} \sqrt{\mu} (n(x) \cdot v) dv.$$

Since by (5.12)

$$\left\| \frac{w_{\varrho} g^{\delta}}{\langle v \rangle} \right\|_{\infty} + |w_{\varrho} r^{\delta}|_{\infty} \lesssim_{m} 1,$$

we can apply Proposition 4.1 to deduce that

$$||w_{\varrho} f_{m}^{\delta}||_{\infty} + |w_{\varrho} f_{m}^{\delta}||_{\infty} \\ \lesssim_{m} \delta ||w_{\varrho} f_{m}^{\delta}||_{\infty} + \delta^{m} ||w_{\varrho} f_{m}^{\delta}||_{\infty}^{2} + \delta ||w_{\varrho} f_{m}^{\delta}||_{\infty} + ||w_{\varrho} g^{\delta}||_{\infty} + ||w_{\varrho} f^{\delta}||_{\infty}.$$

We therefore conclude that  $\|w_{\varrho} f_m^{\delta}\|_{\infty} + |w_{\varrho} f_m^{\delta}|_{\infty} \lesssim 1$  and the expansion is valid.

**Proof of Theorem 1.5.** We now consider a slab  $-\frac{1}{2} \le x \le +\frac{1}{2}$ . We consider the stationary solution  $f_s \in L^{\infty}$  satisfying

$$v_{1}\partial_{x}f_{s} + Lf_{s} = \Gamma(f_{s}, f_{s}), \qquad (x, v) \in (-\frac{1}{2}, +\frac{1}{2}) \times \mathbf{R}^{3},$$

$$f_{s}(x, v) = P_{\gamma}f_{s} + \frac{\mu^{\delta} - \mu}{\sqrt{\mu}} \int_{n(x) \cdot v > 0} f_{s}\sqrt{\mu}(n(x) \cdot v) dv,$$
(5.13)

for  $x = -\frac{1}{2}$  or  $x = +\frac{1}{2}$ . From Theorem 1.1,  $\|w_{\varrho} f_s\|_{\infty} \lesssim \delta$ . We claim that  $\partial_x f_s(x, v) \in L^2([-\frac{1}{2} + \varepsilon, +\frac{1}{2} - \varepsilon] \times \mathbf{R}^3)$  for any small  $\varepsilon > 0$ . In fact, we multiply (5.13) by a 1-dimensional spatial smooth cutoff function  $\chi(x)$ ,  $(\chi \equiv 0 \text{ near } x = -\frac{1}{2} \text{ and } x = +\frac{1}{2})$  so that

$$v_1 \partial_x [\chi f_s] + L[\chi f_s] = \Gamma(\chi f_s, f_s) - v_1 f_s \chi'.$$

We take one spatial derivative (5.13) to get

$$v_1 \partial_{xx} [\chi f_s] + L \partial_x [\chi f_s] = \Gamma(\partial_x [\chi f_s], f_s) + \Gamma(\chi f_s, \partial_x f_s) - v_1 \chi' \partial_x f_s - v_1 \chi'' f_s$$

$$= \Gamma(\partial_x [\chi f_s], f_s) + \Gamma(\chi f_s, \partial_x f_s) + \chi' (L f_s - \Gamma(f_s, f_s)) - v_1 \chi'' f_s$$

$$= \Gamma(\partial_x [\chi f_s], f_s) + \Gamma(f_s, \partial_x [\chi f_s])$$

$$-\Gamma(f_s, \chi' f_s) + \chi' (L f_s - \Gamma(f_s, f_s)) - v_1 \chi'' f_s,$$

where we have used (5.13) to replace  $-v_1\partial_x f_s = Lf_s - \Gamma(f_s, f_s)$ . Letting  $Z = \chi \partial_x f_s$ , then we have

$$v_1 Z_x + LZ = \Gamma(Z, f_s) + \Gamma(f_s, Z) -\Gamma(f_s, \chi' f_s) + \chi'(L f_s - \Gamma(f_s, f_s)) - v_1 \chi'' f_s.$$

Note that  $-\Gamma(f_s, \chi' f_s) + \chi'(L f_s - \Gamma(f_s, f_s)) - v_1 \chi'' f_s \in L^{\infty}$ . Multiply by Z and use Green's identity to have (no boundary contribution)

$$\|(\mathbf{I} - \mathbf{P})Z\|_{\nu}^{2} \lesssim \delta \{ \|Z\|_{\nu}^{2} + 1 \}.$$

Then we repeat the proof of Lemma 3.4 but replacing  $\phi_a$  in (3.37) with the solution of  $-\Delta\phi_a=a$  with  $\phi_a=0$  on  $\partial\Omega$ . Since the boundary condition of Z is  $Z_{\gamma}\equiv 0$ , Lemma 3.4 is valid so that

$$\|\chi \partial_x f_s\|_2^2 \lesssim \|Z\|_2^2 < +\infty,$$

and  $\partial_x f_s$  is locally in  $L^2_{loc}(0,1)$ .

Recall that

$$u_s(x) = \frac{1}{\int_{\mathbf{R}^3} F_s(x, v) dv} \int_{\mathbf{R}^3} v F_s(x, v) dv.$$

Now we are going to show

$$u_s(x) = 0$$
, for any  $x \in [-\frac{1}{2}, +\frac{1}{2}]$ .

For any smooth test function  $\psi \in C^{\infty}([-\frac{1}{2}, +\frac{1}{2}])$ , due to the weak formulation for  $F_s$ , we have for the first component of  $\rho_s u$  that

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi'(x)(\rho_{s}u_{s})_{1}dx$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi'(x) \int_{\mathbf{R}^{3}} v_{1}F_{s}(x,v)dvdx$$

$$= \psi(\frac{1}{2}) \int_{\mathbf{R}^{3}} v_{1}F_{s}(\frac{1}{2}) - \psi(-\frac{1}{2}) \int_{\mathbf{R}^{3}} v_{1}F_{s}(-\frac{1}{2}) - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi(x) \int_{\mathbf{R}^{3}} v_{1}\partial_{x}F_{s}(x,v)$$

$$= -\int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi(x) \int_{\mathbf{R}^{3}} Q(F_{s}, F_{s})(x,v)dvdx$$

$$= 0,$$

where we used the boundary condition as well as the orthogonality of Q to the collision invariants. Hence  $\partial_x[(\rho_s u_s)_1(x)] = 0$  in the distribution sense and hence  $\rho_s u_s \in W^{1,\infty}$ . Therefore  $\rho_s u_s$  is continuous up to the boundary and  $\lim_{x\downarrow -\frac{1}{2}} \rho_s u_s(x) = \rho_s u_s(-\frac{1}{2})$  and  $\lim_{x\uparrow +\frac{1}{2}} \rho_s u_s(x) = \rho_s u_s(+\frac{1}{2})$ . Moreover for all  $\psi \in C^{\infty}([-\frac{1}{2}, +\frac{1}{2}])$ 

$$0 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \psi'(x) (\rho_s u_s)_1 = (\rho_s u_s)_1(\frac{1}{2}) \psi(\frac{1}{2}) - (\rho_s u_s)_1(-\frac{1}{2}) \psi(-\frac{1}{2}),$$

to conclude

$$(\rho_s u_s)_1(x) \equiv 0$$
, for all  $x \in [-\frac{1}{2}, +\frac{1}{2}]$ .

On the other hand

$$\rho_s(x) = \int_{\mathbf{R}^3} \mu(v) + \sqrt{\mu} f_s(x, v) dv \ge \int_{\mathbf{R}^3} \mu(v) dv - \int_{\mathbf{R}^3} \sqrt{\mu} (v) w^{-1}(v) dv \times ||w f_s||_{\infty} > 0,$$

for small  $||wf_s||_{\infty}$  so that  $\rho_s$  is positive. Thus  $(u_s)_1(x) \equiv 0$ . The components  $(u_s)_i$  for i = 2, 3 vanish by symmetry:

$$(u_s)_i(x) = \frac{1}{\rho_s(x)} \int_{\mathbf{R}^3} v_i F_s(x, v) dv$$

$$= \frac{1}{\rho_s(x)} \int_{v_i > 0} v_i F_s(x, v) dv - \frac{1}{\rho_s(x)} \int_{\tilde{v}_i > 0} \tilde{v}_i F_s(x, v) d\tilde{v}$$

$$= \frac{1}{\rho_s(x)} \int_{v_i > 0} v_i F_s(x, v) dv - \frac{1}{\rho_s(x)} \int_{\tilde{v}_i > 0} \tilde{v}_i \tilde{F}_s(x, \tilde{v}) d\tilde{v}, \qquad (5.14)$$

where  $(\tilde{v})_i = -v_i$  and  $\tilde{v}_j = v_j$  for  $j \neq i$ , and we have defined  $\tilde{F}_s(x, v) = F_s(x, \tilde{v})$ . Then  $\tilde{F}$  solves

$$v_1 \partial_{x_1} \tilde{F}_s = Q(\tilde{F}_s, \tilde{F}_s) , \quad \tilde{F}_s(x, v) = \mu^{\theta}(x, v) \int_{n(x) \cdot v > 0} \tilde{F}_s(x, v) (n(x) \cdot v) dv,$$

where  $x \in \partial\Omega$  and  $n(x) \cdot v < 0$ . By the uniqueness, we conclude  $\tilde{F}_s(x,v) = F_s(x,v)$  and hence  $\tilde{F}_s(x,\tilde{v}) = F_s(x,\tilde{v})$  almost everywhere. Therefore it follows that (5.14) vanishes and  $(u_s)_i \equiv 0$  almost everywhere for i = 2, 3.

Recall the definition of temperature  $\theta_s$  in (1.23). Since  $\partial_x f_s \in L^2([-\frac{1}{2} + \varepsilon, +\frac{1}{2} - \varepsilon] \times \mathbf{R}^3)$ , we have  $\partial_x \theta_s \in L^2([-\frac{1}{2} + \varepsilon, +\frac{1}{2} - \varepsilon] \times \mathbf{R}^3)$  so that in the sense of distribution,

$$\kappa(\theta_s)\partial_x\theta_s = \partial_x\{K(\theta_s)\},$$

with  $K' = \kappa > 0$ . Since  $u_s \equiv 0$ , by Green's identity,

$$\partial_x \left\{ \frac{1}{2} \int_{\mathbf{R}^3} v |v|^2 f_s \right\} \equiv \partial_x q_s = 0.$$

By the Fourier law (1.1) we have then, in the sense of distribution

$$\partial_{xx}\{K(\theta_s)\}=0.$$

Therefore, in the sense of distributions

$$K(\theta_s) = Ax + B.$$

But if  $\theta_s = \frac{1}{3\rho_s} \int |v|^2 f_s \sqrt{\mu} \in L^{\infty}$ , then  $K(\theta_s) \in L^{\infty}$ , and this implies that  $K(\theta_s)$  is in  $W^{2,\infty}$  so that  $K(\theta_s)$  is continuous up to the boundary. But if  $K' = \kappa > 0$  we thus deduce that  $\theta_s$  itself is continuous up to the boundary  $x = -\frac{1}{2}, +\frac{1}{2}$ . We then can rewrite

$$A = K(\theta(+\frac{1}{2})) - K(\theta(-\frac{1}{2})), \quad B = \frac{K(\theta(+\frac{1}{2})) + K(\theta(-\frac{1}{2}))}{2}.$$

By the main theorem,  $f_s = \delta f_1 + O(\delta^2)$ , so that

$$\theta_s(x) = \theta_0 + \delta\theta_1(x) + O(\delta^2),$$

and we know that  $\theta_0 = \frac{1}{3} \int |v|^2 \mu$  is constant in our construction. Then we have for all  $-\frac{1}{2} \le x \le +\frac{1}{2}$ ,

$$K(\theta_0 + \delta\theta_1 + O(\delta^2)) = K(\theta_0) + \delta K'(\theta_0)\theta_1 + O(\delta^2). \tag{5.15}$$

We therefore have

$$A = \delta K'(\theta_0) [\theta_1(+\frac{1}{2}) - \theta_1(-\frac{1}{2})] + O(\delta^2),$$
  

$$B = K(\theta_0) + \delta K'(\theta_0) \frac{\theta_1(+\frac{1}{2}) + \theta_1(-\frac{1}{2})}{2} + O(\delta^2).$$

Hence the first order expansion  $\theta_1(x)$  must to be a linear function:

$$\theta_1(x) = \frac{\theta_1(+\frac{1}{2}) + \theta_1(-\frac{1}{2})}{2} + x[\theta_1(+\frac{1}{2}) - \theta_1(-\frac{1}{2})].$$

Here  $\theta_1 = \frac{1}{3 \int dv f_1} \int |v|^2 f_1$  and from (1.21)  $f_1$  satisfies (1.22).

## 6 $L^2$ Decay

The main purpose of this section is to prove the following:

**Proposition 6.1** Suppose that for all t > 0

$$\iint_{\Omega \times \mathbf{R}^3} g(t, x, v) \sqrt{\mu} dv = \int_{\gamma_-} r \sqrt{\mu} d\gamma = 0.$$
 (6.1)

Then there exists a unique solution to the problem

$$\partial_t f + v \cdot \nabla_x f + Lf = g, \quad f(0) = f_0, \quad in \ \Omega \times \mathbf{R}^3 \times \mathbf{R}_+,$$
 (6.2)

with

$$f_{-} = P_{\gamma}f + r, \quad on \ \gamma_{-} \times \mathbf{R}_{+}, \tag{6.3}$$

such that for all t 0,

$$\iint_{\Omega \times \mathbf{R}^3} f(t, x, v) \sqrt{\mu} dx dv = 0.$$
 (6.4)

Moreover, there is  $\lambda > 0$  such that

$$||f(t)||_2^2 + |f(t)|_2^2 \lesssim e^{-\lambda t} \Big\{ ||f_0||_2^2 + \int_0^t e^{\lambda s} ||g(s)||_2^2 ds + \int_0^t e^{\lambda s} |r(s)|_2^2 ds \Big\}.$$

**Lemma 6.2** Assume that g and r satisfy (6.1) and f satisfies (6.2), (6.3) and (6.4). Then there exists a function G(t) such that, for all  $0 \le s \le t$ ,

$$\int_{s}^{t} \|\mathbf{P}f(\tau)\|_{\nu}^{2} \lesssim G(t) - G(s) + \int_{s}^{t} \|g(\tau)\|_{2}^{2} + |r(\tau)|_{2}^{2} + \int_{s}^{t} \|(\mathbf{I} - \mathbf{P})f(\tau)\|_{\nu}^{2} + \int_{s}^{t} |(1 - P_{\gamma})f(\tau)|_{2,+}^{2}.$$

Moreover,  $G(t) \lesssim ||f(t)||_2^2$ .

**Proof.** The key of the proof is to use the same choices of test functions (allowing extra dependence on time) of (3.16), (3.23), (3.30) and (3.37) and estimate the new contribution  $\int_s^t \iint_{\Omega \times \mathbf{R}^3} \partial_t \psi f$  in the time dependent weak formulation

$$\int_{s}^{t} \int_{\gamma_{+}} \psi f d\gamma - \int_{s}^{t} \int_{\gamma_{-}} \psi f d\gamma - \int_{s}^{t} \iint_{\Omega \times \mathbf{R}^{3}} v \cdot \nabla_{x} \psi f$$

$$= -\iint_{\Omega \times \mathbf{R}^{3}} \psi f(t) + \iint_{\Omega \times \mathbf{R}^{3}} \psi f(s) + \int_{s}^{t} \iint_{\Omega \times \mathbf{R}^{3}} -\psi L(\mathbf{I} - \mathbf{P}) f + \psi g$$

$$+ \int_{s}^{t} \iint_{\Omega \times \mathbf{R}^{3}} \partial_{t} \psi f. \tag{6.5}$$

We note that, with such choices

$$G(t) = -\iint_{\Omega \times \mathbf{R}^3} \psi f(t), \quad |G(t)| \lesssim ||f(t)||_2^2.$$

Without loss of generality we give the proof for s = 0.

**Remark:** We note that (6.1), (6.2), (6.3) and (6.4) are all invariant under a standard t-mollification for all t > 0. The estimates in step 1 to step 3 are obtained via a t-mollification so that all the functions are smooth in t. For the notational simplicity we do not write explicitly the parameter of the regularization.

Step 1. Estimate of  $\nabla_x \Delta_N^{-1} \partial_t a = \nabla_x \partial_t \phi_a$  in (6.5)

In the weak formulation (with time integration over  $[t, t+\varepsilon]$ ), if we choose the test function  $\psi = \phi\sqrt{\mu}$  with  $\phi(x)$  dependent only of x, then we get (note that Lf and g against  $\phi(x)\sqrt{\mu}$  are zero)

$$\sqrt{2\pi} \int_{\Omega} [a(t+\varepsilon) - a(t)] \phi(x) = 2\pi \sqrt{2\pi} \int_{t}^{t+\varepsilon} \int_{\Omega} (b \cdot \nabla_{x}) \phi(x) + \int_{t}^{t+\varepsilon} \int_{\gamma_{-}} r \phi \sqrt{\mu},$$

where  $\int_{\mathbf{R}^3} \mu(v) dv = \sqrt{2\pi}$ ,  $\int_{\mathbf{R}^3} (v_1)^2 \mu(v) dv = 2\pi \sqrt{2\pi}$  and we have used the splitting (3.19) and (3.20). Taking difference quotient, we obtain, for almost all t,

$$\int_{\Omega} \phi \partial_t a = \sqrt{2\pi} \int_{\Omega} (b \cdot \nabla_x) \phi + \frac{1}{2\pi} \int_{\gamma_-} r \phi \sqrt{\mu}.$$

Notice that, for  $\phi = 1$ , from (6.1) the right hand side of the above equation is zero so that, for all t > 0,

$$\int_{\Omega} \partial_t a(t) dx = 0.$$

On the other hand, for all  $\phi(x) \in H^1 \equiv H^1(\Omega)$ , we have, by the trace theorem  $|\phi|_2 \lesssim ||\phi||_{H^1}$ ,

$$\left| \int_{\Omega} \phi(x) \partial_t a dx \right| \lesssim |r|_2 |\phi|_2 + ||b||_2 ||\phi||_{H^1}$$

$$\lesssim \{ ||b(t)||_2 + |r|_2 \} ||\phi||_{H^1}.$$

Therefore we conclude that, for all t > 0,

$$\|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2 + |r|_2,$$

where  $(H^1)^* \equiv (H^1(\Omega))^*$  is the dual space of  $H^1(\Omega)$  with respect to the dual pair

$$\langle A, B \rangle = \int_{\Omega} A(x)B(x)dx,$$

for  $A \in H^1$  and  $B \in (H^1)^*$ .

By the standard elliptic theory, we can solve the Poisson equation with the Neumann boundary condition

$$-\Delta\Phi_a = \partial_t a(t) , \qquad \frac{\partial\Phi_a}{\partial n}\Big|_{\partial\Omega} = 0,$$

with the crucial condition  $\int_{\Omega} \partial_t a(t,x) dx = 0$  for all t > 0. Notice that  $\Phi_a = -\Delta_N^{-1} \partial_t a = \partial_t \phi_a$  where  $\phi_a$  is defined in (3.37). Moreover we have

$$\|\nabla_x \partial_t \phi_a\|_2 = \|\Delta_N^{-1} \partial_t a(t)\|_{H^1} = \|\Phi_a\|_{H^1}$$

$$\lesssim \|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2 + |r|_2.$$

Therefore we conclude, for almost all t > 0,

$$\|\nabla_x \partial_t \phi_a(t)\|_2 \lesssim \|b(t)\|_2 + |r|_2. \tag{6.6}$$

Step 2. Estimate of  $\nabla_x \Delta^{-1} \partial_t b^j = \nabla_x \partial_t \phi_b^i$  in (6.5)

In (6.5) we choose a test function  $\psi = \phi(x)v_i\sqrt{\mu}$ . Since  $\mu(v) = \frac{1}{2\pi}e^{-\frac{|v|^2}{2}}$ ,  $\int v_iv_j\mu(v)dv = \int v_iv_j(\frac{|v|^2}{2} - \frac{3}{2})\mu(v)dv = 2\pi\sqrt{2\pi}\delta_{i,j}$  and we get

$$2\pi\sqrt{2\pi} \int_{\Omega} [b_{i}(t+\varepsilon) - b_{i}(t)] \phi$$

$$= -\int_{t}^{t+\varepsilon} \int_{\gamma} f \phi v_{i} \sqrt{\mu} + 2\pi\sqrt{2\pi} \int_{t}^{t+\varepsilon} \int_{\Omega} \partial_{i} \phi [a+c]$$

$$+ \int_{t}^{t+\varepsilon} \iint_{\Omega \times \mathbf{R}^{3}} \sum_{j=1}^{d} v_{j} v_{i} \sqrt{\mu} \partial_{j} \phi (\mathbf{I} - \mathbf{P}) f + \int_{t}^{t+\varepsilon} \iint_{\Omega \times \mathbf{R}^{3}} \phi v_{i} g \sqrt{\mu}.$$

Taking difference quotient, we obtain

$$\begin{split} & \int_{\Omega} \partial_t b_i(t) \phi \\ &= \frac{-1}{2\pi\sqrt{2\pi}} \int_{\gamma} f(t) v_i \phi \sqrt{\mu} + \int_{\Omega} \partial_i \phi [a(t) + c(t)] \\ &+ \frac{1}{2\pi\sqrt{2\pi}} \Big\{ \iint_{\Omega \times \mathbf{R}^3} \sum_{j=1}^d v_j v_i \sqrt{\mu} \partial_j \phi (\mathbf{I} - \mathbf{P}) f(t) + \iint_{\Omega \times \mathbf{R}^3} \phi v_i g(t) \sqrt{\mu} \Big\}. \end{split}$$

For fixed t > 0, we choose  $\phi = \Phi_h^i$  solving

$$-\Delta \Phi_b^i = \partial_t b_i(t), \quad \Phi_b^i|_{\partial \Omega} = 0.$$

Notice that  $\Phi_b^i = -\Delta^{-1}\partial_t b_i = \partial_t \phi_b^i$  where  $\phi_b^i$  is defined in (3.24). The boundary terms vanish because of the Dirichlet boundary condition on  $\Phi_b^i$ . Then we have, for  $t \geq 0$ ,

$$\int_{\Omega} |\nabla_{x} \Delta^{-1} \partial_{t} b_{i}(t)|^{2} dx = \int_{\Omega} |\nabla_{x} \Phi_{b}^{i}|^{2} dx = -\int_{\Omega} \Delta \Phi_{b}^{i} \Phi_{b}^{i} dx 
\lesssim \varepsilon \{ \|\nabla_{x} \Phi_{b}^{i}\|_{2}^{2} + \|\Phi_{b}^{i}\|_{2}^{2} \} + \|a(t)\|_{2}^{2} + \|c(t)\|_{2}^{2} 
+ \|(\mathbf{I} - \mathbf{P}) f(t)\|_{2}^{2} + \|g(t)\|_{2}^{2} 
\lesssim \varepsilon \|\nabla_{x} \Phi_{b}^{i}\|_{2}^{2} + \|a(t)\|_{2}^{2} + \|c(t)\|_{2}^{2} 
+ \|(\mathbf{I} - \mathbf{P}) f(t)\|_{2}^{2} + \|g(t)\|_{2}^{2},$$

where we have used the Poincaré inequality. Hence, for all t>0

$$\|\nabla_x \partial_t \phi_b^i(t)\|_2 = \|\nabla_x \Delta^{-1} \partial_t b_i(t)\|_2$$
  
 
$$\lesssim \|a(t)\|_2 + \|c(t)\|_2 + \|(\mathbf{I} - \mathbf{P})f(t)\|_2 + \|g(t)\|_2.$$
 (6.7)

Step 3. Estimate of  $\nabla_x \Delta^{-1} \partial_t c = \nabla_x \partial_t \phi_c$  in (6.5)

In the weak formulation, we choose a test function  $\phi(x)(\frac{|v|^2}{2}-\frac{3}{2})\sqrt{\mu}$ . Since  $\int_{\mathbf{R}^3}\mu(v)(\frac{|v|^2}{2}-\frac{3}{2})dv=0$ ,  $\int_{\mathbf{R}^3}\mu(v)v_iv_j(\frac{|v|^2-3}{2})=2\pi\sqrt{2\pi}\delta_{i,j}$  and  $\int_{\mathbf{R}^3}\mu(v)(\frac{|v|^2}{2}-\frac{3}{2})^2dv=3\pi\sqrt{2\pi}$  we get

$$3\pi\sqrt{2\pi} \int_{\Omega} \phi(x)c(t+\varepsilon,x)dx - 3\pi\sqrt{2\pi} \int_{\Omega} \phi(x)c(t,x)dx$$

$$= 2\pi\sqrt{2\pi} \int_{t}^{t+\varepsilon} \int_{\Omega} b \cdot \nabla_{x}\phi - \int_{t}^{t+\varepsilon} \int_{\gamma} (\frac{|v|^{2}}{2} - \frac{3}{2})\sqrt{\mu}\phi f$$

$$+ \int_{t}^{t+\varepsilon} \iint_{\Omega \times \mathbf{R}^{3}} (\mathbf{I} - \mathbf{P})f(\frac{|v|^{2}}{2} - \frac{3}{2})\sqrt{\mu}(v \cdot \nabla_{x})\phi$$

$$+ \int_{t}^{t+\varepsilon} \iint_{\Omega \times \mathbf{R}^{3}} \phi g(\frac{|v|^{2}}{2} - \frac{3}{2})\sqrt{\mu},$$

and taking difference quotient, we obtain

$$\int_{\Omega} \phi(x) \partial_t c(t, x) dx$$

$$= \frac{2}{3} \int_{\Omega} b(t) \cdot \nabla_x \phi - \frac{1}{3\pi\sqrt{2\pi}} \int_{\gamma} (\frac{|v|^2}{2} - \frac{3}{2}) \sqrt{\mu} \phi f(t)$$

$$+ \frac{1}{3\pi\sqrt{2\pi}} \iint_{\Omega \times \mathbf{R}^3} (\mathbf{I} - \mathbf{P}) f(t) (\frac{|v|^2}{2} - \frac{3}{2}) \sqrt{\mu} (v \cdot \nabla_x) \phi$$

$$+ \frac{1}{3\pi\sqrt{2\pi}} \iint_{\Omega \times \mathbf{R}^3} \phi g(t) (\frac{|v|^2}{2} - \frac{3}{2}) \sqrt{\mu}.$$

For fixed t > 0, we define a test function  $\phi = \Phi_c$  where  $\phi_c$  is defined in (3.16). The boundary terms vanish because of the Dirichlet boundary condition on  $\Phi_c$ . Then we have, for t > 0,

$$-\Delta \Phi_c = \partial_t c(t), \quad \Phi_c|_{\partial \Omega} = 0.$$

Notice that  $\Phi_c = -\Delta^{-1}\partial_t c(t) = \partial_t \phi_c(t)$  in (3.16). We follow the same procedure of estimates  $\nabla_x \Delta^{-1} \partial_t a$  and  $\nabla_x \Delta^{-1} \partial_t b$  to have

$$\begin{split} &\|\nabla_{x}\Delta^{-1}\partial_{t}c(t)\|_{2}^{2} = \int_{\Omega} |\nabla_{x}\Phi_{c}(x)|^{2}dx \\ &= \int_{\Omega} \Phi_{c}(x)\partial_{t}c(t,x)dx \\ &\lesssim \varepsilon\{\|\nabla_{x}\Phi_{c}\|_{2}^{2} + \|\Phi_{c}\|_{2}^{2}\} + \|b(t)\|_{2}^{2} + \|(\mathbf{I} - \mathbf{P})f(t)\|_{2}^{2} + \|g(t)\|_{2}^{2} \\ &\lesssim \varepsilon\|\nabla_{x}\Phi_{c}\|_{2}^{2} + \|b(t)\|_{2}^{2} + \|(\mathbf{I} - \mathbf{P})f(t)\|_{2}^{2} + \|g(t)\|_{2}^{2}, \end{split}$$

where we have used the Poincaré inequality. Finally we have, for all t > 0,

$$\|\nabla_x \partial_t \phi_c\|_2 \lesssim \|\nabla_x \Delta^{-1} \partial_t c(t)\|_2$$
  
 
$$\lesssim \|b(t)\|_2 + \|(\mathbf{I} - \mathbf{P})f(t)\|_2 + \|g(t)\|_2.$$
 (6.8)

Step 4. Estimate of a, b, c contributions in (6.5)

To estimate c contribution in (6.5), we plug (3.16) into (6.5) to have from (3.20)

$$\int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i} \sqrt{\mu} \partial_{t} \partial_{i} \phi_{c} f$$

$$= \sum_{j=1}^{d} \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i} v_{j} \mu(v) \partial_{t} \partial_{i} \phi_{c} b_{j}$$

$$+ \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i} \sqrt{\mu} \partial_{t} \partial_{i} \phi_{c} (\mathbf{I} - \mathbf{P}) f.$$

The second line has non-zero contribution only for j = i which leads to zero by the definition of  $\beta_c$  in (3.21). We thus have from (6.8), for  $\varepsilon$  small,

$$\left| \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{c}) v_{i} \sqrt{\mu} \partial_{t} \partial_{i} \phi_{c} f \right|$$

$$\lesssim \int_{0}^{t} \left\{ \|b\|_{2} + \|(\mathbf{I} - \mathbf{P})f\|_{2} + \|g\|_{2} \right\} \|(\mathbf{I} - \mathbf{P})f\|_{2}$$

$$\lesssim \varepsilon \int_{0}^{t} \|b\|_{2}^{2} + \int_{0}^{t} \left\{ \|(\mathbf{I} - \mathbf{P})f\|_{2}^{2} + \|g\|_{2}^{2} \right\}.$$
(6.9)

Combining with (3.22) in Lemma 3.4 we conclude, for  $\varepsilon$  small,

$$\int_{0}^{t} \|c(s)\|_{2}^{2} ds \lesssim G(t) - G(0)$$

$$+ \int_{0}^{t} \left\{ \|(\mathbf{I} - \mathbf{P})f(s)\|_{\nu}^{2} + \|g(s)\|_{2}^{2} + |(1 - P_{\gamma})f(s)|_{2,+}^{2} + |r(s)|_{2}^{2} + \varepsilon \|b(s)\|_{2}^{2} \right\} ds.$$
(6.10)

To estimate b contribution in (6.5), we plug (3.23) into (6.5) to have from (3.20)

$$\int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) \sqrt{\mu} \partial_{t} \partial_{j} \phi_{b}^{j} f$$

$$= \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) \mu \partial_{t} \partial_{j} \phi_{b}^{j} \{ \frac{|v|^{2}}{2} - \frac{3}{2} \} c$$

$$+ \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) \sqrt{\mu} \partial_{t} \partial_{j} \phi_{b}^{j} (\mathbf{I} - \mathbf{P}) f, \tag{6.11}$$

where we used (3.28) to remove the a contribution. We thus have from (6.7),

$$\left| \int_{0}^{t} \iint_{\Omega \times \mathbf{R}^{3}} (v_{i}^{2} - \beta_{b}) \sqrt{\mu} \partial_{t} \partial_{j} \phi_{b}^{j} f \right|$$

$$\lesssim \int_{0}^{t} \left\{ \|a\|_{2} + \|c\|_{2} + \|(\mathbf{I} - \mathbf{P})f\|_{2} + \|g\|_{2} \right\} \left\{ \|c\|_{2} + \|(\mathbf{I} - \mathbf{P})f\|_{2} \right\}$$

$$\lesssim \int_{0}^{t} \left\{ \|(\mathbf{I} - \mathbf{P})f\|_{2}^{2} + \|c\|_{2}^{2} + \|g\|_{2}^{2} + \varepsilon \|a\|_{2}^{2} \right\}. \tag{6.12}$$

Next we plug (3.30) into (6.5) and we have from (6.7),

$$\int_{0}^{t} \int_{\Omega \times \mathbf{R}^{3}} |v|^{2} v_{i} v_{j} \sqrt{\mu} \partial_{t} \partial_{j} \phi_{b}^{i} f$$

$$= \int_{0}^{t} \int_{\Omega \times \mathbf{R}^{3}} |v|^{2} v_{i} v_{j} \sqrt{\mu} \partial_{t} \partial_{j} \phi_{b}^{i} (\mathbf{I} - \mathbf{P}) f$$

$$\lesssim \int_{0}^{t} \{ \|a\|_{2} + \|c\|_{2} + \|(\mathbf{I} - \mathbf{P}) f\|_{2} + \|g\|_{2} \} \|(\mathbf{I} - \mathbf{P}) f\|_{2}$$

$$\lesssim \int_{0}^{t} \{ \|(\mathbf{I} - \mathbf{P}) f\|_{\nu}^{2} + \|g\|_{2}^{2} + \varepsilon [\|a\|_{2}^{2} + \|c\|_{2}^{2}] \}.$$
(6.13)

Combining this with (3.36) in Lemma 3.4 we conclude from (6.5) that

$$\int_{0}^{t} \|b(s)\|_{2}^{2} ds \lesssim G(t) - G(0)$$

$$+ \int_{0}^{t} \left\{ \|(\mathbf{I} - \mathbf{P})f(s)\|_{\nu}^{2} + \|g(s)\|_{2}^{2} + |(1 - P_{\gamma})f(s)|_{2,+}^{2} + |r(s)|_{2}^{2} + \varepsilon \|a\|_{2}^{2} + \|c\|_{2}^{2} (1 + \varepsilon) \right\} ds.$$
(6.14)

Finally in order to estimate a contribution in (6.5) we plug (3.37) for into (6.5). We estimate

$$\int_{0}^{t} \int_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{a}) v_{i} \mu \partial_{t} \partial_{i} \phi_{a} f \qquad (6.15)$$

$$= \int_{0}^{t} \int_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{a}) (v_{i})^{2} \mu \partial_{t} \partial_{i} \phi_{a} b_{i}$$

$$+ \int_{0}^{t} \int_{\Omega \times \mathbf{R}^{3}} (|v|^{2} - \beta_{a}) v_{i} \mu \partial_{t} \partial_{i} \phi_{a} (\mathbf{I} - \mathbf{P}) f$$

$$\lesssim \int_{0}^{t} \{ \|b(t)\|_{2,\Omega} + |r|_{2} \} \{ \|b\|_{2,\Omega} + \|(\mathbf{I} - \mathbf{P}) f\|_{2} \}$$

Combining this with Lemma 3.4 we conclude

$$\int_{0}^{t} \|a(s)\|_{2}^{2} ds \lesssim G(t) - G(0)$$

$$+ \int_{0}^{t} \left\{ \|(\mathbf{I} - \mathbf{P})f(s)\|_{\nu}^{2} + \|g(s)\|_{2}^{2} + |(1 - P_{\gamma})f(s)|_{2,+}^{2} + |r(s)|_{2}^{2} + \|b\|_{2}^{2} \right\} ds.$$
(6.16)

From (6.10), (6.14) and (6.16), we prove the lemma by choosing  $\varepsilon$  sufficiently small.

**Proof of Proposition 6.1.** The proof is parallel to the steady case, proof of Proposition 3.1. We start with the approximating sequence. with  $f^0 \equiv f_0$ ,

$$\partial_t f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} + \nu f^{\ell+1} - K f^{\ell} = g, \quad f^{n+1}(0) = f_0,$$

$$f_-^{\ell+1} = (1 - \frac{1}{j}) P_{\gamma} f^{\ell} + r.$$

$$(6.17)$$

Step 1. We first show that for fixed  $j, f^{\ell} \to f^{j}$  as  $\ell \to \infty$ . Notice that by the compactness of K,

$$(Kf^{\ell},f^{\ell+1}) \leq \varepsilon [\|f^{\ell+1}\|_{\nu}^2 + \|f^{\ell}\|_{\nu}^2] + C_{\varepsilon} [\|f^{\ell+1}\|_2^2 + \|f^{\ell}\|_2^2].$$

We first take an inner product (using Green's identity) with  $f^{\ell+1}$  in (6.17). With the same choice of  $C_j$  in (3.6) and using the boundary condition

$$\begin{split} & \|f^{\ell+1}(t)\|_{2}^{2} + (1-\varepsilon) \int_{0}^{t} \|f^{\ell+1}(s)\|_{\nu}^{2} + \int_{0}^{t} |f^{\ell+1}(s)|_{2,+}^{2} ds \\ & \leq \varepsilon \int_{0}^{t} \|f^{\ell}(s)\|_{\nu}^{2} ds + \left[ (1-\frac{1}{j})^{2} + \frac{1}{j^{2}} \right] \int_{0}^{t} |P_{\gamma}f^{\ell}|_{2,-}^{2} \\ & + C_{j} \int_{0}^{t} |r|_{2}^{2} + C_{\varepsilon} \int_{0}^{t} \max_{1 \leq i \leq \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2}. \\ & \leq \varepsilon \int_{0}^{t} \|f^{\ell}(s)\|_{\nu}^{2} ds + \left[ (1-\frac{1}{j})^{2} + \frac{1}{j^{2}} \right] \int_{0}^{t} |f^{\ell}(s)|_{2,+}^{2} ds \\ & + C_{j} \int_{0}^{t} |r|_{2}^{2} + C_{\varepsilon} \int_{0}^{t} \max_{1 \leq i \leq \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2} \\ & \leq \max \left\{ \frac{\varepsilon}{1-\varepsilon}, \left[ (1-\frac{1}{j})^{2} + \frac{1}{j^{2}} \right] \right\} \left\{ (1-\varepsilon) \int_{0}^{t} \|f^{\ell}(s)\|_{\nu}^{2} ds + \int_{0}^{t} |f^{\ell}(s)|_{2,+}^{2} ds \right\} \\ & + C_{j} \int_{0}^{t} |r|_{2}^{2} + C_{\varepsilon} \int_{0}^{t} \max_{1 \leq i \leq \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2}. \end{split}$$

We choose  $\varepsilon > 0$  sufficiently small, so that  $\max \left\{ \frac{\varepsilon}{1-\varepsilon}, \left[ (1-\frac{1}{j})^2 + \frac{1}{j^2} \right] \right\} \le 1-\varepsilon$ . Now we iterate again to have

$$\leq (1-\varepsilon)^{2} \left\{ (1-\varepsilon) \int_{0}^{t} \|f^{\ell-1}(s)\|_{\nu}^{2} ds + \int_{0}^{t} |f^{\ell}(s)|_{2,+}^{2} ds \right\}$$

$$+ [(1-\varepsilon)+1] \left\{ C_{\varepsilon} \int_{0}^{t} |r|_{2}^{2} + C_{\varepsilon} \int_{0}^{t} \max_{1 \leq i \leq \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2} \right\}$$

$$\vdots$$

$$\leq (1-\varepsilon)^{\ell+1} \left\{ (1-\varepsilon) \int_{0}^{t} \|f^{0}(s)\|_{\nu}^{2} ds + \int_{0}^{t} |f^{0}|_{2,+}^{2} \right\}$$

$$+ \frac{1-(1-\varepsilon)^{\ell+1}}{\varepsilon} \left\{ C_{\varepsilon} \int_{0}^{t} |r|_{2}^{2} + C_{\varepsilon} \int_{0}^{t} \max_{1 \leq i \leq \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2} \right\}.$$

$$(6.18)$$

We therefore have, from  $f^0 \equiv f_0$ ,

$$\max_{1 \le i \le \ell+1} \|f^{i}(t)\|_{2}^{2} \lesssim_{\varepsilon, j} \left\{ \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} \max_{1 \le i \le \ell+1} \|f^{i}(s)\|_{2}^{2} ds + \int_{0}^{t} \|g(s)\|_{\nu}^{2} ds + \|f_{0}\|_{2}^{2} + t\|f_{0}\|_{\nu}^{2} + t|f_{0}|_{2,+}^{2} \right\}.$$

By Gronwall's lemma we have

$$\max_{1 \leq i \leq \ell+1} \|f^i(t)\|_2^2 \lesssim_{\varepsilon,j,t} \left\{ \int_0^t |r|_2^2 + \int_0^t \|g(s)\|_\nu^2 ds + \|f_0\|_2^2 + t \|f_0\|_\nu^2 + t |f_0|_{2,+}^2 \right\},$$

where t > 0 is fixed. This in turns leads to

$$\max_{1 \le i \le \ell+1} \left\{ \|f^i(t)\|_2^2 + \int_0^t \|f^i(s)\|_{\nu}^2 + \int_0^t |f^i(s)|_+^2 ds \right\} \\
\lesssim_{\varepsilon,j,t} \left\{ \int_0^t |r|_2^2 + \int_0^t \|g(s)\|_{\nu}^2 ds + \|f_0\|_2^2 + t\|f_0\|_{\nu}^2 + t|f_0|_{2,+}^2 \right\}.$$

Upon taking the difference, we have

$$\partial_t [f^{\ell+1} - f^{\ell}] + v \cdot \nabla_x [f^{\ell+1} - f^{\ell}] + \nu [f^{\ell+1} - f^{\ell}] = K[f^{\ell} - f^{\ell-1}], \tag{6.19}$$

with  $[f^{\ell+1} - f^{\ell}](0) \equiv 0$  and  $f_-^{\ell+1} - f_-^{\ell} = (1 - \frac{1}{j})P_{\gamma}[f^{\ell} - f^{\ell-1}]$ . Applying (6.18) to  $f^{\ell+1} - f^{\ell}$  yields

$$\begin{split} & \|f^{\ell+1}(t) - f^{\ell}(t)\|_{2} + \int_{0}^{t} \|f^{\ell+1}(s) - f^{v}(s)\|_{\nu}^{2} + \int_{0}^{t} |f^{\ell+1}(s) - f^{\ell}(s)|_{2,+}^{2} ds \\ & \leq \quad (1 - \varepsilon)^{\ell} \int_{0}^{t} |f^{1} - f^{0}|_{2,+}^{2} + \frac{1}{\varepsilon} \left\{ \int_{0}^{t} \max_{1 \leq i \leq n} \|f^{i}(s) - f^{i-1}(s)\|_{\nu}^{2} ds \right\}. \end{split}$$

From (6.18), this implies, from the Gronwall's lemma, that, by taking maximum over  $1 \le i \le \ell + 1$ ,

$$||f^{\ell+1}(t) - f^{\ell}(t)||_{2} \lesssim_{\varepsilon,j,t} (1-\varepsilon)^{\ell} \int_{0}^{t} |f^{1} - f^{0}|_{2,+}^{2},$$

$$\int_{0}^{t} ||f^{\ell+1}(s) - f^{\ell}(s)||_{\nu}^{2} + \int_{0}^{t} |f^{\ell+1}(s) - f^{\ell}(s)|_{2,+}^{2} ds \lesssim_{\varepsilon,j,t} (1-\varepsilon)^{\ell} \int_{0}^{t} |f^{1} - f^{0}|_{2,+}^{2}.$$

Therefore,  $f^{\ell}$  is a Cauchy sequence and  $f^{\ell} \to f^{j}$  so that  $f^{j}$  is the solution of

$$\partial_t f + v \cdot \nabla_x f + Lf = g, \quad f(0) = f_0, \qquad f_- = (1 - \frac{1}{i})P_\gamma f + r.$$
 (6.20)

Step 2. We let  $j \to \infty$ . Upon using Green's identity and the boundary condition and (3.8), for any  $\eta > 0$ ,

$$||f^{j}(t)||_{2}^{2} + \int_{0}^{t} ||(\mathbf{I} - \mathbf{P})f^{j}(s)||_{\nu}^{2} ds + \int_{0}^{t} |(1 - P_{\gamma})f^{j}(s)|_{2,+}^{2} ds$$

$$\leq \eta \int_{0}^{t} |P_{\gamma}f^{j}(s)|_{2,+}^{2} ds + C_{\eta} \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds + ||f_{0}||_{2}^{2}.$$
(6.21)

Note that, from Ukai's trace theorem (Lemma 2.1) and (6.20)

$$\int_{0}^{t} |P_{\gamma}f^{j}(s)|_{2,+}^{2} ds \lesssim \int_{0}^{t} |f^{j}(s)\mathbf{1}_{\gamma^{\varepsilon}}|_{2}^{2} ds \qquad (6.22)$$

$$\lesssim \int_{0}^{t} ||f^{j}(s)||_{2}^{2} ds + \int_{0}^{t} ||\partial_{t}[f^{j}]^{2} + v \cdot \nabla_{x}[f^{j}]^{2}||_{1}$$

$$\lesssim \int_{0}^{t} ||f^{j}(s)||_{2}^{2} ds + \int_{0}^{t} |(Lf^{j}, f^{j})| + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds$$

$$\lesssim \int_{0}^{t} ||f^{j}(s)||_{2}^{2} ds + \int_{0}^{t} ||(\mathbf{I} - \mathbf{P})f^{j}(s)||_{\nu}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds.$$

But from (6.22), choosing  $\eta$  small, we obtain from (6.21) and (6.22) that

$$||f^{j}(t)||_{2} + \int_{0}^{t} ||(\mathbf{I} - \mathbf{P})f^{j}(s)||_{\nu}^{2} ds + \int_{0}^{t} |(1 - P_{\gamma})f^{j}(s)|_{2,+}^{2} ds$$

$$\lesssim \int_{0}^{t} ||f^{j}(s)||_{2}^{2} ds + \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds + ||f_{0}||_{2}^{2}.$$

It follows from the Gronwall's lemma that

$$||f^{j}(t)||_{2}^{2} \lesssim_{t} \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds + ||f_{0}||_{2}^{2},$$
 (6.23)

and hence

$$||f^{j}(t)||_{2}^{2} + \int_{0}^{t} ||(\mathbf{I} - \mathbf{P})f^{j}(s)||_{\nu}^{2} ds + \int_{0}^{t} |(1 - P_{\gamma})f^{j}(s)|_{2,+}^{2} ds$$

$$\lesssim_{t} \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds + ||f_{0}||_{2}^{2}.$$
(6.24)

Since  $\|\mathbf{P}f^j(s)\|_{\nu}^2 \lesssim \|f^j(s)\|_2^2$ , integrating (6.23) from 0 to t and combining with (6.22) we have

$$\int_{0}^{t} \|\mathbf{P}f^{j}(s)\|_{\nu}^{2} ds + \int_{0}^{t} |P_{\gamma}f^{j}(s)|_{2,+} ds \qquad (6.25)$$

$$\lesssim_{t} \int_{0}^{t} \|(\mathbf{I} - \mathbf{P})f^{j}(s)\|_{\nu} ds + \int_{0}^{t} \{|r(s)|_{2}^{2} + \|g(s)\|_{2}^{2}\} ds + \|f_{0}\|_{2}^{2}.$$

Add (6.24) and (6.25) together to have

$$||f^{j}(t)||_{2}^{2} + \int_{0}^{t} ||f^{j}(s)||_{\nu}^{2} ds + \int_{0}^{t} |f^{j}(s)|_{2}^{2} ds \lesssim_{t} \int_{0}^{t} |r|_{2}^{2} + \int_{0}^{t} ||g(s)||_{\nu}^{2} ds + ||f_{0}||_{2}^{2}.$$
 (6.26)

By taking a weak limit, we obtain a weak solution f to (6.2) with the same bound (6.26). Taking difference, we have

$$\partial_t [f^j - f] + v \cdot \nabla_x [f^j - f] + L[f^j - f] = 0, \qquad [f^j - f]_- = P_\gamma [f^j - f] + \frac{1}{i} P_\gamma f^j, \qquad (6.27)$$

with  $[f^j - f](0) = 0$ . Applying the same estimate (6.26) with  $r = \frac{1}{j} P_{\gamma} f^j$  we obtain

$$||f^{j}(t) - f(t)||_{2}^{2} + \int_{0}^{t} ||f^{j}(s) - f(s)||_{\nu}^{2} ds + \int_{0}^{t} |f^{j}(s) - f(s)|_{2}^{2} ds$$

$$\lesssim_{t} \frac{1}{j} \int_{0}^{t} |P_{\gamma} f^{j}|^{2} \lesssim_{t} \frac{1}{j} \to 0.$$

We thus construct f as a  $L^2$  solution to (6.2).

**Remark 6.3** In both Step 1 and Step 2, we do not need the zero mass constraint which is instead essential for next step.

Step 3. Decay estimate.

To conclude our proposition, let

$$y(t) \equiv e^{\lambda t} f(t).$$

We multiply (6.2) by  $e^{\lambda t}$ , so that y satisfies

$$\partial_t y + v \cdot \nabla_x y + Ly = \lambda y + e^{\lambda t} g, \qquad y|_{\gamma_-} = P_{\gamma} y_+ + e^{\lambda t} r.$$
 (6.28)

We apply Green's identity and (6.22), (6.21) together with  $\eta \ll \lambda$  to obtain

$$||y(t)||_{2}^{2} + \int_{0}^{t} ||(\mathbf{I} - \mathbf{P})y(s)||_{\nu}^{2} + \int_{0}^{t} |(1 - P_{\gamma})y(s)|^{2,+}$$

$$\leq \lambda \int_{0}^{t} ||y(s)||_{2}^{2} + ||y(0)||_{2}^{2} + C_{\lambda} \int_{0}^{t} e^{\lambda s} ||r|_{2}^{2} + \int_{0}^{t} e^{\lambda s} ||g(s)||_{2}^{2} ds.$$

$$(6.29)$$

From (6.2) we know that

$$\iint_{\Omega\times\mathbf{R}^3}y\sqrt{\mu}=\iint_{\Omega\times\mathbf{R}^3}(\lambda y+e^{\lambda t}g)\sqrt{\mu}=0,\ \int_{\gamma_-}e^{\lambda t}r\sqrt{\mu}d\gamma=0.$$

Applying Lemma 6.2 to (6.28), we deduce

$$\int_{0}^{t} \|\mathbf{P}y(s)\|_{\nu}^{2} ds \lesssim G(t) - G(0)$$
(6.30)

$$+ \int_{0}^{t} \|(\mathbf{I} - \mathbf{P})y(s)\|_{\nu}^{2} ds + \int_{0}^{t} e^{\lambda s} \|g\|_{2}^{2} ds$$

$$+ \lambda \int_{0}^{t} \|y\|_{2}^{2} ds + \int_{0}^{t} \{|(1 - P_{\gamma})y(s)|_{2,+}^{2} + e^{\lambda s} |r|_{2}^{2}\} ds,$$

$$(6.31)$$

where  $G(t) \lesssim \|y(t)\|_2^2$ . Multiplying with a small constant×(6.30)+(6.29) to obtain, for some  $\varepsilon \ll 1$ 

$$\|y(t)\|_{2}^{2} - \varepsilon G(t) \| + \varepsilon \left\{ \int_{0}^{t} \|(\mathbf{I} - \mathbf{P})y(s)\|_{\nu}^{2} + \|\mathbf{P}y(s)\|_{\nu}^{2} \right\} + \varepsilon \int_{0}^{t} \|(I - P_{\gamma})y(s)\|_{2,+}^{2}$$

$$\leq C\lambda \int_{0}^{t} \|y(s)\|_{2}^{2} + \|y(0)\|_{2}^{2} - \varepsilon G(0) \| + C_{\varepsilon,\lambda} \int_{0}^{t} e^{\lambda s} |r|_{2}^{2} + \int_{0}^{t} e^{\lambda s} \|g(s)\|_{2}^{2} ds.$$

By further choosing  $\lambda \ll \varepsilon$ , and  $\|(\mathbf{I} - \mathbf{P})y(s)\|_{\nu}^2 + \|\mathbf{P}y(s)\|_{\nu}^2 \gtrsim \|y(s)\|_2^2$  for hard potentials, we conclude our proposition.

# 7 $L^{\infty}$ Stability and Non-Negativity

To conclude the proof of Theorem 1.3 we need  $L^{\infty}$  estimates. They are provided, in the linear case, by next

**Proposition 7.1** Let  $||w_{\rho}f_0||_{\infty} + |\langle v\rangle w_{\rho}r|_{\infty} + ||w_{\rho}g||_{\infty} < +\infty$  and  $\iint \sqrt{\mu}g = \int_{\gamma} r\sqrt{\mu} = \iint f_0\sqrt{\mu} = 0$ . Then the solution f to (6.2) satisfies

$$||w_{\rho}f(t)||_{\infty} + |w_{\rho}f(t)|_{\infty} \le e^{-\lambda t} \{||w_{\rho}f_0||_{\infty} + \sup e^{\lambda s} ||w_{\rho}g||_{\infty} + \int_0^t e^{\lambda s} |\langle v \rangle w_{\rho}r(s)|_{\infty} ds \}.$$

Furthermore, if  $f_0|_{\gamma_-} = P_{\gamma}f_0 + r_0$ ,  $f_0, r$  and g are continuous, then f(t, x, v) is continuous away from  $\mathfrak{D}$ . In particular, it  $\Omega$  is convex then  $\mathfrak{D} = \gamma_0$ .

**Proof.** We use exactly the same approximating sequence and repeat step 1 to step 3 of Proposition 6.1 to show that  $f^j$ , f are bounded, via Lemma 4.4. We denote  $h^{\ell} = w_{\varrho} f^{\ell}$ , where  $w_{\varrho}$  is scaled weight in (4.1). Rewrite (6.17) as

$$\partial_{t}h^{\ell+1} + v \cdot \nabla_{x}h^{\ell+1} + \nu h^{\ell+1} = K_{w_{\varrho}}h^{\ell} + w_{\varrho}g,$$

$$h^{\ell+1}_{-} = \frac{1 - \frac{1}{j}}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} h^{\ell}(t, x, v') \tilde{w}_{\varrho}(v') d\sigma + w_{\varrho}r.$$
(7.1)

Step 1.We take  $\ell \to \infty$  in  $L^{\infty}$ . Upon integrating over the characteristic lines  $\frac{dx}{dt} = v$ , and  $\frac{dv}{dt} = 0$  and the boundary condition repeatedly, along the stochastic cycles, we obtain (by replacing  $1 - \frac{1}{j}$  with 1 and  $\varepsilon = 0$ ) that (4.8) is valid for  $h^{\ell+1}$ . Therefore, for  $\ell \geq 2\varrho$ , by Lemma 4.4 that

$$\begin{split} \sup_{0 \leq s \leq T_0} e^{\frac{\nu_0}{2}s} |h^{\ell+1}(s,x,v)| \\ &\leq \frac{1}{8} \max_{0 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2}s} \|h^{\ell-l}(s)\|_{\infty}\} + C(k) \max_{1 \leq l \leq 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds \\ &+ \|h_0\|_{\infty} + k \left[ \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu_0}{2}s} |w_{\varrho}r(s)|_{\infty} \right\} + \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \right] \\ &\equiv \frac{1}{8} \max_{0 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2}s} \|h^{\ell-l}(s)\|_{\infty}\} + C(k) \max_{1 \leq l \leq 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds \\ &+ D, \end{split}$$

where  $T_0 = \varrho^{4/5}$ ,  $\varrho = k$  are chosen to be sufficiently large but fixed. Now this is valid for all  $\ell \geq 2k$ . By induction on  $\ell$ , we can iterate such bound for  $\ell + 2, .... \ell + 2k$  to obtain

$$\sup_{0 \le s \le T_{0}} e^{\frac{\nu_{0}}{2}s} \|h^{\ell+i}(s)\|_{\infty} \tag{7.2}$$

$$\le \frac{1}{8} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_{0}} \{e^{\frac{\nu_{0}}{2}s} \|h^{\ell+i-l}(s)\|_{\infty}\} + C(k) \max_{-2k \le l \le 2k} \int_{0}^{T_{0}} \|f^{\ell-l}(s)\|_{2} ds + D$$

$$\le \frac{1}{4} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_{0}} \{e^{\frac{\nu_{0}}{2}s} \|h^{\ell+i-l-1}(s)\|_{\infty}\} + 2C(k) \max_{-2k \le l \le 2k} \int_{0}^{T_{0}} \|f^{\ell-l}(s)\|_{2} ds + 2D$$

$$\vdots$$

$$\le \frac{1}{4} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_{0}} \{e^{\frac{\nu_{0}}{2}s} \|h^{\ell-l}(s)\|_{\infty}\} + (i+1)C(k) \left\{ \max_{-2k \le l \le 2k} \int_{0}^{T_{0}} \|f^{\ell-l}(s)\|_{2} ds + D \right\}.$$

Taking maximum over i = 1, ... 2k, and by induction on  $\ell$ ,

$$\max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} e^{\frac{\nu_0}{2}s} \|h^{\ell+1-l}(s)\|_{\infty}$$

$$\le \frac{1}{4} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} \{e^{\frac{\nu_0}{2}s} \|h^{\ell-l}(s)\|_{\infty}\} + C(k) \Big[\max_{-2k \le l \le 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds\Big] + D$$

$$\le \frac{1}{4^2} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} \{e^{\frac{\nu_0}{2}s} \|h^{\ell-1-l}(s)\|_{\infty}\} + (1 + \frac{1}{4}) \Big\{C(k) \Big[\max_{-2k \le l \le 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds\Big] + D\Big\}$$

$$\vdots$$

$$\le \frac{1}{4^{\ell/2k}} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} \{e^{\frac{\nu_0}{2}s} \|h^{l}(s)\|_{\infty}\} + \Big\{C(k) \Big[\max_{-2k \le l \le 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds\Big] + D\Big\},$$

where we may assume that  $\ell$  is a multiple of k. Now for  $\max_{1 \le l \le 2k} \|h^l\|_{\infty}$ , we can use (4.8) for k = 1 repeatedly for  $h^{2k} \to h^{2k-1} \dots \to h_0$  to obtain

$$\max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2}s} \|h^l(s)\|_{\infty}\} \lesssim_k \|h_0\|_{\infty} + \sup_{0 \leq s \leq T_0} \left\{e^{\frac{\nu_0}{2}s} |w_{\varrho}r(s)|_{\infty}\right\} + \sup_{0 \leq s \leq T_0} \left\{e^{\frac{\nu_0}{2}s} \left\|\frac{w_{\varrho}g(s)}{\langle v \rangle}\right\|_{\infty}\right\}.$$

We therefore conclude that

$$\begin{split} & \max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} e^{\frac{\nu_0}{2}s} \|h^{\ell+1-l}(s)\|_{\infty} \\ & \lesssim_k C(k) \Big\{ \max_{-2k \leq l \leq 2k} \int_0^{T_0} \|f^{\ell-l}(s)\|_2 ds + D \Big\} \\ & + \|h_0\|_{\infty} + \sup_{0 \leq s \leq T_0} \Big\{ e^{\frac{\nu_0}{2}s} |w_{\varrho} r(s)|_{\infty} \Big\} + \sup_{0 \leq s \leq T_0} \Big\{ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\varrho} g(s)}{\langle v \rangle} \right\|_{\infty} \Big\} \,, \end{split}$$

Now  $\max_{1 \leq l \leq \infty} ||f^l||_2$  is bounded by step 1 in the proof of Proposition 6.1,  $||\cdot||_2$  is bounded by  $||w_{\varrho}\cdot||_{\infty}$  for  $\beta > 3$  and  $|r|_2$  is bounded by  $||w_{\varrho}\langle v\rangle r(s)||_{\infty}$ . Hence, there is a limit (unique) solution  $h_{\ell} \to h = w_{\varrho} f \in L^{\infty}$ . Furthermore, h satisfies (4.8) with  $h^{\ell+1} \equiv h$ . By subtracting  $h^{\ell+1} - h$  in (4.8) with D = 0, we obtain from Lemma 4.4 for  $h^{\ell+1} - h$  and (7.2):

$$\max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} e^{\frac{\nu_0}{2}s} \|h^{\ell+1-l}(s) - h(s)\|_{\infty} 
\le \frac{1}{4^{\ell/2k}} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_0} \{e^{\frac{\nu_0}{2}s} \|h^l(s) - h(s)\|_{\infty}\} 
+ \Big\{ C(k) \max_{-2k \le l \le 2k} \int_0^{T_0} \|f^{\ell-l}(s) - f(s)\|_2 ds \Big\}.$$

From step 1 of Lemma 6.1 we deduce  $\sup_{0 \le s \le T_0} \|h^{\ell+1-l}(s) - h(s)\|_{\infty} \to 0$  for  $\ell$  large.

Step 2. We take  $j \to \infty$ . Let  $f^j$  to be the solution to (6.20) and integrate along  $\frac{dx}{dt} = v$ ,  $\frac{dv}{dt} = 0$  repeatedly. Again (4.8) is valid for  $h^{\ell} \equiv h^j$  (replacing  $(1 - \frac{1}{j})$  by 1 and  $\varepsilon = 0$ ) and l = 0.

Lemma 4.4 implies

$$\begin{split} \sup_{0 \leq s \leq T_0} e^{\frac{\nu_0}{2} s} \|h^j(s)\|_{\infty} \\ & \leq \frac{1}{8} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2} s} \|h^j(s)\|_{\infty}\} + C(k) \int_0^{T_0} \|f^j(s)\|_2 ds \\ & + \|h_0\|_{\infty} + k \left[ \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu_0}{2} s} |w_{\ell} r(s)|_{\infty} \right\} + \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu_0}{2} s} \left\| \frac{w_{\ell} g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \right], \end{split}$$

with  $T_0 = \varrho^{4/5}$ ,  $\varrho = k$ , and  $\varrho$  sufficiently large but fixed. Therefore, by an induction over j,

$$\sup_{0 \le s \le T_0} e^{\frac{\nu_0}{2}s} \|h^j(s)\|_{\infty} \le C(k) \int_0^{T_0} \|f^j(s)\|_2 ds + \|h_0\|_{\infty}$$

$$+k \left[ \sup_{0 \le s \le T_0} \left\{ e^{\frac{\nu_0}{2}s} |w_{\ell} r(s)|_{\infty} \right\} + \sup_{0 \le s \le T_0} \left\{ e^{\frac{\nu_0}{2}s} \left\| \frac{w_{\ell} g(s)}{\langle v \rangle} \right\|_{\infty} \right\} \right].$$

Since  $\int_0^{T_0} \|f^j(s)\|_2 ds$  is bounded from step 2 of Proposition 6.1 (with no mass constraint), this implies that  $\|h^j\|_{\infty}$  is uniformly bounded and we obtain a (unique) solution  $h = w_{\varrho} f \in L^{\infty}$ . Taking the difference, we have

$$\partial_{t}[h^{j} - h] + v \cdot \nabla_{x}[h^{j} - h] + \nu[h^{j} - h] = K_{w_{\varrho}}[h^{j} - h]$$

$$[h^{j} - h]_{-} = \frac{1}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} [h^{j} - h](t, x, v') \tilde{w}_{\varrho}(v') d\sigma$$

$$-\frac{1}{j} \frac{1}{\tilde{w}_{\varrho}(v)} \int_{n(x) \cdot v' > 0} h^{j}(t, x, v') \tilde{w}_{\varrho}(v') d\sigma.$$

We regard  $\frac{1}{j} \frac{1}{\tilde{w}_{\rho}(v)} \int_{n(x) \cdot v' > 0} h_{\gamma_{-}}(t, x, v') \tilde{w}_{\rho}(v') d\sigma$  as r in Lemma 4.4. So Lemma 4.4 implies that

$$\sup_{0 \le s \le T_0} e^{\frac{\nu_0}{2}s} \|h^j(s) - h(s)\|_{\infty} 
\lesssim C(k) \int_0^{T_0} \|f^j(s) - f(s)\|_2 ds + k \sup_{0 \le s \le T_0} |w_{\varrho} r(s)|_{\infty} \lesssim \frac{1}{j},$$

which goes to zero as j to  $\infty$ .

We now obtain a  $L^{\infty}$  solution  $h = w_{\rho} f$  to (6.2). Since we have  $L^{\infty}$  convergence at each step, we deduce that h is continuous away from  $\mathfrak{D}$ .

To obtain decay estimate, we integrate (6.2) over the trajectory. The estimate (4.8) is valid for  $h^{\ell} \equiv h$  so that from Lemma 4.4 for  $\rho > 0$  sufficiently large we have:

$$\sup_{0 \le s \le T_{0}} e^{\frac{\nu_{0}}{2}s} \|h(s)\|_{\infty} + \sup_{0 \le s \le T_{0}} e^{\frac{\nu_{0}}{2}s} |h(s)|_{\infty}$$

$$\leq \|h_{0}\|_{\infty} + C(k) \left[ \sup_{0 \le s \le T_{0}} \left\{ e^{\frac{\nu_{0}}{2}s} |w_{\varrho}r(s)|_{\infty} \right\} + \sup_{0 \le s \le T_{0}} \left\{ e^{\frac{\nu_{0}}{2}s} \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\}$$

$$+ \int_{0}^{T_{0}} \|f(s)\|_{2} ds \right],$$

$$(7.3)$$

where  $T_0 = \varrho^{4/5}$ ,  $\varrho = k$  and  $\varrho$  sufficiently large. Letting  $s = T_0$  in (7.3), we obtain

$$||h(T_0)||_{\infty} + |h(T_0)|_{\infty} \le e^{-\frac{\nu_0}{2}T_0} ||h_0||_{\infty}$$

$$+C(k) \left[ \sup_{0 \le s \le T_0} \{|w_{\varrho}r(s)|_{\infty}\} + \sup_{0 \le s \le T_0} \left\{ \left\| \frac{w_{\varrho}g(s)}{\langle v \rangle} \right\|_{\infty} \right\} + \int_0^{T_0} ||f(s)||_2 ds \right].$$

$$(7.4)$$

Let

$$R \equiv \sup_{0 \le s \le \infty} \left\{ e^{\frac{\lambda}{2}s} |\langle v \rangle w_{\varrho} r(s)|_{\infty} \right\} + \sup_{0 \le s \le \infty} \left\{ e^{\frac{\lambda}{2}s} \|w_{\varrho} g(s)\|_{\infty} \right\} + \|w_{\varrho} f_0\|_{\infty}.$$

Since  $\beta > 3/2$ , Proposition 6.1 implies that

$$||f(t)||_2 \lesssim_{\varrho} e^{-\frac{\lambda}{2}t} R.$$

Let m be an integer. We look at times  $t = mT_0$ . By induction on m, we have from the definition of R and (7.4) for  $\lambda \ll \nu_0$ ,

$$\begin{split} &\|h([m+1]T_0)\|_{\infty} + |h([m+1]T_0)|_{\infty} \\ &\leq e^{-\frac{\nu_0}{2}T_0} \|h(mT_0)\|_{\infty} \\ &\quad + C(\varrho)e^{-\frac{\nu_0}{2}T_0} \bigg[ \sup_{0 \leq s \leq T_0} \big\{ |w_\varrho r(s+mT_0)|_{\infty} \big\} + \sup_{0 \leq s \leq T_0} \Big\{ \bigg\| \frac{w_\varrho g(s+mT_0)}{\langle v \rangle} \bigg\|_{\infty} \Big\} \\ &\quad + \int_0^{T_0} \|f(s+mT_0)\|_2 ds \bigg] \\ &\leq e^{-\frac{\nu_0}{2}T_0} \|h(mT_0)\|_{\infty} + C(\varrho)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} \times \\ &\quad \Big[ \sup_{0 \leq s \leq T_0} e^{\frac{m\lambda T_0}{2}} \big\{ |w_\varrho r(s+mT_0)|_{\infty} \big\} + \sup_{0 \leq s \leq t} e^{\frac{m\lambda T_0}{2}} \bigg\| \frac{w_\varrho g(s+mT_0)}{\langle v \rangle} \bigg\|_{\infty} \\ &\quad + \int_0^{T_0} e^{\frac{m\lambda T_0}{2}} \|f(s+mT_0)\|_2 ds \bigg] \\ &\leq e^{-\frac{\nu_0}{2}T_0} \|h(mT_0)\|_{\infty} + C(\varrho)(1+T_0)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} R \\ &\leq e^{-2\frac{\nu_0}{2}T_0} \|h([m-1]T_0)\|_{\infty} + C(\varrho)(1+T_0)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} R \\ &= e^{-2\frac{\nu_0}{2}T_0} \|h([m-1]T_0)\|_{\infty} + C(\varrho)(1+T_0)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} R[1+e^{-\frac{\nu_0-\lambda}{2}T_0}] \\ &\leq e^{-3\frac{\nu_0}{2}T_0} \|h([m-2]T_0)\|_{\infty} \\ &\quad + C(\varrho)(1+T_0)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} R[1+e^{-\frac{\nu_0-\lambda}{2}T_0}] \\ &\vdots \\ &\leq e^{-m\frac{\nu_0}{2}T_0} \|h_0\|_{\infty} + C(\varrho)(1+T_0)e^{-\frac{\nu_0}{2}T_0 - \frac{m\lambda T_0}{2}} R \sum_j e^{-j\frac{\nu_0-\lambda}{2}T_0} \\ &\leq C_{k,\nu_0,\lambda,\varrho} e^{-\frac{m\lambda T_0}{2}} R. \end{split}$$

Combining with (7.3) for  $0 \le t_1 \le T_0$ , we deduce that for any  $t = mT_0 + t_1$ ,  $||h(t)||_{\infty} + |h(t)|_{\infty} \lesssim_{k,\nu_0,\lambda,\varrho} e^{-t}R$ . We deduce our proposition with  $w = w_{\varrho}$  for  $\varrho > 0$  sufficiently large but fixed. A simple scaling of  $\varrho$  concludes the proof of the proposition.

By using the above  $L^{\infty}$  estimate we can conclude the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We consider the following iteration sequence

$$\begin{split} \partial_t f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} + L f^{\ell+1} &= L_{\sqrt{\mu} f_s} f^\ell + \Gamma(f^\ell, f^\ell), \\ f_-^{\ell+1} &= P_\gamma f^{\ell+1} + \frac{\mu_\delta - \mu}{\sqrt{\mu}} \int_{\gamma_+} f^\ell \sqrt{\mu} (n \cdot v) dv. \end{split}$$

Clearly we have

$$\iint_{\Omega \times \mathbf{R}^3} \{ L_{\sqrt{\mu} f_s} f^{\ell} + \Gamma(f^{\ell}, f^{\ell}) \} \sqrt{\mu} dx dv = 0, \qquad \int \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \left\{ \int_{\gamma_{+}} f^{\ell} \right\} d\gamma = 0.$$

Recall  $w_{\varrho}(v) = (1 + \varrho^2 |v|^2)^{\frac{\beta}{2}} e^{\zeta |v|^2}$  in (4.1). Note that for  $0 \le \zeta < \frac{1}{4}$ ,

$$\left\| e^{\frac{\lambda s}{2}} w_{\varrho} \left\{ \frac{1}{\langle v \rangle} [L_{\sqrt{\mu} f_{s}} f^{\ell} + \Gamma(f^{\ell}, f^{\ell})(s)] \right\} \right\|_{\infty}$$

$$\lesssim \delta \sup_{0 \leq s \leq t} \| e^{\frac{\lambda s}{2}} w_{\varrho} f^{\ell}(s) \|_{\infty} + \left\{ \sup_{0 \leq s \leq t} \| e^{\frac{\lambda s}{2}} w_{\varrho} f^{\ell}(s) \|_{\infty} \right\}^{2}.$$

Using (5.2) for  $0 \le \zeta < \frac{1}{4+2\delta}$ 

$$\left| e^{\frac{\lambda t}{2}} w_{\varrho} \langle v \rangle \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \left\{ \int_{\gamma_{+}} f^{\ell} \right\} \right|_{\infty} \lesssim \delta \sup_{0 \leq s \leq t} |e^{\frac{\lambda s}{2}} f^{\ell}(s)|_{\infty}.$$

By Proposition 7.1, we deduce

$$\sup_{0 \le s \le t} \|e^{\frac{\lambda s}{2}} w_{\ell} f^{\ell+1}(s)\|_{\infty} + \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_{\ell} f^{\ell+1}(s)|_{\infty} 
\lesssim \|w_{\ell} f_{0}\|_{\infty} + \delta \sup_{0 \le s \le t} \|e^{\frac{\lambda s}{2}} w_{\ell} f^{\ell}(s)\|_{\infty} + \delta \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_{\ell} f^{\ell}(s)|_{\infty} 
+ \left\{ \sup_{0 \le s \le t} \|e^{\frac{\lambda s}{2}} w_{\ell} f^{\ell}(s)\|_{\infty} \right\}^{2}.$$

For  $\delta$  small, there exists a  $\varepsilon_0$  (uniform in  $\delta$ ) such that, if the initial data satisfy (1.20), then

$$\sup_{0\leq s\leq t}\|e^{\frac{\lambda s}{2}}w_{\varrho}f^{\ell+1}(s)\|_{\infty}+\sup_{0\leq s\leq t}|e^{\frac{\lambda s}{2}}w_{\varrho}f^{\ell+1}(s)|_{\infty}\lesssim \|w_{\varrho}f_{0}\|_{\infty}.$$

By taking difference  $f^{\ell+1} - f^{\ell}$ , we deduce that

$$\begin{split} \partial_t [f^{\ell+1} - f^\ell] + v \cdot \nabla_x [f^{\ell+1} - f^\ell] + L[f^{\ell+1} - f^\ell] \\ &= L_{\sqrt{\mu} f_s} [f^\ell - f^{\ell-1}] + \Gamma(f^\ell - f^{\ell-1}, f^\ell) + \Gamma(f^{\ell-1}, f^\ell - f^{\ell-1}), \\ [f^{\ell+1} - f^\ell]_- &= P_\gamma [f^{\ell+1} - f^\ell] + \frac{\mu_\delta - \mu}{\sqrt{\mu}} \int_{\gamma_+} [f^\ell - f^{\ell-1}] (n(x) \cdot v) dv, \end{split}$$

with  $f^{\ell+1} - f^{\ell} = 0$  initially. Repeating the same argument, we obtain

$$\begin{split} \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_{\varrho}[f^{\ell+1} - f^{\ell}](s) \|_{\infty} + \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_{\varrho}[f^{\ell+1} - f^{\ell}](s)|_{\infty} \\ \lesssim & \left[ \delta + \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_{\varrho} f^{\ell}(s) \|_{\infty} + \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_{\varrho} f^{\ell-1}(s) \|_{\infty} \right] \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_{\varrho}[f^{\ell} - f^{\ell-1}](s) \|_{\infty}. \end{split}$$

This implies that  $f^{\ell+1}$  is a Cauchy sequence. The uniqueness is standard.

Positivity. To conclude the positivity of  $F_s$ , we need to show

$$F_s + \sqrt{\mu} f(t) \ge 0,$$

if initially  $F_s + \sqrt{\mu} f_0 \ge 0$ . To this end, we first assume the cross section B is bounded and hence  $\nu$  is bounded. We note that previous approximating sequence is not well suited to show the positivity. Therefore, we need to design a different iterative sequence. We use the following one:

$$\begin{split} \partial_t F^{\ell+1} + v \cdot \nabla_x F^{\ell+1} + \nu(F^\ell) F^{\ell+1} &= Q_{\mathrm{gain}}(F^\ell, F^\ell), \\ F^{\ell+1}_- &= \mu_\delta \int_{\gamma_+} F^\ell(n(x) \cdot v) dv, \end{split}$$

where  $F^0 = F_s + \sqrt{\mu} f_0$  and

$$\nu(F) = \int_{\mathbf{R}^3} dv_* \int_{\mathbf{S}^2} d\omega B(v - v_*, \omega) F(v_*), \tag{7.5}$$

with initial condition  $F^{\ell+1}(0) = F_s + \sqrt{\mu} f_0 \ge 0$ . Clearly, such an iteration preserves the non-negativity. We need to show  $F^{\ell}$  is convergent to conclude the non-negativity of the (unique!) limit  $F(t) \ge 0$ . Writing  $F^{\ell+1} = F_s + \sqrt{\mu} f^{\ell+1}$ , we have

$$\partial_{t} f^{\ell+1} + v \cdot \nabla_{x} f^{\ell+1} + \nu(v) f^{\ell+1} - K f^{\ell} 
= \Gamma_{\text{gain}}(f^{\ell}, f^{\ell}) - \nu(\sqrt{\mu} f^{\ell}) f^{\ell+1} - \nu(\sqrt{\mu} f_{s}) f^{\ell+1} - \nu(\sqrt{\mu} f^{\ell}) f_{s} 
+ \frac{1}{\sqrt{\mu}} \left\{ Q_{\text{gain}}(\sqrt{\mu} f^{\ell}, \sqrt{\mu} f_{s}) + Q_{\text{gain}}(\sqrt{\mu} f_{s}, \sqrt{\mu} f^{\ell}) \right\}, 
f_{-}^{\ell+1} = P_{\gamma} f^{\ell} + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{\gamma_{+}} f^{\ell} \sqrt{\mu} (n(x) \cdot v) dv.$$
(7.6)

Taking inner product (Green's identity) with  $f^{\ell+1}$ , from Ukai's trace Theorem, and  $(Kf^{\ell}, f^{\ell+1}) \lesssim ||f^{\ell}||_2^2 + ||f^{\ell+1}||_2^2$  (bounded cross section B) and the boundary condition we have

$$||f^{\ell+1}(t)||_{2}^{2} + \int_{0}^{t} ||f^{\ell+1}||_{2}^{2} + \int_{0}^{t} |f^{\ell+1}||_{2,+}^{2}$$

$$\lesssim ||f_{0}||_{2}^{2} + [1 + C\delta^{2}] \int_{0}^{t} |P_{\gamma}f^{\ell}|_{2}^{2} + \int_{0}^{t} ||f^{\ell}||_{2}^{2}$$

$$+ [\delta + ||wf^{\ell}||_{\infty} + ||wf^{\ell}||_{\infty}^{2}] \int_{0}^{t} ||f^{\ell+1}||_{2}^{2}.$$

$$(7.7)$$

By (6.22) and the equation (7.6), assuming  $\max_{1 \le l \le \ell} \|wf^l\|_{\infty} < +\infty$  we have

$$\int_{0}^{t} |P_{\gamma} f^{\ell}|_{2}^{2} \lesssim \int_{0}^{t} ||f^{\ell}||_{2}^{2} + \int_{0}^{t} ||f^{\ell-1}||_{2}^{2} + [\delta + ||wf^{\ell-1}||_{\infty} + ||wf^{\ell-1}||_{\infty}^{2}] \int_{0}^{t} ||f^{\ell}||_{2}^{2} 
\lesssim \max_{\ell-1 \leq l \leq \ell} \int_{0}^{t} ||f^{\ell}||_{2}^{2} ds.$$
(7.8)

Splitting  $1 + C\delta^2 = 1 - \delta + (\delta + C\delta^2)$  in (7.7) and using (7.8) we have

$$||f^{\ell+1}(t)||_{2}^{2} + \int_{0}^{t} ||f^{\ell+1}||_{2}^{2} + \int_{0}^{t} |f^{\ell+1}||_{2,+}^{2}$$

$$\lesssim ||f_{0}||_{2}^{2} + [1 - \delta] \int_{0}^{t} |P_{\gamma}f^{\ell}|_{2}^{2} ds + \max_{\ell-1 \le l \le \ell} \int_{0}^{t} ||f^{l}||_{2}^{2} ds.$$

Then we iterate to obtain

$$||f^{\ell+1}||_{2}^{2} + \int_{0}^{t} ||f^{\ell+1}||_{2}^{2} + \int_{0}^{t} |f^{\ell+1}||_{2,+}^{2}$$

$$\leq [1 - \delta]^{2} \int_{0}^{t} |P_{\gamma} f^{\ell-1}||_{2}^{2} + [1 + (1 - \delta)] \left\{ ||f^{0}||_{2}^{2} + \max_{1 \leq l \leq \ell+1} \int_{0}^{t} ||f^{l}(s)||_{2}^{2} \right\}$$

$$\vdots$$

$$\leq (1 - \delta)^{\ell+1} \int_{0}^{t} |P_{\gamma} f^{0}||_{2}^{2} + \frac{1 - (1 - \delta)^{\ell+1}}{\delta} \left\{ ||f^{0}||_{2}^{2} + \max_{1 \leq l \leq \ell+1} \int_{0}^{t} ||f^{l}(s)||_{2}^{2} \right\}.$$

$$(7.9)$$

Taking maximum over  $1 \le l \le \ell + 1$ ,

$$\max_{1 \leq l \leq \ell+1} \|f^l(t)\|_2^2 \lesssim \|f^0\|_2^2 + \int_0^t |P_\gamma f^0|_2^2 + \int_0^t \max_{1 \leq l \leq \ell+1} \|f^l(s)\|_2^2 ds.$$

By Gronwall's lemma, from  $f^0 = f_0$ ,

$$\max_{1 \le 1 \le n+1} \|f^l(t)\|_2 \lesssim_t \|f_0\|_2 + t|P_\gamma f_0|_2, \tag{7.10}$$

uniformly bounded.

We now apply Lemma 4.4 with

$$g = L_{\sqrt{\mu}f_s} f^{\ell} + \Gamma(f^{\ell}, f^{\ell}), \quad r = \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{\gamma_+} f^{\ell} \sqrt{\mu} d\gamma,$$

for some  $T_0$  large, with  $f^0 = f_0$ , using the same arguments as in proof of (7.4), to get

$$\sup_{0 \le s \le T_{0}} e^{\frac{\nu_{0}}{2}s} |w_{\varrho} f^{\ell+1}(s, x, v)|$$

$$\le \frac{1}{4} \max_{1 \le l \le 2k} \sup_{0 \le s \le T_{0}} \left\{ e^{\frac{\nu_{0}}{2}s} ||w_{\varrho} f^{\ell+1-l}(s)||_{\infty} \right\} + ||w_{\varrho} f_{0}||_{\infty}$$

$$+ T_{0} \left[ \delta \max_{1 \le l \le \ell} \sup_{0 \le s \le T_{0}} \left\{ e^{\frac{\nu_{0}}{2}s} |w_{\varrho} f^{\ell+1-l}(s)|_{\infty} \right\} + \max_{0 \le l \le 2k} \sup_{0 \le s \le T_{0}} \left\{ e^{\frac{\nu_{0}}{2}s} |w_{\varrho} f^{\ell+1-l}(s)|_{\infty} \right\}^{2} \right]$$

$$+ C(T_{0}) \max_{1 \le l \le 2k} \int_{0}^{T_{0}} ||f^{\ell-1}(s)||_{2} ds.$$

where  $\ell \geq 2k$ . For

$$\delta + \max_{0 \le l \le 2k} \sup_{0 \le s \le t} \left\{ e^{\frac{\nu_0}{2}s} |w_{\varrho} f^{\ell+1-l}(s)|_{\infty} \right\} \ll 1, \tag{7.11}$$

small, we obtain from  $\beta > 3$ ,

$$\sup_{0 \le s \le t} e^{\frac{\nu_0}{2}s} |w_{\varrho} f^{\ell+1}(s, x, v)| \le \frac{1}{2} \max_{1 \le l \le 2k} \sup_{0 \le s \le t} \{ e^{\frac{\nu_0}{2}s} ||w_{\varrho} f^{\ell+1-l}(s)||_{\infty} \} + C_{T_0} ||w_{\varrho} f_0||_{\infty}.$$

Hence we obtain

$$\max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} e^{\frac{\nu_0}{2}s} |w_{\varrho} f^{\ell+2-l}(s,x,v)| \leq \frac{1}{2} \max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2}s} \|w_{\varrho} f^{\ell+1-l}(s)\|_{\infty}\} + C_k \|w_{\varrho} f_0\|_{\infty},$$

and

$$\max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} e^{\frac{\nu_0}{2}s} |w_{\varrho} f^{\ell+2-l}(s,x,v)| \leq \frac{1}{2^{\ell/2k}} \max_{1 \leq l \leq 2k} \sup_{0 \leq s \leq T_0} \{e^{\frac{\nu_0}{2}s} \|w_{\varrho} f^l(s)\|_{\infty}\} + 2C_k \|w_{\varrho} f_0\|_{\infty}.$$

But for  $1 \le l \le 2k$ , we use Lemma with k = 1 to get

$$\max_{1 \le l \le 2k} \sup_{0 < s < T_0} \{ e^{\frac{\nu_0}{2} s} \| w_{\varrho} f^l(s) \|_{\infty} \} \le C_k \| w_{\varrho} f_0 \|_{\infty}, \tag{7.12}$$

so that (7.11) is valid as long as  $||w_{\varrho}f_0||_{\infty}$  is sufficiently small. We therefore obtain uniform bound

$$\max_{1 \le l \le \ell} \sup_{0 < s < T_0} \|w_{\varrho} f^l(s)\|_{\infty} \le C_k \|w_{\varrho} f_0\|_{\infty}.$$

This leads to  $w_{\varrho}f^{\ell} \to w_{\varrho}f \in L^{\infty}$ . Furthermore, f satisfies (7.6) with  $f^{\ell+1} = f^{\ell} = f$ . Therefore  $f^{\ell+1} - f$  satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu(v)\}[f^{\ell+1} - f] - K[f^{\ell} - f] = R', \quad [f^{\ell+1} - f](0) \equiv 0,$$

$$[f^{\ell+1} - f]_+ = P_{\gamma}[f^{\ell} - f] + \frac{\mu_{\delta} - \mu}{\sqrt{\mu}} \int_{\gamma_{-}} [f^{\ell} - f] \sqrt{\mu}(n(x) \cdot v) dv,$$

where

$$|(R', f^{\ell+1} - f)| \lesssim \{\delta + \|w_{\varrho}f^{\ell}\|_{\infty} + \|w_{\varrho}f^{\ell}\|_{\infty}^{2} + \|w_{\varrho}f\|_{\infty} + \|w_{\varrho}f\|_{\infty}^{2}\}\{\|f^{\ell+1} - f\|_{2}\}.$$
 (7.13)

Notice that from (7.12)

$$\max_{\ell} \|w_{\ell} f^{\ell}\|_{\infty}, \ \|w_{\ell} f\|_{\infty} \le C_{\ell} \|w_{\ell} f_0\|_{\infty}.$$

With the same proof of (7.8)

$$\int_0^t |P_{\gamma}[f^{\ell} - f]|_2^2 \lesssim \max_{\ell - 1 \le i \le \ell} ||f^i - f||_2^2 ds.$$

Combining the two above estimates we have

$$||f^{\ell+1}(t) - f(t)||_2^2 + \int_0^t ||f^{\ell+1} - f||_2^2 + \int_0^t |f^{\ell+1} - f||_{2,+}^2$$

$$\lesssim [1 - \delta] \int_0^t |P_{\gamma}[f^{\ell} - f]|_2^2 ds + \max_{\ell - 1 \le i \le \ell} \int_0^t ||f^i - f||_2^2 ds.$$

We iterate as for (7.9) to obtain

$$\max_{1 \le i \le \ell+1} \|f^i(t) - f(t)\|_2^2 \lesssim \int_0^t \max_{1 \le i \le \ell+1} \|f^i - f\|_2^2 ds.$$

Then applying Gronwall's lemma and choosing small  $t_0 > 0$  we conclude that  $f^j$  is Cauchy in [0,t] and  $f^j \to f$  by uniqueness and  $F(t) \ge 0$  for  $0 \le t \le t_0$ . We can repeat this process to show  $F(t) \ge 0$ , for all  $t \ge 0$ .

Finally, we take  $F_0 \sim \mu$ , so that  $F_0 \geq 0$  and hence  $F(t) \geq 0$ . Therefore  $\lim_{t \to +\infty} F(t) = F_s \geq 0$ . This completes the proof in the case B bounded. To remove this limitation, we use a cut-off procedure as before and reduce to the bounded case where previous result holds. Then we pass to the limit in the cut-off using the a priori bounds and uniqueness.

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### References

- [1] Aoki, Ke.; Lukkarinen, J.; Spohn, H.: Energy Transport in Weakly Anharmonic Chains Jour. Statistical Physics, 124 (2006), 1105–1129
- [2] Arkeryd, L.; Esposito, R.; Marra, R.; Nouri, A.: Stability for Rayleigh-Benard convective solutions of the Boltzmann equation. Arch. Ration. Mech. Anal. 198 (2010), no. 1, 125–187.
- [3] Arkeryd, L.; Esposito, R.; Marra, R.; Nouri, A.: Ghost effect by curvature in planar Couette flow. Kinet. Relat. Models. 4 (2011), no. 1, 109–138.
- [4] Arkeryd, L.; Nouri, A. :  $L^1$  solutions to the stationary Boltzmann equation in a slab. Ann. Fac. Sci. Toulouse Math. (6) 9 (2000), no. 3, 375–413.
- [5] Arkeryd, L.; Nouri, A.: Asymptotic techniques for kinetic problems of Boltzmann type. Proceedings of the 3rd edition of the summer school in "Methods and Models of kinetic theory", Riv. Mat. Univ. Parma. 7 (2007), 1–74.
- [6] Basile, G.; Olla, S.; Spohn, H.: Wigner functions and stochastically perturbed lattice dynamics, Arch. Rat. Mech. Anal. 195,(2010) 171–203.
- [7] Boltzmann, L.: Further studies on the termal equilibrium of gas molecules, 88–174, in Kinetic Theory 2, ed. S.G. Brush, Pergamon, Oxford (1966)
- [8] Bonetto, F.; Lebowitz, J.L.; Ray-Bellet, L. : Fourier's law: A challenge to theorists, Mathematical Physics, 2000.

- [9] Cercignani, C.: The Boltzmann Equation and its Applications, New York, Springer-Verlag (1987).
- [10] Cercignani, C.; Illner, R.; Pulvirenti, M.: The mathematical theory of dilute gases, Springer-Verlag, (1994)
- [11] Chapman, S.; Cowling, T.G.: The Mathematical Theory of Non-uniform Gases, Cambridge University Press, (1991).
- [12] Desvillettes, L.; Villani, C.: On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. 159, no. 2 (2005) 245–316.
- [13] Di Perna, R. J.; Lions, P. L.: On the Cauchy Problem for Boltzmann Equations: Global Existence and Weak Stability, Ann. Math. 130 (1989), 321–366.
- [14] Di Perna, R. J.; Lions, P. L.: Ordinary differential equations, transport theory and Sobolve spaces. Invent. Math. 98 (1989) 511–547.
- [15] Esposito, R.; Guo, Y.; Marra, R.: Phase transition in a Vlasov-Boltzmann binary mixture. Comm. Math. Phys. 296 (2010), no. 1, 1–33.
- [16] Esposito, R.; Lebowitz, J. L.; Marra, R.: Hydrodynamic limit of the stationary Boltzmann equation in a slab, Comm. Math. Phys. 160 (1994), 49–80.
- [17] Esposito, R.; Lebowitz, J. L.; Marra, R.: The Navier-Stokes limit of stationary solutions of the nonlinear Boltzmann equation, J. Stat. Phys. 78 (1995), 389–412.
- [18] Guiraud, J. P.: An H-theorem for a gas of rigid spheres in a bounded domain. In: Pichon, G. (ed.) Theories cinetique classique et relativistes, CNRS, Paris, (1975) pp. 29–58,
- [19] Guo, Y.: The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. 153 (2003), no. 3, 593–630.
- [20] Guo, Y.: The Vlasov-Landau-Poisson system in a periodic box. http://dx.doi.org/10.1090/S0894-0347-2011-00722-4, J. Amer. Math. Soc. 2011.
- [21] Guo, Y.: Bounded solutions for the Boltzmann equation. Quart. Appl. Math. 68 (2010), no. 1, 143–148.
- [22] Guo, Y.: Decay and continuity of the Boltzmann equation in bounded domains. Arch. Ration. Mech. Anal. 197 (2010), no. 3, 713–809.
- [23] Guo, Y.; Jang, J.: Global Hilbert expansion for the Vlasov-Poisson-Boltzmann system. Comm. Math. Phys. 299 (2010), no. 2, 469–501.
- [24] Guo, Y.; Jang, J.; Jiang, N.: Acoustic limit for the Boltzmann equation in optimal scaling. Comm. Pure Appl. Math. 63 (2010), no. 3, 337–361.
- [25] Gressman, P.; Strain, R.: Global classical solutions of the Boltzmann equation without angular cut-off. J. Amer. Math. Soc. 24 (2011), no. 3, 771–847.
- [26] Kim, C.: Formation and Propagation of Discontinuity for Boltzmann Equation in Non-Convex Domains. Comm. Math. Phys. 308 (2011) no 3, 641–701.

- [27] Kim, C.: Boltzmann equation with a large external field., Comm. PDE (2011) to appear
- [28] Maxwell, J. C.: On the Dynamical Theory of gases, Phil. Trans. Roy. Soc. London, 157, 49–88, (1866).
- [29] Ohwada, T.; Aoki, K.; Sone Y.: Heat transfer and temperature distribution in a rarefied gas between two parallel plates with different temperatures: Numerical analysis of the Boltzmann equation for a hard sphere molecule, in Rarefied Gas Dynamics: Theoretical and Computational Techniques, edited by E. P. Muntz, D. P. Weaver, and D. H. Campbell (AIAA, Washington, 1989)
- [30] Olla, S.: Energy diffusion and superdiffusion in oscillators lattice neworks, New trends in Math. Phys. (2009), 539–547.
- [31] Sone, Y.: Molecular gas dynamics. Theory, techniques, and applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2007.
- [32] Sone, Y.: Kinetic theory and fluid dynamics. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2002.
- [33] Speck, J.; Strain, R.: Hilbert expansion from the Boltzmann equation to relativistic fluids. Comm. Math. Phys. 304 (2011), no. 1, 229–280.
- [34] Strain, R.: Asymptotic stability of the relativistic Boltzmann equation for the soft potentials. Comm. Math. Phys. 300 (2010), no. 2, 529–597.
- [35] Ukai, S.: Solutions to the Boltzmann Equations. Pattern and Waves Qualitative Analysis of Nonlinear Differential Equations, North Holland, Amsterdam (1986) 37–96.
- [36] Ukai, S.: On the existence of global solutions of a mixed problem for the nonlinear Boltzmann equation. Proc. Japan Acad. A 53 (1974) 179–184.
- [37] Vidav, I., Spectra of perturbed semigroups with applications to transport theory. J. Math. Anal. Appl., 30 (1970) 264–279.
- [38] Villani, C.: A Review of Mathematical Problems in Collisional Kinetic Theory. Handbook of Fluid Mechanics (2003) D. Serre, S. Friedlander Ed., vol. 1.
- [39] Villani, C.: Hypocoercivity. Mem. Amer. Math. Soc. 202 (2009), no. 950
- [40] Yu, S.-H.: Stochastic Formulation for the Initial-Boundary Value Problems of the Boltzmann Equation, Arch. Rat. Mech. Anal., 192 (2009), no. 2, 217–274