MORREY TYPE SPACES AND MULTIPLICATION OPERATOR IN SOBOLEV SPACES

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ABSTRACT. The paper deals with the operator $u \to gu$ defined in the Sobolev space $W^{r,p}(\Omega)$ and which takes values in $L^p(\Omega)$ when Ω is an unbounded open subset in R^n . The functions g belong to wider spaces of L^p connected with the Morrey type spaces. L^p estimates and compactness results are stated.

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1. Introduction

Let Ω be an unbounded open subset in \mathbb{R}^n .

In literature there are different results about the study of multiplication operator for a suitable function $q:\Omega\to C$

$$u \longrightarrow gu,$$
 (1.1)

as an operator defined in a Sobolev space (with or without weight) and which takes values in a $L^p(\Omega)$ space.

In $W_0^{1,p}(\Omega)$ or in $W^{1,p}(\Omega)$ with Ω regular enough, reference results are some well-known inequalities which state the boundedness of (1.1): Hardy type inequalities (see H.Brezis [2], A.Kufner [12], J.Nečas [13]) when g(x) is an appropriate power of the distance of x from a subset of $\partial\Omega$, C.Fefferman inequality [10] (see, e.g. F.Chiarenza-M.Franciosi [9], F.Chiarenza-M.Frasca [10]) obtained when g belongs to suitable Morrey spaces.

Our interest is to study multiplication operator in the Sobolev space $W^{r,p}(\Omega)$, $r \in N$, $1 \le p < +\infty$ when g belongs to suitable Morrey type spaces $\mathcal{M}^{p,s}$ introduced in [8].

In the recent paper [8] have been stated an embedding result and a Fefferman type inequality.

This paper, which can be considered as a continuation of the previous work, deepens the study of such spaces and, in addition, introduces meaningful subspaces of these spaces in which L^p estimates and compactness results are stated.

As we noted in [8], one of the aspects of our interest lies in the fact that this type of inequalities are useful tools to prove a priori bounds when studying elliptic equations. These estimates enable us to state existence and uniqueness results. For applications in the study of the a priori bounds see [3], [4], [5], [6], [7]. The spaces considered in some of these papers are connected to the spaces $\mathcal{M}^{p,s}$.

These spaces are wider than L^p spaces, than classical Morrey space and are connected, for suitable values of s, to the well known Morrey spaces defined when Ω is an unbounded open subset in \mathbb{R}^n .

In Section 2 and Section 3 we complete the description of the spaces $\mathcal{M}^{p,s}$ with respect to the previous paper analyzing their inclusion properties and the relations between such spaces and the Morrey spaces introduced in bounded and unbounded domains until now.

In Section 4 we state L^p estimates when the functions g belong to suitable subspaces $\tilde{\mathcal{M}}^{p,s}$ and $\mathcal{M}_0^{p,s}$ defined in Section 2. We explain also the dependence of the constants in the L^p bounds. To this aim it is necessary to introduce a kind of modulus of continuity of functions belonging to the Morrey type spaces. In the process of approximation by functions more regular the dependence of the constants in the estimates plays a key role.

The compactness result for the multiplication operator is stated in Section 5.

In conclusion, one of the main aspects of the paper consists in the study of a class of spaces that generalizes the definition of classical Morrey space, $L^{p,n-sp}$, $s \leq \frac{n}{p}$.

Furthemore we observe that the spaces $\mathcal{M}^{p,s-\frac{n}{p}}$, $s \geq \frac{n}{p}$, provides an intermediate space between $L^p(\Omega)$ and $L^p_{loc}(\overline{\Omega})$ in addition to the fact that includes $L^{\infty}(\Omega)$.

The estimates obtained when the multiplication operator uses functions belonging to Morrey spaces, as we pointed out above, are used in the study of elliptic equations.

We emphasize that our results are obtained using tools different from those used to get some similar estimates in type Morrey spaces in unbounded domains and are of more general type (cfr. [6], [7] where one can also find some references).

2. Notations and Morrey type spaces

Let \mathbb{R}^n be the n-dimensional real euclidean space. We set

$$B_r(x) = \{ y \in \mathbb{R}^n : |y - x| < r \}, \quad B_r = B_r(0) \quad \forall x \in \mathbb{R}^n, \, \forall r \in \mathbb{R}_+.$$

For any $x \in \mathbb{R}^n$, we call open infinite cone having vertex at x every set of the type

$$\{x + \lambda(y - x) : \lambda \in R_+, |y - z| < r\},\$$

where $r \in R_+$ and $z \in R^n$ are such that |z - x| > r.

For all $\theta \in]0, \pi/2[$ and for all $x \in \mathbb{R}^n$ we denote by $C_{\theta}(x)$ an open infinite cone having vertex at x and opening θ .

For a fixed $C_{\theta}(x)$, we set

$$C_{\theta}(x,h) = C_{\theta}(x) \cap B_h(x), \quad \forall h \in R_+.$$

Let Ω be an open set in \mathbb{R}^n . We denote by $\Gamma(\Omega, \theta, h)$ the family of open cones $\mathbb{C} \subset\subset \Omega$ of opening θ and height h.

We assume that the following hypothesis is satisfied:

 h_1) There exists $\theta \in]0, \pi/2[$ such that

$$\forall x \in \Omega \quad \exists C_{\theta}(x) \text{ such that } \overline{C_{\theta}(x,\rho)} \subset \Omega.$$

Let $(\Omega_{\rho}(x))_{x\in\Omega}$ be the family of open sets in \mathbb{R}^n defined as

$$\Omega_{\rho}(x) = B_{\rho}(x) \cap \Omega, \quad x \in \Omega, \quad \rho > 0.$$

If $1 \leq p < +\infty$ and $s \in R$, we denote by $\mathcal{M}^{p,s}(\Omega)$ the space of functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$||g||_{\mathcal{M}^{p,s}(\Omega)} = \sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \left(\rho^{s-n/p} ||g||_{L^p(\Omega_\rho(x))} \right) < +\infty, \qquad d > 0,$$
 (2.1)

equipped with the norm defined by (2.1). Furthermore let $\tilde{\mathcal{M}}^{p,s}(\Omega)$ be the closure of $L^{\infty}(\Omega)$ in $\mathcal{M}^{p,s}(\Omega)$ and $\mathcal{M}^{p,s}_0(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $\mathcal{M}^{p,s}(\Omega)$.

Remark 2.1. We observe that:

- 1. If Ω is a bounded open set, $s \in [0, n/p]$ and $d = diam \Omega$, then $\mathcal{M}^{p,s}(\Omega)$ is the space $L^{p,n-sp}(\Omega)$, the classical Morrey space.
- 2. If $\Omega = \mathbb{R}^n$ then $L^{p,n-sp}(\Omega) \subset \mathcal{M}^{p,s}(\Omega)$, $s \in [0, n/p]$.
- 3. The spaces $\mathcal{M}^{p,s}(\Omega)$ are riduced to the known Morrey space $M^{p,n-sp}(\Omega)$, $s \in]0,n/p]$, introduced in analogy to the classical Morrey spaces when Ω is an unbounded open set.
- 4. For s = 0, the norm (2.1) is a sort of average integral on $B_{\rho}(x)$. In particular this is true if the family $\{\Omega_{\rho}(x)\}$ shrinks nicely to x, that is if $\Omega_{\rho}(x) \subset B_{\rho}(x)$ for any $\rho > 0$ and there is a constant $\alpha > 0$, independent of ρ , such that $|\Omega_{\rho}(x)| > \alpha |B_{\rho}(x)|$. The Theorem 3.1 in Section 3 states that $L^{\infty}(\Omega)$ is continuously embedded in $\mathcal{M}^{p,0}(\Omega)$.
- 5. If s < 0, in analogy to the classical Morrey spaces, one can see that $\mathcal{M}^{p,s}(\Omega) = \{0\}$ by the Lebesgue differentiation Theorem.

For functions that belong to the spaces $\tilde{\mathcal{M}}^{p,s}(\Omega)$ we can state the following alternative definition. Let $\Sigma(\Omega)$ the σ -algebra of Lebesgue measurable subsets of Ω .

Lemma 2.2. A function $g \in \tilde{\mathcal{M}}^{p,s}(\Omega)$ if and only if $g \in \mathcal{M}^{p,s}(\Omega)$ and the function

$$\sigma_g^{p,s}: \quad t \in [0,1] \longrightarrow \sup_{\substack{E \in \Sigma(\Omega) \\ selb,dl}} \|g\chi_E\|_{\mathcal{M}^{p,s}(\Omega)}$$

$$(2.2)$$

is continuous in zero, where χ_E denotes the characteristic function of E.

Proof. Let g be a function belonging to the space $\mathcal{M}^{p,s}(\Omega)$ and let $\sigma_g^{p,s}$ be continuous in zero. We denote by δ_{ϵ} a positive number such that

$$E \in \Sigma(\Omega)$$
, $\sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \left(\rho^{-n} |E \cap B_{\rho}(x)| \right) \le \delta_{\epsilon} \Longrightarrow ||g\chi_{E}||_{\mathcal{M}^{p,s}(\Omega)} \le \epsilon$

and by Ω_r the set

$$\Omega_r(g) = \{ x \in \Omega : |g(x)| \ge r \} \qquad \forall r \in R_+. \tag{2.3}$$

Then there exists a positive constant $c \in R_+$, independent of r and g, such that

$$\sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \left(\rho^{-n} |\Omega_r| \cap B_\rho(x) \right) \le c \frac{\|g\|_{\mathcal{M}^{p,s}}^p(\Omega)}{r^p}.$$

If we set

$$r_{\epsilon} = c \left(\frac{\|g\|_{\mathcal{M}^{p,s}(\Omega)}^p}{\delta_{\epsilon}} \right)^{1/p}$$

we get

$$\sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \left(\rho^{-n} |\Omega_{r_{\epsilon}} \cap B_{\rho}(x)| \right) \le \delta_{\epsilon}$$

and, as a consequence,

$$\|g\chi_{\Omega_{k_{\epsilon}}}\|_{\mathcal{M}^{p,s}(\Omega)} \leq \epsilon.$$

Let us introduce now the function

$$g_{\epsilon} = g - g\chi_{\Omega_{k_{\epsilon}}}.$$

Evidently $g_{\epsilon} \in L^{\infty}(\Omega)$ and

$$||g - g_{\epsilon}||_{\mathcal{M}^{p,s}(\Omega)} = ||g\chi_{\Omega_{k_{\epsilon}}}||_{\mathcal{M}^{p,s}(\Omega)} \le \epsilon.$$

Conversely, let us assume $g \in \tilde{\mathcal{M}}^{p,s}\Omega$). Then there exists $g_{\epsilon} \in L^{\infty}(\Omega)$ such that

$$||g - g_{\epsilon}||_{\mathcal{M}^{p,s}\Omega} \le \epsilon/2.$$

So we get

$$||g\chi_{E}||_{\mathcal{M}^{p,s}(\Omega)} \leq ||(g - g_{\epsilon})\chi_{E}||_{\mathcal{M}^{p,s}(\Omega)} + ||g_{\epsilon}\chi_{E}||_{\mathcal{M}^{p,s}(\Omega)} \leq$$

$$\leq \epsilon/2 + \rho^{s}||g_{\epsilon}||_{L^{\infty}(\Omega)} \sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} (\rho^{-n}|E \cap B_{\rho}(x)|)^{\frac{1}{p}}$$
(2.4)

from which we deduce that

$$||g\chi_E||_{\mathcal{M}^{p,s}(\Omega)} \le \epsilon$$

for any $E \in \Sigma(\Omega)$ such that

$$\sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \rho^{-n} |\Omega_{\rho}(x) \cap E| \le \left(\frac{\epsilon}{2 d^{s} ||g_{\epsilon}||_{L^{\infty}(\Omega)}}\right)^{p}$$

and the lemma is proved.

3. Inclusion properties

The following Theorem states inclusion properties of the spaces $\mathcal{M}^{p,s}(\Omega)$. Some of these, a) and b), introduced in the previous paper [8], will be proved again for sake of completeness showing explicitly the value of the constants in the estimates.

Theorem 3.1. The following inclusions hold:

a)
$$L^{\infty}(\Omega) \hookrightarrow \mathcal{M}^{p,s}(\Omega)$$
 $\forall p \in [1, +\infty[$ and $\forall s \geq 0.$

b)
$$L^q(\Omega) \hookrightarrow \mathcal{M}^{q,s}(\Omega) \qquad \forall s \ge \frac{n}{q}, \qquad 1 \le p \le q < +\infty.$$

c)
$$\mathcal{M}^{q,s}(\Omega) \hookrightarrow \mathcal{M}^{p,s}(\Omega)$$
, $1 \le p \le q < +\infty$.

$$d) \ \mathcal{M}^{q,\mu}(\Omega) \hookrightarrow \mathcal{M}^{p,\lambda}(\Omega) \qquad \ \lambda,\mu>0 \quad \ \frac{\lambda-n}{p} \leq \frac{\mu-n}{q}, \quad \ 1 \leq p \leq q < +\infty.$$

$$e) \ L^q(\Omega) \subset \mathcal{M}^{p,s}_0(\Omega) \subset \tilde{\mathcal{M}}^{p,s}(\Omega) \qquad \forall s \geq \tfrac{n}{q} \qquad \ 1 \leq p \leq q < +\infty.$$

$$f) \ \mathcal{M}^{q,s}(\Omega) \subset \tilde{\mathcal{M}}^{p,s}(\Omega), \qquad 1 \le p < q < +\infty.$$

Proof.

a) If $g \in L^{\infty}(\Omega)$ we get

$$||g||_{\mathcal{M}^{p,s}(\Omega)} = \sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \left(\rho^{s-n/p} ||g||_{L^{p}(\Omega_{\rho}(x))} \right) \leq$$

$$\leq ||g||_{L^{\infty}(\Omega)} \sup_{\substack{x \in \Omega \\ \rho + \in [0,d]}} \rho^{s} \left(\rho^{-n/p} |\Omega_{\rho}(x)|^{1/p} \right) = c||g||_{L^{\infty}(\Omega)}$$
(3.1)

where, if ω_n denotes the volume of the unit ball B(0,1), we get $c = \omega_n^{\frac{1}{p}} d^s$.

b)-c) By Hölder inequality it follows

$$||g||_{\mathcal{M}^{p,s}(\Omega)} \leq \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} (\rho^{s-\frac{n}{p}} ||g||_{L^{q}(\Omega_{\rho}(x))} |\Omega_{\rho}(x)|^{\frac{1}{p}-\frac{1}{q}}) \leq$$

$$\leq c_{1} \sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} \rho^{s-\frac{n}{q}} ||g||_{L^{q}(\Omega_{\rho}(x))} \leq c_{2} ||g||_{L^{q}(\Omega)},$$
(3.2)

with $c_2 = c_1 d^{s - \frac{n}{q}} = \omega_n^{\frac{1}{p} - \frac{1}{q}} d^{s - \frac{n}{q}}$.

d) As above by Hölder inequality, since $\lambda \leq n\left(1-\frac{p}{q}\right)+\frac{\mu p}{q}$, it results

$$\int_{\Omega_{\rho}(x)} |g|^{p} \leq |\Omega_{\rho}(x)|^{1-\frac{p}{q}} \left(\int_{\Omega_{\rho}(x)} |g|^{q} \right)^{\frac{p}{q}} \leq$$

$$\leq \omega_{n}^{1-\frac{p}{q}} \rho^{n\left(1-\frac{p}{q}\right)+\mu\frac{p}{q}} \left(\rho^{-\mu} \int_{\Omega_{\rho}(x)} |g|^{q} \right)^{\frac{p}{q}} \leq$$

$$\leq \omega_{n}^{1-\frac{p}{q}} \rho^{\lambda} d^{n\left(1-\frac{p}{q}\right)+\mu\frac{p}{q}-\lambda} \left(\rho^{-\mu} \int_{\Omega_{\rho}(x)} |g|^{q} \right)^{\frac{p}{q}} \tag{3.3}$$

from which

$$\left(\rho^{-\lambda} \int_{\Omega_{\rho}(x)} |g|^p\right)^{\frac{1}{p}} \le c \left(\rho^{-\mu} \int_{\Omega_{\rho}(x)} |g|^q\right)^{\frac{1}{q}}$$

with $c = \omega_n^{1-\frac{p}{q}} d^{n\left(1-\frac{p}{q}\right)+\mu\frac{p}{q}-\lambda}$. So we can deduce the assertion.

e) Let $g \in L^q(\Omega)$. By density of $C_0^{\infty}(\Omega)$ in L^p it follows that there exists a function $\phi_{\epsilon} \in C_0^{\infty}(\Omega)$ such that

$$\|g - \phi_{\epsilon}\|_{L^{q}(\Omega)} \le \frac{\epsilon}{\omega_{n}^{\frac{1}{p} - \frac{1}{q}} d^{s - \frac{n}{q}}}.$$

Then we get, again by Hölder inequality,

$$\|g - \phi_{\epsilon}\|_{\mathcal{M}^{p,s}(\Omega)} = \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \rho^{s-n/p} \left(\|g - \phi_{\epsilon}\|_{L^{p}(\Omega_{\rho}(x))} \right) \leq$$

$$\leq \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \rho^{s-n/p} \|g - \phi_{\epsilon}\|_{L^{q}(\Omega_{\rho}(x))} |\Omega_{\rho}(x)|^{\frac{1}{p} - \frac{1}{q}} \right) \leq$$

$$\leq \omega_{n}^{\frac{1}{p} - \frac{1}{q}} d^{s - \frac{n}{q}} \|g - \phi_{\epsilon}\|_{L^{q}(\Omega)} < \epsilon.$$

$$(3.4)$$

The second inclusion is obvious.

f) We observe that, if $g \in \mathcal{M}^{q,s}(\Omega)$, from c) we have $g \in \mathcal{M}^{p,s}(\Omega)$. Furthermore

$$||g\chi_{E}||_{\mathcal{M}^{p,s}(\Omega)} = \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \rho^{s-\frac{n}{p}} ||g||_{L^{p}(\Omega_{\rho}(x)\cap E)} \le$$

$$\le \sup_{\substack{x \in \Omega \\ \rho \in]0,d]}} \rho^{s-\frac{n}{p}} |\Omega_{\rho}(x) \cap E|^{\frac{1}{p}-\frac{1}{q}} ||g||_{L^{q}(\Omega_{\rho}(x)\cap E)} \le$$

$$\le ||g||_{\mathcal{M}^{q,s}(\Omega)} \sup_{\substack{x \in \Omega \\ \rho \in [0,d]}} (\rho^{-n} |\Omega_{\rho}(x) \cap E|)^{\frac{1}{p}-\frac{1}{q}},$$
(3.5)

and we deduce that the function $\sigma_g^{p,s}$, defined by (2.2), is continuous in zero. From Lemma 2.2 it follows that $g \in \tilde{\mathcal{M}}^{p,s}(\Omega)$.

The following simple examples show that the inclusion

$$L^{p}(\Omega) \subset \mathcal{M}^{p,\frac{n}{p}}(\Omega). \tag{3.6}$$

is a strict inclusion.

Example 3.2. The constant functions belong to $S^{p,\frac{n}{p}}(\Omega)$ and do not belong to $L^p(\Omega)$.

Example 3.3. The function $\frac{1}{1+|x|^{\alpha}}$ belongs to $S^{p,\frac{n}{p}}$ for any $\alpha > 0$ but does not belong to L^p if $\alpha \in]0,\frac{n}{p}]$.

4. L^p estimates

Let r, p, q be real number with the condition

$$h_2$$
) $r \in \mathbb{N}$, $1 \le p \le q < +\infty$, $q \ge \frac{n}{r}$, $q > \frac{n}{r}$ if $\frac{n}{r} = p > 1$.

The Theorem 4.1 below was stated in [8]. We will derive our L^p estimates using this result. The proof of the Theorem uses the following inequality (see again [8], Lemma 4.2) which states a connection between the functions belonging to $L^1(\Omega)$ and the functions in $L^1(\Omega_{\rho}(x))$:

$$c_1 \|v\|_{L^1(\Omega)} \le \int_{\Omega} \rho^{-n} \|v\|_{L^1(\Omega_{\rho}(x))} dx \le c_2 \|v\|_{L^1(\Omega)}, \quad c_1, c_2 \in R_+, \quad \forall v \in L^1(\Omega).$$
 (4.1)

Theorem 4.1. If h_1) and h_2) hold, then for any $g \in \mathcal{M}^{q,\frac{s}{p}}(\Omega)$, $s \leq p$, and for any $u \in W^{r,p}(\Omega)$ we get $gu \in L^p(\Omega)$ and

$$||gu||_{L^{p}(\Omega)} \le C||g||_{\mathcal{M}^{q,\frac{s}{p}}(\Omega)} ||u||_{W^{r,p}(\Omega)},$$
 (4.2)

where the constant c = c(p, q, r, n) is independent of g and u.

The next two lemma state L^p estimates.

Lemma 4.2. If h_1), h_2) hold and $g \in \tilde{\mathcal{M}}^{q,\frac{s}{p}(\Omega)}$, $s \leq p$, then for any $\epsilon \in R_+$ there exists $c(\epsilon) \in R_+$ such that

$$||gu||_{L^p(\Omega)} \le \epsilon ||u||_{W^{r,p}(\Omega)} + c(\epsilon)||u||_{L^p(\Omega)} \qquad \forall u \in W^{r,p}(\Omega).$$

Proof. Let $\phi_{\epsilon} \in L^{\infty}(\Omega)$ such that

$$\|g - \phi_{\epsilon}\|_{\mathcal{M}^{q,\frac{s}{p}}(\Omega)} \le \epsilon/c, \tag{4.3}$$

where the constant c is the constant in the bound (4.2). Then by Theorem 4.1

$$||gu||_{L^{p}(\Omega)} \leq ||(g - \phi_{\epsilon})u||_{L^{p}(\Omega)} + ||\phi_{\epsilon}u||_{L^{p}(\Omega)} \leq \leq c||(g - \phi_{\epsilon})u||_{\mathcal{M}^{q,\frac{s}{p}}(\Omega)} ||u||_{W^{r,p}(\Omega)} + ||\phi_{\epsilon}||_{L^{\infty}(\Omega)} ||u||_{L^{p}(\Omega)} \leq \leq \epsilon ||u||_{W^{r,p}(\Omega)} + c(\epsilon) ||u||_{L^{p}(\Omega)}$$

$$(4.4)$$

with $c(\epsilon) = \|\phi_{\epsilon}\|_{L^{\infty}(\Omega)}$.

Lemma 4.3. If h_1), h_2) hold and $g \in \mathcal{M}_0^{q,\frac{s}{p}}(\Omega)$, $s \leq p$, then for any $\epsilon \in R_+$ there exist $c(\epsilon) \in R_+$ and an open set $\Omega_{\epsilon} \subset\subset \Omega$ with cone property such that

$$||gu||_{L^p(\Omega)} \le \epsilon ||u||_{W^{r,p}(\Omega)} + c(\epsilon)||u||_{L^p(\Omega_\epsilon)} \qquad \forall u \in W^{r,p}(\Omega). \tag{4.5}$$

Proof. Let $\phi_{\epsilon} \in C_0^{\infty}(\Omega)$ be such that (4.3) holds. Reasoning as in the proof of the Lemma 4.2 we get

$$||gu||_{L^{p}(\Omega)} \le \epsilon ||u||_{W^{r,p}(\Omega)} + c(\epsilon)||u||_{p,supp\phi_{\epsilon}}, \tag{4.6}$$

with $c(\epsilon)$ as in the Lemma 4.2.

Then let us fix $\theta \in]0, \pi/2[$ and $h_{\epsilon} \in]0, \text{dist}(\partial\Omega, \text{supp }\phi_{\epsilon})/2[$. If we denote by Ω_{ϵ} the open set of \mathbb{R}^n union of the cones $C \in \Gamma(\Omega, \theta, h_{\epsilon})$ such that $C \cap \text{supp } g_{\epsilon} \neq \emptyset$, then (4.5) follows from (4.6).

Let us define the modulus of continuity of a function $g \in \tilde{\mathcal{M}}^{p,s}(\Omega)$. First let us introduce the function

$$\tau_s^p[g](t) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup |E \cap B_d(x)| \le t}} \|g \chi_E\|_{\mathcal{M}^{p,s}(\Omega)}, \qquad t \in R_+,$$

where χ_E is the characteristic function of E. It follows by Lemma 2.2 that that $g \in \tilde{\mathcal{M}}^{p,s}(\Omega)$ if and only if $g \in \mathcal{M}^{p,s}(\Omega)$ and

$$\lim_{t\to 0} \tau_s^p[g](t) = 0.$$

We define the modulus of continuity of $g \in \tilde{\mathcal{M}}^{p,s}(\Omega)$ as a function $\tau[g]: R_+ \to R_+$ satisfying

$$\tau_s^p[g](t) \le \tau[g](t) \qquad \forall t \in R_+ \,, \qquad \lim_{t \to 0} \tau[g](t) = 0 \,.$$

If $g \in L^p_{loc}(\overline{\Omega})$, we get

$$\lim_{r \to +\infty} \sup_{x} |\Omega_r(g) \cap B_d(x)| = 0, \qquad (4.7)$$

where $\Omega_r(g)$ is defined in (2.3). Let us denote, for any $k \in R_+$, by $r_k = r_k(g)$ a real number such that

$$\sup_{x} |\Omega_{r_k}(g) \cap B_d(x)| \le \frac{1}{k}$$

and by r[g] the function

$$r[g]: k \in R_+ \to r[g](k) = r_k \in R_+.$$
 (4.8)

Now we state the following lemma in which we emphasize the dependence of the constants in the final bound.

Lemma 4.4. In the same hypotheses of Lemma 4.2 for any $k \in R_+$ we get

$$||g u||_{L^p(\Omega)} \le C \tau[g] \left(\frac{1}{k}\right) ||u||_{W^{r,p}(\Omega)} + r[g](k) ||u||_{L^p(\Omega)} \quad \forall u \in W^1(\Omega),$$

where C is the constant in (4.2), $\tau[g]$ is the modulus of continuity of g in $\tilde{\mathcal{M}}^{q,\frac{s}{p}}(\Omega)$ and r[g] is the function defined by (4.8).

Proof. Let

$$g_k = (1 - \chi_{\Omega_{r_k}}) g.$$

The function g_k so defined belongs to the space $L^{\infty}(\Omega)$. From Theorem 4.1 we get

$$||g u||_{L^{p}(\Omega)} \leq ||(g - g_{k}) u||_{L^{p}(\Omega)} + ||g_{k} u||_{L^{p}(\Omega)} \leq$$

$$\leq C||g - g_{k}||_{\mathcal{M}^{q,\frac{s}{p}}(\Omega)} ||u||_{W^{r,p}(\Omega)} + ||g_{k} u||_{L^{p}(\Omega)} =$$

$$= C||g\chi_{\Omega_{r_{k}}}||_{\mathcal{M}^{p,s}(\Omega)} ||u||_{W^{r,p}(\Omega)} + r[g](k)||u||_{L^{p}(\Omega)}.$$

$$(4.9)$$

Taking in mind (4.7) and modulus of continuity we deduce the result.

5. Compactness result

Now we state the compactness result.

Theorem 5.1. If
$$h_1$$
), h_2) hold and $g \in \mathcal{M}_0^{q,\frac{s}{p}}(\Omega)$, $s \leq p$, then the operator $u \in W^{r,p}(\Omega) \longrightarrow gu \in L^p(\Omega)$

is compact.

Proof. We remark that for any open set $\Omega' \subset\subset \Omega$ the operator

$$u \in W^{r,p}(\Omega) \longrightarrow u|_{\Omega'} \in W^{r,p}(\Omega')$$

is linear and bounded.

On the other hand, if Ω' verifies the cone property too, by Rellich-Kondrachov's theorem the operator

$$u \in W^{r,p}(\Omega') \longrightarrow u \in L^p(\Omega')$$

is compact. Then also the operator

$$u \in W^{r,p}(\Omega) \longrightarrow u|_{\Omega'} \in L^p(\Omega')$$

is compact. Therefore if $\{u_n\} \in W^{r,p}(\Omega')$ is a bounded sequence there exists a subsequence $\{u_{n_k}\}$ converging to u in $L^p(\Omega')$. By Lemma 4.3 we get

$$||g(u_{n_k} - u)||_{L^p(\Omega)} \le \epsilon ||u_{n_k} - u||_{W^{r,p}(\Omega)} + c(\epsilon)||u_{n_k} - u||_{L^p(\Omega')} \quad \forall u \in W^{r,p}(\Omega)$$

from which we can deduce the result.

Remark 5.2. We remark that if $g \in \mathcal{M}^{q,s}(\Omega)$, $s \geq 0$, the results stated in Section 4 and Section 5 are still avalable. In the case s > 0 this is due to the inclusion properties (see Theorem 3.1). Then the bounds stated when $g \in \mathcal{M}^{q,\frac{s}{p}}(\Omega)$ are of more general type.

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