Minimal P-symmetric periodic solutions of nonlinear Hamiltonian systems

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Abstract

In this paper some existence results for the minimal P-symmetric periodic solutions are proved for first order autonomous Hamiltonian systems when the Hamiltonian function is superquadratic, asymptotically linear and subquadratic. These are done by using critical points theory, Galerkin approximation procedure, Maslov P-index theory and its iteration inequalities.

Keywords: Maslov P-index, iteration inequality, minimal P-symmetric periodic solutions, Hamiltonian systems

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1 Introduction and main results

We study the P-boundary problem of first order autonomous Hamiltonian systems:

$$\begin{cases} \dot{x} = JH'(x), \ \forall x \in \mathbb{R}^{2n}, \\ x(\tau) = Px(0), \end{cases} \tag{1.1}$$

where $\tau > 0$, $P \in Sp(2n)$, $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $H(Px) = H(x)$, $\forall x \in \mathbb{R}^{2n}$. $H'(x)$ denote its gradient, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix, I_n is the identity matrix on $\mathbb{R}^n$ and n is the positive integer.

A solution (τ, x) of the problem (1.1) is called *P-solution* of the Hamiltonian systems. It is a kind of generalized periodic solution of Hamiltonian systems. The problem (1.1) has relation with the the closed geodesics on Riemannian manifold (cf.[13]) and symmetric periodic solution or the quasi-periodic solution problem (cf.[14]). In addition, C. Liu in [20] transformed some periodic boundary problem for asymptotically linear delay differential systems and some asymptotically linear delay Hamiltonian systems to P-boundary problems of Hamiltonian systems as above, we

also refer [3, 9, 15, 16] and references therein for the background of P-boundary problems in N-body problems.

Suppose P satisfies $P^k = I$, here k is assumed to be the smallest positive integer such that $P^k = I$ (this condition for P is called $(P)_k$ condition in the sequel), so the P-solution (τ, x) can be extended as a $k\tau$-periodic solution $(k\tau, x^k)$. We say that a T-periodic solution (T, x) of a Hamiltonian system in (1.1) is P-symmetric if $x(\frac{T}{k}) = Px(0)$. T is the P-symmetric period of x. We define T be the minimal P-symmetric period of x if $T = \min\{\lambda > 0 \mid x(t + \frac{\lambda}{k}) = Px(t), \forall t \in \mathbb{R}\}$. Note that T might not be the minimal period of x although it is the minimal P-symmetric period of x.

In recent years, Maslov P-index theory was developed to study the existence and multiplicity of P-solutions (cf.[7, 8, 19, 20]), specially, the corresponding iteration theory was built to estimate the minimality of the period of P-solution (i.e., the minimal P-symmetric period) (cf.[21, 23]) and look for geometrically distinct P-solutions (i.e., subharmonic P-solutions) (cf.[24]). It is meaningful to study the minimal P-symmetric periodic solutions of (1.1). So far there are very few papers about it.

In the following, we always suppose $P \in Sp(2n)$ satisfies the $(P)_k$ condition.

In this paper, combining the Galerkin approximation procedure (cf. [22, 23, 24]) with the method with C. Liu and me (cf. [23]), we study the minimal P-symmetric periodic solutions of (1.1) when the Hamiltonian function H is superquadratic, asymptotically linear and sub-quadratic respectively.

For $\tau > 0$, we define

$$S_\tau(H) = \{x \in C^1([0, \tau], \mathbb{R}^{2n}) : x \neq constant, x \text{ is a } P\text{-solution of } (1.1)\}.$$

We now state the main results as follows.

Theorem 1.1. *Suppose $P \in Sp(2n)$ satisfies the $(P)_k$ condition, and H satisfies the following conditions:*

(H0) $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ with $H(Px) = H(x)$, $\forall x \in \mathbb{R}^{2n}$;

(H1) $H(x) = \frac{1}{2}(h_0 x, x) + o(|x|^2)$ as $|x| \to 0$;

(H2) $H(x) - \frac{1}{2}(h_0 x, x) \geq 0$, $\forall x \in \mathbb{R}^{2n}$,
where h_0 is semi-positive definite symmetric matrix with $P^T h_0 P = h_0$;

(H3) There exist constants $\mu > 2$ and $R_0 > 0$ such that

$$0 < \mu H(x) \leq H'(x) \cdot x, \quad \forall |x| \geq R_0;$$

(HX1) $H''(x(t)) \geq 0$ for every $x \in S_\tau(H)$ and $t \in \mathbb{R}$;

(HX2) $\int_0^\tau H''(x(t))dt > 0$ for every $x \in S_\tau(H)$;

(HX3) $i_P(h_0) + \nu_P(h_0) \leq \dim \ker_{\mathbb{R}}(P - I)$, where $(i_P(h_0), \nu_P(h_0))$ denote the Maslov P-index of h_0.

Then (1.1) possesses a P-solution x with the minimal P-symmetric period $k\tau$ or $\frac{k\tau}{k+1}$.

Remark 1.2. Specially, if $h_0 = 0$, then $i_P(h_0) = 0$, $\nu_P(h_0) = \dim\ker_{\mathbb{R}}(P - I)$, $\forall P \in Sp(2n)$. At the moment, (HX3) holds automatically. Our result generalize the corresponding one in [21].

For the asymptotically linear Hamiltonian systems, we consider the case that the asymptotical matrix may be degenerate and the get the following two theorems:

Theorem 1.3. *Suppose $P \in Sp(2n)$ satisfies the $(P)_k$ condition, and H satisfies (H0),(H1), (H2), (HX1), (HX2) and the following conditions:*

(H4) There exists constant a_1, a_2 and some $s \in (1, \infty)$ such that

$$|H''(x)| \le a_1|x|^s + a_2;$$

(H5) There exists semi-positive definite symmetric matrix h_∞ with $P^T h_\infty P = h_\infty$ such that

$$H'(x) = h_\infty x + o(|x|) \quad as \quad |x| \to \infty;$$

(H6) $h_\infty - h_0$ is positive definite, $h_\infty h_0 = h_0 h_\infty$, where $h_0 \in \mathfrak{L}_s(\mathbb{R}^{2n})$ is the matrix given in (H1) and (H2);

(HX4) $i_P(h_\infty) > i_P(h_0) + \nu_P(h_0)$, $i_P(h_0) + \nu_P(h_0) \le \dim\ker_{\mathbb{R}}(P - I)$, where $(i_P(h_\infty), \nu_P(h_\infty))$ denote the Maslov P-index of h_∞.

Then (1.1) possesses a P-solution x with the minimal P-symmetric period $k\tau$ or $\frac{k\tau}{k+1}$ provided one of the following cases occurs:

(1) $\nu_P(h_\infty) = 0$;

(2) $\nu_P(h_\infty) > 0$ and $G_\infty(x) = H(x) - \frac{1}{2}(h_\infty x, x)$ satisfies

$$|G'_\infty(x)| \le M \quad for \quad x \in \mathbb{R}^{2n}, \quad G_\infty(x) \to +\infty \quad as \quad |x| \to \infty. \tag{1.2}$$

Theorem 1.4. *Suppose $P \in Sp(2n)$ satisfies the $(P)_k$ condition, and H satisfies (H0),(H1), (H2),(H4), (H5), (HX1), (HX2) and the following conditions:*

(H7) $\{x \in \mathbb{R}^{2n} : H'(x) = 0\} = \{0\}$;

(HX5) $i_P(h_\infty) + \nu_P(h_\infty) \le \dim\ker_{\mathbb{R}}(P - I) + 1$, $i_P(h_\infty) + \nu_P(h_\infty) \notin [i_P(h_0), i_P(h_0) + \nu_P(h_0)]$.

Then (1.1) possesses a P-solution x with the minimal P-symmetric period $k\tau$ or $\frac{k\tau}{k+1}$ provided one of the following cases occurs:

(1) $\nu_P(h_\infty) = 0$;

(2) $\nu_P(h_\infty) > 0$ and $G_\infty(x) = H(x) - \frac{1}{2}(h_\infty x, x)$ satisfies

$$|G'_\infty(x)| \le M \quad for \quad x \in \mathbb{R}^{2n}, \quad G_\infty(x) \to +\infty \quad as \quad |x| \to \infty. \tag{1.3}$$

Remark 1.5. In Theorem 1.4, we do not need the condition (H6).

The following theorem studies the minimal P-*symmetric* periodic solutions of subquadratic Hamiltonian systems with P-boundary

$$\begin{cases} \dot{x} = \lambda JH'(x), \ \forall x \in \mathbb{R}^{2n}, \ \lambda \in \mathbb{R}, \\ x(\tau) = Px(0). \end{cases} \tag{1.4}$$

This is motivated by [2, 11].

Theorem 1.6. *Suppose* $P \in Sp(2n)$ *satisfies the* $(P)_k$ *condition, and H satisfies (H0) and*

(H8) $|H'(x)| \leq M$ *for* $x \in \mathbb{R}^{2n}$, *and* $H(x) \to +\infty$ *as* $|x| \to \infty$;

(H9) $H(0) = 0$ *and* $H'(x)$, $H(x) > 0$ *for* $x \neq 0$.

Suppose $\tau > 0$, *(HX1) and (HX2) hold. There exists* $\lambda_\tau > 0$ *such that for any* $\lambda \geq \lambda_\tau$, *(1.4) possesses a P-solution x with the minimal P-symmetric period* $k\tau$ *or* $\frac{k\tau}{k+1}$.

In order to get the information about the Maslov P-index of the *P-solution*, we need the relation between the Maslov P-index and Morse index. This has been done in Section 2 by using the Galerkin approximation procedure and the Maslov P-index theory. The main idea comes from [11] and [21].

2 Maslov P-index and Morse index

Maslov P-index was first studied in [7] and [19] independently for any symplectic matrix P with different treatment, it was generalized by C. Liu and the author in [22, 23]. And then C. Liu used relative index theory to develop Maslov P-index in [21] which is consistent with the definition in [22, 23]. In fact, when the symplectic matrix $P = diag\{-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa\}$, $0 \leq \kappa \in \mathbb{Z} \leq n$, the (P, ω)-index theory and its iteration theory were studied in [8] and then be successfully used to study the multiplicity of closed characteristics on partially symmetric convex compact hypersurfaces in $\mathbb{R}^{2n}$. Here we use the notions and results in [21, 22, 23].

For $\tau > 0$, $P \in Sp(2n)$, $\mathfrak{L}_s(\mathbb{R}^{2n})$ denotes all symmetric real $2n \times 2n$ matrices. For $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ and satisfies $P^T B(t+\tau)P = B(t)$. If γ is the fundamental solution of the linear Hamiltonian systems

$$\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}. \tag{2.1}$$

Then the Maslov P-index pair of γ is defined as a pair of integers

$$(i_P, \nu_P) \equiv (i_P(\gamma), \nu_P(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\},$$

where i_P is the index part and

$$\nu_P = \dim \ker(\gamma(\tau) - P)$$

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is the nullity. We also call (i_P, ν_P) the Maslov P-index of $B(t)$, just as in [21, 22, 23]. If (τ, x) is a *P-solution* of (1.1), then the Maslov P-index of the solution x is defined to be the Maslov P-index of $B(t) = H''(x(t))$ and denoted by $(i_P(x), \nu_P(x))$.

Let $S_{k\tau} = \mathbb{R}/(k\tau\mathbb{Z})$ and $W_P = \{z \in W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n}) \mid z(t + \tau) = Pz(t)\}$, it is a closed subspace of $W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n})$ and is also a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ as in $W^{1/2,2}(S_{k\tau}, \mathbb{R}^{2n})$. Let $\mathfrak{L}_s(W_P)$ and $\mathfrak{L}_c(W_P)$ denote the space of the bounded selfadjoint linear operator and compact linear operator on W_P. We define two operators $A, B \in \mathfrak{L}_s(W_P)$ by the following bilinear forms:

$$\langle Ax, y \rangle = \int_0^\tau (-J\dot{x}(t), y(t))dt, \quad \langle Bx, y \rangle = \int_0^\tau (B(t)x(t), y(t))dt. \tag{2.2}$$

Suppose that $\cdots \le \lambda_{-j} \le \cdots \le \lambda_{-1} < 0 < \lambda_1 \le \cdots \le \lambda_j \le \cdots$ are all nonzero eigenvalues of the operator A (count with multiplicity), correspondingly, e_j is the eigenvector of λ_j satisfying $\langle e_j, e_i \rangle = \delta_{ji}$. We denote the kernel of the operator A by W_P^0 which is exactly the space $\ker_{\mathbb{R}}(P - I)$. For $m \in \mathbb{N}$, we define the finite dimensional subspace of W_P by

$$W_P^m = W_m^- \oplus W_P^0 \oplus W_m^+$$

with $W_m^- = \{z \in W_P | z(t) = \sum_{j=1}^m a_{-j}e_{-j}(t), a_{-j} \in \mathbb{R}\}$ and $W_m^+ = \{z \in W_P | z(t) = \sum_{j=1}^m a_j e_j(t), a_j \in \mathbb{R}\}$.

We suppose P_m be the orthogonal projections $P_m : W_P \to W_P^m$ for $m \in \mathbb{N} \cup \{0\}$. Then $\{P_m \mid m = 0, 1, 2, \cdots\}$ be the Galerkin approximation sequence respect to A.

For $S \in \mathfrak{L}_s(W_P)$, we denote by $M^*(S)$ the eigenspaces of S with eigenvalues belonging to $(0, +\infty)$, $\{0\}$ and $(-\infty, 0)$ with $* = +, 0$ and $* = -$, respectively. Similarly, for any $d > 0$, we denote by $M_d^*(S)$ the d-eigenspaces of S with eigenvalues belonging to $[d, +\infty)$, $(-d, d)$ and $(-\infty, -d]$ with $* = +, 0$ and $* = -$, respectively. We denote $m^*(S) = \dim M^*(S)$, $m_d^*(S) = \dim M_d^*(S)$ and $S^\sharp = (S|_{ImS})^{-1}$.

The following theorem gives the relationship between the Maslov P-index and the Morse index. When P is a symplectic orthogonal matrix, C.Liu in [19] has got corresponding result. Now we generalize it for any symplectic matrix P. It plays a key role in the proof of the main results.

Theorem 2.1. *Suppose $B(t) \in C(\mathbb{R}, \mathfrak{L}_s(\mathbb{R}^{2n}))$ and satisfies $P^T B(t+\tau)P = B(t)$ with the Maslov P-index $(i_P(B), \nu_P(B))$, for any constant $0 < d < \frac{1}{4}\|(A - B)^\sharp\|^{-1}$, there exists an $m_0 > 0$ such that for $m \ge m_0$, there holds*

$$\begin{aligned} m_d^+(P_m(A - B)P_m) &= m + \dim\ker_{\mathbb{R}}(P - I) - i_P(B) - \nu_P(B), \\ m_d^-(P_m(A - B)P_m) &= m + i_P(B), \\ m_d^0(P_m(A - B)P_m) &= \nu_P(B), \end{aligned} \tag{2.3}$$

where B be the operator defined by (2.2) corresponding to $B(t)$.

Proof. Let $x(t) = \gamma_P(t)\xi(t) \in W_P, \xi \in W^{1/2,2}(S_\tau, \mathbb{R}^{2n}), \gamma_P(t)$ is defined in [22, 24] is a symplectic path which satisfies $\gamma_P(0) = I$ and $\gamma_P(\tau) = P$. Then we have

$$\langle Ax, x\rangle = \int_0^\tau (-J\dot{x}(t), x(t))dt$$

$$= \int_0^\tau [(-J\dot{\xi}(t), \xi(t)) - (\gamma_P(t)^T J\dot{\gamma}_P(t)\xi(t), \xi(t))]dt$$

$$= \int_0^\tau [(-J\dot{\xi}(t), \xi(t)) - (\bar{B}_{\gamma_P}(t)\xi(t), \xi(t))]dt,$$

$$\langle (A - B)x, x\rangle = \int_0^\tau [(-J\dot{x}(t), x(t)) - (B(t)x(t), x(t))]dt$$

$$= \int_0^\tau [(-J\dot{\xi}(t), \xi(t)) - (\gamma_P(t)^T J\dot{\gamma}_P(t)\xi(t), \xi(t)) - (\gamma_P(t)^T B(t)\gamma_P(t)\xi(t), \xi(t))]dt$$

$$= \int_0^\tau [(-J\dot{\xi}(t), \xi(t)) - (\widetilde{B}_{\gamma_P}(t)\xi(t), \xi(t))]dt,$$

where $\bar{B}_{\gamma_P}(t) = \gamma_P(t)^T J\dot{\gamma}_P(t), \widetilde{B}_{\gamma_P}(t) = \gamma_P(t)^T J\dot{\gamma}_P(t) + \gamma_P(t)^T B(t)\gamma_P(t)$. By the definitions of $\gamma_P(t)$ and $B(t), \widetilde{B}_{\gamma_P}(t)$ and $\bar{B}_{\gamma_P}(t)$ are both symmetric matrix functions and $\widetilde{B}_{\gamma_P}(0) = \widetilde{B}_{\gamma_P}(\tau), \bar{B}_{\gamma_P}(0) = \bar{B}_{\gamma_P}(\tau)$. The operators A and $A - B$ defined in W_P correspond to the operators $-J\frac{d}{dt} - \bar{B}_{\gamma_P}$ and $-J\frac{d}{dt} - \widetilde{B}_{\gamma_P}$ defined in $W^{1/2,2}(S_\tau, \mathbb{R}^{2n})$. Suppose γ is the fundamental solution of $\dot{z}(t) = JB(t)z(t)$.

Consider the following linear Hamiltonian systems

$$\dot{z}(t) = J\widetilde{B}_{\gamma_P}(t)z(t), \quad z(t) \in \mathbb{R}^{2n}. \tag{2.4}$$

Suppose $\widetilde{\gamma}(t)$ is the fundamental solution of (2.4). Then by direct computation, we obtain

$$\widetilde{\gamma}(t) = \gamma_P(t)^{-1}\gamma(t) = \gamma_2.$$

And similarly, $\gamma_P(t)^{-1}$ is the fundamental solution of $\dot{z}(t) = J\bar{B}_{\gamma_P}(t)z(t)$. By Theorem 7.1 in [25], there exists an $m^* > 0$ such that for $m \geq m^*$ such that

$$m_d^+(P_m(A - B)P_m) = m + i(\bar{B}_{\gamma_P}) - i(\widetilde{B}_{\gamma_P}) + \nu(\bar{B}_{\gamma_P}) - \nu(\widetilde{B}_{\gamma_P}),$$
$$m_d^-(P_m(A - B)P_m) = m - i(\bar{B}_{\gamma_P}) + i(\widetilde{B}_{\gamma_P}), \tag{2.5}$$
$$m_d^0(P_m(A - B)P_m) = \nu(\widetilde{B}_{\gamma_P})$$

where $\bar{B}_{\gamma_P}$ and $\widetilde{B}_{\gamma_P}$ be the compact operator defined by (2.2) corresponding to $\bar{B}_{\gamma_P}(t)$ and $\widetilde{B}_{\gamma_P}(t)$. $(i(\bar{B}_{\gamma_P}), \nu(\bar{B}_{\gamma_P}))$ and $(i(\widetilde{B}_{\gamma_P}), \nu(\widetilde{B}_{\gamma_P}))$ is the Maslov-type index of $\bar{B}_{\gamma_P}(t)$ and $\widetilde{B}_{\gamma_P}(t)$ in [25]. Now by Theorem 3.3 in [22], we have

$$i(\bar{B}_{\gamma_P}) = i_P(0) - i(\gamma_P) - n = -i(\gamma_P) - n, \ i(\widetilde{B}_{\gamma_P}) = i_P(B) - i(\gamma_P) - n. \tag{2.6}$$

Note that

$$\nu(\bar{B}_{\gamma_P}) = \nu(\gamma_P(t)^{-1}) = \dim \ker_\mathbb{R}(P - I), \ \nu(\widetilde{B}_{\gamma_P}) = \nu(\widetilde{\gamma}) = \nu_P(\gamma_2) = \nu_P(\gamma) = \nu_P(B). \tag{2.7}$$

Finally we get (2.3) by (2.5)-(2.7). $\qquad\square$

The following theorem was proved in [21] by relative index theory and iteration theory of Maslov P-index.

Theorem 2.2. *Suppose $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ and $P \in Sp(2n)$ satisfies the $(P)_k$ condition. For $\tau > 0$, let x_0 be a P-solution of (1.1). If the Maslov P-index of x_0 satisfies*

$$i_P(x_0) \leq \dim \ker_{\mathbb{R}}(P - I) + 1,$$

and further satisfies (HX1) and (HX2). Then the minimal P-symmetric period of x_0 is $k\tau$ or $\frac{k\tau}{k+1}$.

In order to estimate the Maslov P-index of a critical point of the functional we considered, we need the following result which was proved in [12, 17, 27].

Theorem 2.3. *Let E be a real Hilbert space with orthogonal decomposition $E = X \oplus Y$, where $\dim X < +\infty$. Suppose $f \in C^2(E, \mathbb{R})$ satisfies the (P.S) condition and the following conditions:*

(F1) There exist ρ and $\alpha > 0$ such that

$$f(w) \geq \alpha, \quad \forall w \in \partial B_\rho(0) \cap Y.$$

(F2) There exist $e \in \partial B_1(0) \cap Y$ and $R > \rho$ such that

$$f(w) < \alpha, \quad \forall w \in \partial Q.$$

where $Q = (\overline{B_R(0)} \cap X) \oplus \{re \mid 0 \leq r \leq R\}$.

Then

1. *f possesses a critical value $c \geq \alpha$, which is given by*

$$c = \inf_{h \in \Lambda} \max_{w \in Q} f(h(w)),$$

   where $\Lambda = \{h \in C(\overline{Q}, E) \mid h = id \ on \ \partial Q\}$.

2. *If $f''(w)$ is Fredholm for $w \in \mathcal{K}_c(f) \equiv \{w \in E : f'(w) = 0, f(w) = c\}$, then there exists an element $w_0 \in \mathcal{K}_c(f)$ such that the negative Morse index $m^-(w_0)$ and nullity $m^0(w_0)$ of f at w_0 satisfies*

$$m^-(w_0) \leq \dim X + 1 \leq m^-(w_0) + m^0(w_0). \tag{2.8}$$

Definition 2.4. [12] *Let E be a C^2-Riemannian manifold, D is a closed subset of E. A family $\mathcal{F}(\alpha)$ is said to be a homological family of dimension q with boundary D if for some nontrival class $\alpha \in H_q(E, D)$ the family $\mathcal{F}(\alpha)$ is defined by*

$$\mathcal{F}(\alpha) = \{G \subset E : \alpha \text{ is in the image of } i_\star : H_q(G, D) \to H_q(E, D)\},$$

where $i_\star$ is the homomorphism induced by the immersion $i : G \to E$.

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Theorem 2.5. *[12] As in the definition 2.4, for given E, D and α, let $\mathcal{F}(\alpha)$ be a homological family of dimension q with boundary D. Suppose that $f \in C^2(E, \mathbb{R})$ satisfies (P.S) condition. Define*

$$c \equiv c(f, \mathcal{F}(\alpha)) = \inf_{G \in \mathcal{F}(\alpha)} \sup_{w \in G} f(w). \tag{2.9}$$

Suppose that $\sup_{w \in D} f(w) < c$ and f' is Fredholm on

$$\mathcal{K}_c = \{x \in E : f'(x) = 0, f(x) = c\}. \tag{2.10}$$

Then there exists $x \in \mathcal{K}_c$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of the functional f at x satisfy

$$q - m^0(x) \leq m^-(x) \leq q.$$

3 Superquadratic Hamiltonian systems

In this section, we study the minimal P-symmetric periodic solution of superquadratic Hamiltonian systems with P-boundary conditions. In order to prove Theorem 1.1, we need the following arguments.

For $z \in W_P$, we define

$$f(z) = \frac{1}{2} \int_0^{k\tau} (-J\dot{z}(t), z(t))dt - \int_0^{k\tau} H(z)dt = k(\frac{1}{2}\langle Az, z\rangle - \int_0^\tau H(z)dt). \tag{3.1}$$

It is well known that $f \in C^2(W_P, \mathbb{R})$ whenever

$$H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \quad \text{and} \quad |H''(x)| \leq a_1|x|^s + a_2; \tag{3.2}$$

for some $s \in (1, \infty)$ and all $x \in \mathbb{R}^{2n}$. Looking for solutions of (1.1) is equivalent to looking for critical points of f.

Proof of Theorem 1.1. We carry out the proof in several steps.

Step 1. Since the growth condition (3.2) has not been assumed for H, we need to truncate the function H at infinite. We follow the method in Rabinowitz's pioneering work [26].

Let $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(y) \equiv 1$ if $y \leq K$, $\chi(y) \equiv 0$ if $y \geq K + 1$, and $\chi'(y) < 0$ if $y \in (K, K + 1)$, where K is free for now. Set

$$H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))R_K|z|^4, \tag{3.3}$$

where the constant R_K satisfies

$$R_K \geq \max_{K \leq |z| \leq K+1} \frac{H(z)}{|z|^4}.$$

Then $H_K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, and there is $K_0 > 0$ such that for $K \geq K_0$, H_K satisfies (H1), (H2) and (3.2) with $s = 2$. Moreover a straightforward computation shows (H3) hold with μ replaced by $\nu = \min\{\mu, 4\}$. Integrating this inequality then yields

$$H_K(z) \geq a_1 |z|^\nu - a_2 \tag{3.4}$$

for all $z \in \mathbb{R}^{2n}$, where a_1, $a_2 > 0$ are independent of K.

Let $G_K(z) = H_K(z) - \frac{1}{2}(h_0 z, z)$, then by (3.4) it is easy to show that

$$G_K(z) \geq a_3 |z|^\nu - a_4 \tag{3.5}$$

for all $z \in \mathbb{R}^{2n}$, where a_3, $a_4 > 0$ are independent of K.

Finally, we set

$$f_K(z) = \frac{1}{2} \int_0^{k\tau} (-J\dot{z}(t), z(t)) dt - \int_0^{k\tau} H_K(z) dt = \frac{k}{2} \langle Az, z \rangle - \int_0^{k\tau} H_K(z) dt, \quad \forall z \in W_P, \tag{3.6}$$

then $f_K \in C^2(W_P, \mathbb{R})$.

Step 2. For $m > 0$, let $f_{K,m} = f_K|_{W_P^m}$. We will show that $f_{K,m}$ satisfies the hypotheses of Theorem 2.3.

By (H1) and (3.3), for any $\epsilon > 0$, there is a $M = M(\epsilon, K) > 0$ such that

$$G_K(z) \leq \epsilon |z|^2 + M|z|^4, \ \forall \ x \in \mathbb{R}^{2n}. \tag{3.7}$$

Let B_0 be the operator defined by (2.2) corresponding to h_0, and let

$$X_m = M^-(P_m(A - B_0)P_m) \oplus M^0(P_m(A - B_0)P_m), \quad Y_m = M^+(P_m(A - B_0)P_m).$$

For $z \in Y_m$, by (3.7) and the fact that $P_n B_0 = B_0 P_n$ for $n \geq 0$, we have

$$f_{K,m}(z) = \frac{k}{2} \langle (A - B_0)z, z \rangle - \int_0^{k\tau} G_K(z) dt$$

$$\geq \frac{k}{2} \| (A - B_0)^\sharp \|^{-1} \| z \|^2 - (\epsilon \alpha_2 + M\alpha_4 \| z \|^2) \| z \|^2.$$

So there are constant $\rho = \rho(K) > 0$ and $\alpha = \alpha(K) > 0$, which are sufficiently small and independent of m, such that

$$f_{K,m}(z) \geq \alpha, \quad \forall z \in \partial B_\rho(0) \cap Y_m. \tag{3.8}$$

Let $e \in \partial B_1(0) \cap Y_m$ and set

$$Q_m = \{re \mid 0 \leq r \leq r_1\} \oplus (B_{r_1}(0) \cap X_m),$$

where r_1 is free for the moment. Let $z = z^- + z^0 \in B_{r_1}(0) \cap X_m$, then

$$f_{K,m}(z + re) = \frac{k}{2} \langle (A - B_0)z^-, z^- \rangle + \frac{k}{2} r^2 \langle (A - B_0)e, e \rangle - \int_0^{k\tau} G_K(z + re) dt$$

$$\leq \frac{k}{2} r^2 \| A - B_0 \| - \frac{k}{2} \| (A - B_0)^\sharp \|^{-1} \| z^- \|^2 - \int_0^{k\tau} G_K(z + re) dt. \tag{3.9}$$

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If $r = 0$, there holds

$$f_{K,m}(z + re) \leq -\frac{k}{2}\|(A - B_0)^\sharp\|^{-1}\|z^-\|^2.$$ (3.10)

If $r = r_1$ or $\|z\| = r_1$, by (3.5), there holds

$$\int_0^{k\tau} G_K(z + re)dt \geq \int_0^\tau G_K(z + re)dt \geq a_3 \int_0^{k\tau} |z + re|^\nu dt - k\tau a_4 \geq a_5(|z^0|^\nu + r^\nu) - a_6 \quad (3.11)$$

Combining (3.9) with (3.11) yields

$$f_{K,m}(z + re) \leq a_7 r^2 - a_8\|z^-\|^2 - a_5(\|z^0\|^\nu + r^\nu) + a_6.$$

So we can choose r_1 large enough which is independent of K and m such that

$$f_{K,m}(z + re) \leq 0, \quad \forall z \in \partial Q_m.$$ (3.12)

Now using the same argument as ([23], Theorem 4.2), we have $f_{K,m}$ has a critical value $c_{K,m} \geq \alpha$, which is given by

$$c_{K,m} = \inf_{g \in \Lambda_m} \max_{w \in Q_m} f_{K,m}(g(w)),$$ (3.13)

where $\Lambda_m = \{g \in C(Q_m, W_P^m) \mid g = id \text{ on } \partial Q_m\}$. Moreover, there is a critical point $x_{K,m}$ of $f_{K,m}$ which satisfies

$$m^-(x_{K,m}) \leq \dim X_m + 1.$$ (3.14)

Step 3. Since $id \in \Lambda_m$, by (3.9) and (H2) we have

$$c_{K,m} \leq \sup_{w \in Q_m} f_{K,m}(w) \leq \frac{k}{2}r_1^2\|A - B_0\|.$$ (3.15)

Then in the sense of subsequence we have

$$c_{K,m} \to c_K, \quad \alpha \leq c_K \leq \frac{k}{2}r_1^2\|A - B_0\|.$$ (3.16)

Using the same argument as (4.40)-(4.43) in [23], we have that f_K satisfies the (P.S)* condition on W_P, i.e., any sequence $\{z_m\} \subset W_P$ satisfying $z_m \in W_P^m$, $f_{K,m}(z_m)$ is bounded and $f'_{K,m}(z_m) \to 0$ possesses a convergent subsequence in W_P. Hence in the sense of the subsequence we have

$$x_{K,m} \to x_K, \quad f_K(x_K) = c_K, \quad f'_K(x_K) = 0.$$ (3.17)

By the standard argument as in [23], x_K is a classical nonconstant P-*solution* of

$$\begin{cases} \dot{x} = JH'_K(x), \ \forall x \in \mathbb{R}^{2n}, \\ x(\tau) = Px(0). \end{cases}$$ (3.18)

Indeed, if $x_K(t)$ is a constant solution of (3.18), by (H2), then

$$f_K(x_K) = \frac{k}{2}\langle Ax_K, x_K \rangle - \int_0^{k\tau} \frac{1}{2}(h_0 x_K, x_K) dt - \int_0^{k\tau} [H_K(x_K) - \frac{1}{2}(h_0 x_K, x_K)] dt \leq 0. \quad (3.19)$$

This contradicts to $f_K(x_K) = c_K \geq \alpha > 0$.

And there is a $K_0 > 0$ such that for all $K \geq K_0$, $\|x_K\|_{L^\infty} < K$. Then $H_K'(x_K) = H'(x_K)$ and x_K is a non-constant P-solution of (3.18). We denote it simply by $x := x_K$.

Step 4. Let $B(t) = H_K''(x(t))$ and B be the operator defined by (2.2) corresponding to $B(t)$. By direct computation, we get

$$\langle f_K''(z)w, w \rangle - k\langle (A - B)w, w \rangle = \int_0^{k\tau} [H_K''(x(t))w, w) - (H_K''(z(t))w, w)] dt, \quad \forall w \in W_P.$$

Then by the continuous of H_K'',

$$\|f_K''(z) - k(A - B)\| \to 0 \quad \text{as} \quad \| z - x \| \to 0. \quad (3.20)$$

Let $d = \frac{1}{4}\|(A - B)^\sharp\|^{-1}$. By (3.20), there exists $r_0 > 0$ such that

$$\|f_K''(z) - k(A - B)\| < \frac{1}{2}d, \quad \forall z \in V_{r_0} = \{z \in W_P : \ \|z - x\| \leq r_0\}.$$

Hence for m large enough, there holds

$$\|f_{K,m}''(z) - kP_m(A - B)P_m\| < \frac{1}{2}d, \quad \forall z \in V_{r_0} \cap W_P^m. \quad (3.21)$$

For $z \in V_{r_0} \cap W_P^m$, $\forall w \in M_d^-(P_m(A - B)P_m) \setminus \{0\}$, from (3.21) we have

$$\langle f_{K,m}''(z)w, w \rangle \leq k\langle P_m(A - B)P_m w, w \rangle + \|f_{K,m}''(z) - kP_m(A - B)P_m\| \cdot \|w\|^2$$
$$\leq -d\|w\|^2 + \frac{1}{2}d\|w\|^2 = -\frac{1}{2}d\|w\|^2 < 0.$$

Then

$$\dim M^-(f_{K,m}''(z)) \geq \dim M_d^-(P_m(A - B)P_m), \quad \forall z \in V_{r_0} \cap W_P^m. \quad (3.22)$$

Similary to the proof of (3.22), for large m, there holds

$$\dim M^+(f_{K,m}''(z)) \geq \dim M_d^-(P_m(A - B)P_m), \quad \forall z \in V_{r_0} \cap W_P^m. \quad (3.23)$$

By (3.14), (3.17), (3.22) and Theorem 2.1, for large m we have

$$m + i_P(h_0) + \nu_P(h_0) + 1 \geq \dim X_m + 1 \geq m^-(x_{K,m})$$
$$\geq \dim M_d^-(P_m(A - B)P_m) = m + i_P(x).$$

Then by (HX3), we have

$$i_P(x) \leq i_P(h_0) + \nu_P(h_0) + 1 \leq \dim \ker_{\mathbb{R}}(P - I) + 1. \quad (3.24)$$

Finally, by (3.24), (HX1), (HX2) and Theorem 2.2, the proof is completed.

$\square$

4 Asymptotically linear Hamiltonian systems

Proof of Theorem 1.4. Let W_P, A, P_m be as in Section 2, and let f be defined by (3.1). Then (H4) implies that $f \in C^2(W_P, \mathbb{R})$. Let B_0 and B_∞ be the operator defined by (2.2) corresponding to h_0 and h_∞ respectively.

For $m > 0$, let $f_m = f|_{W_P^m}$. We carry out the proof in several steps.

Step 1. By (H1), it is easy to prove that

$$f(z) = \frac{k}{2} \langle (A - B_0)z, z \rangle + o(\|z\|^2) \quad \text{as} \quad z \to 0. \tag{4.1}$$

Let

$$X_m = M^-(P_m(A - B_0)P_m) \oplus M^0(P_m(A - B_0)P_m), \quad Y_m = M^+(P_m(A - B_0)P_m).$$

For $z \in Y_m$, by (4.1) and the fact that $P_n B_0 = B_0 P_n$ for $n \geq 0$, there exists $\rho > 0$ small enough that

$$\begin{aligned}
f_{K,m}(z) &= \frac{k}{2} \langle (A - B_0)z, z \rangle + o(\|z\|^2) \\
&\geq \frac{k}{2} \|(A - B_0)^\sharp\|^{-1} \|z\|^2 + o(\|z\|^2) \\
&\geq \alpha = \frac{k}{4} \|(A - B_0)^\sharp\|^{-1} \|\rho\|^2 > 0, \quad \forall \, z \in \partial B_\rho(0) \cap Y_m.
\end{aligned} \tag{4.2}$$

Step 2. Since $P_n B_\infty = B_\infty P_n$ for $n \geq 0$, it is easy to show that there exists $m_0 > 0$ such that

$$\ker(A - B_0) \subset W_P^m.$$

On the other hand, there is $m_1 > 0$ such that for $m \geq m_1$

$$\dim \ker(P_m(A - B_\infty)P_m) \leq \dim \ker(A - B_\infty). \tag{4.3}$$

Then there exists $m_1 \geq m_0$ such that for $m \geq m_1$,

$$\ker P_m(A - B_\infty)P_m = \ker(A - B_\infty). \tag{4.4}$$

This implies that

$$Im P_m(A - B_\infty)P_m \subset Im(A - B_\infty).$$

Then for any $z \in Im P_m(A - B_\infty)P_m$, we have

$$\|P_m(A - B_\infty)P_m\| = \|(A - B_\infty)\| \geq \|(A - B_\infty)^\sharp\|^{-1} \|z\|.$$

Then for any $0 < d \leq \frac{1}{4} \|(A - B_\infty)^\sharp\|^{-1}$,

$$M_d^*(P_m(A - B_\infty)P_m) = M^*(P_m(A - B_\infty)P_m), \quad \text{where} \quad * = +, -, 0. \tag{4.5}$$

By Theorem 2.1, there exist $m_2 \geq m_1$ such that for $m \geq m_2$,

$$\dim M^-(P_m(A - B_\infty)P_m) = m + i_P(h_\infty). \tag{4.6}$$

Similarly, there exists $m_3 > 0$ such that for $m \geq m_3$,

$$\dim M^-(P_m(A - B_0)P_m) = m + i_P(h_0),$$
$$\dim M^0(P_m(A - B_0)P_m) = \nu_P(h_0), \tag{4.7}$$

Let $m_4 = \max\{m_2, m_3\}$. For $m \geq m_4$, by (4.6), (4.7) and (HX4) we have

$$\dim M^-(P_m(A - B_\infty)P_m) > \dim X_m.$$

It implies that there exists

$$y \in M^-(P_m(A - B_\infty)P_m) \cap Y_m, \quad \|y\| = 1. \tag{4.8}$$

By (4.8), we have $(A - B_\infty)y \in Y_m$, $(A - B_0)y \in Y_m$. For any $z = z_- + z_0 \in X_m$,

$$\langle (B_\infty - B_0)y, z \rangle = -\langle (A - B_\infty)y, z \rangle + \langle (A - B_0)y, z \rangle = 0. \tag{4.9}$$

By (H6) we have that $B_\infty - B_0$ is positive definite and

$$\langle (B_\infty - B_0)z_0, z_0 \rangle \geq \lambda_0 \|z_0\|^2, \quad \text{where} \quad \lambda_0 > 0, \tag{4.10}$$

$$[(A - B_\infty) - (A - B_0)]^2 = (B_0 - B_\infty)^2 = (B_\infty - B_0)^2 = [(A - B_0) - (A - B_\infty)]^2. \tag{4.11}$$

(4.11) implies that

$$(A - B_\infty)(A - B_0) = (A - B_0)(A - B_\infty). \tag{4.12}$$

Hence

$$0 = \langle (A - B_\infty)z_-, (A - B_0)z_0 \rangle = \langle (A - B_0)(A - B_\infty)z_-, z_0 \rangle$$
$$= \langle (A - B_\infty)(A - B_0)z_-, z_0 \rangle = \langle (A - B_0)z_-, (A - B_\infty)z_0 \rangle, \tag{4.13}$$

it implies that $\langle z_-, (A - B_\infty)z_0 \rangle = 0$. Hence

$$\langle (B_\infty - B_0)z_-, z_0 \rangle = \langle z_-, (B_\infty - B_0)z_0 \rangle$$
$$= -\langle z_-, (A - B_\infty)z_0 \rangle + \langle z_-, (A - B_0)z_0 \rangle = 0. \tag{4.14}$$

Set

$$Q_m = \{z = ry + z_- + z_0 \in W_P^m : z_- + z_0 \in X_m, \|z_- + z_0\| \leq r_1, 0 \leq r \leq r_1\}, \tag{4.15}$$

$r_1 > 0$ will be determined later. For $z = ry + z_- + z_0 \in Q_m$, by (4.9), (4.10), (4.14) and (H5), we have

$$f_m(z) = \frac{k}{2}\langle (A - B_\infty)z, z \rangle - \int_0^{k\tau} G_\infty(z)dt$$
$$= \frac{k}{2}\langle (A - B_0)z_-, z_- \rangle + \frac{kr^2}{2}\langle (A - B_\infty)y, y \rangle$$
$$\quad - \frac{k}{2}\langle (B_\infty - B_0)z_-, z_- \rangle - \frac{k}{2}\langle (B_\infty - B_0)z_0, z_0 \rangle + o(\|z\|^2)$$
$$\leq -\frac{k}{2}\|(A - B_0)^\sharp\|^{-1}\|z_-\|^2 - \frac{kr^2}{2}\|(A - B_\infty)^\sharp\|^{-1}\|y\|^2 - \frac{k\lambda_0}{2}\|z_0\|^2 + o(\|z\|^2)$$
$$\leq -\frac{k}{2}\min\{\|(A - B_0)^\sharp\|^{-1}, r^2\|(A - B_\infty)^\sharp\|^{-1}, \lambda_0\}\|z\|^2 + o(\|z\|^2).$$

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Then taking $r_1 >$ to be large enough we have

$$f_m(z) \leq 0, \quad \forall\, z \in \partial Q_m. \tag{4.16}$$

Step 2. Using the same arguments as the proof of Lemma 2.1 in [18] and Lemma 7.1 in [28], we have that f_m satisfies (P.S) condition and f satisfies (P.S)* condition either (H5) with $\nu_P(h_\infty) = 0$ or the condition (2) in Theorem 1.3. By (4.2), (4.16) and Theorem 2.3, f_m has a critical value $c_m \geq \alpha$, which is given by

$$c_m = \inf_{g \in \Lambda_m} \max_{w \in Q_m} f_m(g(w)), \tag{4.17}$$

where $\Lambda_m = \{g \in C(Q_m, W_P^m) \mid g = id \text{ on } \partial Q_m\}$. Moreover, there is a critical point x_m of f_m which satisfies

$$m^-(x_m) \leq \dim X_m + 1. \tag{4.18}$$

Since $id \in \Lambda_m$, by (4.17) and (H2) we have

$$c_m \leq \sup_{w \in Q_m} f_m(w) \leq \beta = kr_1^2 \|A - B_0\|. \tag{4.19}$$

Then in the sense of subsequence we have

$$c_m \to c, \quad 0 < \alpha \leq c \leq \beta. \tag{4.20}$$

Since f satisfies the (P.S)* condition on W_P, hence in the sense of the subsequence we have

$$x_m \to x, \quad f(x) = c, \quad f'(x) = 0. \tag{4.21}$$

Now using the same arguments as (3.19)-(3.24), by (4.18)-(4.21) and (HX4), we have that x is a non-constant P-solution of (1.1) with its Maslov P-index $i_P(x)$ satisfying

$$i_P(x) \leq i_P(h_0) + \nu_P(h_0) + 1 \leq \dim \ker_{\mathbb{R}}(P - I) + 1. \tag{4.22}$$

The proof is completed by (4.22), (HX1), (HX2) and Theorem 2.2. $\qquad\square$

Proof of Theorem 1.4. Step 1. Let W_P, A, P_m be as in Section 2, and let f be defined by (3.1). Then (H4) implies that $f \in C^2(W_P, \mathbb{R})$. Let B_∞ be the operator defined by (2.2) corresponding to h_∞.

For $m > 0$, let $f_m = f|_{W_P^m}$. Using the same arguments as the proof of Lemma 2.1 in [18] and Lemma 7.1 in [28], we have that f_m satisfies (P.S) condition and f satisfies (P.S)* condition either (H5) with $\nu_P(h_\infty) = 0$ or the condition (2) in Theorem 1.4. Let

$$X_m = M^-(P_m(A - B_\infty)P_m) \oplus M^0(P_m(A - B_\infty)P_m), \quad Y_m = M^+(P_m(A - B_\infty)P_m).$$

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For $z \in Y_m$, by (1.3) we have

$$
\begin{aligned}
f_m(z) &= \frac{k}{2}\langle (A - B_\infty)z, z \rangle - \int_0^{k\tau} G_\infty(z)dt \\
&\geq \frac{k}{2}\|(A - B_0)^\sharp\|^{-1}\|z\|^2 - M_1\|z\|^2 \\
&\geq \alpha = -\frac{k}{2}\|(A - B_0)^\sharp\|^{-1}M_1^2.
\end{aligned}
\tag{4.23}
$$

For $z = z_- + z_0 \in X_m$, by (1.3) we have

$$
\begin{aligned}
f_m(z) &= \frac{k}{2}\langle (A - B_\infty)z_-, z_- \rangle - \int_0^{k\tau} G_\infty(z)dt \\
&\leq -\frac{k}{2}\|(A - B_0)^\sharp\|^{-1}\|z_-\|^2 + M_1\|z_-\| - \int_0^{k\tau} G_\infty(z_0)dt.
\end{aligned}
\tag{4.24}
$$

Since $B_\infty P_n = P_n B_\infty$, there exist $m_1 > 0$ such that for $m \geq m_1$,

$$
\ker P_m(A - B_\infty)P_m = \ker(A - B_\infty).
$$

Then by (1.3),

$$
\int_0^{k\tau} G_\infty(z_0)dt \to +\infty, \quad \text{as} \;\; \|z_0\| \to \infty.
\tag{4.25}
$$

By (4.24) and (4.25), there exist $r_1 > 0$ and $\beta < \alpha$ such that

$$
f_m(z) \leq \beta, \;\; \forall \;\; z \in \partial Q_m,
\tag{4.26}
$$

where $Q_m = \{z \in X_m : \|z\| \leq r_1\}$. The constants α, β and r_1 in the above are independent of m.

Step 2. Let $S = Y_m$, then ∂Q_m and S homologically link (cf.[4]). Let $D = \partial Q_m$ and $\delta = [Q_m] \in H_l(W_P^m, D)$, where $l = \dim X_m$. Then δ is nontrival and $\mathcal{F}(\delta)$ defined by Definite 2.4 is a homological family of dimension l with boundary D. It is well known that f_m' is Fredholm on $\mathcal{K}_{c_m}$ defined by (2.9) and (2.10). By (4.23) and (4.26), we obtain

$$
\sup_{w \in D} f_m(w) \leq \beta < \alpha \leq c_m = c(f_m, \mathcal{F}(\delta)).
$$

Then by Theorem 2.5, there exists $x_m \in \mathcal{K}_{c_m}$ such that the Morse indices $m^-(x_m)$ and $m^0(x_m)$ of f_m at x_m satisfies

$$
\dim X_m - m^0(x_m) \leq m^-(x_m) \leq \dim X_m.
\tag{4.27}
$$

Since $Q_m \in \mathcal{F}(\delta)$, by (4.24) we have

$$
c_m \leq \sup_{w \in Q_m} f_m(w) \leq \frac{k}{2}r_1^2\|A - B_\infty\| + M_1 r_1 = M_2.
$$

Hence in the sense of subsequence we have

$$
c_m \to c, \;\; \alpha \leq c \leq M_2.
$$

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Since f satisfies (P.S)* condition, in the sense of subsequence,

$$x_m \to x_0, \quad f(x_0) = c, \quad f'(x_0) = 0. \tag{4.28}$$

Using the standard arguments we have x_0 is a classical P-*solution* of (1.1). Now using the same arguments as (3.20)-(3.22), there exists $r_2 > 0$ such that

$$\dim M^{\pm}(f_m''(z)) \geq \dim M_d^{\pm}(P_m(A-B)P_m), \quad \forall z \in \{z \in W_P^m : \|z - x_0\| \leq r_2\}, \tag{4.29}$$

where B be the operator defined by (2.2) corresponding to $B(t) = H''(x_0(t))$.

By (4.5), (4.27)-(4.29) and Theorem 2.1, there exists $m_2 \geq m_1$ such that for $m \geq m_2$,

$$\begin{aligned}
m + i_P(h_\infty) + \nu_P(h_\infty) = \dim X_m &\geq m^-(x_m) \\
&\geq \dim M_d^-(P_m(A-B)P_m) = m + i_P(x_0) \\
m + i_P(h_\infty) + \nu_P(h_\infty) = \dim X_m &\leq m^-(x_m) + m^0(x_m) \\
&\leq \dim(M_d^-(P_m(A-B)P_m) \oplus M_d^0(P_m(A-B)P_m)) \\
&= m + i_P(x_0) + \nu_P(x_0).
\end{aligned}$$

Thus there holds

$$i_P(h_\infty) + \nu_P(h_\infty) - \nu_P(x_0) \leq i_P(x_0) \leq i_P(h_\infty) + \nu_P(h_\infty). \tag{4.30}$$

Combining (4.30) with (HX5) yields that $x_0 \neq 0$, or by (H2) we have

$$B(t) = H''(x_0(t)) = h_0, \text{ and } i_P(x_0) = i_P(h_0), \ \nu_P(x_0) = \nu_P(h_0). \tag{4.31}$$

So (4.30) contradicts to (HX5). Further, we have that x_0 is non-constant by (H7).

Now our conclusion follows from (4.30), (HX5), (HX1), (HX2) and Theorem 2.2. The proof is complete.

$\square$

5   Subquadratic Hamiltonian systems

Proof of Theorem 1.6. Let W_P, A, P_m and W_P^m be defined as in Section 2, let

$$g(z) = \lambda \int_0^{k\tau} H(z)dt - \frac{k}{2}\langle Az, z \rangle, \quad \forall z \in W_P. \tag{5.1}$$

Set $g_m = g|_{W_P^m}$ for $m > 0$, it is easy to prove that g_m satisfies (P.S) condition and g satisfies (P.S)* condition under the condition (H8)(cf.[2]). Let

$$X_m = P_m(M^+(A)), \quad Y_m = P_m(M^-(A) \oplus M^0(A)).$$

For $z \in X_m$, by (H8), (H9) and (5.1),

$$g(z) \leq \lambda M_1 \|z\| - \frac{k}{2} \|A^\sharp\|^{-1} \|z\|^2.$$

So there exists $r_\lambda > 1$ and $Q_m = \{z \in W_P^m : \|z\| \leq r_\lambda\}$ such that

$$g(z) \leq 0, \quad \forall z \in \partial Q_m. \tag{5.2}$$

Let $v \in Q_m$ with $\|z\| = 1$ and $S_m = v + Y_m$. For $z = v + z_- + z_0 \in S_m$,

$$
\begin{aligned}
g(z) &= \lambda \int_0^{k\tau} H(z)dt - \frac{k}{2}\langle Az^-, z^-\rangle - \frac{k}{2}\langle Av, v\rangle \\
&\leq \lambda \int_0^{k\tau} H(z)dt + \frac{k}{2}\|A^\sharp\|^{-1}\|z^-\|^2 - \frac{k}{2}\|A\|.
\end{aligned}
\tag{5.3}
$$

Following [2], three cases are needed to be considered.

Case 1 $\|z_-\|^2 > a_0 = 3\|A^\sharp\|\|A\|$. Then by (H10) and (5.3),

$$g(z) \geq \frac{k}{2}(\|A^\sharp\|^{-1}\|z^-\|^2 - \|A\|) \geq \|A\|.$$

Case 2 $\|z_-\|^2 \leq a_0 = 3\|A^\sharp\|\|A\|$ and $\|z_0\| > a_1$. Then by (H9) and (5.3),

$$g(z) \geq \lambda k\tau H(z_0) - \lambda M_1\|z_- + v\| + \frac{k}{2}\|A^\sharp\|^{-1}\|z^-\|^2 - \frac{k}{2}\|A\| \geq 1$$

if $\lambda \geq 1$ and a_1 is large enough.

Case 3 $\|z_-\|^2 \leq a_0$ and $\|z_0\| \leq a_1$. Let $S = v + M^-(A) \oplus M^0(A)$ and $\Omega = \{z \in S : \|z_-\|^2 \leq a_0, \|z_0\| \leq a_1\}$, then Ω is convex and weakly compact. Since $\int_0^{k\tau} H(z)dt$ is weakly continuous, it achieves its infimum σ on Ω at $\hat{z} = v + \hat{z}_- + \hat{z}_0$. By (H10) and the fact $\hat{z} \neq 0$, we have $\sigma > 0$. Therefore,

$$g(z) \geq \lambda\sigma + \frac{k}{2}\|A^\sharp\|^{-1}\|z^-\|^2 - \frac{k}{2}\|A\| \geq 1$$

if $\lambda \geq \sigma^{-1}(\frac{k}{2}\|A\| + 1)$ and $z \in \Omega$. Hence

$$g(z) \geq 1, \quad \forall z \in \Omega_m = \{z \in S_m : \|z_-\|^2 \leq a_0, \|z_0\| \leq a_1\}.$$

Combining the three cases, we have the constants

$$\lambda_\tau = \sigma^{-1}(\frac{k}{2}\|A\| + 1) + 1, \quad \alpha = \min\{\|A\|, 1\} > 0$$

such that for $\lambda \geq \lambda_\tau$, we have

$$g(z) \geq \alpha, \quad \forall z \in S_m. \tag{5.4}$$

Since ∂Q_m and S_m homologically link, by Theorem II.1.2 and Definition II.1.2 in [4], ∂Q_m and S homologically link. By (5.2) and (5.4), using the same argument as step 2 in the proof of Theorem 1.4, there is a classical P-solution x_0 of (1.4) such that

$$i_P(x_0) \leq \dim \ker_{\mathbb{R}}(P - I) \tag{5.5}$$

$$g(x_0) = c \geq \alpha > 0, \quad g'(x_0) = 0. \tag{5.6}$$

By (H9) and (5.6), x_0 is non-constant. By (5.5), (HX1), (HX2) and Theorem 2.2, we complete the proof. $\square$

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