Greatest lower bounds on the transverse Ricci curvature of some toric Sasaki manifolds

Hong Huang

Abstract

We determine the greatest lower bounds on the transverse Ricci curvature of compact toric Sasaki manifolds with positive basic first Chern class and with the first Chern class of the contact bundle being trivial. This is based on Wang-Zhu's and Futaki-Ono-Wang's works, and is an analogue of C. Li's work on toric Fano manifolds.

Key words: toric Sasaki manifolds; transverse Ricci curvature; Aubin's continuity path; Monge-Ampère equation

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1   Introduction

In [S] Székelyhidi defines the following invariant

$$R(X) := \sup\{t \mid \exists \text{ a Kähler metric } \omega \in c_1(X) \text{ such that } Ric(\omega) > t\omega\}$$

for a Fano manifold X. (See also [T].) In [L] Li determines this invariant for any compact toric Fano manifold X, based on Wang and Zhu's seminal work [WZ] on the existence of Kähler-Ricci soliton on any compact toric Fano manifold. Note that recently Datar and Székelyhidi [DS] recover the main results in [WZ] and [L] among other things, and Yao [Y] extends the result of [L] to the case of homogeneous toric bundles.

In this note we first define an invariant analogous to $R(X)$ above for compact Sasaki manifolds with positive basic first Chern class and with the first Chern class of the contact bundle being trivial. Then, similarly to [L], we determine the greatest lower bounds on the transverse Ricci curvature of compact toric Sasaki manifolds with positive basic first Chern class and with the first Chern class of the contact bundle being trivial, using and adapting Wang-Zhu's and Futaki-Ono-Wang's estimates in [WZ] and [FOW].

As in for example [BGS] and [GZ], for a compact Sasaki manifold S of dimension $2m + 1$ with Sasaki structure (ξ, η, Φ, g) we define the space

$$\mathcal{H} := \{\phi \in C_B^\infty(S, \mathbb{R}) \mid \eta_\phi = \eta + 2d_B^c\phi \text{ is a contact form}\},$$

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where $d_B^c = \frac{\sqrt{-1}}{2}(\bar{\partial}_B - \partial_B)$. Assuming that $c_1^B(S) > 0$ and $c_1(D) = 0$, where $c_1^B(S)$ is the basic first Chern class and $D = \text{Ker } \eta$ is the contact bundle, following [S] we introduce the invariant

$$R(S) := \sup\{t \mid \exists\, \phi \in \mathcal{H} \text{ such that } \rho_\phi^T > t(m+1)d\eta_\phi\},$$

where ρ_ϕ^T is the transverse Ricci form derived from the Sasaki structure constructed in [FOW, Proposition 4.2] with transverse Kähler form $\frac{1}{2}d\eta_\phi$.

Now we turn to the toric Sasaki manifolds; see for example [MS06], [MSY] and [FOW]. Recall (see for example [FOW]) that a toric Sasaki manifold S is a $(2m + 1)$-dimensional Sasaki manifold with an effective action of a $(m + 1)$-dimensional torus $G \cong T^{m+1}$ preserving the Sasaki structure (ξ, η, Φ, g) such that the Reeb field ξ is induced by an element of the Lie algebra $\mathfrak{g}$ of G. Thus the cone $(C(S), \bar{g}) = (\mathbb{R}_+ \times S, dr^2 + r^2 g)$ of a toric Sasaki manifold S is a toric Kähler manifold.

Let S be a $(2m + 1)$-dimensional compact toric Sasaki manifold. The moment map $\mu_\eta : S \to \mathfrak{g}^*$ w.r.t. the contact form η is given by

$$\langle \mu_\eta(x), X \rangle = \eta(X_S(x)), \quad \forall x \in S,$$

where X_S is the vector field on S induced by $X \in \mathfrak{g}$, i.e., $X_S(x) := \frac{d}{dt}|_{t=0} exp(tX) \cdot x$. On the other hand, the complexification $G^c \cong (\mathbb{C}^*)^{m+1}$ acts on $C(S)$ by biholomorphic automorphisms, and the corresponding moment map $\mu : C(S) \to \mathfrak{g}^*$ w.r.t. the Kähler form $\omega = d(\frac{1}{2}r^2\eta)(= dr^2 + r^2 g)$ (here the pull-back of η by the projection $C(S) \to S$ is still denoted by η) is given by

$$\langle \mu(x), X \rangle = r^2\eta(X_S(x)), \quad \forall x \in C(S),$$

where X_S is viewed as a vector field on $C(S)$. We denote the image of μ by $C(\mu)$, which is a convex rational polyhedral cone. So there exist vectors $\lambda_a, a = 1, \cdots, d$, in the integral lattice $\mathbb{Z}_\mathfrak{g} :=\text{Ker}\{\exp : \mathfrak{g} \to G\}$ such that

$$C(\mu) = \{y \in \mathfrak{g}^* \mid l_a(y) = \langle y, \lambda_a \rangle \geq 0, \ a = 1, \cdots, d\}.$$

We also denote the interior of $C(\mu)$ by $\text{Int}C(\mu)$. It is easy to see that the image of μ_η

$$\text{Im}(\mu_\eta) = \{\alpha \in C(\mu) \mid \alpha(\xi) = 1\}.$$

Now we assume further that the compact toric Sasaki manifold S has $c_1^B(S) > 0$ and $c_1(D) = 0$. Then by [FOW, Proposition 4.3], $c_1^B(S)$ is represented by $\tau d\eta$ for some positive constant τ. Using $\mathcal{D}$-homothetic transformation if needed we may and will assume that $(m + 1)d\eta \in 2\pi c_1^B(S)$. Moreover, by [FOW, Proposition 6.7], using transverse Kähler deformation if needed we may and will further assume that the symplectic potential on $(C(S), d(\frac{1}{2}r^2\eta))$ is given by formula (42) in [FOW]. Then by [FOW] there exists a unique rational vector $\gamma \in \mathfrak{g}^*$ such that

$$\langle \gamma, \lambda_a \rangle = -1, \quad a = 1, \cdots, d.$$

Choose a m-dimensional subtorus $H \subset G$ whose Lie algebra is

$$\mathfrak{h} := \{x \in \mathfrak{g} \mid \langle \gamma, x \rangle = 0\}.$$

In particular, since $\langle \gamma, \xi \rangle = -(m+1)$ ([FOW, (49)]), $\mathfrak{h}$ does not contain ξ. (Here we have identified the Reeb field ξ with the element in $\mathfrak{g}$ which induces it.) Let $H^c \cong (\mathbb{C}^*)^m$ be the complexification of H. Fix a point $p \in \mu^{-1}(\mathrm{Int}C(\mu))$, let $Orb_{C(S)}(H^c, p)$ be the orbit through p of the H^c-action on $C(S)$. The moment map $\mu_{\eta,H} : Orb_{C(S)}(H^c, p) \to \mathfrak{h}^*$ on the Kähler manifold $(Orb_{C(S)}(H^c, p), \frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)})$ for the H-action is defined by

$$\langle \mu_{\eta,H}(y), X \rangle = \eta(X)(y), \qquad y \in Orb_{C(S)}(H^c, p), X \in \mathfrak{h},$$

where the X on the RHS of the equality is the vector field on $Orb_{C(S)}(H^c, p)$ induced by $X \in \mathfrak{h}$. It turns out that

$$\mathrm{Im}(\mu_{\eta,H}) = \iota^*(\mathrm{Im}(\mu_\eta)) = \{\iota^*\alpha \mid \alpha \in C(\mu), \alpha(\xi) = 1\},$$

where $\iota : \mathfrak{h} \to \mathfrak{g}$ is the inclusion map. (See [FOW].) This image is a compact convex polyhedron. It is not necessarily rational, since the Sasaki structure on S may not be quasi-regular. (Compare [MS06] and [FOW].)

On $Orb_{C(S)}(H^c, p) \cong (\mathbb{C}^*)^m$ we introduce the affine logarithm coordinates

$$(w^1, \cdots, w^m) = (x^1 + \sqrt{-1}\theta^1, \cdots, x^m + \sqrt{-1}\theta^m)$$

for a point

$$(e^{x^1 + \sqrt{-1}\theta^1}, \cdots, e^{x^m + \sqrt{-1}\theta^m}) \in (\mathbb{C}^*)^m \cong Orb_{C(S)}(H^c, p).$$

Now $\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)}$ is determined by a convex function u^0 on $\mathbb{R}^m$,

$$\frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)} = \sqrt{-1}\partial\bar{\partial}u^0 = \frac{\sqrt{-1}}{4}\frac{\partial^2 u^0}{\partial x^i \partial x^j}dw^i \wedge d\overline{w^j}.$$

It is easy to see (cf. for example [FOW]) that (after translation) the interior $\mathrm{Int}(\mathrm{Im}(\mu_{\eta,H}))$ can be identified with

$$\Sigma := \{Du^0(x) = (\frac{\partial u^0}{\partial x^1}(x), \cdots, \frac{\partial u^0}{\partial x^m}(x)) \mid x \in \mathbb{R}^m\}.$$

We call the closure $\overline{\Sigma}$ the moment polytope of $(Orb_{C(S)}(H^c, p), \frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)})$ for the H-action (compare (57) in [FOW]).

It follows from Proposition 7.3 (or Lemma 7.5) of [FOW] that the origin O of $\mathbb{R}^m$ is contained in Σ. We observe that the barycenter P_c of the moment polytope $\overline{\Sigma}$ coincides with the origin O if and only if the Sasaki-Futaki invariant f of S (for definition see [BGS] and [FOW]) vanishes, see Proposition 3.4.

Similarly to [L, Theorem 1] we have

Theorem 1.1. *Let (S, ξ, η, Φ, g) be a compact $(2m+1)$-dimensional toric Sasaki manifold with positive basic first Chern class and with the first Chern class of the contact bundle being trivial. Let $\overline{\Sigma}$ and P_c be as above.*
If $P_c \neq O$, then $R(S) < 1$, and

$$R(S) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|},$$

where Q is the intersection of the ray $P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_cO}$ with $\partial\overline{\Sigma}$.
If $P_c = O$, then S admits a Sasaki-Einstein metric, and $R(S) = 1$.

That if $P_c = O$ then S admits a Sasaki-Einstein metric follows from [FOW], see also the proof of Proposition 3.4 below; we include it here for completeness. The bridge between $R(S)$ and $\frac{|\overline{OQ}|}{|\overline{P_cQ}|}$ is Aubin's continuity path for finding Sasaki-Einstein metrics. In Section 2, following [S] we show that on a $(2m+1)$-dimensional compact Sasaki manifold (S, ξ) (not necessarily toric) with positive basic first Chern class and with the first Chern class of the contact bundle being trivial, $R(S)$ is equal to the maximum existence time of Aubin's continuity path for finding Sasaki-Einstein metrics on S. In Section 3 we use this continuity path to prove Theorem 1.1.

For the most part of the proof of Theorem 1.1 we follow closely the lines of [L] (see also [Y]), using and/or adapting estimates from [FOW] and [WZ]. However, there is one point where our argument is slightly different from that in [L]: To prove the Claim 1 on p.4929 of [L], Li uses the simple formula (2) on p. 4923 of [L] expressing the initial Kähler potential $\tilde{u}_0$ via the vertices of the moment polytope. In our case, such a simple expression for u^0 is not available in general (when the Sasaki structure is not quasi-regular). Instead we have the formula (81) on p. 621 of [FOW] for u^0, which is somewhat difficult to treat directly. The idea is to use the Legendre transform to convert the Kähler potential u^0 to the symplectic potential $G_0(v)$, and exploit the degenerate behavior of $(\text{Hess } G_0(v))^{-1}$ near the boundary $\partial\overline{\Sigma}$ to prove a result similar to the Claim 1 in [L]. (Compare also [FOW] and [D].)

2 The invariant $R(S)$

Let (S, ξ, η, Φ, g) be a $(2m+1)$-dimensional compact (not necessarily toric) Sasaki manifold with positive basic first Chern class, with $c_1(D) = 0$ ($D = \text{Ker } \eta$) and with $(m+1)d\eta \in 2\pi c_1^B(S)$.

Define (cf. for example [FOW], [GZ], [Z11a]) Mabuchi functional on $\mathcal{H}$ (see the Introduction) via its variation

$$\frac{d}{dt}\mathcal{M}(\phi_t) = \int_S \dot{\phi}_t(2m(m+1) - s_{\phi_t}^T)(\frac{1}{2}d\eta_{\phi_t})^m \wedge \eta$$

and the requirement $\mathcal{M}(0) = 0$, where $s_{\phi_t}^T$ is the transverse scalar curvature derived from the Sasaki structure constructed in [FOW, Proposition 4.2] with transverse Kähler form $\frac{1}{2}d\eta_{\phi_t}$.

Let χ be a transverse Kähler form on S. We also define (cf. for example [VZ]) the $\mathcal{J}_\chi$ functional on $\mathcal{H}$ via its variation

$$\frac{d}{dt}\mathcal{J}_\chi(\phi_t) = 2m(m+1)\int_S \dot{\phi}_t(\chi \wedge (\frac{1}{2}d\eta_{\phi_t})^{m-1} - (\frac{1}{2}d\eta_{\phi_t})^m) \wedge \eta$$

and the requirement $\mathcal{J}_\chi(0) = 0$. Compare also [S].

Given $\psi \in \mathcal{H}$, let h_ψ be determined by

$$\rho_\psi^T - (m+1)d\eta_\psi = \sqrt{-1}\partial_B\bar{\partial}_B h_\psi$$

and

$$\int_S e^{h_\psi}(\frac{1}{2}d\eta_\psi)^m \wedge \eta = \int_S (\frac{1}{2}d\eta_\psi)^m \wedge \eta,$$

Aubin's continuity path for finding Sasaki-Einstein metrics is given by the following transverse Monge-Ampère equation for $\phi_t \in \mathcal{H}$

$$\frac{(d\eta + 2\sqrt{-1}\partial_B\bar{\partial}_B\phi_t)^m \wedge \eta}{(d\eta + 2\sqrt{-1}\partial_B\bar{\partial}_B\psi)^m \wedge \eta} = e^{h_\psi - t(2m+2)\phi_t},$$

or

$$\frac{\det(g_{i\bar{j}}^T + \phi_{i\bar{j}})}{\det(g_{i\bar{j}}^T + \psi_{i\bar{j}})} = \exp(h_\psi - t(2m+2)\phi_t). \qquad (*)_t$$

The equation $(*)_t$ is equivalent to $\rho_{\phi_t}^T = (m+1)(td\eta_{\phi_t} + (1-t)d\eta_\psi)$. When $t = 0$ the equation is solvable by the transverse Yau theorem in [E].

Following [S] we call a functional $\mathcal{F}$ defined on the space $\mathcal{H}$ proper if there exist constants $\epsilon, C > 0$ such that

$$\mathcal{F}(\psi) > \epsilon\mathcal{J}_{\frac{1}{2}d\eta}(\psi) - C$$

for any $\psi \in \mathcal{H}$.

Theorem 2.1. *Let S be as above. The following are equivalent for $0 \le t < 1$.*
  1) Given any $\psi \in \mathcal{H}$ the equation $(*)_t$ can be solved.
  2) There exists $\psi \in \mathcal{H}$ such that $\rho_\psi^T > t(m+1)d\eta_\psi$.
  3) The functional $\mathcal{M} + (1-t)\mathcal{J}_{\frac{1}{2}d\eta_\psi}$ is proper for any $\psi \in \mathcal{H}$.

Proof The proof is along the lines of proof of Theorem 1 in [S]. We only indicate some necessary modifications. We use [JZ] and [vC] to replace [CT08] in the proof of Proposition 3 in [S], and use [NS] to replace [BM87] in the proof of Lemma 5 in [S]. □

3   Proof of Theorem 1.1

Let (S, ξ, η, Φ, g) be a compact $(2m+1)$-dimensional toric Sasaki manifold satisfying the assumptions of Theorem 1.1. Choose H, p and u^0 as in the Introduction.

Choose $\psi = 0$ in $(*)_t$ of Section 2. Then as in [FOW], $(*)_t$ can be converted to the following Monge-Ampère equation for a strictly convex function u

$$\det(u_{ij}) = \exp(-(2m+2)(tu + (1-t)u^0)) \quad \text{on} \quad \mathbb{R}^m. \qquad (**)_t$$

By Theorem 2.1, $(**)_t$ is solvable when $t < R(S)$.

Let u be a solution to $(**)_t$, and

$$w_t = tu + (1-t)u^0.$$

Since $Dw_t(\mathbb{R}^m) = Du(\mathbb{R}^m) = Du^0(\mathbb{R}^m) = \Sigma$ (compare for example, [M1], [WZ], and the proof of Fact 2 in Section 2 of [Hu1]) and $O \in \Sigma$ (as observed in the Introduction), the strictly convex function w_t is proper, and attains its minimum m_t at a unique point $x_t \in \mathbb{R}^m$.

Proposition 3.1.   *1) There exists a constant C independent of $t < R(S)$, such that*

$$|m_t| \leq C.$$

2) There exist $\kappa > 0$ and a constant C, both independent of $t < R(S)$, such that

$$w_t \geq \kappa|x - x_t| - C.$$

Proof The proof is the same as that of Proposition 2 in [L], which uses arguments of [WZ, Lemma 3.2] and [D, Section 3.4, Proposition 1]; compare also the proof of Lemma 3.1 in [Hu1]. $\qquad\qquad\square$

Proposition 3.2.   *Fix t_0. There exists a constant C_1 such that $|x_t| \leq C_1$ for $0 \leq t \leq t_0$, where x_t is the minimum point of $w_t = tu + (1-t)u^0$ with u being any solution to $(**)_t$ if and only if there exists a constant C_2 such that $|\varphi_t| \leq C_2$ for $0 \leq t \leq t_0$, where $u^0 + \varphi_t$ is any solution to $(**)_t$.*

Proof The proof is similar to that of [L, Proposition 3] with the help of Proposition 3.1, 1), [FOW, Proposition 7.3] and [Z11b, Theorem 1.1]. (Alternatively, one can also use Proposition 3.1, 2) and the argument in the last paragraph of Section 3 in [D].) $\qquad\qquad\square$

Proposition 3.3. *If $R(S) < 1$, there exist a sequence $\{t_k\}$ and a point $y_\infty \in \partial\overline{\Sigma}$, such that*

$$\lim_{k\to\infty} t_k = R(S), \quad \lim_{k\to\infty} |x_{t_k}| = \infty, \quad \lim_{k\to\infty} Du^0(x_{t_k}) = y_\infty.$$

Proof The result follows easily from Theorem 2.1, Proposition 3.2, the properness of u^0 and the compactness of $\overline{\Sigma}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Recall [FOW] that

$$\overline{\Sigma} = \cap_{a=1}^{d}\{l'_a(v) \geq 0\}, \tag{3.1}$$

where $l'_a(v) = \langle v, \lambda'_a \rangle + \frac{1}{m+1}$, and $\lambda'_a \in \mathfrak{h} \cong \mathbb{R}^m$ is given by the decomposition

$$\lambda_a = \iota(\lambda'_a) + \frac{1}{m+1}\xi,$$

where λ_a is as in the Introduction.

W.l.o.g. we may assume that
$$l'_a(y_\infty) = 0, \quad a = 1, \cdots, d_0,$$
$$l'_a(y_\infty) > 0, \quad a = d_0 + 1, \cdots, d,$$
where $d_0 \geq 1$.

Note that we have

$$\int_{\mathbb{R}^m} e^{-(2m+2)w_t}dx = \int_{\mathbb{R}^m} \det(u_{ij})dx = \int_\Sigma dy = Vol(\Sigma). \tag{3.2}$$

Since w_t is a proper strictly convex function on $\mathbb{R}^m$, $w_t(x) \to +\infty$ as $|x| \to \infty$. So we have

$$\int_{\mathbb{R}^m} \frac{\partial w_t}{\partial x^i} e^{-(2m+2)w_t}dx = -\frac{1}{2m+2}\int_{\mathbb{R}^m} \frac{\partial e^{-(2m+2)w_t}}{\partial x^i}dx = 0, \quad i = 1, \cdots, m.$$

(Compare also for example [D].) It follows that when $t < R(S)$,

$$\int_{\mathbb{R}^m} (Du^0)e^{-(2m+2)w_t}dx = -\frac{t}{1-t}\int_{\mathbb{R}^m} (Du)e^{-(2m+2)w_t}dx.$$

On the other hand,

$$\int_{\mathbb{R}^m} (Du)e^{-(2m+2)w_t}dx = \int_{\mathbb{R}^m} (Du)\det(u_{ij})dx = \int_\Sigma ydy = Vol(\Sigma)P_c,$$

where P_c is the barycenter of $\overline{\Sigma}$ (as in the statement of Theorem 1.1).

Thus as in [L] we get

$$\frac{1}{Vol(\Sigma)}\int_{\mathbb{R}^m} (Du^0)e^{-(2m+2)w_t}dx = -\frac{t}{1-t}P_c \tag{3.3}$$

when $t < R(S)$.

As in [Y] we define

$$R_\Sigma := \sup\{t \mid 0 \leq t < 1, -\frac{t}{1-t}P_c \in \Sigma\}. \tag{3.4}$$

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Since $e^{-(2m+2)w_t} > 0$, $\frac{1}{Vol(\Sigma)}\int_{\mathbb{R}^m} e^{-(2m+2)w_t}dx = 1$ by (3.2), $Du^0(x) \in \Sigma$ for any $x \in \mathbb{R}^m$, and Σ is convex, we have

$$\frac{1}{Vol(\Sigma)}\int_{\mathbb{R}^m}(Du^0)e^{-(2m+2)w_t}dx \in \Sigma.$$

Combining with (3.3) we get that $-\frac{t}{1-t}P_c \in \Sigma$ for $t < R(S)$. So

$$R(S) \le R_\Sigma. \tag{3.5}$$

Compare [Y]. In particular, if $R(S) = 1$, then $R_\Sigma = 1$.

Let $(w^1, \cdots, w^m)$ be the affine logarithm coordinates on $Orb_{C(S)}(H^c, p) \cong (\mathbb{C}^*)^m$ as in the introduction, let $X_k = -\frac{\sqrt{-1}}{2}\frac{\partial}{\partial w^k}$, and θ_{X_k} be its Hamiltonian function (see p.597 and p.604 of [FOW]), $k = 1, \cdots, m$. By [FOW, Lemma 7.4], $\theta_{X_k} = \frac{\partial u^0}{\partial x^k}$.

The following result is implicitly from [FOW], and is analogous to [M1, Corollary 5.5], [M2, Lemma 6.1], and [F, Theorem 3.4.1].

Proposition 3.4. *(cf. [FOW]) $P_c = O$ if and only if the Sasaki-Futaki invariant of S vanishes.*

Proof For $k = 1, \cdots, m$, we compute as in the proof of [FOW, Lemma 7.5],

$$
\begin{aligned}
f(X_k) &= -\int_S \theta_{X_k}(\tfrac{1}{2}d\eta)^m \wedge \eta \\
&= -\int_S \frac{\partial u^0}{\partial x^k}\det(u^0_{ij})dx \wedge d\theta \wedge \eta \\
&= -\text{const.}\int_\Sigma y_k dy.
\end{aligned}
$$

So that the Sasaki-Futaki invariant f of S vanishes implies that $\int_\Sigma y_k dy = 0$ for $1 \le k \le m$, and $P_c = O$.

On the other hand, if $P_c = O$, then $c_i = 0$ $(1 \le i \le m)$ satisfy the equations in Lemma 7.5 in [FOW]. Since that the c_i $(1 \le i \le m)$ satisfying the equations in Lemma 7.5 in [FOW] are unique (compare [TZ, Lemma 2.2] and the Remark on p. 93 of [WZ]), we see that the vector field X in Proposition 5.3 of [FOW] must be trivial. Then from the proof of [FOW,Theorem 1.1] we see that S admits a Sasaki-Einstein metric, and the Sasaki-Futaki invariant of S vanishes. $\qquad\square$

Proposition 3.5. *Suppose $R(S) < 1$. Let $Q := -\frac{R(S)}{1-R(S)}P_c$, then $Q \in \partial\overline{\Sigma}$. More precisely Q lies on the same faces of $\overline{\Sigma}$ as the point y_∞ does, that is,*

$$
\begin{aligned}
l'_a(Q) &= 0, & a &= 1, \cdots, d_0, \\
l'_a(Q) &> 0, & a &= d_0 + 1, \cdots, d.
\end{aligned}
$$

Consequently in this case $P_c \ne O$.

Proof. Using (3.3), (3.2) and (3.1) we get

$$
\begin{aligned}
l'_a(-\tfrac{t}{1-t}P_c) &= \tfrac{1}{vol(\Sigma)}\int_{\mathbb{R}^m}\langle Du^0, \lambda'_a\rangle e^{-(2m+2)w_t}dx + \tfrac{1}{m+1} \\
&= \tfrac{1}{vol(\Sigma)}\int_{\mathbb{R}^m}(\langle Du^0, \lambda'_a\rangle + \tfrac{1}{m+1})e^{-(2m+2)w_t}dx \ge 0.
\end{aligned}
$$

Since $R(S) < 1$, we can let $t \to R(S)$ and get that

$$l'_a(-\frac{R(S)}{1-R(S)}P_c) \geq 0, \quad a = 1, \cdots, d.$$

Now the rest of the arguments is almost the same as in the proof of Proposition 4 in [L], using Propositions 3.1, 2), and Proposition 3.3, with the Claim 1 in [L] replaced by the Claim below.

Claim (Compare p. 56 of [D]) The derivative of the function $s_a(x) := \log(l'_a(Du^0(x)))$ is bounded on $\mathbb{R}^m$.
Proof of Claim. We compute

$$Ds_a(x) = \frac{D^2u^0(x)\lambda'_a}{l'_a(Du^0(x))} = \frac{(D^2G_0(v))^{-1}\lambda'_a}{l'_a(v)},$$

where $v = Du^0(x)$, and $G_0(v)$ is the Legendre transform of the Kähler potential $u^0(x)$, and is the symplectic potential of the the Kähler manifold
$(Orb_{C(S)}(H^c, p), \frac{1}{2}d\eta|_{Orb_{C(S)}(H^c,p)})$.

By computing the Hessian $D^2G_0(v)$ using formula (82) of [FOW] one sees that as one approaches the $(m-1)$-dimensional face $l'_a(v) = 0$ of $\overline{\Sigma}$ from the interior, the positive definite matrix $(D^2G_0(v))^{-1}$ will tend to be degenerate, and will acquire a kernel that is generated by the normal λ'_a when one reaches the face $l'_a(v) = 0$ at last. (Compare for example [A] and the proof of Fact 3 in Section 2 of [Hu1].) So $\frac{(D^2G_0(v))^{-1}\lambda'_a}{l'_a(v)}$ can be extended to a continuous function on the closure $\overline{\Sigma}$. □

Combining (3.5) with Proposition 3.5 we get that $R(S) = R_\Sigma$. Now we see that if $P_c \neq O$, then $R_\Sigma < 1$ by definition of R_Σ (see (3.4)) and the compactness of $\overline{\Sigma}$, and $R(S) < 1$. If $P_c = O$, then $R_\Sigma = 1$ by definition and the fact $O \in \Sigma$, and $R(S) = 1$.

Now Theorem 1.1 is proved.

Remark 1. The statement that $R(S) < 1$ implies that $P_c \neq O$ can also be proved as follows: If $P_c = O$, then by [FOW] (see the proof of Proposition 3.4 here) there is a Sasaki-Einstein metric on S, which implies $R(S) = 1$. So that $R(S) < 1$ implies that $P_c \neq O$.

Remark 2. In the proof of Proposition 4 in [L], Li uses the fact that in his situation, $P_c \neq O$ implies $R(X_\triangle) < 1$, although he does not state it explicitly. This fact can be easily deduced from the arguments in [L]; compare [Y] (and the above proof of Theorem 1.1). It can also be proved as follows: If $P_c \neq O$, then by [M1] the Futaki invariant of $X_\triangle$ does not vanish, and $X_\triangle$ cannot be K-semistable, so $R(X_\triangle) < 1$ by Corollary 1.1 of [MS].

In our situation, the fact that if $P_c \neq O$ then $R(S) < 1$ (which was proved above) can also be proved as follows: By Proposition 3.4, the Sasaki-Futaki invariant of S does not vanish. So by [BHLT] S can not be K-semistable. In a forthcoming

paper [Hu2] we'll show that for a $(2m + 1)$-dimensional compact (not necessarily toric) Sasaki manifold S with positive basic first Chern class and with $c_1(D) = 0$ ($D = \text{Ker } \eta$), $R(S) = 1$ implies that S is K-semistable. (The proof is along the lines of [MS], uses Sasaki-Ricci flow (cf. [C], [H]), and also uses [CS], [JZ] and [vC].) So that $P_c \neq O$ implies that $R(S) < 1$.

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School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,

Beijing 100875, P.R. China

E-mail address: hhuang@bnu.edu.cn