GROMOV–WITTEN THEORY OF $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$

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ABSTRACT. We compute the relative orbifold Gromov–Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$, with respect to vertical fibers. Via a vanishing property of the Hurwitz–Hodge bundle, 2-point rubber invariants are calculated explicitly using Pixton's formula for the double ramification cycle, and the orbifold quantum Riemann–Roch. As a result parallel to its crepant resolution counterpart for \mathcal{A}_n , the GW/DT/Hilb/Sym correspondence is established for $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$. The computation also implies the crepant resolution conjecture for relative orbifold Gromov–Witten theory of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$.

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1. Introduction

1.1. **Overview.** Upon its emergence, GW/DT correspondence has aroused plenty of interests in mathematical physics. The story begins with the technique of topological vertex invented in [4] to compute Gromov–Witten invariants for toric Calabi–Yau 3-folds, where generating functions for GW invariants are expressed as summation over partitions. As observed by [17, 13, 14], this combinatorial feature can be interpreted in terms of another enumerative theory — the Donaldson–Thomas theory. Lots of works have been done after this discovery, including the GW/DT correspondence for local curves [7, 16] and its generalization to $\mathcal{A}_n \times \mathbb{P}^1$ [12, 15]. Here \mathcal{A}_n is defined as the minimal resolution of the singular quotient $\mathbb{C}^2/\mathbb{Z}_{n+1}$, where the cyclic group

$$\mathbb{Z}_{n+1} := \mathbb{Z}/(n+1)\mathbb{Z} = \{\zeta \in \mathbb{C}|\zeta^{n+1} = 1\}$$

acts on \mathbb{C}^2 in the anti-diagonal manner:

$$\zeta \cdot (x, y) := (\zeta x, \zeta^{-1} y).$$

The resolution $\mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}$ is a crepant resolution, meaning that it preserves the canonical class. On the other hand, there is an obvious resolution of the same singularity in the category of orbifolds, the stacky quotient $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$. In the spirit of the crepant resolution conjecture [19, 6], one expects a GW/DT correspondence for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$, which should be closely related to that for $\mathcal{A}_n \times \mathbb{P}^1$.

1.2. Summary of results. Let $\mathcal{X} := [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$ be our target, and $\mathcal{D} = \coprod_{i=1}^r [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \{z_i\}$ be a disjoint union of vertical fibers, where z_1, \dots, z_r are distinct points on \mathbb{P}^1 . A relative stable map from an orbifold nodal curve C to \mathcal{X} , relative to \mathcal{D} , is a map from C to a modified target $\mathcal{X}[k]$, for some k. $\mathcal{X}[k]$ is defined by gluing \mathcal{X} along \mathcal{D} with k copies of "bubbles", constructed by the projective completion of the normal bundle of \mathcal{D} in \mathcal{X} . The map $C \to \mathcal{X}[k]$ is required to be stable, and satisfy certain transversality conditions. For the precise definition and detailed discussions on orbifold relative GW theory, we refer the readers to [1].

Let m > 0 be a fixed integer. Consider the moduli space of such relative stable maps,

$$\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r),$$

where g is the genus of domains, $\gamma = (\gamma_1, \dots, \gamma_p)$ is a tuple of elements in \mathbb{Z}_{n+1} indicating the monodromies of non-relative marked points, and $\overline{\mu}^1, \dots, \overline{\mu}^r$ are \mathbb{Z}_{n+1} -weighted partitions of m. The partition $\overline{\mu}^i$ records the ramification profile of the stable map with the i-th divisor, where the decoration of each part remembers the monodromy of the corresponding relative marked point.

Let T be the 2-dimensional torus acting on the fiber. Note that \mathcal{X} , and hence the moduli space, are noncompact, but admit a T-action with compact fixed loci. The moduli space of relative stable maps is equipped with a T-equivariant perfect obstruction theory. Hence by T-localization one can define the GW invariants, written as correlation functions

$$\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{g,\gamma}^{\mathcal{X},\circ}.$$

Here we do not allow contracted connected components, and the circle here means the connected theory. One can define the generating function

$$Z'_{\mathrm{GW}}(\mathcal{X})^{\circ,\sim}_{\overline{\mu},\overline{\nu}}:=\sum_{g>0}\sum_{\gamma}(-1)^gz^{2g}\frac{x^{\gamma}}{\gamma!}\langle\overline{\mu},\overline{\nu}\rangle^{\mathcal{X},\circ,\sim}_{g,\gamma}.$$

Another equivalent way to think about this is to identify the moduli with that of relative stable maps to $\mathcal{Y} := B\mathbb{Z}_{n+1} \times \mathbb{P}^1$, and consider the GW theory twisted by the obstruction bundle associated with the normal bundle of $\mathcal{Y} \subset \mathcal{X}$.

By the orbifold GW degeneration formula [1], the computation of r-point functions reduces to that of 3-point functions. If we assume the generation conjecture (see Section 6), one can further reduce it to the case when one of the three partitions is of the form $(1,0)^m, (2,0)(1,0)^{m-2}$, or $(1,k)(1,0)^{m-1}$. In this case, one can reduce the 3-point functions to 2-point rubber invariants by a rigidification argument. It turns out that the obstruction bundle is only nontrivial on certain simple strata of the moduli, and an application of Pixton's formula for the double ramification cycle leads to an explicit formula for $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$. We are able to compare the result with Maulik [12] after a change of variables.

Let S be an smooth orbifold surface. Denote by \mathcal{F}_S the vector space spanned by $H^*_{\text{orb}}(S)$ -weighted partitions of m, which we call the Fock space.

Theorem 1.1 (GW Crepant resolution). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m, \qquad (2,0)(1,0)^{m-2}, \qquad or \qquad (1,k)(1,0)^{m-1}, k \neq 0$$

let $\vec{\mu}, \vec{\nu}, \vec{\rho} \in \mathcal{F}_{\mathcal{A}_n}$ be their correspondents. We have

$$Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu}, \vec{\nu}, \vec{\rho}} = Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1)_{\overline{\mu}, \overline{\nu}, \overline{\rho}},$$

under the change of variables

$$s_j = \zeta \exp\left(\frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a\right), \qquad 1 \le j \le n,$$

where $\zeta = e^{\frac{2\pi\sqrt{-1}}{n+1}}$.

Here $\vec{\mu}$, $\vec{\nu}$ and $\overline{\mu}$, $\overline{\nu}$ are identified via the explicit isomorphism

$$\Phi: H^*_{\mathrm{orb}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]) \cong H^*(\mathcal{A}_n),$$

(1.1)
$$e_0 \mapsto 1, \qquad e_i \mapsto \frac{\zeta^{i/2} - \zeta^{-i/2}}{n+1} \sum_{j=1}^n \zeta^{ij} \omega_j, \qquad 1 \le i \le n,$$

where $\omega_1, \dots, \omega_n \in H^2(\mathcal{A}_n, \mathbb{Q})$ is the dual basis to the exceptional curves in \mathcal{A}_n .

As a byproduct, we observe that the moduli space of genus-0 stable maps to $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ shares a common open substack with the moduli of relative stable maps to \mathcal{X} . Moreover, the obstruction bundles coincide and vanish outside of this open substack. Thus our computation also leads to a formula for the orbifold quantum cohomology of the symmetric product $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$.

Theorem 1.2 (GW/Sym correspondence). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1}, k \neq 0,$

we have

$$z^{l(\mu)+l(\nu)+l(\rho)-m}Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=\langle\overline{\mu},\overline{\nu},\overline{\rho}\rangle_{\mathrm{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])},$$

where the right hand side is the 3-point genus-zero orbifold GW invariants of $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$.

In [22], the first named author proved the crepant resolution conjecture for relative DT invariants of \mathcal{X} , via a further DT/Hilb correspondence to the quantum cohomology of Hilb^m([$\mathbb{C}^2/\mathbb{Z}_{n+1}$]). Combining Theorem 1.1 and these results with the GW/DT correspondence for $\mathcal{A}_n \times \mathbb{P}^1$, we obtain the following.

Theorem 1.3 (GW/DT correspondence). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1}, k \neq 0,$

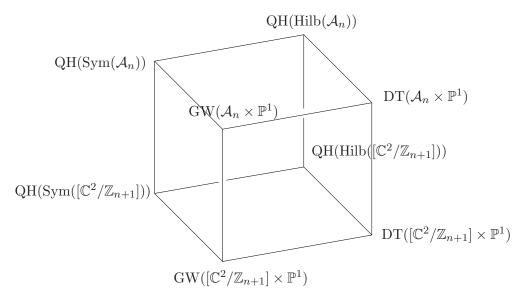
we have

$$(-iz)^{l(\mu)+l(\nu)+l(\rho)-m}Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=(-1)^mZ'_{\mathrm{DT}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}},$$

under the change of variables

$$Q = q_0 q_1 \cdots q_n = -e^{iz}, \qquad q_j = \zeta \exp\left(\frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a\right), \qquad 1 \le j \le n$$

In conclusion, we obtain a GW/DT/Hilb/Sym correspondence on the $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$ level, which can be viewed as a crepant resolution/transformation correspondent to its parallel picture on the \mathcal{A}_n level. The relationship among these theories can be summarized in the following diagram.



The paper is organized as follows. In Section 2 and 3, we explain in detail the definition of relative GW invariants, and how one can reduce the calculation of 3-point functions with one divisor insertion to 2-point rubber invariants. In Section 4, we prove the vanishing property of the obstruction bundle, and use Pixton's formula for the double ramification cycle to calculate the 2-point rubber invariants. Following the calculation of J. Zhou in [21], we apply the change of variables and prove Theorem 1.1. In Section 5, we discuss the orbifold quantum cohomology of symmetric products and obtain the GW/Sym correspondence. Finally, Section 6 is a summary of all existing results, where we prove the GW/DT correspondence for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$.

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2. Geometry

2.1. Geometry of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$. Fix an integer $n \geq 0$. Let \mathbb{P}^1 be the projective line and $\mathcal{O}_{\mathbb{P}^1}$ be the trivial line bundle on it. Let \mathcal{Y} be the trivial \mathbb{Z}_{n+1} -gerbe over \mathbb{P}^1 , coming from the root construction [2, 8] of order n+1 on $\mathcal{O}_{\mathbb{P}^1}$. In other words, \mathcal{Y} is defined by the following Cartesian diagram.

$$\mathcal{Y} \longrightarrow B\mathbb{C}^*$$

$$\downarrow \qquad \qquad \downarrow \qquad \lambda \mapsto \lambda^{n+1}$$

$$\mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}} B\mathbb{C}^*.$$

There is an orbifold line bundle L on \mathcal{Y} associated with the top map in the above diagram. The degree of L is zero, but there is a nontrivial action by \mathbb{Z}_{n+1} on the fibers of L, for which the generator acts by multiplication with $\zeta := e^{\frac{2\pi\sqrt{-1}}{n+1}}$. We will be interested in the relative GW theory of the total space $L \oplus L^{-1} \to \mathcal{Y}$, which is isomorphic to

$$\mathcal{X} := [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$$

where the generator $\zeta \in \mathbb{Z}_{n+1}$ acts on $(x,y) \in \mathbb{C}^2$ by $(\zeta x, \zeta^{-1}y)$.

2.2. Moduli space of stable maps. Let m be a positive integer. A \mathbb{Z}_{n+1} -weighted partition of m

$$\overline{\mu} = \{(\mu_1, k_1), \cdots, (\mu_{l(\mu)}, k_{l(\mu)})\}$$

means the following: $\mu := \{\mu_1, \dots, \mu_{l(\mu)}\}$ is an ordinary partition of m, and each part is decorated with an element $k_i \in \mathbb{Z}_{n+1}, i = 1, \dots, l(\mu)$.

Define the subset $A'(\bar{\mu})$ and $A''(\bar{\mu})$, such that

$$A'(\overline{\mu}) \sqcup A''(\overline{\mu}) = \{1, \cdots, l(\overline{\mu})\},\$$

where $k_i = 0$ if and only if $i \in A'(\overline{\mu})$. Denote $l'(\overline{\mu}) = |A'(\overline{\mu})|$ and $l''(\overline{\mu}) = |A''(\overline{\mu})|$.

For any $\overline{\mu}$, we use the notation $-\overline{\mu}$ to denote $\{(\mu_1, -k_1), \cdots, (\mu_{l(\mu)}, -k_{l(\mu)})\}$. Let $\gamma = (\gamma_1, \cdots, \gamma_p)$ be a vector of nontrivial elements in \mathbb{Z}_{n+1} , with $l(\gamma) = p$.

Let z_1, \dots, z_r be r points on \mathcal{Y} and $\overline{\mu}^1, \dots, \overline{\mu}^r$ be \mathbb{Z}_{n+1} -weighted partitions of m. Define

$$\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu}^1,\cdots,\overline{\mu}^r)$$

to be the moduli space of relative stable maps to $(\mathcal{Y}, z_1, \dots, z_r)$ with ramification profiles $\overline{\mu}^1, \dots, \overline{\mu}^r$. A general point $[f] \in \overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y}, \overline{\mu}^1, \dots, \overline{\mu}^r)$ may not map the domain curve to \mathcal{Y} but to \mathcal{Y} attached with a chain of r copies of \mathcal{Y} at z_1, \dots, z_r . For the precise definition of orbifold relative stable maps, we refer to [1].

There are p non-relative marked points x_1, \dots, x_p on the domain curve, with monodromies $\gamma_1, \dots, \gamma_p \in \mathbb{Z}_{n+1}$. For each $\overline{\mu}^i = \{(\mu_1^i, k_1^i), \dots, (\mu_{l(\mu^i)}^i, k_{l(\mu^i)}^i)\}$, there are $l(\mu^i)$ relative marked points on the domain curve, with monodromies $k_1^i, \dots, k_{l(\mu^i)}^i \in \mathbb{Z}_{n+1}$.

In order for the moduli space $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu}^1,\cdots,\overline{\mu}^r)$ to be non-empty, we must have the condition

$$\sum_{i=1}^{p} \gamma_i + \sum_{i=1}^{r} \sum_{j=1}^{l(\mu^i)} k_j^i = 0 \in \mathbb{Z}_{n+1},$$

or the following parity condition

$$\sum_{i=1}^{p} \gamma_i + \sum_{i=1}^{r} \sum_{j=1}^{l(\mu^i)} k_j^i = 0 \mod(n+1),$$

if we identify \mathbb{Z}_{n+1} with the set $\{0, \dots, n\}$.

The virtual dimension of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu}^1,\cdots,\overline{\mu}^r)$ is equal to

$$\operatorname{vdim}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu}^1,\cdots,\overline{\mu}^r)) := 2g - 2 + p + \sum_{i=1}^r l(\overline{\mu}^i) - (r-2)m.$$

We will also consider the disconnected version $\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu}^{1},\cdots,\overline{\mu}^{r})$, where the domain curve C is allowed to be disconnected, and $\chi:=2(h^{0}(\mathcal{O}_{C})-h^{1}(\mathcal{O}_{C}))$.

2.3. **Torus action.** Let $T = (\mathbb{C}^*)^2$ be the 2-dimensional algebraic torus. Consider the standard action of T on \mathbb{C}^2 , with T-characters $(n+1)t_1$, $(n+1)t_2$, which induces a T-action on $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$ with characters t_1 , t_2 . Identifying $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$ with the total space of $L \oplus L^{-1} \to \mathcal{Y}$, we have a T-action on the fibers of $L \oplus L^{-1} \to \mathcal{Y}$ via characters t_1 , t_2 .

Let $\overline{\mu}$, $\overline{\nu}$ be two \mathbb{Z}_{n+1} -weighted partitions of m. Recall that we have defined the moduli space of relative stable maps $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})$ and $\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu})$. If we choose the two relative points $z_1,z_2\in\mathcal{Y}$ to be $0,\infty$, then we have a \mathbb{C}^* -action on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})$ and $\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu})$ induced by the standard \mathbb{C}^* -action on \mathcal{Y} . Define the quotient spaces $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})$ $\|\mathbb{C}^*$ and $\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu})$ $\|\mathbb{C}^*$ as

$$\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu}) /\!\!/ \mathbb{C}^* := \left(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu}) \middle\backslash \overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})^{\mathbb{C}^*}\right) \middle/ \mathbb{C}^*$$

and

$$\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu}) /\!\!/ \mathbb{C}^* := \left(\overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu}) \middle\backslash \overline{\mathcal{M}}_{\chi,\gamma}^{\bullet}(\mathcal{Y},\overline{\mu},\overline{\nu})^{\mathbb{C}^*} \right) \big/ \mathbb{C}^*$$

respectively. These are the moduli spaces of relative stable maps to the nonrigid \mathcal{Y} , which by convention, are called *rubber* moduli spaces.

2.4. **Obstruction bundle.** Let $\pi: \mathcal{U} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y}, \overline{\mu}^1, \cdots, \overline{\mu}^r)$ be the universal domain curve and \mathcal{T} be the universal target. There is a universal map $F: \mathcal{U} \to \mathcal{T}$ and a contraction map $\tilde{\pi}: \mathcal{T} \to \mathcal{Y}$. Define

$$V_1 = R^1 \pi_* \tilde{F}^* L, \qquad V_2 = R^1 \pi_* \tilde{F}^* L^{-1}$$

where $\tilde{F} = \tilde{\pi} \circ F : \mathcal{U} \to \mathcal{Y}$. The rank of V_1 is

$$g-1+\sum_{i=1}^{p}\frac{\gamma_i}{n+1}+\sum_{i=1}^{r}\sum_{j=1}^{l(\mu^i)}\frac{k_j^i}{n+1}+\delta,$$

where we identify \mathbb{Z}_{n+1} with the set $\{0, \dots, n\}$ and δ is defined to be

$$\delta = \begin{cases} 1, & \text{if all monodromies on the domain curve are trivial,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the rank of V_2 is

$$g - 1 + \sum_{i=1}^{p} \frac{n+1-\gamma_i}{n+1} + \sum_{i=1}^{r} \sum_{j=1}^{l(\mu^i)} \frac{n+1-k_j^i - (n+1)\delta_{0,k_j^i}}{n+1} + \delta,$$

where

$$\delta_{0,x} = \left\{ \begin{array}{l} 1, & x = 0, \\ 0, & x \neq 0. \end{array} \right.$$

So the rank of the bundle $V := V_1 \oplus V_2$ is equal to

$$\operatorname{rk}(V) = 2g - 2 + p + \sum_{i=1}^{r} l''(\overline{\mu}^i) + 2\delta.$$

- 3. Relative Gromov–Witten theory of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$
- 3.1. Relative GW invariants. Recall that $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$, $\mathcal{Y} = B\mathbb{Z}_{n+1} \times \mathbb{P}^1$. Let z_1, \dots, z_r be r points on \mathcal{Y} and $\overline{\mu}^1, \dots, \overline{\mu}^r$ be \mathbb{Z}_{n+1} -weighted partitions of m. We are interested in the GW theory of \mathcal{X} , relative to the r fibers $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \{z_i\}$.

Let $\gamma = (\gamma_1, \dots, \gamma_p)$ be a vector of nontrivial elements in \mathbb{Z}_{n+1} , with $l(\gamma) = p$. Define

$$\overline{\mathcal{M}}_{q,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r)$$

to be the moduli space of relative stable maps to $(\mathcal{X}, [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \{z_1\}, \cdots, [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \{z_r\})$, with ramification profiles $\overline{\mu}^1, \cdots, \overline{\mu}^r$, and p non-relative marked points x_1, \cdots, x_p on the domain curve with monodromies $\gamma_1, \cdots, \gamma_p \in \mathbb{Z}_{n+1}$. The moduli space $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X}, \overline{\mu}^1, \cdots, \overline{\mu}^r)$ is noncompact.

Recall that we have a T-action on the fibers of \mathcal{X} with weights t_1, t_2 , which induces a T-action on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r)$. The fixed loci of this action is

$$\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r)^T = \overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu}^1,\cdots,\overline{\mu}^r).$$

Therefore, although $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r)$ is noncompact, the fixed loci $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{X},\overline{\mu}^1,\cdots,\overline{\mu}^r)^T$ is compact. The relative GW invariants can be defined T-equivariantly as

$$\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{g, \gamma}^{\mathcal{X}, \circ} := \frac{1}{|\operatorname{Aut}(\mu^1)| \cdots |\operatorname{Aut}(\mu^r)|} \int_{[\overline{\mathcal{M}}_{g, \gamma}(\mathcal{X}, \overline{\mu}^1, \cdots, \overline{\mu}^r)^T]^{\operatorname{vir}}} \frac{1}{e_T(N^{\operatorname{vir}})},$$

where N^{vir} is the virtual normal bundle and $e_T(-)$ is the T-equivariant Euler class. Here the factors $|\text{Aut}(\mu^i)|$ come from the convention that we treat the relative marked points as *unordered*.

In other words,

$$\langle \overline{\mu}^{1}, \cdots, \overline{\mu}^{r} \rangle_{g, \gamma}^{\mathcal{X}, \circ} = \frac{1}{|\operatorname{Aut}(\mu^{1})| \cdots |\operatorname{Aut}(\mu^{r})|} \int_{\left[\overline{\mathcal{M}}_{g, \gamma}(\mathcal{Y}, \overline{\mu}^{1}, \cdots, \overline{\mu}^{r})\right]^{\operatorname{vir}}} \frac{e_{T}(R^{1} \pi_{*} \tilde{F}^{*}(L \oplus L^{-1}))}{e_{T}(R^{0} \pi_{*} \tilde{F}^{*}(L \oplus L^{-1}))} \\
= \frac{1}{|\operatorname{Aut}(\mu^{1})| \cdots |\operatorname{Aut}(\mu^{r})|} \int_{\left[\overline{\mathcal{M}}_{g, \gamma}(\mathcal{Y}, \overline{\mu}^{1}, \cdots, \overline{\mu}^{r})\right]^{\operatorname{vir}}} \frac{e_{T}(R^{0} \pi_{*} \tilde{F}^{*}(L \oplus L^{-1}))}{e_{T}(R^{0} \pi_{*} \tilde{F}^{*}(L \oplus L^{-1}))},$$

where V is the obstruction bundle defined in the last section. The rank of $R^0\pi_*\tilde{F}^*(L\oplus L^{-1})$ is equal to 2δ and $e_T(R^0\pi_*\tilde{F}^*(L\oplus L^{-1}))=(t_1t_2)^{\delta}$.

We are also interested in the case when the target is nonrigid. The rubber invariants $\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{g,\gamma}^{\mathcal{X},\circ,\backsim}$ can be similarly defined:

$$\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{g, \gamma}^{\mathcal{X}, \circ, \sim} := \frac{1}{|\operatorname{Aut}(\mu^1)| \cdots |\operatorname{Aut}(\mu^r)|} \int_{\left[\overline{\mathcal{M}}_{g, \gamma}(\mathcal{Y}, \overline{\mu}^1, \cdots, \overline{\mu}^r) /\!/ \mathbb{C}^*\right]^{\operatorname{vir}}} \frac{e_T(V)}{e_T(R^0 \pi_* \tilde{F}^* (L \oplus L^{-1}))}.$$

Here we abuse the notations V, π and \tilde{F} for their counterparts in the nonrigid case. In later sections, invariants with r relative insertions will be called r-point functions.

The notations

$$\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{\chi, \gamma}^{\chi, \bullet}, \qquad \langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{\chi, \gamma}^{\chi, \bullet, \sim}$$

will denote the disconnected r-point correlation functions, where the domain curve C is allowed to be disconnected, and $\chi := 2(h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C))$.

3.2. 2-point Rubber invariants and (t_1+t_2) -divisibility. In this subsection, we consider rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$, where the two relative points on \mathcal{Y} for $\overline{\mu}$ and $\overline{\nu}$ are 0 and ∞ . Recall the definition

$$\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} = \frac{1}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|} \int_{\left[\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})/\!\!/\mathbb{C}^*\right]^{\mathrm{vir}}} \frac{e_T(V)}{e_T(R^0\pi_*\tilde{F}^*(L \oplus L^{-1}))}.$$

The virtual dimension d of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu}) /\!\!/ \mathbb{C}^*$ is equal to

$$vdim = 2g - 3 + p + l(\overline{\mu}) + l(\overline{\nu}).$$

On the other hand, the rank of the obstruction bundle $V = V_1 \oplus V_2$ is

$$\operatorname{rk}(V) = 2g - 2 + p + l''(\overline{\mu}) + l''(\overline{\nu}) + 2\delta.$$

Recall that in orbifold GW theory, the concept of Hodge bundle is generalized to the so-called Hurwitz-Hodge bundle. For each character $\chi: \mathbb{Z}_{n+1} \to \mathbb{C}^*$, there is an associated Hurwitz-Hodge bundle \mathbb{E}_{χ} . Let $\lambda_i^{\chi} = c_i(\mathbb{E}_{\chi})$ be the *i*-th Chern class of the Hurwitz-Hodge bundle \mathbb{E}_{χ} , called the Hurwitz-Hodge class. In our case, the vector bundles V_1 and V_2 are dual to the Hurwitz-Hodge bundles \mathbb{E}_U and $\mathbb{E}_{U^{\vee}}$, where U and U^{\vee} denote respectively the fundamental representation of \mathbb{Z}_{n+1} and its dual. Let r_1 and r_2 be the rank of V_1 and V_2 respectively. Then we have

$$e_T(V_1) = t_1^{r_1} - t_1^{r_1 - 1} \lambda_1^U + \dots + (-1)^{r_1} \lambda_{r_1}^U$$

$$e_T(V_2) = t_2^{r_2} - t_2^{r_2 - 1} \lambda_1^{U^{\vee}} + \dots + (-1)^{r_2} \lambda_{r_2}^{U^{\vee}}.$$

There is an orbifold version of the Mumford relation:

$$e_T(V_1)e_T(V_2)\Big|_{t_1+t_2=0} = t_1^{r_1}t_2^{r_2}.$$

In particular,

$$\lambda_{r_1}^U \lambda_{r_2}^{U^{\vee}} = 0, \quad \text{if } r_1 + r_2 > 0.$$

Lemma 3.1. The rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ vanish unless either

$$\operatorname{rk}(V) = \operatorname{vdim} = 0, \quad or \quad \operatorname{rk}(V) = \operatorname{vdim} + 1 > 0.$$

In particular, the latter case happens only if $\delta = 1$ or $\delta = 0$ and $l(\overline{\mu}) = l''(\overline{\mu})$, $l(\overline{\nu}) = l''(\overline{\nu})$.

Proof. If $\delta = 1$, the lemma holds by computations in [7]. We concentrate in the case $\delta = 0$. By dimensional reason, for the integral not to vanish, one must have $\operatorname{rk}(V) \geq \operatorname{vdim}$. On the other hand, direct comparison shows that $\operatorname{rk}(V) \leq \operatorname{vdim} + 1$. It suffices to show that the invariants vanish when $\operatorname{rk}(V) = \operatorname{vdim} > 0$, which follows from $\lambda_{r_1}^U \lambda_{r_2}^{U^{\vee}} = 0$ and dimension counting in the integral.

We now analyze the rubber invariants in different contexts.

- a) $\delta = 1$, i.e. all monodromies around loops on the domain curve are zero. In other words, $p = l''(\overline{\mu}) = l''(\overline{\nu}) = 0$, and rk(V) = 2g. The invariants simply reduce to the smooth case in [7].
- b) $\delta = 0$ and rk(V) = vdim = 0. There are only several possibilities in this case and one can compute the invariants directly by naive counting.

•
$$g = p = 0$$
, $l(\overline{\mu}) = 2$, $l''(\overline{\mu}) = l''(\overline{\nu}) = l(\overline{\nu}) = 1$.

$$\langle \overline{\mu}, \overline{\nu} \rangle_{0,\emptyset}^{\mathcal{X}, \circ, \sim} = \frac{1}{(n+1)|\mathrm{Aut}(\mu)|}.$$

•
$$g = p = 0$$
, $l(\overline{\mu}) = l''(\overline{\mu}) = 2$, $l(\overline{\nu}) = 1$, $l''(\overline{\nu}) = 0$.

$$\langle \overline{\mu}, \overline{\nu} \rangle_{0,\emptyset}^{\mathcal{X},\circ,\sim} = \frac{1}{(n+1)|\mathrm{Aut}(\mu)|}.$$

• $g = 0, p = 1, l(\overline{\mu}) = 1, l''(\overline{\mu}) = 0, l(\overline{\nu}) = l''(\overline{\nu}) = 1.$

$$\langle \overline{\mu}, \overline{\nu} \rangle_{0,\gamma}^{\mathcal{X},\circ,\sim} = \frac{1}{n+1}.$$

c) $\delta=0$ and $\mathrm{rk}(V)=\mathrm{vdim}+1>0$, in which case $l(\overline{\mu})=l''(\overline{\mu}),\ l(\overline{\nu})=l''(\overline{\nu})$, i.e. all monodromies are nontrivial. Mumford's relation $\lambda_{r_1-1}^U\lambda_{r_2}^{U^\vee}=\lambda_{r_1}^U\lambda_{r_2-1}^{U^\vee}$ implies

$$\begin{split} \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} &= \frac{(-1)^{r_1+r_2-1}}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|} \int_{\left[\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})/\!\!/\mathbb{C}^*\right]^{\mathrm{vir}}} \left(t_1 \lambda_{r_1-1}^U \lambda_{r_2}^{U^{\vee}} + t_2 \lambda_{r_1}^U \lambda_{r_2-1}^{U^{\vee}} \right) \\ &= \frac{(-1)^{r_1+r_2-1} (t_1+t_2)}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|} \int_{\left[\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu})/\!\!/\mathbb{C}^*\right]^{\mathrm{vir}}} \lambda_{r_1}^U \lambda_{r_2-1}^{U^{\vee}}. \end{split}$$

In particular, it is divisible by $(t_1 + t_2)$. This is the main case we will treat in the following sections.

The argument for the $(t_1 + t_2)$ -divisibility is valid in more general contexts. We summarize this feature in the following lemma, whose proof is exactly the same as above.

Lemma 3.2. If $\operatorname{rk}(V) > 0$, then the invariants $\langle \overline{\mu}^1, \dots, \overline{\mu}^r \rangle_{g,\gamma}^{\mathcal{X},\circ}$ and $\langle \overline{\mu}^1, \dots, \overline{\mu}^r \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ are divisible by $(t_1 + t_2)$.

3.3. 3-point functions. According to the degeneration formula [1], r-point functions $\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{g,\gamma}^{\mathcal{X},\circ}$ can be determined by 3-point functions. Hence we are particularly interested in the case r=3. Moreover, under the generation conjecture (see Section 6.3), it suffices to consider the following three special cases.

Let $\overline{\mu}, \overline{\nu}, \overline{\rho}$ be three \mathbb{Z}_{n+1} -weighted partitions of m. In the following three subsections, we will study the relative GW invariants

$$\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{q,\gamma}^{\mathcal{X},\circ}$$

for

$$\overline{\rho} = \{(1,0), \cdots, (1,0)\}, \quad \{(2,0), (1,0), \cdots, (1,0)\}, \quad \text{or} \quad \{(1,k), (1,0), \cdots, (1,0)\},$$

where $k \neq 0$. For simplicity, we abbreviate the notations as

$$(1,0)^m$$
, $(2,0)(1,0)^{m-2}$, $(1,k)(1,0)^{m-1}$.

In this subsection, apart from several exceptional cases, the 3-point functions above can be reduced to the rubber invariants of the previous section. For the exceptional cases, 3-point functions can be easily computed.

3.3.1. Case $\overline{\rho} = (1,0)^m$. By computations in Section 2.2, the virtual dimension of the moduli space $\overline{\mathcal{M}}_{q,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu},\overline{\rho})$ is

$$\operatorname{vdim}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu},\overline{\rho})) = 2g - 2 + p + l(\overline{\mu}) + l(\overline{\nu}).$$

On the other hand, by the computation in Section 2.4, the rank of the obstruction bundle $V = V_1 \oplus V_2$ is

$$\operatorname{rk}(V) = 2g - 2 + p + l''(\overline{\mu}) + l''(\overline{\nu}) + 2\delta.$$

We consider the two possibilities $\delta = 1$ and $\delta = 0$.

a) $\delta = 1$, i.e. all the monodromies around loops on the domain curve are trivial.

In this case
$$l''(\overline{\mu}) = l''(\overline{\nu}) = p = 0$$
. We have

$$\operatorname{rk}(V) \leq \operatorname{vdim}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{Y}, \overline{\mu}, \overline{\nu}, \overline{\rho})).$$

By dimensional reason, for the invariants to be nontrivial, the equality needs to hold, which happens only if $l(\overline{\mu}) = l(\overline{\nu}) = 1$, and hence $\operatorname{vdim} = \operatorname{rk}(V) = 2g$. Either by a $(t_1 + t_2)$ -divisibility argument or by the smooth case [7], the only nontrivial case is g = 0, and $\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{0,\emptyset}^{\mathcal{X}, \circ} = \frac{1}{m(n+1)t_1t_2}$.

The Crepant Resolution Conjecture in this case is easy to show. Let \mathcal{A}_n be the crepant resolution of $\mathbb{C}^2/\mathbb{Z}_{n+1}$. The torus $T=(\mathbb{C}^*)^2$ acts on \mathcal{A}_n with fixed points p_1, \dots, p_{n+1} . The tangent weights at the fixed point p_i are

$$w_i^- := (n+2-i)t_1 - (i-1)t_2, \qquad w_i^+ := (i-n-1)t_1 + it_2.$$

For $H^*(\mathcal{A}_n)$ -weighted partitions $\vec{\mu}^1, \dots, \vec{\mu}^r$ of m and $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$, one can define the T-equivariant relative GW invariants (see [12]) $\langle \vec{\mu}^1, \dots, \vec{\mu}^r \rangle_{g,(\beta,m)}^{\mathcal{A}_n \times \mathbb{P}^1}$. The Crepant Resolution Conjecture in this case is the following lemma:

Lemma 3.3. Let $\overline{\rho} = (1,0)^m, \overline{\mu} = \overline{\nu} = (m,0)$. Under the correspondence (1.1), we have

$$\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{0,\emptyset}^{\mathcal{X}, \circ} = \langle \overrightarrow{\mu}, \overrightarrow{\nu}, \overrightarrow{\rho} \rangle_{0,(0,m)}^{\mathcal{A}_n \times \mathbb{P}^1, \circ},$$

where $\vec{\mu}, \vec{\nu}, \vec{\rho}$ are the correspondents of $\overline{\mu}, \overline{\nu}, \overline{\rho}$.

Proof. The correspondents $\vec{\mu}, \vec{\nu}, \vec{\rho}$ have the same underlying partitions with those of $\overline{\mu}, \overline{\nu}, \overline{\rho}$, and their $H^*(\mathcal{A}_n)$ -weights are all equal to the identity. First we should notice that by Lemma 4.2 of [12] and by dimensional constraints, the only nontrivial invariant for such $\vec{\mu}, \vec{\nu}, \vec{\rho}$ is indeed when $\beta = 0, g = 0$. Therefore, if we consider the map from the domain curve to the \mathcal{A}_n direction, it is a constant map to one of the T-fixed points $p_1 \cdots, p_{n+1}$. Observe that $w_i^- + w_i^+ = t_1 + t_2$ and $w_i^+ = -w_{i+1}^-$ for any i. We have

$$\langle \vec{\mu}, \vec{\nu}, \vec{\rho} \rangle_{0,(0,m)}^{\mathcal{A}_n \times \mathbb{P}^1} = \frac{1}{m} \sum_{i=1}^{n+1} \frac{1}{w_i^- w_i^+} = \frac{1}{m(t_1 + t_2)} \sum_{i=1}^{n+1} \left(\frac{1}{w_i^-} + \frac{1}{w_i^+} \right) = \frac{1}{m(t_1 + t_2)} \left(\frac{1}{w_1^-} + \frac{1}{w_{n+1}^+} \right),$$
 which is $\frac{1}{m(n+1)t_1t_2} = \langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{0,\emptyset}^{\mathcal{X}, \circ}$.

b) $\delta = 0$. The rank of V is $\mathrm{rk}(V) = 2g - 2 + p + l''(\overline{\mu}) + l''(\overline{\nu})$. We also have

$$\operatorname{rk}(V) \leq \operatorname{vdim}(\overline{\mathcal{M}}_{q,\gamma}(\mathcal{Y},\overline{\mu},\overline{\nu},\overline{\rho})) = 2g - 2 + p + l(\overline{\mu}) + l(\overline{\nu}).$$

Again, dimension counting forces the equality to hold.

If vdim = rk(V) > 0, then by Lemma 3.2, the invariant is a polynomial in t_1 , t_2 divisible by $(t_1 + t_2)$, and hence has to be zero by dimensional constraints.

If vdim = rk(V) = 0, we must have p = 0, g = 0, and $l(\overline{\mu}) = l(\overline{\nu}) = l''(\overline{\mu}) = l''(\overline{\nu}) = 1$. The only nontrivial invariant is for $\overline{\rho} = (1,0)^m$, $\overline{\mu} = (m,k)$, $\overline{\nu} = (m,-k)$, $k \neq 0$, which is $\frac{1}{m(n+1)}$.

In general, this case will contribute to disconnected invariants with $\overline{\rho} = (1,0)^m$, $\overline{\mu} = -\overline{\nu}$, $l(\overline{\mu}) = l''(\overline{\mu})$. The partition function is

$$Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=Z_{\overline{\mu}}^{-1},$$

where $Z_{\overline{\mu}} = |\operatorname{Aut}(\overline{\mu})|(n+1)^{l(\overline{\mu})} \prod_{i=1}^{l(\overline{\mu})} \mu_i$. This matches the parallel DT partition function:

Lemma 3.4. For $\overline{\rho} = (1,0)^m, \overline{\mu} = -\overline{\nu}, l(\overline{\mu}) = l''(\overline{\mu}), \text{ we have }$

$$Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=Z'_{\mathrm{DT}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}.$$

3.3.2. Case $\overline{\rho} = (2,0)(1,0)^{m-2}$. In this case, we will reduce the relative GW invariants $\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{g,\gamma}^{\mathcal{X},\circ,\gamma}$ to the rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\gamma}$. The key point is that one can replace a nonrigid invariant with a rigid invariant by imposing the condition that a marked point on the domain curve lies on a fixed fiber of $L \oplus L^{-1} \to \mathcal{Y}$. Consider the following descendent 2-point relative invariants.

$$\langle \overline{\mu} | \tau_1[F] | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \langle \overline{\mu} | \tau_1(\mathbf{1}) | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} = (2g - 2 + p + l(\overline{\mu}) + l(\overline{\nu})) \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim},$$

where [F] is the fiber class and the second equality is the dilaton equation.

On the other hand, $\langle \overline{\mu}|\tau_1[F]|\overline{\nu}\rangle_{g,\gamma}^{\mathcal{X},\circ}$ can be computed by the degeneration formula. Let the base \mathcal{Y} degenerate into two components, such that the two relative marked points lie on one component and the fiber insertion lies on the other. Degeneration formula implies

$$\langle \overline{\mu} | \tau_1[F] | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \sum_{\overline{\eta},\gamma' \sqcup \gamma'' = \gamma,\Gamma_1,\Gamma_2} \langle \overline{\mu}, \overline{\nu}, \overline{\eta} \rangle_{\Gamma_1,\gamma'}^{\mathcal{X},\bullet} Z_{\overline{\eta}} \langle -\overline{\eta} | \tau_1[F] \rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet},$$

where $Z_{\overline{\eta}} = |\operatorname{Aut}(\overline{\eta})|(n+1)^{l(\eta)} \prod_{i=1}^{l(\eta)} \eta_i$, and the summation is over all domain curve configurations Γ_1 , Γ_2 , such that the glued curve over Γ_1 , Γ_2 is connected.

The second factor, which is a priori an integral over the moduli of relative stable maps with disconnected domains, can be written as that over a product of moduli spaces with connected domains. Each such moduli space is either of the form

$$\overline{\mathcal{M}}_{g_i,\gamma^i\sqcup(\mathbf{1})}(\mathcal{Y},-\overline{\eta}^i), \qquad \text{or} \qquad \overline{\mathcal{M}}_{g_i,\gamma^i}(\mathcal{Y},-\overline{\eta}^i),$$

depending on whether the insertion $\tau_1[F]$ is on the particular connected component or not.

The virtual dimensions of $\overline{\mathcal{M}}_{q_i,\gamma^i\sqcup(1)}(\mathcal{Y},-\overline{\eta}^i)$ and $\overline{\mathcal{M}}_{q_i,\gamma^i}(\mathcal{Y},-\overline{\eta}^i)$ are respectively

$$2q_i - 1 + p_i + m_i + l(\overline{\eta}^i), \qquad 2q_i - 2 + p_i + m_i + l(\overline{\eta}^i),$$

where $p_i, m_i, \overline{\eta}^i$ are the corresponding data associated to the component. The rank of the obstruction bundle V over both moduli spaces is equal to

$$rk(V) = 2g_i - 2 + p_i + l''(\overline{\eta}^i) + 2\delta.$$

a) $\delta = 0$. We have

$$\operatorname{vdim}(\overline{\mathcal{M}}_{g_i,\gamma^i\sqcup(\mathbf{1})}(\mathcal{Y},-\overline{\eta}^i))\geq \operatorname{rk}(V)+2, \qquad \operatorname{vdim}(\overline{\mathcal{M}}_{g_i,\gamma^i}(\mathcal{Y},-\overline{\eta}^i))>\operatorname{rk}(V).$$

The only nontrivial invariants come from the first type of components, and since $\deg \tau_1[F] = 2$, the equality holds, i.e. $m_i = 1$ and $l''(\overline{\eta}^i) = l(\overline{\eta}^i) = 1$. Now $\gamma^i \neq \emptyset$ in order for the sum of monodromies at all marked points to vanish $(l''(\overline{\eta}^i) = l(\overline{\eta}^i) = 1)$ implies the only relative marked point has nontrivial monodromy).

However if $g_i > 0$, together with $\gamma^i \neq \emptyset$ it would also imply $\operatorname{rk}(V) > 0$ and the invariant is divisible by $(t_1 + t_2)$, which forces it to vanish by dimensional reasons. In short, the only invariant that survives is when $g_i = 0$, and $p_i = l''(\overline{\eta}^i) = m_i = 1$; so $-\overline{\eta}^i = (1, k), k \neq 0$. The restriction of γ'' on this component is (k), and hence $\gamma' = \gamma \setminus (k)$. The contribution of this component to the invariants $\langle -\overline{\eta}|\tau_1[F]\rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet}$ is $\frac{1}{n+1}$.

b) $\delta = 1$. Counting dimensions, for $\overline{\mathcal{M}}_{g_i,\gamma^i\sqcup(1)}(\mathcal{Y}, -\overline{\eta}^i)$, we must have $m_i = 2$, $p_i = 0$ $l(\overline{\eta}^i) = 1$, and hence $\overline{\eta}^i = (2,0)$; for $\overline{\mathcal{M}}_{g_i,\gamma^i}(\mathcal{Y}, -\overline{\eta}^i)$, we must have $m_i = l(\overline{\eta}^i) = 1$, $p_i = 0$, and hence $\overline{\eta}^i = (1,0)$. The contribution of this component to the invariant $\langle -\overline{\eta}^i|\tau_1[F]\rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\circ}$ is $\frac{1}{2(n+1)}$.

Combining a) and b), for invariants $\langle -\overline{\eta}|\tau_1[F]\rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet}$, relative insertions that could appear in the gluing formula are $\overline{\eta}=(2,0)(1,0)^{m-2}$ or $\overline{\eta}=(1,k)(1,0)^{m-1}$, $k\neq 0$. Moreover, $\langle -\overline{\eta}|\tau_1[F]\rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet}$ are

exactly canceled by the gluing factor Z_{η} . So we have

$$\langle \overline{\mu}|\tau_1[F]|\overline{\nu}\rangle_{g,\gamma}^{\mathcal{X},\circ} = \langle \overline{\mu},\overline{\nu},(2,0)(1,0)^{m-2}\rangle_{g,\gamma}^{\mathcal{X},\circ} + \sum_{k=1}^n \langle \overline{\mu},\overline{\nu},(1,k)(1,0)^{m-1}\rangle_{g,\gamma\setminus(k)}^{\mathcal{X},\circ}.$$

Recall the rigidification result obtained at the beginning of this subsection, and we conclude that

$$\langle \overline{\mu}, \overline{\nu}, (2,0)(1,0)^{m-2} \rangle_{g,\gamma}^{\mathcal{X},\circ} + \sum_{k=1}^{n} \langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{g,\gamma \setminus (k)}^{\mathcal{X},\circ} = (2g-2+p+l(\overline{\mu})+l(\overline{\nu})) \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}.$$

3.3.3. Case $\overline{\rho}=(1,k)(1,0)^{m-1}$, $k\neq 0$. In this case, similar argument still works to reduce $\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{g,\gamma}^{\mathcal{X},\circ}$ to the rubber integral. First there is a similar rigidification argument

$$\langle \overline{\mu} | \tau_0[\iota_* k] | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \langle \overline{\mu} | \tau_0(k) | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} = \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma \sqcup (k)}^{\mathcal{X},\circ,\sim}$$

where $\iota: F \to \mathcal{X}$ is the inclusion of the fiber and we view k as a twisted sector of $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$.

On the other hand, we can still use the degeneration formula. Degenerate the base \mathcal{Y} into two components, such that the two relative marked points lie on one component and the fiber insertion lies on the other. We have the following degeneration formula

$$\langle \overline{\mu} | \tau_0[\iota_* k] | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \sum_{\overline{\eta},\gamma' \sqcup \gamma'' = \gamma,\Gamma_1,\Gamma_2} \langle \overline{\mu}, \overline{\nu}, \overline{\eta} \rangle_{\Gamma_1,\gamma'}^{\mathcal{X},\bullet} Z_{\overline{\eta}} \langle -\overline{\eta} | \tau_0[\iota_* k] \rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet},$$

where $Z_{\overline{\eta}} = |\operatorname{Aut}(\overline{\eta})|(n+1)^{l(\overline{\eta})} \prod_{i=1}^{l(\overline{\eta})} \eta_i$, and we are summing over all domain curve configurations Γ_1 , Γ_2 such that the glued curve over Γ_1 , Γ_2 is connected.

As before, the second factor is an integral over a product of moduli spaces of relative stable maps with connected domains, each of the form

$$\overline{\mathcal{M}}_{g_i,\gamma^i\sqcup(k)}(\mathcal{Y},-\overline{\eta}^i)$$
 or $\overline{\mathcal{M}}_{g_i,\gamma^i}(\mathcal{Y},-\overline{\eta}^i),$

depending on whether the insertion $\tau_0[\iota_*k]$ is on the particular connected component or not.

The virtual dimensions of $\overline{\mathcal{M}}_{g_i,\gamma^i\sqcup(k)}(\mathcal{Y},-\overline{\eta}^i)$ and $\overline{\mathcal{M}}_{g_i,\gamma^i}(\mathcal{Y},-\overline{\eta}^i)$ are respectively

$$2g_i - 1 + p_i + m_i + l(\overline{\eta}^i), \qquad 2g_i - 2 + p_i + m_i + l(\overline{\eta}^i),$$

where p_i , m_i , $\overline{\eta}^i$ are the corresponding data associated to the component. The ranks of the obstruction bundle V are respectively

$$2g_i - 1 + p_i + l''(\overline{\eta}^i) + 2\delta, \qquad 2g_i - 2 + p_i + l''(\overline{\eta}^i) + 2\delta.$$

a) $\delta = 1$, which only happens for the second type of components, since the first type already has a nontrivial marking k. To get nontrivial invariants, one must have $\operatorname{rk}(V) \geq \operatorname{vdim}$, which implies $m_i = l(\overline{\eta}^i) = 1$. Hence $\overline{\eta}^i = (1,0)$.

This also forces that $\operatorname{rk}(V) = \operatorname{vdim}$, and hence $\operatorname{rk}(V) = \operatorname{vdim} = 0$ since otherwise the invariants vanish by $(t_1 + t_2)$ -divisibility. Hence $g_i = p_i = 0$. The invariant $\langle -\overline{\eta} \rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\circ}$ contributed by the component is $\frac{1}{n+1}$.

b) $\delta = 0$. For the two types of components, we always have respectively

$$vdim > rk(V) + 1, vdim > rk(V).$$

Only the first type contributes nontrivially, and the equality must hold. However, in this case we must also need $\operatorname{rk}(V) = 0$, since otherwise the invariants will be divisible by $(t_1 + t_2)$ and therefore vanish by dimensional constraint. The only possibility is $g_i = p_i = 0$, $m_i = l(\overline{\eta}^i) = l''(\overline{\eta}^i) = 1$; so $\overline{\eta}^i = (1, k)$, $k \neq 0$. The invariant $\langle -\overline{\eta} | \tau_0[\iota_* k] \rangle_{\Gamma_2, \gamma''}^{\mathcal{X}, \circ}$ contributed by this component is $\frac{1}{n+1}$.

Combining a) and b), for invariants $\langle -\overline{\eta}|\tau_0[\iota_*k]\rangle_{\Gamma_2,\gamma''}^{\mathcal{X},\bullet}$, relative insertions that could appear are $\overline{\eta} = \overline{\rho} = (1,k)(1,0)^{m-1}, \ k \neq 0$, and we compute

$$\langle \overline{\mu} | \tau_0[\iota_* k] | \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{g,\gamma}^{\mathcal{X},\circ}.$$

Recall the rigidification result obtained at the beginning of this subsection, and we conclude that

$$\langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{g,\gamma}^{\mathcal{X}, \circ} = \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma \sqcup (k)}^{\mathcal{X}, \circ, \sim}.$$

4. Rubber invariants and Crepant Resolution Conjecture

Recall that in Section 3, we have reduced the relative GW invariants $\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{g,\gamma}^{\mathcal{X},\circ}$ for

$$\overline{\rho} = (1,0)^m$$
, $(2,0)(1,0)^{m-2}$, or $(1,k)(1,0)^{m-1}$,

to rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$. In this section, using the orbifold Grothendieck–Riemann–Roch calculation in [20], we will compute the rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$, under the main assumption

(†):
$$\delta = 0$$
, $\operatorname{rk}(V) = \operatorname{vdim} + 1 > 0$;

in particular, $l(\overline{\mu}) = l''(\overline{\mu}), \ l(\overline{\nu}) = l''(\overline{\nu}).$

4.1. Double ramification cycle and Pixton's formula. The double ramification cycle $DR_{g,N}$ in $\overline{\mathcal{M}}_{g,N}$ is defined as the pushforward of the virtual class under the forgetful map

$$\overline{\mathcal{M}}_{g,N-l(\mu)-l(\nu)}(\mathbb{P}^1,\mu,\nu)^{\sim} \to \overline{\mathcal{M}}_{g,N},$$

where μ , ν are ordinary partitions of m.

Consider the Cartesian diagram

$$\overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1} \times \mathbb{P}^1, \overline{\mu}, \overline{\nu})^{\sim} \xrightarrow{p} \overline{\mathcal{M}}_{g,\overline{\mu} \sqcup \overline{\nu} \sqcup \gamma}(B\mathbb{Z}_{n+1}) \\
\downarrow^{\pi} \\
\overline{\mathcal{M}}_{g,p}(\mathbb{P}^1, \mu, \nu)^{\sim} \xrightarrow{p} \overline{\mathcal{M}}_{g,l(\mu)+l(\nu)+p},$$

where $\overline{\mu} \sqcup \overline{\nu} \sqcup \gamma$ means the collection of all (relative and non-relative) marked points with their monodromies, and for simplicity we have abused the notations p and π . Our strategy is to push the virtual class of the rubber moduli space forward to the moduli of curve.

Lemma 4.1.

$$p_* \left[\overline{\mathcal{M}}_{g,\gamma} (B\mathbb{Z}_{n+1} \times \mathbb{P}^1, \overline{\mu}, \overline{\nu})^{\sim} \right]^{\text{vir}} = \pi^* \mathrm{DR}_{g,l(\mu)+l(\nu)+p}.$$

Proof. By definition of the double ramification cycle and the commutativity of π^* and p_* , it suffices to prove

$$\left[\overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1}\times\mathbb{P}^1,\overline{\mu},\overline{\nu})^{\sim}\right]^{\mathrm{vir}} = \pi^* \left[\overline{\mathcal{M}}_{g,p}(\mathbb{P}^1,\mu,\nu)^{\sim}\right]^{\mathrm{vir}}.$$

Hence by functoriality of virtual classes [5], it suffices to prove the perfect obstruction theory on $\overline{\mathcal{M}}_{a,v}(\mathbb{P}^1,\mu,\nu)^{\sim}$ pulls back to the upstairs.

We apply the perfect obstruction theory introduced in [1] for the moduli of relative stable maps. For a moduli point in $\overline{\mathcal{M}}_{g,p}(\mathbb{P}^1,\mu,\nu)^{\sim}$ represented by a relative stable map $f:C\to\mathbb{P}^1[k]$, one needs to add certain extra orbifold structures at nodes and relative divisors of $\mathbb{P}^1[k]$, and also on the domain, such that the resulting map $f':C'\to\mathbb{P}^1[k]'$ is transversal, in the sense of [1]. The obstruction theory is then given by the complex

$$\operatorname{Ext}^{\bullet}\left([(f')^*\Omega_{\mathbb{P}^1[k]'} \to \Omega_{C'}(\Sigma')], \mathcal{O}_{C'}\right),$$

where $\Sigma' \subset C'$ denotes the divisor of all non-relative markings.

Now let $g: \mathcal{C} \to B\mathbb{Z}_{n+1} \times \mathbb{P}^1[k]$ be a moduli point over f. The key observation is that the relative structure is only along the \mathbb{P}^1 direction. The extra orbifold structures are actually introduced by root constructions, which commutes with the base change by $B\mathbb{Z}_{n+1}$. Hence the cotangent complex $[(g')^*\Omega_{B\mathbb{Z}_{n+1}\times\mathbb{P}^1[k]'}\to\Omega_{\mathcal{C}'}(\Sigma')]$ is simply the pull-back of $[(f')^*\Omega_{\mathbb{P}^1[k]'}\to\Omega_{\mathcal{C}'}(\Sigma')]$, and so is the obstruction theory. The same argument works in families.

Now the rubber invariant $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ is equal to

$$\frac{(-1)^{r_1+r_2-1}(t_1+t_2)}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|}\int_{\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})}\pi^*\mathrm{DR}\cdot\lambda_{r_1}^U\lambda_{r_2-1}^{U^\vee}=\frac{1}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|}\int_{\left[\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}([\mathbb{C}^2/\mathbb{Z}_{n+1}])\right]^{\mathrm{vir}}}\pi^*\mathrm{DR}.$$

In order to compute the rubber invariant $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$, we first need to study the double ramification cycle DR. A combinatorial expression for DR is obtained in [11], known as Pixton's formula. This formula will be used to study $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ in this paper and we now give a brief description of it.

Let $G_{g,N}$ be the set of all genus g stable graphs with N leaves. To each $\Gamma \in G_{g,N}$, we associate the moduli space

$$\overline{\mathcal{M}}_{\Gamma} := \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v), n(v)}.$$

Then there is a map

$$\xi_{\Gamma}: \overline{\mathcal{M}}_{\Gamma} \to \overline{\mathcal{M}}_{q,N},$$

whose image is the closure of the boundary stratum associated with Γ .

Let r be a positive integer and $\Gamma \in G_{g,N}$. Fix a double ramification datum $A = (a_1, \dots, a_N)$, where $a_i \in \mathbb{Z}$ and $\sum_{i=1}^N a_i = 0$. A weighting mod r of Γ is a function

$$w: H(\Gamma) \to \{1, \cdots, r\},\$$

satisfying

(1) For each $h_i \in L(\Gamma)$ corresponding to the marking $i \in \{1, \dots, n\}$,

$$w(h_i) = a_i \mod r,$$

(2) For each $e \in E(\Gamma)$ corresponding to two half-edges $h, h' \in H(\Gamma)$,

$$w(h) + w(h') = 0 \mod r,$$

(3) For each $v \in V(\Gamma)$,

$$\sum_{h \text{ incident to } v} w(h) = 0 \mod r.$$

We denote by $W_{\Gamma,r}$ the set of all weightings mod r of Γ , and by $P_g^{d,r}(A)$ the degree d component of the tautological class

$$\sum_{\Gamma \in G_{g,N}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\operatorname{Aut}\Gamma|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \left(\prod_{i=1}^n \exp(a_i^2 \psi_i) \prod_{e=(h,h') \in E(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \right)$$

in $R^*(\overline{\mathcal{M}}_{g,n})$.

Pixton shows that for fixed g, A, and d, the class $P_g^{d,r}(A)$ is a polynomial in r. Denote by $P_g^d(A)$ the constant term of $P_g^{d,r}(A)$. The main result of [11] is the following theorem:

Theorem 4.2 ([11]). For $g \ge 0$ and double ramification data A,

$$\mathrm{DR}_g(A) = 2^{-g} P_g^d(A) \in R^g(\overline{\mathcal{M}}_{g,N}).$$

4.2. Vanishing property of the Hurwitz-Hodge classes.

Proposition 4.3. The Hurwitz-Hodge class $c_{r_1+r_2-1}(\mathbb{E}_U \oplus \mathbb{E}_{U^{\vee}})$ vanishes on the boundary strata of $\overline{\mathcal{M}}_{g,\overline{\mu},\overline{\nu},\gamma}(B\mathbb{Z}_{n+1})$, except those consisting of irreducible singular nodal curves with nontrivial monodromies at nodes.

Proof. We investigate the behavior of Hurwitz–Hodge classes on the boundary. The normalization of a reducible twisted nodal curve C in the boundary has at least two connected components. Let $\nu: \widetilde{C} \to C$ be a partial normalization such that $\widetilde{C} = C_1 \sqcup C_2$ has two connected components. Let f be the number of normalizing nodes. We have

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} \longrightarrow \bigoplus_{i=1}^f \mathcal{O}_{p_i} \longrightarrow 0.$$

Tensor it with $U \oplus U^{\vee}$ and consider the long exact sequence (recall that U is the \mathbb{Z}_{n+1} -representation with weight 1)

$$0 \longrightarrow H^{0}(C, \mathcal{O}_{C} \otimes (U \oplus U^{\vee})) \longrightarrow H^{0}(\widetilde{C}, \mathcal{O}_{\widetilde{C}} \otimes (U \oplus U^{\vee})) \longrightarrow (\mathbb{C}^{f} \otimes (U \oplus U^{\vee}))^{\mathbb{Z}_{n+1}}$$
$$\longrightarrow H^{1}(C, \mathcal{O}_{C} \otimes (U \oplus U^{\vee})) \longrightarrow H^{1}(\widetilde{C}, \mathcal{O}_{\widetilde{C}} \otimes (U \oplus U^{\vee})) \longrightarrow 0,$$

where the first two terms always vanish and we have

$$0 \longrightarrow (\mathbb{C}^f \otimes (U \oplus U^{\vee}))^{\mathbb{Z}_{n+1}} \longrightarrow H^1(C, \mathcal{O}_C \otimes (U \oplus U^{\vee})) \longrightarrow H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}} \otimes (U \oplus U^{\vee})) \longrightarrow 0,$$

The curve is in the image of

$$\iota: \overline{\mathcal{M}}_{q_1,\overline{\mu}^1,\overline{\nu}^1,\gamma^1,\alpha}(B\mathbb{Z}_{n+1}) \times \overline{\mathcal{M}}_{q_2,\overline{\mu}^2,\overline{\nu}^2,\gamma^2,-\alpha}(B\mathbb{Z}_{n+1}) \to \overline{\mathcal{M}}_{g,\overline{\mu},\overline{\nu},\gamma}(B\mathbb{Z}_{n+1}),$$

where $g_1 + g_2 + b - 1 = g$, the union of $(\overline{\mu}^i, \overline{\nu}^i, \gamma^i)$ on the two pieces i = 1, 2 matches the total datum, and $\pm \alpha = \pm (\alpha_1, \dots, \alpha_f)$ here stand for f markings on either factors with opposite monodromies.

Denote by $R := r_1 + r_2$, $\mathbb{V} := \mathbb{E}_U \oplus \mathbb{E}_{U^{\vee}}$ and $\mathbb{V}_i, R_i, i = 1, 2$ the corresponding bundles and their ranks on the two factors. We have the sequence

$$0 \longrightarrow (\mathcal{O}^{\oplus f} \otimes (U \oplus U^{\vee}))^{\mathbb{Z}_{n+1}} \longrightarrow \iota^* \mathbb{V} \longrightarrow \mathbb{V}_1 \boxplus \mathbb{V}_2 \longrightarrow 0.$$

Hence,

$$\iota^* c_{R-1}(\mathbb{V}) = c_{R-1}(\mathbb{V}_1 \boxplus \mathbb{V}_2),$$

and the ranks satisfy

$$R_1 + R_2 = (2g_1 - 2 + l(\overline{\mu}^1) + l(\overline{\nu}^1) + p_1 + f'') + (2g_2 - 2 + l(\overline{\mu}^2) + l(\overline{\nu}^2) + p_2 + f'')$$

$$= 2(g - f + 1) - 4 + l(\mu) + l(\nu) + p + 2f''$$

$$= R - 2(f - f'')$$

$$< R,$$

where f'' is the number of nodal markings with nontrivial monodromies.

If the inequality is strict, then $R_1 + R_2 \leq R - 2$, and $\iota^* c_{R-1}(\mathbb{V})$ simply vanishes by dimensional reasons. If the equality holds, which only happens when all nodal markings have nontrivial monodromies, then

$$\iota^* c_{R-1}(\mathbb{V}) = c_{R_1}(\mathbb{V}_1) c_{R_2-1}(\mathbb{V}_2) + c_{R_1-1}(\mathbb{V}_1) c_{R_2}(\mathbb{V}_2),$$

which vanishes since the \mathbb{Z}_{n+1} -Mumford relation implies $c_{R_1}(\mathbb{V}_1) = c_{R_2}(\mathbb{V}_2) = 0$.

We are left with the case when C is irreducible nodal but with some nodes having trivial monodromies. Similarly one can consider the normalization sequence and the rank inequality would be strict. $\iota^*c_{R-1}(\mathbb{V})$ still vanishes by dimensional reasons.

The vanishing result in Theorem 4.3 will greatly simplify our computation in the next subsection.

- 4.3. Computation of rubber invariants and Crepant Resolution Conjecture. In this subsection, we compute our rubber invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ and prove the corresponding Crepant Resolution Conjecture.
- 4.3.1. Pixton's formula calculation. By Theorem 4.2 and 4.3, in order to compute the rubber invariant

$$\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} = \frac{(-1)^{r_1+r_2-1}(t_1+t_2)}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|} \int_{\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})} \pi^* \mathrm{DR} \cdot \lambda_{r_1}^U \lambda_{r_2-1}^{U^\vee},$$

it suffices to consider the restriction of π^*DR to the main stratum, and to the strata of irreducible singular curves with nontrivial monodromies at nodes. Therefore, in the graph sum expression of $P_g^{d,r}(A)$, we only need to consider those graphs with one vertex and f loops, for $0 \le f \le g$.

Given $\Gamma \in G_{g,l(\overline{\mu})+l(\overline{\nu})+l(\gamma)}$ with $V(\Gamma)=1$ and $E(\Gamma)=f$, we have

$$\prod_{e=(h,h')\in E(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}$$

$$= -\prod_{e=(h,h')\in E(\Gamma)} \left((-w(h)(r - w(h))) + \frac{(-w(h)(r - w(h)))^2}{2} (\psi_h + \psi_{h'}) + \cdots + \frac{(-w(h)(r - w(h)))^k}{k!} (\psi_h + \psi_{h'})^{k-1} + \cdots \right),$$

where the series must terminate after finite terms, by dimensional reasons.

The classical Bernoulli's formula implies

$$1^{k} + 2^{k} + \dots + r^{k} = \frac{1}{k+1} \sum_{i=0}^{k} \frac{(k+1)!}{i!(k+1-i)!} B_{i} r^{k+1-i},$$

where B_i is the *i*-th Bernoulli number. Since $r^{b_1(\Gamma)} = r^f$, in order to pick the constant term of $P_g^{d,r}(A)$, only the term $\frac{w(h)^{2k}}{k!}$ in the factor $\frac{(-w(h)(r-w(h)))^k}{k!}$ survives, because when we sum over $w(h) \in \{1, \dots, r\}$ the terms in

$$1^{2k} + 2^{2k} + \dots + r^{2k} = \frac{1}{2k+1} \sum_{i=0}^{2k} \frac{(2k+1)!}{i!(2k+1-i)!} B_i r^{2k+1-i},$$

is at least r-linear. Hence the product over edges would produce an r^f factor. Furthermore, we can only pick the term $\frac{1}{2k+1}\frac{(2k+1)!}{(2k)!(2k+1-2k)!}B_{2k}r=B_{2k}r$ in the above summation.

Let $[\cdot]_d$ denote the degree d part of a class. Pixton's formula expands as

$$DR = 2^{-g} \sum_{f=0}^{g} \frac{(-1)^{f}}{2^{f} f!} \cdot (\xi_{\Gamma_{f}})_{*} \left(\sum_{M=f}^{g} \left[\exp \left(\sum_{i=1}^{l(\overline{\mu})} \mu_{i}^{2} \psi_{i} + \sum_{j=1}^{l(\overline{\nu})} \nu_{j}^{2} \psi_{j+l(\overline{\mu})} \right) \right]_{g-M}$$

$$\cdot \left(\sum_{m_{1}+\dots+m_{f}=M} \prod_{i=1}^{f} B_{2m_{i}} \left(\sum_{k_{i}=0}^{m_{i}-1} \frac{\psi_{h_{i}}^{k_{i}}}{m_{i}k_{i}!} \frac{\psi_{h_{i}'}^{m_{i}-1-k_{i}}}{(m_{i}-1-k_{i})!} \right) \right) \right)$$

$$= \sum_{f=0}^{g} \frac{(-1)^{f}}{f!} (\xi_{\Gamma_{f}})_{*} \left(\sum_{M=f}^{g} 2^{-M} \cdot \frac{\left(\sum_{i=1}^{l(\overline{\mu})} \mu_{i}^{2} \psi_{i} + \sum_{j=1}^{l(\overline{\nu})} \nu_{j}^{2} \psi_{j+l(\overline{\mu})} \right)^{g-M}}{2^{g-M} (g-M)!} \right)$$

$$\cdot \left(\sum_{m_1 + \dots + m_f = M} \prod_{i=1}^f B_{2m_i} \left(\sum_{k_i = 0}^{m_i - 1} \frac{\psi_{h_i}^{k_i}}{(2m_i)k_i!} \frac{\psi_{h_i'}^{m_i - 1 - k_i}}{(m_i - 1 - k_i)!} \right) \right) \right),$$

where Γ_f is the unique graph in $G_{g,l(\overline{\mu})+l(\overline{\nu})+l(\gamma)}$ with $|V(\Gamma)|=1$ and $|E(\Gamma)|=f$.

4.3.2. Grothendieck–Riemann–Roch calculation. The double ramification cycle has been expressed in terms of ψ classes. Now we compute our rubber invariant $\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ using the Grothendieck–Riemann–Roch theorem.

In [21], J. Zhou obtains the following expression for descendent GW invariants of $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$, using Grothendieck–Riemann–Roch:

$$\left\langle \prod_{j=1}^{N} \tau_{k_{j}}(a_{j}) \right\rangle_{g}^{\left[\mathbb{C}^{2}/\mathbb{Z}_{n+1}\right]}$$

$$= \frac{(t_{1}+t_{2})(n+1)^{2g-1}(-1)^{r_{1}}}{2} \sum_{I \sqcup J=\left[N\right]} \frac{B_{r_{1}+r_{2}}\left(\frac{c(I)}{n+1}\right)}{r_{1}+r_{2}} (-1)^{\left|J\right|} \cdot \frac{1}{4^{g} \prod_{j=1}^{N} (2k_{j}+1)!!}$$

$$+\delta_{N,2} \cdot (t_{1}+t_{2})(n+1)^{2g-1} (-1)^{g} \sum_{c=0}^{n} \frac{B_{2g}\left(\frac{c}{n+1}\right)}{2g} \frac{1}{4^{g-1} \prod_{k_{j}>0} (2k_{j}-1)!!},$$

where $a_j \in \mathbb{Z}_{n+1}$, $c(I) = -\sum_{i \in I} a_i$. Moreover, we assume that all a_j occurring here are nonzero, $\sum a_j = 0$, and $\sum k_j = g$, which hold in our case.

Applying the formula above, the rubber invariant $|\operatorname{Aut}(\mu)||\operatorname{Aut}(\nu)|\langle \overline{\mu}, \overline{\nu}\rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}$ is equal to

$$\begin{split} &(-1)^{r_1+r_2-1}(t_1+t_2)\int_{\overline{M}_{g,\overline{\mu}\cup\overline{\nu}\cup\gamma}(B\mathbb{Z}_{n+1})}\pi^*\mathrm{DR} \cdot \lambda_{r_1}^U\lambda_{r_2-1}^{U^\vee} \\ &= \int_{\left[\overline{M}_{g,\overline{\mu}\cup\overline{\nu}\cup\gamma}(\left[\mathbb{C}^2/\mathbb{Z}_{n+1}\right])\right]^{\mathrm{vir}}}\pi^*\mathrm{DR} \\ &= \sum_{f=0}^g \frac{(-1)^f(n+1)^f}{f!} \sum_{\alpha_f} \int_{\left[\overline{M}_{g-f,\overline{\mu}\cup\overline{\nu}\cup\gamma\cup\alpha_f}(\left[\mathbb{C}^2/\mathbb{Z}_{n+1}\right])\right]^{\mathrm{vir}}} \sum_{M=f}^g 2^{-M} \cdot \frac{\left(\sum_{i=1}^{l(\overline{\mu})} \mu_i^2 \overline{\psi}_i + \sum_{j=1}^{l(\overline{\nu})} \nu_j^2 \overline{\psi}_{j+l(\overline{\mu})}\right)^{g-M}}{2^{g-M}(g-M)!} \\ &\cdot \left(\sum_{m_1+\dots+m_f=M} \prod_{i=1}^f B_{2m_i} \left(\sum_{k_i=0}^{m_i-1} \frac{\overline{\psi}_{k_i}^{k_i}}{(2m_i)k_i!} \frac{\overline{\psi}_{m_i}^{m_i-1-k_i}}{(2m_i)k_i!} \right)\right) \\ &= \sum_{f=0}^g \frac{(-1)^f(n+1)^f}{f!} \sum_{\alpha_f} \int_{\left[\overline{M}_{g-f,\overline{\mu}\cup\overline{\nu}\cup\gamma\cup\alpha_f}(\left[\mathbb{C}^2/\mathbb{Z}_{n+1}\right])\right]^{\mathrm{vir}}} \sum_{M=f}^g 2^{-M} \\ &\cdot \sum_{k_i+\sum_j l_j=g-M} \prod_{i=1}^{l(\overline{\mu})} \frac{\mu_i^{2k_i} \overline{\psi}_{k_i}^{k_i}}{2^{k_i} k_i!} \prod_{j=1}^g \frac{y_j^{2l_i} \overline{\psi}_{j+l(\overline{\mu})}^{l_j}}{2^{l_j} l_j!} \cdot \left(\sum_{m_1+\dots+m_f=M} \prod_{i=1}^f B_{2m_i} \left(\sum_{k_i=0}^{m_i-1} \frac{\overline{\psi}_{k_i}^{k_i}}{(2m_i)k_i!} \frac{\overline{\psi}_{m_i-1-k_i}^{m_i-1-k_i}}{(m_i-1-k_i)!}\right)\right) \\ &= \sum_{f=0}^g \frac{(-1)^f(n+1)^f}{f!} \sum_{\alpha_f} \sum_{M=f} 2^{-M} \frac{(t_1+t_2)(n+1)^{2g-2f-1}(-1)^{r_1}}{2} \\ &\cdot \sum_{k_i+\sum_j l_i=g-M} \prod_{j=1}^{l(\overline{\mu})} \frac{(\underline{\mu}_i)^{2k_i}}{(2k_i+1)!} \prod_{i=1}^{l(\overline{\mu})} \frac{(\underline{\nu}_i)^{2l_i}}{(2l_j+1)!} \end{split}$$

$$\begin{split} & \cdot \frac{1}{4^{M-f}} \sum_{m_1 + \dots + m_f = M} \prod_{i=1}^f \left(\sum_{k_i = 0}^{m_i - 1} \frac{B_{2m_i}}{(2m_i)(2k_i + 1)!!k_i!} \frac{1}{(2(m_i - 1 - k_i) + 1)!!(m_i - 1 - k_i)!} \right) \\ & = \sum_{f = 0}^g \frac{(-1)^f (n+1)^f}{f!} \sum_{\alpha_f} \sum_{M = f}^g 2^{-M} \frac{(t_1 + t_2)(n+1)^{2g-2f-1}(-1)^{r_1}}{2} \\ & \cdot \sum_{I \sqcup J = \overline{\mu} \sqcup \overline{\nu} \sqcup \gamma \sqcup \alpha_f} \frac{B_{r_1 + r_2} \left(\frac{c(I)}{n+1}\right)}{r_1 + r_2} (-1)^{|J|} \sum_{\sum_i k_i + \sum_j l_j = g-M} \prod_{i = 1}^{l(\overline{\mu})} \frac{\left(\frac{\mu_i}{2}\right)^{2k_i}}{(2k_i + 1)!} \prod_{j = 1}^{l(\overline{\nu})} \frac{\left(\frac{\nu_i}{2}\right)^{2l_i}}{(2l_j + 1)!} \\ & \cdot \frac{1}{2^{m-f}} \sum_{m_1 + \dots + m_f = M} \prod_{i = 1}^f \left(\sum_{k_i = 0}^{m_i - 1} \frac{B_{2m_i}}{(2m_i)(2k_i + 1)!} \frac{1}{(2(m_i - 1 - k_i) + 1)!}\right) \\ & = \sum_{f = 0}^g \frac{(-1)^f (n+1)^f}{f!} \sum_{\alpha_f} \sum_{M = f}^g \frac{(t_1 + t_2)(n+1)^{2g-2f-1}(-1)^{r_1}}{2} \cdot \sum_{I \sqcup J = \overline{\mu} \sqcup \overline{\nu} \sqcup \gamma \sqcup \alpha_f} \frac{B_{r_1 + r_2} \left(\frac{c(I)}{n+1}\right)}{r_1 + r_2} (-1)^{|J|} \\ & \cdot [z^{2g-2M}] \left(\prod_{i = 1}^{l(\overline{\mu})} \mathcal{S}(\mu_i z) \prod_{j = 1}^{l(\overline{\nu})} \mathcal{S}(\nu_j z)\right) \cdot \left(\sum_{m_1 + \dots + m_f = M} \prod_{i = 1}^f \frac{B_{2m_i}}{(2m_i)(2m_i)!}\right). \end{split}$$

Here the summation \sum_{α_f} is over all tuples $\alpha_f = (\alpha_{h_1}, -\alpha_{h_1}, \cdots, \alpha_{h_f}, -\alpha_{h_f})$ with $\alpha_{h_1}, \cdots, \alpha_{h_f}$ being *nontrivial* elements in \mathbb{Z}_{n+1} , corresponding to monodromies around the nodes; in the last equality we used the fact that

$$\sum_{k_i=0}^{m_i-1} \frac{(2m_i)!}{(2k_i+1)!} \frac{1}{(2(m_i-1-k_i)+1)!} = 2^{2m_i-1};$$

the function S(z) is defined as

$$\mathcal{S}(z) := \frac{\sinh(z/2)}{z/2}.$$

By the trick in Section 3.1 of [21], we can rewrite the term involving Bernoulli numbers as follows.

$$\begin{split} \sum_{I \sqcup J = [N]} \frac{B_{r_1 + r_2} \left(\frac{c(I)}{n+1}\right)}{r_1 + r_2} (-1)^{|J|} &= \frac{1}{n+1} \sum_{c=0}^n \sum_{l=0}^n \zeta^{lc} \zeta^{l} \sum_{i \in I} a_i \sum_{I \sqcup J = [N]} \frac{B_{r_1 + r_2} \left(\frac{c}{n+1}\right)}{r_1 + r_2} (-1)^{|J|} \\ &= \frac{1}{n+1} \sum_{c=0}^n \sum_{l=0}^n \zeta^{lc} \frac{B_{r_1 + r_2} \left(\frac{c}{n+1}\right)}{r_1 + r_2} \sum_{I \sqcup J = [N]} \zeta^{l} \sum_{i \in I} a_i (-1)^{|J|} \\ &= \frac{1}{n+1} \sum_{l=1}^n \sum_{c=0}^n \zeta^{lc} \frac{B_{r_1 + r_2} \left(\frac{c}{n+1}\right)}{r_1 + r_2} \prod_{j=1}^N (\zeta^{la_j} - 1). \end{split}$$

Therefore, let a, b be the tuples of markings determined by $\overline{\mu}$, $\overline{\nu}$ respectively, and we have

$$= \frac{\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}}{|\operatorname{Aut}(\mu)||\operatorname{Aut}(\nu)|} \sum_{f=0}^{g} \frac{(-1)^{f} (n+1)^{f}}{f!} \sum_{\alpha_{f}} \sum_{M=f}^{g} \frac{(t_{1}+t_{2})(n+1)^{2g-2f-2}(-1)^{r_{1}}}{2}$$

$$\cdot \sum_{l=1}^{n} \sum_{c=0}^{n} \zeta^{lc} \frac{B_{r_1+r_2}\left(\frac{c}{n+1}\right)}{r_1+r_2} \cdot \prod_{i=1}^{l(\overline{\mu})} (\zeta^{la_i}-1) \prod_{j=1}^{l(\overline{\nu})} (\zeta^{lb_j}-1) \prod_{k=1}^{l(\gamma)} (\zeta^{l\gamma_k}-1) \prod_{i=1}^{f} (\zeta^{l\alpha_{h_i}}-1) (\zeta^{-l\alpha_{h_i}}-1) \cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_i z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_j z)\right) \left(\sum_{m_1+\dots+m_f=M} \prod_{i=1}^{f} \frac{B_{2m_i}}{(2m_i)(2m_i)!}\right).$$

Notice that since $l \neq 0$,

$$\sum_{\alpha_{h_i}=1}^{n} (\zeta^{l\alpha_{h_i}} - 1)(\zeta^{-l\alpha_{h_i}} - 1) = \sum_{\alpha_{h_i}=1}^{n} (2 - \zeta^{l\alpha_{h_i}} - \zeta^{-l\alpha_{h_i}}) = 2(n+1).$$

Hence, one can eliminate the sum \sum_{α_f} and obtain

$$\langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim} = \frac{1}{|\mathrm{Aut}(\mu)||\mathrm{Aut}(\nu)|} \sum_{f=0}^{g} \frac{(-1)^{f} 2^{f}}{f!} \sum_{M=f}^{g} \frac{(t_{1}+t_{2})(n+1)^{2g-2}(-1)^{r_{1}}}{2} \cdot \sum_{l=1}^{n} \sum_{c=0}^{n} \zeta^{lc} \frac{B_{r_{1}+r_{2}}\left(\frac{c}{n+1}\right)}{r_{1}+r_{2}} \cdot \prod_{i=1}^{l(\overline{\mu})} (\zeta^{la_{i}}-1) \prod_{j=1}^{l(\overline{\nu})} (\zeta^{lb_{j}}-1) \prod_{k=1}^{l(\gamma)} (\zeta^{l\gamma_{k}}-1) \cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_{i}z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_{j}z)\right) \left(\sum_{m_{1}+\dots+m_{f}=M} \prod_{i=1}^{f} \frac{B_{2m_{i}}}{(2m_{i})(2m_{i})!}\right).$$

4.3.3. Generating functions.

Definition 4.4. Define the generating function for rubber invariants as

$$Z(x,z)_{\overline{\mu},\overline{\nu}}^{\circ,\sim} := \sum_{g \ge 0,\gamma} (-1)^g z^{2g} \frac{x^{\gamma}}{\gamma!} \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim},$$

where $\gamma = (\gamma_1, \dots, \gamma_p)$ with $0 \neq \gamma_i \in \mathbb{Z}_{n+1}$, $x_{\gamma} = x_{\gamma_1} \dots x_{\gamma_p}$, and we use the more intuitive notation $\gamma!$ to denote $|\operatorname{Aut}(\gamma)|$. The factor $\gamma!$ appears because we would like to count those extra marked points as *unordered*. Moreover, the summation is over all rubber invariants satisfying the (\dagger) assumption at the beginning of Section 4.

In order for the moduli space $\overline{\mathcal{M}}_{g,a\sqcup b\sqcup\gamma}([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ to be nonempty, we must have

$$\sum_{i=1}^{l(\overline{\mu})} a_i + \sum_{j=1}^{l(\overline{\nu})} b_j + \sum_{k=1}^{l(\gamma)} \gamma_k = 0 \mod (n+1).$$

Recall that the Bernoulli polynomials $B_k(t)$, and Bernoulli numbers B_k are defined by the following Taylor expansion:

$$\frac{ze^{tz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(t) \frac{z^k}{k!}, \qquad B_k := B_k(0).$$

The only nonzero odd Bernoulli number is $B_1 = -\frac{1}{2}$. Define the generating function

$$F(z) := \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!} = \frac{z}{e^z - 1} + \frac{z}{2} - 1.$$

With the observations above, the generating function $Z(x,z)_{\overline{\mu},\overline{\nu}}^{\circ,\sim}$ can be expressed as

$$|\operatorname{Aut}(\mu)||\operatorname{Aut}(\nu)| \cdot Z(x,z)^{\circ,\sim}_{\overline{\mu},\overline{\nu}}$$

$$\begin{split} &= \sum_{g \geq 0} \sum_{\gamma} (-1)^g z^{2g} \frac{x^{\gamma}}{\gamma!} \cdot \frac{1}{n+1} \sum_{b=0}^{n} \prod_{i=1}^{l(\overline{\mu})} \zeta^{ba_i} \prod_{j=1}^{l(\overline{\nu})} \zeta^{bb_j} \prod_{k=1}^{l(\overline{\nu})} \zeta^{b\gamma_k} \sum_{f=0}^{g} \frac{(-1)^f 2^f}{f!} \\ &\cdot \sum_{M=f}^{g} \frac{(t_1+t_2)(n+1)^{2g-2}(-1)^{r_1}}{2} \sum_{l=1}^{n} \sum_{c=0}^{n} \zeta^{lc} \frac{B_{r_1+r_2}\left(\frac{c}{n+1}\right)}{r_1+r_2} \prod_{i=1}^{l(\overline{\nu})} (\zeta^{la_i}-1) \prod_{j=1}^{l(\overline{\nu})} (\zeta^{lb_j}-1) \prod_{k=1}^{l(\gamma)} (\zeta^{l\gamma_k}-1) \\ &\cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_i z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_j z) \right) \left(\sum_{m_1+\dots+m_f=M} \prod_{i=1}^{f} \frac{B_{2m_i}}{(2m_i)(2m_i)!} \right) \\ &= -\sum_{g \geq 0} z^{2g} \sum_{b=0}^{n} \sum_{l=1}^{n} \prod_{i=1}^{l(\overline{\mu})} \zeta^{ba_i + \frac{a_i}{2}} \prod_{j=1}^{l(\overline{\nu})} \zeta^{bb_j + \frac{b_j}{2}} \cdot \sum_{p \geq 0} \frac{1}{p!} \left(\sum_{a=1}^{n} \zeta^{ba + \frac{a}{2}} (\zeta^{la}-1) x_a \right)^p \sum_{f=0}^{g} \frac{(-1)^f 2^f}{f!} \\ &\sum_{M=f} \frac{(t_1+t_2)(n+1)^{2g-3}}{2} \cdot \sum_{c=0}^{n} \zeta^{lc} \frac{B_{r_1+r_2}\left(\frac{c}{n+1}\right)}{r_1+r_2} \prod_{i=1}^{l(\overline{\mu})} (\zeta^{la_i}-1) \prod_{j=1}^{l(\overline{\nu})} (\zeta^{lb_j}-1) \\ &\cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_i z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_j z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^f, \end{split}$$

where we've fixed a certain square root $\zeta^{\frac{1}{2}} = e^{\frac{\pi i}{n+1}}$ of ζ , and by $\int \frac{F(z)}{z}$ we always mean the power series obtained by termwise integration with constant term 0; we also used the formula for the rank r_1 given in Section 2.4.

We will also need the generating function encoding rubber invariants for all $\overline{\mu}$, $\overline{\nu}$.

Definition 4.5. For any $j \in \{1, \dots, n\}$, we introduce a formal variable y_j and define the following change of variables:

$$y_j = \frac{2\sqrt{-1}}{n+1} \sum_{a=1}^n \sin \frac{a\pi}{n+1} \cdot \zeta^{ja} x_a = \frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a.$$

For any $1 \le s \le t \le n$, define

$$y_{s\to t} = y_s + \dots + y_t.$$

By Lemma 3.3 of [21], we have the following lemma:

Lemma 4.6. For $0 \le b \le n$, $1 \le l \le n$, we have

$$\sum_{a=1}^{n} \zeta^{ba + \frac{a}{2}} (\zeta^{la} - 1) x_a = \begin{cases} (n+1) y_{b+1 \to b+l}, & b+l < n+1, \\ -(n+1) y_{b+l-n \to b}, & b+l \ge n+1, \end{cases}$$

Definition 4.7. For any integer d > 0, $i \in \{1, 2, 3\}$ and any $a \in \{1, \dots, n\}$, we introduce two formal variables $p_{d,a}^i$, $\hat{p}_{d,a}^i$ and define the following change of variables:

$$\hat{p}_{d,j}^i = \frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} p_{d,a}^i.$$

For any $1 \le s \le t \le n$, define

$$\hat{p}_{d,s\to t}^{i} = \hat{p}_{d,s}^{i} + \dots + \hat{p}_{d,t}^{i}.$$

For any \mathbb{Z}_{n+1} -weighted partition $\overline{\mu} = \{(\mu_1, a_1), \cdots, (\mu_{l(\overline{\mu})}, a_{l(\overline{\mu})})\}$, define

$$p^i_{\overline{\mu}} = p^i_{\mu_1,a_1} \cdots p^i_{\mu_{l(\overline{\mu})},a_{l(\overline{\mu})}}, \qquad \hat{p}^i_{\overline{\mu}} = \hat{p}^i_{\mu_1,a_1} \cdots \hat{p}^i_{\mu_{l(\overline{\mu})},a_{l(\overline{\mu})}}.$$

Given any ordinary partition $\mu = \{\mu_1, \dots, \mu_{l(\mu)}\}\$, define

$$\hat{p}_{\mu,s\to t}^i = (\hat{p}_{\mu_1,s}^i + \dots + \hat{p}_{\mu_1,t}^i) \cdots (\hat{p}_{\mu_{l(\mu)},s}^i + \dots + \hat{p}_{\mu_{l(\mu)},t}^i).$$

Similar to Lemma 4.6, we have the following lemma:

Lemma 4.8. For d > 0, $i \in \{1, 2, 3\}$, $0 \le b \le n$, $1 \le l \le n$, we have

$$\sum_{a=1}^n \zeta^{ba+\frac{a}{2}}(\zeta^{la}-1)p^i_{d,a} = \left\{ \begin{array}{ll} (n+1)\hat{p}^i_{d,b+1\to b+l}, & b+l < n+1, \\ -(n+1)\hat{p}^i_{d,b+l-n\to b}, & b+l \geq n+1, \end{array} \right.$$

Definition 4.9. Define the generating function $Z(x,z,p^1,p^2)^{\circ,\sim}$ as

$$Z(x,z,p^1,p^2)^{\circ,\sim} := \sum_{|\overline{\mu}|=|\overline{\nu}|=m} Z(x,z)^{\circ,\sim}_{\overline{\mu},\overline{\nu}} p^1_{\overline{\mu}} p^2_{\overline{\nu}}.$$

Applying Lemma 4.6 and Lemma 4.8 to the generating function $Z(x,z,p^1,p^2)^{\circ,\sim}$, we have $Z(x,z,p^1,p^2)^{\circ,\sim}$

$$= -\sum_{|\overline{\mu}| = |\overline{\nu}| = m} \frac{p_{\overline{\mu}}^1 p_{\overline{\nu}}^2}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \sum_{g \geq 0} z^{2g} \sum_{b = 0}^n \sum_{\substack{1 \leq l \leq n \\ b + l < n + 1}} \prod_{i = 1}^{l(\overline{\mu})} \zeta^{ba_i + \frac{a_i}{2}} \prod_{j = 1}^{l(\overline{\nu})} \zeta^{bb_j + \frac{b_j}{2}} \\ \sum_{p \geq 0} \frac{1}{p!} ((n+1)y_{b+1 \to b + l})^p \sum_{f = 0}^g \frac{(-1)^f 2^f}{f!} \sum_{M = f}^g \frac{(t_1 + t_2)(n+1)^{2g-3}}{2} \cdot \sum_{c = 0}^n \zeta^{lc} \frac{B_{r_1 + r_2}\left(\frac{c}{n+1}\right)}{r_1 + r_2} \\ \cdot \prod_{i = 1}^{l(\overline{\mu})} (\zeta^{la_i} - 1) \prod_{j = 1}^{l(\overline{\nu})} (\zeta^{lb_j} - 1) \cdot [z^{2g-2M}] \left(\prod_{i = 1}^{l(\overline{\mu})} S(\mu_i z) \prod_{j = 1}^{l(\overline{\nu})} S(\nu_j z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^f \\ - \sum_{|\overline{\mu}| = |\overline{\nu}| = m} \frac{p_{\overline{\mu}}^1 p_{\overline{\nu}}^2}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \sum_{g \geq 0} z^{2g} \sum_{b = 0}^n \sum_{\substack{1 \leq l \leq n \\ b + l \geq n+1}} \prod_{i = 1}^{l(\overline{\mu})} \zeta^{ba_i + \frac{a_i}{2}} \prod_{j = 1}^{l(\overline{\nu})} \zeta^{bb_j + \frac{b_j}{2}} \\ \cdot \sum_{p \geq 0} \frac{1}{p!} (-(n+1)y_{b+l-n \to b})^p \sum_{f = 0}^g \frac{(-1)^f 2^f}{f!} \sum_{M = f}^g \frac{(t_1 + t_2)(n+1)^{2g-3}}{2} \cdot \sum_{c = 0}^n \zeta^{lc} \frac{B_{r_1 + r_2}\left(\frac{c}{n+1}\right)}{r_1 + r_2} \\ \cdot \prod_{i = 1}^{l(\overline{\mu})} (\zeta^{la_i} - 1) \prod_{j = 1}^{l(\overline{\nu})} (\zeta^{lb_j} - 1) \cdot [z^{2g-2M}] \left(\prod_{i = 1}^{l(\overline{\mu})} S(\mu_i z) \prod_{j = 1}^{l(\overline{\nu})} S(\nu_j z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^f \\ = - \sum_{|\mu| = |\nu| = m} \sum_{g \geq 0} \sum_{b = 0}^n \sum_{\substack{1 \leq l \leq n \\ b + l < n + 1}} \frac{(n+1)^{l(\mu) + l(\nu)} \hat{p}_{\mu, b + 1 \to b + l}^2 \hat{p}_{\nu, b + 1 \to b + l}^2}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \\ \cdot \sum_{p \geq 0} \frac{1}{p!} ((n+1)y_{b+1 \to b + l})^p \sum_{f = 0}^g \frac{(-1)^f 2^f}{f!} \sum_{M = f}^g \frac{(t_1 + t_2)(n+1)^{2g-3}}{2} \sum_{c = 0}^n \zeta^{lc} \frac{B_{r_1 + r_2}\left(\frac{c}{n+1}\right)}{r_1 + r_2} \\ \cdot [z^{2g-2M}] \left(\prod_{i = 1}^{l(\overline{\mu})} S(\mu_i z) \prod_{i = 1}^{l(\overline{\nu})} S(\nu_j z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^f$$

$$\begin{split} & - \sum_{|\mu| = |\nu| = m} \sum_{g \geq 0} z^{2g} \sum_{b=0}^{n} \sum_{\substack{1 \leq l \leq n \\ b+l \geq n+1}} \frac{(-(n+1))^{l(\mu)+l(\nu)} \hat{p}_{\mu,b+l-n \to b}^{1} \hat{p}_{\nu,b+l-n \to b}^{2}}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \\ & \cdot \sum_{p \geq 0} \frac{1}{p!} (-(n+1)y_{b+l-n \to b})^{p} \sum_{f=0}^{g} \frac{(-1)^{f} 2^{f}}{f!} \sum_{M=f}^{g} \frac{(t_{1}+t_{2})(n+1)^{2g-3}}{2} \sum_{c=0}^{n} \zeta^{lc} \frac{B_{r_{1}+r_{2}}\left(\frac{c}{n+1}\right)}{r_{1}+r_{2}} \\ & \cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_{i}z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_{j}z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^{f} \\ & = -2 \sum_{|\mu| = |\nu| = m} \sum_{g \geq 0} z^{2g} \sum_{1 \leq s \leq t \leq n} \frac{\hat{p}_{\mu,s \to t}^{1} \hat{p}_{\nu,s \to t}^{2}}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \cdot \sum_{p \geq 0} \frac{1}{p!} (y_{s \to t})^{p} \sum_{f=0}^{g} \frac{(-1)^{f} 2^{f}}{f!} \\ & \sum_{M=f} \frac{(t_{1}+t_{2})(n+1)^{2g-3+l(\mu)+l(\nu)+p}}{2} \cdot \sum_{c=0}^{n} \zeta^{c(t-s+1)} \frac{B_{r_{1}+r_{2}}\left(\frac{c}{n+1}\right)}{r_{1}+r_{2}} \\ & \cdot [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_{i}z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_{j}z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^{f}, \end{split}$$

where in the last equality, we have used the fact

$$B_m(1-x) = (-1)^m B_m(x).$$

4.3.4. Crepant Resolution Conjecture. Now we compare the computations above with the results (Proposition 3.6 and 4.3) in [12]. Let $\mathcal{A}_n \to [\mathbb{C}^2/\mathbb{Z}_{n+1}]$ be the crepant resolution. Let $\mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$ denote the space of \mathbb{Z}_{n+1} -weighted partitions, and $\mathcal{F}_{\mathcal{A}_n}$ denote the space of $H^*(\mathcal{A}_n)$ -weighted partitions.

A curve class $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$ is specified by the datum $(\beta, m) := m[\mathbb{P}^1] + \beta$, where m is the fixed integer as before, and $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$. The generating functions for the relative GW theory of $\mathcal{A}_n \times \mathbb{P}^1$ and the rubber theory are defined in [12] as

$$Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)^{\circ, \sim}_{\vec{\mu}, \vec{\nu}} := \sum_{g, \beta} z^{2g} s_1^{(\beta, \omega_1)} \cdots s_n^{(\beta, \omega_n)} \langle \vec{\mu}, \vec{\nu} \rangle_{g, (\beta, m)}^{\circ, \sim},$$

$$Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu}^1, \cdots, \vec{\mu}^r} := \sum_{\chi, \beta} z^{-\chi} s_1^{(\beta, \omega_1)} \cdots s_n^{(\beta, \omega_n)} \langle \vec{\mu}^1, \cdots, \vec{\mu}^r \rangle_{\chi, (\beta, m)}^{\mathcal{A}_n \times \mathbb{P}^1, \bullet},$$

where $\vec{\mu}^i \in \mathcal{F}_{\mathcal{A}_n}$.

Recall the explicit isomorphism between the two Fock spaces. If we identify \mathbb{Z}_{n+1} with the orbifold cohomology $H^*_{\text{orb}}([\mathbb{C}^2/\mathbb{Z}_{n+1}])$, the isomorphism is given by

$$\Phi: H^*_{\text{orb}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]) \cong H^*(\mathcal{A}_n),$$

$$e_0 \mapsto 1, \qquad e_i \mapsto \frac{\zeta^{i/2} - \zeta^{-i/2}}{n+1} \sum_{j=1}^n \zeta^{ij} \omega_j, \qquad 1 \le i \le n,$$

where $\omega_1, \dots, \omega_n \in H^2(\mathcal{A}_n, \mathbb{Q})$ is the dual basis to the exceptional curves in \mathcal{A}_n .

Theorem 4.10 (Rubber GW crepant resolution). Given $\overline{\mu}, \overline{\nu} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, let $\vec{\mu}, \vec{\nu} \in \mathcal{F}_{\mathcal{A}_n}$ be their correspondents under the isomorphism above. Then the $(t_1 + t_2)$ -linear terms of

$$Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)^{\circ, \sim}_{\vec{\mu}, \vec{\nu}} \quad and \quad Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1)^{\circ, \sim}_{\overline{\mu}, \overline{\nu}}$$

coincide under the change of variables

$$s_j = \zeta \exp\left(\frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a\right), \qquad 1 \le j \le n$$

Proof. We make the following observation. For any $1 \le l \le n$ and any formal variable u,

$$1 + \sum_{d=1}^{\infty} (\zeta^l e^u)^d = \frac{1}{1 - \zeta^l e^u} = -\frac{\sum_{c=0}^n \zeta^{cl} e^{cu}}{e^{(n+1)u} - 1} = -\sum_{c=0}^n \zeta^{cl} \sum_{k=1}^{\infty} \frac{B_k(\frac{c}{n+1})}{k!} ((n+1)u)^{k-1}.$$

Taking derivatives on both sides for $2g - 3 + l(\mu) + l(\nu)$ times,

$$\sum_{d=1}^{\infty} d^{2g-3+l(\mu)+l(\nu)} (\zeta^l e^u)^d = -\sum_{c=0}^n \zeta^{cl} \sum_{p=0}^{\infty} \frac{B_{r_1+r_2}\left(\frac{c}{n+1}\right)}{(r_1+r_2)p!} (n+1)^{2g-3+l(\mu)+l(\nu)+p} u^p,$$

where we used the fact $r_1 + r_2 = 2g - 2 + l(\mu) + l(\nu) + p$. Substitute this identity into the generating function,

$$Z(x, z, p^{1}, p^{2})^{\circ, \sim} = (t_{1} + t_{2}) \sum_{|\mu| = |\nu| = m} \sum_{g \geq 0} z^{2g} \sum_{1 \leq s \leq t \leq n} \frac{\hat{p}_{\mu, s \to t}^{1} \hat{p}_{\nu, s \to t}^{2}}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \sum_{d=1}^{\infty} d^{2g-3+l(\mu)+l(\nu)} (\zeta^{t-s+1} \exp(y_{s \to t}))^{d} \cdot \sum_{f=0}^{g} \frac{(-1)^{f} 2^{f}}{f!} \sum_{M=f}^{g} [z^{2g-2M}] \left(\prod_{i=1}^{l(\overline{\mu})} \mathcal{S}(\mu_{i}z) \prod_{j=1}^{l(\overline{\nu})} \mathcal{S}(\nu_{j}z) \right) \cdot [z^{2M}] \left(\int \frac{F(z)}{z} \right)^{f}.$$

A key observation here is that the power series $\int \frac{F(z)}{z}$ starts from the quadratic term (we let the integration constant be zero). Thus we can rewrite the sum $\sum_{M=f}^g$ as $\sum_{M=0}^g$ and the result does not change. Therefore,

$$Z(x, z, p^{1}, p^{2})^{\circ, \sim} = (t_{1} + t_{2}) \sum_{|\mu| = |\nu| = m} \sum_{1 \leq s \leq t \leq n} \frac{\hat{p}_{\mu, s \to t}^{1} \hat{p}_{\nu, s \to t}^{2}}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \sum_{d=1}^{\infty} d^{-3 + l(\mu) + l(\nu)} \cdot (\zeta^{t - s + 1} \exp(y_{s \to t}))^{d} \cdot \prod_{i=1}^{l(\mu)} \mathcal{S}(d\mu_{i}z) \prod_{j=1}^{l(\nu)} \mathcal{S}(d\nu_{j}z) \cdot \exp\left(-2\int \frac{F(dz)}{dz}\right)$$

By Lemma 4.11 below, we obtain the following formula:

$$Z(x, z, p^{1}, p^{2})^{\circ, \sim} = \frac{(t_{1} + t_{2})}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|} \sum_{|\mu| = |\nu| = m} \sum_{1 \leq s \leq t \leq n} \hat{p}_{\mu, s \to t}^{1} \hat{p}_{\nu, s \to t}^{2}$$

$$\sum_{d=1}^{\infty} \frac{d^{l(\mu) + l(\nu) - 3} (\zeta^{t - s + 1} \exp(y_{s \to t}))^{d} \cdot (\prod_{i=1}^{l(\mu)} \mathcal{S}(d\mu_{i}z) \prod_{j=1}^{l(\nu)} \mathcal{S}(d\nu_{j}z))}{\mathcal{S}(dz)^{2}}$$

This coincides with the formula Proposition 3.6 in [12], where the parameters s_j are related to y_j by

$$s_j = \zeta e^{y_j}, \qquad 1 \le j \le n.$$

The following lemma computes the exponential term.

Lemma 4.11.

$$\exp\left(\int \frac{F(z)}{z}\right) = \mathcal{S}(z).$$

Proof. Both sides take the value 1 at z = 0. Thus it makes sense to take logarithms and it suffices to prove

$$\int \frac{F(z)}{z} = \log \mathcal{S}(z),$$

which is clear by checking the derivatives of both sides match with each other.

4.4. Crepart Resolution Conjecture for 3-point functions. Recall that we have the following rigidification results from Section 3, in the case rk(V) = vdim + 1 > 0.

$$\langle \overline{\mu}, \overline{\nu}, (2,0)(1,0)^{m-2} \rangle_{g,\gamma}^{\mathcal{X},\circ} + \sum_{k=1}^{n} \langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{g,\gamma \setminus (k)}^{\mathcal{X},\circ} = (2g-2+p+l(\overline{\mu})+l(\overline{\nu})) \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma}^{\mathcal{X},\circ,\sim}.$$

$$\langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{g,\gamma}^{\mathcal{X},\circ} = \langle \overline{\mu}, \overline{\nu} \rangle_{g,\gamma \sqcup (k)}^{\mathcal{X},\circ,\sim}.$$

We can obtain the following result for the *disconnected* theory, by the same argument as in Proposition 4.4 of [12]. The partition function for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$ is defined as

$$Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu}^1,\dots,\overline{\mu}^r}:=\sum_{\chi,\gamma}z^{-\chi}\frac{x^{\gamma}}{\gamma!}\langle\overline{\mu}^1,\dots,\overline{\mu}^r\rangle_{\chi,\gamma}^{[\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1,\bullet}.$$

Theorem 4.12 (GW crepant resolution). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m, \qquad (2,0)(1,0)^{m-2}, \qquad or \qquad (1,k)(1,0)^{m-1},$$

let $\vec{\mu}, \vec{\nu}, \vec{\rho} \in \mathcal{F}_{A_n}$ be their correspondents. We have

$$Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu}, \vec{\nu}, \vec{\rho}} = Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1)_{\overline{\mu}, \overline{\nu}, \overline{\rho}},$$

under the change of variables

$$s_j = \zeta \exp\left(\frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a\right), \qquad 1 \le j \le n.$$

Proof. The case $\overline{\rho} = (1,0)^m$ is a direct corollary of Lemma 3.3 and 3.4. We now concentrate on $\overline{\rho} = (2,0)(1,0)^{m-2}$ or $(1,k)(1,0)^{m-1}$.

Recall that in Section 3 we have classified all invariants in cases: when $\delta = 1$, invariants reduce to the smooth case [7] and can be matched directly; when $\delta = 0$ and $\mathrm{rk}(V) > 0$, which we call the (†) condition, invariants are all linear in $(t_1 + t_2)$; when $\delta = 0$ and $\mathrm{rk}(V) = \mathrm{vdim} = 0$, invariants are constant in $(t_1 + t_2)$. We try to match the latter two parts separately. The rigidification results can be rewritten as equations

$$Z'_{\mathrm{GW}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}] \times \mathbb{P}^{1})^{\circ}_{\overline{\mu},\overline{\nu},(2,0)(1,0)^{m-2}} + \sum_{k=1}^{n} x_{k} Z'_{\mathrm{GW}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}] \times \mathbb{P}^{1})^{\circ}_{\overline{\mu},\overline{\nu},(1,k)(1,0)^{m-1}}$$

$$= z^{-l(\mu)-l(\nu)} \left(z \frac{\partial}{\partial z} + \sum_{k=1}^{n} x_{k} \frac{\partial}{\partial x_{k}} \right) \left(z^{l(\mu)+l(\nu)-2} Z'_{\mathrm{GW}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}] \times \mathbb{P}^{1})^{\circ,\sim}_{\overline{\mu},\overline{\nu}} \right),$$

$$Z'_{\mathrm{GW}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}] \times \mathbb{P}^{1})^{\circ}_{\overline{\mu},\overline{\nu},(1,k)(1,0)^{m-1}} = z^{-2} \frac{\partial}{\partial x_{k}} Z'_{\mathrm{GW}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}] \times \mathbb{P}^{1})^{\circ,\sim}_{\overline{\mu},\overline{\nu}},$$

and the disconnected version is also true. On the other hand, by the change of variables

$$\frac{\partial}{\partial x_k} = \frac{\zeta^{k/2} - \zeta^{-k/2}}{n+1} \sum_{i=1}^n \zeta^{jk} s_k \frac{\partial}{\partial s_k}.$$

Compared with Proposition 4.4 of [12], we conclude by Theorem 4.10 that the $(t_1 + t_2)$ -linear terms of $Z'_{\text{GW}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu}, \vec{\nu}, \vec{\rho}}$ and $Z'_{\text{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1)_{\overline{\mu}, \overline{\nu}, \overline{\rho}}$ coincide.

It suffices to match the constant terms, which are contributed by invariants satisfying vdim = $\operatorname{rk}(V) = 0$. As discussed in Part b) of Section 3.2, the followings are the only three possibilities for this to give nontrivial connected invariants: $\overline{\mu} = (m_1, 0)(m_2, k), \overline{\nu} = (m, -k); \overline{\mu} = (m_1, k)(m_2, -k), \overline{\nu} = (m, 0); \overline{\mu} = (m, 0), \overline{\nu} = (m, -k)$. Here $m_1 + m_2 = m, k \neq 0$, and $\overline{\mu}, \overline{\nu}$ could be switched, and $\overline{\rho} = (2, 0)(1, 0)^{m-2}$ for the first two and $\overline{\rho} = (1, k)(1, 0)^{m-1}$ for the third possibility. In all these cases we have g = 0 and

$$\langle \overline{\mu}, \overline{\nu}, \overline{\rho} \rangle_{0,\gamma=\emptyset}^{\mathcal{X},\circ} = \langle \mu, \nu, \rho \rangle_0^{\mathbb{P}^1,\circ} \cdot (e_k, e_{-k})_{\mathbb{C}^2/\mathbb{Z}_{n+1}},$$

where $\langle \ \rangle^{\mathbb{P}^1}$ is the relative GW invariant for \mathbb{P}^1 , and $(-,-)_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$ is the orbifold Poincaré pairing.

On the other hand, consider the corresponding $\vec{\mu}$, $\vec{\nu}$ for those cases, and the theory $\langle \vec{\mu}, \vec{\nu}, \vec{\rho} \rangle_{g,\beta}^{\mathcal{A}_n \times \mathbb{P}^1, \circ}$, where $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$. When $\beta \neq 0$, Proposition 4.3 of [12] implies that these invariants always vanish. When $\beta = 0$, we have

$$\langle \vec{\mu}, \vec{\nu}, \vec{\rho} \rangle_{g,\beta=0}^{\mathcal{A}_n \times \mathbb{P}^1, \circ} = \langle \mu, \nu, \rho \rangle_g^{\mathbb{P}^1, \circ} \cdot (\Phi(e_k), \Phi(e_{-k}))_{\mathcal{A}_n},$$

where Φ is the isomorphism (1.1), and $(-,-)_{\mathcal{A}_n}$ is the Poincaré pairing on \mathcal{A}_n . In all three possibilities one can count that the virtual dimension of $\overline{\mathcal{M}}(\mathbb{P}^1,\mu,\nu,\rho)$ is 2g; hence the invariants are only nonzero when g=0. Finally, one can observe that the isomorphism Φ actually preserves the Poincaré pairing, and therefore 3-point functions in the three exceptional cases also match. \square

5. Orbifold quantum cohomology of symmetric products

As a generalization of [6], there is another theory in connection with our picture, the orbifold quantum cohomology of the symmetric products: $\operatorname{Sym}([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ and $\operatorname{Sym}(\mathcal{A}_n)$.

Let X be a DM stack, and $\operatorname{Sym}^m(X) := [X^m/S_m]$ be its m-th symmetric product. We would like to consider the orbifold GW theory of $\operatorname{Sym}^m(X)$. Let $f: \mathcal{C} \to \operatorname{Sym}^m(X)$ be a stable map. Following K. Costello [10], f is equivalent to certain étale cover $C' \to C$ of degree m, together with a map to X, where C is the coarse curve of C in some sense. We will make this picture clear in the case $X = [\mathbb{C}^2/\mathbb{Z}_{n+1}]$.

Connected components of the inertia stack $I \operatorname{Sym}^m(X)$ are indexed by partitions λ of m. For a partition λ , which corresponds to a conjugacy class of S_n , the associated component is

(5.1)
$$\left[(X^n)^{\lambda} / \operatorname{Aut}(\lambda) \right] = X^{l(\lambda)} \times B\operatorname{Aut}(\lambda),$$

where $(X^n)^{\lambda}$ is the fixed loci in X^n under the action of elements in the conjugacy class, $\operatorname{Aut}(\lambda)$ is the stabilizer.

The state space for $\operatorname{Sym}^m(X)$ is its orbifold cohomology $H^*_{\operatorname{orb}}(\operatorname{Sym}^m(X))$, which is the cohomology of $I\operatorname{Sym}^m(X)$, with some degree shift. It is a classical result that the super graded vector space $\bigoplus_{m\geq 0} H^*_{\operatorname{orb}}(\operatorname{Sym}^m(X))$ can be realized as an irreducible highest weight representation of the super-Heisenberg algebra associated with $H^*(X)$. For a reference, see for example Section 5.2 in [3]. As a result, $H^*_{\operatorname{orb}}(\operatorname{Sym}^m(X))$ has a basis indexed by $H^*(X)$ -weighted partitions of m.

In the case when X is a surface, there is an isomorphism

$$H_{\operatorname{orb}}^*(\operatorname{Sym}^m(X)) \cong H^*(\operatorname{Hilb}^m(X)),$$

which respects the gradings and Poincaré pairings. Moreover, if X has trivial canonical bundle, it also preserves the (orbifold) cup product.

5.1. Sym($[\mathbb{C}^2/\mathbb{Z}_{n+1}]$). Precisely, the m-th symmetric product of $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$ is defined as

$$\operatorname{Sym}^{m}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}]) := [[\mathbb{C}^{2}/\mathbb{Z}_{n+1}]^{m}/S_{m}] = [\mathbb{C}^{2m}/(\mathbb{Z}_{n+1} \wr S_{m})],$$

where $\mathbb{Z}_{n+1} \wr S_m := (\mathbb{Z}_{n+1})^m \rtimes S_m$ is the wreath product.

Denote $G := \mathbb{Z}_{n+1} \wr S_m$. Stable maps into $[\mathbb{C}^{2m}/G]$ are the same as those mapping into the origin BG. Evaluation maps land in the orbifold cohomology of BG, which is indexed by conjugacy classes of G, or in other words, indexed by \mathbb{Z}_{n+1} -weighted partitions of m.

The age of a component indexed by such a partition $\overline{\lambda}$ is $m - l'(\overline{\lambda})$, where l' denotes the number of parts with trivial decoration. Components with age 1 are exactly

$$\overline{\rho} = (2,0)(1,0)^{m-2}, \quad \text{or} \quad (1,k)(1,0)^{m-1}, \quad k \neq 0 \in \mathbb{Z}_{n+1}.$$

We aim to compute the 2-point functions, whose moduli space is

$$\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])).$$

Here the datum $(\overline{\mu}, \overline{\nu}; b, \gamma)$ specifies the monodromies around marked points on the domain curve, in which $\overline{\mu}, \overline{\nu}$ are treated as marked points with insertions from the target, and b, γ are extra marked points treated as the analog of "degree class". More precisely, $\overline{\mu}$ and $\overline{\nu}$ are two \mathbb{Z}_{n+1} -weighted partitions, indicating monodromies at two of the marked points; $\gamma = (\gamma_1, \dots, \gamma_{l(\gamma)}) \in \mathbb{Z}_{n+1}$ records extra marked points with monodromies $(1, \gamma_i)(1, 0)^{m-1}, \gamma_i \neq 0, i = 1, \dots, l(\gamma); b \geq 0$ is an integer, recording the number of extra unordered marked points with monodromies $(2, 0)(1, 0)^{m-1}$. The reason why we treat these two types of extra markings differently is that we would like to count only those coming from the Sym operation as unordered.

In other words, let $\bar{D}_{\overline{\mu}}, \bar{D}_{\overline{\nu}}, \bar{D}_0, \dots, \bar{D}_n$ be the connected components in the rigidified inertial stack $\bar{I} \operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ associated with partitions $\overline{\mu}, \overline{\nu}, (2,0)(1,0)^{m-2}$, and $(1,k)(1,0)^{m-1}, k = 1, \dots, n$. Then one can define

$$\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\mathrm{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])):=[\overline{\mathcal{M}}/S_b],$$

where $\overline{\mathcal{M}}$ is the fiber product in the following Cartesian diagram

$$\overline{\mathcal{M}} \xrightarrow{\longrightarrow} \bar{D}_{\overline{\mu}} \times \bar{D}_{\overline{\nu}} \times \bar{D}_{0} \times \cdots \bar{D}_{n}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\overline{\mathcal{M}}_{0,2+b+l(\gamma)}(\operatorname{Sym}^{m}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}])) \xrightarrow{\longrightarrow} (\bar{I} \operatorname{Sym}^{m}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}]))^{2+b+l(\gamma)}$$

The 2-point functions are defined as

$$\langle \overline{\mu}, \overline{\nu} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])} := \sum_{b,\gamma} z^b \frac{x^{\gamma}}{\gamma!} \int_{\left[\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}]))\right]^{\operatorname{vir}}} 1,$$

where the integration is defined by T-localization. Similarly, we can define r-point functions and the following equations directly follow from the definition

$$\langle \overline{\mu}, \overline{\nu}, (2,0)(1,0)^{m-2} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])} = \frac{\partial}{\partial z} \langle \overline{\mu}, \overline{\nu} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])},$$
$$\langle \overline{\mu}, \overline{\nu}, (1,k)(1,0)^{m-1} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])} = \frac{\partial}{\partial x_k} \langle \overline{\mu}, \overline{\nu} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])}.$$

We now apply Costello's construction (e.g. Lemma 2.2.1 in [10]). Let $f: \mathcal{C} \to \operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ be a stable map, representing a closed point in the moduli space. By definition this is equivalent

to a principal S_m -bundle $\mathcal{P} \to \mathcal{C}$, together with an S_m -equivariant map $\mathcal{P} \to [\mathbb{C}^2/\mathbb{Z}_{n+1}]$. Taking $\mathcal{C}' := \mathcal{P} \times_{S_m} \{1, \dots, m\}$, we obtain a diagram

(5.2)
$$\begin{array}{c} \mathcal{C}' \xrightarrow{f'} [\mathbb{C}^2/\mathbb{Z}_{n+1}] \\ \downarrow \\ \mathcal{C}, \end{array}$$

where π is an étale covering. Note that f' is not necessarily representable. The moduli space $\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}]))$ is then isomorphic to the moduli space of such étale coverings.

Let $\underline{\pi}: \underline{\mathcal{C}'} \to \underline{\mathcal{C}}$ be the induced map between coarse moduli spaces, which is a branched covering. The ramification profile is completely determined by the topological datum $(\overline{\mu}, \overline{\nu}; b, \gamma)$. For example, suppose $0, \infty \in \underline{\mathcal{C}}$ are images of the marked points in \mathcal{C} associated with $\overline{\mu}, \overline{\nu}$. Then $\underline{\pi}^{-1}(0)$ consists of $l(\mu)$ points, with ramification degrees $\mu_1, \dots, \mu_{l(\mu)}$, and monodromies given by decorations of $\overline{\mu}$. Similar for $\infty \in \underline{\mathcal{C}}$. Moreover, there are b branched points on $\underline{\mathcal{C}}$ over which the ramification profiles are specified by $(2,0)(1,0)^{m-2}$, and γ_k points on $\underline{\mathcal{C}}$ over which the ramification profiles are specified by $(1,k)(1,0)^{m-1}$. Finally, the genus $g=g(\mathcal{C}')$ can be computed via Riemann–Hurwitz:

$$b = 2g - 2 + l(\mu) + l(\nu).$$

Moreover, the obstruction theories are compatible. Replacing $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$ with $B\mathbb{Z}_{n+1}$, there is a similar description of $\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^m(B\mathbb{Z}_{n+1}))$, where objects are diagrams as above with f' mapping into $B\mathbb{Z}_{n+1}$. Consider the Hurwitz-Hodge bundle V associated with the \mathbb{Z}_{n+1} -representation \mathbb{C}^2 , whose fibers are $H^1(\mathcal{C}, \pi_*(\mathcal{O}_{\mathcal{C}'} \otimes \mathbb{C}^2))$. In the case $\operatorname{rk}(V) > 0$, this is the obstruction bundle, and by the same argument as in previous sections,

$$\int_{\left[\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^{m}(\left[\mathbb{C}^{2}/\mathbb{Z}_{n+1}\right]))\right]^{\operatorname{vir}}} 1 = \int_{\left[\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^{m}(B\mathbb{Z}_{n+1}))\right]^{\operatorname{vir}}} e_{T}(V) \\
= (t_{1}+t_{2}) \int_{\left[\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}(\operatorname{Sym}^{m}(B\mathbb{Z}_{n+1}))\right]^{\operatorname{vir}}} c_{\operatorname{rk}(V)-1}(V).$$

Let $\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ}(\operatorname{Sym}^{m}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}]))$ be the substack parameterizing étale coverings of the form (5.2) with \mathcal{C}' connected. Similar constructions also work. The relationship between connected and disconnected version of invariants are the same as in the relative GW theory.

We now make a key observation: over certain open substacks of the moduli space, the perfect obstruction theories on $\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ}(\operatorname{Sym}^{m}(B\mathbb{Z}_{n+1}))$ and $\overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1}\times\mathbb{P}^{1},\overline{\mu},\overline{\nu})^{\sim}$ are the same, and the Chern class of obstruction bundles vanish on the complements of those open substacks.

Let $\mathcal{U}_1 \subset \overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1} \times \mathbb{P}^1, \overline{\mu}, \overline{\nu})^{\sim}$ be the open substack of relative stable maps with *irreducible* domains. Let $\mathcal{U}_2 \subset \overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ}(\operatorname{Sym}^m(B\mathbb{Z}_{n+1}))$ be the open substack where the \mathcal{C}' as in diagram (5.2) are *irreducible*.

Lemma 5.1. There is an equivalence of stacks

$$\Psi: \mathcal{U}_2 \xrightarrow{\sim} [\mathcal{U}_1/\mathrm{Aut}(\mu) \times \mathrm{Aut}(\nu)].$$

Proof. Given a moduli point [f] in \mathcal{U}_2 represented by a diagram as (5.2), we have a map $\bar{\pi}: \mathcal{C}' \to \mathbb{P}^1$ by composing π with the coarse moduli $\mathcal{C} \to \mathbb{P}^1$. Together with f', this defines a map $(f', \bar{\pi}): \mathcal{C}' \to B\mathbb{Z}_{n+1} \times \mathbb{P}^1$, which might not be representable. We define the image of [f] under Ψ as the relative coarse moduli space $\bar{f}': \mathcal{C}' \to B\mathbb{Z}_{n+1} \times \mathbb{P}^1$ of the map $(f', \bar{\pi})$. By description above, one can see that \bar{f}' is a relative stable map, with the required topological datum. Checking automorphisms, one can see that Ψ is fully faithful.

It is also essentially surjective. In fact, given an object $\bar{f}': C' \to B\mathbb{Z}_{n+1} \times \mathbb{P}^1$ in \mathcal{U}_1 , applying the root construction, one can always add orbifold structures on the branch and ramification divisors, to make the modified map étale, and the associated map to $\operatorname{Sym}^m(B\mathbb{Z}_{n+1})$ representable. Hence we obtain an object in \mathcal{U}_2 whose image is isomorphic to \bar{f}' .

In other words, we have the following diagram:

$$\mathcal{U}_{1} \xrightarrow{j} \overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1} \times \mathbb{P}^{1}, \overline{\mu}, \overline{\nu})^{\sim} \\
\downarrow^{q} \\
\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ}(\operatorname{Sym}^{m}(B\mathbb{Z}_{n+1})) \xrightarrow{p} \overline{\mathcal{M}}_{g,\overline{\mu} \sqcup \overline{\nu} \sqcup \gamma}(B\mathbb{Z}_{n+1}),$$

where j is an open embedding and i is étale, whose image is the open substack \mathcal{U}_2 .

Lemma 5.2. Let $\mathcal{U}_3 \subset \overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})$ be the open substack of irreducible curves. Then the class

$$p_* \left[\overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ} (\operatorname{Sym}^m(B\mathbb{Z}_{n+1})) \right]^{\operatorname{vir}} - \frac{q_* \left[\overline{\mathcal{M}}_{g,\gamma} (B\mathbb{Z}_{n+1} \times \mathbb{P}^1, \overline{\mu}, \overline{\nu})^{\sim} \right]^{\operatorname{vir}}}{|\operatorname{Aut}(\mu)| |\operatorname{Aut}(\nu)|}$$

is supported on the complement $\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})\setminus\mathcal{U}_3$.

Proof. For simplicity, we denote $\overline{\mathcal{M}}_1 := \overline{\mathcal{M}}_{g,\gamma}(B\mathbb{Z}_{n+1} \times \mathbb{P}^1, \overline{\mu}, \overline{\nu})^{\sim}$ and $\overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{0,(\overline{\mu},\overline{\nu};b,\gamma)}^{\circ}(\operatorname{Sym}^m(B\mathbb{Z}_{n+1}))$. Let $\tilde{j}: \tilde{\mathcal{U}}_1 \subset \overline{\mathcal{M}}_1$ be the open substack of relative stable maps with relative dimension 0. Then $\mathcal{U} \subset \tilde{\mathcal{U}}_1$ dense open, and $g^{-1}(\mathcal{U}_3) \subset \tilde{\mathcal{U}}_1$.

Consider the excision exact sequence $A_*(\overline{\mathcal{M}}\backslash \tilde{\mathcal{U}}_1) \to A_*(\overline{\mathcal{M}}) \xrightarrow{\tilde{\jmath}^*} A_*(\tilde{\mathcal{U}}_1) \to 0$. Since obstruction theories are compatible, and $\tilde{\mathcal{U}}_1$ is unobstructed, we have $\tilde{\jmath}^*[\overline{\mathcal{M}}_1]^{\mathrm{vir}} = [\tilde{\mathcal{U}}_1]^{\mathrm{vir}} = [\tilde{\mathcal{U}}_1]$. On the other hand, $\mathcal{U}_1 \subset \tilde{\mathcal{U}}_1$ is dense; so they have the same closure $\bar{\mathcal{U}}_1$ in $\overline{\mathcal{M}}_1$. Hence $[\overline{\mathcal{M}}_1]^{\mathrm{vir}} - [\bar{\mathcal{U}}_1] \in A_*(\overline{\mathcal{M}}_1\backslash \tilde{\mathcal{U}}_1)$. Moreover, q maps $\overline{\mathcal{M}}_1\backslash \mathcal{U}_1$ into the complement of \mathcal{U}_3 , we have $q_*[\overline{\mathcal{M}}_1]^{\mathrm{vir}} - q_*[\bar{\mathcal{U}}_1]$ is supported on $\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})\backslash \mathcal{U}_3$. The same is true for $p_*[\overline{\mathcal{M}}_2]^{\mathrm{vir}} - p_*[\bar{\mathcal{U}}_2]$, since actually $\bar{\mathcal{U}}_2 = \overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_2$ is unobstructed. Finally, looking at generic points, by Lemma 5.1, we have $q_*[\bar{\mathcal{U}}_1] - |\mathrm{Aut}(\mu)| |\mathrm{Aut}(\nu)| \cdot p_*[\bar{\mathcal{U}}_2]$ is supported on the complement of \mathcal{U}_3 . Hence the lemma holds. \square

Recall that the vanishing result Theorem 4.3 states that $c_{r_1+r_2-1}(V_1 \oplus V_2)$, as a class on $\overline{\mathcal{M}}_{g,\overline{\mu}\sqcup\overline{\nu}\sqcup\gamma}(B\mathbb{Z}_{n+1})$, vanishes on the complement of \mathcal{U}_3 . Hence Lemma 5.2 establish an identity between 2-point functions of $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ and rubber GW invariants of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$. Passing back to the disconnected theory, we obtain the following GW/Sym correspondence.

Theorem 5.3 (GW/Sym correspondence). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1}$

where $k \neq 0$, we have

$$z^{l(\mu)+l(\nu)+l(\rho)-m}Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=\langle\overline{\mu},\overline{\nu},\overline{\rho}\rangle_{\mathrm{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])},$$

where the right hand side is the 3-point genus-zero orbifold GW invariants of $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$.

Proof. As explained above, in case $\operatorname{rk}(V) = r_1 + r_2 > 0$, 2-point invariants $\langle \overline{\mu}, \overline{\nu} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])}$ are exactly the same as a 2-point rubber invariants for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$. In other words,

$$\langle \overline{\mu}, \overline{\nu} \rangle_{\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])}$$
 and $z^{l(\mu)+l(\nu)-2} Z'_{\operatorname{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1)^{\sim}_{\overline{\mu},\overline{\nu}}$

have the same $(t_1 + t_2)$ -linear parts.

For the constant terms, i.e. the $\operatorname{rk}(V) = \operatorname{vdim} = 0$ case, the 3-point functions on $\operatorname{Sym}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ reduce to the ordinary orbifold cup product. One can check the identity by direct computation. \square

5.2. Sym (\mathcal{A}_n) . As mentioned before, the orbifold cohomology of $\operatorname{Sym}^m(\mathcal{A}_n)$ are parameterized by $H^*(\mathcal{A}_n)$ -weighted partitions of m. An insertion of partition $\vec{\lambda}$ at a marked point $x \in \mathcal{C}$ indicates that the evaluation map lands into the twisted sector associated with λ . The age of such a component is $m - l(\lambda)$. Hence a basis of $H^2_{\operatorname{orb}}(\operatorname{Sym}^m(\mathcal{A}_n))$ can be taken as the following,

$$(2,1)(1,1)^{m-2}$$
, $(1,\omega_i)(1,1)^{m-1}$, $1 \le i \le n$,

where $\omega_1, \dots, \omega_n \in H^2(\mathcal{A}_n, \mathbb{Q})$ is the dual basis to the exceptional curves in \mathcal{A}_n .

The 3-point function of this theory has been computed by Cheong–Gholampour [9]. By definition the r-point function is

$$\langle \vec{\mu}^{1}, \cdots, \vec{\mu}^{r} \rangle_{\operatorname{Sym}^{m}(\mathcal{A}_{n}), (b, \beta)} := \int_{\left[\overline{\mathcal{M}}_{0, r}(\operatorname{Sym}^{m}(\mathcal{A}_{n}), (b, \beta))\right]^{\operatorname{vir}}} \prod_{i=1}^{r} \operatorname{ev}_{i}^{*} \vec{\mu}^{i},$$

$$\langle \vec{\mu}^{1}, \cdots, \vec{\mu}^{r} \rangle_{\operatorname{Sym}^{m}(\mathcal{A}_{n})} := \sum_{b, \beta} z^{b} s_{1}^{(\beta, \omega_{1})} \cdots s_{n}^{(\beta, \omega_{n})} \langle \vec{\mu}^{1}, \cdots, \vec{\mu}^{r} \rangle_{\operatorname{Sym}^{m}(\mathcal{A}_{n}), (b, \beta)},$$

where the integer b > 0 here stands for the number of unordered extra marked points with monodromies $(2,1)(1,1)^{m-2}$.

Theorem 5.4 (GW/Sym correspondence for \mathcal{A}_n , Theorem 0.3 in [9]). Given $\vec{\mu}, \vec{\nu}, \vec{\rho} \in \mathcal{F}_{\mathcal{A}_n}$, with

$$\vec{\rho} = (1,1)^m, \qquad (2,1)(1,1)^{m-2}, \qquad or \qquad (1,\omega_k)(1,1)^{m-1},$$

we have

$$z^{l(\mu)+l(\nu)+l(\rho)-m} Z'_{\mathrm{GW}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu},\vec{\nu},\vec{\rho}} = \langle \vec{\mu}, \vec{\nu}, \vec{\rho} \rangle_{\mathrm{Sym}^m(\mathcal{A}_n)},$$

where the right hand side is the 3-point genus-zero orbifold GW invariants of $\operatorname{Sym}^m(\mathcal{A}_n)$.

6.1. Relative DT theory and GW/DT correspondence of $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$. Recall the setting of Section 3.1. Let $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$ be the target and z_1, \dots, z_r be r points on $\mathcal{Y} = B\mathbb{Z}_{n+1} \times \mathbb{P}^1$. Consider the DT theory for \mathcal{X} , relative to the fibers $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \{z_i\}$, $i = 1, \dots, r$. For the detailed definition of relative DT theory we refer the readers to [23, 22]. The moduli space of relative DT theory is the relative Hilbert stack

$$\operatorname{Hilb}^{m,\varepsilon}(\mathcal{X}[k],\overline{\mu}^1,\cdots,\overline{\mu}^r),$$

parameterizing 1-dimensional compactly supported closed substacks \mathcal{Z} in the modified "bubbled" target $\mathcal{X}[k]$, for all $k \geq 0$. The stacky curves \mathcal{Z} are required to satisfy some transversality and stability conditions.

Here $m \geq 0$ is an integer, and $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n) \in \mathbb{Z}^{n+1}$. The pair (m, ε) indicates the topological datum

$$[\mathcal{O}_{\mathcal{Z}}] = m[\mathcal{O}_{\mathcal{Y}} \otimes \mathbb{C}_{\text{reg}}] + \sum_{j=0}^{n} \varepsilon_{j} [\mathcal{O}_{\text{pt}} \otimes \mathbb{C}_{\zeta^{j}}] \in K(\mathcal{X}),$$

where \mathbb{C}_{ζ^j} denotes the 1-dimensional \mathbb{Z}_{n+1} -representation with weight j, and \mathbb{C}_{reg} denotes the regular representation.

 $\overline{\mu}^{i}$'s are relative insertions specifying restrictions of \mathcal{Z} to boundary divisors, living in the Fock space

$$\mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]} := H^*(\mathrm{Hilb}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])) \cong H^*(\mathrm{Hilb}^m(\mathcal{A}_n)) = \mathcal{F}_{\mathcal{A}_n}.$$

The relative DT invariants are defined via T-localization with respect to the 2-dimensional torus T acting on the fiber. One can form the generating function

$$\langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_m := \sum_{\varepsilon > 0} q_0^{\varepsilon_0} \cdots q_n^{\varepsilon_n} \langle \overline{\mu}^1, \cdots, \overline{\mu}^r \rangle_{m,\varepsilon},$$

and the corresponding reduced DT r-point function

$$Z'_{\mathrm{DT}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu}^1,\dots,\overline{\mu}^r}:=\frac{\langle\overline{\mu}^1,\dots,\overline{\mu}^r\rangle_m}{\langle\overline{\mu}^1,\dots,\overline{\mu}^r\rangle_0},$$

where we omit the number m which is always fixed and implicit in the formula.

In [22] the following theorem is proved, indicating a close connection between relative reduced DT theory of \mathcal{X} and the quantum cohomology of $\operatorname{Hilb}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$.

Theorem 6.1 (DT/Hilb correspondence, Theorem 1.1 of [22]). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1},$

we have

$$Z'_{\mathrm{DT}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=\langle\overline{\mu},\overline{\nu},\overline{\rho}\rangle_{\mathrm{Hilb}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])},$$

where the right hand side is the 3-point genus-zero GW invariants of $\mathrm{Hilb}^m([\mathbb{C}^2/\mathbb{Z}_{n+1}])$.

In this DT/Hilb correspondence, there is no change of variables or analytic continuation. The Fock spaces on both sides are the same, and the parameters q are identical. Using this result, one can prove a crepant resolution correspondence between the relative DT theories of $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$ and $\mathcal{A}_n \times \mathbb{P}^1$.

Recall that there is an explicit isomorphism between cohomology rings

$$\Phi: H^*_{\text{orb}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]) \cong H^*(\mathcal{A}_n),$$

$$e_0 \mapsto 1, \qquad e_i \mapsto \frac{\zeta^{i/2} - \zeta^{-i/2}}{n+1} \sum_{j=1}^n \zeta^{ij} \omega_j, \qquad 1 \le i \le n,$$

where $\omega_1, \dots, \omega_n \in H^2(\mathcal{A}_n, \mathbb{Q})$ is the dual basis to the exceptional curves in \mathcal{A}_n . Under this isomorphism, we can explicitly identify the Fock spaces $\mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]} \cong \mathcal{F}_{\mathcal{A}_n}$.

For a curve $Z \subset \mathcal{A}_n \times \mathbb{P}^1$, its topological data are specified by the pair $(\chi, (\beta, m))$, where $\chi = \chi(\mathcal{O}_Z) \in \mathbb{Z}$ and $\beta \in H_2(\mathcal{A}_n, \mathbb{Z})$ such that $m[\mathbb{P}^1] + \beta = [Z] \in H_2(\mathcal{A}_n \times \mathbb{P}^1, \mathbb{Z})$. The generating function for the relative DT theory of $\mathcal{A}_n \times \mathbb{P}^1$ is defined in [15] as

$$Z_{\mathrm{DT}}(\mathcal{A}_n \times \mathbb{P}^1)_{\vec{\mu}^1, \cdots, \vec{\mu}^r} := \sum_{\chi, \beta} Q^{\chi} s_1^{(\beta, \omega_1)} \cdots s_n^{(\beta, \omega_n)} \langle \vec{\mu}^1, \cdots, \vec{\mu}^r \rangle_{\chi, (\beta, m)}^{\mathcal{A}_n \times \mathbb{P}^1},$$

and the reduced partition function Z'_{DT} is defined by quotient out the degree 0 contribution. We have the following reselt.

Theorem 6.2 (DT crepant resolution, Theorem 1.2 in [22]). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1},$

let $\vec{\mu}, \vec{\nu}, \vec{\rho} \in \mathcal{F}_{\mathcal{A}_n}$ be their correspondents. We have

$$Q^{-m}Z'_{\mathrm{DT}}(\mathcal{A}_{n}\times\mathbb{P}^{1})_{\vec{\mu},\vec{\nu},\vec{\rho}}=Z'_{\mathrm{DT}}([\mathbb{C}^{2}/\mathbb{Z}_{n+1}]\times\mathbb{P}^{1})_{\overline{\mu},\overline{\nu},\overline{\rho}},$$

under the change of variables

$$Q = q_0 q_1 \cdots q_n, \qquad s_i = q_i, \qquad i \ge 1.$$

The GW/DT correspondence for $\mathcal{A}_n \times \mathbb{P}^1$ was proved in [15].

Theorem 6.3 (GW/DT correspondence for $\mathcal{A}_n \times \mathbb{P}^1$, Theorem 1.1 in [15]). Given $\vec{\mu}, \vec{\nu}, \vec{\rho} \in \mathcal{F}_{\mathcal{A}_n}$, with

$$\vec{\rho} = (1,1)^m,$$
 $(2,1)(1,1)^{m-2},$ or $(1,\omega_k)(1,1)^{m-1},$

we have

$$(-iz)^{l(\mu)+l(\nu)+l(\rho)-m}Z'_{\mathrm{GW}}(\mathcal{A}_n\times\mathbb{P}^1)_{\vec{\mu},\vec{\nu},\vec{\rho}} = (-Q)^{-m}Z'_{\mathrm{DT}}(\mathcal{A}_n\times\mathbb{P}^1)_{\vec{\mu},\vec{\nu},\vec{\rho}},$$

under the change of variables $Q = -e^{iz}$.

Combining with Theorem 4.10 and Theorem 6.2, we obtain the following main theorem.

Theorem 6.4 (GW/DT correspondence). Given $\overline{\mu}, \overline{\nu}, \overline{\rho} \in \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$, with

$$\overline{\rho} = (1,0)^m,$$
 $(2,0)(1,0)^{m-2},$ or $(1,k)(1,0)^{m-1},$

we have

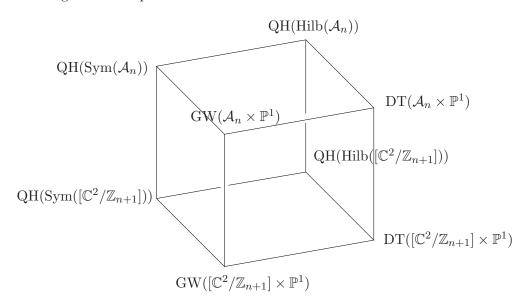
$$(-iz)^{l(\mu)+l(\nu)+l(\rho)-m}Z'_{\mathrm{GW}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times\mathbb{P}^1)_{\overline{\mu},\overline{\nu},\overline{\rho}}=(-1)^mZ'_{\mathrm{DT}}([\mathbb{C}^2/\mathbb{Z}_{n+1}]\times)_{\overline{\mu},\overline{\nu},\overline{\rho}}$$

under the change of variables

$$Q = q_0 q_1 \cdots q_n = -e^{iz}, \qquad q_j = \zeta \exp\left(\frac{1}{n+1} \sum_{a=1}^n (\zeta^{a/2} - \zeta^{-a/2}) \zeta^{ja} x_a\right), \qquad 1 \le j \le n.$$

The formula for change of variables here is exactly the same as the GW/DT correspondence for CY local \mathbb{Z}_{n+1} -gerby curves, proved in [18]. In a future work, we will see that the relative GW/DT theory for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{P}^1$, and more generally, for nontrivial local gerby curves, is closely related to the GW/DT topological vertex in the CY case. The former is also known as the *capped* vertex.

6.2. Equivalences of theories. As a summary of all existing results, we obtain the following diagram, indicating relationships between various theories.



A few remarks on the equivalences:

1) Each line here means a equivalence of theories, in the sense that the 3-point functions of the two theories connected by the line are equal, provided that one insertion is the identity or a divisor. The equality here is possibly up to change of variables and analytic continuation.

- 2) 45° lines indicate GW/Sym and DT/Hilb correspondences. All these correspondences are without change of variables, and are proved via some identification of parts of the moduli's. In
 other words, these are the more geometric correspondences, and the holomorphic symplectic
 structures on \mathcal{A}_n and $[\mathbb{C}^2/\mathbb{Z}_{n+1}]$ play a crucial role.
- Horizontal lines in the front face are GW/DT correspondences. The change of variables involves exponential maps.
- 4) Vertical lines in the front face are crepant resolution correspondences for relative GW and DT theories.
- 5) Lines in the back face indicate the crepant resolution/transformation correspondence for the (partial) Hilbert–Chow morphisms

$$\operatorname{Hilb}(\mathcal{A}_n) \xrightarrow{\cong} \operatorname{Hilb}([\mathbb{C}^2/\mathbb{Z}_{n+1}]) \longrightarrow \operatorname{Sym}(\mathcal{A}_n) \longrightarrow \operatorname{Sym}([\mathbb{C}^2/\mathbb{Z}_{n+1}]),$$

where $\mathrm{Hilb}(\mathcal{A}_n)$ and $\mathrm{Hilb}([\mathbb{C}^2/\mathbb{Z}_{n+1}])$ are mutually symplectic flops, related by wall-crossings.

6.3. Generation conjecture. Consider the Fock spaces $\mathcal{F}_{\mathcal{A}_n} \cong \mathcal{F}_{[\mathbb{C}^2/\mathbb{Z}_{n+1}]}$. Let D_0, \dots, D_n be a basis of the divisors, for example, the obvious ones we have taken in previous sections. 3-point functions of each theory described above define a product structure on \mathcal{F} , and our previous results state that the operators M_{D_i} of multiplication by divisors of each theory are equivalent.

There is a further step one can make to extend the results to general r-point functions. By the degeneration formula, it suffices to know all 3-point functions, with arbitrary insertions, instead of only divisors. In other words, we need to know the multiplication operator M_{γ} for any class $\gamma \in \mathcal{F}$. As in [15], we make the following conjecture.

Conjecture 6.5. The joint eigenspaces for the operators M_{D_i} , $0 \le i \le n$ are 1-dimensional for all m > 0.

Under this conjecture, the ring \mathcal{F} can be generated by divisors D_i , and our results extend.

Corollary* 6.6. Under the above conjecture, all theories are equivalent in the sense that all r-point correlation functions are equal for arbitrary n.

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