Notes on the stability threshold
for radially anisotropic polytropes

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Abstract

We discuss some contradictions found in the literature concerning the problem of stability of collisionless spherical stellar systems which are the simplest anisotropic generalization of the well-known polytrope models. Their distribution function $F(E, L)$ is a product of power-low functions of the energy E and the angular momentum L, i.e. $F \propto L^{-s}(-E)^q$. On the one hand, calculation of the growth rates in the framework of linear stability theory and N-body simulations show that these systems become stable when the parameter s characterizing the velocity anisotropy of the stellar distribution is lower than some finite threshold value, $s < s_{\mathrm{crit}}$. On the other hand Palmer & Papaloizou (1987) showed that the instability remained up to the isotropic limit $s = 0$.

Using our method of determining the eigenmodes for stellar systems, we show that the growth rates in weakly radially-anisotropic systems are indeed positive, but decrease exponentially as the parameter s approaches zero, i.e. $\gamma \propto \exp(-s_*/s)$. In fact, for the systems with finite lifetime this means stability.

Keywords: Galaxy: center, galaxies: kinematics and dynamics.

1   Introduction

Stability properties of stellar spherical clusters determine a set of dynamically allowable equilibrium configurations. Presence of an instability can, for example, lead to ellipsoidal deformation.

For a long time, one believed that any stellar spherical clusters, except for the pathological models, were stable. This belief appeared after the classical works by Antonov (1960, 1962) devoted to isotropic systems, and was reassured in the following papers concerning some particular anisotropic systems

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(e.g., Mikhailovsky et al. 1970, Doremus et al. 1971). So, it needed some time to realize the very possibility of instability of spheres, starting from Polyachenko & Shukhman (1972) until Merritt & Aguilar (1985), Barnes et al. (1986) and May & Binney (1986) have made it widely known.

The radial orbit instability is suppressed for sufficiently rounded orbits, or when the kinetic energy stored in transverse directions $T_\perp$ becomes sufficiently high. Polyachenko & Shukhman (1981) proposed a global anisotropy parameter as a ratio of the radial to transverse kinetic energy of the system, $\xi \equiv 2T_r/T_\perp$. For the Idlis model (Idlis 1956), they found the stability boundary $\xi = 1.59$. Later, Fridman & Polyachenko (1984), using this and two other families of models, proposed a hypothesis that the global anisotropy parameter can give a general stability criterion for anisotropic systems: the system is stable if $\xi < 1.7 \pm 0.25$.[1]

Following studies of spherical models by means of the linear stability analysis (Saha 1991, Weinberg 1991, Bertin et al. 1994) and N-body simulations (Merritt & Aguilar 1985, Barnes et al. 1986, Merritt 1987, Dejonghe & Merritt 1988, Meza & Zamorano 1997) included a variety of models radially anisotropic on the periphery and isotropic in the center, and vice versa. In these works, the stability boundaries in terms of the global anisotropy parameter fall in the broad range $1.2 < \xi < 2.9$. Thus, the hypothesis about universal stabilization in the narrow region of ξ was denied. Note, however, that in each case a certain value of the stability boundary corresponding to the radially anisotropic system ($\xi > 1$) was found. The same is referred to the *generalized polytropes*, with the DF

$$F(E, L) = C(s, q)\, L^{-s}(-E)^q \quad , \tag{1.1}$$

which become stable at $\xi \approx 1.4$ (Fridman & Polyachenko 1984; Barnes et al. 1986). Here E and L denote the energy and the angular momentum of a star, $C(s, q)$ is the normalizing constant, s and q – parameters of the model. Additive constant in gravitational potential $\Phi_0(r)$ is chosen in such a way that $\Phi_0(R) = 0$, where R is the radius of the system.

Generalized polytropes are the simplest generalization of the isotropic polytrope models (the latter correspond to $s = 0$). The polytrope models are classical ones in stability theory of both gaseous and collisionless gravitating systems. One can recall, for example, the work by Antonov (1962), in which stability of polytrope models with decreasing DF was shown. Models with increasing DF can be unstable, but they give exotic mass distribution with increasing density outwards, thus describing unrealistic stellar systems.

The generalized polytropes are more versatile. The global anisotropy parameter for (1.1) can be obtained in a simple form:

$$\xi = \frac{2}{2 - s} \quad . \tag{1.2}$$

Note that for this model the (local) anisotropy parameter $\beta(r)$ (Binney 1980) does not depend on radius, $\beta = 1 - 1/\xi = s/2$. It is possible to evaluate expressions for the radial and the transverse kinetic energy when $s < 2$, the limit $s \to 2$ corresponding to the system in which almost all orbits are radial. Since $s = 0$ case is stable (let us consider only realistic models with $q > 0$), and the case $s \to 2$ is unstable due to the radial orbit instability, there should be a critical value of the parameter $s = s_{\mathrm{crit}}$ which divides stable and unstable systems.

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[1] Results of stability analysis of one of three families of models described in Fridman & Polyachenko (1984) were reconsidered later in Polyachenko (1987) report. Reconsidered stability boundary $2T_r/T_\perp$ fell between 2.05 and 2.10, instead of the previous boundary, $2T_r/T_\perp = 1.62$, i. e. the systems proved to be more stable than it was supposed before. The same boundary for this family was obtained later by Dejonghe and Merritt (1988) with the help of N-body simulations. A new (corrected) boundary is slightly out of the range $2T_r/T_\perp = 1.7 \pm 0.25$ suggested by Fridman & Polyachenko (1984).

Using the matrix method for spheres (Polyachenko & Shukhman 1981), which is analogous to the Kalnajs matrix method for disks (Kalnajs 1977), it was found that growth rates of the instability became small for $s \lesssim 0.6$, almost independently of parameter q (Fridman & Polyachenko 1984). Thus, the critical parameter for generalized polytropes is $s_{\mathrm{crit}} \approx 0.6$ (or $\xi \approx 1.4$). Similar result was obtained by N-body simulations (Barnes et al. 1986).

Palmer & Papaloizou (1987) (henceforth PP87) have investigated the same models using approximate equation for unstable modes with low growth rates. They showed that instability must persist even for models arbitrary close to isotropic limit $s = 0$; this seemingly contradicts previous results mentioned above.

Solutions of the approximate equation form a set of infinite number of unstable modes with decreasing growth rates (eigenvalues) which accumulate near zero frequency $\omega = 0$. These eigenvalues correspond to eigenfunctions with different number of nodes; the nodeless eigenfunction gives the largest growth rate. As was argued in PP87, the largest eigenvalue cannot be caught by the approximate equation, since it is too large to fulfill the assumed condition.

Guided by their models B and C, which correspond to $s = 2/3$ and $s = 1/3$ ($q = 1$), we have calculated the growth rates near the isotropic limit using the approximate equation. The result was paradoxical: the largest eigenvalue kept on grow to infinity as $s \to 0$, while one expects that all modes, including the nodeless one, would cease to zero.

This paper pursues two goals. First, we try to reconcile the results obtained by the matrix method and N-body from one side, and the results of PP87 from the other side. Second, we clarify the paradox about the applicability of the approximate equation by PP87. For simplicity, we shall assume models with $s < 1$ only. This condition provides sufficiently smooth gravitational potential in the center, and linear dependence of the precession velocity on angular momentum for nearly radial orbits (for more details, see below). In Section 2, we derive an approximate equation for modes with low growth rates from the full integral equations for spheres obtained by Polyachenko et al. (2007). It coincides in all but one important detail with the approximate integro-differential equation by PP87 (the equivalence of two equations is demonstrated in Appendix). Numerical results are given in Section 3, Section 4 contains conclusions.

2 The integral equation for modes with low growth rates

Traditional linear stability theories employ matrix methods, expanding the perturbed potential and density in series using special biorthonormal sets of basis functions (Kalnajs 1977, Polyachenko & Shukhman 1981). As a result, one obtains a set of integral equations which incorporates the mode frequency in a complicated nonlinear manner. Thus each frequency is to be obtained separately by, for example, the Cauchy integration in the complex plane.

Recently we have proposed an alternative method for calculation of eigenmodes (Polyachenko 2004, 2005; Polyachenko et al. 2007). The advantages of our method are (i) linear form of the equation for eigenmodes and (ii) absence of the basic biorthonormal set that should be customized for a particular problem. The alternative method is the most adequate to derive the approximate integral equation similar to one used in PP87. We start with the full integral equation for perturbations proportional

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to spherical harmonic with the index l:

$$\phi_{l_1,\,l_2}(E,L) = \frac{4\pi G}{2l+1} \sum_{l_1'=-\infty}^{\infty} \sum_{l_2'=-l}^{l} D_l^{l_2'} \int \int \frac{dE'\,L'\,dL'}{\Omega_1(E',L')} \times$$

$$\times\,\Pi_{l_1,\,l_2;\,l_1',\,l_2'}(E,L;E',L')\,\frac{\phi_{l_1'\,l_2'}(E',L')\mathcal{D}_{l_1',l_2'}F(E',L')}{\omega - \Omega_{l_1'\,l_2'}(E',L')}. \quad (2.1)$$

Integration in (2.1) is over the curved triangle in the phase plane (E',L'): $\Phi_0(0) < E' < 0,\ \ 0 \le L' \le L_{\mathrm{circ}}(E')$; $L_{\mathrm{circ}}(E)$ is the angular momentum on the circular orbit with energy E; ω is the eigenfrequency; $\Omega_{l_1 l_2}(E,L) \equiv l_1\,\Omega_1(E,L) + l_2\,\Omega_2(E,L)$; $\Omega_{1,2}$ are the orbital frequencies; $\mathcal{D}_{l_1',l_2'}F(E',L') \equiv \Omega_{l_1'l_2'}(E',L')\,(\partial F/\partial E') + l_2'\,(\partial F/\partial L')$; the coefficients D_l^k are equal to zero for odd $|l - k|$, otherwise

$$D_l^k = \frac{1}{2^{2l}}\,\frac{(l+k)!(l-k)!}{\left[\left(\tfrac{1}{2}\,(l-k)\right)!\left(\tfrac{1}{2}\,(l+k)\right)!\,\right]^2};$$

l_1 and l_2 are indices of expansion over angular variables w_1 and w_2 in the action–angle formalism (Landau & Lifshitz 1976),

$$\delta\Phi(I_1,I_2,I_3,w_1,w_2) = \sum_{l_1 l_2}(\delta\Phi)_{l_1 l_2}(\mathbf{I})\,\exp[\,i(l_1 w_1 + l_2 w_2)],$$

conjugated to the action variables $I_i\ (i = 1,2,3)$:

$$I_1 = \frac{1}{\pi}\int\limits_{r_{\min}}^{r_{\max}}\sqrt{2E - 2\Phi_0(r) - \frac{L^2}{r^2}}\,dr, \quad I_2 = L,\ \ I_3 = L_z.$$

Due to degeneracy on the azimuthal number m, the eigenfrequency ω can be calculated for axially symmetric perturbations $\delta\Phi(r,\theta;t) = \chi(r)P_l(\theta)e^{-i\omega t}$ only. The kernel of the integral equation is

$$\Pi_{l_1,\,l_2;\,l_1',\,l_2'}(E,L;E',L') = \oint dw_1 \oint dw_1'\,\mathcal{F}_l\,[\,r\,(E,L;w_1),r'(E',L';w_1')] \times$$

$$\times\,\cos\Theta_{l_1 l_2}(E,L,w_1)\,\cos\Theta_{l_1'l_2'}(E',L',w_1'), \quad (2.2)$$

where $\Theta_{l_1\,l_2}(E,L;w_1) = (\Omega_{l_1 l_2}/\Omega_1)\,w_1 - l_2\delta\varphi(E,L;w_1)$,

$$\delta\varphi(E,L,w_1) = L\int\limits_{r_{\min}(E,\,L)}^{r(E,L,w_1)}\frac{dx}{x\,\sqrt{[2E - 2\Phi_0(x)]\,x^2 - L^2}};$$

$$\mathcal{F}_l(r,r') = r_<^l/r_>^{l+1},\quad r_< \equiv \min(r,r'),\quad r_> \equiv \max(r,r').$$

Finally, the eigenfunctions $\phi_{l_1\,l_2}(E,L)$ are connected to the radial part of the perturbed potential as follows:

$$\phi_{l_1\,l_2}(E,L) = \frac{1}{\pi}\int\limits_{0}^{\pi}\cos\Theta_{l_1 l_2}(E,L;w_1)\,\chi\big[r(E,L,w_1)\big]\,dw_1.$$

To obtain the approximate equation by PP87 from (2.1) (rather its full equivalent in the form of the integral equation in E-space), one should make two simplifications, considering (i) low frequencies $\omega = i\gamma$ and even spherical numbers l; (ii) domination of nearly radial orbits.

The denominators of resonance terms $l'_1 = -\frac{1}{2} l'_2$ contain construction proportional to the precession rate

$$\Omega_{l'_1 l'_2}(E', L') = l'_2(\Omega_2 - \tfrac{1}{2}\Omega_1) \equiv l'_2 \Omega_{\mathrm{pr}}(E', L'), \tag{2.3}$$

which is small for nearly radial orbits,[2]

$$\Omega_{\mathrm{pr}}(E, L) = \varpi(E)\, L.$$

Dropping the nonresonance terms and denoting $\phi_{-\frac{1}{2} l_2, l_2}(E, 0) = \Phi(E)$, from (2.1) one can have

$$\Phi(E) = -\frac{8\pi G}{2l+1} \sum_{l'_2=2}^{l} D_l^{l'_2} l'^2_2 \int \int \frac{dE'\, L'\, dL'}{\nu(E')}\, \Phi(E')\, \Pi(E; E') \frac{\Omega_{\mathrm{pr}}(E', L') \dfrac{\partial F}{\partial L'}}{\gamma^2 + l'^2_2 \Omega^2_{\mathrm{pr}}(E', L')}. \tag{2.4}$$

For DF in the form $F(E, L) = g(E)\, L^{-s}$, the approximate equation reads

$$\Phi(E) = \frac{8\pi G}{2l+1} \sum_{k=2}^{l} D_l^k k^2 \int \int \frac{dE'}{\nu(E')}\, \Phi(E')\, g(E')\, \varpi(E')\, \Pi(E; E') \int_0^{} \frac{s\, L^{-s+1} dL}{\gamma^2 + k^2 \varpi^2(E')\, L^2}, \tag{2.5}$$

where $\nu(E) \equiv \Omega_1(E, 0)$, $\Pi(E, E')$ is the result of reduction of $\Pi_{l_1, l_2; l'_1, l'_2}(E, L; E', L')$ for the radial orbits.

Due to singularity of DF, the integral in (2.5) diverge when $\gamma = 0$, and is large when γ is small. This justifies omission of nonresonance terms, and sets constrains on the maximum value of γ.

It is clear that main contribution to the integral comes from a narrow region $L \sim \gamma/\varpi$, thus one can change the variable of integration and replace the upper boundary by infinity:

$$\int_0^{} \frac{L^{-s+1} dL}{\gamma^2 + k^2 \varpi^2\, L^2} = \gamma^{-s}(k\varpi)^{s-2}\, I(s), \tag{2.6}$$

where

$$I(s) \equiv \int_0^{\infty} \frac{x^{-s+1}\, dx}{1 + x^2} = \frac{\pi}{2\, \sin(\frac{1}{2}\pi\, s)}. \tag{2.7}$$

By appropriate change of the eigenfunction, one can reduce the problem to the integral equation

$$\lambda(s)\, \Psi(E) = \int_{-1}^{0} dE'\, \mathcal{R}_s(E, E')\, \Psi(E'), \tag{2.8}$$

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[2]For generalized polytropes, the gravitational potential behaves like $\Phi_0(r) \propto r^{2-s}$ near the center. Thus, for the case of our interest $s < 1$ the gravitational force $-\Phi'_0(r)$ is non-singular at the center and hence the precession velocity of nearly radial orbits is indeed linear with respect to the angular momentum, $\Omega_{\mathrm{pr}}(E, L) \approx \varpi(E)\, L$ (see, e.g., Touma & Tremaine 1997). Note that for singular $\Phi'_0(r)$, (say, $\Phi_0(r) \propto r^p$, with $p < 1$) the dependence of precession rate on L is not linear, $\Omega_{\mathrm{pr}} \propto L^p$.

with

$$\lambda = \gamma^s, \tag{2.9}$$

and the positively defined symmetric kernel function

$$\mathcal{R}_s(E, E') = \alpha(s) \sqrt{h(E)\, h(E')} \; Q(E, E'), \tag{2.10}$$

where $h(E) = g_s(E)\varpi^{s-1}(E)\,\nu(E)$,

$$Q(E, E') = \int\limits_0^{r_{\max}(E)} \int\limits_0^{r_{\max}(E')} \frac{dr\, dr'\, \mathcal{F}_l(r, r')}{\sqrt{2E' - 2\Phi_0(r')}\, \sqrt{2E - 2\Phi_0(r)}}, \tag{2.11}$$

$$\alpha(s) = \frac{(4\pi)^2 G}{(2l+1)} \frac{s}{\sin(\pi s/2)} \sum_{k=2}^{l} D_l^k k^s. \tag{2.12}$$

In Appendix, we show that Eg. (2.8) in E-space is fully equivalent to the approximate integral equation in r-space obtained by PP87.

The kernel (2.10) defines a self-adjoint Hilbert-Schmidt operator in the infinite-dimensional space, so the eigenvalues λ_n ($n = 0, 1, 2...$) must have an accumulation point, $\lim_{n\to\infty} \lambda_n = 0$. Existence of arbitrary small eigenvalues is crucial for PP87 in demonstrating the instability of singular generalized polytropes with $s > 0$.

Note that in the limit $s \ll 1$ Eq. (2.8) with the kernel (2.10) looks unnatural. Let us consider explicitly the case $s = 0$. The integral equation and the kernel then read as

$$\Lambda\Psi(E) = \int\limits_{-1}^{0} dE' \mathcal{R}_0(E, E')\Psi(E'), \tag{2.13}$$

$$\mathcal{R}_0(E, E') = \alpha(0) \sqrt{\frac{g_0(E)\, g_0(E')\, \nu(E)\, \nu(E')}{\varpi(E)\, \varpi(E')}} \; Q(E, E'), \tag{2.14}$$

with $\Lambda \equiv \lambda(0)$, $\alpha(0) = [32\pi G/(2l+1)] \sum_{k=2}^{l} D_l^k$. For example, in the units where $4\pi G = 1$, for $l = 2$ one has $\alpha(0) = 3/5$. A norm of the kernel is of order unity and thus first several eigenvalues, corresponding to eigenfunctions with few nodes, must be of order unity. It is needed to emphasize that there is no small parameter left in the problem (2.13), the only small parameter s in the isotropic limit has disappeared from the equations.

We have calculated several largest eigenvalues Λ_n for spherical indices $l = 2, 4, 6$. The results are summarized in the Table. The eigenvalues exceeding 1 are emphasized by boldface. In particular, for $l = 2$ two eigenvalues are greater than 1. This means that an arbitrary small anisotropy (or arbitrary small s) will produce exponentially high growth rates:

$$\gamma_n = \Lambda_n^{1/s} \propto \exp\left(\frac{1}{s} \ln \Lambda_n\right), \quad n = 0, 1. \tag{2.15}$$

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$l \setminus n$	0	1	2	3	4	5
2	**3.7851**	**1.0301**	0.4762	0.2637	0.1621	0.1068
4	**1.3838**	0.4194	0.2215	0.1370	0.0921	0.0654
6	0.7029	0.2209	0.1232	0.0803	0.0566	0.0419

Table 1: The largest 6 eigenvalues Λ_n (i.e. λ_n at $s = 0$) for quadrupole $l = 2$, and next two spherical harmonics ($l = 4$ and $l = 6$) for parameter $q = 1.0$.

However, the growth rates of other modes are exponentially small:

$$\gamma_n(s) = \Lambda_n^{1/s} \propto \exp\left(-\frac{1}{s} \ln \frac{1}{\Lambda_n}\right), \quad n \geq 2. \tag{2.16}$$

Note that (2.15) contradicts Eq. (2.4), in which the kernel becomes zero at $s = 0$ due to the term $\partial F / \partial L$. The inconsistency evidently comes from changing the upper limit of integration in (2.6) to infinity: for $s = 0$ this integral turns into $\int_0^{L_{\max}} L dL / (\gamma^2 + k^2 \varpi^2 L^2)$ and diverges if $L_{\max} \to \infty$. However, such a form of the integrand is valid for nearly radial orbits only. Besides, we have expanded the integration region up to infinite angular momentum. These are justified for the systems mainly populated by nearly radial orbits, but not for the nearly isotropic ones.

To cope with anomalously growing modes (2.15), one can take into account the finite value of $L_{\max}$ in (2.6). Changing the variable of integration, $L = [\gamma/(k\varpi)]x$, the integral (2.6) can be reduced to

$$\int_0^{L_{\mathrm{circ}}} \frac{L^{-s+1} dL}{\gamma^2 + k^2 \varpi^2 L^2} = \gamma^{-s}(k\varpi)^{s-2} \int_0^{k\varpi L_{\mathrm{circ}}/\gamma} \frac{x^{-s+1} dx}{1 + x^2}.$$

Since $\varpi L_{\mathrm{circ}} \sim \Omega_2(E, L_{\mathrm{circ}}(E)) - \frac{1}{2}\Omega_1(E, L_{\mathrm{circ}}(E)) = \Omega - \frac{1}{2}\varkappa$, where Ω and $\varkappa$ are the circular and radial frequencies, one can replace the upper boundary of integration by Ω/γ, with some characteristic dynamical frequency $\Omega \sim \Omega_D$, $\Omega_D \equiv (GM/(2R^3))^{1/2}$. Then instead of (2.8), we have

$$\bar{I}(s, \gamma/\Omega) = \int_0^{\Omega/\gamma} \frac{x^{1-s} dx}{1 + x^2} = \frac{1}{2} \int_{\gamma^2/\Omega^2}^1 z^{s/2-1}(1 - z)^{-s/2} dz \tag{2.17}$$

(parameter $\gamma/\Omega \ll 1$). At small s, this integral is finite,

$$\bar{I}(s, \gamma/\Omega) = s^{-1}[1 - (\gamma/\Omega)^s] = s^{-1}[1 - \exp(-s\ln(\Omega/\gamma))],$$

so $\bar{I}(0, \gamma/\Omega) = \ln(\Omega/\gamma)$, and the kernel turns to zero at $s = 0$. Unfortunately, it is impossible to take the integral explicitly for arbitrary s. Instead, in our crude approximation, we shall use a model expression obeying necessary features:

$$\bar{I}(s, \Omega/\gamma) = \frac{\pi}{2\sin(\frac{1}{2}\pi s)}\left[1 - \left(\frac{\gamma}{\Omega}\right)^s\right] = \frac{\pi}{2\sin(\frac{1}{2}\pi s)}\left[1 - \exp\left(-s\ln\frac{\Omega}{\gamma}\right)\right]. \tag{2.18}$$

For very small s, $s \ll \left[\ln(\Omega/\gamma)\right]^{-1} \ll 1$, the expression gives $\bar{I}(s, \gamma/\Omega) \approx \ln(\Omega/\gamma)$, for $s \gg \left[\ln(\Omega/\gamma)\right]$ it coincides with old expression (2.7).

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The corrected approximate integral equation can be written in the same form as the old one, (2.8), (2.10), but instead of (2.9) one should write a new relation between λ and γ:

$$\lambda = \frac{\gamma^s}{1 - (\gamma/\Omega)^s}, \quad \gamma = \left(\frac{\lambda}{1 + \lambda/\Omega^s}\right)^{\frac{1}{s}}. \tag{2.19}$$

This relation clearly provides exponentially small growth rates for all modes at $s \to 0$. In this limit, the spectrum of eigenvalues $\lambda_n(s)$ remains the same, but now

$$\gamma_n(s) \propto \left(\frac{\Lambda_n}{1 + \Lambda_n}\right)^{1/s} = \exp\left[-\frac{1}{s}\ln\left(1 + \frac{1}{\Lambda_n}\right)\right] \tag{2.20}$$

asymptotically tends to zero when $s \to 0$.

3 Unstable modes of generalized polytropes

For numerical evaluations of growth rates of anisotropic polytropes, one usually introduces the following units:

$$4\pi G = 1, \quad A(s, q) = 1, \quad \Psi(0) = 1, \tag{3.1}$$

where G is the gravitational constant, $\Psi(r)$ is the relative potential, $\Psi(r) = -\Phi_0(r)$, and $A(s, q)$ is the coefficient in the expression for density distribution,

$$\rho_0(r) = A(s, q)\, r^{-s}\, \Psi^{q + \frac{3-s}{2}}. \tag{3.2}$$

The relation between $A(s, q)$ and $C(s, q)$ is well-known (see, e.g., Fridman & Polyachenko 1984):

$$A(s, q) = C(s, q)\, (2\pi)^{3/2}\, 2^{-s/2} \frac{\Gamma(q + 1)\, \Gamma\left(-\frac{1}{2}s + 1\right)}{\Gamma\left(q + \frac{1}{2}(5 - s)\right)},$$

$\Gamma(x)$ denotes the Gamma function.

In these units, the dimensionless potential $\psi(r) = -\Phi_0(r)$ satisfies the Poisson equation:

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + r^{-s}\psi^{q + \frac{3-s}{2}} = 0, \tag{3.3}$$

with a boundary condition $\psi(0) = 1$. To find a second boundary condition, one can notice that the Poisson equation possesses a solution in terms of a power series in $z = r^{2-s} \equiv r^p$ (see also Hénon 1973). Denoting $\bar{\psi}(z) = \psi(r)$, one has

$$\frac{d^2\bar{\psi}}{dz^2} + \frac{3-s}{2-s}\frac{1}{z}\frac{d\bar{\psi}}{dz} + \frac{1}{(2-s)^2}\frac{1}{z}\bar{\psi}^{q + \frac{3-s}{2}} = 0. \tag{3.4}$$

In the limit $z \ll 1$, the solution is $\bar{\psi}(z) = 1 + D_1 z + \mathcal{O}(z^2)$, where $D_1 = -1/[(2-s)(3-s)]$. Thus, one can obtain the potential by integrating (3.4) from $z = 0$, with the boundary conditions $\bar{\psi}(0) = 1$, $\left(d\bar{\psi}/dz\right)\big|_{z=0} = D_1$. The right boundary of integration z_R is determined from the condition $\bar{\psi}(z_R) = 0$, thus radius of the system is $R = z_R^{1/(2-s)}$.

Evaluation of the radial frequency $\nu(E) \equiv \Omega_1(E, 0)$ is usual. The precession rate at low angular momenta $(\varpi(E) = (\partial \Omega_{\mathrm{pr}}(E, L)/\partial L)|_{L=0})$ is calculated according to Eq. (2.11) by Polyachenko et al. (2007):

$$\varpi(E) = \frac{\nu(E)}{\pi} \left[\int\limits_{0}^{r_{\max}(E)} \frac{dr}{r^2} \left(\frac{1}{\sqrt{2E + 2\Psi(r)}} - \frac{1}{\sqrt{2E + 2}} \right) - \frac{1}{r_{\max}(E)\sqrt{2E + 2}} \right].$$

The following asymptotic formulas, applicable for stars in the very center of the sphere, $\varepsilon \equiv E + 1 \ll 1$, are used for testing of the precision of calculations:

- $s \ll 1$

$$\varpi(\varepsilon) = \frac{1}{40} \left(q + \frac{3}{2} \right) \left[1 + \frac{1}{160} \varepsilon \left(1 - \frac{262}{7} q \right) \right] + \frac{s}{12\,\varepsilon};$$

- otherwise, $s < 1$

$$\nu(\varepsilon) = \sqrt{2\pi} \, \frac{\Gamma(\frac{1}{p} + \frac{1}{2})}{\Gamma(1 + \frac{1}{p})} \, K^{\frac{1}{p}} \, \varepsilon^{\frac{1}{2} - \frac{1}{p}}; \tag{3.5}$$

$$\varpi(\varepsilon) = -\frac{\Gamma(\frac{1}{2} + \frac{1}{p}) \, \Gamma(1 - \frac{1}{p})}{\Gamma(1 + \frac{1}{p}) \, \Gamma(\frac{1}{2} - \frac{1}{p})} \, K^{\frac{2}{p}} \, \varepsilon^{-\frac{2}{p}} > 0, \tag{3.6}$$

where $K^{-1} = p\,(1 + p)$, $p = 2 - s$.

Some problems were experienced in calculating the function $Q(E, E')$ (2.11), which is a part of the kernel (2.10). The following asymptotic formulas are useful (valid for $l = 2$):

- on the diagonal $\varepsilon = \varepsilon' \ll 1$, $s = 0$

$$Q(\varepsilon, \varepsilon) \simeq \sqrt{\frac{3}{2}} (\frac{1}{2} + \mathbf{G}) \, \varepsilon^{-1/2} = 1.734 \, \varepsilon^{-1/2},$$

where $\mathbf{G} = 0.915\,965\,594...$ is the Catalan's constant;

- $\varepsilon' \ll \varepsilon \ll 1$, arbitrary s

$$Q(\varepsilon, \varepsilon') = \frac{5\sqrt{\pi}}{12p} \, \frac{\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p} + \frac{1}{2})} \, K^{-\frac{1}{p}} \, \varepsilon^{-\frac{1}{2}} \, (\varepsilon')^{\frac{1}{p} - \frac{1}{2}}. \tag{3.7}$$

The integral equation (2.8) have been solved for modes with the spherical harmonics $l = 2, 4, 6$. First of all, we were interested in models with small parameter $s \ll 1$ to find out how small the growth rates of weakly anisotropic systems are. Also we have calculated growth rates for models B ($s = 2/3$, $q = 1$) and C ($s = 1/3$, $q = 1$) of PP87. In the numerical work, we restricted ourselves by models within the range $0 \leq s \leq 0.8$.

Fig. 1 shows the behavior of the characteristic frequency $\Omega_D(s) = (GM/2R^3)^{1/2}$ for $q = 1$ and $q = 0.7$. One can see that the dependence is weak and $\Omega_D \sim 0.1$ in the considered range of the parameter s.
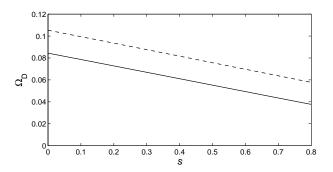
Figure 1: The dependence of characteristic frequency Ω_D on the parameter s for $q = 1$ (solid line) and $q = 0.7$ (dashed line).
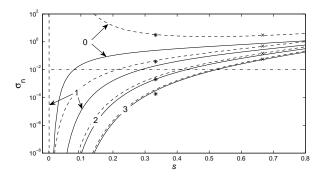

Figure 2: The dependence of the scaled growth rates $\sigma_n \equiv \gamma_n/\Omega_D$ of four most unstable modes on the parameter s for the model $q = 1$: dashed curves show solutions for (2.9), solid curves – for the corrected relation (2.19). Crosses show values of σ_n for model B, stars – for model C from PP87. The characteristic frequency is $\Omega = 0.08$.

Fig. 2 shows the growth rates σ_n in units of characteristic frequency Ω_D v.s. the parameter s for $q = 1$. Dashed lines show growth rates obtained from (2.8) with relation (2.9); this case is equivalent to the approximate equation used by PP87. For $s = 1/3$ and $s = 2/3$, our growth rates agree satisfactorily with one obtained in the cited paper. Numbers denote modes (0 – the nodeless mode, 1 – the mode with one node, etc.). The growth rates of the first two modes increase violently as the model approach the isotropic limit, other modes decrease exponentially to zero.

Solid lines show the growth rates v.s. the parameter s obtained from (2.19). Its behavior complies with intuitive expectations that the unstable modes should be stabilized in the isotropic limit.

A region near $s \approx 0$ is magnified in Fig. 3. Due to fast (exponential) decrease of the growth rates, they become negligibly small at $s = 0.01$.

Fig. 4 shows several first eigenfunctions $\Psi_n(E)$ ($n = 0, 1, 2, 3$) of the integral equation (2.8) for the
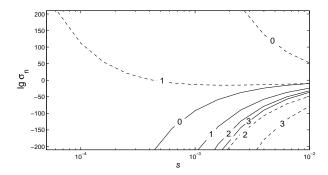
Figure 3: The same as in Fig. 2, in lg – lg axis.

model $s = 1/3$, $q = 1$ and corresponding radial parts of the potential $\chi_n(r)$:

$$\chi_n(r) = \int_{-1}^{0} E \sqrt{h(E)} \, \Psi_n(E) \int_{0}^{r_{\max}(E)} \frac{dr' \, \mathcal{F}_l(r, r')}{\sqrt{2E - 2\Phi_0(r')}} \tag{3.8}$$

(for derivation of this relation, see Appendix). In all cases, the eigenfunctions have equal number of nodes coinciding with n. The form of radial parts of the potential $\chi_n(r)$ is in qualitative agreement with those presented in Fig. 1 of PP87, which are the solutions of the integral equation (2.25).

It is interesting to compare the growth rates obtained with our approximate integral equation with independent calculations of generalized polytropes. In Fig. 5, we show the dependence of growth rates for first seven modes on the parameter s for $q = 0.7$. Crosses mark an "experimental" curve from Fridman & Polyachenko (1984). The curve breaks at $s \approx 0.6$, $\gamma \approx 0.004$ because of the accuracy of the matrix method employed. The numbers poorly agree with each other, so that it is impossible to join the curve marked by crosses with a new curve of the principal mode $n = 0$.

The validity of the approximate integral equation is restricted to very small growth rates. For modes with sufficiently high n, this restriction does not play any role since their growth rates are small for all s, but it is important for modes with few nodes, especially for the nodeless one. Its growth rate increases with anisotropy, and presumably at $s \lesssim 0.15$ achieves the upper boundary. This might be the reason of the evident discrepancy between the two curves. In Fig. 5, the uncertainty region $0.15 \lesssim s < 0.6$ is shown by a dashed line.

4   Conclusions

1. It is difficult to reconcile the results of the linear stability analysis (Fridman & Polyachenko, 1984) and N-body experiments (Barnes et al. 1986) with the results by PP87 for two reasons.

First, the approximate integral equation derived in PP87, on which their analysis is based, is applicable to very small growth rates only. In the isotropic limit corresponding $s \to 0$, these growth rates are exponentially small, i.e. $\gamma \propto \exp(-s_*/s)$. The estimates show, that even for $s \approx 0.5$, the allowed growth rates are much less than $0.01\Omega_D$, that is definitely below any reasonable accuracy of the matrix method and N-body experiments.
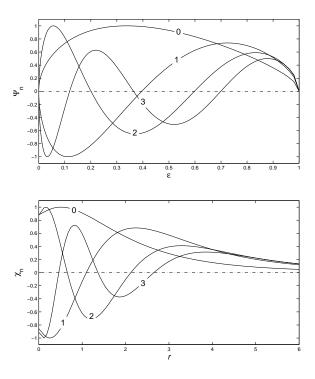
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Figure 4: The first four eigenfunctions of quadrupole harmonics $l = 2$: $\Psi_n(E)$ (upper figure) and the radial part of the potential $\chi_n(r)$ (lower figure). The normalization of the eigenfunction is arbitrary.
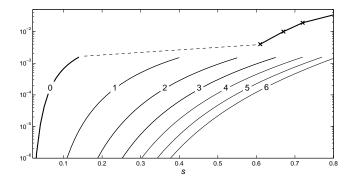

Figure 5: The dependence of the growth rates γ_n from (2.19) of the first seven unstable modes on parameter s for the model $q = 0.7$. Crosses mark the growth rates obtained by Fridman & Polyachenko (1984). The characteristic frequency is taken $\Omega = 0.02$.

Second, strictly speaking, the approximate integral equation is not applicable to unstable modes having eigenfunctions with just few nodes, including the principal mode with maximum growth rate, since for not too large n the eigenvalues λ_n are not very small even in the limit $s \to 0$. However the principal mode plays the major role in determining the stability boundary (see Fridman & Polyachenko (1984), Barnes et al. (1986)).

2. The growth rates formally accurate when they are small. However, it is likely that once $\sigma_n \equiv \gamma_n/\Omega_D < 1$, they give reasonable estimates of actual growth rates in practice (PP87). Then since isotropic systems with decreasing DF are stable (Antonov 1960, 1962), *all* modes should become stable when s approaching zero, and valid approximate equation must describe *all* modes correctly. This fact is in contradiction to our solution of PP87's approximate equation (2.25) (equivalent of our Eq. (2.8) in which the relation $\lambda = \gamma^s$ is assumed). This solution demonstrates explicitly that for the quadrupole harmonic $l = 2$ there are two modes with exponentially increasing growth rates for $s \to 0$.

The reason for such a discrepancy arises from behavior of the terms of the approximate integral equation in the limit $s \to 0$. Using the notation of PP87, the growth rates can be expressed in the form

$$\gamma^s = \frac{A_3}{A_1 - A_2},$$

where A_i are some averages (quadratic forms) of the positively defined operators.[3] In particular, A_3 denotes the average of the integral operator defined by (2.10), A_2 denotes the omitted nonresonance part and A_1 denotes the operator which is a left side of the radial Poisson equation. Since both terms A_1 and A_3 retain in the limit when the system becomes isotropic, Palmer & Papaloizou infer that instability exists no matter how weak the divergence in DF as $L \to 0$ is.

In this paper we argue that the term A_3 *must vanish* when $s \to 0$ in order to comply with stabilization of isotropic models. Using our alternative method of determining the unstable eigenmodes based on solution of the linear eigenvalue problem, we derived the appropriate integral equation. This equation gives a set of unstable modes, all of which become stable in the isotropic limit $s \to 0$.

To summarize, our considerations prove that the instability growth rates of all modes in unbounded models at $L = 0$ indeed do not vanish unless the models are isotropic, but they becomes exponentially small. Actually it means stability if we take into account a finite lifetime of real astronomical objects. Besides, the most probable distributions are non-singular ones, so we have to infer that stable distributions generally become unstable at some *finite* value of radial anisotropy, i.e. finite anisotropy threshold exists. Width of the threshold depends on a particular model.

Acknowledgments

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[3]For convenience, we have changed signs of all three PP87's operators $\mathcal{L}_i$ to make them positively defined.

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APPENDIX A. Equivalence of the integral equation in E-space and the integro-differential equation (2.25) of PP87 in r-space

Actually, the integro-differential equation (2.25) of PP87 presents the radial part of the Poisson equation for l-th harmonic of the potential

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\chi(r)}{dr} - \frac{l(l+1)}{r^2}\chi(r) = 4\pi\,G\,\Pi(r), \tag{A1}$$

where $\Pi(r)$ is the radial part of a density perturbation $\delta\rho(r,\theta;t) = \Pi(r)P_l(\cos\theta)\,e^{-i\omega t}$. We can calculate it using an expression for the perturbed DF δf obtained from the kinetic equation written in the actions–angles variables (Landau & Lifshitz 1976). For details of calculation of δf see Polyachenko & Shukhman (1981) or Fridman & Polyachenko (1984). Thus we have

$$\Pi(r)P_l(\cos\theta) = \int d\boldsymbol{v}\,\delta f(\boldsymbol{r},\boldsymbol{v}) =$$

$$= -\sum_{l_1}\sum_{l_2}\int d\boldsymbol{v}\,P_l^{l_2}(0)P_l^{-l_2}(\sin\theta_0)e^{il_2\pi}\phi_{l_1 l_2}(E,L)\frac{l_1\,\partial F/\partial I_1 + l_2\,\partial F/\partial I_2}{\omega - \Omega_{l_1 l_2}}\,e^{i(l_1 w_1 + l_2 w_2)}, \tag{A2}$$

where $\sin\theta_0 = L_z/L$. Separation of the radial part of perturbed density yields:

$$\Pi(r) = \tfrac{1}{2}\,(2l+1)\int_0^\pi \delta\rho(r,\theta)P_l(\cos\theta)\,\sin\theta\,d\theta. \tag{A3}$$

We can integrate over θ using addition theorem for Legendre polynomials and a relation which connects θ with the angular variable w_2

$$\cos\theta = \cos\theta_0\cos(w_2 - \partial S_1/\partial I_2), \tag{A4}$$

where S_1 is radial action,

$$S_1(r) = \int_{r_{\min}}^r dr'\,\sqrt{2E - 2\Phi_0(r') - L^2/r'^2}.$$

We have

$$P_l(\cos\theta) = \sum_{k=-l}^l e^{-ik(w_2 - \partial S_1/\partial I_2)}P_l^k(\sin\theta_0)\,P_l^{-k}(0)\,e^{-ik\pi}. \tag{A5}$$

Since

$$v_r = \pm\sqrt{2E - 2\Phi_0(r) - \frac{L^2}{r^2}}, \quad v_\theta = \pm\frac{L}{r}\sqrt{1 - \frac{\sin^2\theta_0}{\sin^2\theta}}, \quad v_\varphi = \frac{L}{r}\frac{\sin\theta_0}{\sin\theta},$$

then we obtain for the volume element in the velocity space

$$d\boldsymbol{v} \equiv dv_r dv_\theta dv_\varphi = \frac{4L\,dL\,dE\,d(\sin\theta_0)}{r^2\sqrt{2E - 2\Phi_0(r) - L^2/r^2}\,\sqrt{\cos^2\theta_0 - \cos^2\theta}}.$$

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(The factor 4 appears here because we have to take into account particles having velocities v_r and v_θ of both signs.) To integrate over θ in (A3) it is convenient to go to integration over w_2. We have

$$\sin\theta \, d\theta = \cos\theta_0 \, \sin(w_2 - \partial S_1/\partial I_2) \, dw_2$$

and

$$\int\limits_0^\pi \frac{P_l(\cos\theta)\, e^{il_2 w_2}\, \sin\theta \, d\theta}{\sqrt{\cos^2\theta_0 - \cos^2\theta}} = \frac{1}{2}\int\limits_0^{2\pi} P_l(\cos\theta)\, e^{il_2 w_2} dw_2. \qquad (A6)$$

As a result we obtain

$$\Pi(r) = -\frac{1}{4}\,(2l+1)(2\pi)\sum_{l_1}\sum_{l_2} \int\int 4L\,dLdE\,\frac{\phi_{l_1 l_2}(E,L)\,e^{i(l_1 w_1 + l_2 \partial S_1/\partial I_2)}}{r^2\,\sqrt{2E - 2\Phi_0(r) - L^2/r^2}}\,\frac{l_1\,\partial F/\partial I_1 + l_2\,\partial F\partial I_2}{\omega - \Omega_{l_1 l_2}} \times$$

$$\times \int\limits_{-1}^1 dz P_l^{l_2}(0)\, P_l^{-l_2}(z)\, P_l^{-l_2}(0)\, P_l^{l_2}(z), \quad z \equiv \sin\theta_0. \qquad (A7)$$

Taking into account that the integral over z is

$$\int\limits_{-1}^1 dz\, P_l^{l_2}(0) P_l^{-l_2}(z) P_l^{-l_2}(0) P_l^{l_2}(z) = \frac{2}{2l+1}\,D_l^{l_2},$$

we find the final expression for the radial part of perturbed density $\Pi(r)$:

$$\Pi(r) = -\frac{4\pi}{r^2}\sum_{l_1}\sum_{l_2=-l}^{l} D_l^{l_2}\int\int L\,dL\,dE\,\frac{\phi_{l_1 l_2}(E,L)\,\cos\Theta_{l_1 l_2}}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}}\,\frac{\Omega_{l_1 l_2}\,\partial F/\partial E + l_2\,\partial F/\partial L}{\omega - \Omega_{l_1 l_2}}, \quad (A8)$$

where (see Sec. 2)

$$\phi_{l_1\,l_2}(E,L) = \frac{1}{2\pi}\oint\limits_0^\pi \cos\Theta_{l_1 l_2}(E,L;w_1)\,\chi\big[r(E,L,w_1)\big]\,dw_1 = \frac{\Omega_1}{\pi}\int\limits_{r_{\min}}^{r_{\max}} \frac{dr\,\chi(r)\,\cos\Theta_{l_1 l_2}}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}}.$$

Keeping in the sum over l_1 the resonance summand $l_1 = -\frac{1}{2}\,l_2$ and supposing growth rate to be small, we retain the contribution with $\partial F/\partial L$ only:

$$\Pi(r) \approx \frac{8\pi}{r^2}\sum_{l_2=2}^{l} D_l^{l_2} l_2^2 \int L\,dL\int \frac{\phi_{-l_2/2,\,l_2}(E,L)\,dE}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}}\,\frac{\Omega_{\rm pr}\,\partial F/\partial L}{\gamma^2 + l_2^2\Omega_{\rm pr}^2}\,\cos\Theta_{-l_2/2,\,l_2}. \qquad (A9)$$

Since the leading contribution to the integral over L comes from small L, we put where it is possible, $L = 0$ and suppose $\Omega_{\rm pr} = \varpi(E)\,L$. Then $\Theta_{-l_2/2,\,l_2}(E,0) = l_2\pi$, and finally

$$\Pi(r) = -\frac{8\pi s\,I(s)}{\gamma^s}\frac{1}{r^2}\sum_{l_2=2}^{l} D_l^{l_2} l_2^s \int \frac{g(E)\,\varpi^{s-1}(E)\,dE}{\sqrt{2E - 2\Phi_0(r)}}\left[\frac{\nu(E)}{\pi}\int\limits_0^{r_{\max}(E)} \frac{dr'\chi(r')}{\sqrt{2E - 2\Phi_0(r')}}\right]. \qquad (A10)$$

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Substitution to r.h.s. of (A1) yields integro-differential equation (2.25) of PP87 in r-space:

$$\frac{d}{dr}\, r^2\, \frac{d\chi(r)}{dr} - l(l+1)\,\chi(r) = -\frac{4\pi\,G}{\gamma^s} \int\limits_0^R K(r,r')\,\chi(r')\,dr', \qquad (A11)$$

with the kernel

$$K(r,r') = \frac{4\pi s}{\sin(\pi\,s/2)} \sum_{k=2}^{l} D_l^k k^s \int\limits_{\max\left[\Phi_0(r),\Phi_0(r')\right]} \frac{g(E)\,\varpi^{s-1}(E)\,\nu(E)\,dE}{\sqrt{2E-2\Phi_0(r)}\,\sqrt{2E-2\Phi_0(r')}}. \qquad (A12)$$

Now it is easy to demonstrate that (A11) is equivalent to the integral equation (2.8) in E-space. We write (A10) in the form

$$\Pi(r) = -\frac{4\pi^2 s}{\sin(\pi s/2)}\frac{1}{\gamma^s}\frac{1}{r^2} \sum_{k=2}^{l} D_l^k k^s \int \frac{g(E)\,\varpi^{s-1}(E)\,dE}{\sqrt{2E-2\Phi_0(r)}}\,\Phi(E), \qquad (A13)$$

where

$$\Phi(E) = \frac{\nu(E)}{\pi} \int\limits_0^{r_{\max}(E)} \frac{dr'\chi(r')}{\sqrt{2E-2\Phi_0(r')}} \qquad (A14)$$

is the radial part of perturbed potential $\chi(r)$ averaged over radial orbit. Rewriting the Poisson equation (A1) in the integral form,

$$\chi(r) = -\frac{4\pi G}{2l+1} \int dr'r'^2\,\Pi(r')\,\mathcal{F}_l(r,r'), \qquad (A15)$$

and averaging both part of this equation over radial orbit with the energy E according to (A14), we obtain the integral equation

$$\Phi(E) = \frac{1}{\gamma^s} \int dE'\mathcal{K}(E,E')\,\Phi(E'), \qquad (A16)$$

where

$$\mathcal{K}(E,E') = \frac{(4\pi)^2 G}{(2l+1)}\frac{s}{\sin(\pi s/2)}\, g(E')\,\varpi^{s-1}(E')\,\nu(E)\times$$

$$\times \sum_{k=2}^{l} D_l^k k^s \int\limits_0^{r_{\max}(E)} dr \int\limits_0^{r_{\max}(E')} dr' \frac{\mathcal{F}_l(r,r')}{\sqrt{2E'-2\Phi_0(r')}\,\sqrt{2E-2\Phi_0(r)}}.$$

Symmetrizing this equation with the help of the substitution

$$\Psi(E) = \sqrt{\frac{g(E)\,\varpi^{s-1}(E)}{\nu(E)}}\,\Phi(E),$$

we obtain the integral equation (2.8) with the kernel (2.10) presented in the main text.

Finally, using the expression (A13) for $\Pi(r)$ and the integral form of the Poisson equation (A15), we can easily obtain the relation (3.8), which connects the eigenfunction $\Psi_n(E)$ of the integral equation (2.8) with the eigenfunction $\chi_n(r)$ of PP87's integro-differential equation (2.25) (or Eq. (A11) in our notations).