A Sufficient Condition for Tail Asymptotics of SIR Distribution in Downlink Cellular Networks

Naoto Miyoshi
Department of Mathematical and Computing Science
Tokyo Institute of Technology
2-12-1-W8-52 Ookayama, Tokyo 152-8552, Japan
Email: miyoshi@c.titech.ac.jp

Tomoyuki Shirai
Institute of Mathematics for Industry
Kyushu University
744 Motooka, Fukuoka 819-0395, Japan
Email: shirai@imi.kyushu-u.ac.jp

Abstract—We consider the spatial stochastic model of single-tier downlink cellular networks, where the wireless base stations are deployed according to a general stationary point process on the Euclidean plane with general i.i.d. propagation effects. Recently, Ganti & Haenggi (2016) consider the same general cellular network model and, as one of many significant results, derive the tail asymptotics of the signal-to-interference ratio (SIR) distribution. However, they do not mention any conditions under which the result holds. In this paper, we compensate their result for the lack of the condition and expose a sufficient condition for the asymptotic result to be valid. We further illustrate some examples satisfying such a sufficient condition and indicate the corresponding asymptotic results for the example models. We give also a simple counterexample violating the sufficient condition.

I. INTRODUCTION

In the design and analysis of wireless communication networks, the signal-to-interference ratio (SIR), defined in the next section, is a key quantity. In this paper, we consider the probability distribution of the SIR in the spatial stochastic models of downlink cellular networks, where the wireless base stations (BSs) are deployed according to spatial point processes on the Euclidian plane (see, e.g., [1]–[3]). The SIR distribution in these cellular network models can be provided as a closed-form or a numerically computable form for some restricted cases such as the BSs are deployed according to homogeneous Poisson point processes or kinds of determinantal point processes with specific propagation effects of fading and shadowing (see, e.g., [4], [5]). However, such special cases can not always represent the actual BS deployments and propagation effects in the real cellular networks, so that some approximation and/or asymptotic approaches have been proposed to cope with more general models (see, e.g., [6]–[9]).

In the current paper, we focus on the tail asymptotics of the SIR distribution in the single-tier network models, where the BSs are deployed according to general stationary point processes with general propagation effects. Recently, Ganti & Haenggi [9] consider the same general cellular network models and investigate the asymptotics of the SIR distribution both at the origin and at infinity. In [9], they derive the tail asymptotic result which is just our concern, but they do not mention any conditions under which the result holds. In their proof, a technique of changing the order of the limit and integrals is used, which generally requires a kind of uniform integrability condition. This paper then compensates [9] for the lack of the uniform integrability condition and exposes a sufficient condition for the order change of the limit and integrals. We further give some examples satisfying such a sufficient condition as well as a counterexample violating it.

The rest of the paper is organized as follows: First, we describe the spatial stochastic model of single-tier downlink cellular networks and define the SIR for the typical user in the next section. Section III states the main result, where we give a sufficient condition under which the tail asymptotics of the SIR distribution is properly obtained. In section IV, we illustrate some examples satisfying the condition and indicate the asymptotic results for the corresponding models of the examples. We further give a simple counterexample violating the sufficient condition.

II. NETWORK MODEL

Let $\Phi = \{X_i\}_{i \in \mathbb{N}}$ denote a point process on $\mathbb{R}^2$, which is assumed to be ordered such that $|X_1| \le |X_2| \le \cdots$. Each point X_i, $i \in \mathbb{N}$, represents the location of a BS of the cellular network and we refer to the BS located at X_i as BS i. We assume that the point process Φ is simple and locally finite almost surely in P (P-a.s.) and also stationary with intensity $\lambda \in (0, \infty)$. Assuming further that all BSs transmit at the same power level and each user is associated with the nearest BS, we focus on a typical user located at the origin $o = (0, 0)$. Let H_i, $i \in \mathbb{N}$, denote the random propagation effect representing the fading and shadowing from BS i to the typical user, where H_i, $i \in \mathbb{N}$, are mutually independent and identically distributed (i.i.d.), and independent of the point process Φ as well. The path-loss function representing the attenuation of signals with distance is given by $\ell(r) = r^{-2\beta}$, $r > 0$, for $\beta > 1$. The downlink SIR for the typical user is then given by

$$\mathsf{SIR}_o = \frac{H_1 \, \ell(|X_1|)}{\sum_{i=2}^{\infty} H_i \, \ell(|X_i|)}, \tag{1}$$

where we should recall that X_1 is the nearest point of Φ from the origin.

Our concern in the current paper is the tail asymptotics of the SIR distribution; that is, the asymptotic behavior of $\mathsf{P}(\mathsf{SIR}_o > \theta)$ as $\theta \to \infty$.

III. General asymptotic result

In this and later sections, P^0 and E^0 denote respectively the Palm probability and the corresponding expectation with respect to the marked point process $\Phi_H = \{(X_i, H_i)\}_{i \in \mathbb{N}}$ (see, e.g., [1, Sec. 1.4]). Note that $\mathsf{P}^0(\Phi(\{o\}) = 1) = 1$ while $\mathsf{P}(\Phi(\{o\}) \geq 1) = 0$. Note also that, due to the mutual independence of $\Phi = \{X_i\}_{i \in \mathbb{N}}$ and $\{H_i\}_{i \in \mathbb{N}}$, $\mathsf{P}^0(H_i \in C) = \mathsf{P}(H_i \in C)$ for $C \in \mathcal{B}(\mathbb{R}_+)$. When we consider Φ under the Palm distribution P^0, we use the index 0 for the point at the origin; that is, $X_0 = o = (0, 0)$.

To give the main theorem which is a refinement of Theorem 4 of [9], we first define the typical Voronoi cell and its circumscribed radius. For a point process Φ on $\mathbb{R}^2$ and a point X_i of Φ, the Voronoi cell of X_i with respect to Φ is defined as the set;

$$\mathcal{C}(X_i) = \{x \in \mathbb{R}^2 : |x - X_i| \leq |x - X_j|, X_j \in \Phi\};$$

that is, the set of points in $\mathbb{R}^2$ whose distance to X_i is not greater than that to any other points of Φ. The typical Voronoi cell is then $\mathcal{C}(o)$ under the Palm distribution P^0 and its circumscribed radius, denoted by $R(o)$, is the radius of the smallest disk centered at the origin and containing $\mathcal{C}(o)$ under P^0.

Theorem 1: We suppose the following.
(A) For the point process $\Phi = \{X_i\}_{i \in \mathbb{N}}$, $\mathsf{E}^0(R(o)^2) < \infty$ and $\mathsf{E}^0(|X_k|^2) < \infty$ for any $k \in \mathbb{N}$.
(B) For the sequence of the propagation effects $\{H_i\}_{i \in \mathbb{N}}$, $\mathsf{E}(H_1^{1/\beta}) < \infty$ and there exist $\alpha > 0$ and $c_H > 0$ such that the Laplace transform $\mathcal{L}_H$ of H_i, $i \in \mathbb{N}$, satisfies $\mathcal{L}_H(s) \leq c_H s^{-\alpha}$ for $s \geq 1$.
It then holds that

$$\lim_{\theta \to \infty} \theta^{1/\beta} \, \mathsf{P}(\mathsf{SIR}_o > \theta)$$
$$= \pi \lambda \, \mathsf{E}(H_1^{1/\beta}) \, \mathsf{E}^0 \left[\left(\sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right)^{-1/\beta} \right]. \quad (2)$$

One can show that the right-hand side of (2) does not depend on the intensity λ of the point process Φ (see the remark of Definition 4 in [9]).

Remark 1: The right-hand side of (2) in Theorem 1 is identical to EFIR^δ in Theorem 4 of [9]; that is, that theorem and our Theorem 1 assert the same result. The difference between the two theorems is that we provide a set of conditions (A) and (B), the role of which is discussed in the proof and remarks thereafter.

Proof: Let F_H denote the distribution function of H_i, $i \in \mathbb{N}$, and let $\overline{F_H}(x) = 1 - F_H(x)$. By (1) and $\ell(r) = r^{-2\beta}$, $r > 0$, the tail probability of the downlink SIR for the typical user is expressed as

$$\mathsf{P}(\mathsf{SIR}_o > \theta) = \mathsf{E}\overline{F_H}\left(\theta \, |X_1|^{2\beta} \sum_{i=2}^{\infty} \frac{H_i}{|X_i|^{2\beta}} \right). \quad (3)$$

Applying the Palm inversion formula (see, e.g., [1, Sec. 4.2]) to the right-hand side above,

$$\mathsf{P}(\mathsf{SIR}_o > \theta)$$

$$= \lambda \int_{\mathbb{R}^2} \mathsf{E}^0 \left[\overline{F_H}\left(\theta \, |x|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i - x|^{2\beta}} \right) \mathbf{1}_{\mathcal{C}(o)}(x) \right] \mathrm{d}x$$

$$= \theta^{-1/\beta} \lambda \int_{\mathbb{R}^2} \mathsf{E}^0 \left[\overline{F_H}\left(|y|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i - \theta^{-1/(2\beta)} y|^{2\beta}} \right) \right.$$
$$\left. \times \mathbf{1}_{\mathcal{C}(o)}(\theta^{-1/(2\beta)} y) \right] \mathrm{d}y,$$

where the second equality follows by substituting $y = \theta^{1/(2\beta)} x$. Here, if we can find a random function A satisfying

$$\overline{F_H}\left(|y|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i - \theta^{-1/(2\beta)} y|^{2\beta}} \right) \mathbf{1}_{\mathcal{C}(o)}(\theta^{-1/(2\beta)} y)$$

$$\leq A(y), \quad \mathsf{P}^0\text{-a.s.}, \quad (4)$$

$$\int_{\mathbb{R}^2} \mathsf{E}^0 A(y) \, \mathrm{d}y < \infty, \quad (5)$$

the dominated convergence theorem leads to

$$\lim_{\theta \to \infty} \theta^{1/\beta} \, \mathsf{P}(\mathsf{SIR}_o > \theta)$$
$$= \lambda \int_{\mathbb{R}^2} \mathsf{E}^0 \overline{F_H}\left(|y|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right) \mathrm{d}y. \quad (6)$$

We leave finding such an A and approve (6) for a moment. Substituting $z = \left(\sum_{i \in \mathbb{N}} H_i/|X_i|^{2\beta} \right)^{1/(2\beta)} y$ to the last integral in (6) yields

$$\int_{\mathbb{R}^2} \mathsf{E}^0 \overline{F_H}\left(|y|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right) \mathrm{d}y$$

$$= \mathsf{E}^0 \left[\left(\sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right)^{-1/\beta} \right] \int_{\mathbb{R}^2} \overline{F_H}(|z|^{2\beta}) \, \mathrm{d}z, \quad (7)$$

and furthermore,

$$\int_{\mathbb{R}^2} \overline{F_H}(|z|^{2\beta}) \, \mathrm{d}z = 2\pi \int_0^\infty \overline{F_H}(r^{2\beta}) \, r \, \mathrm{d}r$$

$$= \frac{\pi}{\beta} \int_0^\infty \overline{F_H}(s) \, s^{1/\beta - 1} \, \mathrm{d}s$$

$$= \pi \, \mathsf{E}(H_1^{1/\beta}). \quad (8)$$

Hence, applying (7) and (8) to (6), we obtain (2).

It remains to find a function A satisfying (4) and (5). Since $\overline{F_H}$ is nonincreasing and $|X_i - y| \leq |X_i| + R(o)$ P^0-a.s. for $y \in \mathcal{C}(o)$, we can set the function A satisfying (4) as

$$A(y) = \overline{F_H}\left(|y|^{2\beta} \sum_{i \in \mathbb{N}} \frac{H_i}{(|X_i| + R(o))^{2\beta}} \right).$$

Then, substituting $z = \left(\sum_{i \in \mathbb{N}} H_i/(|X_i| + R(o))^{2\beta} \right)^{1/(2\beta)} y$ and applying (8) again, we have

$$\int_{\mathbb{R}^2} \mathsf{E}^0 A(y) \, \mathrm{d}y$$

$$= \pi \, \mathsf{E}(H_1^{1/\beta}) \, \mathsf{E}^0 \left[\left(\sum_{i \in \mathbb{N}} \frac{H_i}{(|X_i| + R(o))^{2\beta}} \right)^{-1/\beta} \right].$$

For the second expectation on the right-hand side above, applying the identity $x^{-1/\beta} = \Gamma(1/\beta)^{-1} \int_0^\infty e^{-x\,s}\, s^{-1+1/\beta}\, ds$ yields

$$
\mathsf{E}^0\!\left[\left(\sum_{i\in\mathbb{N}} \frac{H_i}{(|X_i|+R(o))^{2\beta}}\right)^{-1/\beta}\right]
$$

$$
= \frac{1}{\Gamma(1/\beta)}\int_0^\infty s^{-1+1/\beta}
$$
$$
\times\, \mathsf{E}^0\!\left[\exp\!\left(-s\sum_{i\in\mathbb{N}} \frac{H_i}{(|X_i|+R(o))^{2\beta}}\right)\right] ds
$$

$$
= \frac{1}{\Gamma(1/\beta)}\int_0^\infty s^{-1+1/\beta}
$$
$$
\times\, \mathsf{E}^0\!\left[\prod_{i\in\mathbb{N}}\mathcal{L}_H\!\left(\frac{s}{(|X_i|+R(o))^{2\beta}}\right)\right] ds,
$$

where Γ denotes Euler's Gamma function. Recall that X_i, $i\in\mathbb{N}$, are ordered such that $|X_1|<|X_2|<\cdots$. Thus, by taking $k\in\mathbb{N}$ such that $\alpha\beta k>1$, and applying $\mathcal{L}_H(s)\le c_H\, s^{-\alpha}$ for $s\ge 1$ from condition (B), we have

$$
\int_0^\infty s^{-1+1/\beta}\, \mathsf{E}^0\!\left[\prod_{i\in\mathbb{N}}\mathcal{L}_H\!\left(\frac{s}{(|X_i|+R(o))^{2\beta}}\right)\right] ds
$$
$$
\le \int_0^\infty s^{-1+1/\beta}\, \mathsf{E}^0\!\left[\mathcal{L}_H\!\left(\frac{s}{(|X_k|+R(o))^{2\beta}}\right)^k\right] ds
$$
$$
\le \mathsf{E}^0\!\left[\int_0^{(|X_k|+R(o))^{2\beta}} s^{-1+1/\beta}\, ds\right]
$$
$$
+\, c_H{}^k\, \mathsf{E}^0\!\left[(|X_k|+R(o))^{2\alpha\beta k}\right.
$$
$$
\left.\times \int_{(|X_k|+R(o))^{2\beta}}^\infty s^{-1+1/\beta-\alpha k}\, ds\right]
$$
$$
= \beta\left(1 + \frac{c_H{}^k}{\alpha\beta k - 1}\right)\mathsf{E}^0\!\left[(|X_k|+R(o))^2\right]. \tag{9}
$$

Hence, the inequality $(a+b)^2 \le 2\,(a^2+b^2)$ ensures (5) under condition (A) of the theorem. ∎

Remark 2: The differences between the proof of [9] and ours are as follows. The first and less essential one is that, in [9], they modify the right-hand side of (3) into an appropriate form and then apply the Campbell-Mecke formula. On the other hand, we apply the Palm inversion formula directly. Second, [9] does not specify any condition under which the result holds. However, equality (6) requires some kind of uniform integrability condition to change the order of the limit and integrals. Our set of conditions (A) and (B) in Theorem 1 gives a sufficient condition for this order change to be valid.

Remark 3: The condition (B) claims that the Laplace transform of H_i, $i\in\mathbb{N}$, decays faster than or equal to the power law. Though this condition excludes the distribution with a mass at the origin, it covers many practical distributions. For example, the Gamma distribution $\mathrm{Gam}(\alpha,b)$, $\alpha>0$, $b>0$, has the Laplace transform $\mathcal{L}_H(s) = (1+b\,s)^{-\alpha}$ and we can take $c_H \ge b^{-\alpha}$.

The asymptotic constant of (2) in Theorem 1 depends on the point process Φ and the distribution F_H of the propagation effects. The following proposition gives the impact of the propagation effects to the asymptotic constant by comparing with the case without propagation effects.

Proposition 1: Let $C(\beta, F_H)$ denote the limit on the right-hand side of (2), specifying the dependence on the value of β and the propagation effect distribution F_H. When $\mathsf{E}H_1 < \infty$, we have

$$
C(\beta, F_H) \ge \frac{\mathsf{E}(H_1{}^{1/\beta})}{(\mathsf{E}H_1)^{1/\beta}}\, C(\beta, \delta_1), \tag{10}
$$

where δ_1 denotes the Dirac measure with the mass at the unity.

Proof: The result immediately follows from Jensen's inequality conditioned on $\Phi = \{X_i\}_{i\in\mathbb{N}}$. On the right-hand side of (2), since $f(x) = x^{-1/\beta}$ is convex for $x>0$,

$$
\mathsf{E}^0\!\left[\left(\sum_{i\in\mathbb{N}} \frac{H_i}{|X_i|^{2\beta}}\right)^{-1/\beta}\right] \ge \mathsf{E}^0\!\left[\left(\sum_{i\in\mathbb{N}} \frac{\mathsf{E}H_1}{|X_i|^{2\beta}}\right)^{-1/\beta}\right]
$$
$$
= \frac{1}{(\mathsf{E}H_1)^{1/\beta}}\, \mathsf{E}^0\!\left[\left(\sum_{i\in\mathbb{N}} \frac{1}{|X_i|^{2\beta}}\right)^{-1/\beta}\right],
$$

and (10) holds. ∎

Remark 4: When $F_H = \mathrm{Exp}(1)$ (Rayleigh fading without shadowing), the result of Proposition 1 coincides with the second part of Theorem 2 in [10]. In the inequality (10), it is easy to see (by Jensen's inequality) that the coefficient $\mathsf{E}(H_1{}^{1/\beta})/(\mathsf{E}H_1)^{1/\beta}$ is smaller than or equal to the unity. Now, suppose that $\mathsf{E}H_1 = 1$. Then, the dominated convergence theorem (due to $H_1{}^{1/\beta} \le 1+H_1$ a.s.) leads to $\mathsf{E}(H_1{}^{1/\beta}) \to 1$ as both $\beta \downarrow 1$ and $\beta \uparrow \infty$, which implies that $C(\beta, F_H) \ge C(\beta, \delta_1)$ might be true when the value of β is close to the unity or sufficiently large.

IV. Examples

A. Poisson process networks

In this section, we consider the homogeneous Poisson point process Φ with finite and nonzero intensity λ. We first confirm that Φ satisfies the condition (A) of Theorem 1.

Lemma 1: Let $\Phi = \{X_i\}_{i\in\mathbb{N}}$ denote the homogeneous Poisson point process with intensity $\lambda \in (0,\infty)$, where X_i, $i\in\mathbb{N}$, are ordered such that $|X_1| < |X_2| < \cdots$. Then, for $\epsilon > 0$,

$$
\mathsf{E}^0 e^{\epsilon|X_k|} < \infty, \quad k\in\mathbb{N}, \tag{11}
$$
$$
\mathsf{E}^0 e^{\epsilon R(o)} < \infty. \tag{12}
$$

This lemma ensures that both $|X_k|$, $k\in\mathbb{N}$, and $R(o)$ have any order of moments.

Proof: Let D_r denote the disk centered at the origin with radius $r>0$. By Slivnyak's theorem (see, e.g., [1, Sec. 1.4], [11]),

$$
\mathsf{P}^0(|X_k|>r) = \mathsf{P}(|X_k|>r) = \mathsf{P}(\Phi(D_r)<k)
$$
$$
= e^{-\lambda\pi r^2}\sum_{j=0}^{k-1} \frac{(\lambda\pi r^2)^j}{j!}.
$$

Therefore, exploiting $\pi\lambda r^2 - \epsilon r \geq \pi\lambda r^2/2$ for $r \geq 2\epsilon/(\pi\lambda)$,

$$
\begin{aligned}
\mathsf{E}^0 e^{\epsilon|X_k|} &= \int_0^\infty e^{\epsilon r}\, \frac{2\pi\lambda r\,(\pi\lambda r^2)^{k-1}}{(k-1)!}\, e^{-\pi\lambda r^2}\,\mathrm{d}r \\
&\leq e^{2\epsilon^2/(\pi\lambda)} \int_0^{2\epsilon/(\pi\lambda)} \frac{2\pi\lambda r\,(\pi\lambda r^2)^{k-1}}{(k-1)!}\, e^{-\pi\lambda r^2}\,\mathrm{d}r \\
&\quad + \int_{2\epsilon/(\pi\lambda)}^\infty \frac{2\pi\lambda r\,(\pi\lambda r^2)^{k-1}}{(k-1)!}\, e^{-\pi\lambda r^2/2}\,\mathrm{d}r \\
&= e^{2\epsilon^2/(\pi\lambda)} P\!\left(k, \frac{4\epsilon^2}{\pi\lambda}\right) + 2^k\, Q\!\left(k, \frac{2\epsilon^2}{\pi\lambda}\right),
\end{aligned}
$$

and (11) holds, where P and Q denote respectively the regularized lower and upper incomplete Gamma functions.

On the other hand, for the circumscribed radius $R(o)$ of the typical Voronoi cell of Φ, Calka [12, Theorem 3] shows that

$$
\mathsf{P}^0(R(o) > r) \leq 4\,\pi\,\lambda\,r^2\,e^{-\pi\lambda r^2} \quad \text{for } r \geq r_0 \approx 0.337,
$$

so that, we have

$$
\begin{aligned}
\mathsf{E}^0 e^{\epsilon R(o)} &= 1 + \epsilon \int_0^\infty e^{\epsilon r}\, \mathsf{P}^0(R(o) > r)\,\mathrm{d}r \\
&\leq 1 + \epsilon \int_0^{r_0 \vee (2\epsilon/(\pi\lambda))} e^{\epsilon r}\,\mathrm{d}r \\
&\quad + \epsilon \int_{r_0 \vee (2\epsilon/(\pi\lambda))}^\infty 4\,\pi\,\lambda\,r^2\,e^{-\pi\lambda r^2/2}\,\mathrm{d}r \\
&= e^{\epsilon r_0 \vee (2\epsilon/(\pi\lambda))} + \frac{8}{(2\pi\lambda)^{1/2}}\,\Gamma\!\left(\frac{3}{2}, \frac{\pi\lambda r_0{}^2}{2} \vee \frac{2\epsilon^2}{\pi\lambda}\right),
\end{aligned}
$$

and (12) holds, where $\Gamma(a,b)$ denotes the upper incomplete Gamma function and $a \vee b = \max(a,b)$. ■

We now apply Theorem 1 and obtain the following.

Corollary 1: Suppose that $\Phi = \{X_i\}_{i\in\mathbb{N}}$ is the homogeneous Poisson point process. When the propagation effect sequence $\{H_i\}_{i\in\mathbb{N}}$ satisfies the condition (B) of Theorem 1, the right-hand side of (2) reduces to $(\beta/\pi)\,\sin(\pi/\beta)$.

Proof: Since the conditions of Theorem 1 are fulfilled, the result follows from the proof of Lemma 6 in [9]. ■

Remark 5: The asymptotic result from Corollary 1 agrees with that in [5, Remark 4], where the Rayleigh fading is considered. Corollary 1 states that the downlink coverage probability in the Poisson cellular network is asymptotically insensitive to the distribution of the propagation effects as far as it satisfies the condition (B) of Theorem 1.

B. Determinantal process networks

In this section, we consider a general stationary and isotropic determinantal point process Φ on $\mathbb{C} \simeq \mathbb{R}^2$ with intensity λ. Let $K\colon \mathbb{C}^2 \to \mathbb{C}$ denote the continuous kernel of Φ with respect to the Lebesgue measure. Then, the joint intensities (correlation functions) ρ_n, $n \in \mathbb{N}$, with respect to the Lebesgue measure are given by

$$
\rho_n(z_1, z_2, \ldots, z_n) = \det\big(K(z_i, z_j)\big)_{i,j=1,2,\ldots,n},
$$

for $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Note that, due to the stationarity and isotropy, it holds that $\rho_1(z) = K(z,z) = \lambda$ and that $\rho_2(0,z) = \lambda^2 - |K(0,z)|^2$ depends only on $|z|$ for $z \in \mathbb{C}$. In order for

the point process Φ to be well-defined, we assume that (i) the kernel K is Hermitian in the sense that $K(z,w) = \overline{K(w,z)}$ for $z, w \in \mathbb{C}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, and (ii) the integral operator on $L^2(\mathbb{C})$ corresponding to K is of locally trace class with the spectrum in $[0,1]$; that is, for a compact set $C \in \mathcal{B}(\mathbb{C})$, the restriction K_C of K on C has eigenvalues $\kappa_{C,i}$, $i \in \mathbb{N}$, satisfying $\sum_{i\in\mathbb{N}} \kappa_{C,i} < \infty$ and $\kappa_{C,i} \in [0,1]$ for each $i \in \mathbb{N}$ (see, e.g., [13, Chap. 4]).

Concerning the condition (A) of Theorem 1, we show the following.

Lemma 2: (i) Let X_i, $i \in \mathbb{N}$, denote the points of Φ such that $|X_1| < |X_2| < \cdots$. Then, there exist $a_1 > 0$ and $a_2 > 0$ such that, for any $k \in \mathbb{N}$, we can take a sufficiently large $r > 0$ satisfying

$$
\mathsf{P}^0(|X_k| > r) \leq a_1\, e^{-a_2\, r^2}. \tag{13}
$$

(ii) Let $R(o)$ denote the circumscribed radius of the typical Voronoi cell $\mathcal{C}(o)$ of Φ. Then, there exist $b_1 > 0$ and $b_2 > 0$ such that, for $r > 0$,

$$
\mathsf{P}^0(R(o) > r) \leq b_1\, e^{-b_2\, r^2}. \tag{14}
$$

By Lemma 2, it is easy to confirm, similar to Lemma 1, that both $|X_k|$, $k \in \mathbb{N}$, and $R(o)$ have any order of moments under P^0. To prove Lemma 2, we use the following supplementary lemma, the proof of which is in the appendix:

Lemma 3: The kernel K of Φ satisfies

$$
\int_{\mathbb{C}} |K(0,z)|^2\,\mathrm{d}z \leq K(0,0) = \lambda. \tag{15}
$$

Proof of Lemma 2: Let $\mathsf{P}^!$ denote the reduced Palm probability with respect to the marked point process $\Phi_H = \{(X_i, H_i)\}_{i\in\mathbb{N}}$ and let C denote a bounded set in $\mathcal{B}(\mathbb{C})$. Since the (reduced) Palm version of a determinantal point process is also determinantal (see, e.g., [14]), $\Phi(C)$ under $\mathsf{P}^!$ has the same distribution as $\sum_{i\in\mathbb{N}} B_{C,i}$ with some kind of mutually independent Bernoulli random variables $B_{C,i}$, $i \in \mathbb{N}$ (see, e.g., [13, Sec. 4.5]). Thus, the Chernoff-Hoeffding bound for an infinite sum with finite mean (see, e.g., [15], [16] for a finite sum) implies that, for any $\epsilon \in [0,1)$, there exists a $c_\epsilon > 0$ such that

$$
\mathsf{P}^!\big(\Phi(C) \leq \epsilon\, \mathsf{E}^! \Phi(C)\big) \leq e^{-c_\epsilon\, \mathsf{E}^! \Phi(C)}, \tag{16}
$$

where $\mathsf{E}^!$ denotes the expectation with respect to $\mathsf{P}^!$. On the other hand, the kernel of the Palm version of Φ is given by (see [14])

$$
K^0(z,w) = \frac{K(z,w)\,K(0,0) - K(z,0)\,K(0,w)}{K(0,0)}, \quad z, w \in \mathbb{C}.
$$

Therefore, the intensity function (1-correlation) of Φ under $\mathsf{P}^!$ reduces to

$$
\rho_1^0(z) = K^0(z,z) = \lambda - \frac{|K(0,z)|^2}{\lambda}, \tag{17}
$$

so that, Lemma 3 leads to

$$
\mathsf{E}^! \Phi(C) = \int_C \rho_1^0(z)\,\mathrm{d}z \geq \lambda\,\mu(C) - 1, \tag{18}
$$

where μ denotes the Lebesgue measure on $\mathbb{C}$.

Proof of (i): Note that $\mathsf{P}^0(|X_k| > r) = \mathsf{P}^!(\Phi(D_r) \leq k-1)$. Since $\mathsf{E}^!\Phi(D_r) \geq \lambda\pi r^2 - 1$ from (18), applying this to (16) yields

$$\mathsf{P}^!\big(\Phi(D_r) \leq \epsilon\,(\lambda\pi r^2 - 1)\big) \leq \mathrm{e}^{c_\epsilon}\,\mathrm{e}^{-c_\epsilon\,\lambda\pi r^2}.$$

Hence, for any $\epsilon \in (0,1)$ and $k \in \mathbb{N}$, we can take $r > 0$ satisfying $\epsilon\,(\lambda\pi r^2 - 1) \geq k-1$, which implies (13).

Proof of (ii): We here derive an upper bound of $\mathsf{P}^0(R(o) > r)$ by using Foss & Zuyev's seven petals [17], which are considered to obtain an upper bound of the tail probability for the circumscribed radius of the typical Poisson-Voronoi cell. Consider a collection of seven disks of common radii r centered at the points $(r, 2\pi k/7)$, $k = 0, 1, \ldots, 6$, in polar coordinates. The petal 0 is given as the intersection of two circles centered at $(r, 0)$, $(r, 2\pi/7)$ and the angular domain between the rays $\phi = 0$ and $\phi = 2\pi/7$. The petal k is the rotation copy of petal 0 by angle $2\pi k/7$ for $k = 1, 2, \ldots, 6$ (see Figure 1). Let $\mathcal{P}_{k,r}$, $k = 0, 1, \ldots, 6$, denote the set formed by petal k on the complex plane $\mathbb{C}$. Then, according to the discussion in the proof of Lemma 1 of [17],

$$\mathsf{P}^0(R(o) > r) \leq \mathsf{P}^!\Big(\bigcup_{k=0}^{6}\{\Phi(\mathcal{P}_{k,r}) = 0\}\Big)$$
$$\leq 7\,\mathsf{P}^!\big(\Phi(\mathcal{P}_{0,r}) = 0\big), \qquad (19)$$

where the second inequality follows from the isotropy of the Palm version of Φ. Now, we can apply the bound (16) with $\epsilon = 0$ and we have

$$\mathsf{P}^!\big(\Phi(\mathcal{P}_{0,r}) = 0\big) \leq \mathrm{e}^{-c_0\,\mathsf{E}^!\Phi(\mathcal{P}_{0,r})}. \qquad (20)$$

Hence, (14) holds since $\mathsf{E}^!\Phi(\mathcal{P}_{0,r}) \geq \lambda\mu(\mathcal{P}_{0,r}) - 1$ and $\mu(\mathcal{P}_{0,r}) = 2\,r^2\left(\pi/7 + \sin(\pi/7)\,\cos(3\pi/7)\right)$. ∎

Remark 6: We can take c_0 in (20) equal to the unity since determinantal point processes are weakly sub-Poisson (in particular, due to the ν-weakly sub-Poisson property) (see [18] for details).

Remark 7: When the kernel K of the determinantal point process is explicitly specified, it may be possible to obtain a tighter upper bound on the tail probability of the circumscribed radius of the typical Voronoi cell. For example, the case of the Ginibre point process is given by the following proposition.

Proposition 2: For the Ginibre point process (with intensity π^{-1}), the circumscribed radius for the typical Voronoi cell $\mathcal{C}(o)$ satisfies

$$\mathsf{P}^0(R(o) > r) \leq 7\,\mathrm{e}^{-(u(r)\vee v(r))}, \qquad (21)$$

where

$$u(r) = \frac{1}{7}\left(4r^2\,\cos^2\frac{2\pi}{7} + \exp\left(-4r^2\,\cos^2\frac{2\pi}{7}\right) - 1\right),$$
$$v(r) = \frac{2r^2}{\pi}\left(\frac{\pi}{7} + \sin\frac{\pi}{7}\,\cos\frac{3\pi}{7}\right)$$
$$\qquad + \frac{1}{7}\left(\exp\left(-4r^2\,\cos^2\frac{\pi}{7}\right) - 1\right).$$

Proof: The kernel of the Ginibre point process is given by

$$K(z,w) = \frac{1}{\pi}\,\mathrm{e}^{-(|z|^2 + |w|^2)/2}\,\mathrm{e}^{z\overline{w}}, \qquad z, w \in \mathbb{C},$$
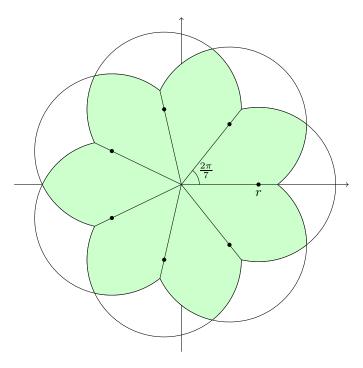

Fig. 1.   Foss & Zuyev's seven petals.

with respect to the Lebesgue measure (see, e.g., [14]). Thus, the intensity function (17) of the (reduced) Palm version of Φ reduces to

$$\rho_1^0(z) = \frac{1}{\pi}\left(1 - \mathrm{e}^{-|z|^2}\right), \quad z \in \mathbb{C}. \qquad (22)$$

Now, we obtain two lower bound of $\mathsf{E}^!\Phi(\mathcal{P}_{0,r})$ as follows. Let $\mathcal{S}_\eta$ denote the circular sector centered at the origin with radius η and angular domain between $\phi = 0$ and $\phi = 2\pi/7$. When we take $\eta_1 = 2r\cos(2\pi/7)$ and $\eta_2 = 2r\cos(\pi/7)$, we have $\mathcal{S}_{\eta_1} \subset \mathcal{P}_{0,r} \subset \mathcal{S}_{\eta_2}$. Therefore, applying (22), we have the first lower bound;

$$\mathsf{E}^!\Phi(\mathcal{P}_{0,r}) \geq \mathsf{E}^!\Phi(\mathcal{S}_{\eta_1}) = \int_{\mathcal{S}_{\eta_1}} \rho_1^0(z)\,\mathrm{d}z$$
$$= \frac{1}{7}\left(\eta_1{}^2 + \mathrm{e}^{-\eta_1{}^2} - 1\right) = u(r).$$

The second lower bound is given by

$$\mathsf{E}^!\Phi(\mathcal{P}_{0,r}) = \int_{\mathcal{P}_{0,r}} \rho_1^0(z)\,\mathrm{d}z$$
$$\geq \frac{1}{\pi}\left(\mu(\mathcal{P}_{r,0}) - \int_{\mathcal{S}_{\eta_2}} \mathrm{e}^{-|z|^2}\,\mathrm{d}z\right) = v(r).$$

Hence, we have (21) from (19) and (20) with $c_0 = 1$. ∎

Indeed, there is $r_* \approx 0.5276\cdots$ such that $u(r) > v(r)$ for $r < r_*$ and $u(r) < v(r)$ for $r > r_*$.

We are now ready to give the tail asymptotics of the SIR distribution when the BSs are deployed according to the Ginibre point process.

Corollary 2: Suppose that $\Phi = \{X_i\}_{i\in\mathbb{N}}$ is the Ginibre point process. When the propagation effect sequence $\{H_i\}_{i\in\mathbb{N}}$

satisfies the condition (B) of Theorem 1, we have

$$
\begin{aligned}
&\lim_{\theta \to \infty} \theta^{1/\beta}\, \mathsf{P}(\mathsf{SIR}_o > \theta) \\
&= \frac{\mathsf{E}(H_1^{1/\beta})}{\Gamma(1+1/\beta)} \int_0^\infty \prod_{i=1}^\infty \frac{1}{i!} \int_0^\infty \mathrm{e}^{-u}\, u^i\, \mathcal{L}_H\!\left(\left(\frac{t}{u}\right)^\beta\right) \mathrm{d}u\, \mathrm{d}t.
\end{aligned}
\tag{23}
$$

Furthermore, when $H_i \sim \mathrm{Gam}(m, 1/m)$ (Nakagami-m fading without shadowing),

$$
\begin{aligned}
&\lim_{\theta \to \infty} \theta^{1/\beta}\, \mathsf{P}(\mathsf{SIR}_o > \theta) \\
&= \frac{\beta}{B(m,\,1/\beta)} \int_0^\infty \prod_{i=1}^\infty \frac{1}{i!} \int_0^\infty \frac{\mathrm{e}^{-u}\, u^i}{\left(1 + (v/u)^\beta\right)^m} \, \mathrm{d}u\, \mathrm{d}v, \quad (24)
\end{aligned}
$$

where B denotes the Beta function.

For the proof of Corollary 2, we use the following proposition which is a consequence of [19] and [20].

Proposition 3: Let X_i, $i \in \mathbb{N}$, denote the points of the reduced Palm version of the Ginibre point process. Then, the set $\{|X_i|\}_{i \in \mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i \in \mathbb{N}}$, where Y_i, $i \in \mathbb{N}$, are mutually independent and $Y_i \sim \mathrm{Gam}(i+1, 1)$ for each $i \in \mathbb{N}$.

Proof of Corollary 2: For the Ginibre point process, we can see from Lemma 2 (or Proposition 2) that $|X_k|$, $k \in \mathbb{N}$, and $R(o)$ have any order of moments with respect to the Palm probability P^0; that is, the condition (A) of Theorem 1 is fulfilled. Thus, applying the identity $x^{-1/\beta} = \Gamma(1/\beta)^{-1} \int_0^\infty \mathrm{e}^{-x\,s}\, s^{-1+1/\beta}\, \mathrm{d}s$ to the right-hand side of (2), we have

$$
\begin{aligned}
&\mathsf{E}^0\!\left[\left(\sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right)^{-1/\beta} \right] \\
&= \frac{1}{\Gamma(1/\beta)} \int_0^\infty s^{-1+1/\beta}\, \mathsf{E}^0\!\left[\exp\!\left(-s \sum_{i \in \mathbb{N}} \frac{H_i}{|X_i|^{2\beta}} \right) \right] \mathrm{d}s \\
&= \frac{1}{\Gamma(1/\beta)} \int_0^\infty s^{-1+1/\beta}\, \mathsf{E}^0\!\left[\prod_{i \in \mathbb{N}} \mathcal{L}_H\!\left(\frac{s}{|X_i|^{2\beta}} \right) \right] \mathrm{d}s \\
&= \frac{1}{\Gamma(1+1/\beta)} \int_0^\infty \mathsf{E}^0\!\left[\prod_{i \in \mathbb{N}} \mathcal{L}_H\!\left(\left(\frac{t}{|X_i|^2} \right)^\beta \right) \right] \mathrm{d}t,
\end{aligned}
$$

where the last equality follows by substituting $t = s^{1/\beta}$. Here, Proposition 3 states that $\{|X_i|^2\}_{i \in \mathbb{N}} \stackrel{d}{=} \{Y_i\}_{i \in \mathbb{N}}$ under P^0 with $Y_i \sim \mathrm{Gam}(i+1, 1)$, so that, applying the density function of $\mathrm{Gam}(i+1, 1)$, $i \in \mathbb{N}$, we have (23).

When $H_i \sim \mathrm{Gam}(m, 1/m)$, then $\mathcal{L}_H(s) = (1 + s/m)^{-m}$ and $\mathsf{E}(H_1^{1/\beta}) = \Gamma(m+1/\beta)/(m^{1/\beta}\,(m-1)!)$. Thus, applying these to the right-hand side of (23),

$$
\begin{aligned}
(23) = {} & \frac{\Gamma(m+1/\beta)}{\Gamma(1+1/\beta)\, m^{1/\beta}\,(m-1)!} \\
& \times \int_0^\infty \prod_{i=1}^\infty \frac{1}{i!} \int_0^\infty \frac{\mathrm{e}^{-u}\, u^i}{\left(1 + m^{-1}\,(t/u)^\beta\right)^m} \, \mathrm{d}u\, \mathrm{d}t.
\end{aligned}
$$

Finally, substituting $v = m^{-1/\beta}\, t$ and applying $B(x,y) = \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$, we obtain (24). ∎

Remark 8: When $m = 1$, (24) reduces to the result of [5, Theorem 2], which considers the Rayleigh fading.

C. A counterexample

Finally, we give a simple counterexample that violates the condition (A) of Theorem 1. Let T denote a random variable with density function $f_T(t) = (a-1)\, t^{-a}$, $t \ge 1$, for $a \in (1, 2)$. Given a sample of T, we consider the mixed and randomly shifted lattice $\Phi = (\mathbb{Z} \times T\,\mathbb{Z}) + U_T$, where U_T denotes a uniformly distributed random variable on $[0, 1] \times [0, T]$. The intensity λ of Φ is then $\lambda = \mathsf{E}(1/T) = (a-1)/a < \infty$. For any nonnegative and measurable function f, the definition of the Palm probability implies

$$
\begin{aligned}
\mathsf{E}^0 f(T) &= \frac{1}{\lambda}\, \mathsf{E}\!\left(f(T)\, \Phi(I) \right) \\
&= \frac{1}{\lambda}\, \mathsf{E}\!\left(f(T)\, \mathsf{E}(\Phi(I) \mid T) \right) = \frac{1}{\lambda}\, \mathsf{E}\!\left(\frac{f(T)}{T} \right),
\end{aligned}
$$

where $I = [0, 1] \times [0, 1]$. Hence, applying $R(o)^2 = (1+T^2)/4$ to the above, we have

$$
\mathsf{E}^0\!\left(R(o)^2 \right) = \frac{1}{4\,\lambda}\, \mathsf{E}\!\left(\frac{1}{T} + T \right) = \frac{1}{4\,\lambda}\,\left(\lambda + \mathsf{E}(T) \right) = \infty.
$$

V. Concluding Remark

More other examples will be investigated in the extended version of the paper.

Appendix

Proof of Lemma 3: For a compact set $C \in \mathcal{B}(\mathbb{C})$, let K_C denote the restriction of K on C. Let also $\kappa_{C,i}$ and $\varphi_{C,i}$, $i \in \mathbb{N}$, denote respectively the eigenvalues of K_C and the corresponding orthonormal eigenfunctions; that is,

$$
\int_C \varphi_{C,i}(z)\, \overline{\varphi_{C,j}(z)}\, \mathrm{d}z = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases}
\tag{25}
$$

Then Mercer's theorem states that the following spectral expansion holds (see, e.g., [21]);

$$
K_C(z, w) = \sum_{i=1}^\infty \kappa_{C,i}\, \varphi_{C,i}(z)\, \overline{\varphi_{C,i}(w)}, \quad z, w \in C.
\tag{26}
$$

Note that $\kappa_{C,i} \in [0, 1]$, $i \in \mathbb{N}$, under the assumption on the kernel K. Thus we have

$$
\begin{aligned}
\int_C |K(0, z)|^2\, \mathrm{d}z &= \int_C |K_C(0, z)|^2\, \mathrm{d}z \\
&= \sum_{i=1}^\infty \kappa_{C,i}^2\, |\varphi_{C,i}(0)|^2
\end{aligned}
$$

$$\leq K_C(0,0) = K(0,0),$$

where the second equality follows from (25) and the inequality holds since $\kappa_{C,i} \in [0,1]$, $i \in \mathbb{N}$, and (26). Hence, letting $C \uparrow \mathbb{C}$, we obtain (15). ■

REFERENCES

[1] F. Baccelli and B. Błaszczyszyn, "Stochastic geometry and wireless networks, Volume I: Theory," *Foundations Trends (R) Networking*, vol. 3, pp. 249–449, 2009.

[2] M. Haenggi, *Stochastic Geometry for Wireless networks*. Cambridge Univ. Press, 2013.

[3] S. Mukherjee, *Analytical Modeling of Heterogeneous Cellular Networks: Geometry, Coverage, and Capacity*. Cambridge Univ. Press, 2014.

[4] J. G. Andrews, F. Baccelli, and R. K. Ganti, "A tractable approach to coverage and rate in cellular networks," *IEEE Trans. Commun.*, vol. 59, pp. 3122–3134, 2011.

[5] N. Miyoshi and T. Shirai, "A cellular network model with Ginibre configured base stations," *Adv. Appl. Probab.*, vol. 46, pp. 832–845, 2014.

[6] H. Nagamatsu, N. Miyoshi, and T. Shirai, "Padé approximation for coverage probability in cellular networks," *SpaSWiN-WiOpt 2014*, pp. 693–700, 2014.

[7] M. Haenggi, "The mean interference-to-signal ratio and its key role in cellular and amorphous networks," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 597–600, 2014.

[8] A. Guo and M. Haenggi, "Asymptotic deployment gain: A simple approach to characterize the SINR distribution in general cellular networks," *IEEE Trans. Commun.*, vol. 63, pp. 962–976, 2015.

[9] R. K. Ganti and M. Haenggi, "Asymptotics and approximation of the SIR distribution in general cellular networks," *IEEE Trans. Wireless Commun.*, vol. 15, pp. 2130–2143, 2016.

[10] N. Miyoshi and T. Shirai, "Downlink coverage probability in a cellular network with Ginibre deployed base stations and Nakagami-m fading channels," *WiOpt 2015*, pp. 483–489, 2015.

[11] I. M. Slivnyak, "Some properties of stationary flows of homogeneous random events," *Theory Probab. Appl.*, vol. 7, pp. 336–341, 1962.

[12] P. Calka, "The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane," *Adv. Appl. Probab.*, vol. 34, pp. 702–717, 2002.

[13] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. American Math. Soc., 2009.

[14] T. Shirai and Y. Takahashi, "Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and Boson processes," *J. Funct. Analysis*, vol. 205, pp. 414–463, 2003.

[15] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Ann. Math. Statist.*, vol. 23, pp. 493–507, 1952.

[16] W. Hoeffding, "Probability inequalities for sums of bounded random variables," *J. American Statist. Assoc.*, vol. 58, pp. 13–30, 1963.

[17] S. G. Foss and S. A. Zuyev, "On a Voronoi aggregate process related to a bivariate Poisson process," *Adv. Appl. Probab.*, vol. 28, pp. 965–981, 1996.

[18] B. Błaszczyszyn and D. Yogeshwaran, "On comparison of clustering properties of point processes," *Adv. Appl. Probab.*, vol. 46, pp. 1–20, 2014.

[19] A. Goldman, "The Palm measure and the Voronoi tessellation for the Ginibre process," *Annals Appl. Probab.*, vol. 20, pp. 90–128, 2010.

[20] E. Kostlan, "On the spectra of Gaussian matrices," *Linear Algebra Appl.*, vol. 162–164, pp. 385–388, 1992.

[21] J. Mercer, "Functions of positive and negative type, and their connection with the theory of integral equations," *Philos. Trans. Royal Soc. London: Ser. A*, vol. 209, pp. 415–446, 1909.