A DENSITY PROPERTY
OF HENSELIAN VALUED FIELDS

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ABSTRACT. We give an elementary proof of a version of the implicit function theorem over Henselian valued fields K. It yields a density property for such fields (introduced in a joint paper with J. Kollár), which is indispensable for ensuring reasonable topological and geometric properties of algebraic subsets of K^n.

Following [6] (see also [7]), we say that a topological field K satisfies the *density property* if the following equivalent conditions hold.

(1) If X is a smooth, irreducible K-variety and $\emptyset \neq U \subset X$ is a Zariski open subset, then $U(K)$ is dense in $X(K)$ in the K-topology.

(2) If C is a smooth, irreducible K-curve and $\emptyset \neq U$ is a Zariski open subset, then $U(K)$ is dense in $C(K)$ in the K-topology.

(3) If C is a smooth, irreducible K-curve, then $C(K)$ has no isolated points.

This property is indispensable for ensuring reasonable topological and geometric properties of algebraic subsets of K^n; see [7] for the case where the ground field K is a Henselian rank one valued field. For Henselian non-trivially valued fields, it can be directly deduced from the Jacobian criterion for smoothness and the implicit function theorem, as stated in [8, Theorem 7.4] or [5, Proposition 3.1.4]. Here we give elementary proofs of some versions of the inverse mapping and implicit function theorems.

We begin with a simplest version of Hensel's lemma in several variables, studied by Fisher [4]. Let $\mathfrak{m}^{\times n}$ stand for the n-fold Cartesian product of $\mathfrak{m}$ and $R^\times$ for the set of units of R. The origin $(0, \ldots, 0) \in R^n$ is denoted by $\mathbf{0}$.

(H) *Assume that a ring R satisfies Hensel's conditions (i.e. it is linearly topologized, Hausdorff and complete) and that an ideal $\mathfrak{m}$ of R is closed. Let $f = (f_1, \ldots, f_n)$ be an n-tuple of restricted power series*

$f_1, \ldots, f_n \in R\{X\}$, $X = (X_1, \ldots, X_n)$, J be its Jacobian determinant and $a \in R^n$. If $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$, then there is a unique $a \in \mathfrak{m}^{\times n}$ such that $f(a) = \mathbf{0}$.

Proposition 1. *Under the above assumptions, f induces a bijection*

$$\mathfrak{m}^{\times n} \ni x \to f(x) \in \mathfrak{m}^{\times n}$$

of $\mathfrak{m}^{\times n}$ onto itself.

Proof. For any $y \in \mathfrak{m}^{\times n}$, apply condition (H) to the restricted power series $f(X) - y$. $\square$

If, moreover, the pair $(R, \mathfrak{m})$ satisfies Hensel's conditions (i.e. every element of $\mathfrak{m}$ is topologically nilpotent), then condition (H) holds by [1, Chap. III, §4.5].

Remark 2. Henselian local rings can be characterized both by the classical Hensel lemma and by condition (H): a local ring $(R, \mathfrak{m})$ is Henselian iff $(R, \mathfrak{m})$ with the discrete topology satisfies condition (H) (cf. [4, Proposition 2]).

Now consider a Henselian local ring $(R, \mathfrak{m})$. Let $f = (f_1, \ldots, f_n)$ be an n-tuple of polynomials $f_1, \ldots, f_n \in R[X]$, $X = (X_1, \ldots, X_n)$ and J be its Jacobian determinant.

Corollary 3. *Suppose that $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^\times$. Then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the $\mathfrak{m}$-adic topology. If, in addition, R is a Henselian valued ring with maximal ideal $\mathfrak{m}$, then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the valuation topology.*

Proof. Obviously, $J(a) \in R^\times$ for every $a \in \mathfrak{m}^{\times n}$. Let $\mathcal{M}$ be the jacobian matrix of f. Then

$$f(a + x) - f(a) = \mathcal{M}(a) \cdot x + g(x) = \mathcal{M}(a) \cdot (x + \mathcal{M}(a)^{-1} \cdot g(x))$$

for an n-tuple $g = (g_1, \ldots, g_n)$ of polynomials $g_1, \ldots, g_n \in (X)^2 R[X]$. Hence the assertion follows easily. $\square$

The proposition below is a version of the inverse mapping theorem.

Proposition 4. *If $f(\mathbf{0}) = \mathbf{0}$ and $e := J(\mathbf{0}) \neq 0$, then f is an open embedding of $e^2 \cdot \mathfrak{m}^{\times n}$ into $e \cdot \mathfrak{m}^{\times n}$.*

Proof. Let $\mathcal{N}$ be the adjugate of the matrix $\mathcal{M}(\mathbf{0})$ and $y = e^2 b$ with $b \in \mathfrak{m}^{\times n}$. Since

$$f(X) = e\mathcal{M}(a) \cdot X + e^2 g(x)$$

for an n-tuple $g = (g_1, \ldots, g_n)$ of polynomials $g_1, \ldots, g_n \in (X)^2 R[X]$, we get the equivalences

$$f(eX) = y \iff f(eX) - y = \mathbf{0} \iff e\mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N}g(X) - \mathcal{N}b) = \mathbf{0}.$$

Applying Corollary 3 to the map $h(X) := X + \mathcal{N}g(X)$, we get

$$f^{-1}(y) = ex \iff x = h^{-1}(\mathcal{N}b) \quad \text{and} \quad f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

This finishes the proof. □

Further, let R be a Henselian valued ring with maximal ideal $\mathfrak{m}$. Let $0 \leq r < n$, $p = (p_{r+1}, \ldots, p_n)$ be an $(n-r)$-tuple of polynomials $p_{r+1}, \ldots, p_n \in R[X]$, $X = (X_1, \ldots, X_n)$, and

$$J := \frac{\partial(p_{r+1}, \ldots, p_n)}{\partial(X_{r+1}, \ldots, X_n)}, \quad e := J(\mathbf{0}).$$

Suppose that

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \ldots = p_n(x) = 0\}.$$

In a similar fashion as above, we can establish the following version of the implicit function theorem.

Proposition 5. *If $e \neq 0$, then there exists a continuous map*

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \to (e \cdot \mathfrak{m})^{\times(n-r)}$$

such that $\phi(0) = 0$ and the graph map

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \to (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times(n-r)}$$

is an open embedding into the zero locus V of the polynomials p.

Proof. Put $f(X) := (X_1, \ldots, X_r, p(X))$; of course, the jacobian determinant of f at $\mathbf{0} \in R^n$ is equal to e. Keep the notation from the proof of Proposition 4, take any $b \in e^2 \cdot \mathfrak{m}^{\times r}$ and put $y := (e^2b, 0) \in R^n$. Then we have the equivalences

$$f(eX) = y \iff f(eX) - y = \mathbf{0} \iff e\mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N}g(X) - \mathcal{N} \cdot (b, 0)) = \mathbf{0}.$$

Applying Corollary 3 to the map $h(X) := X + \mathcal{N}g(X)$, we get

$$f^{-1}(y) = ex \iff x = h^{-1}(\mathcal{N} \cdot (b, 0)) \quad \text{and} \quad f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

Therefore the function

$$\phi(u) := eh^{-1}(\mathcal{N} \cdot (u, 0)/e^2)$$

is the one we are looking for. □

The density property of Henselian non-trivially valued fields follows immediately from Proposition 5 and the Jacobian criterion for smoothness (see e.g. [2, Theorem 16.19]), recalled below for the reader's convenience.

Theorem 6. *Let $I = (p_1, \ldots, p_s) \subset K[X]$, $X = (X_1, \ldots, X_n)$ be an ideal, $A := K[X]/I$ and $V := \mathrm{Spec}\,(A)$. Suppose the origin $\mathbf{0} \in K^n$ lies in V (equivalently, $I \subset (X)K[X]$) and V is of dimension r at $\mathbf{0}$. Then the Jacobian matrix*

$$\mathcal{M} = \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \ i = 1, \ldots, s,\ j = 1, \ldots, n \right]$$

has rank $\leq (n-r)$ and V is smooth at $\mathbf{0}$ iff $\mathcal{M}$ has exactly rank $(n-r)$. Furthermore, if V is smooth at $\mathbf{0}$ and

$$\det \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \ i, j = r+1, \ldots, n \right] \neq 0,$$

then $p_{r+1}, \ldots, p_n$ generate the localization $I \cdot K[X]_{(X_1, \ldots, X_n)}$ of the ideal I with respect to the maximal ideal $(X_1, \ldots, X_n)$.

Let us mention that we are currently preparing a series of papers devoted to geometry of algebraic subsets of K^n, i.al. to the results of our article [7], for the case where the ground field K is an arbitrary Henselian valued field of equicharacteristic zero. Finally, I wish to thank Laurent Moret-Bailly for pointing out the implicit function theorem in the paper [5].

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