HOLOMORPHIC EIGENFUNCTIONS OF THE VECTOR FIELD ASSOCIATED WITH THE DISPERSIONLESS KADOMTSEV-PETVIASHVILI EQUATION

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ABSTRACT. Vector fields naturally arise in many branches of mathematics and physics. Recently it was discovered that Lax pairs for many important multidimensional integrable partial differential equations (PDEs) of hydrodynamic type (also known as dispersionless PDEs) consist of vector field equations. These vector fields have complex coefficients and their analytic, in the spectral parameter, eigenfunctions play an important role in the formulations of the direct and inverse spectral transforms.

In this paper we prove existence of eigenfunctions of the basic vector field associated with the celebrated dispersionless Kadomtsev-Petviashvili equation, which are holomorphic in the spectral parameter λ in the strips $|\operatorname{Im}\lambda| > C_0$.

1. INTRODUCTION

The dispersionless Kadomtsev-Petviashvili equation (dKP)

$$(1) \qquad (u_t + uu_x)_x + u_{yy} = 0, \ u = u(x,y,t) \in \mathbb{R}, \ x,y,t \in \mathbb{R},$$

is a model equation describing the propagation of weakly nonlinear quasi one-dimensional waves, in the absence of dispersion and dissipation, in many physical contexts (see [24], [30], [7], [1]). It arises as commutation condition for the following pair of vector fields:

$$(2) \qquad \begin{aligned} \hat{L}_1 &\equiv \partial_y + \lambda\partial_x - u_x\partial_\lambda, \\ \hat{L}_2 &\equiv \partial_t + (\lambda^2 + u)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \end{aligned}$$

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$\lambda \in \mathbb{C}$ being the spectral parameter [9], [10], [29]. This integrability scheme allows one to construct the formal solution of the Cauchy problem for dKP, and properly defined analytic in λ eigenfunctions of the above vector fields play a crucial role in the formulation of the direct and inverse spectral transforms [13], [14].

In this paper, in the framework of the direct problem, we construct analytic zero-level eigenfunctions of the first vector field $\hat{L}_1$ (therefore we omit the t-dependence in all formulas):

$$(3) \qquad\qquad \hat{L}_1 \Psi = [\partial_y + \lambda \partial_x - u_x(x, y) \partial_\lambda] \Psi = 0,$$

where

$$(4) \qquad\qquad \Psi = \Psi(x, y, \lambda), \quad x, y \in \mathbb{R}, \quad \lambda \in \mathbb{C},$$

with the asymptotic behaviors

$$(5) \qquad\qquad \Psi_1(x, y, \lambda) \to \lambda, \quad \text{as} \quad x^2 + y^2 \to \infty,$$

$$(6) \qquad\qquad \Psi_2(x, y, \lambda) \to x - \lambda y, \quad \text{as} \quad x^2 + y^2 \to \infty,$$

assuming that $u(x, y)$ decays sufficiently fast if $x^2 + y^2 \to \infty$. A basis of common eigenfunctions for both vector fields $\hat{L}_1$ and $\hat{L}_2$ is given by Ψ_1 and by the combination $\Psi_2 - t\Psi_1^2$ [13].

To construct these analytic eigenfunctions, we make use of the complex forced Hopf equation (15), describing the level sets of function Ψ. We remark that equation (3) is equivalent to the Benney system [3] written in terms of the generating function for the momenta, and in the Benney framework, $\Psi - \lambda$ is automatically holomorphic near infinity [11]. The relation between (3) and the complex version of (15) was also introduced and used in the framework of Benney system [11]; in this contest the solution of (15) is also automatically holomorphic near infinity. The vector field $\hat{L}_2$ is equivalent to the symmetry of the Benney system, introduced in [12].

The procedure we propose here to construct analytic eigenfunctions of vector field equations should be applicable also to the vector field Lax pairs of other basic examples of integrable PDEs, like the heavenly [19], the two dimensional dispersionless Toda [6, 28, 5] and the Martinez Alonso - Shabat - Pavlov [15, 18] equations.

We remark that Derchyi Wu has recently proven the unique solvability of the nonlinear Riemann-Hilbert problem characterizing the inverse transform for dKP, under a small-norm assumption, using appropriate Sobolev spaces (private communication).

2. Three approaches to real vector field equations

To start with, let us recall some basic facts from Hamiltonian mechanics. In the **real** framework, $\hat{L}_1$ is a Hamiltonian vector field

$$(7) \quad \hat{L}_1 \equiv \partial_y + \lambda \partial_x - u_x(x, y)\partial_\lambda = \partial_y + \{H, \cdot\}_{\lambda, x}, \quad x, y, \lambda \in \mathbb{R}, \quad u \in \mathbb{R},$$

where

$$(8) \qquad\qquad H(x, \lambda, y) = \frac{\lambda^2}{2} + u(x, y)$$

is the Hamiltonian of a newtonian particle in the time-dependent potential $u(x, y)$ (here time is y). Eigenfunctions Ψ of $\hat{L}_1$,

$$(9) \qquad\qquad \hat{L}_1 \Psi = 0, \quad x, y, \lambda \in \mathbb{R}, \quad \Psi(x, y, \lambda) \in \mathbb{R},$$

are exactly conservation laws for the associated hamiltonian dynamical system:

$$(10) \qquad\qquad \frac{dx}{dy} = \lambda, \quad \frac{d\lambda}{dy} = -u_x(x, y).$$

Hamiltonian systems can also be studied using the Hamilton-Jacobi equation

$$(11) \qquad\qquad \frac{\partial S}{\partial y} + H\left(x, \frac{\partial S}{\partial x}\right) = 0, \quad \lambda = \frac{\partial S}{\partial x},$$

which, in our example, takes this form:

$$(12) \qquad\qquad \frac{\partial S}{\partial y} + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + u(x, y) = 0.$$

Equations (9), (10), (12) give three equivalent formulations of the problem. If we know the trajectories of (10), then any function constant on these trajectories solves (9). And vice versa, if we have two independent constants of motion, the common level sets are the trajectories. The connection between the dynamical system (10) and the Hamilton-Jacobi equation (12) is well-known. Finally, we go from (9) to (12) just considering the level sets

$$(13) \qquad\qquad \Psi(x, y, \lambda) = k;$$

solving (13) with respect to λ:

$$(14) \qquad\qquad \lambda = \Lambda(x, y, k)$$

we construct a function Λ satisfying the forced Hopf equation

$$(15) \qquad\qquad \frac{\partial \Lambda}{\partial y} + \Lambda\frac{\partial \Lambda}{\partial x} = -\frac{\partial u}{\partial x},$$

which is nothing but the x-derivative of the Hamilton-Jacobi equation (12) for $\Lambda = S_x$.

Using the equivalence between (9) and (10), we can easily construct Jost eigenfunctions of (3) (another important ingredient of the dKP direct problem [13]) satisfying the following asymptotics

$$(16) \qquad \begin{aligned} \varphi_1(x, y, \lambda) &\to \lambda, \\ \varphi_2(x, y, \lambda) &\to x - \lambda y, \end{aligned} \qquad \text{as } y \to -\infty.$$

Indeed, consider the solution of (10) with the following initial data

$$(17) \qquad\qquad x = x_0, \quad \lambda = \lambda_0 \;\text{ at }\; y = y_0.$$

For sufficiently regular and well-localized $u(x, y)$, the solution exists globally in $y \in \mathbb{R}$, is unique and has the free particle behavior at $\pm\infty$

$$(18)$$
$$\lambda = \lambda_\pm(x_0, \lambda_0, y_0) + o(1), x = x_\pm(x_0.\lambda_0, y_0) + y\lambda_\pm(x_0, \lambda_0, y_0) + o(1) \text{ as } y \to \pm\infty.$$

Therefore the Jost solutions can be directly expressed in terms of the long-time asymptotics:

$$(19) \quad \varphi_1(x_0, y_0, \lambda_0) = \lambda_-(x_0, \lambda_0, y_0), \quad \varphi_2(x_0, y_0, \lambda_0) = x_-(x_0.\lambda_0, y_0).$$

3. Reduction to the complex Hopf equation

If λ is complex, we loose one of the principal ingredients – the dynamical system (10). One of the reasons is that we do not want to assume that $u(x, y)$ has good analytic continuation to the complex domain. Nevertheless, the connection between equations (3) and (15) is still present and it will be our principal tool.

Lemma 1. *At regular points* $(\partial_\lambda \Psi(x, y, \lambda) \neq 0)$ *the function* $\Lambda(x, y, k)$ *defined by (13), (14) for complex* λ, *satisfies the* **complex** *Hopf equation with source:*

$$(20) \qquad\qquad \Lambda_y + \Lambda\Lambda_x = -u_x.$$

The proof is straightforward:

$$\Psi(x, y, \Lambda(x, y, k)) \equiv k$$

therefore

$$\frac{d}{dx}\Psi(x, y, \Lambda(x, y, k)) = 0, \quad \frac{d}{dy}\Psi(x, y, \Lambda(x, y, k)) = 0,$$

$$\Psi_x + \Lambda_x\Psi_\lambda = 0, \quad \Psi_y + \Lambda_y\Psi_\lambda = 0,$$

$$\Lambda\Psi_x + \Lambda\Lambda_x\Psi_\lambda + \Psi_y + \Lambda_y\Psi_\lambda = 0,$$

$$\Lambda\Psi_x + \Lambda\Lambda_x\Psi_\lambda + \Psi_y + \Lambda_y\Psi_\lambda - u_x\Psi_\lambda + u_x\Psi_\lambda = 0,$$

$$[\Psi_y + \Lambda\Psi_x - u_x\Psi_\lambda] + [\Lambda_y\Psi_\lambda + \Lambda\Lambda_x\Psi_\lambda + u_x\Psi_\lambda] = 0,$$

$$[\Lambda_y + \Lambda\Lambda_x + u_x]\Psi_\lambda = 0.$$

Taking into account that $\Psi_\lambda \neq 0$, we have the Hopf equation. $\square$

Consider the level sets for the function $\Psi_1(x, y, \lambda)$; the asymptotic condition (5) implies:

$$\lambda + o(1) = k, \quad \text{as} \quad x^2 + y^2 \to \infty;$$

therefore

(21) $$\Lambda_1(x, y, k) \to k \quad \text{as} \quad x^2 + y^2 \to \infty.$$

Therefore, to construct Ψ_1, we solve the Hopf equation

(22) $$(\Lambda_1)_y + \Lambda_1(\Lambda_1)_x = -u_x$$

with the boundary condition:

(23) $$\Lambda_1(x, y, k) \to k \quad \text{as} \quad x^2 + y^2 \to \infty.$$

Unfortunately we can not apply the same procedure to the function Ψ_2 because, for $u \equiv 0$, $\Psi_2(x, y, \lambda) \equiv x - \lambda y$ and the levels sets

$$x - \lambda y = k, \quad \lambda = \frac{x - k}{y}$$

are always singular.

Instead, let us consider the level sets for the function

$$\Psi(x, y, \lambda) = \Psi_1(x, y, \lambda) + \varepsilon\Psi_2(x, y, \lambda), \quad \varepsilon \ll 1,$$

$$\Psi(x, y, \Lambda_1 + \varepsilon\Lambda_2 + O(\varepsilon^2)) = k,$$

where $k = \Psi_1(x, y, \Lambda_1)$. Combining the coefficients at order ε^1 we obtain:

(24) $$\Psi_2(x, y, \Lambda_1(x, y, k)) = -\Lambda_2(x, y, k) \, \partial_\lambda \Psi_1(x, y, \lambda)|_{\lambda = \Lambda_1(x,y,k)}.$$

For $x^2 + y^2 \to \infty$ we obtain

$$\Lambda_2(x, y, k) \to -\Psi_2(x, y, \Lambda_1(x, y, k)) \quad \text{as} \quad x^2 + y^2 \to \infty;$$

therefore

(25) $$\Lambda_2(x, y, k) \to -(x - ky) \quad \text{as} \quad x^2 + y^2 \to \infty.$$

Substituting $\Lambda = \Lambda_1 + \varepsilon\Lambda_2 + O(\varepsilon^2)$ into the Hopf equation (20) and combining the coefficients at order ε^1 we obtain:

(26) $$(\Lambda_2)_y + (\Lambda_1\Lambda_2)_x = 0.$$

Therefore, to construct $\Psi_2(x, y, \lambda)$, we have to solve the linearized Hopf equation (26) with the boundary condition (25).

4. A SKETCH OF OUR CONSTRUCTION

Here we present a short description of our work.

(1) Assuming that $|\operatorname{Im} k|$ is sufficiently large, we solve the nonlinear Hopf equation (22), (23), and obtain some estimates on the solution.

   To solve equation (22), and, in general, nonlinear PDEs, it is natural to use Sobolev spaces not only because one can control the derivatives involved in the equations, but also because Sobolev spaces with sufficiently many derivatives are Banach algebras – the product of two elements of that space also belongs to it, and the multiplication is continuous. More precisely, if $f(x,y), g(x,y) \in H^l(\mathbb{R}^2)$, $l \geq 2$, then $f \cdot g \in H^l(\mathbb{R}^2)$ and $\|f \cdot g\|_{H^l(\mathbb{R}^2)} \leq \alpha_l \|f\|_{H^l(\mathbb{R}^2)} \|g\|_{H^l(\mathbb{R}^2)}$ (see Lemma 7 of the Appendix for details).

   Assuming that $u(x,y) \in H^4(\mathbb{R}^2)$, we construct the solution of (22), (23) in the form $\Lambda_1(x,y,k) = k + \phi(x,y,k)$, where $\phi \in H^4(\mathbb{R}^2)$.

(2) To obtain some preliminary estimates on the behavior of $\phi(x,y,k)$ at large $x^2 + y^2$, we show that $\phi(x,y,k)$ lies also in the Banach spaces $W^{2,p}$ for p sufficiently close to 2.

(3) Assuming that $u(x,y)$ decays sufficiently fast at infinity (see Proposition 1 for details), we show that $\phi(x,y,k)$ satisfies an inhomogeneous Beltrami equation. Using this fact and the estimates on $\phi(x,y,k)$ proven in the previous step, we obtain additional estimates on the asymptotics of $\phi(x,y,k)$ for large $x^2 + y^2$.

(4) By interpreting the linearized Hopf equation (26) as an inhomogeneous Beltrami equation, we prove the existence and uniqueness of $\Lambda_2(x,y,k)$.

(5) Using the linearized Hopf equations for the functions $\partial_{\bar{k}}\Lambda_1$, $\partial_{\bar{k}}\Lambda_2$, we show that $\partial_{\bar{k}}\Lambda_1 = \partial_{\bar{k}}\Lambda_2 = 0$; i.e., for fixed (x,y), the functions Λ_1, Λ_2 are analytic in k.

(6) Using the linearized Hopf equations for the function $\partial_k\Lambda_1$, we show that, for sufficiently large $|\operatorname{Im} k|$, we have $|\partial_k\Lambda_1 - 1| \leq C < 1$. Combining this fact with estimates on $\phi(x,y,k)$, we show that the inversion with respect to k of equation (14) is well-defined, and gives us the analytic eigenfunctions $\Psi_1(x,y,\lambda)$, $\Psi_2(x,y,\lambda)$ of the vector field $\hat{L}_1$.

5. Notation, some basic results and definitions

Assuming that k is a fixed complex number, $\operatorname{Im} k \neq 0$, we shall use the following complex notation:

$$
\begin{aligned}
z &= \frac{1}{\bar{k} - k}(x - ky), & \bar{z} &= -\frac{1}{\bar{k} - k}(x - \bar{k}y), \\
x &= \bar{k}z + k\bar{z}, & y &= z + \bar{z}, \\
\partial_{\bar{z}} &= \partial_y + k\partial_x, & \partial_z &= \partial_y + \bar{k}\partial_x, \\
\partial_x &= \frac{1}{\bar{k} - k}(\partial_z - \partial_{\bar{z}}), & \partial_y &= \frac{1}{\bar{k} - k}(\bar{k}\partial_{\bar{z}} - k\partial_z).
\end{aligned}
$$
(27)

To simplify the notation, we shall use the following agreement: unless it generates confusion, $f(x, y)$ and $f(z, \bar{z})$ will denote **the same** function in the plane. Moreover we will often omit the $\bar{z}$ dependence in the argument of functions; therefore writing $f(z)$ instead of $f(z, \bar{z})$ **does not imply** $\partial_{\bar{z}} f(z) = 0$.

We use the following normalization for the Fourier transform:

$$
(28) \quad f(x, y) = \frac{1}{2\pi} \iint \hat{f}(p) e^{i(p_x x + p_y y)} d^2 p, \quad p = (p_x, p_y), \quad d^2 p = dp_x dp_y,
$$

$$
(29) \quad \hat{f}(p) = \frac{1}{2\pi} \iint f(x, y) e^{-i(p_x x + p_y y)} dx dy.
$$

It is clear, that

$$
(30) \quad \widehat{\partial_z f} = i(p_y + \bar{k} p_x)\hat{f}, \quad \widehat{\partial_{\bar{z}} f} = i(p_y + k p_x)\hat{f},
$$

$$
(31) \quad \widehat{fg}(p) = \frac{1}{2\pi}(\hat{f} * \hat{g})(p),
$$

where

$$
(32) \quad (\hat{f} * \hat{g})(p) = \iint \hat{f}(p - q)\hat{g}(q) d^2 q
$$

is the **convolution** operator.

We recall that, if $\hat{f} \in L^1(\mathbb{R}^2)$, then $f(z)$ is continuous, decays for $|z| \to \infty$ and

$$
(33) \quad |f(z)| \leq \frac{1}{2\pi} \iint |\hat{f}(p)| d^2 p.
$$

We also need the following theorem from the Vekua's book [26].

Theorem 1. *Denote by Π the following operator:* $\Pi = \partial_z \partial_{\bar{z}}^{-1}$

$$(34) \qquad (\Pi f)(z) = -\frac{1}{\pi} \iint \frac{f(\xi)}{(\xi - z)^2} d^2\xi$$

(1) *Let $f(\xi) \in L^p(\mathbb{R}^2)$. Then the integral (34) is well-defined, in the sense of principal value, almost everywhere in z and $(\Pi f)(z) \in L^p(\mathbb{R}^2)$.*

(2) *For all $1 < p < \infty$, Π is a bounded operator on $L^p(\mathbb{R}^2)$*

$$(35) \qquad \|\Pi f\|_{L^p} \leq \gamma(p)\|f\|_{L^p},$$

where $\gamma(p)$ is a continuous function of p and $\gamma(2) = 1$.

The proof of this Theorem essentially uses the results of the papers [21], [25]. This statement can also be viewed as a corollary of the Zygmund-Calderon theorem (see [22]).

The Sobolev space $H^l(\mathbb{R}^2) = W^{l,2}(\mathbb{R}^2)$ is the Hilbert space defined using the following scalar product

$$(36) \qquad (f, g)_{H^l} = \iint \overline{\hat{f}(p)}\hat{g}(p)(1 + p_x^2 + p_y^2)^l d^2p$$

$$(37) \qquad \|f\|_{H^l} = \sqrt{(f, f)_{H^l}}$$

Another (equivalent) norm is defined by

$$(38) \qquad \|f\|_{W^{l,2}} = \left[\sum_{k_1 + k_2 \leq l} \iint |\partial_x^{k_1} \partial_y^{k_2} f(x, y)|^2 dx dy \right]^{1/2}$$

The Sobolev space $W^{l,p}(\mathbb{R}^2)$ is a Banach space generated by the following norm:

$$(39) \qquad \|f\|_{W^{l,p}} = \left[\sum_{k_1 + k_2 \leq l} \iint |\partial_x^{k_1} \partial_y^{k_2} f(x, y)|^p dx dy \right]^{1/p}$$

We shall also use the following notations $f \in W^{l,2\pm\epsilon_0}$, $f \in W^{l,2\pm}$:

(1) $f \in W^{l,2\pm\epsilon_0}$ if $f \in W^{l,2+\epsilon}$ for all ϵ such that $-\epsilon_0 \leq \epsilon \leq \epsilon_0$.

(2) $f \in W^{l,2\pm}$ if $f \in W^{l,2\pm\epsilon_0}$ for some $\epsilon_0 > 0$.

6. SOLVING THE HOPF EQUATION FOR Λ_1

Let us describe the iterative procedure for solving (22). Let

$$(40) \qquad \Lambda_1 = k + \phi(x, y, k);$$

then we have:

$$(41) \qquad \phi_y + k\phi_x + \phi\phi_x = -u_x.$$

In the complex notation (27), equation (41) takes the form:

$$(42) \qquad \partial_{\bar{z}}\phi + \frac{1}{\bar{k}-k}\phi(\partial_z - \partial_{\bar{z}})\phi = -\frac{1}{\bar{k}-k}(\partial_z - \partial_{\bar{z}})u,$$

equivalent, for ϕ decaying at infinity, to

$$(43) \qquad \phi = -\frac{1}{2(\bar{k}-k)}(\partial_{\bar{z}}^{-1}\partial_z - 1)(\phi^2) - \frac{1}{\bar{k}-k}(\partial_{\bar{z}}^{-1}\partial_z - 1)(u).$$

Let us check that, for sufficiently small ϕ, the map

$$(44) \qquad \phi \to G(\phi) = \frac{1}{2(\bar{k}-k)}(\partial_{\bar{z}}^{-1}\partial_z - 1)(\phi^2)$$

in the space $H^l(\mathbb{R}^2)$ is contracting.

In the Fourier representation we have:

$$(45) \qquad \widehat{\partial_{\bar{z}}^{-1}\partial_z f} = \frac{p_y + \bar{k}p_x}{p_y + kp_x}\hat{f},$$

therefore $\partial_{\bar{z}}^{-1}\partial_z$ is an unitary operator on $H^l(\mathbb{R}^2)$. Therefore we have

$$(46) \qquad \left\| \frac{1}{2(\bar{k}-k)}(\partial_{\bar{z}}^{-1}\partial_z - 1) \right\|_{H^l} \leq \frac{1}{2|\operatorname{Im}k|}.$$

Consequently, from

$$(47) \qquad G(\phi_1) - G(\phi_2) = \frac{1}{2(\bar{k}-k)}(\partial_{\bar{z}}^{-1}\partial_z - 1)[(\phi_1 + \phi_2)(\phi_1 - \phi_2)]$$

it follows that (see Lemma 7, part 2) of the Appendix)

$$(48) \qquad \|G(\phi_1) - G(\phi_2)\| \leq \frac{\alpha_l}{|\operatorname{Im}k|}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|.$$

Therefore we proved

Lemma 2. *Let $\mathcal{B}$ denote the ball in H^l of radius $\frac{|\operatorname{Im}k|}{\alpha_l}$:*

$$(49) \qquad \phi \in \mathcal{B} \quad if \quad \|\phi\|_{H^l} < \frac{|\operatorname{Im}k|}{\alpha_l}.$$

    (1) *Let ϕ be a function from $\mathcal{B}$. Then*

$$(50) \qquad \|G(\phi)\|_{H^l} \leq \frac{1}{2}\|\phi\|_{H^l}$$

    (2) *Let ϕ_1, ϕ_2 be functions from the interior of $\mathcal{B}$. Then*

$$(51) \qquad \|G(\phi_1) - G(\phi_2)\|_{H^l} < \|\phi_1 - \phi_2\|_{H^l}$$

Lemma 3.   (1) *Let*

(52) $$\|u\|_{H^l} < C \frac{|\operatorname{Im} k|^2}{2\alpha_l}, \quad C < 1.$$

Then the map

(53) $$F(\phi) = -\frac{1}{2(\bar{k} - k)}(\partial_{\bar{z}}^{-1}\partial_z - 1)(\phi^2) - \frac{1}{\bar{k} - k}(\partial_{\bar{z}}^{-1}\partial_z - 1)(u)$$

maps the ball $\mathcal{B}$ onto itself.
   (2) *Let us introduce also the ball $\mathcal{B}_1$ in H^l of radius $\frac{2}{|\operatorname{Im} k|}\|u\|$, which is much smaller than $\mathcal{B}$ for large $\operatorname{Im} k$. Then, if*

(54) $$\|\phi\|_{H^l} < \frac{2}{|\operatorname{Im} k|}\|u\|_{H^l},$$

it follows that $F(\phi)$ maps the ball $\mathcal{B}_1$ onto itself:

(55) $$\|F(\phi)\|_{H^l} < \frac{2}{|\operatorname{Im} k|}\|u\|_{H^l}.$$

Combining all these estimates we obtain

Theorem 2. *Let function $u(x, y)$ satisfy the inequality (52). Then the iteration procedure*
   (1) $\phi_0 = 0$
   (2) $\phi_{j+1} = F(\phi_j), \quad j \geq 0.$
converges in $H^l(\mathbb{R}^2)$, defining the unique localized solution of (42), and

(56) $$\|\phi\|_{H^l} < C \frac{|\operatorname{Im} k|}{\alpha_l}, \quad \max_{z \in \mathbb{C}} |\phi(z, \bar{z})| < C|\operatorname{Im} k|.$$

For the second estimate of (56) see the Lemma 7, parts 1), 2) of the Appendix.
   We finally observe that

(57) $$\partial_{\bar{z}}^{-1} = \frac{1}{k}\left(\partial_x + \frac{1}{k}\partial_y\right)^{-1},$$

and we can formally write, in the large $\operatorname{Im} k$ limit,

(58) $$\partial_{\bar{z}}^{-1} = \frac{1}{k}\partial_x^{-1}\left(\sum_{n \geq 0}(-1)^n \left(\frac{\partial_y \partial_x^{-1}}{k}\right)^n\right).$$

Then equation (43) implies the following formal asymptotics

(59) $$\Lambda_1 = k - \frac{u}{k} + \frac{\partial_y \partial_x^{-1} u}{k^2} - \frac{\partial_y^2 \partial_x^{-2} u + u^2/2}{k^3} + O\left(\frac{1}{k^4}\right).$$

7. Some estimates on the asymptotics of ϕ

Since, as it will be shown in Section 8, the linear equation for Λ_2 can be interpreted as a Beltrami equation, it follows that Hilbert-Sobolev spaces are not adequate to deal with the problem (see [26]). Therefore in this Section we show that ϕ belongs also to some non-Hilbert Sobolev spaces.

In this section we assume that $l = 4$, and the inequality (52) is fulfilled. It means that $\phi \in H^4(\mathbb{R}^2)$ and we have the inequalities (56). In addition, ϕ and its first two derivatives are continuous and bounded (see Lemma 7, part 1) of the Appendix).

Let us prove the following estimates:

Proposition 1. *Let $u(x,y) \in H^4(\mathbb{R}^2) \cap W^{2,2\pm}$ and $(x+iy)^3 u(x,y) \in W^{2,2\pm}$. Let k satisfy the inequality (52).*
   Then
   (1) $\phi \in H^4 \cap W^{2,2\pm}$
   (2)

$$\phi = \frac{c_2}{z^2} + O\left(\frac{1}{z^3}\right), \quad |z| \to \infty, \tag{60}$$

$$\partial_z \phi = O\left(\frac{1}{z^3}\right), \quad |z| \to \infty. \tag{61}$$

The proof consists of a series of steps.

   (1) We want to view one of the functions ϕ in the quadratic term of equation (42) as a known function (ϕ) and the other one as the unknown (Φ). Therefore equation (42) can be rewritten as a linear equation for Φ in two different ways:

$$\partial_{\bar{z}}\Phi + \frac{1}{\bar{k}-k}\phi(\partial_z - \partial_{\bar{z}})\Phi = -\frac{1}{\bar{k}-k}(\partial_z - \partial_{\bar{z}})u, \tag{62}$$

   and

$$\partial_{\bar{z}}\Phi + \frac{1}{2(\bar{k}-k)}(\partial_z - \partial_{\bar{z}})[\phi \cdot \Phi] = -\frac{1}{\bar{k}-k}(\partial_z - \partial_{\bar{z}})u. \tag{63}$$

   Function Φ can be obtained by solving the integral equation

$$\Phi = -\frac{1}{2(\bar{k}-k)}(\partial_{\bar{z}}^{-1}\partial_z - 1)[\phi \cdot \Phi] - \frac{1}{\bar{k}-k}(\partial_{\bar{z}}^{-1}\partial_z - 1)u, \tag{64}$$

   in all spaces $W^{2,2+\epsilon}$, where $|\epsilon|$ is sufficiently small. Now the iteration process converges due to estimates (56), (35); therefore $\Phi \in W^{2,2\pm}$, and the first statement of Proposition 1 is proven.

(2) Consider the auxiliary **homogeneous** equation:

$$\partial_{\bar{z}} w + \frac{1}{\bar{k} - k} \phi (\partial_z - \partial_{\bar{z}}) w = 0, \tag{65}$$

which is equivalent to the Beltrami equation:

$$\partial_{\bar{z}} w - q(z, \bar{z}) \partial_z w = 0, \tag{66}$$

where

$$q(z, \bar{z}) = \frac{\frac{\phi(z,\bar{z})}{k - k}}{1 + \frac{\phi(z,\bar{z})}{k - k}}, \tag{67}$$

$$|q(z, \bar{z})| \le C, \qquad \|q(z, \bar{z})\|_{H^l} \le \frac{C}{\alpha_l}, \quad C < 1. \tag{68}$$

To continue, we need the following lemma from [26]:

Lemma 4. *Let $q(z)$ be a measurable bounded function such that*
(a) $|q(z)| \le C < 1$ for all $z \in \mathbb{C}$.
(b) $q(z) \in L^p$ for some $p < 1$.
Then the Beltrami equation (66) has an unique (up to a constant) solution $w(z)$ such that
(a) $w = z + O(1)$ as $z \to \infty$
(b) $z \to w(z, \bar{z})$ defines a one-to-one continuous map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

We know that $\phi \in H^4(\mathbb{R}^2) \cap W^{2,2\pm}(\mathbb{R}^2)$, therefore we can improve the estimate (a) of Lemma 4.

Lemma 5. *Let q be defined by (67) in terms of $\phi \in H^4(\mathbb{R}^2) \cap W^{2,2\pm}(\mathbb{R}^2)$ satisfying (56), with $C < 1/2$. Then*
(a) $w(z) = z + o(1)$.
(b) All functions

$$\frac{\partial w}{\partial z}, \quad \frac{\partial \bar{w}}{\partial z}, \quad \frac{\partial w}{\partial \bar{z}}, \quad \frac{\partial \bar{w}}{\partial \bar{z}}, \quad \frac{\partial z}{\partial w}, \quad \frac{\partial \bar{z}}{\partial w}, \quad \frac{\partial z}{\partial \bar{w}}, \quad \frac{\partial \bar{z}}{\partial \bar{w}},$$

$$\frac{\partial^2 w}{\partial z^2}, \quad \frac{\partial^2 w}{\partial \bar{z}^2}, \quad \frac{\partial^2 w}{\partial z \partial \bar{z}}, \quad \frac{\partial^2 \bar{w}}{\partial z^2}, \quad \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}, \quad \frac{\partial^2 \bar{w}}{\partial z \partial \bar{z}},$$

$$\frac{\partial^2 z}{\partial w^2}, \quad \frac{\partial^2 z}{\partial \bar{w}^2}, \quad \frac{\partial^2 z}{\partial w \partial \bar{w}}, \quad \frac{\partial^2 \bar{z}}{\partial w^2}, \quad \frac{\partial^2 \bar{z}}{\partial \bar{w}^2}, \quad \frac{\partial^2 \bar{z}}{\partial w \partial \bar{w}},$$

are continuous and bounded.
(c) $Jac((w, \bar{w}), (z, \bar{z}))$ and $Jac((z, \bar{z}), (w, \bar{w}))$ are continuous and bounded functions.

The proof follows from the the following explicit formula

$$(69) \qquad w = z + \partial_{\bar{z}}^{-1} f, \quad f = \left[\sum_{k=0}^{\infty} (q \partial_z \partial_{\bar{z}}^{-1})^k \right] q$$

The convergence of the series in $W^{2,2\pm}$ follows from arguments similar to those in Lemma 9 of the Appendix. In H^4 we have the estimate

$$(70) \qquad \|(q\partial_z \partial_{\bar{z}}^{-1})\| \le \alpha_4 \|q(z,\bar{z})\|_{H^4} \le C < \frac{1}{2}.$$

Therefore

$$\|f\|_{H^4} \le 2\|q(z,\bar{z})\|_{H^4} \le \frac{2C}{\alpha_4},$$

$$(71) \qquad \|\partial_z \partial_{\bar{z}}^{-1} f\|_{H^4} \le 2\|q(z,\bar{z})\|_{H^4} \le \frac{2C}{\alpha_4}.$$

From Lemma 7, part 2 we also have

$$(72) \qquad |f| \le 2C < 1, \quad |\partial_z \partial_{\bar{z}}^{-1} f| \le 2C < 1.$$

Since $w = z + \partial_{\bar{z}}^{-1} f$, where $f \in H^4(\mathbb{R}^2) \cap W^{2,2\pm}(\mathbb{R}^2)$, then $\hat{f} \in L^q(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ for some $q > 2$. From the Hölder inequality it follows that $\widehat{\partial_{\bar{z}}^{-1} f} \in L^1(\mathbb{R}^2)$, and $\partial_{\bar{z}}^{-1} f = o(1)$ for $z \to \infty$.
Since

$$(73) \qquad \partial_z w = 1 + \partial_{\bar{z}}^{-1} f, \quad \partial_{\bar{z}} w = q \partial_z w,$$

one immediately sees that

$$(74) \quad Jac((w,\bar{w}),(z,\bar{z})) = (\partial_z w \cdot \overline{\partial_z w})(1 - q\bar{q}) \ge (1 - 2C)^2 (1 - C^2) > 0.$$

$\square$

It is natural to use w as a new coordinate. It is easy to check that

$$(75) \qquad [\partial_{\bar{z}} - q\partial_z] = [1 - q\bar{q}] \overline{\left(\frac{\partial w}{\partial z} \right)} \partial_{\bar{w}}.$$

Equation (62) is equivalent to:

$$(76) \quad \partial_{\bar{w}} \Phi = \mathcal{U}, \quad \text{where } \mathcal{U} = -[1 - q\bar{q}]^{-1} \overline{\left(\frac{\partial w}{\partial z} \right)}^{-1} \left(1 + \frac{\phi(z,\bar{z})}{k - \bar{k}} \right)^{-1} u_x.$$

Let $\mathcal{U}(w,\bar{w}) = \mathcal{U}_+(w,\bar{w}) + \mathcal{U}_-(w,\bar{w})$, where $\mathcal{U}_+(w,\bar{w})$ has support in the ball $|z| \le 2$ and $\mathcal{U}_-(w,\bar{w})$ has support outside the ball $|z| \le 1$. Then

$$(77) \qquad \Phi = \Phi_+ + \Phi_-, \quad \Phi_+ = \partial_{\bar{w}}^{-1} \mathcal{U}_+, \quad \Phi_- = \partial_{\bar{w}}^{-1} \mathcal{U}_-.$$

The function Φ_+ is holomorphic in $1/w$ outside the ball $|z| \leq 2$, and

$$(78) \qquad \Phi_- = \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \frac{1}{w^3} \partial_{\bar{w}}^{-1} (w^3 \mathcal{U}_-).$$

We know that $\Phi \in W^{2,2\pm}$, therefore, for large $|z|$, we have

$$(79) \qquad \Phi = \frac{\tilde{d}_2}{w^2} + \frac{\tilde{d}_3}{w^3} + \frac{1}{w^3} \partial_{\bar{w}}^{-1} (w^3 \mathcal{U}_-),$$

$$(80) \qquad \partial_w \Phi = O\left(\frac{1}{w^3}\right), \quad \partial_{\bar{w}} \Phi = O\left(\frac{1}{w^3}\right)$$

Here we used, that $(x+iy)^3 u(x,y) \in W^{2,2\pm}$, therefore $w^3 \mathcal{U}_- \in W^{1,2\pm}$, $\partial_{\bar{w}}^{-1}(w^3 \mathcal{U}_-)$ is continuous, bounded and decays at infinity, the functions $w^3 \mathcal{U}_- \in W^{1,2\pm}$ and $\partial_w \partial_{\bar{w}}^{-1}(w^3 \mathcal{U}_-)$ are continuous and bounded (see Lemma 8 of the Appendix).

Taking into account that $\partial_z w$, $\partial_{\bar{z}} w$ are bounded continuous functions, we finish the proof of Proposition 1.

$\square$

8. Solution of the linearized Hopf equation

Now we are ready to construct the function $\Lambda_2(x,y,k)$. In the complex coordinates, equation (22) has the following form

$$(81) \qquad \partial_{\bar{z}} \Lambda_2 + \frac{1}{\bar{k} - k}(\partial_z - \partial_{\bar{z}})(\phi \Lambda_2) = 0,$$

and the boundary condition (25) is equivalent to

$$(82) \qquad \Lambda_2(x,y,k) = (k - \bar{k})z + o(1) \quad \text{as} \quad z \to \infty.$$

We shall use the following simple

Lemma 6. *Let Ξ be a solution of (81), $p(w)$ be an arbitrary* **holomorphic** *function of w, where w is the special solution of (65) defined in the previous section. Then $p(w)\Xi$ also solves (81).*

Let us now construct a special solution of (81) in the form $\Xi = 1+\Xi_1$, where Ξ_1 decays at infinity. Then we have

$$(83) \qquad \partial_{\bar{z}} \Xi_1 + \frac{1}{\bar{k} - k}(\partial_z - \partial_{\bar{z}})(\phi \Xi_1) = -\frac{1}{\bar{k} - k}(\partial_z - \partial_{\bar{z}})(\phi)$$

$$(84) \qquad \Xi_1 + \frac{1}{\bar{k} - k}(\partial_z \partial_{\bar{z}}^{-1} - 1)(\phi \Xi_1) = -\frac{1}{\bar{k} - k}(\partial_z \partial_{\bar{z}}^{-1} - 1)(\phi)$$

$$(85) \qquad \Xi_1 = - \left[1 + \frac{1}{\bar{k} - k} (\partial_z \partial_{\bar{z}}^{-1} - 1) \phi \right]^{-1} \frac{1}{\bar{k} - k} (\partial_z \partial_{\bar{z}}^{-1} - 1)(\phi),$$

and $\Xi_1 \in W^{2,2\pm}$; therefore $\Xi_1 \to 0$ as $|z| \to \infty$, and $\partial_z \Xi_1$ is a bounded function. Equation (84) is equivalent to:

$$\Xi_1 = \frac{1}{k - \bar{k}} (\partial_z \partial_{\bar{z}}^{-1} - 1)(\phi \Xi_1) + \frac{1}{k - \bar{k}} (\partial_z \partial_{\bar{z}}^{-1} - 1)(\phi) =$$

$$= \frac{1}{k - \bar{k}} \left[-\phi - \phi \Xi_1 + \partial_{\bar{z}}^{-1} \left(\phi_z + \phi_z \Xi_1 + \phi (\Xi_1)_z \right) \right] =$$

$$(86) \qquad = O\left(\frac{1}{z^2} \right) + \partial_{\bar{z}}^{-1} O\left(\frac{1}{z^3} \right) + \frac{1}{k - \bar{k}} \partial_{\bar{z}}^{-1} \left(\frac{c_2}{z^2} (\Xi_1)_z \right) =$$

$$= O\left(\frac{1}{z^2} \right) + \frac{d'}{z} + O\left(\frac{1}{z^{2-\epsilon}} \right) + \frac{d''}{z} + + \frac{d'''}{z^2} + \frac{c_2}{z^2} \partial_{\bar{z}}^{-1} (\Xi_1)_z =$$

$$= \frac{d}{z} + O\left(\frac{1}{z^{2-\epsilon}} \right), \quad z \to \infty,$$

where $\epsilon > 0$ is an arbitrary positive constant, and d, d', d'', d''' are some momenta. Since $\Xi_1 \in W^{2,2\pm}$, then $d = 0$ and

$$(87) \qquad \Xi = 1 + O\left(\frac{1}{z^{2-\epsilon}} \right).$$

Using Lemma 6, we finally obtain:

$$(88) \qquad \Lambda_2 = 2 \operatorname{Im} k \cdot \Xi \cdot w$$

At last, from equations (58), (59), (81) we obtain the asymptotics of Λ_2 for large $\operatorname{Im} k$:

$$(89) \quad \Lambda_2 = - (x - ky) + \frac{yu}{k} - \frac{xu + \partial_x^{-1}(2yu_y + u)}{k^2} +$$

$$+ \frac{x\partial_x^{-1} u_y + \partial_x^{-1}(xu_y) + \frac{3}{2} yu^2 + 3\partial_x^{-2}(yu_y)_y}{k^3} + O\left(\frac{1}{k^4} \right).$$

9. From the Hopf equation to the vector fields eigenfunctions

In the previous section we constructed the function $\Lambda_1(x, y, k) = k + \phi(x, y, k)$; i.e. the level sets for the function $\Psi_1(x, y, \lambda)$:

$$(90) \qquad \Psi_1(x, y, \Lambda_1(x, y, k)) = k.$$

Therefore, to construct $\Psi_1(x, y, \lambda)$, we have to solve with respect to k the following equation

$$(91) \qquad \Lambda_1(x, y, k) = \lambda.$$

Let us calculate $\partial_k \Lambda_1(x, y, k)$, $\partial_{\bar{k}} \Lambda_1(x, y, k)$. Differentiating by k, $\bar{k}$ the principal Hopf equation (20) for Λ_1, we obtain that both functions $\partial_k \Lambda_1(x, y, k)$, $\partial_{\bar{k}} \Lambda_1(x, y, k)$ satisfy (81) with the boundary conditions:

(92) $\partial_k \Lambda_1(x, y, k) = 1 + o(1), \quad \partial_{\bar{k}} \Lambda_1(x, y, k) = o(1) \quad \text{as} \quad z \to \infty.$

Therefore

(93) $\partial_k \Lambda_1(x, y, k) = \Xi = 1 + \Xi_1, \quad \partial_{\bar{k}} \Lambda_1(x, y, k) \equiv 0.$

In particular, $\Lambda_1(x, y, k)$ is holomorphic in k. To calculate $\partial_{\bar{k}} \Lambda_2(x, y, k)$ we can differentiate by $\bar{k}$ equation (26). Taking into account that $\Lambda_1(x, y, k)$ is holomorphic in k, we obtain

(94) $(\partial_{\bar{k}} \Lambda_2)_y + (\Lambda_1 \partial_{\bar{k}} \Lambda_2)_x = 0,$

with the boundary condition

(95) $\partial_{\bar{k}} \Lambda_2(x, y, k) = o(1) \quad \text{as} \quad z \to \infty;$

therefore

(96) $\partial_{\bar{k}} \Lambda_2(x, y, k) \equiv 0.$

Let $u(x, y)$ satisfy (52) with $C < 1/2$, $l = 4$. From (85) and (56) we immediately obtain that

(97) $\|\Xi_1\|_{H^4} \leq \dfrac{2C}{\alpha_4}, \quad \max |\Xi_1| \leq 2C, \quad C < \dfrac{1}{2}.$

Let us fix a point (x, y) and consider the map

(98) $k \to \lambda = \Lambda_1(x, y, k)$

from the semi-plane

(99) $\operatorname{Im} k \geq D = \sqrt{\dfrac{2\alpha_4 \|u\|_{H^4}}{C}}$

to the complex plane. We see that, for a pair of points k_1, k_2,

(100) $\Lambda_1(x, y, k_1) - \Lambda_1(x, y, k_2) = k_1 - k_2 + \phi(x, y, k_1) - \phi(x, y, k_2).$

But
(101)
$|\phi(x, y, k_1) - \phi(x, y, k_2)| \leq |k_1 - k_2| \max_{k \in [k_1, k_2]} |\Xi_1(x, y, k)| \leq 2C|k_1 - k_2|,$

where $[k_1, k_2]$ denotes the segment connecting the points k_1 and k_2.
     Therefore

(102) $|\Lambda_1(x, y, k_1) - \Lambda_1(x, y, k_2)| \geq |k_1 - k_2|(1 - 2C),$

and different points have different images.

From (56) it follows that the image completely covers the semi-plane

(103) $$\operatorname{Im}\lambda \geq \sqrt{\frac{2\alpha_4\|u\|_{H^4}}{C}} \cdot (1+C), \quad C < \frac{1}{2}.$$

To check that a point λ_0, belongs to this image it is sufficient to consider the image of the square with the boundaries
(104)
$$\operatorname{Im}k = D, \ \operatorname{Im}k = \frac{2\operatorname{Im}\lambda_0}{1-C}, \ \operatorname{Re}k = \operatorname{Re}\lambda_0 - \frac{2\operatorname{Im}\lambda_0}{1-C}, \ \operatorname{Re}k = \operatorname{Re}\lambda_0 + \frac{2\operatorname{Im}\lambda_0}{1-C}.$$

An easy estimate shows that λ_0 lies inside the region surrounded by the image of the above boundary, therefore λ_0 has a preimage, and, as it was shown above, this preimage is unique.

Therefore, on the semi-plane defined by (103), the functions $\Psi_1(x,y,\lambda)$, $\Psi_2(x,y,\lambda)$ are well-defined and holomorphic in λ.

We end this paper rederiving the well-known formal asymptotics of Ψ_1 and Ψ_2 for large $|\operatorname{Im}\lambda|$.

Combining equations (14) and (59) we obtain the asymptotic formulas:
(105)
$$\lambda = \Lambda_1 = k - \frac{u}{k} + \frac{\partial_y\partial_x^{-1}u}{k^2} - \frac{\partial_y^2\partial_x^{-2}u + u^2/2}{k^3} + O\left(\frac{1}{k^4}\right), \quad |\operatorname{Im}k| \to \infty.$$

Then the inversion with respect to k yields
(106)
$$\Psi_1(x,y,\lambda) = k = \lambda + \frac{u}{\lambda} - \frac{\partial_x^{-1}u_y}{\lambda^2} + \frac{\partial_x^{-2}u_{yy} - u^2/2}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \quad |\operatorname{Im}\lambda| \to \infty.$$

In addition, combining (24), (106) and (89), one obtains
(107)
$$\Psi_2(x,y,\lambda) = x - \lambda y - \frac{yu}{\lambda} + \frac{\partial_x^{-1}(yu)_y}{\lambda^2} + \frac{\frac{yu^2}{2} - \partial_x^{-2}(yu)_{yy}}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \quad |\operatorname{Im}\lambda| \to \infty.$$

10. Appendix. Some basic fact from functional analysis

Let us recall some basic facts about Banach spaces $L^p(\mathbb{R}^2)$.

(1) Hölder inequality (see [20]). Let

(108) $$1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

$f \in L^p$, $g \in L^q$. Then

(109) $$f \cdot g \in L^r, \quad \text{and} \quad \|f \cdot g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

(2) Young's inequality (see [20]). Let,

(110) $$1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

$f \in L^p$, $g \in L^q$. Then

(111) $$f * g \in L^r, \quad \text{and} \quad \|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

(3) Hausdorff-Young inequality (see [20], [8]).

It is well-known that the Fourier transform (28) is a unitary map $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. For Banach spaces the situation is more delicate: Let

(112) $$1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1$$

Then the Fourier transform is a well-defined map $L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ and

(113) $$\left\|\hat{f}\right\|_{L^q} \leq (2\pi)^{(1-2/p)} \|f\|_{L^p},$$

but this map is **not invertible**. If $f(z) \in L^2 \cap L^p$, then its Fourier image belongs to L^2, but this estimate can not be improved.

We need the following properties of the Sobolev spaces (see the books [2], [4], [27])

Lemma 7. *Let $l \geq 2$. Then*

(1) $H^l(\mathbb{R}^2) \subset C^{l-2}(\mathbb{R}^2)$. *Moreover, there exists a constant β_l such that, for $f(x, y) \in H^l(\mathbb{R}^2)$,*

(114) $$\max_{(x,y) \in \mathbb{R}^2} |f(x, y)| \leq \beta_l \|f\|_{H^l}.$$

There are similar estimates also for the first $l - 2$ derivatives.

(2) $H^l(\mathbb{R}^2)$ *is a Banach algebra, i.e. is closed with respect to the multiplication and there exists a constant α_l such that, for any $f, g \in H^l(\mathbb{R}^2)$,*

(115) $$\|fg\|_{H^l} \leq \alpha_l \|f\|_{H^l} \|g\|_{H^l},$$

$\beta_l \leq \alpha_l$.

(3) *Let $f \in H^l(\mathbb{R}^2)$. Then the multiplication operator*

$$h \to f \cdot h$$

is a bounded operator on all spaces $W^{l',p}$, $l' \leq l - 2$.

(4) *The operator $\Pi = \partial_z \partial_{\bar{z}}^{-1}$ is well-defined on all spaces $W^{l,p}$, $1 < p < \infty$, and*

(116) $$\|\partial_z \partial_{\bar{z}}^{-1} f\|_{W^{l,p}} \leq \gamma(p) \|f\|_{W^{l,p}}$$

Some numerical estimates on the constants α_l were obtained in the papers [16], [17]. The property (116) follows immediately from Theorem 1.

Lemma 8. *Consider the space $W = W^{2,2-\epsilon} \cap W^{2,2+\epsilon'}$, $0 < \epsilon < 1$, $\epsilon' > 0$, $\|f\|_W = \|f\|_{W^{2,2-\epsilon}} + \|f\|_{W^{2,2+\epsilon'}}$.*

(1) *Let $f \in W$, then f, f_z, $f_{\bar{z}}$ are Hölder functions for some $\alpha > 0$ (and, as a corollary, continuous), bounded and $f(z) = o(1)$ as $z \to \infty$.*

(2) *W is a Banach algebra, i.e. if $f, g \in W$, then $f \cdot g \in W$ and $\|f \cdot g\|_W \leq C(\epsilon, \epsilon')\|f\|_W\|g\|_W$.*

This Lemma follows immediately from the Theorem 1.21 from Vekua's book [26] stating that:

if $f \in L^p(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)$ with $p > 2$, $1 < p' < 2$, then for $g = \partial_{\bar{z}}^{-1} f$ we have the inequalities:

$$|g(z)| \leq M_{p,p'}(\|f\|_{L_p} + \|f\|_{L'_p}), \quad z \in \mathbb{C},$$

$$|g(z_1) - g(z_2)| \leq M_{p,p'}(\|f\|_{L_p} + \|f\|_{L'_p})|z_1 - z_2|^{\frac{p-2}{p}}.$$

Lemma 9. *Let $q(z, \bar{z})$ be a two times continuously differentiable function, such that all functions q, q_z, $q_{\bar{z}}$, q_{zz}, $q_{z\bar{z}}$, $q_{\bar{z}\bar{z}}$ are bounded, and, in addition, $|q| \leq C < 1$.*

Then there exists an $\epsilon > 0$ such that the operator

(117) $$B = (1 - q(z, \bar{z})\partial_z\partial_{\bar{z}}^{-1})$$

is invertible in all spaces $W^{2,2+\epsilon_1}$, $-\epsilon < \epsilon_1 < \epsilon$, and the inverse operator B^{-1} is uniformly bounded in ϵ_1.

The proof is based on the following trick. Consider the space $W^{2,p}(\mathbb{R}^2)$. In addition to the norm, $\|\|_{W^{2,p}}$, it is convenient to introduce a family of norms $\|\|_{W_\mu^{2,p}}$, which are all equivalent for a fixed k:

(118)

$$\|f\|_{W_\mu^{l,p}} = \left[\iint |f|^p dxdy + \mu^p \iint (|f_z|^p + |f_{\bar{z}}|^p)dxdy + \right.$$

$$\left. + \mu^{2p} \iint (|f_{zz}|^p + 2|f_{z\bar{z}}|^p + |f_{\bar{z}\bar{z}}|^p)dxdy \right]^{1/p}, \quad \mu > 0.$$

Let $q(z, \bar{z})$ be a two times continuously differentiable function, and all functions q, q_z, $q_{\bar{z}}$, q_{zz}, $q_{z\bar{z}}$, $q_{\bar{z}\bar{z}}$ be bounded, $|q| \leq C$. Then the multiplication operator

(119) $$f \to q \cdot f$$

is a well-defined bounded operator on $W^{2,p}$. Moreover, it is easy to check that

$$(120) \qquad \|q \cdot f\|_{W_\mu^{l,p}} \le \left[\max_{z\in\mathbb{C}} |q| + 2\mu \left(\max_{z\in\mathbb{C}} |q_z| + \max_{z\in\mathbb{C}} |q_{\bar{z}}| \right) + \right.$$

$$(121) \qquad \left. + \mu^2 \left(\max_{z\in\mathbb{C}} |q_{zz}| + \max_{z\in\mathbb{C}} |q_{\bar{z}\bar{z}}| + 2\max_{z\in\mathbb{C}} |q_{z\bar{z}|} \right) \right] \|f\|_{W_\mu^{l,p}}$$

We shall use also the following simple

Lemma 10. *Let $z \to w(z, \bar{z})$ be a one-to-one map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*

(1) *Both functions $w(z, \bar{z})$ and $z(w, \bar{w})$ are two times continuously differentiable.*

(2) *All functions*

$$\frac{\partial w}{\partial z}, \quad \frac{\partial \bar{w}}{\partial z}, \quad \frac{\partial w}{\partial \bar{z}}, \quad \frac{\partial \bar{w}}{\partial \bar{z}}, \quad \frac{\partial z}{\partial w}, \quad \frac{\partial \bar{z}}{\partial w}, \quad \frac{\partial z}{\partial \bar{w}}, \quad \frac{\partial \bar{z}}{\partial \bar{w}},$$

$$\frac{\partial^2 w}{\partial z^2}, \quad \frac{\partial^2 w}{\partial \bar{z}^2}, \quad \frac{\partial^2 w}{\partial z \partial \bar{z}}, \quad \frac{\partial^2 \bar{w}}{\partial z^2}, \quad \frac{\partial^2 \bar{w}}{\partial \bar{z}^2}, \quad \frac{\partial^2 \bar{w}}{\partial z \partial \bar{z}},$$

$$\frac{\partial^2 z}{\partial w^2}, \quad \frac{\partial^2 z}{\partial \bar{w}^2}, \quad \frac{\partial^2 z}{\partial w \partial \bar{w}}, \quad \frac{\partial^2 \bar{z}}{\partial w^2}, \quad \frac{\partial^2 \bar{z}}{\partial \bar{w}^2}, \quad \frac{\partial^2 \bar{z}}{\partial w \partial \bar{w}},$$

are bounded.

(3) *$Jac((w, \bar{w}), (z, \bar{z}))$ and $Jac((z, \bar{z}), (w, \bar{w}))$ are bounded functions.*

Then the corresponding map $W^{2,p}(\mathbb{C}) \to W^{2,p}(\mathbb{C})$: $f(w, \bar{w}) \to \tilde{f}(z, \bar{z}) = f(w(z, \bar{z}), \bar{w}(z, \bar{z}))$ is well-defined and bounded in both directions.

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