REMARK ON A THEOREM IN MUMFORD'S RED BOOK OF VARIETIES AND SCHEMES

GUANGLIAN ZHANG

ABSTRACT. In this paper, we firstly point out, by a counter example, that Proposition 6.4 of Section 6 in Bump's book ([Bum]) is error, and then give a correct statement with proof. We finally point out a gap in the proof of Theorem 3, in Chapter I Section 8, of Mumford's red book [Mum], and indicate a way to complete it.

1. INTRODUCTION

In order to show [Mum, Chapter I, § 8, Theorem3], Bump divided his proof into four propositions in [Bum]. But, there is a mistake in one of these propositions ([Bum, Proposition 6.4]). The original proof of Mumford is based on similar ideas, so we find a similar gap in the proof of [Mum, Chapter I, § 6, Theorem 3]. In the following, we first give a counter example of the last statement in [Bum, Proposition 6.4], then present a correct statement and prove it. Finally, we complete the proof of Theorem 3 in Mumford's red book [Mum].

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2. A COUNTER EXAMPLE

Let k be an algebraically closed field. According to [Bum], an algebraic set is a *variety* if it is irreducible. The following proposition can be found in the book of Bump:

Proposition 1 ([Bum] Proposition 6.4)**.** *Let $\phi : X \to Y$ be a finite dominant morphism of affine varieties. Then ϕ is surjective. The fibers of ϕ are all finite. If Z is a closed subset of X, then $\phi(Z)$ is closed, and $\dim(Z) = \dim(\phi(Z))$. If W is closed subvariety of Y, and Z is any irreducible component of $\phi^{-1}(W)$, then $\phi(Z) = W$, and $\dim(Z) = \dim(W)$.*

The last part of this proposition is wrong, and here is a counter example. Assume $k = \mathbb{C}$. Let $X = V(x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}) \subset \mathbb{A}^3$, and Y the image of the morphism below

$$\phi : X \longrightarrow \mathbb{A}^3, \quad (x, y, z) \longmapsto ((1 - 2z)x, (1 - 3z)y, (1 - z)z).$$

Let $W = \phi(X \cap V(z - \frac{1}{4}))$. Note that $X \cap V(z - \frac{1}{4})$ is an irreducible closed subset of X.

Claim: ϕ is a finite morphism, and

$$\phi^{-1}(W) = X \cap V\left(z - \frac{1}{4}\right) \cup \left\{ \left(\frac{\sqrt{3}}{4}, 0, \frac{3}{4} \right), \left(-\frac{\sqrt{3}}{4}, 0, \frac{3}{4} \right) \right\}.$$

In particular, $Z := \{(-\frac{\sqrt{3}}{4}, 0, \frac{3}{4})\}$ is an irreducible component of $\phi^{-1}(W)$ such that $\phi(W) \neq Z$.

Proof. Set $A = k[x, y, z]/(x^2 + y^2 + z^2 - z)$. By abuse of notation, we shall use the same symbols to denote the images of $x, y, z \in k[x, y, z]$ in the quotient A. We first show that ϕ is a finite morphism. Consider the following morphism of k-algebras induced by ϕ

$$\lambda : k[a, b, c] \to A, \quad (a, b, c) \mapsto ((1 - 2z)x, (1 - 3z)y, (1 - z)z),$$

which makes A an algebra over $k[a, b, c]$. We need show that A is integral over $k[a, b, c]$. It is clear that z is integral over $k[a, b, c]$ since $z^2 - z + \lambda(c) = 0$ by the definition of λ. Moreover, as $x^2 + y^2 + z^2 - z = 0$ in A, to see that A is integral over $k[a, b, c]$, it suffices to show that y is integral over the k-subalgebra $k[\lambda(a), \lambda(b), \lambda(c), z] \subseteq A$ of A generated by $\lambda(a), \lambda(b), \lambda(c), z \in A$. We shall do this by finding an integral relation for it. First, by the definition of λ, we have

$$zx = \frac{x - \lambda(a)}{2}, \quad zy = \frac{y - \lambda(b)}{3}, \quad \text{and} \quad x^2 + y^2 = \lambda(c),$$

giving

$$
\begin{aligned}
\lambda(b)^2 &= [(1 - 3z)y]^2 = y^2 - 6zy^2 + 9z^2y^2 \\
&= y^2 - 6zy^2 + 9(z^2\lambda(c) - z^2x^2) \\
&= y^2 - 6y\frac{y - \lambda(b)}{3} - 9(\frac{x - \lambda(a)}{2})^2 + 9z^2\lambda(c) \\
&= -y^2 + 2\lambda(b)y - \frac{9}{4}x^2 + \frac{9}{2}\lambda(a)x - \frac{9}{4}\lambda(a)^2 + 9z^2\lambda(c) \\
&= -y^2 - \frac{9}{4}(\lambda(c) - y^2) + 2\lambda(b)y + \frac{9}{2}\lambda(a)x - \frac{9}{4}\lambda(a)^2 + 9z^2\lambda(c) \\
&= \frac{5}{4}y^2 + 2\lambda(b)y + \frac{9}{2}\lambda(a)x - \frac{9}{4}\lambda(a)^2 + 9z^2\lambda(c) - \frac{9}{4}\lambda(c).
\end{aligned}
$$

Let $w = -\frac{9}{4}\lambda(a)^2 + 9z^2\lambda(c) - \frac{9}{4}\lambda(c) \in k[\lambda(a), \lambda(b), \lambda(c), z] \subseteq A$. Consequently,

$$
\begin{aligned}
(1 - 2z)\lambda(b)^2 &= (1 - 2z)[\frac{5}{4}y^2 + 2\lambda(b)y + \frac{9}{2}\lambda(a)x + w] \\
&= \frac{5}{4}y^2(1 - 2z) + 2\lambda(b)(1 - 2z)y + \frac{9}{2}\lambda(a)^2 + (1 - 2z)w \\
&= \frac{5}{4}y^2 - \frac{5}{2}zy^2 + 2\lambda(b)(1 - 2z)y + \frac{9}{2}\lambda(a)^2 + (1 - 2z)w \\
&= \frac{5}{4}y^2 - \frac{5}{2}y\frac{y - \lambda(b)}{3} + 2\lambda(b)(1 - 2z)y + \frac{9}{2}\lambda(a)^2 + (1 - 2z)w \\
&= \frac{5}{12}y^2 + \frac{5}{6}\lambda(b)y + 2\lambda(b)(1 - 2z)y + \frac{9}{2}\lambda(a)^2 + (1 - 2z)w.
\end{aligned}
$$

In particular, we obtain the following equality in A:

$$\frac{5}{12}y^2 + \frac{5}{6}\lambda(b)y + 2\lambda(b)(1-2z)y + \frac{9}{2}\lambda(a)^2 + (1-2z)w - (1-2z)\lambda(b)^2 = 0.$$

As $w \in k[\lambda(a), \lambda(b), \lambda(c), z] \subseteq A$, we deduce that y is integral over $k[\lambda(a), \lambda(b), \lambda(c), z]$, thus also integral over $k[a, b, c]$. Therefore, ϕ is finite, as claimed.

We now determine $\phi^{-1}(W)$ by computing the fibers of ϕ. For $i = 1, 2$, let $(x_i, y_i, z_i) \in X \subset \mathbb{A}^3$ such that $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$. So

$$\begin{cases} (1 - 2z_1)x_1 = (1 - 2z_2)x_2 \\ (1 - 3z_1)y_1 = (1 - 3z_2)y_2 \\ (1 - z_1)z_1 = (1 - z_2)z_2 =: -a. \end{cases}$$

In particular, z_1, z_2 are roots of $z^2 - z - a = 0$, and

$$\begin{cases} x_1^2 + y_1^2 + a = 0 \\ x_2^2 + y_2^2 + a = 0. \end{cases}$$

We shall distinguish the following four different cases:

- Case 1: $z_1 = z_2 = \frac{1}{3}$. We have

$$\begin{cases} x_1 = x_2 \\ y_1 = \pm y_2 \\ z_1 = z_2 = \frac{1}{3}. \end{cases}$$

- Case 2: $z_1 = z_2 = \frac{1}{2}$. We have

$$\begin{cases} x_1 = \pm x_2 \\ y_1 = y_2 \\ z_1 = z_2 = \frac{1}{2}. \end{cases}$$

- Case 3: $z_1 = z_2 \notin \{\frac{1}{2}, \frac{1}{3}\}$. We have

$$\begin{cases} x_1 = x_2 \\ y_1 = y_2 \\ z_1 = z_2. \end{cases}$$

- Case 4: $z_1 \neq z_2$. Then z_1, z_2 are the two roots of $z^2 - z - a = 0$. Consequently,

$$\begin{cases} z_1 + z_2 = 1 \\ z_1 z_2 = -a, \end{cases}$$

and

$$
\begin{cases}
(1 - 2z_1)x_1 = [1 - 2(1 - z_1)]x_2 = (2z_1 - 1)x_2, \\
(1 - 3z_1)y_1 = [1 - 3(1 - z_1)]y_2 = (3z_1 - 2)y_2 \\
(1 - z_1)z_1 = (1 - z_2)z_2
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
x_1 = -x_2 \\
y_1^2 = y_2^2 \\
(1 - 3z_1)y_1 = (3z_1 - 2)y_2 \\
(1 - z_1)z_1 = (1 - z_2)z_2 = -a.
\end{cases}
$$

In particular, $y_1 = \pm y_2$. If $y_1 = y_2 \neq 0$, we have $1 - 3z_1 = 3z_1 - 2$, thus $z_1 = z_2 = \frac{1}{2}$, which is impossible. If $y_1 = -y_2 \neq 0$, we have $1 - 3z_1 = 2 - 3z_1$, giving also a contradiction. So, we must have $y_1 = y_2 = 0$ in this case. Hence

$$
\begin{cases}
x_1 = -x_2 \\
y_1 = y_2 = 0 \\
z_1 + z_2 = 1, \text{ and } z_1 \neq z_2.
\end{cases}
$$

Based on the above discussion, we deduce

$$
\phi^{-1}(W) = X \cap V(z - \frac{1}{4}) \bigcup \left\{ \left(\frac{\sqrt{3}}{4}, 0, \frac{3}{4} \right), \left(-\frac{\sqrt{3}}{4}, 0, \frac{3}{4} \right) \right\},
$$

and the last statement then follows easily. This completes the proof of our claim. $\square$

We now give a corrected form of the last part of Proposition 1.

Proposition 2. *Let $\phi : X \to Y$ be a finite dominant morphism of affine varieties. Assume Y is* normal. *Let W be an irreducible closed subvariety of Y, and Z an irreducible component of $\phi^{-1}(W)$. Then $\phi(Z) = W$, and $\dim(Z) = \dim(W)$.*

Proof. Let $A = k[X]$ and $B = k[Y]$. As X and Y are affine varieties, one can identify B as a k-subalgebra of A using the dominant morphism ϕ. Write $W = V(\mathfrak{q})$, with $\mathfrak{q} \subset B$ a prime ideal. So $\phi^{-1}(W) = V(\mathfrak{q}A)$. Let $\widetilde{\mathfrak{q}}_1, \widetilde{\mathfrak{q}}_2, \cdots, \widetilde{\mathfrak{q}}_r$ be the minimal prime ideals of $V(\mathfrak{q}A)$. Then $V(\widetilde{\mathfrak{q}}_1), V(\widetilde{\mathfrak{q}}_2), \cdots, V(\widetilde{\mathfrak{q}}_r)$ are the irreducible components of $\phi^{-1}(W)$. Since A is integral over B, thanks to [Bum, Section 1, Proposition 4.3], we have $\mathfrak{q}A \cap B = \mathfrak{q}$. We now claim that $\mathfrak{q} = \widetilde{\mathfrak{q}}_i \cap B$ for all i. Clearly $\widetilde{\mathfrak{q}}_i \cap B \supseteq \mathfrak{q}A \cap B = \mathfrak{q}$. Suppose there exists some i such that $\widetilde{\mathfrak{q}}_i \cap B \supsetneq \mathfrak{q}$. Because Y is normal, by going-down theorem, there exists a prime ideal $\widetilde{\mathfrak{q}}_i'$ of A such that $\widetilde{\mathfrak{q}}_i' \subsetneq \widetilde{\mathfrak{q}}_i$, and $\widetilde{\mathfrak{q}}_i' \cap B = \mathfrak{q}$. In particular, $\widetilde{\mathfrak{q}}_i \supsetneq \widetilde{\mathfrak{q}}_i' \supseteq \mathfrak{q}A$. But this contradicts to the fact that $\widetilde{\mathfrak{q}}_i \in V(\mathfrak{q}A)$ is a minimal ideal, proving our claim. Consequently, ϕ maps the generic point of Z to that of W. So, by the second part of Proposition 1, we find $\phi(Z) = W$ and $\dim(Z) = \dim(W)$.

$\square$

3. The proof

We now in the position to complete the proof of Theorem 3 at Section 6 of Chapter 1 of Mumford's red book. First of all, we need a lemma, which is a special case of a more general well-known statement.

Lemma 1. *Let A be a k-algebra of finite type. Assume A is a domain. Then, there exists some $f \in A \setminus \{0\}$, such that the localisation A_f is normal.*

Proof. Let K denote the fraction field of A, and A' the integral closure of A in K. Since A is a finitely generated over a field, A' is finite as an A-module by [Mat, Chapter 12 Theorem 72]. In particular, there exist $x_1, \cdots, x_n \in A'$ with $A' = \sum_{i=1}^n A \cdot x_i \subseteq K$. As K is the fraction field of A, one can find $f \in A \setminus \{0\}$ such that $f x_i \in A$ for all i. Therefore, $A[1/f] \subset A'[1/f] = \sum_{i=1}^r A[1/f] x_i \subset A[1/f] \subseteq K$. Thus, $A_f = A[1/f] = A'[1/f] = A'_f$ is the localisation of the normal ring A', hence is normal as well. $\square$

Theorem 1 ([Mum] Chapter I § 6 Theorem 3). *Let $f : X \to Y$ be a dominating morphism of varieties and let $r = \dim X - \dim Y$. Then there exists a nonempty open set $U \subset Y$ such that:*

(1) *$U \subset f(X)$, and*

(2) *for all irreducible closed subsets $W \subset Y$ such that $W \cap U \neq \emptyset$, and for all irreducible components Z of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$,*

$$\dim Z = \dim W + r$$

or

$$\operatorname{codim}(Z, X) = \operatorname{codim}(W, Y).$$

Proof. As in the original proof of Mumford, we reduce to the following case: X, Y are affine, and there exists some non-empty open subset $U \subset Y$, such that the induced map $f^{-1}(U) \to U$ is decomposed as

$$f^{-1}(U) \xrightarrow{\pi} U \times \mathbb{A}^r \longrightarrow U,$$

where the first map π is finite and dominant, while the second is the natural projection. So π is surjective by the first part of Proposition 1, and $U \subset f(X)$. Shrinking U if necessary, we further assume U normal according to Lemma 1 above. In particular, the affine variety $U \times \mathbb{A}^r$ is also normal. To finish the proof, let $W \subset Y$ be an irreducible closed subset that meets U, and let $Z \subset X$ be an irreducible component of $f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$. Let $W_0 = W \cap U$, and $Z_0 = Z \cap f^{-1}(U)$. Then $\dim(W) = \dim(W_0)$ and $\dim(Z) = \dim(Z_0)$. Since W_0 is an irreducible closed subset of U, one checks that $W_0 \times \mathbb{A}^r$ is an irreducible closed subset of $U \times \mathbb{A}^r$. Moreover, Z_0 is an irreducible component of $\pi^{-1}(W_0 \times \mathbb{A}^r) = f^{-1}(W_0)$. As $W_0 \times \mathbb{A}^r$ is normal, by Proposition 2, $\pi(Z_0) = W_0 \times \mathbb{A}^r$ and $\dim(Z_0) = \dim(W_0 \times \mathbb{A}^r) = \dim(W_0) + r$. Consequently, $\dim(Z) = \dim(W) + r$, as claimed by (2).

□

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School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China

E-mail address: g.l.zhang@sjtu.edu.cn