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# The Shi arrangement of the type $D_{\ell}$

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**Abstract:** In this paper, we give a basis for the derivation module of the cone over the Shi arrangement of the type  $D_{\ell}$  explicitly.

**Key words:** Hyperplane arrangement; Shi arrangement; Free arrangement.

1. Introduction. Let V be an  $\ell$ -dimensional vector space. An affine arrangement of hyperplanes  $\mathcal{A}$  is a finite collection of affine hyperplanes in V. If every hyperplane  $H \in \mathcal{A}$  goes through the origin, then  $\mathcal{A}$  is called to be central. When  $\mathcal{A}$  is central, for each  $H \in \mathcal{A}$ , choose  $\alpha_H \in V^*$  with  $\ker(\alpha_H) = H$ . Let S be the algebra of polynomial functions on V and let  $\operatorname{Der}_S$  be the module of derivations

$$Der_S := \{ \theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f), f, g \in S, \\ \theta \text{ is } \mathbb{R}\text{-linear} \}.$$

For a central arrangement A, recall

$$D(\mathcal{A}) := \{ \theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}.$$

We say that  $\mathcal{A}$  is a free arrangement if  $D(\mathcal{A})$  is a free S-module. The freeness was defined in [15]. The Factorization Theorem[16] states that, for any free arrangement  $\mathcal{A}$ , the characteristic polynomial of  $\mathcal{A}$  factors completely over the integers.

Let  $E = \mathbb{R}^{\ell}$  be an  $\ell$ -dimensional Euclidean space with a coordinate system  $x_1, \ldots, x_{\ell}$ , and  $\Phi$  be a crystallographic irreducible root system. Fix a positive root system  $\Phi^+ \subset \Phi$ . For each positive root  $\alpha \in \Phi^+$  and  $k \in \mathbb{Z}$ , we define an affine hyperplane

$$H_{\alpha,k} := \{ v \in V \mid (\alpha, v) = k \}.$$

In [10], J.-Y. Shi introduced the Shi arrangement

$$S(A_{\ell}) := \{ H_{\alpha,k} \mid \alpha \in \Phi^+, \ 0 \le k \le 1 \}$$

when the root system is of the type  $A_{\ell}$ . This definition was later extended to the *generalized Shi arrangement* (e.g., [4])

$$\mathcal{S}(\Phi) := \{ H_{\alpha,k} \mid \alpha \in \Phi^+, \ 0 \le k \le 1 \}.$$

Embed E into  $V = \mathbb{R}^{\ell+1}$  by adding a new coordinate z such that E is defined by the equation z = 1 in V. Then, as in [7], we have the cone  $\mathbf{c}\mathcal{S}(\Phi)$  of  $\mathcal{S}(\Phi)$ 

$$\mathbf{c}\mathcal{S}(\Phi) := \{\mathbf{c}H_{\alpha,k} \mid \alpha \in \Phi^+, \ 0 \le k \le 1\} \cup \{\{z = 0\}\}.$$

In [18], M. Yoshinaga proved that the cone  $\mathbf{c}\mathcal{S}(\Phi)$  is a free arrangement with exponents  $(1, h, \ldots, h)$  (h appears  $\ell$  times), where h is the Coxeter number of  $\Phi$ . (He actually verified the conjecture by P. Edelman and V. Reiner in [4], which is far more general.) He proved the freeness without finding a basis.

In [13], for the first time, the authors gave an explicit construction of a basis for  $D(\mathbf{c}\mathcal{S}(A_{\ell}))$ . Then D. Suyama constructed bases for  $D(\mathbf{c}\mathcal{S}(B_{\ell}))$  and  $D(\mathbf{c}\mathcal{S}(C_{\ell}))$  in [14]. In this paper, we will give an explicit construction of a basis for  $D(\mathbf{c}\mathcal{S}(D_{\ell}))$ . A defining polynomial of the cone over the Shi arrangement of the type  $D_{\ell}$  is given by

$$Q := z \prod_{1 \le s < t \le \ell} \prod_{\epsilon \in \{-1,1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t).$$

Note that the number of hyperplanes in  $cS(\mathcal{D}_{\ell})$  is equal to  $2\ell(\ell-1)+1$ . Our construction is similar to the construction in the case of the type  $B_{\ell}$ . The essential ingredients of the recipe are the Bernoulli polynomials and their relatives.

### 2. The basis construction.

**Proposition 2.1.** For  $(p,q) \in \mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq 0}$ , consider the following two conditions for a rational function  $B_{p,q}(x)$ :

1. 
$$B_{p,q}(x+1) - B_{p,q}(x)$$
  

$$= \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)} (x+1)^q (-x)^q,$$
2.  $B_{p,q}(-x) = -B_{p,q}(x).$ 

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Then such a rational function  $B_{p,q}(x)$  uniquely exists. Morever, the  $B_{p,q}(x)$  is a polynomial unless (p,q)=(-1,0) and  $B_{-1,0}(x)=-(1/x)$ .

*Proof.* Suppose  $(p,q) \neq (-1,0)$ . Since the right hand side of the first condition is a polynomial in x, there exists a polynomial  $B_{p,q}(x)$  satisfying the first condition. Note that  $B_{p,q}(x)$  is unique up to a constant term. Define a polynomial  $F(x) = B_{p,q}(x) + B_{p,q}(-x)$ . Since

$$B_{p,q}(-x) - B_{p,q}(-x-1)$$

$$= \frac{(-x)^p - (x+1)^p}{(-x) - (x+1)} (-x)^q (x+1)^q$$

$$= \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)} (x+1)^q (-x)^q$$

$$= B_{p,q}(x+1) - B_{p,q}(x),$$

we have F(x+1) = F(x) for any x. Therefore F(x) is a constant function. Then the polynomial  $B_{p,q}(x) - (F(0)/2)$  is the unique solution satisfying the both conditions. Next we suppose (p,q) = (-1,0). Then we compute

$$B_{-1,0}(x+1) - B_{-1,0}(x)$$

$$= \frac{(x+1)^{-1} - (-x)^{-1}}{(x+1) - (-x)} = -\frac{1}{x+1} + \frac{1}{x}.$$

Thus  $B_{-1,0}(x) = -(1/x)$  is the unique solution satisfying the both conditions.

**Definition 2.2.** Define a rational function  $\overline{B}_{p,a}(x,z)$  in x and z by

$$\overline{B}_{p,q}(x,z) := z^{p+2q} B_{p,q}(x/z).$$

Then  $\overline{B}_{p,q}(x,z)$  is a homogeneous polynomial of degree p+2q except the two cases:  $\overline{B}_{-1,0}(x,z)=-(1/x)$  and  $\overline{B}_{0,q}(x,z)=0$ .

For a set  $I := \{y_1, \dots, y_m\}$  of variables, let

$$\sigma_n^I := \sigma_n(y_1, \dots, y_m), \ \tau_{2n}^I := \sigma_n(y_1^2, \dots, y_m^2),$$

where  $\sigma_n$  stands for the elementary symmetric function of degree n.

**Definition 2.3.** Define derivations

$$\varphi_j := (x_j - x_{j+1} - z) \sum_{i=1}^{\ell} \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left( \prod K_1 \right) \left( \prod K_2 \right)^2$$

$$(-z)^{|K_1|} \sum_{\substack{0 \le n_1 \le |J_1| \\ 0 \le n_2 \le |J_2|}} (-1)^{n_1 + n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z) \frac{\partial}{\partial x_i}$$

for 
$$j = 1, \ldots, \ell - 1$$
 and

$$\varphi_{\ell} := \sum_{i=1}^{\ell} \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left( \prod K_1 \right) \left( \prod K_2 \right)^2 (-z)^{|K_1|}$$
$$(-x_{\ell}) \overline{B}_{-1,k_0}(x_i, z) \frac{\partial}{\partial x_i}$$

for  $j = \ell$ , where

$$J := \{x_1, \dots, x_{j-1}\}, J_1 := \{x_j, x_{j+1}\},$$

$$J_2 := \{x_{j+2}, \dots, x_{\ell}\},$$

$$\prod K_p := \prod_{x_i \in K_p} x_i \ (p = 1, 2),$$

$$k_0 := |J \setminus (K_1 \cup K_2)| \ge 0,$$
  
$$k := (|J_1| - n_1) + 2(|J_2| - n_2) - 1 \ge -1.$$

Note that  $\varphi_j(z) = 0$   $(1 \le j \le \ell)$ . In the rest of the paper, we will give a proof of the following theorem:

**Theorem 2.4.** The derivations  $\varphi_1, \ldots, \varphi_\ell$ , together with the Euler derivation

$$\theta_E := z \frac{\partial}{\partial z} + \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i},$$

form a basis for  $D(\mathbf{c}\mathcal{S}(D_{\ell}))$ .

Note that  $\theta_E(x_i) = x_i \ (1 \le i \le \ell)$  and  $\theta_E(z) = z$ .

**Lemma 2.5.** Let  $1 \le i \le \ell$  and  $1 \le j \le \ell$ . Suppose  $\varphi_j(x_i)$  is nonzero. Then  $\varphi_j(x_i)$  is a homogeneous polynomial of degree  $2(\ell-1)$ .

Proof. Define

$$F_{ij} := (x_j - x_{j+1} - z) \left( \prod K_1 \right) \left( \prod K_2 \right)^2 z^{|K_1|}$$

$$\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z) \qquad (1 \le j \le \ell - 1),$$

$$F_{i\ell} := \left( \prod K_1 \right) \left( \prod K_2 \right)^2 z^{|K_1|} x_{\ell} \overline{B}_{-1,k_0}(x_i, z)$$

when  $K_1, K_2, n_1, n_2$  are fixed. Then  $\varphi_j(x_i)$  is a linear combination of the  $F_{ij}$ 's over  $\mathbb{R}$ .

Note that  $\overline{B}_{k,k_0}(x_i,z)$  is a polynomial unless  $(k,k_0)=(-1,0)$ .

Assume that  $1 \leq j \leq \ell-1$  and  $(k, k_0) = (-1, 0)$ . Then  $J = K_1 \cup K_2$ ,  $n_1 = |J_1|$ ,  $n_2 = |J_2|$ , and  $\overline{B}_{-1,0}(x_i, z) = -1/x_i$ . Therefore each  $F_{ij}$  is a polynomial. Thus  $\varphi_j(x_i)$  is a nonzero polynomial and there exists a nonzero polynomial  $F_{ij}$ . Compute

$$\deg \varphi_j(x_i) = \deg F_{ij}$$
  
= 1 + |K\_1| + 2|K\_2| + |K\_1| + n\_1 + 2n\_2

$$+ \deg \overline{B}_{k,k_0}(x_i, z)$$

$$= 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2 + (2k_0 + k)$$

$$= 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2$$

$$+ 2(|J| - |K_1| - |K_2|) + |J_1| - n_1$$

$$+ 2(|J_2| - n_2) - 1$$

$$= 2(|J| + |J_1| + |J_2|) - |J_1| = 2\ell - 2.$$

Next consider  $\varphi_{\ell}(x_i)$ . If  $k_0 = 0$ , then  $J = K_1 \cup$  $K_2$ . Therefore each  $F_{i\ell}$  is a polynomial. Thus so is  $\varphi_{\ell}(x_i)$ . Compute

$$\deg \varphi_{\ell}(x_i)$$
=  $|K_1| + 2|K_2| + |K_1| + 1 + \deg \overline{B}_{-1,k_0}(x_i, z)$   
=  $2(|K_1| + |K_2|) + 1 + (2k_0 - 1)$   
=  $2(|K_1| + |K_2| + k_0) = 2(\ell - 1)$ .

Let < denote the pure lexicographic order of monomials with respect to the total order

$$x_1 > x_2 > \dots > x_\ell > z.$$

When  $f \in S = \mathbb{C}[x_1, x_2, \dots, x_{\ell}, z]$  is a nonzero polynomial, let in(f) denote the *initial monomial* (e.g., see [6]) of f with respect to the order <.

**Proposition 2.6.** Suppose  $\varphi_i(x_i)$  is nonzero. Then

(1) 
$$\operatorname{in}(\varphi_j(x_i)) \leq x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$
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(1) 
$$\operatorname{in}(\varphi_j(x_i)) \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i},$$
  
(2)  $\operatorname{in}(\varphi_j(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i} \text{ for } i < j,$ 

(3) 
$$\operatorname{in}(\varphi_j(x_i)) < x_1 \quad x_{i-1}x_i \quad \text{for } i < j,$$
  
(3)  $\operatorname{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i} \text{ for } 1 \le i \le \ell.$ 

*Proof.* Recall  $F_{ij}$   $(1 \le j \le \ell - 1)$  and  $F_{i\ell}$  from the proof of Lemma 2.5 when  $K_1, K_2, n_1, n_2$  are fixed. Let  $\deg^{(x_i)} f$  denote the degree of f with respect to  $x_i$  when  $f \neq 0$ .

(1) Since, for every nonzero  $F_{ij}$ , we obtain

$$\deg^{(x_p)} F_{ij} \le 2 \ (1 \le p < i), \quad \deg(F_{ij}) = 2\ell - 2.$$

Hence we may conclude

$$in(F_{ij}) \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

and thus

$$\operatorname{in}(\varphi_j(x_i)) \le \max\{\operatorname{in}(F_{ij})\} \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

(2) Suppose  $i < j < \ell$ . Since  $x_i > x_j > z$ , one has

$$\inf(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) 
\leq x_i^{n_1+2n_2+2k_0+k} = x_i^{2\ell-2j+2k_0-1}$$

when  $\overline{B}_{k,k_0}(x_i,z)$  is nonzero. The equality holds if and only if  $n_1 = n_2 = 0$ .

Suppose that  $F_{ij}$  is nonzero. For  $1 \leq i < j \leq$  $\ell - 1$ , we have

$$\inf(F_{ij}) = \inf(x_j - x_{j+1} - z) \inf((\prod K_1)(\prod K_2)^2 (-z)^{|K_1|}) 
\inf(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) 
\leq x_j \inf((\prod K_1)(\prod K_2)^2 (-z)^{|K_1|}) x_i^{2\ell - 2j + 2k_0 - 1} 
= x_j \inf((\prod K_1)(\prod K_2)^2 (-z)^{|K_1|} x_i^{2k_0}) x_i^{2\ell - 2j - 1} 
\leq x_j (x_1^2 \cdots x_{i-1}^2 x_i^{2j - 2i}) x_i^{2\ell - 2j - 1} (*) 
= x_1^2 \cdots x_{i-1}^2 x_i^{2\ell - 2i - 1} x_j < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell - 2i}.$$

Thus

$$\operatorname{in}(\varphi_i(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

For  $1 < i < j = \ell$ ,

$$\inf(F_{i\ell}) = x_{\ell} \inf((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) \inf(\overline{B}_{-1,k_0}(x_i,z)) 
= x_{\ell} \inf((\prod K_1)(\prod K_2)^2(-z)^{|K_1|}) x_i^{2k_0-1} 
= x_{\ell} \inf((\prod K_1)(\prod K_2)^2(-z)^{|K_1|} x_i^{2k_0}) x_i^{-1} 
\le x_{\ell} (x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}) x_i^{-1} (**) 
= x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i-1} x_{\ell} < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

This proves (2).

Now we only need to prove (3). Let  $i = j < \ell$  in (\*). Then the equality

$$\operatorname{in}(F_{ii}) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

holds if and only if

$$K_1 = \emptyset$$
,  $K_2 = J$ ,  $n_1 = n_2 = k_0 = 0$ ,  $k = 2\ell - 2i - 1$ 

because the leading term of  $\overline{B}_{2\ell-2i-1,0}(x_i,z)$  is equal

$$\frac{x_i^{2\ell - 2i - 1}}{2\ell - 2i - 1}.$$

Next let  $i = \ell$  in (\*\*). Then the equality

$$\operatorname{in}(F_{\ell\ell}) = x_1^2 \cdots x_{\ell-1}^2$$

holds if and only if

$$K_1 = \emptyset, K_2 = J = \{x_1, \dots, x_{\ell-1}\}, k_0 = 0.$$

Therefore, for  $1 \leq i \leq \ell$ ,

$$\operatorname{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

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From Proposition 2.6, we immediately obtain the following Corollary:

Corollary 2.7. (1)

$$\operatorname{in}(\operatorname{det}\left[\varphi_j(x_i)\right]) = \prod_{i=1}^{\ell} \operatorname{in}(\varphi_i(x_i)) = \prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}.$$

(2) Moreover, the leading term of det  $[\varphi_j(x_i)]$  is equal to

$$\frac{1}{(2\ell-3)!!} \prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}.$$

(3) In particular, det  $[\varphi_j(x_i)]$  does not vanish.

Next, we will prove  $\varphi_j \in D(\mathcal{S}(D_\ell))$  for  $1 \leq j \leq \ell$ . We denote  $\mathcal{S}(D_\ell)$  simply by  $\mathcal{S}_\ell$  from now on. Before the proof, we need the following two lemmas:

**Lemma 2.8.** Fix  $1 \le j \le \ell - 1$  and  $\epsilon \in \{-1, 1\}$ . Then (1)

$$\prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) = \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \left( \prod K_1 \right) \times \left( \prod K_2 \right)^2 \left[ -(x_s + \epsilon x_t) \right]^{|K_1|} (\epsilon x_s x_t)^{k_0}.$$

(2)

$$\sum_{\substack{0 \le n_1 \le |J_1| \\ 0 \le n_2 \le |J_2|}} (-1)^{|J_1| + |J_2| - n_1 - n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} (\epsilon x_s)^{k+1}$$

$$= \prod_{x_i \in J_1} (x_i - \epsilon x_s) \prod_{x_i \in J_2} (x_i^2 - x_s^2).$$

Proof. (1) is easy because the left hand side is equal to

$$\prod_{x_i \in I} (x_i^2 - (x_s + \epsilon x_t)x_i + \epsilon x_s x_t).$$

(2) The left handside is equal to

$$\sum_{0 \le n_1 \le |J_1|} (-\epsilon x_s)^{|J_1| - n_1} \sigma_{n_1}^{J_1} \sum_{0 \le n_2 \le |J_2|} (-x_s^2)^{|J_2| - n_2} \tau_{2n_2}^{J_2}$$

which is equal to the right handside.

#### Lemma 2.9.

(1) The polynomial

$$x_s \overline{B}_{k,k_0}(x_s,z) - x_t \overline{B}_{k,k_0}(x_t,z)$$

is divisible by  $x_s^2 - x_t^2$ ,

(2) For  $\epsilon \in \{-1, 1\}$ , the polynomial

$$(x_s - \epsilon x_t)\epsilon x_s x_t \left[ \overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z) \right]$$
$$- (x_s + \epsilon x_t)(\epsilon x_s x_t)^{k_0} \left[ \epsilon x_t x_s^{k+1} - x_s(\epsilon x_t)^{k+1} \right]$$

is divisible by  $x_s + \epsilon x_t - z$ .

*Proof.* (1) follows from the fact that  $-\overline{B}_{k,k_0}(x,z) = \overline{B}_{k,k_0}(-x,z)$  in Proposition 2.1.

(2) follows from the following congruence relation of polynomials modulo the ideal  $(x_s + \epsilon x_t - z)$ :

$$(x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t} \left[\overline{B}_{k,k_{0}}(x_{s}, z) + \epsilon \overline{B}_{k,k_{0}}(x_{t}, z)\right]$$

$$= (x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t}z^{k+2k_{0}} \left[B_{k,k_{0}}(\frac{x_{s}}{z}) - B_{k,k_{0}}(\frac{-\epsilon x_{t}}{z})\right]$$

$$\equiv (x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t}(x_{s} + \epsilon x_{t})^{k+2k_{0}}$$

$$\left[B_{k,k_{0}}(\frac{x_{s}}{x_{s} + \epsilon x_{t}}) - B_{k,k_{0}}(\frac{-\epsilon x_{t}}{x_{s} + \epsilon x_{t}})\right]$$

$$= (x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t}(x_{s} + \epsilon x_{t})^{k+2k_{0}}$$

$$\frac{(\frac{x_{s}}{x_{s} + \epsilon x_{t}})^{k} - (\frac{\epsilon x_{t}}{x_{s} + \epsilon x_{t}})^{k}}{(\frac{x_{s} + \epsilon x_{t}}{x_{s} + \epsilon x_{t}})}(\frac{\epsilon x_{t}}{x_{s} + \epsilon x_{t}})^{k_{0}}(\frac{x_{s}}{x_{s} + \epsilon x_{t}})^{k_{0}}$$

$$= (x_{s} + \epsilon x_{t})(\epsilon x_{s}x_{t})^{k_{0}}\left[\epsilon x_{t}x_{s}^{k+1} - x_{s}(\epsilon x_{t})^{k+1}\right].$$

## **Proposition 2.10.** Every $\varphi_i$ lies in $D(S_\ell)$ .

*Proof.* For  $1 \le j \le \ell - 1, 1 \le s < t \le \ell$ , and  $\epsilon \in \{-1,1\}$ , by Lemma 2.9 and Lemma 2.8, we have the following congruence relation of polynomials modulo the ideal  $(x_s + \epsilon x_t - z)$ :

$$(x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t} \left[\varphi_{j}(x_{s} + \epsilon x_{t} - z)\right]$$

$$= (x_{j} - x_{j+1} - z) \sum_{\substack{K_{1} \cup K_{2} \subseteq J \\ K_{1} \cap K_{2} = \emptyset}} \left(\prod K_{1}\right) \left(\prod K_{2}\right)^{2}$$

$$\times (-z)^{|K_{1}|} \sum_{\substack{0 \le n_{1} \le |J_{1}| \\ 0 \le n_{2} \le |J_{2}|}} (-1)^{n_{1} + n_{2}} \sigma_{n_{1}}^{J_{1}} \tau_{2n_{2}}^{J_{2}}$$

$$\times (x_{s} - \epsilon x_{t})\epsilon x_{s}x_{t} \left[\overline{B}_{k,k_{0}}(x_{s}, z) + \epsilon \overline{B}_{k,k_{0}}(x_{t}, z)\right]$$

$$\equiv (x_{j} - x_{j+1} - z) (x_{s} + \epsilon x_{t})$$

$$\times \sum_{K_{1},K_{2}} \left(\prod K_{1}\right) \left(\prod K_{2}\right)^{2} \left[-(x_{s} + \epsilon x_{t})\right]^{|K_{1}|} (\epsilon x_{s}x_{t})^{k_{0}}$$

$$\times \sum_{n_{1},n_{2}} (-1)^{n_{1} + n_{2}} \sigma_{n_{1}}^{J_{1}} \tau_{2n_{2}}^{J_{2}} \left[\epsilon x_{t}x_{s}^{k+1} - x_{s}(\epsilon x_{t})^{k+1}\right]$$

$$= (x_{j} - x_{j+1} - z) (x_{s} + \epsilon x_{t}) \prod_{x_{i} \in J} (x_{i} - x_{s})(x_{i} - \epsilon x_{t})$$

$$\times (-1)^{|J_{2}|} \left[\epsilon x_{t} \prod_{x_{i} \in J_{1}} (x_{i} - x_{s}) \prod_{x_{i} \in J_{2}} (x_{i}^{2} - x_{s}^{2})\right]$$

$$-x_s \prod_{x_i \in J_1} (x_i - \epsilon x_t) \prod_{x_i \in J_2} (x_i^2 - x_t^2) \bigg] \quad (\dagger).$$

Case 1. When  $x_s \in J$ ,  $(\dagger) = 0$ .

Case 2. When  $x_s \in J_2$  and  $x_t \in J_2$ ,  $(\dagger) = 0$ .

Case 3. When  $x_s \in J_1$  and  $x_t \in J_2$ ,  $(\dagger) = 0$ .

Case 4. When  $x_s \in J_1$ ,  $x_t \in J_1$  and  $\epsilon = 1$ ,  $(\dagger) = 0$ .

Case 5. If  $x_s \in J_1$ ,  $x_t \in J_1$  and  $\epsilon = -1$ , then s = j < t = j + 1. So (†) is divisible by  $x_s + \epsilon x_t - z$ .

We also have the following congruence relation of polynomials modulo the ideal  $(x_s + \epsilon x_t - z)$ :

$$(x_s - \epsilon x_t)\epsilon x_s x_t \left[\varphi_{\ell}(x_s + \epsilon x_t - z)\right]$$

$$= \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_2 \cap K_3 = \emptyset}} \left(\prod K_1\right) \left(\prod K_2\right)^2 (-z)^{|K_1|} (-x_{\ell})$$

$$(x_s - \epsilon x_t) \epsilon x_s x_t [\overline{B}_{-1,k_0}(x_s, z) + \epsilon \overline{B}_{-1,k_0}(x_t, z)]$$
  

$$\equiv (x_s + \epsilon x_t) (-x_\ell) (\epsilon x_t - x_s)$$

$$\sum_{K_1, K_2} \left( \prod K_1 \right) \left( \prod K_2 \right)^2 \left[ -(x_s + \epsilon x_t) \right]^{|K_1|} (\epsilon x_s x_t)^{k_0}$$

$$= (x_s^2 - x_t^2) x_\ell \prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) \qquad (\dagger \dagger).$$

Since  $s < t \le \ell$ , we have  $x_s \in J = \{x_1, \dots, x_{\ell-1}\}$ . Thus  $(\dagger\dagger) = 0$ . Therefore  $\varphi_j(x_s + \epsilon x_t - z)$  is divisible by  $x_s + \epsilon x_t - z$  for  $1 \le j \le \ell, 1 \le s < t \le \ell$ .

For 
$$1 \le j \le \ell$$
,

$$\varphi_i(x_s^2 - x_t^2) = 2x_s \varphi_i(x_s) - 2x_t \varphi_i(x_t)$$

is divisible either by  $x_s \overline{B}_{k,k_0}(x_s,z) - x_t \overline{B}_{k,k_0}(x_t,z)$  or by  $x_s \overline{B}_{-1,k_0}(x_s,z) - x_t \overline{B}_{-1,k_0}(x_t,z)$ , we have

$$\varphi_i(x_s^2 - x_t^2) \equiv 0 \mod(x_s^2 - x_t^2)$$

by Lemma 2.9 (1). This implies  $\varphi_i \in D(\mathcal{S}_{\ell})$ .

Applying Saito's lemma [9] [7, Theorem 4.19], we complete our proof of Theorem 2.4 thanks to Lemma 2.5, Corollay 2.7 (3) and Proposition 2.10. Theorem 2.4 implies that  $\det[\varphi_j(x_i)]$  is a nonzero multiple of (Q/z). By Corollary 2.7 (2) one obtains

Corollary 2.11.

$$\det[\varphi_j(x_i)] = \frac{1}{(2\ell - 3)!!} \prod_{1 \le s < t \le \ell} \prod_{\epsilon \in \{-1, 1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t).$$

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