On the Schrödinger-Poisson system with steep potential well and indefinite potential

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Abstract: In this paper, we study the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V_\lambda(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \\ (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \qquad (\mathcal{SP}_\lambda)$$

where $V_\lambda(x) = \lambda a(x) + b(x)$ with a positive parameter λ, $K(x) \geq 0$ and $f(x,t)$ is continuous including the power-type nonlinearity $|u|^{p-2}u$. By applying the method of penalized functions, the existence of one nontrivial solution for such system in the less-studied case $3 < p \leq 4$ is obtained for λ sufficiently large. The concentration behavior of this nontrivial solution for $\lambda \to +\infty$ are also observed. It is worth to point out that some new conditions on the potentials are introduced to obtain this nontrivial solution and the Schrödinger operator $-\Delta + V_\lambda(x)$ may be strong indefinite in this paper.

Keywords: Schrödinger-Poisson system; Steep potential well; Strong indefinite problem.

1   Introduction

In this paper, we consider the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V_\lambda(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \\ (u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \qquad (\mathcal{SP}_\lambda)$$

where $V_\lambda(x) = \lambda a(x) + b(x)$ with a positive parameter λ, $K(x) \geq 0$ is a weight function and $f(x,t)$ is a continuous nonlinearity.

The Schrödinger-Poisson system $(\mathcal{SP}_\lambda)$ arises in quantum mechanic. It was first introduced in [3] as a physical model describing a charged wave interacting with its own electrostatic field. In such system, the unknowns u and ϕ represent the wave functions associated to the particle and electric potential, and functions $V_\lambda(x)$ and $K(x)$ are respectively an external potential and nonnegative density charge. This system also arises in the electromagnetic field, semiconductor theory, nonlinear optics and plasma physics and sometimes it is called as the Scoründinger-Maxwell system. Due to the important applications in physics, there has been a vast literature on the study of the existence and multiplicity for nontrivial solutions to the system $(\mathcal{SP}_0)$ under various hypotheses on the potentials $b(x)$ and $K(x)$ and the nonlinearities $f(x,t)$. We refer the readers to [1, 2, 9, 10, 16, 17, 20, 22, 25, 26, 28, 31] and the references therein.

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In this paper, we will investigate the Schrödinger-Poisson system $(\mathcal{SP}_\lambda)$ for $\lambda > 0$ and assume $a(x)$ satisfies the following conditions:

(A_1) $a(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $a(x)$ is bounded below.

(A_2) There exists $a_\infty > 0$ such that $|\mathcal{A}_\infty| < +\infty$, where $\mathcal{A}_\infty = \{x \in \mathbb{R}^3 \mid a(x) < a_\infty\}$ and $|\mathcal{A}_\infty|$ is the Lebesgue measure of the set $\mathcal{A}_\infty$.

(A_3) $\Omega_a = \mathrm{int}\, a^{-1}(0)$ is a nonempty open set and has smooth boundary with $\overline{\Omega}_a = a^{-1}(0)$.

(A_4) $\mathrm{dist}(\Omega_a, \mathbb{R}^3 \backslash \mathcal{A}_\infty) > 0$, where $\mathrm{dist}(\Omega_a, \mathbb{R}^3 \backslash \mathcal{A}_\infty) = \inf\{|x - y| \mid x \in \Omega_a, y \in \mathbb{R}^3 \backslash \mathcal{A}_\infty\}$.

Remark 1.1 *If Ω_a is bounded, then by the condition (A_1), the assumption (A_4) is trivial. However, under the conditions (A_1)-(A_4), Ω_a is allowed to be unbounded. Furthermore, $a(x)$ is allowed to be sign-changing under the conditions (A_1)–(A_4).*

Under the conditions (A_1)–(A_3), $\lambda a(x)$ is called as the steep potential well for λ sufficiently large and the depth of the well is controlled by the parameter λ. Such potentials were first introduced by Bartsch and Wang in [4] for the scalar Schrödinger equations. An interesting phenomenon for this kind of Schrödinger equations is that, one can expect to find the solutions which are concentrated at the bottom of the wells as the depth goes to infinity. Due to this interesting property, such topic for the scalar Schrödinger equations was studied extensively in the past decade. We refer the readers to [5, 6, 7, 12, 13, 23, 27, 31] and the references therein. Recently, the steep potential well was also considered for some other elliptic equations and systems, see for example [14, 15, 29, 30, 34, 36] and the references therein. In particular, the steep potential well was introduced to the Schrödinger-Poisson system by Jiang and Zhou in [19], where System $(\mathcal{SP}_\lambda)$ with $a(x) \geq 0$, $b(x) = K(x) = 1$ and $f(x, u) = |u|^{p-2}u$ are considered. They obtained the existence results of one nontrivial solution for the case $p \in (2, 3) \cup [4, 6)$ if λ is sufficiently large and the concentration behavior of the nontrivial solutions for $\lambda \to +\infty$ was also observed in their paper. System $(\mathcal{SP}_\lambda)$ with $b(x) = 0$ and $f(x, u) = |u|^{p-2}u$ if $a(x)$ satisfies the conditions (A_1)–(A_3) was also investigated by Zhao et al. in [36], where one nontrivial solution is founded for the case $p \in (3, 6)$. In the case of $3 < p < 4$, the condition $a(x) \geq 0$ was also needed in [36] and the concentration behavior of the nontrivial solutions for $\lambda \to +\infty$ was also obtained in this case. Zhao et al.'s results for the case $4 < p < 6$ was generalized by Ye and Tang in [34] in the sense that a more general nonlinearity $f(x, u)$ is considered. Ye and Tang also obtained the existence and multiplicity of nontrivial solutions for the System $(\mathcal{SP}_\lambda)$ if the nonlinearity $f(x, u)$ is sub-linear. In the very recent work [30], the first and second authors of the current paper study the System $(\mathcal{SP}_\lambda)$ with $a(x) \geq 0$ when $f(x, u)$ is asymptotically linear, 3-asymptotically linear and 4-asymptotically linear, where some existence and nonexistence results for nontrivial solutions were obtained and the concentration behavior of the nontrivial solutions for $\lambda \to +\infty$ was also observed. These results partially complement the study on the System $(\mathcal{SP}_\lambda)$.

We point out that the case $3 < p < 4$ for the Schrödinger-Poisson System $(\mathcal{SP}_\lambda)$ seems to be more interesting and difficult than other cases under the point of calculus of variations, since it is hard to obtain the boundness of every (PS) sequence in this case. Due to this reason, the Schrödinger-Poisson System $(\mathcal{SP}_\lambda)$ in the case of $3 < p < 4$ are less-studied by the variational methods. In [35], some Pohozaev type conditions on the potentials were introduced to get a special bounded (PS) sequence of the Schrödinger-Poisson System $(\mathcal{SP}_\lambda)$ in the case $3 < p < 4$ by the monotonicity trick of Jeanjean [18]. Similar Pohozaev type conditions on the potentials were also used in [22, 33, 36] for this case. Inspired by the above facts, we wonder whether the Pohozaev type conditions on the potentials are necessary for finding nontrivial solutions of $(\mathcal{SP}_\lambda)$ in the case of $3 < p < 4$? In the current paper, we will explore this question.

Let us first list out our assumptions on $b(x)$, $K(x)$ and $f(x, t)$:

(B_1) $b(x) \in C(\mathbb{R}^3, \mathbb{R})$.

(B_2) There exists $b_0 > 0$ such that $b^-(x) \leq b_0(1 + \sqrt{a(x)})$ for all $x \in \mathbb{R}^3$, where $b^-(x) = \max\{0, -b(x)\}$.

(B_3) Either $b(x) \geq 0$ or $b^{\pm}(x) \not\equiv 0$ on Ω_a with $|\mathcal{B}_0| < +\infty$, where $b^+(x) = \max\{b(x), 0\}$ and $\mathcal{B}_0 = \{x \in \mathbb{R}^3 \mid b(x) < 0\}$.

(K_1) $K(x) \in L^2(\mathbb{R}^3)$.

(K_2) $\Omega_K = \mathrm{int}K^{-1}(0)$ is a nonempty open set and $\mathrm{dist}(\Omega_a, \mathbb{R}^3 \backslash \Omega_K) > 0$.

(F_1) There exist $p \in (2,6)$, $f_0 > 0$ and $f_1 > 0$ such that $-f_1(|t|^2 + |t|^p) \leq f(x,t)t \leq f_0(|t|^2 + |t|^p)$ for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

(F_2) $\lim_{t \to 0} \frac{f(x,t)}{t} = 0$ uniformly for $x \in \mathbb{R}^3$.

(F_3) $\lim_{t \to \infty} \frac{f(x,t)}{t^2} = +\infty$ uniformly for $x \in \mathbb{R}^3$.

(F_4) There exists $f_2 > 0$ such that $f(x,t)t - 2F(x,t) \geq f_2|t|^3$ for all $x \in \mathbb{R}^3$, where $F(x,t) = \int_0^t f(x,s)ds$.

(F_5) There exist $f_3 > 0$ and $\mu > 2$ such that $f(x,t)t - \mu F(x,t) \geq -f_3|t|^3$ for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

Remark 1.2 (1) *If $b(x)$ is bounded below, then the condition (B_2) is trivial. But under the assumptions (B_1)-(B_3), $b(x)$ may be unbounded from below and sign-changing.*

(2) *A typical function satisfying the conditions (F_1)-(F_5) is that $f(x,t) = |t|^{p-2}t$ with $3 < p < 6$. However, $f(x,t)$ may be sign-changing under the conditions (F_1)-(F_5).*

Now, the main result in this paper can be stated as follows.

Theorem 1.1 *Suppose the conditions (A_1)-(A_4), (B_1)-(B_3), (K_1)-(K_2) and (F_1)-(F_5) hold. If $a(x) \geq 0$, $3 < p \leq 4$ and $0 \notin \sigma(-\Delta + b(x), H_0^1(\Omega_a))$, then there exists $\Lambda_* > 0$ and $f_1^* > 0$ such that ($\mathcal{SP}_\lambda$) has a nontrivial weak solution $(u_\lambda, \phi_\lambda)$ for $\lambda \geq \Lambda_*$ and $f_1 < f_1^*$, where $\sigma(-\Delta + b(x), H_0^1(\Omega_a))$ is the spectrum of $-\Delta + b(x)$ on $H_0^1(\Omega_a)$. Furthermore, for every $\{\lambda_n\}$, $(u_{\lambda_n}, \phi_{\lambda_n}) \to (u_0, \phi_0)$ strongly in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, where $\phi_0(x) = \int_{\Omega_a} \frac{K(y)u_0^2(y)}{4\pi|x-y|}dy$ and u_0 is a nontrivial weak solution of the following equation:*

$$-\Delta u + b(x)u + K(x)\left(\int_{\Omega_a} \frac{K(y)u^2(y)}{4\pi|x-y|}dy \right)u = f(x,u), \quad u \in H_0^1(\Omega_a). \quad (1.1)$$

Remark 1.3 *In Theorem 1.1, the conditions (A_4) and (K_2) are introduced to replace the Pohozaev type conditions on the potentials, which seems to be new for the existence of nontrivial weak solutions to the Schrödinger-Poisson system ($\mathcal{SP}_\lambda$) with the steep potential well in the case of $3 < p \leq 4$. Moreover, under the conditions (A_1)–(A_4) and (B_1)–(B_3), the Schrödinger operator $-\Delta + V_\lambda(x)$ may be strong indefinite if λ sufficiently large (see Lemma 2.2 for more details), and it seems to be the first time that such a Schrödinger operator is considered for the Schrödinger-Poisson system ($\mathcal{SP}_\lambda$) in the case of $3 < p \leq 4$.*

The remaining of this paper is devoted to the proof of Theorem 1.1 and it will be organized as follows. In section 2, we will introduce an auxiliary system of the Schrödinger-Poisson system ($\mathcal{SP}_\lambda$) by the method of penalized functions and obtain one nontrivial solution of the auxiliary system by the well-known linking theorem. In section 3, by the Morse iteration, we will prove that some special nontrivial solutions of the auxiliary system are also the nontrivial solutions of the Schrödinger-Poisson system ($\mathcal{SP}_\lambda$) for λ sufficiently large. Theorem 1.1 will also be shown in this section.

In this paper, we will always denote the usual norms in $H^1(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ ($p \geq 1$) by $\|\cdot\|$ and $\|\cdot\|_{L^p(\mathbb{R}^3)}$, respectively. $o_n(1)$ will always denote the quantities tending towards zero as $n \to \infty$.

2 The auxiliary system

In this section, we will introduce an auxiliary system of $(\mathcal{SP}_\lambda)$ by the method of penalized functions. This method was developed by del Pino and Felmer in [11] and also used to find nontrivial solutions of elliptic equations or systems with the steep potential well in several other literatures, see for example [7, 12, 15, 27] and the references therein. For the sake of convenience, we always assume the conditions (A_1)–(A_4), (B_1)–(B_3), (K_1)–(K_2) and (F_1)–(F_5) hold with $a(x) \geq 0$ in this section.

Since the conditions (A_4) and (K_2) hold, we can choose Ω'_a to be a nonempty open set in $\mathbb{R}^3$ which satisfies $\Omega_a \subset \Omega'_a \subset (\mathcal{A}_\infty \cap \Omega_K)$, $\mathrm{dist}(\Omega_a, \mathbb{R}^3 \backslash \Omega'_a) > 0$ and $\mathrm{dist}(\Omega'_a, \mathbb{R}^3 \backslash (\mathcal{A}_\infty \cap \Omega_K)) > 0$. For $\delta \in (0, \min\{1, \frac{1}{2}(S^{-1} \|K\|^{-2}_{L^2(\mathbb{R}^3)} f_2)^{\frac{1}{2}}\})$, we define

$$g_\delta(s) = \begin{cases} -\delta, & s \leq -\delta, \\ s, & |s| \leq \delta, \\ \delta, & s \geq \delta, \end{cases}$$

and $g_\delta(x,t) = 2K(x)(\chi_{\Omega'_a} t + (1 - \chi_{\Omega'_a})g_\delta(t))$, where $\chi_{\Omega'_a}$ is the characteristic function of Ω'_a. Let us consider the following elliptic system

$$\begin{cases} -\Delta u + V_\lambda(x)u + \dfrac{1}{2}\phi g_\delta(x,u) = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = G_\delta(x,u) & \text{in } \mathbb{R}^3, \\ (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3), \end{cases} \qquad (\overline{\mathcal{SP}}_{\lambda,\delta})$$

where $G_\delta(x,u) = \int_0^u g_\delta(x,t)dt$. Clearly, if (u, ϕ) is a weak solution of $(\overline{\mathcal{SP}}_{\lambda,\delta})$ satisfying $|u(x)| \leq \delta$ a.e. on $\mathbb{R}^3 \backslash \Omega'_a$, then (u, ϕ) is also a weak solution of $(\mathcal{SP}_\lambda)$. On the other hand, for every $u \in H^1(\mathbb{R}^3)$, since $|G_\delta(x,u)| \leq K(x)u^2$ for all $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, by the condition (K_1), we have

$$\left(\int_{\mathbb{R}^3} |G_\delta(x,u)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} \leq \left(\int_{\mathbb{R}^3} K(x)^{\frac{6}{5}} u^{\frac{12}{5}} dx\right)^{\frac{5}{6}} \leq \|K\|_{L^2(\mathbb{R}^3)} \|u\|^2_{L^6(\mathbb{R}^3)}. \qquad (2.1)$$

It follows from the Sobolev inequality that $G_\delta(x,u) \in L^{\frac{6}{5}}(\mathbb{R}^3)$. By [8, Lemma 2.1], the Poisson equation $-\Delta \phi = G_\delta(x,u)$ in $\mathbb{R}^3$ has a unique solution in $D^{1,2}(\mathbb{R}^3)$, which is given by

$$\phi_{u,\delta}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{G_\delta(y, u(y))}{|x-y|} dy.$$

Thus, the elliptic system $(\overline{\mathcal{SP}}_{\lambda,\delta})$ can be deduced to the following single Schrödinger equation

$$\begin{cases} -\Delta u + V_\lambda(x)u + \dfrac{1}{2}\phi_{u,\delta} g_\delta(x,u) = f(x,u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \qquad (\overline{\mathcal{S}}_{\lambda,\delta})$$

in the sense that $u \in H^1(\mathbb{R}^3)$ is a solution of $(\overline{\mathcal{S}}_{\lambda,\delta})$ is equivalent to $(u, \phi_{u,\delta}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(\overline{\mathcal{SP}}_{\lambda,\delta})$.

2.1 Variational setting of $(\overline{\mathcal{S}}_{\lambda,\delta})$

For every $\lambda > 0$, we define $E_\lambda = \{u \in D^{1,2}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3}(\lambda a(x) + b^+(x))u^2 dx < +\infty\}$, where $b^+(x) = \max\{b(x), 0\}$. Since the conditions (A_1) and (B_1) hold, E_λ is a Hilbert space with the following inner product

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^3} (\nabla u \nabla v + (\lambda a(x) + b^+(x))uv)dx.$$

The corresponding norm is given by $\|u\|_\lambda = \langle u, u \rangle_\lambda^{\frac{1}{2}}$. By the condition (A_2) and the Hölder and Sobolev inequalities, for every $u \in E_\lambda$ and $\lambda \geq 1$, we have

$$\int_{\mathbb{R}^3} u^2 dx \leq \max\{|\mathcal{A}_\infty|^{\frac{2}{3}} S^{-1}, a_\infty^{-1}\} \int_{\mathbb{R}^3} |\nabla u|^2 + (\lambda a(x) + b^+(x))u^2 dx. \qquad (2.2)$$

Hence, E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda \geq 1$.

Lemma 2.1 $(\overline{\mathcal{S}}_{\lambda,\delta})$ *has a variational structure in E_λ for every $\lambda \geq 1$ and the corresponding functional is given by*

$$\mathcal{E}_{\lambda,\delta}(u) = \frac{1}{2}\mathcal{D}_\lambda(u,u) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

where $\mathcal{D}_\lambda(u,v) = \int_{\mathbb{R}^3}(\nabla u \nabla v + (\lambda a(x) + b(x))uv)dx$ and $F(x,u) = \int_0^u f(x,s)ds$.

Proof. The idea of this proof comes from [25]. We first claim that $\mathcal{E}_{\lambda,\delta}(u)$ is well defined on E_λ for every $\lambda \geq 1$. Indeed, for every $u \in E_\lambda$ with $\lambda \geq 1$, $\mathcal{D}_\lambda(u,u) = \|u\|_\lambda^2 + \int_{\mathbb{R}^3} b^-(x)u^2dx$. By the condition (B_2) and (2.2), we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} b^-(x)u^2dx &\leq b_0 \int_{\mathbb{R}^3}(1+\sqrt{a(x)})u^2dx \\
&\leq b_0(1+\sqrt{a_\infty})\int_{\mathcal{A}_\infty} u^2dx + b_0(\frac{1}{a_\infty} + \frac{1}{\sqrt{a_\infty}})\int_{\mathbb{R}^3\backslash\mathcal{A}_\infty} a(x)u^2dx \\
&\leq b_* \int_{\mathbb{R}^3} |\nabla u|^2 + V_\lambda(x)u^2dx, \quad (2.3)
\end{aligned}
$$

where $b_* = \max\{b_0(1+\sqrt{a_\infty})d_*, b_0(\frac{1}{a_\infty} + \frac{1}{\sqrt{a_\infty}})\}$ and $d_* = \max\{|\mathcal{A}_\infty|^{\frac{2}{3}}S^{-1}, a_\infty^{-1}\}$. It follows that $\mathcal{D}_\lambda(u,u)$ is well defined on E_λ for every $\lambda \geq 1$. On the other hand, since the condition (F_1) holds and E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda \geq 1$, by using the Hölder and Sobolev inequalities in a standard way, we can see that $\int_{\mathbb{R}^3} F(x,u)dx$ is also well defined on E_λ for every $\lambda \geq 1$. It remains to show $\int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx$ is well defined on E_λ for $\lambda \geq 1$. In fact, since $\phi_{u,\delta}$ is the unique solution of the following Poisson equation

$$-\Delta\phi = G_\delta(x,u), \quad \phi \in D^{1,2}(\mathbb{R}^3),$$

we can see that $\|\nabla\phi_{u,\delta}\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx$. Hence, by the Hölder and Sobolev inequalities, we have

$$\|\nabla\phi_{u,\delta}\|_{L^2(\mathbb{R}^3)} \leq S^{-\frac{1}{2}}\left(\int_{\mathbb{R}^3} |G(x,u)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}},$$

where S is the best Sovolev embedding constant and given by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^3)}^2}{\|u\|_{L^6(\mathbb{R}^3)}^2}.$$

This implies that $|\int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx| \leq S^{-1}\left(\int_{\mathbb{R}^3} |G(x,u)|^{\frac{6}{5}}dx\right)^{\frac{5}{3}}$. Now, by (2.1), we can obtain that

$$\left|\int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx\right| \leq S^{-2}\|K\|_{L^2(\mathbb{R}^3)}^2\|u\|_{L^6(\mathbb{R}^3)}^4. \quad (2.4)$$

Since E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda \geq 1$, by the Sobolev inequality, we can see that $\int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx$ is also well defined on E_λ for $\lambda \geq 1$. We complete the proof by showing that $\mathcal{E}_\lambda(u)$ is the corresponding functional of $(\overline{\mathcal{S}}_{\lambda,\delta})$ in E_λ for $\lambda \geq 1$. Indeed, let $\Psi_\delta(u) = \int_{\mathbb{R}^3} \phi_{u,\delta}G_\delta(x,u)dx$. Then $\Psi_\delta(u)$ is well defined on E_λ for $\lambda \geq 1$. For every $u,v \in E_\lambda$, since the condition (K_1) holds, by similar arguments as used in (2.1), we can see that $g_\delta(x,u)v \in L^{\frac{6}{5}}(\mathbb{R}^3)$. It follows from [8, Lemma 2.1] that $\phi_{u,v,\delta}^0 \in D^{1,2}(\mathbb{R}^3)$ is the unique solution of the Poisson equation $-\Delta\phi = g_\delta(x,u)v$ in $\mathbb{R}^3$, which is given by

$$\phi_{u,v,\delta}^0(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{g_\delta(y,u(y))v(y)}{|x-y|}dy.$$

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By similar arguments as used in (2.4), we know that $|\int_{\mathbb{R}^3} G_\delta(x,u)\phi^0_{u,v,\delta}dx| < +\infty$. Therefore, thanks to the Fubini theorem, we have $\Psi_\delta(u) \in C^1(\mathbb{R}^3, \mathbb{R})$ and

$$\langle \Psi'_\delta(u), v \rangle_{E^*_\lambda, E_\lambda} = 2 \int_{\mathbb{R}^3} \phi_{u,\delta} g_\delta(x,u) v dx$$

for every $u, v \in E_\lambda$ with $\lambda \geq 1$, where E^*_λ is the dual space of E_λ. Now, by applying the condition (F_1) and the Hölder and Sobolev inequalities in a standard way and applying a similar argument as used in (2.3), we can see that $\mathcal{E}_{\lambda,\delta}(u)$ is C^1 on E_λ for $\lambda \geq 1$. Furthermore, for every $u, v \in E_\lambda$ with $\lambda \geq 1$, we have

$$\langle \mathcal{E}'_{\lambda,\delta}(u), v \rangle_{E^*_\lambda, E_\lambda} = \mathcal{D}_\lambda(u,v) + \int_{\mathbb{R}^3} \frac{1}{2}\phi_{u,\delta} g_\delta(x,u) v dx - \int_{\mathbb{R}^3} f(x,u)v dx.$$

Hence, if $u \in E_\lambda$ is a critical point of $\mathcal{E}_{\lambda,\delta}(u)$, then u is also a solution of $(\overline{\mathcal{S}}_{\lambda,\delta})$ due to the fact that E_λ is embedded continuously into $H^1(\mathbb{R}^3)$, that is, $\mathcal{E}_{\lambda,\delta}(u)$ is the corresponding functional of $(\overline{\mathcal{S}}_{\lambda,\delta})$ in E_λ for $\lambda \geq 1$. ∎

2.2 Properties of the functional $\mathcal{D}_\lambda(u,u)$

We have known from Lemma 2.1 that $\mathcal{D}_\lambda(u,u)$ is well defined on E_λ for $\lambda \geq 1$. By a similar argument as used in (2.3), we can see that $\mathcal{D}_\lambda(u,u)$ is actually C^2 on E_λ for $\lambda \geq 1$. Furthermore, we also have the following lemma for $\mathcal{D}_\lambda(u,u)$, which is inspired by [7] and [36].

Lemma 2.2 *There exists $\Lambda_0 \geq 1$ such that the augmented Morse index of $\mathcal{D}_\lambda(u,u)$ is less than or equal to k_0 uniformly for $u \in E_\lambda$ with $\lambda \geq \Lambda_0$, where $k_0 = dim\mathcal{V}_b$ and $\mathcal{V}_b = \{u \in H^1_0(\Omega) \mid \int_{\Omega_a}(|\nabla u|^2 + b(x)u^2)dx \leq 0\}$.*

Proof. If $b(x) \geq 0$ on $\mathbb{R}^3$, then it is easy to see that $\mathcal{V}_b = \{0\}$ and $k_0 = 0$. On the other hand, since $b(x) \geq 0$ on $\mathbb{R}^3$, it is also easy to show that the augmented Morse index of $\mathcal{D}_\lambda(u,u)$ is equal to 0 uniformly for $u \in E_\lambda$ with $\lambda \geq 1$, which then implies that Lemma 2.2 holds in this case. It remains to show that Lemma 2.2 is also true if $b(x)$ is sign-changing. Indeed, since the condition (B_3) holds, we can see that $b^{-1}([0,+\infty)) \neq \emptyset$ in this case. It follows from the condition (B_1) that $\mathcal{F}_\lambda \neq \emptyset$ for every $\lambda \geq 1$, where

$$\mathcal{F}_\lambda = \{u \in E_\lambda \mid \text{supp}u \subset b^{-1}([0,+\infty))\}.$$

Let

$$\mathcal{F}^\perp_\lambda = \{u \in E_\lambda \mid \langle u,v \rangle_\lambda = 0 \text{ for all } v \in \mathcal{F}_\lambda\}.$$

Then $E_\lambda = \mathcal{F}_\lambda \oplus \mathcal{F}^\perp_\lambda$. Furthermore, by the condition (B_1), we have $\mathcal{F}^\perp_\lambda \neq \{0\}$ for every $\lambda \geq 1$ since $b(x)$ is sign-changing. For the sake of clarity, the proof is further performed through the following several steps.

Step. 1 We prove that for every $\lambda \geq 1$, the operator $(-\Delta + \lambda a(x) + b^+(x))^{-1}b^-(x)$ has a sequence of positive eigenvalues $\{\alpha_j(\lambda)\}$ in $\mathcal{F}^\perp_\lambda$ satisfying $0 < \alpha_1(\lambda) \leq \alpha_2(\lambda) \leq \cdots \leq \alpha_j(\lambda) \to +\infty$ as $j \to +\infty$, and the corresponding eigenfunctions are a basis of $\mathcal{F}^\perp_\lambda$.

Indeed, it is easy to see that the operator $(-\Delta + \lambda a(x) + b^+(x))^{-1}b^-(x)$ is linear and self-conjugate on $\mathcal{F}^\perp_\lambda$ for all $\lambda \geq 1$. Thanks to the condition (B_3), $(-\Delta + \lambda a(x) + b^+(x))^{-1}b^-(x)$ is also compact on $\mathcal{F}^\perp_\lambda$ for all $\lambda \geq 1$. Thus, by [32, Theorems 4.45 and 4.46], the eigenvalue problem $-\Delta u + (\lambda a(x) + b^+(x))u = \alpha b^-(x)u$ has a sequence of positive eigenvalues $\{\alpha_j(\lambda)\}$ in $\mathcal{F}^\perp_\lambda$ satisfying $0 < \alpha_1(\lambda) \leq \alpha_2(\lambda) \leq \cdots \leq \alpha_j(\lambda) \to +\infty$ as $j \to +\infty$. Furthermore, $\{\alpha_j(\lambda)\}$ can be characterized by

$$\alpha_j(\lambda) = \inf_{\dim M \geq j, M \subset \mathcal{F}^\perp_\lambda} \sup \left\{ \|u\|^2_\lambda \mid u \in M \text{ and } \int_{\mathbb{R}^3} b^-(x)u^2 dx = 1 \right\}, \quad j = 1,2,3,\cdots, \quad (2.5)$$

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and the corresponding eigenfunctions $e_j(\lambda)$ can be chosen so that $\int_{\mathbb{R}^3} b^-(x)e_j^2(\lambda)dx = 1$ for all $j \in \mathbb{N}$ and are a basis of $\mathcal{F}_\lambda^\perp$.

Step. 2 We prove that $\alpha_j(\lambda)$ is nondecreasing for λ and $\alpha_j(\lambda) \to \alpha_j^0$ as $\lambda \to +\infty$, where α_j^0 is a positive eigenvalue of the following problem

$$-\Delta u + b^+(x)u = \alpha b^-(x)u, \quad u \in H_0^1(\Omega_a). \tag{2.6}$$

Indeed, let $\lambda_1 \geq \lambda_2$, then by the definition of E_λ, we have $E_{\lambda_1} = E_{\lambda_2}$. It follows that $\mathcal{F}_{\lambda_2} = \mathcal{F}_{\lambda_1}$, which implies $\mathcal{F}_{\lambda_2}^\perp = \mathcal{F}_{\lambda_1}^\perp$. Note that $\|u\|_{\lambda_1} \geq \|u\|_{\lambda_2}$ for all $u \in E_{\lambda_1}$, so by the definition of $\alpha_j(\lambda_1)$ and $\alpha_j(\lambda_2)$, we can see that $\alpha_j(\lambda_2) \leq \alpha_j(\lambda_1)$, that is, $\alpha_j(\lambda)$ is nondecreasing for λ. In what follows, we will show that $\alpha_j(\lambda) \to \alpha_j^0$ as $\lambda \to +\infty$. By the conditions (B_1) and (B_3), for every $j \in \mathbb{N}$, there exists $\{\varphi_m\}_{1 \leq m \leq j} \subset C_0^\infty(\mathbb{R}^3)$ such that $\text{supp}\varphi_m \subset \Omega_a \cap b^{-1}((-\infty, 0))$ and $\text{supp}\varphi_m \cap \text{supp}\varphi_n = \emptyset$ for $m \neq n$. Let $M_0 = \text{span}\{\varphi_1, \cdots, \varphi_j\}$. Then by (2.5), $\alpha_j(\lambda) \leq \alpha_j^*$, where

$$\alpha_j^* = \sup\left\{ \int_{\Omega_a} (|\nabla u| + b^+(x)u^2)dx \mid u \in M_0 \text{ and } \int_{\Omega_a} b^-(x)u^2 dx = 1 \right\}.$$

It follows from the choice of $e_j(\lambda)$ that $\|e_j(\lambda)\|_\lambda \leq \sqrt{\alpha_j^*}$. Note that by (2.2), we have $\|e_j(\lambda)\|^2 \leq (1 + d_*)\|e_j(\lambda)\|_\lambda^2$, where $d_* > 0$ is a constant given by (2.3). So up to a subsequence, $e_j(\lambda) \rightharpoonup e_j$ weakly in $H^1(\mathbb{R}^3)$ as $\lambda \to +\infty$. By the condition (A_1), $\|e_j(\lambda)\|_\lambda \leq \sqrt{\alpha_j^*}$ once more and the Fatou lemma, we can see that $\int_{\mathbb{R}^3} a(x)e_j(\lambda)^2 dx \to 0$ as $\lambda \to +\infty$ up to a subsequence and $\int_{\mathbb{R}^3} a(x)e_j^2 dx = 0$, which then together with the condition (A_3), implies $e_j \in H_0^1(\Omega_a)$. Since the condition (A_2) holds, we have $|\mathcal{A}_\infty \cap (\mathbb{R}^3 \backslash B_R)| \to 0$ as $R \to +\infty$, where $B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$. It follows from the Sobolev embedding theorem, $\int_{\mathbb{R}^3} a(x)e_j(\lambda)^2 dx \to 0$ as $\lambda \to +\infty$ and $\int_{\mathbb{R}^3} a(x)e_j^2 dx = 0$ that $e_j(\lambda) \to e_j$ strongly in $L^2(\mathbb{R}^3)$ as $\lambda \to +\infty$ up to a subsequence. Now, by the conditions (A_2) and (B_2), for every $\psi \in C_0^\infty(\Omega_a)$, we can see that

$$
\begin{aligned}
\int_{\Omega_a} \nabla e_j \nabla \psi + b^+(x)e_j \psi dx &= \lim_{\lambda \to +\infty} \int_{\mathbb{R}^3} \nabla e_j(\lambda)\nabla\psi + b^+(x)e_j(\lambda)\psi dx \\
&= \lim_{\lambda \to +\infty} \int_{\mathbb{R}^3} \nabla e_j(\lambda)\nabla\psi_\lambda + b^+(x)e_j(\lambda)\psi_\lambda dx \\
&= \lim_{\lambda \to +\infty} \alpha_j(\lambda) \int_{\mathbb{R}^3} b^-(x)e_j(\lambda)\psi_\lambda dx \\
&= \lim_{\lambda \to +\infty} \alpha_j(\lambda) \int_{\mathbb{R}^3} b^-(x)e_j(\lambda)\psi dx \\
&= \alpha_j^0 \int_{\Omega_a} b^-(x)e_j \psi dx,
\end{aligned}
$$

where ψ_λ is the projection of ψ in $\mathcal{F}_\lambda^\perp$. Hence, (e_j, α_j^0) satisfies (2.6). Note that $\alpha_j(\lambda)$ is nondecreasing for λ, so by Step. 1, we can see that α_j^0 is positive.

Now, since $b(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $|\Omega_a| < +\infty$, by a similar argument as used in Step. 1, we can see that the eigenvalue problem (2.5) has a sequence of positive eigenvalues $\{\overline{\alpha}_j\}$ satisfying $0 < \overline{\alpha}_1 \leq \overline{\alpha}_2 \leq \cdots \leq \overline{\alpha}_j \to +\infty$ as $j \to +\infty$, and the corresponding eigenfunctions $\overline{e}_j$ are a basis of $H_0^1(\Omega_a)$. Hence, $k_0 = \dim\mathcal{V}_b < +\infty$. Suppose there exist $j \neq i$ such that $\alpha_j^0 = \alpha_i^0 = \overline{\alpha}_k$ for some $k \in \mathbb{N}$. Then one of the following two cases must happen:

(1) $e_j = e_i$;

(2) $e_j \neq e_i$ and $\int_{\Omega_a} \nabla e_j \nabla e_i + b^+(x)e_j e_i dx = 0$.

If case (1) happen, then by Step. 1 and Step. 2, we have

$$2\overline{\alpha}_k = \lim_{\lambda \to +\infty} (\alpha_j(\lambda) + \alpha_i(\lambda)) = \lim_{\lambda \to +\infty} (\|e_j(\lambda)\|_\lambda^2 + \|e_i(\lambda)\|_\lambda^2) = \lim_{\lambda \to +\infty} (\|e_j(\lambda) - e_i(\lambda)\|_\lambda^2) = 0.$$

It is impossible. Therefore, we must have the case (2). Now, by Step. 2, we can see that there exists $\Lambda_0 \geq 1$ such that $\alpha_{k_0+1}(\lambda) > 1$ for $\lambda \geq \Lambda_0$. It follows from Step. 1 that $\alpha_j(\lambda) > 1$ for all

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$j \geq k_0 + 1$ and $\lambda \geq \Lambda_0$, which implies that the augmented Morse index of $\mathcal{D}_\lambda(u, u)$ is less than or equal to k_0 uniformly for $u \in E_\lambda$ with $\lambda \geq \Lambda_0$. ∎

By Step. 1 of Lemma 2.2, $\mathcal{F}_\lambda^\perp = \mathcal{F}_{\lambda,-}^\perp \oplus \mathcal{F}_{\lambda,+}^\perp$ for every $\lambda \geq 1$, where $\mathcal{F}_{\lambda,-}^\perp = \text{span}\{e_j(\lambda) \mid \alpha_j(\lambda) \leq 1\}$ and $\mathcal{F}_{\lambda,+}^\perp = \text{span}\{e_j(\lambda) \mid \alpha_j(\lambda) > 1\}$, which implies $E_\lambda = \mathcal{F}_{\lambda,-}^\perp \oplus \mathcal{F}_{\lambda,+}^\perp \oplus \mathcal{F}_\lambda$ for every $\lambda \geq 1$. Moreover, due to Lemma 2.2 again, we know that $\dim(\mathcal{F}_{\lambda,-}^\perp) \leq k_0 < +\infty$ for all $\lambda \geq \Lambda_0$ and k_0 is independent of $\lambda \geq \Lambda_0$. Let $l_0 = \max\{j \mid \overline{\alpha}_j < 1\}$. Then $l_0 \leq k_0$ and $\overline{\alpha}_{l_0+1} > 1$ due to the fact that $0 \notin \sigma(-\Delta + b(x), H_0^1(\Omega_a))$. Now, we have the following important estimates for $\mathcal{D}_\lambda(u, u)$ on E_λ.

Lemma 2.3 *Suppose $0 \notin \sigma_p(-\Delta + b(x), H_0^1(\Omega_a))$, then there exists $\Lambda_1 \geq \Lambda_0$ such that for every $u \in E_\lambda$ with $\lambda \geq \Lambda_1$, $\mathcal{D}_\lambda(u, u) \leq -\frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0}})\|u\|_\lambda^2$ on $\mathcal{F}_{\lambda,-}^\perp$ and $\mathcal{D}_\lambda(u, u) \geq \frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})\|u\|_\lambda^2$ on $\mathcal{F}_{\lambda,+}^\perp \oplus \mathcal{F}_\lambda$.*

Proof. By Step. 2 of Lemma 2.2, there exists $\Lambda_1 \geq \Lambda_0$ such that $\frac{1}{\alpha_{l_0}(\lambda)} - 1 \geq \frac{1}{2}(\frac{1}{\overline{\alpha}_{l_0}} - 1)$ and $(1 - \frac{1}{\alpha_{l_0+1}(\lambda)}) \geq \frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})$ for all $\lambda \geq \Lambda_1$. Now, by the definition of $\mathcal{F}_{\lambda,-}^\perp$ and Step. 1 of Lemma 2.2, it is easy to see that $\mathcal{D}_\lambda(u, u) \leq -\frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0}})\|u\|_\lambda^2$ on $\mathcal{F}_{\lambda,-}^\perp$ for all $u \in E_\lambda$ with $\lambda \geq \Lambda_1$. On the other hand, it follows from Step. 1 of Lemma 2.2 once more that $(1 - \frac{1}{\alpha_j(\lambda)}) \geq \frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})$ for all $\lambda \geq \Lambda_1$ and $j \geq l_0 + 1$. Now, for every $u \in \mathcal{F}_{\lambda,+}^\perp \oplus \mathcal{F}_\lambda$, we respectively denote the projections of u on $\mathcal{F}_{\lambda,+}^\perp$ and $\mathcal{F}_\lambda$ by u_λ^* and u_λ^{**}. Then we have

$$
\begin{aligned}
\mathcal{D}_\lambda(u, u) &= \|u\|_\lambda^2 - \int_{\mathbb{R}^3} b^-(x) u^2 dx \\
&= \|u_\lambda^*\|_\lambda^2 + \|u_\lambda^{**}\|_\lambda^2 - \int_{\mathbb{R}^3} b^-(x)(u_\lambda^*)^2 dx \\
&\geq \|u_\lambda^*\|_\lambda^2 + \|u_\lambda^{**}\|_\lambda^2 - \frac{1}{\alpha_{l_0+1}(\lambda)}\|u_\lambda^*\|_\lambda^2 \\
&\geq \frac{1}{2}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})\|u\|_\lambda^2,
\end{aligned}
$$

which completes the proof. ∎

2.3 Critical points of the functional $\mathcal{E}_{\lambda,\delta}(u)$

Since the condition (B_3) holds, we can choose $v_0 \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp} v_0 \subset \Omega_a \cap b^{-1}([0, +\infty))$. For every $R > 0$, we define

$$\mathcal{Q}_{\lambda,R} = \{u = z + t v_0 \mid t \geq 0, z \in \mathcal{F}_{\lambda,-}^\perp, \|u\|_\lambda \leq R\}.$$

Then we have the following lemma, which implies $\mathcal{E}_{\lambda,\delta}(u)$ has a linking structure in E_λ for $\lambda \geq \Lambda_1$.

Lemma 2.4 *There exist $f_1^* > 0$, $\delta_0 \in (0, \min\{1, \frac{1}{2}(S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^{-2} f_2)^{\frac{1}{2}}\})$, $\rho_0 > 0$ and $R_0 > 0$ which are all independent on $\lambda \geq \Lambda_1$ such that*

$$\inf_{\mathcal{S}_{\lambda,\rho_0} \cap (\mathcal{F}_{\lambda,+}^\perp \oplus \mathcal{F}_\lambda)} \mathcal{E}_{\lambda,\delta_0}(u) > \sup_{\partial \mathcal{Q}_{\lambda,R_0}} \mathcal{E}_{\lambda,\delta_0}(u)$$

for $f_1 < f_1^*$ and $\lambda \geq \Lambda_1$, where f_1 is given by the condition (F_1) and $\mathcal{S}_{\lambda,\rho_0} = \{u \in E_\lambda \mid \|u\|_\lambda = \rho_0\}$.

Proof. By the conditions (F_1)-(F_2), there exist $\varepsilon, C_\varepsilon > 0$ such that $|f(x, t)| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1}$. It follows that $|\int_{\mathbb{R}^3} F(x, u) dx| \leq \varepsilon\|u\|_{L^2(\mathbb{R}^3)}^2 + C_\varepsilon\|u\|_{L^p(\mathbb{R}^3)}^p$. Since $\Lambda_1 \geq 1$, by (2.2) and the Hölder and Sobolev inequalities, we have

$$\|u\|_{L^p(\mathbb{R}^3)}^p \leq d_*^{\frac{6-p}{4}} S^{-\frac{3p-6}{4}}\|u\|_\lambda^p, \quad \text{for all } u \in E_\lambda \text{ with } \lambda \geq \Lambda_1, \tag{2.7}$$

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where $d_* > 0$ is a constant given by (2.3). Let $\varepsilon_0 = \frac{1}{8d_*}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})$. Note that $\phi_{u,\delta} \geq 0$ on $\mathbb{R}^3$ by the construction of $g_\delta(x,t)$ and the maximum principle, so by Lemma 2.3 and (2.2) once more, we can see that

$$
\begin{aligned}
\mathcal{E}_{\lambda,\delta}(u) &= \frac{1}{2}\mathcal{D}_\lambda(u,u) + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u,\delta}G_\delta(x,u)dx - \int_{\mathbb{R}^3}F(x,u)dx \\
&\geq \frac{1}{2}\mathcal{D}_\lambda(u,u) - |\int_{\mathbb{R}^3}F(x,u)dx| \\
&\geq \frac{1}{8}(1 - \frac{1}{\overline{\alpha}_{l_0+1}})\|u\|_\lambda^2 - C_*\|u\|_\lambda^p
\end{aligned}
$$

for all $u \in \mathcal{F}_{\lambda,+}^\perp \oplus \mathcal{F}_\lambda$ with $\lambda \geq \Lambda_1$, where $C_* = C_{\varepsilon_0}d_*^{\frac{6-p}{4}}S^{-\frac{3p-6}{4}}$. Let $\rho_0 = \left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{1}{p-2}}$. Then

$$
\inf_{S_{\lambda,\rho_0}\cap(\mathcal{F}_{\lambda,+}^\perp\oplus\mathcal{F}_\lambda)}\mathcal{E}_{\lambda,\delta}(u) \geq \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} \quad \text{for } \lambda \geq \Lambda_1. \tag{2.8}
$$

In what follows, we will prove that there exist $\delta_0 \in (0,1)$ and $R_0 > \rho_0$ independent of $\lambda \geq \Lambda_1$ such that $\sup_{\partial\mathcal{Q}_{\lambda,R_0}}\mathcal{E}_{\lambda,\delta_0}(u) < \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}$ for all $\lambda \geq \Lambda_1$. For the sake of clarity, the proof will be further performed through the following two Claims.

Claim 1 There exists $R_0 > \rho_0$ independent of $\lambda \geq \Lambda_1$ such that

$$
\sup_{\{u=z+tv_0|t\geq 0,z\in\mathcal{F}_{\lambda,-}^\perp,\|u\|_\lambda=R\}}\mathcal{E}_{\lambda,\delta}(u) < 0
$$

for all $\delta \in (0,1)$ and $\lambda \geq \Lambda_1$.

Indeed, since the condition (K_1) holds and E_λ with $\lambda \geq \Lambda_1$ is embedded continuously into $H^1(\mathbb{R}^3)$, for every $u \in E_\lambda$ with $\lambda \geq \Lambda_1$, we have $K(x)|u(x)| \in L^{\frac{6}{5}}(\mathbb{R}^3)$ due to the Hölder inequality. It follows from [8, Lemma 2.1] that the Poisson equation $-\Delta\phi = K(x)|u(x)|$ in $\mathbb{R}^3$ has a unique solution in $D^{1,2}(\mathbb{R}^3)$ which can be given by

$$
\overline{\phi}_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)|u(y)|}{|x-y|}dy.
$$

Moreover, by the Hölder and Sobolev inequalities, $\|\overline{\phi}_u\|_{L^6(\mathbb{R}^3)} \leq S^{-1}\|K\|_{L^2(\mathbb{R}^3)}\|u\|_{L^3(\mathbb{R}^3)}$. By the construction of $G_\delta(x,u)$ and the condition (K_2), we can see that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\phi_{u,\delta}G_\delta(x,u)dx &\leq \int_{\mathbb{R}^3\setminus\Omega_K}2\delta K(x)|u(x)|\int_{\mathbb{R}^3\setminus\Omega_K}\frac{2\delta K(y)|u(y)|}{4\pi|x-y|}dydx \\
&\leq \int_{\mathbb{R}^3}4\delta^2 K(x)|u(x)|\overline{\phi}_u(x)dx \\
&\leq 4\delta^2 S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2\|u\|_{L^3(\mathbb{R}^3)}^2, \tag{2.9}
\end{aligned}
$$

which implies

$$
\sup_{\{u=z+tv_0|t\geq 0,z\in\mathcal{F}_{\lambda,-}^\perp,\|u\|_\lambda=R\}}\mathcal{E}_{\lambda,\delta}(u) \leq \sup_{\{u=z+tv_0|t\geq 0,z\in\mathcal{F}_{\lambda,-}^\perp,\|u\|_\lambda=R\}}J(u) \quad \text{for } 0 < \delta < 1, \tag{2.10}
$$

where $J(u) = \frac{1}{2}\|u\|_\lambda^2 + S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2\|u\|_{L^3(\mathbb{R}^3)}^2 - \int_{\mathbb{R}^3}F(x,u)dx$. Since $\lambda \geq \Lambda_1$, we have $\dim(\mathcal{F}_{\lambda,-}^\perp \oplus \mathbb{R}v_0) \leq k_0 + 1$ by Lemma 2.2 for some k_0 independent of $\lambda \geq \Lambda_1$. Hence, there exists $d_{k_0}, d_{k_0}' > 0$ independent on $\lambda \geq \Lambda_1$ such that $\|u\|_\lambda \leq d_{k_0}\|u\|_{L^3(\mathbb{R}^3)} \leq d_{k_0}'\|u\|_{L^2(\mathbb{R}^3)}$ for all $u \in \mathcal{F}_{\lambda,-}^\perp \oplus \mathbb{R}v_0$. By the condition (F_3) and the Fatou lemma, for every $u \in \mathcal{F}_{\lambda,-}^\perp \oplus \mathbb{R}v_0$ with $\|u\|_\lambda = 1$, we have

$$
\lim_{t\to+\infty}\frac{J(tu)}{t^3} \leq \lim_{t\to+\infty}\frac{d_{k_0}'d_* + d_{k_0}'d_{k_0}^{-1}d_*}{2t} - \lim_{t\to+\infty}\frac{1}{2}\int_{\mathbb{R}^3}\frac{2F(x,tu)}{|tu|^3}u^3dx = -\infty,
$$

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which together with (2.10), implies there exists $R_0 > 0$ independent on $\lambda \geq \Lambda_1$ such that

$$\sup_{\{u=z+tv_0|t\geq 0,z\in\mathcal{F}_{\lambda,-}^{\perp},\|u\|_\lambda=R_0\}} \mathcal{E}_{\lambda,\delta}(u) < 0$$

for all $\delta \in (0,1)$ and $\lambda \geq \Lambda_1$.

Claim 2 There exists $\delta_0 \in (0, \min\{1, \frac{1}{2}(S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^{-2} f_2)^{\frac{1}{2}}\})$ and $f_1^* > 0$ independent of $\lambda \geq \Lambda_1$ such that

$$\sup_{\{u\in\mathcal{F}_{\lambda,-}^{\perp}|\|u\|_\lambda\leq R_0\}} \mathcal{E}_{\lambda,\delta_0}(u) \leq \frac{p-2}{2p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}$$

for $f_1 < f_1^*$ and $\lambda \geq \Lambda_1$.

Indeed, by Lemma 2.3 and (2.9), we have

$$\sup_{\{u\in\mathcal{F}_{\lambda,-}^{\perp}|\|u\|_\lambda\leq R_0\}} \mathcal{E}_{\lambda,\delta}(u) \leq \sup_{\{u\in\mathcal{F}_{\lambda,-}^{\perp}|\|u\|_\lambda\leq R_0\}} (\delta^2 S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2\|u\|_{L^3(\mathbb{R}^3)}^2 - \int_{\mathbb{R}^3} F(x,u)dx).$$

It follows from the condition (F_1), (2.2) and (2.7) that

$$\sup_{\{u\in\mathcal{F}_{\lambda,-}^{\perp}|\|u\|_\lambda\leq R_0\}} (\delta^2 S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2\|u\|_{L^3(\mathbb{R}^3)}^2 - \int_{\mathbb{R}^3} F(x,u)dx)$$

$$\leq \delta^2 S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2 \left(\frac{d_*}{S}\right)^{\frac{1}{2}} R_0^2 + f_1(d_*R_0^2 + d_*^{\frac{6-p}{4}}S^{-\frac{3p-6}{4}}R_0^p).$$

Therefore, there exists $\delta_0 \in (0,1)$ and $f_1^* > 0$ independent of $\lambda \geq \Lambda_1$ such that

$$\sup_{\{u\in\mathcal{F}_{\lambda,-}^{\perp}|\|u\|_\lambda\leq R_0\}} \mathcal{E}_{\lambda,\delta_0}(u) \leq \frac{p-2}{2p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}$$

for $f_1 < f_1^*$ and $\lambda \geq \Lambda_1$.

Now, the conclusion follows immediately from (2.8), Claim 1 and Claim 2. ∎

Since Lemma 2.4 holds, by the well known linking theorem (cf. [24]), $\mathcal{E}_{\lambda,\delta_0}(u)$ has a $(C)_{c_\lambda}$ sequence in E_λ with $\lambda \geq \Lambda_1$. That is, there exists $\{u_{\lambda,n}\} \subset E_\lambda$ such that $\mathcal{E}_{\lambda,\delta_0}(u_{\lambda,n}) = c_\lambda + o_n(1)$ and $(1 + \|u_{\lambda,n}\|_\lambda)\mathcal{E}'_{\lambda,\delta_0}(u_{\lambda,n}) = o_n(1)$ strongly in E_λ^*. Furthermore, we have $c_\lambda \in [\frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}, \frac{1}{2}R_0^2 + \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} + f_1(d_*R_0^2 + d_*^{\frac{6-p}{4}}S^{-\frac{3p-6}{4}}R_0^p)]$.

Lemma 2.5 *There exist $\Lambda_2 \geq \Lambda_1$ and $C^* > 0$ independent of $\lambda \geq \Lambda_2$ such that $\|u_{\lambda,n}\|_\lambda \leq C^* + o_n(1)$ for all $\lambda \geq \Lambda_2$.*

Proof. By the construction of $g_\delta(x,t)$ and the conditions (K_1) and (F_4), we can see that

$$
\begin{aligned}
c_\lambda + o_n(1) &= \mathcal{E}_{\lambda,\delta_0}(u_{\lambda,n}) - \frac{1}{2}\langle\mathcal{E}'_{\lambda,\delta_0}(u_{\lambda,n}), u_{\lambda,n}\rangle_{E_\lambda^*,E_\lambda} \\
&= \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_{\lambda,n},\delta}(G_\delta(x,u_{\lambda,n}) - g_\delta(x,u_{\lambda,n})u_{\lambda,n})dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} f(x,u_{\lambda,n})u_{\lambda,n} - 2F(x,u_{\lambda,n})dx \\
&\geq -2\delta_0^2\int_{\mathbb{R}^3\setminus\Omega_k}\int_{\mathbb{R}^3\setminus\Omega_k} \frac{K(y)K(x)|u_{\lambda,n}(y)||u_{\lambda,n}(x)|}{4\pi|x-y|}dydx + \frac{f_2}{2}\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^3 \\
&\geq -2\delta_0^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)|u_{\lambda,n}(y)||u_{\lambda,n}(x)|}{4\pi|x-y|}dydx + \frac{f_2}{2}\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^3. \quad (2.11)
\end{aligned}
$$

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Since $K(x)|u_{\lambda,n}(x)| \in L^{\frac{6}{5}}(\mathbb{R}^3)$ for every $n \in \mathbb{N}$, by the condition (K_1) once more, we can follow the argument as used in (2.9) to obtain that

$$\left| \int_{\mathbb{R}^3 \setminus \Omega_k} \int_{\mathbb{R}^3 \setminus \Omega_k} \frac{K(y)K(x)|u_{\lambda,n}(y)||u_{\lambda,n}(x)|}{4\pi|x-y|} dy dx \right| \leq S^{-1} \|K\|_{L^2(\mathbb{R}^3)}^2 \|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^2. \qquad (2.12)$$

By (2.11) and (2.12), we have $c_\lambda + o_n(1) + 2\delta_0^2 S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2 \|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^2 \geq f_2\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^3$. It follows from the Young inequality, the choice of δ_0 and the fact $c_\lambda \leq \frac{1}{2}R_0^2 + \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} + f_1(d_*R_0^2 + d_*^{\frac{6-p}{4}} S^{-\frac{3p-6}{4}} R_0^p)$ that $\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)} \leq C_0 + o_n(1)$ for some $C_0 > 0$ independent of $\lambda \geq \Lambda_1$. Now, thanks to the conditions (A_2) and (B_2)-(B_3) and the Hölder inequality, we have

$$\int_{\mathbb{R}^3} b^-(x)|u_{\lambda,n}|^2 dx \leq \int_{\mathcal{B}_0} b_0(1 + \sqrt{a(x)})|u_{\lambda,n}|^2 dx$$

$$\leq b_0(1 + \sqrt{a_\infty})|\mathcal{B}_0|^{\frac{1}{3}}\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^2 + \frac{1}{\lambda\sqrt{a_\infty}}\|u_{\lambda,n}\|_\lambda^2$$

$$\leq b_0(1 + \sqrt{a_\infty})|\mathcal{B}_0|^{\frac{1}{3}}C_0^2 + \frac{1}{\lambda\sqrt{a_\infty}}\|u_{\lambda,n}\|_\lambda^2 + o_n(1), \qquad (2.13)$$

where $\mathcal{B}_0$ is given by the condition (B_3). By the condition (F_5), (2.12) and (2.13), we can obtain that

$$c_\lambda + o_n(1) = \mathcal{E}_{\lambda,\delta_0}(u_{\lambda,n}) - \frac{1}{\mu}\langle \mathcal{E}'_{\lambda,\delta_0}(u_{\lambda,n}), u_{\lambda,n}\rangle_{E_\lambda^*, E_\lambda}$$

$$= (\frac{1}{2} - \frac{1}{\mu})\mathcal{D}_\lambda(u_{\lambda,n}, u_{\lambda,n}) + \int_{\mathbb{R}^3} \phi_{u_{\lambda,n},\delta}(\frac{1}{4}G_\delta(x, u_{\lambda,n}) - \frac{1}{2\mu}g_\delta(x, u_{\lambda,n})u_{\lambda,n})dx$$

$$+ \frac{1}{\mu}\int_{\mathbb{R}^3} f(x, u_{\lambda,n})u_{\lambda,n} - \mu F(x, u_{\lambda,n})dx$$

$$\geq (\frac{1}{2} - \frac{1}{\mu} - \frac{1}{\lambda\sqrt{a_\infty}})\|u_{\lambda,n}\|_\lambda^2 - b_0(1 + \sqrt{a_\infty})|\mathcal{B}_0|^{\frac{1}{3}}C_0^2 - f_3\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^3$$

$$+ \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_{\lambda,n},\delta}(G_\delta(x, u_{\lambda,n}) - g_\delta(x, u_{\lambda,n})u_{\lambda,n})dx$$

$$\geq (\frac{1}{2} - \frac{1}{\mu} - \frac{1}{\lambda\sqrt{a_\infty}})\|u_{\lambda,n}\|_\lambda^2 - b_0(1 + \sqrt{a_\infty})|\mathcal{B}_0|^{\frac{1}{3}}C_0^2 - f_3\|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^3$$

$$- S^{-1}\|K\|_{L^2(\mathbb{R}^3)}^2 \|u_{\lambda,n}\|_{L^3(\mathbb{R}^3)}^2.$$

Therefore, by $c_\lambda \leq \frac{1}{2}R_0^2 + \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} + f_1(d_*R_0^2 + d_*^{\frac{6-p}{4}} S^{-\frac{3p-6}{4}} R_0^p)$, there exists $\Lambda_2 \geq \Lambda_1$ such that $\|u_{\lambda,n}\|_\lambda \leq C^* + o_n(1)$ for $\lambda \geq \Lambda_2$, where C^* is independent of $\lambda \geq \Lambda_2$. ∎

We close this section by the following.

Proposition 2.1 *There exists $\Lambda_3 \geq \Lambda_2$ such that $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ has a nontrivial solution u_λ for $\lambda \geq \Lambda_3$.*

Proof. By Lemma 2.5, $\{u_{\lambda,n}\}$ is bounded in E_λ for $\lambda \geq \Lambda_2$. It follows that $u_{\lambda,n} \rightharpoonup u_\lambda$ weakly in E_λ for some $u_\lambda \in E_\lambda$ as $n \to \infty$. Suppose $u_\lambda = 0$. Then by the condition (K_1) and a similar argument as used in [36, Lemma 2.1], we can see that $\phi_{u_{\lambda,n},\delta_0} \to 0$ strongly in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Since $\{u_{\lambda,n}\}$ is bounded in E_λ for $\lambda \geq \Lambda_2$, by the condition (K_1) once more and the construction of $g_{\delta_0}(x, t)$, we have

$$\int_{\mathbb{R}^3} \phi_{u_{\lambda,n},\delta_0} G_{\delta_0}(x, u_{\lambda,n})dx = \int_{\mathbb{R}^3} \phi_{u_{\lambda,n},\delta_0} g_{\delta_0}(x, u_{\lambda,n})u_{\lambda,n}dx = o_n(1). \qquad (2.14)$$

On the other hand, since the condition (B_3) holds, $|\mathcal{B}_0 \cap (\mathbb{R}^3 \setminus B_R)| \to 0$ as $R \to +\infty$. Therefore, by the Sobolev embedding theorem and the fact that E_λ is embedded continuously into $H^1(\mathbb{R}^3)$

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for $\lambda \geq \Lambda_2$, we can obtain $\int_{\mathbb{R}^3} b^-(x) u_{\lambda,n}^2 dx = o_n(1)$, which together with (2.14) and the fact that $\{u_{\lambda,n}\}$ is a $(C)_{c_\lambda}$ sequence of $\mathcal{E}_{\lambda,\delta_0}(u)$, implies

$$c_\lambda + o_n(1) = \frac{1}{2}\|u_{\lambda,n}\|_\lambda^2 - \int_{\mathbb{R}^3} F(x, u_{\lambda,n}) dx \tag{2.15}$$

and

$$o_n(1) = \|u_{\lambda,n}\|_\lambda^2 - \int_{\mathbb{R}^3} f(x, u_{\lambda,n}) u_{\lambda,n} dx. \tag{2.16}$$

By the conditions (F_1)-(F_2) and (2.2) and (2.16), for every $\varepsilon > 0$, we have

$$\begin{aligned}
\|u_{\lambda,n}\|_\lambda^2 &= \int_{\mathbb{R}^3} f(x, u_{\lambda,n}) u_{\lambda,n} dx + o_n(1) \\
&\leq \varepsilon d_* \|u_{\lambda,n}\|_\lambda^2 + C_\varepsilon \|u_{\lambda,n}\|_{L^p(\mathbb{R}^3)}^p + o_n(1),
\end{aligned} \tag{2.17}$$

where $d_* > 0$ is a constant given by (2.3) and $C_\varepsilon > 0$ is a constant independent of $\lambda \geq \Lambda_2$. Since the condition (A_2) holds, $|\mathcal{A}_\infty \cap (\mathbb{R}^3 \backslash B_R)| \to 0$ as $R \to +\infty$, which together with the Sobolev embedding theorem, $u_{\lambda,n} \rightharpoonup 0$ weakly in E_λ as $n \to \infty$ and the fact that E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda \geq \Lambda_2$, implies $\int_{\mathcal{A}_\infty} u_{\lambda,n}^2 dx = o_n(1)$. Now, by the condition (A_2) and the Hölder and Sobolev inequalities, we can see that

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_{\lambda,n}|^p dx &\leq \left(\int_{\mathbb{R}^3} |u_{\lambda,n}|^2 dx \right)^{\frac{6-p}{4}} \left(\int_{\mathbb{R}^3} |u_{\lambda,n}|^6 dx \right)^{\frac{3p-6}{12}} \\
&\leq \left(\frac{1}{\lambda a_\infty} \int_{\mathbb{R}^3} \lambda a(x) |u_{\lambda,n}|^2 dx + o_n(1) \right)^{\frac{6-p}{4}} S^{-\frac{3p-6}{4}} \|u_{\lambda,n}\|_\lambda^{\frac{3p-6}{2}} \\
&\leq \left(\frac{1}{\lambda a_\infty} \right)^{\frac{6-p}{2}} S^{-\frac{3p-6}{4}} \|u_{\lambda,n}\|_\lambda^p + o_n(1) \|u_{\lambda,n}\|_\lambda^{\frac{3p-6}{2}}.
\end{aligned} \tag{2.18}$$

Thanks to Lemma 2.5, (2.17) and (2.18), we obtain

$$\left(1 - \varepsilon d_* - C_\varepsilon \left(\frac{1}{\lambda a_\infty} \right)^{\frac{6-p}{2}} S^{-\frac{3p-6}{4}} (C^*)^{p-2} \right) \|u_{\lambda,n}\|_\lambda^2 \leq o_n(1).$$

Take $\varepsilon = \frac{1}{2d_*}$. Then there exists $\Lambda_3 \geq \Lambda_2$ such that $\|u_{\lambda,n}\|_\lambda^2 = o_n(1)$ for $\lambda \geq \Lambda_3$. Hence, $u_{\lambda,n} \to 0$ strongly in E_λ with $\lambda \geq \Lambda_3$ as $n \to \infty$. Since E_λ is embedded continuously into $H^1(\mathbb{R}^3)$ for $\lambda \geq \Lambda_3$, we have $\int_{\mathbb{R}^3} F(x, u_{\lambda,n}) dx = o_n(1)$ by the condition (F_1) and the Hölder and Sobolev inequalities, which together with (2.15), implies $c_\lambda = 0$ for $\lambda \geq \Lambda_3$. It is impossible since $c_\lambda \geq \frac{p-2}{p} \left(\frac{2d_*\varepsilon_0}{pC_*} \right)^{\frac{2}{p-2}} > 0$ for $\lambda \geq \Lambda_3$. We close the proof by showing that u_λ is also a solution of $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ for $\lambda \geq \Lambda_2$. Indeed, since $u_{\lambda,n} \rightharpoonup u_\lambda$ weakly in E_λ as $n \to \infty$, by the condition (K_1) and a similar argument as used in [36, Lemma 2.1], we can see that $\phi_{u_{\lambda,n},\delta_0} \to \phi_{u_\lambda,\delta_0}$ strongly in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Now, since the condition (B_3) and (F_1) hold and $\{u_{\lambda,n}\}$ is a $(C)_{c_\lambda}$ sequence of $\mathcal{E}_{\lambda,\delta_0}(u)$, by a standard argument, we can show that $\mathcal{E}'_{\lambda,\delta_0}(u_\lambda) = 0$ for $\lambda \geq \Lambda_3$. It follows from Lemma 2.1 that u_λ is a nontrivial solution of $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ for $\lambda \geq \Lambda_3$. ∎

3 Proof of Theorem 1.1

For the sake of convenience, we also assume the conditions (A_1)–(A_4), (B_1)–(B_3), (K_1)–(K_2) and (F_1)–(F_5) hold with $a(x) \geq 0$ in this section as in section 2. By Proposition 2.1, $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ has a nontrivial solution u_λ for $\lambda \geq \Lambda_3$, where Λ_3 is given by Proposition 2.1. In this section, we will verify that u_λ is also the solution of $(\mathcal{SP}_\lambda)$ for λ sufficiently large. By the choice of Ω'_a, we can find $\Omega_a \subset \Omega''_a \subset \Omega'_a$ such that $\text{dist}(\Omega_a, \mathbb{R}^3 \backslash \Omega''_a) > 0$ and $\text{dist}(\Omega''_a, \mathbb{R}^3 \backslash \Omega'_a) > 0$.

Lemma 3.1 *We have $\int_{\mathbb{R}^3\setminus\Omega_a''} u_\lambda^2 dx \to 0$ as $\lambda \to +\infty$.*

Proof. The idea of this proof comes from [36] and it was also used in [30]. Suppose the contrary, there exists $\{\lambda_n\}$ satisfying $\lambda_n \to +\infty$ as $n \to \infty$ and $\gamma_0 > 0$ such that $\int_{\mathbb{R}^3\setminus\Omega_a''} u_{\lambda_n}^2 dx \geq \gamma_0$. Without loss of generality, we assume $\lambda_n \geq \Lambda_3$ for all $n \in \mathbb{N}$. By Lemma 2.5, we have $\|u_{\lambda_n}\|_{\lambda_n} \leq C^*$, where $C^* > 0$ is independent of $n \in \mathbb{N}$. By (2.2), $\|u_{\lambda_n}\| \leq (1+d_*)C^*$, where $d_* > 0$ is a constant given by (2.3). Therefore, $u_{\lambda_n} \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$ for some $u_0 \in H^1(\mathbb{R}^3)$ as $n \to \infty$. Thanks to the condition (A_1), $\|u_{\lambda_n}\|_{\lambda_n} \leq C^*$ and the Fatou lemma, we can see that $\int_{\mathbb{R}^3} a(x)u_{\lambda_n}^2 dx = o_n(1)$ and $\int_{\mathbb{R}^3} a(x)u_0^2 dx = 0$. It follows from the condition (A_3) that $u_0 \in H_0^1(\Omega_a)$ with $u_0 \equiv 0$ outside Ω_a. Therefore, by the choice of Ω_a'', we have

$$\int_{\mathbb{R}^3} |u_{\lambda_n} - u_0|^2 dx = \int_{\Omega_a''} |u_{\lambda_n} - u_0|^2 dx + \int_{\mathbb{R}^3\setminus\Omega_a''} |u_{\lambda_n}|^2 dx \geq \int_{\mathbb{R}^3\setminus\Omega_a''} |u_{\lambda_n}|^2 dx \geq \gamma_0,$$

which together with the Lions lemma [21], implies that there exist $\{x_n\} \subset \mathbb{R}^3$ satisfying $|x_n| \to +\infty$ as $n \to \infty$ and $r_0, \sigma_0 > 0$ such that

$$\int_{B_{r_0}(x_n)} |u_{\lambda_n} - u_0|^2 dx \geq \sigma_0,$$

where $B_{r_0}(x_n) = \{x \in \mathbb{R}^3 \mid |x - x_n| \leq r_0\}$. Since the condition (A_2) holds and $|x_n| \to +\infty$ as $n \to \infty$, it is easy to see that $|B_{r_0}(x_n) \cap \mathcal{A}_\infty| \to 0$ as $n \to \infty$. Now, by the condition (A_2), $\|u_{\lambda_n}\|_{\lambda_n} \leq C^*$ and the Hölder and Sobolev inequalities once more, we can obtain that

$$
\begin{aligned}
C^* &\geq \int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2 + (\lambda_n a(x) + b^+(x))|u_{\lambda_n}|^2 dx \\
&\geq \lambda_n \int_{B_{r_0}(x_n) \cap (\mathbb{R}^3\setminus\mathcal{A}_\infty)} a(x)|u_{\lambda_n}|^2 dx \\
&\geq \lambda_n a_\infty \int_{B_{r_0}(x_n) \cap (\mathbb{R}^3\setminus\mathcal{A}_\infty)} |u_{\lambda_n} - u_0|^2 dx \\
&\geq \lambda_n a_\infty \left(\int_{B_{r_0}(x_n)} |u_{\lambda_n} - u_0|^2 dx - 4\int_{B_{r_0}(x_n)} |u_0|^2 dx - 4\int_{B_{r_0}(x_n) \cap \mathcal{A}_\infty} |u_{\lambda_n}|^2 dx \right) \\
&\geq \lambda_n a_\infty (\sigma_0 + o_n(1)) \to +\infty,
\end{aligned}
$$

which is a contradiction. ∎

With Lemma 3.1 in hands, we can obtain the following.

Lemma 3.2 *If $3 < p \leq 4$, then there exists $\Lambda_* \geq \Lambda_3$ such that $|u_\lambda(x)| \leq \delta_0$ a.e. on $\mathbb{R}^3\setminus\Omega_a'$ for $\lambda \geq \Lambda_*$.*

Proof. Let $r_* = \min\{\frac{1}{3}\text{dist}(\Omega_a'', \mathbb{R}^3\setminus\Omega_a'), (f_1 + b_0(1+\sqrt{a_\infty}))^{-\frac{1}{2}}, f_1^{-\frac{1}{2}}\}$. Then for every $y \in \mathbb{R}^3\setminus\Omega_a'$, $B_{2r_*}(y) \subset \mathbb{R}^3\setminus\Omega_a''$, where $B_{2r_*}(y) = \{x \in \mathbb{R}^3 \mid |y - x| \leq 2r_*\}$. Let $\rho(x) \in C_0^\infty(\mathbb{R}^3, [0,1])$ be given by

$$\rho(x) = \begin{cases} 1, & x \in B_{r_2}(y), \\ 0, & x \in \mathbb{R}^3\setminus B_{r_1}(y), \end{cases}$$

where $r_* < r_2 < r_1 < 2r_*$. Moreover, $|\nabla\rho(x)| \leq \frac{C_4}{r_1 - r_2}$ and $C_4 > 0$ is independent of r_1 and r_2. For $L > 0$ and $\alpha_0 > 0$, we denote $u_{\lambda,\rho} = \min\{|u_\lambda|^{\alpha_0}, L^{\alpha_0}\}\rho^2 u_\lambda$. Then it is easy to see that $u_{\lambda,\rho}(x) \in E_\lambda$ for $\lambda \geq \Lambda_3$. Since u_λ is a solution of $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ for $\lambda \geq \Lambda_3$, we have

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla u_{\lambda,\rho} + V_\lambda(x)u_\lambda u_{\lambda,\rho} + \phi_{u_\lambda,\delta_0} g_{\delta_0}(x, u_\lambda)u_{\lambda,\rho} dx = \int_{\mathbb{R}^3} f(x, u_\lambda)u_{\lambda,\rho} dx.$$

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By the conditions (A_2) and (B_2) and the Young inequality, there exists $\Lambda_4 \geq \Lambda_3$ such that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla u_{\lambda,\rho} + V_\lambda(x) u_\lambda u_{\lambda,\rho} + \phi_{u_\lambda,\delta_0} g_{\delta_0}(x, u_\lambda) u_{\lambda,\rho} dx$$
$$\geq \frac{1}{2} \int_{\mathbb{R}^3} \min\{|u_\lambda|^{\alpha_0}, L^{\alpha_0}\} \rho^2 |\nabla u_\lambda|^2 dx - 8 \int_{\mathbb{R}^3} |\nabla \rho|^2 |u_\lambda|^{\alpha_0+2} dx$$
$$- b_0(1 + \sqrt{a_\infty}) \int_{\mathbb{R}^3} |u_\lambda|^{\alpha_0+2} dx$$

for $\lambda \geq \Lambda_4$. This together with the condition (F_1) and the choice of r_1, r_2, r_*, implies

$$\int_{\mathbb{R}^3} \min\{|u_\lambda|^{\alpha_0}, L^{\alpha_0}\} \rho^2 |\nabla u_\lambda|^2 dx \leq 2 f_1 \int_{\mathbb{R}^3} \rho^2 |u_\lambda|^{p+\alpha_0} dx + \frac{16 C_4^2 + 2}{(r_1 - r_2)^2} \int_{\mathbb{R}^3} \rho^2 |u_\lambda|^{\alpha_0+2} dx.$$

By the Levi and Sobolev embedding theorems, we obtain

$$\left(\int_{B_{r_2}(y)} |u_\lambda|^{3(\alpha_0+1)} dx \right)^{\frac{1}{3}} \leq \left(\frac{\alpha_0+2}{2} \right)^2 \left(2 f_1 \int_{B_{r_1}(y)} |u_\lambda|^{p+\alpha_0} dx + \frac{16 C_4^2 + 2}{(r_1 - r_2)^2} \int_{B_{r_1}(y)} |u_\lambda|^{2+\alpha_0} dx \right). \tag{3.1}$$

Let $r_n = (1 + k^{-n}) r_*$ and $\alpha_n = 3(\alpha_{n-1} + 1) - p$ with $k^2 < 2$ and $\alpha_0 = \frac{3}{2}$. Then we can replace r_1, r_2, α_0 in (3.1) by $r_{n-1}, r_n, \alpha_{n-1}$ and obtain

$$\left(\int_{B_{r_{n+1}}(y)} |u_\lambda|^{3(\alpha_{n-1}+1)} dx \right)^{\frac{1}{3}}$$
$$\leq \left(\frac{\alpha_{n-1}+2}{2} \right)^2 \left(2 f_1 \int_{B_{r_n}(y)} |u_\lambda|^{p+\alpha_{n-1}} dx + \frac{16 C_4^2 + 2}{(r_n - r_{n+1})^2} \int_{B_{r_n}(y)} |u_\lambda|^{2+\alpha_{n-1}} dx \right). \tag{3.2}$$

Clearly, one of the following two cases must occur:

(1) $\int_{B_{r_n}(y)} |u_\lambda|^{2+\alpha_{n-1}} dx \leq \int_{B_{r_n}(y)} |u_\lambda|^{p+\alpha_{n-1}} dx$ up to a subsequence.

(2) $\int_{B_{r_n}(y)} |u_\lambda|^{p+\alpha_{n-1}} dx \leq \int_{B_{r_n}(y)} |u_\lambda|^{2+\alpha_{n-1}} dx$ up to a subsequence.

If case (1) happen, then by (3.2) and the choice of r_* and r_n, we obtain

$$\|u_\lambda\|_{L^{3(\alpha_{n-1}+1)}(B_{r_{n+1}}(y))}^{\alpha_{n-1}+1} \leq (\alpha_{n-1}+2)^2 \frac{C_5}{(r_n - r_{n+1})^2} \|u_\lambda\|_{L^{3(\alpha_{n-2}+1)}(B_{r_n}(y))}^{\alpha_{n-1}+p}, \tag{3.3}$$

where $C_5 = 16 C_4^2 + 4$. By iterating (3.3), we can see that

$$\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \leq \left(\prod_{n=1}^\infty (\alpha_{n-1}+2)^{\frac{2}{\alpha_{n-1}+1}} \left(\frac{C_5}{(r_n - r_{n+1})^2} \right)^{\frac{1}{\alpha_{n-1}+1}} \|u_\lambda\|_{L^{\frac{3}{2}+p}(B_{2r_*}(y))} \right)^{\prod_{n=1}^\infty \frac{\alpha_{n-1}+p}{\alpha_{n-1}+1}}, \tag{3.4}$$

where $\|\cdot\|_{L^\infty(B_{r_*}(y))}$ is the usual norm in $L^\infty(B_{r_*}(y))$. Since $3 < p \leq 4$, we can obtain from the choice of α_n that $\alpha_n \geq 2\alpha_{n-1}$, which implies that $\alpha_n \geq 2^n$. Note that $r_n = (1 + k^{-n}) r_*$ with $k^{-2} < 2$, we have $\prod_{n=1}^\infty (\alpha_{n-1}+2)^{\frac{2}{\alpha_{n-1}+1}} \left(\frac{C_5}{(r_n - r_{n+1})^2} \right)^{\frac{1}{\alpha_{n-1}+1}} < +\infty$ and $\prod_{n=1}^\infty \frac{\alpha_{n-1}+p}{\alpha_{n-1}+1} < +\infty$. By (3.4), $\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \leq C_6 \|u_\lambda\|_{L^{\frac{3}{2}+p}(B_{2r_*}(y))}^{C_6}$, where $C_6 > 0$ is independent of $\lambda \geq \Lambda_4$ and $y \in \mathbb{R}^3 \backslash \Omega_a'$. If case (2) happen, then by the Hölder inequality and a similar iteration as used in (3.4), we can obtain

$$\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \leq \left(\prod_{n=1}^\infty (\alpha_{n-1}+2)^{\frac{2}{\alpha_{n-1}+1}} \left(\frac{C_5 |B_{r_n}|^{\frac{p-2}{\alpha_{n-1}+p}}}{(r_n - r_{n+1})^2} \right)^{\frac{1}{\alpha_{n-1}+1}} \|u_\lambda\|_{L^{\frac{3}{2}+p}(B_{2r_*}(y))} \right)^{\prod_{n=1}^\infty \frac{\alpha_{n-1}+p}{\alpha_{n-1}+1}}. \tag{3.5}$$

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By similar arguments, we also have from (3.5) that $\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \leq C_6\|u_\lambda\|^{C_6}_{L^{2+p}(B_{2r_*}(y))}$. There-fore, in any case, we can obtain $\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \leq C_6\|u_\lambda\|^{C_6}_{L^{\frac{3}{2}+p}(B_{2r_*}(y))}$, where $C_6 > 0$ is indepen-dent of $\lambda \geq \Lambda_4$ and $y \in \mathbb{R}^3\backslash\Omega'_a$. Note that by Lemma 2.5 and (2.2), we have $\|u_\lambda\| \leq (1+d_*)C^*$. So thanks to Lemma 3.1 and the Hölder and Sobolev inequalities, we can see that $\|u_\lambda\|_{L^\infty(B_{r_*}(y))} \to 0$ as $\lambda \to +\infty$ uniformly on a.e. $\mathbb{R}^3\backslash\Omega'_a$. It follows that there exists $\Lambda_* \geq \Lambda_2$ such that $|u_\lambda(x)| \leq \delta_0$ a.e. on $\mathbb{R}^3\backslash\Omega'_a$ for $\lambda \geq \Lambda_*$. ∎

We close this section by

Proof of Theorem 1.1: By Proposition 2.1, $(\overline{\mathcal{S}}_{\lambda,\delta_0})$ has a nontrivial solution u_λ for $\lambda \geq \Lambda_3$. It follows that $(u_\lambda, \phi_{u_\lambda,\delta_0})$ is a nontrivial solution of $(\overline{\mathcal{SP}}_{\lambda,\delta_0})$ for $\lambda \geq \Lambda_3$. Thanks to Lemma 3.2 and the construction of $g_{\delta_0}(x,t)$, if $3 < p \leq 4$ and $\lambda \geq \Lambda_*$, then $(u_\lambda, \phi_{u_\lambda,\delta_0})$ is also a nontrivial solution of $(\mathcal{SP}_\lambda)$. It remains to show the concentration behavior of $(u_\lambda, \phi_{u_\lambda,\delta_0})$. Suppose $(u_{\lambda_n}, \phi_{u_{\lambda_n},\delta_0})$ is a sequence of solutions with $\lambda_n \to +\infty$. Then u_{λ_n} is also a solution of $(\overline{\mathcal{S}}_{\lambda_n,\delta_0})$, by similar arguments as used in Lemma 3.1, $u_{\lambda_n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ for some $u_0 \in H^1_0(\Omega_a)$ satisfying $u_0 \equiv 0$ outside Ω_a and $\int_{\mathbb{R}^3\backslash\Omega''_a} u^2_{\lambda_n} dx \to 0$ as $n \to \infty$. It follows from the condition (A_2) and the choice of Ω''_a that $u_{\lambda_n} \to u_0$ strongly in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Note that the condition (F_1) holds and $u_{\lambda_n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ as $n \to \infty$, by the Hölder and Sobolev inequalities, we have $\int_{\mathbb{R}^3} f(x, u_{\lambda_n})u_{\lambda_n} - f(x, u_0)u_0 dx \to 0$ as $n \to \infty$, which together with the conditions (A_1), (B_3) and (K_1) and the Fatou lemma, implies

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 + b(x)u^2_0 dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)u^2_0(y)u^2_0(x)}{4\pi|x-y|}dydx$$
$$\leq \liminf_{n\to\infty}(\mathcal{D}_{\lambda_n}(u_{\lambda_n}, u_{\lambda_n}) + \int_{\mathbb{R}^3} K(x)\phi_{u_{\lambda_n},\delta_0}u^2_{\lambda_n}dx)$$
$$= \liminf_{n\to\infty} \int_{\mathbb{R}^3} f(x, u_{\lambda_n})u_{\lambda_n}dx$$
$$= \int_{\mathbb{R}^3} f(x, u_0)u_0 dx. \tag{3.6}$$

Since $u_0 \in H^1_0(\Omega_a)$, it is easy to show that u_0 is a solution of (1.1). Therefore, by (3.6), we must have $\mathcal{D}_{\lambda_n}(u_{\lambda_n}, u_{\lambda_n}) \to \int_{\mathbb{R}^3} |\nabla u_0|^2 + b(x)u^2_0 dx$ as $n \to \infty$. It follows that $u_{\lambda_n} \to u_0$ strongly in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Since $c_{\lambda_n} \in [\frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}, \frac{1}{2}R^2_0 + \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} + f_1(d_*R^2_0 + d_*^{\frac{6-p}{4}}S^{-\frac{3p-6}{4}}R^p_0)]$ and $\frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}}$ and $\frac{1}{2}R^2_0 + \frac{p-2}{p}\left(\frac{2d_*\varepsilon_0}{pC_*}\right)^{\frac{2}{p-2}} + f_1(d_*R^2_0 + d_*^{\frac{6-p}{4}}S^{-\frac{3p-6}{4}}R^p_0)$ are independent of n, we also have $u_0 \neq 0$. Note that $\{\lambda_n\}$ is arbitrary, so u_λ has the concentration behaviors described as in this theorem. ∎

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