# On Seymour's Second Neighborhood Conjecture of m-free Digraphs \*

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#### Abstract

This paper gives an approximate result related to Seymour's Second Neighborhood conjecture, that is, for any m-free digraph G, there exists a vertex  $v \in V(G)$  and a real number  $\lambda_m$  such that  $d^{++}(v) \geq \lambda_m d^+(v)$ , and  $\lambda_m \to 1$  while  $m \to +\infty$ . This result generalizes and improves some known results in a sense.

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## 1 Introduction

Throughout this article, all digraphs are finite, simple and digonless. As usual, for a vertex v of the digraph G, we denote by  $N_G^+(v)$  the set of out-neighbors of v,  $N_G^{++}(v)$  the set of vertices at distance 2 from v. Let  $d_G^+(v) = |N_G^+(v)|$  (the out-degree of v) and  $d_G^{++}(v) = |N_G^{++}(v)|$ . We will omit the subscript if the digraph is clear from the context

In 1990, Seymour [3] proposed the following conjecture.

**Conjecture 1.1** (Seymour's Second Neighborhood Conjecture) For any digraph G, there exists a vertex v in G such that  $d^{++}(v) > d^+(v)$ .

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We call the vertex v in Conjecture 1.1 a Seymour vertex. In 2001, Kaneko and Locke [8] showed that any digraph with the minimum outdegree less than 7 has a Seymour vertex. In 2007, Fisher [5] showed that any tournament has a Seymour vertex; Fidler and Yuster [4] proved that any tournament minus a star or a subtournament, and any digraph G with minimum degree |V(G)| - 2 have Seymour vertices. In 2008, Hamidoune [7] proved that any vertex-transitive digraph has a Seymour vertex. In 2013, Lladó [10] proved that any digraph with large connectivity has a Seymour vertex. In 2016, Cohn et al. [2] gave a probabilistic statement about Seymour's conjecture and proved that almost surely there are a large number of Seymour vertices in random tournaments and even more in general random digraphs. For a general digraph, Conjecture 1.1 is still open.

Another approach to Conjecture 1.1 is to determinate the maximum value of  $\lambda$  such that there is a vertex v in G satisfying  $d^{++}(v) \geq \lambda d^+(v)$  for any digraph G. In 2003, Chen, Shen and Yuster [1] gave  $\lambda = 0.657298 \cdots$ , which is the unique real root of the polynomial  $2x^3 + x^2 - 1$ . Furthermore, they improved this bound to  $0.67815 \cdots$  mentioned in the end of the article [1].

A digraph G is called to be m-free if G contains no directed cycles of G with length at most m. In 2010, Zhang and Zhou [11] showed that for any 3-free digraph G, there exists a vertex v in G such that  $d^{++}(v) \geq \lambda d^+(v)$ , where  $\lambda = 0.6751 \cdots$  is the only real root in the interval (0,1) of the polynomial  $x^3 + 3x^2 - x - 1$ . In this paper, we consider general m-free digraphs and obtain the following result.

**Theorem 1.2** Let m be an arbitrarily fixed integer with  $m \geq 3$  and G be an m-free digraph, then there exists a vertex v in G such that  $d^{++}(v) \geq \lambda_m d^+(v)$ , where  $\lambda_m$  is the only real root in the interval (0,1) of the polynomial

$$g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1).$$
(1.1)

Furthermore,  $\lambda_m$  is increasing with m, and  $\lambda_m \to 1$  while  $m \to +\infty$ .

Since G is simple and digonless, G is 2-free. When m=2, the polynomial defined in (1.1) is exactly  $2x^3+x^2-1$ , and our result can be considered to be a generalization of Chen  $et\ al.$ 's result. When  $m=3,\,\lambda_3=0.6823\cdots$ , which improves Zhang  $et\ al.$ 's value on  $\lambda_3$ . When  $m=4,\,\lambda_4=0.7007\cdots$ . From Theorem 1.2, we immediately get the following corollary.

**Corollary 1.3** For every  $\varepsilon > 0$ , there is a positive integer m such that every m-free digraph contains a vertex v with  $d^{++}(v) \geq (1 - \varepsilon) d^+(v)$ .

The first conclusion in Theorem 1.2 is our main result. The proof proceeds by induction on the number of vertices. In the induction step, we assume to the contrary that  $d^{++}(v) < \lambda_m d^+(v)$  for any vertex v in G, where  $\lambda_m$  is the unique real root of  $g_m(x)$  in the interval (0,1). Then we show that the assumption leads to a contradiction. To this end, we need the following lemmas.

**Lemma 1.4** For  $m \geq 3$ , the polynomial  $g_m(x)$  defined in (1.1) is strictly increasing and has a unique real root in the interval (0,1).

Proof: Since 
$$g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1)$$
, we have  $q'_m(x) = 6x^2 - 2(m-3)x + (2m-4) = 6x^2 + 2x + (2m-4)(1-x)$ .

Clearly,  $g'_m(x) > 0$  when  $m \ge 3$  and  $x \in (0,1)$ , which implies  $g_m(x)$  is strictly increasing in [0,1]. Since  $g_m(0) = -m + 1 < 0$  and  $g_m(1) = 2 > 0$ , it follows that there is a unique real root in the interval (0,1) of the polynomial.

**Lemma 1.5** (Hamburger et al. [6]) If one can delete t edges from a digraph G to make it acyclic, then there exists a vertex v in G such that  $d^+(v) \leq \sqrt{2t}$ .

**Lemma 1.6** (Liang and Xu [9]) If an m-free digraph G is obtained from a tournament by deleting t edges, then one can delete from G an additional t/(m-2) edges so that the resulting digraph is acyclic.

Combining Lemma 1.5 with Lemma 1.6, we can easily get the following lemma.

**Lemma 1.7** If an m-free digraph G is obtained from a tournament by deleting t edges, then there exists a vertex v in G such that  $d^+(v) \leq \sqrt{2t/(m-2)}$ .

*Proof:* From Lemma 1.6, an m-free G is obtained from a tournament by deleting t edges, then we can delete t/(m-2) edges from G to make it acyclic. From Lemma 1.5, there exists a vertex v in G such that  $d^+(v) \leq \sqrt{2t/(m-2)}$ .

## 2 Proof of Theorem 1.2

We first prove the first conclusion by induction on the number of vertices. Theorem 1.2 is trivial for any digraph with 1 or 2 vertices. Assume that Theorem 1.2 holds for all digraphs with less than n vertices. Let G be an m-free digraph with n vertices,  $n \geq 3$  and  $m \geq 3$ . Assume to the contrary that  $d^{++}(v) < \lambda_m d^+(v)$  for any vertex v in G, where  $\lambda_m$  is the unique real root of  $g_m(x)$  in the interval (0,1). Our purpose is to show that the assumption leads to a contradiction.

Let u be a vertex in G with minimum out-degree. Let  $A = N^+(u)$ ,  $B = N^{++}(u)$ , a = |A| and b = |B|. By our assumption, we have

$$b = d^{++}(u) < \lambda_m d^+(u) = \lambda_m a.$$
 (2.1)

For any two disjoint subsets  $X, Y \subseteq V(G)$ , let E(X, Y) denote the edges from X to Y and e(X, Y) = |E(X, Y)|. Since G is simple and digorless, we have that

$$e(X,Y) + e(Y,X) \le |X| \cdot |Y|.$$

For simplicity, for any subset  $S \subseteq V(G)$ , use S to denote the subgraph of G induced by S. By the definitions of A and B, we have

$$\sum_{v \in A} d_G^+(v) = |E(A)| + e(A, B). \tag{2.2}$$

By the choice of  $u, d^+(v) \ge d^+(u) = a$  for any  $v \in V(G)$ , and so

$$\sum_{v \in A} d_G^+(v) \ge |A| \cdot d^+(u) = a^2. \tag{2.3}$$

Since  $|E(A)| \le a(a-1)/2$ , we have

$$e(A, B) = \sum_{v \in A} d_G^+(v) - |E(A)| \ge a^2 - a(a-1)/2 = a(a+1)/2.$$

It follows that there exists  $v \in A$  such that  $e(v, B) \ge e(A, B)/a \ge (a+1)/2$ . Since  $b = |B| \ge e(v, B)$  for any  $v \in A$ , it follows that  $\lambda_m a > b \ge e(v, B) \ge (a+1)/2 > a/2$ , which implies

$$\lambda_m > 1/2. \tag{2.4}$$

The subgraph A can be obtained from a tournament of order a by deleting t edges. Let  $\theta = t/a^2$ . Since  $0 \le t \le a(a-1)/2$ , we have  $0 \le \theta \le (a-1)/2a < 1/2$  and

$$|E(A)| = a(a-1)/2 - t = (1/2 - \theta)a^2 - a/2 < (1/2 - \theta)a^2.$$
(2.5)

Combining (2.2), (2.3) with (2.5), we have that

$$e(A,B) = \sum_{v \in A} d_G^+(v) - |E(A)| > a^2 - (1/2 - \theta)a^2 = (1/2 + \theta)a^2.$$
 (2.6)

Since G is m-free, it follows that the subgraph A is m-free. From Lemma 1.7, there is a vertex  $w_0 \in A$  such that

$$d_A^+(w_0) \le \sqrt{2t/(m-2)} = a\sqrt{2\theta/(m-2)}. (2.7)$$

Let  $d_B^+(w_0) = |N_B^+(w_0)|$ , then  $d_B^+(w_0) \le |B| = b$ . Since  $d_A^+(w_0) + d_B^+(w_0) = d_G^+(w_0)$ , it follows from (2.1) that  $d_A^+(w_0) = d_G^+(w_0) - d_B^+(w_0) \ge d_G^+(w_0) - b \ge a - \lambda_m a = (1 - \lambda_m) a$ , that is,

$$d_A^+(w_0) \ge (1 - \lambda_m) a. \tag{2.8}$$

Combining (2.7) with (2.8), we have  $\sqrt{2\theta/(m-2)}a > (1-\lambda_m)a$ , that is,

$$\theta > (m-2)(1-\lambda_m)^2/2.$$
 (2.9)

Since A is m-free and |A| = a < n, by induction hypothesis there is a vertex  $w_1 \in A$  such that  $|N_A^{++}(w_1)| \ge \lambda_m |N_A^+(w_1)|$ , where  $\lambda_m$  is the unique real root of  $g_m(x)$  in the interval (0,1).

Let  $X = N_A^+(w_1)$ ,  $Y = N_B^+(w_1)$  and |Y| = d. It follows from (2.1) that

$$d = |Y| \le |B| = b < \lambda_m a. \tag{2.10}$$

By the induction hypothesis,  $|A-X| \ge |N_A^{++}(w_1)| \ge \lambda_m |X|$ , that is,  $(1+\lambda_m)|X| \le |A| = a$ . By (2.4)  $\lambda_m > \frac{1}{2}$ , we have

$$|X| \le \frac{a}{1 + \lambda_m} < \frac{2a}{3}.$$

By the choice of u, we have  $d_G^+(w_1) \ge d_G^+(u) = a$ , and so

$$d = |Y| = |N_G^+(w_1)| - |X| > a - \frac{2a}{3} = \frac{a}{3}.$$
 (2.11)

Combining (2.10) with (2.11), we have

$$a/3 < d < \lambda_m a. \tag{2.12}$$

For any  $y \in Y$ , use  $d^+_{V-A-Y}(y)$  to denote the number of out-neighbors of y in G not in  $A \cup Y$ . Since  $d^+_G(w_1) < \lambda_m d^+_G(w_1)$  and  $d^+_A(w_1) \ge \lambda_m d^+_A(w_1)$ , we have

$$d_{V-A-Y}^+(y) \le d_G^{++}(w_1) - d_A^{++}(w_1) < \lambda_m d_G^+(w_1) - \lambda_m d_A^+(w_1) = \lambda_m d.$$

Noting that  $d_G^+(y) \ge d_G^+(u) = a$  and  $\sum_{y \in Y} d_Y^+(y) = |E(Y)| \le d(d-1)/2$ , we obtain

$$\begin{split} e(Y,A) &= \sum_{y \in Y} |N_A^+(y)| \\ &\geq \sum_{y \in Y} (a - d_{V-A-Y}^+(y) - d_Y^+(y)) \\ &> (a - \lambda_m d) \, d - \sum_{y \in Y} d_Y^+(y) \\ &> (a - \lambda_m d) \, d - d(d-1)/2 \\ &> (a - \lambda_m d - d/2) \, d, \end{split}$$

that is

$$e(Y,A) > (a - \lambda_m d - d/2)d. \tag{2.13}$$

Combining (2.1), (2.6), (2.9) with (2.13), we have

$$\lambda_{m}a^{2} \geq ab$$

$$\geq e(A, B) + e(B, A)$$

$$\geq e(A, B) + e(Y, A)$$

$$> (1/2 + \theta) a^{2} + (a - \lambda_{m}d - d/2) d$$

$$> [1/2 + (m - 2)(1 - \lambda_{m})^{2}/2] a^{2} + (a - \lambda_{m}d - d/2) d$$

$$= -(\lambda_{m} + 1/2)d^{2} + ad + [1/2 + (m - 2)(1 - \lambda_{m})^{2}/2] a^{2}$$

that is,

$$\lambda_m a^2 > -(\lambda_m + 1/2) d^2 + ad + [1/2 + (m-2)(1-\lambda_m)^2/2] a^2, \tag{2.14}$$

where  $a/3 < d < \lambda_m a$  (see (2.12)). For  $a/3 \le z \le \lambda_m a$ , let the function

$$f(z) = -(\lambda_m + 1/2)z^2 + az + [1/2 + (m-2)(1-\lambda_m)^2/2]a^2.$$

Since f(z) is a quadratic function with a negative leading coefficient, the following inequality holds.

$$f(z) \ge \min\{f(a/3), f(\lambda_m a)\}$$
 for any  $z \in [a/3, \lambda_m a]$ . (2.15)

Combining (2.14) with (2.15), we have

$$\lambda_m a^2 > f(d) \ge \min\{f(a/3), f(\lambda_m a)\}. \tag{2.16}$$

We first note that, since

$$f(\lambda_m a) = \frac{a^2[-2\lambda_m^3 + (m-3)\lambda_m^2 - (2m-6)\lambda_m + (m-1)]}{2},$$

if  $\lambda_m a^2 > f(\lambda_m a)$ , then

$$\lambda_m a^2 > \frac{a^2[-2\lambda_m^3 + (m-3)\lambda_m^2 - (2m-6)\lambda_m + (m-1)]}{2},$$

that is

$$g_m(\lambda_m) = 2\lambda_m^3 - (m-3)\lambda_m^2 + (2m-4)\lambda_m - (m-1) > 0.$$

This fact shows that  $\lambda_m$  is not a root of the polynomial  $g_m(x)$ , which contradicts our assumption on  $\lambda_m$ .

It follows that  $\lambda_m a^2 \leq f(\lambda_m a)$ , and so  $\lambda_m a^2 > f(a/3)$  by (2.16). Since

$$f(a/3) = \frac{a^2[9(m-2)\lambda_m^2 - (18m-34)\lambda_m + (9m-4)]}{18}.$$

we have

$$\lambda_m a^2 > \frac{a^2 [9(m-2)\lambda_m^2 - (18m-34)\lambda_m + (9m-4)]}{18}.$$

Simplifying this inequality, we obtain

$$9(m-2)\lambda_m^2 - (18m-16)\lambda_m + (9m-4) < 0.$$

This implies

$$\lambda_m > \frac{9m - 8 - \sqrt{54m - 8}}{9(m - 2)}. (2.17)$$

Now we show (2.17) is a contradiction to that  $\lambda_m$  is the only root in the interval (0,1) of the polynomial  $g_m(x)$ . We rewrite the polynomial  $g_m(x)$  as

$$g_m(x) = \frac{1}{9}(p(x) - q(x)),$$
 (2.18)

where

$$p(x) = 18x^3 + 9x^2 - 20x + 5,$$
  

$$q(x) = 9(m-2)x^2 - (18m-16)x + (9m-4).$$

The polynomial q(x) has a real root

$$\varphi_m = \frac{9m - 8 - \sqrt{54m - 8}}{9(m - 2)},\tag{2.19}$$

that is

$$q(\varphi_m) = 0. (2.20)$$

Comparing (2.17) with (2.19), we have

$$\lambda_m \ge \varphi_m \quad \text{for } m \ge 3.$$
 (2.21)

Since

$$\varphi_m = 1 + \frac{10 - \sqrt{54m - 8}}{9(m - 2)}$$

$$= 1 + \frac{108 - 54m}{9(m - 2)(10 + \sqrt{54m - 8})}$$

$$= 1 - \frac{6}{10 + \sqrt{54m - 8}},$$

it is easy to see that  $\varphi_m$  is strictly increasing with m for  $m \geq 3$ . Thus we have

$$\varphi_m \ge \varphi_3 = 1 - \frac{6}{10 + \sqrt{154}} > 1 - \frac{3}{10} = \frac{7}{10}.$$
(2.22)

A simple calculation gives us that p(x) is a strictly increasing function for  $x > \frac{7}{10}$  and  $p(\frac{7}{10}) = 1.584 > 0$ . Noting that  $g_m(x)$  is a strictly increasing function over the interval [0, 1], and by (2.18), (2.20), (2.21), (2.22), we have

$$g_m(\lambda_m) > g_m(\varphi_m) = \frac{1}{9}[p(\varphi_m) - q(\varphi_m)] = \frac{1}{9}p(\varphi_m) > \frac{1}{9}p(\frac{7}{10}) > 0.$$

This fact shows that  $\lambda_m$  is not a root of the polynomial  $g_m(x)$ , a contradiction to our assumption, and so the first conclusion follows.

We now prove the second conclusion. Since  $g_m(x) = 2x^3 - (m-3)x^2 + (2m-4)x - (m-1)$ ,  $g_m(\lambda_m) = 0$  and

$$g_{m+1}(x) = 2x^3 - (m-2)x^2 + (2m-2)x - m$$
  
=  $2x^3 - (m-3)x^2 + (2m-4)x - (m-1) - x^2 + 2x - 1$   
=  $q_m(x) - (1-x)^2$ ,

for any  $m \geq 3$  we have

$$g_{m+1}(\lambda_m) = g_m(\lambda_m) - (1 - \lambda_m)^2 = -(1 - \lambda_m)^2 < 0 = g_{m+1}(\lambda_{m+1}).$$

Since  $g_m(x)$  is strictly increasing in the interval (0,1) for any  $m \geq 3$  by Lemma 1.4, it follows that  $\lambda_m < \lambda_{m+1}$ , which implies that  $\lambda_m$  is increasing with m.

We rewrite  $g_m(x)$  as

$$g_m(x) = 2x(x^2 - 1) + 2x^2 - (m - 1)(1 - x)^2$$

It is easy to check that  $\mu_m = \frac{\sqrt{m-1}}{\sqrt{m-1}+\sqrt{2}} \in (0,1)$  is a real root of the polynomial  $2x^2 - (m-1)(1-x)^2$ . It follows that  $g_m(\mu_m) = 2\mu_m(\mu_m^2 - 1) < 0 = g_m(\lambda_m)$ . Since  $g_m(x)$  is strictly increasing in the interval (0,1) by Lemma 1.4, we have

$$0 < \mu_m < \lambda_m < 1$$
.

Since  $\lim_{m\to +\infty} \mu_m = \lim_{m\to +\infty} \frac{\sqrt{m-1}}{\sqrt{m-1}+\sqrt{2}} = 1$ , it follows that  $\lim_{m\to +\infty} \lambda_m = 1$ .

The proof of Theorem 1.2 is complete.

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