## Non-homogenous disks in the chain of matrices

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Abstract: We investigate the generating functions of multi-colored discrete disks with non-homogenous boundary condition in the context of the Hermitian multi-matrix model where the matrices are coupled in an open chain. We show that the study of the spectral curve of the matrix model allows to solve a set loop equations to get a recursive formula computing mixed trace correlation functions to leading order in the large matrix limit.

## 1 Introduction

The theory of random matrices has many ramifications in different fields of mathematics and physics. In the recent years, following the work of Eynard [11], matrix models have seen significant advances especially through their applications to enumerative geometry. The enumeration of surfaces can indeed be addressed by the evaluation of the result of a saddle point approximation on integrals defined over a set of random matrices as the size of the matrices gets very large. To be more precise, the result of such an approximation can be seen as the generating function of a set of fat graphs composed of ribbons glued together along vertices [4]. This set of fat graphs is in bijection with the set of discrete surfaces, or maps as denoted by combinatorists, i.e. surfaces composed of polygons glued by their edges. The matrix integrals can thus be seen, in this large matrix limit, as the generating function of a set of maps with a weight depending on the probability measure put on the set of matrices under study [2, 10, 25]. Such objects are of prime interest not only to mathematicians but also to physicists for investigating the behavior of statistical systems on random surfaces (or lattices) as well as for understanding quantization of two dimensional gravity or topological string theories from a discrete point of view. Discrete surfaces also provide a very good toy model for developing some intuition to be applied later to string and gauge theories.

From this perspective, a lot of progress have been made by Eynard et al. leading to powerful universal technics for the computation of generating functions of maps of different types. Among the results obtained, one finds:

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- The enumeration of maps of arbitrary topology, i.e. arbitrary genus and number of boundaries, has been achieved in [1, 5, 11];
- The enumeration of bicolored maps of arbitrary genus and arbitrary boundary conditions has been completed in the series of papers [12, 18, 9, 15, 21, 19];
- The enumeration of bicolored maps drawn on non-orientable surfaces has also been solved in [6, 16, 7, 8, 3] with homogenous boundary conditions;
- The enumeration of maps whose faces are colored with more than two colors has been partly solved in [13, 23]. In particular, the generating functions of maps with homogenous boundary conditions are known whatever their genus is.

One big outcome of these works is the universality of the solution found for all these problems. Indeed, for each of the aforementioned enumerative problems, the generating function of maps with homogenous boundary conditions are given by a unique universal inductive formula on the Euler characteristic of the surfaces enumerated. Even better is the fact that this inductive formula, sometimes called topological recursion, goes beyond the field of random matrices and enumeration of maps. The topological recursion [20, 22] seems to be a universal solution to different problems of enumerative geometry consisting in the computation of the volume of the moduli space of Riemann surfaces with respect to different measures opening the way for new insights in mathematics and topological string theories even when this measure does not localize on discrete surfaces.

The topological recursion was inspired by the computation of generating functions of open surfaces with homogenous boundary conditions in the different matrix models encountered. The next step in fully understanding the enumeration of maps in general is to understand the action of boundary operator, that is compute the generating function of surfaces with boundaries along which the boundary condition changes. This has been understood in the hermitian two matrix model case where it has been possible to compute the generating function of surfaces composed of polygons of two different colors with arbitrary boundary conditions [19]. The result points towards a generalization of the topological recursion for these new observables. Going further requires the study of the so-called chain of hermitian matrix model which generates maps colored with an arbitrary number of colors. Even if the homogenous boundary condition case is understood [23], there are only a few result concerning changes of boundary conditions [13].

In these notes, we present a recursive method, similar to the topological recursion, which allows to compute the generating function of discs with mixed boundary conditions. In theorem 4.1 and corollary 4.1 we prove that the corresponding correlation function  $\left\langle \operatorname{Tr} \left[ \frac{1}{x_1 - M_1} \frac{1}{x_2 - M_2} \dots \frac{1}{x_N - M_N} \right] \right\rangle$  can be computed to leading order in the large matrix limit in terms of the spectral curve of the model. The formula takes a very nice form which seems to follow from the all possible degeneracies of the surfaces generated. From the two matrix model experience, it is probably the building block for all the other generating functions and probably one of the most important objects of the theory. We expect to be able to use the techniques developed in this paper to compute any observable of this model.

The computation of these disk amplitudes has thus many interesting features. First of all, it gives an efficient method for computing the generating functions of discs composed of colored polygons with mixed-boundary conditions. In addition, we are convinced that the result as well as the methods used in its derivation are generalizable to the computation of the generating function of surfaces of arbitrary topology. From this perspective, these notes should provide the building block for the computation of all observable of the chain of matrices model. This result is also important for better

understanding conformal field theories. Indeed, when going to a particular limit in the parameter space of matrix models, one reaches critical regimes described by minimal models, a particular type of rational conformal field theories. The knowledge of boundary operators in the matrix model setup thus also gives access to boundary operator in such rational conformal field theories. Finally, the form of the solution points towards a generalization outside of the matrix model's setup generalizing the topological recursion formalism leading to the knowledge of new open amplitudes in topological string theories or may be new relative Gromov-Witten invariants.

This paper is organized as follows:

- In section 2, we introduce the model as well as the notations required in the following and present the result of these notes;
- In section 3, we remind how algebraic geometry appears in this context through the so-called loop equations and introduce all the notations and basic knowledge further needed.
- In section 4, we derive and solve a closed set of loop equation satisfied by the leading order mixed correlations functions.
- In section 5, we specialize this result to the chain of 3 matrices and give explicit examples.
- Section 6 is devoted to the conclusion and a short discussion on the perspectives.

# 2 The open chain of matrices

#### 2.1 The model

We want to generate discrete surfaces, i.e. surfaces composed of polygons glued by their edges. In addition, we want to include some additional information on these random surfaces such as a spin structure or some matter. For this purpose, each polygon carries one color labeled by an index ranging from 1 to  $\mathcal{N}$ .

Generating functions for such discrete colored surfaces can be obtained as correlation functions of a multi-matrix model. The latter studies the distribution of matrix elements with respect to a probability measure  $d\mu$  on the set  $(H_N)^N$  of  $\mathcal{N}$ -uple of hermitian matrices of respective size  $N \times N$  defined as

$$d\mu\left(M_{1},\ldots,M_{\mathcal{N}}\right) := dM_{1}dM_{2}\ldots dM_{\mathcal{N}}\exp\left(-\frac{N}{T}\operatorname{Tr}\left[\sum_{k=1}^{\mathcal{N}}V_{k}(M_{k}) - \sum_{k=2}^{\mathcal{N}}c_{k-1,k}M_{k-1}M_{k}\right]\right) \tag{2-1}$$

where  $M_i$  is a hermitian matrix of size N,  $dM_i$  is the product of Lebesgue measures of the real components of  $M_i$  and

$$V_k(x) = \sum_{i=2}^{d_k+1} \frac{g_i^{(k)}}{i} x^i$$
 (2-2)

are polynomial potentials of respective degrees  $d_k + 1$ . The partition function of this model is defined as

$$\mathcal{Z} := \int_{(H_{\mathcal{N}})^{\mathcal{N}}} d\mu \left( M_1, \dots, M_{\mathcal{N}} \right). \tag{2-3}$$

In this paper, we interpret this random matrix integral as a generating function of discrete surfaces. This interpretation is exact if one considers a formal version of this integral, which is defined as

the result of a saddle point approximation of this one<sup>2</sup>. In the following, all the matrix integrals encountered are formal in this sense.

In a nutshell, the combinatorial interpretation of such formal matrix integral is obtained by expanding the non quadratic part of the action

$$S := \sum_{k=1}^{N} V_k(M_k) - \sum_{k=2}^{N} M_{k-1} M_k$$
 (2-4)

around the extremum  $\{M_i = 0\}_{i=1}^{\mathcal{N}}$  and represent the result by Feynman graphs whose edges are ribbons which cannot be twisted. In the expansion of the action around this saddle, each factor  $\operatorname{Tr} M_i^k$  is represented by a k-valent vertex of color i whose legs are ribbons. This peculiarity of the edges puts an ordering on the legs around every vertex. One can thus replace every vertex by a face (or polygon) whose number of edges equals the valence of the initial vertex. Then, each graph is dual to a map, or discrete surface.

Hence, the partition function of the matrix model is defined as the generating function of closed surfaces made of polygons colored with  $\mathcal{N}$  different colors and glued together following the prescriptions:

- the polygons of color k have at most  $d_k + 1$  edges;
- two polygons of respective colors i and j can be glued by their edges only.

In this representation, the partition function reads

$$\mathcal{Z} = \sum_{v=0}^{\infty} T^{v} \sum_{g=0}^{\infty} N^{2-2g} \sum_{S \in \mathbb{M}_{g}(v)} \frac{1}{\# \text{Aut}(S)} \prod_{i,k} \left( g_{i}^{(k)} \right)^{n_{i,k}(S)} \prod_{i,j} \left( \left[ C^{-1} \right]_{i,j} \right)^{\widetilde{n}_{i,j}(S)}$$
(2-5)

where  $\mathbb{M}_g(v)$  is the set of orientable genus g maps with v vertices,  $\mathrm{Aut}(S)$  is the number of automorphism of S,  $n_{i,k}(S)$  the number of faces of color k with i edges in S,  $\widetilde{n}_{i,j}(S)$  the number of edges between a face of color i and a face of color j in S and  $[C^{-1}]$  the inverse of the matrix of the bilinear terms of the action

$$C := \begin{pmatrix} g_2^{(1)} & -c_{1,2} & & & & 0 \\ -c_{1,2} & g_2^{(2)} & -c_{2,3} & & & & & \\ & -c_{2,3} & g_2^{(3)} & -c_{3,4} & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -c_{\mathcal{N}-2,\mathcal{N}-1} & g_2^{(\mathcal{N}-1)} & -c_{\mathcal{N}-1,\mathcal{N}} \\ 0 & & & & -c_{\mathcal{N}-1,\mathcal{N}} & g_2^{(\mathcal{N})} \end{pmatrix}.$$
 (2-6)

The free energy

$$\mathcal{F} := \ln \mathcal{Z} \tag{2-7}$$

is the generating function of the connected surfaces contributing to  $\mathcal{Z}$ .

For any function  $f:(H_N)^{\mathcal{N}}\to\mathbb{C}$ , one defines a corresponding correlation function by

$$\langle f(M_1, \dots, M_{\mathcal{N}}) \rangle := \frac{1}{\mathcal{Z}} \int_{(H_N)^{\mathcal{N}}} f(M_1, \dots, M_{\mathcal{N}}) \, d\mu(M_1, \dots, M_{\mathcal{N}}). \tag{2-8}$$

The correlation functions of this model can also be interpreted as the generating functions of open surfaces built with the same prescription and the same measure as the closed ones, i.e. surfaces with

 $<sup>^{2}</sup>$ This formal integral is defined as a formal series in T and does not need to converge. More precisions about this definition can be found in [14]

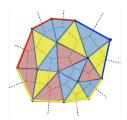


Figure 1: Example of map contributing to  $\langle \operatorname{Tr} M_1^4 M_2^3 M_3^2 \rangle$  where the type 1 condition (resp. type 2 and type 3) is depicted in yellow (resp. red and blue). It is important recalling that the boundary condition corresponds to the color of the outer face which has been removed and not the color of the inner polygons touching the boundary. The dual fat graph is depicted in dotted lines.

boundaries and prescribed boundary conditions on it. Let us briefly recall how to define generating functions of such surfaces. Consider a correlation function of the form:

$$\left\langle \operatorname{Tr} \left[ (M_{i_1})^{j_1} (M_{i_2})^{j_2} \dots (M_{i_k})^{j_k} \right] \right\rangle := \frac{1}{\mathcal{Z}} \int_{(H_N)^{\mathcal{N}}} \operatorname{Tr} \left[ (M_{i_1})^{j_1} (M_{i_2})^{j_2} \dots (M_{i_k})^{j_k} \right] d\mu (M_1, \dots, M_{\mathcal{N}})$$
(2-9)

where  $(i_1, i_2, ..., i_k) \in [1, \mathcal{N}]^k$  and  $(j_1, ..., j_k) \in (\mathbb{N}^*)^k$  are arbitrary positive integers. The factor Tr  $\left[ (M_{i_1})^{j_1} (M_{i_2})^{j_2} ... (M_{i_k})^{j_k} \right]$  in the integrand constrains all the generated graphs to contain one

vertex with  $\sum_{l=1}^{k} i_l$  colored legs organized so that the sequence of colors coincide with the sequence

of matrices in  $(M_{i_1})^{j_1} (M_{i_2})^{j_2} \dots (M_{i_k})^{j_k}$ . Thus, this correlation function is the generating function of all possible fat graphs that can be glued to this vertex, i.e. all fat graphs with one boundary constrained by the condition that it must be glued to this vertex. In the dual representation where k-valent vertices are replaced by k-gones of the same color, this function generates all surfaces with one boundary constrained by the same type of gluing condition.

In order to deal with all the possible exponents  $j_l$  at once one introduces the resolvents

$$W_{i_{1},i_{2},...,i_{k}}(x_{1},x_{2},...,x_{k}) := \frac{1}{N} \sum_{\substack{j_{1},...,j_{k}=1\\ \\ \vdots \\ N}}^{\infty} \frac{\left\langle \operatorname{Tr}\left[\left(M_{i_{1}}\right)^{j_{1}}\left(M_{i_{2}}\right)^{j_{2}}...\left(M_{i_{k}}\right)^{j_{k}}\right]\right\rangle}{x_{1}^{j_{1}+1}x_{2}^{j_{2}+1}...x_{k}^{j_{k}+1}}$$

$$:= \frac{1}{N} \left\langle \operatorname{Tr}\frac{1}{x_{1}-M_{i_{1}}}\frac{1}{x_{2}-M_{i_{2}}}...\frac{1}{x_{k}-M_{i_{k}}}\right\rangle$$

$$(2-10)$$

as formal power series in its parameters  $x_1, x_2, \ldots, x_k$ . It is very important to note that these resolvents are asymptotic series defined only when there arguments are large! Indeed, we will show that these correlation functions are multivalued functions of their variables x in the complex plane<sup>3</sup> but can be promoted to monovalued functions on an Riemann surface. However, one must always keep in mind that the physical quantities are the coefficients of the expansion of these correlation functions around a very particular point.

Using the observation of 't Hooft for QCD [26], one can also select the genus of the generated surfaces. In order to do so, one can show that these correlation functions are series in  $\frac{1}{N^2}$  and that the order of  $\frac{1}{N^2}$  is genus of the generated surface, i.e. one can write:

$$W_{i_1,i_2,\dots,i_k}(z_1,z_2,\dots,z_k) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_{i_1,i_2,\dots,i_k}^{(g)}(z_1,z_2,\dots,z_k)$$
 (2-11)

<sup>&</sup>lt;sup>3</sup>This property can be seen as the result of the finiteness of the radius of convergency of this series.

where  $W_{i_1,i_2,...,i_k}^{(g)}(z_1,z_2,...,z_k)$  is the generating function of genus g surfaces with one boundary with the boundary condition induced by  $W_{i_1,i_2,...,i_k}(z_1,z_2,...,z_k)$ . In this paper, we will be interested in the large N limit of these correlation functions, that is to say, the generating functions of discs with boundary operators of the form  $W_{i_1,i_2,...,i_k}^{(0)}(z_1,z_2,...,z_k)$ .

## 2.2 Main result: disk amplitudes

The main result of this paper is the computation of the mixed disk amplitudes with the use of a recursion relation explained in th.4.1:

$$W_{1,j,j+1,...,\mathcal{N}}(p_1, p_j, ..., p_{\mathcal{N}}) = \operatorname{Res}_{q \to p_1, p_j^{+i,j}} \mathcal{K}_j(p_1, q_j, p_j) W_{1,j+1,...,\mathcal{N}}(q, p_{j+1}, ..., p_{\mathcal{N}})$$
(2-12)

where the ingredients are explained in the following of the paper. In particular, this formula makes a heavy use of the different sheeted structures of the spectral curve of the model under study.

Another important feature of these notes are the explicit computation of these amplitudes as well as their expansion in the 3-matrix model. In section 5, we apply the spectral curve approach as well as th.4.1 to explicitly compute the number of tri-colored discs with given mixed boundary conditions.

#### 2.3 More notations

In this section, we present the notations used in the following of these notes. We used the notations of [23] as long as it was appropriate so that the reader interested in both aspects of this matrix model can go from one paper to the other without being confused.

First of all, let us remind the correlation functions to be computed:

$$W_{i_1,i_2,\dots,i_k}(x_1,x_2,\dots,x_k) := \frac{T}{N} \left\langle \operatorname{Tr} \frac{1}{x_1 - M_{i_1}} \frac{1}{x_2 - M_{i_2}} \dots \frac{1}{x_k - M_{i_k}} \right\rangle.$$
(2-13)

We also define the generating function of surfaces with two boundaries by

$$W_{\vec{i};\vec{j}}(x_1, x_2, \dots, x_k; y_1, \dots, y_l) := \left\langle \operatorname{Tr} \left( \frac{1}{x_1 - M_{i_1}} \dots \frac{1}{x_k - M_{i_k}} \right) \operatorname{Tr} \left( \frac{1}{y_1 - M_{j_1}} \dots \frac{1}{y_l - M_{j_l}} \right) \right\rangle_{c}$$
(2-14)

where the index c stands for the connected part and

$$\vec{i} = (i_1, i_2, \dots, i_k)$$
 ,  $\vec{j} = (j_1, j_2, \dots, j_l)$  (2-15)

are sequences of integers in  $[1, \mathcal{N}]$ .

Following [13], let us now define a set of polynomials  $f_{i,j}(x_i,\ldots,x_j)$  by induction:

$$\begin{cases}
f_{i,j} := 0 & \text{if } j < i - 1 \\
f_{i,i-1} := 1 \\
f_{i,i}(x_i) := \frac{V_i'(x_i)}{c_{i,i+1}} \\
c_{j,j+1} f_{i,j}(x_i, \dots, x_j) := V_j'(x_j) f_{i,j-1}(x_i, \dots, x_{j-1}) - c_{j-1,j} x_{j-1} x_j f_{i,j-2}(x_i, \dots, x_{j-2})
\end{cases} (2-16)$$

For  $1 \le k \le l \le \mathcal{N}$ , let us then define

$$P_{k,l;j_1,\dots,j_m}(x_1,\dots,x_{\mathcal{N}};y_1,\dots,y_m) := \Pr_{x_k,\dots,x_l} f_{k,l}(x_k,\dots,x_l) W_{1,\dots,\mathcal{N};j_1,j_2,\dots,j_m}(x_1,\dots,x_{\mathcal{N}};y_1,\dots,y_m)$$
(2-17)

which is a polynomial in the variables  $x_i$ , i = k, ..., l.

Given a pair of complex numbers  $(x_1, x_2)$ , we define a set of polynomial functions  $\{\hat{x}_i(x_1, x_2)\}_{i=3}^{\mathcal{N}}$  by the recursion relation

$$c_{i,i+1}\hat{x}_{i+1} := V_i'(\hat{x}_i) - c_{i-1}\hat{x}_{i-1}$$
(2-18)

for all  $i = 2, ..., \mathcal{N}$ . The case i = 2 is slightly different and we define

$$Y(x_1) := V_1'(x_1) - W_1(x_1). \tag{2-19}$$

It should be noted that the latter function has a  $1/N^2$  expansion.

Finally, it is convenient to denote any observable whose variables satisfy the constraints Eq. (2-18) by:

$$\widehat{\mathcal{O}}(x_1, x_2) := \mathcal{O}(x_1, x_2, \hat{x}_3(x_1, x_2), \hat{x}_4(x_1, x_2), \dots, \hat{x}_{\mathcal{N}}(x_1, x_2)). \tag{2-20}$$

All the observables of the model admit a topological expansion [26], i.e. they can be written as a formal power series in  $\frac{1}{N^2}$ :

$$\mathcal{O} = \sum_{g=0}^{\infty} N^{-2g} \mathcal{O}^{(g)}.$$
(2-21)

# 3 Master loop equation and spectral curve

Before going into the explicit computation of correlation functions, let us recall the algebraic structure underlying the hermitian multi-matrix model studied by Eynard in [13]. We will particularly focus on the sheeted structure of the spectral curve since it is the most important ingredient needed in the following. The interested reader can find a longer and more complete presentation of this structure in [13] or more recently in [23].

#### 3.1 Loop equations

The partition function and the correlation functions, seen as generating functions of random surfaces, are constrained by an infinite set of equations prescribing how the weight of a surface is changed when one remove one edge from it. These equations where first obtained by Tutte [27, 28] for counting triangulated surfaces and later introduced under the name of loop equation in the context of matrix models by Migdal [24]. These equations proved to be some very powerful tools since they allowed the computation of many observables in different types of hermitian matrix models through the arising of algebraic geometry.

Practically, these equations can be obtained by writing the invariance of the integral in Eq. (2-3) under particular infinitesimal change of variables:

$$Z = \int dM_{1} \dots dM_{j} \dots dM_{N}e \left[ \sum_{k=1}^{N} V_{k}(M_{k}) - \sum_{k=2}^{N} c_{k-1,k} M_{k-1} M_{k} \right] =$$

$$= \int dM_{1} \dots d\tilde{M}_{j} \dots dM_{N} J(M_{j}, \tilde{M}_{j}) \times$$

$$- \frac{N}{T} \operatorname{Tr} \left[ \sum_{k \neq j} V_{k}(M_{k}) + V_{j}(\tilde{M}_{j}) - \sum_{k \neq j, j+1} c_{k-1,k} M_{k-1} M_{k} - c_{j-1,j} M_{j-1} \tilde{M}_{j} - c_{j,j+1} \tilde{M}_{j} M_{j+1} \right]$$

$$e$$

$$(3-1)$$

where  $\tilde{M}_j := M_j + \epsilon \delta M_j$  is an infinitesimal deformation of  $M_j$ . The loop equation just corresponds to saying that the variation of the action is compensated by the Jacobian of this change of variable to first order in  $\epsilon$ .

The set of loop equations corresponding to the changes of variables

$$M_k \to M_k + \epsilon \frac{1}{x_{k+1} - M_{k+1}} \frac{1}{x_{k+2} - M_{k+2}} \dots \frac{1}{x_{N} - M_N} \frac{1}{x_1 - M_1}$$
 (3-2)

supplemented with some algebra allows to show that

**Theorem 3.1** For  $0 < k < \mathcal{N}$ , the correlation functions satisfy

$$(x_{k} - \hat{x}_{k}(x_{1})) W_{1,k,\dots,\mathcal{N}}(x_{1}, x_{k}, \dots, x_{\mathcal{N}}) = W_{1,k+1,\dots,\mathcal{N}}(x_{1}, x_{k+1}, \dots, x_{\mathcal{N}})$$

$$-P_{1,k-1}(x_{1}, \hat{x}_{2}, \dots, \hat{x}_{k-1}, x_{k}, \dots, x_{\mathcal{N}})$$

$$+ \frac{T}{N^{2}} P_{2,k-1;1}(x_{1}, \hat{x}_{2}, \dots, \hat{x}_{k-1}, x_{k}, \dots, x_{\mathcal{N}})$$

$$(3-3)$$

where

$$P_{i,k}(x_1, \dots, x_N) := \Pr_{x_i, \dots, x_k} f_{i,k}(x_1, \dots, x_{k-1}) W_{1, \dots, N}(x_1, \dots, x_N)$$
(3-4)

is a polynomial in the variables  $x_j$ ,  $j=i,\ldots,k$  and the polynomials  $f_{i,k}(x_i,\ldots,x_k)$  are defined by induction through

$$f_{i,k}(x_i,\dots,x_{k-1}) = \frac{V_k'(x_k)f_{i,k-1} - c_{k-1,k}x_{k-1}x_kf_{i,k-2}}{c_{k,k+1}}.$$
(3-5)

For  $k = \mathcal{N}$ , one has

$$(V_{\mathcal{N}}'(\hat{x}_{\mathcal{N}}) - \hat{x}_{\mathcal{N}-1}) W_1(x_1) = P_{1,\mathcal{N}}(x_1, \hat{x}_2, \dots, \hat{x}_{\mathcal{N}}) + \frac{T^2}{N^2} P_{2,\mathcal{N};1}(x_1, \hat{x}_2, \dots, \hat{x}_{\mathcal{N}}; x_1).$$
(3-6)

This theorem was proved in [13, 23]. For completeness, it is derived for the present model with the notations in hand in appendix A.

## 3.2 Leading order and master loop equation

By plugging  $Y(x) = V_1'(x) - TW_1(x)$  into Eq. (3-3), the leading order in the  $1/N^2$  expansion of this equation reads

$$\hat{E}(x_1, y(x_1)) = 0. (3-7)$$

where

$$y(x) = Y^{(0)}(x) (3-8)$$

is the leading order of the  $\frac{1}{N^2}$  expansion of Y(x) and

$$E(x_1, x_2, \dots, x_{\mathcal{N}}) = (V_1'(x_1) - x_2)(V_{\mathcal{N}}(x_{\mathcal{N}}) - x_{\mathcal{N}-1}) - P_{1,\mathcal{N}}^{(0)}(x_1, \dots, x_{\mathcal{N}})$$
(3-9)

is a polynomial in all its variables.

Consider Eq. (3-7) as an equation with unknown  $x_1$  and  $y(x_1)$ . On can see that this is a polynomial equation of degree  $\prod_{i=1}^{\mathcal{N}} d_i$  in  $x_1$  and  $1 + \prod_{i=2}^{\mathcal{N}} d_i$  in  $y(x_1)$ . Thus, it defines an algebraic curve  $\mathcal{E}(x,y) = 0$  in  $C^2$ .

## 3.3 Description of the spectral curve

Let us now consider the algebraic equation

$$\hat{E}(x,y) = 0. \tag{3-10}$$

It defines an algebraic curve in  $\mathbb{C}^2$  which encodes most of the combinatorial properties of the matrix model: it is the spectral curve associated to the model.

In this section, we study the properties the algebraic curve  $\hat{E}(x,y) = 0$  as a compact Riemann surface  $\mathcal{L}$  equipped with two meromorphic functions x and y such that

$$\forall p \in \mathcal{L}, \ \hat{E}(x(p), y(p)) = 0. \tag{3-11}$$

A point  $p \in \mathcal{L}$  is equivalent to a pair  $(x(p), y(p)) \in \mathbb{C}^2$  satisfying  $\hat{E}(x(p), y(p)) = 0$ .

Since, given a pair  $(x, y) \in \mathbb{C}^2$ , all the  $\hat{x}_i(x, y)$  are unambiguously fixed, one can see them as monovalued meromorphic functions  $z_i$  on  $\mathcal{L}$  such that:

$$\forall p \in \mathcal{L}, \ z_i(p) = \hat{x}_i(x(p), y(p)). \tag{3-12}$$

By definition, one has

$$x(p) = z_1(p)$$
 ,  $y(p) = z_2(p)$ . (3-13)

Using these functions, one can write:

$$\forall p \in \mathcal{L} \ , \ E(z_1(p), z_2(p), z_3(p), \dots, z_{\mathcal{N}}(p)) = 0$$
 (3-14)

and one can choose to fix any of the functions  $z_i(p)$  as parameter to study the curve. Depending on which parameter we chose we obtain different descriptions that we can now characterize.

#### 3.3.1 Sheeted structure and points at infinity

Let us first study this curve in terms of the "physical" variable  $x := z_1$ . Since  $\hat{E}(x, y)$  is a polynomial in y of degree  $1 + \prod_{i=2}^{\mathcal{N}} d_i$ , y(x) is a  $1 + \prod_{i=2}^{\mathcal{N}} d_i$ -valued functions of x. This means that, for a generic value

of x, there exist  $1 + \prod_{i=2}^{N} d_i$  points  $p^j$  on  $\mathcal{L}$  such that:

$$\forall i = 0, \dots, \prod_{i=2}^{N} d_i, \ z_1(p^i) = z_1(p).$$
 (3-15)

Hence  $\mathcal{L}$  can be seen as  $1 + s_1$  copies of the Riemann sphere, called  $z_1$ -sheets, glued together by cuts in such a way that  $x(p) = z_1(p)$  is injective in each sheet. This sheeted structure reflects the multi-valuedness of the resolvent  $W_1^{(0)}(x)$  as a function of x. Indeed, once the value of this correlation function is fixed,  $y(p) = z_2(p)$  is fixed and all the  $z_i(p) = \hat{x}_i(x(p), y(p))$ 's are determined. Thus, this fixes one unique point in  $\mathcal{L}$ .

How can we extract the right value of y(x) giving access to the generating functions of random surfaces? In order to distinguish the different x-sheets, one can look at the pre-images of  $\infty$  by x(p), i.e. the points  $\infty^j \in \mathcal{L}$  such that  $x(\infty^j) = \infty$ .

From its definition as an asymptotic series, one knows that the physical solution of the algebraic equation should satisfy<sup>4</sup>:

$$y(p) \sim_{x(p)\to\infty} V'(x(p)) + \frac{1}{x(p)} + O(x^{-2}(p)).$$
 (3-16)

It means that there exists at least one pre-image of infinity, denoted  $\infty^1 \in \mathcal{L}$ , such that

$$z_2(p) \sim_{p \to \infty^1} V_1'(z_1(p)) + \frac{1}{z_1(p)} + O(z_1^{-2}(p))$$
 (3-17)

and more generally

$$\forall k = 2, \dots, \mathcal{N}, \quad z_k(p) \sim_{p \to \infty^1} O(z_1^{s_k})$$
(3-18)

where

$$s_k = \prod i = 1^{k-1} d_i \tag{3-19}$$

One could symmetrically choose  $z_N$  as a variable and use the physical condition that

$$z_{\mathcal{N}-1}(p) \sim_{z_{\mathcal{N}}(p) \to \infty} V_{\mathcal{N}}'(z_{\mathcal{N}}(p)) + \frac{1}{z_{\mathcal{N}}(p)} + O(z_{\mathcal{N}}^{-2}(p)).$$
 (3-20)

This implies that there exist at least one pre-image  $\infty^{\mathcal{N}}$  of infinity by x such that:

$$z_1(p) \sim_{p \to \infty^{\mathcal{N}}} O(z_{\mathcal{N}}(p)^{r_1}) \tag{3-21}$$

where

$$r_k = \prod_i i = k + 1^{\mathcal{N}} d_i. \tag{3-22}$$

This means that  $r_1$  different  $z_1$ -sheets merge at  $\infty^{\mathcal{N}}$ . Since  $r_1 + 1$  is exactly the number of  $z_1$ -sheets, there exist no other pre-image of infinity by  $x = z_1$  and the fiber above infinity reads:  $x^{-1}(\infty) = \{\infty^1, \infty^{\mathcal{N}}\}$ .

One can summarize the structure of  $\hat{E}$  in terms of  $z_1$  as follows<sup>5</sup>:

- $\mathcal{L}$  is composed of  $1 + r_1 z_1$ -sheets;
- $z_1(p)$  has two poles in  $\mathcal{L}$ : one simple pole noted  $\infty^1$  and one pole of degree  $r_1$  noted  $\infty^{\mathcal{N}}$ ;
- $\infty^1$  belongs to only one sheet called the physical sheet since it corresponds to physical solutions. All other sheets merge at  $\infty^{\mathcal{N}}$ .

The unique  $z_1$ -sheet containing  $\infty^1$  is only referred to as the physical sheet. Indeed, in order to recover the physically meaningful quantities Tr  $M_1^k$ , one has to expand y(z) around  $z \to \infty^1$ :

$$y(z) \sim_{z \to \infty^1} V_1'(x(z)) + \sum_{k=0}^{\infty} \frac{1}{x(z)^{k+1}} \lim_{N \to \infty} \frac{1}{N} \langle \operatorname{Tr} M_1^k \rangle.$$
 (3-23)

The expansion of y(z) around  $\infty^{\mathcal{N}}$  doesn't have such a simple combinatorial interpretation.

One can also fix  $z_k(p)$  and look at the  $z_k$ -sheeted structure of  $\mathcal{L}$ . With the same arguments, one can see that:

• There exists  $r_k + s_k z_k$ -sheets where  $z_k(p)$  is injective;

 $<sup>^4</sup>$ We call this solution physical because it is the branch whose expansion in x gives rise to the generating function of surfaces.

<sup>&</sup>lt;sup>5</sup>An example is depicted in fig.2.

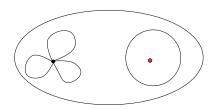


Figure 2: Example of sheeted structure of  $\mathcal{L}$  where one has represented a genus 0 spectral curve as the Riemann sphere, the red dot represents the pole  $\infty^1$  while the black dot is the other pole  $\infty^{\mathcal{N}}$ . The different sheets are represented by the connected components separated by continuous lines. In this case,  $r_1 = 4$  of them join at  $\infty^{\mathcal{N}}$  and only one contains  $\infty^1$ .

- $z_k(p)$  has two poles. One pole of degree  $s_k$  in  $\infty^1$  where  $s_k$   $z_k$ -sheets meet and one pole of degree  $r_k$  in  $\infty^N$  where the other  $r_k$   $z_k$ -sheets merge.
- One notes  $p^{+j,k}$  (resp.  $p^{-j,k}$ ) for  $j=1,\ldots,s_k$  (resp.  $j=1,\ldots,r_k$ ) the pre-images of  $z_k(p)$  belonging to the different sheets which merge at  $\infty^1$  (resp.  $\infty^N$ ).

In particular, one can see that there exists a neighborhood of  $\infty^1$  where  $z_1$  is injective in the  $z_k$ -sheets. This means that:

$$z_k(p^{+i,k}) = z_k(q^{+j,k}) \Rightarrow x(p^{+i,k}) = x(q^{+j,k})$$
 (3-24)

in some neighborhood of  $\infty^1$ .

# 4 Correlation functions

## 4.1 State of the art

Let us first recall the results of [13, 23].

In [23], a very efficient inductive method has been found for computing the whole topological expansion of the free energy as well as the correlation functions involving only the first matrix of the chain. This means that the free energy as well as the correlation functions of the type  $\left\langle \prod_{i=1}^k \operatorname{Tr} \frac{1}{x_k - M_1} \right\rangle$  are known to any order in the large N expansion.

In [13], some other observable were computed. First of all, the leading order in the large N expansion of any two point correlation function of the type  $\left\langle \operatorname{Tr} \frac{1}{x_1 - M_i} \operatorname{Tr} \frac{1}{x_2 - M_j} \right\rangle_c$  was computed and expressed in terms of a fundamental one form on  $\mathcal{L} \times \mathcal{L}$  often called the Bergman kernel. More important to us is the derivation of the simplest disk amplitude with mixed boundary conditions, that is the large N limit of the correlation function  $\left\langle \operatorname{Tr} \frac{1}{x_1 - M_1} \frac{1}{x_2 - M_N} \right\rangle$ .

### 4.2 Mixed correlation functions

With this description of the algebraic curve in hand, we are ready to compute the correlation functions. Let us first promote them to mono-valued differentials  $W_{i_1,...i_k}(p_1,...,p_k)$  on  $\mathcal{L}^k$  defined by:

$$W_{i_1,\dots i_k}(p_1,\dots,p_k) = W_{i_1,\dots i_k}(z_{i_1}(p_1),\dots,z_{i_k}(p_k)) dz_{i_1}(p_1),\dots,dz_{i_k}(p_k). \tag{4-1}$$

We use the same notation for any other function considered in the previous section: the curly letters represent differentials on the spectral curve.

From now on, we will consider only the leading order of the correlation functions in the  $1/N^2$  expansion. We thus abusively denote

$$W_{i_1,i_2,\dots,i_k}(x_1,x_2,\dots,x_k) := W_{i_1,i_2,\dots,i_k}^{(0)}(x_1,x_2,\dots,x_k)$$

$$(4-2)$$

and

$$P_{k,l}(x_1, \dots, x_N) := P_{k,l}^{(0)}(x_1, \dots, x_N)$$
(4-3)

the leading orders of the observables we are studying. Let us also remind that

$$y(x_1) := Y^{(0)}(x_1) = V_1'(x_1) - W_1^{(0)}(x_1).$$
(4-4)

In terms of differentials on the spectral curve, to leading order in the  $1/N^2$  expansion, the loop equation Eq. (3-3) can then be written:

$$(z_{j}(p_{j}) - z_{j}(p_{1}))\mathcal{W}_{1,j,j+1,\dots,\mathcal{N}}(p_{1}, p_{j}, \dots, p_{\mathcal{N}}) = \mathcal{W}_{1,j+1,\dots,\mathcal{N}}(p_{1}, p_{j+1}, \dots, p_{\mathcal{N}})dz_{j}(p_{j}) - \mathcal{P}_{1,j-1}(p_{1}, z_{2}(p_{1}), \dots, z_{j-1}(p_{1}), p_{j}, \dots, p_{\mathcal{N}})$$

$$(4-5)$$

where

$$\frac{\mathcal{P}_{1,j-1}(p_1, z_2(p_1), \dots, z_{j-1}(p_1), p_j, \dots, p_{\mathcal{N}})}{\prod_{i=j}^{\mathcal{N}} dz_i(p_i)} = P_{1,j-1}(z_1(p_1), \dots, z_{j-1}(p_1), z_j(p_j), \dots, z_{\mathcal{N}}(p_{\mathcal{N}})) \quad (4-6)$$

is a polynomial in  $z_1(p_1)$  of degree  $r_j - 1$ .

The main result of these notes is that one can solve this equation by induction on j. Indeed, Eq. (4-5) expresses  $W_{1,j,j+1,\ldots,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  in terms of  $W_{1,j+1,\ldots,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  provided that one knows the polynomial  $\mathcal{P}_{1,j-1}(p_1,z_2(p_1),\ldots,z_{j-1}(p_1),p_j,\ldots,p_{\mathcal{N}})$ . The polynomiality of the later actually allows one to compute it by using only Eq. (4-5) and get

#### Theorem 4.1 Recursion relation for the disk amplitudes.

The mixed disk amplitudes satisfy the recursion relation

$$W_{1,j,j+1,\dots,\mathcal{N}}(p_1, p_j, \dots, p_{\mathcal{N}}) = \operatorname{Res}_{q \to p_1, p_j^{+i,j}} \frac{W_{1,j+1,\dots,\mathcal{N}}(q, p_{j+1}, \dots, p_{\mathcal{N}}) \prod_{k=1}^{s_j} \left[ z_1(p_1) - z_1(p_j^{+k,j}) \right]}{\left[ z_1(q) - z_1(p_1) \right] \left[ z_j(p_j) - z_j(p_1) \right] \prod_{k=1}^{s_j} \left[ z_1(q) - z_1(p_j^{+k,j}) \right]}.$$
(4-7)

Since the initial value

$$W_{1,\mathcal{N}}(p_1, p_{\mathcal{N}}) = \sum_{k=1}^{\mathcal{N}} \frac{E\left(Z_1, ..., Z_k, \tilde{Z}_{k+1}, ..., \tilde{Z}_{\mathcal{N}}\right)}{\left(Z_k - \tilde{Z}_k\right) \left(Z_{k+1} - \tilde{Z}_{k+1}\right)}$$
(4-8)

$$Z_k = z_k \left( p_1^{+,1}(z_1) \right) \quad \tilde{Z}_k = z_k \left( p_{\mathcal{N}}^{-,\mathcal{N}}(z_{\mathcal{N}}) \right) \tag{4-9}$$

was computed in [13], this theorem allows to compute  $W_{1,j,j+1,...,\mathcal{N}}(p_1,p_j,...,p_{\mathcal{N}})$  for any  $j=2,...,\mathcal{N}-1$  and in particular the complete mixed correlation function:

Corollary 4.1 The disk amplitudes read, for j = 2, ..., N-1:

$$W_{1,j,j+1,\dots,\mathcal{N}}(p_1,p_j,\dots,p_{\mathcal{N}}) = \underset{q_j}{\text{Res}} \underset{q_{j+1}}{\text{Res}} \dots \underset{q_{\mathcal{N}-1}}{\text{Res}} \prod_{\alpha=j}^{\mathcal{N}-1} \mathcal{K}_{\alpha}(p_1,q_{\alpha},p_{\alpha}) W_{1,\mathcal{N}}(q_{\mathcal{N}-1},p_{\mathcal{N}})$$
(4-10)

where one defines the recursion kernel

$$\mathcal{K}_{\alpha}(p,q,r) = \frac{\prod_{k=1}^{s_{\alpha}} \left[ z_{1}(p) - z_{1}(r^{+k,\alpha}) \right]}{\left[ z_{1}(q) - z_{1}(p) \right] \left[ z_{\alpha}(r) - z_{\alpha}(p) \right] \prod_{k=1}^{s_{\alpha}} \left[ z_{1}(q) - z_{1}(r^{+k,\alpha}) \right]}$$
(4-11)

and the residue sign stands for

$$\forall i = j, \dots, N-1, \text{ Res}_{q_i} = \underset{q_i \to p_1}{\text{Res}} + \sum_{k=1}^{s_i} \underset{q_i \to p_i^{+k,i}}{\text{Res}}.$$
 (4-12)

In turn, this gives access to any disk amplitude of the form  $W_{i_1,...,i_k}$  with  $1 \le i_1 < i_2 < ... < i_k \le i_N$  by considering the leading order of the expansion the complete disk amplitude as some  $p_i \to \infty^1$ .

#### 4.3 Proof of the recursion relation

For proving th.4.1, one first needs a simple technical lemma.

**Lemma 4.1**  $(z_j(p_j)-z_j(p_1))\mathcal{W}_{1,j,j+1,\ldots,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  vanishes when  $p_1=p_j^{+i,j}$ , for  $i=1,\ldots,s_k$ .

The proof follows from the combinatorial interpretation of  $W_{1,j,j+1,...,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  when  $p_1$  lies in the physical sheet.

Indeed, for  $p_1 \to \infty^1$  (and not in any other patch of the spectral curve), the definition of  $W_{1,j,j+1,\ldots,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  reads

$$N(z_{j}(p_{j}) - z_{j}(p_{1})) W_{1,j,j+1,...,\mathcal{N}}(p_{1}, p_{j}, ..., p_{\mathcal{N}}) =$$

$$= (z_{j}(p_{j}) - z_{j}(p_{1})) \left\langle \operatorname{Tr} \frac{1}{z_{1}(p_{1}) - M_{1}} \frac{1}{z_{j}(p_{j}) - M_{j}} ... \frac{1}{z_{\mathcal{N}}(p_{\mathcal{N}}) - M_{\mathcal{N}}} \right\rangle. \quad (4-13)$$

One can thus write

$$= \left\langle \operatorname{Tr} \frac{1}{z_{1}(p_{1}) - Z_{j}(p_{1})} \mathcal{W}_{1,j,j+1,\dots,\mathcal{N}}(p_{1}, p_{j}, \dots, p_{\mathcal{N}}) \right. \\ \left\langle \operatorname{Tr} \frac{1}{z_{1}(p_{1}) - M_{1}} \frac{1}{z_{j+1}(p_{j+1}) - M_{j+1}} \dots \frac{1}{z_{\mathcal{N}}(p_{\mathcal{N}}) - M_{\mathcal{N}}} \right\rangle \\ \left. - \left\langle \operatorname{Tr} \frac{1}{z_{1}(p_{1}) - M_{1}} \frac{z_{j}(p_{1}) - M_{j}}{z_{j}(p_{j}) - M_{j}} \dots \frac{1}{z_{\mathcal{N}}(p_{\mathcal{N}}) - M_{\mathcal{N}}} \right\rangle$$

$$(4 - 14)$$

which vanishes when  $z_j(p_j) = z_j(p_1)$ , i.e.  $p_1$  belongs to the same  $z_j$ -fiber as  $p_j$ . Since this formula is valid only for  $p_1$  in the  $z_1$  physical sheet, this additional constraints implies that  $(z_j(p_j) - z_j(p_1))\mathcal{W}_{1,j,j+1,\ldots,\mathcal{N}}(p_1,p_j,\ldots,p_{\mathcal{N}})$  vanishes for  $p_1 = p_j^{+i,j}$ .  $\square$ 

#### Proof of theorem 4.1

Thus, one knows that the left hand side of Eq. (4-5), vanishes for  $p_1 = p_j^{+i,j}$ .

On the other hand, the second term of the right hand side,  $P_{1,j-1}$ , is a polynomial in  $z_1(p_1)$  of degree  $s_j - 1$ . Thanks to the vanishing of the left hand side, one knows its value in the  $d_1 \dots d_{j-1}$  points,  $z_1(p_j^{+i,j})$  for  $j = 1, \dots, s_j$ :

$$P_{1,j-1}(z_1(p_j^{+i,j}), z_2(p_j^{+i,j}), \dots, z_{j-1}(p_j^{+i,j}), z_j(p_j), \dots, z_{\mathcal{N}}(p_{\mathcal{N}})) = W_{1,j+1,\dots,\mathcal{N}}(p_j^{+i,j}, p_{j+1}, \dots, p_{\mathcal{N}}).$$
(4-15)

By Lagrange interpolation, one gets an explicit expression for this polynomial which one can plug back into Eq. (3-3) to get

$$(z_{j}(p_{j}) - z_{j}(p_{1}))\mathcal{W}_{1,j,\dots,\mathcal{N}}(p_{1}, p_{j}, \dots, p_{\mathcal{N}}) = \mathcal{W}_{1,j+1,\dots,\mathcal{N}}(p_{1}, p_{j+1}, \dots, p_{\mathcal{N}})$$

$$-\sum_{i=1}^{s_{j}} \frac{\mathcal{W}_{1,j+1,\dots,\mathcal{N}}(p_{j}^{+i,j}, p_{j+1}, \dots, p_{\mathcal{N}}) \prod_{k=1}^{s_{j}} \left[ z_{1}(p_{1}) - z_{1}(p_{j}^{+k,j}) \right]}{\left[ z_{1}(p_{1}) - z_{1}(p_{j}^{+i,j}) \right] \prod_{k \neq j} \left[ z_{1}(p_{j}^{+i,j}) - z_{1}(p_{j}^{+k,j}) \right]}.$$

$$(4-16)$$

This last equation can be written under the form of a residue formula

$$\mathcal{W}_{1,j,\dots,\mathcal{N}}(p_1, p_j, \dots, p_{\mathcal{N}}) = \operatorname{Res}_{q \to p_1, p_j^{+i,j}} \frac{\mathcal{W}_{1,j+1,\dots,\mathcal{N}}(q, p_{j+1}, \dots, p_{\mathcal{N}}) \prod_{k=1}^{s_j} \left[ z_1(p_1) - z_1(p_j^{+k,j}) \right]}{\left[ z_1(q) - z_1(p_1) \right] \left[ z_j(p_j) - z_j(p_1) \right] \prod_{k=1}^{s_j} \left[ z_1(q) - z_1(p_j^{+k,j}) \right]}.$$
(4-17)

# 5 Example of application: the 3-matrix model

In this section, we perform explicit computations in the case  $\mathcal{N}=3$  resulting in the computation of three colored discs with mixed boundary conditions.

# 5.1 Genus 0 spectral curve

For enumerating maps, the spectral curve has to be a rational curve, i.e. its genus has to be vanishing. Hence, there exists a global coordinate p such that the functions  $z_i$  are rational functions [14]

$$\forall i = 1, \dots, 3, \ z_i(p) = \sum_{k = -s_i}^{r_i} \alpha_{i,k} \ p^k$$
 (5-1)

satisfying

$$V_1'(z_1(p)) - z_2(p) \sim_{p \to \infty} \frac{T}{p \alpha_{1,1}} + O(p^{-2})$$
 (5-2)

$$V_3'(z_3(p)) - z_2(p) \sim_{p\to 0} \frac{Tp}{\alpha_{3,-1}} + O(p^2)$$
 (5-3)

$$V_2'(z_2(p)) = z_1(p) + z_3(p), (5-4)$$

where  $\alpha_{1,1}=\alpha_{3,-1}=\gamma$ . The limits of the sums describing the parametrizations are obtained from the degrees of the potentials, and are given by  $s_1=r_3=1, \, r_1=d_2d_3, \, s_2=d_1, \, r_2=d_3$  and  $s_3=d_1d_2$ . The solution branch giving rise to the spectral curve is the one where the coefficients  $\gamma$  and  $\alpha_{i,k}$  are algebraic functions of T such that  $\gamma \to 0$  as  $T \to 0$ . With this global parameterization  $\infty_1$  is located in  $p=\infty$  whereas  $\infty_3$  is p=0.

Once the spectral curve, i.e. the collection of rational functions  $z_i(p)$ , is known, one can study its different sheeted structures. The  $z_i$ -fiber over a point  $Z_i$  is given by the set of solution  $p^{\pm j,i}(Z_i)$  satisfying

$$z_i\left(p^{\pm j,i}(Z_i)\right) = Z_i. \tag{5-5}$$

Among these solutions,  $p^{+j,i}(Z) \to \infty$  and  $p^{-j,i}(Z) \to 0$  as  $Z \to \infty$ .

**Remark 5.1** In the setup of enumerative geometry, the exponent of T is the generating functions is the number of vertices in the maps considered. Thus, as long as one is interested in sufficiently small maps, the preceding equations need being solved only up to a given order in the T expansion.

One can thus compute the fibers above a point Z through its small T expansion.

# 5.2 The complete one loop function

#### 5.2.1 Loop equations and analytic solution

Since the 3-matrix model is the simplest model which carries the whole complexity of the chain of matrices, let us use it as a toy model for deriving more carefully the loop equations involved in the computation of the complete mixed correlation function.

Let us consider the following change of variables

$$M_1 \to M_1 + \epsilon \delta M_1 \tag{5-6}$$

where

$$\delta M_1 = \frac{1}{z_2 - M_2} \frac{1}{z_3 - M_3} \frac{1}{z_1 - M_1}. (5-7)$$

In order to keep the path-integral invariant under such a change, we demand that the contributions from the Jacobian of the change of variables and the contributions from the variation of the action cancel each other. To order  $\epsilon$ , we have

$$\left\langle Tr\left(\frac{\partial \delta M_1}{\partial M_1}\right)\right\rangle = -N\left(\left\langle Tr(V_1'(M_1)\delta M_1)\right\rangle - \left\langle Tr(\delta M_1 M_2)\right\rangle\right) \tag{5-8}$$

The left hand side can be written in terms of the connected correlators

$$\left\langle Tr(\delta M_1)Tr\left(\frac{1}{z_1-M_1}\right)\right\rangle = \left\langle Tr(\delta M_1)\right\rangle_c \left\langle Tr\left(\frac{1}{z_1-M_1}\right)\right\rangle_c + \left\langle Tr(\delta M_1)Tr\left(\frac{1}{z_1-M_1}\right)\right\rangle_c, \tag{5-9}$$

therefore we can write the left hand side in terms of correlators dependent on the points  $p,q,r\in\Sigma$  as

$$N^{2}\mathcal{W}_{1,2,3}(p,q,r)\mathcal{W}_{1}(p) + \mathcal{W}_{1,2,3;1}(p,q,r;p)$$
(5-10)

where

$$z_1(p) = z_1$$
 ,  $z_2(q) = z_2$  ,  $z_3(r) = z_3$ . (5-11)

On the other hand the right hand side can be expressed as

$$-N \langle Tr((V_1'(M_1) - V_1'(z_1))\delta M_1) \rangle + N^2(V'(z_1(p)) + z_2(q))\mathcal{W}_{1,2,3}(p,q,r) - N^2\mathcal{W}_{1,3}(p,r).$$
 (5-12)

By collecting the terms and considering only the order  $N^0$  terms, we have

$$(z_2(q) - z_2(p))\mathcal{W}_{1,2,3}(p,q,r) = \mathcal{W}_{1,3}(p,r) - \operatorname{Pol}_{z_1(p)} V_1'(z_1(p))\mathcal{W}_{1,2,3}(p,q,r),$$
(5-13)

where

$$W_{1,3}(p,r) = \frac{1}{Z_2 - \tilde{Z}_2} \left( \frac{E(z_1, \tilde{Z}_2, z_3)}{z_1 - \tilde{Z}_1} - \frac{E(z_1, Z_2, z_3)}{z_3 - Z_3} \right)$$
 (5-14)

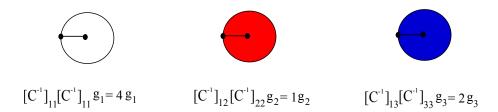


Figure 3: Maps contributing to  $\mathcal{T}_{1,0,0}^2$ . The type 1 (resp. 2 and 3) boundary condition is represented by the color white (resp. red and blue). The outer face is the marked face of type 1.

is known from [13], and where we used the notation:  $Z_2 = V_1'(z_1) - W_1(z_1)$ ,  $\tilde{Z}_2 = V_3'(z_3) - W_3(z_3)$ ,  $Z_3 = V_2'(Z_2) - z_1$  and  $\tilde{Z}_1 = V_2'(\tilde{Z}_2) - z_3$ .

This equation can be solved and its solution is given by th.4.1 under the form

$$\mathcal{W}_{1,2,3}(p_1, p_2, p_3) = -Res_{Q \to p_1, p_3^{+i,3}} \left( \frac{\mathcal{W}_{1,3}(Q, p_3) \prod_{k=1}^{s_2} (z_1(p_1) - z_1(p_2^{+k,2}))}{(z_1(Q) - z_1(p_1))(z_3(p_3) - z_3(p_1))) \prod_{k=1}^{s_2} (z_1(Q) - z_1(p_2^{+k,2}))} \right). \tag{5-15}$$

#### 5.2.2 Triangulations

In this section we present some explicit computations for the three matrix model. We assume that all of the matrices are subject to cubic potentials of the form

$$V_i(x) = \frac{1}{2}g_2^{(i)}x^2 + \frac{1}{3}g_3^{(i)}x^3.$$
 (5-16)

Furthermore, for simplicity we assume  $g_2^{(1)} = g_2^{(3)} = 1$  and  $g_2^{(2)} = 3$  in order to obtain a convenient form for the propagator matrix:

$$C^{-1} := \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}. \tag{5-17}$$

The conditions 5-2, can be used to express the  $z_i$ 's as a  $\sqrt{T}$ -expansion whose coefficients are functions on the spectral curve that also depend on the coefficients of the potentials.

$$z_i(p) = \sum_{n=1}^{\infty} z_i^{(n/2)}(g_3^{(1)}, g_3^{(2)}, g_3^{(3)}; p) T^{n/2}$$
(5-18)

For example, the first few orders of  $z_1(p)$  are given by

$$z_1^{(1/2)}\left(g_3^{(j)};p\right) = -\left(p + \frac{2}{p}\right),$$
 (5-19)

which can be recognized as the gaussian contribution,

$$z_1^{(1)}\left(g_3^{(j)};p\right) = \frac{g_3^{(2)} + 3g_3^{(3)}}{p^2} - 8g_3^{(1)} - 2g_3^{(2)} - 4g_3^{(3)},\tag{5-20}$$

$$\begin{split} z_1^{(3/2)} \left( g_3^{(j)}; p \right) &= p \left( -16 (g_3^{(1)})^2 - 7 g_3^{(1)} g_3^{(2)} - 2 (g_3^{(2)})^2 - 13 g_3^{(1)} g_3^{(3)} - 7 g_3^{(2)} g_3^{(3)} - 16 (g_3^{(3)})^2 \right) \\ &+ \frac{2 g_3^{(2)} g_3^{(3)}}{p^3} + \frac{-32 g_3^{(1)} - 6 g_3^{(1)} g_3^{(2)} - 2 g_3^{(1)} g_3^{(3)} + 16 (g_3^{(3)})^2}{p}, \end{split}$$

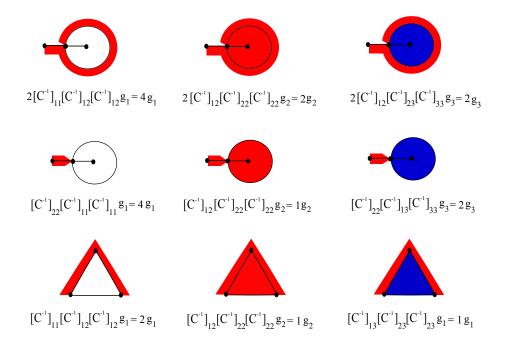


Figure 4: Maps contributing to  $\mathcal{T}_{1,2,0}^2$ . The outer face is the marked face whose boundary has type (1,2,2).

$$\begin{split} z_1^{(2)} \left(g_3^{(j)}; p\right) &=& -384 (g_3^{(1)})^3 - 192 g_3^{(2)} (g_3^{(1)})^2 - 60 g_3^{(1)} (g_3^{(2)})^2 - 12 (g_3^{(2)})^3 - 288 (g_3^{(1)})^2 g_3^{(3)} - \\ &- & 162 g_3^{(1)} g_3^{(2)} g_3^{(3)} - 48 (g_3^{(2)})^2 g_3^{(3)} - 216 g_3^{(1)} (g_3^{(3)})^2 - 120 g_3^{(2)} (g_3^{(3)})^2 + 192 (g_3^{(3)})^3 \\ &+ & \frac{1}{p^4} g_3^{(2)} (g_3^{(3)})^2 + \frac{1}{p^2} \left( 32 (g_3^{(1)})^2 g_3^{(2)} 14 g_3^{(1)} (g_3^{(2)})^2 + 4 (g_3^{(2)})^3 + 96 (g_3^{(1)})^2 g_3^{(3)} \right), \end{split}$$

and so on.

The same treatment can be given to the other meromorphic parametrizations and obtain similar expressions :

$$z_2(p) = -\sqrt{T}\left(p + \frac{1}{p}\right) + T\left(\frac{g_3^{(3)}}{p^2} + g_3^{(1)}p^2 - 4g_3^{(1)} - 2g_3^{(2)} - 4g_3^{(3)}\right) + \dots$$
 (5-21)

$$z_3(p) = -\sqrt{T}\left(2p + \frac{1}{p}\right) + T\left(\left(3g_3^{(1)} + g_3^{(2)}\right)p^2 - 8g_3^{(1)} - 2g_3^{(2)} - 4g_3^{(3)}\right) + \dots$$
 (5-22)

It is important to notice that these coefficients have to be found only once and thereafter they can be used to compute the T- expansions for the correlation functions without any further modification. Using these parametrizations and Eq. (5-15), we compute order by order the terms in the formal T-expansion for the disk amplitude with mixed boundary conditions i.e.

$$W_{1,2,3}(z_1, z_2, z_3) = \sum_{v=0}^{\infty} \sum_{\vec{n} \in \mathbb{N}^3} \frac{\mathcal{T}_{\vec{n}}^v(\vec{g}_3)}{z_1^{n_1+1} z_2^{n_2+1} z_3^{n_3+1}} T^v.$$
 (5-23)

It is important to recall that

$$\sum_{v=0}^{\infty} \mathcal{T}_{\vec{n}}^{v}(\vec{g_3}) T^{v} = \lim_{N \to \infty} \frac{1}{N} \left\langle Tr\left(M_1^{n_1} M_2^{n_2} M_3^{n_3}\right) \right\rangle. \tag{5-24}$$

Here  $\mathcal{T}_{\vec{n}}^v(\vec{g_3})$  denotes the contributions to the correlation function due to disks with v vertices and whose boundary conditions are given by an ordered sequence  $(n_1, n_2, n_3)$  of boundaries of types 1, 2 and 3 respectively. Since  $g_2^{(1)} = g_2^{(3)}$  the symmetry

$$\mathcal{T}^{v}_{(n_1, n_2, n_3)}(g_3^{(1)}, g_3^{(2)}, g_3^{(3)}) = \mathcal{T}^{v}_{(n_3, n_2, n_1)}(g_3^{(3)}, g_3^{(2)}, g_3^{(1)})$$
(5-25)

holds. Moreover, the  $\mathcal{T}_{\vec{n}}^v(\vec{g_3})$ 's are given by power expansions in the couplings of the cubic terms in the potentials, i.e

$$\mathcal{T}_{\vec{n}}^{v}(\vec{g_3}) = \sum_{\vec{m} \in \mathbb{N}^3} \mathcal{T}_{\vec{n}, \vec{m}}^{v}(g_3^{(1)})^{m_1} (g_3^{(2)})^{m_2} (g_3^{(3)})^{m_3}, \tag{5-26}$$

here  $\mathcal{T}^{v}_{\vec{n},\vec{m}}$  has to be understood as the sum of the contributions to  $\mathcal{T}^{v}_{\vec{n}}$  containing  $m_i$  triangles of type i=1,2,3. It is worth seeing this explicitly for a couple of simple examples:

$$\mathcal{T}_{(1,0,0)}^{2}(\vec{g_3}) = -\left(4g_3^{(1)} + g_3^{(2)} + 2g_3^{(3)}\right) \tag{5-27}$$

$$\mathcal{T}_{(1,2,0)}^{3}(\vec{g_3}) = -\left(10g_3^{(1)} + 4g_3^{(2)} + 7g_3^{(3)}\right)$$
 (5-28)

The explicit enumeration of maps giving rise to the coefficients of these numbers is presented in fig.3 and fig.4.

The contributions to  $W_{1,2,3}$  up to three vertices (omitting terms that can be obtained by symmetry) are displayed in the following table. For clarity we use the simplified notation  $g_3^{(1)} = g_1$ ,  $g_3^{(2)} = g_2$  and  $g_3^{(3)} = g_3$ .

| Boundaries \ Vertices | 1 | 2                    | 3  |
|-----------------------|---|----------------------|--|
| (1,0,0)               | 1 | $-4g_3 - g_2 - 2g_3$ | $-128g_1^3 -64g_1^2g_2 -20g_1g_2^2$                                  |
|                       |   |                      | $\begin{vmatrix} -4g_2^3 & -96g_1^2g_3 & -54g_1g_2g_3 \end{vmatrix}$ |
|                       |   |                      | $-16g_2^2g_3 -72g_1g_3^2 -40g_2g_3^2$                                |
|                       |   |                      | $-64g_3^{\bar{3}}$   |
| (2,0,0)               | 0 | 2                    | $64g_1^2 + 24g_1g_2 + 4g_2^2 + 40g_1g_3$                             |
|                       |   |                      | $+12g_2g_3 +16g_3^2$   |
| (3,0,0)               | 0 | 0                    | $-32g1 -7g_2 -13g_3$   |
| (4,0,0)               | 0 | 0                    | 8  |
| (0,1,0)               | 1 | $-2g_1 - g_2 - 2g_3$ | $-64g_1^3 -40g_1^2g_3^2 -16g_1g_2^2$                                 |
|                       |   |                      | $\begin{bmatrix} -4g_2^3 & -56g_1^2g_3 & -42g_1g_2g_3 \end{bmatrix}$ |
|                       |   |                      | $-16g_2^2g_3 -56g_1g_3^2 -40g_2g_3^2$                                |
|                       |   |                      | $-64g_3^2$   |
| (0,2,0)               | 0 | 1                    | $16g_1^3 + 12g_1g_2 + 4g_2^2 + 18g_1g_3$                             |
|                       |   |                      | $+12g_2g_3 +16g_3^3$   |
| (0,3,0)               | 0 | 0                    | $-7g_1 - 4g_2 - 7g_3$  |
| (0,4,0)               | 0 | 0                    | 2  |
| (1,1,0)               | 0 | 1                    | $32g_1^2 + 17g_1g_2 + 4g_2^2 + 27g_1g_3$                             |
|                       |   |                      | $+12g_2g_3 +16g_3^2$   |
| (1,2,0)               | 0 | 0                    | $-10g_1 - 4g_2 - 7g_3$   |
| (1,3,0)               | 0 | 0                    | 2  |
| (2,1,0)               | 0 | 0                    | $-16g_1 -5g_2 -9g_3$   |
| (3,1,0)               | 0 | 0                    | 4  |
| (2,2,0)               | 0 | 0                    | 1  |
| (1,0,1)               | 0 | 0                    | $-32g_1^2$ $-19g_1g_2$ $-5g_2^2$                                     |
|                       |   |                      | $-41g_1g_3 -19g_2g_3 -32g_3^2$                                       |
| (2,0,1)               | 0 | -1                   | $16g_1 + 7g_2 + 14g_3$   |
| (3,0,1)               | 0 | 0                    | -4   |
| (2,0,2)               | 0 | 0                    | 7  |
| (1,1,1)               | 0 | 0                    | $-2g_1 - g_2 - 4g_3$   |
| (2,1,1)               | 0 | 0                    | 1  |
| (1,2,1)               | 0 | 0                    | 2  |

# 6 Conclusion and prospects

In these notes, we found a recursive formula giving access to the so called complete disk amplitude in the multi-matrix model where an arbitrary number of matrices are coupled in chain. As in the two matrix model, this recursive formula only involves the spectral curve of the matrix model. Once the latter is known by the computation of the non-mixed disk amplitude, the recursive procedure presented in this paper explains how to get a disk with mixed boundary conditions by simple series expansion in terms of local variables on the spectral curve. This result proves a conjecture of Eynard [13] and is the first step towards the generalization of the topological recursion formalism for the computation of mixed amplitude in the mutli-matrix setup. From the two matrix model experience [19], it was also anticipated that the mixed correlation functions should follow from recursion relations taking into account all possible degenerations of the maps enumerated.

This work obviously calls for generalizations and the computation of arbitrary mixed amplitude to any order in the  $1/N^2$  expansion will probably be accessible by a recursive procedure similar to this one. This problem will be addressed in a forthcoming paper.

On the other hand, a direct combinatorial understanding of these recursion relations is still missing and would deserve further investigations. Can we give any meaning to the so-called non-physical sheets

of the spectral curve? Can we get a universal factorization formula for arbitrary disk amplitude as in [17]? These problems are definitely of high interest, not only for combinatorial motivations but also to investigate the generic structure underlying the mixed amplitudes in matrix models. This last point might be fundamental to understand the possible applications of these amplitudes to fields other than random matrix theories, such as topological string theories.

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# Appendix A Derivation of the loop equations

## A.1 Loop equation for the complete mixed function

Let us recall the derivation of the loop equations in the setup of [13].

Consider the change of variable  $\delta M_1 = \frac{1}{x_2 - M_2} \dots \frac{1}{x_N - M_N} \frac{1}{x_1 - M_1}$ . The associated loop equation reads

$$[c_{1,2}x_2 - Y(x_1)] W_{1,\dots,\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) - \frac{T}{N^2} W_{1,\dots,\mathcal{N};1}(x_1,\dots,x_{\mathcal{N}};x_1) =$$

$$= -P_{1,1}(x_1,\dots,x_{\mathcal{N}}) + c_{1,2}W_{1,3,4,\dots,\mathcal{N}}(x_1,x_3,\dots,x_{\mathcal{N}}).$$

$$(1-1)$$

One can extract some interesting information from this equation. It can indeed read

$$\operatorname{Res}_{x_{2} \to \infty} W_{1,\dots,\mathcal{N}}(x_{1},\dots,x_{\mathcal{N}}) V_{2}'(x_{2}) dx_{2} = -\operatorname{Res}_{x_{2} \to \infty} \frac{P_{1,1}(x_{1},\dots,x_{\mathcal{N}}) V_{2}'(x_{2}) dx_{2}}{c_{1,2}x_{2} - Y(x_{1})} + c_{1,2} \operatorname{Res}_{x_{2} \to \infty} \frac{W_{1,3,4,\dots,\mathcal{N}}(x_{1},x_{3},\dots,x_{\mathcal{N}}) V_{2}'(x_{2}) dx_{2}}{c_{1,2}x_{2} - Y(x_{1})} + \frac{T}{N^{2}} \operatorname{Res}_{x_{2} \to \infty} \frac{W_{1,\dots,\mathcal{N};1}(x_{1},\dots,x_{\mathcal{N}};x_{1}) V_{2}'(x_{2}) dx_{2}}{c_{1,2}x_{2} - Y(x_{1})}.$$

$$(1-2)$$

One can remark that, for a function f(x) analytic in  $x \to \infty$ 

$$\operatorname{Res}_{x \to \infty} \frac{f(x)dx}{x - x_1} = \operatorname{Pol}_{x_1} f(x_1)$$
(1-3)

which implies that

$$c_{1,2} \underset{x_2 \to \infty}{\operatorname{Res}} W_{1,\dots,\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) V_2'(x_2) dx_2 = c_{1,2} V_2'(\hat{x}_2) W_{1,3,4,\dots,\mathcal{N}}(x_1,x_3,\dots,x_{\mathcal{N}}) + \left[ \frac{T}{N^2} \underset{x_2}{\operatorname{Pol}} W_{1,\dots,\mathcal{N};1}(x_1,\dots,x_{\mathcal{N}};x_1) V_2'(x_2) - \underset{x_2}{\operatorname{Pol}} P_{1,1}(x_1,\dots,x_{\mathcal{N}}) V_2'(x_2) \right]_{c_{1,2} x_2 = Y(x_1)}.$$

$$(1-4)$$

Using the notations of the preceding section, one gets

$$c_{1,2} \mathop{\rm Res}_{x_2 \to \infty} W_{1,\dots,\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) V_2'(x_2) dx_2 = -P_{1,2}(x_1,\hat{x}_2,x_3,\dots,x_{\mathcal{N}}) \\ -c_{1,2} W_{3,4,\dots,\mathcal{N}}(x_3,\dots,x_{\mathcal{N}}) \\ +c_{1,2} V_2'(\hat{x}_2) W_{1,3,4,\dots,\mathcal{N}}(x_1,x_3,\dots,x_{\mathcal{N}}) \\ +\frac{T}{N^2} P_{2,2;1}(x_1,\hat{x}_2,x_3,\dots,x_{\mathcal{N}}).$$

$$(1-5)$$

#### A.2 Hierarchy of loop equations

Let us now consider the change of variable  $\delta M_2 = \frac{1}{x_3 - M_3} \frac{1}{x_4 - M_4} \dots \frac{1}{x_1 - M_1}$ . The corresponding loop equation is

$$0 = \frac{T}{N} \left\langle \operatorname{Tr} \frac{1}{x_1 - M_1} V_2'(M_2) \frac{1}{x_3 - M_3} \dots \frac{1}{x_{\mathcal{N}} - M_{\mathcal{N}}} \right\rangle + c_{1,2} W_{3,4,\dots,\mathcal{N}}(x_3,\dots,x_{\mathcal{N}}) + c_{2,3} W_{1,4,\dots,\mathcal{N}} - (c_{1,2}x_1 + c_{2,3}x_3) W_{1,3,4,\dots,\mathcal{N}}(x_1, x_3, \dots, x_{\mathcal{N}}).$$

$$(1-6)$$

The first term is actually

$$\frac{T}{N} \left\langle \operatorname{Tr} \frac{1}{x_1 - M_1} V_2'(M_2) \frac{1}{x_3 - M_3} \dots \frac{1}{x_N - M_N} \right\rangle = \operatorname{Res}_{x_2 \to \infty} W_{1, \dots, N}(x_1, \dots, x_N) V_2'(x_2) dx_2 \qquad (1-7)$$

which implies, using Eq. (1-5),

$$(x_{3} - \hat{x}_{3}(x_{1})) W_{1,3,\dots,\mathcal{N}}(x_{1}, x_{3}, \dots, x_{\mathcal{N}}) = W_{1,4,\dots,\mathcal{N}}(x_{1}, x_{4}, \dots, x_{\mathcal{N}}) - \left[ \underset{x_{1}, x_{2}}{\text{Pol}} \frac{V'_{1}(x_{1})V'_{2}(X_{2})}{c_{1,2}c_{1,3}} W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right] + \underset{x_{1}, x_{2}}{\text{Pol}} \frac{C_{1,2}x_{1}x_{2}}{c_{2,3}} W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) + \frac{T}{N^{2}} \underset{x_{2}}{\text{Pol}} \frac{V'_{2}(x_{2})}{c_{1,2}c_{2,3}} W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right]_{x_{2} = \hat{x}_{2}}$$

$$(1 - 8)$$

where

$$\hat{x}_3 := \frac{V_2'(\hat{x}_2) - c_{1,2}x_1}{c_{2,3}}. (1-9)$$

Let us now proceed by induction. Let  $0 < k < \mathcal{N}$  and assume that one has

$$(x_{j} - \hat{x}_{j}(x_{1})) W_{1,j,\dots,\mathcal{N}}(x_{1}, x_{j}, \dots, x_{\mathcal{N}}) =$$

$$= W_{1,j+1,\dots,\mathcal{N}}(x_{1}, x_{j+1}, \dots, x_{\mathcal{N}})$$

$$- \left[ \underset{x_{1},\dots,x_{j-1}}{\text{Pol}} f_{1,j-1}(x_{1}, \dots, x_{j-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right]_{\{x_{i} = \hat{x}_{i}\}_{i=2}^{j-1}}$$

$$+ \frac{T}{N^{2}} \left[ \underset{x_{2},\dots,x_{j-1}}{\text{Pol}} f_{2,j-1}(x_{2}, \dots, x_{j-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right]_{\{x_{i} = \hat{x}_{i}\}_{i=2}^{j-1}}$$

$$(1-10)$$

for  $j \leq k$ . We now show that it is also true for j = k + 1.

This property for j = k implies that

$$\operatorname{Res}_{x_{k}} V'_{k}(x_{k}) dx_{k} W_{1,k,\dots,\mathcal{N}}(x_{1}, x_{k}, \dots, x_{\mathcal{N}}) = \\
\operatorname{Res}_{x_{k}} \frac{V'_{k}(x_{k}) dx_{k} W_{1,k+1,\dots,\mathcal{N}}(x_{1}, x_{k+1}, \dots, x_{\mathcal{N}})}{(x_{k} - \hat{x}_{k}(x_{1}))} \\
- \operatorname{Res}_{x_{k}} \frac{V'_{k}(x_{k}) dx_{k} \left[\operatorname{Pol}_{x_{1},\dots,x_{k-1}} f_{1,k-1}(x_{1}, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}})\right]_{\left\{x_{i} = \hat{x}_{i}\right\}_{i=2}^{k-1}}}{(x_{k} - \hat{x}_{k}(x_{1}))} \\
+ \frac{T}{N^{2}} \operatorname{Res}_{x_{k}} \frac{V'_{k}(x_{k}) dx_{k} \left[\operatorname{Pol}_{x_{2},\dots,x_{k-1}} f_{2,k-1}(x_{2}, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}})\right]_{\left\{x_{i} = \hat{x}_{i}\right\}_{i=2}^{k-1}}}{(x_{k} - \hat{x}_{k}(x_{1}))} \\
(1-11)$$

i.e.

$$\operatorname{Res}_{x_{k}} V'_{k}(x_{k}) dx_{k} W_{1,k,\dots,\mathcal{N}}(x_{1}, x_{k}, \dots, x_{\mathcal{N}}) = V'_{k}(\hat{x}_{k}) W_{1,k+1,\dots,\mathcal{N}}(x_{1}, x_{k+1}, \dots, x_{\mathcal{N}}) \\
- \left[ \operatorname{Pol}_{x_{1},\dots,x_{k}} V'_{k}(x_{k}) f_{1,k-1}(x_{1}, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right]_{\{x_{i} = \hat{x}_{i}\}_{i=2}^{k}} \\
+ \frac{T}{N^{2}} \left[ \operatorname{Pol}_{x_{2},\dots,x_{k}} V'_{k}(x_{k}) f_{2,k-1}(x_{2}, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_{1}, \dots, x_{\mathcal{N}}) \right]_{\{x_{i} = \hat{x}_{i}\}_{i=2}^{k}}$$

$$(1-12)$$

Let us consider the induction hypothesis for j = k - 1, it gives

$$\operatorname{Res}_{x_{k-1}} dx_{k-1} x_{k-1} W_{1,k-1,\dots,\mathcal{N}}(x_1, x_{k-1}, \dots, x_{\mathcal{N}}) = 
\hat{x}_{k-1} W_{1,k,\dots,\mathcal{N}}(x_1, x_k, \dots, x_{\mathcal{N}}) 
- \left[ \operatorname{Pol}_{x_1,\dots,x_{k-1}} x_{k-1} f_{1,k-2}(x_1, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_1, \dots, x_{\mathcal{N}}) \right]_{\{x_i = \hat{x}_i\}_{i=2}^{k-1}} 
+ \frac{T}{N^2} \left[ \operatorname{Pol}_{x_2,\dots,x_{k-1}} x_{k-1} f_{2,k-2}(x_1, \dots, x_{k-1}) W_{1,\dots,\mathcal{N}}(x_1, \dots, x_{\mathcal{N}}) \right]_{\{x_i = \hat{x}_i\}_{i=2}^{k-1}} .$$

$$(1-13)$$

The loop equation corresponding to  $\delta M_k = \frac{1}{x_{k+1} - M_{k+1}} \dots \frac{1}{x_{N} - M_{N}} \frac{1}{x_1 - M_1}$  reads (for  $k < \mathcal{N}$ )

$$0 = \frac{T}{N} \left\langle \operatorname{Tr} \frac{1}{x_{1} - M_{1}} V'_{k}(M_{k}) \frac{1}{x_{k+1} - M_{k+1}} \dots \frac{1}{x_{N} - M_{N}} \right\rangle \\ -c_{k-1,k} \operatorname{Res}_{x_{k-1}, x_{k}} dx_{k} dx_{k-1} x_{k-1} W_{1,k-1, \dots, N}(x_{1}, x_{k-1}, \dots, x_{N}) \\ +c_{k,k+1} W_{1,k+2, \dots, N}(x_{1}, x_{k+2}, \dots, x_{N}) \\ -c_{k,k+1} x_{k+1} W_{1,k+1, \dots, N}(x_{1}, x_{k+1}, \dots, x_{N}).$$

$$(1-14)$$

Plugging in the expression found earlier, one gets

$$(x_{k+1} - \hat{x}_{k+1}(x_1)) W_{1,k+1,\dots,\mathcal{N}}(x_1, x_{k+1}, \dots, x_{\mathcal{N}}) =$$

$$= W_{1,k+2,\dots,\mathcal{N}}(x_1, x_{k+1}, \dots, x_{\mathcal{N}})$$

$$- \left[ \underset{x_1,\dots,x_{k-1}}{\text{Pol}} f_{1,k}(x_1,\dots,x_{k-1}) W_{1,\dots,\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) \right]_{\{x_i = \hat{x}_i\}_{i=2}^{k-1}}$$

$$+ \frac{T}{N^2} \left[ \underset{x_2,\dots,x_{k-1}}{\text{Pol}} f_{2,k}(x_2,\dots,x_{k-1}) W_{1,\dots,\mathcal{N}}(x_1,\dots,x_{\mathcal{N}}) \right]_{\{x_i = \hat{x}_i\}_{i=2}^{k-1}}$$

$$(1-15)$$

with

$$\hat{x}_{k+1} = \frac{V_k'(\hat{x}_k) - c_{k-1,k}\hat{x}_{k-1}}{c_{k,k+1}},\tag{1-16}$$

$$f_{1,k}(x_1,\dots,x_{k-1}) = \frac{V_k'(x_k)f_{1,k-1} - c_{k-1,k}x_{k-1}x_kf_{1,k-2}}{c_{k,k+1}}$$
(1-17)

and

$$f_{2,k}(x_1,\ldots,x_{k-1}) = \frac{V_k'(x_k)f_{2,k-1} - c_{k-1,k}x_{k-1}x_kf_{2,k-2}}{c_{k,k+1}}.$$
(1-18)

Together with the initial equation for  $W_{1,\mathcal{N}}(x_1,x_{\mathcal{N}})$ , this ends the proof.

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