EXTERIOR NAVIER-STOKES FLOWS FOR BOUNDED DATA

KEN ABE

ABSTRACT. We prove unique existence of mild solutions on L^{∞}_{σ} for the Navier-Stokes equations in an exterior domain in \mathbb{R}^n , $n \geq 2$, subject to the non-slip boundary condition.

1. Introduction

We consider the initial-boundary value problem of the Navier-Stokes equations in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$:

(1.1)
$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0 & \text{in} \quad \Omega \times (0, T), \\ \operatorname{div} u &= 0 & \text{in} \quad \Omega \times (0, T), \\ u &= 0 & \text{on} \quad \partial \Omega \times (0, T), \\ u &= u_0 & \text{on} \quad \Omega \times \{t = 0\}. \end{aligned}$$

There is a large literature on the solvability of the exterior problem for initial data decaying at space infinity. However, a few results are available for non-decaying data. A typical example of non-decaying flow is a stationary solution of (1.1) having a finite Dirichlet integral, called D-solution [24]. It is known that D-solutions are bounded in Ω and asymptotically constant as $|x| \to \infty$; see Remarks 1.2 (ii). In this paper, we do not impose on u_0 conditions at space infinity.

The purpose of this paper is to establish a solvability of (1.1) for merely bounded initial data. We set the solenoidal L^{∞} -space,

$$L_{\sigma}^{\infty}(\Omega) = \left\{ f \in L^{\infty}(\Omega) \,\middle|\, \int_{\Omega} f \cdot \nabla \varphi \, \mathrm{d}x = 0 \quad \text{for } \varphi \in \hat{W}^{1,1}(\Omega) \right\},\,$$

by the homogeneous Sobolev space $\hat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{loc}(\Omega) \mid \nabla \varphi \in L^1(\Omega) \}$. For exterior domains, the space L^∞_σ agrees with the space of all bounded divergence-free vector fields, whose normal trace is vanishing on $\partial\Omega$ [4]. The L^∞ -type solvability for (1.1) is recently established on $C_{0,\sigma}$ in the previous work of the author [1], where $C_{0,\sigma}$ is the L^∞ -closure of $C^\infty_{c,\sigma}$, the space of all smooth solenoidal vector fields with compact support in Ω . Since the condition $u_0 \in C_{0,\sigma}$ imposes the decay $u_0 \to 0$ as $|x| \to \infty$, we develop an existence theorem for non-decaying space L^∞_σ , which in particular includes asymptotically constant vector fields. Moreover, the space L^∞_σ includes vector fields rotating at space infinity; see Remarks 1.2 (iv). When Ω is the whole space [16] or a half space [33], [7], the existence of mild solutions of (1.1) on L^∞_σ is proved by explicit formulas of the Stokes semigroup. In

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this paper, we prove unique existence of mild solutions on L^{∞}_{σ} for exterior domains based on L^{∞} -estimates of the Stokes semigroup [4], [2].

To state a result, let S(t) denote the Stokes semigroup. It is proved in [4] that S(t) is an analytic semigroup on L^{∞}_{σ} for exterior domains of class C^3 . Let \mathbb{P} denote the Helmholtz projection. We write div $F = (\sum_{i=1}^n \partial_i F_{ij})$ for matrix-valued functions $F = (F_{ij})$. It is proved in [2] that the composition operator S(t)Pdiv satisfies an estimate of the form

(1.2)
$$\|S(t)\mathbb{P}\operatorname{div} F\|_{L^{\infty}(\Omega)} \leq \frac{C_{\alpha}}{t^{\frac{1-\alpha}{2}}} \|F\|_{L^{\infty}(\Omega)}^{1-\alpha} \|\nabla F\|_{L^{\infty}(\Omega)}^{\alpha},$$

for $F\in C_0^1\cap W^{1,2}(\Omega)$, $t\leq T_0$ and $\alpha\in(0,1)$. Here, $W^{1,2}(\Omega)$ denotes the Sobolev space and $C_0^1(\Omega)$ denotes the $W^{1,\infty}$ -closure of $C_c^\infty(\Omega)$, the space of all smooth functions with compact support in Ω . Although the projection $\mathbb P$ may not act as a bounded operator on L^∞ , the L^∞ -estimate (1.2) implies that the composition $S(t)\mathbb P$ div is uniquely extendable to a bounded operator from C_0^1 to $C_{0,\sigma}$. Note that $F\in C_0^1$ imposes a decay condition at space infinity. Thus the extension to C_0^1 is not sufficient for studying non-decaying solutions. In this paper, we prove that the composition $S(t)\mathbb P$ div is uniquely extendable to a bounded operator $\overline{S(t)\mathbb P}$ div from the non-decaying space $W_0^{1,\infty}$ to L_σ^∞ , where $W_0^{1,\infty}$ is the space of all functions in $W^{1,\infty}$ vanishing on $\partial\Omega$.

By means of the new extension, we study the integral equation on L_{σ}^{∞} of the form

(1.3)
$$u(t) = S(t)u_0 - \int_0^t \overline{S(t-s)\mathbb{P}\operatorname{div}}(uu)(s)ds.$$

Here, $uu = (u_i u_j)$ is the tensor product. We call solutions of (1.3) mild solution on L_{σ}^{∞} . Since the projection \mathbb{P} may not be bounded on L^{∞} , the extension S(t)Pdiv is not expressed by the individual operators. We thus prove that mild solutions satisfy (1.1) by using a weak form. Let $C_{c,\sigma}^{\infty}(\Omega \times [0,T])$ denote the space of all smooth solenoidal vector fields with compact support in $\Omega \times [0,T)$. Let C([0,T];X) (resp. $C_w([0,T];X)$) denote the space of all (resp. weakly-star) continuous functions from [0,T] to a Banach space X. Let $BUC_{\sigma}(\Omega)$ denote the space of all solenoidal vector fields in $BUC(\Omega)$ vanishing on $\partial\Omega$, where $BUC(\Omega)$ is the space of all bounded uniformly continuous functions in $\overline{\Omega}$. Let $[\cdot]_{\Omega}^{(\beta)}$ denote the β -th Hölder semi-norm in $\overline{\Omega}$. The main result of this paper is the following:

Theorem 1.1. Let Ω be an exterior domain with C^3 -boundary in \mathbb{R}^n , $n \geq 2$. For $u_0 \in L^\infty_\sigma$, there exist $T \geq \varepsilon/\|u_0\|_\infty^2$ and a unique mild solution $u \in C_w([0,T];L^\infty)$ such that

(1.4)
$$\int_0^T \int_{\Omega} (u \cdot (\partial_t \varphi + \Delta \varphi) + uu : \nabla \varphi) dx dt = -\int_{\Omega} u_0(x) \cdot \varphi(x, 0) dx$$

for all $\varphi \in C^{\infty}_{c,\sigma}(\Omega \times [0,T))$, with some constant $\varepsilon = \varepsilon_{\Omega}$. The solution u satisfies

(1.5)
$$\sup_{0 < t \le T} \left\{ \left\| u \right\|_{L^{\infty}(\Omega)}(t) + t^{\frac{1}{2}} \left\| \nabla u \right\|_{L^{\infty}(\Omega)}(t) + t^{\frac{1+\beta}{2}} \left[\nabla u \right]_{\Omega}^{(\beta)}(t) \right\} \le C_1 \left\| u_0 \right\|_{L^{\infty}(\Omega)},$$

(1.6)
$$\sup_{x \in \Omega} \left\{ \left[u \right]_{[\delta, T]}^{(\gamma)}(x) + \left[\nabla u \right]_{[\delta, T]}^{(\frac{\gamma}{2})}(x) \right\} \le C_2 \left\| u_0 \right\|_{L^{\infty}(\Omega)},$$

for $\beta, \gamma \in (0, 1)$ and $\delta \in (0, T)$ with the constant C_1 , independent of u_0 and T. The constant C_2 depends on γ , δ and T. If $u_0 \in BUC_{\sigma}$, u, $t^{1/2}\nabla u \in C([0, T]; BUC)$ and $t^{1/2}\nabla u$ vanishes at time zero.

Remarks 1.2. (i) (Blow-up rate) By the estimate of the existence time in Theorem 1.1, we obtain a blow-up rate of mild solutions $u \in C_w([0, T_*); L^{\infty})$ of the form

$$||u||_{L^{\infty}(\Omega)} \ge \frac{\varepsilon'}{\sqrt{T_* - t}} \quad \text{for } t < T_*,$$

with $\varepsilon' = \varepsilon^{1/2}$, where $t = T_*$ is the blow-up time. The above blow-up estimate was first proved by Leray [25] for $\Omega = \mathbb{R}^3$. See [16] for $n \geq 3$ and [33] ([27], [7]) for a half space. The statement of Theorem 1.1 is valid also for a half space and improves regularity properties of mild solutions on L_{σ}^{∞} proved in [33], [7].

(ii)(*D*-solutions) In [24], Leray proved the existence of *D*-solutions u satisfying $u - u_{\infty} \in L^{6}(\Omega)$ for $u_{\infty} \in \mathbb{R}^{3}$ in the exterior domain $\Omega \subset \mathbb{R}^{3}$. His construction is based on an approximation for $R \to \infty$ of the problem

$$-\Delta u_R + u_R \cdot \nabla u_R + \nabla p_R = 0 \quad \text{in } \Omega_R,$$

$$\text{div } u_R = 0 \quad \text{in } \Omega_R,$$

$$u_R = 0 \quad \text{on } \partial \Omega,$$

$$u_R = u_\infty \text{ on } \{|x| = R\},$$

for $\Omega_R = \Omega \cap \{|x| < R\}$ ([23, Chapter 5, Theorem 5]). See also [13, Theorem 3.2] ([14, Theorem X.4.1]) for a different construction. If the Dirichlet integral is finite, stationary solutions of (1.1) are locally bounded in $\overline{\Omega}$ (e.g., [14, Theorem X.1.1]). Moreover, *D*-solutions are bounded as $|x| \to \infty$ by $u - u_\infty \in L^6(\Omega)$. Thus, *D*-solutions are elements of L_σ^∞ for n = 3.

When n=2, more analysis is needed for information about the behavior as $|x|\to\infty$ since a finite Dirichlet integral does not imply decays at space infinity (e.g., $u=(\log |x|)^{\alpha}$ for $0<\alpha<1/2$). Leray's construction gives D-solutions also in $\Omega\subset\mathbb{R}^2$. It is proved in [18] ([19]) that Leray's solutions are bounded in $\overline{\Omega}$ and converge to some constant \overline{u}_{∞} in the sense that $\int_0^{2\pi}|u(re_r)-\overline{u}_{\infty}|\mathrm{d}\theta\to0$ as $r\to\infty$, where (r,θ) is the polar coordinate and $e_r=(\cos\theta,\sin\theta)$. Moreover, every D-solutions are bounded and asymptotically constant in the above sense [6, Theorem 12]. Thus, D-solutions are elements of L_{σ}^{∞} also for n=2. Theorem 1.1 yields a local solvability of (1.1) around D-solutions without imposing decay conditions for initial disturbance.

(iii) (Global well-posedness for n=2) It is well known that the exterior problem (1.1) for n=2 is globally well-posed for initial data having finite energy, e.g., [22]. However, global well-posedness is unknown for non-decaying data $u_0 \in L^{\infty}_{\sigma}$. For the whole space, the vorticity $\omega = \partial_1 u^2 - \partial_2 u^1$ satisfies the a priori estimate

$$\|\omega\|_{L^{\infty}(\mathbb{R}^2)} \leq \|\omega_0\|_{L^{\infty}(\mathbb{R}^2)} \quad t > 0.$$

It is proved in [17] that the Cauchy problem of (1.1) for n=2 is globally well-posed for $u_0 \in L^\infty_\sigma$ based on the local solvability result in [16]. We proved a local solvability on L^∞_σ for exterior domains. Note that global solutions exist for rotationally symmetric initial data $u_0 \in L^\infty_\sigma$; see below (iv).

(iv) (Rotating flows) An example of $u_0 \in L^{\infty}_{\sigma}$ which is not asymptotically constant is a vector field rotating at space infinity. For example, we consider the two-dimensional unit

disk Ω^c centered at the origin and a rotationally symmetric initial data $u_0 = u_0^{\theta}(r)e_{\theta}(\theta)$ for $e_{\theta}(\theta) = (-\sin\theta, \cos\theta)$. Observe that u_0 is a solenoidal vector field in Ω and a direction of u_0 varies for $\theta \in [0, 2\pi]$ and $u_0^{\theta} \in L^{\infty}(1, \infty)$. Solutions of (1.1) for u_0 are rotationally symmetric and given by

$$u = e^{t\Delta_D} u_0 \qquad p = \int_1^{|x|} \frac{|u|^2}{r} dr,$$

where Δ_D denotes the Laplace operator subject to the Dirichlet boundary condition. The solution u is bounded in $\Omega \times (0, \infty)$ and non-decaying as $|x| \to \infty$.

(v) (Associated pressure) We invoke that the associated pressure of mild solutions on L^p $(p \ge n)$ is determined by the projection operator $\mathbb{Q} = I - \mathbb{P}$ and

$$\nabla p = \mathbb{Q}\Delta u - \mathbb{Q}(u \cdot \nabla u).$$

Since the projection \mathbb{Q} may not be bounded on L^{∞} , this representation is no longer available for mild solutions on L^{∞}_{σ} . When $\Omega = \mathbb{R}^n$ or \mathbb{R}^n_+ , the projection \mathbb{Q} has explicit kernels and we are able to find associated pressure of mild solutions on L^{∞} ; see [16] for $\Omega = \mathbb{R}^n$ and [33], [27], [7] for $\Omega = \mathbb{R}^n_+$. Although explicit kernels are not available for exterior domains, we are able to find the associated pressure of mild solutions on L^{∞} . We set

$$\nabla p = \mathbb{K}W - \mathbb{Q}\mathrm{div}F$$

for $W = -(\nabla u - \nabla^T u)n_{\Omega}$ and F = uu, where n_{Ω} is the unit outward normal on $\partial\Omega$ and \mathbb{K} is a solution operator of the homogeneous Neumann problem (*harmonic-pressure operator*) [4, Remarks 4.3 (ii)]. Note that $W = -\text{curl } u \times n_{\Omega}$ for n = 3. The operators \mathbb{K} and \mathbb{Q} div act for bounded functions and the associated pressure on L^{∞} is uniquely determined by (1.7) in the sense of distribution; see Remark 3.5 for a detailed discussion.

For asymptotically constant initial data u_0 (i.e., $u_0 \to u_\infty$ as $|x| \to \infty$), local solvability of (1.1) for n=3 is proved in [29, Theorem 5.2] by means of the Oseen semigroup. In the paper, the problem (1.1) is reduced to an initial-boundary problem for decaying data by shifting u by a constant u_∞ . Our analysis is based on the L^∞ -estimates of the Stokes semigroup which yields a local-in-time solvability of (1.1) without conditions for u_0 at space infinity.

The L^{∞} -theory for the Cauchy problem of the Navier-Stokes equations is developed by Knightly [20], [21], Cannon and Knightly [8], Cannone [9] ([10]) and Giga et al. [16]. For the whole space, mild solutions on L^{∞} are smooth and satisfy (1.1) in a classical sense [16]. For a half space, mild solutions on L^{∞} are constructed in [33] (see also [27], [7]). There are a few results on solvability of the exterior problem for non-decaying data. In [15], unique existence of continuous solutions of (1.1) for $n \geq 3$ is proved for non-decaying and Hölder continuous initial data. The result is extended in [28] for merely bounded $u_0 \in L^{\infty}_{\sigma}$ and $n \geq 3$ by using L^{∞} -estimates of the Stokes semigroup [3], [4]. Note that mild solutions on L^{∞}_{σ} are not constructed without the composition operator $\overline{S(t)}$ Pdiv. We proved the unique existence of mild solutions on L^{∞}_{σ} , which in particular yields a local-in-time solvability for n = 2. The integral form (1.3) is fundamental for studying solutions of (1.1). We expect that mild solutions on L^{∞} are sufficiently smooth and satisfy (1.1) in a classical sense.

The article is organized as follows. In Section 2, we extend the composition operator $S(t)\mathbb{P}\mathrm{div}$ to a bounded operator from $W_0^{1,\infty}$ to L_σ^∞ by approximation as we did the Stokes semigroup in [4]. We extend $S(t)\mathbb{P}\mathrm{div}$ as a solution operator $F\longmapsto v(\cdot,t)$ for solutions (v,q) of the Stokes equations for $v_0=\mathbb{P}\mathrm{div}\ F$. Note that $v_0=\mathbb{P}\mathrm{div}\ F$ for $F\in W_0^{1,\infty}$ is not an element of L^∞ in general since the projection \mathbb{P} is not bounded on L^∞ . We understand $v_0=\mathbb{P}\mathrm{div}\ F$ as distribution by using the fact that $\nabla\mathbb{P}\varphi\in L^1$ for $\varphi\in C_c^\infty$ (Lemma A.1). We approximate $F\in W_0^{1,\infty}$ by a sequence $\{F_m\}\subset C_c^\infty$ locally uniformly in Ω and obtain a unique extension $S(t)\mathbb{P}\mathrm{div}: F\longmapsto v(\cdot,t)$ by a limit v of the sequence $v_m=S(t)\mathbb{P}\mathrm{div}\ F_m$.

In Section 3, we prove Theorem 1.1. We approximate initial data $u_0 \in L^{\infty}_{\sigma}$ by a sequence $\{u_{0,m}\} \subset C^{\infty}_{c,\sigma}$ satisfying $u_{0,m} \to u_0$ a.e. in Ω and $\|u_{0,m}\|_{\infty} \leq C\|u_0\|_{\infty}$. Since the property of mild solutions (1.4) may not follow from a direct iteration argument on L^{∞}_{σ} , we construct mild solutions by approximation. We apply an existence theorem on $C_{0,\sigma}$ [1] and construct a sequence of mild solutions $u_m \in C([0,T];C_{0,\sigma})$ satisfying (1.4)-(1.6) for $u_{0,m} \in C_{0,\sigma}$. We prove that u_m subsequently converges to a mild solution u for $u_0 \in L^{\infty}_{\sigma}$ locally uniformly in $\overline{\Omega} \times (0,T]$.

In Appendix A, we show that $\nabla \mathbb{P} \varphi \in L^1$ for $\varphi \in C_c^{\infty}$ by means of the layer potential.

2. An extension of the composition operator

In this section, we prove that the composition operator $S(t)\mathbb{P}\partial$ is uniquely extendable to a bounded operator from $W_0^{1,\infty}$ to L_σ^∞ . We prove unique existence of solutions of the Stokes equations for initial data $v_0 = \mathbb{P}\partial f$, $f \in W_0^{1,\infty}$, and extend the composition as a solution operator $S(t)\mathbb{P}\partial: f \longmapsto v(\cdot,t)$. In what follows, $\partial = \partial_j$ indiscriminately denotes the spatial derivatives for $j=1,\cdots,n$.

2.1. **The Stokes system.** We consider the Stokes equations,

(2.1)
$$\begin{aligned} \partial_t v - \Delta v + \nabla q &= 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} v &= 0 & \text{in } \Omega \times (0, T), \\ v &= 0 & \text{on } \partial \Omega \times (0, T), \\ v &= v_0 & \text{on } \Omega \times \{t = 0\}. \end{aligned}$$

We set the norm

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}} |\nabla v(x,t)| + t |\nabla^2 v(x,t)| + t |\partial_t v(x,t)| + t |\nabla q(x,t)|.$$

Let d(x) denote the distance from $x \in \Omega$ to $\partial \Omega$. Let $(v, \nabla q) \in C^{2+\mu,1+\frac{\mu}{2}}(\overline{\Omega} \times (0,T]) \times C^{\mu,\frac{\mu}{2}}(\overline{\Omega} \times (0,T])$, $\mu \in (0,1)$, satisfy the equations and the boundary condition of (2.1). We say that (v,q) is a solution of (2.1) for $v_0 = \mathbb{P}\partial f$, $f \in W_0^{1,\infty}(\Omega)$, if

(2.2)
$$\sup_{0 < t \le T} \left\{ t^{\gamma} \left\| N(\nu, q) \right\|_{\infty}(t) + t^{\gamma + \frac{1}{2}} \left\| d\nabla q \right\|_{\infty}(t) \right\} < \infty,$$

for some $\gamma \in [0, 1/2)$ and

(2.3)
$$\int_0^T \int_{\Omega} (v \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q \cdot \varphi) dx dt = \int_{\Omega} f \cdot \partial \mathbb{P} \varphi_0 dx$$

for all $\varphi \in C_c^{\infty}(\Omega \times [0,T))$ with $\varphi_0(x) = \varphi(x,0)$. The left-hand side is finite since $\varphi(\cdot,t)$ is supported in Ω and $\gamma < 1/2$. The right-hand side is finite since $\partial \mathbb{P} \varphi_0$ is integrable in Ω for $\varphi_0 \in C_c^{\infty}(\Omega)$ by Lemma A.1. As explained later in Remarks 2.9 (i), the operator $\mathbb{P}\partial$ is uniquely extendable for $f \in W_0^{1,\infty}$ and we are able to define $\mathbb{P}\partial f$ in the sense of distribution. The goal of this section is to prove:

Theorem 2.1. Let Ω be an exterior domain with C^3 -boundary. Let T > 0. For $v_0 = \mathbb{P}\partial f$, $f \in W_0^{1,\infty}(\Omega)$, there exists a unique solution (v,q) of (2.1) satisfying

(2.4)
$$\sup_{0 \le t \le T} \left\{ t^{\gamma} \| N(v, q) \|_{\infty}(t) + t^{\gamma + \frac{1}{2}} \| d\nabla q \|_{\infty}(t) \right\} \le C \| f \|_{\infty}^{1 - \alpha} \| f \|_{1, \infty}^{\alpha},$$

for $\alpha \in (0,1)$ with $\gamma = (1-\alpha)/2$ and some constant C, depending on α , T and Ω .

Theorem 2.1 implies the following:

Theorem 2.2. The composition operator $S(t)\mathbb{P}\partial$ is uniquely extendable to a bounded operator $\overline{S(t)\mathbb{P}\partial}$ from $W_0^{1,\infty}(\Omega)$ to $L_\sigma^\infty(\Omega)$ together with the estimate

$$\sup_{0 < t \le T} t^{\gamma + \frac{|k|}{2} + s} \left\| \partial_t^s \partial_x^k \overline{S(t)} \mathbb{P} \overline{\partial} f \right\|_{\infty} \le C \|f\|_{\infty}^{1 - \alpha} \|f\|_{1, \infty}^{\alpha},$$

for $f \in W_0^{1,\infty}(\Omega)$, $0 \le 2s + |k| \le 2$ and $\alpha \in (0,1)$ with $\gamma = (1 - \alpha)/2$.

2.2. **Hölder estimates and uniqueness.** In order to prove Theorem 2.1, we recall local Hölder estimates and a uniqueness result for the Stokes equations. In the subsequent section, we give a proof for Theorem 2.1 by approximation.

We set the Hölder semi-norm

$$[f]_{Q}^{(\mu,\frac{\mu}{2})} = \sup_{t \in (0,T]} [f]_{\Omega}^{(\mu)}(t) + \sup_{x \in \Omega} [f]_{(0,T]}^{(\frac{\mu}{2})}(x), \quad \mu \in (0,1),$$

for $Q = \Omega \times (0, T]$. We set

$$N = \sup_{\delta \leq t \leq T} \left\| N(v,q) \right\|_{L^{\infty}(\Omega)}(t)$$

for solutions (v, q) of (2.1). The following local Hölder estimate is proved in [3, Proposition 3.2 and Theorem 3.4] based on the Schauder estimates for the Stokes equations [35] ([32], [34]).

Proposition 2.3. Let Ω be an exterior domain with C^3 -boundary. (i) (Interior estimates) For $\mu \in (0,1)$, $\delta > 0$, T > 0, R > 0, there exists a constant $C = C(\mu, \delta, T, R, d)$ such that

$$\left[\nabla^2 v \right]_{O'}^{(\mu, \frac{\mu}{2})} + \left[v_t \right]_{O'}^{(\mu, \frac{\mu}{2})} + \left[\nabla q \right]_{O'}^{(\mu, \frac{\mu}{2})} \le CN$$

holds for all solutions (v, q) of (2.1) for $Q' = B_{x_0}(R) \times (2\delta, T]$ and $x_0 \in \Omega$ satisfying $\overline{B_{x_0}(R)} \subset \Omega$, where d denotes the distance from $B_{x_0}(R)$ to $\partial \Omega$.

(ii) (Estimates up to the boundary) There exists $R_0 > 0$ such that for $\mu \in (0,1)$, $\delta > 0$, T > 0 and $R \le R_0$, there exists a constant C depending on μ , δ , T, R and C^3 -regularity

of $\partial\Omega$ such that (2.6) holds for all solutions (v,q) of (2.1) for $Q' = \Omega_{x_0,R} \times (2\delta,T]$ and $\Omega_{x_0,R} = B_{x_0}(R) \cap \Omega$, $x_0 \in \partial\Omega$.

We observe the uniqueness of solutions for (2.1). The uniqueness of the Stokes equations (2.1) for $v_0 \in L^\infty_\sigma$ in an exterior domain is proved based on the uniqueness result in a half space [33] by a blow-up argument; see [4, Lemma 2.12]. In order to prove Theorem 2.1, we need a stronger uniqueness result since solutions of (2.1) for $v_0 = \mathbb{P}\partial f$, $f \in W_0^{1,\infty}$, may not be bounded at t = 0. The corresponding uniqueness result for a half space is recently proved in [2, Theorem 5.1]. We deduce the result for exterior domains by the same blow-up argument as we did in [4].

Proposition 2.4. Let Ω be an exterior domain with C^3 -boundary. Let $(v, \nabla q) \in C^{2,1}(\overline{\Omega} \times (0,T]) \times C(\overline{\Omega} \times (0,T])$ satisfy the equations and the boundary condition of (2.1), and (2.2) for some $\gamma \in [0,1/2)$. Assume that

$$\int_0^T \int_{\Omega} (v \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q \cdot \varphi) dx dt = 0,$$

for all $\varphi \in C_c^{\infty}(\Omega \times [0, T))$. Then, $v \equiv 0$ and $\nabla q \equiv 0$.

2.3. **Approximation.** We prove Theorem 2.1. We show existence of solutions for the Stokes equations (2.1) for $v_0 = \mathbb{P}\partial f$, $f \in W_0^{1,\infty}$, by approximation. We approximate $f \in W_0^{1,\infty}$ by elements of C_c^{∞} locally uniformly in $\overline{\Omega}$.

Lemma 2.5. Let Ω be an exterior domain with Lipschitz boundary. There exist constants C_1 , C_2 such that for $f \in W_0^{1,\infty}(\Omega)$ there exists a sequence of functions $\{f_m\}_{m=1}^{\infty} \subset C_c^{\infty}(\Omega)$ such that

(2.7)
$$||f_m||_{\infty} \leq C_1 ||f||_{\infty},$$

$$||\nabla f_m||_{\infty} \leq C_2 ||f||_{1,\infty},$$

$$f_m \to f \quad locally \ uniformly \ in \ \overline{\Omega} \quad as \ m \to \infty.$$

The proof of Lemma 2.5 is reduced to the whole space and bounded domains.

Proposition 2.6. The statement of Lemma 2.5 holds when $\Omega = \mathbb{R}^n$ with $C_1 = 1$.

Proof. We cutoff the function $f \in W^{1,\infty}(\mathbb{R}^n)$. Let $\theta \in C_c^{\infty}[0,\infty)$ be a cut-off function satisfying $\theta \equiv 1$ in [0,1], $\theta \equiv 0$ in $[2,\infty)$ and $0 \le \theta \le 1$. We set $\theta_m(x) = \theta(|x|/m)$ for $m \ge 1$ so that $\theta_m \equiv 1$ for $|x| \le m$ and $\theta_m \equiv 0$ for $|x| \ge 2m$. Then, $f_m = f\theta_m$ satisfies (2.7).

Proposition 2.7. Let Ω be a bounded domain with Lipschitz boundary. There exists a constant C_3 such that for $f \in W_0^{1,\infty}(\Omega)$ there exists a sequence of functions $\{f_m\}_{m=1}^{\infty} \subset C_c^{\infty}(\Omega)$ such that

(2.8)
$$\|\nabla f_m\|_{\infty} \le C_3 \|\nabla f\|_{\infty}$$

$$f_m \to f \quad uniformly \text{ in } \overline{\Omega} \quad as \ m \to \infty.$$

Proof. We begin with the case when Ω is star-shaped, i.e., $\lambda\Omega_{x_0} \subset \overline{\Omega}$ for some $x_0 \in \Omega$ and all $\lambda < 1$, where $\lambda\Omega_{x_0} = \{x_0 + \lambda(x - x_0) \mid x \in \Omega\}$. We may assume $x_0 = 0 \in \Omega$ and $\lambda\Omega \subset \overline{\Omega}$ by translation.

For $f \in W_0^{1,\infty}(\Omega)$, we set

$$f_{\lambda}(x) = \begin{cases} f(x/\lambda) & x \in \lambda\Omega, \\ 0 & x \in \Omega \setminus \overline{\lambda\Omega}. \end{cases}$$

Then, f_{λ} is in $W^{1,\infty}(\Omega)$ since f is vanishing on $\partial\Omega$. It follows that

$$\|\nabla f_{\lambda}\|_{\infty} \leq \frac{1}{\lambda} \|\nabla f\|_{\infty},$$

and $f_{\lambda} \to f$ uniformly in $\overline{\Omega}$ as $\lambda \to 1$. By a mollification of f_{λ} , we obtain a sequence $\{f_m\} \subset C_c^{\infty}(\Omega)$ satisfying (2.8) with $C_3 = 2$.

For general Ω , we take an open covering $\{D_j\}_{j=1}^N$ so that $\overline{\Omega} \subset \bigcup_{j=1}^N D_j$ and $\Omega_j = \Omega \cap D_j$ is Lipschitz and star-shaped for some $x_j \in \Omega_j$ [14, Lemma II 1.3]. We take a partition of unity $\{\xi_j\}_{j=1}^N \subset C_c^\infty(\mathbb{R}^n)$ such that $\sum_{j=1}^N \xi_j = 1$, $0 \le \xi_j \le 1$, spt $\xi_j \subset \overline{D_j}$ and set

$$f = \sum_{j=1}^{N} f_j, \quad f_j = f \xi_j.$$

Since spt $f_j \subset \overline{\Omega}_j$, $\xi_j = 0$ on ∂D_j and f = 0 on $\partial \Omega$, f_j is in $W_0^{1,\infty}(\Omega_j)$. Since Ω_j is star-shaped for some $x_j \in \Omega_j$, there exists $\{f_{j,m}\} \subset C_c^\infty(\Omega_j)$ satisfying (2.8) in Ω_j with $C_3 = 2$. We extend $f_{j,m} \in C_c^\infty(\Omega_j)$ to $\Omega \setminus \overline{\Omega_j}$ by the zero extension (still denoted by $f_{j,m}$) and set $f_m = \sum_{j=1}^N f_{j,m}$. Then, $f_m \in C_c^\infty(\Omega)$ converges to f uniformly in $\overline{\Omega}$. We estimate

$$\left\|\nabla f_m\right\|_{L^{\infty}(\Omega)} \leq \sum_{j=1}^{N} \left\|\nabla f_{j,m}\right\|_{L^{\infty}(\Omega_j)} \leq 2 \sum_{j=1}^{N} \left\|\nabla f_j\right\|_{L^{\infty}(\Omega_j)}.$$

Since $\nabla f_j = \nabla f \xi_j + f \nabla \xi_j$ and

$$||f||_{L^{\infty}(\Omega)} \le C_p ||\nabla f||_{L^{\infty}(\Omega)},$$

by the Poincaré inequality (e.g., [11, 5.8.1 Theorem 1]), we obtain

$$\left\|\nabla f_m\right\|_{L^{\infty}(\Omega)} \le C \left\|\nabla f\right\|_{L^{\infty}(\Omega)}.$$

Thus, $\{f_m\} \subset C_c^{\infty}(\Omega)$ satisfies (2.8). The proof is complete.

Proof of Lemma 2.5. The assertion follows from Propositions 2.6 and 2.7.

We recall the a priori estimate of $S(t)\mathbb{P}\partial$ for $f \in C_c^{\infty}(\Omega)$ [2, Theorem 1.2].

Proposition 2.8. There exists a constant C such that

(2.9)
$$\sup_{0 \le t \le T} t^{\gamma + \frac{|k|}{2} + s} \left\| \partial_t^s \partial_x^k S(t) \mathbb{P} \partial f \right\|_{\infty} \le C \|f\|_{\infty}^{1 - \alpha} \|\nabla f\|_{\infty}^{\alpha}$$

for $f \in C_c^{\infty}(\Omega)$, $0 \le 2s + |k| \le 2$ and $\alpha \in (0, 1)$ with $\gamma = (1 - \alpha)/2$.

Proof of Theorem 2.1. For $f \in W_0^{1,\infty}$, we take a sequence $\{f_m\} \subset C_c^{\infty}$ satisfying (2.7). For $v_{0,m} = \mathbb{P}\partial f_m$, there exists a solution of the Stokes equations (v_m, q_m) satisfying

$$\int_0^T \int_{\Omega} (v_m \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q_m \cdot \varphi) dx dt = \int_{\Omega} f_m \cdot \partial \mathbb{P} \varphi_0 dx,$$

for $\varphi \in C_c^{\infty}(\Omega \times [0, T))$. By (2.9) and (2.7), there exists a constant C independent of $m \ge 1$ such that

$$\sup_{0 \le t \le T} \left\{ t^{\gamma} \| N(v_m, q_m) \|_{\infty}(t) + t^{\gamma + \frac{1}{2}} \| d\nabla q_m \|_{\infty}(t) \right\} \le C \| f \|_{\infty}^{1 - \alpha} \| f \|_{1, \infty}^{\alpha}.$$

We apply Proposition 2.3 and observe that there exists a subsequence of (v_m, q_m) such that (v_m, q_m) converges to a limit (v, q) locally uniformly in $\overline{\Omega} \times (0, T]$ together with ∇v_m , $\nabla^2 v_m$, $\partial_t v_m$ and ∇q_m . By sending $m \to \infty$, we obtain a solution (v, q) of (2.1) for $v_0 = \mathbb{P}\partial f$. By Proposition 2.4, the limit (v, q) is unique. We proved the unique existence of solutions of (2.1) for $v_0 = \mathbb{P}\partial f$ and $f \in W_0^{1,\infty}$ satisfying (2.4). The proof is now complete.

Remarks 2.9. (i) By the approximation (2.7) we are able to extend the operator $\mathbb{P}\partial$ for $f \in W_0^{1,\infty}$. We take a sequence $\{f_m\} \subset C_c^{\infty}$ satisfying (2.7) by Lemma 2.5 and observe that $v_{0,m} = \mathbb{P}\partial f_m$ satisfies

$$(v_{0,m},\varphi) = -(f_m,\partial \mathbb{P}\varphi) \quad \text{for } \varphi \in C_c^{\infty}(\Omega).$$

Since $\partial \mathbb{P} \varphi \in L^1(\Omega)$ by Lemma A.1, the sequence $\{v_{0,m}\}$ converges to a limit v_0 in the distributional sense and the limit v_0 satisfies $(v_0, \varphi) = -(f, \partial \mathbb{P} \varphi)$. Since the limit v_0 is unique, the operator $\mathbb{P} \partial$ is uniquely extendable for $f \in W_0^{1,\infty}$.

(ii) We recall that for a sequence $\{v_{0,m}\}_{m=1}^{\infty} \subset L_{\sigma}^{\infty}$ satisfying

$$||v_{0,m}||_{\infty} \le K_1,$$

$$v_{0,m} \to v_0 \quad \text{a.e. } \Omega,$$

with some constant K_1 , there exists a subsequence such that $S(t)v_{0,m}$ converges to $S(t)v_0$ locally uniformly in $\overline{\Omega} \times (0, \infty)$ [4]. From the proof of Theorem 2.1, we observe that for a sequence $\{f_m\} \subset W_0^{1,\infty}$ satisfying

$$\|f_m\|_{1,\infty} \leq K_2,$$

$$f_m \to f$$
 locally uniformly in $\overline{\Omega}$,

 $\overline{S(t)\mathbb{P}\partial}f_m$ subsequently converges to $\overline{S(t)\mathbb{P}\partial}f$ locally uniformly in $\overline{\Omega}\times(0,\infty)$.

(iii) The extension $\overline{S(t)}\mathbb{P}\overline{\partial}$ satisfies the property

$$S(t)\overline{S(s)\mathbb{P}\partial}f = \overline{S(t+s)\mathbb{P}\partial}f$$

for $t \ge 0$, s > 0 and $f \in W_0^{1,\infty}$. In fact, this property holds for $f_m \in C_c^{\infty}$ satisfying (2.7). By choosing a subsequence, $v_m(\cdot,t) = S(t)\mathbb{P}\partial f_m$ converges to $v(\cdot,t) = S(t)\mathbb{P}\partial f$ locally uniformly in $\overline{\Omega} \times (0,\infty)$ as in the proof of Theorem 2.1. For fixed s > 0, sending $m \to \infty$ implies

$$S(t)S(s)\mathbb{P}\partial f_m = S(t)v_m(s) \to S(t)v(s)$$

= $S(t)\overline{S(s)\mathbb{P}\partial}f$ locally uniformly in $\overline{\Omega} \times (0, \infty)$.

Thus the property is inherited to $S(t)\mathbb{P}\partial f$.

3. Mild solutions on L_{σ}^{∞}

We prove Theorem 1.1 by approximation. We show that a sequence of mild solutions $\{u_m\}$ subsequently converges to a limit u locally uniformly in $\overline{\Omega} \times (0, T]$ by the L^{∞} -estimates (1.5) and (1.6). Then, by an approximation argument for linear operators, we show that the limit u satisfies the integral equation (1.3). We first recall the existence of mild solutions on $C_{0,\sigma}$ [1, Theorem 1.1]

Proposition 3.1. For $u_0 \in C_{0,\sigma}$, there exist $T \geq \varepsilon_0/||u_0||_{\infty}^2$ and a unique mild solution $u \in C([0,T];C_{0,\sigma})$ satisfying (1.3)-(1.6).

We approximate $u_0 \in L^{\infty}_{\sigma}$ by elements of $C^{\infty}_{c,\sigma} \subset C_{0,\sigma}$. We take a sequence $\{u_{0,m}\}_{m=1}^{\infty} \subset C^{\infty}_{c,\sigma}(\Omega)$ satisfying

(3.1)
$$\begin{aligned} ||u_{0,m}||_{\infty} &\leq C||u_0||_{\infty} \\ u_{0,m} &\to u_0 \quad \text{a.e. in } \Omega, \end{aligned}$$

with some constant C, independent of $m \ge 1$ [4, Lemma 5.1]. We apply Proposition 3.1 and observe that there exists $T_m \ge \varepsilon_0/\|u_{0,m}\|_{\infty}^2$ and a unique mild solution $u_m \in C([0,T_m];C_{0,\sigma})$ satisfying

(3.2)
$$u_m(t) = S(t)u_{0,m} - \int_0^t \overline{S(t-s)} \mathbb{P} \operatorname{div} F_m(s) ds,$$
$$F_m = u_m u_m.$$

Since T_m is estimated from below by (3.1), we take $T \ge \varepsilon/\|u_0\|_{\infty}^2$ for $\varepsilon = \varepsilon_0 C^{-2}/2$ so that $T_m \ge T$ and $u_m \in C([0,T];C_{0,\sigma})$ for $m \ge 1$.

Proposition 3.2. There exists a subsequence such that u_m converges to a limit u locally uniformly in $\overline{\Omega} \times (0, T]$ together with ∇u_m .

Proof. It follows from (1.5), (1.6) and (3.1) that

(3.3)
$$\sup_{0 \le t \le T} \left\{ ||u_m||_{\infty}(t) + t^{\frac{1}{2}} ||\nabla u_m||_{\infty}(t) + t^{\frac{1+\beta}{2}} [|\nabla u_m||_{\Omega}^{(\beta)}(t)] \right\} \le C_1' ||u_0||_{\infty},$$

(3.4)
$$\sup_{x \in \Omega} \left\{ [u_m]_{[\delta,T]}^{(\gamma)}(x) + [\nabla u_m]_{[\delta,T]}^{(\frac{\gamma}{2})}(x) \right\} \le C_2' \|u_0\|_{\infty},$$

for $\beta, \gamma \in (0, 1)$ and $\delta \in (0, T]$ with some constants C_1' and C_2' , independent of $m \ge 1$. Since u_m and ∇u_m are uniformly bounded and equi-continuous in $\overline{\Omega} \times [\delta, T]$, the assertion follows from the Ascoli-Arzelà theorem.

Proposition 3.3. The limit $u \in C_w([0,T];L^\infty)$ is a mild solution for $u_0 \in L^\infty_\sigma$.

Proof. We observe that the limit u satisfies (1.4) by sending $m \to \infty$. The estimates (3.3) and (3.4) are inherited to u. We prove that u satisfies the integral equation (1.3). By (3.1)

and choosing a subsequence, $S(t)u_{0,m}$ converges to $S(t)u_0$ locally uniformly in $\overline{\Omega} \times (0, T]$ by Remarks 2.9 (ii). It follows from (3.3) and Proposition 3.2 that

$$||F_m||_{\infty} \le K,$$

$$||\nabla F_m||_{\infty} \le \frac{2}{s^{\frac{1}{2}}}K,$$

$$F_m \to F$$
 locally uniformly in $\overline{\Omega} \times (0, T]$ as $m \to \infty$,

for F = uu and $K = C'_1 ||u_0||_{\infty}$. By choosing a subsequence, we have

$$\overline{S(\eta)}\mathbb{P}\mathrm{div}F_m \to \overline{S(\eta)}\mathbb{P}\mathrm{div}F$$
 locally uniformly in $\overline{\Omega} \times (0,T]$,

for each $s \in (0, t)$ as in Remarks 2.9 (ii). It follows from (3.5) and (2.5) that

$$\left\| \overline{S(t-s)\mathbb{P}\operatorname{div}}F_m \right\|_{\infty} \le \frac{C}{(t-s)^{\frac{1-\alpha}{2}}} \left(1 + \frac{2}{s^{\frac{\alpha}{2}}} \right) K^2$$

for 0 < s < t and $\alpha \in (0, 1)$. By the dominated convergence theorem, we have

$$\int_0^t \overline{S(t-s)\mathbb{P}\mathrm{div}} F_m \mathrm{d}s \to \int_0^t \overline{S(t-s)\mathbb{P}\mathrm{div}} F \mathrm{d}s \quad \text{locally uniformly in } \overline{\Omega} \times [0,T].$$

Thus sending $m \to \infty$ implies that the limit u is a mild solution for $u_0 \in L_\sigma^\infty$. Since $S(t)u_0$ is weakly-star continuous on L^∞ at t = 0 [4], so is u.

It remains to show continuity at t = 0 for $u_0 \in BUC_{\sigma}$.

Proposition 3.4. For $u_0 \in BUC_{\sigma}$, $S(t)u_0$, $t^{1/2}\nabla S(t)u_0 \in C([0,T];BUC)$ and $t^{1/2}||\nabla S(t)u_0||_{\infty} \to 0$ as $t \to 0$.

Proof. Since S(t) is a C_0 -analytic semigroup on BUC_{σ} [4], $S(t)u_0 \in C([0,T]; BUC_{\sigma})$. Moreover, $t^{1/2}\nabla S(t)u_0$ is continuous and bounded for $t \in (0,T]$ in BUC. We show that $t^{1/2}\|\nabla S(t)u_0\|_{\infty} \to 0$ as $t \to 0$.

We divide u_0 into two terms by using the Bogovskii operator. For $u_0 \in BUC_{\sigma}$, there exists $u_0^1 \in C_{0,\sigma}$ with compact support in $\overline{\Omega}$ and $u_0^2 \in BUC_{\sigma}$ supported away from $\partial\Omega$ such that $u_0 = u_0^1 + u_0^2$ (see [4, Lemma 5.1]). Let A denote the Stokes operator and D(A) denote the domain of A in BUC_{σ} . Since S(t) is a C_0 -semigroup on BUC_{σ} , D(A) is dense in BUC_{σ} . It follows from the resolvent estimate [5, Theorem 1.3] that

(3.5)
$$\|\nabla v\|_{\infty} \le C(\|v\|_{\infty} + \|Av\|_{\infty}) \quad \text{for } v \in D(A).$$

We take an arbitrary $\epsilon > 0$. For $u_0^1 \in C_{0,\sigma}$, there exists $\{u_{0,m}^1\} \subset C_{c,\sigma}^{\infty}$ such that $||u_0^1 - u_{0,m}^1||_{\infty} \le \epsilon$ for $m \ge N_{\epsilon}^1$. We apply (3.5) and observe that

$$t^{\frac{1}{2}} \|\nabla S(t)u_{0,m}^{1}\|_{\infty} \le t^{\frac{1}{2}} C(\|S(t)u_{0,m}^{1}\|_{\infty} + \|S(t)Au_{0,m}^{1}\|_{\infty})$$

$$\le t^{\frac{1}{2}} C'(\|u_{0,m}^{1}\|_{\infty} + \|Au_{0,m}^{1}\|_{\infty}) \to 0 \quad \text{as } t \to 0.$$

We estimate

$$\begin{split} \overline{\lim}_{t \to 0} t^{\frac{1}{2}} \|\nabla S(t) u_0^1\|_{\infty} &\leq \overline{\lim}_{t \to 0} (t^{\frac{1}{2}} \|\nabla S(t) (u_0^1 - u_{0,m}^1)\|_{\infty} + t^{\frac{1}{2}} \|\nabla S(t) u_{0,m}^1\|_{\infty}) \\ &\leq C'' \epsilon. \end{split}$$

We set $u_{0,m}^2 = \eta_{\delta_m} * u_0^2$ by the mollifier η_{δ_m} so that $u_{0,m}^2$ is smooth in $\overline{\Omega}$ and $\|u_0^2 - u_{0,m}^2\|_{\infty} \le \epsilon$ for $m \ge N_{\epsilon}^2$. Since $u_{0,m}^2$ is supported away from $\partial \Omega$, we have $AS(t)u_{0,m}^2 = S(t)\Delta u_{0,m}^2$ (see [4, Proposition 6.1]). By a similar way as for u_0^1 , we estimate $\overline{\lim}_{t\to 0} t^{1/2} \|\nabla S(t)u_0^2\|_{\infty} \le C'' \epsilon$. We proved

$$\overline{\lim}_{t\to 0} t^{\frac{1}{2}} \|\nabla S(t)u_0\|_{\infty} \le 2C'' \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we proved $t^{1/2} \|\nabla S(t)u_0\|_{\infty} \to 0$ as $t \to 0$.

Proof of Theorem 1.1. The assertion follows from Propositions 3.1-3.4. The proof is now complete. \Box

Remark 3.5. We set the associated pressure of mild solutions on L^{∞} by (1.7) and the harmonic-pressure operator $\mathbb{K}: L^{\infty}_{tan}(\partial\Omega) \longrightarrow L^{\infty}_{d}(\Omega)$, which is a solution operator of the homogeneous Neumann problem,

$$\Delta q = 0 \quad \text{in } \Omega,$$

$$\frac{\partial q}{\partial n} = \text{div}_{\partial \Omega} W \quad \text{on } \partial \Omega.$$

Note that $\Delta u \cdot n = \operatorname{div}_{\partial\Omega} W$ by the divergence-free condition of u. Here, $L^{\infty}_{\operatorname{tan}}(\partial\Omega)$ denotes the space of all bounded tangential vector fields on $\partial\Omega$ and $L^{\infty}_d(\Omega)$ is the space of all functions $f \in L^1_{\operatorname{loc}}(\Omega)$ such that df is bounded in Ω for $d(x) = \inf_{y \in \partial\Omega} |x - y|, \ x \in \Omega$. Since $W = -(\nabla u - \nabla^T u)n$ is bounded on $\partial\Omega$ for mild solutions on L^{∞} , $\nabla q = \mathbb{K}W$ is defined as an element of L^{∞}_d . Moreover, $\mathbb{Q}\operatorname{div} F$ is uniquely defined for $F = uu \in W^{1,\infty}_0$ as a distribution by Remarks 2.9 (i). Thus the associated pressure is defined by (1.7) for mild solutions on L^{∞} .

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Appendix A. L^1 -estimates for the Neumann problem

In Appendix A, we prove that $\nabla \mathbb{P} \varphi \in L^1(\Omega)$, $\varphi \in C_c^{\infty}(\Omega)$, for an exterior domain Ω . We first estimate L^1 -norms of solutions for the Poisson equation in \mathbb{R}^n by using the heat semi-group. Then, we reduce the problem to the homogeneous Neumann problem and estimate solutions by a layer potential.

Lemma A.1. Let Ω be an exterior domain with C^2 -boundary in \mathbb{R}^n , $n \geq 2$. Then, $\nabla \mathbb{P} \varphi \in L^1(\Omega)$ for $\varphi \in C_c^{\infty}(\Omega)$.

We set $\nabla \Phi = \mathbb{Q}\varphi$ for $\mathbb{Q} = I - \mathbb{P}$. It suffices to show that $\nabla^2 \Phi$ is integrable in Ω . We recall that the Φ solves the Neumann problem

(A.1)
$$\Delta \Phi = \operatorname{div} \varphi \quad \text{in } \Omega,$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega.$$

See [14, Lemma III.1.2]. We observe that $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ by the elliptic regularity theory (e.g., [26, Teor. 4.1]) since φ is smooth in Ω and the boundary is C^2 . We may assume that $0 \in \Omega^c$ by translation. We take R > 0 such that $\Omega^c \subset B_0(R)$. Let E denote the fundamental solution of the Laplace equation, i.e., $E(x) = C_n |x|^{-(n-2)}$ for $n \ge 3$ and $E(x) = -(2\pi)^{-1} \log |x|$ for n = 2, where $C_n = (an(n-2))^{-1}$ and a denotes the volume of n-dimensional unit ball. We first show that the statement of Lemma A.1 is valid for $\Omega = \mathbb{R}^n$. In the sequel, we do not distinguish $\varphi \in C_c^{\infty}(\Omega)$ and its zero extension to $\mathbb{R}^n \setminus \Omega$.

Proposition A.2. Set $h = E * \varphi$ and $\Phi_1 = -div h$. Then, $\nabla^3 h$ is integrable in \mathbb{R}^n . In particular, $\nabla^2 \Phi_1 \in L^1(\mathbb{R}^n)$.

Proof. By using the heat semigroup, we transform h into

$$h = \int_0^\infty e^{t\Delta} \varphi dt.$$

We divide *h* into two terms and observe that

$$\partial_x^3 h = \int_0^1 \partial_x e^{t\Delta} \partial_x^2 \varphi dt + \int_1^\infty \partial_x^3 e^{t\Delta} \varphi dt,$$

where $\partial_x = \partial_{x_j}$ indiscriminately denotes the spatial derivatives for $j = 1, \dots, n$. We estimate

$$\begin{aligned} \|\partial_{x}^{3}h\|_{L^{1}(\mathbb{R}^{n})} & \lesssim \int_{0}^{1} \frac{1}{t^{1/2}} \|\partial_{x}^{2}\varphi\|_{L^{1}(\mathbb{R}^{n})} dt + \int_{1}^{\infty} \frac{1}{t^{3/2}} \|\varphi\|_{L^{1}(\mathbb{R}^{n})} dt \\ & \lesssim \|\partial_{x}^{2}\varphi\|_{L^{1}(\mathbb{R}^{n})} + \|\varphi\|_{L^{1}(\mathbb{R}^{n})}. \end{aligned}$$

We proved $\nabla^3 h \in L^1(\mathbb{R}^n)$.

We reduce (A.1) to the homogeneous Neumann problem

(A.2)
$$\begin{aligned} -\Delta \Phi_2 &= 0 & \text{in } \Omega, \\ \frac{\partial \Phi_2}{\partial n} &= g & \text{on } \partial \Omega. \end{aligned}$$

We write connected components of Ω by unbounded Ω_0 and bounded $\Omega_1, \dots, \Omega_N$, i.e., $\Omega = \Omega_0 \cup (\bigcup_{j=1}^N \Omega_j)$.

Proposition A.3. Set $\Phi_2 = \Phi - \Phi_1$. Then, $\Phi_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves (A.2) for $g = div_{\partial\Omega}(An)$ and $A = \nabla h - \nabla^T h$. The function $g \in C(\partial\Omega)$ satisfies

(A.3)
$$\int_{\partial\Omega_j} g d\mathcal{H} = 0 \quad for \ j = 0, 1, \dots, N.$$

Proof. We observe that $\Phi_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $-\Delta \Phi_2 = 0$ in Ω and $\partial \Phi_2/\partial n = \partial(\operatorname{div}h)/\partial n$ on $\partial \Omega$. We take an arbitrary $\rho \in C_c^{\infty}(\mathbb{R}^n)$. Since $An = (\sum_{1 \le j \le n} (\partial_j h^i - \partial_i h^j) n^j)_{1 \le i \le n}$ is a tangential vector field on $\partial \Omega$ (i.e., $An \cdot n = 0$ on $\partial \Omega$), applying integration by parts yields

$$\begin{split} \int_{\partial\Omega} g\rho \mathrm{d}\mathcal{H} &= \int_{\partial\Omega} \mathrm{div}_{\partial\Omega}(An)\rho \mathrm{d}\mathcal{H} \\ &= -\int_{\partial\Omega} (An) \cdot \nabla \rho \mathrm{d}\mathcal{H} \\ &= -\int_{\partial\Omega} (\partial_j h^i - \partial_i h^j) n^j \partial_i \rho \mathrm{d}\mathcal{H} \\ &= -\int_{\partial\Omega} \partial_j h^i n^i \partial_i \rho \mathrm{d}\mathcal{H} + \int_{\partial\Omega} \partial_j h^i n^i \partial_j \rho \mathrm{d}\mathcal{H}, \end{split}$$

where the symbol of summation is suppressed. By integration by parts, we have

$$\begin{split} \int_{\partial\Omega} \partial_{j} h^{i} n^{j} \partial_{i} \rho \mathrm{d}\mathcal{H} &= \int_{\partial\Omega} (\Delta h^{i} \partial_{i} \rho + \nabla h^{i} \cdot \nabla \partial_{i} \rho) \mathrm{d}x \\ &= \int_{\partial\Omega} (\Delta h^{i} \partial_{i} \rho - \nabla \mathrm{div} h \cdot \nabla \rho) \mathrm{d}x + \int_{\partial\Omega} \nabla h^{i} \cdot \nabla \rho n^{i} \mathrm{d}\mathcal{H}. \end{split}$$

Since $-\Delta h = \varphi$ is supported in Ω , it follows that

$$\int_{\partial\Omega} g\rho \mathcal{H} = -\int_{\Omega} (\Delta h - \nabla \operatorname{div} h) \cdot \nabla \rho dx$$

$$= -\int_{\Omega} (\Delta h \cdot \nabla \rho + \Delta \operatorname{div} h\rho) dx + \int_{\partial\Omega} \frac{\partial}{\partial n} \operatorname{div} h\rho d\mathcal{H}$$

$$= \int_{\partial\Omega} \frac{\partial}{\partial n} \operatorname{div} h\rho d\mathcal{H}.$$

Since $\partial\Omega$ is C^2 , n is extendable to a C^1 -function in a tubular neighborhood of $\partial\Omega$. Thus, g is continuous on $\partial\Omega$. Since $\rho\in C_c^\infty(\mathbb{R}^n)$ is arbitrary, we proved $\partial(\operatorname{div} h)/\partial n=g$ on $\partial\Omega$. Since g is a surface-divergence form, by integration by parts, (A.3) follows. The proof is complete.

We estimate Φ_2 by means of the layer potential.

Proposition A.4. (i) For $g \in C(\partial\Omega)$ satisfying (A.3), there exists a moment $h \in C(\partial\Omega)$ satisfying $\int_{\partial\Omega} h d\mathcal{H} = 0$ and

$$-g(x) = \frac{1}{2}h(x) + \int_{\partial\Omega} n(x) \cdot \nabla_x E(x-y)h(y) d\mathcal{H}(y) \quad x \in \partial\Omega.$$

(ii) Set the single layer potential

$$\tilde{\Phi}_2(x) = -\int_{\partial\Omega} E(x - y)h(y)d\mathcal{H}(y).$$

Then, $\tilde{\Phi}_2$ is continuous in $\overline{\Omega}$. Moreover, the normal derivative $\partial_n \tilde{\Phi}_2$ exits and is continuous on $\partial \Omega$. The function $\tilde{\Phi}_2$ satisfies (A.2) and decays as $|x| \to \infty$.

Proof. The assertion (i) is based on the Fredholm's theorem. See [12, (3.40), (3.13), (3.30)]. Since h is bounded on $\partial\Omega$, $\tilde{\Phi}_2$ is continuous in Ω . Moreover, we have

$$-\frac{\partial \tilde{\Phi}_2}{\partial n}(x) = \frac{1}{2}h(x) + \int_{\partial \Omega} n(x) \cdot \nabla E(x - y)h(y) d\mathcal{H}(y) \quad x \in \partial \Omega.$$

See [12, (3.25), (3.28)]. Thus $\tilde{\Phi}_2$ satisfies (A.2) by the assertion (i). When $n \ge 3$, $\tilde{\Phi}_2(x) \to 0$ as $|x| \to \infty$ since the fundamental solution decays as $|x| \to \infty$. Moreover, when n = 2, the average of h on $\partial\Omega$ is zero and we have

$$\tilde{\Phi}_{2}(x) = -\int_{\partial\Omega} (E(x - y) - E(x))h(y)d\mathcal{H}(y)
= \frac{1}{2\pi} \int_{\partial\Omega} \log\left(\frac{|x - y|}{|x|}\right)h(y)d\mathcal{H}(y) \to 0 \quad \text{as } |x| \to \infty.$$

The proof is complete.

Proposition A.5. The function $\tilde{\Phi}_2$ agrees with Φ_2 up to constant.

Proof. Since $\nabla \Phi_2 = \nabla \Phi - \nabla \Phi_1$ is L^p -integrable in Ω for all $p \in (1, \infty)$ (e.g., [31]), we may assume that $\Phi_2 \to 0$ as $|x| \to \infty$ by shifting Φ_2 by a constant. We set $\Psi = \Phi_2 - \tilde{\Phi}_2$ and observe that Ψ is continuous in $\overline{\Omega}$. Moreover, the normal derivative exists and is continuous on $\partial \Omega$ by Proposition A.4. The function Ψ satisfies $-\Delta \Psi = 0$ in Ω , $\partial \Psi / \partial n = 0$ on $\partial \Omega$ and $\Psi \to 0$ as $|x| \to \infty$. By the elliptic regularity theory [26], Ψ is smooth in Ω and continuously differentiable in $\overline{\Omega}$.

We shall show that $\Psi \equiv 0$. Since Ψ decays as $|x| \to \infty$, there exits a point $x_0 \in \overline{\Omega}$ such that $\sup_{x \in \Omega} \Psi(x) = \Psi(x_0)$. Suppose that $x_0 \in \partial \Omega$. Since the boundary of class C^2 satisfies the interior sphere condition, the Hopf's lemma [30, Chapter 2 Theorem 7] implies that $\partial \Psi(x_0)/\partial n > 0$. Thus $x_0 \in \Omega$. We apply the strong maximum principle [30, Chapter 2 Theorem 5] and conclude that Ψ is constant. Since Ψ decays as $|x| \to \infty$, we have $\Psi \equiv 0$. The proof is complete.

Proposition A.6. $\nabla^2 \Phi_2$ is integrable in Ω .

Proof. Since $\nabla^2\Phi_2$ is integrable near the boundary $\partial\Omega$, it suffices to show that $\nabla^2\Phi_2\in L^1(\{|x|\geq 2R\})$. Since $h\in C(\partial\Omega)$ satisfies $\int_{\partial\Omega}h\mathrm{d}\mathcal{H}=0$, we observe that

$$\begin{split} \tilde{\Phi}_2(x) &= -\int_{\partial\Omega} (E(x-y) - E(x))h(y)\mathrm{d}\mathcal{H}(y) \\ &= \int_0^1 \mathrm{d}t \int_{\partial\Omega} y \cdot (\nabla E)(x-ty)h(y)\mathrm{d}\mathcal{H}(y). \end{split}$$

Since $\Omega^c \subset B_0(R)$, for $|x| \ge 2R$ we observe that

$$|x - ty| \ge ||x| - |ty||$$

$$\ge |x| - R$$

$$\ge \frac{|x|}{2}.$$

Since $\tilde{\Phi}_2$ agrees with Φ_2 up to constant, we estimate

$$|\nabla^2 \Phi_2(x)| \lesssim \int_0^1 dt \int_{\partial \Omega} \frac{|h(y)|}{|x - ty|^{n+1}} d\mathcal{H}(y)$$

$$\lesssim \frac{1}{|x|^{n+1}} ||h||_{L^1(\partial \Omega)}.$$

Thus, $\nabla^2 \Phi_2$ is integrable in $\{|x| \ge 2R\}$. The proof is complete.

Proof of Lemma A.1. By Propositions A.2 and A.6, the assertion follows.

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(K. ABE) Department of Mathematics, Faculty of Science, Kyoto University, Kitashirakawa Oiwakecho, Sakyo, Kyoto 606-8502, Japan

E-mail address: kabe@math.kyoto-u.ac.jp