ALBANESE MAPS OF PROJECTIVE MANIFOLDS WITH NEF ANTICANONICAL BUNDLES

JUNYAN CAO

ABSTRACT. Let X be a projective manifold such that the anticanonical bundle $-K_X$ is nef. We prove that the Albanese map $p:X\to Y$ is locally trivial. In particular, p is a submersion.

Résumé. Soit X une variété projective à fibré anticanonique nef. On montre que l'application d'Albanese $p:X\to Y$ est localement triviale. En particulier, p est lisse.

1. Introduction

Let X be a compact Kähler manifold such that the anticanonical bundle $-K_X$ is nef, and let $p: X \to Y$ be the Albanese map. By the work of Q. Zhang [Zha96] and M. Pǎun [Pau12], we know that π is a fibration, i.e. π is surjective and has connected fibres. Conjecturally, the Albanese map has more regularities:

1.1. Conjecture. [DPS96] Let X be a compact Kähler manifold such that $-K_X$ is nef, and let $p: X \to Y$ be the Albanese map. Then p is locally trivial, i.e., for any small open set $U \subset Y$, $p^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fibre of p. In particular, p is a submersion.

This conjecture has been proved under the stronger assumption that T_X is nef, $-K_X$ is hermitian or the anticanonical bundle of the generic fibre is big [CP91, DPS94, DPS96, CDP14, CH17a]. For the general case, [LTZZ10] proved that p is equidimensional and has reduced fibres. In low dimension, [PS98] proved that the Albanese map is a submersion for 3-dimensional projective manifolds.

The aim of this article is to prove the conjecture under the assumption that X is projective :

1.2. Theorem. Let X be a projective manifold with nef anti-canonical bundle and let $p: X \to Y$ be the Albanese map. Then p is locally trivial, i.e., for any small open set $U \subset Y$, $p^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fibre of p.

As an application of Theorem 1.2, we can study the structure of the universal cover of projective manifolds with nef anticanonical bundles. Recalling that, for a compact Kähler manifold with hermitian semipositive anticanonical bundle X, [DPS96, CDP14] proved that, the universal covering \widetilde{X} admits a holomorphic and isometric splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_l,$$

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where Y_j are irreducible Calabi-Yau manifolds, S_k are irreducible hyperkähler manifolds, and Z_l are rationally connected manifolds with irreducible holonomy. They expect a similar splitting result for compact Kähler manifolds with nef anticanonical bundles¹:

1.3. Conjecture. Let X be a compact Kähler manifold with nef anticanonical bundle. Then the universal covering \widetilde{X} of X admits the following splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times Z,$$

where Y_j are irreducible Calabi-Yau manifolds, S_k are irreducible hyperkähler manifolds, and Z is a rationally connected manifold.

This conjecture was proved for 3-dimensional projective manifolds [BP04]. For an arbitrary compact Kähler manifold X with nef anticanonical bundle, thanks to [Cam95, Pau97, Pau12], we know that the fundamental group $\pi_1(X)$ of X is almost abelian (cf. also Proposition4.19). Together with Theorem1.2, we get the following partial result for Conjecture1.3.

1.4. Corollary. Let X be a projective manifold with nef anticanonical bundle. Then the universal cover \widetilde{X} of X admits the following splitting

$$\widetilde{X} \simeq \mathbb{C}^r \times F$$
.

Here F is a compact simply connected projective manifold with nef anticanonical bundle, and $r = \sup h^{1,0}(\widehat{X})$ where the supremum is taken over all finite étale covers $\widehat{X} \to X$.

Let us explain briefly the basic ideas of the proof of Theorem 1.2. Like many works on the study of the manifolds with nef anticanonical bundles (cf. [BC16, CH17a, CPZ03, CZ13, DPS93, Den17a, FG12, LTZZ10, Ou17, Pau12, Zha05] to quote only a few), the proof of Theorem 1.2 is based on the positivity of direct images. More precisely, in the setting of Theorem 1.2, let L be a pseudo-effective line bundle on X and let A be an ample line bundle on X. In general, we don't know about the positivity of $p_{\star}(L+A)$. However, as $-K_{X/Y}$ is nef in our case, we can obtain the positivity of $p_{\star}(L+A)$ by using the following very elegant argument in [Zha05].

Fix a possibly singular metric h_L such that $i\Theta_{h_L}(L) \geq 0$ in the sense of current and let $m \in \mathbb{N}$ large enough such that $\mathcal{J}(h_L^{\frac{1}{m}}) = \mathcal{O}_X^{-2}$. We have

(1.4.1)
$$L + A = mK_{X/Y} + (-mK_{X/Y} + A) + L.$$

As $-K_{X/Y}$ is nef, $(-mK_{X/Y} + A)$ is ample and can be equipped with a smooth metric h_1 with positive curvature. Therefore $h = h_1 + h_L$ defines a possibly singular metric on

(1.4.2)
$$\widetilde{L} := (-mK_{X/Y} + A) + L$$

¹Very recently, [CH17b] proved the conjecture for projective manifolds with nef anticanonical bundles.

 $^{^2 \}text{We}$ refer to the paragraph before Theorem 2.10 for the definition of $\mathcal{J}(h_{\overline{m}}^{\frac{1}{m}}).$

with $i\Theta_h(\widetilde{L}) \geq 0$ and $\mathcal{J}(h^{\frac{1}{m}}) = \mathcal{O}_X$. Then the powerful results on the positivity of direct images (cf.[BP08, BP10, Kaw82, Kaw98, Kol85, Fuj16, PT14, Tsu10, Vie95] among many others) can be used to study the direct image

$$p_{\star}(mK_{X/Y} + \widetilde{L}) = p_{\star}(L + A).$$

We refer to Proposition 2.13 and Corollary 2.14 for some more accurate statements.

Another main ingredient involved in the proof is inspired and very close to [DPS94, 3.D] and [CPZ03]. Recalling that, under the assumption that $-K_{X/Y}$ is p-ample, [DPS94] proved that $p_{\star}(-mK_{X/Y})$ is numerically flat for every $m \in \mathbb{N}$. Thanks to this numerically flatness, we can prove the local trivialness of the Albanese map [DPS94, CH17a]. In the situation of Theorem 1.2, as $-K_{X/Y}$ is not necessarily strictly positive along the fibres, we consider an arbitrary p-ample line bundle L on X to replace $-K_{X/Y}$. By [LTZZ10], we can assume that $p_{\star}(mL)$ is locally free for every $m \in \mathbb{N}$. By combining [DPS94, 3.D] with the positivity of direct images discussed above, we can prove that, $p_{\star}(mL')$ is numerically flat for every $m \in \mathbb{N}$, where $L' := \operatorname{rank} p_{\star}(L) \cdot L - p^{\star} \det p_{\star}(L)$. The fibration p is thus locally trivial by using a criteria proved in [DPS94, CH17a], cf. also Proposition 2.8.

Here are the main steps of the proof of Theorem1.2. Firstly, using the positivity of direct images [BP10], the diagonal method of Viehweg [Vie95, Thm 6.24] as well as the method of Zhang [Zha05], we prove in Proposition3.15 that for any p-ample line bundle A on X, if $p_{\star}(A)$ is locally free, then $rA-p^{\star}$ det $p_{\star}(A)$ is pseudo-effective, where r is the rank of $p_{\star}(A)$. Secondly, after passing to some isogeny of the abelian variety Y, we can assume that $\frac{1}{r} \det p_{\star}(A)$ is a line bundle. By using an isogeny argument [DPS94, Lemma 3.21] and [BP10], we prove that $p_{\star}(A) \otimes (-\frac{1}{r} \det p_{\star}(A))$ is numerically flat. Finally, we use the arguments in [DPS94, CH17a] to conclude that p is locally trivial.

Our paper is organized as follows. In Section 2, after recalling some basic notations and results about the positivity of line bundles and vector bundles, we will review a criteria of the locally trivialness in [CH17a]. We will also gather some results about the positivity of direct images in [BP08, BP10, PT14]. In Section 3, inspired by [DPS94, Section 3.D], we will prove two important propositions which will be the key ingredients in the proof of main theorem 1.2. Both propositions imply in particular that the Albanese map is very rigid. Finally, a complete proof of Theorem 1.2 and Corollary 1.4 is provided in Section 4.

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2. Preparation

We first recall some basic notations about the positivity of line bundles and vector bundles. We refer to [Dem12, DPS94, Laz] for more details.

- **2.5.** Definition. Let X be a projective manifold.
 - (1) We say that a holomorphic line bundle L over X is numerically effective, nef for short, if L · C ≥ 0 for every curve C ⊂ X. Thanks to [Dem12, Prop 6.10], a line bundle L to be nef is equivalent to say that for every ε > 0, there is a smooth hermitian metric h_ε on L such that iΘ_{h_ε}(L) ≥ -εω where ω is a fixed Kähler metric on X.
 - (2) We say that a holomorphic vector bundle E over X is nef, if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.
 - (3) We say that a holomorphic vector bundle E over X is numerically flat if both E and its dual E* are nef. It is easy to see that E is numerically flat if and only if c₁(det E) = 0 and E is nef.
 - (4) Let $p: X \to Y$ be a fibration between two projective manifolds and let L be a line bundle on X. We say that L is p-ample (resp. p-very ample), if there exists a line bundle L_Y on Y such that $L+p^*L_Y$ is ample (resp. very ample).

The following two theorems about the numerically flat vector bundles will be useful for us.

- **2.6. Theorem.** Let X be a compact Kähler manifold and let E be a numerically flat vector bundle on X. Then
 - (1) [DPS94, Thm 1.18] E admits a filtration

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

by vector subbundles such that the quotients E_k/E_{k-1} are hermitian flat.

- (2) [Sim92, Section 3] E is a local system and the natural Gauss-Manin connection D_E on E is compatible with the natural flat connection on the quotients E_k/E_{k-1}, i.e., D_E(E_k) ⊂ E_k ⊗ Ω¹_X and the induced connection D_E on E_k/E_{k-1} coincides with the hermitian flat connection on E_k/E_{k-1} for every k.
- **2.7.** Remark. Recently, Y. Deng [Den17b, Chapter 6] gave an elegant and short proof of Theorem2.6 (2). In the case X is a torus, we refer also to [Ver04, Lem 6.5, Cor 6.6] for a short proof of Theorem2.6 (2).

Theorem 2.6 implies the following criteria, which will be useful for the proof of Theorem 1.2.

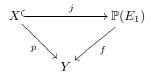
2.8. Proposition. [CH17a, Prop 4.1] Let $p: X \to Y$ be a flat fibration between two compact Kähler manifolds and let L be a p-very ample line bundle (cf. Definition 2.5 (4)). Set $E_m := p_{\star}(mL)$. If E_m is numerically flat for every $m \geq 1$, then p is locally trivial. Moreover, let $\pi: \widetilde{Y} \to Y$ be the universal cover of Y. Then $\widetilde{X} := X \times_Y \widetilde{Y}$ admits the following splitting

$$\widetilde{X} \simeq \widetilde{Y} \times F$$
,

where F is the generic fibre of p.

For the reader's convenience, we give the proof of it.

Proof. As L is p-very ample, we have a p-relative embedding and $L = j^*\mathcal{O}_{\mathbb{P}(E_1)}(1)$.



For m large enough, we have the exact sequence

$$(2.8.1) 0 \to f_{\star}(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m)) \to f_{\star}(\mathcal{O}_{\mathbb{P}(E_1)}(m)) \to p_{\star}(mL) \to 0.$$

As E_1 is numerically flat, Theorem2.6 implies that E_1 is a local system. Let D_{E_1} be the flat connection with respect to this local system. We assume that $n = \operatorname{rank} E_1$. Since \widetilde{Y} is simply connected, π^*E_1 is a trivial vector bundle on \widetilde{Y} and we can take some flat sections (with respect to D_{E_1})

$$\{e_1, e_2, \cdots, e_n\} \subset H^0(\widetilde{Y}, \pi^* E_1)$$

such that $\{e_1, e_2, \cdots, e_n\}$ generates $\pi^* E_1$.

Set $F_m := f_{\star}(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m))$. Since both $f_{\star}(\mathcal{O}_{\mathbb{P}(E_1)}(m)) = \operatorname{Sym}^m E_1$ and $p_{\star}(mL)$ are numerically flat by assumption, F_m is also numerically flat. Then F_m is a local system and let D_{F_m} be the flat connection on it. Then $\pi^{\star}F_m$ is a trivial vector bundle, and we can take some flat sections (with respect to D_{F_m})

$$\{s_1, s_2, \cdots, s_t\} \subset H^0(\widetilde{Y}, \pi^* F_m)$$

such that $\{s_1, s_2, \dots, s_t\}$ generates $\pi^* F_m$, where t is the rank of F_m .

Let $\varphi: \pi^* f_{\star}(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_1)}(m)) \to \pi^* f_{\star}(\mathcal{O}_{\mathbb{P}(E_1)}(m)) = \pi^* \operatorname{Sym}^m E_1$ be the inclusion induced by (2.8.1). Let $D_{\operatorname{Sym}^m E_1}$ be the flat connection on $\operatorname{Sym}^m \pi^* E_1$ induced by D_{E_1} . Thanks to [Cao13, Lemma 4.3.3], we know that for every $i, \varphi(s_i)$ is flat with respect the connection $D_{\operatorname{Sym}^m E_1}$. In particular, for every i, we can find constants $a_{i,\alpha}$ such that

$$\varphi(s_i) = \sum_{\alpha = (\alpha_1, \dots, \alpha_n), |\alpha| = m} a_{i,\alpha} \cdot e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}.$$

In other words, the *p*-relative embedding of \widetilde{X} in $\mathbb{P}^{n-1} \times \widetilde{Y}$:

$$\widetilde{X} \longrightarrow \mathbb{P}^{n-1} \times \widetilde{Y}$$

$$\downarrow^{p} \qquad \downarrow^{p}$$

$$\widetilde{Y} \longrightarrow \widetilde{Y}$$

is defined by the polynomials $\varphi(s_i)$ whose coefficients are independent of \widetilde{Y} . Then p is locally trivial and we have the splitting $\widetilde{X} \simeq \widetilde{Y} \times F$, where F is the generic fibre of p.

In the second part of this section, we would like to recall some results about the positivity of direct images. For more details, we refer to [BP08, BP10, Fuj16, Hor10, Kaw82, Kaw98, Kol85, PT14, Tsu10, Vie83, Vie95] to quote only a few.

To be begin with, we first recall the definition of possibly singular hermitian metrics. We refer to [Dem12] for more details. Let X be a projective manifold and let $L \to X$ be a line bundle on X endowed with a Hermitian metric h_L . We

make the convention that, unless explicitly mentioned otherwise, the metrics in this article are allowed to be singular. Let $\Omega \subset X$ be any trivialization open set for L and e_L be a basis of L over Ω . Then

$$|e_L|_{h_L}^2 = e^{-\varphi}$$

for some function $\varphi \in L^1_{loc}(\Omega)$. We say that φ is the weight of h_L . Thanks to the Lelong-Poincaré formula, we know that

(2.8.2)
$$\frac{i}{\pi}\Theta_{h_L}(L) = dd^c\varphi.$$

We now recall the definition of the multiplier ideal sheaves cf. [Dem12, 5.B] for more details. Let $m \in \mathbb{N}$. Let $\mathcal{J}(h_L^{\frac{1}{m}}) \subset \mathcal{O}_X$ be the germs of holomorphic function $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-\frac{\varphi}{m}}$ is integrable near x. It is well known that $\mathcal{J}(h_L^{\frac{1}{m}})$ is a coherent sheaf. If $\frac{i}{\pi} \Theta_{h_L}(L) \geq 0$ in the sense of current, thanks to (2.8.2), the weight φ is a psh function. Therefore, for $m \in \mathbb{N}$ large enough, we have $\mathcal{J}(h_L^{\frac{1}{m}}) = \mathcal{O}_X$.

The following result is a very special version of the standard Ohsawa-Takegoshi type extension theorem. We refer to for example [Man93, Dem12] among many others for the more general versions.

2.9. Theorem. [Man93, Dem12] Let $p: X \to Y$ be a fibration between two projective manifolds and let L_Y be a very ample line bundle on Y such that the global sections of L_Y separates all 2n-jets, where n is the dimension of Y. Let L be a pseudoeffective line bundle on X with a possibly singular hermitian metric h such that $i\Theta_h(L) \geq 0$ on X. Let $y \in Y$ be a generic point. Then the following restriction $H^0(X, \mathcal{O}_X(K_X + L + p^*L_Y) \otimes \mathcal{J}(h)) \to H^0(X_y, \mathcal{O}_{X_y}(K_X + L + p^*L_Y) \otimes \mathcal{J}(h|_{X_y}))$ is surjective.

We will use the following theorem to study the positivity of direct images in this article. It is a consequence of [BP08, BP10, PT14].

2.10. Theorem.[BP10, PT14] Let $p: X \to Y$ be a fibration between two projective manifolds and let L be a pseudo-effective line bundle on X with a possibly singular metric h_L such that $i\Theta_{h_L}(L) \geq 0$ in the sense of current. Let m be a positive number such that $\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ for a generic fibre X_y . If $p_{\star}(mK_{X/Y} + L) \neq 0$, then $\det p_{\star}(mK_{X/Y} + L)$ is a pseudo-effective line bundle on Y.

Moreover, let A_Y be a very ample line bundle on Y such that the global sections of $A_Y - K_Y$ separates all 2n-jets, where n is the dimension of Y. Then the restriction (2.10.1) $H^0(X, mK_{X/Y} + L + p^*A_Y) \to H^0(X_y, mK_{X/Y} + L + p^*A_Y)$ is surjective for a generic $y \in Y$.

2.11. Remark. Note that the choice of A_Y depends only on Y and is independent of the fibration $p: X \to Y$, L and m. This will be crucial in our article.

Proof. We explain briefly the proof. Since $p_{\star}(mK_{X/Y}+L) \neq 0$, by [BP10, A.2.1], there exists a m-relative Bergman type metric $h_{m,B}$ on $mK_{X/Y}+L$ with respect to h_L such that $i\Theta_{h_{m,B}}(mK_{X/Y}+L) \geq 0$. Then $h:=\frac{m-1}{m}h_{m,B}+\frac{1}{m}h_L$ defines a possibly singular metric on

$$\widetilde{L} := \frac{m-1}{m} (mK_{X/Y} + L) + \frac{1}{m}L,$$

with $i\Theta_h(\widetilde{L}) \geq 0$. By construction, we have

(2.11.1)
$$mK_{X/Y} + L = K_{X/Y} + \widetilde{L}.$$

Let $y \in Y$ be a generic point and let φ_m be the weight of $h_{m,B}$. Then for every $s \in H^0(X_y, mK_{X/Y} + L)$, by the construction of the m-relative Bergman kernel metric, $|s|^2 e^{-\varphi_m}$ is C^0 -bounded. Combining this with the assumption $\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$, we know that

(2.11.2)
$$\int_{X_n} |s|_h^2 < +\infty.$$

Therefore the inclusion

$$p_{\star}(\mathcal{O}_X(K_{X/Y}+\widetilde{L})\otimes\mathcal{J}(h))\subset p_{\star}(K_{X/Y}+\widetilde{L})$$

is generically isomorphic. By applying [PT14, Thm 3.3.5] and [CP17, Cor 2.9], we know that $\det p_{\star}(K_{X/Y}+\widetilde{L})$ is pseudo-effective. Therefore $\det p_{\star}(mK_{X/Y}+L)$ is pseudo-effective.

To prove the surjectivity of (2.10.1), we can first assume that

$$H^0(X_y, mK_{X/Y} + L + p^*A_Y) \neq 0$$
 for a generic $y \in Y$,

which is equivalent to say that $p_{\star}(mK_{X/Y}+L)\neq 0$. Note that

$$K_X + \widetilde{L} + p^*(A_Y - K_Y) = mK_{X/Y} + L + p^*(A_Y).$$

By applying Theorem 2.9 to the line bundle $K_X + \widetilde{L} + p^*(A_Y - K_Y)$, thanks to (2.11.2), we know that s be can extended to a section in $H^0(X, mK_{X/Y} + L + p^*A_Y)$. In other words, the restriction (2.10.1) is surjective.

We need two slight generalizations of the above theorem. The first is a direct consequence of Theorem 2.10 and the argument in [CP17, Lemma 5.4].

2.12. Proposition. Let $p: X \to Y$ be a fibration between two projective manifolds and let L be a line bundle on X with a possibly singular metric h_L such that $i\Theta_{h_L}(L) \geq p^*\alpha$ in the sense of current for some smooth d-closed (1,1)-form α on Y. Let m be a positive number such that $\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ for a generic fibre X_y and $p_\star(mK_{X/Y}+L) \neq 0$. Then h_L induces a metric \hat{h} on $\det p_\star(mK_{X/Y}+L)$ such that

$$i\Theta_{\widehat{h}}(\det p_{\star}(mK_{X/Y}+L)) \ge r \cdot \alpha$$
 on Y

in the sense of current, where $r = \operatorname{rank} p_{\star}(mK_{X/Y} + L)$.

2.13. Proposition. Let $p: X \to Y$ be a fibration between two projective manifolds and let A_Y be a very ample line bundle on Y in Theorem 2.10. Let F be a pseudo-effective line bundle on X with a possibly singular metric h_F such that $i\Theta_{h_F}(F) \ge 0$ in the sense of current and let $m \in \mathbb{N}$ be a number such that $\mathcal{J}(h_F^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ for a generic fibre X_y .

Let $Q \ge 0$ be some effective divisor on X such that the support of Q does not meet the general fibre of p. Let N be a line bundle such that $N + \epsilon F + p^*A_Y$ is

semipositive ³ for some $0 < \epsilon < 1$. Then the restriction

$$H^0(X, mK_{X/Y} + N + F + Q + 2p^*A_Y) \rightarrow H^0(X_y, mK_{X/Y} + N + F + Q + 2p^*A_Y)$$
 is surjective.

Proof. As $N + p^*A_Y + \epsilon F$ is semipositive, it can be equipped with a smooth metric h_1 with semi-positive curvature. Let h_Q be a singular metric on Q such that $i\Theta_{h_Q}(Q) = [Q]$. Then $h = h_1 + (1 - \epsilon)h_F + h_Q$ defines a metric on the line bundle

$$N + F + Q + p^* A_Y = (N + \epsilon F + p^* A_Y) + (1 - \epsilon)F + Q$$

with $i\Theta_h(N+F+Q+p^*A_Y) \geq 0$. Since $\mathcal{J}(h_F^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ and $p(Q) \subsetneq Y$, we have $\mathcal{J}(h^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$. The pair $(L, h_L) := (N+F+Q+p^*A_Y, h)$ satisfies thus the condition in Theorem2.10. By applying Theorem2.10, we know that

$$H^0(X, mK_{X/Y} + N + F + Q + 2p^*A_Y) \to H^0(X_y, mK_{X/Y} + N + F + Q + 2p^*A_Y)$$
 is surjective. \Box

Together with the arguments in [Zha05], we have

2.14. Corollary. Let $p: X \to Y$ be a fibration between two projective manifolds and let A_Y be a very ample line bundle on Y in Theorem 2.10. If $-K_{X/Y}$ is nef, then for every p-ample pseudo-effective line bundle L on X, the restriction map

$$H^0(X, L + 2p^*A_Y) \to H^0(X_y, L + 2p^*A_Y)$$

is surjective for a generic $y \in Y$.

Proof. Since L is pseudo-effective, there exists a possibly singular metric h_L such that $i\Theta_{h_L}(L) \geq 0$. Let $m \in \mathbb{N}$ large enough such that $\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ for a generic fibre X_y . As L is p-ample, $\epsilon L + p^*A_Y$ is ample for some $0 < \epsilon < 1$. Combining this with the nefness of $-mK_{X/Y}$, we know that $-mK_{X/Y} + \epsilon L + p^*A_Y$ is ample. The corollary is thus proved by using Proposition 2.13, where we take $N = -mK_{X/Y}$, F = L and $Q = \mathcal{O}_X$.

3. Two propositions

Let $p: X \to Y$ be a fibration between two projective manifolds and let L be an ample line bundle on X. If p is a trivial fibration⁴ and $p_{\star}(L)$ is non zero, we know that $\det p_{\star}(L)$ is ample and $L - \frac{1}{r}p^{\star} \det p_{\star}(L)$ is semi-ample where r is the rank of $p_{\star}(L)$. The goal of this section is to prove some similar results when p is smooth in codimension 1 and $-K_{X/Y}$ is nef.

To begin with, by combining the diagonal method of Viehweg [Vie95, Thm 6.24] with the method in [Zha05], we can prove the following key proposition of the article. The proof is very close to [Vie95, Thm 6.24][Tsu10, Section 2.6][CP17, Thm 3.13].

³It means that there exists a smooth hermitian metric such that the curvature is semipositive. In particular, if $N + \epsilon F + p^* A_Y$ is \mathbb{R} -semiample, then it is semipositive.

⁴It means that $X \simeq Y \times F$ where F is a generic point of p.

3.15. Proposition. Let $p: X \to Y$ be a fibration between two projective manifolds and we assume that $-K_{X/Y}$ is nef. We suppose that p is smooth in codimension 1, namely p is smooth outside a subvariety Z in X of codimension at least 2. Let A be a p-ample line bundle on X such that $p_{\star}(A)$ is locally free. Then $rA - p^{\star} \det p_{\star}(A)$ is pseudo-effective, where r is the rank of $p_{\star}(A)$.

Sketch of the proof. To explain the idea of the proof, we first sketch the proof under the assumption that p is smooth.

We consider the natural morphism

$$s: \det p_{\star}(A) \to \bigotimes^r p_{\star}(A).$$

Let $X^r = X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiberwise product of the fibration $p: X \to Y$. Let $\operatorname{pr}_i: X^r \to X$ be the i-th directional projection and let $\operatorname{p}^r: X^r \to Y$ be the natural induced fibration. Set $A_r := \bigotimes_{i=1}^r \operatorname{pr}_i^\star A$ and $L := A_r - (\operatorname{p}^r)^\star \det p_\star(A)$.

As
$$(p^r)_{\star}(A_r) = \bigotimes^r p_{\star}(A)$$
, the morphism s induces a non-trivial section (3.15.1) $\tau \in H^0(X^r, L)$.

The idea of Viehweg is as follows. Let $j:X\to X^r$ be the diagonal embedding. We know that

(3.15.2)
$$L|_{j(X)} = rA - p^* \det p_*(A).$$

Note that (3.15.1) implies that L is effective on X^r . If the effectiveness of L on X^r implies the pseudo-effectiveness of $L|_{j(X)}$, thanks to (3.15.2), the proposition is proved. In general, it is not true. However, by using Corollary 2.14, we can prove it in our case.

To be more precise, let A_Y be the ample line bundle on Y in Theorem2.10. For every $q \in \mathbb{N}^*$, qL is effective and p^r-ample. Since $-K_{X^r/Y} = \sum_i \operatorname{pr}_i^*(-K_{X/Y})$ is nef, we can apply Corollary2.14 to $(\operatorname{p}^r: X^r \to Y, qL)$. Then the restriction

(3.15.3)
$$H^0(X^r, qL + 2(p^r)^*A_Y) \to H^0(X_n^r, qL + 2(p^r)^*A_Y)$$

is surjective for a generic fibre X_y^r .

Let $j: X \to X^r$ be the diagonal embedding. By restricting (3.15.3) to j(X), thanks to (3.15.2), we have

$$H^{0}(X^{r}, qL + 2(\mathbf{p}^{r})^{*}A_{Y}) \xrightarrow{\qquad} H^{0}(X_{y}^{r}, qL + 2(\mathbf{p}^{r})^{*}A_{Y})$$

$$\downarrow^{j^{*}} \downarrow$$

$$\downarrow^{j^{*}} \downarrow$$

$$H^{0}(X, qrA - q \cdot p^{*} \det p_{*}(A) + 2p^{*}A_{Y}) \xrightarrow{\qquad} H^{0}(X_{y}, qrA)$$

Since the image of $H^0(X_y^r, qL + 2(\mathbf{p}^r)^*A_Y) = H^0(X_y^r, qA_r) \to H^0(X_y, qrA)$ is non trivial, the surjectivity (3.15.3) and the above commutative diagram implies that the image of

$$H^0(X, qrA - q \cdot p^* \det p_*(A) + 2p^*A_Y) \to H^0(X_y, qrA)$$

is non zero. In particular, $qrA - q \cdot p^* \det p_*(A) + 2p^*A_Y$ is effective on X. Then $rA - p^* \det p_*(A) + \frac{2}{q} \cdot p^*A_Y$ is \mathbb{Q} -effective for every q > 0. By letting $q \to +\infty$, the proposition is proved.

Now we give the complete proof of the proposition, which follows closely [CP17, Thm 3.13].

Proof of the proposition 3.15. Let Y_1 be the flat locus of p. As p is smooth in codimension 1, $p^{-1}(Y \setminus Y_1)$ is of codimension at least two. After replacing Z by $Z \cup p^{-1}(Y \setminus Y_1)$, we can assume that p is flat over $p(X \setminus Z)$ and Z is of still codimension at least 2. Let $X^r = X \times_Y X \times \cdots \times_Y X$ be the r-times fiberwise product of $p: X \to Y$, and let $X^{(r)}$ be a desingularisation of X^r . Let

$$\operatorname{pr}_i: X^{(r)} \to X$$

be the *i*-th directional projection, and $\mathbf{p}^{\mathbf{r}}:X^{(r)}\to Y$ be the natural induced morphism. Set $E:=X^{(r)}\setminus(\bigcap_{i}\mathbf{pr}_{i}^{-1}(X\setminus Z))$. Then the diagonal embedding j:

 $X \setminus Z \hookrightarrow X^{(r)}$ satisfies $j(X \setminus Z) \subset X^{(r)} \setminus E$. Note that for a generic point $y \in Y$, as p is smooth over y, we have

$$(3.15.4) X_u^{(r)} \cap E = \emptyset,$$

where $X_y^{(r)}$ is the fibre over y.

We consider the natural morphism

$$s: \det p_{\star}(A) \to \bigotimes^r p_{\star}(A).$$

Set $A_r := \bigotimes_{i=1}^r \operatorname{pr}_i^* A$ and $L := A_r - (\operatorname{p^r})^* \det p_*(A)$. Since p is flat over $p(X \setminus Z)$ and $p_*(A)$ is locally free, thanks to [Hor10, Lemma 3.15], s induces a non-trivial section $\tau \in H^0(X^{(r)}, L + E')$ for some divisor E' supported in E. Let A_Y be the ample line bundle on Y in Theorem 2.10.

For every $q \ge 1$, we first prove that there exists a divisor E_q supported in E such that the restriction

(3.15.5)
$$H^0(X^{(r)}, qL + E_q + 2(p^r)^*A_Y) \to H^0(X_y^{(r)}, qL + 2(p^r)^*A_Y)$$

is surjective for a generic $y \in Y$.

In fact, there exist some effective divisors E_1 and E_2 supported in E such that

(3.15.6)
$$-K_{X^{(r)}/Y} = \sum_{i=1}^{r} \operatorname{pr}_{i}^{\star}(-K_{X/Y}) + E_{1} - E_{2}.$$

Then for every $m \in \mathbb{N}$, we have (3.15.7)

$$qL+qmE_2+2(p^r)^*A_Y = qmK_{X^{(r)}/Y}+qm(\sum_{i=1}^r pr_i^*(-K_{X/Y}))+qL+qmE_1+2(p^r)^*A_Y.$$

As $-K_{X/Y}$ is nef, for $\epsilon \ll 1$, the line bundle

$$qm(\sum_{i}\operatorname{pr}_{i}^{\star}(-K_{X/Y}))+\epsilon L+(\operatorname{p^{r}})^{\star}A_{Y}=\sum_{i}\operatorname{pr}_{i}^{\star}(-qmK_{X/Y}+\epsilon A)+(\operatorname{p^{r}})^{\star}(A_{Y}-\varepsilon\det p_{\star}(A))$$

is semi-ample on $X^{(r)}$.

We can thus apply Proposition 2.13 to the fibration $p^r: X^{(r)} \to Y$ by taking $N=qm(\sum_{i=1}^r \operatorname{pr}_i^{\star}(-K_{X/Y})), \ F=qL \ \text{and} \ Q=qmE_1, \ \text{where} \ m\gg 1 \ \text{is large}$ enough with respect to qL. Together with (3.15.7), the restriction

 $(3.15.8) \quad H^0(X^{(r)}, qL + qmE_2 + 2(p^r)^*A_Y) \to H^0(X_y^{(r)}, qL + qmE_2 + 2(p^r)^*A_Y)$

is thus surjective, where $y \in Y$ is a generic point. Thanks to (3.15.4), we have

$$H^{0}(X_{y}^{(r)}, qL + qmE_{2} + 2(\mathbf{p}^{r})^{*}A_{Y}) = H^{0}(X_{y}^{(r)}, qL + 2(\mathbf{p}^{r})^{*}A_{Y}).$$

Combining this with (3.15.8), (3.15.5) is proved by taking $E_q = qmE_2$.

Finally, we take the pull-back j^* of (3.15.5), where $j: X \setminus Z \to X^{(r)}$ is the diagonal embedding. As $j(X \setminus Z) \subset X^{(r)} \setminus E$ and E_q is supported in E, we obtain the following commutative diagram

$$H^0\big(X^{(r)},qL+2(\mathbf{p^r})^{\star}A_Y+E_q\big) \xrightarrow{\hspace*{1cm}} H^0\big(X_y^{(r)},qL+2(\mathbf{p^r})^{\star}A_Y\big) \ .$$

$$\downarrow^{j^{\star}} \qquad \qquad \downarrow^{j^{\star}} \downarrow$$

$$H^0(X\setminus Z,qrA-q\cdot p^{\star}\det p_{\star}(A)+2p^{\star}A_Y) \xrightarrow{\hspace*{1cm}} H^0(X_y,qrA)$$

Note that the image of $H^0(X_y^{(r)}, qL + 2(p^r)^*A_Y) \to H^0(X_y, qrA)$ is non zero. Together with the surjectivity (3.15.5) and the above commutative diagram, we know that the image of

$$H^0(X \setminus Z, qrA - q \cdot p^* \det p_*(A) + 2p^*A_Y) \to H^0(X_u, qrA)$$

is non zero. As $\operatorname{codim}_X Z \geq 2$, $\operatorname{qr} A - \operatorname{q} \cdot \operatorname{p}^\star \det \operatorname{p}_\star(A) + 2 \cdot \operatorname{p}^\star A_Y$ is effective. Then $rA - p^* \det p_*(A) + \frac{2}{q} \cdot p^*A_Y$ is Q-effective. The proposition is proved by letting $q \to +\infty$.

Using [Zha05, PT14], we can prove

3.16. Proposition. Let $p: X \to Y$ be a fibration between two projective manifolds and we suppose that $-K_{X/Y}$ is nef. Let L be a pseudo-effective and p-ample line bundle on X. If $p_{\star}(L)$ is not zero, then $\det p_{\star}(L)$ is pseudo-effective.

Proof. Let h_L be a possibly singular metric on L such that $i\Theta_{h_L}(L) \geq 0$ on X and let $m \in \mathbb{N}$ large enough such that $\mathcal{J}(h_L^{\frac{1}{m}}) = \mathcal{O}_X$. Since L is p-ample, there exists an ample line bundle A_Y on Y such that $L + p^*A_Y$ is ample on X. Then for every $\varepsilon > 0$, $-mK_{X/Y} + \varepsilon(L + p^*A_Y)$ is ample. Therefore we can find a smooth metric h_{ε} on $-mK_{X/Y} + \varepsilon L$ such that

$$i\Theta_{h_{\varepsilon}}(-mK_{X/Y}+\varepsilon L) \geq -\varepsilon p^{\star}\omega_{Y},$$

where ω_Y is a (1,1)-form in the class of $c_1(A_Y)$.

Then $h_{\varepsilon} := h_{\varepsilon} + (1 - \varepsilon)h_L$ defines a metric on $-mK_{X/Y} + L$ such that

$$i\Theta_{\widetilde{h}} (-mK_{X/Y} + L) \ge -\varepsilon p^* \omega_Y$$
 and $\mathcal{J}(\widetilde{h}_{\varepsilon}^{\frac{1}{m}}) = \mathcal{O}_X$.

By applying Proposition 2.12, $\widetilde{h}_{\varepsilon}$ induces a metric $\widehat{h}_{\varepsilon}$ on

$$\det p_{\star}(mK_{X/Y} + (-mK_{X/Y} + L))$$

such that

$$i\Theta_{\widehat{h}_{\varepsilon}}(\det p_{\star}(mK_{X/Y}+(-mK_{X/Y}+L))) \geq -\varepsilon\operatorname{rank} p_{\star}(L)\cdot p^{\star}\omega_{Y}.$$

The proposition is proved by letting $\varepsilon \to 0$.

4. Proof of the main theorem

We begin to prove the main theorem:

4.17. Theorem. Let X be a projective manifold with nef anti-canonical bundle and let $p: X \to Y$ be the Albanese map. Then p is locally trivial. Moreover, let $\mathbb{C}^r \to Y$ be the universal cover and set $\widetilde{X} := X \times_Y \mathbb{C}^r$. Then \widetilde{X} admits the following splitting

$$\widetilde{X} \simeq \mathbb{C}^r \times F$$
,

where F is the generic fibre of p.

Proof. First of all, thanks to [LTZZ10], p is flat. We can thus find a very ample line bundle A on X such that $p_{\star}(mA)$ is locally free for every $m \in \mathbb{N}$ and the natural morphism $\operatorname{Sym}^m p_{\star}(A) \to p_{\star}(mA)$ is surjective for every $m \in \mathbb{N}$. Let r be the rank of $p_{\star}(A)$. After passing to some isogeny of Y, we can assume that $\frac{1}{r} \det p_{\star}(A)$ is a line bundle. Set

(4.17.1)
$$L := A - \frac{1}{r} p^* \det p_*(A).$$

Thanks to [LTZZ10], p is smooth in codimension 1. We can thus apply Proposition 3.15 to $(p: X \to Y, A)$. Therefore L is pseudo-effective, and by construction we have

$$(4.17.2) c_1(p_*(L)) = 0.$$

The plan of the rest of the proof is as follows. From Step 1 to Step 3, by combining [DPS94, 3.D] with the results about the positivity of direct images, we will prove that $p_{\star}(L)$ is numerically flat on Y. In Step 4, we will prove the theorem. We remark that, if $-K_X$ is hermitian positive, we can easily prove that $p_{\star}(L)$ is hermitian flat by using [PT14, Thm 3.3.5], [CP17, Thm 5.2] and the arguments in (1.4.1). However, as $-K_X$ is only nef in our case, we can not use directly [PT14]. We use here the isogeny argument [DPS94, Lemma 3.21] to prove the nefness of $p_{\star}(L)$.

Step 1: Construction

Let A_Y be a sufficiently ample line bundle on Y such that $A_Y - \frac{1}{r} \det p_{\star}(A)$ is ample, and A_Y satisfies the condition in Theorem 2.10. We can ask also that, for every $F \in \operatorname{Pic}^0(Y)$, $A_Y + F$ is very ample on Y.

Since Y is a torus, for every $n \in \mathbb{N}$ sufficiently divisible, we can take a n to 1 isogeny $\pi_{Y,n}: Y \to Y$. Let $X_n := X \times_{\pi_{Y,n}} Y$. Then X_n is smooth with nef anticanonical bundle, and we have

(4.17.3)
$$\pi_{Y,n}^{\star}c_1(A_Y) = n \cdot c_1(A_Y) \in H^{1,1}(Y,\mathbb{R}).$$

$$X_n \xrightarrow{\pi_n} X$$

$$\downarrow p$$

$$\downarrow p$$

$$Y \xrightarrow{\pi_{Y,n}} Y$$

Set $V_n := \pi_{Y,n}^{\star} p_{\star}(L)$. As L is proved to be pseudo-effective, by applying Corollary 2.14 to $(p_n : X_n \to Y, \pi_n^{\star} L)$, the restriction

$$(4.17.4) H0(Y, 2A_Y \otimes V_n) \to (2A_Y \otimes V_n)_y$$

is surjective for a generic $y \in Y$.

Step 2: Global surjectivity

We prove in this step that for n sufficiently large and divisible, the restriction

$$(4.17.5) H0(Y, 3A_Y \otimes V_n) \to (3A_Y \otimes V_n)_{y_0}$$

is surjective for every $y_0 \in Y$.

In fact, thanks to the generic surjectivity (4.17.4), we can find

$$\{s_1,\cdots,s_r\}\subset H^0(Y,2A_Y\otimes V_n)$$

such that $s := s_1 \wedge \cdots \wedge s_r \in H^0(Y, 2rA_Y \otimes \det V_n)$ is non zero. By (4.17.2), we know that

$$c_1(\det V_n) = \pi_{Y,n}^* c_1(\det p_*(L)) = 0.$$

Therefore the numerical class

(4.17.6)
$$c_1(\operatorname{div} s) = c_1(2rA_Y) \in H^{1,1}(Y,\mathbb{R})$$

is independent of n.

On the other hand, after a translation, we can suppose without lose of generality that $\pi_{Y,n}(y_0)$ is the origin in Y. Then $\{\pi_{Y,n}^{-1}(\pi_{Y,n}(y_0))\}$ is the set of the *n*-torsion points in Y. Thanks to (4.17.6), the numerical class $c_1(\operatorname{div}(s))$ is independent of n. As a consequence, [MFK, Prop 7.7] implies that for n large enough, the divisor $\operatorname{div}(s)$ could not contain the set $\{\pi_{Y,n}^{-1}(\pi_{Y,n}(y_0))\}^{-5}$. Therefore there exists a point $y_1 \in \pi_{Y,n}^{-1}(\pi_{Y,n}(y_0))$ such that

$$(4.17.7) s(y_1) \neq 0.$$

Finally, let G_n be the Galois group associated to the étale cover $\pi_{Y,n}: Y \to Y$ and let $g \in G_n$ such that $g(y_0) = y_1$. As V_n is G_n -invariant, g induces the isomorphisms

$$H^0(Y, 2A_Y \otimes V_n) \to H^0(Y, 2g^*(A_Y) \otimes V_n)$$

and

$$H^0(Y, 2rA_Y \otimes \det V_n) \to H^0(Y, 2rg^*(A_Y) \otimes \det V_n).$$

Therefore

$$g^{\star}(s_1)(y_0) \wedge \cdots \wedge g^{\star}(s_r)(y_0) = g^{\star}(s)(y_0) = s(y_1) \neq 0.$$

As a consequence, $\{g^{\star}(s_1)(y_0), \cdots, g^{\star}(s_r)(y_0)\}$ generates $(2g^{\star}(A_Y) \otimes V_n)_{y_0}$. Note that $A_Y - g^{\star}A_Y \in \text{Pic}^0(Y)$. The construction of A_Y implies thus that

$$A_Y + 2(A_Y - g^*A_Y)$$

⁵The proof in [MFK, Prop 7.7] is an effective estimate. We can also give non effective estimate proof as follows. In fact, if it is not true, then for every n sufficiently divisible, we can find an ample line bundle in the same class of $2rc_1(A_Y)$, such that the set $\{\pi_{Y,n}^{-1}(\pi_{Y,n}(y_0))\}$ is contained in the zero locus Z_n of a section of this line bundle. As the volumes of $[Z_n]$ is independent of n, by the compactness of cycle spaces, after passing to a subsequence, $[Z_n]$ will tends to a divisor [Z] in Y. However, the torsion sets $\{\pi_{Y,n}^{-1}(\pi_{Y,n}(y_0))\}$ will not converge to a strict subvariety of X when $n \to +\infty$. We get thus a contradiction.

is very ample. Therefore we can find a section $\tau \in H^0(Y, 3A_Y - 2g^*A_Y)$ such that $\tau(y_0) \neq 0$. Then $\{\tau \otimes g^*(s_1)(y_0), \dots, \tau \otimes g^*(s_r)(y_0)\}$ generates $(3A_Y \otimes V_n)_{y_0}$ and (4.17.5) is proved.

Step 3: Numerically flatness of $p_{\star}(L)$

Let $\mathbb{P}(p_{\star}(L))$ (resp. $\mathbb{P}(V_n)$) be the projectivization of $p_{\star}(L)$ (resp. V_n). We have the commutative diagram

$$\mathbb{P}(V_n) \xrightarrow{\pi_n} \mathbb{P}(p_{\star}(L)) \\
\downarrow^{p_n} \qquad \qquad \downarrow^{p} \\
Y \xrightarrow{\pi_{Y,n}} Y$$

Let ω_Y be a Kähler metric in the same class of A_Y . Thanks to (4.17.5), we can find a smooth metric h on $\mathcal{O}_{\mathbb{P}(V_n)}(1)$ such that

$$i\Theta_h(\mathcal{O}_{\mathbb{P}(V_n)}(1)) \ge -3p_n^{\star}\omega_Y = -\frac{3}{n}(\pi_n \circ p)^{\star}\omega_Y.$$

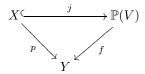
Note that $\pi_n^{\star} \mathcal{O}_{\mathbb{P}(p_{\star}(L))}(1) = \mathcal{O}_{\mathbb{P}(V_n)}(1)$. Then h induces a smooth metric h_n on $\mathcal{O}_{\mathbb{P}(p_{\star}(L))}(1)$ by taking the average of the translates of h by the $\pi_{Y,n}$ -torsion points. We have

$$i\Theta_{h_n}(\mathcal{O}_{\mathbb{P}(p_{\star}(L))}(1)) \ge -\frac{3}{n}p^{\star}\omega_Y.$$

As this holds for every n sufficiently large and divisible, $\mathcal{O}_{\mathbb{P}(p_{\star}(L))}(1)$ is nef by definition. Therefore the vector bundle $p_{\star}(L)$ is nef. Combining this with (4.17.2), $p_{\star}(L)$ is thus numerically flat.

Step 4: Final conclusion

Let $V := p_{\star}(L)$. As A is very ample, V induces a p-relative embedding



We have $L = j^* \mathcal{O}_{\mathbb{P}(V)}(1)$. For m large enough, we have thus the exact sequence

$$(4.17.8) 0 \to f_{\star}(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(V)}(m)) \to f_{\star}(\mathcal{O}_{\mathbb{P}(V)}(m)) \to p_{\star}(mL) \to 0.$$

Thanks to Step 4, $f_{\star}(\mathcal{O}_{\mathbb{P}(V)}(m)) = \operatorname{Sym}^m V$ is numerically flat.

Claim: $p_{\star}(mL)$ is numerically flat for every $m \geq 1$.

We will postpone the proof of the claim to Lemma 4.18 and first finish the proof of the theorem. As $p_{\star}(mL)$ is numerically flat for every $m \geq 1$, by using Proposition 2.8, the theorem is proved.

To complete the proof of the main theorem, it remains to prove the claim

4.18. Lemma. The vector bundle $p_{\star}(mL)$ is numerically flat for every $m \geq 1$.

Proof. As Sym^m V is numerically flat, the exact sequence (4.17.8) implies that $p_{\star}(mL)$ is nef. It remains to prove that $c_1(p_{\star}(mL)) = 0$.

In fact, as mL is p-ample and $p_{\star}(mL)$ is locally free, Proposition 3.15 implies that $mL - \frac{1}{r_m}p^{\star}c_1(p_{\star}(mL))$ is \mathbb{Q} -pseudo-effective, where r_m is the rank of $p_{\star}(mL)$. Then

(4.18.1)
$$\widetilde{L} := L - \frac{1}{m \cdot r_m} p^* c_1(p_*(mL))$$

is Q-pseudo-effective. After passing to some isogeny of Y, we can assume that $\frac{1}{m \cdot r_m} c_1(p_{\star}(mL))$ is a line bundle. Therefore \widetilde{L} is a pseudo-effective line bundle. By taking the determinant of the direct image of (4.18.1), we get

$$c_1(\det p_{\star}(\widetilde{L})) = c_1(\det p_{\star}(L)) - \frac{1}{m \cdot r_m} c_1(p_{\star}(mL)).$$

Combining this with (4.17.2), we have

(4.18.2)
$$\frac{1}{m \cdot r_m} c_1(p_{\star}(mL)) + c_1(\det p_{\star}(\widetilde{L})) = 0 \in H^{1,1}(Y, \mathbb{R}).$$

To conclude, as $p_{\star}(mL)$ is proved to be nef, $c_1(p_{\star}(mL))$ is pseudo-effective. By construction, \widetilde{L} is pseudo-effective and p-ample. Then Proposition3.16 implies that $\det p_{\star}(\widetilde{L})$ is also pseudo-effective. Therefore (4.18.2) implies that $c_1(p_{\star}(mL)) = 0$ and the lemma is proved.

We now discuss the structure of the universal cover of X. Let X be a compact Kähler manifold with nef anticanonical bundle. Thanks to [Pau97, Pau12], we know

4.19. Proposition. [Pau97, Thm 2] [Pau12] Let X be a compact Kähler manifold with nef anticanonical bundle. Then after a finite étale cover of X, the Albanese map

$$p: X \to Y$$

induces an isomorphism of fundamental groups.

Proof. For readers' convenience, we recall briefly the main steps of the proof of [Pau97, Thm 2]. First of all, thanks to [Zha96, Pau12], the albanese map is surjective with connected fibers. Let (G_n) be the descending central series of $\pi_1(X)$, i.e., $G_1 = \pi_1(X)$, $G_{n+1} = [G, G_n]$. Set $G'_n = \sqrt{G_n}$. By applying [Cam95, Thm 2.2] to p, as p is a fibration and $\pi_1(Y)$ is abelian, we know that

$$p_{\star}: \pi_1(X)/G'_n \to \pi_1(Y)$$

is an isomorphism for all n.

[Pau97, Thm 1] shows that $\pi_1(X)$ is virtually nilpotent. Therefore, up to a finite étale cover of X, we can assume that $\pi_1(X)$ is nilpotent and torsion free. Then $\pi_1(X)/G'_n = \pi_1(X)$ for some $n \in \mathbb{N}$. As a consequence,

$$p_{\star}:\pi_1(X)\to\pi_1(Y)$$

is an isomorphism.

As an application, we have the following result.

4.20. Corollary. Let X be a projective manifold with nef anticanonical bundle. Then the universal cover \widetilde{X} of X admits the following splitting

$$\widetilde{X} \simeq \mathbb{C}^r \times F.$$

Here F is a simply connected projective manifold with nef anticanonical bundle and $r = \sup h^{1,0}(\widehat{X})$ where the supremum is taken over all finite étale covers $\widehat{X} \to X$.

Proof. After some finite étale cover of X, we can assume that $h^{1,0}(X) = r$. Thanks to Proposition 4.19, we can assume moreover that the Albanese map $p: X \to Y$ induces an isomorphism of fundamental groups:

$$(4.20.1) p_{\star}: \pi_1(X) \to \pi_1(Y).$$

Let F be the generic fibre of p. Then $\dim Y = r$. Let $\pi : \mathbb{C}^r \to Y$ be the universal cover and set $X_1 := X \times_Y \mathbb{C}^r$. Theorem4.17 implies that p is locally trivial and we have the splitting

$$(4.20.2) X_1 \simeq \mathbb{C}^r \times F.$$

It remains to prove that F is simply connected. As p is a submersion, we have the exact sequence

$$\pi_2(Y) \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to 1.$$

Since Y is a torus, we know that $\pi_2(Y) = 1$. Then the isomorphism (4.20.1) and the above exact sequence imply that $\pi_1(F)$ is trivial. The corollary is proved. \square

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Junyan Cao, Sorbonne Université - Campus Pierre et Marie Curie, Institut de Mathématiques de Jussieu, $4,\,$ Place Jussieu, Paris $75252,\,$ France

 $E ext{-}mail\ address: junyan.cao@imj-prg.fr}$