PERVERSE OBSTRUCTIONS TO FLAT REGULAR COMPACTIFICATIONS

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ABSTRACT. Suppose $\pi:W\to S$ is a smooth, proper morphism over a variety S contained as a Zariski open subset in a smooth, complex variety \bar{S} . The goal of this note is to consider the question of when π admits a regular, flat compactification. In other words, when does there exists a flat, proper morphism $\bar{\pi}:\overline{W}\to \bar{S}$ extending π with \overline{W} regular? One interesting recent example of this occurs in the preprint [9] of Laza, Saccà and Voisin where π is a family of abelian 5-folds over a Zariski open subset S of $\bar{S}=\mathbb{P}^5$. In that paper, the authors construct \overline{W} using the theory of compactified Prym varieties and show that it is a holomorphic symplectic manifold (deformation equivalent to O'Grady's 10-dimensional example).

In this note I observe that non-vanishing of the local intersection cohomology of $R^1\pi_*\mathbb{Q}$ in degree at least 2 provides an obstruction to finding a $\bar{\pi}$. Moreover, non-vanishing in degree 1 provides an obstruction to finding a $\bar{\pi}$ with irreducible fibers. Then I observe that, in some cases of interest, results of Brylinski, Beilinson and Schnell can be used to compute the intersection cohomology [1, 4, 12]. I also give examples involving cubic 4-folds motivated by [9] and ask a question about palindromicity of hyperplane sections.

1. Introduction

Let \bar{S} denote a smooth, quasi-projective, complex (irreducible) variety of dimension d, and let S denote a non-empty Zariski open subset of \bar{S} . Suppose $\pi:W\to S$ is a smooth, proper morphism of relative dimension n. I will call an irreducible, regular scheme \overline{W} equipped with a proper morphism $\bar{\pi}:\overline{W}\to \bar{S}$ a regular compactification of π if

- (i) \overline{W} contains W as a Zariski dense open subset;
- (ii) the restriction of $\bar{\pi}$ to W is π .

Question 1.1. Under what conditions can we find a regular compactification $\bar{\pi}: \overline{W} \to \bar{S}$ of π which is flat over \bar{S} . Also, under what conditions can we find a $\bar{\pi}: \overline{W} \to \bar{S}$ as above with irreducible fibers?

My goal in this note is to write down some necessary conditions in terms of local intersection cohomology. For this, let $j:S\to \bar{S}$ denote the inclusion of S in \bar{S} . Pick an integer k and set $\mathbf{L}=R^k\pi_*\mathbb{Q}$. Then the intersection complex IC \mathbf{L} is a polarizable Hodge module on \bar{S} with underlying perverse sheaf given by the intermediate extension of the underlying local system L to \bar{S} . The intersection complex is also called the IC complex and is also written as $j_{!*}\mathbf{L}[d]$. The underlying perverse sheaf is a complex of sheaves with cohomology in the interval [-d,0). The local intersection cohomology of \mathbf{L} at a point $s\in \bar{S}$ is

$$IH_s^j \mathbf{L} := H^{j-d}(\mathbf{L})_s. \tag{1.2}$$

So $\operatorname{IH}_s^j \mathbf{L}$ is the $(j-d)^{\operatorname{th}}$ cohomology of the stalk of IC \mathbf{L} at s. Clearly $\operatorname{IH}_s^j \mathbf{L} = 0$ unless $j \in [0,d)$. Moreover, $\operatorname{IH}_s^0 \mathbf{L}$ is the space of local invariants of \mathbf{L} at s. So, $\operatorname{IH}_s^0 \mathbf{L}$ is the fiber \mathbf{L}_s for $s \in S$. At points $s \in \bar{S}$, $\operatorname{IH}_s^0 \mathbf{L} = \Gamma(B \cap S, \mathbf{L})$ for a sufficiently small ball B in \bar{S} containing s.

The following theorem, which I believe is a well-known consequence of the decomposition theorem of Beilinson, Bernstein and Deligne [2], gives a way to obtain information about possible compactifications $\bar{\pi}$ from the topology of π . For the convenience of the reader I will prove it in Section 2.

Theorem 1.3. Suppose $\bar{\pi} : \overline{W} \to \bar{S}$ is a regular compactification of π . Then

- (i) the complex $\bigoplus_i IC(R^{n+i}\pi_*\mathbb{Q})[-i]$ includes in $R\bar{\pi}_*\mathbb{Q}[d+n]$ as a direct factor;
- (ii) for every integer m and each point $s \in \bar{S}$, $\bigoplus_{j+k=m} \mathrm{IH}^{J}_{s}(R^{k}\pi_{*}\mathbb{Q})$ includes as a direct factor in the cohomology group $\mathrm{H}^{m}(\bar{W}_{s},\mathbb{Q})$ of the fiber of $\bar{\pi}$ over s.
- (iii) If the inclusion in (i) is an isomorphism, then so is the inclusion in (ii).

Corollary 1.4. If a flat, regular compactification $\bar{\pi}$ of π exists, then, for all $s \in \bar{S}$ and all integers j, k with j + k > 2n, $\operatorname{IH}_s^j(R^k\pi_*\mathbb{Q}) = 0$. If the fibers of $\bar{\pi}$ are irreducible, then $\operatorname{IH}^0(R^{2n}\pi_*\mathbb{Q}) = \mathbb{Q}$ and the groups $\operatorname{IH}_s^j(R^{2n-j}\pi_*\mathbb{Q})$ vanish for j > 0.

Proof. If $\bar{\pi}$ is flat, then dim $\overline{W}_s = n$ for all $s \in \bar{S}$. So $H^m(\overline{W}_s, \mathbb{Q}) = 0$ for m > 2n. Thus Theorem 1.3 (ii) implies the first statement. If the fiber \overline{W}_s is irreducible, then $H^{2n}\overline{W}_s = \mathbb{Q}$. The constant sheaf is a direct factor in $R^{2n}\pi_*\mathbb{Q}$. So dim $IH^0_s(R^{2n}\pi_*\mathbb{Q}) \geq 1$ for all $s \in \bar{S}$. The rest of Corollary 1.4 is now immediate from Theorem 1.3.

In writing this note, I was mainly motivated by a recent preprint of Laza, Saccà and Voisin which concerns the situation where $\pi:A\to S$ is an abelian scheme of relative dimension n [9]. In this case, set $\mathbf{H}:=R^1\pi_*\mathbb{Q}(1)$. It is a polarized variation of Hodge structure of weight -1 on S, which, is isomorphic to $R^{2n-1}\pi_*\mathbb{Q}(n)$ by Hard Lefschetz. We get the following.

Corollary 1.5. Set $k = \max\{j : \operatorname{IH}_s^j \mathbf{H} \neq 0\}$. Suppose a flat regular compactification $\bar{\pi} : \bar{A} \to \bar{S}$ of π exists. Then $k \leq 1$. If \bar{A}_s is irreducible, then k = 0.

Proof. This follows directly from Corollary 1.4 applied to $R^{2n-1}\pi_*\mathbb{Q}$.

Suppose X is a smooth, closed, 2m-dimensional subvariety of $P:=\mathbb{P}^N$ for some positive integers m and N. By cutting X with hyperplanes, we get a family $\mathcal{X} \to P^\vee$ over the dual projective space, which is smooth over a Zariski dense open subset $U \subset P^\vee$. (See §3.) Set n:=2m-1 so that the general member of the family $\mathcal{X} \to P^\vee$ is, by Bertini, a smooth n-dimensional variety. We get a variation of Hodge structure $\mathbf{H}_\mathbb{Z}$ over U such that the fiber over $H \in P^\vee$ is $H^n(X \cap H, \mathbb{Z}(m))$. Let $J(\mathbf{H}_\mathbb{Z}) \to U$ denote the family of Griffiths intermediate Jacobians of $\mathbf{H}_\mathbb{Z}$. In very special cases, it turns out to be an abelian scheme. Write \mathbf{H} for the \mathbb{Q} -variation of Hodge structure obtained by tensoring $\mathbf{H}_\mathbb{Z}$ with \mathbb{Q} . In Section 3, I will prove the following theorem (which, along with Corollary 1.7, assumes the notation of the preceding paragraph).

Theorem 1.6. Suppose that **H** is non-constant. Let $H \in \check{\mathbb{P}}^N$ be a hyperplane and write $Y := H \cap X$ for the hyperplane section. Write $b_k Y := \dim H^k(Y, \mathbb{Q})$ for the k-th Betti number. Then, for k > 0,

$$b_{n+k}Y - b_{n-k}Y = \dim \mathrm{IH}_H^k \mathbf{H}.$$

Call Y palindromic (resp. weakly palindromic) if $b_{n+k}Y = b_{n-k}Y$ for all k (resp. for all k > 1).

Corollary 1.7. Suppose that $\pi: J(\mathbf{H}_{\mathbb{Z}}) \to U$ is a non-constant abelian scheme admitting a flat, regular compactification $\bar{\pi}: \bar{J} \to P^{\vee}$. Fix $H \in P^{\vee}$ and set $Y = X \cap H$. Then Y is weakly palindromic. If the fiber of $\bar{\pi}$ over $H \in P^{\vee}$ is irreducible, then Y is a palindromic

Proof. Since $R^1\pi_*\mathbb{Q}(1)\cong \mathbb{H}$, Corollary 1.7 follows from Theorem 1.6 and Corollary 1.5.

In [9], the authors produce a flat, regular compactification \bar{A} of a family $\pi: A \to U$ of abelian 5-folds over an open subset of \mathbb{P}^5 . In Section 4, I will give examples where Corollary 1.7 can be used to rule out the existence of a flat, regular compactification, or a flat, regular compactification with irreducible fibers. I will also give a consequence (Corollary 4.3) of the main result of [9] and state a conjecture (Conjecture 4.4) about palindromicity partially motivated by the results of [9].

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The interaction between palindromicity and intersection cohomology comes up in a similar way to the way it is used here in my joint paper [3] written with T. Chow. I thank Chow for many conversations about the notion of palindromicity.

2. Proof of Theorem 1.3

Proof. Let d denote the dimension of \bar{S} and let n denote the dimensions of the fibers of π . By the decomposition theorems of Beilinson—Bernstein—Deligne [2] and Saito [11], we have $R\bar{\pi}_*\mathbb{Q}[d+n] = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i[-i]$ where the \mathcal{F}_i are direct sums of intersection complexes coming from local systems on various strata. The restriction of \mathcal{F}_i to S is equal to $R^{n+i}\pi_*\mathbb{Q}[d]$. So each \mathcal{F}_i contains $IC(R^{n+i}\pi_*\mathbb{Q})$ as a direct factor, and this implies Theorem 1.3 (i).

Let $\iota: \{s\} \to \bar{S}$ denote the inclusion of the point s. By proper base change, we have $H^m(\overline{W}_s, \mathbb{Q}) = H^m(\iota^*R\bar{\pi}_*\mathbb{Q}) = H^{m-d-n}(\iota^*R\bar{\pi}_*\mathbb{Q}[d+n]) = \oplus_i H^{m-d-n}(\iota^*\mathcal{F}_i[-i])$. This vector space contains as a direct factor the space

$$\bigoplus_{i} H^{m-d-n}(\iota^* \operatorname{IC}(R^{n+i}\pi_* \mathbb{Q})[-i]) = \bigoplus_{i} H^{m-d-n-i}(\iota^* \operatorname{IC}(R^{n+i}\pi_* \mathbb{Q}))
= \bigoplus_{i} \operatorname{IH}^{m-n-i}(R^{n+i}\pi_* \mathbb{Q}))
= \bigoplus_{i+k=m} \operatorname{IH}^{j}(R^k \pi_* \mathbb{Q}).$$

Moreover, if the inclusion in (i) is an isomorphism, the two spaces are equal. This proves (ii) and (iii). \Box

3. Proof of Theorem 1.6

Now we fix the notation from the introduction that X is a smooth 2m dimensional closed subvariety of $P = \mathbb{P}^N$ and P^\vee is the dual projective space. Let $\mathcal{X} := \{(x,H) \in X \times P^\vee : x \in H\}$ denote the incidence variety. Write q and p for the projections on the first and second factors respectively. Then q is a \mathbb{P}^{N-1} -bundle. So \mathcal{X} is smooth and irreducible of dimensions $d_{\mathcal{X}} = n + N$ with n = 2m - 1. On the other hand, the fiber of $p : \mathcal{X} \to P^\vee$ over a hyperplane H is the hyperplane section $Y_H := H \cap X$. Write U for the locus of hyperplanes H such that Y_H is smooth, and set $\mathcal{X}_U = p^{-1}(U)$. Then the restriction of p to \mathcal{X}_U gives a smooth, proper morphism $p_U : \mathcal{X}_U \to U$.

Set $\mathbf{H} := R^n p_{U*} \mathbb{Q}(m)$. This is a weight -1 variation of pure Hodge structure on U. By weak Lefschetz, it follows that the sheaves $R^{n-k} p_{U*} \mathbb{Q}$ are constant for k > 0. In fact they are the constant sheaves given by $\mathbf{H}^{n-k}(X,\mathbb{Q})$. Then, Hard Lefschetz shows that, for k > 0, $R^{n+k} p_{U*} \mathbb{Q}(k) \cong R^{n-k} p_{U*} \mathbb{Q}$. By Deligne's degeneracy theorem [5], $Rp_{U*} \mathbb{Q}(n) = \bigoplus_k R^k p_{U*} \mathbb{Q}[-k]$. So $Rp_{U*} \mathbb{Q}(n)$ is a direct sum of shifted constant sheaves and $\mathbf{H}[-n]$.

The following theorem, which is Theorem C of C. Schnell's paper [12], shows that an analogous decomposition holds on the level of $Rp_*\mathbb{Q}$ provided that **H** is non-constant. As explained by Beilinson in [1], the result is also a direct consequence of a much older paper of Brylinski on the Radon transform and perverse sheaves [4].

Theorem 3.1 (Beilinson, Brylinski, Schnell). Suppose H is non-constant. Then

$$Rp_*\mathbb{Q}[d_{\mathcal{X}}] = \bigoplus_k \mathrm{IC}(R^{n+k}p_{U*}\mathbb{Q})[-k]$$
$$= \mathrm{IC}(\mathbf{H}) \oplus \bigoplus_{k \neq 0} \mathrm{IC}(R^{n+k}p_{U*}\mathbb{Q})[-k].$$

In particular, $Rp_*Q[d_{\mathcal{X}}]$ is the direct sum of $IC(\mathbf{H})$ and (shifted) constant sheaves on P^{\vee} .

Proof of Theorem 1.6. Pick $H \in P^{\vee}$ and set $Y = X \cap H$. We have $IH_H^j(R^{n+i}\pi_*\mathbb{Q}) = 0$ for $ij \neq 0$. So, using Theorem 1.3 (iii), we have

$$H^{n+j}(Y,\mathbb{Q}) = IH_H^j(\mathbb{H}) \oplus IH^0(R^{n+j}\pi_*\mathbb{Q})$$
(3.2)

for $j \neq 0$. By Hard-Lefschetz, $R^{n+j}\pi_*\mathbb{Q}(j) \cong R^{n-j}\pi_*\mathbb{Q}$. So, since $\operatorname{IH}_H^j(\mathbf{H}) = 0$ for j < 0, we get that $b_{n+j}Y - b_{n-j}Y = \dim \operatorname{IH}_H^j(\mathbf{H})$ for j > 0 as desired.

4. Examples

Terminology Reminder. If *S* is an irreducible complex scheme of finite type and *P* is a property of closed points of *S*, then *P* holds for the *general* point of *S* if *P* holds outside of a countable union of proper subschemes of *S*.

Remark 4.1. I am providing this reminder because there seems to be some confusion in the literature about the words "general" and "very general." (Some authors say "very general" for the above notion and "general" when the property holds outside of a closed proper subscheme.) This is the way that the term is used by Voisin in [13, p. 93], and I believe that it conforms to the usage by Laza, Saccà and Voisin in [9].

Cubic 4-folds. The paper [9] starts with a smooth cubic 4-fold X embedded in $P = \mathbb{P}^5$ and considers the family $p: \mathcal{X} \to P^{\vee}$. The family $p_U: \mathcal{X}_U \to U$ of smooth cubic 3 folds gives rise to a variation of Hodge structure $\mathbf{H}_{\mathbb{Z}}$ and a family $\pi: J(\mathbf{H}_{\mathbb{Z}}) \to U$ which turns out to be a family of 5-dimensional abelian varieties.

Theorem 4.2 (Laza—Saccà—Voisin). Suppose the cubic 4-fold X is general. Then there is flat regular compactification $\bar{\pi}: \bar{J} \to P^{\vee}$ with irreducible fibers.

Explanation. The fact that there exists a regular flat compactification $\bar{\pi}: \bar{J} \to P^{\vee}$ is part of the main theorem of [9]. The irreducibility of the fibers is not explicitly stated in [9], but it is an important part of the construction. Proving it amounts to tracing through several definitions and intermediate results in [9], which I now do.

By [9, Definition 4.11], the compactified relative Prym variety, $\overline{\text{Prym}}\,\widetilde{\mathcal{C}}_B/\mathcal{C}_B$, is irreducible (as it is defined as an irreducible component of a larger scheme). By [9, Proposition 4.14], this definition is stable under base change so that the fiber over a point $b \in B$ is also irreducible. Then [9, Proposition 5.1] states that $\overline{\text{Prym}}\,\widetilde{\mathcal{C}}_B/\mathcal{C}_B \to B$ is flat when B is a certain Fano variety \mathcal{F}^0 of lines.

Section 5 of [9] descends the family of Prym varieties over \mathcal{F}^0 (which maps surjectively to \mathbb{P}^5) to $P^\vee = \mathbb{P}^5$. As explained in the paragraph between Lemmas 5.3 and 5.4 of [9], the result is a family $\bar{I} \to P^\vee$ whose pullback to \mathcal{F}^0 is the above family of compactified Prym varieties. Since the compactified Prym varieties are irreducible, the fibers of $\bar{I} \to P^\vee$ are as well.

Corollary 4.3. Suppose X is a general cubic 4-folds and $H \in P^{\vee}$ is any hyperplane. Then $Y := X \cap H$ is palindromic.

This motivates the following conjecture.

Conjecture 4.4. Suppose X is a general complete intersection in $P = \mathbb{P}^N$ of multi-degree (d_1, d_2, \ldots, d_k) with $d_1 \leq \cdots \leq d_k$. Assume that $d_1 \gg 0$. Then, for any hyperplane $H \in P^{\vee}$, the hyperplane section $Y := X \cap H$ is palindromic.

Remark 4.5. Perhaps "≫" could be replaced with a reasonable lower bound.

Here is a simple argument proving the conjecture when X is a general surface complete intersection of multi-degree not equal to (2), (3) or (2,2). In that case, the Noether-Lefschetz theorem says that the Néron-Severi group of X is \mathbb{Z} with generator $[X \cap H]$ (for any hyperplane H). (See Voisin's book [13, Theorem 3.32] and [8, Theorem 1] for modern proofs.) It follows easily that $Y := X \cap H$ is irreducible (since [Y] cannot be the direct sum of two non-trivial effective divisors). But, since Y is a curve, this implies that Y is palindromic.

Proposition 4.6. Suppose X is a smooth, hypersurface in $P = \mathbb{P}^N$ and $H \in P^{\vee}$ is any hyperplane section. Then $Y := X \cap H$ is weakly palindromic.

Sketch. This is well-known (see [6, Theorem 2.1]). So I only give a sketch. The main point is that Y has isolated singularities. From this one can either use the Clemens-Schmid exact sequence or an argument comparing the intersection cohomology with the ordinary cohomology.

Suppose X is an arbitrary smooth cubic 4-fold. Since every hyperplane section is weakly palindromic, Corollary 1.7 does not rule out the existence of a flat regular compactification $\bar{\pi}: \bar{J} \to P^{\vee}$. However, it can rule out the existence of a flat regular compactification with irreducible fibers: If X is a cubic fourfold containing a non-palindromic cubic 3-fold $Y = X \cap H$, then the fiber $\bar{\pi}^{-1}\{H\}$ is not irreducible. To find such a cubic 4-fold we use the following Lemma.

Lemma 4.7. Suppose Y = V(f) is a degree d hypersurface in \mathbb{P}^{N-1} . Fix an embedding $\mathbb{P}^{N-1} \subset \mathbb{P}^N$. Then there is a smooth degree d hypersurface X in \mathbb{P}^N such that $Y = X \cap \mathbb{P}^N$ if and only if Y has isolated singularities.

Sketch. The "only if" part is easy (and was already used above in the proof of Proposition 4.6). For the "if" part, suppose $f(x_1,\ldots,x_N)$ is a degree d homogeneous polynomial. Consider the linear subspace V in $H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(d))$ spanned by f and x_0h as h runs over all degree d-1 homogeneous polynomials in the N+1 variables. The base locus of the linear system |V| is Y. So the general member of |V| is smooth off of Y by Bertini. But the singularities of $g=f+x_0h$ on Y are contained in the intersection of V(h) with the singularities of Y. Therefore, the general member of |V| is smooth.

Now, I use a result of Segre and Fano as interpreted by Dolgachev.

Theorem 4.8. There exists a cubic 3 fold Y with 10 ordinary double points and $b_4Y = 6$.

Proof. See [7, Proposition 1.1] and the discussion shortly before and shortly after. \Box

Corollary 4.9. There exists a smooth, cubic 4-fold X containing a cubic 3-fold Y with $b_4Y=6$. For such an X, there is no flat regular compactification $\bar{\pi}: \bar{J} \to P^{\vee}$ with irreducible fibers.

Proof. The cubic 3-fold Y with $b_4Y=6$ is not palindromic since $b_2Y=1$ by weak Lefschetz. Using Lemma 4.7, we can find a smooth cubic 4-fold X containing Y. The result then follows from Corollary 1.7.

Quadrics in Cubic 4-**folds.** Suppose X is a smooth 2m-dimensional subvariety in $P = \mathbb{P}^N$ as in the beginning of §3. The family $\pi: J(\mathbf{H}_{\mathbb{Z}}) \to U$ of intermediate Jacobians will be an abelian scheme provided the Hodge structure $H^{2m-1}Y$ of a smooth hyperplane section Y is level ≤ 1 . This means that $H^{2m-1}(Y,\mathbb{C}) = F^mH^{2m-1}(Y,\mathbb{C})$. As in §3, we set n = 2m - 1.

I do not have a very clear idea how often the situation above occurs for arbitrary X. However, in [10], Rapoport has a table of all complete intersections Y for which the Hodge level of the middle dimensional cohomology is 1. Write $V_n(d_1,\ldots,d_k)$ for the family of smooth complete intersections of dimension n coming from intersecting k hypersurfaces of degrees d_1,\ldots,d_k in \mathbb{P}^{n+k} . Then, according to Rapoport's table, the only non-empty families of level 1 with n odd are: $V_n(2,2)$, $V_n(2,2,2)$, $V_3(3)$, $V_3(2,3)$, $V_5(3)$ and $V_3(4)$.

The case where Y is a cubic 3-fold, $V_3(3)$, was the subject of the last subsection. In this, section I want to consider $V_3(2,3)$.

So fix a cubic 4-fold X embedded in \mathbb{P}^5 . Set $\mathcal{L}:=\mathcal{O}_{\mathbb{P}^5}(1)_{|X}$. An easy computation shows that $|\mathcal{L}^2|=\dim H^0(\mathbb{P}^5,\mathcal{O}_{\mathbb{P}^5}(2))-1=20$. So, the complete linear system \mathcal{L}^2 , gives an embedding of X into $P:=\mathbb{P}^{20}$. Cutting X with hyperplanes $H\subset P$, we get a family $p:\mathcal{X}\to P^\vee$ as in the beginning of $\S 3$ which is smooth over an open subset $U\subset P^\vee$. Since the smooth hyperplane sections are complete intersections of type $V_3(2,3)$, they have level 1. Therefore, the family $\pi:J(\mathbf{H}_{\mathbb{Z}})\to U$ is an abelian scheme. In fact, Rapoport's table also gives $b_3Y=40$ for Y of type $V_3(2,3)$. So the family $\pi:J(\mathbf{H}_{\mathbb{Z}})\to U$ is, in fact, a family of 20-dimensional abelian varieties over a 20 dimensional base.

Theorem 4.10. Let X be a cubic 4-fold as above embedded in $P = \mathbb{P}^{20}$. There is no regular flat compactification $\bar{\pi}: \bar{J} \to P^{\vee}$ of the family $\pi: J(\mathbf{H}_{\mathbb{Z}}) \to U$ of intermediate Jacobians.

Proof. The elements $H \in P^{\vee}$ are in 1-1 correspondence with quadrics in \mathbb{P}^5 . Pick two hyperplanes L_1 and L_2 in \mathbb{P}^5 such that the cubic 3 folds $Y_i := X \cap L_i$ are smooth and distinct. Let H be the point in P^{\vee} corresponding to the union $L_1 \cup L_2$. Then $Y := X \cap H$ has two irreducible components. Therefore $b_6Y = 2$. So Y is not weakly palindromic. The result follows from Corollary 1.7.

Remark 4.11. In [9, §1.3], Laza, Saccà and Voisin point out that the total space of the family $J(\mathbf{H}_{\mathbb{Z}}) \to U$ admits a holomorphic symplectic form which would extend to any compactification $\bar{\pi}: \bar{J} \to P^{\vee}$. They also show that this form is non-degenerate above a quadric if and only if the quadric is non-degenerate [9, Lemma 1.20]. Theorem 4.10 can be seen as an amplification of this result.

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