PRIMITIVE FORMS AND FROBENIUS STRUCTURES ON THE HURWITZ SPACES

TODOR MILANOV

ABSTRACT. The main goal of this paper is to introduce the notion of a primitive form for a generic family of Hurwitz covers of \mathbb{P}^1 with a fixed ramification profile over infinity. We prove that primitive forms are in one-to-one correspondence with semi-simple Frobenius structures on the base of the family. Furthermore, we introduce the notion of a polynomial primitive form and show that the corresponding class of Frobenius manifolds contains the Hurwitz Frobenius manifolds of Dubrovin. Finally, we apply our theory to investigate the relation between the Eynard–Orantin recursion and Frobenius manifolds.

Contents

1. Introduction	2
1.1. Motivation	2
1.2. Organization of the paper	4
1.3. Future directions	$\frac{4}{5}$
2. Statement of the main results	5
2.1. Branched coverings with a fixed ramification profile over infinity	5
2.2. Saito structure	6
2.3. Primitive forms and Frobenius structures	8
3. The period map	10
3.1. The Kodaira–Spencer isomorphism	10
3.2. The Gauss–Manin connection	11
3.3. The period isomorphism	13
4. Good basis	16
4.1. Higher-residue pairing	16
4.2. Construction of the good basis	19
4.3. The Gauss–Manin connection	23
5. Primitive forms	26
5.1. Definition	26
5.2. Preliminary notation	27
5.3. Differential and algebraic constraints	28
5.4. Solving the equations for primitive forms	31
6. Polynomial primitive forms	33
6.1. Polynomiality and the sheaf \mathcal{H} .	33
6.2. Primary differentials	34

6.3. Polynomiality of the <i>R</i> -matrix	39
6.4. Two-dimensional Frobenius manifolds	41
7. Topological recursion and semi-simple Frobenius structures	44
7.1. The periods of the Frobenius structure	44
7.2. Stationary phase asymptotic	46
7.3. The local EO recursion	48
7.4. Frobenius manifolds and EO recursion	54
8. Topological recursion for twisted de Rham cohomology	56
8.1. Calibration of the Frobenius structure	57
8.2. The total descendant potential	60
8.3. Topological recursion and descendants	61
8.4. Generalization	63
8.5. Topological recursion for polynomial primitive forms	64
References	66

1. Introduction

1.1. **Motivation.** The main motivation behind this paper is the higher-genus reconstruction of Givental and its applications to the classical Riemann-Hilbert problem. Recall that if X is a smooth projective variety with semi-simple quantum cohomology, then Givental's higher genus reconstruction expresses all Gromov— Witten (GW) invariants of X in terms of the semi-simple Frobenius structure underlying quantum cohomology. The reconstruction was proposed in [16, 17] and proved in full generality by C. Teleman in [30]. Using Givental's reconstruction as a definition we can define the analogues of GW-invariants for any semi-simple Frobenius manifold. The generating function of all genus invariants is called the total descendent potential of the corresponding semi-simple Frobenius manifold. On the other hand, a semi-simple Frobenius manifold can be defined as a solution to a Riemann-Hilbert problem (see [7]). Usually, we have a good knowledge of the corresponding monodromy data, while the semi-simple Frobenius manifold depends on it in a highly transcendental way. We are interested in the question whether we can express the invariants in terms of the monodromy data in an algebraic way, e.g., via some explicit recursions.

One possible way to answer this question was proposed in our joint work [3]. We have defined a W-algebra (depending only on the monodromy data) and proved that each state in the W-algebra provides differential constraints for the total descendent potential of a simple singularity. The constraints can be interpreted as recursion that uniquely determines the coefficients of the total descendent potential (see [21]), so we get a positive answer of the question raised above for the Frobenius manifolds corresponding to simple singularities. The ideas from [3] are straightforward to generalize to any semi-simple Frobenius manifold, but the problem is that it is very hard to find states in the W-algebra.

In the paper [22], we have tested, in the settings of A_N -singularity, a new idea to construct states in the W-algebra based on the topological recursion (see [13, 4]). Let us recall some basic settings for the recursion. The starting pont is a triple (Σ, x, y) , where $x : \Sigma \to \mathbb{P}^1$ is a branched covering and y is a meromorphic function satisfying some genericity assumptions. The Riemann surface Σ is also known as the spectral curve. Using this data Eynard and Orantin have proposed a recursion that produces a set of symmetric forms

$$\omega_{g,n}(q_1,\ldots,q_n) \in T_{q_1}^*\Sigma \otimes \cdots \otimes T_{q_n}^*\Sigma, \quad 2g-2+n>0,$$

defined for all $(q_1, \ldots, q_n) \in \Sigma^n$ such that q_i is not a ramification point of $x : \Sigma \to \mathbb{P}^1$ and having at most finite order poles if some q_i is a ramification point (see [13] for precise definitions). We will refer to the recursion as the topological recursion or the Eynard–Orantin (EO) recursion. We are interested to find other examples of semi-simple Frobenius manifolds for which the topological recursion can be used to construct states in the \mathcal{W} -algebra. This problem can be splited into two parts. The first part is to classify all semi-simple Frobenius manifolds that correspond to topological recursion. The second part is to determine whether the topological recursion has a global contour formulation in a sense to be clarified below.

The problem of describing the correspondence between semi-simple Frobenius manifolds and topological recursion was essentially solved in the recent paper [11]. More precisely, the authors proved that the Hurwitz Frobenius manifolds (see [7]) correspond to an EO recursion, provided we relax the condition that yis a meromorphic function to the condition that the 1-form dy is holomorphic in a neighborhood of the finite ramification points. Although it is not stated explicitly in [11], using the results of this paper it is not very hard to prove that the Hurwitz Frobenius manifolds are the only Frobenius manifolds that correspond to an Eynard-Orantin recursion (with the relaxed condition on y). In particular, the set of Frobenius manifolds that correspond to the original EO recursion (as defined in [13]) is contained in the set of Hurwitz Frobenius manifolds. More precisely, recall that a Hurwitz Frobenius manifold (see [7]) depends on the choice of a primary differential ϕ on Σ (see also Section 6.2). In order, to have the original EO recursion we have to require that $\phi = dy$ for some meromorphic function y. Therefore, from the point of view of the original EO recursion, it is still an open problem to classify primary differential that are exact as meromorphic forms.

The starting point of the current paper is the observation that many of the constructions in [11] have a very natural interpretation in terms of K. Saito's theory of primitive forms [26]. Our main goal is to develop in a systematic way the notion of primitive forms for families of Hurwitz covers. The main outcome of our approach is that we were able to find an interesting generalization of the topological recursion which allows us to extend the correspondence described in [11] to include a wider class of semi-simple Frobenius manifolds. We also found a very interesting identity expressing the so-called descendant correlators in terms of oscillatory integrals whose integrands are defined by the topological recursion.

1.2. Organization of the paper. The main part of the paper is devoted to constructing Frobenius structures on the base of a generic family of Hurwitz covers. Our approach is based on K. Saito's theory of primitive forms (see [19, 27]). The general theory is developed in Sections 2, 3, 4, and 5. Many of the results here can be obtained from the work of A. Douai and C. Sabbah (see [5, 6]), which generalizes the Hodge theoretic approach of M. Saito (see [28]). Our approach has the advantage of being elementary. We were able to prove the existence of primitive forms without relying on Hodge theory thanks to a special set of holomorphic forms that were introduced essentially in the work of V. Shramchenko [29].

Let us point out that if we allow an arbitrary primitive form, then we can obtain any semi-simple Frobenius manifold. However, the analytic properties of the underlying Frobenius manifold would not be captured by the geometry of the underlying spectral curve. On the other hand, our original goal is to understand W-constraints (and integrable systems) using the geometry of the spectral curve. That is why in Section 6, we have proposed the notion of polynomial primitive forms and established several basic properties. It will be interesting to classify all Frobenius manifolds that correspond to polynomial primitive forms. We carry out this classification for Frobenius manifolds of dimension 2. We obtained a discrete set of Frobenius manifolds, which includes the Frobenius manifolds corresponding to A_2 -singularity and quantum cohomology of \mathbb{P}^1 . We also prove that the primary differentials (see [7]) are polynomial primitive forms.

In Section 7 we define the problem of comparing semi-simple Frobenius manifolds and the topological recursion. We prove that a semi-simple Frobenius manifold corresponds to a topological recursion if and only if it is one of the Dubrovin's Hurwitz Frobenius manifolds.

Finally, in Section 8 we express the descendant correlators in terms of oscillatory integrals, whose integrands are precisely the forms defined by the EO recursion. Similar formulas were derived in the settings of equivariant mirror symmetry for \mathbb{P}^1 in [14]. Motivated by these results we propose to think of the EO recursion as defining twisted de Rham cohomology classes. We found a generalization of the EO recursion that deserves a further investigation from the point of view of mirror symmetry.

1.3. Future directions. Let us point out that the EO recursion is defined in terms of sums of residues over the ramification points of $x: \Sigma \to \mathbb{P}^1$ and it is local in a sense that the forms whose residues are computed are defined only locally near each ramification point. Nevertheless, if we require that y satisfies some extra properties, then Bouchard and Eynard proved in [4] that the sum of the local residues of the EO recursion can be replaced with a contour integral of a global meromorphic form on Σ . This special class of EO-recursions that admits a global contour integral presentation will be called Bouchard-Eynard (BE) recursions. It is the BE recursion that was identified with W-constraints in [22]. Although the EO recursion can be generalized in various ways, for our purposes, the interesting

generalizations are the ones that admit a global contour formulation. It is very interesting to investigate the existence of a BE-type recursion for the generalization of the topological recursion proposed in this paper. In fact the existence of a BE recursion is an open problem even for the generalization of the EO recursion (proposed in [11]) corresponding to the Hurwitz Frobenius manifolds.

Our classification in the rank 2 case, shows that most of the Frobenius manifolds corresponding to finite reflection groups do not correspond to polynomial primitive forms. Nevertheless, based on our recent work in [24], it is clear how to generalize the methods of this paper in order to include the case of Frobenius manifolds corresponding to finite reflection groups. Namely, we have to allow for the spectral curve to be an orbifold Riemann surface. It will be interesting to search for a corresponding generalization of the EO and BE recursions.

Acknowledgements. I would like to thank Ilya Karzhemanov and especially Tomoyuki Abe for very useful discussions on sheaf cohomology. Also, I would like to thank Sergey Lando for e-mailing me a draft of his work in progress on a related subject. This work is partially supported by JSPS Grant-In-Aid 26800003 and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. I would like also to thank the mathematical research institute MATRIX in Australia where part of this research was performed.

2. Statement of the main results

Let us begin by introducing the main ingredients of our construction and stating the results that we would like to prove.

2.1. Branched coverings with a fixed ramification profile over infinity. Let $f: \Sigma \to \mathbb{P}^1$ be a branched covering. Recall that a point $p \in \Sigma$ is called a ramification point if df(p) = 0 and that $u \in \mathbb{P}^1$ is called a branch point if $f^{-1}(u)$ contains a ramification point. We will be interested only in branched coverings f that are generic in the following sense: if $u \in \mathbb{P}^1 \setminus \{\infty\}$ is a branch point, then the fiber $f^{-1}(u)$ contains only 1 ramification point p and the local degree of p is 2, i.e., there is a local coordinate p on p near p, s.t., $p(q) = p + \frac{1}{2}t(q)^2$ for all $p \in \Sigma$ sufficiently close to p.

Let u_i° $(1 \leq i \leq N)$ be the set of finite branch points and ∞_i $(1 \leq i \leq d)$ be the points of the fiber $f^{-1}(\infty)$. The Riemann-Hurwitz formula yields

$$N = 2g - 2 + d + \sum_{i=1}^{d} m_i,$$

where g is the genus of Σ and m_i $(1 \leq i \leq d)$ is the local degree of f near the point ∞_i $(1 \leq i \leq d)$. Let us choose a reference point u_0° on $\mathbb{C} \setminus \{u_1^{\circ}, \ldots, u_N^{\circ}\}$. Then we have a monodromy representation

$$\rho: \pi_1(\mathbb{C}\setminus\{u_1^\circ,\ldots,u_N^\circ\},u_0^\circ)\to S_m, \quad m:=m_1+\cdots+m_d,$$

where S_m is the group of permutations of the points in the fiber $f^{-1}(u_0^{\circ})$.

Let us denote by B the universal cover of the configuration space

$$\{u \in (\mathbb{P}^1)^{N+1} \mid u_i \neq u_j \text{ for } i \neq j, \text{ and } u_{N+1} = \infty\}.$$

The points of B consists of pairs $(u, [\gamma])$ of a point u in the configuration space and the homotopy class of a path (in the configuration space) from $u^{\circ} := (u_1^{\circ}, \dots, u_N^{\circ})$ to u. It is known that B is a contractible Stein manifold. Put

$$D_i = \{(\widetilde{u}, \lambda) \in B \times \mathbb{P}^1 \mid \lambda = u_i\}, \quad 1 \le i \le N + 1,$$

where $\widetilde{u} = (u, [\gamma]) \in B$ and u_i is the *i*-th component of u. Note that the projection on the first factor

$$\pi: B \times \mathbb{P}^1 \setminus \{D_1 \cup \cdots \cup D_{N+1}\} \to B$$

is a smooth fibration with fibers diffeomorphic to $\mathbb{C}\setminus\{u_1^\circ,\ldots,u_N^\circ\}$. The latter is identified with the fiber of π over the point $(u^\circ,[1])$. The long exact sequence of homotopy groups and the contractibility of B imply that the natural inclusion of the fiber induces an isomorphism

$$\pi_1(\mathbb{C}\setminus\{u_1^\circ,\ldots,u_N^\circ\})\cong\pi_1(B\times\mathbb{P}^1\setminus\{D_1\cup\cdots\cup D_{N+1}\}),$$

where we suppressed the base points in the above notation, but they are uniquely determined from u_0° and the embedding of the fiber in the total space of the fibration. In particular, the monodromy representation ρ from above defines a representation

$$\rho: \pi_1(B \times \mathbb{P}^1 \setminus \{D_1 \cup \cdots \cup D_{N+1}\}) \to S_m.$$

Let

$$\widetilde{\varphi}: \widetilde{X} \to \mathbb{P}^1 \setminus \{D_1 \cup \cdots \cup D_{N+1}\}$$

be the degree-m covering whose monodromy representation is ρ . Using the Riemann's extension theorem we can extend $\tilde{\varphi}$ to a branched covering (for more details see [15], Proposition 1.2)

$$\overline{\varphi}: \overline{X} \to B \times \mathbb{P}^1.$$

2.2. Saito structure. Let us denote by $\Omega^{p}_{\overline{X}/B}$ (p=0,1) the sheaf of relative holomorphic p-forms on \overline{X} relative to

$$\overline{\pi} := \operatorname{pr}_B \circ \overline{\varphi} : \overline{X} \to B.$$

Put $D_{\infty} := \overline{\varphi}^{-1}(B \times \{\infty\})$ and let

(1)
$$\Omega^{\underline{p}}_{\overline{X}/B}(m) := \Omega^{\underline{p}}_{\overline{X}/B} \otimes \mathcal{O}_{\overline{X}}(m D_{\infty}), \quad m \ge 0$$

be the sheaf of relative p-forms that are holomorphic except for a possible pole of order at most m along the divisor D_{∞} . The set (1) is naturally a filtered directed system. Let us define

$$\Omega^{p,\infty}_{\overline{X}/B} := \varinjlim_{m} \Omega^{p}_{\overline{X}/B}(m), \quad p = 0, 1.$$

Put $X := \overline{X} \setminus D_{\infty}$ and let

$$\varphi := \overline{\varphi}|_X : X \to B \times \mathbb{C}.$$

Following K. Saito [26], we define the twisted de Rham cohomology \mathcal{H} of the holomorphic function

$$F := \operatorname{pr}_{\mathbb{C}} \circ \varphi : X \to \mathbb{C}$$

by the following formula

$$\mathcal{H}:=\overline{\pi}_*\Omega^{1,\infty}_{\overline{X}/B}[z]/(zd_{\overline{X}/B}+d_{\overline{X}/B}F\wedge)\overline{\pi}_*\Omega^{0,\infty}_{\overline{X}/B}[z]),$$

where $d_{\overline{X}/B}$ is the relative de Rham differential. This is a sheaf on B and the completion $\widehat{\mathcal{H}} := \mathcal{H} \otimes \mathbb{C}[\![z]\!]$ turns out to be isomorphic to $\mathcal{T}_B[\![z]\!]$, where \mathcal{T}_B is the sheaf of holomorphic vector fields on B. Every section ω of $\mathcal{H} \otimes \mathbb{C}[z, z^{-1}]$ defines a family of oscillatory integrals

$$\Gamma \mapsto (-2\pi z)^{-1/2} \int_{\Gamma} e^{F/z} \omega,$$

where the integration cycle $\Gamma = \Gamma_{u,z}$ depends on the choice of parameters $(u,z) \in B \times \mathbb{C}^*$ and it is an element of the relative homology group

$$H_1(X_u, \operatorname{Re}(F/z) \ll 0; \mathbb{C}) := \varprojlim_{m \in \mathbb{Z}_{>0}} H_1(X_u, \operatorname{Re}(F/z) < -m; \mathbb{C}) \cong \mathbb{C}^N.$$

The above homology groups form a vector bundle on $B \times \mathbb{C}^*$ equipped with a flat Gauss–Manin connection. The sheaf $\mathcal{H} \otimes \mathbb{C}[z,z^{-1}]$ has an induced connection ∇ , which is also called Gauss–Manin connection, s.t.,

$$\int_{\Gamma} e^{F/z} \nabla_v \omega = v \int_{\Gamma} e^{F/z} \omega,$$

where $v \in \mathcal{T}_B$ is a vector field and Γ is any flat family of semi-infinite cycles. Using the above formula with $v = \partial_z$ we can also extend ∇ in the z-direction.

The sheaf \mathcal{H} is equipped also with a non-degenerate pairing K, which as we will see later on coincides with K. Saito's higher residue pairing. Namely,

$$K: \mathcal{H} \otimes_{\mathcal{O}_B} \mathcal{H} \to \mathcal{O}_B[\![z]\!]z$$

is defined by

(2)
$$K(\omega_1, \omega_2) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{N} \int_{\Gamma_i} e^{F/z} \omega_1 \int_{\Gamma_i^{\vee}} e^{-F/z} \omega_2^*,$$

where * denotes the involution $z \mapsto -z$, $\{\Gamma_i\}_{i=1}^N$ is any basis of flat semi-infinite cycles, $\{\Gamma_j^{\vee}\}_{j=1}^N$ is a dual basis, and we are identifying Γ_j^{\vee} with homology cycles via the intersection pairing

$$H_1(X_u, \operatorname{Re}(F/z) \ll 0; \mathbb{Z}) \times H_1(X_u, \operatorname{Re}(F/z) \gg 0; \mathbb{Z}) \to \mathbb{Z}.$$

Note that this is a perfect pairing, i.e., the intersection matrix in an appropriate integral basis is non-degenerate with determinant ± 1 .

Remark 2.1. The above homology groups and the intersection pairing can be computed via Morse theory. Namely, for fixed $(u, z) \in B \times \mathbb{C}^*$ the function $g := \text{Re}(F/z) : X_u \to \mathbb{R}$ is a real Morse function. Note that the critical points of g are by definition the ramification points of the covering $X_u \to \mathbb{C}$. In particular, the cycles Γ_i (resp. Γ_i^{\vee}) can be constructed as the gradient trajectories of -g (resp. g) that flow out of the i-th critical point.

Remark 2.2. If \mathcal{F} is a sheaf of Abelian groups on a manifold M, then we denote by $\mathcal{F}[z]$ the sheaf on M obtained by sheafification of the presheaf $V \mapsto \mathcal{F}(V)[z]$. Note that if V has finitely many connected components, then $\mathcal{F}[z](V) = \mathcal{F}(V)[z]$. Similarly we define $\mathcal{F}[z]$ and $\mathcal{F}[z, z^{-1}]$.

Remark 2.3. The notation $\mathcal{H} \otimes \mathbb{C}[\![z]\!]$ (resp. $\mathcal{H} \otimes \mathbb{C}[z,z^{-1}]$) is for the sheaf defined in the same way as \mathcal{H} except the we replace the sheaves $\overline{\pi}_*\Omega^{p,\infty}_{\overline{X}/B}[z]$ with $\overline{\pi}_*\Omega^{p,\infty}_{\overline{X}/B}[\![z]\!]$ (resp. $\overline{\pi}_*\Omega^{p,\infty}_{\overline{X}/B}[z,z^{-1}]$).

Both the connection ∇ and the higher-residue pairing K extend uniquely to the completion $\widehat{\mathcal{H}}$. The data $(\widehat{\mathcal{H}}, K, \nabla)$ will be called *Saito structure*. It allows us to introduce the notion of a *primitive form* in $\widehat{\mathcal{H}}$ (see Section 3).

2.3. Primitive forms and Frobenius structures. Let us outline how to construct the Frobenius structure using a primitive form $\omega \in \widehat{\mathcal{H}}(V)$, where $V \subseteq B$ is an open subset (see [19, 27]). The critical values of F, i.e., the branch points u_i , $1 \le i \le N$ will turn out to be canonical coordinates so the multiplication is

$$\partial_{u_i} \bullet \partial_{u_i} := \delta_{i,j} \, \partial_{u_i},$$

the Frobenius pairing is

$$(\partial_{u_i}, \partial_{u_j}) := K^{(0)}(z \nabla_{\partial_{u_i}}[\omega], z \nabla_{\partial_{u_j}}[\omega]),$$

and the Euler vector field is

$$E = u_1 \partial_{u_1} + \dots + u_N \partial_{u_N}.$$

Using this data we define a connection $\widetilde{\nabla}$ on the vector bundle $\operatorname{pr}_V^* TV$, where $\operatorname{pr}_V : V \times \mathbb{C}^* \to V$ is the projection. Namely

$$\widetilde{\nabla}_{\partial_{u_i}} := \nabla^{\text{L.C.}}_{\partial_{u_i}} + z^{-1} \partial_{u_i} \bullet, \quad 1 \leq i \leq N,$$

and

$$\widetilde{\nabla}_{\partial_z} := \partial_z - z^{-1}\theta - z^{-2}E \bullet,$$

where $\nabla^{\text{L.C.}}$ is the Levi–Civita connection of the Frobenius pairing, for a vector field $v \in \mathcal{T}_B(V)$ we denoted by $v \bullet \in \text{End}(\mathcal{T}_B)(V)$ the linear operator of Frobenius multiplication by v, and $\theta \in \text{End}(\mathcal{T}_B)(V)$ is defined by

(3)
$$\theta(v) = \nabla_v^{\text{L.C.}} E - \left(1 - \frac{D}{2}\right) v,$$

where the constant D, known as *conformal dimension*, is such that θ is skew-symmetric with respect to the Frobenius pairing.

Remark 2.4. We do not assume that θ is diagonalizable.

Given a primitive form $\omega \in \widehat{\mathcal{H}}(V)$, we can construct a period isomorphism

$$\mathcal{T}_B(V)[\![z]\!] \cong \widehat{\mathcal{H}}(V), \quad v \mapsto z \nabla_v \omega.$$

The axioms of a primitive form imply that the period isomorphism intertwines the connection $\widetilde{\nabla}$ and the Gauss–Manin connection, i.e.,

$$z\nabla_v\nabla_w(z^{-1/2}\omega) = z\nabla_{\widetilde{\nabla}_v w}(z^{-1/2}\omega)$$

for all $v, w \in \mathcal{T}_{B \times \mathbb{C}^*}$. In particular, the flatness of the Gauss–Manin connection implies that $\widetilde{\nabla}$ is flat. It remain only to recall that the axioms of a Frobenius structure are equivalent to the flatness of the connection $\widetilde{\nabla}$ (see [7]).

Theorem 2.5. Let $V \subset B$ be an open subset, then every semi-simple Frobenius structure on V, such that, the canonical vector fields coincide with ∂_{u_i} , $1 \leq i \leq N$, is the Frobenius structure associated to a primitive form in $\widehat{\mathcal{H}}(V)$.

Theorem 2.5 might look a bit surprising at first, because it essentially says that the branch covering $f: \Sigma \to \mathbb{P}^1$ that we used to set up the entire theory is irrelevant. The reason for this is that if we work with the completion $\widehat{\mathcal{H}}$, then there is an excision principle (see Proposition 3.4, b)) which allows us to replace the complex manifold \overline{X} with a tubular neighborhood of the relative critical variety

$$C = \{ p \in X \mid d_{X/B}F(p) = 0 \}.$$

The variety $C \cong B^N$, which explains why the choice of the branch covering is irrelevant. In order to obtain Frobenius manifolds that depend on the branch covering $f: \Sigma \to \mathbb{P}^1$ in an essential way, we propose to work with primitive forms $\omega \in \mathcal{H}$. Such forms can be represented by holomorphic forms that depend polynomially on z. We refer to them as polynomial primitive forms.

Theorem 2.6. The primary differentials of Dubrovin are polynomial primitive forms.

The primary differentials are divided into 5 types. Except for type IV, they are elements of $\Omega^{1,\infty}_{\overline{X}/B}(\overline{X})$, so each primary differential determines naturally an element in $\mathcal{H}(B)$. The primary differentials of type IV are multivalued, but they have analytic branches in a neighborhood of C. Using the excision principle from Proposition 3.4, b), we can construct cohomology classes in $\widehat{\mathcal{H}}$, which turn out to be in \mathcal{H} . As a byproduct of our argument proving Theorems 2.5 and 2.6 we got the following important result.

Corollary 2.7. The sheaf \mathcal{H} is a free $\mathcal{O}_B[z]$ -module of rank N. Moreover, there exists an $\mathcal{O}_B[z]$ -basis $\{\omega_i\}_{i=1}^N \subset \mathcal{H}(B)$ such that $K(\omega_i, \omega_j) = z\delta_{ij}$.

The formalism of primitive forms allows us to answer the question when does an EO recursion define a Frobenius manifold. More precisely, following [23] we introduce the notion of a local EO recursion. It is proved in [23] that every semi-simple Frobenius manifold provides a solution to a local EO recursion. We will prove that every EO recursion also provides a solution to a local EO recursion. Therefore, we can define the notion of a semi-simple Frobenius structure that is a solution to an EO recursion.

Theorem 2.8. A semi-simple Frobenius manifold is a solution to an EO recursion if and only if it is a Hurwitz Frobenius manifold in the sense of Dubrovin, i.e., the corresponding primitive form is a sum of homogeneous primary differentials of the same degree.

3. The period map

The goal of this section is to introduce the notion of a primitive form in $\widehat{\mathcal{H}}$. Our settings are slightly different from the original ones (see [26]), but the necessary modifications are straightforward. We are going to use the following notation. Let $\pi := \overline{\pi}|_X : X \to B$ and $\Omega^1_{X/B}$ be the sheaf of sections of the relative cotangent bundle $T^*_{X/B} := T^*X/\pi^*T^*B$. The fiber of $T^*_{X/B}$ at a point $p \in X$ is $T^*_pX_{\pi(p)}$. If $\omega \in \Omega^1_{X/B}(U)$ and $p \in U$, then we denote by $\omega(p) \in T^*_pX_{\pi(p)}$ the value of the section ω at p.

3.1. The Kodaira–Spencer isomorphism. Let us define the following sheaves of vector fields on \overline{X} :

$$\mathcal{T}_{\overline{X}}^{\infty} = \varinjlim_{m} \ \mathcal{T}_{\overline{X}}(m), \quad \mathcal{T}_{\overline{X}/B}^{\infty} = \varinjlim_{m} \ \mathcal{T}_{\overline{X}/B}(m),$$

where $\mathcal{T}_{\overline{X}}$ (resp. $\mathcal{T}_{\overline{X}/B}$) is the sheaf of holomorphic (resp. holomorphic relative) vector fields and for every sheaf \mathcal{F} of $\mathcal{O}_{\overline{X}}$ -modules we put $\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_{\overline{X}}(mD_{\infty})$.

Lemma 3.1. Let $U \subset B$ be an open Stein subset. Then every holomorphic vector field $v \in \mathcal{T}_B(U)$ admits a lift $\widetilde{v} \in \mathcal{T}_{\overline{X}}^{\infty}(\overline{\pi}^{-1}(U))$, such that $v = \overline{\pi}_*\widetilde{v}$.

Proof. Recall that we have the following exact sequence

$$0 \to \mathcal{T}_{\overline{X}/B}(m) \to \mathcal{T}_{\overline{X}}(m) \to \overline{\pi}^* \mathcal{T}_B \otimes \mathcal{O}_{\overline{X}}(m) \to 0,$$

for every $m \in \mathbb{Z}$. Let us choose m > 4g-4. Note that the cohomology groups $H^1(\overline{X}_u, \mathcal{T}_{\overline{X}_u}(m)) = 0$, because $\mathcal{T}_{\overline{X}_u}(m)$ is a line bundle on \overline{X}_u of degree 2-2g+m and all line bundles of degree more than the degree of the canonical bundle have vanishing higher cohomologies. Furthermore, the map $\overline{\pi}: \overline{X} \to B$ is a proper regular map between complex manifolds, where regular means that $d_p\overline{\pi}: T_p\overline{X} \to T_{\overline{\pi}(p)}B$ is surjective for all $p \in \overline{X}$. We get that the higher direct image sheaf $R^1\overline{\pi}_*(\mathcal{T}_{\overline{X}}(m)) = 0$, because it is a coherent sheaf on B with vanishing fibers (see [18], Theorem 10.5.5). We get the following exact sequence of \mathcal{O}_B -modules:

$$0 \to \overline{\pi}_* \mathcal{T}_{\overline{X}/B}(m) \to \overline{\pi}_* \mathcal{T}_{\overline{X}}(m) \to \mathcal{T}_B \otimes \overline{\pi}_* \mathcal{O}_{\overline{X}}(m) \to 0.$$

On the other hand, we have an injective map

$$0 \to \mathcal{T}_B(U) \to (\mathcal{T}_B \otimes \overline{\pi}_* \mathcal{O}_{\overline{X}}(m))(U) \cong \mathcal{T}_{\overline{X}}(m)(\overline{\pi}^{-1}(U))/\mathcal{T}_{\overline{X}/B}(m)(\overline{\pi}^{-1}(U)),$$

where the isomorphism holds, because U is Stein and $\overline{\pi}_* \mathcal{T}_{\overline{X}/B}(m)$ is a coherent sheaf. In particular, every holomorphic vector field $v \in \mathcal{T}_B(U)$ admits a lift $\widetilde{v} \in \mathcal{T}_{\overline{X}}(m)(\overline{\pi}^{-1}(U)) \subset \mathcal{T}^{\infty}_{\overline{Y}}(\overline{\pi}^{-1}(U))$.

If $v \in \mathcal{T}_B(U)$, then we cover $U = \bigcup_i U_i$ with open Stein subsets, choose a lift \widetilde{v}_i of $v|_{U_i}$ for every i (see Lemma 3.1), and define the section $\widetilde{v}(F)|_C \in \pi_*\mathcal{O}_C(U)$ by gluing the sections $\widetilde{v}_i(F)|_C \in \pi_*\mathcal{O}_C(U_i)$. It is straightforward to check that the gluing is possible and that the construction is independent of the choices of lifts. Therefore we get a map (of \mathcal{O}_B -modules)

$$\mathcal{T}_B \to \pi_* \mathcal{O}_C, \quad v \mapsto \widetilde{v}(F)|_C,$$

which will be called the *Kodaira–Spencer map*. Note that C is a disjoint union of N analytic varieties, each isomorphic to B via the map $\pi := \overline{\pi}|_X : X \to B$. Using this fact, it is straightforward to check that the Kodaira–Spencer map is an isomorphism and that the induced multiplication \bullet on \mathcal{T}_B takes the form

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{ij} \partial_{u_j}.$$

Indeed, the connected components of C are

$$C_i = \{ p \in C \mid F(p) = u_i \}, \quad 1 \le i \le N.$$

In a tubular neighborhood of C_i we can choose a holomorphic function t_i , s.t.,

$$F(p) = u_i + \frac{1}{2}t_i(p)^2.$$

The image of ∂_{u_i} under the Kodaira–Spencer map is the direct sum of the N functions $\widetilde{\partial}_{u_i} F|_{C_i} = \delta_{ij}$, $1 \leq j \leq N$. The above statements are obvious.

3.2. The Gauss–Manin connection. Let $U \subset B$ be an open connected Stein subset. Note that $\overline{\pi}^{-1}(U) \cap D_{\infty}$ has finitely many (in fact d) connected components.

Lemma 3.2. If $V \subset \overline{X}$ is an open subset such that $V \cap D_{\infty}$ has finitely many connected components, then

$$H^0\left(V\,,\,\varinjlim_{m}\;\Omega^p_{\overline{X}/B}(m)\right)=\varinjlim_{m}\;H^0\left(V\,,\,\Omega^p_{\overline{X}/B}(m)\right).$$

Proof. Note that the RHS is canonically embedded into the LHS. We will prove that the LHS is a subset of the RHS. By definition, a section of $\varinjlim \Omega^p_{\overline{X}/B}$ over the open subset V corresponds to a collection of pairs $\{(V_\alpha, s_\alpha)\}_{\alpha \in \mathcal{A}}$, such that $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is an open covering of V,

$$s_{\alpha} \in \varinjlim H^{0}(V_{\alpha}, \Omega^{p}_{\overline{X}/B}(m)) = \bigcup_{m=0}^{\infty} H^{0}(V_{\alpha}, \Omega^{p}_{\overline{X}/B}(m)),$$

and $s_{\alpha}|_{V_{\alpha}\cap V_{\beta}} = s_{\beta}|_{V_{\alpha}\cap V_{\beta}}$. Let us denote by m_{α} the order of the pole of s_{α} along $V_{\alpha}\cap D_{\infty}$. If V_{α} and V_{β} intersect the same connected component of $V\cap D_{\infty}$, then $m_{\alpha}=m_{\beta}$. Since the connected components of $V\cap D$ are finitely many, there exists an integer m>0, such that $m_{\alpha}< m$ for all α . Therefore $s_{\alpha}\in H^{0}(V_{\alpha},\Omega^{p}_{\overline{X}/B}(m))$ and since $\Omega^{p}_{\overline{X}/B}(m)$ is a sheaf, we get that there exists a section $s\in H^{0}(V,\Omega^{p}_{\overline{X}/B}(m))$ such that $s_{\alpha}=s|_{V_{\alpha}}$. This is exactly what we had to prove.

Lemma 3.3. Let $U \subset B$ be an open connected Stein subset, then the quotient map

$$\mathrm{rel}: H^0(\overline{\pi}^{-1}(U), \Omega^{1,\infty}_{\overline{X}}) \to H^0(\overline{\pi}^{-1}(U), \Omega^{1,\infty}_{\overline{X}/B})$$

is surjective.

Proof. We have the following exact sequence

$$0 \to \overline{\pi}^*\Omega^1_B \otimes \mathcal{O}_{\overline{X}}(m) \to \Omega^1_{\overline{X}}(m) \to \Omega^1_{\overline{X}/B}(m) \to 0,$$

for every $m \in \mathbb{Z}$. Note that $\Omega_B^1 = \mathcal{O}_B^{\oplus N}$ is a free sheaf of rank N, because the cotangent bundle T^*B is trivial. Let us choose m > 2 - 2g. Since $\overline{\pi}^*\Omega_B^1 = \mathcal{O}_{\overline{X}}^{\oplus N}$, we get $R^1\overline{\pi}_*(\overline{\pi}^*\Omega_B^1 \otimes \mathcal{O}_{\overline{X}}(m)) = 0$ and the following exact sequence

$$0 \to \Omega_B^1 \otimes \overline{\pi}_* \mathcal{O}_{\overline{X}}(m) \to \overline{\pi}_* \Omega_{\overline{X}}^1(m) \to \overline{\pi}_* \Omega_{\overline{X}/B}^1(m) \to 0.$$

Let us apply to the above sequence the functor $\varinjlim H^0(U, -)$. The exactness of the sequence will be preserved, because direct limits preserve exact sequences (of Abelian groups), the open subset U is Stein, and the sheaf $\Omega^1_B \otimes \overline{\pi}_* \mathcal{O}_{\overline{X}}(m)$ is coherent. In particular, we get that the map

$$\varinjlim_{m} H^{0}(\overline{\pi}^{-1}(U), \Omega^{1}_{\overline{X}}(m)) \to \varinjlim_{m} H^{0}(\overline{\pi}^{-1}(U), \Omega^{1}_{\overline{X}/B}(m)) \to 0$$

is surjective. Using that U is connected, we get that $\pi^{-1}(U) \cap D_{\infty}$ has finitely many connected components. It remains only to recall Lemma 3.2.

If $\omega \in \overline{\pi}_* \Omega^{1,\infty}_{\overline{X}/B}(U)$ is a relative holomorphic form, then we denote by

$$[\omega] := \int e^{F/z} \omega$$

the equivalence class of ω in $\mathcal{H}(U)$. We have the following formula for the Gauss–Manin connection (see [1])

(4)
$$z\nabla_v[\omega] = [\operatorname{rel} \circ \iota_{\widetilde{v}}((zd_{\overline{X}} + d_{\overline{X}}F \wedge)\widetilde{\omega})],$$

where $\widetilde{v} \in \mathcal{T}_{\overline{X}}^{\infty}(\overline{\pi}^{-1}(U))$ and $\widetilde{\omega} \in \Omega_{\overline{X}}^{1,\infty}(\overline{\pi}^{-1}(U))$ are lifts of $v \in \mathcal{T}_B(U)$ and ω , $\iota_{\widetilde{v}}$ is contraction by the vector field \widetilde{v} , and $d_{\overline{X}}$ is the de Rham differential on \overline{X} .

Let us derive more explicit formula for the Gauss–Manin connection. To begin with, note that (4) is still true if we choose the lift \tilde{v} , s.t.,

$$\widetilde{v}(F) = \iota_{\widetilde{v}} dF = 0.$$

The above condition, together with $\tilde{v}(u_i) = v(u_i)$, $1 \le i \le N$, uniquely determines \tilde{v} . Note that \tilde{v} might have a pole of order 1 along C. Nevertheless such a lift works too. Let us also fix a lift of ω and write it in the form

$$\widetilde{\omega}(p) = h(p)dF(p) + \sum_{j=1}^{N} h_j(p)du_j, \quad p \in \pi^{-1}(U) \setminus C.$$

Since $\widetilde{\omega}$ extends to a holomorphic 1-form on $\pi^{-1}(U)$, using local coordinates (u_1,\ldots,u_N,t_i) near C_i , s.t., $F=u_i+\frac{1}{2}t_i^2$, we get that the functions h(p) and $h_i(p)$ $(1 \leq i \leq N)$ satisfy the following conditions: h(p) has a pole of order at most 1 at C_i for all i, $h_i(p)$ has a pole of order at most 1 along C_i and it is holomorphic along C_j for $j \neq i$, and $h(p) + h_i(p)$ is holomorphic at $p = p_i$. Note that $\omega = hd_{X/B}F$, so only the functions h_i $(1 \leq i \leq N)$ depend on the choice of the lift and the ambiguity of each h_i is up to a holomorphic function on $\pi^{-1}(U)$. The formula for the Gauss–Manin connection takes the form

(5)
$$z\nabla_{\partial_{u_i}}[\omega] = (-h_i(p) + zh_i^{(1)}(p))d_{X/B}F(p),$$

where $h_i^{(1)} = \widetilde{\partial}_{u_i} h - (d_{X/B} h_i / d_{X/B} F)$.

3.3. **The period isomorphism.** The key result of this section can be stated as follows.

Proposition 3.4. Let $U \subset B$ be an open connected Stein subset and

$$\omega = \sum_{n=0}^{\infty} \omega_n z^n \in \overline{\pi}_* \Omega^{1,\infty}_{\overline{X}/B}(U) \llbracket z \rrbracket$$

be such that $\omega_0(p) \neq 0$ for all $p \in C \cap \overline{\pi}^{-1}(U)$.

a) The period map

$$\mathcal{T}_B(U)[\![z]\!] \cong \widehat{\mathcal{H}}(U), \quad v \mapsto z \nabla_v \omega$$

is an isomorphism.

b) Suppose that U=B and that $X'\subset X$ is an open subset that contains C. Then the natural restriction map $\Omega^{i,\infty}_{\overline{X}/B}(\overline{X})\to \Omega^i_{X/B}(X')$, i=0,1, induces an isomorphism

$$\widehat{\mathcal{H}}(B) \cong \Omega^1_{X/B}(X')[\![z]\!]/(zd_{X/B}+d_{X/B}F\wedge)\Omega^0_{X/B}(X')[\![z]\!].$$

Proof. a) It is enough to prove that

$$\mathcal{T}_B(U)\llbracket z\rrbracket\cong\overline{\pi}_*\Omega^{1,\infty}_{\overline{X}/B}(U)\llbracket z\rrbracket/(zd_{\overline{X}/B}+d_{\overline{X}/B}F\wedge)\overline{\pi}_*\Omega^{0,\infty}_{\overline{X}/B}(U)\llbracket z\rrbracket.$$

Since $\mathcal{T}_B[\![z]\!]$ is a sheaf, the above identity shows that the RHS coincides with the space of holomorphic sections over U of the quotient sheaf $\widehat{\mathcal{H}}$. Let us consider only the case when $\omega_k = 0$ for k > 0. The argument in the general case is the same, but the notation is a bit more cumbersome.

First we prove that the period map is surjective. Let

$$\psi = \sum_{k=0}^{\infty} \psi_k z^k \in \overline{\pi}_* \Omega^{1,\infty}_{\overline{X}/B}(U) \llbracket z \rrbracket.$$

We want to prove that there are sequences of holomorphic functions $c_{k,i} \in \mathcal{O}_B(U)$ and $\eta_k \in \mathcal{O}_{\overline{X}}^{\infty}(\overline{\pi}^{-1}(U)), 1 \leq i \leq N, k \geq 0$, such that

(6)
$$\psi = \sum_{k=0}^{\infty} \left((zd_{\overline{X}/B} + d_{\overline{X}/B}F \wedge) \eta_k z^k + \sum_{i=1}^{N} c_{k,i} z^k z \nabla_{\partial_{u_i}} [\omega] \right).$$

Comparing the coefficients in front of z^k for k > 0, we get (7)

$$\psi_k(p) = d_{\overline{X}/B} \eta_{k-1}(p) + \left(\eta_k(p) + \sum_{i=1}^N \left(-h_i(p) c_{k,i}(u) + c_{k-1,i}(u) h_i^{(1)}(p) \right) \right) d_{\overline{X}/B} F(p).$$

Similarly, comparing the coefficients in front of z^0 we get

(8)
$$\psi_0(p) = \eta_0(p) d_{\overline{X}/B} F(p) - \sum_{i=1}^N h_i(p) c_{0,i}(u) d_{\overline{X}/B} F(p).$$

First, we find $c_{0,i}$. Let us fix $u \in U$ and denote by $p_i \in X_u$ the critical points of F. Let (u_1, \ldots, u_N, t_i) be the local coordinates near C_i , s.t., $F = u_i + \frac{1}{2}t_i^2$. Dividing both sides of (8) by $t_i(p)$ and computing the residue at $p = p_i$ we get

$$\operatorname{res}_{p=p_i} \frac{\psi_0(p)}{t_i(p)} = -c_{0,i}(u) \operatorname{res}_{p=p_i} h_i(p) dt_i = c_{0,i}(u) \operatorname{res}_{p=p_i} h(p) dt_i.$$

Since $\omega(p_i) = dt_i \operatorname{res}_{p=p_i} h(p) dt_i \neq 0$, we can solve uniquely for $c_{0,i}(u)$. This choice of $c_{0,i}$ makes the relative 1-form

$$\psi_0(p) + \sum_{i=1}^{N} h_i(p) c_{0,i}(u) d_{\overline{X}/B} F(p)$$

vanish for $p \in C$, which implies that we can write it uniquely in the form $\eta_0(p)d_{\overline{X}/B}F(p)$ for some $\eta_0 \in \mathcal{O}_{\overline{X}}^{\infty}(\overline{\pi}^{-1}(U))$. Assuming that we have found $c_{k',i}$ and $\eta_{k'}$ for all $k' = 0, 1, \ldots, k-1$, then applying the above argument to equation (7), we find first $c_{k,i}$ and then η_k . This completes the proof of the surjectivity.

To prove the injectivity, it is enough to verify that if $\psi = 0$, then the above algorithm gives $c_{k,i} = 0$. This is straightforward to do, which completes the proof of part a).

b) Note that if we assume that $\psi \in \Omega^1_{\overline{X}/B}(X')[\![z]\!]$ is any form, then the equation (6) still has a unique solution $(\eta_k, c_{k,i}), k \geq 0, 1 \leq i \leq N$, with $\eta_k \in \Omega^0_{\overline{X}/B}(X')$ and $c_{k,i} \in \mathcal{O}_B(B)$. The existence and uniqueness of the solution of (6) is equivalent to the statement in part b).

Let us assume that $V \subset B$ is an open subset and that $\omega \in \widehat{\mathcal{H}}(V)$ is a cohomology class. We would like to formulate a condition on ω under which the period map

(9)
$$\mathcal{T}_{V}\llbracket z\rrbracket \to \widehat{\mathcal{H}}|_{V}, \quad v \mapsto z\nabla_{v}\omega$$

is an isomorphism. If $U \subset V$ is an open connected Stein subset, then according to Proposition 3.4 $\omega|_U$ can be represented by a holomorphic form

$$\omega_U = \sum_{n=0}^{\infty} \omega_U^{(n)} z^n, \quad \omega_U^{(n)} \in H^0(\overline{\pi}^{-1}(U), \Omega^{1,\infty}_{\overline{X}/B})$$

We require that $\omega_U^{(0)}(p) \neq 0$ for all $p \in \overline{\pi}^{-1}(U) \cap C$. Note that for each $p \in \overline{\pi}^{-1}(V) \cap C$ the value of $\omega_U^{(0)}(p)$ is independent of the choices of a Stein neighborhood U of p and a form ω_U representing the cohomology class $\omega|_U$. If this condition on ω is satisfied, then we will say that the leading order term of ω is a volume form along $\overline{\pi}^{-1}(V) \cap C$. Since open connected Stein neighborhoods form a basis for the topology of B, we get from Proposition 3.4, Part a) that the period map (9) is an isomorphism for every $\omega \in \widehat{\mathcal{H}}(V)$ whose leading order term is a volume form along C.

Lemma 3.5. There exists $\omega \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{X})$, s.t., $\omega(p) \neq 0$ for all $p \in C$.

Proof. Note that the divisor $D_{\infty} = \sum_{i=1}^{d} m_i \infty_i$. We have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{X}}(k D_{\infty}) \xrightarrow{dF} \Omega^{1}_{\overline{X}/B}(k D_{\infty} + \sum_{i=1}^{d} (m_{i} + 1) \infty_{i}) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where dF denotes the map of multiplication by the meromorphic relative 1-form $d_{\overline{X}/B}F$, which is holomorphic on X and has a pole of order m_i+1 at ∞_i . The sheaf $\mathcal Q$ is defined to be the quotient of the preceding two sheaves, so that the sequence is exact. Finally, the number k is a sufficiently large integer, so that $H^1(\overline{X}_u, \mathcal{O}_{\overline{X}_u}(kD_\infty)) = 0$ (e.g. any k > 2g-2 works). Using the Riemann–Roch formula we get that

$$\dim_{\mathbb{C}} H^0(\overline{X}_u, \mathcal{O}_{\overline{X}_u}(kD_{\infty})) = 1 - g + k \sum_{i=1}^d m_i$$

and

$$\dim_{\mathbb{C}} H^{0}(\overline{X}_{u}, \Omega^{1}_{\overline{X}_{u}}(kD_{\infty} + \sum_{i=1}^{d} (m_{i} + 1)\infty_{i}) = g - 1 + d + (k+1)\sum_{i=1}^{d} m_{i}$$

are independent of u. Furthermore, using that $\overline{\pi}: \overline{X} \to B$ is a regular map, we get that $R^1\overline{\pi}_*(\mathcal{O}_{\overline{X}_n}(kD_\infty)) = 0$ and that

$$0 \longrightarrow \overline{\pi}_* \mathcal{O}_{\overline{X}}(k D_{\infty}) \xrightarrow{dF} \overline{\pi}_* \Omega^{1}_{\overline{X}/B}(k D_{\infty} + \sum_{i=1}^{d} (m_i + 1) \infty_i) \longrightarrow \overline{\pi}_* \mathcal{Q} \longrightarrow 0$$

is an exact sequence of vector bundles. Note that the rank of $\overline{\pi}_* \mathcal{Q}$ is $2g - 2 + d + \sum_{i=1}^d m_i = N$. Since B is contractible and Stein the vector bundles must be trivial. Note that

$$\overline{\pi}_* \mathcal{Q}(B) = H^0(\overline{X}, \Omega^1_{\overline{X}/B}(kD_\infty + \sum_{i=1}^d (m_i + 1)\infty_i))/H^0(\overline{X}, \mathcal{O}_{\overline{X}}(kD_\infty)dF)$$

Let us fix holomorphic forms

$$\omega_i \in H^0(\overline{X}, \Omega^1_{\overline{X}/B}(kD_\infty + \sum_{i=1}^d (m_i + 1)\infty_i)), \quad 1 \le i \le N,$$

inducing a trivialization of the vector bundle $\overline{\pi}_*\mathcal{Q}$. Let us define a square holomorphic matrix $\Phi(u) = (\Phi_{ij}(u))_{i,j=1}^N$ of size N by $\omega_i(p_j) = \Phi_{ij}(u)dt_j$. We claim that $\Phi(u)$ is invertible for every $u \in B$. Otherwise, there is $u \in B$ and constants c_i , such that $\sum_i c_i \omega_i(p)$ vanishes for all $p = p_j$. However, this would imply that $\sum_i c_i \omega_i(p)$ is proportional to dF, which would imply that the projections of ω_i in the fiber over u of the vector bundle $\overline{\pi}_*\mathcal{Q}$ are linearly dependent. This contradicts the fact that the forms induce a trivializing frame in every fiber of $\overline{\pi}_*\mathcal{Q}$. Let us define

$$(c_1(u),\ldots,c_N(u))^T := \Phi(u)^{-1}(1,\ldots,1)^T,$$

then the form $\omega(p) := \sum_{i=1}^{N} c_i(u)\omega_i(p)$ would satisfy $\omega(p_j) = 1 \neq 0$ for all $j = 1, 2, \ldots, N$.

The existence of the form ω in Lemma 3.5 implies that $\widehat{\mathcal{H}}$ is a free $\mathcal{O}_B[\![z]\!]$ -module of rank N and that the excision principle in Proposition 3.4, Part b) holds.

4. Good basis

Following K. Saito, we introduce the so-called higher residue pairing

$$K: \widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}} \to \mathcal{O}_B[\![z]\!]z$$

and prove the existence of a good basis on B. Recall that if $V \subset B$ is an open subset, then a set of cohomology classes $\{\omega_i\}_{i=1}^N \subset \widehat{\mathcal{H}}(V)$ is called a good basis on V, if

- (GB1) $\{\omega_i\}_{i=1}^N$ is a $\mathcal{O}_B[\![z]\!]$ -basis of $\widehat{\mathcal{H}}|_V$.
- (GB2) The higher residues vanish, i.e., $K(\omega_i, \omega_j) \in z \mathcal{O}_B(V)$.
- 4.1. **Higher-residue pairing.** Let us define K locally on every open connected Stein subset $U \subset B$. It is straightforward to check that the local definitions can be glued. Let $\psi^{(i)} \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{\pi}^{-1}(U))[\![z]\!]$, i=1,2 be arbitrary. Following K. Saito [26] we define the higher residue pairing

$$K(\psi^{(1)}, \psi^{(2)}) \in \mathcal{O}_B(U)[\![z]\!]z$$

as the sum of the residues

(10)
$$\sum_{i=1}^{N} \operatorname{res}_{p=p_i(u)} \phi^{(1)}(p,z) \psi^{(2)}(p,-z) z$$

where we fix $u \in U$, denote by $p_i(u) \in X_u$ the critical points of F, and let $\phi^{(1)} \in \Omega^{0,\infty}_{\overline{X}/B}(\overline{\pi}^{-1}(U) \setminus C)[\![z]\!]$ be such that

$$(zd_{X/B} + d_{X/B}F \wedge)\phi^{(1)}(p,z) = \psi^{(1)}(p,z).$$

Note that the above equation has a solution for all $p \in X \setminus C$.

It is easy to check that the pairing induces a pairing on \mathcal{H} satisfying the following properties.

- (HR1) $K(\psi^{(1)}, \psi^{(2)}) = -K(\psi^{(2)}, \psi^{(1)})^*$, where * is the involution $z \to -z$.
- (HR2) $a(z)K(\psi^{(1)},\psi^{(2)}) = K(a(z)\psi^{(1)},\psi^{(2)}) = K(\psi^{(1)},a(-z)\psi^{(2)})$ for all $a \in$
- (HR3) The Leibnitz rule holds

$$zvK(\psi^{(1)}, \psi^{(2)}) = K(z\nabla_v\psi^{(1)}, \psi^{(2)}) - K(\psi^{(1)}, z\nabla_v\psi^{(2)})$$

for all $v \in \mathcal{T}_B$ and $v = z\partial_z$. (HR4) Let $K^{(0)}(\psi^{(1)}, \psi^{(2)}) \in \mathcal{O}_B$ be the coefficient in front of z^1 in $K(\psi^{(1)}, \psi^{(2)})$ and $\psi_0^{(i)}(p)$ be the coefficient of $\psi^{(i)}(p,z)$ in front of z^0 . Then

$$K^{(0)}(\psi^{(1)}, \psi^{(2)}) = \sum_{i=1}^{N} \operatorname{res}_{p=p_i(u)} \frac{\psi_0^{(1)}(p)\psi_0^{(2)}(p)}{d_{X/B}F(p)}.$$

(HR5)
$$K(\psi^{(1)}, \psi^{(2)}) \in \mathcal{O}_B(U)[\![z]\!]z$$
.

Proposition 4.1. The properties (HR1)-(HR5) determine the higher residue pairing uniquely.

Proof. Let $\omega \in \pi_* \Omega^{1,\infty}_{\overline{X}/B}(U)$ be a form such that $\omega(p) \neq 0$ for all $p \in C \cap \pi^{-1}(U)$. The existence of such form is proved in Lemma 3.5. We have the following formulas for the Gauss-Manin connection

(11)
$$z\nabla_{\partial_{u_i}}z\nabla_{\partial_{u_j}}[\omega] = \sum_{k=1}^N C_{ij}^k(u,z)z\nabla_{\partial_{u_k}}[\omega],$$

where $C_{ij}^k(u,z) = \sum_{n=0}^{\infty} C_{ij,n}^k(u) z^n$ for some holomorphic functions $C_{ij,n}^k \in \mathcal{O}_B(U)$ and similarly

(12)
$$z^{2}\nabla_{\partial_{z}}z\nabla_{\partial_{u_{i}}}[\omega] = \sum_{j=1}^{N} C_{i}^{j}(u,z)z\nabla_{\partial_{u_{j}}}[\omega].$$

where $C_i^j(u,z) = \sum_{n=0}^{\infty} C_{i,n}^j(u) z^n$ for some holomorphic functions $C_{i,n}^j \in \mathcal{O}_B(U)$. We are going to introduce several matrices. They will all be square matrices of size $N \times N$. Let us denote by K(u,z) the matrix with entries $K_{ij}(u,z) :=$ $K(z\nabla_{\partial u_i},z\nabla_{\partial u_j})$, by $C_i(u,z)$ $(1 \leq i \leq N)$ the matrix with entries $(C_i)_{kj}:=C^k_{ij}(u,z)$, and by $C_0(u,z)$ the matrix with entries $(C_0)_{ij}=C^j_i(u,z)$. Put $K(u,z)=\sum_{n=0}^{\infty}K_n(u)z^{n+1}$ and $C_i(u,z)=\sum_{n=0}^{\infty}C_{i;n}(u)z^n$ $(0 \leq i \leq N)$. Property (HR4) fixes $K_0(u)$, while a straightforward computation using formula (5) yields that $C_{i;0}=E_{ii}$ $(1 \leq i \leq N)$ and $C_{0;0}=-\sum_{i=1}^{N}u_iE_{ii}$, where E_{ij} denotes the matrix with only one non-zero entry, which is on position (i,j) and it is equal to 1.

Using (HR2) and (HR3) we get

$$z\partial_{u_i}K(u,z) = C_i(u,z)^T K(u,z) - K(u,z)C_i(u,-z)$$

and

$$z^{2}\partial_{z}K(u,z) = C_{0}(u,z)^{T}K(u,z) - K(u,z)C_{0}(u,-z).$$

Let us assume that we have two pairings K', K'' satisfying the axioms (HR1)–(HR5), then the matrix $K(u,z) := K'(u,z)K''(u,z)^{-1}$ satisfies the differential equations

$$z\partial_{u_i}K(u,z) = [C_i(u,z)^T, K(u,z)], \quad 1 \le i \le N,$$

 $z^2\partial_z K(u,z) = [C_0(u,z)^T K(u,z)].$

Arguing by induction on n, we are going to prove that $K_n = 0$ for all n > 0. Comparing the coefficients in front of the powers of z, we get the following system of equations

$$[K_0, E_{ii}] = [K_0, C_{0;0}] = 0, \quad 1 \le i \le N,$$

$$\partial_{u_i} K_n = [E_{ii}, K_{n+1}] + \sum_{m=1}^{n+1} [C_{i;m}^T, K_{n+1-m}], \quad n \ge 0, \quad 1 \le i \le N,$$

and

$$(n+2)K_{n+1} = [C_{0;0}, K_{n+2}] + \sum_{m=1}^{n+2} [C_{0;m}^T, K_{n+2-m}], \quad n \ge -1.$$

The firs set of equations is trivially satisfied, because K_0 is the identity matrix (here we used axiom (HR4), which implies that the leading terms of K' and K'' are fixed and equal. If $K_1 = \cdots = K_n = 0$, then from the 2nd set of equations we get that $[K_{n+1}, E_{ii}] = 0$ for all i, so K_{n+1} must be diagonal. Note that in the last equation $C_{0;0}$ and K_{n+2-m} $(1 \le m \le n+2)$ are diagonal matrices. Comparing the diagonal entries, we get $K_{n+1} = 0$.

Remark 4.2. The same argument can be used to prove the uniqueness of K. Saito's higher residue pairing in the settings of singularity theory.

Note that the pairing on \mathcal{H} defined by formula (2) satisfies all axioms (HR1)–(HR5). Moreover, it can be extended uniquely to a pairing on the completion $\widehat{\mathcal{H}}$, so that the axioms (HR1)–(HR5) still hold.

Corollary 4.3. The higher residue pairing (10) coincides with the pairing on $\widehat{\mathcal{H}}$ defined by (2).

Proof. The only non-trivial part in the proof is to verify that the pairing (2) satisfies (HR4). However, the verification reduces to computing the leading order term of the stationary phase asymptotic of the oscillatory integrals. A standard computation yields

(13)
$$\int_{\Gamma_i} e^{F/z} \omega \sim (-2\pi z)^{1/2} e^{u_i/z} \left. \frac{\omega(p)}{dt_i(p)} \right|_{p=p_i} \left(1 + \cdots \right), \quad z \to 0,$$

where ω is a holomorphic 1-form on X_u , t_i is a local coordinate on X_u near the critical point p_i , s.t., $F(p) = u_i + \frac{1}{2}t_i(p)^2$. The leading order term of the pairing (2) becomes

$$z \sum_{i=1}^{N} \frac{\omega_1(p)\omega_2(p)}{dt_i(p)^2} \bigg|_{p=p_i} = z \sum_{i=1}^{N} \operatorname{res}_{p=p_i} \frac{\omega_1(p)\omega_2(p)}{dF(p)}. \quad \Box$$

4.2. Construction of the good basis. Let us fix a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^g \subset H_1(\Sigma, \mathbb{Z})$. Since B is contractible, we can use the Gauss–Manin connection to construct a symplectic basis in $H_1(\overline{X}_u; \mathbb{Z})$ for all $u \in B$. Recall that by construction, the ramification points over infinity of the branched covering $\overline{X} \to B \times \mathbb{P}^1$ provide d holomorphic sections $\infty_i : B \to \overline{X}$, $1 \le i \le d$ of $\overline{\pi} : \overline{X} \to B$, while the ramifications over the finite branch points provide N holomorphic sections $p_i : B \to X$, $1 \le i \le N$. In particular, the connected component C_i of the relative critical set C of F coincides with $p_i(B)$. Let us fix a holomorphic function t_i defined in a tubular neighborhood of C_i , s.t.,

$$F(p) =: u_i + \frac{1}{2}t_i(p)^2, \quad 1 \le i \le N,$$

where $u_i = F(p_i \circ \pi(p))$ is the critical value of F corresponding to the connected component C_i . The choice of each t_i is unique up to a sign.

For fixed $u \in B$, let us denote by $\omega_p(q)$ the unique meromorphic 1-form of the 3rd kind on \overline{X}_u that has poles only at q = p and $q = \infty_1(u)$ and such that

$$\oint_{q \in \alpha_i} \omega_p(q) = 0, \quad \operatorname{res}_{q=p} \omega_p(q) = -\operatorname{res}_{q=\infty_1} \omega_p(q) = 1.$$

Let us introduce the relative differential forms

$$\omega_i(p) = \operatorname{res}_{q=p_i} t_i(q)^{-1} \omega_p(q) d_{\overline{X}/B} F(p), \quad 1 \le i \le N.$$

If p is sufficiently close to C_i , then using (u_1, \ldots, u_N, t_i) as local coordinates, we get that

(14)
$$\omega_i(p) = (-1 + O(t_i))dt_i(p).$$

In particular, we get that ω_i is holomorphic in a neighborhood of C_i . Clearly, $\omega_i(p)$ is holomorphic in a tubular neighborhood of C_j for $i \neq j$, so $\omega_i \in \Omega^1_{X/B}(X')$, where X' is a tubular neighborhood of C. Let us recall Proposition 3.4, part b) with

X' being the tubular neighborhood of C introduced above and $\omega \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{X})$ a form whose existence is guaranteed by Lemma 3.5. We get that the forms $\omega_i \in \Omega^1_{X/B}(X')$ determine global sections $[\omega_i] \in \widehat{\mathcal{H}}(B)$.

The 1-forms $\omega_i(p)$ are multivalued for $p \in X$, because $\omega_p(q)$ is multivalued:

$$\omega_p(q) = \int_{\infty_1}^p B(p', q),$$

where B(p', p'') is the so-called *Riemann's 2nd fundamental form* or *fundamental bi-differential*. It is defined as the unique symmetric quadratic meromorphic differential on $\overline{X}_u \times \overline{X}_u$ that has a pole of order 2 along the diagonal with no-residues normalized by

$$B(p', p'') = \frac{dt(p')dt(p'')}{(t(p') - t(p''))^2} + \cdots,$$

and

$$\oint_{p' \in \alpha_i} B(p', p'') = 0, \quad 1 \le i \le g,$$

where in the first condition p' and p'' are sufficiently close to some point $p_0 \in \overline{X}_u$, t is a local coordinate in a neighborhood of p_0 , and the dots stand for terms that are holomorphic in a neighborhood of (p_0, p_0) in $\overline{X}_u \times \overline{X}_u$. Note however, that the cohomology class $[\omega_i]$ is independent of the choice of a holomorphic branch of ω_i in a neighborhood of C.

Proposition 4.4. The cohomology classes $\{[\omega_i]\}_{i=1}^N \subset \widehat{\mathcal{H}}(B)$ form a good basis in which the higher residue pairing takes the form

$$K([\omega_i], [\omega_j]) = z\delta_{ij}, \quad 1 \le i, j \le N.$$

Proof. Recalling Proposition 3.4, Part a), we get that condition (GB1) is equivalent to saying that for every open connected Stein subset $U \subset V$ the projection of ω_i to

(15)
$$\widehat{\mathcal{H}}(U)/z\widehat{\mathcal{H}}(U) = \overline{\pi}_* \Omega_{\overline{X}/B}^{1,\infty}(U)/d_{\overline{X}/B} F \wedge \overline{\pi}_* \Omega_{\overline{X}/B}^{0,\infty}(U)$$

is an $\mathcal{O}_B(U)$ -basis. This however is obvious, because $\omega_i(p_j) = -\delta_{ij}dt_j$, so the condition (G1) in the definition of a good basis holds.

Using formula (2) for the higher residue pairing, we get

$$K([\omega_a], [\omega_b]) = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{N} \int_{\Gamma_i} e^{F(p')/z} \omega_a(p') \int_{\Gamma_i^{\vee}} e^{-F(p')/z} \omega_b(p'),$$

where on the RHS the oscillatory integrals should be identified with the corresponding stationary phase asymptotic as $z \to 0$. In particular, only the germs of the integration paths Γ_i and Γ_i^{\vee} near the critical point p_i are relevant. This

justifies why we can replace the holomorphic form in $\Omega^{1,\infty}_{\overline{X}/B}(\overline{X})[\![z]\!]$ that represent the cohomology class $[\omega_a]$ by a holomorphic branch of ω_a defined only in a neighborhood of the critical points.

We are going to proof that condition (G2) from the definition of a good basis is equivalent to a certain identity for the matrix series $R_{\Sigma}(z) = 1 + R_{\Sigma,1}z + \cdots$ whose (a, i)-th entry is defined by

$$[R_{\Sigma}(z)]_i^a := -(-2\pi z)^{-1/2} \int_{p' \in \Gamma_i} e^{(F(p') - u_i)/z} \omega_a(p').$$

Remark 4.5. In the notation of [10], $R_{\Sigma}(z) = \widehat{R}^{-1}(-z)$.

The leading order term of $R_{\Sigma}(z)$ is the identity matrix as it can be seen easily from (13) and (14). The higher residue pairing takes the form

$$K([\omega_a], [\omega_b]) = z \sum_{i=1}^{N} [R_{\Sigma}(z)]_i^a [R_{\Sigma}(-z)]_i^b.$$

We will prove that the series $R_{\Sigma}(z)$ satisfies the symplectic condition $R_{\Sigma}(z)R_{\Sigma}^{T}(-z) = 1$, therefore $K([\omega_{a}], [\omega_{b}]) = z\delta_{ab}$.

The symplectic condition is stated in [10], Lemma 5.1 and it is a consequence of a more general identity proved in the lemma. For the sake of completeness and for the reader's convenience, let us prove directly the symplectic condition. Our argument follows the ideas of [10]. It is more convenient to prove that

(16)
$$\sum_{z=1}^{N} [R_{\Sigma}(z)]_{i}^{a} [R_{\Sigma}(-z)]_{j}^{a} = \delta_{ij}.$$

By definition the LHS of (16) is

$$\sum_{a=1}^{N} \frac{z^{-1}}{2\pi\sqrt{-1}} \int_{\Gamma_i \times \Gamma_i^{\vee}} e^{(F(p')-u_i)/z} e^{-(F(p'')-u_j)/z} \omega_a(p') \omega_a(p'').$$

The key observation is that

$$\omega_a(p')\omega_a(p'') = \left(\operatorname{res}_{q=p_a} \frac{\omega_{p'}(q)\omega_{p''}(q)}{d_{X/B}F(q)}\right) d_{X/B}F(p')d_{X/B}F(p'').$$

Using the residue theorem on \overline{X}_u we get

$$\sum_{q=1}^{N} \operatorname{res}_{q=p_{a}} \frac{\omega_{p'}(q)\omega_{p''}(q)}{d_{X/B}F(q)} = -(\operatorname{res}_{q=p'} + \operatorname{res}_{q=p''}) \frac{\omega_{p'}(q)\omega_{p''}(q)}{d_{X/B}F(q)} = -\frac{\omega_{p''}(p')}{d_{X/B}F(p')} - \frac{\omega_{p'}(p'')}{d_{X/B}F(p'')},$$

where we used the Cauchy theorem

$$\operatorname{res}_{q=p} \omega_p(q) f(q) = f(p)$$

for any function f on \overline{X}_u holomorphic in a neighborhood of p. The LHS of (16) turns into

$$-\frac{z^{-1}}{2\pi\sqrt{-1}}\int_{\Gamma_i\times\Gamma_j^\vee}e^{(F(p')-u_i)/z}e^{-(F(p''-u_j)/z}\Big(\omega_{p''}(p')\,d_{X/B}F(p'')+\omega_{p'}(p'')\,d_{X/B}F(p')\Big).$$

Let us consider first the case when $i \neq j$. In this case $\omega_{p'}(p'')$ and $\omega_{p''}(p')$ have no singularities, so using integration by parts we get

$$-\frac{z^{-1}}{2\pi\sqrt{-1}} \int_{\Gamma_{i}\times\Gamma_{j}^{\vee}} e^{(F(p')-u_{i})/z} e^{-(F(p''-u_{j})/z} \omega_{p''}(p') d_{X/B} F(p'') =$$

$$-\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{i}\times\Gamma_{j}^{\vee}} e^{(F(p')-u_{i})/z} e^{-(F(p''-u_{j})/z} B(p'', p')$$

and

$$-\frac{z^{-1}}{2\pi\sqrt{-1}} \int_{\Gamma_{i}\times\Gamma_{j}^{\vee}} e^{(F(p')-u_{i})/z} e^{-(F(p''-u_{j})/z} \omega_{p'}(p'') d_{X/B}F(p') = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{i}\times\Gamma_{j}^{\vee}} e^{(F(p')-u_{i})/z} e^{-(F(p''-u_{j})/z} B(p',p'').$$

The fundamental bi-differential is symmetric, so the above integrals cancel out, i.e., formula (16) holds for $i \neq j$.

If i = j, then as discussed above, since we are interested only in the asymptotic of the integrals, we may assume that Γ_i is a small path defined in a neighborhood of p_i . Let us split $\omega_{p'}(p'')$ and $\omega_{p''}(p')$ into singular and regular parts:

$$\omega_{p'}(p'') = \frac{dt_i(p'')}{t_i(p'') - t_i(p')} + \omega_{p'}^{\text{reg}}(p'')$$

and

$$\omega_{p''}(p') = \frac{dt_i(p')}{t_i(p') - t_i(p'')} + \omega_{p''}^{reg}(p').$$

The regular parts do not contribute, because they cancel out after integration by parts just like in the case of $i \neq j$. While the singular parts add up to

$$\frac{z^{-1}}{2\pi\sqrt{-1}} \int_{\Gamma_i \times \Gamma_i^{\vee}} e^{(F(p')-u_i)/z} e^{-(F(p''-u_i)/z} dt_i(p') dt_i(p'').$$

Again, since we are interested only in the asymptotic as $z \to 0$, we may assume that $F(p) = u_i + \frac{1}{2}t_i^2$ and

$$\Gamma_i = \{ t_i = \sqrt{-1}s \mid -\infty < s < +\infty \}, \quad \Gamma_i^{\lor} = \{ t_i = s \mid -\infty < s < +\infty \}.$$

The above integral turns into

$$\frac{z^{-1}}{2\pi} \left(\int_{-\infty}^{\infty} e^{-s^2/(2z)} ds \right)^2 = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-s^2/2} ds \right)^2 = 1. \quad \Box$$

4.3. The Gauss–Manin connection. Let us derive a formula for the Gauss–Manin connection in the frame of $\widehat{\mathcal{H}}$ given by the good basis $\{[\omega_i]\}_{i=1}^N$ defined in the previous section. The formulas can be obtained easily from [11], Theorem 7. However, for the sake of completeness, let us derive them directly.

The main tool is the Rauch's variation formula (see [25])

(17)
$$\frac{\delta\omega_p}{\delta u_i}(q) = \operatorname{res}_{q'=p_i} \frac{\omega_p(q')B(q',q)}{d_{X/B}F(q')}.$$

Let us explain the meaning of the LHS. For fixed $u \in B$, let us restrict our family $X \to B$ to a small disc $B_i^*(u) \subset B$ with center u, obtained by varying only the i-th branch point u_i , i.e.,

$$B_i^*(u) = \{u^* \in B \mid u_i^* = u_i \text{ for } i \neq j, |u_i^* - u_i| \ll 1\}.$$

The resulting family $X^* \to B_i^*(u)$ is a deformation of X_u constructed by replacing the holomorphic coordinate t_i on X_u around the critical point p_i in which $F(p) = u_i + \frac{1}{2}t_i(p)^2$ with a holomorphic coordinate t_i^* , s.t., $F(p) = u_i^* + \frac{1}{2}t^*(p)^2$. In particular, by removing a tubular neighborhood of $C_i \cap X^*$ in X^* , we get a holomorphically trivial family. Therefore, we can identify the differentials $\omega_p(q)$, $q \in X^*$ as a family of differentials on \overline{X}_u holomorphic in the complement of a small disc around p_i . The LHS of (17) is interpreted as the usual derivative with respect to u_i^* evaluated at $u_i^* = u_i$.

The Rauch's variational derivative is compatible with the Gauss–Manin connection

$$\partial_{u_i} \int_{\Gamma} e^{F(p)/z} \omega_a(p) = \int_{\Gamma} e^{F(p)/z} \frac{\delta \omega_a}{\delta u_i}(p),$$

where the integrals are interpreted via their asymptotic as $z \to 0$, so we may assume that the integration cycle Γ is supported in a tubular neighborhood of the critical points, where $\omega_a(p)$ is holomorphic. Recalling the definition of $\omega_a(p)$ we get

$$\frac{\delta\omega_a}{\delta u_i}(p) = \operatorname{res}_{q'=p_a} \left(\delta_{ia} \frac{\omega_p(q')}{t_a(q')^3} + \operatorname{res}_{q''=p_i} \frac{\omega_p(q'')B(q'',q')}{t_a(q')d_{X/B}F(q'')} \right) d_{X/B}F(p)$$

where we used the Rauch's variational formula (17) and

$$\frac{\delta(t_a^{-1})}{\delta u_i} = \delta_{ia} t_a^{-3}.$$

Let us consider first the case when $i \neq a$, then the above formula turns into

$$\frac{\delta\omega_a}{\delta u_i}(p) = \beta_{ai}\,\omega_i(p),$$

where

(18)
$$\beta_{ai} = \operatorname{res}_{q'=p_a} \operatorname{res}_{q''=p_i} \frac{B(q', q'')}{t_a(q')t_a(q'')}.$$

If i = a, then we use the residue theorem to compute the residue with respect to q''. Note that the poles are only at $q'' = p_a$, $1 \le a \le N$, q'' = p and q'' = q', so the residue operation with respect to q'' can be replaced by

$$-\sum_{a\neq i} \operatorname{res}_{q''=p_a} - \operatorname{res}_{q''=p} - \operatorname{res}_{q''=q'}.$$

The contribution of the sum of the residues is exactly what we have computed in the case $i \neq a$, i.e.,

$$-\sum_{a\neq i}\beta_{ia}\,\omega_a(p).$$

The contribution of the residue at q'' = p is computed via the Cauchy theorem

$$-\operatorname{res}_{q'=p_i}\frac{B(p,q')}{t_i(q')} = -d_{X/B}\Big(\frac{\omega_i(p)}{d_{X/B}F}\Big).$$

The residue at q'' = q' can be computed using that

$$\operatorname{res}_{q''=q'} f(q'') B(q'', q') = df(q')$$

for every meromorphic function f. Using the above formula we get

$$-\operatorname{res}_{q''=q'}\frac{\omega_p(q'')B(q'',q')}{t_i(q')d_{X/B}F(q'')} = -t_i(q')^{-1}d_{q'}\Big(\frac{\omega_p(q')}{d_{X/B}F(q')}\Big).$$

Since we are interested in the residue at $q' = p_i$, let us use the local coordinate $t_i(q')$. Then $\omega_p(q') = f_i(p, q')dt_i(q')$ for some function f_i holomorphic in q' near p_i and $d_{X/B}F(q') = t_i(q')dt_i(q')$. Therefore

$$-t_i(q')^{-1}d_{q'}\left(\frac{\omega_p(q')}{d_{X/B}F(q')}\right) = -t_i(q')^{-3}\omega_p(q') - d_{q'}f_i(p, q')$$

and we get the following formula

$$\sum_{a=1}^{N} \frac{\delta \omega_a}{\delta u_i} = -d_{X/B} \left(\frac{\omega_i(p)}{d_{X/B} F} \right).$$

For the oscillatory integrals we get the following equations

$$\partial_{u_i} \int_{\Gamma} e^{F(p)/z} \omega_a = \beta_{ai} \int_{\Gamma} e^{F(p)/z} \omega_i, \quad i \neq a,$$

$$\sum_{a=1}^{N} \partial_{u_a} \int_{\Gamma} e^{F(p)/z} \omega_i = z^{-1} \int_{\Gamma} e^{F(p)/z} \omega_i.$$

Similarly, we can compute

$$E \int_{\Gamma} e^{F(p)/z} \omega_i(p),$$

where $E = \sum_{a=1}^{N} u_a \partial_{u_a}$. Using the Rauch's variation formula we get

$$\int_{\Gamma} e^{F(p)/z} \sum_{a=1}^{N} u_a \operatorname{res}_{q'=p_i} \left(\delta_{ia} \frac{\omega_p(q')}{t_i(q')^3} + \operatorname{res}_{q''=p_a} \frac{\omega_p(q'')B(q'',q')}{t_i(q')d_{X/B}F(q'')} \right) d_{X/B}F(p).$$

Exchanging the order of the residues we get

(19)
$$\sum_{a=1}^{N} u_a \operatorname{res}_{q''=p_a} \operatorname{res}_{q'=p_i} \frac{\omega_p(q'')B(q'',q')}{t_i(q')d_{X/B}F(q'')} d_{X/B}F(p).$$

Note that for $a \neq i$ the above form has a pole of order 1 at $q'' = p_a$. Therefore we can rewrite the sum of the terms for which $a \neq i$ as

$$\sum_{a \neq i} \operatorname{res}_{q'' = p_a} \operatorname{res}_{q' = p_i} \frac{F(q'')\omega_p(q'')B(q'', q')}{t_i(q')d_{X/B}F(q'')} \ d_{X/B}F(p).$$

Using the residue theorem we can replace the residue operation $\sum_{a\neq i} \operatorname{res}_{q''=p_a}$ with $-\operatorname{res}_{q''=p_i} - \operatorname{res}_{q''=p}$. Note that

$$\operatorname{res}_{q'=p_i} \frac{B(q'', q')}{t_i(q')} = d_{q''} \left(\frac{\omega_i(q'')}{d_{X/B} F(q'')} \right).$$

Therefore, the sum (19) turns into the sum of

(20)
$$-\operatorname{res}_{q''=p_{i}}\frac{(F(q'')-u_{i})\omega_{p}(q'')}{d_{X/B}F(q'')}d_{q''}\left(\frac{\omega_{i}(q'')}{d_{X/B}F(q'')}\right)d_{X/B}F(p)$$

and

(21)
$$-F(p)d_p\Big(\frac{\omega_i(p)}{d_{X/B}F(p)}\Big).$$

Using that $F(q'') = u_i + \frac{1}{2}t_i(q'')^2$ and integration by parts, we get that (20) is

$$\frac{1}{2} \operatorname{res}_{q''=p_i} \frac{\omega_i(q'')}{d_{X/B}F(q'')} \,\omega_p(q'') \,d_{X/B}F(p) = -\frac{1}{2} \,\omega_i(p),$$

where we used that the 1-form $\frac{\omega_i(q'')}{d_{X/B}F(q'')}dt_i(q'')$ has a pole of order 1 and residue -1 at $q''=p_i$ (see formula (14)). Note also that the contribution of (21) to the oscillatory integral is

$$-\int_{\Gamma} e^{F(p)/z} F(p) d_p \left(\frac{\omega_i(p)}{d_{X/B} F(p)} \right) = \int_{\Gamma} e^{F(p)/z} \left(\frac{F(p)}{z} + 1 \right) \omega_i$$

We get the following equation

$$(z\partial_z + E) \int_{\Gamma} e^{F(p)/z} \omega_i(p) = \frac{1}{2} \int_{\Gamma} e^{F(p)/z} \omega_i(p).$$

In other words we proved the following Proposition.

Proposition 4.6. In the good basis frame $\{[\omega_i]\}_{i=1}^N$ the Gauss–Manin connection takes the form

$$\nabla_{\partial_{u_i}}[\omega_a] = \beta_{ai}[\omega_i], \quad i \neq a,$$

$$\sum_{a=1}^{N} z \nabla_{\partial_{u_a}}[\omega_i] = [\omega_i],$$

$$(z \nabla_{\partial_z} + \nabla_E)[\omega_i] = \frac{1}{2} [\omega_i], \quad 1 \leq i \leq N.$$

5. Primitive forms

The goal in this section is to define a primitive form and prove that their classification reduces to the differential equations that classify semi-simple Frobenius manifolds.

- 5.1. **Definition.** Let $\omega \in \widehat{\mathcal{H}}(V)$ be a cohomology class whose leading order term is a volume form along C. Recall that under this condition the period map (9) is an isomorphism. Furthermore, the form ω is called a *primitive form* on V if the following 5 properties are satisfied
- $(\mathrm{PF1}) \ K^{(n)}(z\nabla_{\partial_{u_i}}\omega,z\nabla_{\partial_{u_i}}\omega)=0 \ \text{for all} \ 1\leq i,j\leq N \ \text{and} \ n\geq 1.$
- $(\mathrm{PF2}) \ K^{(n)}(z\nabla_{\partial_{u_i}}\omega z\nabla_{\partial_{u_j}}\omega, z\nabla_{\partial_{u_k}}\omega) = 0 \text{ for all } 1 \leq i,j,k \leq N \text{ and } n \geq 2.$
- (PF3) $K^{(n)}(z^2\nabla_{\partial_z}\omega z\nabla_{\partial_{u_i}}\omega, z\nabla_{\partial_{u_i}}\omega) = 0$ for all $1 \le i, j \le N$ and $n \ge 2$.
- (PF4) The cohomology class ω is homogeneous in the following sense

$$(z\nabla_{\partial_z} + \nabla_E)\omega = r\,\omega,$$

where $E = \sum_{i=1}^{N} u_i \partial_{u_i}$ and r is some constant independent of z and u. (PF5)

$$\sum_{i=1}^{N} z \nabla_{\partial_{u_i}} \omega = \omega.$$

If ω is a primitive form, then the Gauss–Manin connection (11) and (12) takes a very simple form. Note that the residue pairing

$$(\partial_{u_i}, \partial_{u_j}) := K^{(0)}(z \nabla_{\partial_{u_i}} \omega, z \nabla_{\partial_{u_j}} \omega) = \frac{\delta_{ij}}{\Delta_i},$$

where Δ_i , $1 \leq i \leq N$ are some holomorphic functions, s.t., $\Delta_i(u) \neq 0$ for all $u \in U$. From formula (11) and axiom (PF2) we get

$$C^k_{ij;n}=0, \quad n\geq 2.$$

The leading order terms $C_{ij;0}^k = \delta_{ij}\delta_{ik}$ coincide with the structure constants of the Frobenius multiplication, while the Leibnitz rule for the higher residue pairing

shows that $C_{ij;1}^k$ are the Christophel's symbols for the residue pairing. Similarly, from (12) and axiom (PF3) we get that

$$C_{i:n}^j = 0, \quad n \ge 2.$$

The leading order term $C_{i;0}^j = -u_i \delta_{ij}$ is the matrix of the linear operator of Frobenius multiplication by -E. Axiom (PF4) yields the following formula

$$\nabla^{\text{L.C.}}_{\partial_{u_i}} E = (r+1)\partial_{u_i} - \sum_{j=1}^{N} C_{i;1}^j \partial_{u_j},$$

where $\nabla^{\text{L.C.}}$ is the Levi–Civita connection for the residue pairing. It is straightforward to prove that the gauge transformation $z^{1/2}\nabla z^{-1/2}$ of the Gauss–Manin connection, turns into the deformed flat connection of a semi-simple Frobenius structure on U of conformal dimension D=1-2r (see [27] for more details).

5.2. **Preliminary notation.** Let $\{[\omega_i]\}_{i=1}^N \subset \widehat{\mathcal{H}}(B)$ be the good basis constructed in the previous section. Let us also fix an open subset $V \subset B$. We would like to find all formal series

$$c_i(u,z) \in \mathbb{C}[\![z]\!], \quad 1 \le i \le N,$$

depending analytically on $u \in V$, s.t.,

$$\sum_{i=1}^{N} c_i(u, z) [\omega_i]$$

is a primitive form in $\widehat{\mathcal{H}}(V)$. Put

$$\omega := ([\omega_1], \ldots, [\omega_N]).$$

Then the Gauss-Manin connection (see Proposition 4.6) takes the form

$$\nabla_{i}\omega = \omega \widetilde{B}_{i}(u, z), \quad 1 \leq i \leq N,$$

$$\nabla_{z\partial_{z} + E}\omega = \frac{1}{2}\omega,$$

where $\nabla_i := \nabla_{\partial_{u_i}}$ and the matrix

$$\widetilde{B}_i(u,z) = z^{-1}E_{ii} + B_i(u), \quad B_i(u) := \sum_{j:j \neq i} \beta_{ij}(u)(E_{ij} - E_{ji}),$$

where E_{ij} denotes the square matrix of size N with all entries 0, except for the entry in position (i, j), which is 1.

Let us denote by $c(u, z) := (c_1(u, z), \dots, c_N(u, z))^T$ the column vector whose entries are the functions that we would like to classify. Put

$$\widetilde{\omega} = (\widetilde{\omega}_1, \dots, \widetilde{\omega}_N),$$

where

$$\widetilde{\omega}_i = z \nabla_i(\omega c) = \omega \Big(E_{ii}c + z(B_ic + \partial_{u_i}c) \Big), \quad 1 \le i \le N.$$

Note that $\widetilde{\omega} = \omega \widetilde{R}(u, z)$, where $\widetilde{R}(u, z)$ is a matrix whose *i*-th column is given by

$$E_{ii}c(u,z) + z(B_i(u)c(u,z) + \partial_{u_i}c(u,z)).$$

If $\omega c(u,z)$ is a primitive form for $u \in V$, then since by construction $\omega_i(p_j) = -\delta_{ij}dt_j$, we get that

$$\omega(p_j) c(u, z) = (-c_j(u, 0) + O(z))dt_j.$$

By definition (see Section 3) $c_i(u,0) \neq 0$ for all $u \in U$. Put

$$C(u) := \operatorname{Diag}(c_1(u,0), \dots, c_N(u,0))$$

and $R(u,z) := \widetilde{R}(u,z)C(u,z)^{-1}$. Note that the series

$$R(u,z) = 1 + R_1(u)z + R_2(u)z^2 + \cdots,$$

where $R_i(u)$ are square matrices of size N whose entries depend analytically on $u \in V$.

5.3. **Differential and algebraic constraints.** In this section, we will be using quite frequently the following notation. If A is a matrix, then A_{ij} will be the entry in raw i and column j. In case A is a raw (resp. column), then we denote by A_i the i-th entry of the raw (resp. column).

Axiom (PF1) is equivalent to

$$K(\widetilde{\omega}_i, \widetilde{\omega}_j) = zK^{(0)}(\widetilde{\omega}_i, \widetilde{\omega}_j), \quad 1 \le i, j \le N.$$

Using that

$$\widetilde{\omega}_i = (\omega R(u, z)C(u))_i = \sum_{k=1}^N \omega_k R_{ki}(u, z)c_i(u, 0)$$

and $K(\omega_k, \omega_\ell) = z\delta_{k\ell}$, we get

$$K^{(0)}(\widetilde{\omega}_i, \widetilde{\omega}_j) = c_i(u, 0)c_j(u, 0)\delta_{ij}$$

and

(22)
$$R(u,z)^T R(u,-z) = 1.$$

Put

$$\gamma_{ij} := \partial_{u_j} c_i(u, 0) c_j(u, 0)^{-1}, \quad 1 \le i, j \le N$$

and

$$\Gamma_i = \sum_{j:j \neq i} \gamma_{ij} (E_{ij} - E_{ji}), \quad 1 \le i \le N.$$

Lemma 5.1. The operator series R(u, z) satisfies the following differential equations

$$z\partial_{u_i}R = [R, E_{ii}] + z(R\Gamma_i - B_iR), \quad 1 \le i \le N.$$

Proof. According to Axiom (PF2),

(23)
$$K(z\nabla_{\partial_{u_i}}\widetilde{\omega}_j,\widetilde{\omega}_k) \in \mathbb{C} z + \mathbb{C} z^2.$$

On the other hand

$$z\nabla_{\partial u_i}\widetilde{\omega}_j=(z\nabla_{\partial u_i}(\omega RC))_j=((z\nabla_{\partial u_i}\omega)RC+\omega z\nabla_{\partial u_i}(RC))_j.$$

Recalling the differential equations for ω we get

$$z\nabla_{\partial_{u_i}}\widetilde{\omega}_j = \sum_{\ell=1}^N [\omega_\ell] (E_{ii}\widetilde{R} + z(B_i\widetilde{R} + \partial_{u_i}\widetilde{R}))_{\ell j}.$$

Therefore (23) is equivalent to

$$\sum_{\ell=1}^{N} (E_{ii}\widetilde{R} + z(B_{i}\widetilde{R} + \partial_{u_{i}}\widetilde{R}))_{\ell j}\widetilde{R}_{\ell k}(u, -z) \in \mathbb{C} + \mathbb{C} z.$$

The above condition, written in matrix form, becomes

$$\widetilde{R}(u,-z)^T((E_{ii}\widetilde{R}+z(B_i\widetilde{R}+\partial_{u_i}\widetilde{R}))=A_0^{(i)}+A_1^{(i)}z,$$

where $A_{\alpha}^{(i)}$, $\alpha = 0, 1$, are some matrices independent of z. Recalling $\widetilde{R}(u, z) = R(u, z)C(u)$ we get

(24)
$$E_{ii}R(u,z) + z(B_i(u)R + \partial_{u_i}R(u,z)) = R(u,z)(B_0^{(i)} + B_1^{(i)}z),$$

where

$$B_0^{(i)} = C(u)^{-1} A_0^{(i)} C(u)^{-1}, \quad B_1^{(i)} = C(u)^{-1} A_1^{(i)} C(u)^{-1} - \partial_{u_i} C(u) C(u)^{-1}.$$

Comparing the coefficients in front of z^0 and z^1 in (24) we get

$$B_0^{(i)} = E_{ii},$$

 $B_1^{(i)} = [E_{ii}, R_1] + B_i.$

The commutator $[E_{ii}, R_1]$ can be expressed in terms of B_i and C(u). By definition $R_1 = \widetilde{R}_1 C(u)^{-1}$, where \widetilde{R}_1 is the coefficient in front of z^1 in $\widetilde{R}(u, z)$. Recalling the definition of $\widetilde{R}(u, z)$, we get

$$(\widetilde{R}_1)_{ij} = (B_j(u)c(u,0))_i + \partial_{u_j}c_i(u,0) = \partial_{u_j}c_i(u,0) - \beta_{ij}(u)c_j(u,0), \text{ for } i \neq j.$$

Therefore

$$(R_1)_{ij} = \gamma_{ij} - \beta_{ij}$$

and for the commutator we get

$$[E_{ii}, R_1] = \Gamma_i - B_i.$$

The above formula yields $B_1^{(i)} = \Gamma_i$, so the differential equation (24) turns into the differential equation we wanted to prove.

Lemma 5.2. The following differential equations hold

$$(z\partial_z + E)R(u,z) = 0,$$

$$(z\partial_z + E)c(u,z) = \left(r - \frac{1}{2}\right)c(u,z),$$

$$(\partial_{u_1} + \dots + \partial_{u_N})c(u,z) = 0.$$

Proof. The differential equations are consequences of Axioms (PF2)–(PF5). Let us derive the first one. The computations for the remaining ones are straightforward.

According to Axiom (PF2) and (PF3), we have

(25)
$$K(\nabla_{z\partial_z + E} \widetilde{\omega}_i, \widetilde{\omega}_i) \in \mathbb{C} + \mathbb{C} z.$$

On the other hand

$$\nabla_{z\partial_z + E} \widetilde{\omega}_i = (\nabla_{z\partial_z + E} \,\omega \widetilde{R})_i = \left(\omega \left(\frac{1}{2}\widetilde{R} + (z\partial_z + E)\widetilde{R}\right)\right)_i.$$

Using that $K(\omega_k\omega_\ell)=z\delta_{k\ell}$, $\widetilde{R}(u,z)=R(u,z)C(u)$, and (22), we get that (25) is equivalent to

$$R(u, -z)^T \left(z\partial_z + E + \frac{1}{2}\right) R(u, z) \in \operatorname{Mat}_{N \times N}(\mathbb{C}) z^{-1} + \operatorname{Mat}_{N \times N}(\mathbb{C}).$$

The above condition implies

$$(z\partial_z + E)R(u,z) = R(u,z)(A_0z^{-1} + A_1).$$

Comparing the coefficients in front of z^{-1} and z^0 we get that $A_0 = A_1 = 0$, which is exactly what we need.

Let $c(u,z) \in \mathbb{C}^N[\![z]\!]$ be a formal series depending analytically on $u \in B$, s.t., the *i*-th component $c_i(u,z)$ $(1 \le i \le N)$ of c(u,z) satisfies $c_i(u,0) \ne 0$ for some $u \in B$. Let us define $V \subset B$ to be the open subset of those $u \in B$, such that, $c_i(u,0) \ne 0$ for all $1 \le i \le N$. According to our assumptions $V \ne \emptyset$.

Proposition 5.3. The cohomology class

$$\omega c(u,z) = \sum_{i=1}^{N} c_i(u,z) [\omega_i]$$

is a primitive form in $\widehat{\mathcal{H}}(V)$ if and only if the following equations are satisfied

(26)
$$(E_{ii} + z(B_i(u) + \partial_{u_i}))c(u, z) = R(u, z)C(u)e_i,$$

(27)
$$R(u,z)R(u,-z)^T = 1,$$

(28)
$$z\partial_{u_i}R(u,z) = [R(u,z), E_{ii}] + z(R(u,z)\Gamma_i(u) - B_i(u)R(u,z)),$$

$$(29) (z\partial_z + E)R(u,z) = 0,$$

(30)
$$(z\partial_z + E)c(u, z) = (r - 1/2)c(u, z),$$

(31)
$$(\partial_{u_1} + \dots + \partial_{u_N})c(u, z) = 0,$$

where $1 \le i \le N$ and e_i is the vector column, whose i-th entry is 1 and all other entries are 0.

In one direction, the Proposition is already established. Expecting more carefully the derivation of equations (26)–(31), it is straightforward to check that the equations in Proposition 5.3 are sufficient to guarantee that $\omega c(u,z)$ satisfies all axioms (PF1)–(PF5).

5.4. Solving the equations for primitive forms. We are going to prove that the solutions c(u,z) to (26)–(31) are uniquely determined from c(u,0) and we are going to derive differential equations for c(u,0), which guarantee that the reconstruction of c(u,z) from c(u,0) is a solution to (26)–(31).

Lemma 5.4. Let c(u, z) be a solution to (26)–(31). Then

- a) We have $\gamma_{ij} = \gamma_{ji}$.
- b) The series R(u, z) is uniquely determined from γ_{ij} , $1 \le i, j \le N$.

Proof. a) The symplectic condition $R(u,-z)^T R(u,z) = 1$ implies that $R_1^T = R_1$, so $[E_{ii}, R_1]^T = -[E_{ii}, R_1]$. On the other hand, we already proved (see Lemma 5.1) that

$$[R_1, E_{ii}] + \Gamma_i - B_i = 0.$$

Therefore $\Gamma_i^T = -\Gamma_i$, which is equivalent to $\gamma_{ij} = \gamma_{ji}$. b) Comparing the coefficients in front of z^{k+1} in (27) and (28) we get

(32)
$$\partial_{u_i} R_k = [R_{k+1}, E_{ii}] + R_k \Gamma_i - B_i R_k$$

and

$$\sum_{i=1}^{N} u_i \partial_{u_i} R_{k+1} = -(k+1) R_{k+1}.$$

These recursions can be solved uniquely for R_k , $k \geq 1$, in terms of γ_{ij} . Indeed, we already proved that

$$(R_1)_{ij} = \gamma_{ij} - \beta_{ij}, \quad i \neq j.$$

To determine the diagonal entries of R_1 , we combine the above equations to get

$$-R_1 = \sum_i u_i [R_2, E_{ii}] + R_1 \left(\sum_i u_i \Gamma_i\right) - \left(\sum_i u_i B_i\right) R_1.$$

From here we get

$$(R_1)_{ii} = \sum_{j:j\neq i} (u_i - u_j)(\gamma_{ij} - \beta_{ij})(\gamma_{ij} + \beta_{ij}).$$

In general, we use equations (32) to determine the non-diagonal entries of R_{k+1} , while for the diagonal entries we use

$$-(k+1)R_{k+1} = \sum_{i} u_i [R_{k+2}, E_{ii}] + R_{k+1} \left(\sum_{i} u_i \Gamma_i\right) - \left(\sum_{i} u_i B_i\right) R_{k+1}. \quad \Box$$

Lemma 5.5. If c(u,z) is a solution to (26)–(31), then

$$c(u,z) = R(u,z) \sum_{i=1}^{N} c_i(u,0)e_i.$$

Proof. Equation (28) can be written us

$$(z(\partial_{u_i} + B_i(u)) + E_{ii}) \circ R(u, z) = R(u, z) \circ (z(\partial_{u_i} + \Gamma_i(u)) + E_{ii}),$$

i.e., R(u, z) is a gauge transformation intertwining two connections. Using the above relation, we get that $C(u)e_i$ is

$$R(u,z)^{-1}(z(\partial_{u_i} + B_i(u)) + E_{ii})c(u,z) = (z(\partial_{u_i} + \Gamma_i(u)) + E_{ii})(R(u,z)^{-1}c(u,z)).$$

Summing over all i, we get

$$c(u,0) = \sum_{i=1}^{N} C(u)e_i = R(u,z)^{-1}c(u,z),$$

where we used that $\sum_i \Gamma_i = 0$ and $\sum_i \partial_{u_i} R(u, z) = 0$. The former identity follows from the definition of Γ_i and the symmetry $\gamma_{ij} = \gamma_{ji}$, while the latter is a consequence of (28): sum (28) over all i and recall that $\sum_i \Gamma_i = \sum_i B_i = 0$. Using Lemma 5.4 and Lemma 5.5 we get the following equations for c(u, 0):

(33)
$$\gamma_{ij} := (\partial_{u_i} c_i(u, 0)) c_j(u, 0)^{-1} \quad \text{is symmetric in } i \text{ and } j,$$

$$[\partial_{u_i} + \Gamma_i, \partial_{u_i} + \Gamma_j] = 0, \quad 1 \le i, j \le N,$$

$$(\partial_{u_1} + \dots + \partial_{u_N})c(u,0) = 0,$$

(36)
$$E c_i(u,0) = (r-1/2)c_i(u,0), \quad 1 \le i \le N.$$

Proposition 5.6. Let $c_i(u,0)$, $1 \le i \le N$, be a set of functions analytic for $u \in V$ and such that $c_i(u,0) \ne 0$ for all i. The functions c_i solve the equations (33)–(36) if and only if they can be extended to functions $c_i(u,z)$, $1 \le i \le N$, solving the equations (26)–(31).

Proof. We have already proved that if $c_i(u, z)$, $1 \le i \le N$, solve (26)–(31), then $c_i(u, 0)$, $1 \le i \le N$, solve (33)–(36). In the inverse direction, let us assume that $c_i(u, 0)$, $1 \le i \le N$, solve (33)–(36). We need to check that the series $c_i(u, z)$, $1 \le i \le N$, defined by the reconstructions of Lemma 5.4 and Lemma 5.5 solve (26)–(31). This however is straightforward.

Proof of Theorem 2.5. According to [7], a semi-simple Frobenius structure on V is specified by a set of N functions $c_i(u,0)$, $1 \le i \le N$, satisfying the system of differential equations defined by (33)–(36) and such that $c_i(u,0) \ne 0$ for all $u \in V$. On the other hand, recalling Proposition 5.6 we get that there is a one-to-one correspondence between solutions $c_i(u,0)$ to (33)–(36) satisfying $c_i(u,0) \ne 0$ for $u \in V$ and primitive forms in $\widehat{\mathcal{H}}(V)$.

6. Polynomial primitive forms

Suppose that $V \subset B$ is an open subset. We say that the cohomology class $\omega \in \widehat{\mathcal{H}}(V)$ is a polynomial class if for every $p \in V$ we can find an open connected Stein subset $U \subset V$ such that the restriction $\omega|_U \in \widehat{\mathcal{H}}(U)$ can be represented by a form

$$\sum_{n=0}^{n_0(U)} \omega_U^{(n)}(-z)^n, \quad \omega_U^{(n)} \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{\pi}^{-1}(U)),$$

depending polynomially on z. Note that we do not require the degrees $n_0(U)$ of the polynomials to be uniformly bounded. Nevertheless, we will prove that we can always choose a covering of V with polynomial representatives for which the degree of the polynomials are uniformly bounded. If ω is both polynomial and primitive, then we say that ω is a polynomial primitive form.

6.1. Polynomiality and the sheaf \mathcal{H} . The main goal in this section is to prove that the polynomial classes in $\widehat{\mathcal{H}}$ correspond to cohomology classes in \mathcal{H} . To make this statement precise, let us first prove that the map $i: \mathcal{H} \to \widehat{\mathcal{H}}$ induced by the inclusion

$$\overline{\pi}_*\Omega^1_{\overline{X}/B}[z] \to \overline{\pi}_*\Omega^1_{\overline{X}/B}[\![z]\!]$$

is injective. Recall that the sheaf \mathcal{H} is the sheafification of the presheaf H_F on B defined by

$$H_F(U) := \overline{\pi}_* \Omega^1_{\overline{X}/B}(U)[z]/(zd_{\overline{X}/B} + d_{\overline{X}/B}F \wedge) \overline{\pi}_* \Omega^0_{\overline{X}/B}(U)[z].$$

Similarly, the completion $\widehat{\mathcal{H}}$ is the sheafification of the presheaf

$$\widehat{H}_F(U) := \overline{\pi}_* \Omega^1_{\overline{X}/B}(U) [\![z]\!]/(z d_{\overline{X}/B} + d_{\overline{X}/B} F \wedge) \overline{\pi}_* \Omega^0_{\overline{X}/B}(U) [\![z]\!].$$

The injectivity of i is a corollary of the following lemma, which implies that the induced map on the stalks is injective.

Lemma 6.1. Let $U \subset B$ be an open connected subset, then the natural map $i_F: H_F(U) \to \widehat{H}_F(U)$ is injective.

Proof. We have to prove that if

$$(zd_{\overline{X}/B} + d_{\overline{X}/B}F \wedge)\phi = \psi, \quad \psi \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{\pi}^{-1}(U))[z], \quad \phi \in \Omega^{0,\infty}_{\overline{X}/B}(\overline{\pi}^{-1}(U))[z]$$

then ϕ is polynomial in z. Let us write $\phi = \sum_k \phi_k z^k$ and $\psi = \sum_k \psi_k z^k$ with $\psi_k = 0$ for $k > k_0$ for some positive integer k_0 . We have to prove that $\phi_k = 0$ for $k \gg 0$.

Since U is connected, the intersection $\overline{\pi}^{-1}(U) \cap D_{\infty}$ has finitely many connected components. Therefore

$$H^0(\overline{\pi}^{-1}(U), \Omega^{0,\infty}_{\overline{X}/B}) = \varinjlim_{m} H^0(\overline{\pi}^{-1}(U), \mathcal{O}_{\overline{X}}(m)) = \bigcup_{m=0}^{\infty} H^0(\overline{\pi}^{-1}(U), \mathcal{O}_{\overline{X}}(m)).$$

Therefore, we can choose m_0 such that $\phi_{k_0} \in \overline{\pi}_* \mathcal{O}_{\overline{X}}(m_0)(U)$. Note that

$$\phi_{k_0+\ell} = \left(-\frac{d_{\overline{X}/B}}{d_{\overline{X}/B}F}\right)^{\ell} \phi_{k_0}$$

for all $\ell > 0$. Note that the RHS of the above identity is a section of

$$\overline{\pi}_* \mathcal{O}_{\overline{X}}((m_0 - \ell)D_{\infty}).$$

But the above pushforward is 0 for $\ell > m_0$, so $\phi_k = 0$ for all $k > k_0 + m_0$.

For every open subset $U \subset B$ we have the following commutative diagram

$$H_{F}(U) \xrightarrow{\rho_{F}} \mathcal{H}(U)$$

$$\downarrow i_{F} \qquad \qquad \downarrow i_{F}$$

$$\widehat{H}_{F}(U) \xrightarrow{\widehat{\rho}_{F}} \widehat{\mathcal{H}}(U)$$

where ρ_F and $\widehat{\rho}_F$ are the natural maps from a presheaf to its sheafification. If we assume in addition that U is connected and Stein, then the above lemma implies that i_F is injective, while Proposition 3.4 implies that $\widehat{\rho}_F$ is an isomorphism. We get that both ρ_F and i must be injective. The following Proposition is straightforward to prove by using the properties of the above commutative diagram.

Proposition 6.2. Let $V \subset B$ be an open subset. Then a cohomology class $\omega \in \widehat{\mathcal{H}}(V)$ is polynomial iff $\omega \in \mathcal{H}(V)$.

6.2. **Primary differentials.** Let us recall the definition of Dubrovin's primary differentials. They are splited into five types. Following [11, 29] we express them in terms of the fundamental bi-differential.

Type I. Normalized Abelian differentials of the second kind on \overline{X}_u , $u \in U$ with poles only at ∞_i $(1 \le i \le d)$ of order not exceeding the order of the pole of $d_{X/B}F$

$$\phi_{t_{i,a}}(p) := \frac{1}{a} \operatorname{res}_{q=\infty_i} F(q)^{a/m_i} B(q,p), \quad 1 \le i \le d, \quad 1 \le a \le m_i - 1.$$

Type II. Normalized Abelian differentials of the second kind

$$\phi_{v_i}(p) := \operatorname{res}_{q=\infty_i} F(q)B(q,p), \quad 2 \le i \le d.$$

Note that $\phi_{v_i}(p)$ has pole only at $p = \infty_i$ and the principal part is of the form $-d_{X/B}F(p)$ + regular terms.

Type III. Normalized Abelian differentials of the third kind

$$\phi_{w_i}(p) := \omega_{\infty_i,\infty_1}(p) = \int_{\infty_1}^{\infty_i} B(q,p), \quad 2 \le i \le d$$

having poles of order 1 only at ∞_1 and ∞_i with residues respectively -1 and 1.

Type IV. Multi-valued analytic differentials on \overline{X}_u

$$\phi_{r_i}(p) := \frac{1}{2\pi\sqrt{-1}} \oint_{q \in \alpha_i} F(q)B(q, p), \quad 1 \le i \le g$$

with increment along the cycle β_i equal to

$$\phi_{r_i}(p+\beta_j) - \phi_{r_i}(p) = -\delta_{ij} d_{X/B} F(p).$$

Type V. The holomorphic 1-forms on \overline{X}_u

$$\phi_{s_i}(p) := \frac{1}{2\pi\sqrt{-1}} \oint_{q \in \beta_i} B(q, p), \quad 1 \le i \le g$$

satisfying the normalization condition

$$\oint_{p \in \alpha_i} \phi_{s_j}(p) = \delta_{ij}.$$

Every primary differential ϕ induces a cohomology class $[\phi] \in \widehat{\mathcal{H}}$ as follows. Let $X' \subset \overline{X}$ be a tubular neighborhood of C such that $\phi \in \Omega^1_{\overline{X}/B}(X')$. Then $[\phi]$ is defined via the isomorphism in Proposition 3.4, Part b).

Proposition 6.3. The cohomology classes $[\phi]$ are polynomial in z, i.e., $[\phi] \in \mathcal{H}(B)$.

Proof. All primary differentials, except for Type IV are already in $\Omega_{\overline{X}/B}^{1,\infty}(\overline{X})$, so the statement in the lemma is obvious. Let us assume that $\phi = \phi_{r_i}$ is of type IV.

Let us fix a positive integer m > 2g - 1. Denote by K the coherent sheaf on \overline{X} defined by the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Omega^{\frac{1}{X/B}}(m\infty_1) \xrightarrow{\operatorname{res}_{\infty_1}} \mathbb{C}_{\infty_1} \longrightarrow 0,$$

where $\mathbb{C}_{\infty_1} = (\infty_1)_* \mathbb{C}_B$ is the pushforward of the constant sheaf \mathbb{C}_B on B via the section $\infty_1 : B \to \overline{X}$. On the other hand we have the following exact sequence of sheaves of $\overline{\pi}^{-1}\mathcal{O}_B$ -modules

$$0 \longrightarrow \overline{\pi}^{-1}\mathcal{O}_B \longrightarrow \Omega^0_{\overline{X}/B}((m-1)\infty_1) \stackrel{d_{\overline{X}/B}}{\longrightarrow} \mathcal{K} \to 0.$$

Pushing forward we get

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \overline{\pi}_* \Omega^0_{\overline{X}/B}((m-1)\infty_1) \longrightarrow \overline{\pi}_* \mathcal{K} \longrightarrow R^1 \overline{\pi}_* (\overline{\pi}^{-1} \mathcal{O}_B) \longrightarrow 0.$$

We get that $R^1\overline{\pi}_*(\overline{\pi}^{-1}\mathcal{O}_B)$ is a coherent \mathcal{O}_B -modules, because the remaining sheaves in the above exact sequence are coherent \mathcal{O}_B -modules. Moreover, using the proper base change theorem in sheaf cohomology (see [20], Theorem 6.2), we get that the stalk of $R^1\overline{\pi}_*(\overline{\pi}^{-1}\mathcal{O}_B)$ at a point $u \in B$ is

$$H^1(\overline{X}_u, j_u^{-1}\overline{\pi}^{-1}\mathcal{O}_B) = H^1(\overline{X}_u, \mathbb{C}) \otimes \mathcal{O}_{B,u},$$

where $j_u : \overline{X}_u \to \overline{X}$ is the natural inclusion. Therefore, the sheaf $R^1 \overline{\pi}_*(\overline{\pi}^{-1} \mathcal{O}_B)$ is the sheaf of holomorphic sections of the vector bundle on B whose fiber over

 $u \in B$ is $H^1(\overline{X}_u, \mathbb{C})$. Note that this is a holomorphically trivial bundle because B is Stein and contractible. Let us construct a trivialization by choosing a basis of $H^1(\Sigma, \mathbb{C})$ (recall that $\overline{X}_{u^\circ} := \Sigma$ is our reference fiber) Poincare dual to the basis $\{\alpha_i, \beta_i\}_{i=1}^g \subset H_1(\Sigma, \mathbb{C})$ and using the parallel transport with respect to the Gauss–Manin connection. On the other hand, since B is Stein, the above 4-term exact sequence remains exact when we take global sections. Therefore, we can find global meromorphic forms (with no residues along ∞_1)

$$\phi_{s_j}^{\vee} \in H^0(\overline{X}, \Omega_{\overline{X}/B}^1(m\infty_1)), \quad 1 \le j \le g,$$

that represent the Poincare duals of the cycles β_j , $1 \le j \le g$, i.e.,

$$\oint_{\alpha_i} \phi_{s_j}^{\vee} = 0, \quad \oint_{\beta_i} \phi_{s_j}^{\vee} = \delta_{ij}.$$

Let us fix $u \in B$ and define

$$\psi_i(p) = \int_{p_1(u)}^p \phi_{s_i}^{\vee}, \quad 1 \le i \le g,$$

where $p_1(u) \in \overline{X}_u$ is the critical point of F corresponding to the critical value u_1 . This is a multi-valued analytic function with increment along the β_i cycle

$$\psi_i(p+\beta_j) - \psi_i(p) = \oint_{\beta_j} d\psi_i(p) = \delta_{ij}.$$

It follows that the 1-form

$$\omega_{r_i} := \phi_{r_i}(p) + \psi_i(p)dF(p) + z\phi_{s_i}^{\vee}(p) = \phi_{r_i}(p) + (zd + dF \wedge)\psi_i(p)$$

is holomorphic for all $p \in \overline{X}_u \setminus \{\infty_1\}$ with a finite order pole at ∞_1 . Clearly the point-wise construction for $u \in B$ produces a global form $\omega_{r_i} \in \Omega^{1,\infty}_{\overline{X}/B}(\overline{X})[z]$, which is polynomial in z of degree 1 and under the restriction map from Proposition 3.4, Part b), the cohomology class of ω_{r_i} is mapped to the cohomology class of the primary differential ϕ_{r_i} .

Proposition 6.4. If ϕ is a primary differential, then the corresponding cohomology class $[\phi] \in \widehat{\mathcal{H}}(B)$ is a primitive form on an appropriate open subset $U \subset B$.

Proof. We claim that the higher-residue pairings

$$K([\phi], [\omega_a]) = zc_a, \quad 1 \le i \le N,$$

where $\{[\omega_a]\}_{a=1}^N \subset \widehat{\mathcal{H}}(B)$ is the good basis we have constructed in Section 4 and

$$c_a = -\operatorname{res}_{p=p_a} \frac{\phi(p)}{t_a(p)}$$

are holomorphic functions on B. The notation in the above formula is the one we have introduced in the beginning of Section 4. The proof follows from the

following two formulas

(37)
$$\sum_{a=1}^{N} z \nabla_{\partial_{u_a}} [\phi] = [\phi]$$

and

$$z\nabla_{\partial u_a}[\phi] = c_a[\omega_a]$$

and Proposition 4.4. Let us prove the above two formulas when $\phi = \phi_{t_{i,b}}$ is a Type I differential. The argument in the remaining 4 cases is similar. By definition

$$[\phi] = b^{-1} \int e^{F(p)/z} \operatorname{res}_{q=\infty_i} \Big(F(q)^{b/m_i} B(q, p) \Big).$$

On the other hand, recalling the Rauch's variational formula we get

$$\sum_{a=1}^{N} \delta_{u_a} B(q, p) = \sum_{a=1}^{N} \operatorname{res}_{q'=p_a} \frac{B(q, q') B(q', p)}{d_{X/B} F(q')} = -d_p \left(\frac{B(q, p)}{d_{X/B} F(p)} \right) - d_q \left(\frac{B(q, p)}{d_{X/B} F(q)} \right),$$

where in the second equality we have used the residue theorem for \overline{X}_u and the fact that the integrand has poles only at $q' = p_a$ $(1 \le a \le N)$, q' = q, and q' = p. Using the above formula and

$$-\operatorname{res}_{q=\infty_{i}} F(q)^{b/m_{i}} d_{q} \left(\frac{B(q,p)}{d_{X/B} F(q)} \right) = \frac{b}{m_{i}} \operatorname{res}_{q=\infty_{i}} F(q)^{\frac{b}{m_{i}} - 1} B(q,p) = 0.$$

we get

$$\sum_{a=1}^{N} z \nabla_{\partial_{u_a}} [\phi] = b^{-1} \int e^{F(p)/z} (-z d_p) \operatorname{res}_{q=\infty_i} \left(\frac{F(q)^{b/m_i} B(q, p)}{d_{X/B} F(p)} \right).$$

Integrating by parts the RHS of the above formula we get (37).

To prove (38) we use that

$$\delta_{u_a} B(q, p) = \operatorname{res}_{q'=p_a} \frac{B(q, q') B(q', p)}{d_{X/B} F(q')} = \operatorname{res}_{q'=p_a} \left(\frac{B(q, q')}{t_a(q')} \right) \operatorname{res}_{q'=p_a} \left(\frac{B(q', p)}{t_a(q')} \right)$$

The above formula implies that

$$z\nabla_{u_a}[\phi] = c_a \int e^{F(p)/z} (-zd_p) \operatorname{res}_{q'=p_a} \left(\frac{\omega_p(q')}{t_a(q')}\right) = c_a[\omega_a].$$

We proved that $[\phi] = \sum_{a=1}^{N} c_a(u)[\omega_a]$, where the coefficients c_a are independent of z. We are going to prove that the functions $c_a(u)$ satisfy the differential equations (33)–(34) and that the matrix R(u,z) reconstructed from $c_a(u)$ via Lemma 5.4 is 1. To this end, it is enough to prove that $\gamma_{ij} = \beta_{ij}$ and that $[\phi]$ is homogeneous. Indeed, if we know that $\gamma_{ij} = \beta_{ij}$, then equation (33) is satisfied. Furthermore,

$$\Gamma_i = \sum_{j:j\neq i} \gamma_{ij} (E_{ij} - E_{ji}) = \sum_{j:j\neq i} \beta_{ij} (E_{ij} - E_{ji}) = B_i,$$

so recalling Proposition 4.6 we get that $\partial_{u_i} + \Gamma_i$ is the leading order term of the Gauss-Manin connection on $\widehat{\mathcal{H}}$ (written in the frame $[\omega_i]_{i=1}^N$):

$$\nabla_{\partial_{u_i}} = \partial_{u_i} + B_i + z^{-1} E_{ii}.$$

The flatness of the Gauss-Manin connection implies equations (34). The next equation (35) is a consequence of (37) and Proposition 4.6. Finally, it is easy to see that if $\gamma_{ij} = \beta_{ij}$ then in the reconstruction procedure for R(u, z) we have $R_k(u) = 0$ for all $k \ge 1$.

Let us prove that $\gamma_{aj} = \beta_{aj}$ and that $[\phi]$ is homogeneous. Again, we will do this only for $\phi = \phi_{t_{i,b}}$ a primary differential of Type I, because the argument in the remaining cases is similar. First, we need to prove that

$$\partial_{u_a} c_j(u) = \beta_{aj}(u)c_a(u), \quad a \neq j.$$

By definition

$$c_j = -b^{-1} \operatorname{res}_{q=\infty_i} \operatorname{res}_{p=p_j} \frac{F(q)^{b/m_i} B(q, p)}{t_j(p)}.$$

We are going to use that

$$\partial_{u_a} c_j = -b^{-1} \operatorname{res}_{q=\infty_i} \operatorname{res}_{p=p_j} \delta_{u_a} \Big(\frac{F(q)^{b/m_i} B(q, p)}{t_i(p)} \Big),$$

where δ_{u_a} is the Rauch variational derivative. If $a \neq j$, then we have

$$\delta_{u_a} \left(\frac{F(q)^{b/m_i} B(q, p)}{t_j(p)} \right) = \operatorname{res}_{q'=p_a} \frac{F(q)^{b/m_i} B(q, q') B(q', p)}{t_j(p) d_{X/B} F(q')}.$$

On the other hand.

$$\operatorname{res}_{q'=p_a} \frac{B(q,q')B(q',p)}{d_{X/B}F(q')} = \operatorname{res}_{q'=p_a} \left(\frac{B(q,q')}{t_a(q')} \right) \operatorname{res}_{q'=p_a} \left(\frac{B(q',p)}{t_a(q')} \right).$$

It remains only to recall the definition (18).

For the homogeneity part, we have to compute

$$(z\nabla_{\partial_z} + \nabla_E)[\phi] = \left(z\nabla_{\partial_z} + \sum_{a=1}^N u_a \nabla_{\partial_{u_a}}\right)[\phi].$$

Using the Rauch's variational formula we get that $\sum_a u_a \delta_{u_a} B(q,p)$ is

$$\sum_{q=1}^{N} \operatorname{res}_{q'=p_a} \frac{F(q')B(q,q')B(q',p)}{d_{X/B}F(q')} = -d_p \left(\frac{F(p)B(q,p)}{d_{X/B}F(p)} \right) - d_q \left(\frac{F(q)B(q,p)}{d_{X/B}F(q)} \right).$$

The contribution of the first term to $z\nabla_E[\phi]$ is

$$b^{-1} \int e^{F(p)/z} \operatorname{res}_{q=\infty_i} F(q)^{b/m_i} (-zd_p) \left(\frac{B(q,p)F(p)}{d_{X/B}F(p)} \right) = -z^2 \nabla_{\partial_z} [\phi],$$

while the contribution of the second term is

$$b^{-1} \int e^{F(p)/z} \operatorname{res}_{q=\infty_i} F(q)^{b/m_i} (-zd_q) \left(\frac{B(q,p)F(q)}{d_{X/B}F(q)} \right) = (b/m_i) [\phi].$$

It follows that the form $[\phi_{t_{i,b}}]$ is homogeneous of degree $r_{t_{i,b}} := b/m_i$.

Theorem 2.6 is a direct consequence of Propositions 6.3 and 6.4. For future references, let us list the homogeneous degrees of the primary differentials

$$r_{t_{i,b}} = b/m_i$$
, $1 \le i \le d$, $1 \le b \le m_i - 1$, $r_{v_i} = r_{r_i} = 1$, $r_{w_i} = r_{s_j} = 0$, $2 \le i \le d$, $1 \le j \le g$.

6.3. Polynomiality of the *R*-matrix. Let $V \subset B$ be an open connected subset. Note that every cohomology class $\omega \in \widehat{\mathcal{H}}(V)$ can be written as

$$\omega = \sum_{i=1}^{N} c_i[\omega_i], \quad c_i := K(\omega, [\omega_i])z^{-1},$$

where the coefficients $c_i \in \mathcal{O}_B(V)[\![z]\!]$.

Proposition 6.5. The cohomology class ω is polynomial, if and only if the coefficients $c_i(u, z)$ $(1 \le i \le N)$ depend polynomially on z.

Proof. For fixed $u \in B$ the differentials $\omega_i(p)$ are multi-valued analytic on X_u with increment along the cycle β_i given by

$$\omega_i(p+\beta_j) - \omega_i(p) = c_{ij}(u)dF(p),$$

where

$$c_{ij}(u) = \operatorname{res}_{q=p_i} \frac{\theta_j(q)}{t_i(q)}, \quad \theta_j(q) := \oint_{p \in \beta_i} B(p, q).$$

Recall that in the proof of Proposition 6.3 we have constructed multivalued analytic functions $\psi_i(p)$ on \overline{X} with finite order poles along the divisor D_{∞} , such that the 1-forms $d_{\overline{X}/B}\psi_i \in H^0(\overline{X}, \Omega^{1,\infty}_{\overline{X}/B})$, have vanishing residues along D_{∞} , and

$$\oint_{p \in \alpha_i} d\psi_j(p) = 0, \quad \oint_{p \in \beta_i} d\psi_j(p) = \delta_{ij}, \quad 1 \le i, j \le g.$$

We get that under the restriction map defined by Proposition 3.4, Part b), the cohomology class of $\omega_i(p)$ is identified with the cohomology class of

$$\omega_i - \sum_{j=1}^g c_{ij}(u) \left(zd_{\overline{X}/B} + d_{\overline{X}/B}F\wedge\right) \psi_j(p).$$

Note that the form on the RHS is analytic and polynomial in z of degree at most 1, so the cohomology classes $[\omega_i]$ are polynomial. Therefore, we need only to prove that if ω is polynomial, then $c_i(u, z)$ are polynomial in z.

Let us fix $u_0 \in V$ and an open connected Stein neighborhood $U \subset V$ of u_0 , s.t., ω can be represented by a polynomial form

$$\omega_U = \sum_{n=0}^{n_0(U)} \omega_U^{(n)} z^n, \quad \omega_U^{(n)} \in H^0(\overline{\pi}^{-1}(U), \Omega^{\frac{1}{X/B}}(m D_\infty)),$$

where m is a sufficiently large integer. Note that if $\theta \in H^0(\overline{\pi}^{-1}(U), \Omega^1_{\overline{X}/B}(m D_\infty))$, then since θ has a finite order pole along D_∞ , we can express the singular part of θ along D_∞ in terms of linear combinations of finitely many expressions of the type $F(p)^n \phi(p)$, where $n \geq 0$ and ϕ is a primary differential of type I, II, or III. Therefore, there are polynomials $a_{\phi}(\lambda) \in \mathcal{O}_B(U)[\lambda]$, s.t.,

$$\theta(p) - \sum_{\phi \in I \cup II \cup III} a_{\phi}(F(p))\phi(p),$$

is a holomorphic 1-form on \overline{X}_u and hence it can be written as a linear combination of primary differentials of type V. Using this observation, we get that

$$[\omega_U] = \sum_{\phi} [b_{\phi}(z, F)\phi] = \sum_{\phi} b_{\phi}(z, -z^2 \nabla_{\partial_z})[\phi],$$

where the sum is over primary differentials of Type I, II, III, and V, $b_{\phi} \in \mathcal{O}_B(U)[z,\lambda]$, and for the second equality we used that

$$-z^{2}\nabla_{\partial_{z}}\int e^{F(p)/z}\phi(p) = \int e^{F(p)/z}F(p)\phi(p).$$

We get that

$$[\omega_U] = \sum_{\phi} \sum_{i=1}^{N} c_{\phi,i} b_{\phi}(z, -z^2 \nabla_{\partial_z}) [\omega_i],$$

where the coefficients $c_{\phi,i}$ are independent of z. On the other hand, according to Proposition 4.6 (see the notation in Section 5.2) we have

$$-z^{2}\nabla_{\partial_{z}}\omega^{\mathrm{gb}} = \omega^{\mathrm{gb}}\left(\left(\frac{1}{2} - \sum_{i=1}^{N} u_{i}B_{i}(u)\right)z - \sum_{i=1}^{N} u_{i}E_{ii}\right),$$

where $\omega^{\text{gb}} := ([\omega_1], \dots, [\omega_N])$. Therefore, $b_{\phi}(z, -z^2 \nabla_{\partial_z})[\omega_i]$ is a linear combination of $[\omega_i]$, $1 \le j \le N$, with coefficients depending polynomially on z. We get that

$$\omega_U = \sum_{i=1}^{N} c_i^U \left[\omega_i \right],$$

where $c_i^U \in \mathcal{O}_B(U)[z]$. Therefore, $c_i|_U = c_i^U$ is polynomial in z of some degree n(U). Writing $c_i = \sum_{n=0}^{\infty} c_{i,n} z^n$, we get that $c_{i,n} = 0$ for all n > n(U). \square Note that Proposition 6.5 can be reformulated as follows. The map

$$\mathcal{H} \to \mathcal{O}_B[z]^{\oplus N}, \quad \omega \mapsto (c_1, \dots, c_N), \quad c_i := K(\omega, \omega_i)$$

is an isomorphism of $\mathcal{O}_B[z]$ -modules and $\{[\omega_i]\}_{i=1}^N$ is an $\mathcal{O}_B[z]$ -basis. Therefore, we proved Corollary 2.7.

Let $c(u) = (c_1(u), \ldots, c_N(u))$ be an arbitrary solution to the equations (33)–(36). Let us denote by R(u, z) the matrix reconstructed from c(u) via the algorithm of Lemma 5.4. Then we have the following corollary of Proposition 6.5.

Corollary 6.6. The primitive form corresponding to c(u) is polynomial if and only if the entries of the matrix R(u, z) depend polynomially on z.

In the next section we are going to formulate the problem of relating the correlator forms defined by the Eynard–Orantin recursion with the higher-genus invariants of a semi-simple Frobenius manifold. The case for which R(u,z)=1 plays a key role. Note that if R(u,z)=1, then according to Lemma 5.5 the coefficients $c_i(u,z)=c_i(u,0)$ are independent of z. Polynomial primitive forms for which R(u,z)=1 can be classified in terms of primary differentials. We have the following corollary.

Corollary 6.7. Let $\omega \in \widehat{\mathcal{H}}$ be a cohomology class of homogeneous degree r. Then ω is a primitive form whose matrix R(u,z)=1 if and only if

$$\omega = \sum_{\phi: \deg(\phi) = r} a_{\phi}[\phi],$$

where the sum is over all primary differentials of homogeneous degree r and a_{ϕ} are constants independent of u and z.

Proof. If R(u, z) = 1 then $R_1(u) = 0$, so $\gamma_{ij}(u) = \beta_{ij}(u)$. It remains only to use that the space of solutions $c(u) = (c_1(u), \ldots, c_N(u))$ to the system of differential equations (33)–(35) satisfying the additional constraint $\gamma_{ij} = \beta_{ij}$ is N-dimensional and a basis is given by

$$c_{\phi}(u) = (c_{\phi,1}(u), \dots, c_{\phi,N}(u)), \quad c_{\phi,i}(u) = -\operatorname{res}_{p=p_i} \frac{\phi(p)}{t_i(p)},$$

where ϕ runs through the set of primary differentials.

6.4. **Two-dimensional Frobenius manifolds.** We would like to classify all two dimensional semi-simple Frobenius manifolds that correspond to the polynomial primitive forms of the Hurwitz spaces. To begin with, we need to classify all coverings $f: \Sigma \to \mathbb{P}^1$ for which

$$N = 2g - 2 + d + \sum_{i=1}^{d} m_i = 2.$$

It is easy to see that g = 0, so $\Sigma = \mathbb{P}^1$ and that for the covering map f, up to isomorphism there are only two cases. Either d = 1, $m_1 = 3$, and

$$f(x) = \frac{x^3}{3} + s_1 x + s_2$$

or d = 2, $m_1 = m_2 = 1$, and

$$f(x) = x + s_1 x^{-1} + s_2,$$

where x is the coordinate function on $\mathbb{C} = \Sigma \setminus \{\infty\}$ and $(s_1, s_2) \in \mathbb{C}^* \times \mathbb{C}$ are some parameters.

6.4.1. Case 1. The universal cover $B = \mathbb{C}^2$, the family $X = B \times \mathbb{C}$ and

$$F: X \to \mathbb{C}, \quad F(t, x) = \frac{x^3}{3} + e^{t_1}x + t_2.$$

The critical points and the corresponding critical values of F are respectively

$$p_{1,2} = \pm \sqrt{-1} e^{t_1/2}$$
 and $u_{1,2} = t_2 \pm \frac{2\sqrt{-1}}{3} e^{3t_1/2}$.

The good basis (see Section 4) takes the form

$$\omega_i = -(x+p_i)\frac{dx}{\sqrt{2p_i}}, \quad i = 1, 2,$$

where we have to make an additional choice of a sign of $\sqrt{p_i}$. Let us write

$$p_i = e^{(t_1 + (2i-1)\pi\sqrt{-1})/2}, \quad i = 1, 2,$$

then we define

$$\sqrt{p_i} := e^{(t_1 + (2i-1)\pi\sqrt{-1})/4}, \quad i = 1, 2.$$

Let us assume that

$$c_1(u,z)\omega_1+c_2(u,z)\omega_2$$

is a polynomial primitive form of homogeneous degree r. Equations (35) and (36) imply that

$$c_i(u,0) = c_i^{\circ}(u_1 - u_2)^{r-1/2}, \quad i = 1, 2,$$

where c_i° are some constants. Since

$$\gamma_{1,2}(u) = -\frac{r-1/2}{u_1-u_2} (c_1^{\circ}/c_2^{\circ})$$

$$\gamma_{2,1}(u) = -\frac{r-1/2}{u_1 - u_2} \left(c_2^{\circ}/c_1^{\circ}\right)$$

we get that $c_1^{\circ}/c_2^{\circ} = \pm \sqrt{-1}$. It is enough to classify primitive forms up to a constant factor, so we may assume that $c_1^{\circ} = \sqrt{-1}$ and $c_2^{\circ} = 1$.

It remains only to find the R-matrix reconstructed from $c_i(u,0)$ according to the algorithm of Lemma 5.4. We already know that

$$\Gamma_1 = -\Gamma_2 = \frac{a}{u_1 - u_2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad a := -\sqrt{-1} \left(r - \frac{1}{2} \right).$$

We need to compute the matrices B_1 and B_2 . The Riemann's second fundamental form is

$$B(x_1, x_2) = \frac{dx_1 dx_2}{(x_1 - x_2)^2}.$$

After a straightforward computation we get that the coefficients (18) are

$$\beta_{1,2}(u) = \frac{b}{u_1 - u_2}, \quad b := \sqrt{-1/6}.$$

Therefore,

$$B_1 = -B_2 = \frac{b}{u_1 - u_2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

After a straightforward computation we get

$$R(u,z) = 1 + \sum_{k=1}^{\infty} R_k^{\circ} z^k (u_1 - u_2)^{-k},$$

where

$$R_k^{\circ} = a_k \begin{bmatrix} (a+b(-1)^{k-1} & k \\ (-1)^{k-1}k & -b+a(-1)^{k-1} \end{bmatrix}$$

where the numbers a_k $(k \ge 1)$ are defined recursively by $a_1 = a - b$ and

$$a_{k+1} = \frac{1}{k+1}(a^2 + b^2 + k^2 + 2ab(-1)^{k-1})a_k, \quad k \ge 1.$$

Since the matrix R(u, z) is polynomial in z, there exists an integer m > 0, such that $a_k = 0$ for all $k \ge m$ and $a_{m-1} \ne 0$. Therefore,

$$a = \left(\frac{(-1)^m}{6} \pm m\right)\sqrt{-1}$$

Recalling also that D = 1 - 2r is the conformal dimension, we get that all twodimensional semi-simple Frobenius manifolds of conformal dimension

$$D = \frac{(-1)^n}{3} + 2n, \quad n \in \mathbb{Z},$$

correspond to polynomial primitive forms.

6.4.2. Case 2. The universal cover $B = \mathbb{C}^2$ and the family $X = B \times \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The function

$$F(t,x) = x + e^{t_1}/x + t_2.$$

The critical points and the corresponding critical values are

$$p_{1,2} = \pm e^{t_1/2}, \quad u_{1,2} = t_2 \pm 2e^{t_1/2}.$$

The good basis takes the form

$$\omega_i = -\frac{p_i(x+p_i)}{x^2} \frac{dx}{\sqrt{2p_i}},$$

where we define

$$\sqrt{p_i} := e^{(t_1 + 2\pi(i-1)\sqrt{-1})/4}, \quad i = 1, 2.$$

The coefficient (18) is

$$\beta_{1,2} = \frac{b}{u_1 - u_2}, \quad b := \sqrt{-1/2}.$$

The formulas for the matrix R(u, z) are the same as in the previous case, except that the value of the constant b now is $\sqrt{-1}/2$ (instead of $\sqrt{-1}/6$). We get that the polynomial primitive forms in this case correspond to semi-simple Frobenius manifolds of conformal dimension

$$D = (-1)^n + 2n, \quad n \in \mathbb{Z},$$

i.e., D is an odd integer.

Remark 6.8. The Frobenius structures of A_2 -singularity and of quantum cohomology of \mathbb{P}^1 have conformal dimensions respectively $\frac{1}{3}$ and 1, so they do correspond to polynomial primitive forms. On the other hand, the Frobenius structures on the orbit space of a finite reflection group of type $I_2(k)$ has conformal dimension $D = 1 - \frac{2}{k}$. Therefore, if k > 3 (note that type $I_2(3)$ coincides with type A_2), the Frobenius structure does not correspond to a polynomial primitive form.

7. TOPOLOGICAL RECURSION AND SEMI-SIMPLE FROBENIUS STRUCTURES

Let us recall the notation of Section 2.1. Suppose that $\omega = \sum_{n\geq 0} \omega^{(n)} (-z)^n$ is a polynomial primitive form on $\mathcal{H}(U)$ for some contractible open subset $U \subset B$ containing the point u° . In particular, U is equipped with a semi-simple Frobenius structure. Using the Kodaira–Spencer isomorphism (see Section 3.1) we identify the tangent space $T_{u^{\circ}}U$ with the algebra of functions on the critical scheme of f

$$H := \Gamma(X_{u^{\circ}}, \mathcal{O}_{X_{u^{\circ}}}/\mathcal{O}_{X_{u^{\circ}}}(-p_1^{\circ} - \dots - p_N^{\circ})),$$

where p_i° are the zeros of df in $X_{u^{\circ}} = \Sigma \setminus f^{-1}(\infty)$. Let us trivialize the tangent and the co-tangent bundles

$$T^*U \cong TU \cong U \times T_{u^{\circ}}U \cong U \times H$$
,

where the first isomorphism uses the Frobenius pairing, the second one uses the Levi–Civita connection of the Frobenius pairing, and the last one is the Kodaira–Spencer isomorphism.

7.1. The periods of the Frobenius structure. Given $(u, \lambda) \in B \times \mathbb{C}$, we denote

$$X_{u,\lambda} = \{ p \in X_u \mid \varphi(p) = (u,\lambda) \}.$$

The set of (u, λ) such that the number of points in $X_{u,\lambda}$ is not $m_1 + \cdots + m_d$ (the degree of the covering $X_u \to \mathbb{C}$) form an analytic hypersurface in $B \times \mathbb{C}$,

called the *discriminant*. For every open subset $U \subseteq B$, we put $(U \times \mathbb{C})'$ for the complement to the discriminant in $U \times \mathbb{C}$. The relative homology groups

$$H_1(X_u, X_{u,\lambda}; \mathbb{C}), \quad (u, \lambda) \in (B \times \mathbb{C})'$$

form a rank N vector bundle on $(B \times \mathbb{C})'$ equipped with a flat Gauss–Manin connection.

Let us fix a reference point $(u^{\circ}, \lambda^{\circ}) \in (U \times \mathbb{C})'$. For every relative cycle

$$\alpha \in \mathfrak{h} := H_1(X_{u^{\circ}}, X_{u^{\circ}, \lambda^{\circ}}; \mathbb{C})$$

and every integer n we define the multi-valued analytic function $I_{\alpha}^{(n)}:(U\times\mathbb{C})'\to H$ as follows. First, for $\ell\geq 0$ we define

$$I_{\alpha}^{(-\ell)}(u,\lambda) := -d_u \sum_{n > 0} \int_{\alpha_{u,\lambda}} \frac{(\lambda - F(p))^{n+\ell}}{(n+\ell)!} \omega^{(n)} \in T_u^* U \cong H,$$

where the value of the RHS depends on the choice of a reference path in $(U \times \mathbb{C})'$ and $\alpha_{u,\lambda} \in H_1(X_u, X_{u,\lambda}; \mathbb{C})$ is the relative cycle obtained from α via a parallel transport along the reference path. Note that

$$\partial_{\lambda} I_{\alpha}^{(-\ell)}(u,\lambda) = I_{\alpha}^{(-\ell+1)}(u,\lambda), \quad \forall \ell > 0.$$

For $n \geq 0$, we define

$$I_{\alpha}^{(n)}(u,\lambda) = \partial_{\lambda}^{n} I_{\alpha}^{(0)}(u,\lambda).$$

We will refer to $I_{\alpha}^{(n)}$ as *periods*. Their relation to the oscillatory integrals (see Section 2.2) is the following. Put

$$J_{\Gamma}(u,z) := (-2\pi z)^{-1/2} (zd_u) \int_{\Gamma} e^{F(p)/z} \omega \in T_u^* U \cong H.$$

Let us choose the cycle Γ to be the Lefschetz thimble Γ_i consisting of points $p \in X_u$, such that the gradient trajectory through p of the Morse function $-\operatorname{Re}(F(p)/z)$ flows out of the critical point p_i . The image of Γ_i via the map $F: X_u \to \mathbb{C}$ is a smooth path starting at the critical value u_i and approaching ∞ in such a way that $\operatorname{Re}(\lambda/z) \to -\infty$ as λ approaches ∞ along $F(\Gamma_i)$. If $\lambda \in F(\Gamma_i)$ then let us denote by $\gamma_{\lambda}^{(i)} \in H_1(X_u, X_{u,\lambda}; \mathbb{Z})$ the cycle obtained from Γ_i by truncating all points $p \in \Gamma_i$, such that,

$$\operatorname{Re}(F(p)/z) < \operatorname{Re}(\lambda/z).$$

We have

$$J_{\Gamma_i}(u,z) = (-2\pi z)^{-1/2} \int_{u_i}^{\infty} e^{\lambda/z} I_{\gamma^{(i)}}^{(0)}(u,\lambda) d\lambda.$$

Using integration by parts, we also get that

$$J_{\Gamma_i}(u,z) = \frac{1}{\sqrt{2\pi}} (-z)^{\ell - \frac{1}{2}} \int_{u_i}^{\infty} e^{\lambda/z} I_{\gamma(i)}^{(-\ell)}(u,\lambda) d\lambda$$

for all $\ell \geq 0$.

Let us fix a basis $\{\phi_i\}_{i=1}^N \subset H$ and denote by $t = (t_1, \dots, t_N)$ the flat coordinate system on U, such that $\partial/\partial t_i = \phi_i$ via the identification $T_uU \cong H$ and $t_i(u^\circ) = t_i^\circ$, $1 \leq i \leq N$, where the choice of t_i° will be specified later on. The Frobenius multiplication \bullet in T_uU gives rise (via $T_uU \cong H$) to a Frobenius multiplication \bullet_u in H for every $u \in U$, while the operator θ (see (3)) gives rise to a linear operator in H independent of u. The oscillatory integrals satisfy the following system of differential equations

$$(39) z\partial_{t_i}J(u,z) = \phi_i \bullet_u J(u,z), \quad 1 \le i \le N,$$

$$(40) (z\partial_z + E)J(u,z) = \theta J(u,z).$$

The periods $I_{\alpha}^{(n)}(u,\lambda)$ satisfy the system of differential equations obtained from the above one via the Laplace transform

$$\partial_{t_{i}} I_{\alpha}^{(n)}(u,\lambda) = -\phi_{i} \bullet_{u} I_{\alpha}^{(n+1)}(u,\lambda), \quad 1 \leq i \leq N,$$

$$(\lambda - E \bullet_{u}) \partial_{\lambda} I_{\alpha}^{(n)}(u,\lambda) = \left(\theta - \frac{1}{2} - n\right) I_{\alpha}^{(n)}(u,\lambda).$$

The above formulas define a flat connection known as the $second\ structure\ connection.$

7.2. Stationary phase asymptotic. The primitive form can be written as

$$\omega = \sum_{i=1}^{N} c_i(u, z)[\omega_i],$$

where $c_i(u, z)$ depend polynomially on z (see Proposition 6.5). Let c(u, z) be the vector column with entries $c_i(u, z)$ and $R_{\omega}(u, z)$ be the matrix uniquely determined from $c_i(u, 0)$ according to Lemma 5.4. In particular, $c(u, z) = R_{\omega}(u, z)c(u, 0)$. Recall the stationary phase asymptotic

$$J_{\Gamma_i}(u,z) \sim \left(J_{i,0}(u) + J_{i,1}(u)z + \cdots\right)e^{u_i/z} \quad z \to 0,$$

where $J_{i,k}(u) = \sum_{a=1}^{N} J_{i,k}^{a}(u)\phi_{a}$. Let $J_{k}(u)$ be the matrix whose (a,i)-entry is $J_{i,k}^{a}(u)$. The matrix $\Psi(u) := J_{0}(u)$ is essentially the Jacobian of the change of flat to canonical coordinates, i.e.,

$$\Psi_{ai}(u) = \sum_{b=1}^{N} -c_i(u,0)\eta^{ab} \frac{\partial u_i}{\partial t_b}, \quad 1 \le a, i \le N,$$

where $\eta^{ab} := (\phi^a, \phi^b)$ and $\{\phi^a\}_{a=1}^N \subset H$ is the basis dual to $\{\phi_a\}_{a=1}^N \subset H$ with respect to the Frobenius pairing. The matrices $J_k(u)$ with k>0 can be recovered recursively from Ψ by using the differential equations for the oscillatory integrals. Put

$$R(u,z) = 1 + R_1(u)z + R_2(u)z^2 + \cdots, \quad R_k(u) := \Psi(u)^{-1}J_k(u).$$

This matrix satisfies the symplectic condition $R(u, z)R(u, -z)^T = 1$ and it is the matrix that defines Givental's total ancestor potential of the semi-simple Frobenius manifold (see [16, 17]).

Proposition 7.1. Let $R_{\Sigma}(u,z)$ be the matrix whose (a,i) entry is

$$-(-2\pi z)^{-1/2} \int_{p \in \Gamma_i} e^{(F(p)-u_i)/z} \omega_a(p).$$

Then $R(u, z) = R_{\omega}(u, z)^T R_{\Sigma}(u, z)$.

Proof. We follow the notation in Section 5.2 and 5.3. By definition

$$(J_i(u,z),\partial_{u_j}) = -z\partial_{u_j} \Big(c(u,0)^T R_{\omega}(u,z)^T R_{\Sigma}(u,z) e^{u_i/z} \Big)_i,$$

where we used Lemma 5.5. Let us recall the differential equations for c(u, 0), $R_{\omega}(u, z)$ (see Proposition 5.3), and $R_{\Sigma}(u, z)$ (see Section 5.2)

$$\partial_{u_{j}} c(u,0) = -\Gamma_{j}(u)c(u,0)
\partial_{u_{j}} R_{\omega}(u,z) = z^{-1}[R_{\omega}(u,z), E_{jj}] + (R_{\omega}(u,z)\Gamma_{j}(u) - B_{j}(u)R_{\omega}(u,z)),
\partial_{u_{j}} R_{\Sigma}(u,z) = z^{-1}[E_{jj}, R_{\Sigma}(u,z)] - B_{j}(u)R_{\Sigma}(u,z).$$

Using these differential equations we get

$$(J_i(u,z), \partial_{u_i}) = -c_j(u,0)(R_\omega^T R_\Sigma)_{ji} e^{u_i/z}.$$

On the other hand

$$J_i(u,z) = \sum_{a=1}^{N} (\Psi(u)R(u,z))_{ai} \phi_a e^{u_i/z} = \sum_{a,k=1}^{N} \phi_a \Psi_{ak}(u) R_{ki}(u,z) e^{u_i/z}.$$

It remains only to observe that $\sum_a \phi_a \Psi_{ak}(u) = -c_k(u,0) du_k$ under the identification $T^*U \cong U \times H$.

Following [23], we define the total ancestor potential using the local Eynard–Orantin recursion (see also [8]). The total ancestor potential is a formal series of the type

$$\mathcal{A}_{u}(\hbar, \mathbf{t}) = \exp\Big(\sum_{g, n=0}^{\infty} \frac{\hbar^{g-1}}{n!} \langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}) \rangle_{g, n}\Big),\,$$

where $\mathbf{t}(\overline{\psi}) = \sum_{k=0}^{\infty} \sum_{a=1}^{N} t_{k,a} \phi_a \overline{\psi}^k$ with $t_{k,a}$ formal variables and the correlators

$$\langle \phi_{a_1} \overline{\psi}^{k_1}, \dots, \phi_{a_n} \overline{\psi}^{k_n} \rangle_{g,n}$$

are non-zero only if they are stable (i.e. 2g - 2 + n > 0) and are defined by the following recursion

$$\langle \phi_{a}\overline{\psi}^{k}, \mathbf{t}, \dots, \mathbf{t} \rangle_{g,n+1} = \frac{1}{4} \sum_{i=1}^{N} \operatorname{res}_{\lambda=u_{i}} \frac{(I_{\gamma^{(i)}}^{(-k-1)}(u, \lambda), \phi_{a})}{(I_{\gamma^{(i)}}^{(-1)}(u, \lambda), 1)} \times \left(\langle \phi_{\gamma^{(i)}}^{+}(u, \lambda; \overline{\psi}), \phi_{\gamma^{(i)}}^{+}(u, \lambda; \overline{\psi}), \mathbf{t}, \dots, \mathbf{t} \rangle_{g-1, n+2} + \sum_{\substack{g'+g''=g\\n'+n''=n}} \binom{n}{n'} \langle \phi_{\gamma^{(i)}}^{+}(u, \lambda; \overline{\psi}), \mathbf{t}, \dots, \mathbf{t} \rangle_{g', n'+1} \langle \phi_{\gamma^{(i)}}^{+}(u, \lambda; \overline{\psi}), \mathbf{t}, \dots, \mathbf{t} \rangle_{g'', n''+1} \right)$$

where the insertion **t** should be understood as $\mathbf{t}(\overline{\psi})$,

(41)
$$\phi_{\alpha}(u,\lambda;z) := \sum_{n \in \mathbb{Z}} I_{\alpha}^{(n+1)}(u,\lambda)(-z)^{n},$$

 $\phi_{\alpha}^{+}(u,\lambda;z)$ is obtained from $\phi_{\alpha}(u,\lambda;z)$ by truncating all terms containing negative powers of z, and all unstable correlators are by definition 0, except for

$$\langle \phi_{\alpha}^{+}(u,\lambda;\overline{\psi}), \mathbf{t} \rangle_{0,2} := \sum_{m=0}^{\infty} \sum_{a=1}^{N} (I_{\alpha}^{(-m)}(u,\lambda), \phi_{a}) t_{m,a}.$$

and

$$\langle \phi_{\alpha}^{+}(u,\lambda;\overline{\psi}), \phi_{\alpha}^{+}(u,\lambda;\overline{\psi}) \rangle_{0,2} = \frac{1}{2}((\lambda - E \bullet_{u})I_{\gamma^{(i)}}^{(1)}(u,\lambda), I_{\gamma^{(i)}}^{(1)}(u,\lambda)).$$

We have the following formula for the Laurent series expansion of the periods

(42)
$$I_{\gamma^{(i)}}^{(-n)}(u,\lambda) = \sqrt{2\pi} \sum_{k=0}^{\infty} (-1)^k \Psi R_k(u) e_i \frac{(\lambda - u_i)^{k+n-1/2}}{\Gamma(k+n+1/2)},$$

where e_i is the column vector with 1 on the i-th position and 0s elsewhere and the period on the LHS is identified with a column vector whose entries are the coordinates of the period with respect to the basis $\{\phi_i\}_{i=1}^N \subset H$. Note that the recursion determines the correlators in terms of the matrix R(u, z).

7.3. **The local EO recursion.** Let us fix a point $u^{\circ} \in B$ and a small neighborhood $U \subset B$ of u° . Let us denote by $X_U := \pi^{-1}(U)$ the restriction to U of the family $\pi: X \to B$. Let $X_U^{\text{loc}} \subset X_U$ be a tubular neighborhood of the relative critical set $C \cap X_U$. Using the local Morse coordinates t_i $(1 \le i \le N)$ of $F: X \to \mathbb{C}$ we identify $X_U^{\text{loc}} \cong U \times \left(\widetilde{\Delta}_1 \sqcup \cdots \sqcup \widetilde{\Delta}_N\right)$ where each $\widetilde{\Delta}_i = \{|t_i| < \epsilon_i\} \subset \mathbb{C}$ is a sufficiently small disk. The restriction of F to $U \times \widetilde{\Delta}_i$ takes the form

$$F(u,t_i) = u_i + \frac{1}{2}t_i^2, \quad (u,t_i) \in U \times \widetilde{\Delta}_i.$$

Following [23] we introduce the local EO recursions. It is defined in terms of a set of symmetric holomorphic forms

$$\omega^i \in \Omega^1_{U \times \widetilde{\Delta}_i/U}(U \times \widetilde{\Delta}_i), \quad \omega^{ij} \in \Omega^1_{U \times \widetilde{\Delta}_i/U} \boxtimes \Omega^1_{U \times \widetilde{\Delta}_j/U}(U \times (\widetilde{\Delta}_i \times \widetilde{\Delta}_j - \widetilde{\Delta}_{ij})),$$

where $\widetilde{\Delta}_{ij} = \emptyset$ if $i \neq j$ and $\widetilde{\Delta}_{ii} = \{(s,t) \in \widetilde{\Delta}_i \times \widetilde{\Delta}_i \mid s^2 = t^2\}$ such that

$$P^{i}(u,\lambda) = \sum_{k=0}^{\infty} P_{k}^{i}(u)(\lambda - u_{i})^{k+1/2}$$

and

$$P^{ij}(u,\lambda_1,\lambda_2) = \frac{\delta_{ij}}{(\lambda_1 - \lambda_2)^2} \left(\frac{(\lambda_1 - u_i)^{1/2}}{(\lambda_2 - u_j)^{1/2}} + \frac{(\lambda_2 - u_j)^{1/2}}{(\lambda_1 - u_i)^{1/2}} \right) + \sum_{k,\ell=0}^{\infty} P_{k,\ell}^{ij}(u)(\lambda_1 - u_i)^{k-1/2}(\lambda_2 - u_j)^{\ell-1/2},$$

where the above series are defined by

$$\omega^{i}(u, t_{i}) = P^{i}(u, \lambda)d\lambda, \quad \lambda = u_{i} + t_{i}^{2}/2$$

and

$$\omega^{ij}(u,t_i,t_j) = P^{ij}(u,\lambda_1,\lambda_2)d\lambda_1 \cdot d\lambda_2, \quad \lambda_1 = u_i + t_i^2/2, \quad \lambda_2 = u_j + t_j^2/2$$

where if i = j, then t_j should be interpreted as a second copy of t_i . Note that the requirement that ω^{ij} is symmetric is equivalent to $P^{ij}(u, \lambda_1, \lambda_2) = P^{ji}(u, \lambda_2, \lambda_1)$. We will need also the expansion

$$P^{ij}(u,\lambda_1,\lambda_2) = \frac{2\delta_{ij}}{(\lambda_1 - \lambda_2)^2} + \sum_{k=0}^{\infty} P_k^{ij}(u,\lambda_1)(\lambda_2 - \lambda_1)^k.$$

Put $\Delta_i := \{(u, \lambda) \in U \times \mathbb{C} \mid |\lambda - u_i| < \epsilon_i^2/2\}$ and let $\Delta_i(u) \subset \mathbb{C}$ be the fiber over $u \in U$ of the projection map $\Delta_i \to U$. Both $P^i(u, \lambda)$ and $P^{ij}(u, \lambda_1, \lambda_2)$ are multi-valued holomorphic functions respectively on Δ_i and $\Delta_i \times_U \Delta_j$. In order to keep track of their values we introduce local systems \mathcal{L}_i on

$$\Delta_i^* := \{(u, \lambda) \in U \times \mathbb{C} \mid 0 < |\lambda - u_i| < \epsilon_i^2 / 2\}, \quad 1 \le i \le N.$$

The sections of \mathcal{L}_i over some open neighborhood $V \subset \Delta_i^*$ are given by the holomorphic branches of $(\lambda - u_i)^{1/2}$ on V. Alternatively, we introduce a line bundle $L_i \to \Delta_i^*$ whose fiber over a point $(u, \lambda) \in \Delta_i^*$ is the relative homology group

$$H_1(\widetilde{\Delta}_i, F(u, \cdot)^{-1}(\lambda) \cap \widetilde{\Delta}_i; \mathbb{C}) \cong \mathbb{C}.$$

The line bundle L_i is equipped with a flat Gauss-Manin connection and intersection pairing $(\alpha|\beta) := \partial \alpha \circ \partial \beta$. The local system \mathcal{L}_i is isomorphic to the sheaf of flat sections β_i such that $(\beta_i|\beta_i) = 2$ via

(43)
$$\beta_i \mapsto \frac{1}{2\sqrt{2}} \int_{\beta_i} dt_i = \pm (\lambda - u_i)^{1/2}.$$

The local EO recursion produces a set of multivalued *correlator forms*

$$\omega_{g,n}^{\alpha_1,\dots,\alpha_n}(u;\lambda_1,\dots,\lambda_n), \quad \alpha_i \in \mathcal{L}_{m_i}, \quad (u,\lambda_i) \in \Delta_{m_i}^*, \quad 1 \le m_i \le N.$$

Let us fix a base point in each Δ_i^* $(1 \leq i \leq N)$. Then the values of the correlator forms depend on the choice of reference paths – one for each point $(u, \lambda_i) \in \Delta_{m_i}^*$. The recursion kernel is defined by

$$K^{\beta_i,\beta_j}(u,\lambda_1,\lambda_2) = \frac{\frac{1}{2} \oint_{\lambda \in C_{\lambda_2}} P^{ij}(u,\lambda_1,\lambda) d\lambda}{P^j(u,\lambda_2)} \frac{d\lambda_1}{d\lambda_2},$$

where C_{λ_2} is a small loop around u_i based at λ_2 .

Remark 7.2. The branch of the integrand in the contour integral $\oint_{C_{\lambda_2}}$ is fixed in such a way that the operation

$$(\lambda_2 - u_i)^a \mapsto \frac{1}{2} \oint_{\lambda \in C_{\lambda_2}} (\lambda - u_i)^a = (\lambda_2 - u_i)^{a+1} / (a+1)$$

computes the anti-derivative (or primitive). In other words, using the reference path to (u, λ_2) and the contour C_{λ_2} we first specify the branch of $P^{ij}(u, \lambda_1, \lambda)$ by moving continuously λ along C_{λ_2} and then we integrate the resulting function backwards, i.e. the orientation of the cycle C_{λ_2} used in the contour integral is the opposite to the orientation used to specify the branch of the integrand.

The local EO recursion takes the form

$$\omega_{0,2}^{\beta_i,\beta_j}(u,\lambda_1,\lambda_2) := \begin{cases} P^{ij}(u,\lambda_1,\lambda_2)d\lambda_1d\lambda_2 & \text{if } (\lambda_1,i) \neq (\lambda_2,j), \\ P_0^{ii}(u,\lambda_1)d\lambda_1 d\lambda_1 & \text{otherwise }, \end{cases}$$

and

$$\omega_{g,n+1}^{\alpha_0,\dots,\alpha_n}(\lambda_0,\dots,\lambda_n) = \sum_{i=1}^N \operatorname{res}_{\lambda=u_i} K^{\alpha_0,\beta_i}(\lambda_0,\lambda) \Big(\omega_{g-1,n+2}^{\beta_i,-\beta_i,\alpha_1,\dots,\alpha_n}(\lambda,\lambda,\lambda_1,\dots,\lambda_n) + \sum_{g'+g''=g} \sum_{i'_1,\dots,i'_{s'}} \omega_{g',n'+1}^{\beta_i,\alpha_{i'_1},\dots,\alpha_{i'_{n'}}}(\lambda,\lambda_{i'_1},\dots,\lambda_{i'_{n'}}) \omega_{g'',n''+1}^{-\beta_i,\alpha_{i''_1},\dots,\alpha_{i''_{n''}}}(\lambda,\lambda_{i''_1},\dots,\lambda_{i''_{n''}}) \Big),$$

where $\beta_i \in \mathcal{L}_i$, the second and the third sums are over all splitting g' + g'' = g and all subsets $\{i'_1, \ldots, i'_{n'}\} \subset \{1, 2, \ldots, n\}$, and

$$\{i_1'',\ldots,i_{n''}''\}:=\{1,2,\ldots,n\}-\{i_1',\ldots,i_{n'}'\}.$$

Example 7.3. Let us assume that U is equipped with a semi-simple Frobenius structure such that u_1, \ldots, u_N are canonical coordinates. Although, our definition of the periods $I_{\gamma^{(i)}}^{(-n)}(u,\lambda)$ was given only for Frobenius structures corresponding to polynomial primitive forms, we define the period $I_{\gamma^{(i)}}^{(-n)}(u,\lambda)$ $(1 \leq i \leq N, n \in \mathbb{Z})$ in general as the unique solution to the second structure connection that

has an expansion of the type (42) for all $(u, \lambda) \in \Delta_i$. Note that in general we think of $\gamma^{(i)}$ us a section of the local system \mathcal{L}_i , i.e., a choice of a holomorphic branch of $(\lambda - u_i)^{1/2}$. Moreover, the recursion that we used to define the total ancestor potential still makes sense. Finally, substituting the expansion (42) in the differential equations for the second structure connection yields a recursion that uniquely determines the matrices $R_k(u)$, $k \geq 0$ starting with $R_0(u) = 1$. The main result of [23] is that the correlator forms

$$^{\operatorname{Frob}}\omega_{g,n}^{\alpha_1,\ldots,\alpha_n}(u;\lambda_1,\ldots,\lambda_n) := \langle \phi_{\alpha_1}^+(u,\lambda_1;\overline{\psi}),\ldots,\phi_{\alpha_n}^+(u,\lambda_n;\overline{\psi}) \rangle_{g,n} d\lambda_1 \cdots d\lambda_n$$

where $(u, \lambda_i) \in \Delta_{m_i}$, $\alpha_i \in \mathcal{L}_{m_i}$, and the insertions ϕ_{α_i} are defined by (41), satisfy the local EO recursion with

$$P^{j}(u,\lambda) = 4(I_{\gamma^{(j)}}^{(-1)}(u,\lambda),1)$$

and

$$P_{k,\ell}^{ij}(u) = \frac{2^{k+\ell+1} V_{k,\ell}^{ij}(u)}{(2k-1)!!(2\ell-1)!!},$$

where $V_{k,\ell}^{ij}(u)$ is the (i,j)-entry of the matrix $V_{k,\ell}$ defined by

$$\frac{R(u,z_1)^T R(u,-z_2) - 1}{z_1 + z_2} = \sum_{k,\ell=0}^{\infty} V_{k,\ell}(u) (-z_1)^k (-z_2)^{\ell}.$$

Example 7.4. Let ϕ be a relative meromorphic differential on $\overline{X}_U := \overline{\pi}^{-1}(U)$ such that $\phi|_{X_U^{\text{loc}}}$ is holomorphic and $\phi(p) \neq 0$ for all $p \in C \cap X_U$. Let $\omega_{g,n}(u; q_1, \ldots, q_n)$ be the correlator forms defined by the following EO recursion (see [11]): all unstable correlators (i.e. $2g - 2 + n \leq 0$) are 0 except for

$$\omega_{0,2}(u;p,q) := B_u(p,q), \quad p \neq q,$$

where B_u is the fundamental bi-differential on \overline{X}_u and

$$\omega_{g,n+1}(u; q_0, q_1, \dots, q_n) =
\sum_{i=1}^{N} \frac{1}{2} \operatorname{res}_{p=p_i} \frac{\int_{p}^{\tau_i(p)} B_u(q_0, p')}{dF(p) \int_{p}^{\tau_i(p)} \phi(p')} \Big(\omega_{g-1,n+1}(u; p, \tau_i(p), q_1, \dots, q_n) +
\sum_{g'+g''=g} \sum_{i'_1, \dots, i'_{n'}} \omega_{g',n'+1}(u; p, q_{i'_1}, \dots, q_{i'_{n'}}) \omega_{g'',n''+1}(u; \tau_i(p), q_{i''_1}, \dots, q_{i''_{n''}}) \Big),$$

where τ_i is the local involution defined via the local coordinate $t_i(p)$ in a neighborhood of the ramification point p_i as $t_i(\tau_i(p)) := -t_i(p)$, the last sum is over all subsets $\{i'_1, \ldots, i'_{n'}\} \subset \{1, 2, \ldots, n\}$ and

$$\{i_1'',\ldots,i_{n''}''\} := \{1,2,\ldots,n\} \setminus \{i_1',\ldots,i_{n'}'\}.$$

Note that the above recursion is more general then the recursion proposed originally in [13]. Namely, if $\phi(p) = -dy(p)$ for some meromorphic function y on \overline{X}_u ,

then the above recursion coincides with the EO recursion for the spectral data $(\overline{X}_u, x_u, y_u)$, where $x_u = F|_{X_u}$ and $y_u = y|_{X_u}$.

Let us define the following set of multi-valued symmetric differentials

$${}^{\mathrm{EO}}\omega_{g,n}^{\alpha^1,\dots,\alpha^n}(u;\lambda_1,\dots,\lambda_n) := d_{\lambda_1}\cdots d_{\lambda_n} \int_{q_1\in\alpha_{u,\lambda_1}^1}\cdots \int_{q_n\in\alpha_{u,\lambda_n}^n} \omega_{g,n}(u;q_1,\dots,q_n),$$

where $\alpha^i \in \mathfrak{h}$, $1 \leq i \leq n$ are given relative cycles and the pair (g, n) is stable, i.e., 2g - 2 + n > 0. Here $\alpha^i_{u,\lambda} \in H_1(X_u, X_{u,\lambda}; \mathbb{C})$ is the relative cycle obtained from α^i via a parallel transport along some reference path in $(U \times \mathbb{C})'$ from $(u^{\circ}, \lambda^{\circ})$ to (u, λ) . Suppose that $(u, \lambda_i) \in \Delta_{m_i}$ and that the reference path from $(u^{\circ}, \lambda^{\circ})$ to (u, λ_i) and the cycle α^i are such that the relative cycle $\alpha^i_{u,\lambda}$ is the vanishing cycle, i.e., it can be represented by an arc $\beta^{m_i}_{u,\lambda} \subset \Delta^*_{m_i}$. In particular, $\alpha^i_{u,\lambda} \in \mathcal{L}_{m_i}$ via the isomorphism (43).

Proposition 7.5. The correlator forms ${}^{EO}\omega_{g,n}^{\alpha^1,\dots,\alpha^n}(u;\lambda_1,\dots,\lambda_n)$ satisfy local EO recursion with

$$P^{ij}(u,\lambda_1,\lambda_2) := d_{\lambda_1} \int_{p \in \beta_{u,\lambda_1}^i} d_{\lambda_2} \int_{q \in \beta_{u,\lambda_2}^j} B_u(p,q)$$

and

$$P^{j}(u,\lambda) = 4 \int_{q \in \beta_{u,\lambda}^{j}} \phi(q).$$

Proof. In order to avoid cumbersome notation we drop the superscript "EO" and we suppress the dependence on u in the correlators, i.e., $u \in U$ will be fixed throughout the proof and we put

$$\omega_{g,n}^{\alpha^1,\dots,\alpha^n}(\lambda_1,\dots,\lambda_n) := {}^{EO}\omega_{g,n}^{\alpha^1,\dots,\alpha^n}(u;\lambda_1,\dots,\lambda_n).$$

Let us apply to the EO recursion defining the forms $\omega_{g,n}$ the operations

(44)
$$d_{\lambda_i} \int_{a_i \in \alpha_i^{u, \lambda_i}}, \quad 0 \le i \le n.$$

By definition the LHS becomes ${}^{\mathrm{EO}}\omega_{g,n}^{\alpha^0,\ldots,\alpha^n}(\lambda_0,\ldots,\lambda_n)$. On the other hand on the RHS we get some hybrid correlators that have insertions both from \overline{X}_u and \mathbb{P}^1 , i.e., we have expressions of the form

$$\omega_{g-1,n+2}^{\alpha^1,\dots,\alpha^n}(p,\tau_i(p),\lambda_1,\dots,\lambda_n)$$

and

$$\omega_{g',n'+1}^{\alpha^{i'_1},\dots,\alpha^{i'_{n'}}}(p,\lambda_{i'_1},\dots,\lambda_{i'_{n'}})\omega_{g'',n''+1}^{\alpha^{i''_1},\dots,\alpha^{i''_{n''}}}(\tau_i(p),\lambda_{i''_1},\dots,\lambda_{i''_{n''}}),$$

where each insertion λ_i is paired with a corresponding cycle α^i (appearing in the superscript) and the pair (α^i, λ_i) means that we have applied to the corresponding

EO-correlator form the operation (44). However, note that

$$\omega_{g-1,n+2}^{\alpha}(p,\tau_i(p),\dots) = \frac{1}{4}\omega_{g-1,n+2}^{\beta_i,-\beta_i,\alpha}(\lambda,\lambda,\dots) + \dots,$$

where $\alpha = (\alpha^1, \dots, \alpha^n)$ and

$$\omega_{g',n'+1}^{\alpha'}(p,\ldots)\omega_{g'',n''+1}^{\alpha''}(\tau_i(p),\ldots) = \frac{1}{4}\omega_{g',n'+1}^{\beta_i,\alpha'}(\lambda,\ldots)\omega_{g'',n''+1}^{-\beta_i,\alpha''}(\lambda,\ldots) + \ldots$$

where $\lambda = F(p)$, $\alpha' = (\alpha_{i'_1}, \dots, \alpha_{i'_{n'}})$, $\alpha'' = (\alpha_{i''_1}, \dots, \alpha_{i''_{n''}})$ and the dots that follow the plus sign on the RHS stand for terms holomorphic at $p = p_i$ (here one has to prove by induction on (g, n) that $\omega_{g,n}(p, \dots) + \omega_{g,n}(\tau_i(p), \dots)$ is analytic at $p = p_i$). Using also that

$$\frac{1}{2}\operatorname{res}_{p=p_i} = \operatorname{res}_{\lambda=u_i}$$

we get that the correlator forms $\omega_{g,n}^{\alpha^1,\dots,\alpha^n}$ satisfy a recursion that has the same form as local EO recursion with recursion kernel

$$K^{\beta^{i},\beta^{j}}(u,\lambda_{1},\lambda_{2}) = \frac{d_{\lambda_{1}} \int_{p \in \beta^{i}_{u,\lambda_{1}}} \int_{q \in \beta^{j}_{u,\lambda_{2}}} B_{u}(p,q)}{4 \int_{q \in \beta^{j}_{u,\lambda_{2}}} \phi(q) d\lambda_{2}},$$

except that the initial conditions are given by

$$\omega_{0,2}^{\beta^i,\beta^j}(\lambda_1,\lambda_2) = d_{\lambda_1} d_{\lambda_2} \int_{p \in \beta^i_{u,\lambda_1}} \int_{q \in \beta^j_{u,\lambda_2}} B(p,q)$$

for $(i, \lambda_1) \neq (j, \lambda_2)$ and

$$\widetilde{\omega}_{0,2}^{\beta^i,-\beta^i}(\lambda,\lambda) := -\widetilde{\omega}_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda) = 4B_u(q,\tau_i(q)),$$

where q is sufficiently close to p_i and $\lambda = F(q)$.

By definition (see Remark 7.2)

$$\frac{1}{2} \oint_{\lambda \in C_{\lambda_2}} d_{\lambda} \int_{q \in \beta_{u,\lambda}^j} B_u(p,q) = \int_{q \in \beta_{u,\lambda}^j} B_u(p,q),$$

Therefore, $K^{\beta^i,\beta^j}(u,\lambda_1,\lambda_2)$ coincides with the recursion kernel of the local recursion introduced in the statement of the proposition. The only issue that we have to resolve is that the initial condition $\omega_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda)$ of the local EO recursion is supposed to be $P_0^{ii}(u,\lambda)$. On the other hand the correlator $\omega_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda)$ contributes to the recursion only when we evaluate

$$\omega_{1,1}^{\alpha}(u;\lambda) = -\sum_{i=1}^{N} \operatorname{res}_{\mu=u_i} K^{\alpha,\beta^i}(\lambda,\mu) \omega_{0,2}^{\beta^i,\beta^i}(\mu,\mu).$$

Since $K^{\alpha,\beta_i}(\lambda,\mu)d\mu$ is analytic at $\mu=u_i$ it is sufficient to prove that the difference

$$\omega_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda) - \widetilde{\omega}_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda)$$

is analytic at $\lambda = u_i$. This is a straightforward local computation. Indeed, using that for $p, q \in U \times \widetilde{\Delta}_i$ we have

$$B(p,q) = dt_i(p)dt_i(q) \left(\frac{1}{(t_i(p) - t_i(q))^2} + \sum_{m,n=0}^{\infty} B_{m,n}^{ii}(u)t_i(p)^m t_i(q)^n \right)$$

we get that

$$P_0^{ii}(u,\lambda) = \frac{1}{4}(\lambda - u_i)^{-2} + \sum_{m,n=0}^{\infty} 2^{m+n+1} B_{2m,2n}^{ii}(\lambda - u_i)^{m+n-1}$$

and

$$\widetilde{\omega}_{0,2}^{\beta^i,\beta^i}(\lambda,\lambda) = d\lambda \cdot d\lambda \Big(\frac{1}{4}(\lambda - u_i)^{-2} + \frac{2B_{0,0}^{ii}}{\lambda - u_i} + \cdots\Big),$$

where the dots stand for terms analytic at $\lambda = u_i$. The analyticity that we wanted to prove follows.

Definition 7.6. We say that a semi-simple Frobenius structure on U is a solution to an EO recursion (defined by a relative meromorphic differential on \overline{X}_U) if the correlator forms defined by the corresponding local EO recursions coincide, i.e.,

$$^{\operatorname{Frob}}\omega_{g,n}^{\alpha_1,\ldots,\alpha_n}(u;\lambda_1,\ldots,\lambda_n) = {^{\operatorname{EO}}}\omega_{g,n}^{\alpha_1,\ldots,\alpha_n}(u;\lambda_1,\ldots,\lambda_n), \quad \forall u \in U.$$

7.4. Frobenius manifolds and EO recursion. Let $B_{m,n}^{i,j}$, $1 \le i, j \le N$, $m, n \ge 0$ be the coefficients defined via the expansion of the fundamental bi-differential

(45)
$$B(p,q) = dt_i(p)dt_j(q) \left(\frac{\delta_{ij}}{(t_i(p) - t_j(q))^2} + \sum_{m,n=0}^{\infty} B_{m,n}^{i,j} t_i(p)^m t_j(q)^n \right),$$

where the point (p,q) is sufficiently close to $(p_i,p_j) \in X_u \times X_u$. Let us denote by $B_{m,n}$ the $N \times N$ matrix whose (i,j)-entry is $B_{m,n}^{i,j}$. The key result in comparing the topological recursion with the Dubrovin's Frobenius structure is the following identity.

Lemma 7.7. The following identity holds

$$\frac{R_{\Sigma}(u,z_1)^T R_{\Sigma}(u,z_2) - 1}{z_1 + z_2} = \sum_{m,n=0}^{\infty} B_{2m,2n}(2m-1)!!(2n-1)!!(-z_1)^m (-z_2)^n.$$

This result is due to Eynard [12] (see also Lemma 5.1 in [11]). It can be proved using the same technique that we used in the proof of Proposition 4.4.

Proof of Theorem 2.8. Let us assume first that $\omega = [\phi] \in \mathcal{H}(U)$ where ϕ is a sum of homogeneous primary differentials of the same degree. According to Corollary 6.7 ω is a primitive form and the matrix $R_{\omega}(u,z) = 1$ (see the notation in Section 7.2). Recalling Proposition 7.1, we get that the R-matrix of the Frobenius structure on U corresponding to the primitive form ω is $R(u,z) = R_{\Sigma}(u,z)$. We claim that both sets of correlator forms $^{\text{Frob}}\omega_{g,n}$ and $^{\text{EO}}\omega_{g,n}$ are defined by the

same local EO recursion. Indeed, on the Frobenius side we have (see Example 7.3)

Frob
$$P_{k,\ell}^{ij}(u) = \frac{2^{k+\ell+1} V_{k,\ell}^{ij}(u)}{(2k-1)!!(2\ell-1)!!}$$

and

Frob
$$P^{j}(u,\lambda) = 4(I_{\beta j}^{(-1)}(u,\lambda), 1).$$

While on the EO-recursion side we have (see Example 7.4)

$$^{\mathrm{EO}}P_{k,\ell}^{ij}(u) = 2^{k+\ell+1} B_{2k,2\ell}^{ij}(u)$$

and

$$^{EO}P^{j}(u,\lambda) = 4 \int_{q \in \beta_{u,\lambda}^{j}} \phi(q),$$

where we used that

$$d_{\lambda_{1}}d_{\lambda_{2}} \int_{p \in \beta_{i}^{u,\lambda_{1}}} \int_{q \in \beta_{j}^{u,\lambda_{2}}} B(p,q) = d\lambda_{1} d\lambda_{2} \left(\frac{\delta_{ij}}{(\lambda_{1} - \lambda_{2})^{2}} \left(\frac{\sqrt{\lambda_{2} - u_{i}}}{\sqrt{\lambda_{1} - u_{i}}} + \frac{\sqrt{\lambda_{1} - u_{i}}}{\sqrt{\lambda_{2} - u_{i}}} \right) + \sum_{m,n=0}^{\infty} B_{2m,2n}^{i,j} 2^{m+n+1} (\lambda_{1} - u_{i})^{m-1/2} (\lambda_{2} - u_{i})^{n-1/2} \right),$$

Using Eynard's identity Lemma 7.7 we get $^{\text{Frob}}P_{k,\ell}^{ij}(u) = {}^{\text{EO}}P_{k,\ell}^{ij}(u)$. Recalling the definition of the periods and using that the primitive form is $[\phi]$ we get

^{Frob}
$$P^j(u,\lambda) = 4(I_{\beta_j}^{(-1)}(u,\lambda),1) = 4\int_{q\in\beta_j^{u,\lambda}}\phi(q) = {}^{\mathrm{EO}}P^j(u,\lambda).$$

In the opposite direction. Let us assume that there is a local EO recursion defined by $P^{ij}(u,\lambda)$ $(1 \leq i,j \leq N)$ and $P^j(u,\lambda)$ $(1 \leq j \leq N)$ such that the corresponding correlator forms coincide with $^{\text{Frob}}\omega_{g,n}$ and with $^{\text{EO}}\omega_{g,n}$ for some semi-simple Frobenius structure on U and some EO-recursion defined by a relative meromorphic form ϕ on \overline{X}_U . It is sufficient to prove that $R(u,z) = R_{\Sigma}(u,z)$. Indeed, let ω be the primitive form corresponding to the Frobenius structure on U. According to Proposition 7.1 if $R(u,z) = R_{\Sigma}(u,z)$, then $R_{\omega}(u,z) = 1$. Therefore, according to Corollary 6.7 the primitive form is represented by a linear combination of homogeneous primary differentials.

Both R-matrices can be extracted from the 3-point genus-0 correlators. Let us denote by $P_k^{ij,j}(u,\lambda)$ $(1 \leq i,j \leq N,\ k \geq 0)$ the coefficients that appear in the expansion

$$P^{ij}(u, \lambda_1, \lambda_2) = \sum_{k=0}^{\infty} P_k^{ij,j}(u, \lambda_1)(\lambda_2 - u_j)^{k-1/2}.$$

In particular,

$$P_0^{ij,j}(u,\lambda) = \delta_{i,j}(\lambda - u_i)^{-1-1/2} + \sum_{k=0}^{\infty} P_{k,0}^{ij}(u)(\lambda - u_i)^{k-1/2}.$$

Using the local EO recursion we get

$$\omega_{0,3}^{\alpha_1,\alpha_2,\alpha_3}(u;\lambda_1,\lambda_2,\lambda_3) = -4\sum_{i=1}^N P_0^{i_1j,j}(u,\lambda_1) P_0^{i_2j,j}(u,\lambda_2) P_0^{i_3j,j}(u,\lambda_3) \frac{d\lambda_1\,d\lambda_2\,d\lambda_3}{P_0^j(u)},$$

where $\alpha_a \in \mathcal{L}_{i_a}$ (a = 1, 2, 3). Suppose first that $i_1 = i_2 = i_3 =: i$. Then the coefficient in front of

$$-4(\lambda_1 - u_i)^{-1/2}(\lambda_2 - u_i)^{-1/2}(\lambda_3 - u_i)^{-1/2}d\lambda_1 d\lambda_2 d\lambda_3$$

in the above correlator is $1/P_0^i(u)$. Therefore, $^{\text{Frob}}P_0^i(u)=^{\text{EO}}P_0^i(u)$ for all $i=1,2,\ldots,N$. Suppose now that $i_2=i_3=:i$. Then the coefficient in front of

$$-4(\lambda_2 - u_i)^{-1/2}(\lambda_3 - u_i)^{-1/2}d\lambda_1 d\lambda_2 d\lambda_3$$

is $P_0^{i_1i,i}(u,\lambda_1)/P_0^i(u)$. Therefore

$$^{\text{Frob}}P_0^{i_1i,i}(u,\lambda_1)/^{\text{Frob}}P_0^i(u) = {}^{\text{EO}}P_0^{i_1i,i}(u,\lambda_1)/{}^{\text{EO}}P_0^i(u).$$

Since we already proved that ${}^{\text{Frob}}P^i_0(u)={}^{\text{EO}}P^i_0(u)$ we get ${}^{\text{Frob}}P^{i_1i,i}_0(u,\lambda_1)={}^{\text{EO}}P^{i_1i,i}_0(u,\lambda_1)$. Comparing the coefficients in front of $(\lambda_1-u_i)^{k-1/2}$ we get

$$^{\text{Frob}}P_{k,0}^{i_1i} = {}^{\text{EO}}P_{k,0}^{i_1i},$$

i.e., $V_{k,0}^{ij} = B_{2k,0}^{ij}(2k-1)!!$. Finally we get

$$(R(u,z)^T - 1)/z = \sum_{k=0}^{\infty} V_{k,0}^{ij}(-z)^k = \sum_{k=0}^{\infty} B_{2k,0}^{ij}(2k-1)!!(-z)^k = (R_{\Sigma}(u,z)^T - 1)/z,$$

where the last identity follows from Eynard's identity (Lemma 7.7).

8. Topological recursion for twisted de Rham cohomology

Semi-simple Frobenius manifolds admit also the notion of descendants. Using the forms defined by the topological recursion we will express the descendant correlation functions for Hurwitz Frobenius manifolds as oscillatory integrals. Motivated by this result, we propose to think of the forms defined by the topological recursion as twisted de Rham cohomology classes. This point of view allows us to find a generalization of the EO recursion corresponding to the Frobenius manifolds defined by polynomial primitive forms.

8.1. Calibration of the Frobenius structure. Near $z = \infty$ the system (39)–(40) admits a weak Levelt solution, i.e., a fundamental solution of the form

$$\Phi(t,z) = S(t,z)z^{\delta}z^{\nu},$$

where the matrices S, δ , and ν have the following properties. We have an expansion $S(t,z) = S_0 + S_1(t)z^{-1} + S_2(t)z^{-2} \cdots$ with S_0 constant (independent of t and z) invertible matrix. The matrices $\delta = \text{Diag}(\delta_1, \ldots, \delta_s)$ and $\nu = \text{Diag}(\nu_1, \ldots, \nu_s)$ are block-diagonal with blocks δ_i and ν_i constant matrices of the same size. The block ν_i is an upper-triangular nilpotent matrix and the block δ_i is a diagonal matrix

$$\delta_i = \operatorname{Diag}(m_{i,1} + r_i, \dots, m_{i,p_i} + r_i)$$

such that $-1 < \text{Re } r_i \le 0$ and $m_{i,1} \le m_{i,2} \le \cdots \le m_{i,p_i}$ is an increasing sequence of integers. The matrix representation of the system (39)–(40) depends on the choice of a flat coordinate system. Choosing a different flat coordinate system transforms (39)–(40) via a constant gauge transformation. Therefore without lost of generality we may assume that $S_0 = 1$.

It is convenient to introduce the following notation. Let $\operatorname{spec}(\delta)$ be the set of eigenvalues of the operator

$$ad_{\delta} : \mathfrak{gl}(H) \to \mathfrak{gl}(H), \quad X \mapsto [\delta, X].$$

Let us denote by $\mathfrak{gl}_a(H)$ the eigensubspace of ad_δ with eigenvalue a. Then we have a direct sum decomposition of vector spaces

$$\mathfrak{gl}(H)=\bigoplus_{a\in\operatorname{spec}(\delta)}\mathfrak{gl}_a(H).$$

Let us denote by $X_{[a]}$ the projection of X on $\mathfrak{gl}_a(H)$. The matrices S, δ , and ν are identified with elements of $\mathfrak{gl}(H)$ via the basis $\{\phi_i\}_{i=1}^N \subset H$ that we fixed above.

Substituting the fundamental series $\Phi(t,z)$ in (40) and comparing the coefficients in front of powers of z we get that

$$\theta = \delta + \nu_{[0]}, \quad kS_k + [\theta, S_k] = E \bullet S_{k-1} + \sum_{\ell=1}^k S_{k-\ell} \nu_{[-\ell]}, \quad k > 0.$$

In particular, a weak Levelt solution can be constructed by setting $(S_k)_{[-k]} = 0$ and solving recursively for $\nu_{[-k]}$ and $(S_k)_{[a]}$ for all k > 0 and all $a \neq -k$. This is a standard procedure, so we skip the details.

Proposition 8.1. There exists a weak Levelt solution such that

$$S(t, -z)^T S(t, z) = 1,$$

where T is transposition with respect to the Frobenius pairing on $H = T_{u^{\circ}}U$.

Proposition 8.1 is known if θ is diagonalizable (see [9]). In fact, the polynomiality of the primitive form might be sufficient to prove that θ is daigonalizable. However, at this point this is unknown. Let us modify the argument from [9] in order to cover the case of θ non-diagonalizable.

Lemma 8.2. The eigenvalues of θ are rational numbers.

Proof. The eigenvalues of $e^{2\pi\sqrt{-1}\theta}$ coincide with the eigenvalues of the monodromy matrix of the fundamental solution $\Phi(t,z)$, because $\theta=\delta+\nu_{[0]}$. Therefore, it is enough to prove that the monodromy matrix of $\Phi(t,z)$ has eigenvalues that are roots of 1. On the other hand the system (39)–(40) can be solved in terms of oscillatory integrals. Therefore, there exists a linear isomorphism

$$\Pi: H_1(X_{u^{\circ}}, \operatorname{Re}(F/z) \ll 0, \mathbb{C}) \to H$$

such that

$$J_{\Gamma}(t,z) = \Phi(t,z)\Pi(\Gamma), \quad \forall \Gamma.$$

Therefore, the eigenvalues of the monodromy matrix are the same as the eigenvalues of the monodromy operator

$$M: H_1(X_{u^{\circ}}, \operatorname{Re}(F/z) \ll 0, \mathbb{C}) \to H_1(X_{u^{\circ}}, \operatorname{Re}(F/z) \ll 0, \mathbb{C})$$

corresponding to the parallel transport around $z=\infty$ with respect to the Gauss–Manin connection. Note that the subspace

$$\operatorname{Im}(M-1) \subset H_1(X_{u^{\circ}}, \operatorname{Re}(F/z) \ll 0, \mathbb{C})$$

admits a basis consisting of compact cycles and cycles supported in a neighborhood of a puncture ∞_i . If Γ is a compact cycle then the corresponding integral is analytic near $z = \infty$, so $(M-1)\Gamma = 0$. If Γ is supported near the puncture ∞_i , then we may assume that $X_{u^{\circ}} = \mathbb{C}$ and $F(p) = p^{m_i}$ and an easy computation yields that the integral depends analytically on z^{-1/m_i} , so $(M^{m_i} - 1)\Gamma = 0$. We get that M satisfies the following equation:

$$(M-1)^2 \prod_{i=1}^{a} (M^{m_i} - 1) = 0.$$

The eigenvalues of M must be also solutions of the above equation, so they are roots of unity as claimed.

Let G_{δ} be the subgroup of GL(H) consisting of linear operators C such that

$$C = 1 + \sum_{\ell=1}^{\infty} C_{[-\ell]}.$$

The group G_{δ} acts on the set of weak Levelt solutions

$$\Phi(t,z) \mapsto \Phi(t,z)C =: \widetilde{S}(t,z)z^{\delta}z^{\widetilde{\nu}},$$

where

$$\widetilde{S}(t,z) = S(t,z)z^{\delta}Cz^{-\delta}, \quad \widetilde{\nu} = C^{-1}\nu C.$$

Note that ν belongs to the Lie algebra \mathfrak{g}_{δ} of G_{δ} consisting of matrices x such that $1+x \in G_{\delta}$. In particular, $e^{2\pi\sqrt{-1}\delta}$ commutes with ν . Proposition 8.1 is a corollary of the following lemma.

Lemma 8.3. Let $\Phi(t,z) = S(t,z)z^{\delta}z^{\nu}$ be a weak Levelt solution. There exists a unique $C \in G_{\delta}$ such that $C^T_{[\ell]} = (-1)^{\ell}C_{[\ell]}$ for all $\ell \leq 0$ and $\widetilde{S}(t,z) := S(t,z)z^{\delta}Cz^{-\delta}$ satisfies the symplectic condition $\widetilde{S}(t,-z)^T\widetilde{S}(t,z) = 1$.

Proof. Let us first point out that the projection $X\mapsto X_{[a]}$ commutes with transposition, i.e., $(X_{[a]})^T=(X^T)_{[a]}$ for all $X\in \mathrm{GL}(H)$ and $a\in \mathrm{spec}(\delta)$. This follows from the skew-symmetry of $\theta=\delta+\nu_{[0]}$. Namely, using that there exists an integer m for which $e^{2\pi\sqrt{-1}\theta m}$ is a unipotent operator (recall that θ has rational eigenvalues) and that $\theta+\theta^T=0$, we get $\nu_{[0]}+(\nu_{[0]})^T=0$ and $\delta+\delta^T=0$. The latter implies $[\delta,X]^T=[\delta,X^T]$, so $T:\mathfrak{gl}_a(H)\to\mathfrak{gl}_a(H)$ is a linear isomorphism for all $a\in \mathrm{spec}(\delta)$. Our claim follows easily. In the rest of the proof we put $X_{[a]}^T:=(X_{[a]})^T=(X^T)_{[a]}$.

Let us fix z to be a positive real number and define

$$A := \Phi(t, -z)^T \Phi(t, z),$$

where we define $(-z)^x := e^{x \log(-z)}$ with $\log(-z) := \log z + \pi \sqrt{-1}$. The differential equations (39)–(40) imply that A is a constant matrix independent of t and z. We will also make use of the following two properties of A

$$A^T = Ae^{2\pi\sqrt{-1}\delta}e^{2\pi\sqrt{-1}\nu}.$$

and

$$\nu^T = -A \,\nu \,A^{-1}.$$

The first one is proved by transposing the identity defining A and analytically continuing in z from z to -z along an arc in the upper half-plane. To prove the second one, first we pick an integer m such that $e^{2\pi\sqrt{-1}m\delta}=1$. Let us analytically continue the identity defining A along a loop that goes m times around $z=\infty$. We get

$$e^{2\pi\sqrt{-1}m\nu^T}Ae^{2\pi\sqrt{-1}m\nu} = A \implies e^{2\pi\sqrt{-1}m\nu^T} = e^{2\pi\sqrt{-1}m(-A\nu A^{-1})}.$$

It remains only to use that we can take the logarithm, because ν^T and $-A\nu A^{-1}$ are nilpotent matrices.

Let us try to find $C \in G_{\delta}$, such that $\widetilde{S}(t,-z)^T \widetilde{S}(t,z) = 1$. Using (47) we get

$$S(t, -z)^T S(t, z) = (-z)^{\delta} A e^{\pi \sqrt{-1}\nu} z^{-\delta}.$$

Comparing the coefficients in front of the powers of z, we get that

$$B := e^{\pi\sqrt{-1}\delta} A e^{\pi\sqrt{-1}\nu}$$

belongs to G_{δ} . Moreover, equation (46) implies that $B_{[\ell]}^T = (-1)^{\ell} B_{[\ell]}$ for all $\ell \leq 0$. After a straightforward computation we get the following equation for C:

$$(e^{\pi\sqrt{-1}\delta}C^Te^{-\pi\sqrt{-1}\delta})\,B\,C=1.$$

The projection of this equation onto $\mathfrak{gl}_{-\ell}(H)$ $(\ell > 0)$ yields

$$B_{[-\ell]} + C_{[-\ell]} + (-1)^{\ell} C_{[-\ell]}^T + \sum_{\substack{i+k+j=\ell\\0 \le i, k, j < \ell}} (-1)^i C_{[-i]}^T B_{[-k]} C_{[-j]} = 0.$$

The above equations determine a unique solution satisfying the additional constraint $C_{\lceil -\ell \rceil}^T = (-1)^\ell C_{\lceil -\ell \rceil}$.

The operator series S(t,z) satisfying the conditions of Proposition 8.1 are called *calibrations*, i.e., these are operator series of the form $1 + S_1(t)z^{-1} + \cdots$ such that $S(t,-z)^T S(t,z) = 1$ and the gauge transformation of the Dubrovin's connection takes the form

$$S(t,z)^{-1} \nabla S(t,z) = d - \left(\theta z^{-1} + \sum_{\ell=1}^{\infty} \nu_{[-\ell]} z^{-\ell-1}\right) dz$$

for some $\nu \in G_{\delta}$ where $\delta := \theta_{ss}$ is the semi-simple part of θ in the Jordan–Chevalley decomposition of θ .

8.2. The total descendant potential. Let us fix a calibration S(t,z) and view the coefficients S_k as matrix-valued functions on U. Recall that when we introduced the flat coordinate system on U we had the freedom to choose $t_a^{\circ} = t_a(u^{\circ})$ as we wish. Let us define $t_a^{\circ} := (S_1(u^{\circ})1, \phi^a), 1 \leq a \leq N$. Using the differential equations (39)–(40) it is easy to check that

$$t_a = (S_1(u)1, \phi^a), \quad 1 < a < N.$$

Using the flat coordinates we embed our Frobenius manifold U in \mathbb{C}^N as an open neighborhood of $t^{\circ} = (t_1^{\circ}, \dots, t_N^{\circ})$. We will identify points $u \in U$ with their coordinates $t = (t_1, \dots, t_N) \in \mathbb{C}^N$.

Our goal is to define the following set of descendant correlators

$$\langle \phi_{a_1} \psi^{k_1}, \dots, \phi_{a_n} \psi^{k_n} \rangle_{g,n},$$

where $g, n \geq 0, k_1, \ldots, k_n \geq 0$, and $1 \leq a_1, \ldots, a_n \leq N$ are arbitrary. By definition the descendant correlators are analytic functions on the Frobenius manifold U. If 2g - 2 + n > 0, then the above correlator is defined in terms of the calibration S(t, z) and the ancestor correlators as follows

$$\langle [S(t,\overline{\psi})\phi_{a_1}\overline{\psi}^{k_1}]_+,\ldots,[S(t,\overline{\psi})\phi_{a_n}\overline{\psi}^{k_n}]_+\rangle_{g,n},$$

 $[]_+$ denotes the operation truncating the terms containing negative powers of $\overline{\psi}$ and the ancestor correlator is evaluated at a point $u \in U$ with coordinates $t = (t_1, \ldots, t_N)$.

If $2g-2+n\leq 0$, then there are 4 cases. If (g,n)=(0,2), then

$$\langle \phi_a \psi^k, \phi_b \psi^\ell \rangle_{0,2} := (W_{k\ell} \phi_b, \phi_a),$$

where the linear operators $W_{k\ell}$ are defined by

$$\sum_{k,\ell=0}^{\infty} W_{k\ell} z^{-k} w^{-\ell} = \frac{S(t,z)^T S(t,w) - 1}{z^{-1} + w^{-1}}.$$

If (q, n) = (0, 1), then

$$\langle \phi_a \psi^k \rangle_{0,1} := \langle \phi_a \psi^{k+1}, 1 \rangle_{0,2} = (S_{k+2}(t)\phi_a, 1).$$

If (g, n) = (0, 0), then

$$\langle \rangle_{0,0} := -\frac{1}{2} \langle \psi - t \rangle_{0,1} = \frac{1}{2} ((S_2(t)S_1(t) - S_3(t))1, 1),$$

where we identified $\mathbb{C}^N \cong H$ via $t \mapsto \sum_a t_a \phi_a$. Note that $F^{(0)}(t) := \langle \rangle_{0,0}$ is the primary potential of the Frobenius structure. Finally if (g, n) = (1, 0), then

$$\langle \ \rangle_{1,0} := \frac{1}{2} \int_0^t \sum_{i=1}^N R_1^{ii}(u) du_i - \frac{1}{24} \sum_{i=1}^N \log c_i(u,0),$$

where $R_1(u)$ is the coefficient in front of z in the matrix R(u,z) defined via the stationary phase asymptotic (see Section 7.2), $R_1^{ij}(u)$ is the (i,j) entry of $R_1(u)$, and $c_i(u,z) = K(\omega, [\omega_i])$ (see Section 7.2). The function $F^{(1)}(t) := \langle \rangle_{1,0}$ is also known as the genus-1 primary potential.

The total descendant potential is by definition the following generating series for the descendant correlators

$$\mathcal{D}_t(\hbar, \mathbf{t}) = \exp\Big(\sum_{q, n=0}^{\infty} \frac{\hbar^{g-1}}{n!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{g, n}\Big),\,$$

where $\mathbf{t}(\psi) = \sum_{k=0}^{\infty} \sum_{a=1}^{N} t_{k,a} \phi_a \psi^k$, the correlators are expanded multilinearly in the formal variables $t_{k,a}$, and all descendant correlators are evaluated at the point $t \in U$.

Remark 8.4. The total descendant potential has the following translation symmetry. Put $\mathcal{D}^{\circ} := \mathcal{D}_{t^{\circ}}$, then

$$\mathcal{D}_t(\hbar, \mathbf{t}) = \mathcal{D}^{\circ}(\hbar, \mathbf{t} + t - t^{\circ})$$

for all t sufficiently close to t° .

Remark 8.5. There is an elegant way to write the relation between descendants and ancestors using Givental's quantization formalism (see [17]). The above definition although a bit cumbersome is more convenient for our purposes.

8.3. Topological recursion and descendants. Let ω be the primitive form corresponding to a primary differential. We are going to express the descendant correlators in terms of the forms $\omega_{g,n}$. Let us fix a reference point $(u^{\circ}, z^{\circ}) \in U \times \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, such that z° is a positive real number such that the

fundamental solution $\Phi(u,z) = S(u,z)z^{\delta}z^{\nu}$ is convergent for all (u,z) sufficiently close to (u°,z°) . There exists a unique linear isomorphism

$$\Pi: H_1(X_{u^{\circ}}, \operatorname{Re}(F/z^{\circ}) \ll 0; \mathbb{C}) \to H$$

such that

$$J_{\Gamma}(u,z) = S(u,z)z^{\delta}z^{\nu} \Pi(\Gamma)$$

for all $(u, z) \in U \times \mathbb{C}^*$ sufficiently close to (u°, z°) . Here the value of $z^x := e^{x \log z}$ for $x = \delta, \nu$ is chosen such that $\log z^{\circ} = \ln z^{\circ}$.

Let $\omega_{g,n}(q_1,\ldots,q_n)$ be the correlator forms defined by the Eynard–Orantin recursion. Note that if Γ is a small loop in X_u around one of the ramification points $p_j(u)$, then

$$\oint_{q_i \in \Gamma} e^{F(q_i)/z} \omega_{g,n}(q_1, \dots, q_n) = 0.$$

Indeed, this is easy to check when (g, n) = (0, 2) or n = 1, while for the remaining cases we can argue by induction on (g, n) using that the correlator forms are symmetric.

For a given set of semi-infinite cycles

$$\Gamma_a \in H_1(X_{u^\circ}, \operatorname{Re}(F/z^\circ) \ll 0; \mathbb{C}), \quad 1 \leq a \leq n$$

we define the following integrals

(48)
$$\prod_{a=1}^{n} (-2\pi z_a)^{-1/2} \int_{q_1 \in \Gamma_1} \cdots \int_{q_n \in \Gamma_n} e^{\frac{F(q_1)}{z_1} + \cdots + \frac{F(q_n)}{z_n}} \omega_{g,n}(q_1, \dots, q_n),$$

where each cycle Γ_i is represented by a path avoiding the ramification points (where the integrand might have poles). According to our previous remark, the integral is independent of the choice of representative paths.

Proposition 8.6. If 2g - 2 + n > 0, then the integral (48) coincides with the following descendant correlator

$$\left\langle \frac{z_1^{\delta} z_1^{\nu} \Pi(\Gamma_1)}{\psi - z_1}, \dots, \frac{z_n^{\delta} z_n^{\nu} \Pi(\Gamma_n)}{\psi - z_n} \right\rangle_{g,n}$$

Proof. Let us fix $u \in U$ and z_1, \ldots, z_n . It is enough to prove the formula when each cycle $\Gamma_a = \varprojlim \gamma_{\lambda}^{(i_a)}$ is the limit as $\lambda \to \infty$ of the Lefshetz thimbles corresponding to a critical value u_{i_a} , where λ varies along a path from u_{i_a} to ∞ such that $\operatorname{Re}(\lambda/z_i) < 0$ (see Section 7.1). For such cycles the integral (48) takes the form

$$\prod_{a=1}^{n} (-2\pi z_a)^{-1/2} \int_{u_{i_1}}^{\infty} \cdots \int_{u_{i_n}}^{\infty} e^{\frac{\lambda_1}{z_1} + \cdots + \frac{\lambda_n}{z_n}} \omega_{g,n}^{\gamma^{(i_1)}, \dots, \gamma^{(i_n)}} (\lambda_1, \dots, \lambda_n).$$

Recalling Theorem 2.8 we can write the above integral as an n-pointed genus-g ancestor correlator in which the insertion on the ath place is

$$\sum_{k=0}^{\infty} (-2\pi z_a)^{-1/2} \int_{u_{i_a}}^{\infty} e^{\lambda_a/z_a} I_{\gamma^{(i_a)}}^{(k+1)}(u,\lambda_a) (-\overline{\psi})^k = -J_{\Gamma_a}(u,z_a) \sum_{k=0}^{\infty} \overline{\psi}^k z_a^{-k-1}.$$

The above expression can be written as

$$-\sum_{k=0}^{\infty} [S(u,\overline{\psi})\overline{\psi}^k]_{+} z_a^{-k-1} z_a^{\delta} z_a^{\nu} \Pi(\Gamma_a).$$

To complete the proof it remains only to recall the definition of the descendant correlators (in the stable range). \Box

8.4. **Generalization.** Let $x: \Sigma \to \mathbb{P}^1$ be a branched covering with ramification profile the same as in the EO-recursion. Let us define the twisted de Rham cohomology group $H_{\text{twdR}}(\Sigma^n, x)$ as the following quotient

$$(\Omega^{1,\infty})^{\boxtimes n}(\Sigma^n)[z_1^{\pm 1},\ldots,z_n^{\pm 1}]/\Big(\sum_{i=1}^n z\ d_i + dx_i \wedge \Big)(\Omega^{0,\infty})^{\boxtimes n}(\Sigma^n)[z_1^{\pm 1},\ldots,z_n^{\pm 1}],$$

where d_i is the de Rham differential on the ith copy of Σ in the direct product $\Sigma^n := \Sigma \times \cdots \Sigma$, the restriction of dx_i on the jth slot of the direct product Σ^n is dx if j=i and 0 if $j \neq i$. Using our results in Section 3.3 (for the case n=1) we get that $H_{\text{twdR}}(\Sigma^n, x)$ is a free $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ -module of rank n^N and a basis is given by

$$[\omega_{i_1,\ldots,i_n}] := [\omega_{i_1}] \boxtimes \cdots \boxtimes [\omega_{i_n}], \quad 1 \le i_1,\ldots,i_n \le N,$$

where ω_i is the good basis constructed in 4.2.

We would like to define a recursion that defines classes

$$\omega_{g,n} \in H_{\text{twdR}}(\Sigma^n, x),$$

that are symmetric with respect to simultaneous permutations of $(q_1, \ldots, q_n) \in \Sigma^n$ and (z_1, \ldots, z_n) . The recursion depends on the choice of a differential $\omega \in \Omega^{1,\infty}(\Sigma)[z]$ and a symmetric bi-differential $W \in (\Omega^{1,\infty})^{\boxtimes 2}[z,w]$ satisfying the following conditions. The differential ω should be a holomorphic volume form in a neighborhood of each finite ramification point p_i , i.e., if we represent ω by $\sum_{n=0}^{n_0} \omega^{(n)}(-z)^n$, then $\omega^{(0)}(p_i) \neq 0$ for all finite ramification points p_i . The symmetric bi-differential is required to have the form

$$W = B + \sum_{m,n\geq 0} \sum_{i,j=1}^{N} V_{m,n}^{ij}[\omega_i] \boxtimes [\omega_j] (-z_1)^m (-z_2)^n,$$

where B is the fundamental bi-differential (in the usual EO-recursion) and $V := \sum_{m,n} V_{m,n} (-z_1)^m (-z_2)^n$ is a polynomial whose coefficients $V_{m,n}$ are square matrices of size N, satisfying the symmetry condition $V_{mn}^{ij} = V_{nm}^{ji}$, where for a matrix A we denoted by A^{ij} the entry in position (i,j).

In order to define the recursion, we need to introduce the operator

$$\partial_x: \Omega^{1,\infty}(V) \to \Omega^{1,\infty}(V), \quad \theta \mapsto d\left(\frac{\theta}{dx}\right)$$

defined for all open subsets $V \subset \Sigma$ that do not contain finite ramification points. If θ has finite order poles at some ramification point, then so does $\partial_x \theta$. We will need also to work with an operator inverse to ∂_x

$$\partial_x^{-1}\theta(p) = dx(p) \wedge d_p^{-1}\theta(p).$$

The choice of $d_p^{-1}\theta(p)$ is not unique so the inverse operation yields multi-valued analytic forms. However, note that if $\theta \in \Omega^{1,\infty}(V)$, then the ambiguity in the definition of $\partial_x^{-n}\theta \in \Omega^{1,\infty}(V)$ is up to dg, where g is a polynomial in x of degree at most n.

Let us assume that

$$\theta \in \Omega^{1,\infty}(\Sigma \setminus \{p_1,\ldots,p_N\})$$

has a finite order pole at every ramification point p_i . Then for all $n \gg 0$ the 1-form $\partial_x^{-n}\theta$ is analytic in a neighborhood of all ramification points, so using the excision principle (see Proposition 3.4, Part b)) we can define a twisted de Rham cohomology class $[\partial_x^{-n}\theta] \in H_{\text{twdR}}(\Sigma, x)$. It is easy to check that $[\partial_x^{-n}\theta]$ is independent of the choice of a branch of $\partial_x^{-n}\theta$. Using the relation

$$[\partial_x^{-n}\theta] = (-z)^{-1}[\partial_x^{-n-1}\theta], \quad n \gg 0.$$

we get that we have the following map

$$\Omega^{1,\infty}(\Sigma \setminus \{p_1,\ldots,p_N\}) \to H_{\text{twdR}}(\Sigma,x), \quad \theta \mapsto [\theta] := (-z)^{-n}[\partial_x^{-n}\theta].$$

We would like to generalize the EO-recursion as follows. For initial condition put

$$\omega_{0,2}(q_1, q_2) = W(q_1, q_2).$$

The recursion is defined by the same formula as before except that we replace the recursion kernel

$$\frac{\int_{p}^{\tau_{i}(p)} B(q_{0}, p')}{dx(p) \int_{p}^{\tau_{i}(p)} \phi(p')}$$

by

$$\frac{\int_{p}^{\tau_{i}(p)} \left(B(q_{0}, p') + \sum_{m,n \geq 0} \sum_{i,j=1}^{N} V_{mn}^{ij} [\omega_{i}(q_{0})] (-z_{0})^{m} (\partial_{x}^{-n} \omega_{j})(p')\right)}{dx(p) \int_{p}^{\tau_{i}(p)} \left(\sum_{n=0}^{n_{0}} (\partial_{x}^{-n} \omega^{(n)})(p')\right)}.$$

8.5. Topological recursion for polynomial primitive forms. The generalization of the EO-recursion proposed above allows us to extend the statements of Theorem 2.8 and 8.6 to the case of polynomial primitive forms.

To begin with let us assume that $\omega \in \mathcal{H}(U)$ is a polynomial primitive form represented by $\sum_{n=0}^{n_0} \omega^{(n)}(-z)^n$. Let us recall also the matrix R(u,z) defining the corresponding ancestor invariants (see Proposition 7.1). In order to define a corresponding generalized topological recursion we have to specify a differential

and a bi-differential satisfying the conditions described in the previous section. We choose the differential to be the primitive form. While the bi-differential

$$W(q_1, q_2) = B(q_1, q_2) + \sum_{m, n=0}^{\infty} \sum_{i, j=1}^{N} V_{mn}^{ij} \omega_i(q_1) \omega_j(q_2) (-z_1)^m (-z_2)^n$$

is chosen in such a way that

$$\frac{R(u, z_1)^T R(u, z_2) - 1}{z_1 + z_2} = \sum_{m, n=0}^{\infty} W_{2m, 2n} (2m - 1)!! (2n - 1)!! (-z_1)^m (-z_2)^n,$$

where $W_{m,n}$ is the matrix whose (i,j)-entry W_{mn}^{ij} is defined as the coefficient in front of $t_i(q_1)^m t_j(q_2)^n dt_i(q_1) dt_j(q_2)$ in the Laurent series expansion of

$$B(q_1, q_2) + \sum_{m,n=0}^{\infty} \sum_{i,j=1}^{N} V_{mn}^{ij}(\partial_x^{-m}\omega_i)(q_1) (\partial_x^{-n}\omega_j)(q_2)$$

at the point (p_i, p_j) .

Using the Taylor series expansion at $q = p_i$

$$\partial_x^{-m}\omega_{i'}(q) = dt_i \left(-\frac{t_i^2 m}{(2m-1)!!} + \sum_{k=0}^{\infty} B_{k0}^{ii'} \frac{t_i^{k+2(m+1)}}{(k+1)(k+3)\cdots(k+2m+1)} \right)$$

and Lemma 7.7 we get that the matrices $V_{m,n}$ are uniquely determined from the identity

$$\frac{R_{\omega}(u,z_1)R_{\omega}(u,z_2)^T - 1}{z_1 + z_2} = \sum_{m,n=0}^{\infty} V_{m,n}(-z_1)^m (-z_2)^n.$$

Note that since R_{ω} is polynomial in z only finitely many $V_{m,n} \neq 0$. Let

$$\omega_{g,n} = \sum_{\kappa = (k_1, \dots k_n)} \omega_{g,n;\kappa} (-z_1)^{k_1} \cdots (-z_n)^{k_n}, \quad \omega_{g,n;\kappa} \in (\Omega^{1,\infty})^{\boxtimes n} (\Sigma^n),$$

be the forms defined by the generalized topological recursion. The multivalued correlator forms $\omega_{g,n}^{\beta_1,\ldots,\beta_n}(\lambda_1,\ldots,\lambda_n)$ are define by the same formulas as before, except that we identify $\omega_{g,n}$ with

$$\sum_{\kappa=(k_1,\dots k_n)} (\partial_{x_1})^{-k_1} \cdots (\partial_{x_n})^{-k_n} \omega_{g,n;\kappa}$$

The arguments used in the proof of Theorem 2.8 and 8.6 can be repeated, so the conclusions of both propositions hold.

References

- [1] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of algebraic curves II*. Grundlehren volume 268, Springer–Verlag, Berlin Heidelberg, 2011.
- [2] V.I. Arnol'd, S.M. Gusein-Zade, and A.N. Varchenko. Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals. Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988.
- [3] B. Bakalov, T. Milanov. W-constraints for the total descendant potential of a simple singularity. Compositio Math. 149 (2013), no. 5, 840–888.
- [4] V. Bouchard and B. Eynard. Think globally, compute locally. J. of High Energy Phys. (2013), no. 2, Article 143.
- [5] A. Douai and C. Sabbah. Gauss-Manin systems, Brieskorn lattices and Frobenius structures (I). Ann. Inst. Fourier, vol. 53, no. 4(2003): 1055–1116.
- [6] A. Douai and C. Sabbah. Gauss-Manin systems, Brieskorn lattices and Frobenius structures (II). Frobenius manifolds (Quantum cohomology and singularities), Aspects of Mathematics, vol. E36, Vieweg(2004): 1–18
- [7] B. Dubrovin. Geometry of 2D topological field theories. In: "Integrable systems and quantum groups" (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.
- [8] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz. Identification of the Givental formula with the spectral curve topological recursion procedure. Comm. in Math. Phys. 328 (2014), no. 2, 669–700.
- [9] B. Dubrovin. Painlevé transcendents in two dimensional topological field theory. arXiv: 9803.107
- [10] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin. Dubrovin's superpotential as a global spectral curve. arXiv: 1509.06954.
- [11] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin. Primary invariants of Hurwitz Frobenius manifolds. arXiv: 1605.07644.
- [12] B. Eynard. Invariants of spectral curves and intersection theory of moduli spaces of complex curves. Comm. Numb. Theory and Phys., vol. 8, no. 3(2014): 541–588.
- [13] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. Comm. Numb. Theory and Phys., vol. 1, no. 2(2007): 347–452.
- [14] B. Fang, C.-C. M. Liu, Z. Zong. The Eynard-Orantin recursion and equivariant mirror symmetry for the projective line. Geometry & Topology, vol. 21, no. 4 (2017): 20492092.
- [15] W. Fulton. Hurwitz schemes and the irreducibility of moduli of algebraic curves. Ann. of Math., vol. 90, no. 3(1969), 542–575
- [16] A. Givental. Semisimple Frobenius structures at higher genus. Internat. Math. Res. Notices, vol. 23(2001), 1265–1286.
- [17] A. Givental. Gromov-Witten invariants and quantization of quadratic Hamiltonians. Mosc. Math. J. vol. 1(2001), 551–568.
- [18] H. Grauert and R. Remmert. Coherent analytic sheaves. Springer-Verlag, Berlin Heidelberg, 1984.
- [19] C. Hertling. Frobenius Manifolds and Moduli Spaces for Singularities. Cambridge Tracts in Mathematics, 151. Cambridge University Press, Cambridge, 2002. x+270 pp.
- [20] B. Iversen Cohomology of sheaves. Springer-Verlag, Berlin Heidelberg, 1986.
- [21] Si-Qi Liu and Y. Zhang. Uniqueness theorem of W-constraints for simple singularities. Lett. Math. Phys., Vol. 103, no. 12(2013): 13291345.
- [22] T. Milanov and D. Lewanski. W-algebra constraints and topological recursion for A_N singularity. International Journal of Math., Vol. 27, no. 13(2016) 1650110 (21 pages).
- [23] T. Milanov. The Eynard-Orantin recursion for the total ancestor potential. Duke Math. J. 163 (2014), no. 9, 1795–1824.

- [24] T. Milanov The Eynard-Orantin recursion for simple singularities. Commun. Number Theory Phys., Vol. 9, no. 4(2015): 707–739.
- [25] H. Rauch. Weierstrass points, branch points, and moduli of Riemann surfaces. Comm. on Pure and Appl. Math., vol. 12(1959), 543–560.
- [26] K. Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 775-792 (1982).
- [27] K. Saito and A. Takahashi. From primitive forms to Frobenius manifolds. From Hodge theory to integrability and TQFT tt*-geometry, 31-48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.
- [28] M. Saito. On the structure of Brieskorn lattice. Ann. Inst. Fourier, vol. 39, no. 1(1989): 27–72.
- [29] V. Shramchenko. Deformations of Frobenius structures on Hurwitz spaces. Internat. Math. Res. Notices, vol. 6(2005), 339–387.
- [30] C. Teleman, The structure of 2D semi-simple field theories. Invent. Math., 188 (2012), no. 3, 525–588.

KAVLI IPMU (WPI), UTIAS, THE UNIVERSITY OF TOKYO, KASHIWA, CHIBA 277-8583, JAPAN

E-mail address: todor.milanov@ipmu.jp