THE DIRICHLET PROBLEM FOR A CLASS OF HESSIAN TYPE EQUATIONS

HEMING JIAO AND TINGTING WANG

ABSTRACT. We are concerned with the Dirichlet problem for a class of Hessian type equations. Applying some new methods we are able to establish the C^2 estimates for an approximating problem under essentially optimal structure conditions. Based on these estimates, the existence of classical solutions is proved.

Keywords: Hessian equations, interior second order estimates, classical solutions.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n $(n \geq 2)$ with smooth boundary $\partial\Omega$. In this paper, we are concerned with the regularity for solutions of the Dirichlet problem

(1.1)
$$\begin{cases} f(\lambda[D^2u + \gamma \triangle uI]) = \psi & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\gamma \geq 0$ is a constant, I is the unit matrix and $\lambda[D^2u + \gamma \triangle uI] = (\lambda_1, ..., \lambda_n)$ denote the eigenvalues of the matrix $\{D^2u + \gamma \triangle uI\}$.

Following [1], $f \in C^2(\Gamma) \cap C(\bar{\Gamma})$ is assumed to be defined in an open convex symmetric cone Γ , with vertex at the origin and

$$\Gamma \supseteq \Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{ each component } \lambda_i > 0\},$$

and to satisfy the following structure conditions:

(1.2)
$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, 1 \le i \le n,$$

(1.3)
$$f$$
 is concave in Γ ,

and

(1.4)
$$f > 0 \text{ in } \Gamma, f = 0 \text{ on } \partial \Gamma.$$

A function $u \in C^2(\Omega)$ is called admissible if $\lambda[D^2u + \gamma \Delta uI] \in \overline{\Gamma}$. According to [1], condition (1.2) ensures that equation (1.1) is degenerate elliptic for admissible solutions. While (1.3) implies that the function F defined by $F[A] = f(\lambda[A])$ to be concave for $A \in \mathcal{S}^{n \times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n \times n}$ is the set of n by n symmetric matrices.

We assume that $\psi \geq 0$ in Ω , so the equation (1.1) is degenerate when $\gamma = 0$. In this paper, there are no geometric restrictions to $\partial\Omega$ being made. Instead, we

assume that there exists a subsolution $\underline{u} \in C^2(\bar{\Omega})$ satisfying $\lambda(D^2\underline{u} + \gamma \triangle \underline{u}I) \in \Gamma$ on $\bar{\Omega}$ and

(1.5)
$$\begin{cases} f(\lambda(D^2\underline{u} + \gamma \triangle \underline{u}I)) \ge \psi \text{ in } \Omega, \\ \underline{u} = \varphi \text{ on } \partial\Omega. \end{cases}$$

Theorem 1.1. Let $\gamma > 0$, $\psi \in C^{\infty}(\bar{\Omega})$ and $\varphi \in C^{\infty}(\partial \Omega)$. Under (1.2)-(1.5), there exists a unique admissible solution $u \in C^{\infty}(\bar{\Omega})$ of (1.1).

We first introduce our procedure to prove Theorem 1.1. By (1.4), there exists a positive constant ε_0 such that

(1.6)
$$f(\lambda(D^2\underline{u} + \gamma \triangle \underline{u}I)) \ge \varepsilon_0 \text{ on } \bar{\Omega}$$

since $\lambda(D^2\underline{u}+\gamma\Delta\underline{u}I)\in\Gamma$. We shall establish the a priori C^2 estimates independent of ε for admissible solutions of the approximating problem

(1.7)
$$\begin{cases} f(\lambda(D^2u_{\varepsilon} + \gamma \Delta u_{\varepsilon}I)) = \psi + \varepsilon \eta(\psi) \text{ in } \Omega, \\ u_{\varepsilon} = \varphi \text{ on } \partial \Omega, \end{cases}$$

where $\eta \in C^{\infty}[0,\infty)$ satisfies

$$\eta(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{\varepsilon_0}{4}], \\ 0 & \text{if } t \in [\frac{\varepsilon_0}{2}, \infty), \end{cases}$$

 $0 \le \eta \le 1$, $|\eta'| \le C\varepsilon_0^{-1}$ and $|\eta''| \le C\varepsilon_0^{-2}$. It follows that, by (1.6),

$$f(\lambda(D^2\underline{u} + \gamma \triangle \underline{u}I)) \ge \psi + \varepsilon \eta(\psi)$$

provided $\varepsilon \leq \frac{\varepsilon_0}{2}$ and obviously, $\psi + \varepsilon \eta(\psi) \geq \min\{\varepsilon, \varepsilon_0/4\} > 0$.

We shall use the techniques of Guan [7] (see [8] and [9] also) to establish such estimates. As usual, the main difficulty is from the boundary estimates of pure normal second order derivative for which we use the strategy of Ivochkina, Trudinger and Wang [11] whose idea is originally from Krylov [13, 14, 15, 16] where the Bellman equations are studied. A key step is the construction of barrier functions in which the existence of \underline{u} plays an important role (see Theorem 5.1).

The presence of $\gamma > 0$ is crucial to the interior estimates for second derivatives. An interesting question is to establish the weak interior estimates (see [11]) when $\gamma = 0$.

For the case that $\psi \geq \psi_0 > 0$, the existence of smooth solutions to the Dirichlet problem (1.1) with $\gamma = 0$ was established by Caffarelli, Nirenberg and Spruck [1] under additional assumptions on f in a domain Ω satisfying that there exists a sufficiently large number R > 0 such that, at every point $x \in \partial \Omega$,

$$(1.8) (\kappa_1, \dots, \kappa_{n-1}, R) \in \Gamma,$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial \Omega$ with respect to the interior normal. Their work was further developed and simplified by Trudinger [17].

Guan considered the Hessian equations of the form

$$(1.9) f(\lambda[\nabla^2 u + \gamma \triangle ug + sdu \otimes du - \frac{t}{2}|\nabla u|^2g + A]) = \psi(x, u, \nabla u)$$

on a Riemannian manifold with metric g with $\psi > 0$, which is arising from conformal geometry (see [4] and [5]). In these papers Guan also assumed that f is homogenous of degree one which implies that the equation (1.9) is strictly elliptic. It would be

interesting to prove Theorem 1.1 for the general form (1.9) on manifolds when $\psi \geq 0$ without any additional conditions on f. The case that $\gamma = 0$ seems more complicated. In a recent work [7], Guan proved Theorem 1.1 under (1.2)-(1.5) when $\gamma = 0$ and $\psi \geq \psi_0 > 0$. Another interesting question would be whether we can get a viscosity solution in $C^{1,1}(\bar{\Omega})$ for $\gamma = 0$ when $\psi \geq 0$.

It was shown in [1] that using (1.8) and the condition that for every C > 0 and every compact set K in Γ there is a number R = R(C, K) such that

$$(1.10) f(R\lambda) \ge C \text{ for all } \lambda \in K$$

one can construct admissible strict subsolutions of equation (1.1) with $\gamma = 0$. Obviously $\Gamma \subset \{\lambda \in \mathbb{R}^n : \sum \lambda_i > 0\}$ and we have $\Delta u \geq 0$ for any admissible function u. So we can construct an admissible strict subsolution of (1.1) when $\gamma \geq 0$ satisfying (1.5) under (1.8) and (1.10) by the same way.

Typical examples are given by $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \le l < k \le n$, defined in the Gårding cone

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \dots, k \},$$

where σ_k are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}, \quad k = 1, \dots, n.$$

The case when $f = \sigma_n^{1/n}$ (the Monge-Ampère equation) and $\gamma = 0$ was studied by Guan, Trudinger and Wang [10] and they obtained the $C^{1,1}$ regularity as $\psi^{1/(n-1)} \in C^{1,1}(\bar{\Omega})$. It would be an interesting problem to show whether the result can be improved for the $f = \sigma_k^{1/k}$ (see [11]).

The rest of this paper is organized as follows. In Section 2, we prove Theorem

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 provided the C^2 estimates for (1.7) is established. C^1 estimate is treated in Section 3. The interior second order estimate is proved in Section 4. In section 5, the estimates for second derivatives are established.

2. Beginning of proof

In this Section we explain how to prove Theorem 1.1 when the second order estimates for (1.7) are established. Let $u_{\varepsilon} \in C^4(\bar{\Omega})$ be the admissible solution of (1.7). For simplicity we shall use the notations $U^{\varepsilon} = D^2 u_{\varepsilon} + \gamma \triangle u_{\varepsilon} I$ and $\underline{U} = D^2 \underline{u} + \gamma \triangle \underline{u} I$. Following the literature, unless otherwise noted, we denote throughout this paper

$$F^{ij}[U^{\varepsilon}] = \frac{\partial F}{\partial U_{ij}^{\varepsilon}}[U^{\varepsilon}], \quad F^{ij,kl}[U^{\varepsilon}] = \frac{\partial^2 F}{\partial U_{ij}^{\varepsilon}\partial U_{kl}^{\varepsilon}}[U^{\varepsilon}].$$

The matrix $\{F^{ij}\}$ has eigenvalues f_1, \ldots, f_n and is positive definite by assumption (1.2), while (1.3) implies that F is a concave function of U_{ij}^{ε} (see [1]). Moreover, when U^{ε} is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij}U_{ij}^{\varepsilon} = \sum f_i \lambda_i, \quad F^{ij}U_{ik}^{\varepsilon}U_{kj}^{\varepsilon} = \sum f_i \lambda_i^2, \quad \lambda[U^{\varepsilon}] = (\lambda_1, \dots, \lambda_n).$$

Suppose $\gamma > 0$ and we have proved that there exists a constant independent of ε such that

$$(2.1) |u_{\varepsilon}|_{C^2(\bar{\Omega})} \le C.$$

Therefore, by the concavity of F,

$$F^{ij}[U_{\varepsilon}](A\delta_{ij} - U_{ij}^{\varepsilon}) \ge F[AI] - F[U_{\varepsilon}] \ge c_0 > 0$$

by fixing A sufficiently large. On the other hand, $-F^{ij}U_{ij}^{\varepsilon} \leq C \sum F^{ii}$ by (2.1). Then we get

$$\sum F^{ii} \ge \frac{c_0}{A+C} > 0.$$

Note that

$$\{\frac{\partial F}{\partial u_{ij}^\varepsilon}[U_\varepsilon]\}=\{F^{ij}[U_\varepsilon]\}+\gamma\sum F^{ii}I\geq \frac{\gamma c_0}{A+C}I.$$

Thus, there exists uniform constants $0 < \lambda_0 \le \Lambda_0 < \infty$ such that

$$\lambda_0 I \le \{ \frac{\partial F}{\partial u_{ij}^{\varepsilon}} [U_{\varepsilon}] \} \le \Lambda_0 I.$$

Hence Evans-Krylov theory (see [2] and [12]) assures a bound M independent of ε such that

$$|u_{\varepsilon}|_{C^{2,\alpha}(\bar{\Omega})} \leq M,$$

for some constant $\alpha \in (0,1)$. The higher regularity can be derived by the Schauder theory (see [3] for example). Using standard method of continuity, we can obtain the existence of smooth solution to (1.7). By sending ε to zero (taking a subsequence if necessary), we can prove Theorem 1.1.

In the following sections, we may drop the subscript ε when there is no possible confusion.

3. The gradient estimates

In this section, we consider the gradient estimates for the admissible solution to (1.7). We first observe that $\lambda[U] \in \Gamma \subset \{\sum \lambda_i > 0\}$ and therefore,

$$(3.1) tr[U] = (1 + n\gamma)\Delta u > 0.$$

Thus we have by the maximum principle that

$$u < u < h$$
 in $\bar{\Omega}$

where h is the harmonic function in Ω with $h = \varphi$ on $\partial \Omega$. Then we obtain

(3.2)
$$\sup_{\Omega} |u| + \sup_{\partial \Omega} |Du| \le C,$$

for some positive constant C independent of ε .

To establish the global gradient estimates, we assume that $|Du|e^{\phi}$ achieves a maximum at an interior point $x_0 \in \Omega$, where ϕ is a function to be determined. We may assume D^2u and $\{F^{ij}\}$ are diagonal at x_0 by rotating the coordinates if necessary. Then at x_0 where the function $\log |Du| + \phi$ attains its maximum, we have

$$\frac{u_k u_{ki}}{|Du|^2} + \phi_i = 0$$

and

$$\frac{u_k u_{kii} + u_{ki} u_{ki}}{|Du|^2} - 2 \frac{(u_k u_{ki})^2}{|Du|^4} + \phi_{ii} \le 0$$

for each $i = 1, \dots, n$. Differentiating the equation (1.7), we get, at x_0 ,

(3.5)
$$F^{ii}u_{kii} + \gamma \triangle u_k \sum F^{ii} = \psi_k + \varepsilon \eta' \psi_k.$$

It follows that

(3.6)
$$F^{ii}u_k u_{kii} + \gamma u_k \triangle u_k \sum F^{ii} \ge -C|Du|.$$

Note that

(3.7)
$$U_{ii}^{2} = (u_{ii} + \gamma \Delta u)^{2} \leq 2u_{ii}^{2} + 2\gamma^{2}(\Delta u)^{2} \leq 2u_{ii}^{2} + 2n\gamma^{2} \sum_{j} u_{jj}^{2}$$
$$\leq 2n \max\{\gamma, 1\}(u_{ii}^{2} + \gamma \sum_{j} u_{jj}^{2}).$$

Therefore, by (3.3), (3.4), (3.6) and (3.7), we have

(3.8)
$$c_0 F^{ii} U_{ii}^2 + |Du|^2 \Big(F^{ii} \phi_{ii} + \gamma \triangle \phi \sum F^{ii} \Big)$$
$$\leq C|Du| + 2|Du|^2 \Big(F^{ii} \phi_i^2 + \gamma |D\phi|^2 \sum F^{ii} \Big),$$

where $c_0 = (2n \max\{\gamma, 1\})^{-1}$.

Let $v = \underline{u} - u + \inf_{\bar{\Omega}} (u - \underline{u}) + 1$ and $\phi = \frac{\delta v^2}{2}$, where δ is a positive constant to be determined. Choosing δ sufficiently small, we can guarantee that

$$\delta - 2\delta^2 v^2 > 0$$
 on $\bar{\Omega}$.

Let $c_1 = \min_{x \in \bar{\Omega}} \left(\delta - 2\delta^2 v^2(x) \right) > 0$. It follows from (3.8) that

$$c_0 F^{ii} U_{ii}^2 + |Du|^2 \mathcal{L}(\underline{u} - u)$$

(3.9)
$$\leq C|Du| - (\delta - 2\delta^2 v^2)|Du|^2 \Big(F^{ii}v_i^2 + \gamma|Dv|^2 \sum F^{ii}\Big)$$

$$\leq C|Du| - c_1|Du|^2 \Big(F^{ii}v_i^2 + \gamma|Dv|^2 \sum F^{ii}\Big).$$

Write $\mu(x) = \lambda(D^2\underline{u}(x) + \gamma \Delta \underline{u}(x)I)$ and note that $\{\mu(x) : x \in \overline{\Omega}\}$ is a compact subset of Γ . There exists uniform constant $\beta \in (0, \frac{1}{2\sqrt{n}})$ such that

(3.10)
$$\nu_{\mu(x)} - 2\beta \mathbf{1} \in \Gamma_n, \ \forall x \in \bar{\Omega}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and $\nu_{\lambda} := Df(\lambda)/|Df(\lambda)|$ is the unit normal vector to the level hypersurface $\partial \Gamma^{f(\lambda)}$ for $\lambda \in \Gamma$. We need the following lemma proved by Guan in [7].

Lemma 3.1. For any fixed $x \in \bar{M}$, denote $\tilde{\mu} = \mu(x)$ and $\tilde{\lambda} = \lambda(U(x))$. Suppose that $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| \geq \beta$. Then there exists a uniform constant $\theta > 0$ such that

(3.11)
$$\sum f_i(\tilde{\lambda})(\tilde{\mu}_i - \tilde{\lambda}_i) \ge \theta \Big(1 + \sum f_i(\tilde{\lambda}) \Big).$$

Now let $\mu = \lambda(D^2\underline{u}(x_0) + \gamma \triangle \underline{u}(x_0))$, $\lambda = \lambda(D^2u(x_0) + \gamma \triangle u(x_0))$ and β as in (3.10). Suppose first that $|\nu_{\mu} - \nu_{\lambda}| \geq \beta$. Define the linear operator \mathcal{L} by

$$\mathcal{L}v := F^{ij}v_{ij} + \gamma \triangle v \sum F^{ii}$$

for $v \in C^2(\Omega)$. By Lemma 3.1,

(3.12)
$$\mathcal{L}(\underline{u} - u) \ge \theta \Big(\sum F^{ii} + 1 \Big)$$

for some $\theta > 0$. Thus, we can obtain a bound $|Du(x_0)| \leq C/\theta$ from (3.9).

We now consider the case $|\nu_{\mu} - \nu_{\lambda}| < \beta$ which implies $\nu_{\lambda} - \beta \mathbf{1} \in \Gamma_n$ and therefore

(3.13)
$$F^{ii} \ge \frac{\beta}{\sqrt{n}} \sum F^{kk}, \ \forall 1 \le i \le n.$$

By the concavity of F we know that

$$\mathcal{L}(u-u) > 0.$$

By the concavity of f again, when $|\lambda| \geq R$ for R sufficiently large, we derive as in [7]

(3.15)
$$|\lambda| \sum F^{ii} \ge f(|\lambda| \mathbf{1}) - f(\lambda) + \sum F^{ii} \lambda_i$$
$$\ge f(|\lambda| \mathbf{1}) - f(\mu) - |\lambda| \sum F^{ii}$$
$$\ge 2b_0 - |\lambda| \sum F^{ii}.$$

for some uniform positive constant b_0 . Therefore, by (3.13) and (3.15), we find

$$(3.16) c_0 F^{ii} U_{ii}^2 + c_1 |Du|^2 F^{ii} v_i^2 \ge \frac{\beta}{\sqrt{n}} \left(c_0 |\lambda|^2 \sum_i F^{ii} + \frac{c_1}{2} |Du|^4 \sum_i F^{ii} \right)$$

$$\ge \frac{\sqrt{2c_0 c_1} \beta}{\sqrt{n}} |Du|^2 |\lambda| \sum_i F^{ii}$$

$$\ge c_2 |Du|^2$$

provided |Du| is sufficiently large, where $c_2 = \frac{\sqrt{2c_0c_1}\beta b_0}{\sqrt{n}}$. Thus, from (3.9) and (3.16) we can get a bound $|Du| \leq C/c_2$.

Suppose $|\lambda| \leq R$. By the concavity of F, we have (see [9])

$$2R \sum F^{ii} \ge F^{ii}U_{ii} + F(2RI) - F(U) \ge -R \sum F^{ii} + b_1,$$

where $b_1 = F(2RI) - F(RI) > 0$. It follows that

$$(3.17) \sum F^{ii} \ge \delta_0 \equiv \frac{b_1}{3R}$$

and

$$c_1 |Du|^2 F^{ii} v_i^2 \ge \frac{c_1 \beta}{2\sqrt{n}} |Du|^4 \sum F^{ii} \ge \frac{c_1 \beta \delta_0}{2\sqrt{n}} |Du|^4$$

provided |Du| is sufficiently large. We then obtain from (3.9) that $|Du(x_0)| \le (2\sqrt{n}C/c_1\beta\delta_0)^{1/3}$.

Hence we have proved that

$$(3.18) |u|_{C^1(\bar{\Omega})} \le C$$

for some positive constant C independent of ε .

4. Interior and global estimates for second derivatives

In this section, we prove the interior second order estimate.

Theorem 4.1. Let $\gamma > 0$ and $u \in C^4(\Omega)$ be an admissible solution of (1.7). Then for any $\Omega' \subset\subset \Omega$, there exists a constant C depending on γ^{-1} , $d' \equiv \operatorname{dist}(\Omega', \partial\Omega)$, $|u|_{C^1(\Omega)}$ and other known data such that

$$(4.1) \sup_{\tilde{\Omega}'} |D^2 u| \le C.$$

Proof. Let

$$W = \max_{x \in \bar{\Omega}, |\xi| = 1} \zeta(x) e^{\phi(x)} D_{\xi\xi} u(x)$$

where ζ and ϕ are functions to be determined with ζ satisfying

$$(4.2) 0 \le \zeta \le 1, \quad |D\zeta| \le a_0, \quad |D^2\zeta| \le a_0 \quad \text{on } \bar{\Omega}.$$

Assume that W is achieved at $x_0 \in \Omega$ and $\xi_0 = e_1 = (1, 0, ..., 0)$. We may also assume that D^2u is diagonal at x_0 . We have, at x_0 where the function $\log u_{11} + \log \zeta + \phi$ attains its maximum,

(4.3)
$$\frac{u_{11i}}{u_{11}} + \frac{\zeta_i}{\zeta} + \phi_i = 0 \text{ for each } i = 1, \dots, n,$$

$$(4.4) F^{ii} \left\{ \frac{u_{11ii}}{u_{11}} - \left(\frac{u_{11i}}{u_{11}} \right)^2 - \left(\frac{\zeta_i}{\zeta} \right)^2 + \frac{\zeta_{ii}}{\zeta} + \phi_{ii} \right\} \le 0.$$

and

$$(4.5) \qquad \frac{\triangle u_{11}}{u_{11}} - \sum_{i} \left(\frac{u_{11i}}{u_{11}}\right)^2 - \sum_{i} \left(\frac{\zeta_i}{\zeta}\right)^2 + \frac{\triangle \zeta}{\zeta} + \triangle \phi \le 0.$$

Differentiating equation (1.7) twice, by the concavity of F, we obtain at x_0 ,

(4.6)
$$F^{ii}u_{ii11} + \gamma(\Delta u)_{11} \sum F^{ii} = \psi_{11} + \varepsilon(\eta'\psi_{11} + \eta''\psi_1^2) \ge -C.$$

Let

$$\phi = \frac{\delta |Du|^2}{2},$$

where $\delta > 0$ is a undetermined constant. By straightforward calculation, we have

$$\phi_i = \delta u_i u_{ii}$$

and

$$\phi_{ii} = \delta u_{ii}^2 + \delta u_j u_{jii}.$$

Note that

(4.7)
$$F^{ii}u_ju_{jii} + \gamma u_j \Delta u_j \sum F^{ii} = u_j(\psi_j + \varepsilon \eta'\psi_j) \ge -C$$

and

$$\phi_i^2 \le C\delta^2 u_{ii}^2.$$

We have

(4.9)
$$\mathcal{L}\phi \ge \delta F^{ii} u_{ii}^2 + \gamma \delta \sum u_{jj}^2 \sum F^{ii} - C\delta.$$

Combining (4.3)-(4.9), we get

(4.10)
$$0 \ge -\frac{C}{u_{11}} - C\delta + (\delta - C\delta^2) F^{ii} u_{ii}^2 + \gamma (\delta - C\delta^2) \sum_{i=1}^{\infty} u_{ij}^2 \sum_{i=1}^{\infty} F^{ii} - \frac{C}{\zeta^2} \sum_{i=1}^{\infty} F^{ii}.$$

Choose δ sufficiently small such that $\delta - C\delta^2 > 0$. Let $\lambda = \lambda(D^2u(x_0) + \gamma \triangle u(x_0))$. By (3.7), we find that

$$|\lambda|^2 = \sum U_{ii}^2 \leq 2n^2 \Big(\max\{\gamma,1\}\Big)^2 \sum u_{jj}^2.$$

Thus, it follows from (4.10) that

$$(4.11) 0 \ge -\frac{C}{u_{11}} - C\delta + 2c_3|\lambda|^2 \sum_{i} F^{ii} - \frac{C}{\zeta^2} \sum_{i} F^{ii},$$

where

$$c_3 = \frac{1}{4}(\delta - C\delta^2)\gamma n^{-2} \left(\max\{\gamma, 1\}\right)^{-2} > 0.$$

By (3.15) and (4.11), we have

$$(4.12) 0 \ge \left(b_0 c_3 |\lambda| - \frac{C}{u_{11}} - C\delta\right) + \left(c_3 |\lambda|^2 - Cb^2 - \frac{C}{\zeta^2}\right) \sum_{i=1}^{\infty} F^{ii}$$

provided $|\lambda|$ is sufficiently large. It follows that $|\lambda|\zeta(x_0) \leq C$.

The function ζ may now be chosen as a cutoff function satisfying $\zeta \equiv 1$ on $\Omega' \subset\subset \Omega$ and $|D\zeta| \leq C/d'$, $|D^2\zeta| \leq C/d'^2$. Then

$$|D^2u|\zeta \leq C \text{ on } \bar{\Omega}$$

and
$$(4.1)$$
 holds.

Remark 4.2. We remark that in the proof of Theorem 4.1 we do not need the existence of \underline{u} .

In the proof of Theorem 4.1, setting $\zeta \equiv 1$, we can prove the following maximal principle.

Theorem 4.3. Let $u \in C^4(\bar{\Omega})$ be an admissible solution of (1.7). Then

(4.13)
$$\sup_{\bar{\Omega}} |D^2 u| \le C(1 + \sup_{\partial \Omega} |D^2 u|),$$

where C depends on γ^{-1} , $|u|_{C^1(\bar{\Omega})}$ and other known data.

We are interested in the case that $\gamma = 0$. Now we prove (4.13) under the existence of \underline{u} satisfying (1.5) and we will see that the constant C would not depend on γ^{-1} when γ is small.

Theorem 4.4. Suppose (1.2)-(1.5) hold. Let $u \in C^4(\bar{\Omega})$ be an admissible solution of (1.7). Then we have

(4.14)
$$\sup_{\bar{\Omega}} |D^2 u| \le C \max\{\gamma, 1\} (1 + \sup_{\partial \Omega} |D^2 u|),$$

for some constant C depending on $|u|_{C^1(\bar{\Omega})}$, $|\underline{u}|_{C^2(\bar{\Omega})}$ and other known data. In particular, if $0 \leq \gamma \leq 1$, (4.13) holds for the constant C depending on $|u|_{C^1(\bar{\Omega})}$, $|\underline{u}|_{C^2(\bar{\Omega})}$ and other known data.

Proof. In the proof of Theorem 4.1, let $\zeta \equiv 1$ and $\phi = \frac{\delta}{2}|Du|^2 + b(\underline{u} - u)$, where δ and b are positive constants to be chosen. Note that

$$\phi_i = \delta u_i u_{ii} + b(\underline{u} - u)_i$$

and

$$\phi_{ii} = \delta u_{ii}^2 + \delta u_j u_{jii} + b(\underline{u} - u)_{ii}.$$

We have

$$\phi_i^2 \le C\delta^2 u_{ii}^2 + Cb^2.$$

Therefore, by (4.7),

(4.16)
$$\mathcal{L}\phi \ge \delta F^{ii} u_{ii}^2 + \gamma \delta \sum u_{jj}^2 \sum F^{ii} - C\delta + b\mathcal{L}(\underline{u} - u).$$

We can derive from (4.3)-(4.6) and (4.15) that

(4.17)
$$\mathcal{L}\phi \le \frac{C}{u_{11}} + C\delta^2(F^{ii}u_{ii}^2 + \gamma \sum u_{jj}^2 \sum F^{ii}) + Cb^2 \sum F^{ii}.$$

Combining (4.16) and (4.17), we obtain

$$(4.18) (\delta - C\delta^2)(F^{ii}u_{ii}^2 + \gamma \sum u_{jj}^2 \sum F^{ii}) + b\mathcal{L}(\underline{u} - u) \le C\delta + \frac{C}{u_{11}} + Cb^2 \sum F^{ii}.$$

We may assume that δ is sufficiently small such that $(\delta - C\delta^2) > \delta/2$. let $\mu = \lambda(D^2\underline{u}(x_0) + \gamma \Delta \underline{u}(x_0))$, $\lambda = \lambda(D^2u(x_0) + \gamma \Delta u(x_0))$. As in the gradient estimates, we consider two cases: (i) $|\nu_{\mu} - \nu_{\lambda}| < \beta$ and (ii) $|\nu_{\mu} - \nu_{\lambda}| \ge \beta$, where β is as in (3.10).

In case (i), we see that (3.13) holds. Thus, by (3.7), (4.18), (3.13) and (3.14), we have

$$(4.19) (2n\max\{\gamma,1\})^{-1} \frac{\delta\beta}{2\sqrt{n}} |\lambda|^2 \sum_{i=1}^{\infty} F^{ii} \leq C\delta + \frac{C}{u_{11}} + Cb^2 \sum_{i=1}^{\infty} F^{ii}.$$

We may assume $|\lambda| \geq R$ for R sufficiently large such that (3.15) holds. Therefore, by (3.15) and (4.19), we have

$$(4.20) (2n \max\{\gamma, 1\})^{-1} \left(\frac{\delta \beta}{4\sqrt{n}} |\lambda|^2 \sum_{i=1}^{\infty} F^{ii} + \frac{\delta \beta b_0}{4\sqrt{n}} |\lambda| \right) \leq C\delta + \frac{C}{u_{11}} + Cb^2 \sum_{i=1}^{\infty} F^{ii}.$$

It follows that

$$|\lambda| \le \frac{8n\sqrt{n}\max\{\gamma,1\}}{\delta\beta}\max\Big\{Cb, \frac{C\delta}{b_0} + \frac{C}{Rb_0}\Big\}.$$

Note that we do not determine δ and b right now.

In case (ii), we can choose b sufficiently small such that $\theta b > Cb^2$, where θ is as in (3.11). We can choose such b and a smaller δ such that $\theta b > C\delta$. Applying Lemma 3.1, we can derive from (4.18) that

$$u_{11} \le \frac{C}{\theta b - C\delta}$$

and (4.14) holds.

5. Boundary estimates for second derivatives

In this section we consider the estimates for the second order derivatives on the boundary $\partial\Omega$. As usual, the construction of barrier functions plays a key role. For any fixed $x_0 \in \Omega$, we may assume that x_0 is the origin of \mathbb{R}^n with the positive x_n axis in the interior normal direction to $\partial\Omega$ at the origin. Let d(x) be the distances from $x \in \overline{\Omega}$ to $\partial\Omega$, and set

$$\Omega_{\delta} = \{ x \in \Omega : |x| < \delta \}.$$

Suppose near the origin, the boundary $\partial\Omega$ is represented by

(5.1)
$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_{\alpha} x_{\beta} + O(|x'|^3)$$

for some C^{∞} smooth function ρ , where $x' = (x_1, ..., x_{n-1})$. For $x \in \partial \Omega$ near the origin, let

$$T_{\alpha} = T_{\alpha}(x) = \partial_{\alpha} + \sum_{\beta < n} B_{\alpha\beta}(x_{\beta}\partial_{n} - x_{n}\partial_{\beta}), \text{ for } \alpha < n$$

and $T_n = \partial_n$. We have (see [1])

$$\mathcal{L}T_{\alpha}u = T_{\alpha}(\psi + \varepsilon\eta(\psi)).$$

It follows that

$$(5.2) |\mathcal{L}T_{\alpha}(u-\varphi)| \le C\left(1+\sum F^{ii}\right)$$

and

(5.3)
$$|T_{\alpha}(u-\varphi)| \leq C|x|^2 \text{ on } \partial\Omega \cap \bar{\Omega}_{\delta} \text{ for } \alpha < n$$

when δ is sufficiently small since $u = \varphi$ on $\partial \Omega$.

To proceed we choose smooth unit orthonormal vector fields e_1, \ldots, e_n in Ω_{δ} such that when restricted to $\partial\Omega$, e_1, \ldots, e_{n-1} are tangential and e_n is normal to $\partial\Omega$. Let $e_i(x) = (\xi_1^i(x), \ldots, \xi_n^i(x))$, $\nabla_i u = \xi_k^i D_k u$, $\nabla_{ij} u = \xi_l^i \xi_k^j D_{kl} u$ and $\nabla^2 u = \{\nabla_{ij} u\}$ in Ω_{δ} . We may assume $\xi_i^i(0) = \delta_{ij}$. In particular, $\lambda(D^2 u) = \lambda(\nabla^2 u)$ and

$$f(\lambda(\nabla^2 u + \gamma \triangle uI)) = f(\lambda(D^2 u + \gamma \triangle uI)).$$

By straightforward calculations, we have

$$(5.4) |\mathcal{L}\nabla_k(u-\varphi)| \le C\Big(1+\sum_i F^{ii} + \sum_i f_i|\widehat{\lambda}_i| + \gamma|\widehat{\lambda}|\sum_i F^{ii}\Big),$$

where $\widehat{\lambda} = \lambda(D^2 u) = \lambda(\nabla^2 u)$. Let $\widehat{F}^{ij} = \xi_l^i \xi_k^j F^{kl}$. We see that $\{\widehat{F}^{ij}\}$ is positive definite with eigenvalues f_1, \ldots, f_n and when $\nabla^2 u$ is diagonal so is $\{\widehat{F}^{ij}\}$.

We shall use the following barrier function

$$(5.5) \ \Psi = \frac{1}{\delta^2} \Big(A_1(u - \underline{u}) + td - \frac{N}{2} d^2 + A_3 |x|^2 \Big) - A_2(u - \underline{u}) - A_4 \sum_{l \le n} |\nabla_l(u - \varphi)|^2,$$

where A_1 , A_2 , A_3 , A_4 , t and N are positive constants satisfying $A_1 > 2A_2$.

Theorem 5.1. Suppose that (1.2)-(1.5) hold. Let $h \in C(\bar{\Omega}_{\delta})$ satisfy $h \leq \overline{C}|x|^2$ on $\bar{\Omega}_{\delta} \cap \partial \Omega$ and $h \leq \overline{C}$ on $\bar{\Omega}_{\delta}$. Then for any positive constant K there exist uniform positive constants t, δ sufficiently small, and A_1, A_2, A_3, N sufficiently large such that $\Psi \geq h$ on $\partial \Omega_{\delta}$ and

(5.6)
$$\mathcal{L}\Psi \le -K\left(1 + \sum F^{ii}\right) \text{ in } \Omega_{\delta}.$$

Proof. Let $v = td - \frac{Nd^2}{2}$. Firstly, we note that

(5.7)
$$\mathcal{L}v = (t - Nd)F^{ij}(d_{ij} + \gamma \triangle d\delta_{ij}) - NF^{ij}(d_id_j + \gamma |Dd|^2\delta_{ij})$$
$$\leq C_0(t + Nd)\sum_i F^{ii} - NF^{ij}d_id_j - \gamma N\sum_i F^{ii}$$

since $|Dd| \equiv 1$. Similar to Proposition 2.19 in [6], we have

(5.8)
$$\widehat{F}^{ij} \nabla_{il} u \nabla_{jl} u \ge \frac{1}{2} \sum_{i \neq r} f_i \widehat{\lambda}_i^2$$

and

(5.9)
$$\sum_{l < n} \sum_{k=1}^{n} (\nabla_{lk} u)^2 \ge \frac{1}{2} \sum_{i \neq r} \widehat{\lambda}_i^2$$

for some index r. It follows from (5.4), (5.8) and (5.9) that

$$\sum_{l < n} \mathcal{L}|\nabla_{l}(u - \varphi)|^{2} = 2\sum_{l < n} F^{ij} \Big((\nabla_{l}(u - \varphi))_{i} (\nabla_{l}(u - \varphi))_{j}$$

$$+ \sum_{l < n} \gamma |D\nabla_{l}(u - \varphi)|^{2} \delta_{ij} \Big) + 2\nabla_{l}(u - \varphi)\mathcal{L}\nabla_{l}(u - \varphi)$$

$$\geq -C \Big(1 + \sum_{l < n} F^{ii} + \sum_{l < n} f_{i}|\widehat{\lambda}_{i}| + \gamma |\widehat{\lambda}| \sum_{l < n} F^{ii} \Big)$$

$$+ \sum_{l < n} \widehat{F}^{ij} \nabla_{il} u \nabla_{jl} u + \frac{\gamma}{2} \sum_{l < n} \sum_{k = 1}^{n} (\nabla_{lk} u)^{2} \sum_{l < n} F^{ii}$$

$$\geq \frac{1}{2} \sum_{i \neq r} f_{i} \widehat{\lambda}_{i}^{2} + \frac{\gamma}{4} \sum_{i \neq r} \widehat{\lambda}_{i}^{2} \sum_{l < n} f_{i}$$

$$-C \Big(1 + \sum_{l < n} F^{ii} + \sum_{l < n} f_{i} |\widehat{\lambda}_{i}| + \gamma |\widehat{\lambda}| \sum_{l < n} F^{ii} \Big).$$

For any $x \in \Omega_{\delta}$, let $\mu = \lambda(D^2\underline{u}(x) + \gamma \Delta\underline{u}(x)I)$, $\lambda = \lambda(D^2u(x) + \gamma \Delta u(x)I)$ and β be as in (3.10). We consider two cases: (i) $|\nu_{\mu} - \nu_{\lambda}| < \beta$ and (ii) $|\nu_{\mu} - \nu_{\lambda}| \ge \beta$. In case (i), we see that (3.13) holds. It follows that

(5.11)
$$\sum_{i \neq r} f_i \widehat{\lambda}_i^2 \ge \frac{\beta}{\sqrt{n}} \sum_{i \neq r} \widehat{\lambda}_i^2 \sum f_i$$

and

(5.12)
$$\mathcal{L}v \le -\frac{N\beta}{2\sqrt{n}} \sum f_i$$

provided t and δ are sufficiently small since $Dd \equiv 1$.

We first assume $|\lambda| \geq R$ for R sufficiently large. If $\hat{\lambda}_r \leq 0$, we have $\sum_{i \neq r} \hat{\lambda}_i > -\hat{\lambda}_r$ since $\Delta u > 0$. It follows that

$$\sum_{i \neq r} \widehat{\lambda}_i^2 \ge c_0 \widehat{\lambda}_r^2$$

for some unform constant $c_0 > 0$. Therefore, by (3.7), there exists unform positive constants c_1 and c_2 such that

(5.13)
$$\sum_{i \neq r} \widehat{\lambda}_i^2 \ge c_1 \sum_{i \neq r} \widehat{\lambda}_i^2 \ge c_2 |\lambda|^2.$$

Combining (3.13), (3.14), (3.15), (5.10), (5.12) and (5.13),

$$\mathcal{L}\Psi \leq -\frac{N\beta}{2\sqrt{n}\delta^{2}} \sum f_{i} + \frac{CA_{3}}{\delta^{2}} \sum f_{i} - \frac{\beta c_{2}}{2\sqrt{n}} A_{4} |\lambda|^{2} \sum f_{i}$$

$$+ CA_{4} \left(1 + \sum f_{i} + |\widehat{\lambda}| \sum f_{i}\right)$$

$$\leq -\frac{N\beta}{2\sqrt{n}\delta^{2}} \sum f_{i} - \frac{\beta c_{2}b_{0}}{4\sqrt{n}} A_{4} |\lambda| + \frac{CA_{3}}{\delta^{2}} \sum f_{i} + CA_{4} \left(1 + \sum f_{i}\right)$$

$$\leq \left(-\frac{N\beta}{2\sqrt{n}\delta^{2}} + \frac{CA_{3}}{\delta^{2}} + CA_{4}\right) \sum f_{i} - A_{4}$$

provided $|\lambda| > R$ and R is sufficiently large.

If $\hat{\lambda}_r > 0$, we have

(5.15)
$$\mathcal{L}u = F^{ij}u_{ij} + \gamma \triangle u \sum F^{ii} = \sum f_i \widehat{\lambda}_i + \gamma \sum \widehat{\lambda}_i \sum F^{ii} \\ \geq -\sum_{i \neq r} f_i |\widehat{\lambda}_i| - \gamma \sum_{i \neq r} |\widehat{\lambda}_i| \sum F^{ii} + f_r \widehat{\lambda}_r + \gamma \widehat{\lambda}_r \sum F^{ii}.$$

Note that for each $\sigma > 0$ and each $1 \le i \le n$,

$$(5.16) \widehat{\lambda}_i^2 \ge 2\sigma |\widehat{\lambda}_i| - \sigma^2.$$

It follows that

(5.17)
$$\frac{A_4}{2} \sum_{i \neq r} f_i \widehat{\lambda}_i^2 \ge 2A_2 \sum_{i \neq r} f_i |\widehat{\lambda}_i| - \frac{2A_2^2}{A_4} \sum_{i \neq r} f_i$$

by letting $\sigma = 2A_2/A_4$ and that

(5.18)
$$\frac{A_4}{4} \sum_{i \neq r} \hat{\lambda}_i^2 \ge 2A_2 \sum_{i \neq r} |\hat{\lambda}_i| - \frac{4A_2^2}{A_4}$$

by letting $\sigma = 4A_2/A_4$. Therefore, by (5.10), (5.15), (5.17) and (5.18), we find

$$\mathcal{L}\left(A_{2}(u-\underline{u}) + A_{4} \sum_{l < n} |\nabla_{l}(u-\varphi)|^{2}\right)$$

$$\geq A_{4}\left(\frac{1}{2} \sum_{i \neq r} f_{i} \widehat{\lambda}_{i}^{2} + \frac{\gamma}{4} \sum_{i \neq r} \widehat{\lambda}_{i}^{2} \sum f_{i}\right) - CA_{2} \sum f_{i}$$

$$+ A_{2}(f_{r} \widehat{\lambda}_{r} + \gamma \widehat{\lambda}_{r} \sum f_{i}) - A_{2}\left(\sum_{i \neq r} f_{i}|\widehat{\lambda}_{i}| + \gamma \sum_{i \neq r} |\widehat{\lambda}_{i}| \sum f_{i}\right)$$

$$- CA_{4}\left(1 + \sum_{i \neq r} F^{ii} + \sum_{i \neq r} f_{i}|\widehat{\lambda}_{i}| + \gamma|\widehat{\lambda}| \sum_{i \neq r} f_{i}\right)$$

$$\geq (A_{2} - CA_{4})\left(\sum_{i \neq r} f_{i}|\widehat{\lambda}_{i}| + \gamma|\widehat{\lambda}| \sum_{i \neq r} f_{i}\right) - CA_{4}\left(1 + \sum_{i \neq r} f_{i}\right)$$

$$- \left(CA_{2} + \frac{2A_{2}^{2}}{A_{4}} + \frac{4A_{2}^{2}}{A_{4}}\gamma\right) \sum_{i \neq r} f_{i}.$$

Therefore, by (3.13) and (3.15), we have

$$\mathcal{L}\Psi \leq -\frac{N\beta}{2\sqrt{n}\delta^{2}} \sum f_{i} - (A_{2} - CA_{4}) \Big(\sum f_{i} |\widehat{\lambda}_{i}| + \gamma |\widehat{\lambda}| \sum f_{i} \Big)
+ C \Big(A_{2} + \frac{A_{2}^{2}}{A_{4}} + \frac{A_{3}}{\delta^{2}} \Big) \sum f_{i} + CA_{4} \Big(1 + \sum f_{i} \Big)
\leq -\frac{N\beta}{2\sqrt{n}\delta^{2}} \sum f_{i} - \frac{\beta(A_{2} - CA_{4})}{\sqrt{n}} |\lambda| \sum f_{i}
+ C \Big(A_{2} + \frac{A_{2}^{2}}{A_{4}} + \frac{A_{3}}{\delta^{2}} \Big) \sum f_{i} + CA_{4} \Big(1 + \sum f_{i} \Big)
\leq -\frac{N\beta}{2\sqrt{n}\delta^{2}} \sum f_{i} - \frac{\beta(A_{2} - CA_{4})b_{0}}{\sqrt{n}}
+ C \Big(A_{2} + \frac{A_{2}^{2}}{A_{4}} + \frac{A_{3}}{\delta^{2}} \Big) \sum f_{i} + CA_{4} \Big(1 + \sum f_{i} \Big).$$

Now we assume $|\lambda| \leq R$. We see that (3.17) holds. Thus, by (5.12) and (5.10), we have

(5.21)
$$\mathcal{L}\Psi \leq -\frac{N\beta}{2\sqrt{n}\delta^2} \sum f_i + C\left(A_2 + \frac{A_3}{\delta^2} + A_4\right) \left(1 + \sum f_i\right) \\ \leq -\frac{N\beta}{4\sqrt{n}\delta^2} \sum f_i - \frac{N\beta\delta_0}{4\sqrt{n}\delta^2} + C\left(A_2 + \frac{A_3}{\delta^2} + A_4\right) \left(1 + \sum f_i\right).$$

Now we fix $A_3 > A_4 > K$ such that

(5.22)
$$A_3 > A_4 \sup_{\overline{\Omega}} \sum_{l < n} |\nabla_l (u - \varphi)|^2 + \overline{C}.$$

In case (ii), we see from Lemma 3.1 that (3.12) holds. We deal with two cases as before: $\hat{\lambda}_r > 0$ and $\hat{\lambda}_r \leq 0$.

If $\hat{\lambda}_r > 0$, similar to (5.20), we obtain

$$\mathcal{L}\Psi \leq -\theta \frac{A_1}{\delta^2} \left(1 + \sum f_i \right) + C_0(t + Nd) \sum f_i
+ \frac{CA_3}{\delta^2} \sum f_i + CA_4 \left(1 + \sum f_i \right) + C \left(A_2 + \frac{A_2^2}{A_4} \right) \sum f_i
- (A_2 - CA_4) \left(\sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i \right).$$

If $\hat{\lambda}_r \leq 0$, similar to (5.15),

$$(5.24) - \mathcal{L}u \ge -\sum_{i \ne r} f_i |\widehat{\lambda}_i| - \gamma \sum_{i \ne r} |\widehat{\lambda}_i| \sum_{i \ne r} F^{ii} - f_r \widehat{\lambda}_r - \gamma \widehat{\lambda}_r \sum_{i \ne r} F^{ii}.$$

Similar to (5.19), we have, for any B > 0,

$$\mathcal{L}\left(-B(u-\underline{u}) + A_4 \sum_{l < n} |\nabla_l(u-\varphi)|^2\right)$$

$$(5.25) \qquad \geq (B - CA_4)\left(\sum_{l < n} f_i|\widehat{\lambda}_i| + \gamma|\widehat{\lambda}|\sum_{l < n} f_i\right) - CA_4\left(1 + \sum_{l < n} f_i\right)$$

$$-\left(CB + \frac{2B^2}{A_4} + \frac{4B^2}{A_4}\gamma\right)\sum_{l < n} f_i.$$

Thus, choosing $B = A_2$, we can see from (3.12), (5.7) and (5.25) that

$$\mathcal{L}\Psi \leq -\theta \left(\frac{A_1}{\delta^2} - 2A_2\right) \left(1 + \sum f_i\right) + C_0(t + Nd) \sum f_i
+ \frac{CA_3}{\delta^2} \sum f_i + CA_4 \left(1 + \sum f_i\right) + C\left(A_2 + \frac{A_2^2}{A_4}\right) \sum f_i
- (A_2 - CA_4) \left(\sum f_i |\widehat{\lambda}_i| + \gamma |\widehat{\lambda}| \sum f_i\right).$$

Now we choose $A_2 \gg A_4$ such that

(5.27)
$$\frac{\beta(A_2 - CA_4)b_0}{\sqrt{n}} - CA_4 > K$$

in (5.20) and $A_2 - CA_4 > 0$ in (5.23) and (5.26). Next, we fix N sufficiently large such that

$$\frac{N\beta}{2\sqrt{n}\delta^2} - \frac{CA_3}{\delta^2} - CA_4 > K$$

in (5.14),

(5.29)
$$\frac{N\beta}{2\sqrt{n}\delta^2} - C\left(A_2 + \frac{A_2^2}{A_4} + \frac{A_3}{\delta^2}\right) - CA_4 > K$$

in (5.20) and

(5.30)
$$\frac{N\beta}{4\sqrt{n}\delta^2} \min\{1, \delta_0\} - C\left(A_2 + \frac{A_3}{\delta^2} + A_4\right) > K$$

in (5.21).

We may assume that t and δ is sufficiently small such that $\theta A_1/\delta^2 > 2C_0(t+Nd)$. Therefore, in case (ii), by (5.23) and (5.26), we have

(5.31)
$$\mathcal{L}\Psi \le \left(-\frac{A_1\theta}{2\delta^2} + \frac{CA_3}{\delta^2} + CA_2 + C\frac{A_2^2}{A_4} + CA_4\right) \left(1 + \sum f_i\right).$$

Finally, we may choose A_1 large enough to obtain (5.6). Furthermore, $v \geq 0$ in Ω_{δ} when $\delta \leq 2t/N$. Therefore, we can ensure $\Psi \geq h$ on $\partial \Omega_{\delta}$.

Now we are ready to establish the boundary estimates for second order derivatives. Firstly, it is easy to obtain a bound independent of ε for pure tangential second order derivatives on the boundary

$$(5.32) |u_{\xi\eta}|_{C^0(\partial\Omega)} \le C$$

from the boundary condition in (1.1), where ξ and η are unit tangential vector fields on $\partial\Omega$

For the estimates of mixed second order derivatives, we see from (5.2), (5.3) and (5.6) that

$$\mathcal{L}(\Psi \pm T_{\alpha}(u-\varphi)) \leq 0 \text{ in } \Omega_{\delta}$$

and

$$\Psi \pm T_{\alpha}(u-\varphi) \geq 0$$
 on $\partial \Omega_{\delta}$

for $\alpha < n$. It follows that

$$(5.33) |u_{\xi\nu}|_{C^0(\partial\Omega)} \le C,$$

where ξ is any unit tangential vector on $\partial\Omega$ and ν is the unit inner normal of $\partial\Omega$. It suffices to establish an upper bound for the double normal derivative on the boundary $\partial\Omega$ since $(1+n\gamma)\Delta u \geq 0$.

As in [11], let $T = \{T_i^j\}$ be a skew-symmetric matrix, such that e^T is orthogonal, where T_i^j is the entry of i^{th} row and j^{th} column of T. Let $\tau = (\tau_1, \ldots, \tau_n)$ be a vector field in Ω given by

$$\tau_i = T_i^j x_i + a_i, \quad i = 1, \dots, n,$$

where a_i is a constant. Denote $u_{\tau\tau} = \tau_i \tau_j u_{ij}$ and $u_{(\tau)(\tau)} = (u_{\tau})_{\tau} = \tau_i \tau_j u_{ij} + (\tau_i)_j \tau_j u_i$. Similar to Lemma 2.1 of [11] we can prove the following lemma.

Lemma 5.2. We have

$$\mathcal{L}(u_{(\tau)(\tau)}) \ge (F[U])_{(\tau)(\tau)}.$$

Proof. Similar to Lemma 2.1 of [11], by the skew-symmetry of T, we have

(5.34)
$$F^{ij}(T_i^k u_{kj\tau} + T_j^k u_{ki\tau}) = -F^{ij,st}(T_i^k u_{kj} + T_j^k u_{ki})U_{st\tau}$$

and

(5.35)
$$F^{ij}(2T_i^kT_j^lu_{kl} + T_i^kT_k^lu_{lj} + T_j^kT_k^lu_{li}) = -F^{ij,st}(T_i^ku_{ki} + T_i^ku_{ki})(T_s^ku_{kt} + T_t^ku_{ks}).$$

Note that

$$(u_{(\tau)(\tau)})_{ij} = u_{ij(\tau)(\tau)} - 2T_i^k u_{kj\tau} - 2T_j^k u_{ki\tau} + 2T_i^s T_j^t u_{st} + T_j^t T_t^s u_{si} + T_i^t T_t^s u_{sj}.$$

We find

(5.36)
$$\mathcal{L}(u_{(\tau)(\tau)}) = F^{ij} u_{ij(\tau)(\tau)} + \gamma \sum_{j} F^{ii} (\Delta u)_{(\tau)(\tau)} + F^{ij} \left(2T_i^s T_j^t u_{st} + T_j^t T_t^s u_{si} + T_i^t T_t^s u_{sj} - 2T_i^k u_{kj\tau} - 2T_j^k u_{ki\tau} \right) + \gamma \sum_{j} F^{ii} \left(2T_l^s T_l^t u_{st} + 2T_l^t T_t^s u_{sl} - 4T_l^k u_{kl\tau} \right).$$

Next, since T is skew-symmetric,

$$2T_l^s T_l^t u_{st} + 2T_l^t T_t^s u_{sl} - 4T_l^k u_{kl\tau} = 0.$$

It follows from (5.34), (5.35) and (5.36) that

$$\begin{split} \mathcal{L}(u_{(\tau)(\tau)}) &= F^{ij} u_{ij(\tau)(\tau)} + \gamma \sum_{j} F^{ii} (\Delta u)_{(\tau)(\tau)} \\ &- F^{ij,st} (T^k_i u_{kj} + T^k_j u_{ki}) (T^k_s u_{kt} + T^k_t u_{ks}) \\ &+ F^{ij,st} \Big((T^k_i u_{kj} + T^k_j u_{ki}) U_{st\tau} + (T^k_s u_{kt} + T^k_t u_{ks}) U_{ij\tau} \Big). \end{split}$$

Note that

$$(u_{\tau})_{ij} = u_{ij\tau} - T_i^k u_{kj} - T_j^k u_{ki}.$$

We obtain

$$\mathcal{L}(u_{(\tau)(\tau)}) = (F[U])_{(\tau)(\tau)} - F^{ij,st}(\gamma \delta_{ij}(\Delta u)_{\tau} + (u_{\tau})_{ij})(\gamma \delta_{st}(\Delta u)_{\tau} + (u_{\tau})_{st})$$

$$\geq (F[U])_{(\tau)(\tau)}.$$

Now we establish the double normal derivative estimates. We may assume the origin is a boundary point such that $e_n = (0, \dots, 0, 1)$ is the unit inner normal there and denote

$$M = \sup_{x \in \partial\Omega} D_{\nu\nu} u(x).$$

where ν is the unit inner normal of $\partial\Omega$ at $x\in\partial\Omega$. Without loss of generality, we assume

$$M = \sup_{\partial \Omega} |D^2 u|,$$

and

$$\sup_{\bar{\mathcal{O}}} |D^2 u| \le CM,$$

for some uniform constant $C \geq 1$. By Lemma 5.2, we have

(5.38)
$$\mathcal{L}(T_{\alpha}^{2}u) \geq T_{\alpha}^{2}(F[U]) = T_{\alpha}^{2}(\psi + \varepsilon \eta(\psi)) \geq -C.$$

According to [11], we see

(5.39)
$$w(x) \equiv T_{\alpha}^{2}u(x) - T_{\alpha}^{2}u(0) < C_{0}(|x'|^{2} + M|x'|^{4}) \equiv h(x')$$

for $x \in \partial \Omega$ with $|x'| \leq r_0$.

Let

$$\overline{\Psi} = \frac{1}{\delta^4} \left(A_1(u - \underline{u}) + td - \frac{N}{2} d^2 + A_3 |x|^4 \right) - A_2(u - \underline{u}) - \sum_{l \le n} |\nabla_l(u - \varphi)|^2.$$

Using the same arguments to Theorem 5.1, by (5.38), (5.37) and (5.39), we can show that there exists positive constants A_1 , A_2 , A_3 , N sufficiently large and t, δ sufficiently small such that

$$\mathcal{L}(w - h(x') - M\overline{\Psi}) \ge 0 \text{ in } \Omega_{\delta}$$

and

$$w - h(x') - M\overline{\Psi} \le 0 \text{ on } \partial\Omega_{\delta}.$$

It follows from the maximum principle that

(5.40)
$$w - h(x') - M\overline{\Psi} \le 0 \text{ on } \overline{\Omega}_{\delta}.$$

It follows that

(5.41)
$$w \le CM(u - \underline{u} + d) + CM|x|^4 + C \text{ on } \bar{\Omega}_{\delta}.$$

Therefore, for each small $\sigma > 0$, we can find a positive constant $\delta_1^4 = C\sigma < \delta^4$ such that

(5.42)
$$w \le CM(u - \underline{u} + d) + \frac{\sigma}{2}M + C \text{ on } \bar{\Omega}_{\delta_1}$$

and

(5.43)
$$\mathcal{L}h \leq (\sqrt{\sigma}M + C) \sum F^{ii} \text{ on } \bar{\Omega}_{\delta_1}.$$

Next, there exists a positive constant $\delta_2 < \delta_1$ such that

$$C(u - \underline{u} + d) \le \frac{\sigma}{2}$$
 on $\Omega - \widehat{\Omega}_{\delta_2}$,

where $\widehat{\Omega}_{\delta_2} \equiv \{x \in \Omega : \operatorname{dist}(x, \partial \Omega) > \delta_2\}$, since $|D(\underline{u} - u)| \leq C$ independent of ε . Hence we can derive from (5.42) that

(5.44)
$$w \le \sigma M + C \text{ on } \bar{\Omega}_{\delta_1} \cap (\Omega - \widehat{\Omega}_{\delta_2}).$$

On the other hand, by (4.1), there exists a positive constant C depending on γ^{-1} , δ_2 and $|u|_{C^1(\bar{\Omega})}$ such that

$$(5.45) |w| \le C in \widehat{\Omega}_{\delta_2}.$$

Thus, there exists a positive constant C_{σ} depending on σ and other known data such that

$$|w| \leq \sigma M + C_{\sigma}$$
 on $\bar{\Omega}_{\delta_1}$.

Similar to Theorem 5.1, we can find positive constants A_1 , A_2 , A_3 , t and N such that

$$\mathcal{L}(w - (\sigma M + C_{\sigma})\Psi') \geq 0 \text{ in } \Omega_{\delta_1}$$

and

$$w - (\sigma M + C_{\sigma})\Psi' \leq 0 \text{ on } \partial\Omega_{\delta_1} \cap \Omega,$$

where

$$\Psi' = \frac{1}{\delta_1^2} \left(A_1(u - \underline{u}) + td - \frac{N}{2}d^2 + A_3|x|^2 \right) - A_2(u - \underline{u}) - \sum_{l \le n} |T_l(u - \varphi)|^2.$$

Note that that main terms to control $\sum F^{ii}$ in Theorem 5.1 are $u - \underline{u}$ and $\frac{N}{2}d^2$. We may assume that σ is sufficiently small. Reviewing the proof of Theorem 5.1, we can find constants t' sufficiently small and N' sufficiently large such that

(5.46)
$$\mathcal{L}(u - \underline{u} + t'd - \frac{N'}{2}d^2) \le -\varepsilon_1 \sum_{i} F^{ii} \text{ in } \Omega_{\delta_1}$$

for some positive constant ε_1 and

(5.47)
$$u - \underline{u} + t'd - \frac{N'}{2}d^2 \ge 0 \text{ on } \bar{\Omega}_{\delta_1}.$$

By (5.43), (5.46) and (5.47), we can choose a constant A sufficiently large such that

$$\mathcal{L}(w - (\sigma M + C_{\sigma})\Psi' - A(\sqrt{\sigma}M + C)w - h) \ge 0 \text{ in } \Omega_{\delta_1}$$

and

$$w - (\sigma M + C_{\sigma})\Psi' - A(\sqrt{\sigma}M + C)w - h \leq 0 \text{ on } \partial\Omega_{\delta_1},$$

where $w = u - \underline{u} + t'd - \frac{N'}{2}d^2$. Thus, by the maximum principle again, we have

$$w \le (C\sqrt{\sigma}M + C_{\sigma})(u - \underline{u} + d + |x|^2) + h(x')$$
 on $\bar{\Omega}_{\delta_1}$

Therefore we obtain

(5.48)
$$(T_{\alpha}^{2}u)_{n}(0) \leq C\sqrt{\sigma}M + C_{\sigma} \text{ for each } \alpha < n.$$

It follows that

$$u_{n(\xi)(\xi)} \le C\sqrt{\sigma}M + C_{\sigma} \text{ on } \partial\Omega$$

for any tangential unit vector field ξ on $\partial\Omega$.

Now choose a new coordinate system and suppose the maximum M is attained at the origin $0 \in \partial\Omega$, and near the origin $\partial\Omega$ is given by (5.1). By the Taylor expansion, we have

$$u_n(x) \le u_n(0) + \sum_{\alpha < n} u_{n\alpha}(0)x_\alpha + (C\sqrt{\sigma}M + C_\sigma)|x'|^2$$

for $x \in \partial \Omega$ near the origin, where $u_{n\alpha}(0)$ is bounded by (5.33). Denote

$$g \equiv u_n(x) - u_n(0) - \sum_{\alpha \le n} u_{n\alpha}(0) x_{\alpha} - (C\sqrt{\sigma}M + C_{\sigma})|x'|^2.$$

In (5.5), we may choose another group of positive constants A_1 , A_2 , A_3 , t, N and δ such that

$$\mathcal{L}\left(g - (\sqrt{\sigma}M + C_{\sigma})\Psi\right) \ge 0 \text{ in } \Omega_{\delta}$$

and

$$g - (\sqrt{\sigma}M + C_{\sigma})\Psi \leq 0 \text{ on } \partial\Omega_{\delta}.$$

Applying the maximum principle again we obtain

$$M = u_{nn}(0) \le C\sqrt{\sigma}M + C_{\sigma}.$$

Choosing $\sqrt{\sigma} < 1/2C$, we get a bound $M \le C$ and (2.1) is proved.

Remark 5.3. We remark that in this paper, the condition that $\gamma > 0$ is only used to establish the interior estimate (4.1).

References

- L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of eigenvalues of the Hessians, Acta Math. 155 (1985), 261-301.
- [2] L. C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Applied Math. 35 (1982), 333-363.
- [3] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd Edition; Springer-Verlag: Berlin, 1983.
- [4] B. Guan, Conformal metrics with prescribed curvature function on manifolds with boundary, Amer. J. Math. 129 (2007), 915-942.
- [5] B. Guan, Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary, Int. Math. Res. Not. 2008, rnn105, 25pp. Addendum, IMRN 2009, 4354-4355, rnp166.
- [6] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, Duke Math. J. 163 (2014), 1491-1524.
- [7] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, arXiv:1403.2133.
- [8] B. Guan and H.-M. Jiao, The Dirichlet problem for Hessian type elliptic equations on Riemannian manifolds, Discrete Conti. Dyn. Syst. 36 (2016), 701-714.
- [9] B. Guan, S.-J. Shi and Z.-N. Sui, On estimates for fully nonlinear parabolic equations on Riemannian manifolds, Anal. PDE. 8 (2015), 1145-1164.
- [10] P.-F. Guan, N. S. Trudinger and X.-J. Wang, Dirichlet problems for degenerate Monge-Ampère equations, Acta. Math. 182 (1999), 87-104.
- [11] N. M. Ivochkina, N. S. Trudinger and X.-J. Wang The Dirichlet problem for degenerate Hessian equations, Comm. Partial Diff. Eqns. 29 (2004), 219-235.
- [12] N. V. Krylov, Boundedly nonhomogeneous elliptic and parabolic equations in a domain, Izvestia Math. Ser. 47 (1983), 75-108.
- [13] N. V. Krylov, Barriers for derivatives of solutions of nonlinear elliptic equations on a surface in Euclidean space, Comm. Partial Diff. Eqns. 19 (1984), 1909-1944.
- [14] N. V. Krylov, Weak interior second order derivative estimates for degenerate nonlinear elliptic equations, Diff. Int. Eqns 7 (1994), 133-156.
- [15] N. V. Krylov, A theorem on the degenerate elliptic Bellman equations in bounded domains, Diff. Int. Eqns. 8 (1995), 961-980.
- [16] N. V. Krylov, On the general notion of fully nonlinear second-order elliptic equations, Trans. Amer. Math. Soc. 347 (1995), 857-895.
- [17] N. S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995), 151-164.

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China $E\text{-}mail\ address$: jiao@hit.edu.cn

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China, $Present\ Address:\$ Jiuquan Satellite Launch Center, Jiuquan, 735000, China

E-mail address: ttwanghit@gmail.com