# Combinatorial applications of the special numbers and polynomials

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## Abstract

In this paper, by using some families of special numbers and polynomials with their generating functions, we give various properties of these numbers and polynomials. These numbers are related to the well-known numbers and polynomials, which are the Euler numbers, the Stirling numbers of the second kind, the central factorial numbers and the array polynomials. We also discuss some combinatorial interpretations of these numbers related to the rook polynomials and numbers. Furthermore, we give computation formulas for these numbers and polynomials.

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**Key Words.** Euler numbers; Central factorial numbers; Array polynomials; Stirling numbers; Generating functions; Binomial coefficients; Combinatorial sum.

#### 1. Introduction

The special numbers and their generating functions have many application in Combinatorial Number System and in Probability Theory. There are many advantage of the generating functions. By using generating functions for special numbers and polynomials, one can get not only various properties of these numbers and polynomials, but also enumerating arguments such as counting the number of subsets and the number of total ordering. In this paper, by using generating functions and their functional equations, we derive some identities and relations for the special combinatorial numbers such as the Stirling numbers of the first kind, the central factorial numbers, the Euler numbers, the array polynomials and the other special numbers. In order to give our results, we need some special numbers and polynomials with their generating functions.

The first kind Apostol-Euler polynomials of order k are defined by means of the following generating function:

$$F_{P1}(t, x; k, \lambda) = \left(\frac{2}{\lambda e^t + 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x; \lambda) \frac{t^n}{n!},$$
(1.1)

 $(|t| < \pi \text{ when } \lambda = 1 \text{ and } |t| < |\ln(-\lambda)| \text{ when } \lambda \neq 1), \ \lambda \in \mathbb{C}, \text{ the set of complex numbers, } k \in \mathbb{N}, \text{ the set of natural numbers. By (1.1), we easily see that}$ 

$$E_n^{(k)}(\lambda) = E_n^{(k)}(0;\lambda),$$

which denotes the first kind Apostol-Euler numbers of order k. By substituting  $k = \lambda = 1$  into (1.1), we have

$$E_n = E_n^{(1)}(1)$$

which denotes the first kind Euler numbers (cf. [4]-[23], and the references cited therein).

The second kind Euler numbers  $E_n^*$  of negative order are defined by means of the following generating function:

$$F_{E2}(t,k) = \left(\frac{2}{e^t + e^{-t}}\right)^{-k} = \sum_{n=0}^{\infty} E_n^{*(-k)} \frac{t^n}{n!},\tag{1.2}$$

where  $|t| < \frac{\pi}{2}$  (cf. [5], [19], and the references cited therein).

The  $\lambda$ -Stirling numbers of the second kind  $S_2(n, v; \lambda)$  defined by means of the following generating function:

$$F_S(t, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} = \sum_{n=0}^{\infty} S_2(n, v; \lambda) \frac{t^n}{n!},$$
(1.3)

where  $v \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$  (cf. [12], [16], [22], and the references cited therein).

By using (1.3), we have

$$S_2(n, v) = \frac{1}{v!} \sum_{j=0}^{v} {v \choose j} (-1)^{v-j} \lambda^j j^n$$

(cf. [12], [16], [22]).

Substituting  $\lambda = 1$  into (1.3), we have the Stirling numbers of the second kind  $S_2(n, v)$  which denotes the number of ways to partition a set of n objects into v groups:

$$S_2(n, v) = S_2(n, v; 1).$$

(cf. [1]-[23]; see also the references cited in each of these earlier works).

In [16], we defined the  $\lambda$ -array polynomials  $S_v^n(x;\lambda)$  by means of the following generating function:

$$F_A(t, x, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_v^n(x; \lambda) \frac{t^n}{n!},$$
(1.4)

where  $v \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$  (cf. [6], [4], [7], [16], [17], and the references cited therein).

The central factorial numbers T(n, k) (of the second kind) are defined by means of the following generating function:

$$F_T(t,k) = \frac{1}{(2k)!} \left( e^t + e^{-t} - 2 \right)^k = \sum_{n=0}^{\infty} T(n,k) \frac{t^{2n}}{(2n)!}$$
 (1.5)

(cf. [2], [9], [10], [17], [23], and the references cited therein).

**Remark 1.** The central factorial numbers are used in combinatorial problems. That is the number of ways to place k rooks on a size m triangle board in three dimensions is equal to

$$T(m+1, m+1-k),$$

where  $0 \le k \le m$  (cf. [1]).

In [19], we defined the numbers  $y_1(n,k;\lambda)$  by means of the following generating functions:

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!} \left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda) \frac{t^n}{n!},$$
(1.6)

where  $k \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$ . If we substitute  $\lambda = -1$  into (1.6), then we get the Stirling numbers of the second kind,  $S_2(n, k)$ :

$$S_2(n,k) = (-1)^k y_1(n,k;-1)$$

(cf. [19], [18]). The numbers  $y_1(n,k;\lambda)$  is related to following novel combinatorial sum:

$$B(n,k) = k! y_1(n,k;1) = \sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n} (e^t + 1)^k |_{t=0},$$
 (1.7)

where n = 1, 2, ... (cf. [11], [19]). In the work of Spivey [20, Identity 8-Identity 10], we see that

$$B(0,k) = 2^k, B(1,k) = k2^{k-1}, B(2,k) = k(k+1)2^{k-2},$$

and also

$$B(m,n) = \sum_{j=0}^{n} \binom{n}{j} j! 2^{n-j} S_2(m,j)$$
 (1.8)

(cf. [3, p.4, Eq-(7)], [19]; see also the references cited in each of these earlier works). In [19], we a conjecture and two open questions associated with the numbers B(n, k).

In [19], we defined the numbers  $y_2(n,k;\lambda)$  by means of the following generating functions:

$$F_{y_2}(t,k;\lambda) = \frac{1}{(2k)!} \left(\lambda e^t + \lambda^{-1} e^{-t} + 2\right)^k = \sum_{n=0}^{\infty} y_2(n,k;\lambda) \frac{t^n}{n!}.$$
 (1.9)

In [19], we gave some combinatorial interpretations for the numbers  $y_1(n, k)$ ,  $y_2(n, k)$  and B(n, k) as well as the generalization of the central factorial numbers. We see that these numbers were related to the rook numbers and polynomials.

### 2. Functional equations and related identities

By using generating functions for the Stirling numbers, the Euler numbers, the central factorial numbers, the array polynomials, the numbers  $y_1(n, k; \lambda)$  and the numbers  $y_2(n, k; \lambda)$  with their functional equations, we derive some identities and relations involving binomial coefficients and these numbers and polynomials. We also give computation formulas for the first kind and the second kind Euler numbers and polynomials.

By using (1.6) and (1.3), we obtain the following functional equation:

$$F_{y_1}(2t, k; -\lambda^2) = (-1)^k k! F_{y_1}(t, k; \lambda) F_S(t, k; \lambda).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} 2^n y_1 \left( n, k; -\lambda^2 \right) \frac{t^n}{n!} = (-1)^k k! \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} S_2 \left( n, k; \lambda \right) \frac{t^n}{n!}.$$

By using the Cauchy product in the above equation, we obtain

$$\sum_{n=0}^{\infty} 2^n y_1 \left( n, k; -\lambda^2 \right) \frac{t^n}{n!} = (-1)^k k! \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} S_2(l, k; \lambda) y_1(n-l, k; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation, we arrive the following theorem:

### Theorem 1.

$$y_1(n,k;-\lambda^2) = (-1)^k k! 2^{-n} \sum_{l=0}^n \binom{n}{l} S_2(l,k;\lambda) y_1(n-l,k;\lambda).$$

By combining (1.4) with (1.5) and (1.9), we obtain the following functional equation:

$$F_A(2t, -k, 2k; 1) = (2k)!F_T(t, k)F_{\nu_2}(t, k; 1).$$

Using the above equation, we get

$$\sum_{n=0}^{\infty} 2^n S_{2k}^n(-k) \frac{t^n}{n!} = (2k)! \sum_{n=0}^{\infty} T(n,k) \frac{t^{2n}}{(2n)!} \sum_{n=0}^{\infty} y_2\left(n,k;1\right) \frac{t^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2^n S_{2k}^n(-k) \frac{t^n}{n!} = (2k)! \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} T(j,k) y_2(n-l,k;1) \frac{t^{2n}}{(2n)!}.$$

By using the above equation, we arrive at the following theorem:

# Theorem 2.

$$S_{2k}^{2n}(-k) = (2k)!2^{-2n} \sum_{l=0}^{n} \binom{n}{l} T(l,k) y_2(n-l,k;1).$$

**Lemma 1.** ([14, Lemma 11, Eq-(7)])

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} A(n,n-2k),$$

where [x] denotes the greatest integer function.

By combining (1.4) and (1.5) with (1.6), we get the following functional equation:

$$F_T(t,k) = \frac{k!}{(2k)!} \sum_{l=0}^k \frac{(2l)!}{l!} F_T\left(\frac{t}{2},l\right) F_A\left(-\frac{t}{2},\frac{l}{2},k-l;1\right).$$

By using the above equation, we obtain

$$\begin{split} &\sum_{n=0}^{\infty} T(n,k) \frac{t^{2n}}{(2n)!} \\ &= &\frac{k!}{(2k)!} \sum_{l=0}^{k} \frac{(2l)!}{l!} \sum_{n=0}^{\infty} 2^{-2n} T(n,l) \frac{t^{2n}}{(2n)!} \sum_{n=0}^{\infty} S_{k-l}^{n} \left(\frac{l}{2},1\right) \frac{t^{n}}{n!}. \end{split}$$

By using Lemma 1, we get

$$\begin{split} &\sum_{n=0}^{\infty} T(n,k) \frac{t^{2n}}{(2n)!} \\ &= & \frac{k!}{(2k)!} \sum_{l=0}^{k} \frac{(2l)!}{l!} \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} T\left(j,l\right) S_{k-l}^{n-2j} \left(\frac{l}{2},1\right) \frac{2^{-2j}}{(2j)!} \frac{t^n}{(n-2j)!} \end{split}$$

Comparing the coefficients on both sides of the above equation, we arrive the following theorem:

**Theorem 3.** If n is an even integer, we have

$$T(n,k) = \frac{(2n)!k!}{(2k)!n!} \sum_{l=0}^{k} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} \frac{(2l)!}{2^{2j}l!} T(j,l) S_{k-l}^{n-2j} \left(\frac{l}{2},1\right)$$

and if n is an odd integer, we have

$$\sum_{l=0}^{k} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} \frac{(2l)!}{2^{2j}l!} T(j,l) S_{k-l}^{n-2j} \left(\frac{l}{2},1\right) = 0.$$

By combining (1.4) with (1.2), we obtain

$$F_T(2t,k) = \frac{2^{2k}}{(2k)!} \sum_{j=0}^{k} {k \choose j} (-1)^{k-j} F_{E2}(t,-2j).$$

By using the above functional equation, we get

$$\sum_{n=0}^{\infty} 2^n T(n,k) \frac{t^{2n}}{(2n)!} = \frac{2^{2k}}{(2k)!} \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_n^{*(-2j)} \frac{t^{2n}}{(2n)!}.$$

Comparing the coefficients of  $\frac{t^{2n}}{(2n)!}$  on both sides of the above equation, we arrive at the computation formula for the second kind Euler numbers of negative order which is given by the following theorem:

## Theorem 4.

$$T(n,k) = \frac{2^{2k-n}}{(2k)!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E_n^{*(-2j)}.$$

By using (1.9) and (1.3), we get

$$F_{y_2}(t,k;-\lambda) = \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^k F_S(t,j;\lambda) F_S(-t,k-j;\lambda^{-1}).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda) \frac{t^n}{n!} = \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^k \left( \sum_{n=0}^{\infty} S_2(n,j;\lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} S_2\left(n,k-j;\lambda^{-1}\right) \frac{(-t)^n}{n!} \right).$$

By using the Cauchy product in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{k!}{(2k)!} \sum_{j=0}^{k} \sum_{d=0}^{n} (-1)^{k+n-d} \binom{n}{d} S_2(d,j;\lambda) S_2(n-d,k-j;\lambda^{-1}) \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation, we arrive at the following theorem:

### Theorem 5.

$$y_2(n,k;\lambda) = \frac{k!}{(2k)!} \sum_{j=0}^{k} \sum_{d=0}^{n} (-1)^{k+n-d} \binom{n}{d} S_2(d,j;\lambda) S_2(n-d,k-j;\lambda^{-1}).$$

Computation formula for the first kind Euler polynomials of order -k is given by the following theorem:

#### Theorem 6.

$$y_2(n,k;\lambda) = \frac{2^n \lambda^{-k}}{(2k)!} \sum_{i=0}^k \binom{k}{j} E_n^{(-k)} \left(\frac{k}{2}; \lambda^2\right).$$

*Proof.* By using (1.9) and (1.1), we obtain the following functional equation:

$$F_{y_2}(t,k;\lambda) = \frac{\lambda^{-k}}{(2k)!} \sum_{j=0}^{k} {k \choose j} F_{P1}\left(2t,\frac{k}{2};j,\lambda^2\right).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \lambda^{-k}}{(2k)!} \sum_{i=0}^{k} \binom{k}{i} E_n^{(-k)} \left(\frac{k}{2};\lambda^2\right) \frac{t^n}{n!}.$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on both sides of the above equation, we arrive at the desired result.

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