SIMPLE LOOPS ON PUNCTURED SURFACES AND BOUNDARIES OF CHARACTER VARIETIES

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ABSTRACT. We prove that every relative moduli space of complex special linear local systems of rank two on a punctured surface is log Calabi-Yau, in that it has a normal projective compactification with anticanonical boundary divisor. We connect this to another result that the generating series for multicurves counted by word length on a punctured surface satisfies a universal symmetry.

1. INTRODUCTION

1.1. Let $S_{g,n}$ be a compact oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components and $\chi(S_{g,n}) = 2 - 2g - n < 0$. Let $X_{g,n}$ be the coarse moduli scheme of $\mathrm{SL}_2(\mathbf{C})$-local systems on $S_{g,n}$, also called its SL_2 character variety. The relative character variety $X_{g,n,k}$ for each $k = (k_1, \cdots, k_n) \in \mathbf{C}^n$ is the subvariety of $X_{g,n}$ obtained by fixing the traces of monodromy along the n boundary components of $S_{g,n}$. Our main result is that each $X = X_{g,n,k}$ is log Calabi-Yau.

Theorem 1.1. *Each X has a normal irreducible projective compactification $\overline{X}$ with canonical divisor K and reduced boundary divisor $D = \overline{X} \setminus X$ satisfying $K + D \sim 0$.*

In particular, each $X_{g,n,k}$ is normal and irreducible with trivial canonical divisor. Theorem 1.1 confirms a special case of the folklore conjecture that (relative) character varieties of surfaces are log Calabi-Yau; see remarks below. It is also closely related to the following combinatorial result about curves on surfaces. Consider the standard presentation of the fundamental group

$$\pi_1(S_{g,n}) = \langle a_1, \cdots, a_{2g+n} | [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] a_{2g+1} \cdots a_{2g+n} = \mathbf{1} \rangle.$$

Since $n \geq 1$ by assumption, $\pi_1(S_{g,n})$ is freely generated by $\sigma = \{a_1, \cdots, a_{2g+n-1}\}$. Given a curve $a \subset S_{g,n}$ (connected closed 1-submanifold not bounding a disk), let $\ell_\sigma(a)$ be the minimum σ-word length of any element $b \in \pi_1(S_{g,n})$ freely homotopic to a parametrization of a. We extend this notion additively to any multicurve in $S_{g,n}$ (a finite union of disjoint curves). We say that a multicurve is non-peripheral if no curve is isotopic to a component of $\partial S_{g,n}$. Let us consider the formal power series $Z_{g,n}(t) = \sum_{r=0}^{\infty} c_{g,n}(r) t^r$ where $c_{g,n}(r)$ is the number of isotopy classes of non-peripheral multicurves $a \subset S_{g,n}$ with $\ell_\sigma(a) = r$.

Theorem 1.2. *The series $Z_{g,n}(t)$ is rational, and satisfies the symmetry*

$$Z_{g,n}(1/t) = Z_{g,n}(t).$$

This gives a topological interpretation of the log Calabi-Yau property of the relative character varieties of punctured surfaces. Theorems 1.1 and 1.2 are related by the remarkable result of Charles-Marché [5] which states that the isotopy classes of multicurves in $S_{g,n}$ provide a **C**-linear basis for the coordinate ring of $X_{g,n}$.

1.2. We summarize the contents of our paper. In Section 2, we introduce the notion of a *word compactification* for the SL_2 character variety X_G of an arbitrary group G with a finite set of generators σ. In detail, we use the word length function on G to construct an increasing filtration Fil^σ on the coordinate ring of X_G, and take the projective scheme associated to the Rees algebra $\bigoplus_{r=0}^\infty \mathrm{Fil}_r^\sigma$. Applying this to the free group on $m \geq 2$ generators, we obtain from a result of Le Bruyn [18] on invariants of matrices that the Hilbert series $H_m(t) = \sum_{r=0}^\infty (\dim \mathrm{Fil}_r^\sigma) \, t^r$ of the Rees algebra for the free group satisfies the functional equation

$$H_m(1/t) = (-1)^{3m-2} t^{2m+1} H_m(t).$$

Using classical results of Hochster-Roberts [14] and Stanley [25], we deduce algebraic properties of the Rees algebra from this. In Section 3, we define our compactification of $X_{g,n,k}$, and use results of Demazure [6] and Watanabe [27] on normal graded domains to convert results obtained from Section 2 to geometry, proving Theorem 1.1. In Section 4, we supply the dimension analysis for singularities of representation varieties needed in Section 3. In Section 5, through Charles-Marché [5] we interpret the above functional equation in terms of multicurves to prove Theorem 1.2. In this spirit, in Section 6, we give another proof of Theorem 1.2 for punctured spheres combinatorially (without invariant theory); the case of higher genus surfaces can also be deduced from this case.

1.3. **Remarks.** For k integral, the varieties $X_{g,n,k}$ admit natural integral models. Our main motivation for Theorem 1.1 comes from its direct ramifications for the expected Diophantine analysis of the integral points; see [13] for a general discussion in the case of log K3 surfaces. For the Diophantine study of the case $(g,n) = (1,1)$, we refer to the work of Bourgain-Gamburd-Sarnak [1], [2].

The varieties $X = X_{g,n,k}$ are in general singular. The definition of a (possibly singular) log Calabi-Yau pair $(\overline{X}, D)$ varies somewhat across the literature (see for example [15], [4], [16], [11]), in the type of divisor D, type of triviality of $K + D$, and singularity type of the pair. Our notion coincides, for instance, with that of [4]: we do not impose *a priori* conditions on singularity, but require D to be an effective integral (reduced) Weil divisor and $K + D$ to be linearly trivial.

Given the explicit nature of our compactifications, the results of this paper may be relevant to conjectures surrounding the boundary divisors of relative character varieties for surfaces, as formulated by Simpson [24] (verified therein for punctured spheres; see also [17]), or more generally of log Calabi-Yau varieties, cf. [16]. We leave this interesting topic for a future investigation.

Sean Keel communicated to us that, for Fock-Goncharov moduli spaces of local systems [8] which are related to, but different from, our relative character varieties, log Calabi-Yau compactifications of open subspaces can be constructed by general techniques of cluster varieties established by Gross-Hacking-Keel [11]; see also [12] for an elaboration of this point.

We mention another interesting compactification of the SL_2 (or more general) character variety for free groups due to Manon [20], using different Rees algebras

associated to trivalent graphs, arising in the context of moduli spaces of principal bundles on closed Riemann surfaces and their degenerations.

From Theorem 1.2, the question naturally arises whether an analogous symmetry of the generating series holds for arbitrary compact surfaces, and for other types of generators. We also leave this for a separate investigation.

1.4. **Convention.** Throughout this paper, a ring or an algebra will always mean a commutative algebra with unity over the complex numbers $\mathbf{C}$, unless otherwise specified. All schemes will be defined over $\mathbf{C}$. Given an affine scheme X, we denote by $\mathbf{C}[X]$ its coordinate ring. Let $\mathbf{M}$ denote the affine scheme parametrizing 2×2 matrices. Let $\{x_{ij}\}_{i,j \in \{1,2\}}$ be the regular functions on $\mathbf{M}$ corresponding to the (i,j)th entries of a matrix, giving an identification $(x_{ij}) : \mathbf{M} \simeq \mathbf{A}^4$ with affine space. The standard matrix variable x and its adjugate x^* are the 2×2 matrices with coefficients in $\mathbf{C}[\mathbf{M}]$ given by

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \quad \text{and} \quad x^* = \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}.$$

In particular, $\det(x)$ and $\operatorname{tr}(x)$ are regular functions on $\mathbf{M}$. Let $^* : \mathbf{M} \to \mathbf{M}$ denote the involution on $\mathbf{M}$ determined by $x \mapsto x^*$.

Given 2×2 matrices a and b, we shall denote $\langle a, b \rangle = aba^*b^*$ and $[a, b] = ab - ba$. For t and $k \in \mathbf{C}$, let $\mathbf{M}_t \subset \mathbf{M}$ be the subscheme of matrices with determinant t, and let $\mathbf{M}_{t,k} \subset \mathbf{M}_t$ be the subscheme of matrices with determinant t and trace k. We shall denote $\mathrm{SL}_2 = \mathbf{M}_1$ and $\mathrm{SL}_{2,k} = \mathbf{M}_{1,k}$. Let $\mathbf{1}$ be the identity matrix. We shall say that a matrix a is *scalar* if $a = \lambda \mathbf{1}$ for some $\lambda \in \mathbf{C}$, and *nonscalar* otherwise. As usual, let $\mathfrak{sl}_2(\mathbf{C})$ denote the tangent space to SL_2 at $\mathbf{1}$.

Given an integer $m \geq 1$, consider $\mathfrak{C}_m = \mathbf{Z}/m\mathbf{Z}$ with its natural cyclic ordering. By a cyclic interval in $\mathfrak{C}_m$ we shall mean a sequence of the form $i_0, i_0 + 1, \cdots, i_0 + k$ for some $i_0 \in \mathfrak{C}_m$ and for some $k \in \{0, \cdots, m - 1\}$. If $|I| = k + 1 < m$, then those elements of $\mathfrak{C}_m$ not contained in I together form a cyclic interval, denoted I^c.

1.5. **Acknowledgements.** I thank Peter Sarnak, Sophie Morel, and Phillip Griffiths for guidance, helpful discussions, and remarks on earlier drafts of this paper. I also thank Carlos Simpson and Sean Keel for correspondences illuminating the literature, and Yuchen Liu for useful conversations.

Contents

2. WORD COMPACTIFICATIONS

2.1. **Graded rings.** Given an affine scheme X with an action by the multiplicative group $\mathbf{G}_m$, its coordinate ring $A = \mathbf{C}[X]$ also carries a $\mathbf{G}_m$-action, or equivalently the structure of a graded ring via the $\mathbf{G}_m$-eigenspace decomposition:

$$A = \bigoplus_{r \in \mathbf{Z}} A_r, \quad \text{where} \quad A_r = \{a \in A : \lambda \cdot a = \lambda^{-r} a \text{ for all } \lambda \in \mathbf{G}_m(\mathbf{C})\}.$$

Conversely, the spectrum $\operatorname{Spec} A$ of a graded ring A carries a natural $\mathbf{G}_m$-action. Given a morphism of graded rings $B \to A$, we have an associated $\mathbf{G}_m$-equivariant morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$, and given $b \in \operatorname{Spec} B(\mathbf{C})$ and $\lambda \in \mathbf{G}_m(\mathbf{C})$ the fibers $(\operatorname{Spec} A)_b$ and $(\operatorname{Spec} A)_{\lambda \cdot b}$ are isomorphic via λ.

Example 2.1. For the grading of the polynomial ring $\mathbf{C}[t]$ by degree, the associated $\mathbf{G}_m$-action on the affine line $\mathbf{A}^1 = \operatorname{Spec} \mathbf{C}[t]$ is the usual one $\lambda \cdot a = \lambda a$ on points.

Let A be a graded algebra. An element $t \in A_1$ is non-nilpotent if and only if the morphism $\mathbf{C}[t] \to A$ is injective, or in other words the associated $\mathbf{G}_m$-equivariant morphism of schemes $\pi : X = \operatorname{Spec} A \to \mathbf{A}^1$ is dominant. Let $X_0 = \operatorname{Spec} A/(t)$ and $X_1 = \operatorname{Spec} A/(t-1)$ be the fibers of π over 0 and 1, respectively, and let $X|_{\mathbf{G}_m} = \pi^{-1}(\mathbf{G}_m)$ where $\mathbf{G}_m = \operatorname{Spec} \mathbf{C}[t, 1/t] \subset \mathbf{A}^1$ is the complement of 0.

Lemma 2.2. *Let A be a graded ring, and let $t \in A_1$ be non-nilpotent as above. We have a natural $\mathbf{G}_m$-equivariant isomorphism*

$$X_1 \times \mathbf{G}_m \simeq X|_{\mathbf{G}_m},$$

equipping X_1 with the trivial $\mathbf{G}_m$-action. In particular, $A/(t-1) \simeq A[1/t]_0$.

Proof. The morphisms $X_1 \times \mathbf{G}_m \to X|_{\mathbf{G}_m}$ and $X|_{\mathbf{G}_m} \to X_1 \times \mathbf{G}_m$ given at the level of points by $(a, \lambda) \mapsto \lambda \cdot a$ and $a \mapsto ((\pi(a))^{-1} \cdot a, \pi(a))$, respectively, are inverses to each other and are $\mathbf{G}_m$-equivariant. This proves the first part of the lemma, and gives us an isomorphism of graded rings $A/(t-1) \otimes \mathbf{C}[t, 1/t] \simeq A[1/t]$. Taking the degree 0 parts of both sides, the second assertion follows. $\square$

Lemma 2.3. *Let A be a connected graded Cohen-Macaulay ring of pure dimension, and let $t \in A_1$ be non-nilpotent. Suppose that*

(1) *X_1 is a normal scheme, and*
(2) *X_0 is reduced of dimension $\dim A - 1$.*

Then A is a normal domain.

Proof. Since A is connected, if it is a normal ring then it is automatically a normal domain. Since A is Cohen-Macaulay, in order to show that A is normal it suffices by Serre's criterion for normality [21, Theorem 23.8, p.183] to show that A is regular in codimension 1.

Since t is not nilpotent, the $\mathbf{G}_m$-equivariant morphism $\pi : X \to \mathbf{A}^1$ is dominant, with isomorphic fibers of dimension $\dim A - 1$ over $\mathbf{G}_m$. By condition (2), in fact π has equidimensional fibers everywhere. By the miracle flatness theorem [21, Theorem 23.1, p.179], since A is Cohen-Macaulay and $\mathbf{C}[t]$ is regular it follows that A is flat over $\mathbf{C}[t]$. In particular, t is not a zero divisor in A.

By Lemma 2.2, we have $X|_{\mathbf{G}_m} \simeq X_1 \times \mathbf{G}_m$, which is regular in codimension 1 by condition (1). Thus, given any prime ideal $\mathfrak{p} \subset A$ of height ≤ 1 such that $t \notin \mathfrak{p}$, the localization $A_{\mathfrak{p}}$ is regular. Next, suppose that $\mathfrak{p} \subset A$ is a prime ideal of height 1

with $t \in \mathfrak{p}$. Since $t \in A$ is not a zero divisor, $A/(t)$ is Cohen-Macaulay of dimension $\dim A/(t) = \dim A - 1$. Hypothesis (2) implies that $(A/tA)_{\mathfrak{p}}$ is regular, and hence $A_{\mathfrak{p}}$ is regular. This proves the result. $\qquad\square$

2.2. **Rees algebras.** By a *filtered algebra* (A, Fil) we mean an algebra A and an increasing filtration Fil of A by complex vector spaces, indexed by $\mathbf{Z}_{\geq 0}$, such that:

(1) $\mathbf{C} \subseteq \mathrm{Fil}_0 A$, and
(2) $\mathrm{Fil}_r \cdot \mathrm{Fil}_s \subseteq \mathrm{Fil}_{r+s}$ for every $r, s \geq 0$.

A morphism of filtered algebras $(B, \mathrm{Fil}) \to (A, \mathrm{Fil})$ is a morphism of algebras $B \to A$ mapping Fil_r into Fil_r for every $r \geq 0$. Given a $\mathbf{Z}_{\geq 0}$-graded algebra A, we have an associated filtered algebra (A, Fil) where Fil is given by $\mathrm{Fil}_r A = \bigoplus_{s=0}^{r} A_s$.

Example 2.4. For the grading of $\mathbf{C}[t]$ by degree, or more generally the grading of a polynomial ring $\mathbf{C}[x_1, \cdots, x_N]$ in several variables by (total) degree, we shall denote by $\mathrm{Fil}^{\deg}$ the associated filtration.

Given a filtered algebra (A, Fil), the Rees algebra A^{Fil} is the graded algebra

$$A^{\mathrm{Fil}} = \bigoplus_{r=0}^{\infty} \mathrm{Fil}_r.$$

A morphism of filtered algebras $(B, \mathrm{Fil}) \to (A, \mathrm{Fil})$ induces a morphism $B^{\mathrm{Fil}} \to A^{\mathrm{Fil}}$ of Rees algebras. Given a filtered algebra (A, Fil), we shall denote by t the element of degree 1 in A^{Fil} corresponding to $1 \in \mathrm{Fil}_1 A = A_1^{\mathrm{Fil}}$. We have identifications

$$A^{\mathrm{Fil}}/(t-1) = A \quad \text{and} \quad A^{\mathrm{Fil}}/(t) = \mathrm{Gr}^{\mathrm{Fil}} A$$

where the latter preserves gradings. Note that $\mathrm{Spec}\, A^{\mathrm{Fil}}$ is connected if and only if $\mathrm{Spec}\, A_0^{\mathrm{Fil}}$ is connected. We have a natural identification $A = A^{\mathrm{Fil}}[1/t]_0$ by Lemma 2.2. Thus, if A^{Fil} is a finitely generated graded algebra, then $\mathrm{Proj}\, A^{\mathrm{Fil}}$ is a projective scheme containing $\mathrm{Spec}\, A$ as an open affine complement to the closed subscheme $\mathrm{Proj}\, \mathrm{Gr}^{\mathrm{Fil}} A$. For simplicity of notation, given a filtered algebra (A, Fil^x) where x is a symbol, we shall often write $A^x = A^{\mathrm{Fil}^x}$ for the Rees algebra.

Example 2.5. We have the following.

(1) Equipping $\mathbf{C}$ with the trivial filtration Fil given by $\mathrm{Fil}_r = \mathbf{C}$ for all $r \geq 0$, we have $\mathbf{C}^{\mathrm{Fil}} = \mathbf{C}[t]$ with grading by degree. For any filtered algebra (A, Fil), the obvious morphism $(\mathbf{C}, \mathrm{Fil}) \to (A, \mathrm{Fil})$ induces a $\mathbf{G}_m$-equivariant morphism of schemes $\pi : \mathrm{Spec}\, A^{\mathrm{Fil}} \to \mathbf{A}^1$.
(2) Given a filtered algebra (B, Fil) and a surjective algebra homomorphism $\varphi : B \to A$, we have an induced filtration Fil on A defined by $\mathrm{Fil}_r = \varphi(\mathrm{Fil}_r)$ for every $r \geq 0$ making φ a morphism of filtered algebras. We thus have an induced morphism $\varphi : B^{\mathrm{Fil}} \to A^{\mathrm{Fil}}$.
(3) For $(A = \mathbf{C}[x_1, \cdots, x_N], \mathrm{Fil}^{\deg})$ the polynomial ring with the filtration by degree, let $A^{\deg}$ be the Rees algebra. We then have an isomorphism $A^{\deg} \simeq \mathbf{C}[X_1, \cdots, X_N, T]$ given by

$$f(x_1, \cdots, x_N) \mapsto T^r f(X_1/T, \cdots, X_N/T)$$

for each $f \in \mathrm{Fil}_r^{\deg} A = A_r^{\deg}$. The projective space $\mathrm{Proj}\, A^{\deg} \simeq \mathbf{P}^N$ is the usual compactification of the affine space $\mathrm{Spec}\, A = \mathbf{A}^N$.

Lemma 2.6. *Given a filtered algebra* (A, Fil)*, we have the following.*

(1) A^{Fil} *is an integral domain if and only if A is.*
(2) *If A^{Fil} is a normal domain, then so is A.*

Proof. (1) If A^{Fil} is an integral domain, then so is $A = A^{\mathrm{Fil}}[1/t]_0$. Conversely, suppose that A is an integral domain. If $ab = 0$ in A^{Fil}, then replacing a and b by nonzero terms (if any) of highest degree we may assume that a and b are homogeneous of degree r and s, respectively. We then have $ab = 0 \in A^{\mathrm{Fil}}_{r+s} = \mathrm{Fil}_{r+s}$ within A, and hence $a = 0$ or $b = 0$. This proves that A^{Fil} is an integral domain.

(2) Given a normal $\mathbf{Z}$-graded domain B, its degree 0 component B_0 is also a normal domain. Indeed, it is obvious that B_0 is a domain; to see that B_0 is also normal, note that if $x \in B$ is integral over B_0 then we must have $x \in B_0$ by degree reasons. Applying this to $A = A^{\mathrm{Fil}}[1/t]_0$, we see that if A^{Fil} is a normal domain then so is A. $\qquad\square$

2.3. **Character varieties.** Let G be a finitely generated group. The SL_2 representation variety Rep_G is the complex affine scheme representing the functor

$$\mathrm{Rep}_G(A) = \mathrm{Hom}(G, \mathrm{SL}_2(A))$$

for every commutative algebra A. By functoriality, given a morphism of finitely generated groups $H \to G$ we have an induced morphism of schemes $\mathrm{Rep}_G \to \mathrm{Rep}_H$. For F_m a free group of rank m, we identify $\mathrm{Rep}_m = \mathrm{Rep}_{F_m}$ with SL_2^m. In general, given a presentation of G by m generators σ, we have a presentation of Rep_G as a closed subscheme $\sigma^* : \mathrm{Rep}_G \hookrightarrow \mathrm{Rep}_m = \mathrm{SL}_2^m$ cut out by equations arising from relations among the generators. The group SL_2 acts on Rep_G via conjugation. We define the character variety

$$X_G = \mathrm{Rep}_G \sslash \mathrm{SL}_2$$

of G as the geometric invariant theory quotient of this action. In other words, it is the spectrum of the ring of invariants $R_G = \mathbf{C}[\mathrm{Rep}_G]^{\mathrm{SL}_2}$. The set $X_G(\mathbf{C})$ parametrizes the isomorphism classes of semisimple representations $G \to \mathrm{SL}_2(\mathbf{C})$. The ring R_G admits a presentation (cf. [23, Theorem 7.1])

$$R_G = \mathbf{C}[[a], a \in G]/([\mathbf{1}] - 2, [a][b] - [ab] - [ab^{-1}])$$

where $[a]$ for each $a \in G$ is the regular function on X_G given by $\rho \mapsto \mathrm{tr}(\rho(a))$ for any representation ρ. Note that each element of R_G can be written as a finite linear combination $\sum_i \lambda_i[a_i]$ for some $\lambda_i \in \mathbf{C}$ and $a_i \in G$. We record the following properties of R_G, as found in Goldman [9, Sections 5.1 and 5.7], where the second part of Lemma 2.7 is attributed to Vogt [26].

Lemma 2.7. *We have the following.*

(1) *Given any $a, b, c \in G$, we have*

$$[abc] + [acb] = [ab][c] + [ac][b] + [bc][a] - [a][b][c], \quad and$$

$$\begin{aligned}[abc][acb] = {}&([a]^2 + [b]^2 + [c]^2) + ([ab]^2 + [bc]^2 + [ac]^2) \\ &- ([a][b][ab] + [b][c][bc] + [a][c][ac]) + [ab][bc][ac] - 4.\end{aligned}$$

(2) *Given any $a, b, c, d \in G$, we have*

$$\begin{aligned}2[abcd] = {}&[a][b][c][d] + [a][bcd] + [b][cda] + [c][dab] + [d][abc] \\ &+ [ab][cd] + [da][bc] - [ac][bd] \\ &- [a][b][cd] - [ab][c][d] - [d][a][bc] - [da][b][c].\end{aligned}$$

2.4. **Word compactifications.** Let G be a finitely generated group. Recall that a *length function* on G is a function $\ell : G \to \mathbf{R}_{\geq 0}$ satisfying the conditions:

(1) $\ell(\mathbf{1}) = 0$,
(2) $\ell(g^{-1}) = \ell(g)$ for every $g \in G$, and
(3) $\ell(gh) \leq \ell(g) + \ell(h)$ for every $g, h \in G$.

A length function equips G with a pseudo-metric via the formula $d_\ell(g, h) = \ell(gh^{-1})$. Given a generating set $\sigma \subseteq G$, we have the σ-word length function $\ell_\sigma : G \to \mathbf{Z}_{\geq 0}$ which is given by $\ell_\sigma(a) = \min\{r : a = w_1 \cdots w_r \text{ for some } w_i \in \sigma \cup \sigma^{-1}\}$. Let Fil^σ be the increasing filtration of the coordinate ring R_G of the character variety of G given by

$$\mathrm{Fil}_r^\sigma R_G = \mathrm{Span}\{[a] : \ell_\sigma(a) \leq r\}.$$

We have $\mathbf{C} = \mathbf{C} \cdot [\mathbf{1}] = \mathrm{Fil}_0 R_G$, and the relations $[a][b] = [ab] + [ab^{-1}]$ on R_G show that $\mathrm{Fil}_r^\sigma \cdot \mathrm{Fil}_s^\sigma \subseteq \mathrm{Fil}_{r+s}^\sigma$ for every $r, s \in \mathbf{Z}_{\geq 0}$. Thus, $(R_G, \mathrm{Fil}^\sigma)$ is a filtered algebra. Let R_G^σ denote the associated Rees algebra.

Definition 2.8. The σ-*word compactification* of X_G is the scheme $X_G^\sigma = \mathrm{Proj}\, R_G^\sigma$.

Lemma 2.9. *Let G be a group generated by a finite set σ.*

(1) *We have $(R_G^\sigma)_0 = \mathbf{C}$. For each $r \geq 1$, we have $\dim_{\mathbf{C}}(R_G^\sigma)_r < \infty$.*
(2) *R_G^σ is finitely generated by homogeneous elements of degree ≤ 3.*

In particular, X_G^σ is projective and can be presented as a closed subscheme of some weighted projective space $\mathbf{P}(w_1, \cdots, w_s)$ with $w_i \in \{1, 2, 3\}$.

Proof. (1) We have $\ell_\sigma(g) = 1$ if and only if $g = \mathbf{1}$, and there are at most finitely many σ-words of length $\leq r$ in G for any given r.

(2) Applying part (2) of Lemma 2.7 and induction, we see that the class $[w] \in R_G$ of every word w of length $r \geq 4$ in G is a linear combination of terms of the form $[w_1] \cdots [w_s]$ with $\sum_{i=1}^s \ell_\sigma(w_i) \leq r$ and $\ell_\sigma(w_i) \leq 3$ for each $i = 1, \cdots, s$. This shows that R_G^σ is generated by $(R_G^\sigma)_1 \oplus (R_G^\sigma)_2 \oplus (R_G^\sigma)_3$. Applying part (1) of this lemma, we obtain the desired result. $\square$

2.5. **An alternate description.** Following the notation from Section 1.4, let $\mathbf{M}$ be the scheme parametrizing 2×2 matrices. Using the isomorphism $(x_{ij}) : \mathbf{M} \simeq \mathbf{A}^4$ via matrix entries, we equip $\mathbf{C}[\mathbf{M}^m]$ for each $m \geq 1$ with grading by degree and associated filtration $\mathrm{Fil}^{\mathrm{deg}}$. We have SL_2 acting on $\mathbf{M}^m$ by simultaneous conjugation, preserving the grading (and hence the filtration) on $\mathbf{C}[\mathbf{M}^m]$. Therefore, SL_2 acts on the Rees algebra $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$ of $(\mathbf{C}[\mathbf{M}^m], \mathrm{Fil}^{\mathrm{deg}})$ preserving the grading. By Example 2.5.(3), we have an identification

$$\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}} = \mathbf{C}[(X_1)_{ij}, \cdots, (X_m)_{ij}, T]$$

with the polynomial ring on $4m+1$ generators, so that $\mathrm{Spec}\, \mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}} = \mathbf{M}^m \times \mathbf{A}^1$. The SL_2-action is the product of the conjugation action on $\mathbf{M}^m$ and the trivial action on $\mathbf{A}^1$. The morphism $\pi : \mathbf{M}^m \times \mathbf{A}^1 \to \mathbf{A}^1$ from Example 2.5.(1) is just the projection onto the second factor.

Let G be a group with finite generating set σ and $|\sigma| = m$. The choice of σ gives us a SL_2-equivariant closed immersion $\sigma^* : \mathrm{Rep}_G \hookrightarrow \mathrm{Rep}_m = \mathrm{SL}_2^m \subset \mathbf{M}^m$. Via the surjective morphism $\mathbf{C}[\mathbf{M}^m] \to \mathbf{C}[\mathrm{Rep}_G]$, the filtration $\mathrm{Fil}^{\mathrm{deg}}$ on $\mathbf{C}[\mathbf{M}^m]$ induces a filtration on $\mathbf{C}[\mathrm{Rep}_G]$ which we shall denote Fil^σ.

Proposition 2.10. *We have* $\mathrm{Fil}_r^\sigma R_G = (\mathrm{Fil}_r^\sigma \mathbf{C}[\mathrm{Rep}_G])^{\mathrm{SL}_2}$ *for every* $r \geq 0$. *In particular, there is a surjective morphism of graded rings*

$$\Phi_\sigma : (\mathbf{C}[\mathbf{M}^m]^{\deg})^{\mathrm{SL}_2} \to R_G^\sigma.$$

Proof. We begin by considering the filtration $\mathrm{Fil}^{\deg}$ on $\mathbf{C}[\mathbf{M}^m]$. For each $r \geq 0$, a close inspection of the work of Procesi [22] (Theorems 1.1, 1.2, and 1.3 *loc.cit.*) shows that the space of invariants $(\mathbf{C}[\mathbf{M}^m]_r)^{\mathrm{SL}_2}$ is spanned by elements of the form

$$\mathrm{tr}(w_1) \, \mathrm{tr}(w_2) \cdots \mathrm{tr}(w_s)$$

where each w_i is a noncommutative monomial in the standard matrix variables $x_1, \cdots, x_m$ on $\mathbf{M}^m$ such that we have $\sum_{i=1}^s \deg(w_i) = r$. Here, the degree $\deg(w_i)$ refers to the total multiplicity of matrices $x_1, \cdots, x_m$ appearing in the monomial w_i. Since SL_2 is linearly reductive, we see that the map

$$(\mathrm{Fil}_r^{\deg} \mathbf{C}[\mathbf{M}^m])^{\mathrm{SL}_2} \to (\mathrm{Fil}_r^\sigma \mathbf{C}[\mathrm{Rep}_G])^{\mathrm{SL}_2},$$

obtained from the surjective morphism $\mathrm{Fil}_r^{\deg} \mathbf{C}[\mathbf{M}^m] \to \mathrm{Fil}_r^\sigma \mathbf{C}[\mathrm{Rep}_G]$ of finite-dimensional complex representations of SL_2, remains surjective. In particular, $(\mathrm{Fil}_r^\sigma \mathbf{C}[\mathrm{Rep}_G])^{\mathrm{SL}_2}$ is spanned by elements of the form

$$[w_1][w_2] \cdots [w_s]$$

where each w_i is a product of elements of σ (with multiplicities, but without inverses) such that the sum total number of elements appearing is $\leq r$. This shows that $(\mathrm{Fil}_r^\sigma \mathbf{C}[\mathrm{Rep}_G])^{\mathrm{SL}_2} \subseteq \mathrm{Fil}^\sigma R_G$ since each element of the above form lies in $\mathrm{Fil}^\sigma R_G$, recalling that the filtration Fil^σ is compatible with multiplication on R_G. The other containment $\mathrm{Fil}^\sigma R_G \subseteq (\mathrm{Fil}^\sigma \mathbf{C}[\mathrm{Rep}_G])^{\mathrm{SL}_2}$ follows from the observation that, using the relation $[a][b] = [ab] + [ab^{-1}]$ in R_G, one can write any $[a]$ with $\ell_\sigma(a) \leq r$ as a linear combination of elements of the above form. Finally, since the surjection $\mathbf{C}[\mathbf{M}^m] \to \mathbf{C}[\mathrm{Rep}_G]$ is compatible with the filtrations and the SL_2-actions, we obtain a surjection

$$(\mathbf{C}[\mathbf{M}^m]^{\deg})^{\mathrm{SL}_2} \to (\mathbf{C}[\mathrm{Rep}_G]^\sigma)^{\mathrm{SL}_2} = (\mathbf{C}[\mathrm{Rep}_G]^{\mathrm{SL}_2})^\sigma = R_G^\sigma$$

which proves the second part of the proposition. $\square$

2.6. **Free groups.** Let F_m be the free group on $m \geq 1$ generators σ. As before, let $\mathrm{Rep}_m = \mathrm{SL}_2^m$ be the representation variety of F_m, and let $X_m = \mathrm{Rep}_m /\!\!/ \mathrm{SL}_2$ be the SL_2 character variety of F_m with $R_m = \mathbf{C}[\mathrm{Rep}_m]^{\mathrm{SL}_2}$.

Example 2.11. We have the following examples, following Goldman [9].

(1) Let F_1 the free group of rank 1 generated by a. We have

$$[a] : X_1 \simeq \mathbf{A}^1.$$

(2) Let F_2 be the free group of rank 2 on generators $\{a, b\}$. By a result of Fricke (see Goldman [9, Section 2.2] for details), we have an isomorphism

$$([a], [b], [ab]) : X_2 \simeq \mathbf{A}^3.$$

(3) Let F_3 be the free group of rank 3 on generators $\{a, b, c\}$. The coordinate ring R_3 of X_3 is the quotient of the polynomial ring on 8 variables

$$\mathbf{C}[[a], [b], [c], [ab], [bc], [ac], [abc], [acb]]$$

by the ideal generated by two elements

$$[abc] + [acb] - ([ab][c] + [ac][b] + [bc][a] - [a][b][c]), \quad \text{and}$$
$$[abc][acb] - (([a]^2 + [b]^2 + [c]^2) + ([ab]^2 + [bc]^2 + [ac]^2)$$
$$- ([a][b][ab] + [b][c][bc] + [a][c][ac]) + [ab][bc][ac] - 4)$$

arising in Lemma 2.7. In particular, X_3 is finite of degree 2 over the affine 6-space $\mathbf{A}^6$ under the projection to the first 6 variables listed above.

Let Fil^σ be the filtration on the coordinate ring $\mathbf{C}[\mathrm{Rep}_m]$ of $\mathrm{Rep}_m = \mathrm{SL}_2^m$ defined in Section 2.4. In other words, it is the filtration obtained by the projection of $\mathrm{Fil}^{\mathrm{deg}}$ under the surjective ring homomorphism $\mathbf{C}[\mathbf{M}^m] \to \mathbf{C}[\mathrm{Rep}_m]$. For $i \in \{1, \cdots, m\}$, let us write $E_i = \det(X_i) - T^2$ for the homogeneous element in $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$ of degree 2, with the identification $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}} = \mathbf{C}[(X_1)_{ij}, \cdots, (X_m)_{ij}, T]$ from Section 2.5.

Lemma 2.12. *We have the following.*
  (1) $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_m)$ *is a normal graded Cohen-Macaulay domain.*
  (2) $E_1, \cdots, E_m$ *is a regular sequence in* $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$.
  (3) *We have* $\mathbf{C}[\mathrm{Rep}_m]^\sigma \simeq \mathbf{C}[\mathbf{M}^m]/(E_1, \cdots, E_m)$.

Proof. Let us write $\mathrm{Rep}'_m = \mathrm{Spec}\, \mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_m)$. Our proof of (1) and (2) will proceed by induction on m. Both statements are clear when $m = 1$ (we note by the Jacobian criterion that the singular locus of Rep'_1 consists of a single point). So let $m \geq 2$. For any $s \in \{1, \cdots, m\}$, note that we have

$$\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_s) = \mathbf{C}[\mathrm{Rep}'_s] \otimes \mathbf{C}[(X_{s+1})_{ij}, \cdots, (X_m)_{ij}].$$

In particular, each $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_s)$ is an integral domain, showing that $E_1, \cdots, E_m$ is a regular sequence in $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$, and $\mathbf{C}[\mathrm{Rep}'_m]$ is a graded Cohen-Macaulay ring of pure dimension $3m + 1$. Consider the $\mathbf{G}_m$-equivariant morphism

$$\pi : \mathrm{Rep}'_m \to \mathbf{A}^1$$

associated to the morphism of graded rings $\mathbf{C}[t] \to \mathbf{C}[\mathrm{Rep}'_m]$ given by $t \mapsto T$. The fiber $\pi^{-1}(1) = \mathrm{SL}_2^m$ is smooth, and $\pi^{-1}(0) = \mathbf{M}_0^m$ is reduced of dimension $3m$. We thus conclude by Lemma 2.3 that $\mathbf{C}[\mathrm{Rep}'_m]$ is a normal domain. This completes the induction, and we have proven (1) and (2).

It remains to prove (3). Note that we have a natural surjective ring homomorphism $\mathbf{C}[\mathrm{Rep}'_m] \to \mathbf{C}[\mathrm{Rep}_m]^\sigma$. Note that $\mathbf{C}[\mathrm{Rep}_m]^\sigma$ is integral by Lemma 2.6, and has dimension at least $3m + 1$. Thus, we see that $\mathbf{C}[\mathrm{Rep}'_m] \to \mathbf{C}[\mathrm{Rep}_m]^\sigma$ must be an isomorphism, as the former is also integral of dimension $3m + 1$. This completes the proof of the lemma. □

Let Fil^σ be the σ-word filtration on R_m defined in Section 2.5, and let R_m^σ be the associated Rees algebra.

Proposition 2.13. *The ring R_m^σ is a normal graded Cohen-Macaulay domain.*

Proof. First, by a result of Hochster-Roberts [14, Main Theorem], we see that the ring $(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2}$ is Cohen-Macaulay. We next claim that the surjective homomorphism $\Phi_\sigma : (\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2} \to R_m^\sigma$ constructed in Proposition 2.10 induces an isomorphism

$$(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2}/(E_1, \cdots, E_m) \simeq R_m^\sigma.$$

Indeed, since $(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2}$ is a pure subring of $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$, the left hand side is isomorphic to $(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_m))^{\mathrm{SL}_2} = (\mathbf{C}[\mathrm{Rep}_m]^{\mathrm{deg}})^{\mathrm{SL}_2} = R_m^\sigma$ by Lemma 2.12, which is the desired result. Since $\mathbf{C}[\mathrm{Rep}_m]^{\mathrm{deg}}$ is a normal graded domain by Lemma 2.12, so is R_m^σ. To show that R_m^σ is Cohen-Macaulay, it suffices to show that $E_1, \cdots, E_m$ is a regular sequence in $(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2}$. But this follows from the fact that $E_1, \cdots, E_m$ is a regular sequence in $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$ and that $(\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}})^{\mathrm{SL}_2}$ is a pure subring of $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$. The desired result follows. $\square$

The Hilbert series $H_m(t)$ of the graded algebra R_m^σ is by definition the following formal power series. Note that $H_m(t)$ is well-defined by Lemma 2.9.

$$H_m(t) = \sum_{r=0}^\infty (\dim_{\mathbf{C}}(R_m^\sigma)_r)\, t^r = \sum_{r=0}^\infty (\dim_{\mathbf{C}} \mathrm{Fil}_r^\sigma R_m)\, t^r \in \mathbf{Z}[[t]].$$

Theorem 2.14. *Assume that $m \geq 2$. Then $H_m(t)$ is rational and satisfies*

$$H_m(1/t) = (-1)^{3m-2} t^{2m+1} H_m(t).$$

In particular, R_m^σ is Gorenstein with canonical module $R_m^\sigma(-2m-1)$.

Proof. For $m = 2$ and F_2 a free group on generators $\sigma = \{a_1, a_2\}$, we may verify the functional equation by direct computation as follows. Recall the isomorphism $([a_1], [a_2], [a_1a_2]) : X_2 \simeq \mathbf{A}^3$ from Example 2.11. It follows $R_2^\sigma \simeq \mathbf{C}[X_1, X_2, X_3, T]$ with each X_1, X_2, T given degree 1 and X_3 given degree 2, so that the Krull dimension of R_2^σ is 4 and

$$H_2(t) = \frac{1}{(1-t)^3(1-t^2)}$$

from which the desired symmetry follows. For $m \geq 3$, we have the following result of Le Bruyn [18] (based on rational expressions due to Weyl and Schur, cf. *loc.cit.*): the Hilbert series $h_m(t)$ of the graded algebra $\mathbf{C}[\mathbf{M}^m]^{\mathrm{SL}_2}$ has the functional equation $h_m(1/t) = -t^{4m}h_m(t)$. Our Hilbert series $H_m(t)$ is related to the series $h_m(t)$ by

$$H_m(t) = \frac{(1-t^2)^m}{1-t} h_m(t),$$

where the factor $(1-t^2)^m/(1-t)$ arises as we take the SL_2-invariants of

$$\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}/(E_1, \cdots, E_m)$$

instead of $\mathbf{C}[\mathbf{M}^m]$, and the fact that $E_1, \cdots, E_m$ is a regular sequence of homogeneous elements of degree 2 in $\mathbf{C}[\mathbf{M}^m]^{\mathrm{deg}}$. Thus, the functional equation of $H_m(t)$ follows from that of $h_m(t)$. Now, the Krull dimension of R_m^σ is $(3m-3)+1$. Hence, our last claim follows from the functional equation and Proposition 2.13, by the result of Stanley [25, Theorem 4.4] (see also [3, Corollary 4.4.6, p.177]). $\square$

3. Relative character varieties

3.1. Let $S_{g,n}$ be a compact oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components such that $\chi(S_{g,n}) = 2 - 2g - n < 0$. We fix a standard presentation of the fundamental group

$$\pi_1(S_{g,n}) = \langle a_1, \cdots, a_{2g+n} | [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] a_{2g+1} \cdots a_{2g+n} = \mathbf{1} \rangle.$$

By our assumption, $\pi_1(S_{g,n})$ is a free group of rank $m = 2g + n - 1 \geq 2$. We shall refer to the set $\sigma = \{a_1, \cdots, a_{2g+n-1}\}$ of free generators as *standard*.

Let $\mathrm{Rep}_{g,n} = \mathrm{Rep}_{\pi_1(S_{g,n})}$ be the representation variety of $S_{g,n}$ (or of $\pi_1(S_{g,n})$), as considered in Section 2.3, and let $X_{g,n} = \mathrm{Rep}_{g,n} /\!\!/ \mathrm{SL}_2$ be the character variety of $S_{g,n}$. Note that σ gives us isomorphisms

$$\sigma^* : \mathrm{Rep}_{g,n} \simeq \mathrm{Rep}_m = \mathrm{SL}_2^m,$$
$$\sigma^* : X_{g,n} \simeq X_m$$

where Rep_m and X_m are as in Section 2.6. For each $k = (k_1, \cdots, k_n) \in \mathbf{A}^n(\mathbf{C})$, the corresponding fibers $\mathrm{Rep}_{g,n,k}$ and $X_{g,n,k}$ of the morphisms

$$(\mathrm{tr}(x_{2g+1}), \cdots, \mathrm{tr}(x_{2g+n-1}), \mathrm{tr}(x_{2g+n})) : \mathrm{Rep}_{g,n} \to \mathbf{A}^n,$$
$$([a_{2g+1}], \cdots, [a_{2g+n}]) : X_{g,n} \to \mathbf{A}^n$$

(where $x_1, \cdots, x_{2g+n-1}$ are the matrix variables of SL_2^m and we abbreviated here $x_{2g+n} = \langle x_1, x_2 \rangle \cdots \langle x_{2g-1}, x_{2g} \rangle x_{2g+1} \cdots x_{2g+n-1}$ for simplicity) are called the relative representation variety and relative character variety of $S_{g,n}$. Let us denote by $R_{g,n,k} = \mathbf{C}[X_{g,n,k}]$ the coordinate ring of the relative character variety.

Example 3.1. We review the presentations of some relative character varieties for surfaces of small Euler characteristic, following Goldman [9].

(1) $(g,n) = (0,3)$. By Example 2.11.(2), we have $([a_1], [a_2], [a_1 a_2]) : X_{0,3} \simeq \mathbf{A}^3$. Since the 3 boundary components of $S_{0,3}$ correspond to a_1, a_2, and $a_1 a_2$, we see that $X_{0,3,k}$ is simply a point for every $k \in \mathbf{A}^3(\mathbf{C})$.

(2) $(g,n) = (1,1)$. By Example 2.11.(2), we have

$$([a_1], [a_2], [a_1 a_2]) : X_{1,1} \simeq \mathbf{A}^3_{(x,y,z)}$$

where (x, y, z) denotes the sequence of standard coordinate functions on $\mathbf{A}^3$. The boundary component of $S_{1,1}$ corresponds to $\langle a_1, a_2 \rangle$, which defines the function given by the expression

$$[\langle a_1, a_2 \rangle] = f(x,y,z) = x^2 + y^2 + z^2 - xyz - 2$$

obtained using the identities $[a][b] = [ab] + [ab^{-1}]$ in the coordinate ring R_2. Thus, $X_{1,1,k}$ for $k \in \mathbf{A}^1(\mathbf{C})$ is an affine cubic surface which is given by a level set $f^{-1}(k)$ in $\mathbf{A}^3$ of the above function, also called a (generalized) Markoff surface. Looking at the partial derivatives of the above function, its critical locus is given by the conditions

$$2x = yz, \quad 2y = xz, \quad \text{and} \quad 2z = xy.$$

The crticial locus is thus $\{(0,0,0), (s_1 2, s_2 2, s_3 2) : s_1 s_2 s_3 = 1, s_i \in \{\pm 1\}\}$, and every relative character variety $X_{1,1,k}$ is normal.

(3) $(g,n) = (0,4)$. Using Example 2.11.(3), given $k = (k_1, \cdots, k_4) \in \mathbf{A}^4(\mathbf{C})$ we have a closed immersion

$$([a_1 a_2], [a_2 a_3], [a_1 a_3]) : X_{0,4} \hookrightarrow \mathbf{A}^3_{(x,y,z)}$$

with the image given by the affine cubic surface

$$f_k(x,y,z) = x^2 + y^2 + z^2 + xyz - ax - by - cz - d = 0$$

where we define $a = k_1 k_2 + k_3 k_4$, $b = k_2 k_3 + k_1 k_4$, $c = k_1 k_3 + k_2 k_4$, and $d = 4 - k_1^2 - k_2^2 - k_3^2 - k_4^2 - k_1 k_2 k_3 k_4$. The critical locus of the function f is given by the conditions

$$2x + yz - a = 0, \quad 2y + xz - b = 0, \quad 2z + xy - c = 0.$$

We see that the locus $\subset \mathbf{A}^3$ defined by the above three equations is finite. Hence, every relative character variety $X_{0,4,k}$ is normal.

3.2.   From Section 2, we have the σ-word filtration Fil^σ on the coordinate ring $R_{g,n}$ with Rees algebra $R_{g,n}^\sigma = (\mathbf{C}[\mathrm{Rep}_{g,n}]^\sigma)^{\mathrm{SL}_2}$. Let $F_1, \cdots, F_{n-1}$ and F_n respectively be the homogeneous elements in $R_{g,n}^\sigma$ of degree $1, \cdots, 1$ and $4g + n - 1$ given by

$$F_1 = [a_{2g+1}] - k_1 t, \quad \cdots, \quad F_{n-1} = [a_{2g+n-1}] - k_{n-1} t, \quad \text{and}$$

$$F_n = [a_{2g+n}] - k_n t^{4g+n-1}.$$

Here, F_n is homogeneous since $\ell_\sigma(a_{2g+n}) = 4g + n - 1$. Let us define the quotient $R_{g,n,k}^\sigma = R_{g,n}/(F_1, \cdots, F_n)$. Note that $R_{g,n,k}^\sigma/(t - 1) = R_{g,n,k}$. The pair

$$(\overline{X}, D) = (\mathrm{Proj}\, R_{g,n,k}^\sigma, \mathrm{Proj}\, R_{g,n,k}^\sigma/(t))$$

is our projective compactification and boundary divisor of $X = X_{g,n,k}$ mentioned in Theorem 1.1. The goal of this section is to verify that $(\overline{X}, D)$ satisfies the desired properties.

Theorem 3.2. *The ring $R_{g,n,k}^\sigma$ is a normal graded Gorenstein domain of dimension $6g + 2n - 5$, with canonical module $R_{g,n,k}^\sigma(-1)$.*

Proof. We have an identification $R_{g,n}^\sigma = R_m^\sigma$. We claim that $F_1, \cdots, F_n$ is a regular sequence in $R_m^\sigma = (\mathbf{C}[\mathrm{Rep}_m]^\sigma)^{\mathrm{SL}_2}$. Indeed, using the fact that $(\mathbf{C}[\mathbf{M}^m]^{\deg})^{\mathrm{SL}_2}$ is a pure subring of $\mathbf{C}[\mathbf{M}^m]^{\deg}$, it suffices to note that $\mathbf{C}[\mathrm{Rep}_m]^\sigma/(F_1, \cdots, F_s)$ is an integral domain (of successively lower dimension) for $1 \le s \le n - 1$. This follows from an easy application of Lemma 2.3 (as in the proof of Lemma 2.12). Now, by Theorem 2.14, R_m^σ is a graded Gorenstein ring of dimension $3m - 2$ with canonical module $R_m^\sigma(-2m - 1)$. Hence, it follows that

$$R_{g,n,k}^\sigma = R_{g,n}^\sigma/(F_1, \cdots, F_n)$$

is a graded Gorenstein ring of dimension $3m - 2 - n = 6g + 2n - 5$ with canonical module

$$R_{g,n,k}^\sigma(-2m - 1 + (n - 1 + 4g + n - 1)) = R_{g,n,k}^\sigma(-1).$$

It remains to show that $R_{g,n,k}^\sigma$ is a normal domain. By Lemma 2.3, it suffices to show the following:

   (1) $R_{g,n,k}$ is a normal ring, and
   (2) $R_{g,n,k}^\sigma/(t)$ is reduced of dimension $6g + 2n - 6$.

If $\chi(S_{g,n}) \le -3$, then (1) and (2) follow from Propositions 4.7 and 4.9. Let us now consider the cases where $|\chi(S_{g,n})| \le 2$. If $\chi(S_{g,n}) = -2$, then (1) and (2) follow from Propositions 4.8 and 4.9 for $(g, n) = (1, 2)$, and they follow from Example 3.1.(3) and Proposition 4.9 for $(g, n) = (0, 4)$. The case $\chi(S_{g,n}) = -1$ remains. The case $(g, n) = (0, 3)$ is trivial, and for $(g, n) = (1, 1)$ Example 3.1.(2) verifies (1). It thus remains to verify (2) for $(g, n) = (1, 1)$. But we find that

$$R_{1,1,k}^\sigma/(t) \simeq \mathbf{C}[x, y, z]/(z(z - xy)),$$

whose spectrum is the union of the plane $\{z = 0\}$ with the surface $\{z = xy\}$ in $\mathbf{A}^3_{(x,y,z)}$, and hence is reduced and of dimension 2. This completes the proof. $\square$

3.3. Let $\overline{X}$ be the compactification of $X = X_{g,n,k}$ as above. Note that $\overline{X}$ is a normal irreducible projective scheme, since $R^\sigma_{g,n,k}$ is a normal domain by Theorem 3.2. By the result of Demazure [6, Théorème (3.5) p.51], there is an ample $\mathbf{Q}$-divisor E on $\overline{X}$ such that

$$R^\sigma_{g,n,k} = \bigoplus_{r=0}^{\infty} H^0(X, \mathcal{O}_X(rE))t^r$$

where $t \in (R^\sigma_{g,n,k})_1$ is the image of $t \in (R^\sigma_{g,n})_1$ under the projection $R^\sigma_{g,n} \to R^\sigma_{g,n,k}$. Following Demazure (*loc.cit.*), the $\mathbf{Q}$-divisor E is given explicitly as follows. Let us write $\mathrm{Div}(t) = \sum p_F F$ for the Weil divisor on $\mathrm{Spec}\, R^\sigma_{g,n,k}$ determined by t. Note that each F is given by $\mathrm{Spec}\, B_F$ where B_F is a graded integral quotient of $R^\sigma_{g,n,k}$. Defining $q_F \geq 0$ by

$$(B_F)_n \neq 0 \iff n \in q_F \mathbf{Z},$$

the $\mathbf{Q}$-divisor E on $\overline{X}$ is given by

$$E = \sum_{q_F \neq 0} \frac{p_F}{q_F} \mathrm{Proj}(B_F).$$

Lemma 3.3. *We have $p_F \in \{0, 1\}$ and $q_F \in \{0, 1, 2\}$ for every prime Weil divisor F of $\mathrm{Spec}\, R^\sigma_{g,n,k}$.*

Proof. The claim $p_F \in \{0, 1\}$ follows from the fact that

$$\mathrm{Spec}(R^\sigma_{g,n,k}/t) = (\mathbf{C}[\mathrm{Rep}'_{g,n,k}]/(t))^{\mathrm{SL}_2}$$

is reduced by Proposition 4.9. To see that $q_F \in \{0, 1, 2\}$, note that B_F is generated in degrees 1, 2, and 3 by Lemma 2.9 so we must have $q_F \in \{0, 1, 2, 3\}$. Suppose toward contradiction that $q_F = 3$, so that $(B_F)_1 = (B_F)_2 = 0$. Given any $a, b, c \in \pi_1(S_{g,n})$, on $R_{g,n,k}$ we have the relations

$$[abc] + [acb] = [ab][c] + [ac][b] + [bc][a] - [a][b][c], \quad \text{and}$$
$$[abc][acb] = ([a]^2 + [b]^2 + [c]^2) + ([ab]^2 + [bc]^2 + [ac]^2)$$
$$- ([a][b][ab] + [b][c][bc] + [a][c][ac]) + [ab][bc][ac] - 4.$$

Since the right hand sides of both equations are zero on B_F which is an integral domain, this shows that $[abc] = [acb] = 0$ on B_F. This shows that we must have $(B_F)_3 = 0$ as well, a contradiction. Hence, we must have $q_F \in \{0, 1, 2\}$. $\square$

Let $D = \overline{X} \setminus X$ be the reduced boundary divisor on $\overline{X}$. As a Weil divisor, D is a formal sum $\sum_V V$ of prime divisors V where $V = \mathrm{Proj}\, B_F$ runs over integral quotients B_F of $R^\sigma_{g,n,k}/(t)$ by minimal homogeneous prime ideals. We are ready to prove the main part of Theorem 1.1, restated as follows.

Theorem 3.4. *The canonical divisor K of $\overline{X}$ satisfies $K + D \sim 0$.*

Proof. Combining Corollary 3.2 with the result of Watanabe [27, Corollary 2.9], we find that $K + E' + E = \mathrm{Div}(f)$ for some $f \in \mathbf{C}(\overline{X})$ (the function field of $\overline{X}$) and

$$E' = \sum_{q_F \neq 0} \frac{q'_F - 1}{q'_F} \mathrm{Proj}\, B_F,$$

where we have set $q'_F = q_F / \gcd(p_F, q_F)$ for $q_F \neq 0$, so that in particular $q'_F = 1$ if $p_F = 0$. But in light of Lemma 3.3, we have

$$0 \sim K + E' + E$$

$$= K + \sum_{\substack{p_F=1 \\ q_F=2}} \frac{1}{2} \operatorname{Proj} B_F + \left\{ \sum_{\substack{p_F=1 \\ q_F=2}} \frac{1}{2} \operatorname{Proj} B_F + \sum_{\substack{p_F=1 \\ q_F=1}} \operatorname{Proj} B_F \right\}$$

$$= K + D$$

from which we conclude the result. $\qquad\square$

4. Singularities of representation varieties

4.1. **Introduction.** Let $m \geq 1$ be an integer. Recall from Section 2.5 that we have an identification

$$\mathbf{C}[\mathbf{M}^m]^{\deg} = \mathbf{C}[(X_1)_{ij}, \cdots, (X_m)_{ij}, T] = \mathbf{C}[\mathbf{M}^m \times \mathbf{A}^1].$$

As before, for $i = 1, \cdots, m$, let $E_i = \det(X_i) - T^2$ be the element of degree 2 in $\mathbf{C}[\mathbf{M}^m]^{\deg}$. Recall that $\mathbf{C}[\operatorname{Rep}_m]^\sigma = \mathbf{C}[\mathbf{M}^m]^{\deg}/(E_1, \cdots, E_m)$, where Rep_m is the representation variety of the free group on m generators σ.

Let $g \geq 0$ and $n \geq 0$ be integers such that $2g+n-1 = m$. Fix $k = (k_1, \cdots, k_n) \in \mathbf{A}^n(\mathbf{C})$. Let $F_1, \cdots, F_{n-1}$ and F_n respectively be the homogeneous elements in $\mathbf{C}[\mathbf{M}^m]^{\deg}$ of degrees $1, \cdots, 1$ and $4g+n-1$ given by

$$F_1 = \operatorname{tr}(X_{2g+1}) - k_1 T, \quad \cdots, \quad F_{n-1} = \operatorname{tr}(X_{2g+n-1}) - k_{n-1} T, \quad \text{and}$$

$$F_n = \operatorname{tr}(\langle X_1, X_2 \rangle \cdots \langle X_{2g-1}, X_{2g} \rangle X_{2g+1} \cdots X_{2g+n-1}) - k_n T^{4g+n-1}.$$

Let us define $\operatorname{Rep}'_m = \operatorname{Spec} \mathbf{C}[\operatorname{Rep}_m]^\sigma$ and $\operatorname{Rep}'_{g,n,k} = \operatorname{Spec} \mathbf{C}[\operatorname{Rep}'_m]/(F_1, \cdots, F_n)$. The singular locus of $\operatorname{Rep}'_{g,n,k}$ is the intersection of $\operatorname{Rep}'_{g,n,k}$ with the locus of critical points $\operatorname{Crit}(E, F)$ of the morphism $(E_1, \cdots, E_m, F_1, \cdots, F_n) : \mathbf{M}^m \times \mathbf{A}^1 \to \mathbf{A}^{m+n}$. Consider now the $\mathbf{G}_m$-equivariant morphism

$$\pi : \operatorname{Rep}'_{g,n,k} \to \mathbf{A}^1$$

of affine schemes induced by the the morphism of graded rings $\mathbf{C}[t] \to \mathbf{C}[\operatorname{Rep}'_{g,n,k}]$ sending $t \mapsto T$. Note that $\pi^{-1}(1) \simeq \operatorname{Rep}_{g,n,k}$ is the relative representation variety of the compact surface $S_{g,n}$ of genus g with n boundary components corresponding to k, as introduced in Section 3.1 (at least when $n \geq 1$). Let us define $H_{g,n} = \pi^{-1}(0)$. Note that it is independent of the choice of k.

4.2. **Critical points.** Fix integers $g \geq 0$ and $n \geq 1$ such that $m = 2g+n-1 \geq 1$. Fix a complex number $t \in \mathbf{C}$. For each $s \in \mathbf{C}$ such that $s^2 = t$, the fiber of the morphism (considered in the proof of Lemma 2.12)

$$\pi : \operatorname{Rep}'_m \to \mathbf{A}^1$$

above s is the scheme $\mathbf{M}_t^m$. By definition, a point $a \in \mathbf{M}_t^m(\mathbf{C})$ is represented by an m-tuple of matrices $a = (a_1, \cdots, a_m)$, each a_i having determinant t. Given such a point, let us introduce the following supplementary notation.

(1) Let $b_i = \langle a_{2i-1}, a_{2i} \rangle$ for each $i \in \{1, \cdots, g\}$.
(2) Let $b_{g+j} = a_{2g+j}$ for each $j \in \{1, \cdots, n-1\}$.

(3) In generality, for a cyclic interval $I = (i_0, i_0 + 1, \cdots, i_0 + k)$ in $\mathfrak{C}_{g+n-1}$ (cf. Section 1.4), let $b_I = b_{i_0} \cdots b_{i_0+k}$ where the indices are considered modulo $g + n - 1$. Also, let $b^I = b_{I^c}$ when I^c is defined. Let us write $b^i = b^{(i)}$ for simplicity.

For example, we have

$$b_{(1,\cdots,g+n-1)} = b_1 \cdots b_{g+n-1} = \langle a_1, a_2 \rangle \cdots \langle a_{2g-1}, a_{2g} \rangle a_{2g+1} \cdots a_{2g+n-1}.$$

We shall attempt to describe, in terms of these parameters, the critical locus of the restriction to $\mathbf{M}_t^m \subset \mathrm{Rep}'_m$ of the morphism $F = (F_1, \cdots, F_n) : \mathrm{Rep}'_m \to \mathbf{A}^n$ defined in Section 4.1. We begin with a lemma on tangent spaces.

Lemma 4.1. *Let $a \in \mathbf{M}(\mathbf{C})$ be given.*

(1) $T_a \mathbf{M}_{\det a} = \{v \in \mathbf{M}(\mathbf{C}) : \mathrm{tr}(va^*) = 0\}$. *We have $\mathfrak{sl}_2(\mathbf{C}) \cdot a \subseteq T_a \mathbf{M}_{\det a}$ and $a \cdot \mathfrak{sl}_2(\mathbf{C}) \subseteq T_a \mathbf{M}_{\det a}$. The containments are equalities if $\det(a) \neq 0$.*

(2) $T_a \mathbf{M}_{\det a, \mathrm{tr}\, a} = \{v \in \mathbf{M}(\mathbf{C}) : \mathrm{tr}(v) = \mathrm{tr}(va^*) = 0\}$. *We have a containment $[\mathfrak{sl}_2(\mathbf{C}), a] \subseteq T_a \mathbf{M}_{\det a, \mathrm{tr}\, a}$ which is an equality if a is nonscalar.*

Proof. (1) For any $a, v \in \mathbf{M}(\mathbf{C})$, we have $\det(a + \varepsilon v) = \det(a) + \varepsilon \, \mathrm{tr}(va^*)$ within $\mathbf{C}[\varepsilon]/(\varepsilon^2)$, from which the first statement follows. In particular, $\dim T_a \mathbf{M}_{\det a} = 3$ provided that $a \neq 0$. Since $\mathrm{tr}(uaa^*) = \mathrm{tr}(aua^*) = \det(a) \, \mathrm{tr}(u) = 0$ for every $u \in \mathfrak{sl}_2(\mathbf{C})$, we have $\mathfrak{sl}_2(\mathbf{C}) \cdot a \subseteq T_a \mathbf{M}_{\det a}$ and $a \cdot \mathfrak{sl}_2(\mathbf{C}) \subseteq T_a \mathbf{M}_{\det a}$. Lastly, if $\det(a) \neq 0$ then $\dim \mathfrak{sl}_2(\mathbf{C}) \cdot a = \dim a \cdot \mathfrak{sl}_2(\mathbf{C}) = 3$ from which the equalities follow by dimension reasons.

(2) For any $a, v \in \mathbf{M}(\mathbf{C})$ we have $\mathrm{tr}(a + \varepsilon v) = \mathrm{tr}(a) + \varepsilon \, \mathrm{tr}(v)$ within $\mathbf{C}[\varepsilon]/(\varepsilon^2)$, from which the first statement follows. We have $\mathrm{tr}([v, a]a^*) = \mathrm{tr}(vaa^* - ava^*) = \det(a) \, \mathrm{tr}(v) - \det(a) \, \mathrm{tr}(v) = 0$ and $\mathrm{tr}([v, a]) = 0$ for any $v \in \mathfrak{sl}_2(\mathbf{C})$, and hence $[\mathfrak{sl}_2(\mathbf{C}), a] \subseteq T_{a,\mathrm{char}}$. Finally, if a is nonscalar, then we have $\dim_{\mathbf{C}} T_a \mathbf{M}_{\det a, \mathrm{tr}\, a} = 2$ since $v \mapsto \mathrm{tr}(va^*)$ and $w \mapsto \mathrm{tr}(v)$ are linearly independent on $\mathbf{M}(\mathbf{C})$. On the other hand, $\dim_{\mathbf{C}}[\mathfrak{sl}_2(\mathbf{C}), a] = 2$ if a is nonscalar, and hence $T_a \mathbf{M}_{\det a, \mathrm{tr}\, a} = [\mathfrak{sl}_2(\mathbf{C}), a]$ by dimension reasons. $\square$

Lemma 4.2. *A critical point $(a_1, \cdots, a_m)$ of $F : \mathbf{M}_t^m \to \mathbf{A}^n$ must satisfy at least one of the following two conditions.*

(1) a_{2g+j} *is scalar for some $j \in \{1, \cdots, n-1\}$.*

(2) $[b_i, b^i] = 0$ *for every $i \in \{1, \cdots, g+n-1\}$, and*

$$[a_{2i-1}, a_{2i} a_{2i-1}^* a_{2i}^* b^i] = [a_{2i}, a_{2i-1}^* a_{2i}^* b^i a_{2i-1}]$$
$$= [a_{2i-1}^*, a_{2i}^* b^i a_{2i-1} a_{2i}] = [a_{2i}^*, b^i a_{2i-1} a_{2i} a_{2i-1}^*] = 0$$

for every $i \in \{1, \cdots, g\}$.

Proof. Assume that $a \in \mathbf{M}_t^m(\mathbf{C})$ satisfies neither of the conditions (1) and (2). We must show that $d_a F : T_a \mathbf{M}_t^m \to \mathbf{C}^n$ is surjective. Now, the first $n-1$ coordinates of $d_a F$ are given by

$$(u_1, \cdots, u_m) \mapsto (\mathrm{tr}(u_{2g+1}), \cdots, \mathrm{tr}(u_{2g+n-1}))$$

for each $u = (u_1, \cdots, u_m) \in T_a \mathbf{M}_t^m$. Since a_{2g+j} is nonscalar for $j \in \{1, \cdots, n-1\}$ by hypothesis, each of the functionals $u \mapsto \mathrm{tr}(u_{2g+j})$ is nonzero. Thus, to show surjectivity of $d_a F$ it remains to show that $(0, \cdots, 0, 1) \in \mathrm{Im}\, d_a F$.

Let $u \in \mathfrak{sl}_2(\mathbf{C})$ be given. We then have $[u, b_{g+j}] \in T_{b_{g+j}}\mathbf{M}_t$ for $j \in \{1, \cdots, n-1\}$, and the composition $T_{b_{g+j}}\mathbf{M}_t \hookrightarrow T_a\mathbf{M}_t^m \to \mathbf{C}^n$ with d_aF sends

$$[u, b_{g+j}] \mapsto (0, \cdots, 0, \operatorname{tr}([u, b_{g+j}]b^{g+j})) = (0, \cdots, 0, \operatorname{tr}(u[b_{g+j}, b^{g+j}])).$$

Similarly, after a short computation (using the fact $u^* = -u$ for $u \in \mathfrak{sl}_2(\mathbf{C})$) we also find that for $i \in \{1, \cdots, g\}$ the composition $T_{(a_{2i-1}, a_{2i})}\mathbf{M}_t^2 \hookrightarrow T_a\mathbf{M}_t^m \to \mathbf{C}^n$ with d_aF sends

$$(ua_{2i-1} - a_{2i-1}u, -a_{2i}u) \mapsto (0, \cdots, 0, \operatorname{tr}(u[a_{2i-1}, a_{2i}a_{2i-1}^*a_{2i}^*b^i])),$$
$$(a_{2i-1}u, 0) \mapsto (0, \cdots, 0, \operatorname{tr}(u[a_{2i}, a_{2i-1}^*a_{2i}^*b^i a_{2i-1}])),$$
$$(0, a_{2i}u) \mapsto (0, \cdots, 0, \operatorname{tr}(u[a_{2i-1}^*, a_{2i}^*b^i a_{2i-1}a_{2i}])),$$
$$(-a_{2i-1}u, ua_{2i} - a_{2i}u) \mapsto (0, \cdots, 0, \operatorname{tr}(u[a_{2i}^*, b^i a_{2i-1}a_{2i}a_{2i-1}^*])).$$

In particular, summing up the above tangent vectors in $T_a\mathbf{M}_t^m$ we find that

$$(ua_{2i-1} - a_{2i-1}u, ua_{2i} - a_{2i}u) \mapsto (0, \cdots, 0, \operatorname{tr}(u[b_i, b^i])).$$

Thus, if a does not satisfy condition (2), at least one of the expressions above must be nonzero for some $u \in \mathfrak{sl}_2(\mathbf{C})$. This implies that $(0, \cdots, 0, 1) \in \operatorname{Im} d_aF$, and hence d_aF is surjective, as desired. $\square$

4.3. **Dimension estimates.** We collect a number of estimates for the dimensions of certain schemes parametrizing sequences of matrices with prescribed conditions. We will later use them to analyze the singularities of $\operatorname{Rep}_{g,n,k}$ and $H_{g,n}$. Throughout Section 4.3, fix integers $g \geq 0$ and $n \geq 0$ such that $m = 2g + n - 1 \geq 1$.

4.3.1. Fix $k = (k_1, \cdots, k_n) \in \mathbf{A}^n(\mathbf{C})$, and let $\Phi : \operatorname{SL}_2^{2g} \times \prod_{i=1}^n \operatorname{SL}_{2,k_i} \to \operatorname{SL}_2$ be the morphism given by

$$(a_1, \cdots, a_{2g+n}) \mapsto \langle a_1, a_2 \rangle \cdots \langle a_{2g-1}, a_{2g} \rangle a_{2g+1} \cdots a_{2g+n}.$$

Lemma 4.3. Φ *is flat above* $\operatorname{SL}_2 \backslash \{\pm\mathbf{1}\}$ *with fibers of dimension* $6g + 2n - 3$.

Proof. First, Φ is dominant by Lemma A.3. Note that $\operatorname{SL}_2^{2g} \times \prod_{i=1}^n \operatorname{SL}_{2,k_i}$ is a complete intersection scheme of dimension $6g + 2n$, and in particular Cohen-Macaulay. The target SL_2 of Φ is regular of dimension 3. By the miracle flatness theorem, to prove that Φ is flat it suffices to show that the fibers of Φ are equidimensional, of dimension $6g + 2n - 3$. Consider the composition $\operatorname{tr} \circ \Phi : \operatorname{SL}_2^{2g} \times \prod_{i=1}^n \operatorname{SL}_{2,k_i} \to \mathbf{A}^1$. Since $\operatorname{tr} \circ \Phi$ is dominant, it is flat. In particular, the fibers of $\operatorname{tr} \circ \Phi$ are all of pure dimension $6g + 2n - 1$.

Next, note that SL_2 acts by conjugation on the domain and target of Φ, and Φ is equivariant with respect to this action. Hence, given any $l \in \mathbf{C}$ the fibers of the restriction $\Phi : \Phi^{-1}(\operatorname{SL}_{2,l}) \to \operatorname{SL}_{2,l}$, away from the fibers over $\pm\mathbf{1}$, are isomorphic hence equidimensional of dimension $6g + 2n - 3$. $\square$

Lemma 4.4. *If* $2g + n - 1 \geq 2$, *then* $\Phi^{-1}(\pm\mathbf{1})$ *has dimension at most* $6g + 2n - 3$.

Proof. Let us denote by F_+ (reps. F_-) the fiber of Φ above $\mathbf{1}$ (resp. $-\mathbf{1}$). We shall proceed by induction on (g, n). Consider first the case where $n = 0$. We must then have $g \geq 2$. Consider the projection

$$\pi : F_{\pm} \subset \operatorname{SL}_2^{2g} \to \operatorname{SL}_2^2$$

onto the last two copies of SL_2. Given $(b_1, b_2) \in \operatorname{SL}_2^2(\mathbf{C})$, the dimension of the fiber $\pi^{-1}(b_1, b_2)$ is estimated as follows.

(1) If $\langle b_1, b_2 \rangle \neq \pm\mathbf{1}$, then $\dim \pi^{-1}(b_1, b_2) = 6(g-1) - 3$ by Lemma 4.3.
(2) If $\langle b_1, b_2 \rangle = \pm\mathbf{1}$, then $\dim \pi^{-1}(b_1, b_2) \leq 6(g-1) - 2$.

Here, part (2) follows from the inductive hypothesis and the fact that, when $g = 2$ so that $g - 1 = 1$, the fiber of $\langle -, - \rangle : \mathrm{SL}_2^2 \to \mathrm{SL}_2$ above $\pm\mathbf{1}$ has dimension at most 4 by Lemma A.4. Since the locus of $(b_1, b_2) \in \mathrm{SL}_2^2$ defined by part (1) is 6-dimensional, and the locus defined by condition (2) is at most 4-dimensional by Lemma A.4, we see that $\dim F_\pm \leq 6g - 3$ as desired. Consider next the case where $n \geq 1$. If $g = 0$, then we must have $n \geq 3$. Consider the projection

$$\pi : F_\pm \subset \mathrm{SL}_2^{2g} \times \prod_{i=1}^{n} \mathrm{SL}_{2,k_i} \to \mathrm{SL}_{2,k_n}$$

onto the last factor of the product. Given $b \in \mathrm{SL}_{2,k_n}(\mathbf{C})$, the dimension of the fiber $\pi^{-1}(b)$ is estimated as follows.

(1) If $b \neq \pm\mathbf{1}$, then $\dim \pi^{-1}(b) = 6g + 2(n-1) - 3$ by the Lemma 4.3.
(2) If $b = \pm\mathbf{1}$, then $\dim \pi^{-1}(b) \leq 6g + 2(n-1) - 2$.

Here, part (2) follows from the inductive hypothesis, from Lemma A.4 in the case $(g, n) = (1, 1)$, and from the observation that the fiber of $\mathrm{SL}_{2,l} \times \mathrm{SL}_{2,\pm l} \to \mathrm{SL}_2$, $(a_1, a_2) \mapsto a_1 a_2$ over $\pm\mathbf{1}$ has dimension 2 in the case $(g, n) = (0, 3)$. Since the locus of b satisfying (1) has dimension 2, we find that $\dim F_\pm \leq 6g + 2n - 3$ as desired. $\square$

4.3.2. Let $\Psi : \mathbf{M}_0^{2g} \times \mathbf{M}_{0,0}^n \to \mathbf{M}_0$ be the morphism given by

$$(a_1, \cdots, a_{2g+n}) \mapsto \langle a_1, a_2 \rangle \cdots \langle a_{2g-1}, a_{2g} \rangle a_{2g+1} \cdots a_{2g+n}.$$

Lemma 4.5. Ψ *is flat above* $\mathbf{M}_0 \setminus \{0\}$ *with fibers of dimension* $6g + 2n - 3$.

Proof. First, Ψ is dominant by Lemma A.8. Note that $\mathbf{M}_0^{2g} \times \mathbf{M}_{0,0}^n$ is a complete intersection scheme of dimension $6g + 2n$, and in particular Cohen-Macaulay. The target $\mathbf{M}_0$ of Ψ is regular away from $\{0\}$, of dimension 3. By the miracle flatness theorem, to prove that Ψ is flat over $\mathbf{M}_0 \setminus \{0\}$ it suffices to show that the fibers of Ψ are equidimensional, of dimension $6g + 2n - 3$. Consider the composition $\mathrm{tr} \circ \Psi_{g,n} : \mathbf{M}_0^{2g} \times \mathbf{M}_{0,0}^n \to \mathbf{A}^1$. Since $\mathrm{tr} \circ \Psi$ is dominant, it is flat. In particular, the fibers of $\mathrm{tr} \circ \Psi$ are all of pure dimension $6g + 2n - 1$.

Next, note that SL_2 acts by conjugation on the domain and target of Ψ, and Ψ is equivariant with respect to this action. Hence, given any $l \in \mathbf{C}$ the fibers of the restriction $\Psi : \Psi^{-1}(\mathbf{M}_{0,l}) \to \mathbf{M}_{0,l}$ are isomorphic, hence equidimensional of dimension $6g + 2n - 3$, away from the fiber over 0. $\square$

Lemma 4.6. $\Psi^{-1}(0)$ *has dimension at most* $6g + 2n - 1$.

Proof. Since the domain $\mathbf{M}_0^{2g} \times \mathbf{M}_{0,0}^n$ of Ψ has dimension $6g + 2n$ and is integral, if $\dim \Psi^{-1}(0) \geq 6g + 2n$ then Ψ must be identically zero, contradicting the fact that Ψ is dominant by Lemma A.8. $\square$

4.4. **Singularities.** Let $S_{g,n}$ be a compact oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components and $\chi(S_{g,n}) = 2 - 2g - n < 0$. The fundamental group of $S_{g,n}$ is free of rank $m = 2g + n - 1 \geq 2$. Let $k = (k_1, \cdots, k_n) \in \mathbf{A}^n(\mathbf{C})$, and let $F_1, \cdots, F_n \in \mathbf{C}[\mathbf{M}^m]^{\deg}$ be as in Section 4.1.

Proposition 4.7. *If* $m \geq 4$, *the scheme* $\mathrm{Rep}_{g,n,k}$ *is normal.*

Proof. Since $\text{Rep}_{g,n,k}$ is a complete intersection of dimension $6g + 2n - 3$ in the regular scheme $\mathbf{M}^m$, it is Cohen-Macaulay. As a consequence of Serre's criterion for normality [21, Theorem 23.8, p.183], a Cohen-Macaulay ring is normal if and only if it is regular in codimension 1. It thus suffices to demonstrate that $\text{Rep}_{g,n,k}$ is regular in codimension 1. Consider the restriction of $F = (F_1, \cdots, F_n)$ above to $\text{SL}_2^m \subset \mathbf{M}^m \times \mathbf{A}^1$. Since SL_2^m is regular, the singular locus of $\text{Rep}_{g,n,k} = F^{-1}(0)$ is the intersection of $\text{Rep}_{g,n,k}$ with the critical locus of F. It suffices to show that the locus in $\text{Rep}_{g,n,k}$ defined by each of the conditions (1) and (2) of Proposition 4.2 has dimension at most $(6g + 2n - 3) - 2$. We shall represent points of $\text{Rep}_{g,n,k}$ by sequences $(a_1, \cdots, a_{2g+n-1})$ of matrices, and adopt the same notations b_I and b^I for the various products of matrices as used in Proposition 4.2.

First, consider the locus $W \subset \text{Rep}_{g,n,k}$ defined by condition (1) of Proposition 4.2, which plays a role only when $n \geq 2$. For each $i = 1, \cdots, n-1$ and $s \in \{\pm 1\}$, the locus $\{b_{g+i} = s\mathbf{1}\} \subset \text{Rep}_{g,n,k}$ is isomorphic to $\text{Rep}_{g,n-1,k'}$ where k' is the $(n-1)$-tuple obtained from k by omitting k_i and replacing k_n by sk_n. By our assumption on (g, n), we have $2g + (n-1) - 1 \geq 2$, and we find by Lemma 4.4

$$\dim\{b_{g+i} = s\mathbf{1}\} = \dim \text{Rep}_{g,n-1,k'} = 6g + 2(n-1) - 3 = (6g + 2n - 3) - 2.$$

As $W = \bigcup_{i=1}^{n-1}\{b_{g+i} = \pm\mathbf{1}\}$, it has codimension 2 in $\text{Rep}_{g,n,k}$ as desired.

Next, let $Z \subset \text{Rep}_{g,n,k}$ be the locus defined by condition (2) of Proposition 4.2. We stratify Z further into three subloci Z_1, Z_2, and Z_3, and estimate their dimensions as follows.

(1) Let $Z_1 \subset Z$ be the sublocus consisting of $(a_1, \cdots, a_m)$ such that $b_i = \pm\mathbf{1}$ for some $i \in \{1, \cdots, g\}$ or $b_{g+j}b_{g+j+1} = \pm\mathbf{1}$ for some $j \in \{1, \cdots, n-2\}$. We claim that $\dim Z_1 \leq (6g + 2n - 3) - 2$.

Given $i \in \{1, \cdots, g\}$ and $s \in \{\pm 1\}$, consider the locus where $b_i = s\mathbf{1}$. By Lemma A.4, (a_{2i-1}, a_{2i}) must vary over a locus of dimension at most 4. For fixed (a_{2i-1}, a_{2i}) the remaining matrices in the sequence $(a_1, \cdots, a_m)$ vary over a locus isomorphic to $\text{Rep}_{g-1,n,k'}$ for $k = (k_1, \cdots, k_{n-1}, sk_n)$, which has dimension $\dim \text{Rep}_{g-1,n,k'} = 6(g-1) + 2n - 3$ by Lemma 4.4 since $2(g-1) + n - 1 \geq 2$ by our assumption. Thus, the locus in Z_1 where where $b_i = s\mathbf{1}$ for $i \in \{1, \cdots, g\}$ has dimension bounded by

$$4 + (6(g-1) + 2n - 3) = (6g + 2n - 3) - 2.$$

Given $j \in \{1, \cdots, n-2\}$ and $s \in \{\pm 1\}$, consider the locus where $b_{g+j}b_{g+j+1} = s\mathbf{1}$. The pair (b_{g+j}, b_{g+j+1}) then varies over a locus of dimension at most 2. Tor fixed (b_{g+j}, b_{g+j+1}) the remaining matrices in the sequence $(a_1, \cdots, a_m)$ vary over a locus isomorphic to $\text{Rep}_{g,n-2,k'}$ where k' is obtained from k by removing k_j and k_{j+1}, and replacing k_n by sk_n. We have $\dim \text{Rep}_{g,n-2,k'} = 6g + 2(n-2) - 3$ by Lemma 4.4 since $2g + (n-2) - 1 \geq 2$ by our assumption. Thus, the given locus has dimension bounded by

$$2 + (6g + 2(n-2) - 3) = (6g + 2n - 3) - 2.$$

This shows that $\dim Z_1 \leq (6g + 2n - 3) - 2$.

(2) Let $Z_2 \subset Z \setminus Z_1$ be the sublocus consisting of $(a_1, \cdots, a_m)$ with $\text{tr}(b_i) = \pm 2$ for some $i \in \{1, \cdots, g\}$ or $\text{tr}(b_{g+j}b_{g+j+1}) = \pm 2$ for some $j \in \{1, \cdots, n-2\}$. We claim that $\dim Z_2 \leq (6g + 2n - 3) - 2$.

Given $i \in \{1, \cdots, g\}$, consider the locus $\text{tr}(b_i) = \pm 2$. By Lemma 4.3, the pair (a_{2i-1}, a_{2i}) varies over a locus of dimension at most 5. For fixed

(a_{2i-1}, a_{2i}), by Lemma A.2 the product b^i must vary over a locus of dimension at most 1. We have two possibilities.

(a) We have $b^i \in \{\pm \mathbf{1}\}$. We must have $2(g-1) + (n-1) - 1 \geq 2$ since Z_2 lies in the complement of Z_1. By Lemma 4.4, for fixed (a_{2i-1}, a_{2i}) the remaining matrices in $(a_1, \cdots, a_m)$ vary over a locus of dimension at most $6(g-1) + 2(n-1) - 3$.

(b) We have $b^i \neq \pm \mathbf{1}$. By Lemma 4.3, for fixed (a_{2i-1}, a_{2i}) and b^i the remaining matrices in $(a_1, \cdots, a_m)$ vary over a locus of dimension at most $6(g-1) - 2(n-1) - 3$.

Thus, we see that the locus of $(a_1, \cdots, a_m)$ in Z satisfying $\mathrm{tr}(b_i) = \pm 2$ for some $i \in \{1, \cdots, g\}$ has dimension bounded by

$$5 + 1 + (6(g-1) + 2(n-1) - 3) = (6g + 2n - 3) - 2.$$

Given $j \in \{1, \cdots, n-2\}$, consider the locus $\mathrm{tr}(b_{g+j} b_{g+j+1}) = \pm 2$. By Lemma 4.3, the pair (b_{g+j}, b_{g+j+1}) varies over a locus of dimension 3. For fixed (b_{g+j}, b_{g+j+1}), by Lemma A.2 the product $b^{\{g+j, g+j+1\}}$ must vary over a locus of dimension at most 1. We have two possibilities.

(a) We have $b^{\{g+j,g+j+1\}} \in \{\pm \mathbf{1}\}$. We must have $2g + (n-3) - 1 \geq 2$ since Z_2 lies in the complement of Z_1. By Lemma 4.4, for fixed (b_{g+j}, b_{g+j+1}) the remaining matrices in $(a_1, \cdots, a_m)$ vary over a locus of dimension at most $6g + 2(n-3) - 3$.

(b) We have $b^{\{g+j,g+j+1\}} \neq \pm \mathbf{1}$. By Lemma 4.4, for fixed (b_{g+j}, b_{g+j+1}) and $b^{\{g+j,g+j+1\}}$ the remaining matrices in $(a_1, \cdots, a_m)$ vary over a locus of dimension at most $6g - 2(n-3) - 3$.

Thus, we see that the locus of $(a_1, \cdots, a_m)$ in Z with $\mathrm{tr}(b_{g+j} b_{g+j+1}) = \pm 2$ for some $j \in \{1, \cdots, n-2\}$ has dimension bounded by

$$3 + 1 + (6g + 2(n-3) - 3) = (6g + 2n - 3) - 2.$$

This shows that $\dim Z_2 \leq (6g + 2n - 3) - 2$.

(3) We claim that $Z_3 = Z \setminus (Z_1 \cup Z_2)$ must have $\dim Z_3 \leq (6g + 2n - 3) - 2$. Suppose first that $g \geq 1$. The pair (a_1, a_2) varies over a locus of dimension 6. For fixed (a_1, a_2), since $\mathrm{tr}(b_1) \neq \pm 2$ we see by Lemma A.2 that there are only finitely many possible values of b^i. For fixed (a_1, a_2) and value of b^i, the remaining matrices in $(a_1, \cdots, a_m)$ vary over a locus of dimension $6(g-1) + 2(n-1) - 3$ by Lemma 4.3 (noting that $2(g-1) + (n-1) - 1 \geq 2$ if $b^i \in \{\pm \mathbf{1}\}$ by assumption that $Z_3 \cap Z_1 = \emptyset$). Hence, for $g \geq 1$, Z_3 has dimension bounded by

$$6 + (6(g-1) + 2(n-1) - 3) = (6g + 2n - 3) - 2.$$

Arguing similarly, if $g = 0$ so that $n \geq 5$, Z_3 has dimension bounded by

$$4 + (6g + 2(n-3) - 3) = (6g + 2n - 3) - 2.$$

Therefore, $\dim Z_3 \leq (6g + 2n - 3) - 2$.

From the above computations, we conclude that $\dim Z \leq (6g + 2n - 3) - 2$. Hence, $\mathrm{Rep}_{g,n,k}$ is normal, as desired. $\qquad \square$

Proposition 4.8. $\mathrm{Rep}_{1,2,k}$ *is normal.*

Proof. Since $\mathrm{Rep}_{1,2,k}$ is a complete intersection of dimension 7 in the regular scheme $\mathbf{M}^m$, it is in particular Cohen-Macaulay. By Serre's criterion, it suffices to show

that $\mathrm{Rep}_{1,2,k}$ is regular in dimension 1. It suffices to show that locus defined by each of conditions (1) and (2) of Proposition 4.2 is of dimension at most 5. We shall represent a point of $\mathrm{Rep}_{1,2,k}$ by a triple of matrices $(a_1, a_2, a_3) \in \mathrm{SL}_2^3$ satisfying $\mathrm{tr}(a_3) = k_1$ and $\mathrm{tr}(\langle a_1, a_2 \rangle a_3) = k_2$, and adopt the notations of Section 4.2.

Consider first the locus of $\mathrm{Rep}_{1,2,k}$ where $a_i = \pm\mathbf{1}$ for some $i \in \{1, 2, 3\}$. By Lemmas 4.3 and A.4, we see that this locus has dimension at most $3 + 2 - 1 = 4$ when $i \in \{1, 2\}$, and at most $3 + 3 - 1 = 5$ when $i = 3$. The union $Y = \bigcup_{i=1}^3 \{a_i = \pm\mathbf{1}\}$ includes the locus of $\mathrm{Rep}_{1,2,k}$ defined by condition (1) of Proposition 4.2, and is at most 5-dimensional.

It remains to estimate the dimension of the locus $Z \subset \mathrm{Rep}_{1,2,k} \setminus Y$ defined by condition (2) of Proposition 4.2. In what follows, we stratify Z into subloci and estimate their respective dimensions.

(1) Let $Z_1 \subset Z$ be the locus where $\mathrm{tr}(a_1) \neq \pm 2$. From the condition

$$[a_1^{-1}, a_2^{-1} a_3 a_1 a_2] = 0,$$

and Lemma A.6, we see that for given a_1 there are only finitely many possible values of $a_2^{-1} a_3 a_1 a_2$. The sublocus where $a_3 a_1 = \pm\mathbf{1}$ has dimension at most $2 + 3 = 5$, so we may restrict our attention to the sublocus of Z_1 where $a_3 a_1 \neq \pm\mathbf{1}$. For fixed a_1, there are only finitely many possible values of $\mathrm{tr}(a_3 a_1)$, and hence a_3 varies over a locus of dimension 1, and for fixed a_1 and a_3 since there are only finitely many possible values of $a_2^{-1} a_3 a_1 a_2$ showing that a_2 varies over a locus of dimension 1 (the centralizer of $a_1 a_3 \neq \pm\mathbf{1}$ being 1-dimensional). Thus, the sublocus of Z_1 where $a_3 a_1 \neq \pm\mathbf{1}$ has dimension at most $3 + 1 + 1 = 5$. Therefore, we conclude that $\dim Z_1 \leq 5$.

(2) Let $Z_2 \subset Z \setminus Z_1$ be the locus where $\mathrm{tr}(a_2) \neq \pm 2$. We then consider

$$[a_2, a_1^{-1} a_2^{-1} a_3 a_1] = 0,$$

and by the same argument as in part (1) we conclude that $\dim Z_2 \leq 5$.

(3) Let $Z_3 \subset Z \setminus (Z_1 \cup Z_2)$ be the locus where $\mathrm{tr}(a_1 a_2) \neq \pm 2$. By the condition

$$[a_1 a_2, a_1^{-1} a_2^{-1} a_3] = 0$$

and Lemma A.6, for fixed a_1 and a_2 there are at most finitely many possible values of $a_1^{-1} a_2^{-1} a_3$ and hence of a_3. Thus, $\dim Z_4 \leq 2 + 2 = 4$.

(4) Let $Z_4 = Z \setminus (Z_1 \cup Z_2 \cup Z_3)$. We must have $\mathrm{tr}(a_1), \mathrm{tr}(a_2), \mathrm{tr}(a_1 a_2) \in \{\pm 2\}$, and hence (a_1, a_2) varies over a locus of dimension at most 3. Hence, $\dim Z_4 \leq 3 + 2 = 5$.

This completes the proof that $\dim Z \leq 5$, and hence $\mathrm{Rep}_{1,2,k}$ is normal. $\square$

Proposition 4.9. *If $m \geq 3$, the intersection $\mathrm{Crit}(E, F) \cap H_{g,n}$ has codimension at least 1 in $H_{g,n}$, and in particular $H_{g,n}$ is reduced.*

Proof. First, the locus in $H_{g,n}$ determined by the condition $a_i = 0$ has dimension

$$
\begin{aligned}
3 + 6(g-1) + 2(n-1) = (6g + 2n - 3) - 2 & \quad \text{if } i \in \{1, \cdots, 2g\}, \\
6g + 2(n-2) = (6g + 2n - 3) - 1 & \quad \text{if } i \in \{2g + 1, \cdots, 2g + n - 1\}.
\end{aligned}
$$

Therefore, the union $Y = \bigcup_{i=1}^m \{a_i = 0\}$ has dimension at most $(6g + 2n - 3) - 1$. Since $\mathrm{Crit}(E) \cap H_{g,n}$ and the locus defined by condition (1) of Lemma 4.2 both lie in Y, their dimensions also cannot be more than $(6g + 2n - 3) - 1$.

Next, let $Z \subset H_{g,n} \setminus Y$ be the locus defined by condition (2) of Lemma 4.2. We must show that $\dim Z \leq (6g + 2n - 3) - 1$. Let us first treat the cases $(g, n) = (0, 4)$ and $(1, 2)$ separately.

(1) Let $(g, n) = (0, 4)$. The locus Z consists of $(a_1, a_2, a_3) \in \mathbf{M}_{0,0}^3(\mathbf{C})$ with a_i nonzero and $[a_1, a_2 a_3] = [a_2, a_3 a_1] = [a_3, a_1 a_2] = 0$. Suppose that one of $a_1 a_2$, $a_2 a_3$, and $a_3 a_1$ is nonzero; say $a_1 a_2 \neq 0$. Then we must have $\mathrm{tr}(a_1 a_2) \neq 0$ by Lemma A.8.(1). But the condition $[a_3, a_1 a_2] = 0$ implies that $a_1 a_2$ is a scalar multiple of a_3 and hence $\mathrm{tr}(a_1 a_2) = 0$, a contradiction. Arguing similarly for the other cases, we conclude that

$$a_1 a_2 = a_2 a_3 = a_3 a_1 = 0.$$

The condition $\mathrm{tr}\, a_2 = \mathrm{tr}\, a_3 = 0$ then implies that a_2 and a_3 are both scalar multiples of a_1. Thus, Z is 4-dimensional.

(2) Let $(g, n) = (1, 2)$. The locus Z consists of $(a_1, a_2, a_3) \in \mathbf{M}_0^2 \times \mathbf{M}_{0,0}(\mathbf{C})$ with a_i nonzero satisfying, among other things, $[\langle a_1, a_2 \rangle, a_3] = 0$. By Lemma A.2.(1), this implies that $\langle a_1, a_2 \rangle$ is a scalar multiple of a_3 and in particular $\mathrm{tr}\langle a_1, a_2 \rangle = 0$. The condition $\mathrm{tr}(\langle a_1, a_2 \rangle a_3) = 0$ then shows that in fact $\langle a_1, a_2 \rangle a_3 = 0$ by Lemma A.8(1). The sublocus of A where $\langle a_1, a_2 \rangle \neq 0$ has dimension

$$2 + 1 + 3 = 6.$$

Indeed, a_3 varies over a locus of dimension 2, for fixed a_3 the value of $\langle a_1, a_2 \rangle$ varies over a locus of dimension 1 by Lemma A.2, and for fixed value of $\langle a_1, a_2 \rangle \neq 0$ the pair (a_1, a_2) varies over a locus of dimension 3 by Lemma 4.5.

It remains to consider the locus $\langle a_1, a_2 \rangle = 0$. This condition implies that at least one of $a_1 a_2, a_2 a_1^*, a_1^* a_2^*$ is zero, by Lemma A.6.(2). The locus (a_1, a_2) where at least two of them are zero has dimension at most 4 by Lemma A.7, and hence (a_1, a_2, a_3) would vary over a locus of dimension at most $4 + 2 = 6$, as desired. Thus, we may assume that exactly one of $a_1 a_2, a_2 a_1^*, a_1^* a_2^*$ is zero. We thus have the following possibilities.

(a) Cosider $a_1 a_2 = 0$ and $a_2 a_1^*, a_1^* a_2^* \neq 0$. We must have $a_2 a_1^* a_2^* \neq 0$ by Lemma A.6.(1), and $a_1 \neq 0$. The condition

$$[a_2 a_1^* a_2^*, a_3 a_1] = 0$$

shows that $\mathrm{tr}(a_3 a_1) = 0$ since $\mathrm{tr}(a_2 a_1^* a_2^*) = 0$. For fixed a_3, the locus of $a_1 \in \mathbf{M}_0(\mathbf{C})$ with $\mathrm{tr}(a_3 a_1) = 0$ has dimension 2 (as easily verified when $a_3 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, and deduced from this in the other cases by conjugation). For fixed nonzero a_1 the locus of of $a_2 \in \mathbf{M}_0(\mathbf{C})$ with $a_1 a_2 = 0$ has dimension 2 by Lemma A.5(1). Thus, the locus of $(a_1, a_2, a_3) \in Z(\mathbf{C})$ with $a_1 a_2 = 0$ has dimension at most $2 + 2 + 2 = 6$, as desired.

(b) Consider $a_2 a_1^* = 0$ and $a_1 a_2, a_1^* a_2^* \neq 0$, so that (a_1, a_2) varies over a locus of dimension at most 5. We have $a_1(\mathrm{tr}(a_2)\mathbf{1} - a_2) = (a_2 a_1^*)^* = 0$ which implies that $\mathrm{tr}\, a_2 \neq 0$ and similarly $\mathrm{tr}\, a_1 \neq 0$. Now, we have

$$a_1^* a_2^* a_3 a_1 a_2 = -[a_2, a_1^* a_2^* a_3 a_1] = 0.$$

Since $a_1^* a_2^*, a_1 a_2 \neq 0$ and $\mathrm{tr}(a_3) = 0$, the above condition implies that, for fixed (a_1, a_2), a_3 varies over a locus of dimension at most 1 by Lemma A.5.(3). Hence, (a_1, a_2, a_3) varies over a locus of dimension $5 + 1 = 6$, as desired.

(c) Consider $a_1^* a_2^* = 0$. We have $a_2^* \neq 0$ and $a_1 a_2 a_1^* \neq 0$, and from the condition $[a_2^* a_3, a_1 a_2 a_1^*] = 0$ we argue as in (a) to see that this locus has dimension at most 6, as desired.

Thus, we have shown that Z has dimension 6.

In the remainder of the proof, we thus assume that $2g + n - 1 \geq 4$. Let us use the notations of Section 4.2. For a cyclic interval I in $\mathfrak{C}_{g+n-1}$, let $Z_I \subset Z$ be the sublocus defined by the condition $b_I \neq 0$. Note that if $I \subseteq J$ are two cyclic intervals (containment meaning that I is a subsequence of J) then $Z_J \subseteq Z_I$. Let

$$W_I = Z_I \setminus \bigcup_{I \subsetneq J} Z_J$$

where the union on the right hand side runs over all cyclic intervals J in $\mathfrak{C}_{g+n-1}$ containing I and distinct from I. Note that we have a stratification

$$Z = \coprod_I W_I$$

where the union runs over the finitely many cyclic intervals I of $\mathfrak{C}_{g+n-1}$. Hence, it suffices to show that each W_I has dimension at most $(6g + 2n - 3) - 1$.

First, consider the case where $|I| = g + n - 1$, and write $I = (i_0, \cdots, i_0 + g + n - 2)$. The condition $[b_{i_0}, b^{i_0}] = 0$ on Z with $b_{i_0}, b^{i_0} \neq 0$ then implies that, by Lemmas A.2 and 4.5, the dimension of W_I is at most

$$6 + 1 + 6(g - 1) + 2(n - 1) - 3 = (6g + 2n - 3) - 1 \qquad \text{if } g \geq 1, \text{ or}$$
$$2 + 1 + 2(n - 2) - 3 = (6g + 2n - 3) - 1 \qquad \text{if } g = 0$$

as desired.

Now, suppose that $|I| < g + n - 1$. Let us further stratify $W_I = W_I' \sqcup W_I''$ where W_I' (resp. W_I'') is the sublocus defined by $b^I \neq 0$ (resp. $b^I = 0$). The subloci W_I' can be handled as above and shown to have dimension at most $(6g + 2n - 3) - 1$. Thus, it remains to consider the subloci W_I'' for cyclic intervals I with $|I| < g + n - 1$. There are two possibilities to consider.

(1) We have $|I^c| = 1$. Since $Z \subset H_{g,n} \setminus Y$, we must then have $I^c = \{i\}$ for some $i \in \{1, \cdots, g\}$. Since $b_i = \langle a_{2i-1}, a_{2i} \rangle = 0$ on W_I'', by Lemma A.6 at least one of $a_{2i-1} a_{2i}$, $a_{2i} a_{2i-1}^*$, and $a_{2i-1}^* a_{2i}^*$ must be zero. By Lemma A.7, the sublocus of W_I'' where at least two of the products is zero has dimension

$$4 + 6(g - 1) + 2(n - 1) = (6g + 2n - 3) - 1.$$

Thus, we are left to consider the sublocus W_I''' of W_I'' where exactly one of the products $a_{2i-1} a_{2i}$, $a_{2i} a_{2i-1}^*$, and $a_{2i-1}^* a_{2i}^*$ is be zero. We thus have the following possibilities.

(a) Consider $a_{2i-1} a_{2i} = 0$ and $a_{2i} a_{2i-1}^*, a_{2i-1}^* a_{2i}^* \neq 0$. We must then have $a_{2i} a_{2i-1}^* a_{2i}^* \neq 0$ by Lemma A.6, and $a_{2i-1} \neq 0$. The condition

$$[a_{2i} a_{2i-1}^* a_{2i}^*, b^i a_{2i-1}] = 0$$

shows that $\operatorname{tr}(b^i a_{2i-1}) = 0$ since $\operatorname{tr}(a_{2i} a_{2i-1}^* a_{2i}^*) = 0$. For fixed $b^i = b_I$ (which is nonzero on W_I), we see that the locus of $a_{2i-1} \in \mathbf{M}_0(\mathbf{C})$ with $\operatorname{tr}(b^i a_{2i-1}) = 0$ has dimension 2, and for fixed nonzero a_{2i-1} the locus of $a_{2i} \in \mathbf{M}_0(\mathbf{C})$ with $a_{2i-1} a_{2i} = 0$ has dimension at most 2. Thus,

the locus of $(a_1, \cdots, a_m) \in W_I''(\mathbf{C})$ with $a_{2i-1}a_{2i} = 0$ has dimension at most

$$2 + 2 + 6(g - 1) + 2(n - 1) = (6g + 2n - 3) - 1$$

as desired.

(b) Consider $a_{2i}a_{2i-1}^* = 0$ and $a_{2i-1}a_{2i}, a_{2i-1}^*a_{2i}^* \neq 0$, so that (a_{2i-1}, a_{2i}) varies over a locus of dimension at most 5. Now, we have

$$a_{2i-1}^* a_{2i}^* b^i a_{2i-1} a_{2i} = -[a_{2i-1}^*, a_{2i}^* b^i a_{2i-1} a_{2i}] = 0.$$

Since $a_{2i-1}^* a_{2i}^*, a_{2i-1}a_{2i} \neq 0$, the above condition implies that, for fixed (a_1, a_2), the value of b^i varies over a locus of 2 by Lemma A.5(3). Hence, by Lemma 4.5, the the sublocus of W_I'' with $a_{2i}a_{2i-1}^* = 0$ has dimension at most

$$5 + 2 + 6(g - 1) + 2(n - 1) - 3 = (6g + 2n - 3) - 1$$

as desired.

(c) Consider $a_{2i-1}^* a_{2i}^* = 0$ and $a_{2i-1}a_{2i}, a_{2i}a_{2i-1}^* \neq 0$. We must then have $a_{2i-1}a_{2i}a_{2i-1}^* \neq 0$ by Lemma A.6, and $a_{2i}^* \neq 0$. From the condition $[a_{2i}^* b^i, a_{2i-1}a_{2i}a_{2i-1}^*] = 0$ we argue as in (a) to see that this locus has dimension at most $(6g + 2n - 3) - 1$, as desired.

(2) We have $|I^c| \geq 2$. Let us write $I^c = \{i_0, \cdots, i_0 + k\}$ for some $k \geq 1$. Let $J = \{i_0 + 1, \cdots, i_0 + k\}$, so that we have, on W_I'',

$$b_{i_0}b_J = 0, \quad b_I b_{i_0} = 0, \quad b_J b_I = 0.$$

Let us further stratify $W_I'' = V_I \sqcup V_I'$ where V_I (resp. V_I') is the sublocus defined by the condition $b_J \neq 0$ (resp. $b_J = 0$). It remains to show that V_I and V_I' each have dimension at most $(6g + 2n - 3) - 1$.

(a) Let us define

$$g_1 = |I \cap \{1, \cdots, g\}|,$$
$$n_1 = |I \cap \{g + 1, \cdots, g + n - 1\}|,$$
$$g_2 = |J \cap \{1, \cdots, g\}|, \quad \text{and}$$
$$n_2 = |J \cap \{g + 1, \cdots, g + n - 1\}|.$$

Now, if $i_0 \in \{1, \cdots, g\}$, then by (a minor variant of) Lemmas 4.5 and A.5 and condition $b_{i_0}b_J = b_I b_{i_0} = 0$, V_I has dimension bounded by

$$6 + (6g_1 + 2n_1 - 1) + (6g_2 + 2n_2 - 1) = 6g + 2(n - 1) - 2 = (6g + 2n - 3) - 1$$

where $g_1 + g_2 = g - 1$ and $n_1 + n_2 = n - 1$. On the other hand, if $i_0 \in \{g+1, \cdots, g+n-1\}$, then by a similar argument V_I has dimension bounded by

$$2 + (6g_1 + 2n_1 - 1) + (6g_2 + 2n_2 - 1) = 6g + 2(n - 2) = (6g + 2n - 3) - 1$$

where $g_1 + g_2 = g$ and $n_1 + n_2 = n - 2$. This proves that V_I has dimension at most $(6g + 2n - 3) - 1$, as desired.

(b) In addition to g_i and n_i as above, let us set

$$g_1' = |(I \cup \{i_0\}) \cap \{1, \cdots, g\}|, \quad \text{and}$$
$$n_1' = |(I \cup \{i_0\}) \cap \{g + 1, \cdots, g + n - 1\}|.$$

Since $g_1' + g_2 = g$ and $n_1' + n_2 = n - 1$, by Lemma 4.6 the dimension of V_I' is bounded by

$$(6g_1' + 2n_1' - 1) + (6g_2 + 2n_2 - 1) = 6g + 2(n-1) - 2 = (6g + 2n - 3) - 1$$

proving the desired result.

This completes the proof that $\dim W_I'' \leq (6g + 2n - 3) - 1$. Therefore, we have $\dim Z \leq (6g+2n-3)-1$, completing the proof that the intersection $\mathrm{Crit}(E, F) \cap H_{g,n}$ has codimension at least 1 in $H_{g,n}$. Finally, since $H_{g,n}$ is Cohen-Macaulay, the second statement of the proposition follows from the first. $\square$

5. Curves on surfaces

5.1. Let $S_{g,n}$ be a compact oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components satisfying $\chi(S_{g,n}) = 2 - 2g - n < 0$. As in Section 3.1, let us fix a sandard presentation of the fundamental group

$$\pi_1(S_{g,n}) = \langle a_1, \cdots, a_{2g+n} | \langle a_1, a_2 \rangle \cdots \langle a_{2g-1}, a_{2g} \rangle a_{2g+1} \cdots a_{2g+n} = \mathbf{1} \rangle$$

and let $\sigma = \{a_1, \cdots, a_{2g+n-1}\}$ be the standard set of free generators. (In Section 5.2, we will give a construction of $S_{g,n}$ making these standard choices concrete.)

Let us denote by $\chi_{g,n}$ the set of conjugacy classes of $\pi_1(S_{g,n}) \setminus \{\mathbf{1}\}$, and let $\bar{\chi}_{g,n}$ be the quotient of $\chi_{g,n}$ obtained by setting the cojugacy class of $a \in \pi_1(S_{g,n})$ to be equivalent to that of a^{-1}. We shall refer to elements of $\bar{\chi}_{g,n}$ as reduced homotopy classes. We may identify $\chi_{g,n}$ with the set of nontrivial (i.e. not null-homotopic) free homotopy classes of loops $S^1 \to S_{g,n}$, and $\bar{\chi}_{g,n}$ with the set of nontrivial free homotopy classes of loops considered up to (possibly orientation-reversing) reparametrizations of S^1. Let $\mathrm{Fin}\,\bar{\chi}_{g,n}$ denote the collection of all finite multisets of elements in $\bar{\chi}_{g,n}$.

By a *curve* in $S_{g,n}$ we shall mean a connected closed 1-submanifold of $S_{g,n}$ not bounding a disk. A *multicurve* $a = \coprod_i a_i \subset S_{g,n}$ is a finite union of disjoint curves in $S_{g,n}$. We shall say that a multicurve a is non-peripheral if no curve is isotopic to a boundary component of $S_{g,n}$. Given a curve $a \subset S_{g,n}$, there is a well-defined element $a \in \bar{\chi}_{g,n}$ obtained by taking the reduced homotopy class of any choice of parametrization $S^1 \to a$. This extends to a well-defined assignment

(∗) $\{\text{isotopy classes of multicurves in } S_{g,n}\} \to \mathrm{Fin}\,\bar{\chi}_{g,n}$

sending $a = \coprod_i a_i \mapsto \{a_i\}$. Let $X_{g,n}$ be the character variety of $S_{g,n}$ as in Section 3.1. In the coordinate ring $R_{g,n} = \mathbf{C}[X_{g,n}]$, we have $[a^{-1}] = [a]$ and $[aba^{-1}] = [b]$ for every $a, b \in \pi_1(S_{g,n})$. In particular, the function $\pi_1(S_{g,n}) \to R_{g,n}$ given by $a \mapsto [a]$ factors through the projection $\pi_1(S_{g,n}) \to \bar{\chi}_{g,n}$. We extend the function $\bar{\chi}_{g,n} \to R_{g,n}$ to an assignment

$$\mathrm{Fin}\,\bar{\chi}_{g,n} \to R_{g,n}$$

sending $a = \{a_1, \cdots, a_s\} \mapsto [a] = [a_1] \cdots [a_s] \in R_{g,n}$. In particular, for each isotopy class of a multicurve $a = \coprod_i a_i$ in $S_{g,n}$, there is a well-defined regular function $[a] = \prod_i [a_i] \in R_{g,n}$ (the empty multicurve $\emptyset$ corresponding to 1). We have the following remarkable result due to Charles and Marché [5, Theorem 1.1].

Theorem 5.1 (Charles-Marché [5]). *The regular functions $[a]$, as a runs over the isotopy classes of multicurves in $S_{g,n}$, form a $\mathbf{C}$-linear basis of $R_{g,n}$.*

Remark. In particular, the assignment (∗) considered above is injective.

5.2. We give an explicit presentation of $S_{g,n}$ as follows. For any compact oriented surface S, we shall denote by ∂S its boundary and $S^\circ = S \setminus \partial S$. Let

$$\mathbf{D} = \{z \in \mathbf{C} : |z| \le 1\}$$

be the closed unit disk. Fix integers $g \ge 0$ and $n \ge 1$ with $m = 2g + n - 1 \ge 2$. Let $\mu_m = \{p_1, \cdots, p_m\}$, where $p_k = e^{2\pi ik/m}$, be the set of mth roots of unity. Let us fix $2m$ open intervals $\{\mathbf{I}'_k, \mathbf{I}''_k\}_{k=1,\cdots,m}$ on $\partial \mathbf{D}$ with disjoint closures in $\partial \mathbf{D} \setminus \{p_1, \cdots, p_m\}$, cyclically ordered counterclockwise on $\partial \mathbf{D}$ relative to the points $\{p_1, \cdots, p_m\}$ in the following way:

$$\mathbf{I}'_1 \to p_1 \to \mathbf{I}''_2 \to \mathbf{I}''_1 \to p_2 \to \mathbf{I}'_2 \to \cdots$$
$$\to \mathbf{I}'_{2g-1} \to p_{2g-1} \to \mathbf{I}''_{2g} \to \mathbf{I}''_{2g-1} \to p_{2g} \to \mathbf{I}'_{2g}$$
$$\to \mathbf{I}'_{2g+1} \to p_{2g+1} \to \mathbf{I}''_{2g+1} \to \cdots \to \mathbf{I}'_m \to p_m \to \mathbf{I}''_m.$$

Let us write $\mathbf{I}_k = \mathbf{I}'_k \sqcup \mathbf{I}''_k$ and set $\mathbf{I} = \coprod_{k=1}^m \mathbf{I}_k$. For $k = 1, \cdots, m$, we attach a compact rectangular strip $R_k = [0,1] \times [0,1]$ to $\mathbf{D}$, so that $\{0\} \times [0,1]$ is identified with $\mathbf{I}'_k$ and $\{1\} \times [0,1]$ is identified with $\mathbf{I}''_k$, in such a way that the resulting surface remains orientable. Note that the surface $S_{g,n}$ thus obtained has genus g with $n \ge 1$ boundary components. Let $0 \in \mathbf{D}$ be the fixed base point of $S_{g,n}$.

For $k = 1, \cdots, m$, let $a_k \in \pi_1(S_{g,n})$ be (the class of) a based loop which passes from 0 to a point in $\mathbf{I}'_k$ via a straight line segment in $\mathbf{D}$, then to a point of $\mathbf{I}''_k$ via a simple path in R_k, then back to 0 via a straight line segment in $\mathbf{D}$. Finally, let us fix a point $q \in \partial \mathbf{D} \setminus \overline{\mathbf{I}}$ situated so that we have a cyclic ordering counterclockwise

$$\overline{\mathbf{I}''_m} \to q \to \overline{\mathbf{I}'_1}$$

along $\partial \mathbf{D}$, and let $a_{2g+n} \in \pi_1(S_{g,n})$ be (the class of) the loop passing from 0 to q via a straight line segment, then traveling once clockwise along the boundary component of $\partial S_{g,n}$ containing q, and then returning to 0 via the same line segment. Under these choices, we have the standard presentation

$$\pi_1(S_{g,n}) = \langle a_1, \cdots, a_{2g+n} | [a_1, a_2] \cdots [a_{2g-1}, a_{2g}] a_{2g+1} \cdots a_{2g+n} = \mathbf{1} \rangle$$

and we set $\sigma = \{a_1, \cdots, a_m\}$ to be the set of standard free generators. Note that each a_k for $k = 1, \cdots, m$ is (the class of) a simple loop in $S_{g,n}$. Given a collection of nontrivial (i.e. not null-homotopic) loops $a : \coprod_{i=1}^s S^1 \to S_{g,n}$, there exists an immersion $a_0 : \coprod_{i=1}^s S^1 \to S_{g,n}$ homotopic to a such that

- $\mathrm{Im}(a_0)$ intersects $\mathbf{I}$ only finitely many times,
- $\mathrm{Im}(a_0)$ contains no intersections outside of $\mathbf{D}^\circ$, and
- $|\mathrm{Im}(a_0) \cap \mathbf{I}'_k| = |\mathrm{Im}(a_0) \cap \mathbf{I}''_k|$ for every $k \in \{1, \cdots, m\}$.

This motivates the following definition. By a *chord* in $\mathbf{D}$ we mean a straight line segment in $\mathbf{D}$ joining two distinct points of $\partial \mathbf{D}$. Given two distinct points $v_1, v_2 \in \partial \mathbf{D}$, we shall write $\{v_1, v_2\}$ to denote the chord joining them.

Definition 5.2. Let $r \ge 0$ be an integer. We denote by $D_{g,n}(r)$ the collection of graphs $\Gamma = (V_\Gamma, E_\Gamma)$ drawn (but not necessarily embedded) in $\mathbf{D}$, consisting of a collection $E_\Gamma = \{e_1, \cdots, e_r\}$ of chords in $\mathbf{D}$ for edges, such that the set of vertices $V_\Gamma = \bigcup_{i=1}^r \partial e_i$ consists of $2r$ distinct points lying in $\mathbf{I} \subset \partial \mathbf{D}$ and satisfying $|\mathbf{I}'_k \cap V_\Gamma| = |\mathbf{I}''_k \cap V_\Gamma|$ for every $k = 1, \cdots, m$. We set $D_{g,n} = \bigcup_{r \ge 0} D_{g,n}(r)$. Let $C_{g,n}(r)$ be the set of planar graphs in $D_{g,n}(r)$, and set $C_{g,n} = \bigcup_{r=0}^\infty C_{g,n}(r)$.

A graph $\Gamma \in D_{g,n}$ is considered only up to isotopy of the vertex set V_Γ within $\mathbf{I}$. By a generic choice of vertex positions, we may assume that each intersection point among the chords lies on exactly two chords, and denote by $I(\Gamma)$ the number of intersection points on Γ under such configuration. Let us call $\Gamma \in D_{g,n}$ *reduced* if no edge of Γ joins two vertices lying on the same connected component of $\mathbf{I}$.

Given $\Gamma \in D_{g,n}$, we obtain a finite collection $a(\Gamma)$ of loops in $S_{g,n}$ (considered up to reparametrization of each loop) by joining vertices of Γ in each $\mathbf{I}'_k$ to $\mathbf{I}''_k$ by pairwise non-intersecting segments in R_k. This gives rise to an assignment

$$D_{g,n} \to \operatorname{Fin} \bar\chi_{g,n}$$

which is surjective by the previous discussion. If $\Gamma \in C_{g,n}$, the construction $a(\Gamma)$ gives us a multicurve in $S_{g,n}$, and the assignment $D_{g,n} \to \operatorname{Fin} \bar\chi_{g,n}$ restricts (via injectivity of the assignment $(*)$ in Section 5.1) to

$$C_{g,n} \to \{\text{isotopy classes of multicurves in } S_{g,n}\}$$

which is surjective (but not injective in general).

5.3.   As in Section 2.3, the standard generating set $\sigma = \{a_1, \cdots, a_m\} \subset \pi_1(S_{g,n})$ gives rise to the word length function $\ell_\sigma : \pi_1(S_{g,n}) \to \mathbf{Z}_{\geq 0}$. This gives rise to a length function $\ell_\sigma : \overline\chi_{g,n} \to \mathbf{Z}_{\geq 0}$ given by

$$\ell_\sigma(a) = \min\{\ell_\sigma(b) : b \in \pi_1(S_{g,n}) \text{ lies in class } a\}.$$

We extend this additively to a function $\ell_\sigma : \operatorname{Fin} \bar\chi_{g,n} \to \mathbf{Z}_{\geq 0}$. In other words, given $b = \{b_1, \cdots, b_s\} \in \operatorname{Fin} \bar\chi_{g,n}$ we define $\ell_\sigma(b) = \sum_{i=1}^s \ell_\sigma(b_i)$. In particular, the length of a multicurve is well-defined via the assignment $(*)$ from Section 5.1. We remark that, by definition, the empty multicurve $\emptyset \subset S_{g,n}$ has length $\ell_\sigma(\emptyset) = 0$, and it is the unique multicurve with this property.

Lemma 5.3. *For each isotopy class of a multicurve $a \subset S_{g,n}$, there is a unique reduced $\Gamma \in C_{g,n}$ such that $a = a(\Gamma)$. Furthermore, if $\ell_\sigma(a) = r$, then $\Gamma \in C_{g,n}(r)$.*

Proof. Observe that, if $\Gamma, \Gamma' \in D_{g,n}$ are reduced and there is a homotopy from $a(\Gamma)$ to $a(\Gamma')$ within $S_{g,n}$ (up to reparametrizations of the loops), then there is a homotopy $(a(t))_{t \in [0,1]}$ with $a(0) = a(\Gamma)$ and $a(1) = a(\Gamma')$ such that

$$|\operatorname{Im} a(t) \cap \mathbf{I}| \leq |\operatorname{Im} a(\Gamma) \cap \mathbf{I}|$$

for all $t \in [0,1]$. Similarly, given $\Gamma, \Gamma' \in C_{g,n}$ reduced such that there is an isotopy from $a(\Gamma)$ to $a(\Gamma')$ within $S_{g,n}$, there is an isotopy $(a(t))_{t \in [0,1]}$ from $a(\Gamma)$ to $a(\Gamma')$ satisfying the above inequality and in particular we must have $\Gamma = \Gamma'$. Further, given a multicurve a it is clear that there is a reduced $\Gamma \in C_{g,n}$ such that $a(\Gamma)$ is isotopic to a. This shows that the assignment

$$\{\Gamma \in C_{g,n} : \Gamma \text{ reduced}\} \to \{\text{isotopy classes of multicurves in } S_{g,n}\}$$

given by $\Gamma \mapsto a(\Gamma)$ is bijective, proving the first statement.

To prove the second statement, note first that if $\Gamma \in D_{g,n}(r)$ for some $r \geq 0$ then $\ell_\sigma(a(\Gamma)) \leq r$. Thus, given a multicurve $a \subset S_{g,n}$ with $\ell_\sigma(a) = r$, by our observation in the previous paragraph it suffices to show that there exists $\Gamma \in D_{g,n}(r)$ such that $a(\Gamma)$ is homotopic to a. In fact, we are reduced to the case where a is a curve. But for a curve, this statement is obvious. $\square$

We prove the following refinement of Theorem 5.1. Recall that $\operatorname{Fil}^\sigma$ is the σ-word length filtration on $R_{g,n}$.

Lemma 5.4. *Let $r \geq 0$. The regular functions $[a]$, as a runs over the isotopy classes of multicurves in $S_{g,n}$ with $\ell_\sigma(a) \leq r$, form a $\mathbf{C}$-linear basis of $\mathrm{Fil}_r^\sigma R_{g,n}$.*

Proof. By Theorem 5.1, the collection $B_r \subset \mathrm{Fil}_r^\sigma$ of functions $[a]$, as a runs over the isotopy classes of multicurves in $S_{g,n}$ with $\ell_\sigma(a) \leq r$, is $\mathbf{C}$-linearly independent. Therefore, it remains to show that B_r spans Fil_r^σ. Given any $a \in \pi_1(S_{g,n})$ with $\ell_\sigma(a) \leq r$, it is easy to see that there exists $\Gamma \in D_{g,n}(s)$ with $s \leq r$ such that $a = a(\Gamma)$. Thus, we are reduced to showing that $[a(\Gamma)] \in \mathrm{Span}\, B_r$ for each $\Gamma \in D_{g,n}(r)$.

So let $\Gamma \in D_{g,n}(r)$ be given. We shall proceed by induction on the intersection number $I(\Gamma)$. Let us choose the vertex positions for Γ so that each of the $I(\Gamma)$ intersection points lies on exactly two of the chords. If $I(\Gamma) = 0$, then $[a(\Gamma)] \in B_r$ and we are done. Suppose next that $I(\Gamma) \geq 1$, and let $e = \{v_1, v_2\}$ and $e' = \{v'_1, v'_2\}$ be two edges of Γ intersecting at a point p. From this we can construct two graphs $\Gamma', \Gamma'' \in D_{g,n}(r)$ as follows.

- $V_{\Gamma'} = V_{\Gamma''} = V_\Gamma$, and
- we have

$$E_{\Gamma'} = (E_\Gamma \setminus \{e, e'\}) \cup \{\{v_1, v'_1\}, \{v'_1, v'_2\}\},$$
$$E_{\Gamma''} = (E_\Gamma \setminus \{e, e'\}) \cup \{\{v_1, v'_2\}, \{v'_1, v_2\}\}.$$

The fact that $\Gamma, \Gamma'' \in D_{g,n}(r)$ implies that $[a(\Gamma')], [a(\Gamma'')] \in \mathrm{Fil}_r^\sigma$. We verify easily that $I(\Gamma'), I(\Gamma'') < I(\Gamma)$, so by the inductive hypothesis $[a(\Gamma')], [a(\Gamma'')] \in \mathrm{Span}\, B_r$. But note that

$$[a(\Gamma)] = \epsilon'[a(\Gamma')] + \epsilon''[a(\Gamma'')]$$

for some $\epsilon', \epsilon'' \in \{\pm 1\}$, as easily seen by the condition that $[a][b] = [ab] + [ab^{-1}]$ in $R_{g,n}$ (applied after changing our base point to the intersection point p). We thus have $[a(\Gamma)] \in \mathrm{Span}\, B_r$, completing the induction. $\square$

For $r \geq 0$, let $c'_{g,n}(r)$ be the number of isotopy classes of multicurves $a \subset S_{g,n}$ with $\ell_\sigma(a) = r$, and let $c_{g,n}(r)$ be the number of isotopy classes of non-peripheral multicurves $a \subset S_{g,n}$ with $\ell_\sigma(a) = r$. Note that we have

$$(1 - t)^{n-1}(1 - t^{4g+n-1}) \sum_{r=0}^\infty c'_{g,n}(r)\, t^r = \sum_{r=0}^\infty c_{g,n}(r)\, t^r.$$

Indeed, the n boundary components of $S_{g,n}$ have lengths $1, \cdots, 1$, and $4g + n - 1$, and any multicurve a' in $S_{g,n}$ can be written uniquely as a disjoint union $a' = a \coprod a''$ in $S_{g,n}$ where a is a non-peripheral multicurve and a'' is a finite disjoint union of curves each of which is isotopic to a boundary component of $S_{g,n}$. We are ready to prove Theorem 1.2, restated below.

Theorem 5.5. *The series $Z_{g,n}(t) = \sum_{r=0}^\infty c_{g,n}(r)\, t^r$ is a rational function, and*

$$Z_{g,n}(1/t) = Z_{g,n}(t).$$

Proof. Note that $R_{g,n}^\sigma \simeq R_m^\sigma$ where $m = 2g + n - 1$ and R_m is the coordinate ring of the character variety X_m of the free group on m generators σ. By Lemma 5.4, its Hilbert series $H_m(t) = \sum_{r=0}^\infty (\dim \mathrm{Fil}_r^\sigma)\, t^r$ is given by

$$H_m(t) = \frac{1}{1-t} \sum_{r=0}^\infty c'_{g,n}(r)\, t^r = \frac{Z_{g,n}(t)}{(1-t)^n(1-t^{4g+n-1})}.$$

By Theorem 2.14, we have $H_m(1/t) = (-1)^{3m-2} t^{2m+1} H_m(t)$. We then have

$$
\begin{aligned}
Z_{g,n}(1/t) &= (1 - t^{-1})^n (1 - t^{-(4g+n-1)}) H_m(1/t) \\
&= (-1)^{3m-2} (1 - t^{-1})^n (1 - t^{-(4g+n-1)}) t^{2m+1} H_m(t) \\
&= (-1)^{3m-2+(n+1)} (1 - t)^n (1 - t^{4g+n-1}) H_m(t) = (-1)^{3m-2+(n+1)} Z_{g,n}(t)
\end{aligned}
$$

noting that $2m + 1 = 2(2g + n - 1) + 1 = 4g + 2n - 1$. But lastly, we note that $3m - 2 + (n+1) = 3(2g + n - 1) - 2 + (n+1)$ is even, and hence $Z_{g,n}(1/t) = Z_{g,n}(t)$, as desired. $\qquad \square$

Remark. As the above proof shows, given $m \geq 2$ the functional equation in Theorem 2.14 is equivalent to Theorem 1.2 for any punctured surface $S_{g,n}$ with $m = 2g+n-1$. In the next section, we shall give an independent, combinatorial proof of Theorem 1.2 for the surface $S_{0,m+1}$, which then implies Theorem 1.2 in the general case by this observation.

6. Combinatorics of planar graphs

6.1.   Let $m \geq 2$ be a fixed integer. Let $S_m = S_{0,m+1}$ be a compact oriented surface of genus 0 with $m + 1$ boundary components. Under the standard presentation of the fundamental group, the standard set $\sigma = \{a_1, \cdots, a_m\}$ freely generates $\pi_1(S_m) \simeq F_m$. As in Section 5, let us consider the generating series

$$
Z_m(t) = Z_{0,m+1}(t) = \sum_{r=0}^{\infty} c_m(r)\, t^r
$$

where $c_m(r) = c_{0,m+1}(r)$ denotes the number of non-peripheral multicurves $a \subset S_m$ with $\ell_\sigma(a) = r$. The goal of this section is to give a combinatorial proof of the following:

Theorem 6.1. *The series $Z_m(t)$ is rational, and satisfies $Z_m(1/t) = Z_m(t)$.*

We adopt the notations of Section 5.2. In particular, S_m is viewed as the closed unit disk $\mathbf{D}$ with $2m$ rectangular strips suitably attached to the boundary $\partial \mathbf{D}$ along the $2m$ intervals $\mathbf{I}$, situated in a particular way around the set $\mu_m = \{p_1, \cdots, p_m\}$, where $p_k = e^{2\pi i k/m}$, of mth roots of unity. Let us adopt the notations $D_m(r) = D_{0,m+1}(r)$, $D_m = D_{0,m+1}$, and similarly $C_m(r) = C_{0,m+1}(r)$ and $C_m = C_{0,m+1}$.

Given a reduced graph $\Gamma \in C_m(r)$, note that one can identify the vertices in each $\mathbf{I}_k$ to a single vertex p_k, so as to obtain a planar multigraph in $\mathbf{D}$ with vertices $\mu_m = \{p_1, \cdots, p_m\} \subset \partial \mathbf{D}$ and r edges such that every vertex has even degree. Conversely, given such a planar multigraph, we can construct a unique reduced graph in $C_m(r)$ that gives rise to it. Let $B_m(r)$ denote the set of planar multigraphs Γ without self-loops in $\mathbf{D}$ having vertex set μ_m and r edges such that every vertex in Γ has even degree. In light of Lemma 5.3 and our discussion, we have

$$
Z_m(t) = (1 - t^m) \sum_{r=0}^{\infty} |B_m(r)|\, t^r.
$$

Here, the factor $(1-t^m)$ is to cancel from our count the contribution of the $(m+1)$th boundary component of S_m, which has σ-length m.

6.2. Let $\mathbf{D}$ and $\partial\mathbf{D}$ be the closed unit disk and circle. Fix an integer $m \geq 3$, and let $\mu_m = \{p_1, \cdots, p_m : p_k = e^{2\pi ik/m}\} \subset \partial\mathbf{D}$ be the set of mth roots of unity. We shall say that two points $p_i, p_j \in \mu_m$ are *contiguous* if $i - j \in \{0, \pm 1\} \mod m$.

For each integer $r \geq 0$, let $A_m(r)$ denote the collection of planar multigraphs in $\mathbf{D}$ with r edges on the m vertices μ_m, such that no edge joins contiguous vertices. Let $A_m^s(r)$ be the set of multigraphs in $A_m(r)$ which are simple (i.e. having no multiple edges). Note that we have $|A_m^s(r)| = 0$ for $r > m - 3$. Let $A_m = \bigcup_{r=0}^{\infty} A_m(r)$ and $A_m^s = \bigcup_{r=0}^{\infty} A_m^s(r)$. Define the generating series

$$F_m(t) = \sum_{r=0}^{\infty} |A_m(r)| t^r.$$

Lemma 6.2. *We have* $\sum_{r=1}^{m-3} (-1)^{r-1} |A_m^s(r)| = 1 + (-1)^{m-4}$.

Proof. The statement is obvious for $m \leq 4$, and hence we treat the case $m \geq 5$. We construct a simplicial complex $\mathbf{L}_m$ as follows. For each $r \geq 0$, the r-simplices of $\mathbf{L}_m$ are labeled by $A_m^s(r + 1)$; we let $\Delta(\Gamma)$ denote the r-simplex associated to $\Gamma \in A_m^s(r + 1)$. Whenever a graph $\Gamma \in A_m(r + 1)$ is obtained by deleting a number of edges from some $\Gamma' \in A_m(r' + 1)$ with $r < r'$, we glue the corresponding r-simplex $\Delta(\Gamma)$ to the r'-simplex $\Delta(\Gamma')$ compatibly. From this construction, the lemma is just the statement

$$\chi(\mathbf{L}_m) = 1 + (-1)^{m-4}$$

where the left hand side is the Euler characteristic of $\mathbf{L}_m$. In fact, it is a relatively well known result (first published by Lee [19, Theorem 1], also found independently by Haiman (cf. Lee *loc.cit.*)) that $\mathbf{L}_m$ is isomorphic to the boundary complex of a convex polytope of dimension $m - 3$, and therefore the following stronger result holds:

$$H_0(\mathbf{L}_m, \mathbf{Z}) = H_{m-4}(\mathbf{L}_m, \mathbf{Z}) = \mathbf{Z}, \quad \text{and} \quad H_r(\mathbf{L}_m, \mathbf{Z}) = 0 \quad \forall 1 \leq r < m - 4.$$

This finishes the proof of the lemma. $\qquad\square$

Let $r_0 \leq m - 3$ be a nonnegative integer, and let $\Gamma \in A_m^s(r_0)$. For $r_0 \leq r \leq m - 3$, let us define $A_m^s(r, \Gamma) = \{\Gamma' \in A_m^s(r) : \Gamma \subseteq \Gamma'\}$. Here, the containment $\Gamma \subseteq \Gamma'$ means that the edge set of Γ' contains the edge set of Γ. Both graphs have the same vertex set, namely μ_m.

Corollary 6.3. *Let $r_0 \leq m - 3$ be a nonnegative integer, and $\Gamma \in A_m^s(r_0)$. Then*

$$\sum_{r=r_0}^{m-3} (-1)^{r-r_0} |A_m^s(r, \Gamma)| = (-1)^{m-3-r_0}.$$

Proof. Note that Γ divides $\mathbf{D}$ into a number of polygons with numbers of sides $m_1, \cdots, m_{r_0+1} \geq 3$ such that $\sum_{k=1}^{r_0+1} (m_k - 3) + r_0 = m - 3$. Here, we are counting each arc of $\partial\mathbf{D}$ joining two contiguous vertices also as sides of a polygon. Using this and Lemma 6.2, we find

$$\sum_{r=r_0}^{m-3} (-1)^{r-r_0} |A_m^s(r, \Gamma)| = \prod_{k=1}^{r_0+1} \left\{ 1 - \sum_{r=1}^{m_k-3} (-1)^{r-1} |A_{m_k}(r)| \right\}$$

$$= \prod_{k=1}^{r_0+1} (-1)^{m_k-3} = (-1)^{m-3-r_0},$$

which is the desired result.   □

Proposition 6.4. *The generating series* $F_m(t) = \sum_{r=0}^{\infty} |A_m(r)| t^r$ *satisfies*

$$F_m(t) = \frac{f_m(t)}{(1-t)^{m-3}}$$

where $f_m(t)$ *is a polynomial of degree* $m - 3$ *such that* $t^{m-3} f_m(1/t) = f_m(t)$.

Proof. For each multigraph $\Gamma \in A_m(r)$, there is a unique simple graph $\Gamma^s \in A_m^s$ that has the same adjacent vertices as Γ (i.e. vertices joined by an edge in Γ are still joined in Γ^s). Note that Γ is obtained from Γ^s by adding edges to adjacent vertices in Γ^s. Thus, we have

$$F_m(t) = \sum_{\Gamma \in A_m^s} \frac{t^{e(\Gamma)}}{(1-t)^{e(\Gamma)}}$$

where $e(\Gamma)$ denotes the number of edges of Γ. Since $|A_m^s(r)| = 0$ for $r > m - 3$ and $e(\Gamma) \le m - 3$ for all $\Gamma \in A_m^s$, we have $F_m(t) = f_m(t)/(1-t)^{m-3}$ where $f_m(t) = \sum_{r=0}^{m-3} |A_m^s(r)| t^r (1-t)^{m-3-r}$ is a polynomial of degree at most $m - 3$. As we have $f_m(0) = F_m(0) = 1$ from the interpretation of $F_m(t)$ as a generating function, to prove that $f_m(t)$ has degree exactly $m - 3$ it suffices to prove the remaining assertion $f_m(t) = t^{m-3} f_m(1/t)$, or equivalently

$$f_m(t) = \sum_{r=0}^{m-3} |A_m^s(r)| (t-1)^{m-3-r}.$$

To prove this, we compute $F_m(t)$ in another way by a type of inclusion-exclusion principle. For each $\Gamma \in A_m^s$, note that the generating function for the number of graphs $\Gamma' \in A_N$ with $(\Gamma')^s \subseteq \Gamma$ is given by $1/(1-t)^{e(\Gamma)}$. Thus, we may write $F_m(t)$ by first adding the contributions from $A_m^s(m-3)$, then adding contributions from $A_m^s(m-4)$ and subtracting away the corresponding "overcount" from the previous step, and so on:

$$F_m(t) = \frac{|A_m^s(m-3)|}{(1-t)^{m-3}} + \frac{1}{(1-t)^{m-4}} \sum_{\Gamma \in A_m^s(m-4)} (1 - |A_m^s(m-3,\Gamma)|) + \cdots .$$

Therefore, we see upon reflection that

$$F_m(t) = \sum_{r=0}^{m-3} \frac{1}{(1-t)^r} \sum_{\Gamma \in A_m^s(r)} \left\{ 1 + \sum_{r'=r+1}^{m-3} \sum_{k=1}^{r'-r} (-1)^k \binom{r'-r}{k-1} |A_m^s(r',\Gamma)| \right\}$$

$$= \sum_{r=0}^{m-3} \frac{1}{(1-t)^r} \sum_{\Gamma \in A_m^s(r)} \sum_{r'=r}^{m-3} (-1)^{r'-r} |A_m^s(r',\Gamma)|$$

$$= \sum_{r=0}^{m-3} \frac{(-1)^{m-3-r} |A_m^s(r)|}{(1-t)^r} = \frac{1}{(1-t)^{m-3}} \sum_{r=0}^{m-3} |A_m^s(r)| (t-1)^{m-3-r}$$

(where the first equality on the last line follows by Corollary 6.3) from which we obtain $f_m(t) = \sum_{r=0}^{m-3} |A_m^s(r)| (t-1)^{m-3-r}$, showing that f_m has degree $m-3$ and that $t^{m-3} f(1/t) = f(t)$ as desired.   □

6.3. Let $m \geq 1$ be an integer. As introduced in Section 6.1, for each $r \geq 0$ let $B_m(r)$ denote the set of planar multigraphs Γ without self-loops in the closed unit disk $\mathbf{D}$ having vertices $\mu_m = \{p_1, \cdots, p_m : p_k = e^{2\pi i k/m}\}$ and r edges such that every vertex in Γ has even degree. Let $B_m^s(r)$ denote the set of those $\Gamma \in B_m(r)$ that are simple graphs, i.e. between any two vertices there is at most one edge. Let $B_m = \bigcup_{r=0}^{\infty} B_m(r)$ and $B_m^s = \bigcup_{r=0}^{\infty} B_m^s(r)$. We are interested in the generating function

$$G_m(t) = \sum_{r=0}^{\infty} |B_m(r)| t^r.$$

By a *bigon* in a multigraph we shall mean a pair of edges whose endpoints coincide. Note that for every $\Gamma \in B_m$ there exists a unique $\Gamma^s \in B_m^s$ such that Γ is obtained by adding bigons to Γ^s without losing planarity. More precisely, Γ^s is constructed from Γ by repeatedly removing bigons from Γ until none remain. Thus, $G_m(t)$ is the sum of the contribution of each $\Gamma \in B_m^s$ which is analyzed as follows. Let $\Gamma \in B_m^s(r)$ be given. Let $e(\Gamma)$ denote the number of edges of Γ, and $e^\circ(\Gamma)$ the number of edges of Γ joining non-contiguous vertices. Note that the edges of Γ joining non-contiguous vertices gives a decomposition of $\mathbf{D}$ into $e^\circ(\Gamma) + 1$ polygons (as in the proof of Corollary 6.3) with numbers of sides $m_1, \cdots, m_{e^\circ(\Gamma)+1}$ such that

$$\sum_{k=1}^{e^\circ(\Gamma)+1} (m_k - 3) + e^\circ(\Gamma) = m - 3.$$

In this setting, the contribution from Γ to the generating function $G_m(t)$ is

$$\frac{t^{e(\Gamma)}}{(1-t^2)^{m+e^\circ(\Gamma)}} \prod_{k=1}^{e^\circ(\Gamma)+1} F_{m_k}(t^2) = \frac{t^{e(\Gamma)}}{(1-t^2)^{m+e^\circ(\Gamma)}} \prod_{k=1}^{e^\circ(\Gamma)+1} \frac{f_{m_k}(t^2)}{(1-t^2)^{m_k-3}}$$

$$= \frac{t^{e(\Gamma)} \prod_{k=1}^{e^\circ(\Gamma)+1} f_{m_k}(t^2)}{(1-t^2)^{2m-3}}$$

where each $f_{m_k}(t^2)$ is a polynomial of degree $2(m_k - 3)$ by Proposition 6.4. Thus, defining $d(\Gamma) = 2\sum_{k=1}^{e^\circ(\Gamma)+1}(m_k - 3)$ and $g_\Gamma(t) = \prod_{k=1}^{e^\circ(\Gamma)+1} f_{m_k}(t^2)$ for each $\Gamma \in B_m^s$ with the m_k's as above, we see that $g_\Gamma(t)$ is a polynomial of degree $d(\Gamma)$ with the symmetry $t^{d(\Gamma)} g_\Gamma(1/t) = g_\Gamma(t)$. We may thus write

$$G_m(t) = \frac{1}{(1-t^2)^{2m-3}} \sum_{\Gamma \in B_m^s} t^{e(\Gamma)} g_\Gamma(t),$$

and we are ready to prove our main result of this section.

Theorem 6.5. *The series* $G_m(t) = \sum_{r=0}^{\infty} |B_m(r)| t^r$ *satisfies the symmetry*

$$G_m(1/t) = (-1)^{2m-3} t^m G_m(t).$$

Proof. Given $\Gamma \in B_m^s$, we define its *dual* $\Gamma^\vee$ as the graph obtained from Γ by removing (resp. adding) one edge between contiguous vertices that were adjacent (resp. not adjacent) in Γ, and retaining any edges between non-contiguous vertices in Γ. It is easy to see that in fact $\Gamma^\vee \in B_m^s$, and $(\Gamma^\vee)^\vee = \Gamma$. Note that we have $g_\Gamma(t) = g_{\Gamma^\vee}(t)$ and $d(\Gamma) = d(\Gamma^\vee)$; these follow from the fact that the construction of $g_\Gamma(t)$ only depended on the edges of Γ joining non-contiguous vertices. Furthermore, note that we have

$$d(\Gamma) + e(\Gamma) + e(\Gamma^\vee) = 3m - 6.$$

Hence, writing

$$G_m(t) = \frac{1}{2} \frac{1}{(1-t^2)^{2m-3}} \sum_{\Gamma \in B_m^s} (t^{e(\Gamma)} + t^{e(\Gamma^\vee)}) g_\Gamma(t),$$

we obtain

$$G_m(1/t) = \frac{1}{2} \frac{1}{(1-t^{-2})^{2m-3}} \sum_{\Gamma \in B_m^s} (t^{-e(\Gamma)} + t^{-e(\Gamma^\vee)}) g_\Gamma(1/t)$$

$$= \frac{1}{2} \frac{(-1)^{2m-3} t^{4m-6}}{(1-t^2)^{2m-3}} \sum_{\Gamma \in B_m^s} \frac{(t^{e(\Gamma^\vee)} + t^{e(\Gamma)}) g_\Gamma(t)}{t^{3m-6}} = (-1)^{2m-3} t^m G_m(t)$$

which gives us the result. $\qquad\square$

6.4. Returning to the problem of Section 6.1, we have $Z_m(t) = (1 - t^m) G_m(t)$. Hence, using Theorem 6.5, we find that

$$Z_m(1/t) = (1 - t^{-m}) G_m(1/t) = (1 - t^{-m})(-1)^{2m-3} t^m G_m(t)$$

$$= (1 - t^m) G_m(t) = Z_m(t),$$

which concludes the proof of Theorem 6.1.

Appendix A. Auxiliary results on matrices

A.1. **Identities.** Let $\mathbf{M}$ be the complex affine scheme parametrizing 2×2 matrices. Let x denote the standard matrix variable for $\mathbf{M}$. Viewing x and x^* as 2×2 matrices with coefficients in $\mathbf{C}[\mathbf{M}]$, we have

$$(1) \qquad\qquad x + x^* = \operatorname{tr}(x)\mathbf{1}, \quad \text{and} \quad xx^* = x^*x = \det(x)\mathbf{1}.$$

Note that $\det(a^*) = \det(a)$ and $\operatorname{tr}(a^*) = \operatorname{tr}(a)$ for any 2×2 matrix a. Multiplying the matrix identity $b + b^* = \operatorname{tr}(b)\mathbf{1}$ by the matrix a and taking the trace, we also obtain the identity

$$(2) \qquad\qquad \operatorname{tr}(a)\operatorname{tr}(b) = \operatorname{tr}(ab) + \operatorname{tr}(ab^*).$$

A.2. **Invariant theory.** We have an action of SL_2 on $\mathbf{M}$ by conjugation, and hence a diagonal conjugation action of SL_2 on $\mathbf{M}^2$. It is classical that the ring of invariants $\mathbf{C}[\mathbf{M}^2 /\!\!/ \mathrm{SL}_2] = \mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2}$ is a polynomial ring on 5 generators

$$\mathbf{C}[\operatorname{tr}(x_1), \operatorname{tr}(x_2), \operatorname{tr}(x_1 x_2), \det(x_1), \det(x_2)],$$

where x_1 and x_2 are the standard matrix variables on $\mathbf{M}^2$. (See for example [7, Theorem 5.3.1(ii), p.68]) In particular, any SL_2-invariant regular function in matrix variables x_1 and x_2 is a polynomial combination of the functions $\operatorname{tr}(x_1)$, $\operatorname{tr}(x_2)$, $\operatorname{tr}(x_1 x_2)$, $\det(x_1)$, and $\det(x_2)$. For example, using the identities (1) and (2), we find that

$$\operatorname{tr}(\langle a_1, a_2 \rangle) = \operatorname{tr}(a_1)^2 \det(a_2) + \operatorname{tr}(a_2)^2 \det(a_1) + \operatorname{tr}(a_1 a_2)^2$$
$$- \operatorname{tr}(a_1)\operatorname{tr}(a_2)\operatorname{tr}(a_1 a_2) - 2\det(a_1)\det(a_2).$$

For each $t \in \mathbf{C}$, the action of SL_2 on $\mathbf{M}^2$ preserves the closed subscheme $\mathbf{M}_t^2 \subset \mathbf{M}^2$, and since SL_2 is linearly reductive we have

$$\mathbf{C}[\mathbf{M}_t^2 /\!\!/ \mathrm{SL}_2] = \mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2}/(\det(x_1) - t, \det(x_2) - t) \simeq \mathbf{C}[\operatorname{tr}(x_1), \operatorname{tr}(x_2), \operatorname{tr}(x_1 x_2)].$$

In particular, $\mathrm{SL}_2^2 /\!\!/ \mathrm{SL}_2 \simeq \mathbf{A}^3$, which is due to Fricke (see Goldman [9, Section 2.2] for details).

Lemma A.1. *The quotient morphism* $\mathbf{M}^2 \to \mathbf{M}^2 /\!\!/ \mathrm{SL}_2 \simeq \mathbf{A}^5$ *is surjective.*

Proof. By general theory, $\mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2}$ is a pure subring of $\mathbf{C}[\mathbf{M}^2]$. In particular, for any ideal I of $\mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2}$ we have $I\mathbf{C}[\mathbf{M}^2] \cap \mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2} = I$. In particular, if I is a proper ideal of $\mathbf{C}[\mathbf{M}^2]^{\mathrm{SL}_2}$ then $I\mathbf{C}[\mathbf{M}^2]$ is a proper ideal of $\mathbf{C}[\mathbf{M}^2]$. This implies the lemma. □

A.3. Commutators.

Lemma A.2. *We have the following.*

  (1) *For $b \in \mathbf{M}_0(\mathbf{C})$ nonscalar, $\{a \in \mathbf{M}_0(\mathbf{C}) : [a, b] = 0\}$ has dimension 1.*
  (2) *For $b \in \mathrm{SL}_2(\mathbf{C})$ nonscalar and $k \in \mathbf{C}$, the locus of $a \in \mathrm{SL}_2(\mathbf{C})$ with $[a, b] = 0$ and $\mathrm{tr}(ab) = k$ has dimension 1 if $\mathrm{tr}(b), k \in \{\pm 2\}$, and is finite otherwise.*

Proof. Since b is nonscalar, any $a \in \mathbf{M}(\mathbf{C})$ with $[a, b] = 0$ must be of the form $a = \lambda_1 \mathbf{1} + \lambda_2 b$ with $\lambda_i \in \mathbf{C}$. In such a case, we have

(A) $$\det(a) = \det(\lambda_1 \mathbf{1} + \lambda_2 b) = \lambda_1^2 + \lambda_1 \lambda_2 \operatorname{tr}(b) + \lambda_2^2 \det(b),$$

(B) $$\operatorname{tr}(ab) = \operatorname{tr}((\lambda_1 \mathbf{1} + \lambda_2 b)b) = \lambda_1 \operatorname{tr}(b) + \lambda_2 \operatorname{tr}(b^2).$$

We first prove (1). If $\det(a) = \det(b) = 0$, then we have $\lambda_1^2 + \lambda_1 \lambda_2 \operatorname{tr}(b) = 0$. It follows that the desired locus is the union $\{\lambda b : \lambda \in \mathbf{C}\} \cup \{\lambda b^* : \lambda \in \mathbf{C}\}$, and hence is 1-dimensional. It remains to prove (2). Let S be the locus of $a \in \mathrm{SL}_2(\mathbf{C})$ determined by the conditions of (2). Under the assumptions, by the Cayley-Hamilton theorem we have $\operatorname{tr}(b^2) - \operatorname{tr}(b)^2 + 2 = 0$ and hence $\operatorname{tr}(b^2) \neq 0$ or $\operatorname{tr}(b)^2 \neq 0$. Thus, equation (B) defines a line in the (λ_1, λ_2)-plane. In particular, S is at most 1-dimensional. Suppose it is not finite. The conic in the (λ_1, λ_2)-plane defined by equation (A) must then be degenerate, i.e. the discriminant

$$-\det(a) \det \begin{bmatrix} 1 & \operatorname{tr}(b)/2 \\ \operatorname{tr}(b)/2 & \det(b) \end{bmatrix} = -1(1 - \operatorname{tr}(b)^2/4)$$

is zero. Thus, we must have $4 = \operatorname{tr}(b)^2$, i.e. $\operatorname{tr}(b) = \pm 2$. Equations (A) and (B) then become

(A') $$1 = (\lambda_1 + \lambda_2 \operatorname{tr}(b)/2)^2,$$

(B') $$k = \operatorname{tr}(b)(\lambda_1 + \lambda_2 \operatorname{tr}(b)/2).$$

The degenerate conic defined by equation (A') is a union of two disjoint lines. For S to be infinite, one of the two lines must coincide with the line defined by equation (B'). In other words, we must have $(k/\operatorname{tr}(b))^2 = 1$, or $k = \pm \operatorname{tr}(b) \in \{\pm 2\}$. If this happens, then S is one-dimensional. Thus, we have proved the lemma. □

A.4. Matrices of determinant one.

Lemma A.3. *Fix $(k_1, k_2, k_3) \in \mathbf{C}^3$. The morphism:*

  (1) $\prod_{i=1}^{2} \mathrm{SL}_{2,k_i} \to \mathrm{SL}_2$ *given by $(a_1, a_2) \mapsto a_1 a_2$ is surjective over $\mathrm{SL}_2 \setminus \{\pm \mathbf{1}\}$.*
  (2) $\prod_{i=1}^{3} \mathrm{SL}_{2,k_i} \to \mathrm{SL}_2$ *given by $(a_1, a_2, a_3) \mapsto a_1 a_2 a_3$ is surjective.*
  (3) $\langle -, - \rangle : \mathrm{SL}_2^2 \to \mathrm{SL}_2$ *is surjective.*

Proof. (1) By Lemma A.1, given any $b \in \mathrm{SL}_2(\mathbf{C})$ there exist $a_i \in \mathrm{SL}_{2,k_i}(\mathbf{C})$ such that $\operatorname{tr}(a_1 a_2) = \operatorname{tr}(b)$. If $\operatorname{tr}(b) \neq \pm 2$, then this implies $(ga_1 g^{-1})(ga_2 g^{-1}) = b$ for some $g \in \mathrm{SL}_2(\mathbf{C})$. Consider now the case $\operatorname{tr}(b) = 2s$ for some $s \in \{\pm 1\}$ but $b \neq s\mathbf{1}$. If $k_1 \neq sk_2$, then $a_1 a_2$ cannot be $s\mathbf{1}$ and hence $(ga_1 g^{-1})(ga_2 g^{-1}) = b$ for some

$g \in \mathrm{SL}_2(\mathbf{C})$ as before. The case remains that $k_1 = sk_2 = k$. Let $\lambda \in \mathbf{C}^\times$ be a root of the polynomial $x^2 - kx + 1 = 0$. We then have

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} s\lambda^{-1} & s \\ 0 & s\lambda \end{bmatrix} = \begin{bmatrix} s & 2s\lambda \\ 0 & s \end{bmatrix},$$

and letting a_1 and a_2 respectively be the two matrices on the left hand side, there exists $g \in \mathrm{SL}_2(\mathbf{C})$ such that $(ga_1g^{-1})(ga_2g^{-1}) = b$.

(2) By part (1), given any $b \in \mathrm{SL}_2(\mathbf{C})$ different from $\pm\mathbf{1}$ and any $k' \in \mathbf{C} \setminus \{\pm 2\}$, there exist $b' \in \mathrm{SL}_{2,k'}(\mathbf{C})$ and $a_3 \in \mathrm{SL}_{2,k_3}(\mathbf{C})$ such that $b'a_3 = b$. Since $b' \neq \pm\mathbf{1}$ due to the condition $k' \neq \pm 2$, again by part (1) there exist $a_1 \in \mathrm{SL}_{2,k_1}(\mathbf{C})$ and $a_2 \in \mathrm{SL}_{2,k_2}(\mathbf{C})$ such that $a_1a_2 = b'$, and thus $a_1a_2a_3 = b$, as desired. If $b = s\mathbf{1}$ for some $s \in \{\pm 1\}$, then choosing $a_3 \in \mathrm{SL}_{2,k_3}(\mathbf{C})$ different from $\pm\mathbf{1}$, there exist $a_1 \in \mathrm{SL}_{2,k_1}(\mathbf{C})$ and $a_2 \in \mathrm{SL}_{2,k_2}(\mathbf{C})$ such that $a_1a_2 = sa_3^{-1}$ and hence $a_1a_2a_3 = b$.

(3) From Lemma A.1, given any $b \in \mathrm{SL}_2(\mathbf{C})$ there exist $a_1, a_2 \in \mathrm{SL}_2(\mathbf{C})$ such that $\mathrm{tr}\langle a_1, a_2 \rangle = \mathrm{tr}\, b$. If $b \neq \pm\mathbf{1}$, then $\langle ga_1g^{-1}, ga_2g^{-1} \rangle = g\langle a_1, a_2 \rangle g^{-1} = b$ for some $g \in \mathrm{SL}_2(\mathbf{C})$, as desired. Finally, in the case where $b = \pm\mathbf{1}$, we have

$$\langle \mathbf{1}, \mathbf{1} \rangle = \mathbf{1} \quad \text{and} \quad \left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = -\mathbf{1}$$

which proves the desired result. $\qquad\square$

Lemma A.4. *The fiber of the morphism* $\langle -, - \rangle : \mathrm{SL}_2^2 \to \mathrm{SL}_2$ *above* $\mathbf{1}$ *has dimension* 4. *The fiber above* $-\mathbf{1}$ *has dimension* 3.

Proof. For $s \in \{+, -\}$, let F_s be the fiber of $\langle -, - \rangle$ over $s\mathbf{1}$. Consider the projection

$$\pi : F_\pm \subset \mathrm{SL}_2^2 \to \mathrm{SL}_2$$

onto the second factor. We first consider F_+. The projection $\pi : F_+ \to \mathrm{SL}_2$ is surjective, since for any $b \in \mathrm{SL}_2(\mathbf{C})$ we have $\langle \mathbf{1}, b \rangle = \mathbf{1}$. Given any $b \in \mathrm{SL}_2(\mathbf{C})$, we identify the fiber $\pi^{-1}(b)$ with the scheme of matrices $a \in \mathrm{SL}_2(\mathbf{C})$ such that $\langle a, b \rangle = \mathbf{1}$, or in other words $[a, b] = 0$. If $b = \pm\mathbf{1}$, then $\pi^{-1}(b) = \mathrm{SL}_2$ has dimension 3. If $b \neq \pm\mathbf{1}$, then any $a \in \pi^{-1}(b)(\mathbf{C})$ must be of the form $a = \lambda_1\mathbf{1} + \lambda_2 b$ for some $\lambda_1, \lambda_2 \in \mathbf{C}$. The condition $\det(a) = 1$ then identifies $\pi^{-1}(b)$ with a curve in the (λ_1, λ_2)-plane. Hence, the fibers of $\pi : F_+ \to \mathrm{SL}_2$ above $\mathrm{SL}_2 \setminus \{\pm\mathbf{1}\}$ are 1-dimensional. Therefore F_+ is $3 + 1 = 4$-dimensional.

Consider next F_-. Given $b \in \mathrm{SL}_2(\mathbf{C})$, we identify the fiber $\pi^{-1}(b)$ with the scheme of matrices $a \in \mathrm{SL}_2(\mathbf{C})$ such that $\langle a, b \rangle = -\mathbf{1}$, or in other words $ab + ba = 0$. Writing $a = (a_{ij})$ and $b = (b_{ij})$, the condition $ab + ba = 0$ amounts to

$$\begin{bmatrix} 2b_{11} & b_{21} & b_{12} & \\ b_{12} & b_{11} + b_{22} & & b_{12} \\ b_{21} & & b_{11} + b_{22} & b_{21} \\ & b_{21} & b_{12} & 2b_{22} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = 0.$$

Hence, for the fiber $\pi^{-1}(b)$ to be nonempty, we need the determinant of the 4×4 matrix on the left hand side to be zero, or in other words $\mathrm{tr}(b) = b_{11} + b_{22} = 0$. Thus, the image of π is $\mathrm{SL}_{2,0}$. Given any $b \in \mathrm{SL}_{2,0}(\mathbf{C})$, the 4×4 matrix above has rank 2, and hence its kernel is 2-dimensional. Given the additional determinant condition $a_{11}a_{22} - a_{12}a_{21} = 1$, the fiber $\pi^{-1}(b)$ has dimension 1. Since $\mathrm{SL}_{2,0}$ is 2-dimensional, we see that F_- is $2 + 1 = 3$-dimensional. $\qquad\square$

A.5. **Matrices of determinant zero.**

Lemma A.5. *Let $b, b' \in \mathbf{M}_0(\mathbf{C})$ both be nonzero.*
1. *The locus of $a \in \mathbf{M}_0(\mathbf{C})$ with $ab = 0$ or $ba = 0$ is 2-dimensional.*
2. *The locus of $a \in \mathbf{M}_0(\mathbf{C})$ with $ab = ba = 0$ is 1-dimensional.*
3. *The locus of $a \in \mathbf{M}_0(\mathbf{C})$ with $bab' = 0$ is 2-dimensional. The intersection of this locus with $\mathbf{M}_{0,0}$ is at most 1-dimensional.*

Proof. We first prove (1) and (2). Let $a = (a_{ij}) \in \mathbf{M}(\mathbf{C})$. Without loss of generality, after conjugation we may assume that

$$b = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ for some } \lambda \in \mathbf{C}^\times, \quad \text{or} \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In the former case: $ab = 0$ if and only if $a_{11} = a_{21} = 0$, and $ba = 0$ if and only if $a_{11} = a_{12} = 0$. In the latter case: $ab = 0$ if and only if $a_{11} = a_{21} = 0$, and $ba = 0$ if and only if $a_{21} = a_{22} = 0$. In both cases, if $ab = 0$ or $ba = 0$ then we automatically have $\det(a) = 0$. Parts (1) and (2) follow immediately from these.

(3) Let us write $b = (b_{ij})$, and $b' = (b'_{ij})$. The linear map $a \mapsto bab'$ on $\mathbf{M}(\mathbf{C})$ is given in terms of matrix coefficients $a = (a_{ij})$ by

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} \mapsto \begin{bmatrix} b_{11}b'_{11} & b_{11}b'_{21} & b_{12}b'_{11} & b_{12}b'_{21} \\ b_{11}b'_{12} & b_{11}b'_{22} & b_{12}b'_{12} & b_{12}b'_{22} \\ b_{21}b'_{11} & b_{21}b'_{21} & b_{22}b'_{11} & b_{22}b'_{21} \\ b_{21}b'_{12} & b_{21}b'_{22} & b_{22}b'_{12} & b_{22}b'_{22} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$$

and since $b, b' \in \mathbf{M}_0(\mathbf{C})$ are both nonzero, the 4×4 matrix above has rank 1. Thus, the kernel of the linear map $a \mapsto bab'$ on $\mathbf{M}(\mathbf{C})$ is a linear subspace of dimension 3. Since the hypersurface $\mathbf{M}_0(\mathbf{C}) \subset \mathbf{M}(\mathbf{C})$ is integral (and not linear), its intersection with the above kernel has dimension at most 2, proving the first assertion.

We now prove the last assertion. We claim that the linear subspace V consisting of $a \in \mathbf{M}(\mathbf{C})$ satisfying $\mathrm{tr}(a) = 0$ and $bab' = 0$ is 2-dimensional. Indeed, otherwise the second and third columns of the 4×4 matrix above must be identically zero, contradicting the hypothesis that b and b' are both nonzero matrices. Now, since V and $\mathbf{M}_{0,0}$ are both integral subschemes of dimension 2 in $\mathbf{M}$, they must coincide or have intersection with dimension at most 1. But $\mathbf{M}_{0,0}$ is not linear, and hence $V \cap \mathbf{M}_{0,0}$ has dimension at most 1, proving the last assertion. $\square$

Lemma A.6. *Given any $a, b \in \mathbf{M}_0(\mathbf{C})$, we have the following.*
1. *$aba^* = 0$ if and only if at least one of ab, ba^* is zero.*
2. *$\langle a, b \rangle = 0$ if and only if at least one of ab, ba^*, a^*b^* is zero.*

Proof. For both parts (1) and (2), one implication is clear: if one of ab, ba^*, a^*b^* is zero, then $\langle a, b \rangle = 0$, and if one of ab, ba^* is zero, then $aba^* = 0$. We shall now prove the converses. Since the lemma is trivial if a or b is zero, we may assume $a, b \in \mathbf{M}_0(\mathbf{C})$ are both nonzero. Without loss of generality, we shall assume after conjugation that a is in Jordan normal form, so that

$$a = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ for some } \lambda \in \mathbf{C}^\times, \quad \text{or} \quad a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In the former case, i.e. when $\mathrm{tr}(a) = \lambda \neq 0$, writing $b = (b_{ij})$ we have

$$ab = \begin{bmatrix} \lambda b_{11} & \lambda b_{12} \\ 0 & 0 \end{bmatrix}, \quad ba^* = \begin{bmatrix} 0 & \lambda b_{12} \\ 0 & \lambda b_{22} \end{bmatrix}, \quad a^*b^* = \begin{bmatrix} 0 & 0 \\ -\lambda b_{21} & \lambda b_{11} \end{bmatrix},$$

and
$$aba^* = \begin{bmatrix} 0 & \lambda^2 b_{12} \\ 0 & 0 \end{bmatrix}, \quad \langle a, b \rangle = \begin{bmatrix} -\lambda^2 b_{12} b_{21} & \lambda^2 b_{12} b_{11} \\ 0 & 0 \end{bmatrix}.$$

In the latter case, i.e. when $\operatorname{tr}(a) = 0$, writing $b = (b_{ij})$ we have
$$ab = \begin{bmatrix} b_{21} & b_{22} \\ 0 & 0 \end{bmatrix}, \quad ba^* = \begin{bmatrix} 0 & -b_{11} \\ 0 & -b_{21} \end{bmatrix}, \quad a^* b^* = \begin{bmatrix} b_{21} & -b_{11} \\ 0 & 0 \end{bmatrix},$$

and
$$aba^* = \begin{bmatrix} 0 & -b_{21} \\ 0 & 0 \end{bmatrix}, \quad \langle a, b \rangle = \begin{bmatrix} b_{21}^2 & -b_{21} b_{11} \\ 0 & 0 \end{bmatrix}.$$

We now proceed with our proof.

(1) Suppose that ab and ba^* are nonzero yet $aba^* = 0$. The condition $aba^* = 0$ would imply that $b_{12} = 0$ if $\operatorname{tr}(a) \neq 0$, and $b_{21} = 0$ if $\operatorname{tr}(a) = 0$. In both cases, since $ab, ba^* \neq 0$ we must have $b_{11}, b_{22} \neq 0$, contradicting the assumption that $\det(b) = 0$. Thus, we must have $aba^* = 0$.

(2) Assume toward contradiction that $ab, ba^*, a^* b^*$ are all nonzero yet $\langle a, b \rangle = 0$. Consider the case $\operatorname{tr}(a) = \lambda \neq 0$. The condition $\langle a, b \rangle = 0$ would imply that $b_{12} = 0$ or $b_{21} = b_{11} = 0$. If $b_{12} = 0$, then since $ab, ba^* \neq 0$ we must have $b_{11}, b_{22} \neq 0$, contradicting the assumption that $\det(b) = 0$. If $b_{21} = b_{11} = 0$, then $a^* b^* = 0$ contradicting our assumption. Thus, we must have $\langle a, b \rangle \neq 0$.

It remains to treat case $\operatorname{tr}(a) = 0$. The condition $\langle a, b \rangle = 0$ would imply that $b_{21} = 0$. But if $b_{21} = 0$, then since $ab, ba^* \neq 0$ we must have $b_{11}, b_{22} \neq 0$, contradicting the assumption that $\det(b) = 0$. Thus, we must have $\langle a, b \rangle \neq 0$, as desired. This finishes the proof. $\qquad\square$

Lemma A.7. *The locus of $(a, b) \in \mathbf{M}_0^2(\mathbf{C})$ such that at least two of ab, ba^*, and $a^* b^*$ are zero has dimension at most 4.*

Proof. Since the locus where $a = 0$ or $b = 0$ has dimension at most 3, we may restrict our attention to the locus with $a, b \neq 0$.

(1) Consider the locus $ab = ba^* = 0$. We then have $ab^* = (ba^*)^* = 0$ and hence $\operatorname{tr}(b)a = ab + ab^* = 0$ and hence $\operatorname{tr}(b) = 0$, since $a \neq 0$ by assumption. Thus, b varies over a locus of dimension 2. For each fixed value of b, the condition $ab = 0$ implies that a varies over a locus of dimension 2 by Lemma A.5. Hence, the locus where $ab = ba^* = 0$ has dimension at most $2 + 2 = 4$.

(2) Consider the locus $ab = a^* b^* = 0$. We then have $ba = (a^* b^*)^* = 0$ and hence $[a, b] = ab - ba = 0$. By Lemma A.2, for fixed a (assumed nonzero) we see that b varies over a locus of dimension at most 1. Hence, the locus where $ab = a^* b^* = 0$ has dimension at most $3 + 1 = 4$.

(3) Consider the locus $ba^* = a^* b^* = 0$. We then have $ba = 0$ and $ab^* = 0$. Replacing the role of b and a, we reduce to case (1), showing that the said locus has dimension at most 4.

This concludes the proof of the lemma. $\qquad\square$

Lemma A.8. *The morphism:*

(1) $\mathbf{M}_{0,0}^2 \to \mathbf{M}_0$ *given by* $(a_1, a_2) \mapsto a_1 a_2$ *is surjective over* $\mathbf{M}_0 \setminus \mathbf{M}_{0,0}$. *The preimage of this morphism over* $\mathbf{M}_{0,0} \setminus \{0\}$ *is empty.*

(2) $(\mathbf{M}_0 \setminus \mathbf{M}_{0,0}) \times \mathbf{M}_{0,0} \to \mathbf{M}_0$ *given by* $(a_1, a_2) \mapsto a_1 a_2$ *is surjective.*

(3) $\langle -, - \rangle : \mathbf{M}_0^2 \to \mathbf{M}_0$ *is surjective.*

Proof. (1) By Lemma A.1, Given any $b \in \mathbf{M}_0(\mathbf{C})$, there exist $a_1, a_2 \in \mathbf{M}_{0,0}(\mathbf{C})$ such that $\operatorname{tr}(a_1 a_2) = \operatorname{tr}(b)$. If $\operatorname{tr}(b) \neq 0$, then this implies that $(g a_1 g^{-1})(g a_2 g^{-1}) = b$ for some $g \in \mathrm{SL}_2(\mathbf{C})$, and $g a_1 g^{-1}, g a_2 g^{-1} \in \mathbf{M}_{0,0}(\mathbf{C})$. For the last statement, note first that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix}$$

and, provided that $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \mathbf{M}_{0,0}(\mathbf{C})$, if the right hand side has trace zero then it must in fact be zero. Since any pair $(a_1, a_2) \in \mathbf{M}_{0,0}^2(\mathbf{C})$ with a_1 nonzero is conjugate to a pair of the form $(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right])$, the last statement follows.

(2) The same argument as in the proof of part (1) goes through when $b \in \mathbf{M}_0(\mathbf{C})$ satisfies $\operatorname{tr} b \neq 0$ or $b = 0$. Suppose that $\operatorname{tr} b = 0$ and $b \neq 0$. Up to conjugation by SL_2, we may assume that $b = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Note then that we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This proves that the morphism $(\mathbf{M}_0 \setminus \mathbf{M}_{0,0}) \times \mathbf{M}_{0,0} \to \mathbf{M}_0$ is surjective, as desired.

(3) By Lemma A.1, given any $b \in \mathbf{M}_0(\mathbf{C})$ there exist $a_1, a_2 \in \mathbf{M}_0(\mathbf{C})$ such that $\operatorname{tr}\langle a_1, a_2 \rangle = \operatorname{tr} b$. If $\operatorname{tr} b \neq 0$, then $\langle g a_1 g^{-1}, g a_2 g^{-1} \rangle = b$ for some $g \in \mathrm{SL}_2(\mathbf{C})$. Consider now $\operatorname{tr} b = 0$. We have $\langle 0, 0 \rangle = 0$. Furthermore, we have

$$\left\langle \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

which shows that, letting a_1 and a_2 respectively be the two matrices on the left hand side, there exists $g \in \mathrm{SL}_2(\mathbf{C})$ such that $\langle g a_1 g^{-1}, g a_2 g^{-1} \rangle = b$, as desired. $\square$

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