# Existence of Rosseland equation

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Abstract The global boundness, existence and uniqueness are presented for the kind of Rosseland equation with a small parameter. This problem comes from conduction-radiation coupled heat transfer in the composites; it's with coefficients of high order growth and mixed boundary conditions. A linearized map is constructed by fixing the function variables in the coefficients and the right-hand side. The solution to the linearized problem is uniformly bounded based on De Giorgi iteration; it is bounded in the Hölder space from a Sobolev-Campanato estimate. This linearized map is compact and continuous so that there exists a fixed point. All of these estimates are independent of the small parameter. At the end, the uniqueness of the solution holds if there is a big zero-order term and the solution's gradient is bounded. This existence theorem can be extended to the nonlinear parabolic problem.

**Keywords:** nonlinear elliptic equation, well-posedness, fixed point, mixed boundary conditions, without growth conditions, Rosseland equation

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#### 1 Introduction

Our original motivation is the Rosseland equation in the conduction-radiation coupled heat transfer [1,2]. Find  $(u_{\varepsilon} - u_b) \in W_0^{1,2}(G)$  (Definition 2.3), such that

$$\int_{C} a_{ij}(u_{\varepsilon}(x), x, \frac{x}{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_{i}} \frac{\partial \varphi}{\partial x_{i}} + \int_{\Gamma} \alpha(u_{\varepsilon} - u_{gas}) \varphi = \int_{C} f(u_{\varepsilon}(x), x, \frac{x}{\varepsilon}) \varphi, \quad \forall \varphi \in W_{0}^{1,2}(G),$$

where  $a_{ij} = k_{ij}(x, \frac{x}{\varepsilon}) + 4u_{\varepsilon}^3 b_{ij}(x, \frac{x}{\varepsilon})$ ;  $(k_{ij}), (b_{ij})$  are symmetric positive definite;  $k_{ij}(x, y), b_{ij}(x, y)$  are 1-periodic in y. The small parameter  $\varepsilon$  is the period of the composite structure.  $\Gamma$  is the natural boundary part of  $\partial G$ . There may be no ellipticity for  $A = (a_{ij})$  without considering physical conditions; uniform estimates independent of  $\varepsilon$  are also needed. This open problem (the existence theory for the equation with coefficients like  $k + 4u^3b$ , without  $\varepsilon$ ) was proposed by Laitinen in 2002 (Remark 3.4 [3]).

There are several steps: firstly describe the physical conditions and find a suitable temperature interval by the global boundness in  $L^{\infty}$  (Lemma 3.1); then construct a linearized map with a fixed point in this interval (Theorem 3.4); the fixed point is unique if there is a big zero-order term and the solution's gradient is bounded (Theorem 4.3).

The novelty is we don't need any growth conditions in [4]: this method can be used for coefficients like  $k + 4u^m b$ ,  $\forall m > 0$ . More specifically,  $\forall C_1, C_2$ ,  $0 < C_1 \le C_2$ ,

$$A(u_{\varepsilon}(x), x, \frac{x}{\varepsilon}) \in [C_3, C_4], \quad \text{if } u_{\varepsilon} \in [C_1, C_2]; \quad 0 < C_3 = C_3(C_1), C_4 = C_4(C_1, C_2).$$
 (1.1)

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Our main tool is the regularity established by Griepentrog and Recke in the Sobolev-Campanato space [5]. Their work asserted that linear elliptic equation of second order with non-smooth data  $(L^{\infty}$ -coefficients, Lipschitz domain, regular sets, non-homogeneous mixed boundary conditions) has a unique solution in  $C^{\beta}(\overline{\Omega})$ ; this Hölder norm smoothly depends on the data.

Note that the well-posedness is still valid if the ellipticity is a priori known or only Dirichlet boundary condition is considered (the famous De Giorgi-Nash estimate holds; see Theorem 8.29 [4]). We present a local gradient estimate for a simplified problem (only the righthand side is nonlinear) in Lemma 4.1; it can be used in the error estimate of the homogenization [6]. All of these results can be extended to the nonlinear parabolic equation if we use the regularity in the parabolic Sobolev-Morrey space [7].

Throughout this paper,  $C, C_i$  denote positive constants independent of the solution and the small parameter  $\varepsilon$ . The unit cell  $Y = (0,1)^n$ . B(x,r) is the open ball of radius r centered at x.  $\varphi \in [C_0, C_1]$  means that  $\varphi \in L^{\infty}$  in the relevant domain if without confusion and  $C_0 \leqslant \varphi \leqslant C_1$ . For a real symmetric matrix  $A = (a_{ij}(u(x), x, y)), A \in [C_0, C_1]$  implies

$$a_{ij}(u(x),x,y)\xi_i\xi_j\geqslant C_0|\xi|^2,\quad \sum |a_{ij}(u(x),x,y)|^2\leqslant C_1^2,\quad \forall (x,y)\in\Omega\times Y\,,\xi\in\mathbb{R}^n.$$

 $\|\varphi\|_q$  is an abbreviation of the norm in the relevant  $L^q$  space.  $T_{min}, T_{max}$  are positive physical constants (the range of the environmental temperature);  $0 < T_{min} \le T_{max}$ .

#### 2 Regular sets, Campanato space and model problem

### 5 Conclusions

The well-posedness is given for the Rosseland equation with a small parameter  $\varepsilon$ . The physical conditions are included in (A1)-(A5). Based on the boundness in  $L^{\infty}$ , we construct a closed convex set  $[T_{min}, T_*]$ . Then, we prove the linearized map is compact and continuous from the Sobolev-Campanato estimate established by Griepentrog and Recke. So there exists a fixed point; the solution to the original nonlinear problems has almost the same estimates as the linear one. These estimates are independent of the small parameter. So there is a subsequence which converges in  $C^0(\overline{\Omega})$  (or  $H^1(\Omega)$ ), if  $\varepsilon \to 0$ . A local gradient estimate of the solution is given for a simplified problem; it can be used to the error estimate of the same type of equation's homogenization. The uniqueness is also based on a linearized map; see (??). Similar results on the nonlinear parabolic problem based on the same method and Sobolev-Morrey estimate [7] will appear elsewhere.

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## References

1 Zhang Q. F., Cui J. Z., Multi-scale analysis method for combined conduction-radiation heat transfer of

- periodic composites, Advances in Heterogeneous Material Mechanics(eds. Fan J. H., Zhang J. Q., Chen H. B., et al), Lancaster: DEStech Publications, 2011, 461-464
- 2 Modest M. F., Radiative heat transfer, 2nd, San Diego: McGraw-Hill, 2003
- 3 Laitinen M. T., Asymptotic analysis of conductive-radiative heat transfer, Asymptotic Analysis, 2002, 29(3): 323-342
- 4 Gilbarg D., Trudinger N. S., Elliptic Partial Differential Equations of Second Order, Berlin: Springer, 2001
- 5 Griepentrog J.A., Recke L., Linear elliptic boundary value problems with non-smooth data: normal solvability on Sobolev-Campanato spaces, Math Nachr, 2001, 225(1): 39-74
- 6 Zhang Q. F., Cui J. Z., Regularity of the correctors and local gradient estimate of the homogenization for the elliptic equation: linear periodic case, 2011, arXiv:1109.1107v1 [math.AP]
- 7 Griepentrog J.A., Sobolev-Morrey spaces associated with evolution equations, Adv Differential Equations, 2007, 12: 781-840
- 8 Gröger K, A  $W^{1,p}$ -estimate for solutions to mixed boundary value problems for second order elliptic differential equations, Math Ann, 1989, 283(4): 679-687
- 9 Brezis H., Vázquez J. L., Blow-up solutions of some nonlinear elliptic problems, Rev Mat Univ Complut Madrid, 1997, 10(2): 443-469
- 10 Wu Z. Q., Yin J. X. and Wang C. P., Elliptic and Parabolic Equations, Singapore: World Scientific, 2006
- 11 Bensoussan A., Lions J. L., Papanicolaou G., Asymptotic Analysis for Periodic Structures, Amsterdam: North-Holland, 1978
- 12 Avellaneda M., Lin F. H., Compactness method in the theory of homogenization, Comm Pure Appl Math, 1987, 40(6): 803-847
- 13 Kenig C. E., Lin F. H. and Shen Z. W., Homogenization of elliptic systems with Neumann boundary conditions, 2010, arXiv: 1010.6114v1 [math.AP]