THE GRAPHS OF JOIN-SEMILATTICES AND THE SHAPE OF CONGRUENCE LATTICES OF PARTICLE LATTICES

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ABSTRACT. We attach to each $\langle 0, \vee \rangle$ -semilattice S a graph G_S whose vertices are join-irreducible elements of S and whose edges correspond to the reflexive dependency relation. We study properties of the graph G_S both when S is a join-semilattice and when it is a lattice. We call a $\langle 0, \vee \rangle$ -semilattice S particle provided that the set of its join-irreducible elements join-generates S and it satisfies DCC. We prove that the congruence lattice of a particle lattice is anti-isomorphic to the lattice of hereditary subsets of the corresponding graph that are closed in a certain zero-dimensional topology. Thus we extend the result known for principally chain finite lattices.

1. Introduction

The structure of congruences of a finite lattice can be understood via the study of covers of their join-irreducible elements. The main tool to do so is the dependency relation on the set of join-irreducible elements of the lattice (cf. [3, page 39] or [7, page 113]). The idea of the use of the dependency relation goes back to [8], its original definition is due to A. Day [1]. The dependency relation found a wide range of applications, aside of studying congruences of finite [3], resp. principally chain finite [7], lattices, let us mention characterization of finite lower bounded lattices. In particular, a finite lattice is lower bounded if and only if it contains no *D*-cycle [3, Corollary 2.39].

A full description of a congruence lattice of a lattice via the dependency relation is established for finite lattice in [3] and generalized to principally chain finite lattices in [7]. In this paper we extend these results to a wider class of particle lattices, i.e., the lattices whose join-irreducible elements join-generate the lattice and they satisfy DCC (the decreasing chain condition). We also study properties of the dependency relation for $\langle 0, \vee \rangle$ -semilattices. In the $\langle 0, \vee \rangle$ -semilattice case one cannot expect such a nice connection with the structure of the congruence lattice of the $\langle 0, \vee \rangle$ -semilattice as in the case of lattices. Indeed, even when a $\langle 0, \vee \rangle$ -semilattice is a small finite lattice, the lattice of its $\langle 0, \vee \rangle$ -semilattice

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congruences may be much richer than the lattice of its lattice congruences (cf. Example 6.1).

Let us sum up the content of the paper. Firstly we study consequences of DCC in posets. In particular we show that for posets satisfying DCC, saturated families of finite subsets of the posets satisfy certain minimality properties. This result is based on the fact that if a poset satisfies DCC, the set of its anti-chains quasi-ordered by the *join-refining relation* \ll (see [3, page 30]) satisfies DCC as well (cf. [7, Exercise 10.4]). In connection with these finiteness properties we study join-covers in join-semilattices and lattices.

We define a graph of a $\langle 0, \vee \rangle$ -semilattice. The set of vertices of the graph is the set of all join-irreducible elements of the $\langle 0, \vee \rangle$ -semilattice and the edges correspond to the *reflexive dependency relation* \underline{D} defined in [7, page 113]. We prove that the graph has no edges other than loops for distributive semilattices and that it is symmetric for modular or relatively complement semilattices. This is a mild generalization of the corresponding results known for lattices (cf. [7, Theorem 10.9]).

We study how the congruence lattices of $\langle 0, \vee \rangle$ -semilattices, resp. lattices are related to the lattices of hereditary subsets of the corresponding graphs. We show that there is a Galois connection between the congruence lattice of a $\langle 0, \vee \rangle$ -semilattice and the lattice of hereditary subsets of its graph. This connection is proved to be particularly nice for particle lattices. We define a zero-dimensional topology on the set of join-irreducible elements of a lattice, and we show that the Galois connection induces an anti-isomorphism between the congruence lattice of a particle lattice and the lattice of closed hereditary subsets of its graph. We apply this result to characterize congruence lattices of atomistic lattices.

2. Basic Concepts

Given a set X, we denote by $[X]^{<\omega}$ the set of all finite subsets of X. We denote by $\mathbf{1}_X$ the identity map on the set X.

2.1. **Posets.** By a *poset* we mean a partially ordered set. Given posets P and P', a map $f: P \to P'$ is said to be *monotone*, resp. *antitone*, provided that $p \le q$ implies $f(p) \le f(q)$, resp. $p \le q$ implies $f(q) \le f(p)$, for all $p, q \in P$. An antitone bijection will be called *anti-isomorphism*.

We say that a poset P satisfies DCC (the descending chain condition) provided that there is no infinite decreasing sequence in P, equivalently, provided that each non-empty subset of P has a minimal element. Dually, we say that P satisfies ACC (the ascending chain condition), if it does not contain an infinite increasing sequence. A subset $A \subseteq P$ is called an anti-chain provided that the elements of A are pairwise incomparable.

A subset O of a poset P is said to be an order ideal provided that $x \leq y \in O$ implies that $x \in O$, for all $x, y \in P$. All order ideals of a poset form a sublattice of the lattice of all subsets of the poset. We denote the lattice of all order ideals of the poset P by O(P) (cf. Subsection 2.3).

Given $X \subseteq \mathbf{P}$, we set

$$\uparrow(X) := \{ p \in \mathbf{P} \mid x \le p \text{ for some } x \in X \}, \text{ and dually }$$

$$\downarrow(X) := \{ p \in \mathbf{P} \mid p \le x \text{ for some } x \in X \}.$$

For a singleton set $X = \{x\}$, we abbreviate the notation writing $\downarrow(x)$ and $\uparrow(x)$.

A tree is a poset T such that $\uparrow(x)$ is well-ordered for each $x \in T$. The order type o(x) of an element $x \in T$ is the order type of $\uparrow(x)$. For an ordinal α , the α th-level of T is the set $T_{\alpha} = \{x \in T \mid o(x) = \alpha\}$. A branch of a tree is its maximal well-ordered subset.

A quasi-order on a set Q is a binary relation, say \ll , on the set Q such that \ll is reflexive and transitive. Given $p,q\in Q$, we define $p\equiv q$ if both $p\ll q$ and $q\ll p$. It is straightforward to verify that \equiv is an equivalence relation on the set Q; we will call the relation \equiv the equivalence induced by the quasi-order \ll . For each $q\in Q$, we denote by \overline{q} the block of the equivalence \equiv containing q and we set $Q=\{\overline{q}\mid q\in Q\}$. It is easy to see that the binary relation \leq defined by $\overline{p}\leq \overline{q}$ if and only if $p\ll q$ is well defined (i.e, it does not depend on the choice of the representatives of the blocks) partial order on the set Q. The poset Q will be called the maximal antisymmetric quotient of Q.

Let P, Q be posets. By a Galois connection (between the posets P and Q) we mean a pair of antitone maps, $F: P \to Q$ and $G: Q \to P$, such that

(2.1)
$$p \leq G(q)$$
 if and only if $q \leq F(p)$, for all $p \in P, q \in Q$.

Property (2.1) is easily seen to be equivalent to

(2.2)
$$p \leq GF(p)$$
 for all $p \in \mathbf{P}$ and $q \leq FG(q)$ for all $q \in \mathbf{Q}$.

2.2. **Join-semilattices and lattices.** Let S be a join-semilattice. The least element of S, if it exists, will be denoted by 0 and called *zero* of the join-semilattice S. We will refer to join-semilattices with zero as $\langle 0, \vee \rangle$ -semilattice.

Let S be a join-semilattice (resp. a lattice). We say that $X \subseteq S$ join-generates S provided that each element of S is the join of a finite subset of X.

Let S be a $\langle 0, \vee \rangle$ -semilattice. We say that $u \in S$ is join-irreducible provided that $u = \bigvee F$ implies that $u \in F$, for all finite subsets F of the $\langle 0, \vee \rangle$ -semilattice S. Observe that since $0 = \bigvee \emptyset$, a join-irreducible element is necessarily non-zero. We denote by J_S the set of all join-irreducible elements of the $\langle 0, \vee \rangle$ -semilattice S.

An element u of $\langle 0, \vee \rangle$ -semilattice S is join-prime provided that $u \leq \bigvee F$ implies $u \leq x$ for some $x \in F$, for all $F \in [S]^{<\omega}$. Clearly every join-prime element is join-irreducible, while join-irreducible elements may not be join-prime in general. We denote by P_S the set of all join-prime elements of S. A non-zero element u of $\langle 0, \vee \rangle$ -semilattice S is completely join-prime provided that $u \leq \bigvee X$ implies $u \leq x$ for some $x \in X$, for all (not necessarily finite) subsets X of S.

Let S be a $(0, \vee)$ -semilattice, let $a, b \in S$. We say that b covers a, and we write $a \prec b$, if a < b and $a \le x \le b$ implies $x \in \{a, b\}$ for all $x \in S$. An element u of a $(0, \vee)$ -semilattice S is an atom provided that $0 \prec u$. A $(0, \vee)$ -semilattice S is

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called *atomistic* provided that each element of S is the join of a set of atoms (cf. [4, p. 234]).

An element a of a complete lattice L is *compact* provided that for all $X \subseteq L$, the inequality $a \leq \bigvee X$ implies that $a \leq \bigvee F$ for some finite $F \subseteq X$. An *algebraic lattice* is a complete lattice whose each element is a (possibly infinite) join of compact elements.

An *ideal* of a join-semilattice S is its subset, say I, such that $x \lor y \in I$ if and only if both $x, y \in I$, i.e., an ideal of the semilattice S is its order ideal closed under finite joins. We denote by $\mathrm{Id}(S)$ the lattice (necessarily algebraic) of all ideals of S.

Given a join-semilattice S, we denote by $\operatorname{con}_{\vee}(S)$ the lattice of all congruences of S, i.e., equivalence relations Θ on S such that for all $a, b, c \in S$, $a \equiv_{\Theta} b$ implies that $a \vee c \equiv_{\Theta} b \vee c$. Give a lattice L, we denote by $\operatorname{con}(L)$ the lattice of all congruence of L, i.e., equivalence relations on L respecting both the join and the meet.

2.3. Strongly distributive lattices. We say that a lattice L is strongly distributive provided that it is isomorphic to the lattice O(P) for some poset P. A strongly distributive lattice is necessarily algebraic and distributive, however not every algebraic distributive lattice is strongly distributive. Combining [7, Lemma 10.6] and [7, Exercise 10.7] we get that

Lemma 2.1. The following are equivalent for a distributive algebraic lattice L:

- (1) L is isomorphic to the lattice of order ideals of a poset.
- (2) Every element of L is a join of completely join-prime elements.
- (3) Every compact element of L is a join of (finitely many) join irreducible compact elements.
- (4) The lattice \mathbf{L} is dually algebraic.
- 2.4. **Graphs.** A graph is a pair G = (J, E) where J is a set (of vertices) and $E \subseteq J \times J$ (is a set of edges). Given $u, v \in J$, we will write $u \to v$ to denote that $(u, v) \in E$. We say that $H \subseteq J$ is hereditary provided that if $u \in H$ and $u \to v$, then $v \in H$, i.e., a hereditary subset H contains with each vertex $u \in H$ all vertices reachable by an oriented path starting at u.

We denote by $\operatorname{hrd}(G)$ the lattice of all hereditary subsets of J. Given $X \subseteq J$ we denote by $\partial(X)$ the largest hereditary subset of X; equivalently, the union of all hereditary subsets of X. A subset $Y \subseteq J$ will be called *co-hereditary* provided that its complement $J \setminus Y$ is hereditary.

3. Posets Satisfying DCC and minimal covers

Let P be a poset, let X, Y be subsets of P. We say that X join-refines Y, which we denote by $X \ll Y$, provided that $X \subseteq \downarrow(Y)$ [3, p. 15]. By [3, Lemma 1.15], the relation \ll forms a quasi-order on $[S]^{<\omega}$ and for each $X \in [P]^{<\omega}$, there is a unique anti-chain $A \in [P]^{<\omega}$ such that $A \equiv X$ (where \equiv is the equivalence

induced by the quasi-order \ll). Furthermore, if A is an anti-chain with $A \equiv X$, then $A \subset X$.

Let \mathcal{C} be a set of finite subsets of P. We call the set \mathcal{C} saturated provided that for each non-empty $X \in \mathcal{C}$ there exists an anti-chain $A \in \mathcal{C}$ such that $A \subseteq X$. We say that $X \in \mathcal{C}$ is \mathcal{C} -minimal provided that $Y \ll X$ implies $X \subseteq Y$, for all $Y \in \mathcal{C}$.

Lemma 3.1. A poset $P = (P, \leq)$ satisfies DCC if and only if each non-empty saturated $\mathfrak{C} \subseteq [P]^{<\omega}$ contains a \mathfrak{C} -minimal element.

Proof. (\Leftarrow) Suppose that the poset \boldsymbol{P} does not satisfy DCC. Then there is an infinite strictly decreasing sequence $b_0 > b_1 > \cdots$ in \boldsymbol{P} . Set $B = \{b_0, b_1, \cdots\}$ and $\mathcal{C} = [B]^{<\omega}$. One easily checks that \mathcal{C} is saturated without a \mathcal{C} -minimal element. (\Rightarrow) Suppose that \boldsymbol{P} satisfies DCC and there is a non-empty saturated $\mathcal{C} \subseteq [\boldsymbol{P}]^{<\omega}$ with no \mathcal{C} -minimal element.

Claim 1. There is an infinite sequence of pairwise distinct anti-chains A_0, A_1, A_2, \ldots in \mathcal{C} such that

$$(3.1) A_0 \gg A_1 \gg A_2 \gg \cdots.$$

Proof of Claim. We construct the sequence inductively. Since $\mathbb C$ is nonempty and saturated, there is a nonempty anti-chain $A_0 \in \mathbb C$. Let n be a positive integer and suppose that we have already constructed a sequence $A_0, A_1, \cdots, A_{n-1}$ of pairwise distinct anti-chains satisfying (3.1). By the assumption, there is no $\mathbb C$ -minimal element. Therefore there is X_n with $X_n \ll A_{n-1}$ and $A_{n-1} \not\subseteq X_n$. Since A_{n-1} is an anti-chain, we have that $A_{n-1} \not\ll X_n$, hence $A_i \not \subseteq X_n$, whence $A_i \not\subseteq X_n$ for all $i = 0, \ldots, n-1$. The set $\mathbb C$ is saturated, therefore X_n contains a nonempty anti-chain $A_n \in \mathbb C$. From $A_n \subseteq X_n \ll A_{n-1}$, we get that $A_n \ll A_{n-1}$. Since $A_n \subseteq X_n$, we have that $A_n \neq A_i$ for all $i = 0, \ldots, n-1$. \square Claim 1.

Put $Q = \bigcup_{n=0}^{\infty} A_n$ and observe that the set Q is infinite. Indeed, it has infinitely many pairwise distinct subsets A_n . Let \prec_Q be the covering relation corresponding to the restriction of the order \leq to Q, i.e., $x \prec_Q z$ for $x, z \in Q$ if x < z and x < y < z for no $y \in Q$. Let T be a set of all finite subsets $\{x_0, x_1, \dots, x_k\}$ of Q such that $x_0 \prec_Q x_1 \prec_Q \dots \prec_Q x_k$, ordered by inclusion. Clearly $T = (T, \subseteq)$ is an infinite tree and $o(\{x_0, x_1, \dots, x_k\}) = k$ for all $\{x_0, x_1, \dots, x_k\} \in T$. Since the sets A_n are anti-chains, $x \prec_Q y$ implies that there is a (necessarily unique) positive integer n with $y \in A_{n-1} \setminus A_n$ and $x \in A_n$. It follows that $\{x \in Q \mid x \prec_Q y\} \subseteq A_n$, in particular, the set is finite. Observing that $T_0 = A_0$, it follows by induction that each level of the tree T is finite. By König's lemma [6], the tree T contains an infinite branch. This branch corresponds to an infinite strictly decreasing chain in T, which contradicts the assumption that T0 satisfies DCC.

Let S be a $(0, \vee)$ -semilattice, $I \subseteq S$, and $a \in S$. An I-cover of a is a finite $F \subseteq I$ such that $a \leq \bigvee F$. Let \mathcal{C}_a denote the set of all I-covers of a. By a minimal I-cover of a we mean a \mathcal{C}_a -minimal element. A minimal join-cover of a is a minimal S-cover of a.

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Observe that given a poset P and $\mathcal{C} \subseteq [P]^{<\omega}$, each \mathcal{C} -minimal element is an anti-chain. This is true because a \mathcal{C} -minimal element cannot contain a proper subset in \mathcal{C} , therefore it is an inti-chain due to \mathcal{C} being saturated. In particular, if I is a subset of a $\langle 0, \vee \rangle$ -semilattice S and $a \in S$, each I-minimal cover of a is an anti-chain.

Lemma 3.2. Let S be a $(0, \vee)$ -semilattice and let I be its join-generating subset. Then all minimal I-covers are minimal covers.

Proof. Let $a \in S$ and let F be a minimal I-cover of a. Let G be a finite subset of S such that $a \leq \bigvee G$ and $G \ll F$. Since I is join-generating, G refines to an I-cover of a, say H. Since F is I-minimal, it follows that $F \subseteq H$, and so $F \ll H$. By the transitivity of \ll we get that $F \ll G$, and since F is an anti-chain, we conclude that $F \subseteq G$. Thus F is a minimal join-cover of a.

We say that a $(0, \vee)$ -semilattice S has the weak minimal join-cover refinement property provided that each join-cover of an element $a \in S$ refines to a minimal join-cover of a. This is the minimal join-cover refinement property [3, p. 30] weakened by removing the assumption that each element of S has only finitely many minimal join-covers.

Lemma 3.3. Let S be a $(0, \vee)$ -semilattice, let I be its join-generating subset. Assume that I, viewed as a poset with the ordering inherited from S, satisfies DCC. Then S satisfies the weak minimal join-cover refinement property.

Proof. Let $a \in S$ and let F be a join cover of a. Let \mathcal{C}_a be the set of all I-covers of a refining F. Since I join-generates S, the set \mathcal{C}_a is nonempty. Applying Lemma 3.1 and the assumption that I satisfies DCC, we infer that there is a \mathcal{C}_a -minimal element. It is clearly an I-minimal cover of a. By Lemma 3.2 all minimal I-covers of a are minimal join-covers of a, and so we found a minimal join-cover of a refining F.

Let us call a $\langle 0, \vee \rangle$ -semilattice S particle provided that J_S is join-generating subset of S satisfying DCC. A lattice is particle if its join-semilattice reduct is a particle $\langle 0, \vee \rangle$ -semilattice. We get readily from Lemma 3.3 that

Corollary 3.4. A particle $(0, \vee)$ -semilattice satisfies the weak minimal join-cover refinement property.

Lemma 3.5. $A \langle 0, \vee \rangle$ -semilattice satisfying DCC is particle.

Proof. It suffices to prove that if S is a $(0, \vee)$ -semilattice with DCC then J_S join-generates S. It is an easy exercise (see [7, Lemma 2.8]).

Note that the converse does not hold in general, indeed, any atomistic $\langle 0, \vee \rangle$ -semilattice is particle but not all atomistic $\langle 0, \vee \rangle$ -semilattice satisfy DCC. Let us finish this section with a partial converse of Lemma 3.3, particularly implying that if $J_{\mathbf{S}}$ join-generates the $\langle 0, \vee \rangle$ -semilattice \mathbf{S} , the minimal join-covers and minimal $J_{\mathbf{S}}$ -covers coincide.

Lemma 3.6. In a $\langle 0, \vee \rangle$ -semilattice S each minimal cover is a minimal J_S -cover.

Proof. To prove that each minimal cover is a minimal J_{S} -cover, it suffices to show that elements of minimal covers are join-irreducible. So let F be a minimal join-cover of $a \in S$ and let $u \in F$. Suppose that u is not join-irreducible. Then there exist $x, y \in S$ both strictly smaller than u with $u = x \vee y$. Then $G = (F \setminus \{u\}) \cup \{x,y\}$ is a join-cover of a such that $G \ll F$ and $F \not\subseteq G$. This contradicts the minimality of F.

Corollary 3.7. If a $\langle 0, \vee \rangle$ -semilattice S has the weak minimal join-cover refinement property, then J_S join-generates S.

4. The Graph of a Join-Semilattice

We define a graph of the $\langle 0, \vee \rangle$ -semilattice S to be the graph $G_S = (J_S, E_S)$, where the set E_S of its edges is defined as follows: given $u, v \in J_S$, then $u \to v$ if there is $x \in S$ such that $u \leq x \vee v$ but $u \nleq x \vee y$ for all y < v. Note that for principally join-finite lattices the relation E_S corresponds to the reflexive dependency relation denoted in [7, p. 113] as \underline{D} . The next lemma is the semilattice version of [3, Lemma 2.31].

Lemma 4.1. Let S be $(0, \vee)$ -semilattice satisfying the weak minimal join-cover refinement property, let $u, v \in J_S$. Then $u \to v$ if and only if v belongs to a minimal join-cover of u.

Proof. (\Leftarrow) Let F be a minimal join-cover of u containing v. Put $x = \bigvee (F \setminus \{v\})$. Then $u \leq \bigvee F = x \vee v$. The minimality of F implies that $u \nleq x \vee y$ for all y < v. (\Rightarrow) Suppose that $u \to v$ with $u \leq x \vee v$ and $u \nleq x \vee y$ for all y < v. Then $\{x, v\}$ is a join-cover of u and since S satisfies the weak minimal join cover refinement property, $\{x, v\}$ refines to a minimal join cover F of u. Put $y = \bigvee \{z \in F \mid z \leq v\}$ (note that this set is non-empty since $u \nleq x$). From $u \leq x \vee y$, we get that y = v. Since v is join-irreducible, we conclude that $v \in F$.

Lemma 4.2. Let S be a $(0, \vee)$ -semilattice, let $p \in P_S$ and $v \in J_S$. Then $p \to v$ implies that p = v.

Proof. Suppose that $p \to v$. By the definition there is $x \in S$ such that $p \le x \lor v$ and $p \nleq x \lor y$ for all y < v. Since p is join-prime, either $p \le x$ or $p \le v$. The first is impossible since $p \nleq x \lor 0$, the latter implies that p = v.

Recall that a join-semilattice is distributive provided that $a \leq b \vee c$ implies that $a = y \vee z$ for some $y, z \in \mathbf{S}$ with $y \leq b$ and $z \leq c$ (see e.g. [4, p. 131]). The terminology comes from the fact that a join-semilattice is distributive if and only if its ideals form a distributive lattice [4, Lemma II.5.1].

It is straightforward that in a distributive $\langle 0, \vee \rangle$ -semilattice, join-irreducible elements are join-prime. In fact, if the set of join-irreducible elements of a $\langle 0, \vee \rangle$ -semilattice is join-generating, then join-prime and join-irreducible elements coincide if and only if the join-semilattice is distributive. Applying Lemma 4.2, we get readily that

Proposition 4.3. Let S be a distributive $(0, \vee)$ -semilattice. Then $u \to v$ implies u = v for all $u, v \in J_S$, i.e, the graph G_S has no edges distinct from loops.

Following [5] we say that a join-semilattice S is modular provided that $a \le b \le a \lor c$ implies that there is $x \le c$ in S such that $b = a \lor x$ (see [9] for alternative definitions of modularity of semilattices). Similarly as in the previous case, a join-semilattice is modular if and only if its ideal lattice is modular.

Proposition 4.4. The graph of a modular $(0, \vee)$ -semilattice S is symmetric, i.e, $u \to v$ implies $v \to u$ for all $u, v \in J_S$.

Proof. Let u, v be join-irreducible elements of S such that $u \to v$. By the definition of edges, there is $x \in S$ such that $u \le x \lor v$ but $u \not\le x \lor y$ for all y < v.

Claim 1. $v \leq x \vee u$.

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Proof of Claim. Since $x \le x \lor u \le x \lor v$, there is $y \le v$ with $x \lor u = x \lor y$, by the modularity. It follows that $u \le x \lor y$, hence y = v. We conclude that $v \le x \lor u$.

Let $z \leq u$ be such that $v \leq x \vee z$. Then $z \leq u \leq x \vee v = x \vee z$, hence, by modularity, there is $w \leq x$ with $u = w \vee z$. Since u is join-irreducible, either $u = w \leq x$, which is not the case, or $u \leq z$. The latter means u = z. Thus we have proved that $v \to u$.

There is an alternative way how to prove Proposition 4.4. Each $\langle 0, \vee \rangle$ -semilattice S embeds into the lattice $\mathrm{Id}(S)$ via the correspondence $a \mapsto \downarrow(a)$, sending each element of S to the corresponding principal ideal. It is straightforward to observe that $a \in S$ is join-irreducible if and only if the principal ideal $\downarrow(a)$ is join-irreducible in $\mathrm{Id}(S)$.

Lemma 4.5. Let S be a $\langle 0, \vee \rangle$ -semilattice, let $u, v \in J_S$. Then $u \to v$ if and only if $\downarrow(u) \to \downarrow(v)$.

Proof. (\Rightarrow) Suppose that $u \to v$. By the definition there is $x \in S$ such that $u \leq x \vee v$ but $u \nleq x \vee y$ for all y < v. It follows that $\downarrow(u) \subseteq \downarrow(x) \vee \downarrow(v)$ and let $I \subseteq \downarrow(v)$ be an ideal of S such that $\downarrow(u) \subseteq \downarrow(x) \vee I$. Then there is $z \in I$ with $u \leq x \vee z$. Since $I \subseteq \downarrow(v)$, we have that $z \leq v$, hence z = v. It follows that $I = \downarrow(v)$, and so we have proved that $\downarrow(u) \to \downarrow(v)$.

 (\Leftarrow) Suppose that $\downarrow(u) \to \downarrow(v)$. Then there is an ideal of S, say I, such that $\downarrow(u) \subseteq I \lor \downarrow(v)$ and $\downarrow(u) \nsubseteq I \lor J$ for every ideal $J \subsetneq \downarrow(v)$. The first inequality implies that there is $x \in I$ with $u \le x \lor v$. Suppose that there is y < v with $u \le x \lor v$. Then $\downarrow(y) \subsetneq \downarrow(v)$ and $\downarrow(u) \subseteq I \lor \downarrow(y)$. This is not the case, and so $u \to v$.

It follows from Lemma 4.5 that Proposition 4.4 reduces to the case when S is a modular lattices. In this case we can argue as in [7, Theorem 10.9].

Finally, we say that a $\langle 0, \vee \rangle$ -semilattice S is relatively complemented if for all $x \leq y \leq z$ in S there is $c \in S$ such that the meet $y \wedge c$ exists, $x = y \wedge c$, and

 $z=y\lor c$; we view the $\langle 0,\vee\rangle$ -semilattice S as a partial lattice, assuming that the meet is defined whenever it exists. Note that S is not necessarily a lattice: Take for example the lattice of all subspaces of an infinitely dimensional vector space V, consider its proper infinitely-dimensional subspace, say W, and remove all infinitely dimensional subspaces of W. The result is a relatively complemented $\langle 0,\vee\rangle$ -semilattice that is not a lattice.

Lemma 4.6. Let S be a relatively complemented $(0, \vee)$ -semilattice, let $u, v \in J_S$. Then

$$u \to v \implies v \to u$$
.

Proof. We can argue as in the proof of [7, Theorem 10.9], observing that all join-irreducible elements of S are atoms.

5. Congruences and join-irreducible elements

Let S be a $\langle 0, \vee \rangle$ -semilattice, let Θ be its congruence, and let $a, b \in S$. We write $a \equiv_{\Theta} b$ when $(a, b) \in \Theta$, and $a \leq_{\Theta} b$ when $(a \vee b, b) \in \Theta$. Observe that $a \leq_{\Theta} b$ is equivalent to $a \vee b \equiv_{\Theta} b$. Let us state and verify simple properties of these relations.

Lemma 5.1. Let S be a $(0, \vee)$ -semilattice, let Θ be its congruence. Then the following holds true:

- (1) For all $a, b \in S$, $a \equiv_{\Theta} b$ if and only if both $a \leq_{\Theta} b$ and $b \leq_{\Theta} a$.
- (2) The binary relation \leq_{Θ} is a quasi-order on S.

Proof. (1) It is clear from the definition of the congruence relation on the semi-lattice that $a \equiv_{\Theta} b$ implies both $a \equiv_{\Theta} a \vee b$ and $b \equiv_{\Theta} a \vee b$. On the other hand $a \leq_{\Theta} b$ is equivalent to $b \equiv_{\Theta} a \vee b$ and $b \leq_{\Theta} a$ is equivalent to $a \equiv_{\Theta} a \vee b$. We conclude that $a \equiv_{\Theta} a \vee b \equiv_{\Theta} b$. (2) Let $a \leq_{\Theta} b$ and $b \leq_{\Theta} c$. It follows that $a \vee b \equiv_{\Theta} b$ and $b \vee c \equiv_{\Theta} c$, hence

$$a \lor c \equiv_{\Theta} a \lor (b \lor c) = (a \lor b) \lor c \equiv_{\Theta} b \lor c \equiv_{\Theta} c$$

whence $a \leq_{\Theta} c$. Thus we proved transitivity of \leq_{Θ} , its reflexivity is clear from the definition.

Let S be a $(0, \vee)$ -semilattice and let Θ be a congruence on S. We put

$$(5.1) J_{\Theta}^{\vee} := \{ u \in J_{\mathbf{S}} \mid u \leq_{\Theta} x \implies u \leq x, \text{ for all } x \in \mathbf{S} \}.$$

Lemma 5.2. Let S be a $(0, \vee)$ -semilattice, let $\Theta \in con_{\vee}(S)$, and let $a, b \in S$. Then

$$(5.2) a \leq_{\Theta} b \implies \downarrow(a) \cap J_{\Theta}^{\vee} \subseteq \downarrow(b) \cap J_{\Theta}^{\vee}.$$

Proof. Let $u \in \downarrow(a) \cap J_{\Theta}^{\vee}$. Since $u \leq a$ and $a \leq_{\Theta} b$, by the assumption, we infer, applying Lemma 5.1(2), that $u \leq_{\Theta} b$. Since $u \in J_{\Theta}^{\vee}$, we conclude that $u \leq b$, and so $u \in \downarrow(b) \cap J_{\Theta}^{\vee}$.

Combining Lemmas 5.1(1) and 5.2, we conclude that given a $\langle 0, \vee \rangle$ -semilattice S, a congruence relation $\Theta \in \text{con}_{\vee}(S)$, and elements $a, b \in S$, the implication

$$(5.3) a \equiv_{\Theta} b \implies \downarrow(a) \cap J_{\Theta}^{\vee} = \downarrow(b) \cap J_{\Theta}^{\vee}$$

holds true.

Given a $(0, \vee)$ -semilattice S and a congruence relation $\Theta \in \operatorname{con}_{\vee}(S)$, we set

$$(5.4) J_{\Theta} := \{ u \in J_{\mathbf{S}} \mid x < u \implies x \not\equiv_{\Theta} u, \text{ for all } x \in \mathbf{S} \}.$$

Lemma 5.3. Let S be a particle $(0, \vee)$ -semilattice, let Θ be a congruence on S, and let $a, b \in S$. Then

$$\downarrow(a) \cap J_{\Theta} \subseteq \downarrow(b) \cap J_{\Theta} \implies a \leq_{\Theta} b.$$

Proof. The statement is clear when a = 0. Suppose that 0 < a and $\downarrow(a) \cap J_{\Theta} \subseteq \downarrow(b) \cap J_{\Theta}$. Put

$$\mathfrak{C}_a := \left\{ A \in \left[J_{\mathbf{S}} \right]^{<\omega} \mid a \equiv_{\Theta} \bigvee A \right\}.$$

One readily sees that \mathcal{C}_a is saturated. Since S is a particle $\langle 0, \vee \rangle$ -semilattice, the poset J_S join-generates S and it satisfies DCC. Since J_S is join-generating in S, the set \mathcal{C}_a is non-empty. Since the poset J_S satisfies DCC, there is an \mathcal{C}_a -minimal element, say F, due to Lemma 3.1.

Claim 1. The inclusion $F \subseteq J_{\Theta}$ holds true.

Proof of Claim. Suppose the contrary. Then there is $y \in F$ such that $x \equiv_{\Theta} y$ for some x < y. If x = 0, then $\bigvee F \equiv_{\Theta} \bigvee (F \setminus \{y\})$, hence $F \setminus \{y\} \in \mathcal{C}_a$, which contradicts the \mathcal{C}_a -minimality of F. If 0 < x, then there is a finite $X \subseteq J_S$ with $x = \bigvee X$, (recall that J_S join-generates S). Put $G = (X \cup F) \setminus \{x\}$. From $x = \bigvee X$ we infer that $\bigvee G = \bigvee F \equiv_{\Theta} a$. It follows that $G \in \mathcal{C}_a$. Observing that $G \ll F$ and $F \nsubseteq G$, (since $x \in F \setminus G$), we get the contradiction with the \mathcal{C}_a -minimality of F.

From Claim 1 we conclude that $F \subseteq \downarrow(a) \cap J_{\Theta} \subseteq \downarrow(b) \cap J_{\Theta} \subseteq \downarrow(b)$. It follows that $b \vee a \equiv_{\Theta} b \vee \bigvee F = b$, hence $a \leq_{\Theta} b$.

Comparing the definitions of the sets J_{Θ} and J_{Θ}^{\vee} , we easily observe that $J_{\Theta}^{\vee} \subseteq J_{\Theta}$. Indeed, J_{Θ} corresponds to the set of all join-irreducible elements minimal in their block Θ -blocks, while J_{Θ}^{\vee} is the set of all join-irreducible elements which are minimum elements of their Θ -blocks. Observing that minimal elements of blocks of lattice congruences are necessarily unique in the blocks, hence minimum, we conclude that $J_{\Theta} = J_{\Theta}^{\vee}$ when \boldsymbol{L} is a lattice and $\Theta \in \operatorname{con}(\boldsymbol{L})$.

Corollary 5.4. Let L be particle lattice, let $\Theta \in con(L)$. Then for all $x, y \in L$,

$$\downarrow(x) \cap J_{\Theta} \subseteq \downarrow(y) \cap J_{\Theta} \iff x \leq_{\Theta} y.$$

Proof. Apply Lemmas 5.2 and 5.3.

Lemma 5.5. Let S be a $\langle 0, \vee \rangle$ -semilattice, let $\Theta \in \text{con}_{\vee}(S)$, and let $u, v \in J_S$. Then the implication

$$(5.6) (u \in J_{\Theta}^{\vee} \text{ and } u \to v) \implies v \in J_{\Theta}$$

holds true.

Proof. Suppose that that there are $u,v \in J_{\mathbf{S}}$ such that $u \to v, u \in J_{\Theta}^{\vee}$, and $v \notin J_{\Theta}$. Since $u \to v$, there is $x \in \mathbf{S}$ with $u \le x \lor v$ and $u \not\le x \lor y$ for all y < v. Since $v \notin J_{\Theta}$, there is $y \in \mathbf{S}$ such that y < v and $y \equiv_{\Theta} v$. The latter gives that $u \le x \lor v \equiv_{\Theta} x \lor y$. From $u \in J_{\Theta}^{\vee}$ and $u \le_{\Theta} x \lor y$ we obtain that $u \le x \lor y$. This is a contradiction.

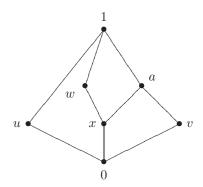


FIGURE 1. The $(0, \vee)$ -semilattice S.

Example 5.1. Consider the $\langle 0, \vee \rangle$ -semilattice S depicted in Figure 1. Let $\Theta \in \operatorname{con}_{\vee}(S)$ be the least congruence identifying elements u and x. The congruence Θ has exactly one non-singular block, namely $\{a, u, w, x, 1\}$. One easily observes that $J_{\Theta} = \{u, x, v\}$ and $J_{\Theta}^{\vee} = \{v\}$. Since $u \leq v \vee w$ but $u \nleq w$, we have that $u \to w$, $u \in J_{\Theta}$ and $w \notin J_{\Theta}$. Similarly, since $v \leq u \vee x$ and $v \nleq x$, we have that $v \to u$, $v \in J_{\Theta}^{\vee}$, and $u \notin J_{\Theta}^{\vee}$. Therefore, the implication (5.5) cannot be strengthen by either assuming that $u \in J_{\Theta}$ or concluding that $v \in J_{\Theta}^{\vee}$.

Of course, the situation simplifies when Θ is a lattice congruence. In this case Lemma 5.5 corresponds to one implication of [7, Theorem 10.5], (see also [3, Lemma 2.33]).

Corollary 5.6. Let L be a lattice, let $u, v \in J_L$. Then for all $\Theta \in con(L)$:

$$(u \in J_{\Theta} \text{ and } u \to v) \implies v \in J_{\Theta}.$$

Lemma 5.7. Let S be $(0, \vee)$ -semilattice satisfying the weak minimal join-cover refinement property. Let H be a hereditary subset of J_S . Let Θ_H be a binary relation on S defined by

$$(5.7) a \equiv_{\Theta_{\mathbf{H}}} b \iff \downarrow(a) \cap H = \downarrow(b) \cap H (for all \ a, b \in \mathbf{S}).$$

Then Θ_H is a congruence of the $\langle 0, \vee \rangle$ -semilattice \mathbf{S} preserving all existing meets. In particular, if \mathbf{S} is a lattice, then $\Theta_H \in \mathrm{con}(\mathbf{S})$.

Proof. It is clear from the definition, that the binary relation Θ_H is reflexive, transitive, and symmetric, thus Θ_H is an equivalence relation on S. Let $a,b,c \in S$ and suppose that $a \equiv_{\Theta_H} b$. This, by (5.7), means that $\downarrow(a) \cap H = \downarrow(b) \cap H$. Let $u \in \downarrow(a \vee c) \cap H$. Since J_S join-generates S due to Corollary 3.7, we can find finite subsets A and C of J_S such that $a = \bigvee A$ and $c = \bigvee C$. Observe that $A \cup C$ is a join-cover of u and since S satisfies the minimal join-cover refinement property, $A \cup C$ refines to a minimal join-cover of u, say F. From Lemma 3.6 we get that $F \subseteq J_S$ and by Lemma 4.1 we have that $u \to v$ for every $v \in F$. Since H is hereditary, we infer that $F \subseteq H$. Since $F \ll A \cup C$, either $v \subseteq a$ or $v \subseteq c$ for every $v \in F$. Since $\downarrow(a) \cap H = \downarrow(b) \cap H$, we have that $v \subseteq a$ implies $v \subseteq b$, hence $v \subseteq b \vee c$, for all $v \in F$. It follows that $u \subseteq \bigvee F \subseteq b \vee c$. We conclude that $u \in \downarrow(b \vee c) \cap H$.

Verifying that $\Theta_{\rm H}$ preserves existing meets is straightforward, indeed, for all $c \in S$,

$$\downarrow\!(c \land a) \cap H = \downarrow\!(c) \cap \downarrow\!(a) \cap H = \downarrow\!(c) \cap \downarrow\!(b) \cap H = \downarrow\!(c \land b) \cap H.$$

6. The Galois connection

In this final section we study the connections between the congruence lattices of $\langle 0, \vee \rangle$ -semilattices (resp. lattices) and the lattices of all hereditary subsets of their graphs. We define a topology on the set of join-irreducible elements of a lattice L, induced by the ordering of J_L , and we prove that the congruence lattice of a particle lattice is anti-isomorphic to the lattice of all closed hereditary subsets of its graph. Thus we generalize [7, Corollary of Theorem 10.5]. We apply this result to characterize the congruence lattices of atomistic lattices.

Lemma 6.1. Let S be a $(0, \vee)$ -semilattice. The pair (F, G) of maps defined as

(6.1)
$$F: \operatorname{con}_{\vee}(S) \to \operatorname{hrd}(G_{S}) \qquad \text{and} \qquad G: \operatorname{hrd}(G_{S}) \to \operatorname{con}_{\vee}(S)$$
$$\Theta \mapsto \partial(J_{\Theta}^{\vee}). \qquad H \mapsto \Theta_{H},$$

forms a Galois connection.

Proof. First, let us carry out an easy verification of the antitonity of the maps F and G. If $\Theta \subseteq \Theta'$ for some $\Theta, \Theta' \in \operatorname{con}_{\vee}(S)$, then $J_{\Theta}^{\vee} \supseteq J_{\Theta'}^{\vee}$ readily from definition (5.1). Consequently, $\partial(J_{\Theta}^{\vee}) \supseteq \partial(J_{\Theta'}^{\vee})$. To establish the latter, let $H \subseteq H'$ be hereditary subsets of J_{S} . It follows from definition (5.7) that

$$a \equiv_{\Theta_{\mathrm{H}'}} b \implies \downarrow(a) \cap H' = \downarrow(b) \cap H' \implies \downarrow(a) \cap H = \downarrow(b) \cap H \implies a \equiv_{\Theta_{\mathrm{H}}} b$$
 for all $a, b \in S$. Thus $\Theta_{\mathrm{H}} \supseteq \Theta_{\mathrm{H}}'$.

It remains to prove that for all $\Theta \in \text{con}_{\vee}(S)$ and all $H \in \text{hrd}(G_S)$:

$$\Theta \subseteq G(H) \iff H \subseteq F(\Theta).$$

- (\Rightarrow) Suppose that $\Theta \subseteq G(H) = \Theta_H$. We are to prove that $H \subseteq F(\Theta) = \partial(J_{\Theta}^{\vee})$. Since H is a hereditary subset of J_{S} , it suffice to verify that $H \subseteq J_{\Theta}^{\vee}$. So let $u \in H$, and let $a \in S$ be such that $u \leq_{\Theta} a$. The assumption $\Theta \subseteq \Theta_H$ gives $u \leq_{\Theta_H} a$, hence $\downarrow(u) \cap H \subseteq \downarrow(a) \cap H$. Since $u \in H$, we infer that $u \in \downarrow(a)$, that is, $u \leq a$. We conclude that $u \in J_{\Theta}^{\vee}$.
- (\Leftarrow) Suppose that $H \subseteq F(\Theta) = \partial(J_{\Theta}^{\vee})$. Applying Lemma 5.2, we have for all $a, b \in S$ that

$$a \equiv_{\Theta} b \implies \downarrow(a) \cap J_{\Theta}^{\vee} = \downarrow(b) \cap J_{\Theta}^{\vee} \implies \downarrow(a) \cap H = \downarrow(b) \cap H \implies a \equiv_{\Theta_{\mathbf{H}}} b.$$
 It follows that $\Theta \subseteq \Theta_{\mathbf{H}} = G(H)$.

Let L be a lattice and let $\Theta \in \operatorname{con}(L)$ be its congruence. As noted above $J_{\Theta} = J_{\Theta}^{\vee}$, and by Corollary 5.6, $J_{\Theta}^{\vee} = \partial(J_{\Theta}^{\vee})$. On the other hand $G(H) \in \operatorname{con}(L)$ for every hereditary subset H of J_L due to Lemma 5.7. It follows that the Galois connection (F, G), defined by (6.1) above (now with the $\langle 0, \vee \rangle$ -semilattice S replaced by the lattice L), consists of the pair of maps

(6.2)
$$F: \operatorname{con}(\boldsymbol{L}) \to \operatorname{hrd}(\boldsymbol{G}_{\boldsymbol{L}}) \qquad \text{and} \qquad G: \operatorname{hrd}(\boldsymbol{G}_{\boldsymbol{L}}) \to \operatorname{con}(\boldsymbol{L})$$
$$\Theta \mapsto J_{\Theta} \qquad \qquad H \mapsto \Theta_{H}.$$

Lemma 6.2. Let (F,G) be the Galois connection defined by (6.2). If L is a particle lattice, then $GF = \mathbf{1}_{con(L)}$.

Proof. By the definition, $GF(\Theta) = \Theta_{J_{\Theta}}$ for all $\Theta \in con(L)$. Recalling definition (5.7) and applying Corollary 5.4 we obtain the sequence of equivalences

$$a \equiv_{\Theta_{J_{\Theta}}} b \iff \downarrow(a) \cap J_{\Theta} = \downarrow(b) \cap J_{\Theta} \iff a \equiv_{\Theta} b, \text{ for all } a, b \in \mathbf{L}.$$

It follows that $\Theta = \Theta_{J_{\Theta}}$, which was to prove.

Example 6.1. Even simple examples show that Lemma 6.1 cannot be similarly extended when we consider the Galois connection between the congruence lattice of a particle (even finite) $\langle 0, \vee \rangle$ -semilattice and the lattice of hereditary subsets of its graph. As an example consider the $\langle 0, \vee \rangle$ -semilattice reduct of five-element modular non-distributive lattice M_3 . The $\langle 0, \vee \rangle$ -semilattice M_3 has a graph with no proper non-trivial hereditary subset but its join-preserving congruences form the twelve element lattice on Figure 2.

Let L be a lattice, let $x, v \in L$ be such that v is join irreducible and x < v. For each such a pair of elements put

(6.3)
$$U(v,x) = \{ u \in J_L \mid u \nleq x \text{ and } u \leq v \} = (\downarrow(v) \setminus \downarrow(x)) \cap J_L.$$

For each $v \in J_L$ set

$$\mathcal{B}(v) = \{ U(v, x) \mid x < v \text{ in } \mathbf{L} \}.$$

Lemma 6.3. Let **L** be a lattice. The collection $\{\mathfrak{B}(v)\}_{v\in J_{\mathbf{L}}}$ is a neighborhood system of a topological space.

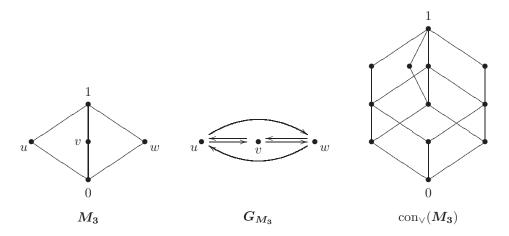


FIGURE 2. The $\langle 0, \vee \rangle$ -semilattice M_3 .

Proof. We shall verify that the collection $\{\mathcal{B}(v)\}_{v\in J_{\mathbf{S}}}$ satisfies properties (BP1-BP3) of [2, page 12]. Let $v\in J_{\mathbf{S}}$. Then $\mathcal{B}(v)$ is non-empty as $U(v,0)\in \mathcal{B}(v)$ and clearly $v\in U$ for each $U\in \mathcal{B}(v)$; thus (BP1) holds true. If $u\in U(v,x)$ for some x< v in \mathbf{S} and some $u\in J_{\mathbf{S}}$, then $u\nleq x$ due to definition (6.3), hence $u\wedge x< u$ and we have that $U(u,u\wedge x)\in \mathcal{B}(u)$. Since $u\le v$, we conclude that $U(u,u\wedge x)\subseteq U(v,x)$. This settles property (BP2). Finally, let $u\in U(v_1,x_1)\cap U(v_2,x_2)$, for some $u,v_i\in J_{\mathbf{L}}$ and $x_i< v_i$ in $\mathbf{L},\ i=1,2$. Then, by definition (6.3), we have that $u\nleq x_i$ and $u\le v_i$ for both i=1,2. It follows that $u\wedge x_i< u$, i=1,2, and since u is join-irreducible, we conclude that $(u\wedge x_1)\vee (u\wedge x_2)< u$, and so $U(u,(u\wedge x_1)\vee (u\wedge x_2))\in \mathcal{B}(u)$ is the desired neighborhood of u with $U(u,(u\wedge x_1)\vee (u\wedge x_2))\subseteq U(v_1,x_1)\cap U(v_2,x_2)$. This proves property (BP3). \square

Let $\mathcal{T}_{\boldsymbol{L}}$ be the topology on the set $J_{\boldsymbol{L}}$ generated by the neighborhood system $\bigcup_{v \in J_{\boldsymbol{L}}} \mathcal{B}(v)$ (cf. [2, Proposition 1.2.3]).

Lemma 6.4. Let L be a lattice, let $v \in J_L$ and let x < v in L. Then the neighborhood U(v,x) is closed in the topology \mathfrak{T}_L .

Proof. Let $u \in J_{\mathbf{S}}$ satisfy $u \notin U(v,x)$. If $u \nleq v$, then $u \wedge v < u$ and it is straightforward from (6.3) that $U(u,u \wedge v) \cap U(v,x) = \emptyset$. Suppose that $u \leq v$. From $u \notin U(v,x)$, we conclude that $u \leq x$, hence $U(u,0) \cap U(v,x) = \emptyset$. It follows that $J_{\mathbf{L}} \setminus U(v,x)$ is open, and so U(v,x) is closed.

Recall that a topological space is zero-dimensional provided that it is T_1 and it has a basis consisting of clopen sets, i.e., sets that are both closed and open, (see [2, p. 360]). Let \mathbf{L} be a lattice. It follows readily from definition (6.3) that $\{v\} = \bigcap \mathcal{B}(v)$ for each $v \in J_L$, thus the topology \mathcal{T}_L is T_1 . It follows from Lemma 6.4 that the topology \mathcal{T}_L has a basis of clopen sets, and so the topology is zero-dimensional.

Given a lattice L, let $\overline{\text{hrd}}(G_L)$ denote the lattice of all closed (w.r.t. the topology \mathcal{T}_L) hereditary subsets of J_L .

Lemma 6.5. Let L be a lattice and let (F,G) be the Galois connection defined by (6.2). Then the following holds true:

- (1) For every $\Theta \in \text{con}(\mathbf{L})$, $F(\Theta) = J_{\Theta} \in \overline{\text{hrd}}(\mathbf{G}_{\mathbf{L}})$.
- (2) For every $H \in \overline{\operatorname{hrd}}(G_L)$, $H = J_{\Theta_H} = FG(H)$.
- Proof. (1) Let $\Theta \in \operatorname{con}(\mathbf{L})$ and let $v \in J_{\mathbf{L}} \setminus J_{\Theta}$. By definition (5.4), there is $x \in \mathbf{L}$ such that x < v and $x \equiv_{\Theta} v$. We are going to show that the neighborhood U(v, x) is included in $J_{\mathbf{L}} \setminus J_{\Theta}$. Let $u \in U(v, x)$, i.e., u is a join-irreducible element of \mathbf{L} such that $u \leq v$ and $u \nleq x$. Since $u \nleq x$, the inequality $x \wedge u < u$ holds true. From $x \equiv_{\Theta} v$ and $u \leq v$, we infer that $x \wedge u \equiv_{\Theta} v \wedge u = u$. We conclude that $u \in J_{\mathbf{L}} \setminus J_{\Theta}$, and so the set $J_{\mathbf{L}} \setminus J_{\Theta}$ is open, hence its complement J_{Θ} is closed. Recall that J_{Θ} is a hereditary subset of $J_{\mathbf{L}}$ due to Corollary 5.6.
- (2) Let $H \in \overline{\mathrm{hrd}}(G_L)$. Since, by Lemma 6.2, the maps (F,G) form a Galois connection, formula (2.2) says that $H \subseteq J_{\Theta_H}$. In order to prove that $J_{\Theta_H} \subseteq H$, pick $u \in J_{\Theta_H}$ and let $x \in L$ be such that x < u. Then $x \not\equiv_{\Theta_H} u$, and so $\downarrow(x) \cap H \subsetneq \downarrow(u) \cap H$ due to (5.7), hence $U(u,x) = (\downarrow(u) \setminus \downarrow(x)) \cap H \neq \emptyset$. It follows that u belongs to the closure of H. Since H is supposed to be closed, we conclude that $u \in H$. Thus we have proved the opposite inclusion $J_{\Theta_H} \subseteq H$. \square

Theorem 6.6. Let L be a particle lattice and let (F,G) be the Galois connection defined by (6.2). Then the image of the map F is $\overline{\operatorname{hrd}}(G_L)$, and the maps $F:\operatorname{con}(L)\to\overline{\operatorname{hrd}}(G_L)$ and $G':=G\upharpoonright\overline{\operatorname{hrd}}(G_L):\overline{\operatorname{hrd}}(G_L)\to\operatorname{con}(L)$ are mutually inverse lattice anti-isomorphisms. In particular, the lattice $\operatorname{con}(L)$ is isomorphic to the lattice of all open co-hereditary subsets of J_L .

<u>Proof.</u> It follows from Lemma 6.5 that the image of the map F corresponds to $\overline{\operatorname{hrd}}(G_L)$ and that $FG'=\mathbf{1}_{\overline{\operatorname{hrd}}(G_L)}$. If L is a particle lattice, we apply Lemmas 6.2 and 6.5(1) to infer that $G'F=\mathbf{1}_{\operatorname{con}(L)}$. We conclude that F and G' are mutually inverse lattice anti-isomorphisms. The last statement of the theorem easily follows.

Proposition 6.7. Let L be a lattice such that the set J_L join-generating and $\downarrow(v)$ is finite for every $v \in J_L$. Then the lattice con(L) is strongly distributive.

Proof. The lattice L is clearly particle. Let $v \in J_L$. Since $\downarrow(v)$ is finite, we infer that the singleton $\{v\} = \bigcap_{x < v} U(v, x)$ is open. It follows that the topology \mathfrak{T}_L is discrete. By Theorem 6.6, the lattice $\operatorname{con}(L)$ is anti-isomorphic to the lattice of all hereditary subsets of the graph G_L . Let \ll be a transitive and reflexive closure of E_L . Now it is straightforward to see that the lattice of all hereditary subsets of the graph G_L is anti-isomorphic to the lattice of all order ideals of the maximal antisymmetric quotient of the quasi-ordered set (J_L, \ll) . It follows that the lattice $\operatorname{con}(L)$ is strongly distributive (cf. Lemma 2.1(4)).

It was proved by M. Tischendorf [10] that every finite lattice has a congruence preserving extension into an atomistic lattice. The construction of [5], adapted in [7, Theorem 10.8], provides a representation of each algebraic strongly distributive lattice as $\operatorname{con}(\boldsymbol{L})$ for a principally chain finite (i.e. no principal ideal of \boldsymbol{L} contains an infinite chain [7, p. 112]) atomistic lattice. Applying Proposition 6.7 we prove that

Corollary 6.8. Congruence lattice of atomistic lattice are exactly strongly distributive lattices.

In particular, Tischendorfs result cannot be extended beyond finite lattices.

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