MINIMIMAL HYPERSURFACES AND BORDISM OF POSITIVE SCALAR CURVATURE METRICS

BORIS BOTVINNIK AND DEMETRE KAZARAS

ABSTRACT. Let (Y,g) be a compact Riemannian manifold of positive scalar curvature (psc). It is well-known, due to Schoen-Yau, that any closed stable minimal hypersurface of Y also admits a psc-metric. We establish an analogous result for stable minimal hypersurfaces with free boundary. Furthermore, we combine this result with tools from geometric measure theory and conformal geometry to study psc-bordism. For instance, assume (Y_0,g_0) and (Y_1,g_1) are closed psc-manifolds equipped with stable minimal hypersurfaces $X_0 \subset Y_0$ and $X_1 \subset Y_1$. Under natural topological conditions, we show that a psc-bordism $(Z,\bar{g}): (Y_0,g_0) \leadsto (Y_1,g_1)$ gives rise to a psc-bordism between X_0 and X_1 equipped with the psc-metrics given by the Schoen-Yau construction.

Contents

1. Introduction	2
1.1. Schoen-Yau minimal hypersurface technique	2
1.2. Stable minimal hypersurfaces with free boundary	2
1.3. Positive scalar curvature bordism and minimal hypersurfaces	3
2. Preliminaries and Theorem 3	5
2.1. Stable minimal hypersurfaces with free boundary	5
2.2. Conformal Laplacian with minimal boundary conditions	6
2.3. Proof of Theorem 3	7
3. Cheeger-Gromov convergence of minimizing hypersurfaces	8
3.1. Convergence of hypersurfaces	8
3.2. Main convergence result	9
3.3. Proof of the Main Lemma: outline	10
3.4. Proof of Claim 1	11
3.5. Proof of Claim 2	12
3.6. Proof of Claim 3	13
4. Proof of Theorem 5	16
4.1. Positive conformal bordism	16
4.2. Long collars	17
Appendix A.	18
A.1. The minimal graph equation	18
A.2. Details on Theorem 4	20
A.3. Doubling minimal hypersurfaces with free boundary	22
A.4. Second fundamental form bounds	24
A.5. Generic regularity in dimension 8	25
References	28

Date: February 7, 2017.

 $^{2000\} Mathematics\ Subject\ Classification.\ 53C27,\ 57R65,\ 58J05,\ 58J50.$

Key words and phrases. Positive scalar curvature metrics, minimal surfaces with free boundary, conformal Laplacian.

1. Introduction

1.1. Schoen-Yau minimal hypersurface technique. For a Riemannian metric g on a smooth manifold, we denote by R_g the scalar curvature function and by H_g the mean curvature of the boundary (if it is not empty). The Schoen-Yau minimal hypersurface technique [27] provides well-known geometric obstructions to the existence of positive scalar curvature. Here is the first fundamental result:

Theorem 1. [27, Proof of Theorem 1] Let (Y,g) be a compact Riemannian manifold with $R_g > 0$, and dim $Y = n \geq 3$. Let $X \subset Y$ be a smoothly embedded stable minimal hypersurface with trivial normal bundle. Then X admits a metric \tilde{h} with $R_{\tilde{h}} > 0$. Furthermore, the metric \tilde{h} could be chosen to be conformal to the restriction $g|_X$.

We note that Theorem 1 is proven by analyzing the conformal Laplacian of the hypersurface X. It it crucial that X is stable minimal. For arbitrary (Y,g) it is a non-trivial (and possibly obstructed) problem to find a stable minimal hypersurface. However, in low dimensions, geometric measure theory can provide a source of stable minimal hypersurfaces.

Theorem 2. (See [20, Chapter 8], [13, Theorem 5.4.15]) Let (Y,g) be a compact orientable Riemannian manifold with $3 \le \dim Y = n \le 7$. Assume $\alpha \in H_{n-1}(Y; \mathbf{Z})$ is a nontrivial element. Then there exists a smoothly embedded hypersurface $X \subset Y$ such that

- (i) up to multiplicity, X represents the class α ;
- (ii) X minimizes volume among all hypersurfaces which represent α up to multiplicity. In particular, the hypersurface X is stable minimal.

There are several important results based on Theorems 1 and 2. In particular, this gives a geometric proof that the torus T^n does not admit a metric of positive scalar curvature for $n \leq 7$, see [27]. This method was also crucial to provide first counterexample to the Gromov-Lawson-Rosenberg conjecture, see [22]. In this paper we extend these ideas and techniques to the case of manifolds with boundary.

1.2. Stable minimal hypersurfaces with free boundary. Let (M, \bar{g}) be a manifold with nonempty boundary ∂M and $W \subset M$ be an embedded hypersurface. We say that a hypersurface W is properly embedded if, in addition, $\partial W = \partial M \cap W$. Such a hypersurface $W \subset M$ is stable minimal with free boundary if W is a local minimum of the volume functional among properly embedded hypersurfaces, see Section 2.1. We establish the following analogue of Theorem 1 for manifolds with boundary in Section 2.3.

Theorem 3. Let (M,\bar{g}) be a compact Riemannian manifold with non-empty boundary ∂M , $R_{\bar{g}} > 0$, $H_{\bar{g}} \equiv 0$, and dim $M = n + 1 \geq 3$. Let $W \subset M$ be an embedded stable minimal hypersurface with free boundary and trivial normal bundle. Then W admits a metric \tilde{h} with $R_{\tilde{h}} > 0$ and $H_{\tilde{h}} \equiv 0$. Furthermore, the metric \tilde{h} could be chosen to be conformal to the restriction $\bar{g}|_W$.

The proof of Theorem 3 is similar to the case of closed manifolds. In particular, we have to analyze the conformal Laplacian on W with $minimal\ boundary\ conditions$. This boundary condition works well with the free boundary stability assumption.

For a compact oriented (n+1)-dimensional manifold M, we consider the relative integral homology group $H_n(M, \partial M; \mathbf{Z})$. Let $\bar{\alpha} \in H_n(M, \partial M; \mathbf{Z})$ be a non-trivial class which we may assume to be represented by a properly embedded hypersurface $W \subset M$. We notice that the boundary ∂W (which may possibly be empty) represents the class $\partial(\bar{\alpha}) \in H_{n-1}(\partial M; \mathbf{Z})$, where ∂ is the connecting homomorphism in the exact sequence

$$(1.1) \cdots \to H_n(\partial M; \mathbf{Z}) \to H_n(M; \mathbf{Z}) \to H_n(M, \partial M; \mathbf{Z}) \xrightarrow{\partial} H_{n-1}(\partial M; \mathbf{Z}) \to \cdots$$

There is an analog of Theorem 2 which relies on a different regularity result, see Appendix A.2 for more details.

Theorem 4. (See [16, Theorem 5.2]) Let (M, \bar{g}) be a compact orientable Riemannian manifold with non-empty boundary ∂M and $3 \leq \dim M = n + 1 \leq 7$. Assume $\bar{\alpha} \in H_n(M, \partial M; \mathbf{Z})$ is a nontrivial element. Then there exists a smooth properly embedded hypersurface $W \subset M$ such that

- (i) up to multiplicity, W represents the class $\bar{\alpha}$;
- (ii) W minimizes volume with respect to \bar{g} among all hypersurfaces which represent $\bar{\alpha}$ up to multiplicity. In particular, W is stable minimal with free boundary.
- 1.3. Positive scalar curvature bordism and minimal hypersurfaces. The main result of this paper is an application of Theorems 3 and 4 to provide new obstructions for psc-metrics to be psc-bordant.

Definition 1. Let (Y_0, g_0) and (Y_1, g_1) be closed oriented n-dimensional manifolds with psc-metrics. Then (Y_0, g_0) and (Y_1, g_1) are psc-bordant if there is a compact oriented (n+1)-dimensional manifold (Z, \bar{g}) such that

- the manifold Z is an oriented bordism between Y_0 and Y_1 , i.e., $\partial Z = Y_0 \sqcup -Y_1$;
- \bar{g} is a psc-metric which restricts to $g_i + dt^2$ near the boundary $Y_i \subset \partial Z$ for i = 0, 1.

We write $(Z, \bar{g}): (Y_0, g_0) \leadsto (Y_1, g_1)$ for a psc-bordism as above.

Remark. Sometimes we consider bordisms $(Z, \bar{g}) : (Y_0, g_0) \leadsto (Y_1, g_1)$ as above where the metrics do not necessarily have positive scalar curvature. However, we always assume that the metric \bar{g} restricts to a product metric near the boundary.

Now we would like to enrich the psc-bordism relation with an extra structure, namely with a choice of homology classes $\alpha_i \in H_{n-1}(Y_i; \mathbf{Z})$, i = 0, 1. Recall the following elementary observation.

Let $\alpha \in H_{n-1}(Y; \mathbf{Z})$, where Y is an oriented closed n-dimensional manifold. Then the cohomology class $D\alpha \in H^1(Y; \mathbf{Z})$ Poincare-dual to α can be represented by a smooth map $\gamma : Y \to B\mathbf{Z} = S^1$. Furthermore, we can assume that a given point $s_0 \in S^1$ is a regular value for γ . It is easy to see that

the inverse image $X_{\gamma} := \gamma^{-1}(s_0) \subset Y$ is an embedded hypersurface which represents the homology class α .

If M is an oriented (n+1)-dimensional manifold with a map $\bar{\gamma}: M \to S^1$, let $\gamma: \partial M \to S^1$ be the restriction $\bar{\gamma}|_{\partial M}$. There is a simple relation between the classes $[\bar{\gamma}] \in H^1(M; \mathbf{Z})$ and $[\gamma] \in H^1(\partial M; \mathbf{Z})$:

Lemma 1. Let $\bar{\alpha} \in H_n(M, \partial M; \mathbf{Z})$ and $\alpha \in H_{n-1}(\partial M; \mathbf{Z})$ be Poincare dual to the classes $[\bar{\gamma}] \in H^1(M; \mathbf{Z})$ and $[\gamma] \in H^1(\partial M; \mathbf{Z})$. Then $\partial(\bar{\alpha}) = \alpha$, where $\partial: H_n(M, \partial M; \mathbf{Z}) \to H_{n-1}(\partial M; \mathbf{Z})$ is the connecting homomorphism. In particular, if $W = \bar{\gamma}^{-1}(s_0) \subset M$ is a smooth properly embedded hypersurface representing $\bar{\alpha}$, then the boundary ∂W represents the class α .

Definition 2. Let (Y_0, g_0) and (Y_1, g_1) be closed oriented n-dimensional Riemannian manifolds with given maps $\gamma_0 : Y_0 \to S^1$ and $\gamma_1 : Y_1 \to S^1$. We say that the triples (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are bordant if there exists a bordism $(Z, \bar{g}) : (Y_0, g_0) \leadsto (Y_1, g_1)$ and a map $\bar{\gamma} : Z \to S^1$ such that $\bar{\gamma}|_{Y_i} = \gamma_i$ for i = 0, 1.

If the metrics g_0 , g_1 and \bar{g} are psc-metrics, we say that the triples (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are *psc-bordant*. In both cases we use the notation $(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$ for such a bordism.

Theorem 5. Let (Y_0, g_0) and (Y_1, g_1) be closed oriented connected n-dimensional manifolds with psc-metrics, $3 \le n \le 7$, and maps $\gamma_0 : Y_0 \to S^1$ and $\gamma_1 : Y_1 \to S^1$. Assume that (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are psc-bordant.

Then there exists a psc-bordism $(Z, \bar{g}, \bar{\gamma}): (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$ and a properly embedded hypersurface $W \subset Z$ such that

- (i) the hypersurface W represents the class $\bar{\alpha} \in H_n(Z, \partial Z; \mathbf{Z})$ Poincare-dual to $[\bar{\gamma}] \in H^1(Z; \mathbf{Z})$;
- (ii) the hypersurface $X_i := \partial W \cap Y_i \subset Y_i$ represents the class $\alpha_i \in H_{n-1}(Y_i; \mathbf{Z})$ Poincare-dual to $[\gamma_i] \in H^1(Y_i; \mathbf{Z}), i = 0, 1;$
- (iii) there exists a metric \bar{h} on W such that $R_{\bar{h}} > 0$ and $H_{\bar{h}} \equiv 0$ along ∂W , and $R_{h_i} > 0$, where $h_i = \bar{h}|_{X_i}$, in particular, $(W, \bar{h}) : (X_0, h_0) \rightsquigarrow (X_1, h_1)$ is a psc-bordism;
- (iv) the metric \bar{h} on W could be chosen to be conformal to the restriction $\bar{g}|_{W}$.

Remark. The psc-bordism $(Z, \bar{g}, \bar{\gamma})$ and hypersurface W may be chosen so that ∂W is arbitrarily C^k -close to a desired homologically volume minimizing representative of $\alpha_0 - \alpha_1$ for any k and i = 0, 1.

Recall few definitions. We say that a conformal class C of metrics is positive if it contains a metric with positive scalar curvature. It is equivalent to the condition that the Yamabe constant Y(X;C) > 0. Now let W be a bordism with $\partial W = X_0 \sqcup X_1$, and C_0 , C_1 be positive conformal classes on X_0 , X_1 respectively. Then we say that the conformal manifolds (X_0, C_0) and (X_1, C_1) are positively conformally cobordant if the relative Yamabe invariant $Y(W, X_0 \sqcup X_1; C_0 \sqcup C_1) > 0$, see Section 4 for details. In these terms, the remark following Theorem 5 can be used to show the following:

Corollary 1. Let (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) be as in Theorem 5. Assume $X_i \subset Y_i$ are volume minimizing hypersurfaces representing homology classes Poincarè-dual to $[\gamma_i] \in H^1(X_i; \mathbf{Z})$, i = 0, 1. Then the conformal manifolds $(X_0, [g_0|_{X_0}])$ and $(X_1, [g_1|_{X_1}])$ are positively conformally cobordant.

The first step in the proof of Theorem 5 is to apply Theorem 4 to $\bar{\alpha}$, obtaining a minimal representative W. The main difficulty is that ∂W is, in general, not a minimal representative of $\partial \bar{\alpha}$ and so we may not apply Theorem 1 to conclude that ∂W even admits a psc-metric. However, in Section 3 we prove the Main Lemma, which states that ∂W becomes closer to minimizing $\partial \bar{\alpha}$ as longer collars are attached to the psc-bordism Z.

This work was motivated by intense discussions with D. Ruberman and N. Saveliev during and after the PIMS Symposium on Geometry and Topology of Manifolds held in Summer 2015. The authors are grateful to D. Ruberman and N. Saveliev for their help and inspiration. It is a pleasure to also thank C. Breiner, A. Fraser, and T. Schick for very helpful comments.

2. Preliminaries and Theorem 3

2.1. Stable minimal hypersurfaces with free boundary. Let (M, \bar{g}) be a compact oriented (n+1)-dimensional Riemannian manifold with nonempty boundary ∂M . Assume $W \subset M$ is a properly embedded hypersurface.

Let \bar{h} denote the restriction metric $\bar{h} = \bar{g}|_W$ and fix a unit normal vector field ν^W on W which is compatible with the orientation. This determines the second fundamental form A^W on W given by the formula $A_{\bar{g}}^W(X,Y) = \bar{g}(\nabla_X Y, \nu^W)$ for vector fields X and Y tangential to W. The trace of $A_{\bar{g}}^W$ with respect to the metric \bar{h} gives the mean curvature $H_{\bar{g}}^W = \operatorname{tr}_{\bar{h}} A_{\bar{g}}^W$. We will often omit the sub- and super-scripts, writing ν, A , and H if there is no risk of ambiguity.

Definition 3. Let $W \subset M$ be a properly embedded hypersurface. A variation of the hypersurface $W \subset M$ is a smooth one-parameter family $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ of proper embeddings $F_t:W\to M$, $t\in(-\epsilon,\epsilon)$ such that F_0 coincides with the inclusion $W\subset M$. A variation $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ is said to be normal if the curve $t\mapsto F_t(x)$ meets W orthogonally for each $x\in W$.

The vector field $X = \frac{d}{dt} F_t|_{t=0}$ is called the *variational vector field* associated to $\{F_t\}_{t\in(-\epsilon,\epsilon)}$. For normal variations, the associated variational vector field takes the form $\phi \cdot \nu^W$ for some function $\phi \in C^{\infty}(W)$. Clearly, a variation $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ gives a smooth function $t \mapsto \operatorname{Vol}(F_t(W))$.

Definition 4. A properly embedded hypersurface $W \subset (M, \bar{g})$ is minimal with free boundary if

$$\frac{d}{dt} \operatorname{Vol}(F_t(W))\big|_{t=0} = 0$$

for all variations $\{F_t\}_{t\in(-\epsilon,\epsilon)}$.

More notation: we denote by $d\sigma$ and $d\mu$ the volume forms of (W, \bar{h}) and $(\partial W, h)$, where $h = \bar{h}|_{\partial W}$ is the induced metric. We denote the outward-pointing unit length normal to ∂M by ν^{∂} . Below, Lemmas 2 and 3 contain well-known formulas, see [14].

Lemma 2. Let (M, \bar{g}) be an oriented Riemannian manifold and let $W \subset M$ be a properly embedded hypersurface. If $\{F_t\}_{t \in (-\epsilon, \epsilon)}$ is a variation of W with variational vector field X, then

(2.1)
$$\frac{d}{dt} \operatorname{Vol}(F_t(W)) \Big|_{t=0} = -\int_W H^W \bar{g}(X, \nu^W) d\mu + \int_{\partial W} \bar{g}(X, \nu^{\partial M}) d\sigma.$$

In particular, a hypersurface W is minimal with free boundary if and only if $H_{\bar{g}}^W \equiv 0$ and W meets the boundary ∂M orthogonally.

Definition 5. A properly embedded minimal hypersurface with free boundary W is stable if

$$\frac{d^2}{dt^2} \operatorname{Vol}(F_t(W))\Big|_{t=0} \ge 0$$

for all variations $\{F_t\}_{t\in(-\epsilon,\epsilon)}$.

If a hypersurface W is minimal with free boundary, then any variational vector field must be parallel to ν^W on ∂W since the variation must go through proper embeddings. Hence, it is enough to consider only normal variations to analyze the second variation of the volume functional.

Lemma 3. Let (M, \bar{g}) be an oriented Riemannian manifold and let $W \subset M$ be a properly embedded minimal hypersurface with free boundary. Let $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ be a normal variation with variational vector field $\phi \cdot \nu^W$. Then

$$(2.2) \quad \frac{d^2}{dt^2} \text{Vol}(F_t(W)) \bigg|_{t=0} = \int_W \left(|\nabla \phi|^2 - \phi^2 (\text{Ric}_{\bar{g}}(\nu^W, \nu^W) + |A^W|^2) \right) d\mu - \int_{\partial W} \phi^2 A^{\partial M}(\nu^W, \nu^W) d\sigma ,$$

where $\operatorname{Ric}_{\bar{q}}$ denotes the Ricci tensor of (M, \bar{q}) .

It will be useful to rewrite equation (2.2). The Gauss-Codazzi equations for a minimal hypersurface $W \subset M$ imply

$$R_{\bar{g}}^M = R_{\bar{h}}^W + 2\mathrm{Ric}_{\bar{g}}(\boldsymbol{\nu}^W, \boldsymbol{\nu}^W) + |\boldsymbol{A}^W|^2$$

on W. Here $R_{\bar{g}}^M$ and $R_{\bar{h}}^W$ are the scalar curvatures of (M, \bar{g}) and (W, \bar{h}) , respectively. It follows that the inequality $\frac{d^2}{dt^2} \operatorname{Vol}(F_t(W))\Big|_{t=0} \geq 0$ is equivalent to

(2.3)
$$\int_{W} |\nabla \phi|^{2} d\mu \geq \int_{W} \frac{1}{2} \phi^{2} \left(R_{\bar{g}}^{M} - R_{\bar{h}}^{W} + |A^{W}|^{2} \right) d\mu - \int_{\partial W} \phi^{2} A^{\partial M} (\nu^{W}, \nu^{W}) d\sigma.$$

2.2. Conformal Laplacian with minimal boundary conditions. The proof of Theorem 3 will rely on some basic facts about the conformal Laplacian on manifolds with boundary. Let (W, \bar{h}) be an n-dimensional manifold with non-empty boundary $(\partial W, h)$ where $h = \bar{h}|_{\partial W}$. We consider the following pair of operators acting on $C^{\infty}(W)$:

$$\begin{cases} L_{\bar{h}} = -\Delta_{\bar{h}} + c_n R_{\bar{h}}^W & \text{in } W \\ B_{\bar{h}} = \partial_{\nu} + 2c_n H_{\bar{h}}^{\partial W} & \text{on } \partial W, \end{cases}$$

where ν is the outward pointing normal vector to ∂W and $c_n = \frac{n-2}{4(n-1)}$

Recall that if $\phi \in C^{\infty}(W)$ is a positive function, then the scalar and boundary mean curvatures of the conformal metric $\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}$ are given by

(2.4)
$$\begin{cases} R_{\tilde{h}} = c_n^{-1} \phi^{-\frac{n+2}{n-2}} \cdot L_{\bar{h}} \phi & \text{in } W \\ H_{\tilde{h}} = \frac{1}{2} c_n^{-1} \phi^{-\frac{n}{n-2}} \cdot B_{\bar{h}} \phi & \text{on } \partial W. \end{cases}$$

We consider a relevant Rayleigh quotient and take the infimum:

(2.5)
$$\lambda_{1} = \inf_{\phi \neq 0 \in H^{1}(W)} \frac{\int_{W} (|\nabla \phi|^{2} + c_{n} R_{\bar{h}}^{W} \phi^{2}) d\mu + 2c_{n} \int_{\partial W} H_{\bar{h}}^{\partial W} \phi^{2} d\sigma}{\int_{W} \phi^{2} d\mu}.$$

According to standard elliptic PDE theory, we obtain an elliptic boundary problem, denoted by $(L_{\bar{h}}, B_{\bar{h}})$, and the infimum $\lambda_1 = \lambda_1(L_{\bar{h}}, B_{\bar{h}})$ is the *principal eigenvalue of the minimal boundary problem* $(L_{\bar{h}}, B_{\bar{h}})$. The corresponding Euler-Lagrange equations are the following:

(2.6)
$$\begin{cases} L_{\bar{h}}\phi = \lambda_1 \phi & \text{in } W \\ B_{\bar{h}}\phi = 0 & \text{on } \partial W. \end{cases}$$

This problem was first studied by Escobar [8] in the context of the Yamabe problem on manifolds with boundary.

Let ϕ be a solution of (2.6). It is well-known that the eigenfunction ϕ is smooth and can be chosen to be positive. A straight-forward computation shows that the conformal metric $\tilde{h} = \phi^{\frac{4}{n-2}} \bar{h}$ has the following scalar and mean curvatures:

(2.7)
$$\begin{cases} R_{\tilde{h}} = \lambda_1 \phi_1^{-\frac{4}{n-2}} & \text{in } W \\ H_{\tilde{h}} \equiv 0 & \text{on } \partial W. \end{cases}$$

In particular, the sign of the eigenvalue λ_1 is a conformal invariant, see [8, 11].

2.3. **Proof of Theorem 3.** Let (M, \bar{g}) and $W \subset M$ be as in Theorem 3. From the assumption $H^{\partial M} \equiv 0$, one can use the Gauss equations to show that $A^{\partial M}(\nu, \nu) = -H^{\partial W}$ where $H^{\partial W}$ is the mean curvature of ∂W as a hypersurface of W. Now, using the condition $R_{\bar{g}}^M > 0$, the stability inequality (2.3) implies

(2.8)
$$\int_{W} \left(|\nabla \phi|^2 + \frac{1}{2} R_{\bar{h}}^W \right) d\mu + \int_{\partial W} \phi^2 H^{\partial W} d\sigma \ge 0$$

for all functions $\phi \in H^1(W)$ with strict inequality if $\phi \not\equiv 0$. By simple manipulation, the inequality (2.8) may be written as

(2.9)
$$\int_{W} (|\nabla \phi|^{2} + c_{n} R_{\bar{h}}^{W}) d\mu + 2c_{n} \int_{\partial W} \phi^{2} H^{\partial W} d\sigma > (1 - 2c_{n}) \int_{W} |\nabla \phi|^{2} d\mu$$

for all $\phi \neq 0 \in H^1(W)$. The right hand side of (2.9) is non-negative since $1 - 2c_n = \frac{n}{2(n-1)} > 0$. Furthermore, the left hand side of (2.9) coincides with the numerator of the Rayleigh quotient in equation (2.5). We conclude that the principal eigenvalue $\lambda_1 = \lambda_1(L_{\bar{h}}, B_{\bar{h}})$ is positive. Let ϕ be an eigenfunction corresponding to λ_1 . Then, according to (2.7), the metric $\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}$ has positive scalar curvature and zero mean curvature on the boundary. This completes the proof of Theorem 3.

3. Cheeger-Gromov convergence of minimizing hypersurfaces

3.1. Convergence of hypersurfaces. Here we introduce the notion of smooth convergence of hypersurfaces we require for the proof of Theorem 5. First, we consider the case when the hypersurfaces are embedded in the same ambient (n+1)-dimensional manifold M. Below we use coordinate charts $\Phi_j: U_j \to M$, where U_j is an open subset of $\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$.

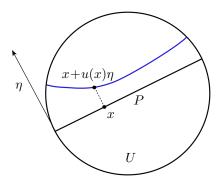


FIGURE 1. graph(u)

Let $P \subset \mathbb{R}^{n+1}$ be a hyperplane equipped with a normal unit vector η , and $U \subset \mathbb{R}^{n+1}_+$ be an open subset. Then for a function $u: P \cap U \to \mathbb{R}$, we denote by graph(u) its graph, see Fig. 1:

$$graph(u) = \{x + u(x)\eta \mid x \in P \cap U \}.$$

Definition 6. Let $k \geq 1$ be an integer. Let (M, \bar{g}) be an (n+1)-dimensional compact Riemannian manifold and let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of smooth, properly embedded hypersurfaces. Then we say that the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface Σ_{∞} C^k -locally as graphs if there exist

- (i) coordinate charts $\Phi_j: U_j \to M$ for $j = 1, \ldots, N$;
- (ii) hyperplanes $P_j \subset \mathbb{R}^{n+1}$ equipped with unit normal vectors η_j for $j = 1, \dots, N$;
- (iii) smooth functions $u_{i,j}: P_j \cap U_j \to \mathbb{R}$ for j = 1, ..., N, i = 1, 2, ..., and $i = \infty$,

which satisfy the following conditions:

(a)
$$\bigcup_{j=1}^{N} \Phi_{j}(\operatorname{graph}(u_{i,j}) \cap U_{j}) = \Sigma_{i} \text{ for } i = 1, 2, \dots \text{ and } i = \infty;$$

(b) for each $j=1,\ldots,N,\ u_{i,j}\to u_{\infty,j}$ in the $C^k(P_j\cap U_j)$ topology as $i\to\infty$.

We say the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface Σ_{∞} smoothly locally as graphs if it converges C^k -locally as graphs for all $k = 1, 2, \ldots$

Next, we consider a sequence $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$, where (M_i, \bar{g}_i) is a Riemannian manifold, $\Sigma_i \subset M_i$ is a properly embedded smooth hypersurface, and $\mathsf{S}_i \subset M_i$ a compact subset, playing a role of a base-point or a finite collection of base points.

Definition 7. Let $k \geq 1$ be an integer, and $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ be a sequence as above, where $\dim M_i = n+1$. We say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ C^k -converges to $(M_{\infty}, \Sigma_{\infty}, \bar{g}_{\infty}, \mathsf{S}_{\infty})$ if there is an

exhaustion of M_{∞} by precompact open sets

$$\mathsf{S}_{\infty} \subset \mathsf{U}_1 \subset \mathsf{U}_2 \subset \cdots \subset M_{\infty}, \quad M_{\infty} = \bigcup_{i=1}^{\infty} \mathsf{U}_i$$

and maps $\Psi_i: \mathsf{U}_i \to M_i$ which are diffeomorphisms onto their images for each $i=1,2,\ldots,$ such that

- (1) $\operatorname{dist}_{H}^{M_{\infty}}(\mathsf{S}_{\infty}, \Psi_{i}^{-1}(\mathsf{S}_{i})) \to 0$ as $i \to \infty$, where $\operatorname{dist}_{H}^{M_{\infty}}$ is the Hausdorff distance for subsets of the manifold M_{∞} ;
- (2) the sequence $\{\Psi_i^*\bar{g}_i\}$ of metrics converges to \bar{g}_{∞} in the $C^k(\mathsf{U}_i)$ -topology as $i\to\infty$;
- (3) the sequence of hypersurfaces $\{\Psi_j^{-1}(\Sigma_i)\}_{i=1}^{\infty}$ converges C^k -locally as graphs in the manifold M_{∞} to $\Sigma_{\infty} \cap \mathsf{U}_j$ as $i \to \infty$ for each $j = 1, \ldots, N$.

Remark. We notice that the conditions (1) and (2) imply that the sequence $\{(M_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ C^k -converges to $(M_{\infty}, \bar{g}_{\infty}, \mathsf{S}_{\infty})$ in the Cheeger-Gromov topology.

We say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ smoothly converges to $(M_{\infty}, \Sigma_{\infty}, \bar{g}_{\infty}, \mathsf{S}_{\infty})$ if it C^k -converges for all $k \geq 1$. Then we say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ sub-converges to $(M_{\infty}, \Sigma_{\infty}, \bar{g}_{\infty}, \mathsf{S}_{\infty})$ if it has a subsequence which converges to $(M_{\infty}, \Sigma_{\infty}, \bar{g}_{\infty}, \mathsf{S}_{\infty})$. In this case we write

$$(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i) \longrightarrow (M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty).$$

3.2. Main convergence result. We are ready to set the stage for the main result of this section. Let (Y,g) be a closed, oriented n-dimensional Riemannian manifold with a homology class $\alpha \in H_{n-1}(Y;\mathbb{Z})$. As we discussed in Section 1, the class α gives the Poincarè dual class $D\alpha = [\gamma] \in H^1(Y;\mathbb{Z})$ represented by some map $\gamma: Y \to S^1$. Furthermore, we assume that there is a bordism

$$(3.1) (M, \bar{g}, \bar{\gamma}) : (Y, g, \gamma) \leadsto (Y', g', \gamma')$$

for some triple (Y', g', γ') . In the above, $\bar{\gamma}: M \to S^1$ represents a class $[\bar{\gamma}] \in H^1(M; \mathbb{Z})$ Poincarè dual to a class $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$.

Recall that $Y \subset \partial M$ and $\bar{g} = g + dt^2$ near Y. For a real number $L \geq 0$, we consider the following Riemannian manifold

$$(M_L, \bar{g}_L) := (M \cup_{Y \times \{-L\}} (Y \times [-L, 0]), \bar{g}_L),$$

where \bar{g}_L restricts to \bar{g} on M and to the product-metric $g + dt^2$ on $Y \times [-L, 0]$. We obtain another bordism

$$(3.2) (M_L, \bar{g}_L, \bar{\gamma}_L) : (Y, g, \gamma) \leadsto (Y', g', \gamma'),$$

where $[\bar{\gamma}_L]$ is the image of $[\bar{\gamma}]$ under the isomorphism $H^1(M;\mathbb{Z}) \cong H^1(M_L;\mathbb{Z})$. We refer to the bordism $(M_L, \bar{g}_L, \bar{\gamma}_L)$ as the L-collaring of $(M, \bar{g}, \bar{\gamma})$. Below we will take L be an integer $i = 1, 2, \ldots$, and write $\bar{\alpha}_L \in H_n(M, \partial M; \mathbb{Z})$ for the class Poincarè dual to $[\bar{\gamma}_L]$.

Main Lemma. Let $(M, \bar{g}, \bar{\gamma}): (Y, g, \gamma) \leadsto (Y', g', \gamma')$ be a bordism as in (3.1) and denote by $(M_i, \bar{g}_i, \bar{\gamma}_i)$ the i-collaring of $(M, \bar{g}, \bar{\gamma})$ as in (3.2) for $i = 0, 1, 2, \ldots$ Fix a basepoint in each component of Y, denote their union by S, and let S_i be the image of S under the inclusion

$$Y \cong Y \times \{0\} \subset Y \times [-i, 0] \subset M_i$$
.

Assume $W_i \subset M_i$ is an oriented homologically volume minimizing representative of $\bar{\alpha}_i$ for $i = 0, 1, 2, \ldots$ If $X \subset Y$ is an embedded hypersurface which is the only volume minimizing representative of $\alpha \in H_{n-1}(Y; \mathbb{Z})$, then there is smooth subconvergence

$$(M_i, W_i, \bar{g}_i, \mathsf{S}_i) \longrightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, \mathsf{S}_{\infty})$$

as $i \to \infty$ where $S_{\infty} \subset Y \times \{0\}$ is the inclusion of S.

Remark. In Main Lemma, we allow the manifold Y' to be empty.

3.3. **Proof of the Main Lemma: outline.** Consider the limiting space $Y \times (-\infty, 0]$, with the exhaustive sequence $U_i = Y \times (-i - 1, 0]$ and maps $\Psi_i : U_i \to M_i$ taking U_i identically onto $Y \times (-i - 1, 0] \subset M_i$. Our choice of U_i and Ψ_i satisfy the conditions (1) and (2) from Definition 7 for obvious reasons.

It will be useful to equip M with a height function $F: M \to [-1, 0]$ satisfying $Y = F^{-1}(0)$ and $Y' = F^{-1}(-1)$. Extend this function to M_i by

$$F_i(x) = \begin{cases} t & \text{if } x = (y, t) \in Y \times [-i, 0] \\ F(x) - i & \text{if } x \in M. \end{cases}$$

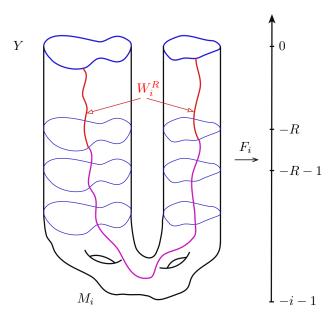


FIGURE 2. The hypersurface $W_i^R \hookrightarrow M_i$. In this figure, $Y' = \emptyset$.

For any positive integer i and heights $0 \le R < R' \le i$, we write

$$W_i^R = F_i^{-1}([-R, 0])$$
 and $W_i[-R', -R] = F_i^{-1}([-R', -R]).$

Let $\alpha \in H_{n-1}(Y; \mathbf{Z})$ be the class from the statement of Main Lemma. For L > 0 let

$$\alpha \times [-L, 0] \in H_n(Y \times [-L, 0], Y \times \{-L, 0\}; \mathbf{Z})$$

be the product of α and the fundamental class of $([-L, 0], \{-L, 0\})$. We will break up the proof of Main Lemma into three claims.

Claim 1. Let L > 0. The hypersurface $X \times [-L, 0] \subset Y \times [-L, 0]$ is the only homologically volume-minimizing representative of $\alpha \times [-L, 0]$.

Claim 2. For each R > 0, $Vol(W_i^R) \to R \cdot Vol(X)$ as $i \to \infty$.

Claim 3. For each R > 0, there is a sequence $\{a_i^R\}_{i=1}^{\infty}$ such that, for each j = 1, 2, ..., the hypersurfaces $\{\Psi_j^{-1}(W_{a_i^R}^R)\}_{i=1}^{\infty}$ converge smoothly locally as graphs in $Y \times (-\infty, 0]$.

Now we show how Main Lemma follows from Claims 1, 2, and 3. Indeed, by Claim 3, for each $k=1,2,\ldots$, there is a sequence $\{a_i^k\}_{i=1}^{\infty}$ such that, for each $j=1,2,\ldots$, the hypersurfaces $\{\Psi_j^{-1}(W_{a_i^k}^k)\}_{i=1}^{\infty}$ converges smoothly locally as graphs to some hypersurface

$$W_{\infty,k} \subset Y \times (-\infty,0].$$

We notice that the hypersurface $W_{\infty,k}$ is contained in $Y \times [-k,0]$ and represents the class $\alpha \times [-k,0]$. Since the convergence is smooth, we have

$$\operatorname{Vol}(\Psi_j^{-1}(W_{\infty,k})) = \lim_{i \to \infty} \operatorname{Vol}(\Psi_j^{-1}(W_{a_i^k}^k)) = k \cdot \operatorname{Vol}(X),$$

where the last equality follows from Claim 2. However, according to Claim 1, the only volume minimizing representative of $\alpha \times [-k,0]$ is the hypersurface $X \times [-k,0]$ which has the volume $k \cdot \operatorname{Vol}(X)$. Thus $W_{\infty,k}$ must be $X \times [-k,0]$. Evidently, the diagonal sequence $\{\Phi_j^{-1}(W_{a_i^i})\}_{i=1}^{\infty}$ has the property that, for each k > 0, $\Phi_j^{-1}(W_{a_i^i}^k)$ converges smoothly locally as graphs to $X \times [-k,0]$. This then completes the proof of Main Lemma.

3.4. **Proof of Claim 1.** Let $\Sigma \subset Y \times [-L, 0]$ be a properly embedded hypersurface representing the class $\alpha \times [-L, 0]$. Consider the projection function $P : \Sigma \to [-L, 0]$. The coarea formula [18, Theorem 5.3.9] applied to P yields

(3.3)
$$\int_{\Sigma} |\nabla P| d\mu = \int_{-L}^{0} \mathcal{H}^{n-1}(P^{-1}(t)) dt ,$$

where \mathcal{H}^{n-1} denotes the (n-1)-dimensional Hausdorff measure associated to the metric $h + dt^2$ on $Y \times [-L, 0]$. Notice that P is weakly contractive in the sense that

$$|P(x) - P(y)| \le \operatorname{dist}^{\Sigma}(x, y)$$

for all $x, y \in \Sigma$. Thus we have the pointwise bound $|\nabla P| \le 1$. Furthermore, since $P^{-1}(t)$ represents the class $\alpha \in H_{n-1}(Y \times \{t\}; \mathbf{Z})$ for each $t \in [-L, 0]$,

$$\mathcal{H}^{n-1}(P^{-1}(t)) \ge \operatorname{Vol}(X)$$

with equality if and only if $P^{-1}(t)$ is X. Combining this observation with (3.3), we conclude

$$Vol(\Sigma) \ge L \cdot Vol(X)$$

with equality if and only if $\Sigma = X \times [-L, 0]$. This completes the proof of Claim 1.

3.5. **Proof of Claim 2.** Before we begin, we will construct particular (in general, non-minimizing) properly embedded hypersurfaces $N_L \subset M_L$ representing α_L with which to compare $Vol(W_L)$ against.

Let $X \subset Y$ and $W_0 \subset M_0$ be as in Main Lemma. Since $\partial W_0 \cap Y$ and X represent the same homology class, they are bordant via a smooth, properly embedded hypersurface $\iota: U \hookrightarrow Y \times [0,1]$. We identify $[0,1] \cong [-L, -L+1]$ to obtain the embedding

$$\iota_L: U \stackrel{\iota}{\hookrightarrow} Y \times [0,1] \cong Y \times [-L, -L+1] \hookrightarrow M_L.$$

Clearly the embedding $\iota: U \hookrightarrow Y \times [0,1]$ may be chosen so that

$$N_L := W_0 \cup_{\partial W_0} U_L \cup (X \times [-L+1,0]),$$

where $U_L = \iota_L(U)$, is a smooth properly embedded hypersurface of M_L .

Evidently, $\operatorname{Vol}(N_L) = \operatorname{Vol}(W_0) + \operatorname{Vol}(U_L) + (L-1)\operatorname{Vol}(X)$ and N_L represents the same homology class as W_L . Since W_L is homologically area-minimizing, we have $\operatorname{Vol}(W_L) \leq \operatorname{Vol}(N_L)$.

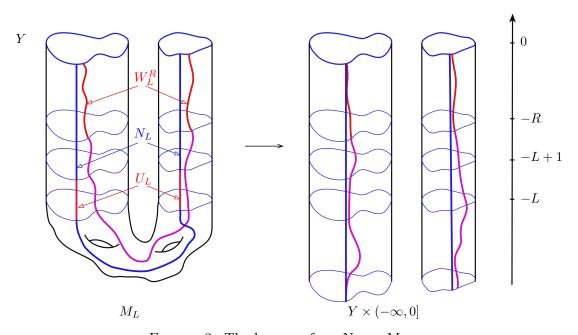


FIGURE 3. The hypersurface $N_L \hookrightarrow M_L$.

In other words, we obtain the inequality

$$(3.4) \operatorname{Vol}(W_L^R) + \operatorname{Vol}(W_L \setminus W_L^R) \le \operatorname{Vol}(W_0) + \operatorname{Vol}(U_L) + (L-1)\operatorname{Vol}(X)$$

for any 0 < R < L - 1.

Now we are ready to prove Claim 2. Assume it fails. Then there exist $\epsilon_0, R_0 > 0$ and an increasing sequence of whole numbers $\{a_i\}_{i=1}^{\infty}$ such that the inequality

$$(3.5) \operatorname{Vol}(W_{a_i}^{R_0}) > R_0 \cdot \operatorname{Vol}(X) + \epsilon_0$$

holds for all i. Combining the inequality (3.4) with the assumption (3.5), we have

(3.6)
$$\operatorname{Vol}(W_0) + \operatorname{Vol}(U_{a_i}) + (a_i - 1)\operatorname{Vol}(X) > \operatorname{Vol}(W_{a_i} \setminus W_{a_i}^{R_0}) + \epsilon_0 + R_0\operatorname{Vol}(X).$$

Now we will inspect the first term in the right hand side of (3.6):

$$\operatorname{Vol}(W_{a_{i}} \setminus W_{a_{i}}^{R_{0}}) = \operatorname{Vol}(W_{a_{i}}[a_{i-1} - a_{i}, -R_{0}]) + \operatorname{Vol}(W_{a_{i}}[-a_{i} - 1, a_{i-1} - a_{i}])$$

$$\geq (a_{i} - a_{i-1} - R_{0})\operatorname{Vol}(X) + \operatorname{Vol}(W_{a_{i-1}})$$

$$> (a_{i} - a_{i-1})\operatorname{Vol}(X) + \epsilon_{0} + \operatorname{Vol}(W_{a_{i-1}} \setminus W_{a_{i-1}}^{R_{0}}).$$
(3.7)

Here we use Claim 1 in the first inequality and the assumption (3.5) in the second.

Combining (3.6) with (3.7), we obtain

$$Vol(W_0) + Vol(U_{a_i}) + (a_i - 1)Vol(X) > (a_i - a_{i-1} + R_0)Vol(X) + 2\epsilon_0 + Vol(W_{a_{i-1}} \setminus W_{a_{i-1}}^{R_0}).$$

We iterate the argument to find

(3.8)
$$Vol(W_0) + Vol(U_{a_i}) + (a_1 - R_0 - 1)Vol(X) > i \cdot \epsilon_0 + Vol(W_{a_1})$$

for every $i = 1, 2, \ldots$ Since the left hand side of (3.8) is independent of i, we arrive at a contradiction by taking i to be sufficiently large.

3.6. **Proof of Claim 3.** While the proof of Claim 3 is rather technical, it is essentially a consequence of standard tools used in the study of stable minimal hypersurfaces. For instance, see [7] for a similar result in a 3-dimensional context. We divide the proof into three steps, referring to Appendix A when necessary.

To begin, we require the following straight-forward volume bound.

Step 1. For each R > 0, there is a constant $V_R > 0$ such that

$$Vol(W_i[-\lambda - R, -\lambda]) \le V_R$$

holds for all i and all $\lambda \in [0, i-R]$. In particular, $\operatorname{Vol}(W_i \cap B_R^{M_i}(x)) \leq V_R$ for all i and $x \in M_i$.

The next key ingredient is the following uniform bound on the second fundamental form A^{W_L} .

Step 2. There is a constant $C_1 > 0$, depending only on the geometry of (M, \bar{g}) , such that

$$\sup_{x \in W_L} |A^{W_L}(x)|^2 \le C_1 \quad \text{for } L \ge 0.$$

Step 2 is a consequence of [25, Corollary 1.1]. See Appendix, Section A.4 for more details.

Step 3. For each R > 0 and j = 1, 2, ..., the sequence of hypersurfaces $\Psi_j^{-1}(W_i^R)$ sub-converges smoothly locally as graphs as $i \to \infty$.

Proof of Step 3. We restrict our attention to the tail of the sequence $\{W_i^R\}_{i=1}^{\infty}$, where $i \geq R+1$. This allows us to consider each W_i^R and W_i^{R+1} as hypersurfaces of $Y \times (-\infty, 0]$ which is where we will show the convergence. By rescaling the original metric \bar{g} , we will assume that $\operatorname{inj}_g \geq 1$ and the bounds

$$\sup_{x \in B_1(y)} |\bar{g}_{ij}(x) - \delta_{ij}| \le \mu_0, \quad \sup_{x \in B_1(y)} \left| \frac{\partial \bar{g}_{ij}}{\partial x^k}(x) \right| \le \mu_0$$

hold for $1 \le i, j, k \le n + 1$ in geodesic normal coordinates centered about any $y \in Y \times (-\infty, 0]$ where μ_0 is the constant from Lemma 5. Let $r = \min(\frac{1}{24}, \frac{1}{6\sqrt{20C_0}})$ where C_0 is the constant from Step 2.

We cover $Y \times [-R, 0]$ by a finite collection of open balls $\mathcal{U} = \{B_r(y_l)\}_{l=1}^N$. Notice that each $B_r(y_l) \subset\subset Y \times [-R-1, 0]$. Consider a ball $B_r(y_l)$ in \mathcal{U} with the property that

$$W_i^{R+1} \cap B_r(y_l) \neq \emptyset$$

for infinitely many i. Unless explicitly stated, we will continue to denote all subsequences by W_i^{R+1} . Our next goal is to show that the sequence of hypersurfaces $\{W_i^R \cap B_r(y_l)\}_{i=1}^{\infty}$ sub-converges smoothly locally as graphs.

We choose a subsequence of W_i^{R+1} and points $x_i \in W_i^{R+1} \cap B_r(y_l)$ which converge to some point $x_\infty \in \overline{B_r(y_l)}$. Now it will be convenient to work in the tangent space to the point x_∞ . We use the short-hand notation $\phi = \exp_{x_\infty}^{\bar{g}}$ and let

$$B = \phi^{-1}(B_1(x_\infty)) \subset T_{x_\infty}(Y \times [-L - 1, 0]).$$

Consider the properly embedded hypersurfaces $\Sigma_i \subset B$ with base points $p_i \in \Sigma_i$, given by

$$\Sigma_i = \phi^{-1}(B_1(x_\infty) \cap W_i^R), \quad p_i = \phi^{-1}(x_i).$$

We also write $Z = \phi^{-1}(y_l)$ Since $W_i^R \subset M_i$ are minimal, Σ_i are minimal hypersurfaces in B with respect to the metric $\bar{g}_B = (\phi^{-1})^*(\bar{g})$.

Notice that the choice of r allows us to apply Corollary 2 to each $\Sigma_i \subset B$ at p_i with s=3r. For each $i=1,2,\ldots$, we obtain an open subset $U_i \subset T_{p_i}\Sigma_i \cap B$, a unit normal vector $\eta_i \perp T_{p_i}\Sigma_i$, and a function $u_i:U_i \to \mathbb{R}$ satisfying the bounds (A.4) and such that $\operatorname{graph}(u_i)=B_{6r}^{\Sigma_i}(p_i)$. Moreover, the connected component of $B_{3r}^{\bar{g}_B}(p_i) \cap \Sigma_i$ containing x_0 lies in $B_{6r}^{\Sigma_i}(p_i)$.

We use compactness of S^n and pass to a subsequence so that the vectors η_i converge to some vector $\eta_\infty \in S^n$. Let $P_\infty \subset T_{x_\infty}(Y \times [-L-1,0])$ be the hyperplane perpendicular to η_∞ . For large

enough i, we may translate and rotate the sets U_i to obtain open subsets $U_i' \subset P_{\infty}$ and functions $u_i' : U_i' \to \mathbb{R}$ such that

- (1) graph $(u_i') = B_{4r}^{\Sigma_i}(p_i);$
- (2) the ball $B_{2r}^{P_{\infty}}(0) \subset U_i'$;
- (3) for each $k \ge 1$ and $\alpha \in (0,1)$ there is a constant C' > 0, depending only on n, k, α , and the geometry of q, such that

$$||u_i'||_{C^{k,\alpha}(U_i')} \le C',$$

see Fig. 4. In particular, writing $u_i'' = u_i'|_{B_{2r}^{P_{\infty}}(0)}$, the sequence $\{u_i''\}_i$ is uniformly bounded in $C^{k,\alpha}(B_{2r}^{P_{\infty}}(0))$. Moreover, the connected component of $B_{2r}(p_i) \cap \Sigma_i$ containing p_i is contained in $\operatorname{graph}(u_i'')$. It follows that Σ_i' , the connected component of $B_r(Z) \cap \Sigma_i$ containing p_i , lies in $\operatorname{graph}(u_i'')$.

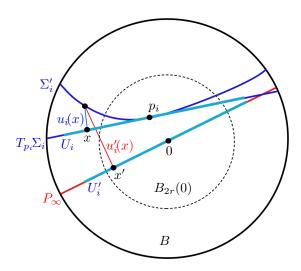


FIGURE 4. The functions u_i' and hypersurfaces Σ_i'

By Arzela-Ascoli, one can find a subsequence of u_i'' converging in $C^k(B_{2r}^{P_\infty}(0))$ to a function $u_\infty: B_{2r}^{P_\infty}(0) \to \mathbb{R}$. In particular, u_∞ is a strong solution to the minimal graph equation on $B_{2r}^{P_\infty}(0)$ with respect to \bar{g}_B and Σ_i' converge as graphs to graph (u_∞) . To summarize our current progress, the components of $W_i^{R+1} \cap B_r(y_l)$ containing x_i sub-converge smoothly to $\phi(\operatorname{graph}(u_\infty))$. This finishes our work with the hypersurfaces Σ_i' .

Now suppose that there is a second sequence of connected components within $W_i^{R+1} \cap B_r(y_l)$. We can repeat the above process to obtain a second limiting hypersurface. Observe that the number of components of $W_i^{R+1} \cap B_r(y_l)$ uniformly bounded in i. Indeed, using the notation above, for any component $\bar{\Sigma}_i \subset W_i^{R+1} \cap B_r(y_l)$, we have

$$\operatorname{Vol}_{\bar{g}_B}(\Sigma_i') \ge \operatorname{Vol}_{\bar{g}_B}(B_r^{P_{\infty}}(0)),$$

which is uniformly bounded below in terms of r and the geometry of g. However, Step 1 implies that $Vol(W_i^R \cap B_r(y_l))$ is bounded above uniformly in i so the number of connected components

 $W_i^R \cap B_r(y_l)$ is uniformly bounded in i. Hence the above process terminates after finitely many iterations. We conclude that the sequence $\{W_i^R \cap B_r(y_l)\}_{i=1}^{\infty}$ sub-converges smoothly locally as graphs to a minimal hypersurface $\Sigma_{\infty,l}$.

Now, restricting to this subsequence, we turn our attention to another ball $B_r(y_{l'})$ in the cover \mathcal{U} . We repeat the above argument to obtain a further subsequence and limiting minimal hypersurface $\Sigma_{\infty,l'}$. Repeating this process for each element of \mathcal{U} produces a subsequence converging to a minimal hypersurface $W_{\infty}^R = \bigcup_{l=0}^N \Sigma_{\infty,l}$ smoothly locally as graphs. This completes the proof of Claim 3, and consequently, the proof of Main Lemma.

4. Proof of Theorem 5

4.1. **Positive conformal bordism.** In order to prove Theorem 5, we have to use fundamental facts relating conformal geometry and psc-bordism. We briefly recall necessary results, following the conventions in [1]. Let Y be a compact closed manifold with $\dim Y = n$ given together with a conformal class C of Riemannian metrics. Then the Y amabe constant of (Y, C) is defined as

$$Y(Y;C) = \inf_{g \in C} \frac{\int_{Y} R_g d\mu_g}{\operatorname{Vol}_g(Y)^{\frac{n-2}{n}}}.$$

We say that a conformal class C is *positive* if Y(Y;C) > 0. It is well-known that C is positive if and only if there exists a psc-metric $g \in C$.

Now let $Z: Y_0 \leadsto Y_1$ be a bordism between closed manifolds Y_0 and Y_1 . Suppose we are given conformal classes C_0 and C_1 on Y_0 and Y_1 , respectively. Let \bar{C} be a conformal class on Z, such that $\bar{C}|_{Y_0} = C_0$ and $\bar{C}|_{Y_1} = C_1$, i.e. $\partial \bar{C} = C_0 \sqcup C_1$. Denote by $\bar{C}^0 = \{\bar{g} \in \bar{C} : H_{\bar{g}} \equiv 0\}$ the subclass of those metrics with vanishing mean curvature of the boundary. Then the relative Yamabe constant of $((Z, \bar{C}), (Y_0 \sqcup Y_1, C_0 \sqcup C_1))$ is defined as

$$Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) = \inf_{\bar{g} \in \bar{C}^0} \frac{\int_Z R_{\bar{g}} d\mu_{\bar{g}}}{\operatorname{Vol}_{\bar{g}}(Z)^{\frac{n-2}{n}}}.$$

This gives the relative Yamabe invariant

$$Y(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) = \sup_{\bar{C}, \, \partial \bar{C} = C_0 \sqcup C_1} Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1).$$

Now we assume that the conformal classes C_0 and C_1 are positive. Then we say that positive conformal manifolds (Y_0, C_0) and (Y_1, C_1) are positive-conformally bordant if there exists a conformal manifold (Z, \bar{C}) and a bordism $Z: Y_0 \rightsquigarrow Y_1$ between Y_0 and Y_1 such that $\partial \bar{C} = C_0 \sqcup C_1$ and $Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) > 0$. In this case, we write $(Z, \bar{C}): (Y_0, C_0) \rightsquigarrow (Y_1, C_1)$. We need the following result which relates the above notions to psc-bordisms.

Theorem 6. [1, Corollary B] Let Y_0 and Y_1 be closed manifolds of dimension $n \geq 3$, $Z: Y_0 \rightsquigarrow Y_1$ be a bordism between Y_0 and Y_1 , and g_0 and g_1 be psc-metrics on Y_0 and Y_1 , respectively. Then $Y(Z, Y_0 \sqcup Y_1; [g_0] \sqcup [g_1]) > 0$ if and only if the boundary metric $g_0 \sqcup g_1$ on $Y_0 \sqcup Y_1$ may be extended to a psc-metric \bar{g} on Z such that $\bar{g} = g_j + dt^2$ near Y_j for j = 0, 1.

4.2. Long collars. We are ready to prove Theorem 5 for $n \le 6$. The adjustments required to adapt the following proof to the case n = 7 are provided in Appendix A.5.

Let (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) be the manifolds from Theorem 5 and let $\alpha_0 \in H_{n-1}(Y_0; \mathbf{Z})$ and $\alpha_1 \in H_{n-1}(Y_1; \mathbf{Z})$ be the classes Poincarè dual to γ_0 and γ_1 , respectively. It is convenient to use the notation $Y_0 = Y_0 \cup Y_1$ and

$$\alpha = (\iota_0)_* \alpha_0 - (\iota_1)_* \alpha_1 \in H_{n-1}(Y; \mathbf{Z}),$$

where $\iota_j: Y_j \hookrightarrow Y$ is the inclusion map for j=0,1. Then we consider hypersurfaces $X_0 \subset Y_0$ and $X_1 \subset Y_0$ which are homologically volume minimizing representatives of the classes α_0 and $-\alpha_1$. The existence of such smooth X_0 and X_1 is guaranteed in this range of dimensions, see [27]. Notice that, by a small conformal change which does not effect the assumptions on (Y_j, g_j, γ_j) , we may assume that X_j is the only representative of α_j with minimal volume for j=0,1, see [24, Lemma 1.3]. We write (X,h_X) for the Riemannian manifold $(X_0 \sqcup X_1, g_0|_{X_0} \sqcup g_1|_{X_1})$.

Now we choose a psc-bordism $(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$. We will use $(Z, \bar{g}, \bar{\gamma})$ to construct a psc-bordism which satisfies the conclusion of Theorem 5. We denote by $\bar{\alpha} \in H_n(Z; \mathbf{Z})$ the homology class Poincarè dual to $\bar{\gamma}$. Then $\partial \bar{\alpha} = \alpha$, see Lemma 1.

Now for each $i=1,2,\ldots$, we consider the *i*-collaring of the bordism $(Z,\bar{g},\bar{\gamma})$, denoted by $(Z_i,\bar{g}_i,\bar{\gamma}_i)$, as in Section 3.2. By Theorem 4, there exists properly embedded hypersurfaces $W_i \subset Z_i$ which are homologically volume minimizing and represents $\bar{\alpha}_i$. The restrictions of \bar{g}_i to W_i and ∂W_i are denoted by \bar{h}_i and h_i , respectively.

In preparation to apply Main Lemma, we fix basepoints $x_j \in X_j$ for each j = 0, 1 and set $S = \{x_0, x_1\} \subset X$. Naturally, the set S is identified with the subsets S_i in $(X \times \{0\}) \subset \partial Z_i$ for $i = 1, 2, \ldots$ and with S_{∞} in the boundary of the cylinder $(X \times \{0\}) \subset (Y \times (-\infty, 0])$. According to Main Lemma we may find a subsequence $\{a_i\}_{i=1}^{\infty}$ such that

$$(Z_{a_i}, W_{a_i}, \bar{g}_{a_i}, \mathsf{S}_{a_i}) \longrightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, \mathsf{S}_{\infty})$$

smoothly as $i \to \infty$ and the Riemannian manifolds $(\partial W_{a_i}, h_{a_i})$ converge to (X, h_X) in the smooth Cheeger-Gromov topology as $i \to \infty$.

Remark. We note that the manifolds $(\partial W_{a_i}, h_{a_i})$, (X, h_X) are compact and so there is no need to specify base points for this convergence.

The following is a special case of a much more general fact on the behavior of elliptic eigenvalue problems under smooth Cheeger-Gromov convergence (see [5]).

Lemma 4. Let $\{(M_i, g_i')\}_{i=1}^{\infty}$ be a sequence of compact Riemannian manifolds smoothly converging to a compact Riemannian manifold $(M_{\infty}, g_{\infty}')$ in the Cheeger-Gromov sense. If $Y(M_{\infty}; [g_{\infty}']) > 0$, then, upon passing to a subsequence, $Y(M_i; [g_i']) > 0$ for all sufficiently large i.

¹ Here we emphasize a proper orientation on Y_0 and Y_1

Proof. For each i = 1, 2, ..., we denote by $\lambda_{1,i} = \lambda_1(L_{g'_i})$ the principal eigenvalue of the conformal Laplacian on (M_i, g'_i) . Let $\phi_i \in C^{\infty}(M_i)$ be the eigenfunction satisfying

$$(4.1) L_{g_i'}\phi_i = \lambda_{1,i}\phi_i, \sup_{M_i} \phi_i = 1.$$

Since $\{(M_i, g_i')\}_{i=1}^{\infty}$ is converging in the Cheeger-Gromov topology to a compact manifold, the coefficients of the operator $L_{g_i'}$ are bounded in the C^1 -norm uniformly in i. In particular, there is a constant $C_1 > 0$, independent of i, such that $|R_{g_i'}| \leq C_1$ on M_i . An obvious estimate on the Rayleigh quotient (2.5) shows that the sequence $\{\lambda_{1,i}\}_{i=1}^{\infty}$ is uniformly bounded above and below.

This allows us to apply the Schauder estimate Theorem 7 to ϕ_i uniformly in i. Using Arzelá-Ascoli, we can find a subsequence, still denoted by $\{(M_i, g_i')\}_{i=1}^{\infty}$, $\{\phi_i\}_{i=1}^{\infty}$, and $\{\lambda_{1,i}\}_{i=1}^{\infty}$, a function $\phi_{\infty} \in C^{\infty}(M_{\infty})$, and a number $\lambda_{1,\infty}$ such that

$$\phi_i \to \phi_{\infty} \quad \lambda_{1,i} \to \lambda_{1,\infty}$$

where the former convergence is in the $C^{2,\alpha}$ -topology. This allows us to take the limit of equation (4.1) as $i \to \infty$. Namely, ϕ_{∞} is a non-zero solution of the equation

$$L_{q_{\infty}}\phi_{\infty}=\lambda_{1,\infty}\phi_{\infty}$$

and so $\lambda_{1,\infty} \geq \lambda_1(L_{g_{\infty}})$. On the other hand, we have assumed that $\lambda_1(L_{g_{\infty}}) > 0$. Hence $\lambda_{1,i} > 0$ for all sufficiently large i.

Now we return to the proof of Theorem 5. Since X is a stable minimal hypersurface of Y with trivial normal bundle, Theorem 1 implies that $Y(X,[g_X])>0$. Now we may apply Lemma 4 to find $Y(\partial W_{a_i},[h_{a_i}])>0$ for sufficiently large i. Fix such an i and let $h'_{a_i}\in[h_{a_i}]$ be a psc metric on ∂W_{a_i} . Since each W_{a_i} is a stable minimal hypersurface with free boundary and trivial normal bundle, Theorem 3 states that $Y(W_{a_i},\partial W_{a_i};[\bar{h}_{a_i}])>0$ for all $i\in\mathbb{N}$. Finally, we use Theorem 6 to find a psc-metric \tilde{h}_{a_i} on W_{a_i} which restricts to $h'_{a_i}+dt^2$ near ∂W_{a_i} . This completes the proof of Theorem 5 for n<6.

Appendix A.

The main goal here is to provide technical details we used in the main body of the paper. In Section A.1, we recall relevant facts on the minimal graph equation and provide the Schauder estimates we use in the proof of Main Lemma. Section A.2 is dedicated to Theorem 4. Here we recall necessary results on currents and state well-known facts on their compactness and regularity, adapted to our setting. Section A.3 describes a simple doubling method which is a convenient technical tool in the remaining sections. In Section A.4, we justify Step 2 from the proof of Claim 3. In Section A.5, we discuss regularity issues in dimension 8 and prove Theorem 5 for n = 7.

A.1. The minimal graph equation. This section is concerned with local properties of hypersurfaces in Riemannian manifolds. Throughout this section we will consider the unit ball in Euclidian space $B = B_1(0) \subset \mathbb{R}^{n+1}$ equipped with a Riemannian metric g and a hypersurface $\Sigma^n \subset B$. The

balls of radius s > 0 centered at $x \in \Sigma$ induced by g and $g|_{\Sigma}$ are denoted by $B_s^g(x) \subset B$ and $B_s^{\Sigma}(x) \subset \Sigma$, respectively. Assume there is a point $x_0 \in \Sigma \cap B_{1/4}(0)$.

The following straight-forward Riemannian version of [6, Lemma 2.4] allows us to consider Σ locally as a graph over $T_{x_0}\Sigma$.

Lemma 5. There is a constant $\mu_0 > 0$ so that if g satisfies

(A.1)
$$\sup_{x \in B} |g_{ij}(x) - \delta_{ij}| \le \mu_0, \quad \sup_{x \in B} \left| \frac{\partial g_{ij}}{\partial x^k}(x) \right| \le \mu_0$$

for $1 \le i, j, k \le n+1$ in standard Euclidian coordinates, then the following holds: If s > 0 satisfies

$$\operatorname{dist}^{\Sigma}(x_0, \partial \Sigma) \ge 3s, \quad \sup_{\Sigma} |A_g|^2 \le \frac{1}{20s^2},$$

then there is an open subset $U \subset T_{x_0}\Sigma \subset \mathbb{R}^{n+1}$, a unit vector η normal to $T_{x_0}\Sigma$, and a function $u: U \to \mathbb{R}$ such that

- (1) graph(u) = $B_{2s}^{\Sigma}(x_0)$;
- (2) $|\nabla u| \le 1$ and $|\nabla \nabla u| \le \frac{1}{s\sqrt{2}}$ hold pointwise.

Moreover, the connected component of $B_s^g(x_0) \cap \Sigma$ containing x_0 lies in $B_{2s}^{\Sigma}(x_0)$.

Now we will give a useful expression for the mean curvature of a graph. Let $U \subset \mathbb{R}^n$ be an open set with standard coordinates $x' = (x^1, \dots, x^n)$ and let g be a Riemannian metric on $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$. For a function $u: U \to \mathbb{R}$, consider its graph

$$graph(u) = \{(x', u(x')) \in \mathbb{R}^{n+1} : x' \in U\}.$$

For i = 1, ..., n, we have the tangential vector fields $E_i = \frac{\partial}{\partial x^i} + \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^{n+1}}$ and the upward-pointing unit vector field ν normal to graph(u). Writing $h_{ij} = g(E_i, E_j)$ for the restriction metric, the mean curvature of graph(u) can be written

$$H_g = h^{ij}g(\nu, \nabla_{E_i}E_j)$$

$$(A.2) = \left(g^{ij} - \frac{\nabla^{i} u \nabla^{j} u}{1 + |\nabla u|^{2}}\right) \left[\frac{\partial^{2} u}{\partial x_{i} \partial x_{j}} + \Gamma^{n+1}_{ij} + \frac{\partial u}{\partial x_{i}} \Gamma^{n+1}_{n+1}{}_{j} + \frac{\partial u}{\partial x_{j}} \Gamma^{n+1}_{n+1}{}_{i} + \frac{\partial u}{\partial x_{i}} \frac{\partial u}{\partial x_{j}} \Gamma^{n+1}_{n+1}{}_{n+1} - \frac{\partial u}{\partial x_{i}} \left(\Gamma^{r}_{ij} + \frac{\partial u}{\partial x_{i}} \Gamma^{r}_{n+1}{}_{j} + \frac{\partial u}{\partial x_{i}} \Gamma^{r}_{i}{}_{n+1} + \frac{\partial u}{\partial x_{i}} \frac{\partial u}{\partial x_{j}} \Gamma^{r}_{n+1}{}_{n+1}\right)\right],$$

see [6, Section 7.1] for a detailed exposition in the 3-dimensional case.

Next, we will state a general version of the Schauder estimates for elliptic operators on Euclidian space. It is applied to the geometric setting in Section 3.

Theorem 7. [15, Corollary 6.3] Let $U \subset \mathbb{R}^n$ be an open set and let $\alpha \in (0,1)$. Suppose $u \in C^{2,\alpha}(U)$ satisfies a uniformly elliptic equation

$$Lu = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0$$

with $a^{ij}, b^i, c \in C^{\alpha}(U)$ and ellipticity constant $\lambda > 0$. If $U' \subset U$ with $\operatorname{dist}^U(U', \partial/u) = d$, then there is a constant C > 0, depending on $d, \lambda, ||a^{ij}||_{C^{\alpha}(U)}, ||b^i||_{C^{\alpha}(U)}, ||c||_{C^{\alpha}(U)}, n$, and α , such that

(A.3)
$$||u||_{C^{2,\alpha}(U')} \le C||u||_{C^0(U)}.$$

Corollary 2. Suppose the unit ball $B = B_1(0) \subset \mathbb{R}^{n+1}$ is equipped with a Riemannian metric g satisfying

$$\sup_{x \in B} |g_{ij}(x) - \delta_{ij}| \le \mu_0, \quad \sup_{x \in B} \left| \frac{\partial g_{ij}}{\partial x^k}(x) \right| \le \mu_0$$

in Euclidian coordinates for all $1 \leq i, j, k \leq n+1$ where μ_0 is the constant from Lemma 5. Let C > 0 be given and set $r = \min(\frac{1}{8}, \frac{1}{\sqrt{80C}})$. Assume that $\Sigma \subset B$ is a properly embedded minimal hypersurface with respect to g such that $\sup_{B} |A^g|^2 \leq C$ and there is a point $x_0 \in B_r(0) \cap \Sigma$. Then there is a smooth function $u: U \to \mathbb{R}$ on $U \subset T_{x_0}\Sigma$ and a unit normal vector to $T_{x_0}\Sigma$ such that

- (1) graph $(u) = B_{2r}^{\Sigma}(x_0)$;
- (2) $|\nabla u| \le 1$ and $|\nabla \nabla u| \le \frac{1}{s\sqrt{2}}$ hold pointwise;
- (3) for each $k \geq 1$ and $\alpha \in (0,1)$ there is a constant C' > 0, depending only on n, k, α , and $||g||_{C^{k,\alpha}(B)}$, such that

$$(A.4) ||u||_{C^{k,\alpha}(U)} \le C'.$$

Moreover, the connected component of $B_r(x_0) \cap \Sigma$ containing x_0 is contained in $B_{2r}^{\Sigma}(x_0)$.

Proof. The choice of radius r allows us to apply Lemma 5 to obtain an open subset $U \subset T_{x_0}\Sigma \subset \mathbb{R}^{n+1}$, a unit vector η normal to $T_{x_0}\Sigma$, and a smooth function $u:U\to\mathbb{R}$ such that $\operatorname{graph}(u)=B_{2s}^\Sigma(x_0),\ |\nabla u|\leq 1$, and $|\nabla\nabla u|\leq \frac{1}{s\sqrt{2}}$ on U. Since Σ is minimal, u solves equation H=0. Now since $||u||_{C^{1,\alpha}(U)}$ is bounded for any fixed $\alpha\in(0,1)$, one can inspect the expression A.2 to see that u solves a linear elliptic equation with coefficients bounded in C^α in terms of μ_0 and r. This allows us to apply Theorem 7 to obtain the estimate $||u||_{C^{2,\alpha}(U')}\leq C||u||_{C^0(U)}$ for some C>0 depending only on μ_0 and r. Standard elliptic estimates [15, Section 6] give a similar estimate in the $C^{k,\alpha}$ -norm for any k.

A.2. **Details on Theorem 4.** Let us recall some basic notions from theory of integer multiplicity currents. The main reference for this material is [13, Chapter 4].

For an open subset $U \subset \mathbb{R}^{n+k}$, let $\Omega^n(U)$ denote the space of all n-forms on \mathbb{R}^{n+k} with compact support in U. An n-current on U is a continuous linear functional $T:\Omega^n(U)\to\mathbb{R}$ and collection of such T for a vector space $\mathcal{D}_n(U)$. The boundary of an n-current T is the (n-1)-current ∂T defined by

$$(\partial T)(\omega) = T(d\omega), \qquad \omega \in \Omega^{n-1}(U).$$

The mass of $T \in \mathcal{D}_n(U)$ is given by $\mathbf{M}(T) = \sup\{T(\omega) : \omega \in \Omega^n(U), |\omega| \leq 1\}$. For example, if T is given by integration along a smooth oriented submanifold M, then $\mathbf{M}(T) = \operatorname{Vol}(M)$.

Let \mathcal{H}^n denote the *n*-dimensional Hausdorff measure on \mathbb{R}^{n+k} . A current $T \in \mathcal{D}_n(U)$ is called integer multiplicity rectifiable (or simply rectifiable) if it takes the form

(A.5)
$$T(\omega) = \int_{M} \omega(\xi(x))\theta(x)d\mathcal{H}^{n}(x), \qquad \omega \in \Omega^{n}(U), \text{ where}$$

- (1) $M \subset U$ is \mathcal{H}^n -measurable and countably n-rectifiable, see [13, Section 3.2.14];
- (2) $\theta: M \to \mathbf{Z}$ is locally \mathcal{H}^n -integrable;
- (3) for \mathcal{H}^n -almost every $x \in M$, $\xi : M \to \Lambda^n T \mathbb{R}^{n+k}$ takes the form $\xi(x) = e_1 \wedge \ldots \wedge e_n$ where $\{e_i\}_{i=1}^n$ form an orthonormal basis for the approximate tangent space $T_x M$, see [13, Section 3.2.16].

Remark. The above definition of integer multiplicity rectifiable currents can also be extended to Riemannian manifolds (M, g) – one defines the mass of a current using the Hausdorff measure given by the metric g.

The regular set $\operatorname{reg}(T)$ of a rectifiable n-current T is given by the set of points $x \in \operatorname{spt}(T)$ for which there exists an oriented n-dimensional oriented C^1 -submanifold $M \subset U$, r > 0, and $m \in \mathbf{Z}$ satisfying

$$T|_{B_r(x)}(\omega) = m \cdot \int_{M \cap B_r(x)} \omega, \quad \forall \omega \in \Omega^n(U).$$

The singular set sing(T) is given by $spt(T) \setminus reg(T)$. The abelian group of n-dimensional integral flat chains on U is given by

$$\mathcal{F}_n(U) = \{R + \partial S \colon R \in \mathcal{D}_n(U) \text{ and } S \in \mathcal{D}_{n+1}(U) \text{ are rectifiable}\}.$$

Now we consider subsets $B \subset A \subset U$. We have the group of integral flat cycles

$$C_n(A, B) = \{T \in \mathcal{F}_n(U) : \operatorname{spt}(T) \subset A, \operatorname{spt}(\partial T) \subset B, \text{ or } n = 0\}$$

and the subgroup of integral flat boundaries

$$\mathcal{B}_n(A,B) = \{T + \partial S \colon T \in \mathcal{F}_n(U), \operatorname{spt}(T) \subset B, S \in \mathcal{F}_{n+1}(U), \operatorname{spt}(S) \subset A\}.$$

The quotient groups $\mathbf{H}_n(A, B) = \mathcal{C}_n(A, B)/\mathcal{B}_n(A, B)$ are the *n*-dimensional integral current homology groups.

There is a natural transformation between the integral singular homology functor and the integral current homology functor which induces an isomorphism $H_n(A, B; \mathbf{Z}) \cong \mathbf{H}_n(A, B)$ in the category of local Lipschitz neighborhood retracts, see [13, Section 4.4.1]. This isomorphism can be combined with a basic compactness result for rectifiable currents to find volume minimizing representatives of homology classes.

Lemma 6. Let (M, \bar{g}) be a compact (n+1)-dimensional Riemannian manifold with boundary and consider an integral homology class $\alpha \in H_n(M, \partial M; \mathbf{Z})$. Let $\tilde{\alpha} \in \mathbb{H}_n(M, \partial M)$ be the image of α under the isomorphism $H_n(M, \partial M; \mathbf{Z}) \to \mathbf{H}_n(M, \partial M)$. Then there exists a homologically volume minimizing integer multiplicity rectifiable current $T \in \tilde{\alpha}$.

Proof. By the Nash embedding theorem there is an isometric embedding $\iota: M \to \mathbb{R}^{n+k}$ for some sufficiently large k. Let \hat{M} be the image of this embedding and set $\hat{\alpha} = \iota_* \tilde{\alpha} \in \mathbf{H}_n(\hat{M}, \partial \hat{M})$. Applying the compactness result in [13, Section 5.1.6], we obtain a homologically volume minimizing current $\hat{T} \in \mathcal{C}_n(\hat{M}, \partial \hat{M})$ representing $\hat{\alpha}$. Since ι is an isometry, $(\iota^{-1})_* \hat{T}$ is the desired current.

Since Lemma 6 guarantees the existence of homologically volume minimizing representative for the homology class α from the hypothesis of Theorem 4, the final ingredient is regularity theory for volume minimizing rectifiable currents with free boundary. The following is a regularity theorem due to M. Grünter [16, Theorem 4.7] adapted to the context of an ambient Riemannian metric. See [19, 14, 26] for Riemannian adaptations of similar results.

Theorem 8. Let $S \subset \mathbb{R}^{n+1}$ be an n-dimensional smooth submanifold, $U \subset \mathbb{R}^{n+1}$ an open set with $\partial S \cap U = \emptyset$, and g a Riemannian metric on U with bounded injectivity radius and sectional curvature. Suppose $T \in \mathcal{F}_n(U)$ with $\operatorname{spt}(\partial T) \subset S$ satisfies $\mathbf{M}_g(T) \leq \mathbf{M}_g(T+R)$ for all open $W \subset U$ and all $R \in \mathcal{F}_n(U)$ with $\operatorname{spt}(R) \subset W$ and $\operatorname{spt}(\partial R) \subset S$. Then we have

- $sing(T) = \emptyset$ if $n \le 6$
- sing(T) is discrete for n = 7
- $\dim_{\mathcal{H}}(\operatorname{sing}(T)) \le n 7$ if n > 7

where $\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of a subset $A \subset U$.

We will briefly explain how Theorem 4 follows from Theorem 8. Let T be the volume minimizing representative of $\bar{\alpha}$ from Theorem 4. For a point $x \in \operatorname{spt}(T)$, set $\phi = \exp_x^{\bar{g}}$ and consider

$$U = \phi^{-1}(B_{r'}^{\bar{g}}(x)) \subset T_x M, \quad S = \phi^{-1}(\partial M \cap B_r^{\bar{g}}(x)),$$
$$T' = (\phi^{-1})_* T \in \mathcal{D}_n(U), \quad g = (\phi^{-1})_* \bar{g},$$

where $0 < r' < r \le \operatorname{inj}(\bar{g})$. By Theorem 8, the singular set of T' is empty and so there is a neighborhood V of $0 \in U$ such that $T'|_V$ is given by an integer multiple of integration along a C^1 -submanifold $M \subset V$. Locally, M can be written as the graph of a C^1 -function which weakly solves the minimal surface equation. Standard elliptic PDE methods imply that M is smooth, see, for instance the proof of Lemma 7 below.

A.3. Doubling minimal hypersurfaces with free boundary. In this section we consider the reflection of a free boundary stable minimal hypersurface over its boundary. To fix the setting, let (M, \bar{g}) be an (n+1)-dimensional compact oriented Riemannian manifold with boundary ∂M and restriction metric $g = \bar{g}|_{\partial M}$. Assume that there is a neighborhood of the boundary on which $\bar{g} = g_{\partial M} + dt^2$. The double of (M, \bar{g}) is the smooth closed manifold $M_{\mathcal{D}}$ given by $M_{\mathcal{D}} = M \cup_{\partial M} (-M)$. Notice that the double $M_{\mathcal{D}}$ comes equipped with an involution $\iota : M_{\mathcal{D}} \to M_{\mathcal{D}}$ which interchanges the two copies of M and fixes the doubling locus $\partial M \subset M_{\mathcal{D}}$. Since \bar{g} splits as a product near the boundary, one can also form the smooth doubling of \bar{g} , denoted by $\bar{g}_{\mathcal{D}}$, by setting $\bar{g}_{\mathcal{D}} = \bar{g}$ on M and $\bar{g}_{\mathcal{D}} = \iota_*\bar{g}$ on -M.

Lemma 7. Let (M, \bar{g}) be a compact oriented Riemannian manifold with boundary with $\bar{g} = g + dt^2$ near ∂M . If $\Sigma \subset M$ be a properly embedded minimal hypersurface with free boundary, then double of Σ , given by $\Sigma_{\mathcal{D}} = \Sigma \cup_{\partial \Sigma} \iota(\Sigma)$ is a smooth minimal hypersurface of $(M_{\mathcal{D}}, \bar{g}_{\mathcal{D}})$. Moreover, if Σ is stable, then so is $\Sigma_{\mathcal{D}}$.

Proof. First, we will show that $\Sigma_{\mathcal{D}}$ is a smooth hypersurface. Clearly, $\Sigma_{\mathcal{D}}$ is smooth away from the doubling locus $\partial \Sigma \subset M_{\mathcal{D}}$. Let $x_0 \in \partial \Sigma$ and let r > 0 be less than the injectivity radius of $\bar{g}_{\mathcal{D}}$. Set $\phi = \exp^{\bar{g}_{\mathcal{D}}}$ and consider

$$\hat{\Sigma} = \phi^{-1}(\Sigma \cap B_r(x_0)), \quad \hat{\Sigma}_{\mathcal{D}} = \phi^{-1}(\Sigma_{\mathcal{D}} \cap B_r(x_0)), \quad \hat{g} = \phi^* \bar{g}_{\mathcal{D}}$$

and ν , the unit normal vector field to $\hat{\Sigma}$ with respect to \hat{g} . Evidently, $\hat{\Sigma}$ is a minimal hypersurface in $T_{x_0}M_{\mathcal{D}}$ with free boundary contained in $T_{x_0}\partial M \subset T_{x_0}M_{\mathcal{D}}$ with respect to \hat{g} . We choose an orthonormal basis for $T_{x_0}M_{\mathcal{D}}$ so that, writing $x \in T_{x_0}M$ as (x^1, \ldots, x^{n+1}) in this basis,

- (1) $T_{x_0}\partial\hat{\Sigma} = \{(x^1,\ldots,x^{n-1},0,0)\};$
- (2) $T_{x_0}\hat{\Sigma} = \{(x^1, \dots, x^n, 0)\};$
- (3) $T_{x_0}\partial M = \{(x^1, \dots, x^{n-1}, 0, x^{n+1})\}.$

This can be accomplished since Σ meets ∂M orthogonally. In these coordinates, the involution ι now takes the form $(x^1, \ldots, x^n, x^{n+1}) \mapsto (x^1, \ldots, -x^n, x^{n+1})$. Notice that, because the second fundamental form of ∂M vanishes, $\phi^{-1}(\partial M \cap B_r(x_0))$ is contained in the hyperplane $\{(x^1, \ldots, x^{n+1}) : x^n = 0\}$.

For a radius r' < r, we consider the *n*-dimensional ball

$$B_{r'}^{n}(0) = \{x \in T_{x_0}M : x^{n+1} = 0, ||x|| < r'\},\$$

the *n*-dimensional half-ball $B^n_{r',+}(0) = \{x \in B^n_{r'}(0) : x^n \ge 0\}$, and the cylinder

$$C_{r'}(0) = \{x \in T_{x_0}M \colon (x^1, \dots, x^n, 0) \in B_{r'}^n(0)\}.$$

For small enough r', we may write $\hat{\Sigma} \cap C_{r'}(0)$ as the graph of a function

$$u: B_{r',+}^n(0) \to \mathbb{R}, \quad \operatorname{graph}(u) = \hat{\Sigma} \cap C_{r'}(0)$$

where graph $(u) = \{(x^1, \dots, x^n, u(x^1, \dots, x^n)) : (x^1, \dots, x^n, 0) \in B^n_{r'}(0)\}$. Now we may form the doubling of u to a function $u_{\mathcal{D}} : B^n_{r'}(0) \to \mathbb{R}$, setting

$$u_{\mathcal{D}}(x^{1},\dots,x^{n}) = \begin{cases} u(x^{1},\dots,x^{n}) & \text{if } x^{n} \ge 0\\ u(x^{1},\dots,x^{n-1},-x^{n}) & \text{if } x^{n} < 0. \end{cases}$$

To show $\Sigma_{\mathcal{D}}$ is smooth at x_0 , it suffices to show that $u_{\mathcal{D}}$ is smooth along $\{x \in B^n_{r'}(0) : x^n = 0\}$.

From the free boundary condition, we have $\frac{\partial u}{\partial x^n} \equiv 0$ on $\{x^n = 0\}$ and so $u_{\mathcal{D}}$ has a continuous derivative on all of $B^n_{r'}(0)$. Since $\hat{\Sigma}$ is smooth and minimal, $u_{\mathcal{D}}$ is smooth and solves the minimal graph equation (A.2) with respect to the metric $\hat{g}_{\mathcal{D}}$ in the strong sense on $\{x \in B^n_{r'}(0): x^n \neq 0\}$. Moreover, it follows from $\frac{\partial u}{\partial x^n} \equiv 0$ on $\{x^n = 0\}$ and the ι -invariance of $\bar{g}_{\mathcal{D}}$ that $u_{\mathcal{D}}$ solves the minimal graph equation weakly on the entire ball $B^n_{r'}(0)$.

From this point, the smoothness of $u_{\mathcal{D}}$ is a standard application of tools from nonlinear elliptic PDE theory, so we will be brief (see [6, Lemma 7.2]). Standard estimates for minimizers implies $u_{\mathcal{D}} \in H^2(B_{r'}^n(0))$ (see [12, Section 8.3.1]). Writing the equation (A.2) in divergence form, we have

(A.6)
$$\frac{\partial}{\partial x^i} \left(a^{ij} \frac{\partial u_{\mathcal{D}}}{\partial x^j} + b^i u_{\mathcal{D}} \right) = 0$$

where the coefficients a^{ij} and b^i depend on $u_{\mathcal{D}}$ and are only differentiable. Since $u_{\mathcal{D}}$ weakly solves equation (A.6),

$$\int_{B_{-l}^{n}(0)} \left(a^{ij} \frac{\partial u_{\mathcal{D}}}{\partial x^{j}} + b^{i} u_{\mathcal{D}} \right) \frac{\partial \psi}{\partial x^{i}} dx = 0,$$

for any test function $\psi \in C_0^{\infty}(B_{r'}^n(0))$ Taking ψ to be of the form $-\frac{\partial w}{\partial x^k}$ for some function w and integrating by parts, one finds $\frac{\partial u_{\mathcal{D}}}{\partial x^k}$ is a weak solution of a uniformly elliptic linear equation with L^{∞} coefficients for each $k = 1, \ldots, n$.

Now we may apply the DeGiorgi-Nash theorem (see [15, Theorem 8.24]) to conclude that, for each r'' < r' there is an $\alpha \in (0,1)$ such that $\frac{\partial u_{\mathcal{D}}}{\partial x^k} \in C^{0,\alpha}(B^n_{r''}(0))$ for each $k=1,\ldots,n$. Now $u^D \in C^{1,\alpha}(B^n_{r''}(0))$ and the functions $\frac{\partial u_{\mathcal{D}}}{\partial x^k}$ solve a uniformly elliptic linear equation with Hölder coefficients. The Schauder estimates from Theorem 7 allow us to conclude that $\frac{\partial u_{\mathcal{D}}}{\partial x^k} \in C^{2,\alpha}(B_{r'}(0))$. This argument may be iterated, see [15, Section 8], to conclude $u_{\mathcal{D}} \in C^{k,\alpha}(B^n_{r''}(0))$ for any k. This finishes the proof that $u_{\mathcal{D}}$ is a smooth solution to the mean curvature equation across the doubling locus $\{x^n=0\}$ and hence $\Sigma_{\mathcal{D}}$ is a smooth minimal hypersurface.

The last step is to show that $\Sigma_{\mathcal{D}}$ is stable. Let $\phi \in C^{\infty}(\Sigma_{\mathcal{D}})$ define a normal variation and write $\phi = \phi_0 + \phi_1$ where ϕ_0 is invariant under the involution and ϕ_1 is anti-invariant under the involution. Now we will consider the second variation of the volume of $\Sigma_{\mathcal{D}}$ with respect to ϕ .

$$\begin{split} \delta_{\phi}^{2}(\Sigma_{\mathcal{D}}) &= \int_{\Sigma_{\mathcal{D}}} |\nabla \phi|^{2} - \phi^{2}(\operatorname{Ric}(\nu, \nu) + |A|^{2}) d\mu \\ &= \int_{\Sigma_{\mathcal{D}}} |\nabla \phi_{0}|^{2} + 2g(\nabla \phi_{0}, \nabla \phi_{1}) + |\nabla \phi_{1}|^{2} - (\phi_{0}^{2} + 2\phi_{0}\phi_{1} + \phi_{1}^{2})(\operatorname{Ric}(\nu, \nu) + |A|^{2}) d\mu \\ &= \delta_{\phi_{0}}^{2}(\Sigma_{\mathcal{D}}) + \delta_{\phi_{1}}^{2}(\Sigma_{\mathcal{D}}) + \int_{\Sigma_{\mathcal{D}}} 2g(\nabla \phi_{0}, \nabla \phi_{1}) - 2\phi_{0}\phi_{1}(\operatorname{Ric}(\nu, \nu) + |A|^{2}) d\mu \\ &= 2\delta_{\phi_{0}|\Sigma}^{2}(\Sigma) + 2\delta_{\phi_{1}|\Sigma}^{2}(\Sigma) \geq 0 \end{split}$$

where the last equality follows from the fact that $g(\nabla \phi_0, \nabla \phi_1)$ and $\phi_0 \phi_1$ are anti-invariant under the involution. This completes the proof of Lemma 7.

A.4. Second fundamental form bounds. In this section, we will prove Step 2 in Section 3.6. Let (M_i, \bar{g}_i) and W_i be as in Main Lemma. The uniform second fundamental form bounds for the stable minimal hypersurfaces $W_i \subset M_i$ can be reduced to a classical estimate due to Schoen-Simon [25] for stable minimal hypersurfaces in Riemannian manifolds. In the following, (M, \bar{g}) is a complete (n+1)-dimensional Riemannian manifold, $x_0 \in M$, $\rho_0 \in (0, \text{inj}_{\bar{q}}(x_0))$, and μ_1 is a constant satisfying

(A.7)
$$\sup_{B_{\rho}(0)} \left| \frac{\partial \bar{g}_{ij}}{\partial x^k} \right| \le \mu_1, \quad \sup_{B_{\rho}(0)} \left| \frac{\partial^2 \bar{g}_{ij}}{\partial x^k \partial x^l} \right| \le \mu_1^2,$$

on the metric ball $B_{\rho_0}(x_0)$ in geodesic normal coordinates (x^1,\ldots,x^{n+1}) centered at x_0 .

Theorem 9 (Corollary 1 [25]). Suppose Σ is an oriented embedded C^2 -hypersurface in an (n+1)dimensional Riemannian manifold (M, \bar{g}) with $x_0 \in \overline{\Sigma}$, μ_1 satisfies (A.7), and μ satisfies the bound $\rho_0^{-n}\mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) \leq \mu. \text{ Assume that } \mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) < \infty \text{ and } \mathcal{H}^{n-2}(\operatorname{sing}(\Sigma) \cap B_{\rho_0}(x_0)) = 0. \text{ If } n \leq 6 \text{ and } \Sigma \text{ is stable in } B_{\rho_0}(x_0), \text{ then}$

$$\sup_{B_{\rho_0}(x_0)} |A^{\Sigma}| \le \frac{C}{\rho_0},$$

where C depends only on n, μ , and $\mu_1 \rho_0$.

Proof of Step 2. By Lemma 7, the doubling $(W_i)_{\mathcal{D}}$ is a smooth stable minimal hypersurface of $(M_i)_{\mathcal{D}}$. In particular, the singular set of $(W_i)_{\mathcal{D}}$ is empty. Moreover, the manifolds $(M_i)_{\mathcal{D}}$ have uniformly bounded geometry so that the injectivity radius is uniformly bounded from below by some $\rho_0 > 0$, and there is a constant μ_1 so that the bounds (A.7) hold in normal coordinates about any $x \in (M_i)_{\mathcal{D}}$, any $\rho \in (0, \rho_0)$, and all $i = 1, 2, \ldots$ According to Step 1, there is a constant μ such that

$$\rho_0^{-n} \operatorname{Vol}(W_i \cap B_{\rho}(x)) \le \mu$$

for all $i=1,2,\ldots$ Hence, we may uniformly apply Theorem 9 on any ball $B_{\rho_0}(x_0)\subset (M_i)_{\mathcal{D}}$ intersecting W_i to obtain the bound in Step 2.

A.5. Generic regularity in dimension 8. It is well known that codimension one volume minimizing currents, in general, have singularities if the ambient space is of dimension 8 or larger. However, in [24] N. Smale developed a method for removing these singularities in 8-dimensional Riemannian manifolds by making arbitrarily small conformal changes. In this section, we will describe the modifications necessary to adapt his method to the case of Theorem 5 with n = 7.

First, we will describe the perturbation result we will use. Let M be a compact (n+1)-dimensional manifold. For $k=3,4,\ldots$, let \mathcal{M}_0^k denote the class of C^k metrics on M which split isometrically as a product on some neighborhood of ∂M . Fix a relative homology class $\alpha \in H_n(M, \partial M; \mathbb{Z})$. We will show the following.

Theorem 10. Let $g_0 \in \mathcal{M}_0^k$ and n = 7. For $\epsilon > 0$, there exists a metric $g \in \mathcal{M}_0^k$ and a g_0 -volume minimizing current T representing α such that $||g - g_0||_{C^k} < \epsilon$ and $\operatorname{spt}(T)$ is smooth.

The proof of Theorem 10 follows by showing the constructions in [24] can be performed on the doubled manifold $M_{\mathcal{D}}$ (see Appendix A.3) in an involution-invariant manner. We proceed in two lemmas. The first lemma holds in any dimension.

Lemma 8. Let $g_0 \in \mathcal{M}_0^k$ and suppose T is a homologically g_0 -volume minimizing current representing α . For $\epsilon > 0$, there is a metric $g \in \mathcal{M}_0^k$ such that $||g - g_0||_{C^k} < \epsilon$ and T is the only g-volume minimizing current representative of α .

Proof. Let A, $d\mu = \theta d\mathcal{H}^n$, and ξ be the underlying rectifiable set, measure, and choice of orientation for the approximate tangent space of A associated to the current T (see Section A.2). We may write $A = \bigcup_{j=1}^{N} A_j$ where each A_j are connected. Choose $p_j \in \operatorname{reg}(A_j) \setminus \partial M$ and $\rho > 0$ so that

$$(B_{\rho}(p_i) \cap A_i) \subset (\operatorname{reg}(A) \setminus \partial M), \quad j = 1, \dots, N.$$

Perhaps restricting to smaller ρ , let $x = (x^1, ..., x^n)$ be geodesic normal coordinates for $B_{\rho}(p_j) \cap A_j$ and let t be the signed distance on $B_{\rho}(p_j)$ from A_j determined by ξ . This gives Fermi coordinates (t, x) on $B_{\rho}(p_j)$. Now fix a bump function $\eta : A \to [0, 1]$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B_{\rho/2}(p_j) \cap A_j \\ 0 & \text{for } x \in B_{\rho}(p_j) \setminus B_{3\rho/4}(p_j) \end{cases}$$

for each $j=1,\ldots N$. Also fix a smooth function $\phi:\mathbb{R}\to\mathbb{R}$ with $\operatorname{spt}(\phi)\subset[-3/4,3/4]$,

$$\phi(t) \ge 0$$
 on $[-1, 1], \phi(0) = 1$, and $\phi(r) < 1$ if $r \ne 0$.

Consider the function $\phi_{\bar{\epsilon}}: M \to \mathbb{R}$ given by

$$\phi_{\bar{\epsilon}}(y) = \begin{cases} 1 - \bar{\epsilon}^{k+1} \phi(t/\bar{\epsilon}) \eta(x) & \text{if } y = (x, t) \in B_{\rho}(p_j) \text{ for some } j \\ 1 & \text{otherwise} \end{cases}$$

for $\bar{\epsilon} > 0$ satisfying $\operatorname{spt}(\phi_{\bar{\epsilon}}) \subset \bigcup_{j=1}^N B_{3\rho/4}(p_j)$. We have the perturbed metrics $g_{\bar{\epsilon}} = \phi_{\epsilon}^{\frac{2}{n}} g_0 \in \mathcal{M}_0^k$. It is straight-forward to show that there exists $\epsilon_1 \in (0, \epsilon)$ such that, for any $\bar{\epsilon} \in (0, \epsilon_1]$, T is the only $g_{\bar{\epsilon}}$ -volume minimizing representative of α (see [24]). Perhaps restricting to smaller values of $\bar{\epsilon}$, we may also arrange for $||g - g_{\bar{\epsilon}}||_{C^k} < \epsilon$. This completes the proof of Lemma 8.

Lemma 9. Let n = 7, $k \ge 3$, $g_0 \in \mathcal{M}^k$, and $\epsilon > 0$. Suppose T is the only g_0 -volume minimizing representative of α , then there exists $g \in \mathcal{M}^k$ such that $||g - g_0||_{C^k} < \epsilon$ and α may be represented (up to multiplicity) by a smooth g-volume minimizing hypersurface.

Proof. Following [24], we construct a conformal factor which will slide the minimizing current off itself in one direction and appeal to a perturbation result for isolated singularities which allows us to conclude that this new current has no singularity. Write $(M_{\mathcal{D}}, g_{0,\mathcal{D}})$ for the doubling of (M, g_0) (see Section A.3) with involution $\iota: M_{\mathcal{D}} \to M_{\mathcal{D}}$. The current T may also be doubled to obtain an involution-invariant current $T_{\mathcal{D}}$ on $M_{\mathcal{D}}$. Similarly to Section A.3, $T_{\mathcal{D}}$ is locally $g_{0,\mathcal{D}}$ -volume minimizing. Let $A = \bigcup_{j=1}^{N} A_j$, $d\mu = \theta d\mathcal{H}^7$, and ξ be the underlying set, measure, and orientation associated to T, as in the proof of Lemma 8.

Let $\rho_0 > 0$ and fix a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

- (1) $\phi(-t) = -\phi(t)$,
- (2) $\phi(t) \ge 0 \text{ for } t \ge 0$,
- (3) $\phi(t) = t \text{ for } t \in [0, \frac{\rho_0}{4}],$
- (4) $\phi(t) = \frac{\rho_0}{2} \text{ for } t \in [\frac{\rho_0}{2}, \frac{3\rho_0}{4}],$

(5)
$$\phi(t) = 0 \text{ for } t \ge \rho_0.$$

Let $\{B_{\rho}(p_j)\}_{j=1}^N$ be a collection of disjoint metric balls in \mathring{M} centered at regular points $p_j \in A_j$. Choose $\rho_0 > 0$ small enough to ensure that, in Fermi coordinates (t,x) for A_j with ξ pointing into the side corresponding to t > 0, the function $(t,x) \mapsto \phi(t)\eta(x)$ is supported in $\bigcup_{j=1}^N B_{\rho}(p_j)$. For a fixed $s \in (0,1)$ and a parameter $\bar{\epsilon} \in (0,1)$, consider the functions $u_{\bar{\epsilon}} : \Sigma \to \mathbb{R}$ given by

$$u_{\bar{\epsilon}}(y) = \begin{cases} 1 - \bar{\epsilon}^s \phi(t) \eta(x) & \text{if } y = (t, x) \in \bigcup_{j=1}^N B_{\rho}(p_j) \\ 1 & \text{otherwise.} \end{cases}$$

The conformal metrics $g_{\bar{\epsilon}} = u_{\bar{\epsilon}}^{\frac{2}{n}} g_0$ will be used to find the desired smooth representative. Since $g_{\bar{\epsilon}}$ splits as a product near ∂M , we may consider the corresponding ι -invariant metric $g_{\bar{\epsilon},\mathcal{D}}$ on $M_{\mathcal{D}}$.

For sake of contradiction, suppose that there is a sequence $\bar{\epsilon}_i \to 0$ and homologically $g_{\bar{\epsilon}_i}$ -volume minimizing currents T_i representing α with $\operatorname{sing}(T_i) \neq \emptyset$ for all $i=1,2,\ldots$ Since $\mathbf{M}(T_i)$ is uniformly bounded in i, T_i weakly converges to some homologically g_0 -volume minimizing current T_∞ which also represents α . Since T is assumed to be the unique such current, we must have $T_\infty = T$. Write P_i , $d\mu_i$, and ξ_i for the set, measure, and orientation corresponding to T_i for $i=1,2,\ldots$ Let Q_i be a connected component of P_i with $\operatorname{sing}(Q_i) \neq \emptyset$ for each $i=1,2,\ldots$ Now Q_i converges in the Hausdorff sense to some sheet Q of T. By the Allard regularity theorem [2], this convergence is smooth away from $\operatorname{sing}(Q)$. Hence, after passing to a subsequence, y_i converges to some $y \in \operatorname{sing}(Q)$.

In terms of the doubled manifold, the ι -invariant currents $T_{i,\mathcal{D}}$ are homologically $g_{\bar{\epsilon}_i,\mathcal{D}}$ -volume minimizing, $T_{i,\mathcal{D}}$ weakly converge to $T_{0,\mathcal{D}}$, and the doubled sets $Q_{i,\mathcal{D}}$ converge to $Q_{\mathcal{D}}$ smoothly away from $\operatorname{sing}(Q_{\mathcal{D}})$. Now let $\mathcal{N} \subset M_{\mathcal{D}}$ be a small distance neighborhood of $Q_{\mathcal{D}}$ so that $\mathcal{N} \setminus Q_{\mathcal{D}}$ consists of two disjoint, open sets \mathcal{N}_- and \mathcal{N}_+ on which the signed distance to $Q_{\mathcal{D}}$ is negative and positive, respectively. In the doubled manifold, we may directly apply the following results from [24].

Lemma 10. [24, Proposition 1.6] For large i, we have

- (1) $Q_{i,\mathcal{D}} \cap \mathcal{N}_- = \emptyset$
- (2) $Q_{i,\mathcal{D}} \cap \mathcal{N}_+ \setminus \operatorname{spt}(\phi_{\epsilon_i} \eta)_{\mathcal{D}} \neq \emptyset$.

In light of Lemma 10, the Simon maximum principle [28] shows

$$(Q_{i,\mathcal{D}} \setminus \operatorname{spt}(\phi_i \eta)_{\mathcal{D}}) \subset (\mathcal{N}_+ \setminus \operatorname{spt}(\phi_i \eta)_{\mathcal{D}})$$

for each $i = 1, 2, \ldots$ Recalling that $Q_{i,\mathcal{D}}$ converges to $Q_{\mathcal{D}}$ in the Hausdorff distance, we may apply the perturbation result [17, Theorem 5.6] to conclude that $Q_{i,\mathcal{D}}$ is smooth for sufficiently large i. This contradiction finishes the proof of Lemma 9.

Theorem 10 follows by first applying Lemma 8 to approximate g_0 with a metric g_1 supporting a unique minimizing representative of α then applying Lemma 9 to approximate g_1 with a metric g_2 and obtain a g_2 -volume minimizing representative of α .

Proof of Theorem 5 for n=7. We will closely follow the argument presented in Section 4. Let $(Z, \bar{g}, \bar{\gamma}): (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$ be a psc-bordism and let $(Z_i, \bar{g}_i, \bar{\gamma}_i)$ be the corresponding *i*-collaring for $i=1,2,\ldots$ As usual, we denote by $\bar{\alpha}_i \in H_7(Z_i, \partial Z_i; \mathbb{Z})$ the Poincaré dual to $\bar{\gamma}_i$.

For each i = 1, 2, ..., we apply Theorem 10 to obtain a metric \hat{g}_i on Z_i so that

$$||\hat{g}_i - \bar{g}_i||_{C^i_{\bar{g}_i}} \le \frac{1}{i}$$

and $\bar{\alpha}_i$ can be represented by a smooth \hat{g}_i -volume minimizing hypersurface W_i . It follows from the proofs of Lemmas 8 and 9 that \hat{g}_i can and will be chosen so that $\{\hat{g}_i \neq \bar{g}_i\} \subset M_1 \subset M_i$ for $i=1,2,\ldots$ Indeed, the perturbations required to form \hat{g}_i are supported on balls centered about chosen regular points of \bar{g}_i -volume minimizing currents and one can always find regular points of minimizers of $\bar{\alpha}_i$ in $M_1 \subset M_i$. Evidently, \hat{g}_i has positive scalar curvature for all sufficiently large i. Since $\hat{g}_i = \bar{g}_i$ on $Y \times [-i, 0] \subset Z_i$, the proof of the Main Lemma shows that there is a subconvergence

$$(Z_i, W_i, \hat{g}_i, S_i) \rightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, S)$$

where Y, X, g, S_i , and S_{∞} are defined as in Section 4. One can now directly apply the argument from 4.2 to finish the proof of Theorem 5 for n = 7.

References

- [1] K. Akutagawa, B. Botvinnik, Manifolds of positive scalar curvature and conformal cobordism theory, Math. Ann. 324 (4) (2002) 817–840.
- [2] W. K. Allard, On the first variation of a varifold, Ann. Math. 95 (1972) 417-491.
- [3] H. Araújo, Critical points of the total scalar curvature plus total mean curvature functional, Indiana Univ. Math. J. 52 (1) (2003) 85–107.
- [4] B. Botvinnik, P. B. Gilkey, Metrics of positive scalar curvature on spherical space forms, Canad. J. Math. 48 (1) (1996) 64–80.
- [5] B. Botvinnik, O. Müller, Cheeger-gromov convergence in a conformal setting, ArXiv:1512.07651
- [6] T. H. Colding, W. P. Minicozzi, II, A course in minimal surfaces, Vol. 121 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2011.
- [7] H. I. Choi, R. Schoen, The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature, Invent. Math. 81 (3) (1985) 387–394.
- [8] J. F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. (2) 136 (1) (1992) 1–50.
- [9] J. F. Escobar, The Yamabe problem on manifolds with boundary, J. Diff. Geom. 35 (1) (1992) 21–84.
- [10] J. F. Escobar, Addendum: "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary" [Ann. of Math. (2) **136** (1992), no. 1, 1–50; Ann. of Math. (2) 139 (3) (1994) 749–750.
- [11] J. F. Escobar, Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary, Indiana Univ. Math. J. 45 (4) (1996) 917–943.
- [12] L. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 2010, second edition.
- [13] H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

- [14] A. Fraser, M. M.-c. Li, Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary, J. Diff. Geom. 96 (2) (2014) 183–200.
- [15] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.
- [16] M. Grüter, Optimal regularity for codimension one minimal surfaces with a free boundary, Manuscripta Math. 58 (3) (1987) 295–343.
- [17] R. Hardt, L. Simon, Area minimizing hypersurfaces with isolated singularities, J. Reine Angew. Math. 362 (1985) 102–129
- [18] S. G. Krantz, H. R. Parks, Geometric integration theory, Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2008.
- [19] M. M.-c. Li, A general existence theorem for embedded minimal surfaces with free boundary, Comm. Pure Appl. Math. 68 (2) (2015) 286–331.
- [20] F. Morgan, Geometric measure theory. A beginner's guide. Fourth edition. Elsevier/Academic Press, Amsterdam, 2009. viii+249 pp.
- [21] P. Petersen, Riemannian geometry, 2nd Edition, Vol. 171 of Graduate Texts in Mathematics, Springer, New York, 2006.
- [22] T. Schick, A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. Topology 37 (1998), no. 6, 1165-1168
- [23] R. Schoen, Minimal submanifolds in higher codimension, Mat. Contemp. 30 (2006) 169–199, xIV School on Differential Geometry (Portuguese).
- [24] N. Smale, Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds, Comm. Anal. Geom. 1 (1993), no. 2, 217-228
- [25] R. Schoen, L. Simon, Regularity of stable minimal hypersurfaces, Comm. Pure Appl. Math. 34 (6) (1981) 741–797.
- [26] R. Schoen, L. Simon, S. T. Yau, Curvature estimates for minimal hypersurfaces, Acta Math. 134 (3-4) (1975) 275–288.
- [27] R. Schoen, S. T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1-3) (1979) 159–183.
- [28] L. Simon, A strict maximum principle for area minimizing hypersurfaces, J. Diff. Geom. 26 (2) (1987) 327–335.
- [29] R. Ye, Construction of embedded area-minimizing surfaces via a topological type induction scheme, Calc. Var. Partial Differ. Equ. 19 (4) (2004) 391–420.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR, 97405, USA *E-mail address*: botvinn@uoregon.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR, 97405, USA E-mail address: demetre@uoregon.edu