# A Problem Concerning Nonincident Points and Lines in Projective Planes

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#### Abstract

In this paper, we study the problem of finding the largest possible set of s points and s lines in a projective plane of order q, such that that none of the s points lie on any of the s lines. We prove that  $s \le 1 + (q+1)(\sqrt{q}-1)$ . We also show that equality can be attained in this bound whenever q is an even power of two.

### 1 Introduction

Suppose  $\Pi = (X, \mathcal{L})$  is a projective plane of order q, where X is the set of points and  $\mathcal{L}$  is the set of lines in  $\Pi$ . For  $Y \subseteq X$  and  $\mathcal{M} \subseteq \mathcal{L}$ , we say that  $(Y, \mathcal{M})$  is a *nonincident* set of points and lines if  $y \notin M$  for every  $y \in Y$  and every  $M \in \mathcal{M}$ .

Define  $f(\Pi)$  to be the maximum integer s such that there exists a nonincident set of s points and s lines in  $\Pi$ . Equivalently,  $f(\Pi)$  is the size of the largest square submatrix of zeroes in the incidence matrix of  $\Pi$ . We use a simple combinatorial argument to prove the upper bound  $f(\Pi) \leq 1 + (q+1)(\sqrt{q}-1)$ , which holds for any projective plane  $\Pi$  of order q. We also show that this bound is tight in certain cases, namely, for the desarguesian plane PG(2,q) when q is an even power of two. This is done by utilising maximal arcs.

### 2 Main Results

**Theorem 1.** For any set Y of s points in a projective plane of order q, the number of lines disjoint from Y is at most

$$\frac{q^3 + q^2 + q - qs}{q + s}.$$

*Proof.* Suppose that  $(X, \mathcal{L})$  is a projective plane of order q. For a subset  $Y \subseteq X$  of s points, define  $\mathcal{L}_Y = \{L \in \mathcal{L} : L \cap Y \neq \emptyset\}$  and define  $\mathcal{L}_Y' = \mathcal{L} \setminus \mathcal{L}_Y$ . Furthermore, for every  $L \in \mathcal{L}_Y$ , define  $L_Y = L \cap Y$ , and then define  $\mathcal{B} = \{L_Y : L \in \mathcal{L}_Y\}$ . Observe that  $\mathcal{B}$  consists of the nonempty intersections of the lines in  $\mathcal{L}$  with the set Y. Denote  $b = |\mathcal{B}| = |\mathcal{L}_Y|$ .

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We will study the set system  $(Y, \mathcal{B})$ . We have the following equations:

$$\sum_{B \in \mathcal{B}} 1 = b$$

$$\sum_{B \in \mathcal{B}} |B| = (q+1)s$$

$$\sum_{B \in \mathcal{B}} {|B| \choose 2} = {s \choose 2}.$$

From the above equations, it follows that

$$\sum_{B \in \mathcal{B}} |B|^2 = s(q+s).$$

The q+1 blocks in  $\mathcal{B}$  through any point  $y \in Y$  contain the s-1 points in  $Y \setminus \{y\}$  once each, as well as q+1 occurrences of y. So the average size of a block in  $\mathcal{B}$  that contains any given point  $y \in Y$  is (q+s)/(q+1). Therefore we define  $\beta = (q+s)/(q+1)$  and compute as follows:

$$0 \leq \sum_{B \in \mathcal{B}} (|B| - \beta)^2$$
$$= s(q+s) - 2\beta(q+1)s + \beta^2 b,$$

from which it follows that

$$b \geq \frac{s(2\beta(q+1) - (q+s))}{\beta^2}$$
$$= \frac{(q+1)^2s}{q+s}.$$

Therefore,

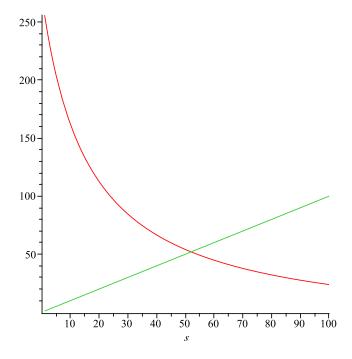
$$|\mathcal{L}'_Y| = q^2 + q + 1 - b$$
  
 $\leq q^2 + q + 1 - \frac{(q+1)^2 s}{q+s}$   
 $= \frac{q^3 + q^2 + q - qs}{q+s}.$ 

Remark. The inequality  $b \ge (q+1)^2 s/(q+s)$  that we proved above is in fact a well-known result that has been proven in many different guises over the years. For example, Mullin and Vanstone [4] proved that  $b \ge r^2 v/(r + \lambda(v-1))$  in any  $(r,\lambda)$  design on v points. If we let r = q+1,  $\lambda = 1$  and v = s, then we obtain  $b \ge (q+1)^2 s/(q+s)$ .

**Corollary 2.** If there exists a nonincident set of s points and t lines in a projective plane of order q, then

$$t \le \frac{q^3 + q^2 + q - qs}{q + s}.$$

Figure 1: Nonincident points and lines when q = 16



Before proving our next general result, we look at a small example. In Figure 1, we graph the functions  $(q^3 + q^2 + q - qs)/(q + s)$  and s for q = 16 and  $s \le 100$ . The point of intersection is (52,52) and it is then easy to see that  $f(\Pi) \le 52$  for any projective plane  $\Pi$  of order 16.

In general, it is easy to compute the point of intersection of these two functions as follows:

$$\frac{q^3 + q^2 + q - qs}{q + s} = s \Leftrightarrow s^2 + 2qs - (q^3 + q^2 + q) = 0$$
$$\Leftrightarrow s = -q \pm \sqrt{q^3 + 2q^2 + q}$$
$$\Leftrightarrow s = -q \pm (q + 1)\sqrt{q}.$$

Since s > 0, the point of intersection occurs when

$$s = -q + (q+1)\sqrt{q} = 1 + (q+1)(\sqrt{q} - 1).$$

The following result is now straightforward.

**Theorem 3.** For any projective plane  $\Pi$  of order q, it holds that  $f(\Pi) \leq 1 + (q+1)(\sqrt{q}-1)$ .

*Proof.* Suppose there is a nonincident set of s points and s lines in a projective plane of order q. Theorem 1 implies that  $s \leq (q^3 + q^2 + q - qs)/(q + s)$ . However, for  $s > 1 + (q + 1)(\sqrt{q} - 1)$ , we have that  $s > (q^3 + q^2 + q - qs)/(q + s)$ , just as in the example considered above. It follows that  $s \leq 1 + (q + 1)(\sqrt{q} - 1)$ .

Next, we examine the case of equality in Theorem 1. This will involve maximal arcs, which we now define. A  $maximal\ (s,\beta)$ -arc in a projective plane of order q is a set Y of s points such that every line meets Y in 0 or  $\beta$  points. It is well-known that a maximal  $(s,\beta)$ -arc has  $s=1+(q+1)(\beta-1)$  and the number of lines that intersect the maximal arc is precisely  $s(q+1)/\beta=s(q+1)^2/(q+s)$ . For additional information on maximal arcs, see [2, §VI.41.3].

Corollary 4. Suppose we have a set Y of s points in a projective plane of order q such that the number of lines disjoint from Y is equal to

$$\frac{q^3 + q^2 + q - qs}{q + s}.$$

Then Y is a maximal  $(s,\beta)$ -arc, where  $s=(q+1)(\beta-1)-1$ . Conversely, if Y is a maximal  $(s,\beta)$ -arc in a projective plane of order q, then number of lines disjoint from Y is equal to  $(q^3+q^2+q-qs)/(q+s)$ .

*Proof.* From the proof of Theorem 1, it is easy to see that equality holds if and only if every line in  $\mathcal{L}_Y$  meets Y in exactly  $\beta$  points, where  $\beta = (q+s)/(q+1)$ . It immediately follows that every line in the plane meets Y in 0 or  $\beta$  points, and therefore Y is a maximal arc. The converse follows from the basic properties of maximal arcs mentioned above.

**Theorem 5.** When q is an even power of 2, there exist nonincident sets of s points and s lines in PG(2,q), where  $s=1+(q+1)(\sqrt{q}-1)$ .

*Proof.* Denniston [3] proved that there is a maximal  $(s, 2^u)$ -arc in PG(2,  $2^v)$  whenever 0 < u < v. Suppose v is even and we take u = v/2. Therefore we have a maximal  $(s, \beta)$ -arc, where  $q = 2^v$ ,  $\beta = \sqrt{q}$  and  $s = 1 + (q+1)(\sqrt{q}-1)$ .

Suppose we take Y to be the s points in the arc and we apply Corollary 4. Since  $s = 1 + (q + 1)(\beta - 1)$ , there are exactly s lines in  $\mathcal{L}'_Y$ . Therefore we have a nonincident set of s points and s lines in PG(2,q).

Corollary 6. If q is an even power of 2, then  $f(PG(2,q)) = 1 + (q+1)(\sqrt{q}-1)$ .

Proof. From Theorem 5, we have  $f(PG(2,q)) \ge 1 + (q+1)(\sqrt{q}-1)$ . However,  $f(PG(2,q)) \le 1 + (q+1)(\sqrt{q}-1)$  from Theorem 3. It follows that  $f(PG(2,q)) = 1 + (q+1)(\sqrt{q}-1)$ .

In the case where q is odd, it was shown by Ball, Blokhuis and Mazzocca [1] that there is no nontrivial maximal arc in the desarguesian plane PG(2, q). The existence of maximal arcs for nondesarguesian projective planes of odd order is unresolved at the present time.

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## References

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