Generalized evolutionary equations with imposed symmetries

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Abstract

The paper intends to propose an algorithm which could identify a general class of pdes describing dynamical systems with similar symmetries. The way that will be followed starts from a given group of symmetries, the determination of the invariants and, then, of the compatible equations of evolution. The algorithm will be exemplified by two classes of equations which describe the Fokker-Planck model and the "backward" Kolgomorov one.

1   Introduction

Lie theory of symmetry groups for differential equations is one of the most important methods for studying nonlinear problems which appear in physics or in other fields of applied mathematics. The main idea of Lie's theory is the investigation of integrability starting from the invariance of the equation under some linear transformations of independent and dependent variables, transformations which define the so called Lie group of symmetries. There are different types of symmetries which can be identified for different differential equations: point-like symmetries, contact symmetries, classical, generalized or non-local symmetries. Many studies have been devoted to this topic, some of them being [1, 2, 3]. The existence of the Lie symmetry generators for differential equations often allows the reduction of those equations to simpler ones. The similarity reduction method for example may be a way to follow for transforming partial derivative equations (pdes) defined in $2D$ space in ordinary differential equations (odes).

Usually, the direct symmetry problem of evolutionary equations is considered. It consists in determining the symmetries of a given evolutionary equation. The aim of this paper is to investigate the inverse problem: what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries. We propose an algorithm which allows finding a general class of pdes describing dynamical systems with similar symmetries. The problem will be effectively formulated in the section 2 of the paper for an equation of the form $u_t(x,t) = F(t,x,u,u_x,u_{2x})$. A general algorithm for identifying the most general equation with a given symmetry group is presented. It is also pointed out how this pde defined in $(1+1)$ dimensions can be reduced to an equivalent ode by using the similarity reduction procedure. The general results will be particularized in the section 3 of the paper for two important models represented by the Fokker-Planck equation and "backward" Kolgomorov one. Generalized Fokker-Planck equations can be derived from generalized linear non-equilibrium

thermodynamics [4]. It was already shown that the Fokker-Planck equation in $2D$ could be transformed, by appropiate change of coordinates, in heat equation or Schrodinger equation [5]. As far as Kolmogorov equation is concerned, it has been considered even in infinite dimensions and interesting applications to stochastic generalized Burgers equations have been noted [6]. Some concluding remarks will end the paper.

2   The general setting of the problem

Let us consider a general dynamical system described in a $(1+1)$ dimensional space (x, t) by a second order partial derivative equation of the form:

$$u_t = A(x, t, u)u_{2x} + B(x, t, u)u_x + C(x, t, u)u, \ A \neq 0 \tag{1}$$

The coefficient functions $A(x, t, u)$, $B(x, t, u)$, $C(x, t, u)$ are arbitrary C^1 functions. Any equation Δ_1 of the form (1) is invariant under the infinitesimal transformations:

$$\bar{x} = x + \xi(t, x, u)\varepsilon + ...; \ \bar{t} = t + \varphi(t, x, u)\varepsilon + ...; \ \bar{u} = t + \eta(t, x, u)\varepsilon + ...$$

if and only if it satisfies [7]:

$$U^{(n)}(\Delta_1) \mid_{\Delta_1=0} = 0 \tag{2}$$

where $U^{(n)}$ is the $n-$th extension of the symmetry operator U. In the case of the equation (1) defined in $(1+1)$ dimensions, the operator U, also known as Lie operator, has the general form:

$$U(x, t, u) = \varphi(x, t, u)\frac{\partial}{\partial t} + \xi(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial u} \tag{3}$$

As in our case the equation (1) is a second order differential equation, the invariance condition (2) will be in fact:

$$U^{(2)}[u_t - A(x, t, u)u_{2x} - B(x, t, u)u_x - C(x, t, u)u] = 0 \tag{4}$$

where $U^{(2)}$ is the second order extension of the Lie symmetry operator U. A concrete computation leads to the equation:

$$\begin{aligned} 0 &= (\varphi A_t + \xi A_x + \eta A_u)u_{2x} + (\varphi B_t + \xi B_x + \eta B_u)u_x + \varphi u C_t + \xi C_x u + \\ &+ C\eta + \eta C_u u + B\eta^x - \eta^t + A\eta^{2x} \end{aligned} \tag{5}$$

By substituting in (5) the general expressions given in [7] for η^x, η^t, η^{2x} and asking for the vanishing of the coefficients of each monomial in the derivatives of $u(t, x)$, one obtains the following differential system:

$$\varphi_x = 0; \ \varphi_u = 0; \ \xi_u = 0; \ \eta_{2u} = 0; -\varphi A_t - \xi A_x - \eta A_u - A\varphi_t + 2A\xi_x = 0;$$

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$$- \varphi B_t - \xi B_x - \eta B_u + B\xi_x - \xi_t - B\varphi_t - 2A\eta_{xu} + A\xi_{2x} = 0 \tag{6}$$
$$-\varphi C_t u - \xi C_x u - C\eta - \eta C_u u - B\eta_x + \eta_t + C\eta_u u - \varphi_t C u - A\eta_{2x} = 0$$

It is important to note that this system can be seen as a general system with the unknown functions $A(x,t,u)$, $B(x,t,u)$, $C(x,t,u)$, $\varphi(x,t,u)$, $\xi(x,t,u)$, $\eta(x,t,u)$. Two completely different situations can be considered:

- one fix the equation (1) by choosing concrete expressions for $A(x,t,u)$, $B(x,t,u)$, $C(x,t,u)$ and one determines the form of the symmetries, that is $\varphi(x,t,u)$, $\xi(x,t,u)$, $\eta(x,t,u)$;

- one looks for the class of equations which have a given form of the symmetry, which means to solve (6) with given $\varphi(x,t,u)$, $\xi(x,t,u)$, $\eta(x,t,u)$ for the unknown functions $A(x,t,u)$, $B(x,t,u)$, $C(x,t,u)$.

Usually the first approach is considered. What we are doing here is to perform the second approach. We will start from the already known symmetries of the Fokker-Planck equation [4], an equation which belongs to the general class of equations (1) and describes various evolutionary processes in quantum optics, solid state physics or statistical physics. Let us therefore consider for the Lie symmetry operator (3) the following form:

$$U(x,t,u) = e^{2t}\frac{\partial}{\partial t} + xe^{2t}\frac{\partial}{\partial x} - x^2 e^{2t}u\frac{\partial}{\partial u} \tag{7}$$

With these choices for $\varphi(x,t,u)$, $\xi(x,t,u)$, $\eta(x,t,u)$, the system (6) reduces to the equations:

$$\begin{aligned}
A_t + xA_x - x^2 A_u u &= 0 \\
-B_t - xB_x + x^2 uB_u - B - 2x + 4Ax &= 0 \\
C_t u + C_x ux - C_u u^2 x^2 - 2Bxu + 2x^2 u + 2Cu - 2Au &= 0
\end{aligned} \tag{8}$$

It is a system with the unknown functions $A(x,t,u)$, $B(x,t,u)$, $C(x,t,u)$ which leads to the general solution of the form:

$$\begin{aligned}
A &= f\left(t - \ln x, ue^{x^2/2}\right), \\
B &= \left[-1 + 2f\left(t - \ln x, ue^{x^2/2}\right)\right]x + \frac{g\left(t - \ln x, ue^{x^2/2}\right)}{x}, \\
C &= \left[-1 + f\left(t - \ln x, ue^{x^2/2}\right)\right]x^2 + f\left(t - \ln x, ue^{x^2/2}\right) + g\left(t - \ln x, ue^{x^2/2}\right) + \\
&\quad + \frac{h\left(t - \ln x, ue^{x^2/2}\right)}{x^2}.
\end{aligned} \tag{9}$$

The solution (9) is given in terms of 3 arbitrary functions $f(u,x,t)$, $g(u,x,t)$, $h(u,x,t)$. In fact, two types of combinations of the variables (u,x,t) have to be considered:

$$\widetilde{z} \equiv (t - \ln x), \phi(z) \equiv ue^{x^2/2} \tag{10}$$

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The significance of these variables becomes clear if one considers the characteristic equations associated with the symmetry operator (7):

$$\frac{dt}{e^{2t}} = \frac{dx}{xe^{2t}} = \frac{du}{-x^2 e^{2t} u} \tag{11}$$

By integrating the equations (11) the following similarity variables are obtained:

$$z = xe^{-t}, \phi(z) = ue^{x^2/2} \tag{12}$$

It is simple now to notice the direct connection between (10) and (12), that is the general solution (9) is given in terms of similarity variables. Moreover, it is also important to note that, by using the similarity transformations (12), the class of the evolutionary systems which are described by the $(1+1)$ equation (1) can be reduced to the $1D$ systems with the evolution given by the following differential equation:

$$z^2 \frac{d^2\phi(z)}{dz^2} f\ [\phi(z), -\ln z] + z\frac{d\phi(z)}{dz} g\ [\phi(z), -\ln z] + \phi(z)h\ [\phi(z), -\ln z] = 0,\ \forall\ f, g, h \tag{13}$$

As a matter of fact, let us note that for $f = 1$, $g = h = 0$, the equation (13) becomes:

$$\frac{d^2\phi(z)}{dz^2} = 0 \tag{14}$$

One obtains a wave type equation in one dimension with the solution of the form:

$$\phi(z) = c_1 z + c_2 \tag{15}$$

where c_1, c_2 are arbitrary constants.

As a conclusion of this section, let us mention that the solution (9) gives the most general class of $2D$ equations of the form (1) which admits a symmetry operator of the form (7) and, by that, accepts a similarity reduction to the $1D$ equation (13). In the next section we will identify two concrete examples of systems which belong to this class of evolutionary equations.

3   Applications

3.1   The Fokker-Planck model

Let us come back to the choice $f = 1$, $g = h = 0$ in the equation (13), choice already considered before. We have to note that for this case the solution (9) of the system (8) transforms into very simple expressions:

$$A = 1,\ B = x,\ C = 1 \tag{16}$$

The equation (1) takes, it too, a very simple form known as the $(1+1)$ Fokker-Planck equation [8]:

$$u_t = u_{2x} + xu_x + u \tag{17}$$

One recovers so that the equation (17) is one of the particular form of the equation (1) which accepts the symmetry (7). As we mentioned, this is an already known result and, in fact, it justifies the choice of the symmetry operator in the form (7). The equation (17) is used in various fields of natural sciences, as physical chemistry or theoretical biology [9]. In the similarity variables (12) it can be reduced to form (14) and can be solved. Coming back to the original variables x, t and $u = u(x, t)$ by inverting (12), the solution (15) of the equation becomes:

$$u(x, t) = c_1 x e^{-t} e^{-x^2/2} + c_2 e^{-x^2/2} \tag{18}$$

So, it was possible to obtain a solution of the equation (18) as a simple case of the general results obtained for the equation (1). We have to mention that, by computational methods, one can obtain a general solution of the equation (18) in a very complicated form. Such results are mentioned in [10], [11].

3.2   The "backward" Kolmogorov model

Let us consider now another particular case of solution in (9): $f = 1$, $g = -1, h = 0$. In this case the equation (1) takes the form of an evolutionary equation of the type "backward" Kolmogorov [12]:

$$u_t = u_{2x} + (x - \frac{1}{x})u_x \tag{19}$$

As the whole class of equations belonging to (1) and as the Fokker-Planck equation, the equation (19) accepts the Lie symmetry (7) and can be reduced to a one-dimensional ordinary differential equation of the form:

$$z \frac{d^2\phi(z)}{dz^2} - \frac{d\phi(z)}{dz} = 0 \tag{20}$$

The equation (20) has a solution of the form:

$$\phi(z) = c_3 + c_4 z^2 \tag{21}$$

Using again (12) one transforms (21) into a solution of (19) of the form:

$$u(x, t) = e^{-x^2/2}(c_3 + c_4 x^2 e^{-2t}) \tag{22}$$

4   Concluding remarks

The results of this paper can be synthesized as follow: (i) the class of $(1 + 1)$ generalized evolutionary equations of type (1) can allow the same concrete Lie symmetry operator (7) if and only if the coefficient functions $A(x, t, u), B(x, t, u), C(x, t, u)$ have the expressions (9); (ii) the knowledge of symmetry operators enabled to introduce, by integrating the characteristic equations (11), the so called similarity variables. Through this similarity approach, the general form for reduced odes (13) associated to the analyzed equations (1) has been obtained; (iii) the results are particularized for two dynamical equations, the Fokker-Planck model and the

"backward" Kolgomorov one. The general algorithm allows deriving simple solutions of the evolutionary equations which describe the two models.

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