ON ALGEBRAIC SURFACES ASSOCIATED TO LINE ARRANGEMENTS

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ABSTRACT. For a line arrangement in the complex projective plane $\mathbb{P}^2$, we investigate the compactification $\overline{F}$ of the affine Milnor fiber in $\mathbb{P}^3$ and its minimal resolution $\widetilde{F}$. We compute the Chern numbers in terms of the combinatorics of the line arrangement, then we show that the minimal resolution is never a quotient of a ball; in addition, we also prove that $\widetilde{F}$ is of general type when the arrangement has only nodes or triple points as singularities.

CONTENTS

1. INTRODUCTION

A **ball quotient** is a smooth projective surface that is biholomorphic to $\mathbf{B}/\Gamma$, where
$$\mathbf{B} = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$$
equipped with the Kähler metric whose Kähler form is given by
$$\omega_{\mathbf{B}} = -\sqrt{-1}\partial\overline{\partial}\log(1 - |z|^2 - |w|^2),$$
and Γ is a discrete cocompact subgroup of isometries of $\mathbf{B}$. A smooth projective surface X is called of **general type** if its canonical divisor K_X is big; and for such surfaces, we have the celebrated Miyaoka-Yau inequality, namely,

$$(1) \qquad\qquad c_1^2(X) \leq 3c_2(X),$$

where the equality holds if and only if X is a ball quotient, see [27].

Let $\mathcal{A} = \{L_1, \cdots, L_d\}$ be a line arrangement in the complex projective plane $\mathbb{P}^2$, where $L_i : \ell_i(x, y, z) = 0$, $i = 1, \cdots, d$ and let $Q = \ell_1 \ell_2 \cdots \ell_d$ be the defining polynomial of $\mathcal{A}$. In [12], F. Hirzebruch considers the Kummer covers of $\mathbb{P}^2$ branched over the arrangement $\mathcal{A}$, and by desingularization, he obtains smooth algebraic surfaces of general type; in addition, he also shows that for some line arrangements, the associated surfaces are ball quotients. Later, Hirzebruch's method was extended to a more general setup, see for instance [14].

In this article, we consider another construction of surfaces associated to the arrangement $\mathcal{A}$. Let $F : Q = 1$ be the affine **Milnor fiber** in $\mathbb{C}^3$ of $\mathcal{A}$, $V(Q) : Q = 0$ be the union of lines contained in $\mathcal{A}$ and $M = \mathbb{P}^2 \setminus V(Q)$ be the complement of $\mathcal{A}$. Then there is a natural Galois covering map $\rho : F \to M$ of degree d. Let $h : F \to F$ be given by $h(x) = \exp(2\pi\sqrt{-1}/d) \cdot x$ be the multiplication by a primitive d-root of unity. As is shown in [8], $H^1(F)$ admits a mixed Hodge structure (MHS) with only two two weights: 1 and 2, for which the induced morphism by h, namely, $h^* : H^1(F) \to H^1(F)$ is a morphism of MHSs. Moreover, in [8], the authors show that

$$W_1 H^1(F) = H^1(F)_{\neq 1}$$

and

$$Gr_2^W H^1(F) = H^1(F)_1 = \rho^* H^1(M),$$

where

$$H^1(F)_{\neq 1} = \ker(h^{*d-1} + \cdots + h^* + \mathrm{Id})$$

while

$$H^1(F)_1 = \ker(h^* - \mathrm{Id}).$$

Note that $\dim Gr_2^W H^1(F) = \dim H^1(M) = d - 1$ depending only on the number of lines in the arrangement $\mathcal{A}$. For many known examples, $\dim W_1 H^1(F) = \dim H^1(F)_{\neq 1}$, which depends on the monodromy h, is very small: for instance, for Hesse arrangement, $\dim H^1(F)_{\neq 1}$ is 6 (see [2]) and for the arrangement $\mathcal{A}(m, m, 3)$ in [7], this number is at most 4. The interested reader may find more such examples in [7] and [4].

To explore the deep reasons for the smallness of $\dim H^1(F)_{\neq 1}$ and also try to find more ball quotients, we consider the natural compactification of F in $\mathbb{P}^3$, namely,

$$\overline{F} \quad : \quad Q(x, y, z) + t^d = 0.$$

By resolving the singularities of $\overline{F}$, we obtain a smooth projective surface. Let $\pi : \widetilde{F} \to \overline{F}$ be the **minimal resolution** of $\overline{F}$. We say that $\mathcal{A}$ is a *pencil* if $V(Q)$ has a singularity of multiplicity $d = |\mathcal{A}|$. We prove the following.

Theorem 1.1. *Assume $d = |\mathcal{A}| \geq 2$. Then*

   (i) *if $d \geq 4$ and $\mathcal{A}$ is a pencil, $c_1^2(\widetilde{F}) > 3c_2(\widetilde{F})$.*
   (ii) *if $d \geq 3$ and $\mathcal{A}$ is not a pencil or $d = 2$, $c_1^2(\widetilde{F}) < 3c_2(\widetilde{F})$.*

   In particular, if $d = 2$ or $d \geq 3$ and $\mathcal{A}$ is not a pencil, then $\widetilde{F}$ is not a ball quotient.

The non-ball-quotient property of $\widetilde{F}$ is always true for $d \geq 2$, see Remark 6.1 below. In fact, if $\mathcal{A}$ is not a pencil, $\widetilde{F}$ contains several rational curves according to Theorem 4.2 below, thus it is not surprising that $\widetilde{F}$ is not a ball quotient because a ball quotient cannot contain any rational curve, see [3], Proposition 19.

However, our results above give much preciser numerical properties about the Chern numbers. In particular, $c_1^2(\widetilde{F}) < 3c_2(\widetilde{F})$ holds when $\mathcal{A}$ is not a pencil. So it is natural to discuss whether $\widetilde{F}$ is a surface of general type. We prove the following result in Section 7.

Theorem 1.2. *Assume $d = |\mathcal{A}| \geq 7$ and $V(Q)$ contains only nodes or triple points as singularities, then $\widetilde{F}$ is of general type.*

In fact, the Chern numbers of $\widetilde{F}$ are uniquely determined by the combinatorics of $\mathcal{A}$. Let t_r be the number of singular points in $V(Q)$ of multiplicity r. Then we have the following.

Theorem 1.3. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ consisting of $d = |\mathcal{A}|$ lines. Then*

(i) *the first Chern number of the associated surface $\widetilde{F}$ is given by*

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d}$$

where $K_{\overline{F}}^2 = d(d-4)^2$ and for $d \not\equiv 1 \bmod r$, we have

$$DCI_{r,d} = -d(r-2)^2 - r\sum_{i=1}^{\lambda}(n_i - 2) + 2(r-2)(r - \gcd(r,d)) + (r-b);$$

for $d \equiv 1 \bmod r$, we have

$$DCI_{r,d} = -(d-1)(r-2)^2.$$

(ii) *the second Chern number of the associated surface $\widetilde{F}$ is given by*

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d}$$

where $\chi(\overline{F})$ is the topological Euler number of $\overline{F}$

$$\chi(\overline{F}) = d(d^2 - 4d + 6) - (d-1)\sum_r t_r(r-1)^2.$$

In addition, for $d \not\equiv 1 \bmod r$, we have

$$DCII_{r,d} = 1 + r\lambda - (r-2)(\gcd(r,d) - 1);$$

for $d \equiv 1 \bmod r$, we have

$$DCII_{r,d} = d - 1.$$

In the above formulae, the numbers λ, b, n_i's are uniquely determined by r and d only from Theorem 4.2 below.

One of motivation of our work is to understand whether the Hodge numbers of $\widetilde{F}$ are combinatorially determined, one of the main open question in the theory of line arrangements, see [20]. As is explained in Section 8, we have the following formulae

$$
\begin{cases}
h^{0,0} & = h^{2,2} = 1, \\
h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q, \\
h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q), \\
h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q.
\end{cases}
$$

where $h^{p,q}$'s denote the Hodge numbers for a given smooth projective surface, c_1^2, c_2 denote the Chern numbers and q the irregularity. For the associated surface $\widetilde{F}$, the Chern numbers c_1^2, c_2 are determined by the combinatorics of $\mathcal{A}$ by Theorem 1.3. On the other hand, it follows from [8] that $2q = \dim H^1(F)_{\neq 1}$, which is known to many line arrangements, see [4]. In fact, in [20], a combinatorial formula for q is given when $\mathcal{A}$ has only double or triple points; more examples are given in [24] where q is computed. [25] is a good and recent survey on the monodromy computations and in a recent preprint [9], an effective algorithm to compute q is provided.

To illustrate more numerical properties of the surface $\widetilde{F}$, we will give some examples in which we compute all the Hodge numbers of $\widetilde{F}$ in the end of this paper. We also compute the Chern ratio. This numerical invariant is of special interest for algebraic surfaces, see for instance [16],[17],[18],[22].

Note that in [12], some combinatorial inequalities about the number of multiple points of line arrangement $\mathcal{A}$ were obtained by applying the Miyaoka-Yau inequality; these inequalities also play a key role in our proof of Theorem 1.1.

In our situation, it still remains a question on how to decide whether $\widetilde{F}$ is of general type in a more general situation; it also remain open whether $\widetilde{F}$ is a minimal surface, i.e., whether $\widetilde{F}$ contains (-1)-curves that are not contracted by π. All these issues will be addressed in subsequent papers.

2. General setting

In this section, we first present some basic facts about normal (not necessarily smooth) surfaces that will be used in the sequel. Although none of them is new, it is hard to find a good reference. Then we focus on surfaces associated to line arrangements.

2.1. **Intersection theory for normal surfaces.** Let X be a projective variety of dimension n over $\mathbb{C}$. When X is smooth, then the intersection theory on X is classical and quite well-known. But if X is not smooth, there are technical problems about defining the intersection number of two divisors on X, see for instance [10], Section 5.5 and also [11]. However, we can always have a well-defined intersection number of n **Cartier divisors** on the projective variety X and such an intersection theory admits similar properties as in the smooth case, see [5], Chapter 3. Notation: the intersection number of n Cartier divisors $D_1, \cdots, D_n$ will be denoted by $D_1 \cdots \cdot D_n$; if $D_1 = \cdots = D_n = D$, the intersection number is denoted by D^n.

In addition, given a morphism $f : C \to X$ from projective curve to a quasi-projective variety, and D a Cartier divisor (class) on X, we define

$$(2) \qquad D \cdot C = \deg(f^*D).$$

We will mainly be concerned with intersection theory on normal surfaces.

Let $X, \widetilde{X}$ be two normal projective surfaces and $\pi : \widetilde{X} \to X$ a proper surjective morphism. Then we have the **pull-back formula**

$$(3) \qquad \pi^*C \cdot \pi^*D = C \cdot D$$

for any two Cartier divisors on X, see [5], Proposition 3.16. Moreover, the pull-back formula (3) together with Formula (2) implies the **projection formula**:

$$(4) \qquad \pi^*D \cdot E = D \cdot \pi_*E,$$

where D is a Cartier divisor on X while E is a curve on $\widetilde{X}$. In particular, if E is contracted by π to a point, then $\pi^*D \cdot E = 0$ for any Cartier divisor D on X.

2.2. **Canonical divisors of normal surfaces.** In the sequel of this section, let X be a normal surface on a smooth projective threefold Y. If X is smooth, then we have a well-defined canonical bundle and hence the canonical divisor K_X; moreover, we have the **adjunction formula**

$$(5) \qquad K_X = (K_Y + X)|_X.$$

When X is not smooth, then we have a canonical bundle on the smooth locus $X \setminus \mathrm{Sing}(X)$ of X, and hence the associated Cartier divisor $K_{X \setminus \mathrm{Sing}(X)}$ on $X \setminus \mathrm{Sing}(X)$. The canonical divisor of X, still denoted by K_X, is the closure in X is $K_{X \setminus \mathrm{Sing}(X)}$ and K_X is a Weil divisor on X. Furthermore, the adjunction formula (5) still holds. Indeed, the equality clearly holds on the smooth locus $X \setminus \mathrm{Sing}(X)$; but $\mathrm{Sing}(X)$ has codimension 2 in X since X is normal, it follows that any Weil divisor on X is uniquely determined by its restriction on $X \setminus \mathrm{Sing}(X)$, and thus Formula (5) is valid on X.

Note that Y being smooth, it follows that K_Y, X are both Cartier divisors on Y, hence $K_X = (K_Y + X)|_X$ is a Cartier divisor on X, and we have the intersection number K_X^2. By [5], Proposition 3.15, we obtain

$$(6) \qquad K_X^2 = (K_Y + X) \cdot (K_Y + X) \cdot X.$$

2.3. **Miyaoka-Yau number.** When X is smooth, then $c_1^2(X) = K_X^2$ and $c_2(X) = \chi(X)$, the topological Euler number. If X is not smooth, K_X^2 and $\chi(X)$ are still well-defined numbers for X. In view of the Miyaoka-Yau inequality, we give the following definition.

Definition 2.4. Let $X \subseteq Y$ be a normal surface on a smooth projective threefold Y, the *Miyaoka–Yau number* of X is defined by

$$MY(X) = 3\chi(X) - K_X^2.$$

Let the normal surface X have an isolated singularity $0 \in X$, and $\pi : \widetilde{X} \to X$ be a minimal resolution of the singularity 0, given by successive embedded blowups. Let $\pi' : \widetilde{Y} \to Y$ be the effect of the successive blowups on Y. Then $\widetilde{Y}$ is a smooth projective threefold, on which $\widetilde{X}$ is a normal surface, hence we have the canonical divisor $K_{\widetilde{X}} = (K_{\widetilde{Y}} + \widetilde{X})|_{\widetilde{X}}$ and the Miyaoka-Yau number of $\widetilde{X}$:

$$MY(\widetilde{X}) = 3\,\chi(\widetilde{X}) - K_{\widetilde{X}}^2.$$

Definition 2.5. Three numerical invariant differences for the minimal resolution $\pi : \widetilde{X} \to X$ are defined as follows:

(i) The difference for the first Chern number is

$$DCI = K_{\widetilde{X}}^2 - K_X^2;$$

(ii) The difference for the second Chern number is

$$DCII = \chi(\widetilde{X}) - \chi(X);$$

(iii) The difference for the Miyaoka-Yau number is

$$DMY = MY(\widetilde{X}) - MY(X) = 3\,DCII - DCI.$$

When X is given by $(X,0) : G_r(u,v) + t^d = 0$ around the local coordinates (u,v,t) centered at 0 on Y where G_r is a product of r distinct linear forms, it turns out that the three differences defined above are determined by r and d, and thus will be denoted by $DCI_{r,d}, DCII_{r,d}$ and $DMY_{r,d} = 3\,DCII_{r,d} - DCI_{r,d}$.

3. Surfaces associated to line arrangements

Let $\mathcal{A} = \{L_1, \cdots, L_d\}$ with $L_i : \ell_i = 0, i = 1, \cdots, d$, be a line arrangement in $\mathbb{P}^2$ with defining polynomial $Q(x,y,z) = \ell_1 \ell_2 \cdots \ell_d$.

Given $r \geq 2$. If a point $x \in \mathbb{P}^2$ lies on exactly r lines in $\mathcal{A}$, or equivalently, x is a singular point of multiplicity r of the curve $V(Q) : Q = 0$ in $\mathbb{P}^2$, we say that x is of multiplicity r. The number of points of multiplicity r will be denoted by t_r.

Consider the affine Milnor fiber $F : Q = 1$ in $\mathbb{C}^3$, for which we have a natural compactification

$$\overline{F} \quad : \quad Q(x,y,z) + t^d = 0$$

in $\mathbb{P}^3$. $\overline{F}$ is a singular normal surface in $\mathbb{P}^3$; a singular point of multiplicity r of $V(Q)$ gives a singular point of multiplicity r of $\overline{F}$, and vice versa. Moreover, since Q is a product of distinct linear forms, around a singular point of $\overline{F}$ of multiplicity r, we have $\overline{F} : G_r(u,v) + t^d = 0$ with $G_r(u,v)$ a product of r distinct linear forms, whose resolution will be detailed investigated in next section.

For later convenience, we first compute the Chern numbers and Miyaoka–Yau number of the singular surface $\overline{F}$.

Example 3.1. The adjunction formula (5) gives

$$K_{\overline{F}} = (K_{\mathbb{P}^3} + \overline{F})|_{\overline{F}} \sim (d - 4)H|_{\overline{F}},$$

where H is a hyperplane section of $\mathbb{P}^3$. Indeed, we have $\overline{F} \sim dH$ and $K_{\mathbb{P}^3} \sim -4H$ (where $\sim$ denotes rational equivalence). Therefore, using Formula (6), we have

$$(7) \qquad\qquad K_{\overline{F}}^2 = d(d-4)^2.$$

Moreover, there is a natural projection

$$p : \overline{F} \to \mathbb{P}^2, \quad (x, y, z, t) \mapsto (x, y, z),$$

which is a branched covering of degree d with ramification locus $V(Q) \subseteq \mathbb{P}^2$, hence

$$\chi(\overline{F}) = 3d - (d-1)\chi(V(Q)).$$

The Euler characteristic number of the singular curve $V(Q)$ is

$$\chi(V(Q)) \;=\; d(3-d) + \sum_r t_r(r-1)^2,$$

which implies that

$$(8) \qquad\qquad \chi(\overline{F}) \;=\; d(d^2 - 4d + 6) - (d-1)\sum_r t_r(r-1)^2$$

Consequently,

$$(9) \qquad MY(\overline{F}) \;=\; 3\chi(\overline{F}) - K_{\overline{F}}^2 = (d-1)\sum_r t_r(r-1)(3-r). \qquad \square$$

Remark 3.2. To deduce (9), we have used the following well-known equality

$$\frac{d(d-1)}{2} = \sum_r t_r \frac{r(r-1)}{2}.$$

Let $\pi : \widetilde{F} \to \overline{F}$ be a minimal resolution of $\overline{F}$, namely, the following three conditions hold:

   (i) $\widetilde{F}$ is a smooth surface and π is proper birational morphism;
   (ii) $\pi : \widetilde{F} \setminus \pi^{-1}(\operatorname{Sing}(\overline{F})) \to \overline{F} \setminus \operatorname{Sing}(\overline{F})$ is an isomorphism;
   (iii) there is no exceptional (-1)-curves on $\widetilde{F}$, i.e., a rational curve E on $\widetilde{F}$ such that $E^2 = -1$ and E is contracted to a point by π.

Such a resolution π can be obtained by successive embedded blowups, namely, by blowing up along submanifolds of $\mathbb{P}^3$ as well as the resulting manifolds in each step. Note that $K_{\widetilde{F}}$ is a Cartier divisor since $\widetilde{F}$ is smooth.

3.3. **Determination of the canonical divisor.** Let $p_1, \cdots, p_s$ be all the singular points of $\overline{F}$ and r_i be the multiplicity of p_i. Let $E_{i,1}, \cdots, E_{i,v_i}$ be the irreducible components of $\pi^{-1}(p_i)$ and $M_{i,j,k} = E_{i,j} \cdot E_{i,k}$ be the intersection product of $E_{i,j}$ and $E_{i,k}$. Let moreover,

$$\mathbf{M_i} = (M_{i,j,k})$$

be the intersection matrix of $E_{i,j}$'s for any fixed i. It is a $v_i \times v_i$ matrix. Set

$$\mathbf{E_i} = (E_{i,1}, E_{i,2} \cdots, E_{i,v_i})$$

as a $1 \times v_i$ matrix.

Then the canonical divisor $K_{\widetilde{F}}$ is of the following form

$$K_{\widetilde{F}} = \pi^* K_{\overline{F}} + \sum_{i=1}^{s} \sum_{j=1}^{v_i} a_{i,j} E_{i,j}.$$

Let

$$\mathbf{A_i} = (a_{i,1}, a_{i,2}, \cdots, a_{i,v_i})^T$$

be a $v_i \times 1$ matrix, where $(\quad)^T$ denotes the transpose of a matrix, then $K_{\widetilde{F}}$ can be written as

$$(10) \qquad\qquad K_{\widetilde{F}} = \pi^* K_{\overline{F}} + \sum_{i=1}^{s} \mathbf{E_i} \mathbf{A_i}.$$

Let

$$\mathbf{E_i} \cdot K_{\widetilde{F}} = (E_{i,1} \cdot K_{\widetilde{F}}, \cdots, E_{i,v_i} \cdot K_{\widetilde{F}}),$$

be a $1 \times v_i$ matrix. By Theorem 4.2 below, each $E_{i,j}$ is a smooth complete curve, and by [15], each $\mathbf{M_i}$ is a symmetric, negative definite $v_i \times v_i$ matrix. Set $\mathbf{N_i} = -\mathbf{M_i}^{-1}$.

Taking intersection product of $K_{\widetilde{F}}$ with the exceptional divisors $E_{i,j}$'s, it follows that

$$\mathbf{A_i} = -\mathbf{N_i}(\mathbf{E_i} \cdot K_{\widetilde{F}})^T,$$

and hence by equality (10), we have

$$(11) \qquad\qquad K_{\widetilde{F}} = \pi^* K_{\overline{F}} - \sum_{i=1}^{s} \mathbf{E_i} \mathbf{N_i}(\mathbf{E_i} \cdot K_{\widetilde{F}})^T.$$

Moreover, it follows from the adjunction formula

$$(12) \qquad\qquad E_{i,j} \cdot K_{\widetilde{F}} = 2g(E_{i,j}) - 2 - E_{i,j}^2$$

that $K_{\widetilde{F}}$ is uniquely determined once we know the genera $g(E_{i,j})$'s and the intersection matrices $\mathbf{M_i}$'s.

Finally, by the pull-back formula (3) and the projection formula (4), we have, by Formula (11), that

$$K_{\widetilde{F}}^2 - K_{\overline{F}}^2 = \sum_{i=1}^{s} \left(\mathbf{E_i} \mathbf{N_i}(\mathbf{E_i} \cdot K_{\widetilde{F}})^T \right)^2.$$

Observe that the term $\mathbf{E_i} \mathbf{N_i}(\mathbf{E_i} \cdot K_{\widetilde{F}})^T$ involves only the resolution of the point p_i, hence the above formula motivates us to study the resolution of only one singularity of a normal surface or more specifically, resolution of a normal surface germ.

3.4. **Miyaoka–Yau number.** Recall that p_i is a singular point of $\overline{F}$ of multiplicity r_i. By definition, we have, with the above notations,

$$DCI_{r_i,d} = \left(\mathbf{E_i} \mathbf{N_i}(\mathbf{E_i} \cdot K_{\widetilde{F}})^T \right)^2,$$

and hence

$$K_{\widetilde{F}}^2 - K_{\overline{F}}^2 = \sum_{i=1}^{s} DCI_{r_i,d} = \sum_{r} t_r DCI_{r,d}.$$

Similarly, for the Euler number, we have

$$\chi(\widetilde{F}) - \chi(\overline{F}) = \sum_{i=1}^{s} DCII_{r_i,d} = \sum_r t_r DCII_{r,d},$$

therefore, it follows from (9) that

$$MY(\widetilde{F}) = MY(\overline{F}) + \sum_r t_r DMY_{r,d}$$

(13)
$$= \sum_r t_r((d-1)(r-1)(3-r) + DMY_{r,d}).$$

For later convenience, we set

$$\mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d}.$$

4. RESOLUTION OF SINGULARITIES

We consider singularities of the type $(X,0) : f(u,v,t) = 0$ with $f(u,v,t) = G_r(u,v) + t^d$, where $G_r(u,v)$ is a product of r distinct linear forms in u,v. Such a type of singularity in fact belongs to a special class of singularities, namely weighted homogeneous singularities, whose resolutions are explicitly known.

4.1. Weighted homogenous singularities. Consider the $\mathbb{C}^*$ action on $\mathbb{C}^3$ given by

$$a \cdot (z_1, z_2, z_3) = (a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3),$$

where the weights $w_i = \text{weight}(z_i)$ are strictly positive integers satisfying

$$\gcd(w_1, w_2, w_3) = 1.$$

An isolated surface singularity $(X', 0) : f'(z_1, z_2, z_3) = 0$ is called *weighted homogeneous* of degree N for the weights w_i if

$$a \cdot f'(z_1, z_2, z_3) = f'(a^{w_1} z_1, a^{w_2} z_2, a^{w_3} z_3) = a^N f'(z_1, z_2, z_3), \qquad \forall a \in \mathbb{C}^*.$$

Theorem 4.2 (see [19] and also [6], Section 4.10). *Let $(X,0) : f(u,v,t) = G_r(u,v) + t^d = 0, r \leq d$ be an isolated weighted homogeneous singularity of degree $N = rd/\gcd(r,d)$, where $G_r(u,v)$ is a product of r distinct linear forms in u,v, for the weights*

$$\begin{cases} w_1 & = \text{weight}(u) = d/\gcd(r,d), \\ w_2 & = \text{weight}(v) = d/\gcd(r,d), \\ w_3 & = \text{weight}(t) = r/\gcd(r,d). \end{cases}$$

Then there is a resolution $\pi : \widetilde{X} \to X$ such that:
- *(i) there is a $\mathbb{C}^*$ action on $\widetilde{X}$ under which the morphism π is equivariant.*
- *(ii) the exceptional divisor $\pi^{-1}(0)$ has exactly one component, denoted by E_0, which is fixed pointwise by the $\mathbb{C}^*$ action on $\widetilde{X}$.*

(iii) $\pi^{-1}(0)$ *has the following form*

$$\pi^{-1}(0) = E_0 \cup E_1 \cup \cdots \cup E_\lambda,$$

where for $k = 1, \cdots, \lambda$,
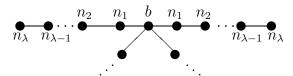
$$E_k = E_k^1 \cup \cdots \cup E_k^r$$

is a disjoint union of r *curves, corresponding to vertices at distance* k *from the center in the dual graph below.*

(iv) *For each* $k = 1, \cdots, \lambda$ *and* $j = 1, \cdots, r$, *the curve* E_k^j *is a smooth rational irreducible curve and has self-intersection* $(E_k^j)^2 = -n_k \leq -2$ *(independent of* j).

(v) E_0 *is a smooth complete curve of genus*

$$
\begin{aligned}
g(E_0) &= \frac{1}{2}\left[\frac{N^2}{w_1 w_2 w_3} - \sum_{i<j} \frac{N \gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{\gcd(N, w_i)}{w_i} - 1\right] \\
&= \frac{1}{2}(r-2)(\gcd(r,d) - 1).
\end{aligned}
$$

(14)

(vi) *The components* E_0, E_k^j's *meet transversally according to the following star-shaped graph*

where the central vertex corresponds to E_0 *and there are exactly* r *arms, which have the same length* λ *and the same weight sequences* $n_1, \cdots, n_\lambda$.

(vii) *Moreover, the above dual graph satisfies the following: if we index the arms* $1, 2, \cdots, r$ *from leftmost to right by the anticlockwise order and go along the arm indexed by* j *from the end closest to* E_0 *to the one farthest to* E_0, *we get, in order, the vertices corresponding to the curves* $E_1^j, E_2^j, \cdots, E_\lambda^j$.

(viii) *Let* $\alpha = w_1 = d/\gcd(r,d)$ *and* $b' = w_3 = r/\gcd(r,d)$. *When* $\alpha = 1$, *then there are in fact no arms, i.e.,* $\lambda = 0$ *and in this case, let* $\beta = 0$. *When* $\alpha > 1$, *choose* $0 < \beta < \alpha$ *such that* $\beta b' \equiv -1 \bmod \alpha$. *Then the weights of the vertices of the dual graph are determined as follows:*

  • *The weight of the central vertex is*

$$b = \frac{N}{w_1 w_2 w_3} + r\beta/\alpha = \frac{\gcd(r,d)(1 + b'\beta)}{\alpha}.$$

  • *The weight sequence* $(n_1, \cdots, n_\lambda)$ *along each arm is given by the following continued fraction decomposition*

$$\frac{\alpha}{\beta} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\cdots - \cfrac{1}{n_\lambda}}}.$$

4.3. **Numerical invariants.** Let $(X, 0) : f(u, v, t) = G_r(u, v) + t^d = 0$ be a surface germ in $(\mathbb{C}^3, 0)$, where G_r is a product of r distinct linear binary forms. Let $\pi : \widetilde{X} \to X$ be the resolution given in Theorem 4.2. With the notations in the theorem, we shall write the divisors on $\widetilde{X}$,

$$E_k = E_k^1 + E_k^2 \cdots + E_k^r, \qquad k = 1, \cdots, \lambda.$$

Clearly, each E_k is a Cartier divisor with compact support on $\widetilde{X}$, and

$$E_k^2 = -rn_k, \quad k = 1, \cdots, \lambda,$$

and

$$E_k \cdot E_{k'} = \begin{cases} r, & \text{if } k' = k \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also, we can see that

$$\chi(\widetilde{X}) - \chi(X) = -1 + \chi(E_0) + r\lambda.$$

Indeed, essentially $\tilde{X}$ is obtained from X by replacing 0 by $(1+r\lambda)$ curves intersecting according to the dual graph; E_0 contributes to $\chi(E_0)$ for $\chi(\widetilde{X})$; each arm in the dual graph gives rise to a disjoint union of λ copies of $\mathbb{P}^1 \setminus \{\text{one point}\} \cong \mathbb{C}$, and hence contributes λ for $\chi(\widetilde{X})$.

Moreover, $K_{\widetilde{X}}$ has the following form

$$K_{\widetilde{X}} = \pi^* K_X + a_0 E_0 + \sum_{k,j} a_k^j E_k^j.$$

Set

$$\mathbf{E^j} = (E_1^j, \cdots, E_\lambda^j), \qquad \mathbf{a^j} = (a_1^j, \cdots, a_\lambda^j),$$

then

$$K_{\widetilde{X}} = \pi^* K_X + a_0 E_0 + \sum_j \mathbf{E^j}(\mathbf{a^j})^T.$$

By considering the adjunction formula, we have

$$E_0 \cdot K_{\widetilde{X}} = g(E_0) - 2 - E_0^2 = 2g(E_0) - 2 + b$$

and for all k, j,

$$E_k^j \cdot K_{\widetilde{X}} = g(E_k^j) - 2 - (E_k^j)^2 = -2 + n_k,$$

hence, by the projection formula and Theorem 4.2, we get a systems of equations

(15) $$\begin{cases} -ba_0 + (a_1^1 + \cdots + a_1^r) & = (r-2)(\gcd(r,d) - 1) - 2 + b \\ -n_k a_k^j + (a_{k-1}^j + a_{k+1}^j) & = -2 + n_k, \quad \forall k, j \end{cases}$$

where we have denoted $a_0^j = a_0$ and $a_{\lambda+1}^j = 0$ for all j.

The intersection matrix of E_0, E_k^l's is negative definite (see [15]), so from (15) we can uniquely solve $a_0, \mathbf{a^j}$'s. Moreover, we can see that if $(a_0, \mathbf{a^1}, \cdots, \mathbf{a^r})$ is a solution of the system (15), $(a_0, \mathbf{a^j}, \mathbf{a^2}, \cdots, \mathbf{a^{j-1}}, \mathbf{a^j}, \mathbf{a^{j+1}}, \cdots, \mathbf{a^r})$ is also a solution for any $j > 1$, hence from uniqueness of the solution, it follows that

$$a_k^1 = a_k^2 = \cdots = a_k^r$$

for all k, hence

$$K_{\widetilde{X}} = \pi^* K_X + a_0 E_0 + \sum_{k=1}^{\lambda} a_k E_k = \pi^* K_X + \sum_{k=0}^{\lambda} a_k E_k,$$
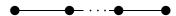
satisfying (following from (15))

(16)
$$\begin{cases} -ba_0 + ra_1 & = (r-2)(\gcd(r,d)-1) - 2 + b \\ -n_k a_k + (a_{k-1} + a_{k+1}) & = -2 + n_k, \quad k = 1, \cdots, \lambda, \end{cases}$$

where $a_{\lambda+1} = 0$.

4.4. **Examples of resolutions.** Now we apply the notations above and consider the resolution given in Theorem 4.2 of the surface germ $(X,0) : G_r(u,v) + t^d = 0$.

Example 4.5. When $r = 2$, then $(X,0)$ is a singularity of type A_{d-1}, and its minimal resolution $\pi : \widetilde{X} \to X$ is well-known: the dual graph is

where there are $(d-1)$ vertices and each vertex has weight 2. Moreover, $K_{\widetilde{X}} = \pi^* K_X$ (see [21]), so we have $DCI_{r,d} = 0, DCII_{r,d} = d - 1$, hence

$$DMY_{r,d} = 3(d-1), \qquad \mathcal{E}_{r,d} = (d-1)(r-1)(3-r) + DMY_{r,d} = 4(d-1). \qquad \square$$

Note that when $r = 2$ and $d = rp + 1$ for $p \geq 1$, the resolution given in Theorem 4.2 is not minimal. Indeed, the central curve E_0 is a (-1)-curve, i.e. $g(E_0) = 0$ and $b = 1$ in Theorem 4.2. Indeed, we have the following.
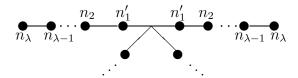
Proposition 4.6. *The resolution given in Theorem 4.2 is* **not** *minimal if and only if $d \equiv 1 \mod r$.*

Proof. The resolution is not minimal only if E_0 is a (-1)-curve, since other exceptional irreducible curves all have self-intersection ≤ -2. This is the case if and only if that $g(E_0) = 0$ and $b = 1$, namely,

$$\begin{cases} 0 & = g(E_0) = \frac{1}{2}(r-2)(\gcd(r,d)-1), \\ 1 & = b = \gcd(r,d)(b'\beta + 1)/\alpha. \end{cases}$$

From the second equality, it follows that $\gcd(r,d) = 1$ and $b'\beta + 1 = \alpha$. Now from $\gcd(r,d) = 1$, we have by definition $\alpha = d/\gcd(r,d) = d$ and $b' = r/\gcd(r,d) = r$, so $d = r\beta + 1$. $\qquad \square$

Consequently, if d cannot be written as $d = rp + 1$ for some $p \geq 1$, the resolution given in Theorem 4.2 is already minimal. If, on the other hand, $d = rp + 1$ for some $p \geq 1$, the resolution given in Theorem 4.2 is not minimal and E_0 is a (-1)-curve. By blowing down E_0, we get another resolution $\widetilde{X}'$ of X, and moreover, since in this case $\alpha = b'\beta + 1 = r\beta + 1$, by performing the continued fraction decomposition of $\alpha/\beta = (r\beta + 1)/\beta$, we have $n_1 = r + 1 \geq 3$, hence $\widetilde{X}'$ is a minimal resolution of X. But now the dual graph is

where $n_1' = n_1 - 1$ and there is no central vertex, meaning that for the r exceptional curves $E_1^1, \cdots, E_1^r$ corresponding to the vertices of weight n_1', we have $E_1^j \cdot E_1^{j'} = 1$ for $j \neq j'$. In particular, the new exceptional divisor does not have normal crossings.

In the sequel, by abuse of notation, we will not distinguish $\widetilde{X}$ and $\widetilde{X}'$ and always denote $\widetilde{X}$ the minimal resolution of X obtained, by blowing down the central curve E_0 if necessary, from the resolution given in Theorem 4.2.

For later convenience, we consider specifically the case $d \equiv 1 \bmod r$.

Example 4.7. Let $r \geq 3$ and $d = rp + 1, p \geq 1$. Then by Proposition 4.6, we have $g(E_0) = 0$ and $b = 1$. According to Theorem 4.2, we have $\alpha = d/\gcd(r, d) = d$ and $b' = r/\gcd(r, d) = r$, so $\alpha = b'p + 1$; since $0 < \beta < \alpha$ is chosen so that $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = p$. In addition,

$$\alpha/\beta = (rp + 1)/p,$$

so considering the continued fraction decomposition, we have

$$\lambda = p, \qquad n_1 = r + 1, \qquad n_2 = n_3 = \cdots = n_\lambda = 2.$$

Blowing E_0 down, we get the minimal resolution $\pi : \widetilde{X} \to X$. The canonical divisor $K_{\widetilde{X}}$ has the following form

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{k=1}^{\lambda} a_k E_k,$$

where $E_1 = E_1^1 + \cdots + E_1^r$ such that $(E_1^l)^2 = -n_1' = -(n_1 - 1)$ and $E_1^l \cdot E_1^{l'} = 1$ for $l < l'$.

Now taking the intersection product of $K_{\widetilde{X}}$ with E_k^l's and applying the adjunction formula and projection formula, we have

$$\begin{cases} -n_1' a_1 + a_2 + (r - 1)a_1 & = -2 + n_1' \\ -n_2 a_2 + (a_1 + a_3) & = -2 + n_2 \\ -n_3 a_2 + (a_2 + a_4) & = -2 + n_3 \\ \quad \vdots \\ -n_{\lambda-1} a_{\lambda-1} + (a_{\lambda-2} + a_\lambda) & = -2 + n_{\lambda-1} \\ -n_\lambda a_\lambda + a_{\lambda-1} & = -2 + n_\lambda \end{cases}$$

that is,

$$
\begin{cases}
-ra_1 + a_2 + (r-1)a_1 & = -2 + r \\
-2a_2 + (a_1 + a_3) & = 0 \\
-2a_2 + (a_2 + a_4) & = 0 \\
\quad\vdots & \\
-2a_{\lambda-1} + (a_{\lambda-2} + a_\lambda) & = 0 \\
-2a_\lambda + a_{\lambda-1} & = 0.
\end{cases}
$$

By considering from bottom equation to the second top one, we have

$$
a_k = (\lambda + 1 - k)a_\lambda, \quad k = 1, \cdots, \lambda - 1;
$$

hence from the first equation, we get $a_\lambda = -(r-2)$. It follows that

$$
K_{\widetilde{X}} = \pi^* K_X - (r-2)(E_p + 2E_{p-1} + \cdots + pE_1),
$$

and thus,

$$
DCI_{r,d} = (r-2)^2 (E_p + 2E_{p-1} + \cdots + pE_1)^2.
$$

Note that $E_k^2 = -rn_k = -2r$ for $k > 1$ and

$$
E_1^2 \quad = \quad (E_1^1 + \cdots + E_1^r)^2 = -r;
$$

moreover, $E_k \cdot E_{k'} = r$ for $k' = k \pm 1$, $=0$ otherwise. Hence, we have

$$
DCI_{r,d} = (r-2)^2(E_p + 2E_{p-1} + \cdots + pE_1)^2 = -(r-2)^2 rp = -(d-1)(r-2)^2.
$$

In addition, we have

$$
DCII_{r,d} = r\lambda = rp = d - 1,
$$

thus

$$
DMY_{r,d} = 3\, DCII_{r,d} - DCI_{r,d} = 3(d-1) + (d-1)(r-2)^2.
$$

Consequently,

$$
\mathcal{E}_{r,d} = DMY_{r,d} + (d-1)(r-1)(3-r) = 4(d-1). \qquad \square
$$

5. Numerical invariants for minimal resolutions

Now we consider the general case of Theorem 4.2. Although our method applies for more general situations, we assume $r \geq 3$ and $d \not\equiv 1 \bmod r$, since otherwise we are done by Example 4.5 and Example 4.7. In particular, the resolution $\pi : \widetilde{X} \to X$ given in Theorem 4.2 is a minimal resolution.

5.1. **Continued fraction decomposition.** In order to apply Theorem 4.2, we first deal with the continued fraction decomposition

$$\frac{\alpha}{\beta} = n_1 - \frac{1}{n_2 - \frac{1}{\cdots - \frac{1}{n_\lambda}}}.$$

Recall that β is chosen such that $b'\beta \equiv -1 \bmod \alpha$, hence $\gcd(\alpha, \beta) = 1$. Let

$$\alpha_0, \alpha_1, \cdots, \alpha_{\lambda-1}, \alpha_\lambda = 1, \alpha_{\lambda+1} = 0$$

be a sequence of natural numbers such that $\gcd(\alpha_i, \alpha_{i+1}) = 1$ for $i = 0, 1, \cdots, \lambda$ and

$$(17) \qquad \frac{\alpha_i}{\alpha_{i+1}} = n_{i+1} - \frac{1}{n_{i+2} - \frac{1}{\cdots - \frac{1}{n_\lambda}}}, \qquad i = 0, 1, \cdots, \lambda - 1.$$

Clearly, the numbers α_i's are uniquely determined by the continued fraction decomposition above, and $\alpha_i > 0$ for $i < \lambda + 1$.

Moreover, we have by definition (17)

$$\frac{\alpha_{i-1}}{\alpha_i} = n_i - \frac{1}{\alpha_i/\alpha_{i+1}} = \frac{n_i\alpha_i - \alpha_{i+1}}{\alpha_i},$$

hence

$$\alpha_{i-1} = n_i\alpha_i - \alpha_{i+1},$$

or in another more convenient formulation

$$(18) \qquad \begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}.$$

Set for $i = 1, \cdots, \lambda$,

$$(19) \qquad \mathbf{G_i} = \begin{pmatrix} n_i & -1 \\ 1 & 0 \end{pmatrix}$$

be a 2×2 matrix. Then the relation (18) can be formulated as

$$\begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G_i} \begin{pmatrix} \alpha_i \\ \alpha_{i+1} \end{pmatrix}.$$

Thus, we have

$$(20) \qquad \begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} = \mathbf{G_i G_{i+1}} \cdots \mathbf{G_\lambda} \begin{pmatrix} \alpha_\lambda \\ \alpha_{\lambda+1} \end{pmatrix} = \mathbf{G_i G_{i+1}} \cdots \mathbf{G_\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for all $i \geq 1$.

Note also that by definition (17) and our conventions, $\alpha_0 = \alpha$ and $\alpha_1 = \beta$.

Let

$$\mathbf{G} = \mathbf{G_1 G_2} \cdots \mathbf{G_\lambda},$$

then by (20), we have

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \mathbf{G} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So $\mathbf{G}$ is of the form

$$\mathbf{G} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

for some integers γ, δ. In fact, we have the following more precise result.

Proposition 5.2. *With the notations as above and in Theorem 4.2, we have*

$$\mathbf{G} = \begin{pmatrix} \alpha & b' - \alpha \\ \beta & \frac{1+b'\beta}{\alpha} - \beta \end{pmatrix},$$

namely, $\gamma = b' - \alpha$ and $\delta = -\beta + (1 + b'\beta)/\alpha$.

Proof. First, we show

Claim 5.3. $-\alpha < \gamma \leq 0$ *and* $-\beta < \delta \leq 0$.

Assuming the claim, note that by definition,

$$\det \mathbf{G} = \alpha\delta - \beta\gamma = 1,$$

hence $\beta\gamma \equiv -1 \bmod \alpha$. Recall also the $b'\beta \equiv -1 \bmod \alpha$, so we have $\gamma = b' - \alpha$ since $\gamma, b' - \alpha \in (-\alpha, 0]$ and the equation $\beta x \equiv -1 \bmod \alpha$ admits a unique solution satisfying $x \in (-\alpha, 0]$. In addition,

$$\delta = \frac{1 + \beta\gamma}{\alpha} = \frac{1 + \beta(b' - \alpha)}{\alpha} = \frac{1 + b'\beta}{\alpha} - \beta.$$

Proof of Claim 5.3: For $i \geq 1$, let

$$\begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} = \mathbf{G_1}\mathbf{G_2} \cdots \mathbf{G_i},$$

then $\xi_i, \eta_i, \gamma_i, \delta_i$ are all integers. It suffices to show the following:

(i) $\xi_i, \eta_i > 0$ for all i.
(ii) $\gamma_i \in (-\xi_i, 0]$ and $\delta_i \in (-\eta_i, 0]$ for all i.

We prove this by induction on i. When $i = 1$, then we have

$$\begin{pmatrix} \xi_1 & \gamma_1 \\ \eta_1 & \delta_1 \end{pmatrix} = \mathbf{G_1} = \begin{pmatrix} n_1 & -1 \\ 1 & 0 \end{pmatrix},$$

and the conclusion obviously holds. Now assuming the validity of the result for i, we have

$$\begin{pmatrix} \xi_{i+1} & \gamma_{i+1} \\ \eta_{i+1} & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \mathbf{G_{i+1}} = \begin{pmatrix} \xi_i & \gamma_i \\ \eta_i & \delta_i \end{pmatrix} \begin{pmatrix} n_{i+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

i) $\xi_{i+1} = n_{i+1}\xi_i + \gamma_i > 2\xi_i - \xi_i > 0$ since $n_{i+1} \geq 2$ and by inductive hypothesis, $\xi_i > 0$ and $\gamma_i \in (-\xi_i, 0]$. Similarly, $\eta_{i+1} = n_{i+1}\eta_i + \gamma_i > 0$ by the inductive hypothesis $\eta_i > 0$ and $\gamma_i \in (-\eta_i, 0]$.

ii) $\gamma_{i+1} = -\xi_i < 0$ since $\xi_i > 0$; in addition,

$$\gamma_{i+1} + \xi_{i+1} = -\xi_i + (n_{i+1}\xi_i + \gamma_i) > (n_{i+1} - 2)\xi_i \geq 0,$$

since $n_{i+1} \geq 2$ and $\gamma_i > -\xi_i$ by the inductive hypothesis. Similarly, $\delta_{i+1} = -\eta_i < 0$ and

$$\delta_{i+1} + \eta_{i+1} = -\eta_i + (n_{i+1}\eta_i + \delta_i) > (n_{i+1} - 2)\eta_i \geq 0.$$

We are done. $\square$

5.4. **Formulae for the canonical divisor.** As before, we assume

$$K_{\widetilde{X}} = \pi^* K_X + \sum_{i=0}^{\lambda} a_i E_i.$$

Therefore,

$$DCI_{r,d} = \sum_{i=0}^{\lambda} a_i^2 E_i^2 + 2 \sum_{i=0}^{\lambda-1} a_i a_{i+1} E_i \cdot E_{i+1}.$$

Recall that $E_0^2 = -b$ and $E_i^2 = -r n_i$ for $i > 0$. In addition, $E_i \cdot E_{i'} = r$ for $i' = i \pm 1$, $=0$ otherwise. Hence, we have

$$DCI_{r,d} = -b a_0^2 + r \sum_{i=1}^{\lambda} a_i (2 a_{i-1} - n_i a_i).$$

By (16), we have $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$, so

$$(21) \qquad DCI_{r,d} \;=\; a_0(-b a_0 + r a_1) + r \left(\sum_{i=1}^{\lambda} n_i a_i - 2 \sum_{i=1}^{\lambda} a_i \right).$$

By (16), we have $-n_i a_i + a_{i-1} + a_{i+1} = -2 + n_i$, so

$$\sum_{i=1}^{\lambda} n_i a_i - 2 \sum_{i=1}^{\lambda} a_i \;=\; \sum_{i=1}^{\lambda} \left(a_{i-1} + a_{i+1} - (n_i - 2) \right) - 2 \sum_{i=1}^{\lambda} a_i$$

$$=\; -\sum_{i=1}^{\lambda} (n_i - 2) + (a_0 - a_1 - a_\lambda);$$

consequently, by (21), we get

$$(22) \quad DCI_{r,d} \;=\; a_0(-b a_0 + r a_1) - 2 + b) - r \sum_{i=1}^{\lambda} (n_i - 2) + r(a_0 - a_1 - a_\lambda).$$

Now we compute a_0, a_1 and a_λ. By equation (16), we have

$$\begin{cases} -b a_0 + r a_1 & = (r-2)(\gcd(r,d) - 1) - 2 + b \\ -n_1 a_1 + (a_0 + a_2) & = -2 + n_1 \\ -n_2 a_2 + (a_1 + a_3) & = -2 + n_2 \\ \quad \vdots & \\ -n_{\lambda-1} a_{\lambda-1} + (a_{\lambda-2} + a_\lambda) & = -2 + n_{\lambda-1} \\ -n_\lambda a_\lambda + a_{\lambda-1} & = -2 + n_\lambda. \end{cases}$$

Let $a_i^* = a_i + 1$ for $i = 0, 1, \cdots, \lambda + 1$. Recall also that $a_{\lambda+1} = 0$. Then the above equations can be reformulated into a more convenient form:

$$
\begin{cases}
-ba_0^* + ra_1^* & = \gcd(r,d)(r-2) \\
-n_1 a_1^* + (a_0^* + a_2^*) & = 0 \\
-n_2 a_2^* + (a_1^* + a_3^*) & = 0 \\
\quad\vdots \\
-n_{\lambda-1} a_{\lambda-1}^* + (a_{\lambda-2}^* + a_\lambda^*) & = 0 \\
-n_\lambda a_\lambda^* + (a_{\lambda-1}^* + a_{\lambda+1}^*) & = 0.
\end{cases}
$$

With the help of the matrices $\mathbf{G_i}$ defined in (19), we have

$$
\begin{pmatrix} a_{i-1}^* \\ a_i^* \end{pmatrix} = \mathbf{G_i} \begin{pmatrix} a_i^* \\ a_{i+1}^* \end{pmatrix},
$$

hence

$$
\begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix} = \mathbf{G_1} \cdots \mathbf{G_\lambda} \begin{pmatrix} a_\lambda^* \\ a_{\lambda+1}^* \end{pmatrix} = \mathbf{G} \begin{pmatrix} a_\lambda^* \\ 1 \end{pmatrix}.
$$

By Proposition 5.2, we thus have

$$
(23) \qquad
\begin{cases}
a_0^* & = \alpha a_\lambda^* + (b' - \alpha) = \alpha a_\lambda + b' \\
a_1^* & = \beta a_\lambda^* + (\frac{1+b'\beta}{\alpha} - \beta) = \beta a_\lambda + (1 + b'\beta)/\alpha.
\end{cases}
$$

Furthermore, it also holds $-ba_0^* + ra_1^* = \gcd(r,d)(r-2)$, thus we obtain three equations in a_0, a_1, a_λ. The solution is as follows, whose proof involves only direct computations and is left to the reader.

Lemma 5.5.

$$
\begin{cases}
a_0 & = (2-r)\alpha + b' - 1, \\
a_1 & = (2-r)\beta + \frac{1+b'\beta}{\alpha} - 1, \\
a_\lambda & = -(r-2).
\end{cases}
$$

As a corollary, it follows from (22) that

$$
(24) \qquad DCI_{r,d} = -d(r-2)^2 - r\sum_{i=1}^{\lambda}(n_i - 2) + 2(r-2)(r - \gcd(r,d)) + (r-b).
$$

It is clear, in the above formula, that the first term $-d(r-2)^2$ is obviously negative, and also $-r\sum_{i=1}^{\lambda}(n_i - 2) \leq 0$ since $n_i \geq 2$. Moreover, $2(r-2)(r - \gcd(r,d)) \geq 0$ since $r \geq 3$ and $r \geq \gcd(r,d)$; in addition,

$$
r - b = \gcd(r,d)\left(b' - \frac{1+b'\beta}{\alpha}\right) = \frac{\gcd(r,d)}{\alpha}(b'(\alpha - \beta) - 1)
$$

is nonnegative since $\alpha - \beta > 0$ and $b' \geq 1$.

5.6. **Estimations of the Miyaoka-Yau numbers.** We consider the Miyaoka-Yau number of the minimal resolution $\pi : \widetilde{F} \to \overline{F}$. Then we have

(25) $$DCII_{r,d} = -1 + \chi(E_0) + r\lambda = 1 + r\lambda - (r-2)(\gcd(r,d) - 1).$$

By definition, $DMY_{r,d} = 3\,DCII_{r,d} - DCI_{r,d}$. Hence, in view of (24), we get

$$
\begin{aligned}
DMY_{r,d} \ &= \ \left(3(1+r\lambda) + r\sum_{i=1}^{\lambda}(n_i - 2) \right) + (d-1)(r-2)^2 \\
&\quad - \left((r-2)(\gcd(r,d) + r - 1) + (r-b) \right).
\end{aligned}
$$

In addition, $\mathcal{E}_{r,d} = DMY_{r,d} + (d-1)(r-1)(3-r)$ by definition, it follows that

(26)
$$
\begin{aligned}
\mathcal{E}_{r,d} \ &= \ \left(r\sum_{i=1}^{\lambda}(n_i + 1) \right) + (d+2) \\
&\quad - \left((r-2)(\gcd(r,d) + r - 1) + (r-b) \right).
\end{aligned}
$$

Now we begin to estimate $\mathcal{E}_{r,d}$. First, we have

$$(r-2)(\gcd(r,d) + r - 1) + (r-b) \leq (r-2)(2r-1) + r = 2(r-1)^2,$$

so the following hold:

(27) $$\mathcal{E}_{r,d} \ \geq \ \left(r\sum_{i=1}^{\lambda}(n_i + 1) \right) + (d+2) - 2(r-1)^2 > -2r(r-1).$$

Remark 5.7. The above estimate is also true when $d \equiv 1 \bmod r$ by Example 4.7 and when $r = 2$ by Example 4.5.

As an application of the above calculations, we give new examples of computing Chern numbers and $\mathcal{E}_{r,d}$ by directly using Formulae (24), (25) and (26).

Example 5.8. Let $r \geq 3$ and $d = rp, p \geq 1$. Then the resolution $\pi : \widetilde{X} \to X$ in Theorem 4.2 is minimal. Then we have

(i) $\gcd(r,d) = r$, so $\alpha = d/\gcd(r,d) = p$ and $b' = r/\gcd(r,d) = 1$. Since $\alpha = b'p$ and by assumption β is chosen so that $0 \leq \beta < \alpha$ satisfying $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = p - 1$.

(ii) We get
$$b = \frac{\gcd(r,d)(1 + b'\beta)}{\alpha} = r.$$

(iii) We have
$$\frac{\alpha}{\beta} = \frac{p}{p-1},$$
doing the continued fraction decomposition, we see that
$$\lambda = p - 1, \qquad n_1 = \cdots = n_\lambda = 2.$$

Therefore,

$$r\sum_{i=1}^{\lambda}(n_i+1)=3r\lambda=3r(p-1)=3d-3r.$$

(iv) Eventually, by (24), we have

$$DCI_{r,d} = -d(r-2)^2;$$

by (25), we have

$$DCII_{r,d}=1+r(p-1)-(r-2)(r-1)=d-(r-1)^2;$$

by (26), we obtain

$$\mathcal{E}_{r,d} = (3d-3r)+(d+2)-\Big((r-2)(2r-1)+0\Big)$$
$$= 4d-2r(r-1).$$

Example 5.9. Let $r \geq 3$ and $d = r(p-1)+(r-1) = rp-1$ for $p \geq 2$. Then we have

(i) $\gcd(r,d) = 1$, so $\alpha = d/\gcd(r,d) = d$ and $b' = r/\gcd(r,d) = r$. Since $\alpha = b'p-1$ and by assumption β is chosen so that $0 \leq \beta < \alpha$ satisfying $b'\beta \equiv -1 \bmod \alpha$, we have $\beta = \alpha-p = p(r-1)-1$.

(ii) We get

$$b = \frac{\gcd(r,d)(1+b'\beta)}{\alpha} = r-1.$$

(iii) We have

$$\frac{\alpha}{\beta} = \frac{d}{r(p-1)-1} = \frac{rp-1}{r(p-1)-1},$$

doing the continued fraction decomposition, we see that $\lambda = p+r-3$, and

$$n_1 = \cdots = n_{p-2} = 2, \qquad n_{p-1} = 3, \qquad n_p = n_{p+1} = \cdots = n_{p+r-3} = 2.$$

Therefore,

$$r\sum_{i=1}^{\lambda}(n_i+1)=r(3\lambda+1)=3r(p+r)-8r=3(d+1)+3r^2-8r.$$

(iv) Eventually, by (24), we have

$$DCI_{r,d} = -d(r-2)^2+(2r-5)(r-1);$$

by (25), we have

$$DCII_{r,d}=1+r(p+r-3)=d+(r-1)(r-2);$$

by (26), we obtain

$$\mathcal{E}_{r,d} = \Big(3(d+1)+3r^2-8r\Big)+(d+2)-\Big(r(r-2)+1\Big)$$
$$= 4(d+1)+2r(r-3).$$

Consequently, when $r = 3$, we have the following:

(1) when $3|d$, we have $\mathcal{E}_{3,d} = 4d - 12$ by Example 5.8;
(2) when $d \equiv 1 \bmod 3$, we have $\mathcal{E}_{3,d} = 4(d-1)$ by Example 4.7;
(3) when $d \equiv 2 \bmod 3$, we have $\mathcal{E}_{3,d} = 4(d+1)$ by the results above.

In particular, when $d \geq 4$, it is always true that $\mathcal{E}_{3,d} \geq 4(d-3)$.

6. Proof of Theorem 1.1

Let $\pi : \widetilde{F} \to \overline{F}$ be the minimal resolution obtained in previous sections. We prove that $MY(\widetilde{F}) \neq 0$ under the assumption of Theorem 1.1.

The proof will be divided into three cases with respect to the values of t_d and t_{d-1}.

i) When the lines in $\mathcal{A}$ form a pencil, namely, $t_d = 1$, we have, $d \neq 3$ by the assumption of Theorem 1.1; moreover, by Example 4.5, Example 5.8 and Formula (13),

$$MY(\widetilde{F}) = \mathcal{E}_{d,d} = 4d - 2d(d-1) = 2d(3-d).$$

ii) If $t_d = 0$ while $t_{d-1} \neq 0$, then we have $t_{d-1} = 1$ and $t_2 = d - 1$ (if $d = 3$, $t_2 = d = 3$). Moreover, by Example 4.5 and Example 4.7, in view of (13), we have

$$
\begin{aligned}
MY(\widetilde{F}) &= t_2 \mathcal{E}_{2,d} + t_{d-1} \mathcal{E}_{r,d} \\
&= (d-1)\left(4(d-1)\right) + 4(d-1) \\
&= 4d(d-1) > 0.
\end{aligned}
$$

iii) Now we consider the case $t_d = 0, t_{d-1} = 0$. Then by the estimation (27), we have

$$
\begin{aligned}
MY(\widetilde{F}) &= t_2 \mathcal{E}_{2,d} + t_3 \mathcal{E}_{3,d} + \sum_{r \geq 4} t_r \mathcal{E}_{r,d} \\
&\geq t_2 \mathcal{E}_{2,d} + t_3 \mathcal{E}_{3,d} - 2 \sum_{r \geq 4} t_r r(r-1) \\
&= \left(t_2(\mathcal{E}_{2,d} + 4) + t_3(\mathcal{E}_{3,d} + 12) \right) - 2 \sum_r t_r r(r-1).
\end{aligned}
$$

From Remark 3.2, we have $\sum_r t_r r(r-1) = d(d-1)$; moreover, from Example 4.5 and the end of Example 5.9, we deduce that

$$(28) \qquad MY(\widetilde{F}) \geq 4d(t_2 + t_3) - 2d(d-1) = 2d\left(2(t_2 + t_3) - (d-1)\right).$$

Now we use the celebrated inequality in the second remark added in proof of [12], which states that

$$t_2 + \frac{3}{4} t_3 \geq d + \sum_{r \geq 5} (r-4) t_r,$$

see also [23] or Appendix A of [26]. In particular, $t_2 + t_3 \geq d$. It follows immediately, by (28), that

$$MY(\widetilde{F}) > 0.$$

The proof now is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 6.1. When $d = |\mathcal{A}| = 3$ and $\mathcal{A}$ is a pencil, i.e., $t_3 = 1$, then $MY(\widetilde{F}) = 0$. Moreover, from Example 5.8, we obtain $DCI_{3,3} = -3$. Hence, by Example 3.1, we have

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2 = K_{\overline{F}}^2 + DCI_{3,3} = 3 \times (3-4)^2 - 3 = 0.$$

Moreover, $c_2(\widetilde{F}) = 0$ since $MY(\widetilde{F}) = 3c_2(\widetilde{F}) - c_1^2(\widetilde{F}) = 0$.

Since $c_2 > 0$ for a smooth projective surface of general type (see [1], Chapter VII), it follows that $\widetilde{F}$ is not of general type. In particular, $\widetilde{F}$ is not a ball quotient.

7. Surfaces of general type associated to line arrangements

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$. By Theorem 1.1, $MY(\widetilde{F}) > 0$ when $\mathcal{A}$ is not a pencil; it is natural to ask whether $\widetilde{F}$ is a surface of general type. Inspired by [12], it is natural to conjecture that $\widetilde{F}$ is of general type if $\mathcal{A}$ is not too singular, i.e., $t_r = 0$ for r large compared with d.

7.1. A general type criterion. We first provide a criterion for a surface to be of general type.

Proposition 7.2. *Let X be a smooth projective surface. If $c_1^2(X) > 9$, then X is of general type.*

Proof. Let X' be a minimal model of X. Then X' is obtained by successively blowing down (-1)-curves. Note that once we blow down a (-1)-curve, c_1^2 increases by 1, so $c_1^2(X') \geq c_1^2(X) > 9$, hence by the Enriques-Kodaira classification of surfaces (see [1], Chapter VI), X' is of general type, and thus so is X. $\qquad$ $\square$

7.3. Surfaces associated to line arrangements with only nodes and triple points. In the sequel, we consider surfaces associated to line arrangements such that $t_r = 0$ whenever $r \geq 4$, and we prove Theorem 1.2.

For $r = 2$, by Example 4.5, we have $DCI_{2,d} = 0$ and $DCII_{2,d} = d - 1$.

When $r = 3$, we have

(i) If $3|d$, $DCI_{3,d} = -d$, $DCII_{3,d} = d - 4$ by Example 5.8;
(ii) If $d \equiv 1 \bmod 3$, $DCI_{3,d} = -(d-1)$, $DCII_{3,d} = d - 1$ by Example 4.7;
(iii) If $d \equiv 2 \bmod 3$, we have $DCI_{3,d} = -(d-2)$, $DCII_{r,d} = d + 2$ by Example 5.9

7.3.1. *Case $3|d$.* When $d = 3p$, we have by (7) that

$$c_1^2(\widetilde{F}) = K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - dt_3 = d((d-4)^2 - t_3),$$

and by (8),

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3t_3 d.$$

By Remark 3.2, we have $2t_2 + 6t_3 = d(d-1)$, hence $t_3 \leq d(d-1)/6$ and thus when $d = 3p \geq 9$,

$$c_1^2(\widetilde{F}) \geq d\left((d-4)^2 - \frac{d(d-1)}{6}\right) > 9.$$

Therefore $\widetilde{F}$ is of general type by Proposition 7.2. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{(d-4)^2 - t_3}{d^2 - 4d + 6 - 3t_3} = \frac{1}{3}\left(1 + \frac{2(d-3)(d-7)}{d^2 - 4d + 6 - 3t_3}\right).$$

7.3.2. *Case $d \equiv 1 \bmod 3$.* When $d = 3p + 1$, we have

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - (d-1)t_3,$$

and

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d-1)t_3.$$

Since $2t_2 + 6t_3 = d(d-1)$, we have $t_3 \leq d(d-1)/6$ so, when $p \geq 2$ or equivalently $d \geq 7$,

$$c_1^2(\widetilde{F}) \geq d(d-4)^2 - \frac{1}{6}d(d-1)^2 > 9,$$

hence, $\widetilde{F}$ is of general type by Proposition 7.2. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{d(d-4)^2 - (d-1)t_3}{d(d^2 - 4d + 6) - 3(d-1)t_3} = \frac{1}{3}\left(1 + \frac{2d(d-3)(d-7)}{d(d^2 - 4d + 6) - 3(d-1)t_3}\right).$$

7.3.3. *Case $d \equiv 2 \bmod 3$.* When $d = 3p + 2$, we have

$$c_1^2(\widetilde{F}) = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = d(d-4)^2 - (d-2)t_3,$$

and

$$c_2(\widetilde{F}) = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = d(d^2 - 4d + 6) - 3(d-2)t_3.$$

Since $2t_2 + 6t_3 = d(d-1)$, we have $t_3 \leq d(d-1)/6$ so, when $p \geq 2$ or equivalently $d \geq 8$,

$$c_1^2(\widetilde{F}) \geq d(d-4)^2 - \frac{1}{6}d(d-1)(d-2) > 9,$$

hence, $\widetilde{F}$ is of general type by Proposition 7.2. In addition,

$$\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} = \frac{d(d-4)^2 - (d-2)t_3}{d(d^2 - 4d + 6) - 3(d-2)t_3} = \frac{1}{3}\left(1 + \frac{2d(d-3)(d-7)}{d(d^2 - 4d + 6) - 3(d-2)t_3}\right).$$

Conclusion: in any case, $\frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})}$ is an increasing function in t_3 with fixed $d \geq 7$. As $t_3 \leq d(d-1)/6$, it follows that

$$1 \leq \liminf_{d \to \infty} \frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} \leq \limsup_{d \to \infty} \frac{c_1^2(\widetilde{F})}{c_2(\widetilde{F})} \leq \frac{5}{3}.$$

Moreover, Theorem 1.2 follows from the above discussions.

8. Chern numbers and Hodge numbers

In [12], some line arrangements are given so that they give ball quotients through the construction via Kummer covers; such arrangements includes the Hesse arrangement and the arrangement $\mathcal{A}(2,2,3) : (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0$. By Theorem 1.1, these arrangements do not give ball quotient through our approach. Instead, we compute the Hodge numbers of the associated surfaces.

8.1. Relations between Hodge numbers and Chern numbers.
In this section, we shall fix a smooth projective surface X. Denote $q = h^{0,1}(X)$ its irregularity and $p = h^{0,2}(X)$ its geometric genus. Denote also $b_i, i = 1, 2, 3, 4$ the Betti numbers of X and c_1^2, c_2 the Chern numbers, as well as $h^{p,q}$ the Hodge numbers.

Then by Noether's formula (see [1]), we first have

$$(29) \qquad\qquad 1 - q + p = \frac{1}{12}(c_1^2 + c_2);$$

secondly, from the formula for Euler characteristic, we have

$$(30) \qquad\qquad 2 - 2b_1 + b_2 = c_2.$$

Moreover, from Hodge decomposition and Serre duality, we have

$$(31) \qquad b_1 = 2q, \quad b_2 = h^{0,2} + h^{2,0} + h^{1,1} \quad h^{p,q} = h^{q,p} = h^{2-p,2-q}, p, q = 0, 1, 2.$$

We may see the equalities (29), (30),(31) as equations for the Hodge numbers $h^{p,q}$'s, assuming known c_1, c_2, q, and we have the following solution:

$$(32) \qquad \begin{cases} h^{0,0} & = h^{2,2} = 1, \\ h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q, \\ h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q), \\ h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q. \end{cases}$$

8.2. Computing Hodge numbers via Chern numbers.
In this section, we denote $\mathcal{A}$ a line arrangement in $\mathbb{P}^2$ and $\widetilde{F}$ be the associated surface. All the Hodge numbers $h^{p,q}$ and Chern numbers c_1^2, c_2 are for $\widetilde{F}$, namely, we abbreviate the notations $h^{p,q}(\widetilde{F})$ by $h^{p,q}$ etc.

The irregularity q is closely related to the monodromy of $h^* : H^1(F) \to H^1(F)$, and indeed, $2q = \dim H^1(F)_{\neq 1}$ by [8]

We first give the formulae for the Chern numbers of $\overline{F}$. In the first two examples below, $t_r \neq 0$ only if $r|d$. By example 5.8, we have

$$DCI_{r,d} = -d(r-2)^2, \qquad DCII_{r,d} = d - (r-1)^2.$$

Example 8.3. (Hesse arrangement) The Hesse arrangement is defined by

$$Q = xyz((x^3 + y^3 + z^3)^3 - 27x^3y^3z^3)$$

with $d = 12$ with $t_2 = 12, t_4 = 9$. Moreover, we have $q = 3$, see [2].

For the Chern numbers, we first have by (7) that $K_{\widetilde{F}}^2 = d(d-4)^2 = 768$. Since $DCI_{2,12} = 0$ and $DCI_{4,12} = -48$, we have

$$c_1^2 = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = 336.$$

So by Proposition 7.2, $\widetilde{F}$ is of general type. Moreover, by (8),

$$\chi(\overline{F}) = d(d^2 - 4d + 6) - (d-1)\sum_r t_r(r-1)^2 = 201.$$

Since $DCII_{2,12} = 11$ and $DCII_{4,12} = 3$, so

$$c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = 360.$$

Finally, by Formula (32), we obtain

$$\begin{cases} h^{0,0} &= h^{2,2} = 1, \\ h^{0,1} &= h^{1,0} = h^{1,2} = h^{2,1} = q = 3, \\ h^{0,2} &= h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) = 60, \\ h^{1,1} &= -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 250. \end{cases}$$

In addition, the Chern ratio

$$\frac{c_1^2}{c_2} = \frac{336}{360} = \frac{14}{15}.$$

Example 8.4. Consider the arrangement $\mathcal{A}(m, m, 3)$ defined by

$$Q = (x^m - y^m)(y^m - z^m)(z^m - x^m) = 0.$$

Then if $m = 3$, we have $t_3 = 12$ and for $m \neq 3$, we have $t_3 = m^2, t_m = 3$. In addition, if $m \equiv 0 \mod 3$, then $q = 2$, otherwise $q = 1$, see [7].

Moreover, by Example 5.8, the following hold:

$$DCI_{3,d} = -d = -3m, \qquad DCII_{3,d} = d - 4 = 3m - 4$$

and

$$DCI_{m,d} = -d(m-2)^2 = -3m(m-2)^2, \qquad DCII_{m,d} = d - (m-1)^2 = 3m - (m-1)^2.$$

Therefore, by (7),

$$(33) \qquad c_1^2 = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = 3m(m-2)(5m-2).$$

and by (8),

$$(34) \qquad c_2 = \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = 9m(m^2 - 2m + 2).$$

(i) First consider the case where $m = 2$. Then $q = 1$ and we have from (33), $c_1^2 = 0$ and from (34), $c_2 = 36$. Therefore, by Formula (32), we have

$$\begin{cases} h^{0,0} & = h^{2,2} = 1, \\ h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q = 1, \\ h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) = 3, \\ h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 32. \end{cases}$$

(ii) Second, consider the case where $m = 3$, then $q = 2$ and we have from (33) that $c_1^2 = 117$ implying that $\widetilde{F}$ is of general type by Proposition 7.2 and from (34), $c_2 = 135$. Therefore, by Formula (32), we have

$$\begin{cases} h^{0,0} & = h^{2,2} = 1, \\ h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q = 2, \\ h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) = 22, \\ h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q = 97. \end{cases}$$

(iii) Consider the case where $m > 3$ and $m \not\equiv 0 \bmod 3$. Then $q = 1$. From (33), we get $c_1^2 = 3m(m-2)(5m-2)$. Note that in our situation $m \geq 4$, hence $c_1^2 \geq 3 \cdot 4 \cdot 2 \cdot 18 = 432 > 9$, hence $\widetilde{F}$ is of general type by Proposition 7.2.

In addition, from (34) we have $c_2 = 9m(m^2 - 2m + 2)$. Therefore, by Formula (32), we have

$$\begin{cases} h^{0,0} & = h^{2,2} = 1, \\ h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q = 1, \\ h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) \\ & = \frac{1}{2}m(m-1)(4m-5) \\ h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q \\ & = 5m^3 - 9m^2 + 13m + 2 \end{cases}$$

(iv) Finally, we consider the case where $m > 3$ and $m \equiv 0 \bmod 3$. Then $q = 2$ and form (33), we have

$$c_1^2 = 3m(m-2)(5m-2) \geq 3 \cdot 6 \cdot (6-2) \cdot (5 \cdot 6 - 2) > 9,$$

hence $\widetilde{F}$ is of general type by Proposition 7.2. Moreover, from (34), we have $c_2 = 9m(m^2 - 2m + 2)$, hence by Formula (32), we have

$$\begin{cases} h^{0,0} & = h^{2,2} = 1, \\ h^{0,1} & = h^{1,0} = h^{1,2} = h^{2,1} = q = 2, \\ h^{0,2} & = h^{2,0} = \frac{1}{12}(c_1^2 + c_2) - (1 - q) \\ & = \frac{1}{2}m(m-1)(4m-5) + 1 \\ h^{1,1} & = -\frac{1}{6}c_1^2 + \frac{5}{6}c_2 + 2q \\ & = 5m^3 - 9m^2 + 13m + 4 \end{cases}$$

Conclusion: for $m \geq 3$, the surface $\widetilde{F}$ is of general type. Furthermore, as $m \to \infty$, one can show that $h^{0,2}, h^{1,1} \to \infty$ while other Hodge numbers remains 1 or 2. In addition, the Chern ratio

$$\frac{c_1^2}{c_2} = \frac{3m(m-2)(5m-2)}{9m(m^2-2m+2)} \to \frac{5}{3} \qquad \text{as } m \to \infty.$$

Example 8.5. Now we consider line arrangements which arise from restriction of higher dimensional hyperplane arrangements. The braid arrangement in $\mathbb{P}^n$ is given by

$$\mathfrak{B}_n : \prod_{0 \leq i < j \leq n} (x_i - x_j) = 0,$$

consisting of $\binom{n+1}{2}$ hyperplanes. Let $E \subseteq \mathbb{P}^n$ be a *generic* projective plane and let $\mathcal{A}_n = \mathfrak{B}_n|_E$ the restriction of $\mathfrak{B}_n$ to E. Then $\mathcal{A}_n$ is a line arrangements in the projective plane with only nodes and triple points such that

$$d = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

and

$$t_3 = \binom{n+1}{3}.$$

Indeed, any triple points of $\mathcal{A}_n$ corresponds to the intersection of exactly three hyperplanes in $\mathfrak{B}_n$, which is thus of the form $\{x_{i_1} = x_{i_2} = x_{i_3}\}$ for some $i_1 < i_2 < i_3$. Hence

$$t_3 = \#\{(i_1, i_2, i_3) \ : \ i_1, i_2, i_3 \in [0, n], \ i_1 < i_2 < i_3\} = \binom{n+1}{3}.$$

From Remark 3.2, we have

$$2t_2 + 6t_3 = d(d-1) = \binom{n+1}{2}\left(\binom{n+1}{2} - 1\right),$$

hence

$$t_2 = \frac{d(d-1)}{2} - 3t_3 = \frac{(n+1)n(n-1)(n-2)}{8}.$$

Note that if $n \equiv 1 \bmod 3$, then $d \equiv 1 \bmod 3$, otherwise $3|d$, so we consider the following two cases:

(i) If $n \not\equiv 1 \bmod 3$, we have $3|d$. Moreover, if $n = 2, 3$, then $q = 1$, otherwise $q = 0$ by [16]. In addition,

$$DCI_{2,d} = 0, \qquad DCII_{2,d} = d - 1$$

and

$$DCII_{3,d} = -d, \qquad DCII_{3,d} = d - 4.$$

Hence,

$$c_1^2 = K_{\widetilde{F}}^2 + \sum_r t_r DCI_{r,d} = \frac{1}{24}n(n+1)(n-2)(n-3)(3n^2 + 19n + 32),$$

so if $n \geq 4$, $c_1^2 > 9$ and thus $\widetilde{F}$ is of general type by Proposition 7.2. Moreover,

$$c_2 \;=\; \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8}n(n+1)(n-2)(n^3+2n^2-3n-12).$$

(ii) If $n \equiv 1 \bmod 3$, then $d \equiv 1 \bmod 3$ and $q = 0$ by [16]. In addition,

$$DCI_{2,d} = 0, \qquad DCII_{2,d} = d - 1$$

and by Example 4.7, we have

$$DCII_{3,d} = -(d-1), \qquad DCII_{3,d} = d - 1.$$

Thus,

$$c_1^2 \;=\; K_{\overline{F}}^2 + \sum_r t_r DCI_{r,d}$$

$$\;=\; \frac{1}{24}n(n+1)\left(3n^2(n^2-15) + 2n(2n^2-21) + 188\right)$$

so if $n \geq 4$, $c_1^2 > 9$ and thus $\widetilde{F}$ is of general type by Proposition 7.2. Moreover,

$$c_2 \;=\; \chi(\overline{F}) + \sum_r t_r DCII_{r,d} = \frac{1}{8}n(n+1)(n^4-7n^2-2n+20).$$

The concrete formulae for the Hodge numbers by applying (32) are left to the reader. For the Chern ratio, we have

$$\lim_{n\to\infty} \frac{c_1^2}{c_2} = 1.$$

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