Discrete determinants and the Gel'fand–Yaglom formula

J.S.Dowker[1]

Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England

I present a partly pedagogic discussion of the Gel'fand–Yaglom formula for the functional determinant of a linear, one–dimensional, second order difference operator, in the simplest settings. The formula is a textbook one in discrete Sturm–Liouville theory and orthogonal polynomials. A two by two matrix approach is developed and applied to Robin boundary conditions. Euler–Rayleigh sums of eigenvalues are computed. A delta potential is introduced as a simple, non-trivial example and extended, in an appendix, to the general case. The continuum limit is considered in a non–rigorous way and a rough comparison with zeta regularised values is made. Vacuum energies are also considered in the free case. Chebyshev polynomials act as free propagators and their properties are developed using the two–matrix formulation, which appears to be novel and has some advantages. A trace formula, rather than the more usual determinant one, is derived for the Gel'fand–Yaglom function.

[1]dowker@man.ac.uk

1. Introduction

Finite structures are very common in science either as approximations to some continuous arrangement, perhaps for numerical purposes, or because of some inherent discreteness or, again, for regularisation. They have also gained a certain currency in elementary particle models.

In this communication I wish to make some rather elementary computations of one or two quantum field theory quantities using finite difference notions.[2] I restrict myself to the simplest one–dimensional systems *i.e.* fields on the interval or circle.

Although these have been discussed, almost *ad nauseam*, I could not find this particular development completely in the literature. The interesting work by Actor, Bender and Reingruber, [4], contains a detailed treatment of the Casimir effect on the lattice, and, while I cannot add too much to their extensive results, I will recover some of the formulae for completeness. I will also compute the discrete determinants for the free field case and I will include a mass here too. Although the explicit results are rather trivial, and just examples of general expressions, I believe they have some didactic merit. As something more substantial, I also treat the case of a delta potential.

Functional determinants appear in many areas and their computation is important physically. An early method is the Gel'fand–Yaglom technique which is a means of finding an operator determinant without knowing the eigenvalues explicitly. The continuum case (originating with Gel'fand and Yaglom, [5], and Levit and Smilansky, [6]) has been analysed fairly extensively. The work by Kirsten and McKane, [7], contains a brief historical survey plus a contour integral proof of the theorem and a discussion of the zero mode problem. In the quantum field theory context, Dunne, [8], can be consulted for orientation and further references.

There has been less work on discrete systems, although there is a considerable body of work concerned with graphs, which, though relevant, I will not consider, *per se*. Very general theorems have been derived by Forman, [9], for the situation when a potential is present. He proves and employs a discrete Gel'fand–Yaglom theorem. In the following sections, I give a simple justification of the formulae by standard spectral means. The original treatment by Gel'fand and Yaglom involves a limit process from a discretisation approach to functional integration, which, in content, is partly equivalent to the remarks here.

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[2] There are numerous texts on finite difference equations. An unusual one is Bleich and Melan, [1] and a modern one is Elaydi, [2]. The classic work by Atkinson, [3], is a central reference.

I treat, at least initially, the simplest set–up that allows me to illustrate the essentials. This will be the uniform continuous string of length L vibrating transversally. An approximation by (equal) mass points takes us back to the precursor of Fourier analysis, the subject of countless historical surveys and textbook explanations. For reference I mention only the classic Rayleigh, [10], and Morse and Feshbach, [11]. The modes of this discrete system are, therefore, ancient but I will develope them again. Some are given, relevantly, in the basic finite difference text by Fort, [12]. There will necessarily be a certain amount of repetition.

A summary of the discretization, of relevance to the present topic, is given by de Verdiére, [13], §9.2.

In the course of the calculation, I encounter Chebyshev polynomials and develope their properties via a two-matrix technique which is convenient, and might be novel.

2. The discrete Gel'fand–Yaglom theorem.

To make the situation precise, replace the interval $[0, L]$ by $\nu + 2$ equally spaced points, or vertices, two being end, or boundary, points. Label the points by j, $0 \leq j \leq \nu + 1$ and consider some scalar function, $y(j)$, satisfying either Dirichlet (D) or Neumann (N) conditions at the ends, (*e.g.* [11]),[3]

$$
\begin{aligned}
y(0) = y(\nu + 1) = 0, \quad D \\
y(0) = y(1), \ y(\nu) = y(\nu + 1), \quad N.
\end{aligned}
\tag{1}
$$

I discuss the Sturm–Liouville problem which, in its simplest formulation, involves the eigenvalue recurrence, (*e.g.* [12]),

$$
y(j + 1) + \big(\lambda - V(j) - 2 \big) y(j) + y(j - 1) = 0,
\tag{2}
$$

subject to boundary conditions, say (1).

I refer to $V(j)$ as the potential because (2) can be rewritten as the more familar looking Laplacian eigenvalue equation,[4]

$$
\left[-\frac{1}{h^2} \nabla \Delta + \overline{V}(j) \right] y(j) = \overline{\lambda} y(j).
\tag{3}
$$

---

[3] For convenience, I will assume that all my functions, eigenfunctions *etc.* are real, except when considering a twisted periodic field later.

[4] ∇ is the backwards difference operator.

The lattice spacing, $h = L/(\nu + 1)$, has been introduced by scaling to give a 'physical' Laplacian and one has the dimensionless quantities, $\lambda = h^2 \overline{\lambda}$ and $V = h^2 \overline{V}$.

The procedure is textbook. Taking D conditions for definiteness, iteration from the $j = 0$ end point, assuming any value of $y(1)$, except zero, yields all the $y(j)$ as polynomials in λ, in particular, the terminal value, $y(\nu + 1, \lambda)$. The eigenvalues are thus the roots of this polynomial, $y(\nu + 1, \lambda) = 0$ (*e.g.* Atkinson, [3]) and the determinant (*i.e.* the product of all the λ) of the operator is its constant term, $y(\nu + 1, 0)$, up to a factor, which is the essence of the Gel'fand–Yaglom formula[5]. The factor involved is unity if the starting term is chosen to be $y(1) = 1$, as can be seen by looking at the $\lambda \to \infty$ limit (see later).

The product of all the physical $\overline{\lambda}$ is only a scaling factor different and one arrives at the discrete Dirichlet result, *e.g.* [9],

$$\mathrm{Det}_D = \frac{1}{h^{2\nu}} \, y(\nu + 1, 0) \,. \tag{4}$$

This formula is thus part and parcel of the standard eigenvalue problem. The resolvent of (2) is,

$$R(\lambda) = \frac{d}{d\lambda} \log y(\nu + 1, \lambda) \,,$$

with the usual machinery. For example, the sums of the inverse powers of the roots follow, *à la* Euler and Rayleigh, [10] I, p.279 , as,

$$-R(\lambda) = \sum_n \frac{1}{\lambda_n} + \lambda \sum_n \frac{1}{\lambda_n^2} + \lambda^2 \sum_n \frac{1}{\lambda_n^3} + \dots \,. \tag{5}$$

The rigorous proof that the discrete formula leads to the original continuous one of Gel'fand and Yaglom and of Levit and Smilansky, [6], is given by Forman. de Verdiére, [13], also discusses the nature of this limit.

3. Dirichlet constant potential

Before continuing to other boundary conditions, I give the simplest application of (4) which is when the potential is constant and equivalent to a mass term, μ^2. I then rewrite (2),

$$y(j + 1) - 2 \cosh 2\gamma \, y(j) + y(j - 1) = 0 \,, \tag{6}$$

---

[5] The nature of this constant is where the problem lies in the continuum case.

where I have set $\mu = h\overline{\mu}$ and $\mu^2 - \lambda = 4\sinh^2\gamma$ and which I must solve subject to the initial conditions $y(0) = 0$, $y(1) = 1$. The roots of the auxiliary equation are,

$$m_\pm = \cosh 2\gamma \pm \sinh 2\gamma = e^{\pm 2\gamma},$$

and the general solution is,

$$y(j) = Am_+^j + Bm_-^j,$$

with

$$A = -B = \frac{1}{2\sinh 2\gamma}.$$

This implies that the discrete Gel'fand–Yaglom function (more conventionally called the fundamental solution) is,

$$y(j) = \frac{\sinh 2\gamma j}{\sinh 2\gamma}, \tag{7}$$

evaluated at the terminal point, $j = \nu + 1$, which is, perhaps, no surprise in view of the textbook continuum analogue. The functions, $y(j)$, written $y(j, \lambda)$, are polynomials in $4\sinh^2\gamma$ (and hence in λ) which can be proved in many ways, one of which is the direct iteration of (6). Equation (2) is a recursion formula for these polynomials, which are Chebyshev polynomials, as is well known, the definition being,

$$U_\nu(\cosh 2\gamma) \equiv \frac{\sinh 2\gamma(\nu + 1)}{\sinh 2\gamma}. \tag{8}$$

Pursuing the calculation, the determinant is obtained by setting $\lambda = 0$,

$$\mathrm{Det}_D(\overline{\mu}) = \frac{1}{h^{2\nu}} \frac{\sinh 2\gamma_0(\nu + 1)}{\sinh 2\gamma_0}. \quad \mu = 2\sinh\gamma_0. \tag{9}$$

The constant of proportionality is settled by the infinite λ limit when the Gel'fand–Yaglom function has the explicit behaviour,

$$\frac{\sinh 2\gamma(\nu + 1)}{\sinh 2\gamma} \to (2\cosh 2\gamma)^\nu \sim (-\lambda)^\nu.$$

The eigenvalues themselves are determined by,

$$\sinh 2\gamma(\nu + 1) = 0$$

or

$$\gamma = \gamma_n \equiv \frac{n\pi i}{2(\nu + 1)},$$

4

and so

$$\overline{\lambda}_n = \overline{\mu}^2 + \frac{4}{h^2} \sin^2 \frac{\pi n}{2(\nu+1)}\,, \quad n = 1, \ldots, \nu\,, \tag{10}$$

which is the textbook result, *e.g.* [12]. Equating the determinant (9) to $\prod_n \lambda_n$ gives a standard product formula, *e.g.* Bromwich, [14], p.211, which comes up later in §13. Furthermore, the sums of inverse powers of the roots, (5), yields finite summations for powers of cosecants, *e.g.*, typically,

$$\sum_{n=1}^{p-1} \operatorname{cosec}^2 \frac{\pi n}{2p} = \frac{2}{3}(p^2 - 1)\,, \tag{11}$$

which are very old and are simple examples of a wide class of trigonometric summations obtainable in many ways.[6] As $p \to \infty$ (the continuum limit) this sum becomes Euler's result, $\zeta_R(2) = \pi^2/6$.

The eigenfunctions follow by noting that the fundamental solution, $y(j, \lambda)$, (7), satisfies the equation (6) with $\lambda = \lambda_n$ and obeys the Dirichlet conditions. The eigenfunctions are therefore,

$$y_n(j) = \sin \frac{j\pi n}{\nu+1}\,, \quad n = 1, \ldots, \nu\,,$$

of which there are ν, this being the number of 'dynamical' points. We therefore reach the standard mode properties, *e.g.* Fort, [12], Spiegel, [17]. This route is not a novel one.

4. Neumann conditions

As a warm–up for the Robin case, I consider Neumann boundary conditions (1) which can be written, $\Delta y(0) = 0$, $\Delta y(\nu) = 0$. If $V(j)$ were uniform, (2) would be satisfied by $\Delta y(j)$ and the problem translated into a Dirichlet one (see §8) but, because of the j dependence, this is not possible and it is necessary to treat (2) and its difference together.[7] This is most neatly expressed using 2×2 matrices as in [3], [9] and elsewhere, *e.g.* [7], although a little differently. It might be considered a 'phase space' representation.

Defining,

$$\Upsilon(j) = \begin{pmatrix} y(j) \\ y(j+1) \end{pmatrix}\,, \tag{12}$$

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[6] I attempted a few comments and gave some references in [15]. See also Berndt and Yeap, [16].

[7] This is a common device in the theory of ordinary differential equations. For difference equations see *e.g.* Porter, [18], Goldberg, [19], p.233 Ex.4., Elaydi, [2].

the recurrence under study is the first order one,

$$\Upsilon(j) - M(j)\Upsilon(j-1) = 0\,, \tag{13}$$

where

$$
\begin{aligned}
M(j) &= \begin{pmatrix} 0 & 1 \\ -1 & V(j) + 2 - \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 2\cosh 2\gamma_j \end{pmatrix}\,,
\end{aligned} \tag{14}
$$

whereby γ_j is defined. One of the equations is just an identity. Note that $\det M(j) = 1$.

The Sturm-Liouville Neumann boundary condition is $\Delta y(0) = 0$, $\Delta y(\nu) = 0$ which means, choosing a normalisation,

$$\Upsilon(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,, \quad \Upsilon(\nu) \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{15}$$

The eigenvalue procedure is to iterate (13) up to $\Upsilon(\nu)$ starting from, $\Upsilon(0)$ *i.e.*
,

$$\Upsilon(\nu) = M(\nu)M(\nu-1)\ldots M(1)\Upsilon(0)\,, \tag{16}$$

and then impose the condition, (15), on $\Upsilon(\nu)$. (Remember, the Ms are functions of λ.)

The roots of the polynomial,

$$P(\nu, \lambda) = \begin{pmatrix} 1, & -1 \end{pmatrix}\Upsilon(\nu) = \Delta y(\nu, \lambda)\,,$$

are then the eigenvalues and the determinant is $P(\nu, 0)$, which is the required theorem, [9],

$$\mathrm{Det}_N = \frac{1}{h^{2\nu}}\Delta y(\nu, 0)\,.$$

5. Robin boundary conditions

Having treated pure Neumann, it is not much more difficult to sort out Robin conditions, called General Local in [9]. The recurrence is still (13) but with the boundary conditions,

$$\Upsilon(0) = \begin{pmatrix} 1 \\ 1+\alpha \end{pmatrix} \equiv \Upsilon_{in}\,, \quad \Upsilon(\nu) \propto \begin{pmatrix} 1+\beta \\ 1 \end{pmatrix} \equiv \Upsilon_{out}\,, \tag{17}$$

where α and β are the Robin parameters defined by,

$$\Delta y(0) = \alpha\, y(0)\,, \quad \Delta y(\nu) = -\beta\, y(\nu + 1)\,.$$

The first condition in (17) is chosen, and the second is imposed.

Defining an 'adjoint',

$$\Upsilon^\dagger(j) = \widetilde{\Upsilon}(j)\, J\,,$$

in terms of the symplectic metric, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the eigenvalue polynomial is, therefore, the matrix element,

$$
\begin{aligned}
(-1,\ 1 + \beta\,)\, \Upsilon_{in}(\nu) &= \Upsilon^\dagger_{out}\Upsilon_{in}(\nu) \\
&= \Upsilon^\dagger_{out}\, M(\nu)M(\nu - 1)\ldots M(1)\Upsilon_{in}\,,
\end{aligned}
\tag{18}
$$

and the product of the λ eigenvalues is proportional to $\Upsilon^\dagger_{out}\Upsilon_{in}(\nu)$, evaluated at $\lambda = 0$, the constant of proportionality being the inverse of the coefficient of the highest power of λ in (18). For very large λ, M, (14), approximates to,

$$M(j) \sim \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}\,, \tag{19}$$

when the right–hand side of (18) becomes $(1 + \beta)(1 + \alpha)(-\lambda)^\nu$ so the determinant is, after scaling to the physical eigenvalues, λ,

$$\mathrm{Det}_R(\overline{\alpha}, \overline{\beta}) = \frac{1}{h^{2\nu}}\, \frac{1}{(1 + \beta)(1 + \alpha)}\, \Upsilon^\dagger_{out} \prod_{j=1}^{\nu} M_0(j)\, \Upsilon_{in}\,, \tag{20}$$

where M_0 is the matrix M evaluated at $\lambda = 0$. For future use I have also introduced the physical constants, $\overline{\alpha} = \alpha/h$, $\overline{\beta} = \beta/h$.

I note that the symplectic product is just the Casoratian (discrete Wronskian), [2],

$$\Upsilon^\dagger_1(j)\, \Upsilon_2(j) \equiv \widetilde{\Upsilon}_1(j)\, J\, \Upsilon_2(j)\,, \tag{21}$$

of two solutions, Υ_1 and Υ_2, of (13), which is a symplectic development because,

$$\widetilde{M}\, J\, M = J\,,$$

and so (21) is uniform, *i.e.* independent of j. This is a neater proof of this fact than the usual ones, *e.g.* Fort, [12].

The equivalence of a 2×2 real matrix formulation and a three–term recurrence relation is expounded by Atkinson, [3], §3.5 involving a geometrical interpretation of symplectic action.

I take up the general formalism again in §10 and turn now to an elementary case.

6. Constant potential

As the simplest example, I again take that of a constant potential. Then $M(j)$ is,

$$M(j) = M = \begin{pmatrix} 0 & 1 \\ -1 & 2\cosh 2\gamma \end{pmatrix}, \qquad (22)$$

with γ as before, *i.e.* $4\sinh^2 \gamma = \mu^2 - \lambda$.

It is shown in Appendix 2 that the power M^ν is given by,

$$M^\nu = \begin{pmatrix} -U_{\nu-2} & U_{\nu-1} \\ -U_{\nu-1} & U_\nu \end{pmatrix}$$

in terms of Chebyshev polynomials, (8), of argument $\cosh 2\gamma$. Then,

$$\Upsilon^\dagger_{out} M^\nu \Upsilon_{in} = (-1, 1+\beta) \begin{pmatrix} -U_{\nu-2} & U_{\nu-1} \\ -U_{\nu-1} & U_\nu \end{pmatrix} \begin{pmatrix} 1 \\ 1+\alpha \end{pmatrix}$$

$$= (\alpha + \beta + \alpha\beta) U_\nu - (\alpha + \beta - \lambda) U_{\nu-1}, \qquad (23)$$

which is to be substituted into (20), after setting $\lambda = 0$ to give the determinant. The formula is symmetric under interchange of α and β, as it should be by geometric symmetry.

As another check, the Dirichlet choice,

$$\Upsilon^D_{in} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \Upsilon^D_{out} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (24)$$

reproduces (9).

Incidentally, if M were more general, rather than diagonalisation, it would be easier, for iteration purposes, to set,

$$M^\nu = a\mathbf{1} + b\, M, \qquad (25)$$

and compute a and b in terms of the eigenvalues of M, *e.g.* Goldberg, [19].

7. The characteristic polynomial and Euler–Rayleigh sums

The Euler–Rayleigh sums, analogous to (11), arise from the expansion of (23) in powers of λ, *i.e.* of $-4\sinh^2\gamma$ (for simplicity I set μ to zero) which is easily accomplished by, say, using Bromwich, [14] chap.IX or the relation to Chebychev polynomials. I find,

$$
\begin{aligned}
\Upsilon_{out}^\dagger \, M^\nu \, \Upsilon_{in} = & \\
= (\alpha\,\beta + \alpha + \beta) & \sum_{s=0}^{\nu} \frac{\nu}{2\nu - s} \binom{2\nu - s}{s} (-\lambda)^{\nu - s} \\
+ \left(\alpha\,\beta - \frac{1}{2}(\alpha\,\beta + \alpha + \beta + 2)\lambda\right) & \sum_{s=0}^{\nu-1} \binom{2\nu - s - 1}{s}(-\lambda)^{\nu - s - 1} .
\end{aligned}
\tag{26}
$$

The ν eigenvalues, λ_n, are the roots of this characteristic polynomial and it is next required to expand its logarithm, which, for low powers, can be done directly by hand. As the simplest case I give

$$
\sum_{n=0}^{\nu-1} \operatorname{cosec}^2(\theta_n) = 2\frac{3\nu^2(\alpha\beta + \alpha + \beta) + \nu(\nu^2 - 1)\alpha\beta + 3\nu(\alpha\beta + \alpha + \beta + 2)}{3\big((1 + \nu)\alpha\beta + \alpha + \beta\big)} ,
\tag{27}
$$

where I have set

$$
\lambda_n = 4\sin^2\theta_n \,,
$$

which defines θ_n.

The continuum limit $h \to 0$, $\nu \to \infty$ is not without interest and is discussed in a section 9.

8. Neumann conditions revisited

As the free N case is not given in Fort, [12], I give, for pedagogic completeness, the conventional calculation by noting, first, that the N conditions, (1), can be written $\Delta y(0) = \Delta y(\nu) = 0$. So, defining $\phi(j) = \Delta y(j)$, one has, from (3),

$$
-\frac{1}{h^2} \nabla\Delta \, \phi(j) = \overline{\lambda} \, \phi(j)
\tag{28}
$$

with $\phi(0) = \phi(\nu) = 0$, which is a D problem on $\nu + 1$ vertices but with the original spacing, h. The D–eigenfunctions are

$$
\phi(j) = \sin\frac{n\pi j}{\nu} \,, \quad n = 1, \ldots, \nu - 1 \,,
$$

and hence the N–eigenfunctions are,[8]

$$
\begin{aligned}
y(j) = \Delta^{-1}\,\phi(j) &= \Delta^{-1}\,\sin\frac{n\pi j}{\nu} \\
&\approx \cos\frac{n\pi(2j-1)}{2\nu}\,, \quad n = 0, 1, \ldots, \nu - 1\,,
\end{aligned}
\tag{29}
$$

up to a numerical factor and possible additional constant. The eigenvalues are,

$$
\lambda_n = \frac{4}{h^2}\,\sin^2\frac{\pi n}{2\nu}\,, \quad n = 0, \ldots, \nu - 1\,.
\tag{30}
$$

Again, there are ν modes, including the uniform zero one, $n = 0$, which corresonds to a constant of integration in (29). ($n = \nu$ gives a vanishing mode.)

Before going on, it would be best to see, as a check, if the pure Neumann determinant, for the free case with mass, agrees with the above mode structure and the Robin expression, (23). Effectively I start again. The initial condition that fixes the Gel'fand–Yaglom function is $z(0) = 1$ and $z(1) = 1$, *i.e.* $\Delta z(0) = 0$. (This is Forman's z.) The general solution is again,

$$
z(j) = Ae^{2\gamma j} + Be^{-2\gamma j}\,,
$$

and the conditions imply,

$$
A + B = 1
$$
$$
Ae^{2\gamma} + Be^{-2\gamma} = 1\,,
$$

which solve to,

$$
A = \frac{e^{-\gamma}}{2\cosh\gamma}\,, \quad B = \frac{e^{\gamma}}{2\cosh\gamma}\,,
$$

so that,

$$
\begin{aligned}
z(j) &= \frac{\cosh(2j-1)\gamma}{\cosh\gamma} = V_{j-1} = \nabla U_{j-1} \\
\Delta z(j) &= 4\sinh^2\gamma\frac{\sinh 2\gamma j}{\sinh 2\gamma} = \Delta\nabla U_{j-1}\,.
\end{aligned}
\tag{31}
$$

where V_j is a Chebyshev polynomial of the third kind, [21], and all Chebyshev arguments are $\cosh 2\gamma$. Appendix 1 contains some relations for Chebyshev polynomials couched in the two–matrix language and (31) can be obtained more rapidly using this.

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[8] If you use Jordan, [20], be aware that there is an error on p.117 that is carried forward. For example, on p.124 the sum of $\cos(x + b)\phi$ is incorrect. The upper limit should be $n - 1$.

Applying the eigenvalue restriction, $\Delta z(\nu, \lambda) \equiv \Delta z(\nu) = 0$, yields the condition $\gamma = \gamma_n = n\pi i/\nu$, $n = 0, 1, \ldots \nu - 1$, and the eigenvalues are,

$$\lambda_n = \mu^2 + \frac{4}{h^2} \sin^2 \frac{\pi n}{2\nu}, \quad n = 0, \ldots, \nu - 1, \tag{32}$$

consistent with (30). The eigenfunctions, (29), also follow trivially from (31).

One sees from (31) that the eigenvalue condition is the same as the Dirichlet one, except for the replacement $\nu \to \nu - 1$ and for the factor $4 \sinh^2 \gamma = \mu^2 - \lambda$. This factor is responsible for the $n = 0$ mode which, in the massless case, is a zero mode.

The Neumann determinant is then,[9]

$$\mathrm{Det}_N(\mu) = \frac{1}{h^{2\nu}} \Delta z(\nu, 0),$$

where the numerical factor follows on the limit,

$$4 \sinh^2 \gamma \frac{\sinh 2\gamma\nu}{\sinh 2\gamma} \to 4 \sinh^2 \gamma (2 \cosh 2\gamma)^{\nu-1} \sim (-\lambda)^\nu.$$

Equating the two forms of the determinant yields the same product formula as in the D case.

The determinant also agrees with the Robin formula, from (20), for $\alpha = \beta = 0$. (This is, of course, simply a check of algebra.)

9. The continuum limit

Comparisons with known results can also be obtained by considering the continuum limit, an historical motivation for discretisation. Again as an example, I consider the Robin determinant (20) with (23) in the limit $h \to 0$. To get the leading divergence, the lowest power of h is required in the expression multiplying $1/h^{2\nu}$. As $h \to 0$ one has the limiting behaviours,

$$2 \sinh \gamma_0 \sim 2\gamma_0 \sim \mu = h\overline{\mu}, \quad 2\gamma_0\nu \sim h\overline{\mu}\nu \sim \overline{\mu}L,$$

and therefore by inspection of (23), one sees that the leading term is of order h. Extracting this gives,

$$\begin{aligned} \Upsilon_{out}^\dagger \, M_0^\nu \, \Upsilon_{in} & \\ & \to (\overline{\alpha} + \overline{\beta}) \cosh \overline{\mu}L + (\overline{\alpha}\overline{\beta} + \overline{\mu}^2) \frac{\sinh \overline{\mu}L}{\overline{\mu}}, \end{aligned} \tag{33}$$

---

[9] This appears to differ by a factor of $1/4$ from Forman's formula, [9].

which agrees with an expression in [22] for the continuum case.

Related is the limit of the simplest Euler–Rayleigh eigenvalue sum, (27). Reverting to physical quantities,

$$h^2 \sum_{n=0}^{\nu-1} \frac{1}{\lambda_n} = \sum_{n=0}^{\nu-1} \frac{1}{\overline{\lambda}_n} \to \frac{3(\overline{\alpha} + \overline{\beta}) + \overline{\alpha}\overline{\beta} + 6}{6(\overline{\alpha}\overline{\beta} + \overline{\alpha} + \overline{\beta})} \,, \tag{34}$$

which is also given in [22].

10. Non–uniform potential. The propagator

Difference equation Sturm–Liouville theory is well developed and can be pursued by analogy to the continuum version, *e.g.* Fort, [12], Levy and Baggott, [23]. In fact Sturm obtained many continuum results via a discrete route, although this was never published.

In this section I wish to develope and summarize the previous matrix formulation, see (12), (13), (22). I consider (13) as a Schrödinger equation for a two–state system with a discrete time labelled by j, and rewrite it by defining a matrix 'propagator' $K(\lambda; j, j')$,

$$K(\lambda; j, j') = \theta(j, j') \prod_{k=j'+1}^{j} M(k) \tag{35}$$

as

$$\Upsilon(j) = K(\lambda; j, j')\Upsilon(j'), \quad j \geq j', \tag{36}$$

which propagates forwards from j' to j and acts as a transfer 2×2 matrix. In the simplest case, the matrix M is given by (22). The form, (35), is an equivalent of the time–ordered exponential solution in time–dependent perturbation theory, but here 'vertex–ordered'. The propagator, $K(\lambda; j, 0)$ is sometimes referred to as the state transition matrix. I denote it by $K(\lambda; j)$. The basic theory is given by Elaydi, [2] §3.2, but my treatment is modified a little and also deals, particularly, with a symplectic invariant propagation.

For consistency, the initial condition, (*i.e.* the first empty product in (35)),

$$K(\lambda; j, j) = \mathbf{1}\,, \tag{37}$$

has to be taken. The step function θ ensures that $K(j, j') = 0$ for $j < j'$, corresponding to causal propagation. The semi–group property,

$$K(\lambda; j, j')K(\lambda; j', j'') = K(\lambda; j, j'')\,,$$

12

(no sum on j') and symplectic invariance,

$$\widetilde{K}(\lambda; j, j') \, J \, K(\lambda; j, j') = J \,, \tag{38}$$

also hold.

K satisfies the equation of motion,

$$K(\lambda; j, j') \equiv E K(\lambda; j - 1, j') = \mathbf{1}\delta_{j,j'} + M(j)K(\lambda; j - 1, j') \,, \tag{39}$$

where the first term arises from the θ factor in (35). A matrix which satisfies (39) is a *fundamental matrix*.

Iteration of (39) gives a power series expansion,

$$K(\lambda; j, j') = \mathbf{1}\delta_{j,j'} + M(j)\delta_{j,j'+1} + M(j)M(j-1)\delta_{j,j'+2} + \dots \,, \tag{40}$$

which is quite equivalent to (35). It also follows from the decomposition,

$$\begin{aligned}
\theta(j, j') &= \delta_{j,j'} + \delta_{j,j'+1} + \delta_{j,j'+2} + \dots \\
&= \Delta^{-1}\delta_{j,j'} \,,
\end{aligned} \tag{41}$$

obvious graphically, arithmetically and in $(\nu + 1) \times (\nu + 1)$ matrix form. It is the discrete version of the distributional operator statement that the θ–function is the integral of the δ–function.

If $M(j)$ is constant, then, trivially $K(\lambda; j, j') = \theta(j, j')M^{j-j'}$, either from (35) or read off from (40).

If an 'unperturbed' propagator, K_0, is defined by,

$$K_0(j, j') = \mathbf{1}\delta_{j,j'} + M_0(j)K_0(j - 1, j') \,, \tag{42}$$

then,

$$K(\lambda; j, j'') = K_0(j, j'') + K_0(j, j')\big(M(j') - M_0(j')\big)K(\lambda; j' - 1, j'') \,,$$

where j' is summed over from 1 to ν, can be considered as a perturbation expansion. If M_0 is constant,

$$K(\lambda; j, j'') = M_0^{j-j''} + M_0^{j-j'}\big(M(j') - M_0\big)K(\lambda; j' - 1, j'') \,. \tag{43}$$

The propagator, $K(\lambda; j, j')$ is defined independently of any boundary conditions which are incorporated, in my approach, by constructing the symplectic scalar products,

$$\begin{aligned}
P(\lambda) &= \Upsilon_{out}^\dagger(j) \, K(\lambda; j, j')\Upsilon_{in}(j') \\
&= \Upsilon_{out}^\dagger(j) \, \Upsilon_{in}(j) \,, \quad \forall j \,.
\end{aligned} \tag{44}$$

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These are polynomials in λ and, because of the uniformity of the Casoratian, are independent of j. The boundary conditions are given by $\Upsilon_{in}(0) = \Upsilon_{in}$ and $\Upsilon_{out}(\nu) = \Upsilon_{out}$, as given in (17). $\Upsilon_{in}(j)$ is the solution of (36) for the 'in' condition and $\Upsilon_{out}(j)$ that for the 'out' one.

The vanishing of $P(\lambda)$ determines the ν eigenvalues, λ_n, $(n = 0, \ldots, \nu - 1)$. This characteristic polynomial reads, in the extreme cases,

$$P(\lambda) = \Upsilon_{out}^{\dagger}(\nu)\, \Upsilon_{in}(\nu) = \Upsilon_{out}^{\dagger}(0)\, \Upsilon_{in}(0)\,.$$

The λ dependence is contained in $\Upsilon_{in}(\nu)$ or in $\Upsilon_{out}^{\dagger}(0)$.

The full determinant is the normalised $P(0)$,[10]

$$\mathrm{Det} = \frac{P(0)}{\Upsilon_{out}^{\dagger} A \Upsilon_{in}}\,. \tag{45}$$

All this we have had before in particular cases.

To expose the parameter λ, and to enlarge on the formalism, it is helpful to split the driving matrix M as,

$$M(j) = B(j) - \lambda\, A(j)\,, \tag{46}$$

where,

$$B(j) = \begin{pmatrix} 0 & 1 \\ -1 & V(j) + 2 \end{pmatrix}\,, \quad A(j) = A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{47}$$

with

$$\widetilde{B}\, J\, B = J\,, \quad \widetilde{A}\, J\, A = 0\,, \quad \widetilde{A}\, J\, B = -A\,. \tag{48}$$

Then consider two fundamental matrices, $K(\lambda; j)$ and $K(\mu; j)$, and make the usual construction, cf [3],

$$\widetilde{K}(\mu; j + 1)\, J\, K(\lambda; j + 1) - \widetilde{K}(\mu; j)\, J\, K(\lambda; j)$$
$$= \widetilde{K}(\mu; j)\big((\widetilde{B}(j) - \mu\widetilde{A})\, J\, (\widetilde{B}(j) - \lambda A) - J\big) K(\lambda; j)$$
$$= (\lambda - \mu)\widetilde{K}(\mu; j)\, A\, K(\lambda; j)\,.$$

Summing over j from 0 to $\nu - 1$ (i.e. performing the inverse Δ^{-1}), one gets

$$\widetilde{K}(\mu; \nu)\, J\, K(\lambda; \nu) - J = (\lambda - \mu)\sum_{j=0}^{\nu-1} \widetilde{K}(\mu; j)\, A\, K(\lambda; j)\,. \tag{49}$$

---

[10] $P(\lambda)$ is the analogue of an S–matrix element.

14

In this equation, λ and μ are any two parameters. I now restrict them to being eigenvalues, that is, solutions of the polynomial equation $P(\lambda) = 0$, or,

$$\Upsilon^{\dagger}_{out}(j)\Upsilon_{in}(j) = 0 , \quad \forall j ,$$

which implies that the 'out' eigenvector $\Upsilon_{out}(j)$ is the same (up to a constant factor) as the 'in' one, $\Upsilon_{in}(j)$, for each eigenvalue[11] and I can denote both of them by $\Upsilon(j, \lambda)$. In particular, $\Upsilon_{out}(0)$ and $\Upsilon_{in}(\nu)$ are independent of the eigenvalue.

In this case, mutiplying (49) by the boundary (eigenvalue independent) vectors Υ_{in}, on the right, and $\widetilde{\Upsilon}_{out}$ on the left, the left–hand side vanishes,

$$\begin{aligned}
\widetilde{\Upsilon}_{out}\, \widetilde{K}(\mu; \nu)\, J\, K(\lambda; \nu)\, \Upsilon_{in} &- \widetilde{\Upsilon}_{out}\, J\, \Upsilon_{in} \\
&= \widetilde{\Upsilon}_{out}(0, \mu)\, J\, \Upsilon_{in}(\nu, \lambda) - \widetilde{\Upsilon}_{out}(\nu)\, J\, \Upsilon_{in}(0) \\
&= \widetilde{\Upsilon}(0)\, J\, \Upsilon(\nu) - \widetilde{\Upsilon}(\nu)\, J\, \Upsilon(0) \\
&= 0
\end{aligned}$$

and so,

$$(\lambda - \mu) \sum_{j=0}^{\nu-1} \widetilde{\Upsilon}(j, \mu)\, A\, \Upsilon(j, \lambda) = 0 ,$$

with the usual conclusion that eigenvectors with different eigenvalues are orthogonal,

$$\sum_{j=0}^{\nu-1} \widetilde{\Upsilon}(j, \mu)\, A\, \Upsilon(j, \lambda) = 0 , \quad \mu \neq \lambda.$$

In the circumstances of the present paper, the matrix A is the projection onto the lower components of $\Upsilon(j)$, *i.e.* onto $y(j+1)$, and so I have regained the usual orthogonality,

$$\sum_{j=1}^{\nu} y(j, \lambda_n)\, y(j, \lambda_m) = \rho_n \delta_{nm} , \quad 0 \leq n, m \leq \nu - 1 , \tag{50}$$

where ρ_n is a normalisation.

A standard procedure then yields the completeness (or dual orthogonality) relation,

$$\sum_{n=0}^{\nu-1} y(j, \lambda_n)\, y(j', \lambda_n)\rho_n^{-1} = \delta_{jj'} , \quad 1 \leq j, j' \leq \nu .$$

---

[11] Equivalently, the eigenvalues are simple.

The appearance in (50) of a sum over the vertices, j, leads us to the traditional matrix approach, (*e.g.* Rayleigh, [10], Atkinson, [3], Chap.6, Gantmacher and Krein, [24]), which takes the entire set of (dynamic) values, $y(j)$ $(1 \leq j \leq \nu)$ as the components of a column ν–vector, y, and writes the collection of difference equations, (2), as a $\nu \times \nu$ matrix equation of the familiar eigenproblem form,

$$L \, y = \lambda y \, .$$

The polynomials, $y(j, \lambda)$, are then related to the Jacobi determinant $\det (L - \lambda \mathbf{1})$. I extend my present formalism a little to reflect this perspective, which has already shown up in (40) and (41).

It is sometimes convenient to employ the operator formalism as in finite dimensional quantum mechanics, due to Schwinger and Weyl, [25], *cf* Floratos [26], and set *e.g.*,

$$\langle j \, | \, K \, | \, j' \rangle = K(j, j') \, .$$

I retain K as a 2×2 *matrix* in phase space. Then the recurrence is,

$$\Upsilon = M \Upsilon \, ,$$

with M a subdiagonal matrix,

$$\langle j \, | \, M \, | \, j' \rangle = M(j) \, \delta_{j, j'+1} \, ,$$

and the series (40) translates into the simple operator equation,

$$K = \mathbf{1} + M \, K \, ,$$

or, formally,

$$K = \frac{\mathbf{1}}{\mathbf{1} - M} = \frac{\mathbf{1}}{\mathbf{1} - B - \lambda A} \, .$$

The elements (which are matrices) of the powers of M are correctly vertex ordered.

I will not pursue this formulation any further at this time except to say that the stepping matrix has just ones along the subdiagonal and represents the translation operator, often denoted by E in finite difference calculus. The Heaviside matrix, Θ, having $\theta(j, j')$ as elements, is triangular with ones in the left–hand part, and on the diagonal. It is related to E by $E\Theta = \Theta - \mathbf{1}$.

11. The δ potential on the interval

A very basic example of a variable potential is one that is non–zero at only one vertex, *i.e.* $V(j) = v\,\delta_{jk'}$. Then, in the product form, (35), of the propagator, only one term $k = k'$ will be different from the rest. The remaining products (powers) can be dealt with as before, in §6, and an explicit expression found for the transition operator $K(\lambda; \nu)$, say.

As this is just meant for illustrative purposes, I choose a value of k', namely $k' = 2$, that results in a simple formula.

In this case, for Dirichlet conditions with (24), I find the polynomial in λ,

$$
\begin{aligned}
\Upsilon^\dagger_{out} K(\lambda; \nu)\Upsilon_{in} &= U_\nu(1 - \lambda/2) + v(2 - \lambda)U_{\nu-2}(1 - \lambda/2) \\
&= U_\nu(1 - \lambda/2) + v\big(U_{\nu-1}(1 - \lambda/2) + U_{\nu-3}(1 - \lambda/2)\big),
\end{aligned}
\tag{51}
$$

in terms of Chebychev polynomials (the 'unperturbed' functions), (8) with $2\cosh 2\gamma = 2 - \lambda$. The eigenvalues are easily determined numerically and, for a small number of vertex points, even analytically as functions of the strength of the potential.

Equation (51) is proved and explored in Appendix 2 where it is extended to the case when all the v_j are populated.

As particular quantities, the sums of the inverse eigenvalue powers can again be computed by expanding the logarithm of this polynomial, which, to lowest orders is,

$$
\Upsilon^\dagger_{out} K(\lambda; \nu)\Upsilon_{in} = (\nu+1)+2v(\nu-1)-\frac{\nu+1}{6}\Big(\nu(\nu+2)+2v(\nu^2-3\nu+3)\Big)\lambda +\dots
\tag{52}
$$

on using,

$$
U_\nu(\cosh 2\gamma) = (\nu + 1)\left(1 - \frac{1}{6}\nu(\nu + 2)\,\lambda + \dots\right).
$$

One then finds, exact in v,

$$
\sum_{n=0}^{\nu-1}\frac{1}{\lambda_n} = \frac{\nu+1}{6}\frac{\big(\nu(\nu+2)+2v(\nu^2-3\nu+3)\big)}{\nu+1+2v(\nu-1)},
$$

which generalises (11).

This identity is an example of a general class of identities discussed in the interesting work by Annaby and Asharabi, [27], where other references can be found.

The determinant is just the constant term in the polynomial, (52),

$$
\mathrm{Det}_D = \nu + 1 + 2v(\nu - 1),
$$

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and a zero mode, $\lambda_0 = 0$, occurs when $v = -(\nu + 1)/2(\nu - 1)$. When v takes the same value with the opposite sign, the final eigenvalue, $\lambda_{\nu-1}$, equals 4.

I remark that perturbation theory on the lattice has been considered by Actor *et al*, [4].

12. The vacuum energy

A rather different technical eigenvalue problem is the calculation of the Casimir energy and I present a quick treatment as a simple and explicit use of the eigenvalues.

The Dirichlet vacuum energy of a free scalar field on $T \times I_\nu$ can be evaluated in closed form as,

$$
\begin{aligned}
E_D &\equiv \frac{1}{2} \sum_{\overline{\lambda}} \overline{\lambda}^{1/2} \\
&= \frac{1}{h} \sum_{n=1}^{\nu} \sin \frac{\pi n}{2(\nu + 1)} \\
&= \frac{1}{h} \left(\cot \frac{\pi}{4(\nu + 1)} - 1 \right) \\
&= \frac{2L}{\pi h^2} - \frac{1}{2h} - \frac{\pi}{24L} + \dots, \quad h \to 0.
\end{aligned}
\tag{53}
$$

If one views the lattice calculation as a regularisation of the continuum one, the $-\pi/24L$ term is recognised as the value given by the ζ–function technique while the first two, ultimately divergent terms, being non–universal, dependent on the regularisation, should be discarded in some way, if one is concerned just with the interval, $[0, L]$ on its own.

The paper [4] contains a full discussion of the expression (53) and I will not enter into any more details. This reference also contains other arrangements, including a discrete version of the Casimir piston.

The Neumann energy is, likewise,

$$
\begin{aligned}
E_N &= \frac{1}{h} \sum_{n=1}^{\nu-1} \sin \frac{\pi n}{2\nu} \\
&= \frac{1}{h} \left(\cot \frac{\pi}{4\nu} - 1 \right) \\
&= \frac{2L}{\pi h^2} - \frac{2}{\pi h} - \frac{1}{2h} - \frac{\pi}{24L} + \dots, \quad h \to 0.
\end{aligned}
\tag{54}
$$

The other boundary condition usually considered is the periodic one. This is given in Fort, [12], Chap.XV. It is convenient, this time, to arrange the ν points,

$0 \leq j \leq \nu - 1$ on the unit circle and impose the periodicity conditions $y(\nu) = y(0)$, $y(-1) = y(\nu - 1)$ which relate values outside the proper range of j to those inside.

The analysis is slightly different depending on whether ν is even, $\nu = 2k + 2$, or odd, $\nu = 2k + 1$. In both cases there are degenerate modes, $\cos(2n\pi j/\nu)$ and $\sin(2n\pi j/\nu)$, for $0 \leq n \leq k$ with eigenvalues,

$$\overline{\lambda} = \frac{4}{h^2} \sin^2 \frac{\pi n}{\nu},$$

where the gap, $h = 2\pi/\nu$. The value $n = 0$ gives the one uniform zero mode. If ν is even, the single mode $\cos \pi j$ must also be added. (This alternates between plus and minus one as the points around the circle are traversed and corresponds to a wave of infinite frequency in the continuum limit.) The total number of modes is always ν.

In exactly the same way as above, the vacuum energies are,

$$E_{2k+2} = \frac{2k+2}{\pi} \left(\sum_{n=1}^{k} \sin \frac{\pi n}{2k+2} + \frac{1}{2} \right)$$

$$= \frac{2k+2}{2\pi} \cot \frac{\pi}{2(2k+2)}$$

$$E_{2k+1} = \frac{2k+1}{\pi} \sum_{n=1}^{k} \sin \frac{\pi n}{2k+1}$$

$$= \frac{2k+1}{2\pi} \cot \frac{\pi}{2(2k+1)}$$

or

$$E_P = \frac{1}{h} \cot \frac{h}{4} \to \frac{4}{h^2} - \frac{1}{12} + \dots, \tag{55}$$

in both cases, as expected. Again, one sees the continuum zeta value of $\zeta_R(-1/2) = -1/12$ appearing as h tends to zero.[12]

Fort, [12], also discusses anti–periodic (real) functions. However I will be a little more general and analyse a system that, in the continuous limit, amounts to an Aharonov-Bohm flux running through the circle. This is mimicked by imposing a phase change on circulating the flux and leaving the equations of motion unchanged.

---

[12] There is a puzzle here. In the continuous case the periodic modes on a circle are the union of Dirichlet and Neumann modes on an interval of size *half* the circumference. One might, therefore, expect to see evidence of this, even in the discrete case, as $h \to 0$. In fact this works for the terms of order h^{-2} and h^0 in (53), (54) and (55) but not for those of order h^{-1}. In order for it to work, the relevant term in (54) should read just $1/2h$ to cancel that in (53), on addition, to give (55) but I could not achieve this.

In the quantum case, the wave function is complex and exponential functions are very convenient.[13] I therefore consider a function, $\psi(j)$, defined on the points, j, and satisfying the twisted periodicity condition,

$$\psi(\nu) = e^{2\pi i \alpha}\psi(0)\,, \quad \psi(\nu - 1) = e^{2\pi i \alpha}\psi(-1)\,. \tag{56}$$

The modes on the discrete circle are,

$$\psi_n^\alpha(j) = e^{2\pi i(n+\alpha)j/\nu}\,, \quad n = 0, \ldots, \nu - 1\,, \quad 0 < \alpha \leq 1\,,$$

with corresponding eigenvalues $(h = 2\pi/\nu)$,

$$\overline{\lambda} = \frac{4}{h^2} \sin^2 \frac{\pi(n+\alpha)}{\nu}\,, \tag{57}$$

and vacuum energy (with a factor of two from the complexification),

$$\begin{aligned}
E(\alpha) &= \frac{2}{h} \sum_{n=0}^{\nu-1} \sin \frac{\pi(n+\alpha)}{\nu} \\
&= \frac{2}{h} \operatorname{cosec} \frac{h}{4} \; \cos \frac{h}{4}(2\alpha - 1) \\
&= \frac{8}{h^2} - \left(\frac{1}{6} - \alpha + \alpha^2\right) + \ldots\,, \quad h \to 0\,.
\end{aligned} \tag{58}$$

$E(\alpha)$ must be extended beyond $\alpha = 1$ using periodicity, *i.e.* $E(1 + \alpha) = E(\alpha)$.

The constant term agrees with the result (the periodic Bernoulli polynomial, $\widetilde{B}_2$) that arises in the continuous circle limit, [28]. When $\alpha = 0$ one regains twice the real periodic value (55). It might be of interest to give the full formal expansion,

$$E(\alpha) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \widetilde{B}_{2m}(\alpha) \left(\frac{h}{2}\right)^{2m-2}\,,$$

which I have not seen elsewhere.

---

[13] It is, of course, possible to retain a real description by doubling up the fibre to an SO(2) one.

13. Direct determination of determinants

It is helpful to have specific values for comparison or limit purposes and I proceed to evaluate the determinants of the free systems directly from the eigenvalues which have just been used. I also look at the continuum limit and some zeta regularised values.

An organisational point is perhaps required. In deriving the answers, I have used known expressions for some infinite products (*e.g.* Bromwich, [14]), which could be obtained, according to the results of the previous sections, *from* the determinants and so these evaluations might be considered superfluous, at one level. However I present them as independent checks. I also include the twisted periodic values.

To repeat, Dalembert's equation is,

$$-\frac{1}{h^2}\nabla\Delta y(j) + \overline{\mu}^2 y(j) - \overline{\lambda}\, y(j) = 0\,. \tag{59}$$

For the interval, I will again use the real form of the eigenfunctions and the calculation is immediate.

The D–determinant on the L–interval using the eigenvalues (10) is,

$$\begin{aligned}
\mathrm{Det}_D(\overline{\mu}) &= \left(\frac{2}{h}\right)^{2\nu}\prod_{n=1}^{\nu}\left(\sin^2\frac{\pi n}{2(\nu+1)} + \frac{1}{4}\mu^2\right), \quad \mu = h\overline{\mu}\\
&= \left(\frac{2}{h}\right)^{2\nu}\left[\prod_{n=1}^{2\nu+1}{}'\left(\sin^2\frac{\pi n}{2(\nu+1)} + \frac{1}{4}\mu^2\right)\right]^{1/2} \tag{60}\\
&= \frac{1}{h^{2\nu}}\frac{\sinh(\nu+1)2\gamma}{\sinh 2\gamma}\,.
\end{aligned}$$

The dash on the second product means that the $n = \nu+1$ term is to be excluded and I have set $\mu = 2\sinh\gamma$.

The massless values are,

$$\begin{aligned}
\mathrm{Det}_D(0) &= \frac{1}{h^{2\nu}}\,(\nu+1)\\
&= \frac{1}{h^{2\nu+1}}\frac{1}{2}\,2L\,.
\end{aligned}$$

The N–determinant is, using (30),

$$\begin{aligned}
\mathrm{Det}_N(\overline{\mu}) &= \left(\frac{2}{h}\right)^{2\nu}\prod_{n=0}^{\nu-1}\left(\sin^2\frac{\pi n}{2\nu} + \frac{1}{4}\mu^2\right)\\
&= \left(\frac{2}{h}\right)^{2\nu}\mu^2\left[\prod_{n=1}^{2\nu-1}{}'\left(\sin^2\frac{\pi n}{2\nu} + \frac{1}{4}\mu^2\right)\right]^{1/2} \tag{61}\\
&= \frac{1}{h^{2\nu}}2\tanh\gamma\sinh 2\gamma\nu\,.
\end{aligned}$$

In the massless limit the determinant vanishes and it is conventional to remove the offending zero mode giving the modified determinant,

$$\mathrm{Det}'_N(0) = \frac{1}{h^{2\nu-2}} \, \nu = \frac{1}{h^{2\nu-2}} \frac{\nu}{L} \frac{1}{2} \, 2L$$

$$\rightarrow \frac{1}{h^{2\nu-1}} \frac{1}{2} \, 2L \,, \quad h \rightarrow 0 \,.$$

Before discussing these results, I give the twisted periodic expressions.

From (57) $(h = 2\pi/\nu)$,

$$\mathrm{Det}_P^{1/2}(\alpha, \overline{\mu}) = \left(\frac{2}{h}\right)^{2\nu} \prod_{n=0}^{\nu-1} \left(\sin^2 \frac{\pi(n+\alpha)}{\nu} + \frac{1}{4}\mu^2\right)$$

$$= \frac{2}{h^{2\nu}} \left(\cosh 2\gamma\nu - \cos 2\pi\alpha\right) \tag{62}$$

$$\mathrm{Det}_P^{1/2}(\alpha, 0) = \frac{1}{h^{2\nu}} \, 4\sin^2 \pi\alpha \,, \quad 0 \le \alpha \le 1 \,,$$

$$\mathrm{Det}'_P(0, 0) = \frac{1}{h^{2\nu+2}} \, 4\,(2\pi)^2 \,, \quad \alpha = 0 \,.$$

For the $\alpha = 0$ case, I have removed the complexification squaring and, for extra generality, I have included the mass term.

For comparison, some determinants, computed from the bare ζ–function regularisation, are well known to be,

$$\mathrm{Det}_{\zeta D}(\overline{\mu}) = 2\frac{\sinh \overline{\mu} L}{\overline{\mu}}$$

$$\mathrm{Det}'_{\zeta N}(0) = 2L$$

$$\mathrm{Det}_{\zeta P}^{1/2}(\alpha, 0) = 4\sin^2 \pi\alpha$$

$$\mathrm{Det}'_{\zeta P}(0, 0) = (2\pi)^2 \,,$$

and one sees that the lattice determinants are proportional to the zeta values, in the continuous limit. In particular,

$$\mathrm{Det}_D(\overline{\mu}) = \frac{1}{h^{2\nu}} \frac{\sinh(\nu+1)2\gamma}{\sinh 2\gamma}$$

$$\rightarrow \frac{1}{h^{2\nu+1}} \frac{\sinh \overline{\mu} L}{\overline{\mu}} \,, \quad h \rightarrow 0$$

$$= \frac{1}{h^{2\nu+1}} \frac{1}{2} \, \mathrm{Det}_{\zeta D}(\overline{\mu}) \,.$$

Forman introduces a twisted periodic condition, denoted B_δ in [9], which has a complex multiplying factor δ instead of $e^{2\pi i\alpha}$ in (56) and Theorem 2.6 gives its

determinant. Evaluating [9] equation, (2.23), in the free case I find,

$$\mathrm{Det}_{B_\delta} = -\frac{1}{h^{2\nu}} \frac{\delta(1-\delta)^2}{(1+|\delta|^2)},$$

which vanishes when $\delta = 1$, as a check, but does not agree, apart from the lattice scaling factor, with (62) when $\delta = e^{2\pi i \alpha}$.[14]

14. Conclusion

A good deal of this paper is expository but there are some novelties. It has been emphasised that the Gel'fand–Yaglom formula for the determinant in the discrete case is a standard component of Sturm–Liouville and orthogonal polynomial theory. I have rewritten this in a neat 2×2 symplectic matrix formulation, slightly different from the usual one, and have calculated the determinant for Robin boundary conditions for a constant potential, as a basic example. The continuum limits have been discussed in a simple–minded way and comparisons made with the work of Forman, [9], revealing some minor discrepancies. For Dirichlet conditions, the determinant for a δ potential was evaluated exactly, highlighting the significance of Chebychev polynomials as 'unperturbed' Sturm–Liouville solutions which is further explored in the Appendices and which contain some technical advances.

The calculations could be broadened to include the general Sturm–Liouville operator, and higher order equations.

Added note

The Green functions for free propagation on the discrete interval (or 'path'), for various boundary conditions, have been obtained in terms of Chebyshev polynomials by Bass, [29], Chung and Yau, [30] and Bendito, Encinas and Carmona, [31].

---

[14] Curiously, if one of the δs is replaced by δ^{-1}, then agreement is found, apart from a factor of two, which seems too much of a coincidence.

Appendix 1. Chebyshev Polynomials

I give some basic results for Chebyshev polynomials in a way that reflects the procedures of this paper. These polynomials occur in the Chebyshev–Gauss scheme for mechanical quadratures. Probably the most economical way of defining them is through the recursion,

$$P_{n+1}(x) - 2x P_n(x) + P_{n-1}(x) = 0 \,, \tag{63}$$

or

$$\nabla \Delta P_n(x) = -(2 - 2x) P_n(x) \,, \tag{64}$$

the different kinds being selected by the 'initial' values, P_0 and P_1, $e.g.$ [21].

As a slight novelty, I use the matrix description adopted in the main body of this paper. So, introducing the two–vector,

$$\Pi_{n+1}(x) \equiv \begin{pmatrix} P_n(x) \\ P_{n+1}(x) \end{pmatrix} \,,$$

the three–term recursion (63) becomes the two–term matrix one,

$$\Pi_n(x) = C(x)\, \Pi_{n-1}(x) \,,$$

$i.e.$

$$\Pi_n(x) = C^n(x)\, \Pi_0(x) \,,$$

where[15]

$$C(x) = \begin{pmatrix} 0 & 1 \\ -1 & 2x \end{pmatrix} \,. \tag{65}$$

This allows me to specify the initial conditions neatly. For example, the 'Dirichlet' vector,

$$\Pi_0^D(x) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,,$$

produces Chebyshev polynomials of the second kind, $P_n(x) = U_n(x)$ while the Neumann one,

$$\Pi_0^N(x) \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,,$$

gives third kind polynomials, $P_n(x) = V_n(x)$, [21], [32].

---

[15] C is what I call M in the main body of this paper.

Many relations between the various kinds can be obtained by combining initial conditions.

The first iteration of the initial vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ *i.e.* minus the Dirichlet one and therefore yields $-U_{n-1}$ after n iterations. So, writing $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, one obtains the relation $V_n = U_n - U_{n-1} = \Delta U_{n-1}$. From which, for example, $\Delta V_n = \nabla \Delta U_n$, helpfully relating Neumann and Dirichlet.

The Robin choice,

$$\Pi_0^R(x) \equiv \begin{pmatrix} 1 \\ 1 + \alpha \end{pmatrix},$$

likewise generates the D–N combination $V_n + \alpha U_n = (1 + \alpha)U_n - U_{n-1}$.

The different polynomial combinations are encapsulated in the form of the power, C^n,

$$C^n = \begin{pmatrix} -U_{n-2} & U_{n-1} \\ -U_{n-1} & U_n \end{pmatrix}, \tag{66}$$

($U_{-1} = 0$) which can be written,

$$C^n = U_n \, A + U_{n-1} \, J + U_{n-2} \, A' , \tag{67}$$

with $A' \equiv A - \mathbf{1}$. From (66), by taking the determinant, one finds the Chebyshev identity,

$$U_{n-1}^2 - U_n \, U_{n-2} = 1 > 0 ,$$

which is a (known) statement of a Turán inequality.

From the $\mathbb{Z}$ group composition rule, $C^m \, C^n = C^{m+n}$, the combination relation,

$$U_{m+n} = U_m \, U_n - U_{m-1} \, U_{n-1} ,$$

can be deduced and the SU(2) character Clebsch–Gordan series,

$$U_m \, U_n = \sum_{k=|m-n|}^{m+n} U_k ,$$

readily follows therefrom by iteration. All these relations have trigonometric derivations.

The generating function can also be transcribed as follows. The matrix equation,

$$(\mathbf{1} - C\,t)^{-1} = \begin{pmatrix} 1 & -t \\ t & 1 - 2tx \end{pmatrix}^{-1}$$

$$= \frac{1}{1 - 2tx + t^2} \begin{pmatrix} 1 - 2tx & t \\ -t & 1 \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} C^n\,t^n\,,$$

and (66), yield the normal generating function for U_n. (Incidentally, to check the top left entry, one has to use $U_{-2} = -1$, $U_{-1} = 0$.) Other identities can be deduced in a similar fashion.

Appendix 2. Calculation of matrix elements

I give some details of the computation of the matrix element polynomial, $P(\lambda) = \Upsilon_{out}^{\dagger} K(\lambda; \nu) \Upsilon_{in}$, whose vanishing determines the eigenvalues.

I write, in dyadic form,

$$P(\lambda) = \mathrm{Tr}\left(K\,\Upsilon_{in}\otimes\Upsilon_{out}^{\dagger}\right) \equiv \mathrm{Tr}\left(K\,Q\right), \tag{68}$$

where the matrix Q takes the specific forms in the D and N cases,

$$Q_D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\,, \quad Q_N = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\,.$$

This is a trace formula for the Gel'fand–Yaglom function. Other expressions are determinant ones, [3], p.95, and also, *e.g.*, [9], [7].

The general form of $K(\lambda; \nu)$ is a product of ν matrices, (35), but as a first example I treat the case discussed in §11 where all matrices are identical, except one. I use the notation of the previous Appendix and write K as,

$$K(\lambda; \nu) = C^m(x)\,C(y)\,C^n(x)\,, \quad m + n + 1 = \nu\,,$$

where $C(x)$ and $C(y)$ are given by (65) with $2x = 2 - \lambda$ and $2y = v + 2 - \lambda$. Then,

$$K = C^m(x)\,C(y)\,C^n(x) = C^{\nu}(x) + C^m(x)\left(C(y) - C(x)\right)C^n(x)$$
$$= C^{\nu}(x) + v\,C^m(x)\,A\,C^n(x)$$

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and hence for (68) in the case of D conditions,

$$
\begin{aligned}
P_D(\lambda) &= \mathrm{Tr}\left(C^\nu(x)\,A\right) + v\mathrm{Tr}\left(C^m(x)\,A\,C^n(x)A\right) \\
&= U_\nu + vU_n\,U_m \\
&= U_\nu + v\big(U_{n+m} + U_{n-1}\,U_{m-1}\big) \\
&= U_\nu + v\big(U_{\nu-1} + U_{n-1}\,U_{\nu-n-2}\big),
\end{aligned}
\tag{69}
$$

where I have used (66). Setting $n = 1$ gives the result (51).

Equation (69) gives the first terms of a perturbation expansion, to elucidate the general nature of which, in a direct way, I consider the case when there are two distinguished matrices in the product (35), and a more systematic notation is required.

The vertices of the interval have been labelled by j which runs from 0 to $\nu + 1$ with the vertices 1 to ν being dynamic.[16] In this range, let the vertices j_1 and j_2 be singled out to correspond to matrices $C(y_{j_1})$ and $C(y_{j_2})$. Then, taking $j_1 > j_2$, and noting that $C(y_{j_i}) = C + v_{j_i}\,A$, with $C \equiv C(x)$, I find,

$$
\begin{aligned}
K(\lambda;\nu)A &= C^{\nu-j_1}\,C(y_{j_1})\,C^{j_1-j_2-1}C(y_{j_2})\,C^{j_2-1}A \\
&= C^{\nu-j_1}\,(C + v_{j_1}A)\,C^{j_1-j_2-1}(C + v_{j_2})\,C^{j_2-1}A \\
&= C^\nu\,A + v_{j_1}\,C^{\nu-j_1}\,A\,C^{j_1-1}\,A + v_{j_2}\,C^{\nu-j_2}\,A\,C^{j_2-1}A \\
&\quad + v_{j_1}v_{j_2}\,C^{\nu-j_1}\,A\,C^{j_1-j_2-1}\,A\,C^{j_2-1}A.
\end{aligned}
$$

I have added the post factor of A to bring out the fact that it is the combination $C^n\,A$ that enters, and this takes the form,

$$
C^n\,A = \begin{pmatrix} 0 & U_{n-1} \\ 0 & U_n \end{pmatrix},
$$

which is preserved under multiplication,

$$
C^n\,A\,C^m A = \begin{pmatrix} 0 & U_m\,U_{n-1} \\ 0 & U_m\,U_n \end{pmatrix}.
$$

The final act of taking the trace picks out just the lower right corner term and so,

$$
\begin{aligned}
\mathrm{Tr}\left(K(\lambda;\nu)A\right) &= U_\nu + U_{\nu-j_1}\,v_{j_1}\,U_{j_1-1} + U_{\nu-j_2}\,v_{j_2}\,U_{j_2-1} \\
&\quad + U_{\nu-j_1}\,v_{j_1}\,U_{j_1-j_2-1}\,v_{j_2}\,U_{j_2-1}.
\end{aligned}
\tag{70}
$$

---

[16] In graph theory language these would be internal vertices.

The structure when more vertices are marked is clear and the general expression is,

$$
\mathrm{Tr}\left(K(\lambda;\nu)A\right) = U_\nu + \sum_{j_1=1}^{\nu} U_{\nu-j_1}\, v_{j_1}\, U_{j_1-1} + \sum_{j_1>j_2=1}^{\nu} U_{\nu-j_1}\, v_{j_1}\, U_{j_1-j_2-1}\, v_{j_2}\, U_{j_2-1}
$$
$$
+ \sum_{j_1>j_2>j_3=1}^{\nu} U_{\nu-j_1}\, v_{j_1}\, U_{j_1-j_2-1}\, v_{j_2}\, U_{j_2-j_3-1}\, v_{j_3}\, U_{j_3-1}
$$
$$
+ \ldots
$$
$$
+ v_1 v_2 \ldots v_\nu\,.
\tag{71}
$$

This equation also follows (equivalently) from the complete iteration of the conventional–looking (43), which I rewrite here, in the notation of these Appendices,

$$
K(\lambda;j) = C^j(x) + \sum_{j_1=1}^{\nu} C^{j-j_1}(x)\, A\, v_{j_1}\, K(\lambda;j_1-1,0)\,, \quad 2x = 2 - \lambda.
\tag{72}
$$

This can be given the usual propagation interpretation of a perturbation series (although finite and exact) with the Chebyshev polynomials acting as free propagators. An obvious graphical representation can be set up.

Setting λ to zero in (71) gives the determinant,

$$
\mathrm{Det}_D = \nu + 1 + \sum_{j_1=1}^{\nu} (\nu - j_1 + 1)\, j_1\, v_{j_1} + \sum_{j_1=1}^{\nu}\sum_{j_2=1}^{j_1} (\nu - j_1 + 1)\,(j_1 - j_2)\, j_2\, v_{j_1}\, v_{j_2}
$$
$$
+ \sum_{j_1=1}^{\nu}\sum_{j_2=1}^{j_1}\sum_{j_3=1}^{j_2} (\nu - j_1 + 1)\,(j_1 - j_2)\,(j_2 - j_3)\, j_3\, v_{j_1}\, v_{j_2}\, v_{j_3}
$$
$$
+ \ldots
$$
$$
+ v_1 v_2 \ldots v_\nu\,.
\tag{73}
$$

The upper limits have been extended to the diagonal values using the vanishing of the summands there, but I make no use of this at the present time.

As a numerical illustration, the characteristic polynomial when $\nu = 3$ is,

$$
P(\lambda) = -\lambda^3 + \lambda^2(6 + S_1) - \lambda\big(10 + 4S_1 + S_2\big) + 4 + v_2 + 3S_1 + 2S_2 + S_3\,,
$$

where

$$
S_1 = v_1 + v_2 + v_3\,, \quad S_2 = v_1 v_2 + v_1 v_3 + v_2 v_3\,, \quad S_3 = v_1 v_2 v_3\,.
$$

It is interesting to note that, for a symmetric potential ($v_1 = v_3$), $P(\lambda)$ has the linear factor, $(\lambda - v_1 - 2)$, which is related, presumably, to Borg's reconstruction theorem.

Neumann boundary conditions can be handled in a like manner. The required polynomial turns out to be

$$
\begin{aligned}
\text{Tr}\left(K(\lambda;\nu)\,Q_N\right) = \Delta V_{\nu-1} + \sum_{j_1=1}^{\nu} V_{\nu-j_1}\,v_{j_1}\,V_{j_1-1} \\
+ \sum_{j_1>j_2=1}^{\nu} V_{\nu-j_1}\,v_{j_1}\,U_{j_1-j_2-1}\,v_{j_2}\,V_{j_2-1} \\
+ \sum_{j_1>j_2>j_3=1}^{\nu} V_{\nu-j_1}\,v_{j_1}\,U_{j_1-j_2-1}\,v_{j_2}\,U_{j_2-j_3-1}\,v_{j_3}\,V_{j_3-1} \\
+ \ldots\ldots + (v_1 v_2 \ldots v_\nu)\,.
\end{aligned}
\tag{74}
$$

The end point propagators are third kind polynomials, V_n, while internal ones are second kind, U_n. The determinant is

$$
\begin{aligned}
\text{Det}_N = \sum_{j_1=1}^{\nu} v_{j_1} + \sum_{j_1>j_2=1}^{\nu} v_{j_1}\,(j_1-j_2)\,v_{j_2} \\
+ \sum_{j_1>j_2>j_3=1}^{\nu} v_{j_1}\,(j_1-j_2)\,v_{j_2}\,(j_2-j_3)\,v_{j_3} + \ldots\ldots + (v_1 v_2 \ldots v_\nu)\,,
\end{aligned}
\tag{75}
$$

since $V_n(x)$ is unity when $\lambda = 0$. Det_N correctly vanishes when all the v_j do due to the resulting zero mode.

These expressions can be used to discuss a Borg–Levinson inverse theorem (*e.g.* Hald, [33]) and Dikii trace identities, but such further analysis and manipulations must be postponed.

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