Wigner-PDC description of photon entanglement as a local-realistic theory

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(Revised, additions: existence of $f(), \Gamma()$ now guaranteed for more stringent restrictions)

The Wigner picture of Parametric Down Conversion (works by Casado et al) can be interpreted as a local-realistic formalism, without the need to depart from quantum mechanical predictions at any step, at least for the relevant subset of QED-states. This involves reinterpreting the expressions for the detection probabilities, by means of an additional mathematical manipulation; though such manipulation seemingly provides enough freedom to guarantee consistency with the expectable, experimentally testable behavior of detectors, this is, in any case, irrelevant in relation to our main result, of a purely mathematical nature. We also include an overview of the consequences of this framework in relation to: (i) typical Bell experiments; (ii) perhaps the most relevant recent related test (Phys.Rev.Lett. 108, 2012). Additionally, we also propose an interpretation on that apparently awkward "subtraction" of the average ZPF intensity at the detection process.

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Previous comments

Before anything else, the tests by Brida et al [17] do not disprove the core of the Wigner-PDC approach as developed in [2–8]; it only does so with some detection models proposed to be able to interpret the expressions in consistency with local-realism (LR). Precisely what we prove here is that such (local-realist) interpretation does not need to depart from QM at any step (at least for the relevant subset of quantum mechanical states), a property that former proposals like [16] did not satisfy (not to mention other problems of their own).

This said, neither is ours here the ultimate proof that everything in regard to the Wigner-PDC approach works perfectly fine. The one and only value of this work is to show that, even still in need to determine further choices both at the physical and purely mathematical levels, there is room to accommodate all necessary restrictions. Once more, without any need to depart from quantum mechanical predictions, provided we stay inside the subset of QED-states compatible with LR. Further extensions of this formalism, giving rise to new predictions such as the "Spontaneous Parametric Up Conversion" proposed in [11] (and apparently also disproved by experimental work in [18], only to regain credit again following some recent reports from two different experimental groups [51]) are irrelevant here.

I am still convinced that no conclusive proof of the violation of local-realism (LR) has ever been obtained: we have the absence of proper space-like separation or "locality loophole" in experiments with massive particles, and the so-called "detection loophole" in experiments with photons, and something related to this last in the latest important development ([34], see Secs.V).

Each of these "loopholes" places the corresponding experimental observations within the frontiers of LR, and once there nothing is happening that cannot (and should not try to) be understood from a purely classical framework, or at least from an intermediate one such as the one we deal with here. I use "intermediate" in the sense that what we have is a quantum formalism, but we are also incorporating, in an implicit way by considering the vacuum state which admits a well defined joint probability density as follows from the positivity of the Wigner function, the limits imposed by LR.

Indeed, those latest developments in [34] are, in a way, even more compelling from the point of view of my claims than the usual Bell experiments are: while these last are usually interpreted as proof of the "non-local" nature of QM, which is a way out to respect pure realism ("something is there even if we do not look at it"), results in [34] would suggest the non-existence of a well defined joint probability density for the results of a set of measurements performed locally upon a system, which is also that (the non-existence) of realism itself.

After all, non-localities, such as action-at-a-distance, are also present in other branches of physics like for instance electromagnetism, but then always reconciled with LR once the full problem is considered (in regard to e.m., the apparition of retarded potentials), and at least showing reasonable properties such a decrease with growing distance between the parties. In view of [34], it is not any longer a matter of just some instantaneous interaction that does not fade away with distance, but something clearly more fundamental: "things are not there until we look at them".

The widespread willingness to accept such a view of physical reality before exhausting other alternatives is difficult to understand: those alternatives have been proposed [2–8], including basic elements (a random background of vacuum field fluctuations) that do not look at all alien from the perspective of classical electromag-

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netism, and that even some recent works within the orthodox approach to the field now acknowledge [35]. Besides, and to my knowledge, [34] is the only amongst other recent related tests [36, 37] which does not acknowledge, explicitly, one or other loophole leading to compatibility with the local-realistic interpretation.

In arxiv:1111.4092 I have already commented upon other recent developments; to the best of my knowledge the situation remains the same as when this was written: there is no loophole-free evidence of non-locality.

But moreover: even if the locality loophole was overcome, in order to be convincing violations (of one or other inequalities) should be high enough to exclude other loopholes (local coincidences, see arXiv:1111.4092) or possible systematic errors. If it is actually true that local-realism (LR) does not impose any limit on the quantum mechanical states that can be prepared in a lab, then there is no reason to expect any special "resistance" of detection rates to go over the critical values (which is a determined by LR), neither for technological limitations (detectors) to become so important at precisely that region of the spectrum (detection rates could saturate at any other point well beyond the LR-frontier, and the issue would be more than clear by now).

Perhaps it is time to consider the possibility that not all states allowed by QM may have a physical counterpart, in particular those that would yield correlations defying LR. Perhaps is also time to start giving credit to models that, even though being quantum-mechanical, may include that restriction built within their formulation; here I propose a step in this direction.

I. INTRODUCTION

The Wigner picture of Quantum Optics of photonentanglement generated by Parametric Down Conversion [1] was developed some years ago in a series of papers [2– 8]; recently, the approach has been revitalized producing another stream of very interesting results [9, 10]. Starting from an stochastic electrodynamical description (hence, based on continuous variables: electromagnetic fields defined at each and every point of space), by use of the so-called Wigner transformation [12] this model acquires a form where all expectation values depend on a probability distribution (hence one objectively defined) for the value of those fields in the vacuum.

Such a picture clearly differs from the usual one in Quantum Information (QInf), based on a discrete description of the photon (a particle) and not on a set of fields in a continuum space. For instance, in the Wigner-PDC picture empty polarization channels at either of the exits of a polarizing beam splitter (PBS) become filled with random components of the vacuum Zero-Point field (ZPF); these random components can for instance give rise to the enhancement of the detection probability for certain realizations of the state of the fields (certain photons, in the QED language): see note [13]. Phenomena

such as this last come as a big difference with the customary model of the set polarizer-detector is treated just as a "black box", able to extract polarization information in principle without the (explicit) intervention of any additional noise.

The Wigner-PDC framework provides an alternative, apparently local-realistic explanation of the results of many typically quantum experiments, the result of a measurement depending on (and only on) the set of hidden variables (HV) inside its light cone: in this case, both the signal propagating from the source and the additional noise introduced by the ZPF at intermediate devices and detectors. Within the first series of papers [2–8] those experiments included: frustrated two photon creation via interference [2]; induced coherence and indistinguishableness in two-photon interference [2, 3]; Rarity and Tapster's 1990 experiment with phase-momentum entanglement [3]; Franson's (original, 1989) experiment [3]; quantum dispersion cancellation and Kwiat, Steinberg and Chiao's quantum eraser [4].

From the most recent one [9, 10] we can also add, on one side, amongst quantum cryptography experiments based on PDC: two-qubit entanglement and cryptography [9] and quantum key distribution and eavesdropping [9]; on the other, an interpretation for the experimental partial measurement of the Bell states generated from a single degree of freedom (polarization) in [10]. Recently there had been other promising developments, which I do not know in detail though [14].

Perhaps needless to say, this "local realistic" picture of the quantum experiments takes place within the limits (detection efficiencies) where it is already well acknowledged that such an explanation may exist; moreover, those limits arise as natural consequences of the theory: they are simply limiting values of detection rates for states of light that are ultimately generated from the vacuum. This last feature is what makes the Wigner approach so interesting; on the other hand, above we have used "apparently" because that local-realistic interpretation, even within the corresponding limits of detection rates, was until now not devoid of difficulties, in particular related to the detection model (so far merely a one-toone counterpart with Glauber's original expressions [15]): as a result of the normal order of operators, there average intensity due to vacuum fluctuations is "subtracted", a subtraction that seems to introduce problems related to the appearance of what could be interpreted as "negative probabilities".

That last is the issue that interests us here, one that, as to be expected, did motivate the proposal of several modifications upon the expressions for the detection probabilities: for instance, as early as in [5], "our theory is also in almost perfect one-to-one correspondence with the standard Hilbert-space theory, the only difference being the modification in the detection probability that we proposed in relation...".

Such modifications ranged from the mere inclusion of temporal and spatial integration [5] to the proposal of much more complicated functional dependencies [8, 16, 19], all of them seeming to pose their own problems; for instance in [16], a departure from the quantum predictions at low or high intensities, experimentally disproved for instance in [17].

Our route here is a different one however: we do not propose any modification of the initial expressions for the detection probabilities (in one-to-one correspondence with the initial quantum electrodynamical model), but just explore the possibility of performing some convenient mathematical manipulation that casts them in a form consistent with the axioms of probability, hence one consistent with local-realism. Following for instance [5] (see also [15]), quantum mechanical detection probabilities, single and joint, can respectively be expressed as

$$P_{i} \propto \langle I_{i} - I_{0,i} \rangle$$

$$= \int_{\alpha,\alpha^{*}} (I_{i}(\alpha,\alpha^{*}) - I_{0,i}) W(\alpha,\alpha^{*}) d\alpha d\alpha^{*}, \quad (1)$$

$$P_{i,j} \propto \int_{\alpha,\alpha^*} (I_i(\alpha,\alpha^*) - I_{0,i}) \cdot (I_j(\alpha,\alpha^*) - I_{0,j}) \times W(\alpha,\alpha^*) \, d\alpha d\alpha^*,$$
 (2)

where α, α^* are vacuum amplitudes of the (relevant set of) frequency modes [20] at the entrance of the crystal, $W(\alpha, \alpha^*)$ is obtained as the Wigner transform of the vacuum state, $I_i(\alpha, \alpha^*)$ is the field intensity (for that mode) and $I_{0,i}$ the mean intensity due to the vacuum amplitudes, both at the entrance of the *i*-th detector (see [20]):

$$I_0 = \int_{\alpha \, \alpha^*} I_0(\alpha, \alpha^*) \, W(\alpha, \alpha^*) \, d\alpha d\alpha^*. \tag{3}$$

The last three expressions would in principle allow us to identify the vacuum amplitudes with a vector of hidden variables $\lambda \in \Lambda$ (Λ is the space of events or probabilistic space), with an associated density function $\rho(\lambda)$,

$$\lambda \equiv \alpha, \alpha^*, \tag{4}$$

$$\rho(\lambda) \equiv W(\alpha, \alpha^*). \tag{5}$$

Now, for instance in [5] it is already acknowledged that (1)–(2) cannot, because of the possible negativity of the difference $I_i(\alpha, \alpha^*) - I_{0,i}$, be written as

$$P_i = \int_{\Lambda} P_i(\det|\lambda) \ \rho(\lambda) \ d\lambda, \tag{6}$$

$$P_{i,j} = \int_{\Lambda} P_{i,j}(det|\lambda) \ \rho(\lambda) \ d\lambda,$$
 (7)

where naturally $P_i(det|\lambda)$, $P_{i,j}(det|\lambda)$ should stay positive (or zero) always. This last is our point of departure: in this paper we propose a reinterpretation of the former marginal and joint detection probabilities based on a certain manipulation of expressions (1)–(2).

The paper is organized as follows. In Sec.II we will consider a setup with just a source and two detectors; once that is understood, the interposition of other devices between the source and the detectors poses no additional conceptual difficulty though it is nevertheless convenient to address it in some detail: this will be done in Sec. III. The calculations in these two sections find support on the proofs provided in Appendix 1, and stand for our main result in this paper. Sec. IV explores the question of " α -factorability" in the model, not only from the mathematical point of view but also providing some more physical insights on its implications.

Up to that point the novel points of the paper are made and its results are self-contained; it is nevertheless natural to extend our analysis to some of their further implications (amongst these, the consequences for Bell tests of local-realism, and also a recent related proposal), in Sec. V, where we also include a preliminary approach to questions regarding the physics of the real detectors. Finally, overall conclusions are presented in Sec. VI, and some supplementary material is provided in Appendix 2, which may not only help make the paper self-contained but perhaps also contribute to clarify some of the questions addressed, in particular the non-factorability issue.

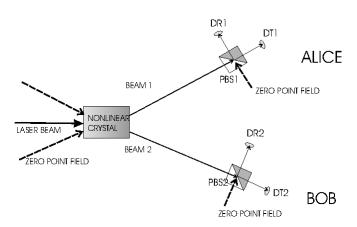


FIG. 1: Wigner-PDC scheme: photon pair generation, polarizing beam splitters (PBS) and detectors. Only relevant inputs of Zero-Point vacuum field (ZPF) are represented in the picture: "relevant" can be understood, in a classical wavelike approach, as "necessary to satisfy energy-momentum conservation" for the (set of) frequency modes of interest; in a purely quantum electrodynamical framework we would be talking about conservation of the commutation relations at the empty exit channels of the devices. Besides, those new ZPF components introduced at the empty exit channels of PBS's 1 and 2 can alter the detection probability for the signal arriving to the detectors (for instance giving rise to its "enhancement"); that signal is on the other hand determined (amongst other hidden variables) by the ZPF components entering the crystal. Figure: courtesy of A. Casado.

II. REINTERPRETING DETECTION PROBABILITIES

Our aim is to adopt here an approach that is deliberately as abstract as it can be chosen to be, because what we are concerned about is an (apparent) problem of the mathematical structure of the theory, rather than other details regarding its connection with physical reality which should in principle be addressed at another stage of the investigation; nevertheless, of course for the sake of credibility some of these details need to be brought up, and so we will do when necessary.

We now propose a reinterpretation of expressions (1)–(2) which is based on the idea that it is legitimate to play with internal degrees of freedom of a theory, as far as its observable predictions remain all of them invariant to these transformations. We intend to make a progressive exploration of the possible consistency conditions, from less to more demanding, but we advance that in no way this threatens the validity of our main result here, as we will be able to provide proof of the existence of the required solution for all of them.

A. Single detections

Knowing that the following equality holds (from here on we drop detector indexes when unnecessary), for some real constant $K_{(m)}$ ("marginal"),

$$K_{(m)} \int_{\alpha,\alpha^*} (I(\alpha,\alpha^*) - I_0) \ W(\alpha,\alpha^*) \ d\alpha d\alpha^*$$
$$= \int_{\Lambda} P(\det|\lambda) \ \rho(\lambda) \ d\lambda, \qquad (8)$$

we realize we do not need to assume

$$P(\det|\lambda) \equiv K_{(m)} \cdot (I(\alpha, \alpha^*) - I_0), \tag{9}$$

as a necessary, compulsory choice; it would be enough to find some $f(x) \ge 0$, satisfying

$$K_{(m)} \int_{\alpha,\alpha^*} (I(\alpha,\alpha^*) - I_0) W(\alpha,\alpha^*) d\alpha d\alpha^*$$

$$= \int_{\alpha,\alpha^*} f(I(\alpha,\alpha^*)) W(\alpha,\alpha^*) d\alpha d\alpha^*,$$
(10)

so we can then safely identify

$$P(det|\lambda) \equiv f(I(\alpha, \alpha^*)),$$
 (11)

with $f(I(\alpha, \alpha^*)) \geq 0$, $\forall I(\alpha, \alpha^*)$. This last is nothing but solving a linear system with only one restriction and an infinite number of free parameters, whose subspace of solutions intersects the region $0 \leq f(I(\alpha, \alpha^*)) \leq 1$ $\forall \alpha, \alpha^*$: see Appendix 1.

B. Joint detections

From the perspective of our approach and regarding joint detections, it is not very clear which are really the minimum conditions of consistence that one would have to enforce; in order to proceed with the maximum generality, let us simply define, for detectors i, j and a constant $K_{(j)}$ ("joint"), a new function $\Gamma(x, y)$, so that

$$K_{(j)} \int (I_{i}(\alpha, \alpha^{*}) - I_{0,i}) \cdot (I_{j}(\alpha, \alpha^{*}) - I_{0,j}) \times W(\alpha, \alpha^{*}) \ d\alpha d\alpha^{*}$$

$$= \int \Gamma(I_{i}(\alpha, \alpha^{*}), I_{j}(\alpha, \alpha^{*})) \ W(\alpha, \alpha^{*}) \ d\alpha d\alpha^{*},$$
(12)

which we will now attempt to interpret as

$$P_{i,j}(det|\lambda) \equiv \Gamma(I_i(\alpha, \alpha^*), I_j(\alpha, \alpha^*)).$$
 (13)

Of course, at a first look it seems very desirable to guarantee that the detection probabilities depend solely on the amount of intensity at the entrance of each detector; hence it would seem natural to add the condition

$$\Gamma(I_i(\alpha, \alpha^*), I_j(\alpha, \alpha^*)) = f(I_i(\alpha, \alpha^*)) \cdot f(I_j(\alpha, \alpha^*)),$$
(14)

i.e., some "factorability" on the incoming intensities. Again, while such an additional condition seems necessary in order to preserve the "physical interpretability" of the "inner structure" of the theory, it is not at all something necessary from the point of view of its observable predictions, as long as these remain invariant; as we will see later, $I_j(\alpha, \alpha^*)$ may not only be unobservable as corresponds to a particular realization of the random fields... it may also happens that it does not actually represent a real intensity, but just an average promediated over another relevant random variable.

In Appendix 1 we have shown that it is always possible to find some suitable Γ satisfying all necessary conditions to be interpreted as a probability distribution and consistent with the observable prediction of the theory regarding joint detections, eq. (2); and, furthermore, that a solution can be found satisfying also (14).

As said, our choice here is to proceed without loss of generality: for this purpose, it is convenient to redefine now $\alpha \equiv \alpha, \alpha^*$, as well as

$$\hat{f}_i(\alpha) \equiv f(I_i(\alpha)),$$
 (15)

$$\hat{\Gamma}_{i,j}(\alpha) \equiv \Gamma(I_i(\alpha), I_j(\alpha)). \tag{16}$$

where we now assume that in general,

$$\Gamma_{i,j}(\alpha) \neq f_i(\alpha) \cdot f_j(\alpha),$$
 (17)

The absence of factorability on the α 's may come perhaps as a surprise to some, given than Clauser-Horne factorability [21] on the hidden variable λ is usually taken for granted; this is a mistake [22], that we have tried to clarify in Appendix 2.

III. ADDING INTERMEDIATE DEVICES: POLARIZERS, PBS'S...

Once we place one or more devices between the crystal and the detectors, typically polarizers, polarizing beam splitters (PBS) or other devices to allow polarization measurements (such as in Fig. 1), in general we cannot any longer describe the fields between both with only one set $\{\alpha\}$ of mode-amplitudes; we need to redefine our α 's as now associated to a particular position \mathbf{r} (they do not any longer determine a frequency mode for all space [20]). Hence, we will now have

$$\alpha(\mathbf{r}) \equiv \{\alpha_{\mathbf{k},\gamma}(\mathbf{r}), \alpha_{\mathbf{k},\gamma}^*(\mathbf{r})\},\tag{18}$$

and, letting \mathbf{r}_s be the position of the source (the crystal), and \mathbf{r}_i the position of the *i*-th polarizer or PBS (or any other intermediate device), we will also redefine

$$\alpha_s \equiv \alpha(\mathbf{r}_s), \quad \alpha_i \equiv \alpha(\mathbf{r}_i),$$
 (19)

with α_i including the *relevant* amplitudes at the (empty) exit channels of that *i*-th intermediate device. With α_s , α_i corresponding each to (a set of) modes with different (sets of relevant) wavevectors $\{\mathbf{k}_s\}$ and $\{\mathbf{k}_i\}$ [23], we can then regard them as two (sets of) statistically independent random variables, with all generality. The intensity at the entrance of detector *i*-th will therefore depend now not only on α_s but also on α_i :

$$P_{i} \propto \langle I_{i}(\alpha_{s}, \alpha_{i}) - I_{0,i} \rangle$$

$$= \int_{\alpha_{s}} \int_{\alpha_{i}} (I_{i}(\alpha_{s}, \alpha_{i}) - I_{0,i}) W(\alpha_{s}) W(\alpha_{i}) d\alpha_{s} d\alpha_{i}.$$
(20)

Once more, something like that can always be rewritten (see former section), for some suitable and positively defined f'(x), as

$$P_{i} = \int_{\alpha_{s}} \int_{\alpha_{i}} f'(I_{i}(\alpha_{s}, \alpha_{i})) W(\alpha_{s}) W(\alpha_{i}) d\alpha_{s} d\alpha_{i}.$$
(21)

Integrating on α_i we would obtain

$$P_i = \int_{\alpha_s} \hat{f}_i'(\alpha_s) \ W(\alpha_s) \ d\alpha_s, \tag{22}$$

from where we define a new function $\hat{f}'_i(\alpha_s)$. On the other hand, for joint detections we would have

$$P_{i,j} \propto \int_{\alpha_s} \int_{\alpha_i} \int_{\alpha_j} (I_i(\alpha_s, \alpha_i) - I_{0,i}) \cdot (I_j(\alpha_s, \alpha_j) - I_{0,j}) \times W(\alpha_s) W(\alpha_i) W(\alpha_j) d\alpha_s d\alpha_i d\alpha_j,$$
(23)

which again can always be rewritten (again see former section), for some positively defined $\Gamma'(x,y)$, as

$$P_{i,j} = \int_{\alpha_s} \int_{\alpha_i} \int_{\alpha_j} \Gamma'(I_i(\alpha_s, \alpha_i), I_j(\alpha_s, \alpha_j)) \times W(\alpha_s) W(\alpha_i) W(\alpha_j) d\alpha_s d\alpha_i d\alpha_j,$$
(24)

and integrating on α_i, α_j we would obtain

$$P_{i,j} = \int_{\alpha_s} \hat{\Gamma}'_{i,j}(\alpha_s) W(\alpha_s) d\alpha_s, \qquad (25)$$

from where we can again define yet another new probability density function $\hat{\Gamma}'_{i,j}(\alpha_s)$.

It is interesting for the sake of clarity to compare the two situations: with (primed functions) and without polarizers (unprimed). It is easy to see that, because the detector only sees the intensity at its entrance channel, clearly (we drop the "s" subscript for simplicity),

$$f'(x) = f(x), \quad \forall x, \tag{26}$$

$$\Gamma'(x,y) = \Gamma(x,y), \ \forall x,y.$$
 (27)

while, in consistency with our approach, in general $\hat{f}'_i(\alpha) \neq \hat{f}_i(\alpha)$, as well as $\hat{\Gamma}'_{i,j}(\alpha) \neq \hat{\Gamma}_{i,j}(\alpha)$.

IV. ON NON-FACTORABILITY

A. Mathematical analysis

Let us go back to the case with just the source and the detectors; we will soon see the following does nevertheless also apply when polarizers or other devices are added to the setup, just the same. According to our reasonings in App. 2, and using (15)–(16), we now realize that there is no way to avoid

$$\hat{\Gamma}_{i,j}(\alpha) = \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha), \tag{28}$$

unless we introduce some additional dependence of the kind $\hat{f}(\alpha) \to \hat{f}(\alpha, \mu)$, so that then

$$\hat{\hat{\Gamma}}_{i,j}(\alpha,\mu) \neq \hat{\hat{f}}_i(\alpha,\mu) \cdot \hat{\hat{f}}_j(\alpha,\mu), \tag{29}$$

where we add a second "hat" to avoid an abuse of notation, and where μ stands for a new set of random variables. This $\hat{f}(\alpha,\mu)$ should be interpreted as a detection probability conditioned to the new vector of random variables μ , i.e,

$$\hat{\hat{f}}(\alpha, \mu) \equiv P(\det|\alpha, \mu). \tag{30}$$

We will impose further demands on $\hat{f}(\alpha, \mu)$, defining

$$\hat{\hat{f}}(\alpha, \mu) \equiv f(I(\alpha, \mu)), \tag{31}$$

something forced by strictly physical arguments: the choice $f(I(\alpha,\mu))$ must prevail over other possible ones - for instance $f(I(\alpha),\mu)$ - due to the need to respect the dependence of the probabilities of detection (conditioned to α or not) alone on the intensity that arrives to the detector, and nothing else.

Now, with the density function $\rho_{\mu}(\mu)$, we could write

$$P(det|\alpha) = \int_{\mu} P(det|\alpha, \mu) \ \rho_{\mu}(\mu) \ d\mu, \qquad (32)$$

$$P_{i,j}(det|\alpha) = \int_{\mu} P_{i,j}(det|\alpha,\mu) \ \rho_{\mu}(\mu) \ d\mu, \quad (33)$$

allowing us to recover our former definitions (15)–(16):

$$\hat{f}(\alpha) \equiv P(\det|\alpha),\tag{34}$$

$$\hat{\Gamma}_{i,j}(\alpha) \equiv P_{i,j}(det|\alpha). \tag{35}$$

For joint detections, the additional variable μ is particularly relevant because, we will always have that while

$$P_{i,j}(\det|\alpha,\mu) = P_i(\det|\alpha,\mu) \cdot P_j(\det|\alpha,\mu), \quad (36)$$

in general

$$P_{i,j}(det|\alpha) \neq P_i(det|\alpha) \cdot P_j(det|\alpha),$$
 (37)

or we could equivalently say that while necessarily

$$\hat{\hat{\Gamma}}_{i,j}(\alpha,\mu) = \hat{\hat{f}}_i(\alpha,\mu) \cdot \hat{\hat{f}}_j(\alpha,\mu), \tag{38}$$

in general

$$\hat{\Gamma}_{i,j}(\alpha) \neq \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha), \tag{39}$$

where of course

$$\hat{\Gamma}_{i,j}(\alpha) = \int_{\mu} \hat{f}_i(\alpha,\mu) \cdot \hat{f}_j(\alpha,\mu) \ \rho_{\mu}(\mu) \ d\mu. \tag{40}$$

To conclude this section, we recover the case with intermediate devices: due to α_i, α_j being, as defined, independent from one another and also from α_s , our hypothetical "flag" μ cannot be associated with none of them. Therefore, in general, and in principle, not only

$$\hat{\Gamma}'_{i,j}(\alpha) \neq \hat{f}'_{i}(\alpha) \cdot \hat{f}'_{i}(\alpha), \tag{41}$$

but also $\hat{\Gamma}_{i,j}(\alpha) \neq \hat{f}_i(\alpha) \cdot \hat{f}_j(\alpha)$ either. We use "in principle" because this question is not yet analyzed in detail; we now see clear, though, that this possible nonfactorability on α 's is nothing more than an internal feature of the model's mathematical structure, bearing no relevance in regard to its double-sided compatibility (or absence of it) both with local-realism and the quantum predicted correlations. A possible candidate for that additional hidden variable would be, at least in my opinion, the phase of the laser μ [25].

V. COMPLEMENTARY QUESTIONS

A. Wigner-PDC's local realism vs. quantum correlations

Though former mathematical developments are fully meaningful and self-contained on their own, yet it would be convenient to give some hints on how a local-realist (LR) model can account for typically quantum correlations, which are known to defy that very same local realism (LR). In the first place and as a general answer, what the Wigner-PDC picture proves is that LR is respected by a certain subset of all the possible quantum states, specifically the ones that can be generated from a non-linear mix of the QED-vacuum (which therefore acts as an "input" for the model) with a quasi-classical (a high-intensity coherent state), highly directional signal, the laser "pump" (which indeed enters in the model as a non-quantized, external potential [2]). Moreover, such a restriction is clearly not arbitrary at all, since it arises from a very simple quantum electrodynamical model of the process of generation of polarization-entangled pairs of photons from Parametric Down Conversion (PDC): see for instance eq. (4.2) in [2].

B. Detection rates and "efficiencies"

Aside from subscripts, we will now also drop "hats" and "primes" for simplicity; of course the fact that $\Gamma_{i,j}(\alpha)$ may not be in general α -factorisable,

$$\Gamma_{i,j}(\alpha) \neq f_i(\alpha) \cdot f_j(\alpha),$$
 (42)

does not at all mean that it cannot well satisfy

$$\int \Gamma_{i,j}(\alpha) \ W(\alpha) \ d\alpha = \left[\int f_i(\alpha) \ W(\alpha) \ d\alpha \right] \cdot \left[\int f_j(\alpha') \ W(\alpha') \ d\alpha' \right], \tag{43}$$

i.e. (let us from now use superscripts "W" and "exp" to denote, respectively, theoretical and experimental detection rates):

$$P_{i,j}^{(W)}(det) = P_i^{(W)}(det) \cdot P_j^{(W)}(det), \tag{44}$$

which is indeed the sense in which the hypothesis of "error independence" is introduced, to our knowledge, in every work on LHV models [27]. This sort of conditions over "average" probabilities (average in the sense that they are integrated in the hidden variable, may that be α alone or also some other one) are the only ones that can be tested in the actual experiment; there, we can just rely on the number of counts registered on a certain time-window ΔT , and the corresponding estimates of the type

$$P_i^{(exp)}(det) \approx \frac{n.\ joint\ det.\ (i,j)\ in\ \Delta T}{n.\ marg.\ det.\ (j)\ in\ \Delta T}.$$
 (45)

Now, if we wish to include some additional uncertainty element reflecting the technological limitations (a "detection efficiency" parameter), what we have to do is to redefine the overall detection probabilities as

$$P_i^{(exp)}(det) \equiv \hat{\eta}_i \cdot P_i^{(W)}(det), \tag{46}$$

$$P_{i,j}^{(exp)}(det) \equiv \hat{\eta}_i \hat{\eta}_j \cdot P_{i,j}^{(W)}(det), \tag{47}$$

as well as

$$P_i^{(exp)}(det|\alpha) \equiv \hat{\eta}_i \cdot P_i^{(W)}(det|\alpha),$$
 (48)

$$P_{i,j}^{(exp)}(det|\alpha) \equiv \hat{\eta}_i \hat{\eta}_j \cdot P_{i,j}^{(W)}(det|\alpha), \tag{49}$$

where $0 \leq \hat{\eta}_i \leq 1$ would play the role of such an (alleged) detection efficiency, the "hats" remarking the fact that the customary definition of the analogous quantity in QInf involves not only our $\hat{\eta}$'s but also the nontechnological contribution. From the point of view of the experimenter it is very difficult to isolate both components (Glauber's theory [15] does not predict a unit detection probability even for high intensity signals); we should perhaps then confine ourselves to the term "observable detection rate" instead of using the clearly misleading one of "detector inefficiency".

C. Consequences on Bell tests, their supplementary assumptions and critical efficiencies

That said and going to a lowest level of detail, states in such an (LR) subset of QED can still indeed exhibit correlations of the class that is believed to collide with LR, yet the procedure through which they are extracted from the experimental set of data does not meet one of the basic assumptions required by every test of a Bell inequality: they do not keep statistical significance with respect to the physical set of "states" or hidden instructions [27]). To guarantee that statistical significance we must introduce some of the following two hypothesis:

(i) all coincidence detection probabilities are independent of the polarizers' orientations ϕ_i, ϕ_j (this is what we call "fair-sampling" [28], for a test of an homogeneous inequality [29]), which implies

$$P_{ij}(\det|\phi_i,\phi_j,\alpha) = P_{ij}(\det|\alpha_s), \tag{50}$$

where of course (see Sec. III) $\alpha \equiv \alpha_s \oplus \alpha_i \oplus \alpha_j$, and where we recall that ϕ_i, ϕ_j would determine which vacuum modes amongst the sets α_i, α_j would intervene in the detection process,

(ii) the interposition of an element between the source and the detector cannot in any case enhance the probability of detection (the "no-enhancement" hypothesis [30], needed to test the Clauser-Horne inequality [21], and presumably every other inhomogeneous one [31]),

$$P_i(det|\phi_i,\alpha) \le P_i(det|\infty,\alpha_s).$$
 (51)

with ∞ denoting the absence of polarizer and with $\alpha \equiv \alpha_s \oplus \alpha_i$ this time.

Following our developments in Sec. III one can easily see that (i) is not in general true, and according to [13] neither is (ii). I.e., whenever states of light are prepared so as to produce the sort of quantum correlations that are known to defy LR, these last come supplemented with the necessary features that prevent it from happening... how could it not?

Yet, a mere breach of (i) or (ii) is not enough to assert the existence of a Local Hidden Variables (LHV) model. which is an equivalent way of saying that the results of the experiment respect LR: it is more than well known that this can only happen for certain values of the observed detection rates [27, 32]. However, from the point of view of (48)–(49), and given the fact that, as proven in Secs II and III, the Wigner-PDC is in all circumstances in accordance with expressions (71)–(72), and hence to all possible Bell inequalities whether our $\hat{\eta}$'s are equal or less than unity, the so-called "critical efficiencies" would merely stand, at least as far as PDC-generated photons are concerned, for bounds on the detection rates that we can experimentally observe (these last in turn constrained by the only subset of quantum states that we can physically prepare).

D. A recent related test

The recent test in [34] would seem formally equivalent to a Bell test of an homogeneous inequality [29], but for two questions: (i) they do not require remote observers; (ii) the use of analogical measurements, that would seem to exclude the so-called "detection loophole" that is widely recognized in other tests. The purpose of this section is to show that such loophole still remains, though making it evident requires a subtler approach, according to our approach here one perhaps more realistic than the usual assumption of "random errors". Besides, and to my knowledge, [34] is the only amongst other recent related tests [36, 37] which does not acknowledge, explicitly, one or other loophole leading to compatibility with the local-realistic interpretation.

Basically, Kot et al's proposal in [34] rests on the probing of some "test function" $F \equiv F(Q_1, Q_2, \dots Q_N)$, where Q_m is an outcome obtained when an observable \hat{Q}_m is measured, and where $\{\hat{Q}_m\}$ is a set of mutually exclusive observables (more precisely, according to eq. (8) also in [34], F involves a set of powers $\{(Q_m)^{2n}, m, n = 1, \dots N\}$). Later, they will be concerned with an "average" $\beta = \langle F \rangle$, which, if a local hidden variables (LHV) model turned out to exist, would have to be expressible as

$$\langle F \rangle_{\Lambda} = \int_{\Lambda} F(\{Q_m\}) \cdot P(\{Q_m\}|\lambda) \ \rho(\lambda) \ d\lambda,$$
 (52)

with λ as usual a vector of hidden variables and Λ the overall space of events. The quantity $P(\{Q_m\}|\lambda) = f(Q_1, \ldots, Q_N|\lambda)$ would stand for a joint probability density function (conditioned to the state λ) for any set of

(predetermined) results for any set of possible measurements upon the observables $\{\hat{Q}_m\}$.

Each \hat{Q}_m will be then sampled on a different subset $\Lambda_m \subset \Lambda$... which would be no problem as long as all $\{\Lambda_m\}$ are statistically faithful to Λ ; however, the fact that measurements may not always give a result (or that these measurements are effectively completed at different timestamps) may destroy that statistical significance, hence invalidating that of the estimate of β . To show that this may happen even in view of (ii), we must go to Glauber's expression in (87). The key is that, for a given timestamp, in general P(t) < 1, though of course after a certain interval Δt the accumulated probability,

$$P_{det}(\Delta t) = \int_{t}^{t+\Delta t} P_{det}(\tau) d\tau, \tag{53}$$

may approach unity, something irrelevant for our argument: the hypothetical 0-instructions in a hypothetical LHV model (see for instance [27]) would no longer have anything to do with some "detection inefficiency", but simply express the fact that for some given time-stamp and observable, P(t) may be less than unity.

Other models of "detection" may involve a timeintegral of the intensity at the entrance of the detector, or other sophistications; they would not affect our argument as long as we recognize (87) as a time-dependent quantity, in general satisfying $P(\mathbf{r},t) < 1$. Once here, the physical connection with the test in [34] can be established by assigning to each time stamp t a set \mathcal{M}_t of hidden instructions of the usual kind (i.e., a different LHV model \mathcal{M}_t for each t): again see [27].

In particular, following [38], a detection on the "signal" arm, at a time-stamp t, prompts the analysis of the signal (coming from the homodyne setup and entering a high frequency oscilloscope) at the "idler" one, over a fixed time window. Yet, a correlation between the detection time-stamp at the signal arm and the choice of observable at the idler would seem to require communication or "signaling" between the two measurement setups.

This last difficulty, which would burden our argument with the need for an additional "locality loophole", can, however, be overcome too: consider an observable \hat{Q} that (always) produces two possible results $Q = \{q_1, q_2\}$. For simplicity let us this time denote $P_Q(q|\lambda)$ the probability of an outcome q when Q is measured on the state λ . Then,

$$\langle Q \rangle_{\Lambda} = \frac{q_1 \cdot P_Q(q_1|\Lambda) + q_2 \cdot P_Q(q_2|\Lambda)}{P_Q(det|\Lambda)}; \tag{54}$$

however, if each result is associated to a detection at a different set of time-stamps, $\{t_{i,1}\}, \{t_{i,2}\}$, then the experimentally accessible quantity is

$$\langle Q \rangle_{ob} = \frac{q_1 \cdot P_Q(q_1 | \Lambda_1^{(Q)}) + q_2 \cdot P_Q(q_2 | \Lambda_2^{(Q)})}{P_Q(det | \Lambda)}.$$
 (55)

In this last expression, $\Lambda_1^{(Q)}$, $\Lambda_2^{(Q)} \subset \Lambda$ are again two different sets selected by the correlation between the time-

stamp at one arm and the result of the measurement at the other when \hat{Q} is measured.

According to the approach in [2–8], the set of relevant vacuum electromagnetic modes inserted at the source are still contained in the fields arriving at each detector, and this will indeed induce some correlation between detection time-stamp at one side and a measurement's outcome at the other, and in general between any two events which are related to the intensity of the incoming signal. This induced correlation is also at the core of other recent results in [9, 10]. A more detailed argument can be obtained by e-mail from the author.

E. An approach to realistic detectors and average ZPF subtraction

This section approaches some physical considerations with the aim of showing that there is plenty of room for a suitable physical interpretation of the model, even when that is not strictly necessary for the coherence of our results, at least from the purely mathematical point of view. Indeed, we have already descended to the physical level when we established, in former sections, the dependence of our f's and Γ 's solely on the intensity arriving at the detector. Expectable behavior for a physical device would typically include a "dead-zone", an approximately linear range and a "saturation" at high intensities (this is indeed the kind of behavior suggested for instance in [16]); amongst other restrictions this would imply, for instance, $f(I(\alpha)) = 0$, when $I(\alpha) \leq \bar{I}_0$ (this last a threshold that may even surpass the expectation value of the ZPF intensity), as well as $\Gamma(I_i(\alpha), I_i(\alpha)) = 0$ either for $I_i(\alpha)$ or $I_i(\alpha)$ below \bar{I}_0 .

Neither these restrictions nor other similar ones would in principle invalidate our proofs in Appendix 1, which seem to provide room enough to simulate a wide range of possible behaviors; however, we must remark that none of our f's and Γ 's can ultimately be considered as fully physical models, due to the fact that they represent point-like detectors (the implications of such an over-simplification may become clearer in a moment).

In close relation to the former, we also propose here a simple physical interpretation of the term $-I_0$ appearing in the expressions for the detection probabilities. From the mathematical point of view such subtraction arises from a mere manipulation of Glauber's original expression [15]. From the physical one, a realistic interpretation would be more than desirable, as that subtraction of ZPF intensity is for instance crucial to explain the absence of an observable contribution of the vacuum field on the detectors' rates [33]; of course we mean "explain in physical terms"; from the mathematical point of view our model here already predicts a vanishing detection probability for the ZPF alone.

My suggestion is that the $-I_0$ term must be (at least) related to the average flux of energy going through the surface of the detector in the opposite direction to the

signal (therefore *leaving* the detector). That interpretation fits the picture of a detector as a physical system producing a signal that depends (with more or less proportionality on some range) on the total energy (intensity times surface times time) that it accumulates. I am just saying "at least related", bearing in mind that to establish such association we would first have to refine the point-like model of a detector which stems directly from the original Glauber's expression [15].

VI. CONCLUSIONS

We have shown that the Wigner description of PDC-generated (Parametric Down Conversion) photonentanglement, so enthusiastically developed in the late nineties [2–8] but then ignored in recent years, can be reformulated as entirely local-realistic (LR). A formalism that is one-to-one with a quantum (field-theoretical) model of the experimental setup can be cast, thanks to an additional manipulation (also one-to-one), into a form that respects all axiomatic laws of probability [39], and therefore LR, as defined for instance in Sec. 2a. The original quantum electrodynamical model takes as an input the vacuum state, which accepts a well defined probabilistic description through the so-called Wigner transform: this is the fact that the analogy with a localrealistic theory is conditioned to. What we call Wigner-PDC accounts, then, for a certain subset within the space of all possible QED-states, determined by a particular set of initial conditions and a certain Hamiltonian governing the time-evolution [48], restrictions that seem to guarantee (according to us here) the compatibility with LR.

Aiming for the maximum generality, as well as to avoid some possible (still under examination) difficulties with factorisable expressions, we have renounced to what we call α -factorability of the joint detection expressions. Such a choice is not only perfectly legitimate [22], but may also be supported by a well feasible interpretation (see Appendix IV); nevertheless, further implications of that non-factorability on α will also be left to be examined elsewhere. Again, whatever they finally turn out to be, they are also irrelevant for our main result in this paper: the Wigner-PDC formalism can be cast into a form that respects all axiomatic laws of probability for space-like separated events.

Neither does the explicit distinction between the cases where polarizers (or other devices placed between the source and the detectors) are or not included in the setup introduce any conceptual difference from the point of view of our main result; however, the question opens room to remark some of the main differences of the Wigner-PDC model (actually, also its Hilbert-space analogue) with the customary description used in the field of Quantum Information (QInf): here, each new device introduces noise, new vacuum amplitudes that fill each of its empty polarization channels at each of its exists, in contrast to the usual QInf "black box", able to extract

polarization information from a photon without any indeterministic component.

As a matter of fact, those additional random components may hold the key to explain the variability of the detection probability (see note [13]) that is necessary, from the point of view of Bell inequalities, to reconcile quantum predictions and LR (see Sec. VA). In particular, the immediacy with which the phenomenon of "detection probability enhancement" arises in the Wigner-PDC framework would suggest that this may be after all the right track to understand why after several decades the minimum detection rates (or, in QInf terminology, critical detection efficiencies) that would lead to obtain conclusive evidence of non-locality are still out of reach (see an explanation of what these critical rates are in [32], for comments on the current state of the question see somewhere else).

In Sec.VD we have also addressed a recent no-go test that I consider particularly relevant, amongst other things because it does not explicitly acknowledge any loophole. In Sec. VE we have done a first, general approach on the question of whether our reinterpretation of the detection probabilities is consistent with the actual physical behavior of detectors. A closer look to this question is out of the scope of the paper; former proposals [16] in regard to this issue aimed perhaps too straightforwardly to the physical level, while they did not even guarantee consistency with the framework we have settled here (proof of this is that it introduces divergences in relation to the purely quantum predictions, divergences later experimentally disproved for instance in [17]). Besides, we have also suggested a possible simple physical interpretation of I_0 -subtraction taking place in the expressions for the detection probabilities: work in any of these directions would anyway seem to require a departure from the point-like model of a detector.

Summarizing, we have shown that a whole family of detection models

$$\mathcal{M}_{det} \equiv \{ f(I(\alpha, \mu)), \Gamma(I_i(\alpha, \mu), I_j(\alpha, \mu)) \}, \tag{56}$$

can be found, consistent both with the quantum mechanical expressions from the Wigner-PDC model and LR. A close examination of the constraints coming from the physical behavior of the detectors and other experimentally testable features is left as a necessary step for the future, with the aim of establishing a subset physically feasible ones; nevertheless, we have the guarantee that all of them produce suitable predictions, as so does their quantum electrodynamical counterpart.

Yet, even at the purely theoretical level some other features remain open too: as a fundamental one, to what extent the model requires what we have called non-factorability on α 's. Once more and after all, QM is just a theory, a theory that provides a formalism upon which to build models, models than can (and should) be refined based on experimental evidence, something that (again) also applies just as well to the one we are dealing with here [49]. Perhaps the particle properties of light

are not enough to assume that the current model of "a photon" is the best representation (and most complete) that we can achieve of light; many of those properties can be understood, I believe, from entirely classical models [50], as well. Quantum entanglement seems to manifest in many of its "reasonable" features but at least as (this particular model of) Parametric Down Conversion is concerned and so far, whenever local-realism would seem to be challenged new phenomena can be invoked so as to prevent, at least potentially, that possibility. Such are "unfair sampling", "enhancement" (as a particular case of variable detection probability) and over all detection rates low enough to open room for the former two.

Those phenomena find theoretical support in the Wigner-PDC picture, but definitely not in the usual, based merely on the correspondence between the $\frac{1}{2}$ -spin algebra for massive particles and the polarization states of a plane wave (a photon then looks just as a $\frac{1}{2}$ -spin particle, what some like to call a two-level system, but for the magnitude of the angular momentum it carries, and its statistics of course). To explore, and exhaust if that is the case, alternative routes such as the one here is not only sensible but also necessary.

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Appendix

1. Auxiliary proofs

Lemma 1: There always exists f(x) satisfying (10), with $0 \le f(x) \le 1$, for $x = I(\alpha, \alpha^*)$ over the space of all possible pairs $\{\alpha, \alpha^*\}$.

Proof: We have a linear system with one restriction and infinite variables: the set of values $\{f(I(\alpha, \alpha^*))\}$; coefficients are given by the $W(\alpha, \alpha^*)$'s, and the independent term is the left-side term of (10), $0 \le P_s \le 1$.

Then let us consider the real, vectorial space $\mathcal V$ where we associate each value $f(I(\alpha,\alpha^*))$ per coordinate: we can assume $\dim(\mathcal V)=N^2$ $(N\to\infty)$ if we consider a direct dependence on the α 's, or we can also say $\dim(\mathcal V)=N$ (again $N\to\infty$) if the dependence is defined upon the intensities, in principle a real number (depends on whether we choose to formulate the model upon instantaneous values, therefore real, or as a complex amplitude, this is not essential here). Both options, f's depending either on $\{\alpha\}$ or intensities $\{I(\alpha,\alpha^*)\}$ serve right for our purposes, the second being perhaps more

appropriate from the point of view of physical interpretation. If they do exist, compatible solutions $f(I(\alpha, \alpha^*))$ for the (unique) restriction will conform a linear manifold $\mathcal{M} \subset \mathcal{V}$. Now we have to see:

- (i) Due to $W(\alpha, \alpha^*) > 0$ for all pairs α, α^* [40], and $P_s \geq 0$ too, \mathcal{M} cannot be parallel with any of the coordinate hyper-planes in \mathcal{V} ; i.e., \mathcal{M} intersects all of them, defined each (each one for a pair α, α^*) by the equation $f(I(\alpha, \alpha^*)) = 0$.
- (ii) Moreover, for $P_s > 0$, \mathcal{M} has a non-trivial intersection (more than one point, the origin) with all coordinate planes inside the first hyper-quadrant \mathcal{V}_{1q} (the subregion $\mathcal{V}_{1q} \subset \mathcal{V}$ given by restricting \mathcal{V} to $f(I(\alpha, \alpha^*)) \geq 0$). This can be seen, for instance, determining the point of crossing with the axes: doing all f's zero except one (always possible because all the W's are strictly above zero), we can see the crossings always take place at the positive half of the corresponding coordinate axis, therefore at the boundary of \mathcal{V}_{1q} . On the other hand, for $P_s = 0$ the solution for the system is trivial.

With (i) and (ii), it is clear that under the restriction (under the set of inequalities) $f(I(\alpha, \alpha^*)) \leq 1 \,\forall \alpha, \alpha^*$, the set of admissible solutions is still not empty, as guaranteed by

$$\int_{\alpha} W(\alpha, \alpha^*) \ d\alpha d\alpha^* = 1, \tag{57}$$

and $P_s \leq 1$, as we will now show. Here we give an inductive reasoning; let us consider the equivalent problem in 3 dimensions, with

$$ax + by + cz = d; (58)$$

clearly, for $a, b, c, d \ge 0$ the former plane always intersects with x = y = z at a point of coordinates

$$x = y = z = d/(a+b+c) = d,$$
 (59)

which is obviously in the first quadrant $(x,y,z\geq 0)$, and moreover, for a+b+c=1 and $0\leq d\leq 1$, clearly also inside of the region $\{0\leq x\leq 1,0\leq y\leq 1,0\leq z\leq 1\}$. The extension to infinite dimensions, topological abnormalities all absent, is direct, with $x\equiv\alpha,\alpha^*$ (or alternatively $x\equiv I(\alpha,\alpha^*)$, $a,b,c\equiv W's$ and $d\equiv P_s\leq 1$, the left hand side of (10).

Lemma 2: There always exists $\Gamma(x,y)$ satisfying (12), with $0 \leq \Gamma(x,y) \leq 1$, for $x = I_i(\alpha,\alpha^*)$, $y = I_j(\alpha,\alpha^*)$ over the space of $\{\alpha,\alpha^*\}$.

Proof: formally identical with Lemma 1.

Lemma 3: There always exists f(x) satisfying simultaneously (10) for a finite collection of i = 1, ..., N detectors, with $0 \le f(x) \le 1$, for $x = I_i(\alpha, \alpha^*)$, and satisfying, for any two pair of detectors, condition (14) as well.

Proof: First we notice that condition (10) for each additional detector stands for a new linear restriction on the problem already treated in *Lemma 1*; due to the system

being under-determined with an infinite number of free parameters this poses no problem.

However, condition (14) is not a linear but a non-linear restriction. Consider first the case N=2 (number of detectors), and let us solve the system under the 2 restrictions (one per detector) of the kind (10), obtaining a solution in accordance with Lemma~1; such solution, as seen, depends on an infinite set of free parameters. Let us give values to all these free parameters (from Lemma 1, such values can be chosen so as to guarantee $0 \le f(x) \le 1, \forall x$) but one, which we will define as

$$f(I_{\Theta}) = \Theta, \tag{60}$$

for some particular (randomly chosen) value $I(\alpha) = I_{\Theta}$ (the same value of $I(\alpha)$ can be produced by several α 's, so it is clearer to associate "labels" to I instead of α , though conceptually botch choices are equivalent); condition (14) between the two detectors stands now for a quadratic equation,

$$A \cdot \Theta^2 + B(I_{\Theta}) \cdot \Theta + C(I_{\Theta}) = D; \tag{61}$$

before we make the former terms explicit, let Ω be the set configured by all possible values of α , and define the subsets

$$\hat{\bar{\Omega}}(I_{\Theta}) \equiv \{\alpha; I_1(\alpha) = I_2(\alpha) = I_{\Theta}\}, \tag{62}$$

$$\bar{\Omega}(I_{\Theta}) \equiv \{\alpha; I_1(\alpha) \neq I_{\Theta}, I_2(\alpha) \neq I_{\Theta}\},$$
 (63)

$$\hat{\Omega}(I_{\Theta}) \equiv \Omega - \hat{\bar{\Omega}}(I_{\Theta}) - \bar{\Omega}(I_{\Theta}), \tag{64}$$

so that now we can write

$$A(I_{\Theta}) = \int_{\hat{\Omega}(I_{\Theta})} d\alpha \ W(\alpha), \tag{65}$$

$$B(I_{\Theta}) = \int_{\hat{\Omega}(I_{\Theta})} d\alpha \ W(\alpha) \cdot [f(I_1(\alpha)) + f(I_2(\alpha))],$$
(66)

$$C(I_{\Theta}) = \int_{\bar{\Omega}(I_{\Theta})} d\alpha \ W(\alpha) \cdot f(I_i(\alpha) \cdot f(I_j(\alpha)). \tag{67}$$

Taking now into account that intensities are continuous functions of α it is reasonable to assume

$$A(I_{\Theta}) \approx 0;$$
 (68)

and now with $0 \le B, C, D \le 1$ and clearly $C \le D$ (just look at eq.61), and finally the equivalence of B to a mere single detection probability (eq. 10 for instance), we can now write

$$B(I_{\Theta}) > D > D - C(I_{\Theta}) > 0, \tag{69}$$

from where it is not difficult to see that eq.(61) always admits a solution $0 \le \Theta \le 1$, where we recall the definition of the free parameter as $\Theta \equiv f(I_{\Theta})$.

Extension to N detectors is inmediate by considering N-1 free parameters and repeating the former reasoning on each one of them at a time, for each corresponding pair of detectors.

Lemmas 1,2 and 3 are directly applicable, aside from Sec. II, to Sec. III.

2. Basic concepts revisited

We first briefly revisit the concepts of locality, determinism and factorability; a good understanding on these concepts is crucial for the main results of the paper, what makes this review not only convenient but almost unavoidable, especially given the presence of some confusion in the literature. In any case, it shall be clearly understood that Clauser and Horne's factorability [21] is not a requisite for local-realism.

a. Locality and realism

A theory predicting the results of two measurements A and B that take place under causal disconnection (relativistic space-like separation) can be defined as local if and only if we can write

$$A = A(\lambda, \phi_A), \quad B = B(\lambda, \phi_B),$$
 (70)

where λ is a (set of) hidden (or explicit) variables defined inside the intersection of both light cones, and ϕ_A , ϕ_B are another two other sets of variables (amongst them the configurable parameters of the measuring devices) defined locally at A and B, respectively, and causally disconnected from each other, i.e.,

$$P(A = a | \lambda, \phi_A, \phi_B) = P(A = a | \lambda, \phi_A), \tag{71}$$

$$P(B = b|\lambda, \phi_A, \phi_B) = P(B = b|\lambda, \phi_B). \tag{72}$$

These last two expressions are usually taken as a definition of *local causality* [41].

Now, realism simply stands for λ (and ϕ_A, ϕ_B as well) having a well defined probability distribution. A set of physical observables corresponding to a particular quantum state can be sometimes described by a well defined joint probability density (such is the case of field amplitudes in any point of space for the vacuum state in QED); for other quantum states that is not possible though.

b. Determinism

A measurement M upon a certain physical system, with k possible outcomes m_k , is *deterministic* on a hidden variable (HV) λ (summarizing the state of that system), if (and only if)

$$P(M = m_k | \lambda) \in \{0, 1\}, \ \forall k, \lambda, \tag{73}$$

which allows us to write

$$M \equiv M(\lambda), \tag{74}$$

and indeterministic iff, for some λ , some k',

$$P(M = m_{k'}|\lambda) \neq \{0, 1\},\tag{75}$$

i.e., at least for some (at least two) of the results for at least one (physically meaningful) value of λ .

Now, indeterminism can be turn into determinism, i.e., (75) can into (73), by defining a new hidden variable μ , so that now, with $\gamma \equiv \lambda \oplus \mu$:

$$P(M = m_k | \gamma) \in \{0, 1\}, \ \forall k, \gamma, \tag{76}$$

which means we can write,

$$M \equiv M(\lambda, \mu), \tag{77}$$

a proof that such a new hidden variable μ can always be found (or built) given in [42].

So far, then, our determinism and indeterminism are conceptually equivalent, though of course they may correspond to different physical situations, depending for instance on whether γ is experimentally accessible or not.

c. Factorability

Let now $\mathcal{M} = \{M_i\}$ be a set of possible measurements, each with a set $\{m_{i,k}\}$ of possible outcomes, not necessarily isolated from each other by a space-like interval. We will introduce the following distinction: we will say \mathcal{M} is

(a) λ -factorisable, iff we can find a set $\{\xi_i\}$ of random variables, independent from each other and from λ too, such that

$$\mu = \bigoplus_{i} \xi_i, \tag{78}$$

and (73) holds again for each M_i on $\gamma_i \equiv \lambda \oplus \xi_i$:

$$P(M_i = m_{i,k}|\gamma_i) \in \{0,1\}, \ \forall i, \ \forall k, \gamma_i,$$
 (79)

this last expression meaning of course that we can write, again for any of the M_i 's,

$$M_i \equiv M_i(\lambda, \xi_i). \tag{80}$$

(b) non λ -factorisable, iff (79) is not possible for any set of statistically independent ξ_i 's.

We will restrict, for simplicity, our reasonings to just two possible measurements $A, B \in \mathcal{M}$, with two possible outcomes, $A, B \in \{+1, -1\}$, all without loss of generality. We have seen that, as the more general formulation, we can always write something like $A = A(\lambda, \xi_A)$, $B = B(\lambda, \xi_B)$.

Lemma 1-

(i) If A and B are deterministic on λ , i.e., (73) holds for A and B, then they are also λ -factorisable, i.e.,

$$P(A = a, B = b|\lambda) = P(A = a|\lambda) \cdot P(B = b|\lambda), \quad (81)$$

for any $a,b \in \{+1,-1\}$. Eq. (81) is nothing but the so-called Clauser-Horne factorability condition [21].

(ii) If A and B are indeterministic on λ , i.e., if (73) does not hold for λ , then: for some μ (always possible to find [42]) such that now (76) holds for $\gamma \equiv \lambda \oplus \mu$, A, B are γ -factorisable,

$$P(A = a, B = b|\gamma) = P(A = a|\gamma) \cdot P(B = b|\gamma), \quad (82)$$

i.e., (81) holds for γ , this time not necessarily for λ .

(iii) Let (79) hold for A, B, on λ, ξ_A, ξ_B : if λ, ξ_A, ξ_B are statistically independent, (hence, A and B are what we have called λ -factorisable), then (81) holds for λ , not necessarily on the contrary.

Proof-

- (i) When (73) holds, for any λ and any $a, b \in \{+1, -1\}$, $P(A = a|\lambda), P(B = b|\lambda) \in \{0, 1\}$, from where we can, trivially, get to (81).
- (ii) It is also trivial that, if (76) holds, (81) can be recovered for γ . That the same is not necessary for λ can be seen with the following counterexample: suppose, for instance, that for $\lambda = \lambda_0$, either A = B = 1 or A = B = -1 with equal probability. It is easy to see that

$$P(A = B = 1|\lambda_0) \neq P(A = 1|\lambda_0) \cdot P(B = 1|\lambda_0),$$
(83)

numerically: $\frac{1}{2} \neq \frac{1}{4}$.

(iii) We have, from independence of λ, ξ_A, ξ_B , and working with probability densities ρ 's: $\rho_{\lambda}(\lambda, \xi_A, \xi_B) = \rho_{\lambda}(\lambda) \cdot \rho_{A}(\xi_A) \cdot \rho_{B}(\xi_B)$, which we can use to write

$$P(A = a, B = b|\lambda) = \int P(A = a, B = b|\lambda, \xi_A, \xi_B)$$

$$\times \rho_A(\xi_A) \cdot \rho_B(\xi_B) \ d\xi_A d\xi_B.$$
(84)

and now with the fact that we can recover (76) for A (B) on $\gamma_A = \lambda \oplus \xi_A$ ($\gamma_B = \lambda \oplus \xi_B$),

$$P(A = a, B = b|\lambda)$$

$$= \int P(A = a|\lambda, \xi_A) \cdot P(B = b|\lambda, \xi_B)$$

$$\times \rho_A(\xi_A) \cdot \rho_B(\xi_B) \ d\xi_A d\xi_B$$

$$= \int P(A = a|\lambda, \xi_A) \cdot \rho_A(\xi_A) \ d\xi_A$$

$$\times \int P(B = b|\lambda, \xi_B) \cdot \rho_B(\xi_B) \ d\xi_B$$

$$= P(A = a|\lambda) \cdot P(B = b|\lambda). \tag{85}$$

On the other hand, let λ, ξ_A, ξ_B be not statistically independent: we can set for instance, as a particular case, $\xi_i \equiv \mu$, $\forall i$, therefore reducing our case to that of (76). Once this is done, our previous counterexample in (ii) is also valid to show that factorability is not necessary for λ here.

- [1] Parametric Down Conversion (PDC): a pair of entangled photons is obtained by pumping a laser beam into a nonlinear crystal.
- [2] A. Casado, T.W. Marshall and E. Santos. J. Opt. Soc. Am. B 14, 494 (1997).
- [3] A. Casado, A. Fernández-Rueda, T.W. Marshall, R. Risco-Delgado, E. Santos. Phys. Rev. A, 55, 3879 (1997).
- [4] A. Casado, A. Fernández-Rueda, T.W. Marshall, R. Risco-Delgado, E. Santos. Phys. Rev. A, 56, 2477 (1997).
- [5] A. Casado, T.W. Marshall, E. Santos. J. Opt. Soc. Am. B 15, 1572 (1998).
- [6] A. Casado, A. Fernández-Rueda, T.W. Marshall, J. Martínez, R. Risco-Delgado, E. Santos. Eur. Phys. J. D 11, 465 (2000),
- [7] A. Casado, T.W. Marshall, R. Risco-Delgado, E. Santos. Eur. Phys. J. D 13, 109 (2001).
- [8] A. Casado, R. Risco-Delgado, E. Santos. Z. Naturforsch. 56a, 178 (2001).
- [9] A. Casado, S. Guerra, J. Plácido. J. Phys. B: At. Mol. Opt. Phys. 41, 045501 (2008).
- [10] A. Casado, S. Guerra, J. Plácido. Advances in Mathematical Physics 501521 (2010).
- [11] K. Dechoum, T.W. Marshall, E. Santos, J. mod. Optics 47, 1273 (2000).
- [12] The Wigner transform applies a quantum state ρ in a real, multivariable function:

$$W[\rho]: \rho \to W(\{x, p\}_{\mathcal{H}}) \in \mathcal{R}, \tag{86}$$

where the set $\{x,p\}_{\mathcal{H}}$ of variables upon which $W(\cdot)$ takes values depends on the structure of \mathcal{H} , the (quantum mechanical) Hilbert space of the system. Under certain restrictions (for a subset of all possible quantum states), $W[\rho]$ can be interpreted as a (joint) probability density function. For instance see:

Y.S. Kim, M.E. Noz. "Phase space picture of Quantum Mechanics (Group theoretical approach)", *Lecture Notes in Physics Series* - Vol **40**, ed. World Scientific.

- [13] For instance, from E. Santos, "Photons Are Fluctuations of a Random (Zeropoint) Radiation Filling the Whole Space", in *The Nature of Light: What Is a Photon?*, p.163, edited by Taylor & Francis Group, LLC (2008): "one of the additional hypotheses used, introduced by Clauser and Horne with the name of "no-enhancement", is naturally violated in SO because a light beam crossing a polarizer may increase its intensity, due to the insertion of ZPF in the fourth channel (...), which is the possibility excluded by the no-enhancement assumption".
- [14] See in arxiv, Casado et al.
- [15] The expressions for the detection probabilities in the Wigner-PDC picture can be obtained by mere manipulation of Glauber's original expressions for marginal and joint detections, respectively:

$$P_i \propto \langle \psi | \hat{E}_i^{(-)} \hat{E}_i^{(+)} | \psi \rangle,$$
 (87)

$$P_{i,j} \propto \langle \psi | \hat{E}_i^{(-)} \hat{E}_j^{(-)} \hat{E}_j^{(+)} \hat{E}_i^{(+)} | \psi \rangle.$$
 (88)

with $|\psi\rangle$ representing the state of the fields in all space, the field operator $\hat{E}_i^{(-)}$ ($\hat{E}_i^{(+)}$) defined at the position of the *i*-th detector \mathbf{r}_i and containing only creation (annihilation) operators, therefore giving rise to expressions

- in the so-called $normal\ order$ of quantum operators. See $[2,\ 5]$ for details.
- [16] A. Casado, T.W. Marshall, R. Risco-Delgado, E. Santos. "A local hidden variables model for experiments involving photon pairs produced in parametric Down Conversion", arXiv:quant-ph/0202097v1 (2002). Roughly speaking, their proposal can be understood as

Roughly speaking, their proposal can be understood as a substitution of our function $f(I(\alpha)) \equiv f(I(\alpha) - I_0)$ by a new one

$$f_{trial}(I(\alpha) - I_0(\alpha)) \equiv \hat{f}_{trial}(\alpha) \equiv (1 - e^{-\zeta \cdot \left[\bar{I}(\alpha) - \bar{I}_0(\alpha)\right]}) \cdot \Theta \left[\bar{I}(\alpha) - I_m\right], \quad (89)$$

where ζ, I_m are parameters, Θ is the Heaviside function, $I_m > I_0$ represents a "threshold" intensity and $\bar{I}(\alpha), \bar{I}_0(\alpha)$ correspond to our intensities $I(\alpha), I_0(\alpha)$, except for the fact that they are calculated performing a previous spatial and temporal integration (on the respective angular and temporal windows of observation) of the field complex amplitude.

The differences between the former proposal and our approach are two. First, the inclusion of spatial/temporal integration; regardless of its relevance in relation to the physical behavior of detectors [33] (we insist that that is not our focus here), the (quantum electrodynamical) model from where we depart (for instance see eq. 4.2 in [2]) involves indeed a "point-like" model of a detector: it is therefore at that stage were proper refinements should be done, and we justified in leaving the question aside for further works. As a second one, f_{trial} deviates from the quantum predictions both at low and high intensities [24], consistently with the fact that neither f_{trial} nor the models in [19] belong to our class of acceptable functions f (guaranteeing a one-to-one correspondence with the initial quantum electrodynamical model).

Indeed, we are left to wonder how our much less ambitious but certainly necessary step was not properly attacked before; it is our view that a project which involves unsolved problems both at the purely mathematical and physical levels should be approached (at least) in two steps. Here we have taken the first of them (regarding the purely mathematical issues); the second would stand for applying all the necessary constraints to reproduce the actual physical behavior of detectors.

- [17] G. Brida, M.Genovese, M. Gramegna, C. Novero, E. Predazzi, arxiv.org/abs/quant-ph/0203048 (2002).
- [18] G. Brida, M.Genovese, J. mod. Optics **50** 1757 (2003).
- [19] T.W. Marshall, E. Santos, "A classical model for a photodetector in the presence of electromagnetic vacuum fluctuations", arXiv:0707.2137.
- [20] For all t, \mathbf{r} , one particular mode of the ZPF, with \mathbf{k} the wave-vector and γ the two possible orthogonal polarization states, is always expressible as

$$\mathbf{E}_{0}(\alpha, \alpha^{*}) = \sum_{\mathbf{k}, \gamma} \left[\alpha_{\mathbf{k}, \gamma} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} + \alpha_{\mathbf{k}, \gamma}^{*} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \right], \quad (90)$$

where the (stochastic) amplitude $\alpha_{\mathbf{k},\gamma}$ determines, for all space, a mode (\mathbf{k},γ) . Of course, we can always define

$$\alpha_{\mathbf{k},\gamma}(\mathbf{r}) = \alpha_{\mathbf{k},\gamma}(\mathbf{r} = 0)e^{i\mathbf{k}\cdot\mathbf{r}},$$
(91)

so that now, for a certain mode ω and at a particular point of space:

$$\mathbf{E}_{0}(\mathbf{r}) = \sum_{\mathbf{k},\gamma} \left[\alpha_{\mathbf{k},\gamma}(\mathbf{r}) e^{-i\omega t} + \alpha_{\mathbf{k},\gamma}^{*}(\mathbf{r}) e^{i\omega t} \right], \tag{92}$$

where it is obvious that the amplitudes $\alpha_{\mathbf{k},\gamma}(\mathbf{r})$ follow exactly the same probability distribution as $\alpha_{\mathbf{k},\gamma}$, i.e., their distribution is governed by $W(\alpha, \alpha^*)$. This all said and for the sake of simplicity, we shall abuse notation (as is done customarily in [2–8]) with

$$\alpha, \alpha^* \equiv \{\alpha_{\mathbf{k},\gamma}, \alpha_{\mathbf{k},\gamma}^*\} \tag{93}$$

i.e., α , α^* denotes a set of mode amplitudes (therefore in principle defined for all space) for a given ω , (one or a set of) relevant wave-vectors \mathbf{k} and (the relevant) polarizations components γ . Finally, of course the overall vacuum field is obtained as a sum of all modes.

- [21] J.F. Clauser, M.A. Horne. Phys. Rev. D 10, 526 (1974).
- [22] The so-called *factorability* condition [21], which for whatever two space-like separated observables A, B would read, in our notation,

$$P(A = a, B = b|\lambda) = P(A = a|\lambda) \cdot P(B = b|\lambda), \quad (94)$$

is not, as sometimes assumed, a necessary one $(\forall a,b,\lambda)$, unless the outcomes of the measurements (either numerical ones or simply whether there is going to be a detection of not) are completely deterministic on λ .

For instance, suppose that for $\lambda = \lambda_0$, either A = B = 1 or A = B = 0 with equal probability; it is easy to see that $P(A = B = 1 | \lambda_0) \neq P(A = 1 | \lambda_0) \cdot P(B = 1 | \lambda_0)$ $(\frac{1}{2} \neq \frac{1}{4})$.

We have addressed thoroughly this question in App. 2; anyway and as shown there, the assumption of (94) in the case of [21] is nevertheless not criticizable, since one can always find (or define) a new hidden variable that guarantees it.

- [23] In the present framework, each wave-vector \mathbf{k} defines a plane wave propagating through the entire coordinate space, which obliges us to take some precautions. Let $\{\mathbf{k}_s\}$ and $\{\mathbf{k}_i\}$ be the sets of relevant wavevectors for the sets of relevant amplitudes $\{\alpha_s\}$ and $\{\alpha_i\}$, respectively, then we will suppose that $\{\mathbf{k}_s\} \cap \{\mathbf{k}_i\} = \emptyset$; in the (really very unlikely) case that this did not happen, we can always redefine α_i so the independence with $\{\alpha_s\}$ still holds.
- [24] For low intensities, the departure from the quantum mechanical model stands for a certain "dead-zone" and a much higher dark count rate.
 At first order in perturbation theory, the expectation value of the number of photons is zero and therefore such

value of the number of photons is zero and therefore such rate vanishes: experimentally observed dark counts are usually associated to thermal effects or electronic noise. Besides, a saturation effect is introduced, which on the other hand is a feature of all real detectors, but in this case it appears at much lower intensities (for energies lower than one photon). Consistency with experimental data demands compliance with certain relations provided in that same paper [16].

[25] Indeed, the laser is described by a coherent state, which for a high intensity signal means that it can be regarded as quasi-classical wave with well defined amplitude and phase (and introduced as a non-quantized external potential, see [2] for instance).

The phase of this complex amplitude is for instance relevant in expressions (4.10) from [2], and has the potential to interfere constructive or destructively with the α 's, increasing or decreasing the overall intensity of the signal, therefore modifying the detection probabilities. From this point of view, such phase cannot be at all excluded from the vector of relevant hidden variables and therefore non-factorability (of detection probabilities) in α is the most natural feature to expect (though not strictly necessary), see our detailed analysis in Sec. IV.

- [26] R. Risco-Delgado, private communication.
- [27] For reference on how, and under which hypothesis LHV models are built:
 - (i) A. Cabello, D. Rodríguez, I. Villanueva. Phys. Rev. Lett. **101**, 120402 (2008),
 - (ii) A. Cabello, J. -Å. Larsson, D. Rodríguez. Phys. Rev. A 79, 062109 (2009).
- [28] The fair sampling assumption is already formulated in [44]: "given a pair of photons emerges from the polarizers, the probability of their joint detection is independent of the polarizer orientations".
- [29] We believe the distinction between homogeneous and inhomogeneous inequalities was first introduced by M.Horne, then later updated by E.Santos (ref). We will supersede previous definitions with our own here: we will call inhomogeneous inequalities all the ones involving coincidence rates of different order (for instance, marginal and joint ones), homogeneous on the contrary.
 - Of course, if for instance a certain subset of the observables involved in the inequality does not have any uncertainty associated to its detection, the inclusion of inhomogeneous terms may not lead to the requirement of supplementary assumptions; this is not the case of experiments with photons, anyway.
- [30] The no-enhancement assumption [21]: "for every emission λ , the probability of a count with a polarizer in place is less or equal to the probability with the polarizer removed", where λ is the (hypothetical) hidden variable expressing the state (at least the initial one) of the pair of particles. We remark: "for every emission λ ".
- [31] An inhomogeneous Bell inequality [29] requires the estimation of coincidence rates of different order (for instance single and double, or double and triple); due to the fact that there is no way to identify if the whole set of particles have been simultaneously emitted and then gone undetected, or they were simply never emitted at all, the test would always require supplementary assumptions; of course from a wave-like perspective, the issue is even more evident. In any case, marginal detection rates would not be experimentally accessible unless we assume "independent errors" at the detector for every and each of the "states" in the model (i.e., detection probabilities cannot depend on the hidden variables: α's, other such as the phase of the pump. etc).
- [32] Let us consider a Bell experiment and let η be what people in QInf define as "detection efficiency" [27] (from our point of view this is clearly a misleading term, we should call it simply "detection rate", whether is reduced or not by technological imperfections, see Sec. V A). A possible Local Hidden Variables (LHV) model for the experiment is then composed by a set of states $\{s\}$ with probabilistic weights (or frequencies) ρ_s : all restrictions the model must satisfy (either regarding the behavior of detectors or the quantum mechanical predictions) are linear in $\{\rho_s\}$

- and the trivial solution $\eta=0$ (all $\rho_s=0$ but for the one that predetermines no detections at all) is always admissible, implying there is always some $\eta_{\rm crit} \geq 0$ such that for all $\eta \leq \eta_{\rm crit}$ the desired LHV can be built. The reasoning applies whether if the LHV just simulates a violation of a particular Bell inequality or also every other quantum prediction for a given state and set of observables (the addition of more restrictions would in principle lower $\eta_{\rm crit}$).
- [33] Though from the mathematical point of view our model here already predicts a vanishing detection probability for the ZPF alone, a physical interpretation of the absence of an observable detection rate as a result of the vacuum fluctuations, or at least that of a significant one, is still an open question. Following Santos's work, the absence of the ZPF from the observational spectrum could be justified, for realistic detectors, on the combination of the following properties:
 - (i) the already commented subtraction of the average ZPF intensity as a result of Glauber's expression in the normal order [15],
 - (ii) spatial and temporal integration, involving the autocorrelation properties of the ZPF,
 - (iii) a "low band pass" frequency response.
 - See, for instance: Emilio Santos, "How photon detectors remove vacuum fluctuations", http://arxiv.org/abs/quant-ph/0207073v2 (2008).
 - We remark again that this issue is in any case irrelevant for our main results here: it concerns models built outside of the quantum framework (which is not our case). Nevertheless, we conjecture that our proposed interpretation of (i) in Sec. VI may be of use to make progress on the question.
- [34] E. Kot, N. Grønbech-Jensen, B.M. Nielsen, J.S. Neergaard-Nielsen, E.S. Polzik, A.S. Sørensen. Phys.Rev.Lett. 108, 233601 (2012).
- [35] V. Handchen et al, NATURE PHOTONICS 6 (2012).
- [36] Peruzzo et al, SCIENCE, Reports, **338** (2012).
- [37] Kaiser et al, SCIENCE, Reports, 338 (2012).
- [38] J.S. Neergard-Nielsen *et al*, OPTIC EXPRESS **15**, 7940 (2007).
- [39] Let Λ be a space of events, all possible probability assignations within the formalism, $F(\cdot): \lambda \in \Lambda \to F(\lambda)$, satisfy $0 \leq F(\lambda) \leq 1 \ \forall \lambda$ and, for any partition $\{\Lambda_i\}$ of Λ (assuming of course additivity on disjoint subsets of events), $\sum_i F(\Lambda_i) = 1$.
 - All Bell inequalities (at least those written in terms of probabilities) can be obtained as a derivation from these laws, plus some very simple supplementary assumptions: statistical independence of variables defined at distant locations. Of course, Bell inequalities written in terms of correlations can be also written in terms of probabilities. Many do no even need such supplementary assumptions (space-like separation ones or of another kind): such is the case of both the CH [21] and the CHSH [44] inequalities. To illustrate this see for instance [45]'s derivation of the CH inequality; the CHSH inequality can on the other hand also be interpreted as a mere algebraic inequality on whatever four quantities taking values ± 1 .
- [40] $W(\alpha, \alpha^*)$ is a Gaussian [2]; hence $W(\alpha, \alpha^*) > 0 \ \forall \alpha, \alpha^*$.

- [41] For a recent work reviewing these concepts see for instance: T. Norsen, "J.S. Bell's Concept of Local Causality", http://arxiv.org/abs/0707.0401.
- [42] A possible (not unique) procedure to build μ is this: for each M_i , we define a new random variable σ_i and assign values for each pair $(\lambda, m_{i,k})$: $\sigma_i \equiv \sigma_i(\lambda, m_{i,k})$, and now simply do $\mu \equiv \bigoplus_i \sigma_i$. As built, σ_i 's are not necessarily independent from one another, nor are they necessarily independent from λ .
- [43] A. Casado, private communication.
- [44] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt. Phys. Rev. Lett. 23, 880 (1969).
- [45] J.-Å. Larsson and J. Semitecolos, Phys. Rev. A 63, 022117 (2001).
- [46] R. Risco-Delgado. PHD Thesis. Universidad de Sevilla (1997).
- [47] A.Casado, PHD Thesis: "Estudio de los experimentos de conversión paramétrica a la baja con el formalismo de la función de Wigner" (Universidad de Sevilla, 1998).
- [48] The PDC model is by construction restricted to a certain subset of QED-states, obtained directly from a mix of the vacuum state (hence one with positive Wigner function) with a quasi-classical signal (the laser), and a time-evolution governed by a quadratic Hamiltonian (hence one that preserves the positivity of the Wigner function); for this last point, see [12] or [47].
- [49] For instance, the description of the "pump" (laser beam) as a non-quantized, external potential is just an approximation (one that allows for a Hamiltonian that is quadratic in creation/annihilation operators, and hence preserves positivity of the Wigner function); in any case, we should not forget that further refinements may include not only quantization of the laser but also additional terms expressing the interaction with further ZPF modes neglected in the present formulation.
- [50] For instance, the inclusion of particle sub-structure could also be the way forward to explain other peculiarities of the quantum world that seem so far alien to a purely classical framework. In particular, two very crucial points:

 (i) the appearance of a discrete spectrum of observable energy exchanges between matter and the electromagnetic field, due to the presence of meta-stable states ("attractors") in the classical phase space of the system (for instance the Hydrogen atom), once observations are assumed as some time-average that smears out transients, being therefore reduced to a discrete spectrum of states and transitions between them;
 - (ii) the possibility of highly directional radiation patterns (emission of energy) and resonances (absorption of energy), in the same way as they arise in macroscopic systems with rich spatial structure.
 - Point (i) can potentially accommodate the famous "quanta" $E=h\omega$ relation, attempts at which we have left at an intermediate step elsewhere; point (ii) would make possible the transfer of energy through long distances with no spread.
- [51] See for instance "Can we celebrate defeat for the photon by Maxwell-Planck theory?", in http://crisisinphysics.wordpress.com/2011/08/01.