## Degenerate Bose Gas: A New Tool for Accurate Frequency Measurement

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We propose a new way of detecting frequencies using superradiant Rayleigh scattering from degenerate Bose gases. A measurement of the time evolution of population at the initial momentum state could determine an unknown frequency with respect to a known one at which the pump laser's frequency modulates. A range of frequencies from kHz to  $\sim$ MHz could be determined with a fractional uncertainty  $10^{-6}$ .

PACS numbers: 03.75.-b, 03.75.Kk, 06.30.Ft, 32.80.Qk

Bose Einstein condensate (BEC) atoms have long-range spatial coherence [1] which offers the possibility to study quantum optics in a new domain where the atom-photon interactions could be altered [2]. The interest in superradiant Rayleigh scattering and matter-wave amplification originates due to long coherence time of the condensate [3–5]. The first photon scattered from the condensate leaves a perturbed BEC, which induce the next photons to scatter along the same direction. Superradiant radiation occurs due to spontaneous Rayleigh scattering of photons from an elongated BEC [3]. Due to momentum conservation atoms end up in a different momentum state after each scattering [6]. These scatterings are equally probable in all directions so atoms could go to various momentum states. However the elongated shape of the condensate introduces self amplification of a particular mode and forms a matter wave grating [4, 7]. This redistribute atoms in different momentum states [4, 5]. Reverse avalanche occurs in the scattered photon mode which is the well known Dicke superradiance [8]. This mechanism is also known as collective atomic recoil lasing (CARL). That was first described for a free electron laser system [9, 10], however it was first observed in an atomic vapor cell [11]. Damping of photons limit intensity of the superradiance [12, 13]. This can be improved by storing the BEC in an optical cavity which will enhance the atom-photon interactions. The influence of atomic motion on the superradiant light scattering from a moving BEC was investigated theoretically and experimentally [14, 15]. In this letter we propose a new way of detecting frequencies using the technique of population transfer to a different momentum state in superradiance. An analytic model is described for detection of an unknown frequency with respect to a reference one. Also we give an experimental scheme for realization of our proposal.

We consider the superradiant radiation occurs when a off-resonant pump laser beam impinges along the axial direction of a cigar-shaped BEC. The atom-photon interaction initiates after the condensate is released from its confining potential in an optical dipole trap (ODT), as shown in Fig. 1 (a). An ensemble of degenerate atoms maintain its elongated shape even after 1-10 ms time of flight (TOF). Atom-photon interactions are fast processes than the time at which superradiant dynamics develop. This is limited by the decay of pump photons from the condensate and is faster than the expansion rate of the BEC. We consider BEC is very dilute after expansion and hence intra-atomic interactions are ignored. In an anisotropic condensate, correlation between scattered photons is enhanced when scattering occurs along the elongated direction (z-axis) of the condensate. This is known as end-fire mode in the regime of Dicke superradiance [16]. Here an atom scatters photon from the pump laser beam of wave-vector  $k_p$  and propagates along the z-direction. The recoil photon along the opposite direction results in a net momentum change  $\Delta p = 2\hbar k_p$  of the atom as shown in Fig. 1 (b). Internal state of atoms do not change but they spread between two momentum states separated by  $\Delta p$ . Hence atomic center of mass motion changes but different momentum states get coupled.

The Hamiltonian describing the superradiant Rayleigh scattering is

$$\hat{H} = \hat{H}_a + \hat{H}_p + \hat{H}_{a-p},\tag{1}$$

which consists of energy associated with free atoms  $\hat{H}_a$ , pump laser field  $\hat{H}_p$  and interaction of atoms with the photons in the pump laser light  $\hat{H}_{a-p}$ . We will explire dynamics along the z-axis of the elongated condensate. For a two level atom of mass m the atomic Hamiltonian is

$$\hat{H}_{a} = \int dz \left[ \hat{\psi}_{|g\rangle}^{\dagger}(z) \left( -\frac{\hbar^{2}}{2m} \frac{\partial^{2}}{\partial z^{2}} \right) \hat{\psi}_{|g\rangle}(z) + \hat{\psi}_{|e\rangle}^{\dagger}(z) \left( -\frac{\hbar^{2}}{2m} \frac{\partial^{2}}{\partial z^{2}} + \hbar \omega_{a} \right) \hat{\psi}_{|e\rangle}(z) \right], \tag{2}$$

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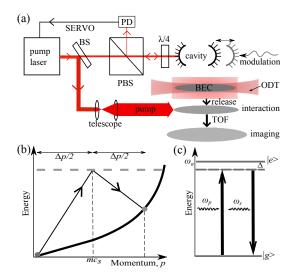


Figure 1: In superradiant scattering: (a) schematic of the experiment, (b) population transfer to the higher momentum state and (c) interaction of a two level atom with a far off-resonant photon, where  $c_s$  is speed of phonons in the BEC.

where  $\hat{\psi}_{|g\rangle}$  ( $\hat{\psi}_{|e\rangle}$ ) and  $\hat{\psi}_{|g\rangle}^{\dagger}$  ( $\hat{\psi}_{|e\rangle}^{\dagger}$ ) are annihilation and creation operators of the atom in its ground (excited) states which are separated by a frequency  $\omega_a$  and  $\hbar$  is the Planck constant. The bosonic field operators satisfy the equal time commutation relations  $[\hat{\psi}_j(z), \hat{\psi}_{j'}^{\dagger}(z')] = \delta_{j,j'}\delta(z,z')$  also  $[\hat{\psi}_j(z), \hat{\psi}_{j'}(z')] = [\hat{\psi}_j^{\dagger}(z), \hat{\psi}_{j'}^{\dagger}(z')] = 0$ , where subscripts j,j' refers to both  $|g\rangle$  and  $|e\rangle$  states respectively. In this case one can neglect the excited state population in Eq. (2) and spontaneous emission since frequency of the pump laser  $\omega_p \ll \omega_a$ . Hamiltonian of the optical pump field of strength  $\eta$  is

$$\hat{H}_{n} = -i\hbar\eta(\hat{c}_{n} - \hat{c}_{n}^{\dagger}),\tag{3}$$

where  $\hat{c}_p$  ( $\hat{c}_p^{\dagger}$ ) are the annihilation (creation) operator of photons, which satisfy the commutation relation  $[\hat{c}_p, \hat{c}_p^{\dagger}] = 1$ . The interaction Hamiltonian of atoms with the photons in the pump laser light is

$$\hat{H}_{a-p} = -i\hbar g c_p \left[ \int dz \, \hat{\psi}^{\dagger}_{|e\rangle}(z) e^{ik_p z} e^{i\phi(t)} \hat{\psi}_{|g\rangle}(z) \right.$$

$$\left. + \int dz \, \hat{\psi}^{\dagger}_{|e\rangle}(z) e^{-ik_p z} e^{-i\phi(t)} \hat{\psi}_{|g\rangle}(z) \right] + \text{h. c.} ,$$

$$(4)$$

where h. c. is the Hermitian conjugate. The coupling strength  $g = d\sqrt{\omega_a/(2\hbar\epsilon_o V)}$  between atoms and photons in the pump light [15] depends on induced electric dipole moment d of the atom, volume V of the condensate interacting with pump beam and permittivity  $\epsilon_o$ . The phase difference between the pump and the scattered photons is

$$\phi(t) = \delta(1 - \epsilon \sin \Omega t)t,\tag{5}$$

where  $\delta = \omega_p - \omega_s$  is their frequency difference. Frequency of the pump laser modulates at  $\epsilon \sin \Omega t$  which evolves  $\phi(t)$  during the atom-photon interaction time. The perturbing modulation amplitude is small,  $\epsilon \delta < \delta$ .

Modulation of  $\omega_p$  can be obtained by coupling the pump light to an unstable Febry-Perot cavity. One mirror of the cavity is firmly fixed and the external modulation is coupled to the other mirror, which changes the cavity length as shown in Fig. 1 (a). A small fraction of purely linear polarized pump light can be coupled to the cavity through a combination of polarizing beam splitter (PBS) and  $\lambda/4$ -wave plate. The transmitted light leaked out of the cavity will have  $\pi/2$ -rotated linear polarization after passing twice through the  $\lambda/4$ -wave plate and hence reflects off from the PBS. This reflected light can be detected by a fast photodiode (PD) whose response time needs to be greater  $(2\pi/\Omega)^{-1}$ . The PD detected signal can be used for frequency stabilization of the pump laser by Pound-Drever-Hall locking technique [17]. The moving mirror of the cavity will shift the lock point of  $\omega_p$ , which then will be modulated. After releasing from ODT, the expanded BEC will fall through a large cross section pump light after a delay of 1-2

ms. The atom-photon interaction will be controlled by pulsing the pump light. After some time of flight the BEC will be imaged for quantitative measurement of the population in the ground state.

Since the pump light is approximately -2 GHz detuned from  $\omega_a$  (Fig. 1 c), the excited state population can be eliminated adiabatically using the Heisenberg equation of motion  $\hat{\psi}_{|e\rangle} = i/\hbar \left[\hat{H}, \hat{\psi}_{|e\rangle}\right]$ . In this case the ground state population is

$$N = \int dz \, \hat{\psi}_{|g\rangle}^{\dagger}(z) \hat{\psi}_{|g\rangle}(z), \tag{6}$$

which is the total number of atoms in the condensate. This yields an equation of motion for the atoms

$$\frac{d}{dt}\hat{\psi}_{|g\rangle}(z) = i\frac{\hbar}{2m}\nabla^2\hat{\psi}_{|g\rangle}(z) - i\frac{2g^2}{\Delta}\hat{c}_p^{\dagger}\hat{c}_p 
\left[1 + \cos\left(2k_pz + \phi(t)\right)\right]\hat{\psi}_{|g\rangle}(z),$$
(7)

where  $\Delta = \omega_p - \omega_a$  is the detuning of the pump laser light and  $\hat{\tilde{c}}_p = \hat{c}_p e^{i\omega_p t}$ . Photons in the pump laser beam acquire an equation of motion

$$\frac{d}{dt}\hat{\tilde{c}}_{p} = -i\frac{2g^{2}}{\Delta}\hat{\tilde{c}}_{p} \int dz \,\hat{\psi}_{|g\rangle}^{\dagger}(z)\cos\left(2k_{p}z + \phi(t)\right)\hat{\psi}_{|g\rangle}(z) 
-\kappa\hat{\tilde{c}}_{p} + \eta,$$
(8)

where  $\kappa \leq c/2L$  is the damping rate of photons in the BEC of length L and c is the velocity of light. Now in the following, the bosonic operators  $\hat{\psi}_{|g\rangle}$  and  $\hat{\tilde{c}}_p$  are substituted by the coherent condensate wavefunction  $\langle \hat{\psi}_{|g\rangle} \rangle = \psi_{|g\rangle}$  and the classical light field amplitude  $\tilde{c}_p$  respectively.

Since L is much longer than the radiation wavelength, one can apply periodic boundary condition. We also assume homogeneous density distribution for simplifying the calculation. In that case evolution of the ground state wavefunction can be written in the basis of eigenfunctions

$$\psi_{|g\rangle}(z,t) = \sum_{n=0}^{\infty} U_n(t) e^{2ink_p z} e^{in\phi(t)}, \tag{9}$$

where eigenvalues are  $n \cdot \Delta p$  for  $n = 0, 1, 2 \dots$  and population at the  $n^{\text{th}}$  eigenstate is  $\rho_n(t) = U_n(t)^* U_n(t)$ . This atomic motion arises under the assumption that the atoms in the BEC are delocalized and their momentum uncertainty is negligible. Using  $\psi_{|g\rangle}(z,t)$  from Eq. (9) in the Eqs. (7) and (8), three coupled ordinary differential equations are obtained. Equation of motion of atoms in the  $n^{\text{th}}$  level with momentum  $p_0$  is

$$\frac{d}{dt}U_n = -i4n^2\omega_r U_n - in\dot{\phi}(t)U_n$$

$$-i\frac{g^2N}{\Delta}\alpha[2U_n + U_{n+1}].$$
(10)

Similarly equation of motion of atoms shifted to the  $(n+1)^{\text{th}}$  level with momentum  $p_0 + \Delta p$  is

$$\frac{d}{dt}U_{n+1} = -i4(n+1)^2 \omega_r U_{n+1} - i(n+1)\dot{\phi}(t)U_{n+1} - i\frac{g^2 N}{\Lambda} \alpha [2U_{n+1} + U_n].$$
(11)

Equation of motion of photons in the pump laser beam is

$$\frac{d}{dt}\widetilde{c}_p = -i\frac{2g^2N}{\Delta}\widetilde{c}_p\rho_{n,n+1} - \kappa\widetilde{c}_p + \eta, \tag{12}$$

where  $\omega_r = \hbar k_p^2/2m$  is the single photon recoil frequency,  $\alpha = \tilde{c}_p^{\dagger} \tilde{c}_p$  is the intensity of the light field and  $\rho_{i,j} = U_i^{\star} U_j$  is the coherence between  $i^{\rm th}$  and  $j^{\rm th}$  eigenstates. The quantity  $\rho_{i,j}$  reduces to the density of a state when i = j. We have introduced rescaled light amplitude  $\tilde{c}_p \to \sqrt{N} \tilde{c}_p$  in the Eq. (12).

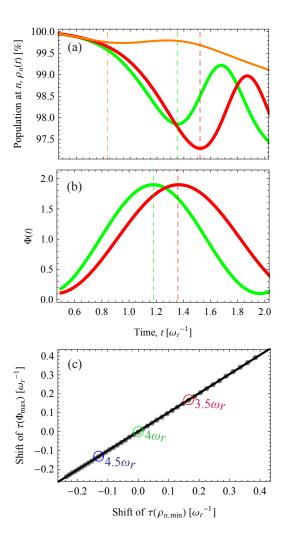


Figure 2: (Color online) The solid lines show (a) population at the initial momentum state, (b)  $\Phi(t)$  at modulation frequencies  $\Omega = 4\omega_r$  (green),  $3.45\omega_r$  (red) and  $1.2\omega_r$  (orange). A strong modulation of the pump laser at  $\epsilon = 0.8$  is considered for all spectra. The dashed lines show  $\tau_i$  where the first minima in the  $\rho_{n,\,\text{min}}$  and the first maxima  $\Phi_{\text{max}}$  appears, which are distinguished by their respective colors. (c) Shift of  $\tau(\Phi_{\text{max}})$  as  $\tau(\rho_{n,\,\text{min}})$  shifts with change of  $\Omega$  from blue (blue) to red (red) detuning relative to  $4\omega_r$  (green) and the solid line is a linear fit to it.

For simplicity we consider the mass flow occurs from  $n^{\text{th}}$  to a  $(n+1)^{\text{th}}$  level. That gives a population difference between two momentum states

$$\Delta \rho(t) = \rho_n(t) - \rho_{n+1}(t), \tag{13}$$

and  $\rho_n + \rho_{n+1} = N/V$  remains conserved. A new set of equations can be obtained for the rate of change of coherence function

$$\frac{d}{dt}\rho_{n,n+1} = i[4(1+2n)\omega_r + \dot{\phi}(t)]\rho_{n,n+1} 
-\mathcal{D}\rho_{n,n+1} + i\alpha\frac{g^2N}{\Delta}\Delta\rho,$$
(14)

and for the population difference

$$\frac{d}{dt}\Delta\rho = -4\alpha \frac{g^2N}{\Delta}\Im\left[\rho_{n,n+1}\right],\tag{15}$$

where,  $\Im$  indicates imaginary part of the correlation function. The decoherence rate of atoms  $\mathcal{D}$  arises due to Doppler broadening, inhomogeneity in the condensate and phase diffusion. The phase diffusion mechanism depends on  $\delta$  and  $p_o$ . Equations (14)-(15), are then equivalent to the well known Maxwell-Bloch equations for a two-level system [6, 18].

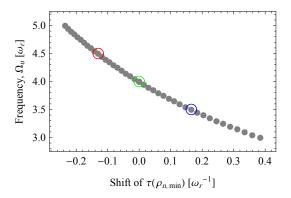


Figure 3: At  $\Omega_r = 4\omega_r$  (green) the relation between  $\Omega_u$  and shift of  $\tau(\rho_{n,\,\text{min}})$ . Example blue and red detuned  $\Omega_u$  are indicated by their colors.

Solution of the Eqns. (14) and (15) identify quantum and semiclassical regimes of the superradiance scattering [15]. In the quantum regime,  $g^2\sqrt{N}/\omega_r\Delta < 2\sqrt{\kappa/\omega_r}$ , momentum of each atom changes by  $\Delta p$  due to scattering of single photon at small  $\kappa$ . At larger  $\kappa$ , multiple scattering of photons give rise to momentum change of atoms greater than  $\Delta p/2$ . This semiclassical regime starts at  $g^2\sqrt{N}/\omega_r\Delta > 2\sqrt{\kappa/\omega_r}$ . We focus on quantum superradiance for detection of a frequency. The CARL phenomena starts when the build up laser power  $\alpha(t) = \tilde{c}_p^{\dagger} \tilde{c}_p$  in the condensate reaches the threshold, which results in the population transfer to  $(n+1)^{\text{th}}$ -level. Coherence time of the matter wave grating determines the efficiency of the superradiant process. The matter wave grating has a finite lifetime which depends on  $p_0$  and the superradiance fades off with the decay of the matter wave grating.

Detection of an unknown frequency  $\Omega_u$  requires measurement of the time evolution of  $\rho_n(t)$  as shown in Fig. 2 (a), when the pump laser frequency is modulated at  $\Omega_u$ . A measurement of the  $\rho_n(t)$ -spectrum at a known modulation frequency  $\Omega_r$  while keeping rest of the conditions unchanged is required. Antinodes in these spectra corresponds to the maximum population transfer to the  $(n+1)^{\text{th}}$  level. The time  $\tau_i(\rho_{n,\text{min}})$  corresponding to the first antinode appears at later times for a higher  $\Omega_i$ , where the subscript i refers to the different modulation frequencies. For our interest, measurement of the  $n^{\text{th}}$  level population at least up to the time  $\tau_i(\rho_{n,\text{min}})$  is necessary. The phase modulating part  $\Phi(\Omega, t) = (1 - \epsilon \sin \Omega t)$ , from Eq. (5), evolves in time as shown in Fig. 2 (b) for two different  $\Omega_i$ . Like  $\tau_i(\rho_{n,\text{min}})$ , the time  $\tau_i(\Phi_{\text{max}})$  at which the first antinode appears, also shifts by the same amount with  $\Omega_i$ . Figure 2 (c) shows one-to-one shifts of  $\tau_i(\rho_{n,\text{min}})$  and  $\tau_i(\Phi_{\text{max}})$  with the change of  $\Omega_i$  with respect to  $\Omega_r = 4\omega_r$ .

For frequency detection, we first need to determine  $\tau_r(\rho_{n,\min})$  and  $\tau_u(\rho_{n,\min})$  from experimentally measured  $\rho_n(t)$ -spectra at known and unknown modulations  $\Omega_r$  and  $\Omega_u$  respectively. For modulations at  $\Omega_r$ , the condition of minima of the  $\rho_n(t)$ -spectra

$$\frac{d\Phi(\Omega_i, t)}{dt} = 0,\tag{16}$$

determines  $\tau_r(\Phi_{\text{max}}) = \pi/2\Omega_r$  for the first antinode. With this,  $\tau_u(\Phi_{\text{max}})$  for the unknown frequency is

$$\tau_u(\Phi_{\text{max}}) = \tau_r(\Phi_{\text{max}}) \pm |\tau_u(\rho_{n,\text{min}}) - \tau_r(\rho_{n,\text{min}})|, \tag{17}$$

where  $\tau_i(\rho_{n,\min})$  are measured experimentally. In this equation "+" appears when  $\tau_u(\rho_{n,\min}) > \tau_r(\rho_{n,\min})$  and "-" when the reverse condition is true. Equation (16) determines the unknown frequency  $\Omega_u = \pi/2\tau_u(\Phi_{\max})$  using  $\tau_u(\Phi_{\max})$  from Eq. (17). Figure 3 shows sample detection of  $\Omega_u$  with respect to the shifts of  $\tau(\rho_{n,\min})$  estimated from measured time evaluation of  $\rho_n(t)$ . The rage of  $\Omega_u$  which can be detected depends on fluctuation limited experimental resolution for measuring the change of population in the  $n^{\text{th}}$  level. As for example Fig. 2 (a) shows a  $\rho_n(t)$ -spectrum at  $\Omega = 1.2\omega_r$  from which the  $\tau(\rho_{n,\min})$  will be difficult to extract due to statistical fluctuations in the measurement. Assuming an experiment can resolve  $\sim 1\%$  atom number fluctuation, a range of frequencies from  $2.5\omega_r - 6\omega_r$  would be possible to detect by this method. As for example recoil frequency  $\omega_r = 2\pi \times 3.7$  kHz and  $2\pi \times 63$  kHz for the D2-transition of <sup>87</sup>Rb and <sup>7</sup>Li isotopes respectively. This estimates a range of frequencies from few kHz to about a MHz could be detected by this technique.

Resolutions of the measured  $\tau_i(\rho_{n,\min})$  could be accurate to  $\sim 2$  ns using precise pulse train generators [19] as a timing system. This gives a frequency resolution  $\Delta\Omega = s\omega_r^2/\pi \times 10^{-9}$ , where  $s \simeq 10/3$  is approximate slope of the spectrum shown in Fig. 3. As for example  $\Delta\Omega = 2\pi \times 0.1$  Hz for <sup>87</sup>Rb and which gives a fractional uncertainty  $\sim 10^{-6}$ . A state-of-the-art frequency standard requires flywheel oscillators [20], since the atomic frequency standards

have limited run time. The proposed techniques could be incorporated for frequency stabilization of the flywheel oscillators.

The authors thank Amitava Sen Gupta for useful discussions. A. Bhattacherjee acknowledges financial support from the Department of Science and Technology, India (grant SR/S2/LOP-0034/2010) and S. De carried out this work as a part of the STIOS program funded by CSIR-NPL.

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