AN NECESSARY CONDITION FOR ISOPERIMETRIC INEQUALITY IN WARPED PRODUCT SPACE

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ABSTRACT. In this note, we show that there is some counterexample for isoperimetric inequality if the condition $(\phi')^2 - \phi'' \phi \leq 1$ does not hold in warped product space.

1. Introduction and preliminary

The isoperimetric inequality is one of the central problem in classical differential geometry. For simple closed curves on plane, it is well known that

$$L^2 > 4\pi A$$
.

where L and A are length of the curve and area of the domain bounded by curve. There is a lot of generalization for isoperimetric inequality. (See [3] for a quick survey.) One aspect is that one can consider these inequalities in general ambient space. The simplest ambient space maybe the space form. For dimension 2 case, suppose K is its Gauss curvature, then, we have the following type isoperimetric inequality

$$L^2 > 4\pi A - KA^2.$$

Here the meaning of L, A is same as previous. The equality in the above will hold, if the curve is a circle.

The more general ambient space may be warped product space since their geodesic manifold are same. In fact, in n + 1 dimensional space \mathbb{R}^{n+1} , the warped product metric is defined by

(1.1)
$$ds^{2} = \frac{1}{f^{2}(r)}dr^{2} + r^{2}(\sum_{i=1}^{n}\cos^{2}\theta_{i-1}\cdots\cos^{2}\theta_{1}d\theta_{i}^{2}).$$

Here, $(r, \theta_1, \dots, \theta_n)$ is the polar coordinate. We always denote the ambient space with the above metric by N^{n+1} . The meaning of the paremeter r is the Euclidean distant of the point $(r, \theta_1, \dots, \theta_n)$. Let S(r) be a level set of r and B(r) be the bounded domain enclosed by S(r). We also define the area of S(r) and volume of B(r) by A(r) and V(r). Then we can define some single valuable function $\xi(r)$ by

$$A(r) = \xi(V(r)).$$

Thus, an appropriate isoperimetric inequality in this space has been proved by Guan-Li-Wang [1].

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Theorem 1 (Guan-Li-Wang). Let $\Omega \subset N^{n+1}$ be a domain bounded by a smooth graphical hypersurface M and $S(r_0)$. We assume that $\phi(r)$ and \tilde{g} satisfy condition

$$0 \le (\phi')^2 - \phi'' \phi \le 1$$
,

then

$$Area(M) \ge \xi(Vol(\Omega)),$$

where Area(M) is the area of M and $Vol(\Omega)$ is the volume of Ω .

The ambient metric of the above theorem is

(1.2)
$$ds^{2} = d\tilde{r}^{2} + \phi^{2}(\tilde{r})dS_{n-1}^{2}.$$

(1.1) can be reformulated by the above form. In fact, we let

$$\tilde{r} = \int \frac{dr}{f(r)}, \quad \phi(\tilde{r}) = r.$$

Thus, we have

$$1 \ge \left(\frac{d\phi}{d\tilde{r}}\right)^2 - \phi \frac{d^2\phi}{d\tilde{r}^2} = f^2 - rff',$$

which implies

(1.3)
$$\frac{1-f^2}{r^2} + \frac{ff'}{r} \ge 0.$$

In this note, we will show that the above condition is necessary condition. Explicitly, we have

Theorem 2. If the condition (1.3) does not hold for some r, there is some hypersurface which is isometric to the geodesic r-sphere and the volume of the convex body bounded by the hypersurface is bigger than the volume of r-geodesic ball.

Using the above theorem, we have the following corollary for metric (1.2).

Corollary 3. For the ambient space equipped with metric (1.2), if at some \tilde{r} , $\frac{d\phi}{d\tilde{r}} \neq 0$, there is some hypersurface which is isometric to the geodesic $\phi(\tilde{r})$ -sphere and the volume of the convex body bounded by the hypersurface is bigger than the volume of $\phi(\tilde{r})$ -geodesic ball. If at some \tilde{r} , $\frac{d\phi}{d\tilde{r}} = 0$, there is some \tilde{r}' closed to \tilde{r} such that we can find some hypersurface which is isometric to the geodesic $\phi(\tilde{r}')$ -sphere and the volume of the convex body bounded by the hypersurface is bigger than the volume of $\phi(\tilde{r}')$ -geodesic ball. In a word, in any case, the condition $(\phi')^2 - \phi''\phi \leq 1$ is necessary for the existence of isoperimetric inequality in warped product space.

Proof. For the case $\frac{d\phi}{d\vec{r}} > 0$, by the discussion between (1.2) and (1.3), it is obvious from Theorem 2. For the case $\frac{d\phi}{d\vec{r}} < 0$, we let

$$r = \phi(\tilde{r}), \quad f = -\frac{d\phi}{d\tilde{r}},$$

we still can obtain condition (1.3), which implies that we also have same result for this case. For the case $\frac{d\phi}{d\tilde{r}} = 0$, without loss of generality, we can assume $\phi(\tilde{r}) > 0$. Thus the condition $(\phi')^2 - \phi''\phi > 1$ implies $\phi'' < 0$ at \tilde{r} . Hence, there is some \tilde{r}'

close to \tilde{r} such that $\phi' > 0$ and $(\phi')^2 - \phi'' \phi > 1$ at \tilde{r}' , which is the first case we have discussed.

we always use \cdot to present the metric in the ambient space. At first, the following formula is valid for any $n \geq 1$.

Lemma 4. Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an open set. Let Σ be $\partial \Omega$ and $r \frac{\partial}{\partial r} = (z^1, z^2, \dots, z^{n+1})$ be its position vector. We also let $X = f(r)r \frac{\partial}{\partial r}$ be conformal Killing vector. Then the support function of Σ is defined by

$$\varphi = X \cdot \nu,$$

where ν is the outward unit normal on Σ . Hence we have

(1.4)
$$\operatorname{Vol}(\Omega) = \int_{\Sigma} g(r)\varphi dS,$$

where dS is the volume element of Σ and

(1.5)
$$g(r) = \frac{1}{r^{n+1}} \int_0^r \frac{t^n}{f(t)} dt.$$

2. An example

The aim of this section is to study the isoperimetric inequality using the example obtained in [2]. At first, let's briefly review our example. For given metric g on \mathbb{S}^n , the isometric embedding problem is trying to find some map

$$\vec{r}: \mathbb{S}^n \to N^{n+1}$$

such that

$$d\vec{r} \cdot d\vec{r} = g.$$

The homogenous linearized problem is

$$dr \cdot D\tau = 0.$$

Here D is the Levi-Civita connection corresponding to ds^2 . For our case, along the geodesic sphere, the metric does not change. Hence, any fixed vector field is the solution of the above system.

We use $(z^1, z^2, \dots, z^{n+1})$ as the coordinate of the ambient space N^{n+1} . We let (u_1, u_2, \dots, u_n) be the sphere coordinate and $(r, \theta_1, \theta_2, \dots, \theta_n)$ be the polar coordinate in ambient space. We present the geodesic sphere of radius r in warped product space by the map

$$(2.1) \vec{r}(u_1, \dots, u_n)$$

$$= r(\cos u_1 \cos u_2 \dots \cos u_n, \cos u_1 \cos u_2 \dots \cos u_{n-1} \sin u_n, \dots, \sin u_1).$$

For any sufficient small constant $\epsilon < \epsilon_0$, we can define some C^k hypersurface Y

$$Y = \vec{r} + (\epsilon + \epsilon^2 h^1(\sin u_1) + \epsilon^3 h^*(\sin u_1)) \frac{\partial}{\partial z^{n+1}},$$

where h^* is a single valuable function and h^1 is defined by

$$h^{1}(\sin u_{1}) = \frac{f^{2}(r) - 1}{2rf^{2}(r)} \sin u_{1}.$$

We can proved that

$$dY \cdot dY = d\vec{r} \cdot d\vec{r}$$
.

More detail of the above discussion can be found in section 8 and section 11 of [2]. We will denote the domain bounded by hypersurface Y by M_{ϵ} .

Thus, the hypersurface Y is isometric to the r geodesic sphere. Their area are same. In what follows, we calculate the volume of M_{ϵ} , namely expanding it for parameter ϵ . That is

$$Y * \frac{\partial}{\partial z^{n+1}} = r \sin u_1 + \epsilon + \frac{\alpha}{r} \epsilon^2 \sin u_1 + o(\epsilon^2).$$

Here we always use * to present the Euclidean inner product in \mathbb{N}^{n+1} and we denote

$$2\alpha = \frac{f^2(r) - 1}{f^2(r)}.$$

Thus, we get

$$r^{2}(\epsilon) = Y * Y = r^{2} + 2\epsilon r \sin u_{1} + \epsilon^{2}(1 + 2\alpha \sin^{2} u_{1}) + o(\epsilon^{2}).$$

Then, we have

(2.2)
$$r(\epsilon) = r + \epsilon \sin u_1 + \frac{\epsilon^2}{2r} (\cos^2 u_1 + 2\alpha \sin^2 u_1) + o(\epsilon^2).$$

We also let $2\rho(\epsilon) = r^2(\epsilon)$. By the discussion in paper [2], we know that the support function can be formulated by

$$\varphi^2 = 2\rho(\epsilon) - \frac{|\nabla \rho(\epsilon)|^2}{f^2(r(\epsilon))}$$
$$= r^2 + 2\epsilon r \sin u_1 + \epsilon^2 (1 + 2\alpha \sin^2 u_1) - \frac{\epsilon^2 \cos^2 u_1}{2f^2(r)} + o(\epsilon^2).$$

Here the norm $|\cdot|$ is respect to the canonical metric on r geodesic sphere. Thus, we have

(2.3)
$$\varphi(\epsilon) = r + \epsilon \sin u_1 + \frac{\alpha \epsilon^2}{r} + o(\epsilon^2).$$

On the other hand, by (1.5), we obviously have

$$\int_0^r \frac{t^n}{f(t)} dt = g(r)r^{n+1}.$$

Then we have

(2.4)
$$g'(r) = \frac{1}{rf(r)} - (n+1)\frac{g(r)}{r}$$

and

$$(2.5) g''(r) = -\frac{f(r) + rf'(r)}{r^2 f^2(r)} - (n+1)(\frac{g'(r)}{r} - \frac{g(r)}{r^2})$$
$$= -\frac{f(r) + rf'(r)}{r^2 f^2(r)} - (n+1)\frac{1}{r^2 f(r)} + (n+1)(n+2)\frac{g(r)}{r^2}.$$

Using (2.2)-(2.5), we have

$$(2.6) \qquad \frac{d(g(r(\epsilon))\varphi(\epsilon))}{d\epsilon}\bigg|_{\epsilon=0} = g'(r)\varphi(0) \frac{dr(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} + g(r) \frac{d\varphi(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$
$$= (\frac{1}{f(r)} - ng(r)) \sin u_1,$$

and

$$\begin{aligned} &(2.7) \quad \frac{d^{2}(g(r(\epsilon))\varphi(\epsilon))}{d\epsilon^{2}}\bigg|_{\epsilon=0} \\ &= \left.g''(r)\varphi(0)\left(\frac{dr(\epsilon)}{d\epsilon}\right)^{2}\bigg|_{\epsilon=0} + g'(r)\varphi(0)\left.\frac{d^{2}r(\epsilon)}{d\epsilon^{2}}\bigg|_{\epsilon=0} + 2g'(r)\left.\frac{dr(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}\left.\frac{d\varphi(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} \right. \\ &\left. + g(r)\left.\frac{d^{2}\varphi(\epsilon)}{d\epsilon^{2}}\bigg|_{\epsilon=0} \right. \\ &= \left.\left[-\frac{f(r) + rf'(r)}{rf^{2}(r)} - \frac{n+1}{rf(r)} + (n+1)(n+2)\frac{g(r)}{r} + 2\alpha(\frac{1}{rf(r)} - (n+1)\frac{g(r)}{r})\right. \\ &\left. + \frac{2}{rf(r)} - 2(n+1)\frac{g(r)}{r}\right]\sin^{2}u_{1} + \left[\frac{1}{rf(r)} - (n+1)\frac{g(r)}{r}\right]\cos^{2}u_{1} + \frac{2\alpha g(r)}{r}. \end{aligned}$$

Hence using Taylor expansion and (2.6),(2.7), we have

$$(2.8) \quad g(r(\epsilon))\varphi(\epsilon)$$

$$= g(r)r + \epsilon \left(\frac{1}{f(r)} - ng(r)\right)\sin u_1 + \frac{1}{2}\epsilon^2 \left[\left(-\frac{f'(r)r + nf(r)}{f^2(r)r} + \frac{2\alpha}{f(r)r}\right)\sin^2 u_1 + \left(\frac{1}{f(r)r} - \frac{(n+1)g(r)}{r}\right)\cos^2 u_1 + \frac{g(r)}{r}\left((n(n+1) - 2\alpha(n+1))\sin^2 u_1 + 2\alpha\right)\right] + o(\epsilon^2).$$

It is obvious that $ds_n^2 = du_1^2 + \cos^2 u_1 ds_{n-1}^2$, where ds_n^2 and ds_{n-1}^2 are the canonical metric of the n and n-1 dimensional unit sphere $\mathbb{S}^n, \mathbb{S}^{n-1}$. We also denote their volume form by dS_n and dS_{n-1} . Thus using the map \vec{r} defined by (2.1), we have

$$\int_{\mathbb{S}^n} \cos^2 u_1 dS_n = \int_{-\pi/2}^{\pi/2} \cos^2 u_1 \cos^{n-1} u_1 du_1 \int_{\mathbb{S}^{n-1}} dS_{n-1}.$$

If we let $t = \sin u_1$, it is clear that

$$\int_{-\pi/2}^{\pi/2} \cos^{n+1} u_1 du_1 = \int_{-1}^{1} (1 - t^2)^{n/2} dt = n \int_{-1}^{1} t^2 (1 - t^2)^{n/2 - 1} dt,$$

where the last equality comes from integral by parts. Thus we obtain

$$\int_{-1}^{1} (1 - t^2)^{n/2} dt = \frac{n}{n+1} \int_{-1}^{1} (1 - t^2)^{n/2 - 1} dt,$$

which implies

$$\int_{\mathbb{S}^n} \cos^2 u_1 dS_n = \frac{n}{n+1} \int_{\mathbb{S}^n} dS_n; \text{ and } \int_{\mathbb{S}^n} \sin^2 u_1 dS_n = \frac{1}{n+1} \int_{\mathbb{S}^n} dS_n.$$

It is also obvious that

$$\int_{\mathbb{S}^n} \sin u_1 dS_n = 0.$$

Thus integral both side of (2.8) on the r geodesic sphere $\mathbb{S}^n(r)$ and use the previous two formulas, then we have

$$(2.9) \int_{\mathbb{S}^{n}(r)} g(r(\epsilon))\varphi(\epsilon)dS = \int_{\mathbb{S}^{n}} g(r(\epsilon))\varphi(\epsilon)r^{n}dS_{n}$$

$$= g(r)r^{n+1} \int_{\mathbb{S}^{n}} dS_{n} - \frac{r^{n+1}}{f^{3}(r)} \left(\frac{1-f^{2}(r)}{r^{2}} + \frac{f(r)f'(r)}{r}\right) \frac{\epsilon^{2}}{2(n+1)} \int_{\mathbb{S}^{n}} dS_{n} + o(\epsilon^{2}).$$

Let us denote

$$\Phi(r) = \frac{f(r)f'(r)}{r} + \frac{1 - f^2(r)}{r^2}.$$

Then, using (1.4), we rewrite (2.9) to be

$$Vol(M_{\epsilon}) = Vol(\mathbb{B}^n(r)) - \frac{r^{n+1} \epsilon^2 \Phi(r)}{2(n+1) f^3(r)} Aera(\mathbb{S}^n) + o(\epsilon^2).$$

Here $\mathbb{B}^n(r)$ is the r geodesic ball. Hence, if at some r, $\Phi(r) < 0$, the volume of the perturbated convex body M_{ϵ} is bigger than the volume of the r geodesic ball. Thus we have proved our main Theorem 2.

At last, let's give some special examples for our theorem. For space form, $\Phi = 0$, the volume is invariant. For ADS space, since

$$f^2 = 1 - \frac{m}{r} + \kappa r^2,$$

we have $\Phi(r) > 0$ which implies that volume will decrease. If we take

$$f^2 = 1 + \frac{m}{r+1},$$

then, we get

$$\Phi(r) = -\frac{m}{2r(r+1)^2} - \frac{m}{r^2(r+1)} < 0,$$

for positive constant m. Thus, the volume will increase for perturbation convex body, which implies the isoperimetric inequality will not hold for arbitrary warped product space.

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