# Elliptic Problems in $\mathbb{R}^N$ with Critical and Singular Discontinuous Nonlinearities

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**Abstract**

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary, $a > 0, \lambda > 0$ and $0 < \delta < 3$ be real numbers. Define $2^* := \dfrac{2N}{N-2}$ and the characteristic function of a set $A$ by $\chi_A$. We consider the following critical problem with singular and discontinuous nonlinearity:

$$(P_\lambda^a) \qquad \begin{cases} -\Delta u & = \lambda \left( u^{2^*-1} + \chi_{\{u<a\}} u^{-\delta} \right), u > 0 \ \text{ in } \ \Omega, \\ u & = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We study the existence and the global multiplicity of solutions to the above problem.

# 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary, $a > 0, \lambda > 0$ and $0 < \delta < 3$. Define $2^* := \dfrac{2N}{N-2}$. Denote by $\chi_A$ the characteristic function of a set $A$. Consider the following elliptic problem with singular and discontinuous nonlinearity:

$$(P_\lambda^a) \quad \begin{cases} -\Delta u = \lambda \left( u^{2^*-1} + \chi_{\{u<a\}} u^{-\delta} \right), u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

**Definition 1.1.** *We say that $u \in H_0^1(\Omega)$ is a weak solution of $(P_\lambda^a)$ if $\operatorname{ess\,inf}_K u > 0$ for any compact set $K \subset \Omega$ and*

$$\int_\Omega \nabla u \nabla \varphi = \lambda \int_\Omega (\chi_{\{u<a\}} u^{-\delta} + u^{2^*-1}) \varphi \quad \text{holds for all } \varphi \in C_0^\infty(\Omega). \qquad (1.1)$$

**Remark 1.1.** *Any solution $u$ of $(P_\lambda^a)$ belongs to $L_{loc}^q(\Omega)$ for all $q < \infty$ (see for instance, Lemma B.3 of [25]). Thus $u \in W_{loc}^{2,q}(\Omega)$ for all $q < \infty$ and hence $u \in C_{loc}^{1,\alpha}(\Omega)$ for all $\alpha \in (0,1)$.*

The formal energy functional $E_\lambda^a(u)$ associated with the problem $(P_\lambda^a)$ is given by

$$E_\lambda^a(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \int_\Omega G(u) - \frac{\lambda}{2^*} \int_\Omega |u|^{2^*}$$

where

$$G(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ (1-\delta)^{-1} u^{1-\delta} & \text{if } 0 < u < \frac{a}{2}, \\ (1-\delta)^{-1}(a/2)^{1-\delta} + \displaystyle\int_{a/2}^u \chi_{\{t<a\}} t^{-\delta} dt & \text{if } u \geq a/2, \end{cases}$$

for $0 < \delta < 3, \delta \neq 1$ and for $\delta = 1$ we replace the terms of the form $(1-\delta)^{-1} x^{1-\delta}$ in the above definition with the term $\log x$.

We show the following multiplicity result for the problem $(P_\lambda^a)$.

**Theorem 1.1.** *For any $a > 0$, there exists $\Lambda^a > 0$ such that*

*(i) $(P_\lambda^a)$ has no solution for any $\lambda > \Lambda^a$,*

*(ii) $(P_\lambda^a)$ admits at least two solutions for any $\lambda \in (0, \Lambda^a)$.*

The study of such problems with discontinuous nonlinearities has increased remarkably in the last few years due to their occurrence in the modeling of various physical problems like the obstacle problem, the seepage surface problem and the Elenbass equation (see [9],[10]). The singular nature of the nonlinearity in $(P_\lambda^a)$ is motivated by the celebrated work of Crandall, Rabinowitz and Tartar in [13] which is further

studied extensively in [14],[15] [16],[17], [18], [21] and [23].

In the pioneering work of Ambrosetti-Brezis-Cerami[3], it was shown that a combination of convex and concave nonlinearities results in multiple positive solutions for the Dirichlet problem with the model nonlinearity $\lambda u^q + u^\alpha$, $0 < q < 1 < \alpha \leq \frac{N+2}{N-2}$. In [19] and [22], the authors have proved similar multiplicity results when the nonlinearity in $(P_\lambda^a)$ has no jump discontinuity and the exponent $\delta$ on the singular term satisfies $0 < \delta < 1$. This range for $\delta$ was extended to $0 < \delta < 3$ in [1] and [14] where the critical and singular nonlinear problem (again without the jump discontinuity) is discussed in $\mathbb{R}^2$.

The problem with jump discontinuity but without the singular term have been studied in [2], [4], [20] and [24].

In all the above mentioned works, the main methods used are variational techniques and the generalized gradient theory for locally Lipschitz functionals as developed in [11] and [12]. But, due to the discontinuous and singular nature of the nonlinear term in our problem, the associated functional is neither differentiable nor locally Lipschitz in $H_0^1(\Omega)$ and hence both these techniques can not be used directly. Therefore, in section 2, we first regularize the discontinuity in $(P_\lambda^a)$ to make the corresponding functional differentiable and then obtain a first solution for $(P_\lambda^a)$ as a limit of the solutions of the regularized problem. Here we give only the outline of the proof for the existence of the first solution, which is discussed thoroughly in [15], indicating only the requi modifications. We then prove that this solution is also a local minimum of the functional $E_\lambda^a$ associated with $(P_\lambda^a)$ in $H_0^1(\Omega)$ topology. Since $E_\lambda^a$ is not in general Fréchet differentiable in $H_0^1(\Omega)$, the "$H^1$ versus $C^1$" result of [8] can not be used. Instead, an appropriate use of Hopf's Lemma helps to handle the discontinuity. In section 3, we prove the existence of a second solution by considering the translate of the problem $(P_\lambda^a)$ by the first solution and then showing the existence of a solution to the translated problem. The functional $I_\lambda$ associated with the translated problem turns out to be locally Lipschitz and hence the theory of generalized gradients can be applied to prove the existence of the second solution. In this section, we employ Ekeland's variational principle and concentration-compactness ideas to show the existence of the second positive solution.

## 2 Existence of a first solution for $(P_\lambda^a)$

In this section, we obtain a solution of $(P_\lambda^a)$ using the regularizing techniques similar to that in [15]. Nevertheless, we give an outline of the arguments here for completeness. Define

$$\Lambda^a = \sup\{\lambda > 0 : (P_\lambda^a) \text{ has at least one solution}\}, \tag{2.1}$$

and

$$\phi_\delta = \begin{cases} e_1 & 0 < \delta < 1, \\ e_1 \left(-\log e_1\right)^{\frac{1}{2}} & \delta = 1, \\ e_1^{\frac{2}{\delta+1}} & 1 < \delta, \end{cases} \tag{2.2}$$

where $e_1$ is the first positive eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ with $\|e_1\|_{L^\infty(\Omega)}$ fixed as a number less than 1.

**Lemma 2.1.** $0 < \Lambda^a < \infty$.

**Proof:** Let $(P_\lambda^a)$ admit a solution $u_\lambda$. Since the nonlinearity on the right hand side of $(P_\lambda^a)$ is superlinear near infinity, there exists a constant $K = K(a) > 0$ such that for all $t > 0$, we have $t^{2^*-1} + \chi_{\{t<a\}}t^{-\delta} > Kt$. Let $\lambda_1$ be the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ with the corresponding eigenfunction $e_1$. Then multiplying $(P_\lambda^a)$ by $e_1$ we get

$$\lambda_1 \int_\Omega u_\lambda e_1 = \lambda \int_\Omega (u_\lambda^{2^*-1} + \chi_{\{u_\lambda<a\}}u_\lambda^{-\delta})e_1$$

$$\geq \lambda K \int_\Omega u_\lambda e_1.$$

This implies $\Lambda^a < \infty$. Now we show that $0 < \Lambda^a$. Consider the following singular problem without the jump-discontinuous term:

$$(P_\lambda^\infty) \qquad \left\{ \begin{array}{rl} -\Delta u &= \lambda(u^{-\delta} + u^{2^*-1}) \ \text{ in } \ \Omega, \\ u &> 0 \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{array} \right.$$

The existence and multiplicity of solutions of a problem in $\mathbb{R}^2$ analogous to $(P_\lambda^\infty)$ has been studied in [1], [14] and [15]. For $N \geq 3$, a similar approach works as we show now. From Theorems 1.1, 2.2 and 2.5 in [13] we can find a unique $v_\lambda \in H_0^1(\Omega)$ solving the following purely singular problem for all $\lambda > 0$:

$$\left. \begin{array}{rl} -\Delta v &= \lambda v^{-\delta}, \ v > 0 \ \text{ in } \ \Omega, \\ v &= 0 \ \text{ on } \ \partial\Omega. \end{array} \right\} \tag{2.3}$$

It can also be shown (see [13] again) that $v_\lambda \to 0$ uniformly in $\Omega$ as $\lambda \to 0^+$. Clearly, $v_\lambda$ is a subsolution to $(P_\lambda^\infty)$. Let $z_\lambda \in H_0^1(\Omega)$ solve

$$\left. \begin{array}{rl} -\Delta z_\lambda &= \lambda, \ \ z_\lambda > 0 \ \text{ in } \ \Omega, \\ z_\lambda &= 0 \ \text{ on } \ \partial\Omega. \end{array} \right\} \tag{2.4}$$

Define $\tilde{w}_\lambda = v_\lambda + z_\lambda$. Note that, if $\lambda_0 > 0$ is small enough, $\tilde{w}_\lambda$ is a supersolution to $(P_\lambda^\infty)$ for all $\lambda < \lambda_0$. Furthermore, $\tilde{w}_\lambda \to 0$ uniformly on $\Omega$ as $\lambda \to 0^+$. Let $\mathcal{M}_\lambda = \{u \in H_0^1(\Omega) : v_\lambda \leq u \leq \tilde{w}_\lambda \text{ in } \Omega\}$. It is easy to see that $\mathcal{M}_\lambda$ is a closed convex (hence weakly closed) set in $H_0^1(\Omega)$. Now, define the following iterative scheme for all $\lambda < \lambda_0$ :

$$\left\{ \begin{array}{rl} u_0 &= v_\lambda; \\ -\Delta u_n - \lambda u_n^{-\delta} &= \lambda u_{n-1}^{2^*-1}, \ u_n > 0, \ \text{ in } \ \Omega, \\ u_n &= 0 \ \text{ on } \ \partial\Omega, \ \ n = 1, 2, 3, \cdots \end{array} \right.$$

The above scheme is well defined as we can solve for $u_n$ in the closed convex set $\mathcal{M}_\lambda$ using the Perron's method in variational guise (see [25]). As $-\Delta u - \lambda u^{-\delta}$ is

a monotone operator in $\mathcal{M}_\lambda$, we get that $\{u_n\}$ is a non-decreasing sequence. Thus by standard compactness, we can find $u_\lambda \in H_0^1(\Omega) \cap C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$ such that $u_n \rightharpoonup u_\lambda$ in $H_0^1(\Omega)$ and $u_n \to u_\lambda$ in $C^\alpha(\bar{\Omega})$. Clearly, since the iteration above started from $v_\lambda$, we obtain that the solution $u_\lambda$ obtained is infact a minimal solution. We note that $v_\lambda \leq u_\lambda \leq \tilde{w}_\lambda$. Also, since $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ for $\lambda \to 0$, $u_\lambda$ solves $(P_\lambda^a)$ for $\lambda > 0$ small and hence $\Lambda^a > 0$. $\qquad\square$

**Corollary 2.1.** *From the proof above, it follows that $(P_\lambda^a)$ admits a solution for $\lambda > 0$ small.*

**Proposition 2.1.** *Let $\delta > 0$. Given $t_0 > 0$, there exists a function $p \in C^2((0,t_0)) \cap C([0,t_0])$ satisfying:*
*(i)$-p'' = p^{-\delta}$ in $(0,t_0)$,*
*(ii) $p(0) = 0$,*
*(iii) $p(s) > 0$ in $(0,t_0)$,*
*(iv) For $t$ small, $p(t) \sim t$ if $0 < \delta < 1$, $p(t) \geq ct$ if $\delta = 1$ and $p(t) \sim t^{\frac{2}{\delta+1}}$ if $\delta > 1$.*

*Proof.* Let $0 < \delta \leq 1$. Existence of a positive function $p \in C^2((0,t_0+1)) \cap C([0,t_0+1])$ satisfying the equation $-p'' = p^{-\delta}$ in $(0,t_0+1)$ and the boundary condition $p(0) = p(t_0+1) = 0$ follows from theorem 1.1 in [13]. Denote by $\rho_1$ the first (positive) eigenfunction of the interval $(0,t_0+1)$. An easy comparison shows that $p(t) \geq c\rho_1(t)$ and hence $p(t) \geq ct$ for all small $t > 0$ and some $c > 0$. If $\delta < 1$ it follows that $p^{-\delta} \in L^p((0,t_0+1))$ for some $p > 1$ and hence by regularity $p \in C^1([0,t_0+1])$. Therefore, $p(t) \sim t$ near $t = 0$ for $\delta < 1$.

If $\delta > 1$, we take $p(t) = [\frac{(1+\delta)^2}{2(\delta-1)}]^{\frac{1}{1+\delta}} t^{\frac{2}{\delta+1}}$ for $t > 0$. $\qquad\square$

We now consider the following two purely singular discontinuous problems:

$$(S_\lambda^a) \quad \begin{cases} -\Delta w = \lambda \chi_{\{w<a\}} w^{-\delta}, & w > 0 \ \text{ in } \ \Omega, \\ w = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

and

$$(S_\lambda^{a,\epsilon}) \quad \begin{cases} -\Delta w_\epsilon = \lambda \chi_\epsilon(w_\epsilon - a) w_\epsilon^{-\delta}, & w_\epsilon > 0 \ \text{ in } \ \Omega, \\ w_\epsilon = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where

$$\chi_\epsilon(t) = \begin{cases} 1 & \text{if } t \leq -\epsilon, \\ -(\frac{t}{\epsilon}) & \text{if } -\epsilon < t < 0, \\ 0 & \text{if } t \geq 0. \end{cases}$$

The existence of $w_\epsilon \in H_0^1(\Omega)$ solving $(S_\lambda^{a,\epsilon})$ and satisfying $w_\epsilon \geq c\phi_\delta$ ($c$ independent of $\epsilon$) follows from proposition 2.1 and theorem 2.2 of [13]. For all $\lambda > 0$, a solution $w_\lambda$ to $(S_\lambda^a)$ is obtained as the weak limit of the sequence of solutions $\{w_\epsilon\} \subset H_0^1(\Omega)$ of $(S_\lambda^{a,\epsilon})$ (for details, see lemma 2.3 in [15]).

**Theorem 2.1.** *$(P_\lambda^a)$ admits a solution $u_\lambda$ for all $\lambda \in (0,\Lambda^a)$. Furthermore, $u_\lambda$ is global minimum of $E_\lambda^a$ in the convex set $\overline{\mathcal{M}} := \{u \in H_0^1(\Omega) : w_\lambda \leq u \leq \overline{u}\} \subset H_0^1(\Omega)$ where $w_\lambda$ is a solution to $(S_\lambda^a)$ and $\overline{u}$ is a suitable super solution of $(P_\lambda^a)$.*

*Proof.* We note that $w_\epsilon$ is a subsolution of the following problem associated to $(P_\lambda^a)$.

$$(P_\lambda^{a,\epsilon}) \qquad \begin{cases} -\Delta u &= \lambda(\chi_\epsilon(u-a)u^{-\delta} + u^{2^*-1}), u > 0 \ \ \text{in} \ \ \Omega, \\ u &= 0 \ \ \text{on} \ \ \partial\Omega. \end{cases}$$

As before, for $0 < \delta < 3$, $\delta \neq 1$, define the following primitive:

$$G_\epsilon(u) = \begin{cases} 0 & \text{if } u \leq 0, \\ (1-\delta)^{-1}u^{1-\delta} & \text{if } 0 < u < \frac{a}{2}, \\ (1-\delta)^{-1}(a/2)^{1-\delta} + \displaystyle\int_{a/2}^u \chi_\epsilon(t-a)t^{-\delta}dt & \text{if } u \geq a/2. \end{cases}$$

If $\delta = 1$ we replace the terms of the form $(1-\delta)^{-1}x^{1-\delta}$ in the above definition with the term $\log x$. Then the formal energy functional on $H_0^1(\Omega)$ associated with the problem $(P_\lambda^{a,\epsilon})$ is

$$E_\lambda^{a,\epsilon}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega G_\epsilon(u) - \frac{\lambda}{2^*}\int_\Omega |u|^{2^*}.$$

Given any $\lambda \in (0, \Lambda^a)$, there exists $\bar\lambda > \lambda$ such that $(P_{\bar\lambda}^a)$ admits a solution $\overline{u}$ and by the definition of $\chi_\epsilon$, $\overline{u}$ is a supersolution of $(P_\lambda^{a,\epsilon})$. Since $-\Delta(w_\epsilon - \overline{u}) \leq \lambda(\chi_\epsilon(w_\epsilon - a)w_\epsilon^{-\delta} - \chi_\epsilon(\overline{u} - a)\overline{u}^{-\delta})$, from the non-increasing nature of the map $t \mapsto \chi_\epsilon(t-a)t^{-\delta}$, $t > 0$, we get $w_\epsilon \leq \overline{u}$. Then the existence of a solution $u_\epsilon$ of $(P_\lambda^{a,\epsilon})$ is obtained as local minimizer of $E_\lambda^{a,\epsilon}$ over the convex set $\mathcal{M}_\epsilon = \{u \in H_0^1(\Omega) : w_\epsilon \leq u \leq \overline{u}\}$. Also, using the same arguments as in [15] and [19], it can be proved that $u_\epsilon$ is a local minimizer of $E_\lambda^{a,\epsilon}$ in $H_0^1(\Omega)$. As $u_\epsilon$ solves $(P_\lambda^{a,\epsilon})$, it is easy to check that $\{u_\epsilon\}$ is bounded in $H_0^1(\Omega)$ and hence weakly converges to some $u_\lambda \in H_0^1(\Omega)$. Then by following the convergence arguments of Lemma 2.3 in [15], it is easy to check that $u_\lambda$ satisfies $(P_\lambda^a)$.

Then as in Proposition 3.1 and Lemma 3.4 of [15], we infer that $u_\lambda$ is a minimizer of $E_\lambda^a$ in $\overline{\mathcal{M}} := \{u \in H_0^1(\Omega) : w_\lambda \leq u \leq \overline{u}\}$. $\qquad\square$

Now we claim that $u_\lambda$ is a local minimum of $E_\lambda^a$ in $H_0^1(\Omega)$. Here we will follow the same approach as in [15] and [24] and thus be sketchy in the proof.

For $A \subset \mathbb{R}^N$ we denote $d(x, A) = \text{dist}(x, A)$ and by $|A|$ the $N-$dimensional Lebesgue measure of $A$.

**Theorem 2.2.** *Let $a > 0$. For $\lambda \in (0, \Lambda^a)$, $u_\lambda$ is a local minimum for $E_\lambda^a$ in $H_0^1(\Omega)$.*

*Proof.* We assume that $u_\lambda$ is not a local minimum of $E_\lambda^a$ in $H_0^1(\Omega)$ and derive a contradiction. Let $\{u_n\} \subset H_0^1(\Omega)$ be such that $u_n \to u_\lambda$ in $H_0^1(\Omega)$ and $E_\lambda^a(u_n) < E_\lambda^a(u_\lambda)$. For $\underline{u} = w_\lambda$ and solution $\bar u$ of $(P_{\bar\lambda}^a)$ where $0 < \lambda < \bar\lambda < \Lambda^a$, define $v_n = \max\{\underline{u}, \min\{u_n, \bar u\}\}$, $\bar w_n = (u_n - \bar u)^+$, $\underline{w}_n = (u_n - \underline{u})^-$, $\overline{S}_n = \text{support } (\bar w_n)$ and $\underline{S}_n = \text{support } (\underline{w}_n)$.

**Claim:** $|\overline{S}_n|, |\underline{S}_n|$ and $\|\bar w_n\|_{H_0^1(\Omega)} \to 0$ as $n \to \infty$.

Proof of claim: First to estimate $|\overline{S}_n|$, we set $\Omega_\sigma = \{x \in \Omega : d(x, \partial\Omega) > \sigma\}$ and $\Omega_{\sigma_1} = \{x \in \Omega_\sigma : d(x, \partial\Omega_\sigma) > \sigma_1\}$. For a given $\epsilon > 0$, we choose $\sigma, \sigma_1 > 0$ sufficiently

small such that $|\Omega \setminus \Omega_\sigma| < \frac{\epsilon}{3}$ and $|\Omega_\sigma \setminus \Omega_{\sigma_1}| < \frac{\epsilon}{3}$. First we prove that there exists a constant $C > 0$ such that

$$\bar{u}(x) - u_\lambda(x) > Cd(x, \partial\Omega_\sigma) \text{ and } \underline{u} + Cd(x, \partial\Omega_\sigma) < u_\lambda \text{ in } \Omega_{\sigma_1}. \qquad (2.5)$$

For proving (2.5) note that as $\bar{u}$ is not a solution of $(P_\lambda^a)$, we have $\bar{u} \not\equiv u_\lambda$ and hence we can choose a small enough ball $B \subset\subset \Omega_{\sigma_1}$ such that $\bar{u} \geq u_\lambda + 2\gamma$ in $B$ for some $\gamma > 0$. Now consider a solution $v$ of the following problem.

$$\left. \begin{array}{rl} -\Delta v &= \lambda u_\lambda^{-\delta} \Psi(v - (\bar{u} - u_\lambda)) \text{ in } \Omega_\sigma \setminus B, \\ v &= \gamma \text{ on } \partial B, \ v = 0 \text{ on } \partial\Omega_\sigma \end{array} \right\} \qquad (2.6)$$

where $\Psi(s) = 1$ if $s \leq 0, \Psi(s) = -1$ if $s > 0$. Then by the elliptic regularity $v \in W^{2,p}(\Omega_\sigma \setminus B) \cap C^{1,\beta}(\overline{\Omega_\sigma \setminus B})$ for some $\beta \in (0,1)$ and for all $p \geq 1$. Also taking $v^-$ as the test function in the above problem and noting that $\bar{u} \geq u_\lambda$ in $\Omega_\sigma \setminus B$, we have $v \geq 0$ in $\Omega_\sigma \setminus B$. Furthermore,

$$-\Delta(\bar{u} - u_\lambda) \geq \lambda(\bar{u}^{-\delta}\chi_{\{\bar{u}<a\}} - u_\lambda^{-\delta}\chi_{\{u_\lambda<a\}}) \geq -\lambda u_\lambda^{-\delta} \text{ in } \Omega$$

and $\bar{u} - u_\lambda \geq 2\gamma$ on $\partial B$, $\bar{u} - u_\lambda \geq 0$ on $\partial\Omega_\sigma$. Thus,

$$\left. \begin{array}{rl} -\Delta(\bar{u} - u_\lambda - v) &\geq \lambda(-u_\lambda^{-\delta} - u_\lambda^{-\delta}\Psi(v - (\bar{u} - u_\lambda)) \text{ in } \Omega_\sigma \setminus B, \\ \bar{u} - u_\lambda - v &\geq \gamma \text{ on } \partial B, \\ \bar{u} - u_\lambda - v &\geq 0 \text{ on } \partial\Omega_\sigma. \end{array} \right\} \qquad (2.7)$$

Taking $(\bar{u} - u_\lambda - v)^-$ as the test function in (2.7) and integrating over $\Omega_\sigma \setminus B$, we have

$$-\int_{\Omega_\sigma \setminus B} |\nabla(\bar{u} - u_\lambda - v)^-|^2 \geq \lambda \int_{\Omega_\sigma \setminus B} [-u_\lambda^{-\delta} - u_\lambda^{-\delta}\Psi(v - (\bar{u} - u_\lambda))](\bar{u} - u_\lambda - v)^-.$$

Now $\Psi(v - (\bar{u} - u_\lambda)) = -1$ if $(\bar{u} - u_\lambda - v)^- > 0$ and thus the right hand side in the above inequality is zero. This implies $(\bar{u} - u_\lambda - v)^- \equiv 0$ in $\Omega_\sigma \setminus B$, i.e., $v \leq \bar{u} - u_\lambda$ in $\Omega_\sigma \setminus B$. Therefore, $-\Delta v = \lambda u_\lambda^{-\delta}$ in $\Omega_\sigma \setminus B$, $v \in C^{1,\beta}(\Omega_\sigma \setminus B)$, $v > 0$ in $\Omega_\sigma \setminus B$ and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega_\sigma$ where $\nu$ is the outward unit normal on $\partial\Omega_\sigma$. Thus we can find $C > 0$ small enough such that $v(x) \geq Cd(x, \partial\Omega_\sigma)$ for all $x \in \Omega_\sigma \setminus B$ and hence $(\bar{u} - u_\lambda)(x) \geq Cd(x, \partial\Omega_\sigma)$ for all $x \in \Omega_\sigma$. A similar argument can be used to show that $\underline{u} + Cd(x, \partial\Omega_\sigma) < u_\lambda$ in $\Omega_{\sigma_1}$. This proves (2.5).

Now using (2.5) we estimate $|\bar{S}_n|$ as

$$|\bar{S}_n| \leq |\Omega \setminus \Omega_\sigma| + |\Omega_\sigma \setminus \Omega_{\sigma_1}| + |\bar{S}_n \cap \Omega_{\sigma_1}|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{1}{(C\sigma_1)^2} \int_{\bar{S}_n \cap \Omega_{\sigma_1}} (u_n - u_\lambda)^2$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{1}{(C\sigma_1)^2} \|u_n - u_\lambda\|_{H_0^1(\Omega)}^2.$$

Therefore we get $|\bar{S}_n| \to 0$ as $n \to \infty$ and

$$\|\bar{w}_n\|_{H_0^1(\Omega)}^2 = \int_{\bar{S}_n} |\nabla(u_n - \bar{u})|^2 \leq 2\left(\int_{\bar{S}_n} |\nabla(u_n - u_\lambda)|^2 + \int_{\bar{S}_n} |\nabla(u_\lambda - \bar{u})|^2\right) \to 0$$

as $n \to \infty$. Using the same approach as above, we get $|\underline{S}_n|$ as $n \to \infty$. This proves the claim.

Note that $v_n \in \overline{\mathcal{M}} = \{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \bar{u}\}$ and $u_n = v_n - \underline{w}_n + \bar{w}_n$. Also, $E_\lambda^a(u_n) = E_\lambda^a(v_n) + A_n + B_n$ where

$$A_n = \frac{1}{2} \int_{\bar{S}_n} (|\nabla u_n|^2 - |\nabla \bar{u}|^2) - \lambda \int_{\bar{S}_n} (G(u_n) - G(\bar{u})) - \frac{\lambda}{2^*} \int_{\bar{S}_n} \left( |u_n|^{2^*} - \bar{u}^{2^*} \right),$$

$$B_n = \frac{1}{2} \int_{\underline{S}_n} (|\nabla u_n|^2 - |\nabla \underline{u}|^2) - \lambda \int_{\underline{S}_n} (G(u_n) - G(\underline{u})) - \frac{\lambda}{2^*} \int_{\underline{S}_n} \left( |u_n|^{2^*} - \underline{u}^{2^*} \right).$$

As $u_\lambda$ is minimizer of $E_\lambda^a$ over $\overline{\mathcal{M}}$ (see theorem 2.1) and $v_n \in \overline{\mathcal{M}}$ we have $E_\lambda^a(u_n) \geq E_\lambda^a(u_\lambda) + A_n + B_n$. Now we claim that $A_n, B_n \geq 0$ for all large $n$ which is a contradiction to our assumption that $E_\lambda^a(u_n) < E_\lambda^a(u_\lambda)$ for all $n$. Note that

$$A_n = \frac{1}{2} \int_{\bar{S}_n} (|\nabla u_n|^2 - |\nabla \bar{u}|^2) - \lambda \int_{\bar{S}_n} (G(u_n) - G(\bar{u})) - \frac{\lambda}{2^*} \int_{\bar{S}_n} \left( |u_n|^{2^*} - \bar{u}^{2^*} \right)$$

$$= \frac{1}{2} \int_{\bar{S}_n} |\nabla \bar{w}_n|^2 + \int_{\bar{S}_n} \nabla \bar{u} \cdot \nabla \bar{w}_n - \lambda \int_{\bar{S}_n} (G(\bar{u} + \bar{w}_n) - G(\bar{u}))$$

$$- \frac{\lambda}{2^*} \int_{\bar{S}_n} \left( (\bar{u} + \bar{w}_n)^{2^*} - \bar{u}^{2^*} \right)$$

$$\geq \frac{1}{2} \int_{\bar{S}_n} |\nabla \bar{w}_n|^2 + \lambda \int_{\bar{S}_n} \left( \chi_{\{\bar{u} < a\}} \bar{u}^{-\delta} \bar{w}_n - (G(\bar{u} + \bar{w}_n) - G(\bar{u})) \right)$$

$$+ \lambda \int_{\bar{S}_n} \left( \bar{u}^{2^*-1} \bar{w}_n - \frac{1}{2^*} ((\bar{u} + \bar{w}_n)^{2^*} - \bar{u}^{2^*}) \right).$$

Now by dividing $\bar{S}_n$ into three subdomains, viz., $\bar{S}_n \cap \{x \in \Omega : a < \bar{u}(x)\}$, $\bar{S}_n \cap \{x \in \Omega : \bar{u}(x) \leq a \leq (\bar{u} + \bar{w}_n)(x)\}$ and $\bar{S}_n \cap \{x \in \Omega : (\bar{u} + \bar{w}_n)(x) < a\}$, one can check that the second integral in the right hand side of the above inequality is nonnegative. Also by the mean value theorem, for some $\theta = \theta(x) \in (0, 1)$ and appropriate positive constants $c_1, c_2, c_3$ we have

$$\int_{\bar{S}_n} \bar{u}^{2^*-1} \bar{w}_n - \frac{1}{2^*} \left( (\bar{u} + \bar{w}_n)^{2^*} - \bar{u}^{2^*} \right) = - \int_{\bar{S}_n} ((\bar{u} + \theta \bar{w}_n)^{2^*-1} - \bar{u}^{2^*-1}) \bar{w}_n$$

$$\geq - \int_{\bar{S}_n} (\bar{u} + \bar{w}_n)^{2^*-2} \bar{w}_n^2$$

$$\geq -c_1 \int_{\bar{S}_n} (\bar{u}^{2^*-2} + \bar{w}_n^{2^*-2}) \bar{w}_n^2$$

$$\geq -c_2 \left( \int_{\bar{S}_n} \bar{u}^{2^*} \right)^{\frac{2^*-2}{2^*}} \|\bar{w}_n\|_{H_0^1(\Omega)}^2$$

$$- c_3 \|\bar{w}_n\|_{H_0^1(\Omega)}^{2^*}. \tag{2.8}$$

Thus using (2.8) we have the following estimation for $A_n$:

$$A_n \geq \frac{1}{2} \|\bar{w}_n\|_{H_0^1(\Omega)}^2 - \lambda c_2 \left( \int_{\bar{S}_n} \bar{u}^{2^*} \right)^{\frac{2^*-2}{2^*}} \|\bar{w}_n\|_{H_0^1(\Omega)}^2 - \lambda c_3 \|\bar{w}_n\|_{H_0^1(\Omega)}^{2^*}. \tag{2.9}$$

Also following the arguments as in [15] and using (2.8) we estimate $B_n$ as

$$B_n \geq \frac{1}{2}\|\underline{w}_n\|^2_{H^1_0(\Omega)} - C\left(\int_{\underline{S}_n} \underline{u}^{2^*}\right)^{\frac{2^*-2}{2^*}} \|\underline{w}_n\|^2_{H^1_0(\Omega)}. \qquad (2.10)$$

Since $|\bar{S}_n|, |\underline{S}_n|$ and $\|\bar{w}_n\|_{H^1_0(\Omega)} \to 0$ as $n \to \infty$, we get $A_n, B_n \geq 0$. This completes the proof of the theorem. $\qquad \square$

## 3  Existence of the second solution for $(P^a_\lambda)$

In this section we obtain a second solution for $(P^a_\lambda)$ for $\lambda \in (0, \Lambda^a)$ by translating the problem to the solution $u_\lambda$ obtained in the previous section. We consider the following translated problem $(\tilde{P}^a_\lambda)$ :

$$(\tilde{P}^a_\lambda) \quad \begin{cases} -\Delta u &= \lambda\left(\chi_{\{u+u_\lambda<a\}}(u+u_\lambda)^{-\delta} - \chi_{\{u_\lambda<a\}}u_\lambda^{-\delta}\right) \\ &\quad + \lambda\left((u+u_\lambda)^{2^*-1} - u_\lambda^{2^*-1}\right) \ \text{ in } \ \Omega, \\ u &> 0 \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

**Remark 3.1.** *It is easy to see that if $v_\lambda \in H^1_0(\Omega)$ weakly solves $(\tilde{P}^a_\lambda)$, then $u_\lambda + v_\lambda$ weakly solves $(P^a_\lambda)$.*

Let us define, for $x \in \Omega$,

$$\tilde{g}(x,s) = (\chi_{\{s+u_\lambda(x)<a\}}(s+u_\lambda(x))^{-\delta} - \chi_{\{u_\lambda(x)<a\}}u_\lambda(x)^{-\delta})\chi_{\mathbb{R}^+}(s) \ \text{ and}$$

$$\tilde{f}(x,s) = \left((s+u_\lambda(x))^{2^*-1} - (u_\lambda(x))^{2^*-1}\right)\chi_{\mathbb{R}^+}(s).$$

Let $\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds$ and $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$. Let $I_\lambda : H^1_0(\Omega) \to \mathbb{R}$ be the energy functional associated with $(\tilde{P}^a_\lambda)$ defined as below:

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda\int_\Omega \tilde{G}(x,u(x))dx - \lambda\int_\Omega \tilde{F}(x,u(x))dx. \qquad (3.1)$$

**Proposition 3.1.** *$I_\lambda$ is locally Lipschitz on $H^1_0(\Omega)$.*

*Proof.* Note that as $H^1_0(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda\int_\Omega \tilde{F}(x,u(x))dx$ is a $C^1$ map, it is sufficient to prove that the map : $H^1_0(\Omega) \ni u \mapsto \int_\Omega \tilde{G}(x,u(x))dx \in \mathbb{R}$ is Lipschitz. We have,

$$\left|\int_\Omega \tilde{G}(x,u+v) - \tilde{G}(x,u)\right| \leq \int_\Omega \left|\int_{u(x)}^{(u+v)(x)} \tilde{g}(x,s)\,ds\right| \leq 2\int_\Omega u_\lambda^{-\delta}|v|.$$

Since $u_\lambda(x) \geq \underline{u} := w_\lambda \geq k_1\phi_\delta \geq Cd(x,\partial\Omega)^{\frac{2}{1+\delta}}$ for some $k_1, C > 0$ where $w_\lambda$ is a solution to $(S^a_\lambda)$ (see the proof of Theorem 2.1) and the remarks immediately above

this theorem), thanks to Hardy's inequality, it can be easily checked that

$$\int_\Omega u_\lambda^{-\delta}|v|\,dx \le \int_\Omega \frac{|v|}{Cd(x,\partial\Omega)^{\frac{2\delta}{1+\delta}}}\,dx = \int_\Omega \frac{|v|}{Cd(x,\partial\Omega)}d(x,\partial\Omega)^{\frac{1-\delta}{1+\delta}} \le C_1\,||v||_{H_0^1(\Omega)}.$$

Hence, $I_\lambda$ is locally Lipschitz. $\square$

**Definition 3.1.** *Let $I : H_0^1(\Omega) \to \mathbb{R}$ be a locally Lipschitz map. The generalized derivative of $I$ at $u$ in the direction of $\phi$ (denoted by $I^0(u,\phi)$) is defined as:*

$$I^0(u,\phi) = \limsup_{h\to 0, t\downarrow 0} \frac{I(u+h+t\phi) - I(u+h)}{t}; \quad u,\phi \in H_0^1(\Omega).$$

*We say that $u$ is a "generalized" critical point of $I$ if $I^0(u,\phi) \ge 0$ for all $\phi \in H_0^1(\Omega)$. See [11] (page 103) for more details.*

**Remark 3.2.** *From Lemma 4.1 of [15], for $u \ge 0$ and $\phi \in H_0^1(\Omega)$, we have the following inequality:*

$$I_\lambda^0(u,\phi) \le \int_\Omega \nabla(u_\lambda + u)\cdot\nabla\phi - \lambda\int_\Omega (u_\lambda + u)^{2^*-1}\phi - \lambda\int_\Omega w^\phi(u_\lambda + u)^{-\delta}\phi \quad (3.2)$$

*for some measurable function $w^\phi \in [\chi_{\{u_\lambda+u<a\}}, \chi_{\{u_\lambda+u\le a\}}]$.*

**Remark 3.3.** *From Remark 4.4 of [15], suppose for some nontrivial, nonnegative $v_\lambda \in H_0^1(\Omega)$ we have $I_\lambda^0(v_\lambda,\phi) \ge 0$ for all $\phi \in H_0^1(\Omega)$, i.e., $v_\lambda$ is a "generalized" critical point of $I_\lambda$. Then, from (3.2),*

$$\lambda(u_\lambda + v_\lambda)^{2^*-1} \le -\Delta(u_\lambda + v_\lambda) \le \lambda[(u_\lambda + v_\lambda)^{2^*-1} + (u_\lambda + v_\lambda)^{-\delta}] \quad (3.3)$$

*in the weak sense. Let us show (3.3). Indeed, as $v_\lambda \ge 0$ and $I_\lambda^0(v_\lambda,\phi) \ge 0$, using (3.2), we have for all $\phi \in H_0^1(\Omega)$,*

$$0 \le I_\lambda^0(v_\lambda,\phi) \le \int_\Omega \nabla(u_\lambda + v_\lambda)\cdot\nabla\phi - \lambda\int_\Omega (u_\lambda + v_\lambda)^{2^*-1}\phi - \lambda\int_\Omega w^\phi(u_\lambda + v_\lambda)^{-\delta}\phi. \quad (3.4)$$

*Let $\phi \ge 0$, then we have from (3.4),*

$$\int_\Omega \nabla(u_\lambda + v_\lambda)\cdot\nabla\phi \ge \lambda\int_\Omega (u_\lambda + v_\lambda)^{2^*-1}\phi + \lambda\int_\Omega w^\phi(u_\lambda + v_\lambda)^{-\delta}\phi.$$

*Since $w^\phi \ge 0$ and given that $\phi \ge 0$, we have*

$$\int_\Omega \nabla(u_\lambda + v_\lambda)\cdot\nabla\phi \ge \lambda\int_\Omega (u_\lambda + v_\lambda)^{2^*-1}\phi \quad (3.5)$$

*or in other words, $-\Delta(u_\lambda + v_\lambda) \ge \lambda(u_\lambda + v_\lambda)^{2^*-1}$ in the weak sense.*

*Next let us consider a $\phi \in H_0^1(\Omega)$ which is non-positive, so that $\psi = -\phi \geq 0$. Again using (3.4) we have,*

$$\int_\Omega \nabla(u_\lambda + v_\lambda) \cdot \nabla(-\psi) \geq \lambda \int_\Omega (u_\lambda + v_\lambda)^{2^*-1}(-\psi) + \lambda \int_\Omega w^{-\psi}(u_\lambda + v_\lambda)^{-\delta}(-\psi).$$

*Multiplying by $-1$ on both sides and using the fact that $w^{-\psi} \in [0,1]$ we get,*

$$\int_\Omega \nabla(u_\lambda + v_\lambda) \cdot \nabla\psi \leq \lambda \int_\Omega (u_\lambda + v_\lambda)^{2^*-1}\psi + \lambda \int_\Omega (u_\lambda + v_\lambda)^{-\delta}\psi.$$

*Since $\psi = -\phi$ is any arbitrary non-negative function in $H_0^1(\Omega)$, the previous expression implies*

$$-\Delta(u_\lambda + v_\lambda) \leq \lambda(u_\lambda + v_\lambda)^{2^*-1} + \lambda(u_\lambda + v_\lambda)^{-\delta} \text{ in the weak sense.} \qquad (3.6)$$

*From (3.5) and (3.6) we conclude that*

$$\lambda(u_\lambda + v_\lambda)^{2^*-1} \leq -\Delta(u_\lambda + v_\lambda) \leq \lambda[(u_\lambda + v_\lambda)^{2^*-1} + (u_\lambda + v_\lambda)^{-\delta}]. \qquad (3.7)$$

*Note that $-\Delta(u_\lambda + v_\lambda)$ is a positive distribution and hence it is given by a positive, regular Radon measure say $\mu$. Then using (3.7) we can show that $\mu$ is absolutely continuous with respect to the Lebesgue measure. Now by Radon Nikodyn theorem there exists a locally integrable function $g$ such that $-\Delta(u_\lambda + v_\lambda) = g$ and hence $g \in L_{loc}^p(\Omega)$ for some $p > 1$. Now using Lemma B.3 of [25] and elliptic regularity we can conclude that $u_\lambda + v_\lambda \in W_{loc}^{2,q}(\Omega)$ for all $q < \infty$ and for almost every $x \in \Omega$, $-\Delta(u_\lambda + w_\lambda)(x) = g(x) \geq \lambda(u_\lambda + v_\lambda)^{2^*-1}(x) > 0$. In particular,*

$$-\Delta(u_\lambda + v_\lambda) > 0 \text{ for a.e on } \{x \in \Omega : (u_\lambda + v_\lambda)(x) = a\}. \qquad (3.8)$$

*On the other hand, we have $-\Delta(u_\lambda + v_\lambda) = 0$ a.e on the set $\{x : (u_\lambda + v_\lambda)(x) = a\}$. This contradicts (3.8) unless the Lebesgue measure of the set $\{x : (u_\lambda + v_\lambda)(x) = a\}$ is zero.*

*Therefore, $w^\phi = \chi_{\{u_\lambda + v_\lambda < a\}}$ a.e. in $\Omega$ for any $\phi \in H_0^1(\Omega)$ and hence $u_\lambda + v_\lambda$ is a second solution for $(P_\lambda^a)$.*

**Remark 3.4.** *Note that as $I_\lambda(u) = E_\lambda^a(u^+ + u_\lambda) - E_\lambda^a(u_\lambda) + \frac{1}{2}\int_\Omega |\nabla u^-|^2$ for any $u \in H_0^1(\Omega)$ and $u_\lambda$ is a local minimum of $E_\lambda^a$ in $H_0^1(\Omega)$, it follows that $0$ is a local minimum of $I_\lambda$ in $H_0^1(\Omega)$.*

Using the Mountain Pass theorem and Ekeland variational principle we show the existence of a generalized critical point for $I_\lambda$ which yields a second solution to $(P_\lambda^a)$. The method of the proof is along lines similar to those of [15]. Let us define $H^+ = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e in } \Omega\}$. Since $0$ is a local minimum of $I_\lambda$, there exists a $\rho_0 > 0$ such that $I_\lambda(0) \leq I_\lambda(u)$ for $||u||_{H_0^1(\Omega)} \leq \rho_0$. The following two cases arise:
**1.**$ZA$ (Zero altitude): $\inf\{I_\lambda(u) : ||u||_{H_0^1(\Omega)} = \rho, u \in H^+\} = I_\lambda(0) = 0$ for all $\rho \in (0, \rho_0)$.
**2.**$MP$ (Mountain Pass): There exists $\rho_1 \in (0, \rho_0)$ such that $\inf\{I_\lambda(u) : ||u||_{H_0^1(\Omega)} = \rho_1, u \in H^+\} > I_\lambda(0)$.

**Lemma 3.1.** *Let ZA hold for some $\lambda \in (0, \Lambda^a)$. Then there exists a nontrivial "generalized" critical point $v_\lambda \in H^+$ for $I_\lambda$.*

*Proof.* Fix $\rho \in (0, \rho_0)$. Then there exists a sequence $\{z_n\} \subset H^+$ with $\|z_n\|_{H_0^1(\Omega)} = \rho$ and $I_\lambda(z_n) \leq 1/n$. Fix $0 < r < \frac{1}{2}\min\{\rho_0 - \rho, \rho\}$ and define $R = \{u \in H^+ : \rho - r \leq \|u\|_{H_0^1(\Omega)} \leq \rho + r\}$. Clearly $R$ is closed and $I_\lambda$ is Lipschitz continuous on $R$ from Proposition 3.1. Thus by Ekeland's variational principle there exists $\{v_n\} \subset R$ such that

(i) $I_\lambda(v_n) \leq I_\lambda(z_n) \leq \dfrac{1}{n}$,

(ii) $\|z_n - v_n\|_{H_0^1(\Omega)} \leq \dfrac{1}{n}$ and

(iii) $I_\lambda(v_n) \leq I_\lambda(v) + \dfrac{1}{n}\|v - v_n\|_{H_0^1(\Omega)}$ for all $v \in R$.

We note that

$$\rho - \frac{1}{n} = \|z_n\|_{H_0^1(\Omega)} - \frac{1}{n} \leq \|v_n\|_{H_0^1(\Omega)} \leq \|z_n\|_{H_0^1(\Omega)} + \frac{1}{n} = \rho + \frac{1}{n}. \qquad (3.9)$$

Therefore, for $\xi \in H^+$ we can choose $\epsilon > 0$ sufficiently small such that $v_n + \epsilon(\xi - v_n) \in R$ for all large $n$. Then by $(iii)$ we get

$$\frac{I_\lambda(v_n + \epsilon(\xi - v_n)) - I_\lambda(v_n)}{\epsilon} \geq -\frac{1}{n}\|\xi - v_n\|_{H_0^1(\Omega)}.$$

Letting $\epsilon \to 0^+$, we conclude

$$I_\lambda^0(v_n, \xi - v_n) \geq -\frac{1}{n}\|\xi - v_n\|_{H_0^1(\Omega)} \text{ for all } \xi \in H^+.$$

From Remark 3.2, for any $\xi \in H^+$, there exists $w_n^{\xi - v_n} \in [\chi_{\{u_\lambda + v_n < a\}}, \chi_{\{u_\lambda + v_n \leq a\}}]$ such that

$$\int_\Omega \nabla(u_\lambda + v_n) \cdot \nabla(\xi - v_n) - \lambda \int_\Omega (u_\lambda + v_n)^{2^*-1}(\xi - v_n)$$

$$- \lambda \int_\Omega w_n^{\xi - v_n}(u_\lambda + v_n)^{-\delta}(\xi - v_n) \geq -\frac{1}{n}\|\xi - v_n\|_{H_0^1(\Omega)}. \qquad (3.10)$$

Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, we may assume $v_n \rightharpoonup v_\lambda \in H^+$ weakly in $H_0^1(\Omega)$. Now by following the same arguments as in Lemma 4.2 of [15] we can show that $v_\lambda$ is a generalized critical point for $I_\lambda$. It remains to show that $v_\lambda \not\equiv 0$. Note that if $I_\lambda(v_\lambda) \neq 0$ we are done. So assume $I_\lambda(v_\lambda) = 0$. Since $\|v_n\|_{H_0^1(\Omega)} \geq \rho/2$ for all large $n$ (see (3.9)), it is sufficient to show that $v_n \to v_\lambda$ strongly in $H_0^1(\Omega)$. Taking $\xi = v_\lambda$ in (3.10) we get

$$\int_\Omega \nabla(u_\lambda + v_\lambda) \cdot \nabla(v_\lambda - v_n) - \lambda \int_\Omega (u_\lambda + v_n)^{2^*-1}(v_\lambda - v_n)$$

$$- \lambda \int_\Omega w_n^{v_\lambda - v_n}(u_\lambda + v_n)^{-\delta}(v_\lambda - v_n) + \frac{1}{n}\|v_\lambda - v_n\|_{H_0^1(\Omega)} \geq \|v_\lambda - v_n\|_{H_0^1(\Omega)}^2.$$

$$(3.11)$$

For any measurable set $E \subset \Omega$, as $u_\lambda \geq k_1 \phi_\delta$ and $v_n \in H^+$, thanks to Hardy's inequality, we have

$$\int_E w_n^{v_\lambda - v_n} |v_n - v_\lambda| (v_n + u_\lambda)^{-\delta} \leq C \int_E \frac{|v_n - v_\lambda|}{u_\lambda^\delta}$$

$$\leq C \int_E \frac{|v_n - v_\lambda|}{d(x, \partial\Omega)^{\frac{2\delta}{1+\delta}}}$$

$$\leq C \int_E \frac{|v_n - v_\lambda|}{d(x, \partial\Omega)} d(x, \partial\Omega)^{\frac{1-\delta}{1+\delta}}$$

$$\leq C \|v_n - v_\lambda\|_{H_0^1(\Omega)} \|d(x, \partial\Omega)^{\frac{1-\delta}{1+\delta}}\|_{L^2(E)}. \quad (3.12)$$

Since $v_n \to v_\lambda$ pointwise a.e. in $\Omega$, by Vitali's convergence theorem,

$$\int_\Omega w_n^{v_\lambda - v_n} |v_n - v_\lambda| (v_n + v_\lambda)^{-\delta} \to 0 \text{ as } n \to \infty. \quad (3.13)$$

Also from Brezis-Lieb lemma ([5]) we have

$$\int_\Omega (u_\lambda + v_n)^{2^*-1}(v_\lambda - v_n) = \int_\Omega (u_\lambda + v_n)^{2^*-1}(u_\lambda + v_\lambda) - \int_\Omega (u_\lambda + v_n)^{2^*}$$

$$= -\|v_\lambda - v_n\|_{L^{2^*}(\Omega)}^{2^*} + o_n(1). \quad (3.14)$$

Now using (3.13) and (3.14) in (3.11) we get

$$\|v_\lambda - v_n\|_{H_0^1(\Omega)}^2 - \lambda \|v_\lambda - v_n\|_{L^{2^*}(\Omega)}^{2^*} \leq o_n(1). \quad (3.15)$$

Also taking $\xi = 2v_n$ in (3.10) and using the fact that $u_\lambda$ solves $(P_\lambda^a)$ we get

$$-\frac{1}{n}\|v_n\|_{H_0^1(\Omega)} \leq \int_\Omega \nabla u_\lambda \cdot \nabla v_n + \int_\Omega |\nabla v_n|^2 - \lambda \int_\Omega (u_\lambda + v_n)^{2^*-1} v_n$$

$$-\lambda \int_\Omega w_n^{v_n} v_n (u_\lambda + v_n)^{-\delta}$$

$$= \|v_n\|_{H_0^1(\Omega)}^2 - \lambda \int_\Omega ((u_\lambda + v_n)^{2^*-1} - u_\lambda^{2^*-1}) v_n$$

$$+\lambda \int_\Omega \left( \chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} - w_n^{v_n}(u_\lambda + v_n)^{-\delta} \right) v_n$$

$$= \|v_\lambda\|_{H_0^1(\Omega)}^2 + \|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 - \lambda \int_\Omega \tilde{f}(v_n) v_n$$

$$+\lambda \int_\Omega \left( \chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} - w_n^{v_n}(u_\lambda + v_n)^{-\delta} \right) v_n + o_n(1).$$

Now as $v_\lambda$ solves $(\tilde{P}_\lambda^a)$ we have

$$\|v_\lambda\|_{H_0^1(\Omega)}^2 = \lambda \int_\Omega \tilde{f}(v_\lambda) v_\lambda + \lambda \int_\Omega \left( \chi_{\{u_\lambda + v_\lambda < a\}}(u_\lambda + v_\lambda)^{-\delta} - \chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} \right) v_\lambda.$$

Using this identity in above inequality we get,

$$-\frac{1}{n}\|v_n\|_{H_0^1(\Omega)} \leq \|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 - \lambda \int_\Omega \left(\tilde{f}(v_n)v_n - \tilde{f}(v_\lambda)v_\lambda\right)$$

$$+ \lambda \int_\Omega \left(\chi_{\{u_\lambda+v_\lambda<a\}}(u_\lambda+v_\lambda)^{-\delta} - \chi_{\{u_\lambda<a\}}u_\lambda^{-\delta}\right)v_\lambda$$

$$+ \lambda \int_\Omega \left(\chi_{\{u_\lambda<a\}}u_\lambda^{-\delta} - w_n^{v_n}(u_\lambda+v_n)^{-\delta}\right)v_n + o_n(1). \quad (3.16)$$

Using again Brezis-Lieb lemma it is easy to check that

$$\int_\Omega \tilde{f}(v_n)v_n - \tilde{f}(v_\lambda)v_\lambda = \|v_n - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o_n(1).$$

Also as $v_n \to v_\lambda$ pointwise a.e. in $\Omega$ and $|\{x \in \Omega : (u_\lambda + v_\lambda)(x) = a\}| = 0$, using estimates similar to the one in (3.12) we have

$$\int_\Omega \left(\chi_{\{u_\lambda+v_\lambda<a\}}(u_\lambda+v_\lambda)^{-\delta} - \chi_{\{u_\lambda<a\}}u_\lambda^{-\delta}\right)v_\lambda$$

$$+ \int_\Omega \left(\chi_{\{u_\lambda<a\}}u_\lambda^{-\delta} - w_n^{v_n}(u_\lambda+v_n)^{-\delta}\right)v_n = o_n(1).$$

Thus (3.16) implies

$$o_n(1) \leq \|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 - \lambda\|v_n - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*}. \quad (3.17)$$

Also as $I_\lambda(v_n) \leq \frac{1}{n}$ and $\tilde{F}(v_n) = \frac{(u_\lambda+v_n)^{2^*}}{2^*} - \frac{u_\lambda^{2^*}}{2^*} - u_\lambda^{2^*-1}v_n$, we have

$$I_\lambda(v_n) = \frac{1}{2}\|v_n\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{2^*}\|u_\lambda+v_n\|_{L^{2^*}(\Omega)}^{2^*} + \frac{\lambda}{2^*}\|u_\lambda\|_{L^{2^*}(\Omega)}^{2^*}$$

$$+ \lambda \int_\Omega u_\lambda^{2^*-1}v_n - \lambda \int_\Omega \tilde{G}(v_n)$$

$$\leq \frac{1}{n}.$$

From the fact that $v_n \rightharpoonup v_\lambda$ weakly in $H_0^1(\Omega)$, this implies

$$\frac{1}{2}\|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{2^*}\|v_n - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + I_\lambda(v_\lambda) + \lambda \int_\Omega \tilde{G}(v_\lambda) - \lambda \int_\Omega \tilde{G}(v_n) \leq o_n(1). \quad (3.18)$$

Now using the Hardy's inequality and Vitali's convergence theorem as in (3.12) one can check that $\int_\Omega \tilde{G}(v_n) \to \int_\Omega \tilde{G}(v_\lambda)$ as $n \to \infty$. Also as $I_\lambda(v_\lambda) = 0$, (3.18) implies

$$\frac{1}{2}\|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{2^*}\|v_n - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} \leq o_n(1). \quad (3.19)$$

Now from (3.15), (3.17) and (3.19) we get $(\frac{1}{2} - \frac{1}{2^*})\|v_n - v_\lambda\|_{H_0^1(\Omega)}^2 \leq o_n(1)$ and hence $v_n \to v_\lambda$ in $H_0^1(\Omega)$. $\qquad\square$

Next we consider the case (MP). As the nonlinearity is critical, we use the following Talenti functions to study the critical level:

$$V_\epsilon(x) = \frac{C_N \epsilon^{(N-2)/2}}{(\epsilon^2 + |x|^2)^{(N-2)/2}}, \quad C_N, \epsilon > 0.$$

Fix any $y \in \Omega_a = \{x \in \Omega : u_\lambda(x) < a\}$. Choose $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\overline{B_r(y)}$ where $r > 0$ is chosen small enough such that $\overline{B_r(y)} \subset \Omega_a$. Define $U_\epsilon(x) = \eta(x)V_\epsilon(x - y)$. Then as $\epsilon \to 0$, a standard computation (see [6]) gives

$$\int_\Omega |U_\epsilon|^{2^*} = \int_{\mathbb{R}^N} |V_1|^{2^*} + o(\epsilon^N) = A + o(\epsilon^N) \tag{3.20}$$

and

$$\int_\Omega |\nabla U_\epsilon|^2 = \int_{\mathbb{R}^N} |\nabla V_1|^2 + o(\epsilon^{N-2}) = B + o(\epsilon^{N-2}). \tag{3.21}$$

We have the following lemma.

**Lemma 3.2.** *There exist $\epsilon_0 > 0$ and $R_0 \ge 1$ such that*

(i) $I_\lambda(RU_\epsilon) < I_\lambda(0) = 0$ *for all $\epsilon \in (0, \epsilon_0)$ and $R \ge R_0$.*

(ii) $I_\lambda(tR_0 U_\epsilon) < \dfrac{S^{\frac{N}{2}}}{N\lambda^{(N-2)/2}}$ *for all $t \in (0,1], \epsilon \in (0, \epsilon_0)$ where $S = \dfrac{B}{A^{2/2^*}}$ is the best constant of the Sobolev embedding.*

*Proof.* Noting that for $v \in H^+$, $E_\lambda^a(u_\lambda + v) = E_\lambda^a(u_\lambda) + I_\lambda(v)$, this is equivalent to show that

(i) $E_\lambda^a(u_\lambda + RU_\epsilon) < E_\lambda^a(u_\lambda)$ for all $\epsilon \in (0, \epsilon_0)$ and $R \ge R_0$.

(ii) $E_\lambda^a(u_\lambda + tR_0 U_\epsilon) < E_\lambda^a(u_\lambda) + \dfrac{S^{\frac{N}{2}}}{N\lambda^{(N-2)/2}}$ for all $t \in (0,1], \epsilon \in (0, \epsilon_0)$.

Now using the fact that $u_\lambda$ solves $(P_\lambda^a)$, first we estimate $E_\lambda^a(u_\lambda + tRU_\epsilon)$, $t > 0$, as follows.

$$
\begin{aligned}
E_\lambda^a(u_\lambda + tRU_\epsilon) &= \frac{1}{2}\int_\Omega |\nabla(u_\lambda + tRU_\epsilon)|^2 - \lambda\int_\Omega G(u_\lambda + tRU_\epsilon) \\
&\quad - \frac{\lambda}{2^*}\int_\Omega (u_\lambda + tRU_\epsilon)^{2^*} \\
&= \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 + \frac{R^2 t^2}{2}\int_\Omega |\nabla U_\epsilon|^2 + tR\int_\Omega \nabla u_\lambda \cdot \nabla U_\epsilon \\
&\quad - \lambda\int_\Omega G(u_\lambda + tRU_\epsilon) - \frac{\lambda}{2^*}\int_\Omega (u_\lambda + tRU_\epsilon)^{2^*} \\
&= \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 + \frac{R^2 t^2}{2}\int_\Omega |\nabla U_\epsilon|^2 + \lambda tR\int_\Omega (\chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} + u_\lambda^{2^*-1}) U_\epsilon \\
&\quad - \lambda\int_\Omega G(u_\lambda + tRU_\epsilon) - \frac{\lambda}{2^*}\int_\Omega (u_\lambda + tRU_\epsilon)^{2^*}.
\end{aligned}
$$

Now we estimate the critical term $\int_\Omega (u_\lambda + tRU_\epsilon)^{2^*}$ using the one-dimensional inequality in Lemma 4 of [7] as:

$$\int_\Omega (u_\lambda + tRU_\epsilon)^{2^*} = \int_\Omega u_\lambda^{2^*} + (tR)^{2^*} \int_\Omega U_\epsilon^{2^*} + 2^* tR \int_\Omega u_\lambda^{2^*-1} U_\epsilon$$
$$+ 2^*(tR)^{2^*-1} \int_\Omega u_\lambda U_\epsilon^{2^*-1} + (R_\epsilon + S_\epsilon).$$

The terms $R_\epsilon$ and $S_\epsilon$ are given by the following expressions:

$$R_\epsilon = \begin{cases} O_\epsilon(\int_\Omega U_\epsilon^2) & \text{if } N < 6, \\ O_\epsilon(\int_{\{u_\lambda \geq tRU_\epsilon\}} u_\lambda (tRU_\epsilon)^{2^*-1}) & \text{if } N \geq 6, \end{cases} \tag{3.22}$$

and

$$S_\epsilon = \begin{cases} O_\epsilon(\int_\Omega U_\epsilon^{2^*-2}) & \text{if } N < 6, \\ O_\epsilon(\int_{\{u_\lambda \leq tRU_\epsilon\}} u_\lambda^{2^*-1}(tRU_\epsilon)) & \text{if } N \geq 6. \end{cases} \tag{3.23}$$

Now $R_\epsilon$ and $S_\epsilon$ can be estimated as in [7] depending on whether $2^* > 3$ or $2^* \leq 3$ as follows:

$$R_\epsilon, S_\epsilon = \begin{cases} O(\epsilon^{(N/2)\theta}) \, \forall \, \theta < 1 & \text{if } N \geq 6, \\ O(\epsilon^2) & \text{if } N = 5, \\ O(\epsilon^{2\theta}) \, \forall \, \theta < 1 & \text{if } N = 4, \\ O(\epsilon) & \text{if } N = 3. \end{cases} \tag{3.24}$$

Thus for all $N \geq 3$, we get $R_\epsilon, S_\epsilon = o(\epsilon^{(N-2)/2})$. Therefore

$$E_\lambda^a(u_\lambda + tRU_\epsilon) = \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 + \frac{R^2 t^2}{2}\int_\Omega |\nabla U_\epsilon|^2 + \lambda \int_\Omega \chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} tRU_\epsilon$$
$$- \lambda \int_\Omega G(u_\lambda + tRU_\epsilon) - \frac{\lambda}{2^*}\int_\Omega u_\lambda^{2^*} - \frac{\lambda t^{2^*} R^{2^*}}{2^*}\int_\Omega U_\epsilon^{2^*}$$
$$- \lambda R^{2^*-1} t^{2^*-1} \int_\Omega U_\epsilon^{2^*-1} u_\lambda + o_\epsilon(\epsilon^{(N-2)/2})$$
$$= E_\lambda^a(u_\lambda) + \frac{R^2 t^2 B}{2} - \frac{\lambda t^{2^*} R^{2^*} A}{2^*} - \lambda R^{2^*-1} t^{2^*-1} \int_\Omega U_\epsilon^{2^*-1} u_\lambda$$
$$+ \lambda \int_\Omega \left(\chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} tRU_\epsilon + G(u_\lambda) - G(u_\lambda + tRU_\epsilon)\right) + o_\epsilon(\epsilon^{(N-2)/2}). \tag{3.25}$$

Now we estimate the last integral, which we denote by $T$, on the right hand side of (3.25) as follows:

$$T = \int_\Omega \left(\chi_{\{u_\lambda < a\}} u_\lambda^{-\delta} tRU_\epsilon + G(u_\lambda) - G(u_\lambda + tRU_\epsilon)\right)$$
$$\leq \int_{A_1} u_\lambda^{-\delta} tRU_\epsilon + \int_{A_2} \left(u_\lambda^{-\delta} tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right)$$

where $A_1 = \{x \in \Omega : u_\lambda(x) < a \le (u_\lambda + tRU_\epsilon)(x)\}$ and $A_2 = \{x \in \Omega : (u_\lambda + tRU_\epsilon)(x) < a\}$. Note that as $U_\epsilon \to 0$ uniformly in $\{x \in \Omega : |x - y| > r\}$, we get $|A_1 \setminus \overline{B_r(y)}| \to 0$ as $\epsilon \to 0$. Also as $u_\lambda$ is continuous and $\overline{B_r(y)} \subset \Omega_a := \{x \in \Omega : u_\lambda < a\}$, there exists $\gamma > 0$ such that $u_\lambda < a - \gamma$ in $B_r(y)$. Thus for $x \in A_1 \cap B_r(y)$, $tRU_\epsilon(x) > \gamma$, i.e.,

$$\eta(x)\frac{C_N \epsilon^{(N-2)/2}}{(\epsilon^2 + |x - y|^2)^{(N-2)/2}} \ge \frac{\gamma}{tR}.$$

Therefore $|x - y| \le \sqrt{\epsilon}(\frac{tR}{\gamma})^{1/(N-2)}C_N^{1/(N-2)} = r_\epsilon$. Thus $A_1 \cap B_r(y) \subset B_{r_\epsilon}(y)$ and

$$\int_{A_1 \cap B_r(y)} U_\epsilon \le \int_{B_{r_\epsilon}(y)} U_\epsilon \le O_\epsilon(1)\epsilon^{(N-2)/2} \int_0^{r_\epsilon} r\,dr = O_\epsilon(1)\epsilon^{N/2}. \tag{3.26}$$

If $x \in A_1 \setminus B_r(y)$, then $U_\epsilon(x) \le O_\epsilon(1)\frac{\epsilon^{(N-2)/2}}{r^{N-2}}$ and hence

$$\int_{A_1 \setminus B_r(y)} U_\epsilon = O_\epsilon(1)\frac{\epsilon^{(N-2)/2}}{r^{N-2}}|A_1 \setminus B_r(y)|. \tag{3.27}$$

Hence from (3.26) and (3.27) we get

$$\int_{A_1} u_\lambda^{-\delta} tRU_\epsilon = o(\epsilon^{(N-2)/2}). \tag{3.28}$$

Also as in page 176 of [17]

$$\int_{A_2} \left(u_\lambda^{-\delta} tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right) \le o(\epsilon^{(N-2)/2}). \tag{3.29}$$

Indeed, fix $0 < \tau < 1/4$. Then

$$\begin{aligned}
D_\epsilon &= \int_{A_2} \left(u_\lambda^{-\delta} tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right) \\
&= \int_{|x-y| \le \epsilon^\tau} \left(u_\lambda^{-\delta} tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right) \\
&\quad + \int_{|x-y| > \epsilon^\tau} \left(u_\lambda^{-\delta} tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right). \tag{3.30}
\end{aligned}$$

Now for $\epsilon$ small, we have the following estimate for the first term on the right hand

side of (3.30).

$$\int_{|x-y|\leq\epsilon^\tau} \left(u_\lambda^{-\delta}tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right)$$

$$\leq c_1 R \int_{|x-y|\leq\epsilon^\tau} U_\epsilon\, dx$$

$$= c_2 R \int_{|x-y|\leq\epsilon^\tau} \frac{\epsilon^{(N-2)/2}}{(\epsilon^2 + |x-y|^2)^{(N-2)/2}}dx$$

$$\leq c_2 R\epsilon^{(N-2)/2} \int_0^{\epsilon^\tau} r\, dr$$

$$\leq c_3 R\epsilon^{(N-2)/2+2\tau} \tag{3.31}$$

Now using the fact that $u_\lambda$ is bounded below in the support of $\eta$ and the mean value theorem, we have the following estimate for the second term on the right hand side of (3.30).

$$\int_{|x-y|>\epsilon^\tau} \left(u_\lambda^{-\delta}tRU_\epsilon + \frac{u_\lambda^{1-\delta}}{1-\delta} - \frac{(u_\lambda + tRU_\epsilon)^{1-\delta}}{1-\delta}\right)$$

$$\leq \int_{|x-y|>\epsilon^\tau} \left(u_\lambda^{-\delta} - (u_\lambda + \theta_1 tRU_\epsilon)^{-\delta}\right) tRU_\epsilon\, dx$$

$$\leq c_4 \int_{supp\ \eta \cap |x-y|>\epsilon^\tau} (u_\lambda + \theta_2 tRU_\epsilon)^{-1-\delta}(tRU_\epsilon)^2\, dx$$

$$\leq c_5 R^2 \int_{|x-y|>\epsilon^\tau} U_\epsilon^2\, dx$$

$$\leq c_6 R^2 \int_{|x-y|>\epsilon^\tau} \frac{\epsilon^{(N-2)}}{(\epsilon^2 + |x-y|^2)^{(N-2)}}dx$$

$$\leq c_7 R^2 \epsilon^{N-2-2\tau(N-2)} \tag{3.32}$$

for some $0 < \theta_1, \theta_2 < 1$. Thus (3.31) and (3.32) gives (3.29). Thus substituting (3.28) and (3.29) in (3.25) we get

$$E_\lambda^a(u_\lambda + tRU_\epsilon) \leq E_\lambda^a(u_\lambda) + \frac{R^2 t^2 B}{2} - \frac{\lambda t^{2^*} R^{2^*} A}{2^*} - \lambda R^{2^*-1}t^{2^*-1}\int_\Omega U_\epsilon^{2^*-1}u_\lambda$$

$$+ o_\epsilon(\epsilon^{(N-2)/2}).$$

Now the lemma follows using the arguments of Section 3 of [26]. □

**Lemma 3.3.** *Let $(MP)$ hold. Then there exists a solution $v_\lambda \in H^+$ of $(\tilde{P}_\lambda^a)$ and hence a second solution for $(P_\lambda^a)$.*

*Proof.* Here we argue as in Lemma 3.5 of [4]. Define a complete metric space $(X, d)$ as

$$X = \{\eta \in C([0,1], H^+) : \eta(0) = 0, \|\eta(1)\|_{H_0^1(\Omega)} > \rho_1,\ I_\lambda(\eta(1)) < 0\},$$

$$d(\eta,\chi) = \max_{t\in[0,1]} \|\eta(t)-\chi(t)\|_{H_0^1(\Omega)}.$$

From $(i)$ of Lemma $(3.2)$, if $R$ is chosen large, it is clear that $X$ is non-empty. Let $\gamma_0 = \inf_{\eta\in X} \max_{s\in[0,1]} I_\lambda(\eta(s))$. Then $(ii)$ of Lemma $(3.2)$ and $(MP)$ implies that

$$0 < \gamma_0 < \frac{S^{\frac{N}{2}}}{N\lambda^{(N-2)/2}}. \tag{3.33}$$

Define

$$\Psi(\eta) = \max_{t\in[0,1]} I_\lambda(\eta(t)), \eta\in X.$$

Thus applying Ekeland's variational principle to the above functional we get a sequence $\{\eta_n\}\subseteq X$ such that

(i) $\max_{t\in[0,1]} I_\lambda(\eta_n(t)) < \gamma_0 + \frac{1}{n}.$

(ii) $\max_{t\in[0,1]} I_\lambda(\eta_n(t)) \le \max_{t\in[0,1]} I_\lambda(\eta(t)) + \frac{1}{n}\max_{t\in[0,1]}\|\eta(t)-\eta_n(t)\|_{H_0^1(\Omega)}$ for all $\eta\in X$.

Set $\Lambda_n = \{t\in[0,1]|I_\lambda(\eta_n(t)) = \max_{s\in[0,1]} I_\lambda(\eta_n(s))\}$. Then as in the Claim on page 659 of [4] we get $t_n\in\Lambda_n$ such that for $v_n = \eta_n(t_n)$ and $\xi\in H^+$ we have

$$I_\lambda^0\left(v_n, \frac{\xi-v_n}{max\{1,\|\xi-v_n\|_{H_0^1(\Omega)}\}}\right) \ge -\frac{1}{n} \tag{3.34}$$

and

$$I_\lambda(v_n)\to\gamma_0 \quad \text{as } n\to\infty. \tag{3.35}$$

From (3.35) we have

$$\frac{1}{2}\|v_n\|_{H_0^1(\Omega)}^2 - \lambda\int_\Omega \tilde{F}(v_n) - \lambda\int_\Omega \tilde{G}(v_n) \le \gamma_0 + o_n(1).$$

As $\tilde{G}(v_n)\le 0$ and $\tilde{F}(v_n) = \frac{(v_n+u_\lambda)^{2^*}}{2^*} - u_\lambda^{2^*-1}v_n - \frac{u_\lambda^{2^*}}{2^*}$, this implies

$$\frac{1}{2}\|v_n\|_{H_0^1(\Omega)}^2 - \lambda\int_\Omega \frac{(v_n+u_\lambda)^{2^*}}{2^*} + \lambda\int_\Omega u_\lambda^{2^*-1}v_n \le \gamma_0 + o_n(1). \tag{3.36}$$

Also substituting $\xi = 2v_n + u_\lambda$ in (3.34), by Remark 3.2 we obtain (by abuse of notation)

$$w_n^{v_n}(x) \in [\chi_{\{v_n(x)+u_\lambda(x)<a\}}, \chi_{\{v_n(x)+u_\lambda(x)\le a\}}]$$

such that

$$\|v_n+u_\lambda\|_{H_0^1(\Omega)}^2 - \lambda\int_\Omega ((v_n+u_\lambda)^{2^*} - w_n^{v_n}(v_n+u_\lambda)^{1-\delta}) \ge -\frac{1}{n}\max\{1,\|v_n+u_\lambda\|_{H_0^1(\Omega)}\}. \tag{3.37}$$

From (3.36) and (3.37) we derive

$$\frac{1}{2}\|v_n\|^2_{H_0^1(\Omega)} - \frac{1}{2^*}\|v_n\|^2_{H_0^1(\Omega)} \le c_1 + c_2\|v_n\|_{H_0^1(\Omega)}$$

where $c_1, c_2 > 0$. Thus $\|v_n\|_{H_0^1(\Omega)} \le C$ for all $n \in \mathbb{N}$. Hence $v_n \rightharpoonup v_\lambda$ weakly in $H_0^1(\Omega)$ and as in case of zero altitude $v_\lambda$ solves $(\tilde{P}_\lambda^a)$. Now we claim that $v_n \to v_\lambda$ in $H_0^1(\Omega)$ and thus $v_\lambda \not\equiv 0$. Without loss of generality we assume $I_\lambda(v_\lambda) = 0$, otherwise it would imply that $v_\lambda \not\equiv 0$ and we are done. As $\|v_n\|_{H_0^1(\Omega)} \le C$, from (3.34), for $\xi \in H^+$ we have $I_\lambda^0(v_n, \xi - v_k) \ge -\frac{C_1}{n}(1 + \|\xi\|_{H_0^1(\Omega)}) = o_n(1)$. Then as in zero altitude case we get

$$\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} - \lambda\|v_n - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)} = o_n(1). \tag{3.38}$$

Also by Brezis-Lieb lemma,

$$
\begin{aligned}
I_\lambda(v_n) &= \frac{1}{2}\|v_n\|^2_{H_0^1(\Omega)} - \lambda\int_\Omega \tilde{F}(v_n) - \lambda\int_\Omega \tilde{G}(v_n) \\
&= \frac{1}{2}\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} + \frac{1}{2}\|v_\lambda\|^2_{H_0^1(\Omega)} + \int_\Omega \nabla(v_n - v_\lambda)\cdot\nabla v_\lambda \\
&\quad - \lambda\left(\frac{1}{2^*}\int_\Omega(v_n + u_\lambda)^{2^*} - \frac{1}{2^*}\int_\Omega u_\lambda^{2^*} - \int_\Omega u_\lambda^{2^*-1}v_n\right) - \lambda\int_\Omega \tilde{G}(v_n) \\
&= \frac{1}{2}\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2^*}\|v_n - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)} + \frac{1}{2}\|v_\lambda\|^2_{H_0^1(\Omega)} - \lambda\int_\Omega \tilde{G}(v_n) \\
&\quad - \lambda\left(\frac{1}{2^*}\int_\Omega(v_\lambda + u_\lambda)^{2^*} - \frac{1}{2^*}\int_\Omega u_\lambda^{2^*} - \int_\Omega u_\lambda^{2^*-1}v_n\right) \\
&\quad + \int_\Omega \nabla(v_n - v_\lambda)\cdot\nabla v_\lambda + o_n(1) \\
&= \frac{1}{2}\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2^*}\|v_n - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)} + I_\lambda(v_\lambda) \\
&\quad + \lambda\int_\Omega\left(\tilde{G}(v_\lambda) - \tilde{G}(v_n)\right) + o_n(1) \\
&= \frac{1}{2}\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2^*}\|v_n - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)} + I_\lambda(v_\lambda) + o_n(1). \tag{3.39}
\end{aligned}
$$

Now as $I_\lambda(v_\lambda) = 0$, using (3.33), (3.35), (3.38) and (3.39), we get

$$\|v_n - v_\lambda\|^2_{H_0^1(\Omega)} = N\gamma_0 + o_n(1) < \frac{S^{\frac{N}{2}}}{\lambda^{(N-2)/2}} + o_n(1). \tag{3.40}$$

Also by the Sobolev embedding we have

$$\|v_n - v_\lambda\|^2_{H_0^1(\Omega)}\left(1 - \lambda S^{\frac{-2^*}{2}}\|v_n - v_\lambda\|^{2^*-2}_{H_0^1(\Omega)}\right) \le \|v_n - v_\lambda\|^2_{H_0^1(\Omega)} - \lambda\|v_n - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)}$$
$$= o_n(1). \tag{3.41}$$

Thus combining (3.40) and (3.41) we obtain $v_n \to v_\lambda$ in $H_0^1(\Omega)$. This completes the proof of the lemma. $\square$

We are now ready to give the

**Proof of Theorem 1.1:** The existence of the first solution $u_\lambda$ for all $\lambda \in (0, \Lambda^a)$ follows from lemma 2.1 and theorem 2.2 . The existence of the second solution $v_\lambda$ for the same range of $\lambda$ follows from lemma 3.1 and lemma 3.3 keeping in view the remark 3.3. $\square$

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