Rigidity results with applications to best constants and symmetry of Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

Jean Dolbeault · Maria J. Esteban · Stathis Filippas · Achilles Tertikas

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Abstract We take advantage of a rigidity result for the equation satisfied by an extremal function associated with a special case of the Caffarelli-Kohn-Nirenberg inequalities to get a symmetry result for a larger set of inequalities. The main ingredient is a reparametrization of the solutions to the Euler-Lagrange equations and estimates based on the rigidity result. The symmetry results cover a range of parameters which go well beyond the one that can be achieved by symmetrization methods or comparison techniques so far.

Keywords Caffarelli-Kohn-Nirenberg inequalities; Hardy-Sobolev inequality; extremal functions; ground state; bifurcation; branches of solutions; Emden-Fowler transformation; radial symmetry; symmetry breaking; rigidity; Keller-Lieb-Thirring inequalities

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Ceremade, Univ. Paris-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cédex 16, France E-mail: dolbeaul@ceremade.dauphine.fr

M.J. Esteban

Ceremade, Univ. Paris-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cédex 16, France E-mail: esteban@ceremade.dauphine.fr

S. Filippas

Department of Mathematics, Univ. of Crete, Knossos Avenue, 714 09 Heraklion & Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Crete, Greece Email: filippas@tem.uoc.gr

A. Tertikas

Department of Mathematics, Univ. of Crete, Knossos Avenue, 714 09 Heraklion & Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Crete, Greece Email: tertikas@math.uoc.gr

J. Dolbeault

1 Introduction and main results

Let $2^* := \infty$ if d = 1, 2, and $2^* := 2d/(d-2)$ if $d \ge 3$. Define

$$\vartheta(p,d) := \frac{d(p-2)}{2p}, \qquad a_c := \frac{d-2}{2},$$

and consider the space $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with respect to the norm $v\mapsto \||x|^{-a}\,\nabla v\,\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$. We will be concerned with the following two families of inequalities

Caffarelli-Kohn-Nirenberg Inequalities (CKN) [2] Let $d \ge 1$. For any $p \in [2, 2^*]$ if $d \ge 3$ or $p \in [2, 2^*]$ if d = 1, 2, for any $\theta \in [\vartheta(p, d), 1]$ with $\theta > 1/2$ if d = 1, there exists a positive constant $\mathsf{C}_{\mathrm{CKN}}(\theta, p, a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}} \, dx\right)^{\frac{2}{p}} \le \mathsf{C}_{\mathsf{CKN}}(\theta, p, a) \left(\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}} \, dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}} \, dx\right)^{1-\theta}$$

holds true for any $v \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$. Here a,b and p are related by $b=a-a_c+d/p$, with the restrictions $a \leq b \leq a+1$ if $d \geq 3$, $a < b \leq a+1$ if d=2 and $a+1/2 < b \leq a+1$ if d=1. Moreover, the constants $\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)$ are uniformly bounded outside a neighborhood of $a=a_c$.

In [4], a new class of inequalities, called weighted logarithmic Hardy inequalities, was considered. These inequalities can be obtained from (1) by taking $\theta = \gamma (p-2)$ and passing to the limit as $p \to 2_+$.

Weighted Logarithmic Hardy Inequalities (WLH)[4] Let $d \geq 1$, $a < a_c$, $\gamma \geq d/4$ and $\gamma > 1/2$ if d = 2. Then there exists a positive constant $\mathsf{C}_{\mathrm{WLH}}(\gamma, a)$ such that, for any $v \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ normalized by

$$\int_{\mathbb{R}^d} |x|^{-2(a+1)} |v|^2 dx = 1 ,$$

we have

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2(a+1)}} \log \left(|x|^{2(a_c-a)} |v|^2 \right) dx \le 2\gamma \log \left[\mathsf{C}_{\mathrm{WLH}}(\gamma, a) \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx \right]. \tag{2}$$

Moreover, the constants $C_{WLH}(\gamma, a)$ are uniformly bounded outside a neighborhood of $a = a_c$.

It is very convenient to reformulate the Caffarelli-Kohn-Nirenberg inequality in cylindrical variables as in [3]. By means of the Emden-Fowler transformation

$$s = \log |x| \in \mathbb{R} , \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1} , \quad y = (s, \omega) , \quad u(y) = |x|^{a_c - a} v(x) ,$$

Inequality (1) for v is equivalent to a Gagliardo-Nirenberg-Sobolev inequality for the function u on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$:

$$\mathsf{K}_{\mathrm{CKN}}(\theta, p, \Lambda) \|u\|_{\mathrm{L}^{p}(\mathcal{C})}^{2} \leq \left(\|\nabla u\|_{\mathrm{L}^{2}(\mathcal{C})}^{2} + \Lambda \|u\|_{\mathrm{L}^{2}(\mathcal{C})}^{2}\right)^{\theta} \|u\|_{\mathrm{L}^{2}(\mathcal{C})}^{2(1-\theta)} \quad \forall u \in \mathrm{H}^{1}(\mathcal{C}).$$

$$(3)$$

Here and throughout the rest of the work we set

$$\Lambda := (a_c - a)^2 .$$

Similarly, with $u(y) = |x|^{a_c - a} v(x)$, Inequality (2) is equivalent to

$$\int_{\mathcal{C}} |u|^2 \log |u|^2 dy \le 2\gamma \log \left[\frac{1}{\mathsf{K}_{\mathrm{WLH}}(\gamma, \Lambda)} \left(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C})}^2 + \Lambda \right) \right] , \qquad (4)$$

for any $u \in H^1(\mathcal{C})$ such that $||u||_{L^2(\mathcal{C})} = 1$. In both cases, we consider on \mathcal{C} the measure $d\mu = |\mathbb{S}^{d-1}|^{-1}d\omega \, ds$ obtained by normalizing the surface of \mathbb{S}^{d-1} to 1 (that is, the uniform probability measure), tensorized with the usual Lebesgue measure on the axis of the cylinder.

We are interested in *symmetry* and *symmetry breaking* issues: when do we know that equality in (1) and (2) is achieved by radial functions or, alternatively, by functions depending only on s in (3) and (4)? Related with inequality (3) is the Rayleigh quotient:

$$\mathcal{Q}_{\Lambda}^{\theta}[u] := \frac{\left(\|\nabla u\|_{2}^{2} + \Lambda \|u\|_{2}^{2}\right)^{\theta} \|u\|_{2}^{2(1-\theta)}}{\|u\|_{p}^{2}}.$$

Here $||u||_q := \left(\int_{\mathcal{C}} |u|^q \ d\mu\right)^{1/q}$. Then (3) and (4) are equivalent to state that

$$\begin{split} \mathsf{K}_{\mathrm{CKN}}(\theta,p,\Lambda) &= \inf_{u \in \mathrm{H}^1(\mathcal{C}) \backslash \{0\}} \mathcal{Q}_{\Lambda}^{\theta}[u] \ , \\ \mathsf{K}_{\mathrm{WLH}}(\gamma,\Lambda) &= \inf_{\substack{u \in \mathrm{H}^1(\mathcal{C}) \backslash \{0\} \\ \|u\|_2 = 1}} \left(\|\nabla u\|_2^2 + \Lambda \right) \, e^{-\frac{1}{2\gamma} \int_{\mathcal{C}} |u|^2 \, \log |u|^2 \, d\mu} \ . \end{split}$$

Let $\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda)$ and $\mathsf{K}^*_{\mathrm{WLH}}(\gamma, \Lambda)$ be the corresponding values of the infimum when the set of minimization is restricted to functions depending only on s. The main interest of introducing the measure $d\mu$ is that $\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda)$ and $\mathsf{K}^*_{\mathrm{WLH}}(\gamma, \Lambda)$ are independent of the dimension and can be computed for d=1 by solving the problem on the real line \mathbb{R} .

Radial symmetry of v = v(x) means that $u = u(s, \omega)$ is independent of ω . Up to translations in s and a multiplication by a constant, the optimal functions in the class of functions depending only on $s \in \mathbb{R}$ solve the equation

$$-u_*'' + \Lambda u_* = u_*^{p-1} \quad \text{in} \quad \mathbb{R}$$

if $\theta = 1$. See Section 2 if $\theta < 1$. Up to translations in s, non-negative solutions of this equation are all equal to the function

$$u_*(s) := \frac{A}{\left[\cosh(B\,s)\right]^{\frac{2}{p-2}}} \quad \forall \, s \in \mathbb{R} \,\,, \tag{5}$$

with $A^{p-2} = \frac{p}{2} \Lambda$ and $B = \frac{1}{2} \sqrt{\Lambda} (p-2)$. The uniqueness up to translations is a standard result (see for instance [11, Proposition B.2] for a proof).

The symmetry breaking issue is now reduced to the question of knowing whether the inequalities

$$\mathsf{K}_{\mathsf{CKN}}(\theta, p, \Lambda) \le \mathsf{K}_{\mathsf{CKN}}^*(\theta, p, \Lambda) \quad \text{and} \quad \mathsf{K}_{\mathsf{WLH}}(\gamma, \Lambda) \le \mathsf{K}_{\mathsf{WLH}}^*(\gamma, \Lambda)$$
 (6)

are strict or not, when $d \geq 2$. Symmetry breaking occurs if the inequality is strict and then optimal functions are not symmetric (symmetric means: depending only on s in the setting of the cylinder, or on |x| in the case of the Euclidean space). In [4, pp. 2048 and 2057], the values of the symmetric constants have been computed. They are given by

$$\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda) := \left[\frac{2p\theta + 2 - p}{(p - 2)^2}\right]^{\frac{p - 2}{2p}} \left[\frac{2p\theta}{2p\theta + 2 - p}\right]^{\theta} \left[\frac{p + 2}{4}\right]^{\frac{6 - p}{2p}} \left[\frac{\sqrt{\pi} \Gamma\left(\frac{2}{p - 2}\right)}{\Gamma\left(\frac{2}{p - 2} + \frac{1}{2}\right)}\right]^{\frac{p - 2}{p}} \Lambda^{\theta - \frac{p - 2}{2p}}$$
(7)

and

$$\begin{split} \mathsf{K}^*_{\mathrm{WLH}}(\gamma, \varLambda) &= \frac{\gamma \left(8\pi^{d+1} \, e\right)^{\frac{1}{4\gamma}}}{\Gamma\left(\frac{d}{2}\right)^{\frac{1}{2\gamma}}} \, \left(\frac{4\varLambda}{4\gamma-1}\right)^{\frac{4\gamma-1}{4\gamma}} \, \, \mathrm{if} \, \, \gamma > \frac{1}{4} \, \, , \\ \mathsf{K}^*_{\mathrm{WLH}}(\gamma, \varLambda) &= \frac{2\pi^{d+1} \, e}{\Gamma\left(\frac{d}{2}\right)^{\frac{2}{2}}} \qquad \qquad \mathrm{if} \quad \gamma = \frac{1}{4} \, \, . \end{split}$$

Let

$$\Lambda_{\text{FS}}(\theta, p, d) := 4 \frac{d-1}{p^2 - 4} \frac{(2 \theta - 1) p + 2}{p + 2} \quad \text{and} \quad \Lambda_{\star}(1, p, d) := \frac{1}{4} (d - 1) \frac{6 - p}{p - 2} \ . \tag{8}$$

We will define $\Lambda_{\star}(\theta,p,d)$ for $\theta<1$ later in the Introduction. Symmetry breaking occurs for any $\Lambda>\Lambda_{\rm FS}$ according to a result of V. Felli and M. Schneider in [15] for $\theta=1$ and in [4] for $\theta<1$ (also see [3] for previous results and [14] if d=2 and $\theta=1$). This symmetry breaking is a straightforward consequence of the fact that for $\Lambda>\Lambda_{\rm FS}$, the symmetric optimals are saddle points of an energy functional, and thus cannot be even local minima. As a consequence, we know that $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)<\mathsf{K}^*_{\rm CKN}(\theta,p,\Lambda)$ if $\Lambda>\Lambda_{\rm FS}(\theta,p,d)$.

Concerning the log Hardy inequality, it was shown in [4] that symmetry breaking occurs, that is, $\mathsf{K}_{\mathrm{WLH}}(\gamma, \Lambda) < \mathsf{K}_{\mathrm{WLH}}^*(\gamma, \Lambda)$, when either d=2 and $\gamma > 1/2$ or $d \geq 3$ and $\gamma \geq d/4$ provided that

$$\Lambda > (d-1)\left(\gamma - \frac{1}{4}\right)$$
.

Concerning symmetry, if $\theta=1$, from [12], we know that symmetry holds for CKN for any $\Lambda \leq \Lambda_{\star}(1,p,d)$. The precise statement goes as follows.

Theorem 1 [12] Let $d \ge 2$. For any $p \in [2, 2^*]$ if $d \ge 3$ or $p \in [2, \infty)$ if d = 2, under the conditions

$$0 < \mu \le \Lambda_{\star}(1, p, d)$$
 and $\mathcal{Q}_{\mu}^{1}[u] \le \mathsf{K}_{\mathrm{CKN}}^{*}(1, p, \mu)$,

the solution of

$$-\Delta u + \mu u = u^{p-1} \quad on \quad \mathcal{C} \tag{9}$$

is given by the one-dimensional equation, written on \mathbb{R} . It is unique, up to translations.

Theorem 1 is a rigidity result. In [12], the proof is given for a minimizer of \mathcal{Q}^1_μ , which therefore satisfies $\mathcal{Q}^1_\mu[u] \leq \mathsf{K}^*_{\mathsf{CKN}}(1,p,\mu)$, but the reader is invited to check that only the latter condition is used in the proof. The proof is based on a chain of estimates which involve optimal interpolation inequalities on the sphere and the Keller-Lieb-Thirring inequality. These inequalities turn out to be equalities, and equality in each of the inequalities is shown to imply that the solution only depends on s (no angular dependence). The result of Theorem 1 gives a sufficient condition for symmetry when $\theta=1$. We shall say that any minimizer is symmetric if it is given by (5), up to multiplications by constants and translations.

Theorem 2 [12] Let $d \geq 2$. For any $p \in [2,2^*]$ if $d \geq 3$ or any $p \in [2,\infty)$ if d = 2, if $0 < \Lambda \leq \Lambda_{\star}(1,p,d)$, then $\mathsf{K}_{\mathrm{CKN}}(1,p,\Lambda) = \mathsf{K}_{\mathrm{CKN}}^*(1,p,\Lambda)$ and any minimizer is symmetric.

In [12], the case $\theta < 1$ is also considered. According to [12, Theorem 9], for any $d \ge 3$, any $p \in (2, 2^*)$ and any $\theta \in [\vartheta(p, d), 1)$, we have the estimate

$$\mathfrak{C}(\theta, p)^{-\frac{2\theta}{q+2}} \mathsf{K}_{\mathrm{CKN}}^*(\theta, \Lambda, p) \le \mathsf{K}_{\mathrm{CKN}}(\theta, \Lambda, p) \le \mathsf{K}_{\mathrm{CKN}}^*(\theta, \Lambda, p) \tag{10}$$

where $q := \frac{2(p-2)}{(2\theta-1)p+2}$ and

$$\mathfrak{C}(\theta,p) := \frac{(p+2)\frac{p+2}{(2\theta-1)\frac{p+2}{p+2}}}{(2\theta-1)\frac{p+2}{p+2}} \, \left(2 - \frac{p}{2}\left(1-\theta\right)\right)^{1-\frac{q}{2}} \, \cdot \left(\frac{\Gamma(\frac{p}{p-2})}{\Gamma(\frac{\theta p}{p-2})}\right)^{2q} \, \left(\frac{\Gamma(\frac{2\theta p}{p-2})}{\Gamma(\frac{2p}{p-2})}\right)^{q}$$

under the condition $a_c^2 < \Lambda \le \frac{(d-1)}{\mathfrak{C}(\theta,p)} \frac{(2\,\theta-3)\,p+6}{4\,(p-2)}$. If $\theta=1$, the equality case in the last inequality characterizes $\Lambda_\star(1,p,d)$ as defined in (8). However (10) does not give a range for symmetry unless $\theta=1$.

Much more is known. According to [13,5], there is a continuous curve $p\mapsto \Lambda_{\rm s}(\theta,p,d)$ with $\lim_{p\to 2_+}\Lambda_{\rm s}(\theta,p,d)=\infty$ and $\Lambda_{\rm s}(\theta,p,d)>a_c^2$ for any $p\in(2,2^*)$ such that symmetry holds for any $\Lambda\leq\Lambda_{\rm s}(1,p,d)$ and there is symmetry breaking if $\Lambda>\Lambda_{\rm s}(1,p,d)$, for any $\theta\in[\vartheta(p,d),1)$. Additionally, we have that $\lim_{p\to 2^*}\Lambda_{\rm s}(1,p,d)=a_c^2$ if $d\geq 3$ and, if d=2, $\lim_{p\to\infty}\Lambda_{\rm s}(1,p,d)=0$ and $\lim_{p\to\infty}p^2\Lambda_{\rm s}(1,p,d)=4$. The existence of this function $\Lambda_{\rm s}$ has been proven in an indirect way, and it is not explicitly known. It has been a long-standing question to decide whether the curves $p\to\Lambda_{\rm s}(\theta,p,d)$ and the curve $p\to\Lambda_{\rm FS}(\theta,p,d)$ coincide or not. This is still an open question, at least for $\theta=1$. For $\theta<1$, and for some specific values of p, it has been shown that, in some cases, $\Lambda_{\rm s}(\theta,p,d)<\Lambda_{\rm FS}(\theta,p,d)$; see [5] for more details, as well as some symmetry results based on symmetrization techniques. A scenario based on numerical computations and asymptotic expansions at the point where non-symmetric positive solutions bifurcate from the symmetric ones has been proposed; see [7,9,10] for details.

Our interest in this work is to establish symmetry of the minimizers of CKN for $\theta < 1$ as well as of the log Hardy inequalities, thus identifying the corresponding sharp constants.

Our first result is an extension of Theorem 2 to the case $\theta < 1$. Our goal is to give explicit estimates of the range for which symmetry holds. This requires some notations and a preliminary result. We set

$$\Pi^*(\theta, p, q) := \left(\frac{\mathsf{K}_{\mathrm{CKN}}^*(\theta, p, 1)}{\mathsf{K}_{\mathrm{CKN}}^*(1, q, 1)^{\frac{q(p-2)}{p(q-2)}}}\right)^{\frac{1}{\theta - \frac{q(p-2)}{p(q-2)}}} .$$
(11)

Next we define

$$q^* = q^*(\theta, p) := \frac{2p\theta}{2 - p(1 - \theta)}. \tag{12}$$

The condition $\theta > \frac{q \, (p-2)}{p \, (q-2)}$ is equivalent to $q > q^*(\theta,p)$ and we can notice that $p < q^*(\theta,p) < 2^*$ for any $\theta \in (\vartheta(p,d),1)$. For $d \geq 3$ we define

$$\Lambda_1(\theta, p, d) := \max_{q \in (q^*, 2^*)} \min \left\{ \Lambda_{\star}(1, q, d), \frac{\theta \Lambda_{\star}(1, p, d)}{(1 - \theta) \Pi^*(\theta, p, q) + \theta} \right\} ,$$

whereas for d=2

$$\Lambda_1(\theta, p, 2) := \max_{q \in (q^*, 6)} \min \left\{ \Lambda_{\star}(1, q, 2), \frac{\theta \, \Lambda_{\star}(1, p, 2)}{(1 - \theta) \, \Pi^*(\theta, p, q) + \theta} \right\} .$$

Next, we can also define

$$N(\theta, p) := \frac{\left(K_{CKN}^*(\theta, p, 1)\right)^{1/\theta}}{K_{CKN}^*(1, q^*(\theta, p), 1)} . \tag{13}$$

We refer to Section 3 for an explicit expression of $N(\theta, p)$. We introduce the exponent

$$\beta = \beta(\theta, p) := 1 - \frac{p-2}{2 p \theta}$$
 (14)

For $2 and <math>\theta \in (\vartheta(p,3),1)$ we denote by $\mathsf{x}^* = \mathsf{x}^*(\theta,p)$ the unique root of the equation

$$\theta (6-p)(x^{\beta}-N)x - (2p\theta-3(p-2))(\theta (x^{\beta}-N) + (1-\theta)(x-1)N) = 0$$

in the interval $(N^{1/\beta}, \infty)$ for $N = N(\theta, p)$, see Lemma 2 in Section 3. Next we define

$$\Lambda_2(\theta, p, d) := \frac{\Lambda_{\star}(1, q^*, d)}{\mathsf{x}^*(\theta, p)} = \frac{1}{4} \left(d - 1 \right) \frac{2 \, p \, \theta - 3 \, (p - 2)}{(p - 2) \, \mathsf{x}^*(\theta, p)} \; ,$$

and

$$\varLambda_{\star}(\theta,p,d) := \max \Big\{ \varLambda_1(\theta,p,d), \varLambda_2(\theta,p,d) \Big\} \ .$$

Theorem 3 Suppose that either d=2 and $p\in(2,6)$ or else $d\geq 3$ and $p\in(2,2^*)$. Then

$$\mathsf{K}_{\mathrm{CKN}}(\theta, p, \Lambda) = \mathsf{K}_{\mathrm{CKN}}^*(\theta, p, \Lambda)$$

and any minimizer of CKN (3) is symmetric provided that one of the following conditions is satisfied:

- (i) d = 2, $\theta \in (\vartheta(p, 2), 1)$ and $0 < \Lambda \le \Lambda_1(\theta, p, 2)$.
- (ii) d = 2, $\theta \in (\vartheta(p,3), 1)$ and $0 < \Lambda \le \Lambda_{\star}(\theta, p, 2)$,
- (iii) $d \geq 3$, $\theta = \vartheta(p, d)$ and $0 < \Lambda \leq \Lambda_2(\theta, p, d)$,
- (iv) $d \geq 3$, $\theta \in (\vartheta(p,d), 1)$ and $0 < \Lambda \leq \Lambda_{\star}(\theta, p, d)$

Our definition of $\Lambda_{\star}(\theta, p, d)$ for $\theta < 1$ is consistent with the definition of $\Lambda_{\star}(1, p, d)$ given in (8) because

$$\lim_{\theta \to 1} \Lambda_1(\theta, p, d) = \lim_{\theta \to 1} \Lambda_2(\theta, p, d) = \Lambda_{\star}(1, p, d) .$$

One of the drawbacks in the definition of $\Lambda_2(\theta, p, d)$ is that $\mathsf{x}^*(\theta, p)$ given by Lemma 2 is not explicit. For an explicit estimate of $\Lambda_2(\theta, p, d)$ see Proposition 2 in Section 5.

By passing to the limit as $p \to 2_+$ in the criterion $\Lambda \le \Lambda_2(\theta, p, d)$, we also obtain an explicit condition for symmetry in the weighted logarithmic Hardy inequalities. For any $N_0 > 1$, consider the smallest root $x > N_0^{1/\beta_0}$ of

$$4 \gamma x^{\beta_0+1} - (8\gamma - 3) N_0 x + (4\gamma - 3) N_0 = 0$$
 with $\beta_0 = 1 - \frac{1}{4\gamma}$

and denote it by $\mathsf{x}_0^*(\gamma)$ if $\mathsf{N}_0 = \mathsf{N}_0(\gamma) := \lim_{p \to 2_+} \mathsf{N}(\gamma(p-2), p)$. An elementary but tedious computation shows that

$$\mathsf{N}_{0}(\gamma) = 2^{1 - \frac{3}{4\gamma}} e^{\frac{1}{4\gamma}} \frac{(2\gamma - 1)^{1 - \frac{1}{\gamma}}}{(4\gamma - 1)^{1 - \frac{3}{4\gamma}}} \left(\frac{\Gamma(2\gamma - \frac{1}{2})}{\Gamma(2\gamma - 1)} \right)^{\frac{1}{2\gamma}}. \tag{15}$$

Let us define

$$\Lambda_0(\gamma, d) := \frac{(d-1)(\gamma - 3/4)}{\mathsf{x}_0^*(\gamma)} \ . \tag{16}$$

We then have

Theorem 4 Assume that either d=2 or 3 and $\gamma>3/4,$ or $d\geq 4$ and $\gamma\geq d/4.$ Then

$$\mathsf{K}_{\mathrm{WLH}}(\gamma, \Lambda) = \mathsf{K}_{\mathrm{WLH}}^*(\gamma, \Lambda) ,$$

and any minimizer of (4) is symmetric provided that

$$0 < \Lambda \le \Lambda_0(\gamma, d)$$
.

For an explicit estimate of $\Lambda_0(\gamma, d)$ see Proposition 3 in Section 5.

Theorem 3 provides us with a *rigidity result*, which is stronger than a simple symmetry result. As a consequence, our estimates of Theorem 3 for the symmetry region cannot be optimal.

Theorem 5 Suppose that either d=2 and $p\in(2,6)$ or else $d\geq 3$ and $p\in(2,2^*)$. If $\theta>\vartheta(p,\min\{3,d\})$, then

$$\Lambda_{\star}(\theta, p, d) < \Lambda_{\rm s}(\theta, p, d) \le \Lambda_{\rm FS}(\theta, p, d)$$
.

If either d=3 and $\theta=\vartheta(p,3)$, or d=2 and $\theta>0$, then

$$\Lambda_2(\theta, p, d) < \Lambda_s(\theta, p, d) \le \Lambda_{FS}(\theta, p, d)$$
.

It can be conjectured that $\Lambda_{\rm s}(\theta,p,d) = \Lambda_{\rm FS}(\theta,p,d)$ holds in the limit case $\theta=1$, and probably also for θ close enough to 1, on the basis of the numerical results of [9] and the formal computations of [10]. On the other hand, it is known from [5] that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$ when $\theta-\vartheta(p,d)$ is small enough, at least for some values of p and d.

The expressions involved in the statement of Theorem 3 look quite technical, but they are interesting for two reasons:

- Theorem 3 determines a range for symmetry which goes well beyond what can be achieved using standard methods and is somewhat unexpected in view of the estimate of [12, Theorem 9]. It is a striking observation that the reparametrization method which has been extensively used in [9,10] allows us to extend to $\theta < 1$ results which were known only for $\theta = 1$.
- Even if they cannot be optimal as shown in Theorem 5, the estimates of Theorem 3 are rather accurate from the numerical point of view, as will be illustrated in Section 5.

This paper is organized as follows. Section 2 is devoted to the reparametrization and the proof of symmetry when $\Lambda \leq \Lambda_1(\theta,p,d)$ in the subcritical case $\vartheta(p,d) < \theta < 1$. To the price of some additional technicalities, the range $\Lambda \leq \Lambda_2(\theta,p,d)$ and $\vartheta(p,\min\{3,d\}) \leq \theta < 1$ is covered in Section 3. The proofs of Theorems 3 and 5 are established in Section 4. The last section is devoted to an explicit approximation of Λ_0 and Λ_2 , and some numerical results which illustrate Theorems 3 and 5. The reader interested in the strategy of the proofs as well as the origin of the expressions of $\Lambda_1(\theta,p,d)$ and $\Lambda_2(\theta,p,d)$ is invited to read first Section 2 and the proof of Lemma 5 in Section 3.

2 Reparametrization and a first symmetry result

We begin by a reparametrization of the branches of the solutions which allows us to reduce the case corresponding to $\theta < 1$ and Λ to the case corresponding to $\theta = 1$ and some related μ , as in Theorem 1. Consider an optimal function u for (3), which therefore satisfies

$$\mathsf{K}_{\mathrm{CKN}}(\theta, p, \Lambda) = \mathcal{Q}_{\Lambda}^{\theta}[u] = (t + \Lambda)^{\theta} \frac{\|u\|_{2}^{2}}{\|u\|_{p}^{2}} \quad \text{with} \quad t := \frac{\|\nabla u\|_{2}^{2}}{\|u\|_{2}^{2}}.$$

According to [5, Theorem1], such a function u exists for any $\theta > \vartheta(p, d)$. As a critical point of \mathcal{Q}_A^{θ} , u solves (9) with

$$\theta \mu = (1 - \theta) t + \Lambda$$

if it has been normalized by the condition

$$\|\nabla u\|_2^2 + \Lambda \|u\|_2^2 = \theta \|u\|_p^p$$
.

Because of the zero-homogeneity of $\mathcal{Q}_{\Lambda}^{\theta}$, such a condition can be imposed without restriction and is equivalent to

$$||u||_{2}^{2} = \frac{\theta}{t+A} ||u||_{p}^{p}. \tag{17}$$

Proposition 1 Let us assume that u is a solution of (9), satisfying $Q_{\Lambda}^{\theta}[u] = \mathsf{K}_{CKN}(\theta, p, \Lambda)$ and (17), with $\theta \mu = (1 - \theta) t + \Lambda$. Then we have

$$Q_{\mu}^{1}[u] \le \mathsf{K}_{\mathrm{CKN}}^{*}(1, p, \mu) . \tag{18}$$

Proof From (6) we know that

$$(t+\Lambda)^{\theta} \frac{\|u\|_2^2}{\|u\|_p^2} \le \mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda) .$$

Using (17), we rewrite this estimate as

$$\theta (t + \Lambda)^{\theta - 1} \|u\|_p^{p-2} \le \mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda)$$
.

Using (17) again and the expression of μ , we obtain

$$Q_{\mu}^{1}[u] = \frac{\theta\left(t + \mu\right)}{t + \Lambda} \|u\|_{p}^{p-2} = \|u\|_{p}^{p-2} \le f(t, \theta, \Lambda, p) \, \mathsf{K}^{*}_{\mathrm{CKN}}(1, p, \mu)$$

with

$$f(t,\theta,\Lambda,p) := \frac{1}{\theta (t+\Lambda)^{\theta-1}} \frac{\mathsf{K}^*_{\mathrm{CKN}}(\theta,p,\Lambda)}{\mathsf{K}^*_{\mathrm{CKN}}(1,p,\mu)} \; .$$

Using the expression of μ and (7), we find that

$$f(t,\theta,\Lambda,p) = \frac{(p+2)^{\frac{p+2}{2p}}}{(2p)^{1-\theta}} \left(\frac{\Lambda\theta}{2+(2\theta-1)p}\right)^{\theta-\frac{p-2}{2p}} (t+\Lambda)^{1-\theta} \left((1-\theta)t+\Lambda\right)^{-\frac{p+2}{2p}}$$

achieves its maximum at $t_0 := \Lambda \left(\frac{2p\theta}{p-2} - 1\right)^{-1} > 0$. Hence $f(t) \le f(t_0) = 1$, which concludes the proof.

Using the notations (11) and (12), we obtain our first symmetry result, which goes as follows.

Lemma 1 Suppose that either d=2 and $p\in(2,6)$ or else $d\geq 3,\ p\in(2,2^*)$. If $\theta\in(\vartheta(p,d),1)$ and

$$\Lambda \le \min \left\{ \Lambda_{\star}(1, q, d), \frac{\theta \Lambda_{\star}(1, p, d)}{(1 - \theta) \Pi^{*}(\theta, p, q) + \theta} \right\}$$

for some $q \in (q^*(\theta, p), 6)$ when d = 2, or for some $q \in (q^*(\theta, p), 2^*)$ when $d \geq 3$, then any optimal function for (3) is symmetric.

Proof Let u be a solution as in Proposition 1. From (6), we know that

$$\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda) \ge (t + \Lambda)^{\theta} \frac{\|u\|_2^2}{\|u\|_p^2}.$$

For $p < q < \min\{6, 2^*\}$ we have by Hölder's inequality, $||u||_p \le ||u||_2^{\delta} ||u||_q^{1-\delta}$ provided $\delta = \frac{2}{p} \frac{q-p}{q-2}$, and thus $1 - \delta = \frac{q}{p} \frac{p-2}{q-2}$. Hence

$$\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda) \ge (t + \Lambda)^{\theta} \left(\frac{\|u\|_2^2}{\|u\|_q^2} \right)^{1-\delta}$$
.

Now, for any $\lambda \in (0, \Lambda_{\star}(1, q, d)]$, we know from Theorem 2 that

$$||u||_q^2 \le \frac{||\nabla u||_2^2 + \lambda ||u||_2^2}{\mathsf{K}_{CKN}^*(1, q, \lambda)}$$
,

which shows that

$$\mathsf{K}^*_{\mathrm{CKN}}(\theta, p, \Lambda) \ge (t + \Lambda)^{\theta} \left(\frac{\mathsf{K}^*_{\mathrm{CKN}}(1, q, \lambda)}{t + \lambda} \right)^{1 - \delta}$$
.

Summarizing, we have found that

$$\frac{(t+\Lambda)^{\theta}}{(t+\lambda)^{1-\delta}} \le \frac{\mathsf{K}_{\mathrm{CKN}}^*(\theta, p, \Lambda)}{(\mathsf{K}_{\mathrm{CKN}}^*(1, q, \lambda))^{1-\delta}} \quad \text{if} \quad \lambda \le \Lambda_{\star}(1, q, d) \ . \tag{19}$$

Next we can make the ansatz $\lambda = \Lambda$. Provided $\Lambda \leq \Lambda_{\star}(1,q,d)$, we get that

$$(t+\varLambda)^{\theta+\delta-1} \leq \frac{\mathsf{K}^*_{\mathrm{CKN}}(\theta,p,\varLambda)}{(\mathsf{K}^*_{\mathrm{CKN}}(1,q,\varLambda))^{1-\delta}} = \left(\varPi^*(\theta,p,q) \varLambda \right)^{\theta+\delta-1} \,,$$

so that $t \leq (\Pi^*(\theta, p, q) - 1) \Lambda$. According to Theorem 1, u is symmetric if

$$\frac{1}{\theta} \left((1 - \theta) t + \Lambda \right) = \mu \le \Lambda_{\star} (1, p, d) , \qquad (20)$$

because (18) holds by Proposition 1. This completes the proof.

In the next section we shall consider an alternative ansatz for which $\lambda \neq \Lambda$.

3 Another symmetry result

In this section we establish an estimate similar to the one of Lemma 1 but based on a different ansatz, which moreover covers the critical case $\theta = \vartheta(p,d)$. We recall that $\beta = \beta(\theta,p) = 1 - \frac{p-2}{2\,p\,\theta}$ has been defined in (14). The proof is slightly more technical than the one of Lemma 1. We start with an auxiliary result.

Lemma 2 For any N > 1, p < 6 and $\theta \in (\vartheta(p,3),1)$, if $\beta = \beta(\theta,p)$ is given by (14), the equation

$$\theta(6-p)(x^{\beta}-N)x - (2p\theta-3(p-2))(\theta(x^{\beta}-N) + (1-\theta)(x-1)N) = 0$$

has a unique root in the interval $(N^{1/\beta}, \infty)$.

When $N = N(\theta, p) > 1$ is given by (13), we denote this root by $x^* = x^*(\theta, p)$.

Proof Consider the function

$$f(x) := \theta (6 - p)(x^{\beta} - N) x - (2 p \theta - 3 (p - 2)) [\theta (x^{\beta} - N) + (1 - \theta)(x - 1)N],$$

and notice first that $f(\mathsf{N}^{1/\beta}) < 0$ because $\theta > \vartheta(p,3)$ and $\mathsf{N}^{1/\beta} > 1$. Next we observe that $\alpha := 2\,p\,\theta - 3\,(p-2) = 2\,p\,\big(\theta - \vartheta(p,3)\big) = 6 - p - 2\,p\,(1-\theta)$ and compute

$$f'(x) = (6 - p) \theta \left[(1 + \beta) x^{\beta} - \mathsf{N} \right] - 2 p \left(\theta - \vartheta(p, 3) \right) \left[\beta \theta x^{\beta - 1} + (1 - \theta) \mathsf{N} \right]$$

and

$$f''(x) = \beta \theta x^{\beta-2} \left[(6-p)(1+\beta)x - (\beta-1)(6-p-2p(1-\theta)) \right] > 0$$

for any x > 1. Using the fact that N > 1, we find that

$$f'(N^{1/\beta}) \ge 2(p-2)(1-\theta)N > 0$$
.

It follows that the function f(x) is increasing and convex for $x > N^{1/\beta}$. Since $f(N^{1/\beta}) < 0$ we conclude that f(x) has a unique root for $x > N^{1/\beta}$.

When $\mathsf{N} = \mathsf{N}(\theta, p)$ we only need to check that $\mathsf{N}(\theta, p) > 1$. This is shown in Lemma 4. Before, we need a preliminary estimate. Consider the Digamma function $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Lemma 3 For all z > 0, we have

$$\frac{1}{2z} < \psi\left(z + \frac{1}{2}\right) - \psi(z) < \ln\left(1 + \frac{1}{2z}\right) + \frac{1}{z} - \frac{2}{2z+1} \ .$$

Proof We use the following representation formula (cf. $[1, \S 6.3.21, p. 259]$):

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-z\,t}}{1-e^{-t}}\right)\,dt$$

and elementary manipulations to get the lower bound

$$\psi\left(z + \frac{1}{2}\right) - \psi(z) = \int_0^\infty \frac{e^{-zt}}{1 + e^{-t/2}} dt > \frac{1}{2} \int_0^\infty e^{-zt} dt = \frac{1}{2z}.$$

As for the upper bound, we have the equivalences

$$\ln\left(1 + \frac{1}{2z}\right) + \frac{1}{z} - \frac{2}{2z+1} - \int_0^\infty \frac{e^{-zt}}{1 + e^{-t/2}} dt > 0$$

$$\iff \ln\left(1 + \frac{1}{2z}\right) + \int_0^\infty e^{-zt} dt - \frac{2}{2z+1} - \int_0^\infty \frac{e^{-zt}}{1 + e^{-t/2}} dt > 0$$

$$\iff \ln\left(1 + \frac{1}{2z}\right) + \int_0^\infty \frac{e^{-t/2} e^{-zt}}{1 + e^{-t/2}} dt - \frac{2}{2z+1} > 0.$$

The result follows from

$$\int_0^\infty \frac{e^{-(z+\frac{1}{2})\,t}}{1+e^{-t/2}}\;dt > \frac{1}{2}\int_0^\infty e^{-(z+\frac{1}{2})\,t}\;dt = \frac{1}{2\,z+1}$$

and, by monotonicity of the function $z \mapsto \ln\left(1 + \frac{1}{2z}\right) - \frac{1}{2z+1}$,

$$\ln\left(1+\frac{1}{2z}\right) > \frac{1}{2z+1}$$
.

Lemma 4 Assume that $2 and <math>\vartheta(p,2) < \theta \le 1$. Then the function $\theta \mapsto \mathsf{N}(\theta,p)$ is decreasing and $\mathsf{N}(1,p) = 1$.

Proof N(1,p) = 1 is a consequence of the definition of N. Using the precise value pf $K_{CKN}^*(\theta, p, \Lambda)$, we obtain the following explicit expression of the function $N(\theta, p)$, namely

$$\left(\frac{2}{2-p\left(1-\theta\right)}\right)^{\frac{p-2}{2\,p\,\theta}} \left(\frac{p+2}{4}\right)^{\frac{6-p}{2\,p\,\theta}} \left(\frac{2\,(2-p\,(1-\theta))}{(2\,\theta-1)\,p+2}\right)^{\frac{2\,p\,\theta-3\,(p-2)}{2\,p\,\theta}} \left\lceil \frac{\Gamma\!\left(\frac{2}{p-2}\right)\Gamma\!\left(\frac{2-p\,(1-\theta)}{p-2}+\frac{1}{2}\right)}{\Gamma\!\left(\frac{2}{p-2}+\frac{1}{2}\right)\Gamma\!\left(\frac{2-p\,(1-\theta)}{p-2}\right)}\right\rceil^{\frac{p-2}{p\,\theta}}.$$

Let us define $G := \mathsf{N}^{\theta}$ and compute

$$\begin{split} \frac{1}{G} \frac{\partial G}{\partial \theta} &= \frac{p \, \theta - 2 \, (p - 2)}{2 - p \, (1 - \theta)} - \frac{2 \, p \, \theta - 3 \, (p - 2)}{\left(2 \, \theta - 1\right) \, p + 2} + \ln \left(\frac{2 \, (2 - p \, (1 - \theta))}{\left(2 \, \theta - 1\right) \, p + 2} \right) \\ &+ \frac{\Gamma' \left(\frac{2 - p \, (1 - \theta)}{p - 2} + \frac{1}{2} \right)}{\Gamma \left(\frac{2 - p \, (1 - \theta)}{p - 2} + \frac{1}{2} \right)} - \frac{\Gamma' \left(\frac{2 - p \, (1 - \theta)}{p - 2} \right)}{\Gamma \left(\frac{2 - p \, (1 - \theta)}{p - 2} \right)} \, . \end{split}$$

By Lemma 3 we get that

$$\frac{1}{G} \frac{\partial G}{\partial \theta} < \frac{p \theta - 2 (p - 2)}{2 - p (1 - \theta)} - \frac{2 p \theta - 3 (p - 2)}{(2 \theta - 1) p + 2} + \ln \left(\frac{2 (2 - p (1 - \theta))}{(2 \theta - 1) p + 2} \right) + \ln \left(\frac{(2 \theta - 1) p + 2}{2 (2 - p (1 - \theta))} \right) + \frac{p - 2}{2 - p (1 - \theta)} - \frac{2 (p - 2)}{(2 \theta - 1) p + 2} = 0.$$

Since

$$\frac{1}{G}\frac{\partial G}{\partial \theta} = \ln \mathsf{N} + \frac{\theta}{\mathsf{N}}\frac{\partial \mathsf{N}}{\partial \theta} < 0 \ ,$$

for $\theta \in (\vartheta(p,2),1]$ and $\mathsf{N}(1,p)=1,$ it follows that $\frac{\partial}{\partial \theta}\mathsf{N}<0.$

After these preliminaries, we can now state the main result of this section.

Lemma 5 Assume that

$$\begin{split} 2$$

Then any optimal function for (3) is symmetric if $\Lambda < \Lambda_2(\theta, p, d)$. Moreover, we have $\lim_{\theta \to 1} \Lambda_2(\theta, p, d) = \Lambda_{\star}(1, p, d)$.

Proof As in the proof of Lemma 1, the starting point of our estimate is inequality (19), which becomes

$$\frac{t+\Lambda}{t+\lambda} \le \mathsf{N}(\theta,p) \left(\frac{\Lambda}{\lambda}\right)^{\beta} \quad \text{if} \quad \lambda < \Lambda_{\star}(1,q,d) \;,$$

under the restriction that we choose $q = q^*(\theta, p)$ given by (12), that is $1 - \delta = \theta$ with δ as in (11). Remarkably, we observe that, for this specific value of q, we have

$$\theta - \frac{p-2}{2p} = \left(1 - \frac{q-2}{2q}\right)(1-\delta)$$

and, as a consequence,

$$\frac{t+\varLambda}{t+\lambda} \leq \mathsf{N}\left(\frac{\varLambda}{\lambda}\right)^{\beta}$$

where $\beta := 1 - \frac{p-2}{2p\theta}$ and $\mathsf{N} = \mathsf{N}(\theta, p)$. Hence we get that

$$t \leq \frac{\mathsf{N}\,\varLambda^\beta\,\lambda - \varLambda\,\lambda^\beta}{\lambda^\beta - \mathsf{N}\,\varLambda^\beta} =: \bar{t} \;.$$

As in the proof of Lemma 1, we can apply Theorem 1 if

ullet Condition (20) holds and a sufficient condition is therefore given by the condition

$$(1-\theta)\bar{t} + \Lambda \leq \theta \Lambda_{\star}(1, p, d)$$
,

that is,

$$\left(\theta\, \varLambda_{\star}(1,p,d) - \varLambda\right) \left(\lambda^{\beta} - \mathsf{N}\, \varLambda^{\beta}\right) \geq \left(1 - \theta\right) \left(\mathsf{N}\, \varLambda^{\beta}\, \lambda - \varLambda\, \lambda^{\beta}\right).$$

• Condition $\lambda < \Lambda_{\star}(1, q, d)$, which is required to get (19), holds, *i.e.*,

$$\lambda < \Lambda_{\star}(1, q, d) = \frac{1}{4} (d - 1) \frac{6 - q}{q - 2} = \frac{1}{4} (d - 1) \frac{2 p \theta - 3 (p - 2)}{p - 2}.$$

For a suitable $x = \lambda/\Lambda > N^{1/\beta}$, to be chosen, these two conditions amount to

$$\begin{split} & \boldsymbol{\Lambda} \leq \phi(\boldsymbol{x}) := \frac{\theta \, \boldsymbol{\Lambda}_{\star}(1, p, d) \left(\boldsymbol{x}^{\beta} - \mathsf{N}\right)}{\theta \left(\boldsymbol{x}^{\beta} - \mathsf{N}\right) + \left(1 - \theta\right) \left(\boldsymbol{x} - 1\right) \mathsf{N}} \,\,, \\ & \boldsymbol{\Lambda} < \chi(\boldsymbol{x}) := \frac{1}{4} \left(d - 1\right) \frac{2 \, p \, \theta - 3 \left(p - 2\right)}{p - 2} \, \frac{1}{x} \,\,. \end{split}$$

After replacing $\Lambda_{\star}(1, p, d)$ by its value according to (8), we get that $\phi(x) - \chi(x)$ has the sign of f(x) as defined in the proof of Lemma 2. By Corollary 4, we know that $\mathbb{N} \geq 1$ and conclude henceforth that any minimizer is *symmetric* if $\Lambda < \chi(\mathsf{x}^*(\theta, p)) = \Lambda_2(\theta, p, d)$.

In the limiting regime corresponding to as $\theta \to 1_-$, we observe that $\phi(x) = \Lambda_{\star}(1, p, d)$ and $\chi(x) = \Lambda_{\star}(1, p, d)/x$, so that $\lim_{\theta \to 1} \Lambda_2(\theta, p, d) = \chi(1) = \Lambda_{\star}(1, p, d)$.

4 Proof of the main results

Proof (Theorem 3) It is a straightforward consequence of Lemma 1 and Lemma 5. Notice that $\lim_{\theta \to 1} \Lambda_1(\theta, p, d) = \Lambda_{\star}(1, p, d)$ because

$$\lim_{\theta \to 1} \frac{\theta \, \varLambda_\star(1,p,d)}{(1-\theta) \, \varPi^*(\theta,p,q) + \theta} = \varLambda_\star(1,p,d) \ .$$

Proof (Theorem 5) The function $q \mapsto \Lambda_1(1,q,d)$ is monotone decreasing and

$$q^*(\theta, p) - p = \frac{p(p-2)(1-\theta)}{2 - p(1-\theta)} \ge 0$$

so that, for i = 1, 2,

$$\Lambda_i(\theta, p, d) \le \Lambda_{\star}(1, q^*(\theta, p), d) \le \Lambda_{\star}(1, p, d) < \Lambda_{FS}(\theta, p, d)$$
.

By definition of $\Lambda_{\rm s}(\theta,p,d)$, we know that $\Lambda_{\star}(\theta,p,d) \leq \Lambda_{\rm s}(\theta,p,d)$. By Theorem 3, if $\Lambda = \Lambda_{\star}(\theta,p,d)$ any minimizer for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)$ is symmetric. On the other hand, by continuity, we know that

$$\mathsf{K}_{\mathrm{CKN}}(\theta, p, \Lambda_{\mathrm{s}}(\theta, p, d)) = \mathsf{K}_{\mathrm{CKN}}^*(\theta, p, \Lambda_{\mathrm{s}}(\theta, p, d))$$
.

Let us assume that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$ and consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to $\Lambda_{\rm s}(\theta,p,d)$ with $\lambda_n > \Lambda_{\rm s}(\theta,p,d)$. If u_n is a non-symmetric minimizer of $\mathsf{K}_{\rm CKN}(\theta,p,\lambda_n)$, we can pass to the limit: up to the extraction of a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges in $\mathrm{H}^1(\mathcal{C})$ towards a minimizer u for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda_{\rm s}(\theta,p,d))$. The function u cannot only depend on s, because any symmetric minimizer for $\mathsf{K}_{\rm CKN}^*(\theta,p,\Lambda)$ is a strict local minimum in $\mathrm{H}^1(\mathcal{C})$ due to the fact that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm FS}(\theta,p,d)$. Hence, for $\Lambda = \Lambda_{\rm s}(\theta,p,d)$ there are two distinct minimizers for $\mathsf{K}_{\rm CKN}(\theta,p,\Lambda)$: one is symmetric and the other one is not symmetric. This proves that $\Lambda_{\rm s}(\theta,p,d) < \Lambda_{\rm s}(\theta,p,d)$ if $\theta > \vartheta(p,\min\{3,d\})$.

In the other cases, that is, if either d=3 and $\theta=\vartheta(p,3)$, or d=2 and $\theta>0$, the same method applies if we replace $\Lambda_{\star}(\theta,p,d)$ by $\Lambda_{2}(\theta,p,d)$.

Proof (Theorem 4) Let us consider f(x) as in the proof of Lemma 2 and assume that $\theta = \gamma (p-2)$. As $p \to 2_+$, f(x)/(p-2) converges towards

$$f_0(x) := 4 \gamma x^{\beta_0 + 1} - (8\gamma - 3) N_0 x + (4\gamma - 3) N_0$$
 with $\beta_0 = 1 - \frac{1}{4\gamma}$.

We easily check that the function $f_0(x)$ is convex for x>0, $f_0(\mathsf{N}_0^{1/\beta_0})<0$ and $f_0'(\mathsf{N}_0^{1/\beta_0})=2\,\mathsf{N}_0>0$. We conclude that $f_0(x)$ has a unique root for $x>\mathsf{N}_0^{1/\beta_0}$. We denote this unique root by $\mathsf{x}_0^*=\mathsf{x}_0^*(\gamma)$. It follows that $\mathsf{x}^*(\gamma\,(p-2),p)$ converges to $\mathsf{x}_0^*(\gamma)$ as $p\to 2_+$. Symmetry then is established by passing to the limit for any $\Lambda\in(0,\Lambda_0(\gamma,d))$ with $\Lambda_0(\gamma,d)$ given by (16).

5 An approximation and some numerical results

The functions $x^*(\theta, p)$ and $x_0^*(\gamma)$ which enter in the results of Theorem 3 and Theorem 4 are not explicit but easy to estimate, which in turn gives explicit estimates of $\Lambda_2(\theta, p, d)$ and $\Lambda_0(\gamma, d)$. Let

$$\alpha = 2 p (\theta - \vartheta(p, 3)) = 2 p \theta - 3 (p - 2),$$

 $\beta = \beta(\theta, p) = 1 - \frac{p - 2}{2 p \theta},$

and

$$\Lambda_{2,\mathrm{approx}}(\theta,p,d) := \frac{(d-1)\,\alpha}{4\,(p-2)}\, \frac{\beta\,\theta\,(6-p) - \alpha\,(1-\theta+\beta\,\theta\,\mathsf{N}^{-1/\beta})}{\beta\,\theta\,(6-p)\,\mathsf{N}^{1/\beta} - \alpha\,(\beta\,\theta+1-\theta)} \ .$$

Proposition 2 Suppose that either d=2 and $p \in (2,6)$ or else $d \geq 3$ and $p \in (2,2^*)$. Then for any $\theta \in (\vartheta(p,3),1)$, we have the estimate

$$\Lambda_2(\theta, p, d) > \Lambda_{2,approx}(\theta, p, d)$$
.

Proof Let us consider the function f defined in the proof of Lemma 2 and recall that f''(x) is positive for any $x \ge N^{1/\beta} > 1$. Moreover we verify that

$$\begin{split} f(\mathsf{N}^{1/\beta}) &= -\left(1-\theta\right)\alpha\,\mathsf{N}\,(\mathsf{N}^{1/\beta}-1) < 0 \ , \\ f'(\mathsf{N}^{1/\beta}) &= \mathsf{N}\left[\beta\,\theta\,(6-p) - \alpha\left(1-\theta+\beta\,\theta\,\mathsf{N}^{-1/\beta}\right)\right] > 0. \end{split}$$

which provides the estimate

$$\mathsf{x}^*(\theta,p) < \mathsf{N}^{1/\beta} - \frac{f(\mathsf{N}^{1/\beta})}{f'(\mathsf{N}^{1/\beta})} = \frac{\beta\,\theta\,(6-p)\,\mathsf{N}^{1/\beta} - \alpha\,(\beta\,\theta + 1 - \theta)}{\beta\,\theta\,(6-p) - \alpha\,\left(1 - \theta + \beta\,\theta\,\mathsf{N}^{-1/\beta}\right)} \ ,$$

and the result follows.

Next we give an estimate of $\Lambda_0(\gamma, d)$ in Theorem 4. Let

$$\varLambda_{0,\mathrm{approx}}(\gamma,d) := \frac{\left(d-1\right)\left(\gamma-\frac{3}{4}\right)}{2\left(\gamma-\frac{1}{4}\right)\mathsf{N}_{0}^{\frac{4\gamma}{4\gamma-1}}-2\left(\gamma-\frac{3}{4}\right)} \ ,$$

with $N_0(\gamma)$ as defined by (15).

Proposition 3 Assume that $d \ge 2$ and $\gamma > 3/4$. Then

$$\Lambda_0(\gamma, d) > \Lambda_{0, \text{approx}}(\gamma, d)$$
.

Proof Recall that $\beta_0=1-\frac{1}{4\gamma}$. Let us consider the function f_0 defined in the proof of Theorem 4. We note that $f_0''(x)$ is positive for x>0. Moreover we verify that $f_0'(\mathsf{N}_0^{1/\beta_0})=2\,\mathsf{N}_0>0$ and $f_0(\mathsf{N}_0^{1/\beta_0})=-(4\,\gamma-3)\,\mathsf{N}_0\,(\mathsf{N}_0^{1/\beta_0}-1)<0$, which provides the estimates

$$\mathbf{X}_{0}^{*}(\gamma) < \mathbf{N}_{0}^{1/\beta_{0}} - \frac{f(\mathbf{N}_{0}^{1/\beta_{0}})}{f'(\mathbf{N}_{0}^{1/\beta})} = 2\left(\gamma - \frac{1}{4}\right)\mathbf{N}_{0}^{\frac{4\gamma}{4\gamma - 1}} - 2\left(\gamma - \frac{3}{4}\right) \;,$$

and the result follows.

To conclude this paper, let us illustrate Theorems 3 and 5 with some numerical results. First we address the case of subcritical $\theta \in (\vartheta(p,d),1)$ and compare Λ_{\star} with $\Lambda_{\rm FS}$: Fig. 1 corresponds to the particular case d=5 and $\theta=0.5$.

The expression of $\Lambda_{\star}(\theta, p, d)$ is not explicit but easy to compute numerically. We recall that Λ_{\star} is the maximum of Λ_{1} and Λ_{2} , both of them being non-explicit. In practice, for low values of the dimension d, the relative difference of Λ_{1} and Λ_{2} is in the range of a fraction of a percent to a few percents, depending on θ and on the exponent p. Moreover, we numerically observe that $\Lambda_{1} \leq \Lambda_{2}$, at least for the values of the parameters considered in Fig. 1. The estimate $\Lambda_{2,\text{approx}}(\theta, p, d)$ of Proposition 2 is remarkably good.

In Fig. 2, we consider the critical case $\theta = \vartheta(p,d)$. The plot corresponds to d=5 and all p in the interval (2,10/3). The exponent $\vartheta(p,d)$ is the one which enters in the Gagliardo-Nirenberg inequality

$$||u||_{\mathrm{L}^p(\mathbb{R}^d)}^2 \le \mathsf{C}_{\mathrm{GN}}(p,d) ||\nabla u||_{\mathrm{L}^2(\mathbb{R}^d)}^{2\vartheta(p,d)} ||u||_{\mathrm{L}^2(\mathbb{R}^d)}^{2(1-\vartheta(p,d))} \quad \forall \, u \in \mathrm{H}^1(\mathbb{R}^d)$$

on the Euclidean space \mathbb{R}^d , without weights. Here $\mathsf{C}_{\mathsf{GN}}(p,d)$ denotes the optimal constant and $p \in (2,\infty)$ if d=1 or $2, p \in (2,2^*]$ if $d \geq 3$. The optimizers are radially symmetry but not known explicitly.

It has been shown in [8, Theorem 1.4] that optimal functions for (1) exist if $C_{\rm GN}(p,d) < C_{\rm CKN}(\theta,p,a)$. On the other hand, optimal functions cannot be symmetric $C_{\rm GN}(p,d) > C_{\rm CKN}^*(\theta,p,a)$: see [5, Section 5] for further details and consequences. This symmetry breaking condition determines a curve $p \mapsto \Lambda_{\rm GN}(p,d)$ which has been computed numerically in [6,7]: there are values of p and d for which the condition $\Lambda > \Lambda_{\rm GN}(p,d)$, which guarantees symmetry breaking (but not existence), is weaker than the condition $\Lambda > \Lambda_{\rm FS}(\theta,p,d)$,

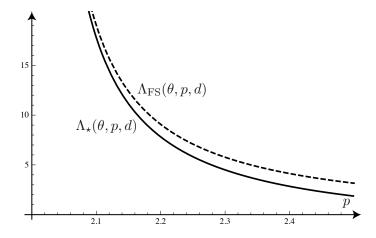


Fig. 1 Curves $p \mapsto \Lambda_{\star}(\theta, p, d)$ and $\Lambda \mapsto \Lambda_{\mathrm{FS}}(\theta, p, d)$ with $\theta = 0.5$ and d = 5. Symmetry holds for $\Lambda \leq \Lambda_{\star}(\theta, p, d)$, while symmetry is broken for $\Lambda \geq \Lambda_{\mathrm{FS}}(\theta, p, d)$. The relative difference of Λ_1 and Λ_2 , *i.e.*, $\Lambda_2(\theta, p, d)/\Lambda_1(\theta, p, d) - 1$, is below 4%. The estimate of Proposition 2 is such that $1 - \Lambda_{2,\mathrm{approx}}(\theta, p, d)/\Lambda_2(\theta, p, d)$ is of the order of 5×10^{-3} .

that is $\Lambda_{\rm GN}(p,d) < \Lambda_{\rm FS}(\theta,p,d)$. See Fig. 2. A rather complete scenario of explanations, based on numerical computations and some formal expansions, has been established in [9,10]. As it had to be expected, we numerically observe that $\Lambda_{\star}(\theta,p,d) \leq \min\{\Lambda_{\rm FS}(\theta,p,d),\Lambda_{\rm GN}(p,d)\}$ when $\theta = \vartheta(p,d)$, for any $p \in (2,2^*)$.

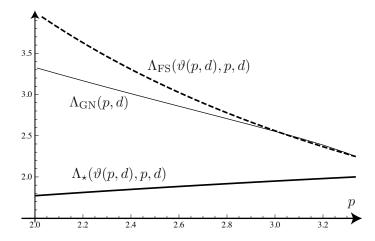


Fig. 2 With $\theta = \vartheta(p,d)$, the curve $p \mapsto \Lambda_{\star}(\theta,p,d)$ is always below the curves $p \mapsto \Lambda_{\rm FS}(\theta,p,d)$ and $p \mapsto \Lambda_{\rm GN}(p,d)$ for any $p \in (2,2^*)$, although $\Lambda_{\rm FS}$ and $\Lambda_{\rm GN}$ are not ordered. The plot corresponds to d=5 and we may notice that $\Lambda_{\rm GN}(p,d) < \Lambda_{\rm FS}$ if p is small enough.

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