TECHNIQUES OF CONSTRUCTIONS OF VARIATIONS OF MIXED HODGE STRUCTURES

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ABSTRACT. We give a way of constructing real variations of mixed Hodge structures over compact Kähler manifolds by using mixed Hodge structures on Sullivan's 1-minimal models of certain differential graded algebras associated with real variations of Hodge structures.

1. Introduction

A real variation of mixed Hodge structure (\mathbb{R} -VMHS) over a complex manifold M is $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ so that:

- (1) **E** is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{W}_* is an increasing filtration of the local system \mathbf{E} .
- (3) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (4) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (5) For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by \mathbf{F}^* is a real variation of Hodge structure of weight k.

The purpose of this paper is to give a way of constructing \mathbb{R} -VMHSs over compact Kähler manifolds starting from real variations of Hodge structures.

1.1. **Prototype.** We first introduce our main results for the simplest case. We briefly review Sullivan's 1-minimal model (see [4], [11], [26] for more details). We fix a ground field \mathbb{K} of characteristic zero. A differential graded algebra (DGA) \mathcal{M}^* is 1-minimal if \mathcal{M}^* is the increasing union of sub-DGAs

$$\mathbb{K} = \mathcal{M}^*(0) \subset \mathcal{M}^*(1) \subset \dots$$

such that $\mathcal{M}^*(k)$ is the exterior algebra $\bigwedge(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k)$ of the direct sum $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ of vector spaces $\mathcal{V}_1, \ldots, \mathcal{V}_k$ with $d\mathcal{V}_k \subset \mathcal{M}^2(k-1) = \bigwedge^2(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{k-1})$ for each $k \geq 1$ where the degree of any element in each V_i is of degree 1. On the dual space of \mathcal{M}^1 , the dual map of the differential $d: \mathcal{M}^1 \to \mathcal{M}^1 \wedge \mathcal{M}^1$ is a Lie bracket. Such Lie algebra is called the *dual Lie algebra* of \mathcal{M}^* . A 1-minimal model of a DGA A^* is a 1-minimal DGA \mathcal{M}^* with a morphism $\phi: \mathcal{M}^* \to A^*$ which induces isomorphisms on 0th and first cohomologies and an injection on second cohomology.

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For any cohomologically connected DGA A^* , a 1-minimal model \mathcal{M}^* of A^* exists and unique up to isomorphisms of DGAs where cohomologically connected means $H^0(A^*) \cong \mathbb{K}$.

Let M be a compact complex manifold with a Kähler metric g. Then, by Morgan's work ([18]), there exists an \mathbb{R} -mixed Hodge structure (W_*, F^*) on the 1-minimal model \mathcal{M}^* of the de Rham complex $A^*(M)$ of M (Morgan's \mathbb{R} -mixed Hodge structure). We notice that there are various choices of Morgan's \mathbb{R} -mixed Hodge structures. On the dual Lie algebra \mathfrak{u} of \mathcal{M}^* with the dual \mathbb{R} -mixed Hodge structure induced by one Morgan's \mathbb{R} -mixed Hodge structure, we define:

Definition 1.1.1. A mixed Hodge \mathfrak{u} -representation is (V, W_*, F^*, Ω) so that:

- V is a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*)
- $\Omega:\mathfrak{u}\to \mathrm{End}(V)$ is a representation and a morphism of \mathbb{R} -mixed Hodge structures.

We notice that Ω is regarded as $\Omega \in \mathcal{M}^1 \otimes \operatorname{End}(V)$ satisfying the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. Thus, $\Omega_{\phi} = \phi(\Omega) \in A^*(M) \otimes \operatorname{End}(V)$ gives a flat connection on the trivial \mathcal{C}^{∞} -vector bundle $M \times V$.

We state a prototype of the main result.

Theorem(Prototype). We can take canonical maps $\phi: \mathcal{M}^* \to A^*(M)$ and $\varphi': \mathcal{M}_{\mathbb{C}}^* \to A^*(M) \otimes \mathbb{C}$ which induce isomorphisms on 0th and first cohomologies and injections on second cohomology and define a special \mathbb{R} -mixed Hodge structure (W_*, F^*) on the 1-minimal model \mathcal{M}^* of $A^*(M)$ so that for any mixed Hodge \mathfrak{u} -representation $\mathfrak{V} = (V, W_*, F^*, \Omega)$, we can construct an \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}_*}, \mathbf{F}_{\mathfrak{V}}^*)$ satisfying the following conditions:

- (1) $\mathbf{E}_{\mathfrak{V}}$ is the trivial \mathcal{C}^{∞} -vector bundle $M \times V$ with the flat connection $d + \Omega_{\phi}$ where $\Omega_{\phi} = \phi(\Omega) \in A^*(M) \otimes \operatorname{End}(V)$.
- (2) $\mathbf{W}_{\mathfrak{D}*}$ is the filtration of the vector bundle $M \times V$ induced by the weight filtration W_* on V.
- (3) For some gauge transformation a of the vector bundle $M \times V_{\mathbb{C}}$, we have $\mathbf{F}_{\mathfrak{V}}^* = a\mathbf{F}^*$ such that \mathbf{F}^* is the filtration of the vector bundle $M \times V_{\mathbb{C}}$ induced by the Hodge filtration F^* on $V_{\mathbb{C}}$.
- (4) $\varphi'(\Omega) = a^{-1}da + a^{-1}\Omega_{\phi}a$.

Remark 1.1.2. For constructing Morgan's \mathbb{R} -mixed Hodge structure, in [18, Section 6 and 7], Morgan makes:

- (i) A bigraded complex 1-minimal model $\mathcal{N}^* = \bigoplus \mathcal{N}^*(p,q)$.
- (ii) A filtered real 1-minimal model (\mathcal{M}^*, W_*) .
- (iii) A filtration preserving isomorphism $\mathcal{I}: (\mathcal{M}^* \otimes \mathbb{C}, W_*) \to (\mathcal{N}^*, W_*)$ where

$$W_r(\mathcal{N}^*) = \bigoplus_{p+q \le r} \mathcal{N}^*(p,q).$$

For the filtration \mathcal{F}^* on \mathcal{M}^* so that

$$F^r(\mathcal{M}^*) = \mathcal{I}^{-1} \left(\bigoplus_{p \ge r} \mathcal{M}^*(p,q) \right),$$

 (W_*, F^*) is an \mathbb{R} -mixed Hodge structure on \mathcal{M}^* ([18, Section 8]). In this construction an isomorphism \mathcal{I} is not unique and Morgan does not give explicit one.

Our construction of "a special \mathbb{R} -mixed Hodge structure" in this theorem is to give an explicit \mathcal{I} canonically determined by a Kähler metric g without using a base point.

Remark 1.1.3. \mathbb{R} -VMHSs ($\mathbf{E}_{\mathfrak{V}}$, $\mathbf{W}_{\mathfrak{V}*}$, $\mathbf{F}_{\mathfrak{V}}^*$) in this theorem are unipotent in the sense of Hain-Zucker [14]. In [14], by using the mixed Hodge structure on the fundamental group derived from iterated integrals, Hain and Zucker constructed unipotent \mathbb{R} -VMHSs associated with unipotent mixed Hodge representations of the fundamental group. Our construction is very similar to Hain-Zucker's construction since Morgan's mixed Hodge structure can be regarded as a mixed Hodge structure on the "tensor product" of the fundamental group and the field \mathbb{R} by Sullivan's de Rham homotopy theory (see [18]). But they are different since Morgan's mixed Hodge structure on the 1-minimal model is different from the mixed Hodge structure on the fundamental group derived from iterated integrals (see [20, Section 8 and 9]). In fact, for our construction, we do not use "base point" unlike Hain-Zucker's construction. Our construction depends on the choice of a Kähler metric. The maps $\phi: \mathcal{M}^* \to A^*(M)$ and $\varphi': \mathcal{M}_{\mathbb{C}}^* \to A^*(M) \otimes \mathbb{C}$ canonically determined by a Kähler structure. An advantage of our construction is that we obtain an explicit globally defined connection forms $\Omega_{\phi} \in A^*(M) \otimes \mathrm{End}(V)$ and $\varphi'(\Omega) \in A^*(M) \otimes \mathrm{End}(V \otimes \mathbb{C})$.

A more precise comparison between Hain-Zucker's construction and our construction is given in Section 10. We can show that any unipotent \mathbb{R} -VMHS is isomorphic to one which is constructed in Theorem (Prototype).

1.2. **Main construction.** In this paper we give an extended version of Theorem (Prototype) for obtaining non-unipotent \mathbb{R} -VMHSs. Let M be a compact Kähler manifold and $\rho: \pi_1(M,x) \to GL(V_0)$ be a real valued representation. Consider the real local system $\mathbf{E}_0 = (\tilde{M} \times V_0)/\pi_1(M,x)$ where \tilde{M} is the universal covering of M. We assume that \mathbf{E}_0 admits an \mathbb{R} -VHS $(\mathbf{E}_0, \mathbf{F}^*)$ with a polarization \mathbf{S} . Consider the bilinear form $\mathbf{S}_x: V_0 \times V_0 \to \mathbb{R}$. Then we have $\rho(\pi_1(M,x)) \subset T = \operatorname{Aut}(V_0,\mathbf{S}_x)$. We assume that $\rho(\pi_1(M,x))$ is Zariski-dense in T.

In this assumption, we will set up the main construction by the following way.

- Corresponding to $\rho : \pi_1(M, x) \to T$, we consider the DGA $A^*(M, \mathcal{O}_{\rho})$ of differential forms on M with values in a certain local system equipped with the T-action defined by Deligne and Hain [13].
- We construct the canonical T-equivariant real 1-minimal model $\phi: \mathcal{M}^* \to A^*(M, \mathcal{O}_{\rho})$ and the canonical T-equivariant complex 1-minimal model $\varphi: \mathcal{N}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$.
- We construct an \mathbb{R} -mixed Hodge structure on \mathcal{M}^* by using $\varphi: \mathcal{N}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ which relates to the T-action.

- For the \mathbb{R} -mixed Hodge structure on the dual Lie algebra \mathfrak{u} of \mathcal{M}^* , we define a mixed Hodge (T,\mathfrak{u}) -representation (V,W_*,F^*,Ω) as a mixed Hodge \mathfrak{u} -representation as in Definition 1.1.1 with some conditions on T-actions.
- For any mixed Hodge (T, \mathfrak{u}) -representation $\mathfrak{V} = (V, W_*, F^*, \Omega)$, we construct an \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ satisfying similar conditions as in Theorem (Prototype).

Obtained \mathbb{R} -VMHSs ($\mathbf{E}_{\mathfrak{D}}, \mathbf{W}_{\mathfrak{D}*}, \mathbf{F}_{\mathfrak{D}}^*$) can be non-unipotent. For each irreducible representation V_{α} of T, we have a \mathbb{R} -VHS on $\mathbf{E}_{\alpha} = (\tilde{M} \times V_{\alpha})/\pi_1(M, x)$. Such variations appear in $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{D}})$. We notice that our construction is closely related to Eyssidieux-Simpson's construction in [6]. By our construction, we can construct \mathbb{R} -VMHSs which are very similar to Eyssidieux-Simpson's VMHSs.

1.3. Arrangement of the paper. In Section 2–5, we will give basics of the main objects of this paper. The main part of this paper is Section 6–10. In Section 6, we will give details of constructions of canonical 1-minimal models and mixed Hodge structures. In Section 7, we will give the definition of mixed Hodge (T,\mathfrak{u}) -representations $\mathfrak{V}=(V,W_*,F^*,\Omega)$ and details of constructions of \mathbb{R} -VMHSs $(\mathbf{E}_{\mathfrak{V}},\mathbf{W}_{\mathfrak{V}*},\mathbf{F}_{\mathfrak{V}}^*)$. In Section 8 and Section 9, we will give techniques of producing mixed Hodge (T,\mathfrak{u}) -modules. In section 9, inspired by Eyssidieux-Simpson's work in [6], we will give \mathbb{R} -VMHSs starting from any T-module, by using the deformation theory of differential graded Lie algebras. In section 10, we show that any unipotent \mathbb{R} -VMHS is isomorphic to one which is constructed in Theorem (Prototype). Acknowledgements.

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2. Variations of mixed Hodge structures

2.1. Mixed Hodge structures.

Definition 2.1.1. An \mathbb{R} -Hodge structure of weight n on a \mathbb{R} -vector space V is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$\overline{V^{p,q}} = V^{q,p}$$
,

or equivalently, a finite decreasing filtration F^* on $V_{\mathbb{C}}$ such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})} = V_{\mathbb{C}}$$

for each p.

An \mathbb{R} -Hodge structure of weight n corresponds to a rational representation $h: U(1) \to GL(V)$. This correspondence is given by

$$V^{p,q} = \{ v \in V_{\mathbb{C}} | h(t)v = t^{n-2q}v \}$$

for p+q=n.

Definition 2.1.2. A polarization of an \mathbb{R} -Hodge structure of weight n is a $(-1)^n$ -symmetric bilinear form $S: V \times V \to \mathbb{R}$ so that:

- (1) The decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is orthogonal for the sesquilinear form $S: V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}} \to \mathbb{C}$.
- (2) $h: V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ defined as $h(u, v) = S(Cu, \bar{v})$ is a positive-definite hermitian form where C is the Weil operator $C_{|V^{p,q}|} = (\sqrt{-1})^{p-q}$.

Suppose V is finite-dimensional. Consider the homomorphism $h: U(1) \to GL(V)$ associated with an \mathbb{R} -Hodge structure of weight n. Then for a polarization S, we have $h(U(1)) \subset \operatorname{Aut}(V, S)$.

Definition 2.1.3. An \mathbb{R} -mixed Hodge structure on an \mathbb{R} -vector space V is a pair (W_*, F^*) such that:

- (1) W_* is an increasing filtration on V.
- (2) F^* is a decreasing filtration on $V_{\mathbb{C}}$ such that the filtration on $Gr_n^W V_{\mathbb{C}}$ induced by F^* is an \mathbb{R} -Hodge structure of weight n.

We call W_* the weight filtration and F^* the Hodge filtration.

Proposition 2.1.4 ([3],[2],[18]). Let (W_*, F^*) be an \mathbb{R} -mixed Hodge structure on an \mathbb{R} -vector space V. Define $V^{p,q} = \mathbb{R}^{p,q} \cap L^{p,q}$ where

$$R^{p,q} = W_{p+q}(V_{\mathbb{C}}) \cap F^p(V_{\mathbb{C}})$$

and

$$L^{p,q} = W^{p+q}(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})} + \sum_{i \geq 2} W_{p+q-i}(V_{\mathbb{C}}) \cap \overline{F^{q-i+1}(V_{\mathbb{C}})}.$$

Then we have the bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ such that

$$\overline{V^{p,q}} = V^{q,p} \mod \bigoplus_{r+s < p+q} V^{r,s},$$

$$W_i(V_{\mathbb{C}}) = \bigoplus_{p+q \le i} V^{p,q}$$
 and $F^i(V_{\mathbb{C}}) = \bigoplus_{p \ge i} V^{p,q}$.

The bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ is called the bigrading of an \mathbb{R} -mixed Hodge structure (W_*, F^*) . If $\overline{V^{p,q}} = V^{q,p}$, then we say that (W_*, F^*) is \mathbb{R} -split. We have the converse statement.

Proposition 2.1.5 ([2],[18]). Let V be an \mathbb{R} -vector space. We suppose that we have a bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ such that $\bigoplus_{p+q \geq n} V^{p,q}$ is an \mathbb{R} -subspace and

$$\overline{V^{p,q}} = V^{q,p} \quad \mod \bigoplus_{r+s < p+q} V^{r,s}.$$

Then the filtrations W and F such that $W_i(V_{\mathbb{C}}) = \bigoplus_{p+q \leq i} V^{p,q}$ and $F^i(V_{\mathbb{C}}) = \bigoplus_{p>i} V^{p,q}$ give an \mathbb{R} -mixed Hodge structure on V.

Indeed, these two propositions give an equivalence of categories between \mathbb{R} -mixed Hodge structures and bigradings as in Proposition 2.1.5 see [2].

For a vector space V with a filtration W_* , we denote by $\operatorname{Aut}(V, W_*)$ the group of automorphisms preserving the filtration W_* and by $\operatorname{Aut}_1(V, W_*)$ the subgroup of $\operatorname{Aut}(V, W_*)$ consisting of $b \in \operatorname{Aut}(V, W_*)$ which induces the identity map on $\operatorname{Gr}_W^n V$ for each n. For an \mathbb{R} -mixed Hodge structure (W_*, F^*) on an \mathbb{R} -vector space V, for any $b \in \operatorname{Aut}_1(V_{\mathbb{C}}, W_*)$ the pair (W_*, bF^*) is also an \mathbb{R} -mixed Hodge structure.

Proposition 2.1.6 ([2]). Let (W_*, F^*) be an \mathbb{R} -mixed Hodge structure on an \mathbb{R} -vector space V. Then there exists $b \in \operatorname{Aut}_1(V_{\mathbb{C}}, W_*)$ so that (W_*, bF^*) is \mathbb{R} -split.

In fact, in this paper our important \mathbb{R} -mixed Hodge structures appear as $(W_*, b^{-1}F_{sp}^*)$ for \mathbb{R} -split \mathbb{R} -mixed Hodge structures (W_*, F_{sp}^*) .

2.2. Variations of Hodge structures. Let M be a complex manifold.

Definition 2.2.1. A real variation of Hodge structure (\mathbb{R} -VHS) of weight $n \in \mathbb{Z}$ over M is a pair $(\mathbf{E}, \mathbf{F}^*)$ so that:

- (1) **E** is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (3) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (4) For any $x \in M$, (E_x, F_x^*) is a \mathbb{R} -Hodge structure of weight n.

For an \mathbb{R} -VHS of weight n, we have the decomposition $\mathbf{E}_{\mathbb{C}} = \bigoplus_{p+q=n} \mathbf{E}^{p,q}$ of \mathcal{C}^{∞} vector bundles so that $\mathbf{F}^r = \bigoplus_{p \geq r} \mathbf{E}^{p,q}$. By the Griffiths transversality, the differential D on $A^*(M, \mathbf{E}_{\mathbb{C}})$ decomposes $D = \partial + \theta + \bar{\partial} + \bar{\theta}$ so that:

$$\partial: A^{a,b}(\mathbf{E}^{c,d}) \to A^{a+1,b}(\mathbf{E}^{c,d}),$$
$$\bar{\partial}: A^{a,b}(\mathbf{E}^{c,d}) \to A^{a,b+1}(\mathbf{E}^{c,d}),$$
$$\theta: A^{a,b}(\mathbf{E}^{c,d}) \to A^{a+1,b}(\mathbf{E}^{c-1,d+1})$$

and

$$\bar{\theta}: A^{a,b}(\mathbf{E}^{c,d}) \to A^{a,b+1}(\mathbf{E}^{c+1,d-1}).$$

We define

$$A^*(M,\mathbf{E}_{\mathbb{C}})^{P,Q} = \bigoplus_{a+c=P,b+d=Q} A^{a,b}(\mathbf{E}^{c,d}),$$

 $D' = \partial + \bar{\theta}$ and $D'' = \bar{\partial} + \theta$. Then we have the double complex

$$(A^*(M,\mathbf{E}_{\mathbb{C}})^{P,Q},D',D'')$$

as the usual Dolbeault complex.

Definition 2.2.2. A polarization of an \mathbb{R} -VHS is a non-degenerate pairing \mathbf{S} : $\mathbf{E} \times \mathbf{E} \to \mathbb{R}$ so that for any $x \in M$ \mathbf{S}_x is a polarization of the \mathbb{R} -hodge structure (E_x, F_x^*) .

2.3. Variations of mixed Hodge structures.

Definition 2.3.1. A real variation of mixed Hodge structure (\mathbb{R} -VMHS) over M is $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ so that:

- (1) **E** is a local system of finite-dimensional \mathbb{R} -vector spaces.
- (2) \mathbf{W}_* is an increasing filtration of the local system \mathbf{E} .
- (3) \mathbf{F}^* is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
- (4) The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
- (5) For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by \mathbf{F}^* is an \mathbb{R} -VHS of weight k.

Example 2.3.2. We introduce essentially trivial cases:

- Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure. Regarding V as the trivial vector bundle over M, V is an \mathbb{R} -VMHS.
- Let $\mathbf{E}_1, \dots, \mathbf{E}_k$ be \mathbb{R} -VHSs. Then the direct sum $\mathbf{E}_1 \oplus \dots \oplus \mathbf{E}_k$ is an \mathbb{R} -VMHS.
- Let V_1, \ldots, V_k be finite-dimensional \mathbb{R} -vector spaces with \mathbb{R} -mixed Hodge structures and $\mathbf{E}_1, \ldots, \mathbf{E}_k$ be \mathbb{R} -VHSs. Then $\bigoplus V_i \otimes \mathbf{E}_i$ is an \mathbb{R} -VMHS.

3. Representations of reductive algebraic groups

3.1. Coordinate rings. Let T be a reductive algebraic group over \mathbb{R} and $\mathbb{R}[T]$ the coordinate ring of T. Let $\{V_{\alpha}\}$ be the set of isomorphism classes of irreducible representations of T. Then as a $T \times T$ -module, we have an isomorphism $\Theta : \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$ sending $f \otimes v \in V_{\alpha}^* \otimes V_{\alpha}$ to the function $T \ni t \to f(tv) \in \mathbb{R}$ (see [13, Section 3] or [27, Theorem 27.3.9]). Hence we have the algebra structure on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$.

The multiplication on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}$ is given by decompositions of tensor products as the following way. For irreducible representations V_{α} , V_{β} , take an irreducible decomposition $V_{\alpha} \otimes V_{\beta} = \bigoplus_{i} V_{\gamma_{i}}$. We should remark that representations $\{\gamma_{i}\}$ have multiplicities. For $f \otimes v \in V_{\alpha}^* \otimes V_{\alpha}$ and $g \otimes w \in V_{\beta}^* \otimes V_{\beta}$, for this decomposition, we take $f \otimes g = \sum h_{i}$ and $v \otimes w = \sum u_{i}$ so that $h_{i} \in V_{\gamma_{i}}^{*}$ and $u_{i} \in V_{\gamma_{i}}$. Then, for $t \in T$, we have

$$(\Theta(f \otimes v)\Theta(g \otimes w))(t) = f(tv)g(tw) = (f \otimes g)(t(v \otimes w))$$

$$= \left(\sum_{i} h_{i}\right) \left(\sum_{j} t u_{j}\right) = \sum_{i} h_{i}(t u_{i}).$$

Take the element

$$\sum_{\alpha} \left(\sum_{\gamma_i = \alpha} h_i \otimes u_i \right) \in \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}.$$

Then, we have

$$\Theta\left(\sum_{\alpha} \left(\sum_{\gamma_i = \alpha} h_i \otimes u_i\right)\right) = \sum_i h_i(tu_i)$$

Thus, the multiplication induced by the isomorphism $\Theta: \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$ is given by

$$(f \otimes v) \cdot (g \otimes w) = \sum_{\alpha} \left(\sum_{\gamma_i = \alpha} h_i \otimes u_i \right) \in \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha}.$$

Example 3.1.1. Let $T = SL_2(\mathbb{R})$ and V be the standard representation on \mathbb{R}^2 . We have

$$\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} = \bigoplus_{k} S^k V^* \otimes S^k V$$

Then $V \otimes V = \wedge^2 V \oplus S^2 V$. For $f \otimes v \in V^* \otimes V$ and $g \otimes w \in V^* \otimes V$

$$(f \otimes v) \cdot (g \otimes w)$$

$$= (f \wedge g) \otimes (v \wedge w) + (f \times g) \otimes (g \times w) \in \wedge^2 V^* \otimes \wedge^2 V \oplus S^2 V^* \otimes S^2 V$$

where $\wedge^2 V = \mathbb{R}$.

Example 3.1.2. Let $T = SL_n(\mathbb{R})$ with $n \geq 3$ and V be the standard representation on \mathbb{R}^n . It is known that any irreducible representation V_{α} is given by

$$\mathbb{S}_{\lambda}V$$

so that \mathbb{S}_{λ} is the Schur functor associated with a partition λ of d see [7]. For certain set of partitions of numbers Λ , we have a bijection

$$\Lambda \ni \lambda \mapsto \mathbb{S}_{\lambda} V \in \{V_{\alpha}\}.$$

Precisely, Λ is the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with k < n(see [7, Section 15]). Thus, each irreducible representation V_{α} can be subscriptable as $V_{(\lambda_1, \lambda_2, \dots, \lambda_k)}$. If $n \geq 5$, by the LittlewoodRichardson rule, we have

$$V_{(2,1)} \otimes V_{(2,1)} = V_{(4,2)} \oplus V_{(4,1,1)} \oplus V_{(3,3)} \oplus V_{(3,2,1)}^{2\oplus} \oplus V_{(3,1,1,1)} \oplus V_{(2,2,2)} \oplus V_{(2,2,1,1)}$$

(see [7, Section 15, Appendix A]). For $f \otimes v \in V_{(2,1)}^* \otimes V_{(2,1)}$ and $g \otimes w \in V_{(2,1)}^* \otimes V_{(2,1)}$, we take

$$f\otimes g=h_{(4,2)}+h_{(4,1,1)}+h_{(3,3)}+h^1_{(3,2,1)}+h^2_{(3,2,1)}+h_{(3,1,1,1)}+h_{(2,2,2)}+h_{(2,2,1,1)}$$
 and

$$v \otimes w = u_{(4,2)} + u_{(4,1,1)} + u_{(3,3)} + u_{(3,2,1)}^1 + u_{(3,2,1)}^2 + u_{(3,1,1,1)} + u_{(2,2,2)} + u_{(2,2,1,1)}$$

corresponding to this decomposition. Then the multiplication induced by the isomorphism $\Theta: \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$ is given by

$$(f \otimes v) \cdot (g \otimes w)$$

$$= h_{(4,2)} \otimes u_{(4,2)} + h_{(4,1,1)} \otimes u_{(4,1,1)} + h_{(3,3)} \otimes u_{(3,3)}$$

$$+ h_{(3,2,1)}^{1} \otimes u_{(3,2,1)}^{1} + h_{(3,2,1)}^{2} \otimes u_{(3,2,1)}^{2}$$

$$+ h_{(3,1,1,1)} \otimes u_{(3,1,1,1)} + h_{(2,2,2)} \otimes u_{(2,2,2)} + h_{(2,2,1,1)} \otimes u_{(2,2,1,1)} \in \bigoplus_{\alpha} V_{\alpha}^{*} \otimes V_{\alpha}.$$

Since there exists a multiplicity on the irreducible representation $V_{(3,2,1)}$, we should remark

$$h_{(3,2,1)}^1 \otimes u_{(3,2,1)}^1 + h_{(3,2,1)}^2 \otimes u_{(3,2,1)}^2 \in V_{(3,2,1)}^* \otimes V_{(3,2,1)}.$$

3.2. Automorphism groups of polarizations. Let V be a real vector space and F^* be an \mathbb{R} -Hodge structure of weight n with a polarization S. Take the automorphism group $T = \operatorname{Aut}(V, S)$. Consider T as an algebraic group over \mathbb{R} . We have $T(\mathbb{C}) = Sp_{2m}(\mathbb{C})$ when the weight n is odd or $T(\mathbb{C}) = O(m, \mathbb{C})$ when n is even. Precisely, for the decomposition $V \otimes \mathbb{C} = \oplus V^{p,q}$ with $k = \sum_{p \text{ odd}} \dim V^{p,q}$ and $l = \sum_{p \text{ even}} \dim V^{p,q}$, we have $T = Sp_{2k}(\mathbb{R})$ when n is odd or T = O(k, l) when n is even. It is known that any irreducible representation V_{α} is given by

$$\mathbb{S}_{\lambda}V \cap V^{[d]}$$

so that \mathbb{S}_{λ} is the Schur functor associated with a partition λ of d and $V^{[d]}$ is the intersection of the kernels of all contractions $V^{\otimes d} \to V^{\otimes (d-2)}$ see [7]. Moreover, for certain set of partitions of numbers Λ , we have a bijection

$$\Lambda \ni \lambda \mapsto \mathbb{S}_{\lambda} V \cap V^{[d]} \in \{V_{\alpha}\}.$$

Precisely, Λ is the set of partitions λ so that the Young diagrams of λ have at most m rows in case $T(\mathbb{C}) = Sp_{2m}(\mathbb{C})$ (see [7, Section 17.3]) or so that the sum of the lengths of the first two columns of the Young diagrams of λ is at most m in case $T(\mathbb{C}) = O(m,\mathbb{C})$ (see [7, Section 19.5]). By this, V_{α} admits an \mathbb{R} -Hodge structure which is induced by the \mathbb{R} -Hodge structure on V. These Hodge structures are described by the following way. Consider the homomorphism $h: U(1) \to GL(V)$ associated with the \mathbb{R} -Hodge structure on V. Then we have $h(U(1)) \subset T$. Thus, the \mathbb{R} -Hodge structure on V_{α} is determined by the homomorphism $\alpha \circ h: U(1) \to GL(V_{\alpha})$.

For an irreducible representation V_{α} of T, we consider the $T \times T$ -module $V_{\alpha}^* \otimes V_{\alpha}$. Take a presentation $V_{\alpha} = \mathbb{S}_{\lambda} V \cap V^{[d]}$. Then V_{α}^* admits an \mathbb{R} -Hodge structure of weight -nd induced by F^* and V_{α} admits an \mathbb{R} -Hodge structure of weight 0. By the isomorphism $O: \bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$, we obtain the \mathbb{R} -Hodge structure of weight 0 on $\mathbb{R}[T]$. Consider the homomorphisms $h:U(1)\to T$ associated with the \mathbb{R} -Hodge structure F^* on V. Then, the \mathbb{R} -Hodge structure of weight 0 on $\mathbb{R}[T]$ is determined by the U(1)-action induced by the homomorphism $h \times h: U(1) \to T \times T$ and the $T \times T$ -module structure on $\mathbb{R}[T]$. Since the multiplication $\mathbb{R}[T] \otimes \mathbb{R}[T] \to \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structures. Moreover, for each irreducible representation V_{α} with the \mathbb{R} -Hodge structure induced by F^* , the corresponding co-module structure $V_{\alpha} \to V_{\alpha} \otimes \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structure induced by T^* , the corresponding co-module structure $V_{\alpha} \to V_{\alpha} \otimes \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structure induced by T^* and T^* -Hodge structure T^* -Hodge structures.

Let $t \in T$. We consider the varied \mathbb{R} -Hodge structure tF^* on V. This \mathbb{R} -Hodge structure corresponds to the homomorphism $tht^{-1}: U(1) \to T$. On $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$, we change the \mathbb{R} -Hodge structure on each left V_{α}^* (resp. right V_{α}) to the one associated with the varied \mathbb{R} -Hodge structure tF^* . Then, by the $T \times T$ -equivariance of the multiplication $\mathbb{R}[T] \otimes \mathbb{R}[T] \to \mathbb{R}[T]$, we can also say that the multiplication $\mathbb{R}[T] \otimes \mathbb{R}[T] \to \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structures for such alternative \mathbb{R} -Hodge structure on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$. This is important for considering \mathbb{R} -VHSs.

We notice that T acts on transitively on the classifying space of polarized \mathbb{R} -Hodge structures on V with the fixed Hodge numbers (see [1] for instance).

4. DGA
$$A^*(M, \mathcal{O}_{\rho})$$

Let M be a C^{∞} -manifold, T a reductive algebraic group over \mathbb{R} and $\rho: \pi_1(M,x) \to T$ be a real valued representation. Assume that $\rho(\pi_1(M,x))$ is Zariski-dense in T. Let $\{V_{\alpha}\}$ be the set of isomorphism classes of irreducible representations of T. Consider the local systems $\mathbf{E}_{\alpha} = (\tilde{M} \times V_{\alpha})/\pi_1(M,x)$. Denote by $A^*(M,\mathbf{E}_{\alpha})$ the space of \mathbf{E}_{α} -valued C^{∞} -differential forms. Consider the cochain complex

$$A^*(M, \mathcal{O}_{\rho}) = \bigoplus_{\alpha} A^*(M, \mathbf{E}_{\alpha}^*) \otimes V_{\alpha}$$

with the differential $D = \bigoplus_{\alpha} D_{\alpha}$. Then by the wedge product and the multiplication on $\bigoplus_{\alpha} V_{\alpha}^* \otimes V_{\alpha} \cong \mathbb{R}[T]$, $(A(M, \mathcal{O}_{\rho}), D)$ is a cohomologically connected DGA with the T-action given by each irreducible representation V_{α} .

Let V a finite-dimensional T-module. We consider the algebra $A^*(M, \mathcal{O}_{\rho}) \otimes \operatorname{End}(V)$ with the T-action and the T-module $V \otimes \mathbb{R}[T]$. Then, by the isomorphism $(V \otimes \mathbb{R}[T])^T \cong V$, the space $(A^*(M, \mathcal{O}_{\rho}) \otimes \operatorname{End}(V))^T$ is identified with $A^*(M, \operatorname{End}(\mathbf{E}))$ where \mathbf{E} is the local system associated with $\rho : \pi_1(M, x) \to T$ and the T-module structure on V. Hence, an element $\Omega \in (A^1(M, \mathcal{O}_{\rho}) \otimes \operatorname{End}(V))^T$ satisfying the Maurer-Cartan equation $D\Omega + \frac{1}{2}[\Omega, \Omega] = 0$ gives the deformed flat connection $D + \Omega$ on the C^{∞} vector bundle \mathbf{E} .

5. Hodge theory on compact Kähler manifolds

Let M be a compact complex manifold. We assume that M admits a Kähler metric g. Let $(\mathbf{E}, \mathbf{F}^*)$ be an \mathbb{R} -VHS of weight n over M with a polarization \mathbf{S} . We consider the double complex $(A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}, D', D'')$ as in Subsection 2.2. We define the differential operator $D^c = \sqrt{-1}(D'' - D')$ on the real valued differential forms $A^*(M, \mathbf{E})$. By the Hermitian metric on $\mathbf{E}_{\mathbb{C}}$ associated with the polarization \mathbf{S} and the Kähler metric g, we define the adjoints D^* , $(D')^*$, $(D'')^*$ and $(D^c)^*$ of differential operators. For the Kähler form ω associated with g, we consider the adjoint operator Λ of the Lefschetz operator $A^*(M, \mathbf{E}) \ni \alpha \mapsto \omega \land \alpha \in A^{*+2}(M, \mathbf{E})$. In the same way as the usual Kähler identity, we have

$$[\Lambda, D] = -(D^c)^*$$

and this equation gives

$$\Delta_D = 2\Delta_{D'} = 2\Delta_{D''}$$

where Δ_D , $\Delta_{D'}$ and $\Delta_{D''}$ are the Laplacian operators (see [29]). Write

$$\mathcal{H}^r(M,\mathbf{E}) = \ker(\Delta_D)_{|A^r(M,\mathbf{E}_{\mathbb{C}})} \quad \text{and} \quad \mathcal{H}^{P,Q}(M,\mathbf{E}_{\mathbb{C}}) = \ker(\Delta_{D''})_{|(A^*(M,\mathbf{E}_{\mathbb{C}}))^{P,Q}}.$$

Then we have the Hodge decomposition

$$\mathcal{H}^{r}(M, \mathbf{E}_{\mathbb{C}}) = \bigoplus_{P+Q=n+r} \mathcal{H}^{P,Q}(M, \mathbf{E}_{\mathbb{C}}).$$

Since Λ is a map of degree -2, by the Kähler identity, we have the following useful equations

$$\mathcal{H}^1(M, \mathbf{E}) = \ker D_{|A^1(M, \mathbf{E})} \cap \ker D_{|A^1(M, \mathbf{E})}^c$$

and

$$\mathcal{H}^{P,Q}(M,\mathbf{E}_{\mathbb{C}}) = \ker D'_{|A^*(M,\mathbf{E}_{\mathbb{C}})^{P,Q}} \cap \ker D''_{|A^*(M,\mathbf{E}_{\mathbb{C}})^{P,Q}}$$

for P + Q = 1 + n. By these equations, we can say that the \mathbb{R} -Hodge structure on $\mathcal{H}^1(M, \mathbf{E})$ is independent of the Kähler metric g and the polarization on $(\mathbf{E}, \mathbf{F}^*)$. As the usual way, we have the polarization

$$\mathcal{H}^1(M, \mathbf{E}) \times \mathcal{H}^1(M, \mathbf{E}) \ni (\alpha, \beta) \mapsto \int \alpha \wedge \beta \wedge \omega^{\dim_{\mathbb{C}} M - 1} \in \mathbb{R}$$

which depends on the Kähler metric g and the polarization S on (E, F^*) .

By the same argument as [4, Section 5], we have the following DD^c -Lemma and D'D''-Lemma.

Theorem 5.0.1. (DD^c -Lemma): $On A^*(M, \mathbf{E})$,

$$\operatorname{im} D \cap \ker D^c = \ker D \cap \operatorname{im} D^c = \operatorname{im} DD^c.$$

Moreover, there exists a linear map $F_g : \operatorname{im} D \cap \ker D^c \to A^{*-2}(M, \mathbf{E})$ so that $\alpha = DD^c F_q \alpha$ for $\alpha \in \operatorname{im} D \cap \ker D^c$.

(D'D"-Lemma): On
$$A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}$$
,

$$\operatorname{im} D' \cap \ker D'' = \ker D' \cap \operatorname{im} D'' = \operatorname{im} D' D''.$$

Moreover, there exists a linear map $F'_g: \operatorname{im} D' \cap \ker D'' \to A^*(M, \mathbf{E}_{\mathbb{C}})^{P-1,Q-1}$ so that $\alpha = D'D''F'_g\alpha$ for $\alpha \in \operatorname{im} D' \cap \ker D''$.

In fact, we can write $F_g = D^*G_D(D^c)^*G_{D^c}$ and $F'_g = (D')^*G_{D'}(D'')^*G_{D''}$ where G_D , G_{D^c} , $G_{D'}$ and $G_{D''}$ are the Green operators (see [4, Proof of (5.11)]).

We consider the sub-complexes

$$\ker D^c \subset A^*(M, \mathbf{E})$$
 and $\ker D' \subset A^*(M, \mathbf{E}_{\mathbb{C}})$.

The DD^c -Lemma and D'D''-Lemma imply the following "formality" results (see [4, Section 6], also [9, Section 7]).

Corollary 5.0.2. • The inclusions

$$\ker D^c \subset A^*(M, \mathbf{E})$$
 and $\ker D' \subset A^*(M, \mathbf{E}_{\mathbb{C}})$

induce cohomology isomorphisms.

• The quotients

$$\ker D^c \to H^*(A^*(M, \mathbf{E}), D^c)$$
 and $\ker D' \to H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D')$
induce cohomology isomorphisms.

• We have isomorphisms

$$H^*(A^*(M, \mathbf{E}), D) \cong H^*(A^*(M, \mathbf{E}), D^c)$$

and

$$H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D) \cong H^*(A^*(M, \mathbf{E}_{\mathbb{C}}), D').$$

Remark 5.0.3. By DD^c -Lemma and D'D''-Lemma, on 0-forms $A^0(M, \mathbf{E}_{\mathbb{C}})$, we have

$$\ker D = \ker D' = \ker D'' = \ker D^c = \ker D'D''$$

By this, if $\alpha \in A^0(M, \mathbf{E}_{\mathbb{C}})$ satisfies $D'D''\alpha = 0$ (resp. $DD^c\alpha = 0$), then we have $D'\alpha = 0$ (resp. $D^c\alpha = 0$) and so for $\beta \in \mathrm{im}D \cap \mathrm{ker}D^c \cap A^2(M, \mathbf{E})$ (resp. $\mathrm{im}D' \cap \mathrm{ker}D'' \cap A^2(M, \mathbf{E}_{\mathbb{C}})^{P,Q}$), a D^c -exact (resp. D'-exact) 1-form $D^c\alpha$ (resp. $D'\alpha$) satisfying $\beta = DD^c\alpha$ (resp. $\beta = D'D''\alpha$) is unique. Hence the maps

$$D^c F_q : \operatorname{im} D \cap \ker D^c \cap A^2(M, \mathbf{E}) \to A^1(M, \mathbf{E})$$

and

$$D'F'_q: \operatorname{im} D' \cap \ker D'' \cap A^2(M,\mathbf{E}_{\mathbb{C}})^{P,Q} \to A^1(M,\mathbf{E}_{\mathbb{C}})^{P,Q-1}$$

do not depend on the choice of a Kähler metric g.

Let $f: M' \to M$ be a holomorphic map from a compact Kähler manifold M'. Then we have the pull-back \mathbb{R} -VHS $(f^*\mathbf{E}, f^*\mathbf{F}^*)$ over M'. Thus, we also define the maps F_g and F'_g for the differential forms on M' with values in $f^*\mathbf{E}$. Consider the pull-back map $f^*: A^*(M, \mathbf{E}) \to A^*(M', f^*\mathbf{E})$. Then, by the above argument, we can say

$$f^* \circ D^c F_g = D^c F_g \circ f^*$$
 on $\operatorname{im} D \cap \ker D^c \cap A^2(M, \mathbf{E})$

and

$$f^* \circ D'F'_q = D'F'_q \circ f^*$$
 on $\operatorname{im} D' \cap \ker D'' \cap A^2(M, \mathbf{E}_{\mathbb{C}})^{P,Q}$.

6. 1-MINIMAL MODELS ON COMPACT KÄHLER MANIFOLDS

- 6.1. **Assumptions.** Let M be a compact Kähler manifold and $\rho: \pi_1(M, x) \to GL(V_0)$ be a real valued representation. Consider the real local system $\mathbf{E}_0 = (\tilde{M} \times V_0)/\pi_1(M, x)$ where \tilde{M} is the universal covering of M. We assume that \mathbf{E}_0 admits an \mathbb{R} -VHS $(\mathbf{E}_0, \mathbf{F}^*)$ of weight N_0 with a polarization \mathbf{S} . Consider the bilinear form $\mathbf{S}_x: V_0 \times V_0 \to \mathbb{R}$. Then we have $\rho(\pi_1(M, x)) \subset T = \operatorname{Aut}(V_0, \mathbf{S}_x)$. We assume that $\rho(\pi_1(M, x))$ is Zariski-dense in T.
- 6.2. Summary of this section. In this section, we will give the canonical mixed Hodge structure on the 1-minimal model of the DGA $A^*(M, \mathcal{O}_{\rho})$ by the following way:
 - (1) We consider the DGA $A^*(M, \mathcal{O}_{\rho})$ associated with the representation ρ : $\pi_1(M, x) \to T$. We take the structure of bidifferential bigraded algebra on $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ by using the variation on \mathbf{E}_0 . (Subsection 6.3)
 - (2) We construct a T-equivariant "real" 1-minimal model $\phi : \mathcal{M}^* \to A^*(M, \mathcal{O}_{\rho})$ with a grading $\mathcal{M}^* = \bigoplus \mathcal{M}_k^*$. (Subsection 6.4)
 - (3) We construct a T-equivariant "complex" 1-minimal model $\varphi : \mathcal{N}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ with a bigrading $\mathcal{N}^* = \bigoplus (\mathcal{N}^*)^{P,Q}$. (Subsection 6.5)
 - (4) We take a T-equivariant isomorphism $\mathcal{I}: \mathcal{M}_{\mathbb{C}}^* \to \mathcal{N}$ which is compatible with filtrations $W_k(\mathcal{M}^*) = \bigoplus_{i \leq k} \mathcal{M}_i^*$ and $W_k(\mathcal{N}^*) = \bigoplus_{P+Q \leq k} (\mathcal{N}^*)^{P,Q}$ and a T-equivariant homotopy $H: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi \circ \mathcal{I}$ to φ . (Subsection 6.7)

(5) By $F^r(\mathcal{M}_{\mathbb{C}}^*) = \mathcal{I}^{-1}(\bigoplus_{P \geq r} (\mathcal{N}^*)^{P,Q})$, we define an \mathbb{R} -mixed Hodge structure on \mathcal{M}^* . (Subsection 6.8)

Avoiding the arguments on T, we can find these constructions in [4] and [18] for the usual de Rham complex $A^*(M)$.

6.3. **DGAs** $A^*(M, \mathcal{O}_{\rho})$ **on compact Kähler manifolds.** Let $\{V_{\alpha}\}$ be the set of isomorphism classes of irreducible representations of T. Consider the local systems $\mathbf{E}_{\alpha} = (\tilde{M} \times V_{\alpha})/\pi_1(M, x)$. Then, for certain set Λ of partitions of numbers, we have a bijection

$$\Lambda \ni \lambda \mapsto \mathbb{S}_{\lambda} \mathbf{E}_0 \cap \mathbf{E}_0^{[d]} \in \{\mathbf{E}_{\alpha}\} \text{ (resp. } \mathbb{S}_{\lambda} V_0 \cap V_0^{[d]} \in \{V_{\alpha}\}).$$

Precisely, Λ is the set of partitions λ so that the Young diagrams of λ have at most m rows in case $T(\mathbb{C}) = Sp_{2m}(\mathbb{C})$ (see [7, Section 17.3]) or so that the sum of the lengths of the first two columns of the Young diagrams of λ is at most m in case $T(\mathbb{C}) = O(m, \mathbb{C})$ (see [7, Section 19.5]). Hence, any \mathbf{E}_{α} (resp. V_{α}) admits a polarized \mathbb{R} -VHS (resp. \mathbb{R} -Hodge structure) induced by $(\mathbf{E}_0, \mathbf{F}^*)$ (resp. (V_0, \mathbf{F}^*_x)). We put the \mathbb{R} -Hodge structure on $\mathbb{R}[T]$ of weight 0 induced by (V_0, \mathbf{F}^*_x) as in Section 3.2. Then, for each irreducible representation V_{α} , the corresponding co-module structure $V_{\alpha} \to V_{\alpha} \otimes \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structures. By the \mathbb{R} -Hodge structure on $\mathbb{R}[T]$, we have the bigrading $\mathbb{C}[T] = \bigoplus_{p} \mathbb{C}[T]^{p,-p}$.

We consider the DGA $A^*(M, \mathcal{O}_{\rho})$ as in Section 4. We define the bigrading on $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ as the following way.

Definition 6.3.1.

$$A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q} = \bigoplus_{\alpha} \bigoplus_{S+U=P, T+V=Q} A^*(M, \mathbf{E}_{\alpha}^* \otimes \mathbb{C})^{S,T} \otimes (V_{\alpha} \otimes \mathbb{C})^{U,V}.$$

By the argument in Subsection 3.2, on each case, we can say that the product on $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ is compatible with the bigrading. Hence, we have the bidifferential bigraded algebra structure

$$(A^*(M,\mathcal{O}_{\rho}\otimes\mathbb{C})^{P,Q},D',D'').$$

We have

$$A^{R}(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) = \bigoplus_{R=P+Q} A^{*}(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}.$$

Since the co-module structure $V_{\alpha} \to V_{\alpha} \otimes \mathbb{R}[T]$ is a morphism of \mathbb{R} -Hodge structures, the co-module structure

$$A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes \mathbb{C}[T]$$

corresponding to the T-action preserves the bigradings on $\mathbb{C}[T]$ and $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$.

Example 6.3.2. Let M be a compact Riemann surface of genus $g \geq 2$. Then M is a compact quotient \mathbb{H}/Γ of the upper-half plane \mathbb{H} by a discrete subgroup Γ in $PSL_2(\mathbb{R})$. Take a lifting $\rho: \Gamma \to SL_2(\mathbb{R})$ of the embedding of Γ into $PSL_2(\mathbb{R})$. By the Borel density, $\rho(\Gamma)$ is Zariski-dense in $SL_2(\mathbb{R})$. Consider the local system $\mathbf{E}_0 = (\mathbb{H} \times \mathbb{R}^2)/\Gamma$. We regard \mathbb{H} as the classifying space of polarized \mathbb{R} -Hodge structures of weight 1 on \mathbb{R}^2 . Then, considering the identity map on \mathbb{H} as a period

map, the local system \mathbf{E}_0 admits a polarized \mathbb{R} -VHS by taking certain decomposition $\mathbf{E}_0 \otimes \mathbb{C} = \mathbf{E}_0^{1,0} \oplus \mathbf{E}_0^{0,1}$. It is known that we can take $\mathbf{E}_0^{1,0} = K^{\frac{1}{2}}$ and $\mathbf{E}_0^{0,1} = K^{-\frac{1}{2}}$ where $K^{\frac{1}{2}}$ is a square-root of the canonical bundle K on M and $D'' = \bar{\partial} + \theta$ such that θ is $1 \in K \otimes \text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \cong \mathbb{C}$ (see [16], [25] and [8]).

We consider the DGA $A^*(M, \mathcal{O}_{\rho})$. By Example 3.1.1, we have

$$A^*(M, \mathcal{O}_{\rho}) = \bigoplus_{k=0}^{\infty} A^*(M, S^k \mathbf{E}_0^*) \otimes S^k V_0$$

where $V_0 = (\mathbf{E}_0)_x$. By the multiplication on the coordinate ring $\mathbb{R}[SL_2(\mathbb{R})]$ (see Example 3.1.1), we have

$$(A^*(M, \mathbf{E}_0^*) \otimes V_0) \wedge (A^*(M, \mathbf{E}_0^*) \otimes V_0) \subset (A^*(M, \mathbb{R}) \otimes \mathbb{R}) \oplus \left(A^*(M, S^2 \mathbf{E}_0^*) \otimes S^2 V_0\right).$$

By the decomposition $\mathbf{E}_0 \otimes \mathbb{C} = \mathbf{E}_0^{1,0} \oplus \mathbf{E}_0^{0,1}$, we have the decompositions $S^k \mathbf{E}_0^* = \mathbf{E}_0^{k,0} \oplus \mathbf{E}_0^{k-1,1} \cdots \oplus \mathbf{E}_0^{0,k}$ and $S^k V_0^* = V_0^{k,0} \oplus V_0^{k-1,1} \cdots \oplus V_0^{0,k}$ where $\mathbf{E}_0^{p,q} = (\mathbf{E}_0^{1,0})^p \times (\mathbf{E}_0^{0,1})^q$ and $V_0^{p,q} = (V_0^{1,0})^p \times (V_0^{0,1})^q$ such that the multiplications are symmetric products. We have

$$A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) = \bigoplus_{k=0}^{\infty} \bigoplus_{0 < i, j < k} A^{*,*}(M, (\mathbf{E}_0^{i,k-i})^*) \otimes V_0^{j,k-j}.$$

For the bigrading $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}$ as the above argument, an element in

$$A^{a,b}(M, (\mathbf{E}_0^{i,k-1})^*) \otimes V_0^{j,k-j}$$

is of type (a-i+j,b+i-j). By the multiplication on $\mathbb{R}[SL_2(\mathbb{R})]$, we have

$$\left(A^{a,b}(M, (\mathbf{E}_0^{1,0})^*) \otimes V_0^{0,1}\right) \wedge \left(A^{c,d}(M, (\mathbf{E}_0^{0,1})^*) \otimes V_0^{1,0}\right) \\
\subset \left(A^{a+c,b+d}(M, \mathbb{C}) \otimes \mathbb{C}\right) \oplus \left(A^{a+c,b+d}(M, (\mathbf{E}_0^{1,1})^*) \otimes V_0^{1,1}\right).$$

6.4. The 1-minimal model associated with DD^c -Lemma. On the DGA $A^*(M, \mathcal{O}_{\rho})$ with the T-action, we also consider another differential $D^c = \sqrt{-1}(D'' - D')$. Then, by Theorem 5.0.1 and the last subsection, we can say that on $A^*(M, \mathcal{O}_{\rho})$, we have the equality

$$\operatorname{im} D \cap \ker D^c = \ker D \cap \operatorname{im} D^c = \operatorname{im} DD^c$$

and there exist a T-equivariant linear map $F_g: \operatorname{im} D \cap \ker D^c \to A^{*-2}(M, \mathcal{O}_\rho)$ so that $\alpha = DD^cF_g\alpha$ for $\alpha \in \operatorname{im} D \cap \ker D^c$. By using this, we construct the DGAs $\mathcal{M}^*(n) = \bigwedge (\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n)$ generated by elements of degree 1 and the homomorphisms $\phi_n : \mathcal{M}^*(n) \to \ker D^c \subset A^*(M, \mathcal{O}_\rho)$ by the inductive way. More precisely, we will construct the components \mathcal{V}_n and the maps $d: \mathcal{V}_n \to \sum_{i+j=n} \mathcal{V}_i \wedge \mathcal{V}_j$ by induction in the following.

• $\mathcal{V}_1 = \ker D \cap \ker D^c \cap A^1(M, \mathcal{O}_\rho)$, the homomorphism $\phi_1 : \bigwedge \mathcal{V}_1 \to \ker D^c$ so that on \mathcal{V}_1 , ϕ_1 is the natural inclusion $\mathcal{V}_1 \hookrightarrow \ker D^c$.

• For the quotient map $q: \ker D \cap \ker D^c \to H^*(\ker D^c)$,

$$\mathcal{V}_2 = \ker \left(q \circ \phi_1 : \bigwedge^2 \mathcal{V}_1 \to H^2(\ker D^c) \right)$$

Define the DGA $\mathcal{M}^*(2) = \bigwedge (\mathcal{V}_1 \oplus \mathcal{V}_2)$ with the differential d so that d is 0 on \mathcal{V}_1 and d on \mathcal{V}_2 is the natural inclusion $\mathcal{V}_2 \hookrightarrow \bigwedge^2 \mathcal{V}_1$. Define the homomorphism $\phi_2 : \mathcal{M}^*(2) \to \ker D^c$ which is an extension of ϕ_1 so that $\phi_2(v) = D^c F_q(\phi_1(dv))$.

• For $n \geq 2$, consider the DGA $\mathcal{M}^*(n) = \bigwedge (\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n)$ with the homomorphism $\phi_n : \mathcal{M}^*(n) \to \ker D^c$ we have constructed. We can say that $\phi_n(v) \in \ker D \cap \ker D^c$ for $v \in \mathcal{V}_1$ and as an inductive hypothesis we assume $\phi_n(v) \in \operatorname{im} D^c$ for $v \in \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$.

$$\mathcal{V}_{n+1} = \ker d_{|\sum_{i+j=n+1} \mathcal{V}_i \wedge \mathcal{V}_j}.$$

Define the extended DGA $\mathcal{M}^*(n+1) = \mathcal{M}^*(n) \otimes \bigwedge \mathcal{V}_{n+1}$ so that the differential d is defined on \mathcal{V}_{n+1} as the natural inclusion $\mathcal{V}_{n+1} \hookrightarrow \sum_{i+j=n+1} \mathcal{V}_i \wedge \mathcal{V}_j$. The homomorphism $\phi_{n+1} : \mathcal{M}^*(n+1) \to \ker D^c$ is defined by $\phi_{n+1}(v) = D^c F_q(\phi_n(dv))$ for $v \in \mathcal{V}_{n+1}$.

Let $\lim \mathcal{M}^*(n) = \mathcal{M}^*$ and $\phi = \lim \phi_n : \mathcal{M}^* \to \ker D^c \subset A^*(M, \mathcal{O}_\rho)$.

By the construction, each \mathcal{V}_i is \overline{dT} -module so that the map $\phi: \mathcal{M}^* \to A^*(M, \mathcal{O}_{\rho})$ is T-equivariant. Our construction is in fact the construction of a 1-minimal model of $\ker D^c$ (see [11, Theorem 13.1]). Since the inclusion $\ker D^c \subset A^*(M, \mathcal{O}_{\rho})$ induces a cohomology isomorphism, we can say that the map $\phi: \mathcal{M}^* \to A^*(M, \mathcal{O}_{\rho})$ induces isomorphisms on 0-th and first cohomology and an injection on second cohomology. Define the multiplicative grading \mathcal{M}_k^* so that elements in \mathcal{V}_k are of type k. Then, each \mathcal{M}_k^* is a T-module and we have $d\mathcal{M}_k^* \subset \mathcal{M}_k^*$. Define the increasing filtration W_* so that $W_n(\mathcal{M}^*) = \bigoplus_{k \leq n} \mathcal{M}_k^*$.

Remark 6.4.1. By the isomorphism $H^*(M, \mathcal{O}_{\rho}) \cong H^*(\ker D^c)$ as in Theorem 5.0.2, the map $q \circ \phi_1 : \bigwedge^2 \mathcal{V}_1 \to H^2(\ker D^c)$ is identified with the cup product

$$\bigwedge^2 H^1(M, \mathcal{O}_\rho) \to H^2(M, \mathcal{O}_\rho).$$

For $n \geq 2$, forgetting the construction of the homomorphism ϕ_{n+1} , the DGA $\mathcal{M}^*(n+1) = \mathcal{M}^*(n) \otimes \bigwedge \mathcal{V}_{n+1}$ is only determined by the DGA $\mathcal{M}^*(n)$. Thus, the DGA \mathcal{M}^* is only determined by $H^1(M, \mathcal{O}_{\rho})$ with the cup product. This property is called 1-formality.

Consider the \mathbb{R} -Hodge structures on $H^1(M, \mathcal{O}_{\rho})$ and $H^2(M, \mathcal{O}_{\rho})$. Then the cup product on $H^1(M, \mathcal{O}_{\rho})$ is a morphism of \mathbb{R} -Hodge structures. Thus, \mathcal{V}_2 admits a \mathbb{R} -Hodge structure of weight 2 and $d: \mathcal{V}_2 \to \mathcal{V}_1 \wedge \mathcal{V}_1$ is a morphism of Hodge structure. Inductively, we can easily say that each \mathcal{V}_n admits an \mathbb{R} -Hodge structure of weight n and the restriction $d: \mathcal{V}_n \to \sum_{i+j=n} \mathcal{V}_i \wedge \mathcal{V}_j$ is a morphism of \mathbb{R} -Hodge structures.

6.5. The 1-minimal model associated with D'D''-Lemma. Consider the bidifferential bigraded algebra

$$(A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}, D', D'')$$
.

Then, by Theorem 5.0.1, we can say that on $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}$,

$$\operatorname{im} D' \cap \ker D'' = \ker D' \cap \operatorname{im} D'' = \operatorname{im} D' D''.$$

and there exist a T-equivariant linear map $F_g': \operatorname{im} D' \cap \ker D'' \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P-1,Q-1}$ so that $\alpha = D'D''F_g'\alpha$ for $\alpha \in \operatorname{im} D' \cap \ker D''$. By using this, we construct the DGAs $\mathcal{N}^*(n) = \bigwedge(\bigoplus_{1 \leq P+Q \leq n} \mathcal{V}^{P,Q})$ generated by elements of degree 1 and the homomorphisms $\varphi_n : \mathcal{N}^*(n) \to \ker D' \subset A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ by the inductive way. More precisely, we will construct the components $\mathcal{V}^{P,Q}$ and the maps $d : \mathcal{V}^{P,Q} \to \bigoplus_{S+U=P,T+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}$ by induction on P+Q in the following.

- For $P, Q \in \mathbb{Z}$ with P + Q = 1, let $\mathcal{V}^{P,Q} = \ker D' \cap \ker D'' \cap A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}$. Define the homomorphism $\varphi_1 : \bigwedge (\bigoplus_{P+Q=1} \mathcal{V}^{P,Q}) \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ so that on $\mathcal{V}^{P,Q}$, φ_1 is the natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow A^1(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$.
- For the quotient map $q: \ker D' \cap \ker D'' \to H^*(\ker D')$, for $P, Q \in \mathbb{Z}$ with P+Q=2, define

$$\mathcal{V}^{P,Q} = \ker \left(q \circ \varphi_1 : \bigoplus_{S+U=P,T+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V} \to H^2(\ker D') \right).$$

Define the DGA $\mathcal{N}^*(2) = \bigwedge(\bigoplus_{1 \leq P+Q \leq 2} \mathcal{V}^{P,Q})$ with the differential d so that d is 0 on $\mathcal{V}^{P,Q}$ for P+Q=1 and d on $\mathcal{V}^{P,Q}$ is the natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow \bigoplus_{S} \mathcal{V}^{P,Q} \hookrightarrow \mathcal{V}^{P,Q} \hookrightarrow$

• For $n \geq 2$, consider the DGA $\mathcal{N}^*(n) = \bigwedge(\bigoplus_{1 \leq P+Q \leq n} \mathcal{V}^{P,Q})$ with the homomorphism $\varphi_n : \mathcal{N}^*(n) \to A^*(M, \mathcal{O}_\rho)$ we have constructed. We can say that $\varphi_n(v) \in \ker D \cap \ker D^c$ for $v \in \mathcal{V}^{P,Q}$ with P + Q = 1 and as an inductive hypothesis we assume $\varphi_n(v) \in \operatorname{im} D'$ for $v \in \mathcal{V}^{P,Q}$ with $P + Q \geq 2$.

For
$$P + Q = n + 1$$
, let

$$\mathcal{V}^{P,Q} = \mathrm{ker} d_{|\sum_{S+U=P,T+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}}.$$

Define the extended DGA $\mathcal{N}^*(n+1) = \mathcal{N}^*(n) \otimes \bigwedge(\bigoplus_{P+Q=n+1} \mathcal{V}^{P,Q})$ so that the differential d is defined on $\mathcal{V}^{P,Q}$ with P+Q=n+1 as the natural inclusion $\mathcal{V}^{P,Q} \hookrightarrow \sum_{S+U=P,T+V=Q} \mathcal{V}^{S,T} \wedge \mathcal{V}^{U,V}$. The homomorphism $\varphi_{n+1}: \mathcal{N}^*(n+1) \to A^*(M,\mathcal{O}_\rho)$ is defined by $\varphi_{n+1}(v) = D'F'_g(\varphi_n(dv))$ for $v \in \mathcal{V}^{P,Q}$ with P+Q=n+1.

Let $\varinjlim \mathcal{N}^*(n) = \mathcal{N}^*$ and $\varphi = \varinjlim \varphi_n : \mathcal{N}^* \to \ker D'' \subset A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}).$

By the construction, each $\bigoplus_{P+Q=k} \mathcal{V}^{P,Q}$ is a T-module so that the map $\varphi: \mathcal{N}^* \to A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ is T-equivariant. Our construction is in fact the construction of a 1-minimal model of kerD''. Since the inclusion ker $D'' \subset A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ induces a

cohomology isomorphism, we can say that the map $\varphi : \mathcal{N}^* \to A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})$ induces isomorphisms on 0-th and first cohomology and an injection on second cohomology.

Define the multiplicative bigrading $(\mathcal{N}^*)^{P,Q}$ so that elements in $\mathcal{V}^{P,Q}$ are of type (P,Q). By the construction, we have $d(\mathcal{N}^*)^{P,Q} \subset (\mathcal{N}^*)^{P,Q}$. Define the increasing filtration W_* so that $W_n(\mathcal{N}^*) = \bigoplus_{P+Q \leq n} (\mathcal{N}^*)^{P,Q}$.

Remark 6.5.1. Let $W_n = \bigoplus_{P+Q=n} \mathcal{V}^{P,Q}$ for each n. Then we have $\mathcal{V}_1 \otimes \mathbb{C} = \mathcal{W}_1$ By the isomorphism $H^*(M, \mathcal{O}_{\rho}) \cong H^*(\ker D')$ as in Theorem 5.0.2, the map $q \circ \varphi_1 : \bigwedge^2 \mathcal{W}_1 \to H^2(\ker D^c)$ is identified with the cup product

$$\bigwedge^2 H^1(M, \mathcal{O}_{\rho}) \to H^2(M, \mathcal{O}_{\rho}).$$

Thus, forgetting the homomorphism φ_n , each \mathcal{W}_n is constructed in the same manner as \mathcal{V}_n . By this we obtain the isomorphism $\mathcal{I}_{sp}: \mathcal{M} \otimes \mathbb{C} \to \mathcal{N}^*$ so that \mathcal{I}_{sp} is the identity map on $\mathcal{V}_1 \otimes \mathbb{C} = \mathcal{W}_1$ and $\mathcal{I}_{sp}(\mathcal{V}_n \otimes \mathbb{C}) = \mathcal{W}_n$ for each n. But, this isomorphism \mathcal{I}_{sp} is not involved with the homomorphisms ϕ and φ but T-equivariant.

Via $\mathcal{V}_1 \otimes \mathbb{C} = \mathcal{W}_1$, $\mathcal{W}_1 = \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$ is in fact the bigrading of the \mathbb{R} -Hodge structure on $\mathcal{V}_1 \cong H^1(M, \mathcal{O}_\rho)$. We can easily check inductively that via $\mathcal{I}_{sp}(\mathcal{V}_n \otimes \mathbb{C}) = \mathcal{W}_n$, $\mathcal{W}_n = \bigoplus_{P+Q=n} \mathcal{V}^{P,Q}$ is considered as the bigrading of the \mathbb{R} -Hodge structure on \mathcal{V}_n as in Remark 6.4.1.

By the construction, we have $\mathcal{V}_n \subset \sum_{i+j=n} \mathcal{V}_i \wedge \mathcal{V}_j$. By this, we can inductively say that \mathcal{V}_n is a T-submodule of a direct sum of copies of the n-th tensor power $\mathcal{V}_1^{\otimes n}$ of \mathcal{V}_1 and the \mathbb{R} -Hodge structure on \mathcal{V}_n is an \mathbb{R} -Hodge substructure of a direct sum of $\mathcal{V}_1^{\otimes n}$. As we saw in Section 5, the \mathbb{R} -Hodge structure on $\mathcal{V}_1 = \bigoplus \mathcal{H}^1(M, E_\alpha^*) \otimes V_\alpha$ is polarizable. Thus the \mathbb{R} -Hodge structure on each \mathcal{V}_n is polarizable (see [20, Corollary 2.12]). Since the co-module structure $\mathcal{V}_1 \to \mathcal{V}_1 \otimes \mathbb{R}[T]$ corresponding to the T-module structure is a morphism of \mathbb{R} -Hodge structures, inductively we can say that each co-module structure $\mathcal{V}_n \to \mathcal{V}_n \otimes \mathbb{R}[T]$ is also a morphism of \mathbb{R} -Hodge structures. In particular, the co-module structure $\mathcal{N}^* \to \mathcal{N}^* \otimes \mathbb{C}[T]$ preserves the bigradings on $\mathbb{C}[T]$ and \mathcal{N}^* .

6.6. Morgan's mixed Hodge structures and the split mixed Hodge structure. By the Hodge decomposition as in Section 5, the r-th cohomology $H^r(M, \mathcal{O}_{\rho})$ admits a Hodge structure of weight r which induced by the Hodge filtration F^* on $A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ so that

$$F^p(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})) = \bigoplus_{P>p} A^*(M, \mathcal{O}_\rho \otimes \mathbb{C})^{P,Q}.$$

Define the increasing filtration W_* on $A^*(M, \mathcal{O}_{\rho})$ so that $W_{-1}(A^*(M, \mathcal{O}_{\rho})) = 0$ and $W_0(A^*(M, \mathcal{O}_{\rho})) = A^*(M, \mathcal{O}_{\rho})$. Then we can say that id: $(A^*(M, \mathcal{O}_{\rho}), W)_{\mathbb{C}} \to (A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}), W_*, F^*)$ is a mixed Hodge diagram over \mathbb{R} in the sense of Morgan ([18]). In [18], Morgan proves that there exists an \mathbb{R} -mixed Hodge structure (Morgan's mixed Hodge structure) on the 1-minimal model of a mixed Hodge diagram over \mathbb{R} by the way in Remark 1.1.2. Hence the 1-minimal model \mathcal{M}^* admits an \mathbb{R} -mixed Hodge structure. We should remark that Morgan's mixed Hodge structure is not unique in general. We give the simplest \mathbb{R} -mixed Hodge structure on \mathcal{M}^* .

Example 6.6.1. We put the \mathbb{R} -Hodge structure of weight n on \mathcal{V}_n for each n as in Remark 6.4.1 (see also Remark 6.5.1). Then, we obtain a split \mathbb{R} -mixed Hodge structure (W_*, F_{sp}^*) on \mathcal{M}^* where W_* is the increasing filtration defined in Section 6.4.

By Remark 6.5.1, we have
$$F_{sp}^r(\mathcal{M}_{\mathbb{C}}^*) = \mathcal{I}_{sp}^{-1}(\bigoplus_{P \geq r} (\mathcal{N}^*)^{P,Q})$$
.

But this \mathbb{R} -mixed Hodge structure is not interesting for studying the geometry of M since this structure is completely determined by the cup product on $H^1(M, \mathcal{O}_{\varrho})$.

6.7. The canonical isomorphism $\mathcal{I}: \mathcal{M}_{\mathbb{C}}^* \to \mathcal{N}^*$ with the homotopy $H: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$. The purpose of this subsection is to construct a T-equivariant isomorphism $\mathcal{I}: \mathcal{M}_{\mathbb{C}}^* \to \mathcal{N}^*$ which is compatible with the increasing filtrations

$$W_k(\mathcal{M}^*) = \bigoplus_{i \le k} \mathcal{M}_i^*$$
 and $W_k(\mathcal{N}^*) = \bigoplus_{P+Q \le k} (\mathcal{N}^*)^{P,Q}$

and T-equivariant homotopy $H: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi \circ \mathcal{I}$ to ϕ which are canonically determined by a Kähler metric g where [t, dt] means the tensor product of polynomials on t with the exterior algebra of dt (see e.g. [11, Chapter 11]).

For the construction, we will trace the arguments in [18, (5.5)~(5.9), (7.3)~(7.5)]. We show by induction. We will use the homotopy theory of DGAs as in [18, Section 5] and [11, Chapter 11]. We assume that we have a T-equivariant isomorphism $\mathcal{I}: \mathcal{M}^*_{\mathbb{C}}(n) \to \mathcal{N}^*(n)$ so that there exists a homotopy $H: \mathcal{M}^*_{\mathbb{C}}(n) \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi_n \circ \mathcal{I}$ to φ_n . For $v \in \mathcal{V}_{n+1}$, we have

$$\phi_n(dv) - \varphi_n(\mathcal{I}(dv)) = D \int_0^1 H(dv)$$

by applying [18, Proposition 5.5] to dv. By the construction of ϕ_{n+1} , we have $\phi_n(dv) = d\phi_{n+1}(v)$ and so $[\varphi_n(\mathcal{I}(dv))] = 0$ in $H^2(A^*(M, \mathcal{O}_\rho \otimes \mathbb{C}))$. By the construction in Section 6.5, we have $\mathcal{I}(dv) \in d\left(\bigoplus_{P+Q \leq n+1} \mathcal{V}^{P,Q}\right)$. By $\ker d_{|\bigoplus_{P+Q \leq n+1} \mathcal{V}^{P,Q}} = \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$, we can take a unique $a(v) \in \bigoplus_{2 \leq P+Q \leq n+1} \mathcal{V}^{P,Q}$ such that $da(v) = \mathcal{I}(dv)$. Consider

$$\phi_{n+1}(v) - \varphi_{n+1}(a(v)) - \int_0^1 H(dv).$$

Then it is closed and so we have a unique $a'(v) \in \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$ such that

$$[\varphi_{n+1}(a'(v))] = \left[\phi_{n+1}(v) - \varphi_{n+1}(a(v)) - \int_0^1 H(dv)\right]$$

in $H^1(A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}))$. We define the extended DGA homomorphism $\mathcal{I}: \mathcal{M}^*_{\mathbb{C}}(n+1) = \mathcal{M}^*_{\mathbb{C}}(n) \otimes \bigwedge \mathcal{V}_{n+1} \to \mathcal{N}^*(n+1)$ so that $\mathcal{I}(v) = a(v) + a'(v)$ for $v \in \mathcal{V}_{n+1}$. We take

$$b(v) = D^*G_D\left(\phi_{n+1}(v) - \varphi_{n+1}\left(\mathcal{I}(v)\right) - \int_0^1 H(dv)\right).$$

where D^* is the adjoint of the differential operator D and G_D is the Green operator. Then we have $Db(v) = \phi_{n+1}(v) - \varphi_{n+1}(\mathcal{I}(v)) - \int_0^1 H(dv)$. We define the extended DGA homomorphism $H: \mathcal{M}_{\mathbb{C}}^*(n+1) = \mathcal{M}_{\mathbb{C}}^*(n) \otimes \bigwedge \mathcal{V}_{n+1} \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ so that for $v \in \mathcal{V}_{n+1}$

$$H(v) = \varphi_{n+1}(\mathcal{I}(v)) + \int_0^t H(dv) + D(b(v)t).$$

Thus, we obtain $\mathcal{I}: \mathcal{M}_{\mathbb{C}}^*(n+1) \to \mathcal{N}^*(n+1)$ so that there exists a homotopy $H: \mathcal{M}_{\mathbb{C}}^*(n+1) \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ from $\varphi_{n+1} \circ \mathcal{I}$ to φ_{n+1} . We can easily check that extensions \mathcal{I} and H are T-equivariant inductively. In fact, since the involved maps d, φ , φ , D^*G_D , etc. are T-equivariant, by the induction hypothesis, the above a(v), a'(v) and b(v) commute with the T-action and hence we san say the T-equivariance of the extensions \mathcal{I} and H.

For each step, b(v) depends on the choice of a Kähler metric. Thus, our construction of I and H depends on the choice of a Kähler metric.

Remark 6.7.1. Let $d\mu_g$ be the volume form associated with the Kähler metric g. Define

$$C_g = \left\{ \Psi \in A^0(M) \otimes \mathbb{C} : \int \Psi d\mu_g = 0 \right\}.$$

Then $D^*G_D: D(A^0(M, \mathcal{O}_{\rho} \otimes \mathbb{C})) \to A^0(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$ is in fact the inverse map of

$$D: C_g \oplus \bigoplus_{\alpha \neq 1} A^0(M, \mathbf{E}_{\alpha}^* \otimes \mathbb{C}) \otimes V_{\alpha} \otimes \mathbb{C} \to D(A^0(M, \mathcal{O}_{\rho} \otimes \mathbb{C})).$$

We notice that C_g is not closed under the multiplication. Thus our construction may be different from the construction by the reduced bar construction of an augmented multiplicative mixed Hodge complex as in [12].

Remark 6.7.2. It is not obvious that we can take $\mathcal{I}(\mathcal{V}_n \otimes \mathbb{C}) \subset \bigoplus_{P+Q=n} \mathcal{V}^{P,Q}$ for $k \geq 3$. For n = 1, by the constructions, we can take $\mathcal{I} : \mathcal{V}_1 \otimes \mathbb{C} \cong \bigoplus_{P+Q=1} \mathcal{V}^{P,Q}$ so that $\phi_1 = \varphi_1 \circ \mathcal{I}$. For n = 2, we can take $\mathcal{I} : \mathcal{V}_2 \otimes \mathbb{C} \cong \bigoplus_{P+Q=2} \mathcal{V}^{P,Q}$ so that the homotopy $H : \mathcal{M}^*_{\mathbb{C}}(n) \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ is given by

$$H(v) = -2\sqrt{-1}D'\alpha + \sqrt{-1}D\alpha t + \sqrt{-1}\alpha dt$$

for $v \in \mathcal{V}_2$ satisfying $\phi(v) = D^c \alpha$.

6.8. The main mixed Hodge structure. We obtain the canonical \mathbb{R} -mixed Hodge structure on \mathcal{M}^* .

Theorem 6.8.1. For the isomorphism $\mathcal{I}: \mathcal{M}_{\mathbb{C}}^* \to \mathcal{N}^*$ as in the last subsection, taking the filtration $F^r(\mathcal{M}_{\mathbb{C}}^*) = \mathcal{I}^{-1}(\bigoplus_{P \geq r} (\mathcal{N}^*)^{P,Q})$, $(\mathcal{M}^*, W_*, F^*)$ is an \mathbb{R} -mixed Hodge structure which is compatible with the differential and the multiplication. The co-module structure $\mathcal{M}^* \to \mathbb{R}[T] \otimes \mathcal{M}^*$ corresponding to the T-module structure on \mathcal{M}^* is a morphism of \mathbb{R} -mixed Hodge structures.

Proof. Since each $W_n(\mathcal{M}^*)$ is a T-submodule, the T action preserves the filtration W_* . Since the co-module structure $\mathcal{N}^* \to \mathcal{N}^* \otimes \mathbb{C}[T]$ preserves the bigradings on $\mathbb{C}[T]$ and \mathcal{N}^* , by the T-equivariance of $\mathcal{I}: \mathcal{M}^*_{\mathbb{C}} \to \mathcal{N}^*$, the co-module structure $\mathcal{M}^*_{\mathbb{C}} \to \mathcal{M}^*_{\mathbb{C}} \otimes \mathbb{C}[T]$ preserves the filtrations F^* on $\mathcal{M}^*_{\mathbb{C}}$ and $\mathbb{C}[T]$. Thus the second assertion follows when $(\mathcal{M}^*, W_*, F^*)$ is an \mathbb{R} -mixed Hodge structure.

It is sufficient to show that for any n, $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n, W_*, F^*)$ is an \mathbb{R} -mixed Hodge structure.

Consider the functor Gr^W from the category of filtered DGAs to the category of DGAs. Then by the definitions

$$W_k(\mathcal{M}^*) = \bigoplus_{i \leq k} \mathcal{M}_i^*$$
 and $W_k(\mathcal{N}^*) = \bigoplus_{P+Q \leq k} (\mathcal{N}^*)^{P,Q}$,

we have $Gr^W(\mathcal{M}^*) = \mathcal{M}^*$ and $Gr^W(\mathcal{N}^*) = \mathcal{N}^*$. We consider the map $\mathcal{I}' : \mathcal{M}^*_{\mathbb{C}} \to \mathcal{N}^*$ which corresponds to $Gr^W(\mathcal{I}) : Gr^W(\mathcal{M}^*_{\mathbb{C}}) \to Gr^W(\mathcal{N}^*)$. We will show $\mathcal{I}' = \mathcal{I}_{sp}$.

We prove this equality inductively on each $\mathcal{I}': \mathcal{M}_{\mathbb{C}}^*(n) \to \mathcal{N}^*(n)$. On n=1, it is obvious. We suppose this on n. Then, by the construction of \mathcal{I} , for $v \in \mathcal{V}_{n+1}$, we have $d\mathcal{I}'(v) = \mathcal{I}_{sp}(dv)$. For $n \geq 1$, by the construction of \mathcal{N}^* , on $\mathcal{W}_n = \bigoplus_{P+Q=n} \mathcal{V}^{P,Q}$ the differential operator d is injective. Thus, we have $\mathcal{I}'(v) = \mathcal{I}_{sp}(v)$.

This implies that for each k, on $Gr_k^W(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n) = \mathcal{V}_k$ the filtration F^* induces the Hodge structure of weight k as in Remark 6.4.1. Hence the theorem follows. \square

We say that an \mathbb{R} -mixed Hodge structure is *graded-polarizable* if the \mathbb{R} -Hodge structure on each k-th graded quotient Gr_k^W is polarizable. We can easily show the following statement.

Proposition 6.8.2. The \mathbb{R} -mixed Hodge structure on \mathcal{M}^* as in Theorem 6.8.1 is graded-polarizable.

Proof. It is sufficient to show that for any n, $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n, W_*, F^*)$ is graded-polarizable. By the above proof, for each k, on $Gr_k^W(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n) = \mathcal{V}_k$ the filtration F^* induces the \mathbb{R} -Hodge structure of weight k as in Remark 6.4.1. Such \mathbb{R} -Hodge structure is polarizable. Hence the proposition follows.

Remark 6.8.3. By the proof of Theorem 6.8, for each n, we can say that the linear map $\mathcal{I} - \mathcal{I}_{sp} : W_n(\mathcal{M}^* \otimes \mathbb{C}) \to W_{n-1}(\mathcal{N}^* \otimes \mathbb{C})$ is an obstruction to splitting of the \mathbb{R} -mixed Hodge structure (W_*, F^*) on \mathcal{M}^* .

It is not obvious that the direct sum $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n$ is a splitting of \mathbb{R} -mixed Hodge structures for $n \geq 3$ since the map \mathcal{I} does not satisfy $\mathcal{I}(\mathcal{V}_n) \subset \bigoplus_{P+Q=n} \mathcal{V}_{P,Q}$.

6.9. **Pro-nilpotent Lie algebra u.** We consider the pro-nilpotent Lie algebra \mathfrak{u} which is dual to the 1-minimal model \mathcal{M}^* . Denote by \mathcal{V}_i^* the dual space of \mathcal{V}_i . Then, we have $\mathfrak{u} = \bigoplus \mathcal{V}_i^*$ and the Lie bracket is dual to the differential $d: \mathcal{V}_k \to \sum_{i+j=k} \mathcal{V}_i \wedge \mathcal{V}_j$. Hence, $\mathfrak{u} = \bigoplus \mathcal{V}_i^*$ is a graded pro-nilpotent Lie algebra so that $\mathfrak{u}_k = \bigoplus_{i \geq k} \mathcal{V}_i^*$ is the k-th term of the lower central series of \mathfrak{u} . By $\mathcal{M}^1 = \bigoplus \mathcal{V}_i$ and the definition of the differential d on \mathcal{M}^1 as in Section 6.4, the Lie bracket on \mathfrak{u} is the dual to the differential $d: \mathcal{M}^1 \to \mathcal{M}^2 = \mathcal{M}^1 \wedge \mathcal{M}^1$. By the \mathbb{R} -mixed Hodge structure on \mathcal{M}^1 as in Theorem 6.8.1, we obtain the \mathbb{R} -mixed Hodge structure (W_i, F^i) on \mathfrak{u} . Since the differential $d: \mathcal{M}^1 \to \mathcal{M}^1 \wedge \mathcal{M}^1$ is a morphism of \mathbb{R} -mixed Hodge structures by Theorem 6.8.1, the \mathbb{R} -mixed Hodge structure on \mathfrak{u} is compatible with its Lie bracket. We have $W_{-k}(\mathfrak{u}) = \bigoplus_{i \geq k} \mathcal{V}_i^* = \mathfrak{u}_k$ and so this filtration is the natural filtration which is given by the lower central series of \mathfrak{u} . In particular, we have $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}] \cong \mathcal{V}_1^*$. We recall that \mathcal{V}_1 is identified with the space of harmonic forms

in $A^*(M, \mathcal{O}_{\rho})$ and hence it is isomorphic to $H^1(M, \mathcal{O}_{\rho})$. As in Remark 6.8.3, the decomposition $\mathfrak{u} = \bigoplus \mathcal{V}_i^*$ does not give a splitting of the \mathbb{R} -mixed Hodge structure and the \mathbb{R} -mixed Hodge structure on \mathfrak{u} is not \mathbb{R} -split.

6.10. **Functoriality.** Let (M_1, g_1) and (M_2, g_2) be compact Kähler manifolds. Let $f: M_2 \to M_1$ be a holomorphic map. For an \mathbb{R} -VHS $(\mathbf{E}, \mathbf{F}^*)$ over M_1 with a polarization \mathbf{S} , we have the pull-back \mathbb{R} -VHS $(f^*\mathbf{E}, f^*\mathbf{F}^*)$ over M_2 with the polarization $f^*\mathbf{S}$. We have the homomorphism $f^*: A^*(M_1, \mathbf{E}) \to A^*(M_2, f^*\mathbf{E})$ which is compatible with bigradings and commutes with all differentials as in Section 5. Moreover, by Remark 5.0.3, we have

$$f^* \circ D^c F_{g_1} = D^c F_{g_2} \circ f^*$$
 on $\operatorname{im} D \cap \ker D^c \cap A^2(M_1, \mathbf{E})$

and

$$f^* \circ D'F'_{g_1} = D'F'_{g_2} \circ f^*$$
 on $\operatorname{im} D' \cap \ker D'' \cap A^2(M_1, \mathbf{E}_{\mathbb{C}})^{P,Q}$.

Let $\rho: \pi_1(M_1, f(x)) \to GL(V_0)$ be a real valued representation for $x \in M_2$. Consider the real local system $\mathbf{E}_0 = (\tilde{M}_1 \times V_0)/\pi_1(M_1, f(x))$ where \tilde{M}_1 is the universal covering of M_1 . We assume that \mathbf{E}_0 admits an \mathbb{R} -VHS $(\mathbf{E}_0, \mathbf{F}^*)$ over M_1 of weight N_0 with a polarization \mathbf{S} . Consider the bilinear form $\mathbf{S}_{f(x)}: V_0 \times V_0 \to \mathbb{R}$. Then we have $\rho(\pi_1(M,x)) \subset T = \operatorname{Aut}(V_0,\mathbf{S}_{f(x)})$. For the induced map $f_*: \pi_1(M_2,x) \to \pi_1(M_1,f(x))$ we assume that $\rho(f_*(\pi_1(M_2,x)))$ is Zariski-dense in T. Let $\rho' = \rho \circ f_*$ In this assumption, we can apply the constructions in this section to $A^*(M_1,\mathcal{O}_{\rho})$ and $A^*(M_2,\mathcal{O}_{\rho'})$. Moreover, we have the T-equivariant homomorphism $f^*: A^*(M_1,\mathcal{O}_{\rho}) \to A^*(M_2,\mathcal{O}_{\rho'})$ which is compatible with bigradings and commutes with all differentials.

Take the canonical 1-minimal models $_1\phi: _1\mathcal{M}^* \to A^*(M_1, \mathcal{O}_\rho)$ and $_2\phi: _2\mathcal{M}^* \to A^*(M_2, \mathcal{O}_{\rho'})$ as in Subsection 6.4 and $_1\varphi: _1\mathcal{N}^* \to A^*(M_1, \mathcal{O}_\rho \otimes \mathbb{C})$ and $_2\varphi: _2\mathcal{N}^* \to A^*(M_2, \mathcal{O}_{\rho'} \otimes \mathbb{C})$ as in Subsection 6.5 (We use the notation $_1\mathcal{M}^* = \bigwedge(_1\mathcal{V}_1 \oplus _1\mathcal{V}_2 \dots)$ etc.). By the property of f^* , the map f^* can be restricted as $f^*: \ker D \cap \ker D^c \cap A^*(M_1, \mathcal{O}_\rho) \to \ker D \cap \ker D^c \cap A^*(M_2, \mathcal{O}_{\rho'})$. Thus we get the map $_1\mathcal{V}_1 \to _2\mathcal{V}_1$. Since the map $_1\mathcal{V}_1 \to _2\mathcal{V}_1$. Since the map $_1\mathcal{V}_1 \to _2\mathcal{V}_1$ is injective for $_1\mathcal{V}_1 \to _2\mathcal{V}_1$. Since the map $_1\mathcal{V}_1 \to _2\mathcal{V}_1$ is injective for $_1\mathcal{V}_1 \to _2\mathcal{V}_1$. Since the map $_1\mathcal{V}_2 \to _2\mathcal{V}_3$ is injective for $_1\mathcal{V}_3 \to _2\mathcal{V}_3$. By the commutativity between $_1\mathcal{V}_1 \to _2\mathcal{V}_2$ and the map $_1\mathcal{V}_2 \to _2\mathcal{V}_3$ as above, we can easily check that we have $_1\mathcal{V}_2 \to _2\mathcal{V}_3 \to _2\mathcal{V}_3$. By the similar way, we have the canonical $_1\mathcal{V}_2 \to _2\mathcal{V}_3 \to _2\mathcal{V}_3$. Moreover, since $_1\mathcal{V}_3 \to _2\mathcal{V}_3 \to _2\mathcal{V}_3$ for each $_1\mathcal{V}_3 \to _2\mathcal{V}_3$ is compatible with the bigradings $_1\mathcal{V}_1 \to _2\mathcal{V}_3$ and $_1\mathcal{V}_3 \to _2\mathcal{V}_3$.

Take the isomorphisms ${}_{1}\mathcal{I}: {}_{1}\mathcal{M}_{\mathbb{C}}^{*} \to {}_{1}\mathcal{N}^{*}$ and ${}_{2}\mathcal{I}: {}_{2}\mathcal{M}_{\mathbb{C}}^{*} \to {}_{2}\mathcal{N}^{*}$ and the homotopies ${}_{1}H: {}_{1}\mathcal{M}_{\mathbb{C}}^{*} \to A^{*}(M_{1}, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ and ${}_{2}H: {}_{2}\mathcal{M}_{\mathbb{C}}^{*} \to A^{*}(M_{2}, \mathcal{O}_{\rho'} \otimes \mathbb{C}) \otimes [t, dt]$ as in Subsection 6.7.

Let $d\mu_1$ and $d\mu_2$ be the volume forms associated with the Kähler metrics g_1 and g_2 respectively. We consider the following condition on $f: M_2 \to M_1$.

Condition (V). For any $\Psi \in A^0(M_1) \otimes \mathbb{C}$ satisfying $\int_{M_1} \Psi d\mu_1 = 0$, we have

$$\int_{M_2} (f^* \Psi) d\mu_2 = 0.$$

Let

$$C_i = \left\{ \Psi \in A^0(M) \otimes \mathbb{C} : \int_{M_i} \Psi d\mu_i = 0 \right\}$$

for i = 1, 2. Then the Condition (V) holds if and only if $f^*(C_1) \subset C_2$. By this, the Condition (V) implies the commutativity

$$f^* \circ D^*G_D = D^*G_D \circ f^*$$

on $D(A^0(M_1, \mathcal{O}_{\rho} \otimes \mathbb{C}))$.

Proposition 6.10.1. If f satisfies the Condition (V), then

$$_{2}\mathcal{I}\circ f_{\mathcal{M}}=f_{\mathcal{N}}\circ {_{1}\mathcal{I}}$$

and

$$_{2}H \circ f_{\mathcal{M}} = (f^* \otimes \mathrm{id}_{[t,dt]}) \circ {_{1}H}.$$

Proof. For each step $_1\phi_n: {}_1\mathcal{M}_n^* \to A^*(M_1, \mathcal{O}_{\rho} \otimes \mathbb{C}), \; {}_1\varphi_n: {}_1\mathcal{N}_n^* \to A^*(M_1, \mathcal{O}_{\rho'} \otimes \mathbb{C}), \; {}_2\phi_n: {}_2\mathcal{M}_n^* \to A^*(M_2, \mathcal{O}_{\rho} \otimes \mathbb{C}), \; {}_2\varphi_n: {}_2\mathcal{N}_n^* \to A^*(M_2, \mathcal{O}_{\rho'} \otimes \mathbb{C}) \; (\text{see Subsection 6.4, 6.5)}, \text{ we will prove the statement inductively. For } n=1, \text{ the claim is obvious.}$ Assuming the claim for n, we prove for n+1. It is sufficient to show that $f_{\mathcal{N}}({}_1\mathcal{I}(v))=2\mathcal{I}(v)(f_{\mathcal{M}}(v)) \; \text{and} \; {}_2H(f_{\mathcal{M}}(v))=(f^*\otimes \mathrm{id}_{[t,dt]})({}_1H(v)) \; \text{for any} \; v\in {}_1\mathcal{V}_{n+1}. \; \text{For } v\in {}_1\mathcal{V}_{n+1}, \text{ the form}$

$$_{1}\phi_{n+1}(v) - {}_{1}\varphi_{n+1}(_{1}\mathcal{I}(v))) - \int_{0}^{1} {}_{1}H(dv)$$

is an exact form. By the inductive assumption, we can say that ${}_2\mathcal{I}(df_{\mathcal{M}}(v)) = df_{\mathcal{N}}({}_1\mathcal{I}(v))$ and

$$_{2}\phi_{n+1}(f_{\mathcal{M}}(v)) - {_{2}\varphi_{n+1}(f_{\mathcal{N}}(_{1}\mathcal{I}(v)))} - \int_{0}^{1} {_{2}H(df_{\mathcal{M}}(v))}$$

is an exact form. This implies $f_{\mathcal{N}}({}_{1}\mathcal{I}(v)) = {}_{2}\mathcal{I}(f_{\mathcal{M}}(v))$ (see the construction of Subsection 6.7). For

$$b(f_{\mathcal{M}}(v)) = D^*G_D\left(2\phi_{n+1}(f_{\mathcal{M}}(v)) - 2\varphi_{n+1}(f_{\mathcal{N}}(1\mathcal{I}(v))) - \int_0^1 2H(df_{\mathcal{M}}(v))\right),$$

we have

$$_{2}H(f_{\mathcal{M}}(v)) = {_{2}}\varphi_{n+1}({_{2}}\mathcal{I}(f_{\mathcal{M}}(v))) + \int_{0}^{t} {_{2}}H(df_{\mathcal{M}}(v)) + D(b(f_{\mathcal{M}}(v))t).$$

By the commutativity between f^* and the map D^*G_D as above, the inductive assumption and $\mathcal{I}_2(f_{\mathcal{M}}(v)) = f_{\mathcal{N}}(\mathcal{I}_1(v))$, we can say that $b(f_{\mathcal{M}}(v)) = f^*(b(v))$ and

$$_2H(f_{\mathcal{M}}(v))$$

$$= f^*({}_{1}\varphi_{n+1}(\mathcal{I}_{1}(v)) + (f^* \otimes \mathrm{id}_{[t,dt]}) \int_0^t {}_{1}H(dv) + (f^* \otimes \mathrm{id}_{[t,dt]})D(b(v)t)$$
$$= (f^* \otimes \mathrm{id}_{[t,dt]})({}_{1}H(v)).$$

Hence the proposition follows.

Since the map $f_{\mathcal{N}}$ is compatible with the bigradings, we have the following result.

Corollary 6.10.2. If f satisfies the condition (V), then $f_{\mathcal{M}}: {}_{1}\mathcal{M}^* \to {}_{2}\mathcal{M}^*$ is a morphism of \mathbb{R} -mixed Hodge structures.

Since $C_i = \Delta_D(A^0(M_i) \otimes \mathbb{C})$, if f^* commutes with the laplacian Δ_D on the 0-forms $A^0(M_1) \otimes \mathbb{C}$, then the condition (V) holds. It is known that a differential map f between compact Riemannian manifolds satisfies the commutativity between the pull-back f^* and the laplacian on the 0-forms if and only if f is a harmonic Riemannian submersion (see [28]). Since a holomorphic map between Kähler manifolds is harmonic, a Kähler submersion (i.e. holomorphic Riemannian submersion between Kähler manifolds) $f: M_2 \to M_1$ satisfies the condition (V).

Remark 6.10.3. In [18], Morgan did not obtain the functoriality on his mixed Hodge structures. (see [19]).

6.11. Constructions for alternate volume forms. Let $d\mu$ be a volume form which is different from the volume form $d\mu_g$ on M associated with a Kähler metric g. Let

$$C_{\mu} = \left\{ \Psi \in A^0(M) \otimes \mathbb{C} : \int_M \Psi d\mu = 0 \right\}.$$

Define δ_{μ} as the inverse map of

$$D: C_{\mu} \oplus \bigoplus_{\alpha \neq 1} A^{0}(M, \mathbf{E}_{\alpha}^{*} \otimes \mathbb{C}) \otimes V_{\alpha} \otimes \mathbb{C} \to D(A^{0}(M, \mathcal{O}_{\rho} \otimes \mathbb{C})).$$

Then we rewrite the constructions of \mathcal{I} and H in Subsection 6.7. In each inductive step, we only replace

$$b(v) = D^*G_D\left(\phi_{n+1}(v) - \varphi_{n+1}\left(\mathcal{I}(v)\right) - \int_0^1 H(dv)\right)$$

with

$$b(v) = \delta_{\mu} \left(\phi_{n+1}(v) - \varphi_{n+1} \left(\mathcal{I}(v) \right) - \int_{0}^{1} H(dv) \right).$$

As the result of this replacement, we obtain another isomorphism \mathcal{I}_{μ} and another homotopy H_{μ} . Then, by the same proof of Theorem 6.8.1, we also have:

Theorem 6.11.1. For the isomorphism $\mathcal{I}_{\mu}: \mathcal{M}_{\mathbb{C}}^* \to \mathcal{N}$, taking the filtration $F_{\mu}^r(\mathcal{M}_{\mathbb{C}}^*) = \mathcal{I}_{\mu}^{-1}(\bigoplus_{P \geq r}(\mathcal{N}^*)^{P,Q})$, $(\mathcal{M}^*, W_*, F_{\mu}^*)$ is an \mathbb{R} -mixed Hodge structure which is compatible with the differential and the multiplication. The co-module structure $\mathcal{M}^* \to \mathbb{R}[T] \otimes \mathcal{M}^*$ corresponding to the T-module structure on \mathcal{M}^* is a morphism of \mathbb{R} -mixed Hodge structures.

We notice that C_{μ} is not closed under the multiplication. Thus our construction may be different from the construction by the reduced bar construction of an augmented multiplicative mixed Hodge complex as in [12].

This alternate construction is interesting for the functoriality. Let (M_1, g_1) and (M_2, g_2) be compact Kähler manifolds and $f: M_2 \to M_1$ a holomorphic map. Excepting the Condition (V), we consider the same situation as in Subsection 6.10.

We have the T-equivariant DGA maps $f_{\mathcal{M}}: {}_{1}\mathcal{M}^* \to {}_{2}\mathcal{M}^*$ and $f_{\mathcal{N}}: {}_{1}\mathcal{N}^* \to {}_{2}\mathcal{N}^*$. Let $d\mu'_1$ and $d\mu'_2$ be volume forms on (M_1, g_1) and (M_2, g_2) respectively which are different from the volume forms associated with the Kähler metrics. We consider the following condition on $f: M_2 \to M_1$.

Condition (V'). For any $\Psi \in A^0(M_1) \otimes \mathbb{C}$ satisfying $\int_{M_1} \Psi d\mu'_1 = 0$, we have

$$\int_{M_2} (f^*\Psi) d\mu_2' = 0.$$

Let

$$C_{\mu'_i} = \left\{ \Psi \in A^0(M) \otimes \mathbb{C} : \int_{M_i} \Psi d\mu'_i = 0 \right\}$$

for i=1,2. We define the maps $\delta_{\mu'_i}$ as above. The Condition (V') holds if and only if $f^*(C_{\mu'_1}) \subset C_{\mu'_2}$. By this, the Condition (V') implies the commutativity

$$f^* \circ \delta_{\mu_1'} = \delta_{\mu_2'} \circ f^*$$

on $D(A^0(M_1, \mathcal{O}_{\rho} \otimes \mathbb{C}))$. For the isomorphisms $\mathcal{I}_{\mu'_i}$ and homotopies $H_{\mu'_i}$ associated with the volume forms μ'_i as above, by the same proof of Proposition 6.10.1, we have:

Proposition 6.11.2. If f satisfies the Condition (V'), then

$$\mathcal{I}_{\mu_2'} \circ f_{\mathcal{M}} = f_{\mathcal{N}} \circ \mathcal{I}_{\mu_1'}$$

and

$$H_{\mu'_2} \circ f_{\mathcal{M}} = (f^* \otimes \mathrm{id}_{[t,dt]}) \circ H_{\mu'_1}.$$

Thus we have the functoriality as Corollary 6.10.2 for the \mathbb{R} -mixed Hodge structures as in Theorem 6.11.1.

Suppose $f: M_2 \to M_1$ is a holomorphic submersion. Then we can define the push-forward $f_*: A^*(M_2) \to A^{*-r}(M_1)$ for $r = \dim M_2 - \dim M_1$. We have

$$\int_{M_2} (f^* \Psi) d\mu_2' = \int_{M_1} \Psi f_* d\mu_2'$$

for any $\Psi \in A^0(M_1) \otimes \mathbb{C}$. Thus, if $f_*d\mu_2' = d\mu_1'$, then the Condition (V') holds.

7. Constructing VMHSs

In this section, we assume the same settings and use same notations as in the previous section. We notice that the arguments in this section are valid for the constructions associated with arbitrary volume forms as in Subsection 6.11.

7.1. Mixed Hodge (T, \mathfrak{u}) -representations.

Definition 7.1.1. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . Then a T-module structure on V is called *mixed Hodge* if the corresponding $\mathbb{R}[T]$ -co-module structure $V \to V \otimes \mathbb{R}[T]$ is a morphism of \mathbb{R} -mixed Hodge structures.

Lemma 7.1.2. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . We suppose that V admits a mixed Hodge T-module structure. Then, for any irreducible representation V_{α} of T, $(V_{\alpha}^* \otimes V)^T$ is an \mathbb{R} -mixed Hodge substructure of $V_{\alpha}^* \otimes V$.

Proof. For the T-module $V_{\alpha}^* \otimes V$, we consider the corresponding $\mathbb{R}[T]$ -co-module structure $V_{\alpha}^* \otimes V \to (V_{\alpha}^* \otimes V) \otimes \mathbb{R}[T]$. Then the T-fixed part $(V_{\alpha}^* \otimes V)^T$ is the kernel of the map

$$V_{\alpha}^* \otimes V \to (V_{\alpha}^* \otimes V) \otimes \mathbb{R}[T]/(V_{\alpha}^* \otimes V) \otimes \langle 1 \rangle.$$

By the assumption, the map

$$V_{\alpha}^* \otimes V \to (V_{\alpha}^* \otimes V) \otimes \mathbb{R}[T]/(V_{\alpha}^* \otimes V) \otimes \langle 1 \rangle$$

is a morphism of \mathbb{R} -mixed Hodge structures. Thus $(V_{\alpha}^* \otimes V)^T$ is an \mathbb{R} -mixed Hodge substructure of $V_{\alpha}^* \otimes V$ (see [20, Section 3.1]).

As in Proposition 2.1.6, for some $b \in \operatorname{Aut}_1(V_{\mathbb{C}}, W_*)$ we can write $F^* = b^{-1}F_{sp}^*$ such that (W_*, F_{sp}^*) is an \mathbb{R} -split \mathbb{R} -mixed Hodge structure. Now we should remark that the V's occurring here are not \mathcal{V} 's that were used in the construction of \mathcal{M}^* . In particular this b is different from the one in the previous section.

We consider the canonical 1-minimal model \mathcal{M}^* and the dual Lie algebra \mathfrak{u} with the \mathbb{R} -mixed Hodge structures (W_*, F^*) as in the last section.

Definition 7.1.3. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . Let $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ be a representation. The representation Ω is called *mixed Hodge* if $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ is a morphism of \mathbb{R} -mixed Hodge structures.

For a finite-dimensional \mathbb{R} -vector space V with an \mathbb{R} -mixed Hodge structure (W_*, F^*) , $\mathfrak{n} = W_{-1}(\operatorname{End}(V))$ is a nilpotent Lie algebra. By $W_{-1}\mathfrak{u} = \mathfrak{u}$, if $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ is compatible with the weight filtrations W_* , then $\Omega(\mathfrak{u}) \subset \mathfrak{n}$. A representation $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ is identified with an element $\Omega \in \mathcal{M}^1 \otimes \operatorname{End}(V)$ satisfying the Maurer-Cartan equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. For this identification, if $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ is compatible with the weight filtrations W_* , then $\Omega \in \mathcal{M}^1 \otimes \mathfrak{n}$.

Remark 7.1.4. Let V a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . As in Proposition 2.1.6, for some $b \in \operatorname{Aut}_1(V_{\mathbb{C}}, W_*)$ we can write $F^* = b^{-1}F_{sp}^*$ such that (W_*, F_{sp}^*) is an \mathbb{R} -split \mathbb{R} -mixed Hodge structure on V. Consider the bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ for this \mathbb{R} -split \mathbb{R} -mixed Hodge structure and write $V_r = \left(\bigoplus_{p+q=r} V^{p,q}\right) \cap V$. Then, a representation $\Omega : \mathfrak{u} \to \operatorname{End}(V)$ is a morphism of \mathbb{R} -mixed Hodge structures if

$$\Omega\left(\bigoplus_{i\leq r}\mathcal{V}_i^*\right)\left(\bigoplus_{i\leq s}V_i\right)\subset\bigoplus_{i\leq r+s}V_i$$

and

$$\mathcal{I}(\Omega)\left(\bigoplus_{-P\geq r}(\mathcal{V}^{P,Q})^*\right)b^{-1}\left(\bigoplus_{p\geq s}V^{p,q}\right)\subset b^{-1}\left(\bigoplus_{p\geq r+s}V^{p,q}\right).$$

Definition 7.1.5. A mixed Hodge (T, \mathfrak{u}) -representation is (V, W_*, F^*, Ω) so that:

- (1) V is a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) .
- (2) V is a T-module and it is mixed Hodge as in Definition 7.1.1.
- (3) Ω is a mixed Hodge representation as in Definition 7.1.3.
- (4) $\Omega: \mathfrak{u} \to \operatorname{End}(V)$ is T-equivariant.

7.2. Flat bundles associated with a mixed Hodge (T, \mathfrak{u}) -representation. For a mixed Hodge (T, \mathfrak{u}) -representation (V, W_*, F^*, Ω) , we consider the following flat bundles.

- Define the \mathcal{C}^{∞} -vector bundle $\mathbf{E} = \bigoplus_{\alpha} (V_{\alpha}^* \otimes V)^T \otimes \mathbf{E}_{\alpha}$ with the flat connection $D = \bigoplus_{\alpha} D_{\alpha}$.
 - We can identify $(A^*(M, \mathcal{O}_{\rho}) \otimes \operatorname{End}(V))^T$ with $A^*(M, \operatorname{End}(\mathbf{E}))$.
 - By the arguments after Definition 7.1.3, we can regard $\Omega \in (\mathcal{M}^1 \otimes \operatorname{End}(V))^T$ satisfying the Maurer-Cartan equation.
 - By the maps $\phi: \mathcal{M}^* \to A^*(M, \mathcal{O}_{\rho})$ and $\varphi \circ \mathcal{I}: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$, we obtain the Maurer-Cartan elements $\Omega_{\phi} = \phi(\Omega) \in A^1(M, \operatorname{End}(\mathbf{E}))$ and $\Omega_{\varphi} = \varphi(I(\Omega)) \in A^1(M, \operatorname{End}(\mathbf{E}_{\mathbb{C}}))$.
- Define the flat bundle $\mathbf{E}_{\Omega_{\phi}}$ as the vector bundle \mathbf{E} with the flat connection $D + \Omega_{\phi}$.
- Define the flat bundle $\mathbf{E}_{\Omega_{\varphi}}$ as the vector bundle $\mathbf{E}_{\mathbb{C}}$ with the flat connection $D + \Omega_{\omega}$.

By Lemma 7.1.2, each $(V_{\alpha}^* \otimes V)^T$ admits an \mathbb{R} -mixed Hodge structure. Each \mathbf{E}_{α} is an \mathbb{R} -VHS by $\mathbf{E}_{\alpha} = \mathbb{S}_{\lambda} \mathbf{E}_{0} \cap \mathbf{E}_{0}^{[d]}$. Thus, we obtain the increasing filtration \mathbf{W}_{*} on \mathbf{E} induced by the weight filtrations of $(V_{\alpha}^* \otimes V)^T$ and the weights of \mathbf{E}_{α} and decreasing filtration \mathbf{F}^* on $\mathbf{E}_{\mathbb{C}}$ induced by Hodge filtrations on $(V_{\alpha}^* \otimes V)^T$ and \mathbf{E}_{α} .

Lemma 7.2.1. On any one of the flat bundles \mathbf{E} , $\mathbf{E}_{\Omega_{\phi}}$ and $\mathbf{E}_{\Omega_{\varphi}}$, the filtration \mathbf{W}_* is a filtration of flat bundles. Moreover, for any i, the identity map on \mathbf{E} induces isomorphisms of flat bundles $Gr_i^{\mathbf{W}}\mathbf{E} \cong Gr_i^{\mathbf{W}}\mathbf{E}_{\Omega_{\phi}}$ and $Gr_i^{\mathbf{W}}\mathbf{E}_{\mathbb{C}} \cong Gr_i^{\mathbf{W}}\mathbf{E}_{\Omega_{\varphi}}$.

Proof. On **E**, the first assertion is obvious.

By the arguments after Definition 7.1.3, we have $\Omega \in (\mathcal{M}^1 \otimes \mathfrak{n})^T$ where $\mathfrak{n} = W_{-1}(\operatorname{End}(V))$. By this, we can say that $\Omega_{\phi} \wedge \mathbf{W}_i \subset A^1(M, \mathbf{W}_{i-1})$ and $\Omega_{\varphi} \wedge \mathbf{W}_i \subset A^1(M, \mathbf{W}_{i-1})$. This implies the lemma.

Proposition 7.2.2. (cf. [6, Lemma 3.11])

$$(D+\Omega_{\varphi})^{1,0}\mathbf{F}^r\subset A^{1,0}(M,\mathbf{F}^{r-1})$$

and

$$(D + \Omega_{\varphi})^{0,1} \mathbf{F}^r \subset A^{0,1}(M, \mathbf{F}^r).$$

Thus the filtration \mathbf{F}^* is a filtration on the holomorphic vector bundle $\mathbf{E}_{\Omega_{\varphi}}$ and the Griffiths transversality holds.

Proof. It is sufficient to show

$$(\Omega_{\varphi})^{1,0} \wedge \mathbf{F}^r \subset A^{1,0}(M, \mathbf{F}^{r-1})$$
 and $(\Omega_{\varphi})^{0,1} \wedge \mathbf{F}^r \subset A^{0,1}(M, \mathbf{F}^r)$.

As in Proposition 2.1.6, we have $b_{\alpha} \in \operatorname{Aut}_{1}((V_{\alpha}^{*} \otimes V)^{T}, W_{*})$ which makes the \mathbb{R} -mixed Hodge structure on $(V_{\alpha}^{*} \otimes V)^{T}$ \mathbb{R} -split. Take $b = \sum b_{\alpha} \otimes \operatorname{id}_{V_{\alpha}} \in \operatorname{Aut}_{1}(V, W_{*})$. Then (W_{*}, bF^{*}) is an \mathbb{R} -split \mathbb{R} -mixed Hodge structure on V and b commutes with the T-action. We take the bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ for the \mathbb{R} -split \mathbb{R} -mixed Hodge structure (W_{*}, bF^{*}) on V and the bigrading $\operatorname{End}(V_{\mathbb{C}}) = \bigoplus \operatorname{End}(V)^{p,q}$ which is induced by $V_{\mathbb{C}} = \bigoplus V^{p,q}$.

Let $\Omega' = b\mathcal{I}(\Omega)b^{-1}$ and $\Omega'_{\varphi} = \varphi(\Omega')$. By the \mathbb{R} -split \mathbb{R} -mixed Hodge structures on $(V_{\alpha}^* \otimes V)^T$ and \mathbb{R} -VHSs \mathbf{E}_{α} , we take the bigrading $\mathbf{E}_{\mathbb{C}} = \bigoplus \mathbf{E}^{P,Q}$. Then our goal is to show

$$(\Omega_{\varphi}')^{1,0} \wedge \left(\bigoplus_{P \geq r} \mathbf{E}^{P,Q}\right) \subset \bigoplus_{P \geq r-1} A^{1,0}(M, \mathbf{E}^{P,Q})$$

and

$$(\Omega'_{\varphi})^{0,1} \wedge \left(\bigoplus_{P \geq r} \mathbf{E}^{P,Q}\right) \subset \bigoplus_{P \geq r} A^{0,1}(M, \mathbf{E}^{P,Q}).$$

We take the bigrading

$$A^*(M, \operatorname{End}(\mathbf{E}_{\mathbb{C}})) = \bigoplus A^*(M, \operatorname{End}(\mathbf{E}))^{P,Q}$$

as in Section 2.2. Then it is sufficient to show $\Omega'_{\varphi} \in \bigoplus_{P \geq 0} A^*(M, \operatorname{End}(\mathbf{E}))^{P,Q}$. Write $\Omega' = \omega'_1 + \cdots + \omega'_l$ such that $\omega'_k \in \bigoplus_{P+Q=k} (\mathcal{N}^*)^{P,Q} \otimes \operatorname{End}(V)$. Then we have $d\omega'_k = -\frac{1}{2} \sum_{i+j=k} [\omega'_i, \omega'_j]$. Thus we obtain

$$\varphi(\omega_k') = -\frac{1}{2} \sum_{i+j=k} D' F_g'[\varphi(\omega_i'), \varphi(\omega_j')].$$

Since $D'F'_g: A^*(M,\operatorname{End}(\mathbf{E}))^{P,Q} \to A^*(M,\operatorname{End}(\mathbf{E}))^{P,Q-1}$, if $\varphi(\omega_1') \in \bigoplus_{P \geq 0} A^*(M,\operatorname{End}(\mathbf{E}))^{P,Q}$, then we can say $\varphi(\omega_k') \in \bigoplus_{P \geq 0} A^*(M,\operatorname{End}(\mathbf{E}))^{P,Q}$ for each k inductively. Thus it is sufficient to show $\varphi(\omega_1') \in \bigoplus_{P \geq 0} A^*(M,\operatorname{End}(\mathbf{E}))^{P,Q}$.

As in Remark 7.1.4, we have

$$\Omega'\left(\bigoplus_{-P\geq r}(\mathcal{V}^{P,Q})^*\right)\left(\bigoplus_{p\geq s}V^{p,q}\right)\subset\bigoplus_{p\geq r+s}V^{p,q}.$$

By this, for any P, Q, we have

$$\omega'_{P+Q}(\mathcal{V}^{P,Q}) = \Omega'\left((\mathcal{V}^{P,Q})^*\right) \subset \bigoplus_{s \geq p-P} (V^{p,q})^* \otimes V^{s,t} \subset \bigoplus_{p \geq -P} \operatorname{End}(V)^{p,q}$$

and hence

$$\omega_k' \in \bigoplus_{p+P \ge 0} (\mathcal{N}^1)^{P,Q} \otimes \operatorname{End}(V)^{p,q}.$$

By the construction of $\varphi: \mathcal{N}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})$, if P + Q = 1, then $\varphi(\mathcal{V}^{P,Q}) \subset A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}$ and hence we have

$$\varphi(\omega_1') \in \bigoplus_{P \ge 0} A^*(M, \operatorname{End}(\mathbf{E}))^{P,Q}.$$

Thus the proposition follows.

Proposition 7.2.3. There exists a weight preserving gauge transformation a of $\mathbf{E}_{\mathbb{C}}$ so that:

- $a(\Omega_{\phi}) = \Omega_{\varphi}$ where $a(\Omega_{\phi}) = a^{-1}Da + a^{-1}\Omega_{\phi}a$.
- a induces the identity map on each $Gr_i^{\mathbf{W}} \mathbf{E}$.

Moreover, such transformation a can be determined by $H(\Omega) \in A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ where $H: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ is the T-equivariant homotopy $H: \mathcal{M}_{\mathbb{C}}^* \to A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes [t, dt]$ as in Subsection 6.7. We call this a canonical.

Proof. In [17, Section 5], it is shown that on a nilpotent DGLA (differential graded Lie algebra) L^* , for two Maurer-Cartan elements $x, y \in L^*$, the following two conditions are equivalent:

- x and y are homotopy equivalent i.e. there exists Maurer-Cartan element $x(t) \in L^* \otimes [t, dt]$ so that x(0) = x and x(1) = y.
- x and y are gauge equivalent i.e. there exists $A \in L^0$ so that $y = \exp(A) * x$ (see Subsection 9.2).

Consider the DGLA $(A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes \mathfrak{n}_{\mathbb{C}})^T$. We can say the homotopy equivalence of Ω_{ϕ} and Ω_{φ} by $H(\Omega)$. Hence we obtain $A \in (A^0(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes \mathfrak{n}_{\mathbb{C}})^T$ so that $\exp(A) * (\Omega_{\phi}) = \Omega_{\varphi}$. By $\mathfrak{n} = W_{-1}(\operatorname{End}(V))$, regarding $A \in \operatorname{End}(\mathbf{E}_{\mathbb{C}})$, we have $A(\mathbf{W}_i) \subset \mathbf{W}_{i-1}$. Thus $a = \exp(A)$ is a desired gauge transformation.

We will find more explicit A. Let $H(\Omega) = \alpha(t) + \beta(t)dt$. We use the techniques in [17, Lemma 5.6, Proposition 5.7 and the proof of Theorem 5.5]. We will find $A(t) \in (A^0(M, \mathcal{O}_{\rho} \otimes \mathbb{C}) \otimes \mathfrak{n}_{\mathbb{C}})^T \otimes [t, dt]$ such that $\alpha(t) = \exp(A(t))(\Omega_{\phi})$. As explained around [17, Proposition 5.7], it is sufficient to solve the differential equation

$$A'(t) + \gamma^A(t) = \beta(t)$$

with A(0) = 0 for certain $\gamma^A(t) \in (A^0(M, \mathcal{O}_\rho \otimes \mathbb{C}) \otimes \mathfrak{n}_{\mathbb{C}})^T \otimes [t, dt]$. We will explain $\gamma^A(t)$ in detail. By using the Baker-Campbell-Hausdorff formula $\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) \dots)$ and the Taylor expansion $A(t+h) = A(t) + A'(t)h + \dots$, we have

$$\exp\left(A(t+h)\right)\exp\left(-A(t)\right) = \exp\left((A'(t)+\gamma^A(t))h + \delta(t,h)h^2\right).$$

Since the filtration W_* on V is a filtration of T-module and T is reductive, we can take a decomposition $V = \bigoplus_i V_i$ of T-modules so that $\bigoplus_{i \leq k} V_i = W_k((V))$. For $\mathfrak{n} = W_{-1}(\operatorname{End}(V))$, this decomposition induces the decomposition $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{n}_{-i}$ of T-modules such that $\bigoplus_{i>k} \mathfrak{n}_{-i} = W_{-k}(\operatorname{End}(V))$ and $[\mathfrak{n}_{-i}, \mathfrak{n}_{-j}] \subset \mathfrak{n}_{-i-j}$. We write

 $A(t) = \sum_{i} A_{-i}(t)$, $\gamma^{A}(t) = \sum_{i} \gamma^{A}_{-i}(t)$ and $\beta(t) = \sum_{i} \beta_{-i}(t)$ associated with this decomposition. Then each $\gamma^{A}_{-i}(t)$ is a linear combination of iterated products of $A_{-j}(t)$ and $A'_{-j}(t)$ with j < i. Thus, inductively, we can solve $A(t) = \sum_{i} A_{-i}(t)$ as

$$A_{-i}(t) = \int_0^t (\beta_{-i}(t) - \gamma_{-i}^A(t)).$$

7.3. Main construction. For a mixed Hodge (T, \mathfrak{u}) -representation $\mathfrak{V} = (V, W_*, F^*, \Omega)$, we construct an \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}^*_{\mathfrak{V}})$. We take:

- $\mathbf{E}_{\mathfrak{V}} = \mathbf{E}_{\Omega_{\phi}}$.
- $\mathbf{W}_{\mathfrak{N}*}$ is the increasing filtration \mathbf{W} on the \mathbb{C}^{∞} -vector bundle \mathbf{E} .
- $\mathbf{F}_{\mathfrak{V}}^* = a\mathbf{F}^*$ where a is the canonical weight preserving gauge transformation as in Proposition 7.2.3.

Theorem 7.3.1. $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{N}}^*)$ is an \mathbb{R} -VMHS.

Proof. By Lemma 7.2.1, $\mathbf{W}_{\mathfrak{I}^*}$ is a filtration of the local system $\mathbf{E}_{\mathfrak{I}}$. By Proposition 7.2.2, $\mathbf{F}_{\mathfrak{I}^*}^*$ is a filtration of the holomorphic vector bundle $\mathbf{E}_{\mathfrak{I}^*}\otimes \mathcal{O}_M$ and the Griffiths transversality holds. We show that $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{I}^*})$ with the filtration induced by $\mathbf{F}_{\mathfrak{I}^*}^*$ is an \mathbb{R} -VHS of weight k. We notice that $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ is an \mathbb{R} -VMHS as in Example 2.3.2. By Lemma 7.2.1, as a local system, we have $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{I}^*}) = Gr_k^{\mathbf{W}}(\mathbf{E})$. By Proposition 7.2.3, $\mathbf{F}_{\mathfrak{I}^*}^* = a\mathbf{F}^*$ induces \mathbb{R} -VHS on $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{I}^*})$. Hence the theorem follows. \square

Remark 7.3.2. Our \mathbb{R} -VMHS ($\mathbf{E}_{\mathfrak{D}}, \mathbf{W}_{\mathfrak{D}*}, \mathbf{F}_{\mathfrak{D}}^*$) is not necessarily graded-polarizable. Since each \mathbf{E}_{α} is polarized \mathbb{R} -VHS, by the construction, if the mixed Hodge structure on $(V_{\alpha}^* \otimes V)^T$ is graded-polarizable for every α , then the \mathbb{R} -VMHS ($\mathbf{E}_{\mathfrak{D}}, \mathbf{W}_{\mathfrak{D}*}, \mathbf{F}_{\mathfrak{D}}^*$) is graded-polarizable.

For a simple case, we explain a way of describing a gauge transformation a as in Proposition 7.2.3 explicitly.

Example 7.3.3. We assume that the weight filtration W_* on V is of length 2 i.e. for some k, $W_{k-2}(V)=0$ and $W_k(V)=V$. Then, by $W_{-3}(\operatorname{End}(V))=0$, $\mathfrak n$ is 2-step i.e. $[\mathfrak n,[\mathfrak n,\mathfrak n]]=0$ and we have $\Omega(W_{-3}(\mathfrak u))=0$. By this, we have $\Omega=\omega_1+\omega_2$ such that $\omega_1\in\mathcal V_1\otimes\mathfrak n$ and $\omega_2\in\mathcal V_2\otimes W_{-2}(\operatorname{End}(V))$. By the Maurer-Cartan equation $d\Omega+\frac12[\Omega,\Omega]=0$, we have $d\omega_2+\frac12[\omega_1,\omega_1]=0$. By constructions of the maps ϕ and φ , we have $\Omega_\phi=\omega_1+D^cA$ and $\Omega_\varphi=\omega_1-2\sqrt{-1}D'A$ where $A=-\frac12F_g[\omega_1,\omega_1]$. In this case, a gauge transformation as in Proposition 7.2.3 is $a=\exp(\sqrt{-1}A)$.

Remark 7.3.4. Replace $A^*(M, \mathcal{O}_{\rho})$, $\left(A^*(M, \mathcal{O}_{\rho} \otimes \mathbb{C})^{P,Q}, D', D''\right)$ and T by the usual de Rham complex $A^*(M)$, usual Dolbeault complex $(A^{*,*}(M), \partial, \bar{\partial})$ and the trivial group respectively. By the arguments as in Section 6 and 7, we obtain Theorem (Prototype). In this case, each \mathbb{R} -VHS on $Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathfrak{V}})$ is constant and so the \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ is unipotent in the sense of Hain-Zucker as in [14]. In this case, we can avoid the argument 6.3 and so we do not need a base point.

Remark 7.3.5. In [21], Hain-Zucker's construction is extended by very interesting but complicated techniques.

7.4. Functoriality. Let (M_1, g_1) and (M_2, g_2) be compact Kähler manifolds and $f: M_2 \to M_1$ a holomorphic map satisfying the condition (V) as in Subsection 6.10. We consider the same situation as in Subsection 6.10. Let ${}_{1}\mathfrak{V} = (V, W_*, F^*, \Omega)$ be a mixed Hodge $(T, {}_{1}\mathfrak{u})$ -representation where ${}_{1}\mathfrak{u}$ (resp. ${}_{2}\mathfrak{u}$) is the pro-nilpotent Lie algebra as in Subsection 6.9 for ${}_{1}\mathcal{M}^*$ (resp. ${}_{2}\mathcal{M}^*$). Now we obtain the \mathbb{R} -VMHS $(\mathbf{E}_{1}\mathfrak{V}, \mathbf{W}_{1}\mathfrak{V}_*, \mathbf{F}^*_{1}\mathfrak{V})$ by the above way. Since f is holomorphic, we have the pull-back \mathbb{R} -VMHS $(f^*\mathbf{E}_{1}\mathfrak{V}, f^*\mathbf{W}_{1}\mathfrak{V}_*, f^*\mathbf{F}^*_{1}\mathfrak{V})$. Otherwise, by the result in Subsection 6.10, ${}_{2}\mathfrak{V} = (V, W_*, F^*, f_{\mathcal{M}}(\Omega))$ is a mixed Hodge $(T, {}_{2}\mathfrak{u})$ -representation and hence we obtain the \mathbb{R} -VMHS $(\mathbf{E}_{2}\mathfrak{V}, \mathbf{W}_{2}\mathfrak{V}_*, \mathbf{F}^*_{2}\mathfrak{V})$ by the above way.

Proposition 7.4.1.

$$(f^*\mathbf{E}_{1}\mathfrak{V},f^*\mathbf{W}_{1}\mathfrak{V}_*,f^*\mathbf{F}_{1}^*\mathfrak{V})=(\mathbf{E}_{2}\mathfrak{V},\mathbf{W}_{2}\mathfrak{V}_*,\mathbf{F}_{2}^*\mathfrak{V}).$$

Proof. Now we have $\phi_2(f_{\mathcal{M}}(\Omega)) = f^*(\phi_1(\Omega))$ (see Subsection 6.10). Thus we have $(f^*\mathbf{E}_{1}\mathfrak{V}, f^*\mathbf{W}_{1}\mathfrak{V}_*) = (\mathbf{E}_{2}\mathfrak{V}, \mathbf{W}_{2}\mathfrak{V}_*)$. Now we write $\mathbf{F}_{1}^*\mathfrak{V} = {}_{1}a_{1}\mathbf{F}^*$ and $\mathbf{F}_{2}^*\mathfrak{V} = {}_{2}a_{2}\mathbf{F}^*$ as above. Then we can easily check that ${}_{2}\mathbf{F}^* = f^*{}_{1}\mathbf{F}^*$ by the construction. Thus it is sufficient to show ${}_{2}a = f^*({}_{1}a)$. We write ${}_{1}a = e^{{}_{1}A}$ and ${}_{2}a = e^{{}_{2}A}$ as in the proof of Proposition 7.2.3. We prove ${}_{2}A = f^*({}_{1}A)$. Write ${}_{1}H(\Omega) = {}_{1}\alpha(t) + {}_{1}\beta(t)dt$ and ${}_{2}H(f_{\mathcal{M}}(\Omega)) = {}_{2}\alpha(t) + {}_{2}\beta(t)dt$. Then, ${}_{1}A = {}_{1}A(1)$ and ${}_{2}A = {}_{2}A(1)$ for the solutions ${}_{1}A(t)$ and ${}_{2}A(t)$ of the differential equations

$$_{1}A'(t) + \gamma^{_{1}A}(t) = _{1}\beta(t)$$

and

$$_2A'(t) + \gamma^{_2A}(t) = _2\beta(t)$$

with $_1A(0) = 0$ and $_2A(0) = 0$ respectively (see the proof of Proposition 7.2.3). By the result of Subsection 6.10, we have $_2H(f_{\mathcal{M}}(\Omega)) = f^* \otimes \mathrm{id}_{[t,dt]}(_1H(\Omega))$. Thus we have $_2\beta(t) = f^* \otimes \mathrm{id}_{[t,dt]}(_1\beta(t))$ and this implies $_2A(t) = f^* \otimes \mathrm{id}_{[t,dt]}(_1A(t))$. Hence we obtain $_2A = f^*(_1A)$.

8. Constructing VMHSs by substructures of 1-minimal models

8.1. Representations of nilpotent Lie algebras. The reference of this subsection is [22].

Let \mathfrak{n} be a nilpotent Lie algebra and $\mathfrak{n} = \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \mathfrak{n}_3 \ldots$ the lower central series of \mathfrak{n} (i.e. $[\mathfrak{n},\mathfrak{n}_i]=\mathfrak{n}_{i+1}$). It is known that $[\mathfrak{n}_i,\mathfrak{n}_j]\subset \mathfrak{n}_{i+j}$. We say that \mathfrak{n} is k-step if $\mathfrak{n}_k\neq 0$ and $\mathfrak{n}_{k+1}=0$. Define the increasing filtration W_* of \mathfrak{n} so that $W_{-k}=\mathfrak{n}_k$ for k>0.

Let $U(\mathfrak{n})$ be the universal enveloping algebra. That is $U(\mathfrak{n}) = T(\mathfrak{n})/I$ where $T(\mathfrak{n})$ is the tensor algebra of \mathfrak{n} and I is the ideal which is generated by

$$\{X\otimes Y-Y\otimes X-[X,Y]|X,Y\in\mathfrak{n}\}.$$

Then we have the natural increasing filtration W_* of $U(\mathfrak{n})$ induced by the above increasing filtration of \mathfrak{n} . This filtration is compatible with the multiplication of $U(\mathfrak{n})$. We suppose \mathfrak{n} is k-step. Let $J = W_{-k}(U(\mathfrak{n}))$. Then J is an ideal and the map $\mathfrak{n} \to U(\mathfrak{n})/J$ induced by the natural inclusion $\mathfrak{n} \hookrightarrow U(\mathfrak{n})$ is an injection. Thus, for the endomorphisms $\operatorname{End}(U(\mathfrak{n})/J)$ of the finite-dimensional vector space $U(\mathfrak{n})/J$,

we have a finite-dimensional faithful representation $\tau : \mathfrak{n} \to \operatorname{End}(U(\mathfrak{n})/J)$. For any $x \in \mathfrak{n}$ and any integer i,

$$\tau(x)(W_{-i}(U(\mathfrak{n})/J)) \subset W_{-i-1}(U(\mathfrak{n})/J)$$

and so τ is a nilpotent representation.

8.2. Sub-structures of 1-minimal models.

Definition 8.2.1. A k-step sub-structure of \mathcal{M}^* is a sub-vector space $\mathcal{X} \subset \mathcal{M}^1$ so that

- $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_k$ such that for each $1 \leq i \leq k$, $\mathcal{X}_i \neq 0$ and $\mathcal{X}_i \subset \mathcal{V}_i$.
- $d: \mathcal{X}_r \to \sum_{i+j=r} \mathcal{X}_i \wedge \mathcal{X}_j$ (thus $\bigwedge \mathcal{X}$ is a sub-DGA of \mathcal{M}^*).
- \mathcal{X} is an \mathbb{R} -mixed Hodge substructure of \mathcal{M}^1 .
- \mathcal{X} is a T-submodule of \mathcal{M}^1 .

We assume that a k-step sub-structure \mathcal{X} is finite-dimensional. Consider the nilpotent Lie algebra \mathcal{X}^* which is dual to the DGA $\bigwedge \mathcal{X}$. Take the dual \mathbb{R} -mixed Hodge structure (W_*, F^*) on \mathcal{X}^* . Then, for $\mathcal{X}^* = \mathcal{X}_1^* \oplus \cdots \oplus \mathcal{X}_k^*$, the bracket on \mathcal{X}^* is dual to the differential $d: \mathcal{X}_r \to \sum_{i+j=r} \mathcal{X}_i \wedge \mathcal{X}_j$. Hence, we can easily check that $\mathcal{X}^* = \mathcal{X}_1^* \oplus \cdots \oplus \mathcal{X}_k^*$ is a graded nilpotent Lie algebra so that the filtration $W_{-n}(\mathcal{X}^*) = \bigoplus_{i \geq n} \mathcal{X}_i^*$ is the natural filtration which is given by the lower central series of \mathcal{X}^* .

Consider the universal enveloping algebra $U(\mathcal{X}^*)$ of the nilpotent Lie algebra \mathcal{X}^* . Then, we obtain the \mathbb{R} -mixed Hodge structure (W_*, F^*) on $U(\mathcal{X}^*)$ which is induced by the \mathbb{R} -mixed Hodge structure on \mathcal{X}^* . By the above argument, the weight filtration W_* is the natural filtration induced by the lower central series of \mathcal{X}^* . Hence, for $J = W_{-k}(U(\mathcal{X}))$, we obtain the \mathbb{R} -mixed Hodge structure (W_*, F^*) on the quotient space $U(\mathcal{X}^*)/J$.

Consider the faithful representation $\mathcal{X}^* \to \operatorname{End}(U(\mathcal{X}^*)/J)$ as in Subsection 8.1. Since the multiplication on $U(\mathcal{X}^*)$ is a morphism of \mathbb{R} -mixed Hodge structures, we can easily show that the representation $\mathcal{X}^* \to \operatorname{End}(U(\mathcal{X}^*)/J)$ is a morphism of \mathbb{R} -mixed Hodge structure. Consider the composition $\Omega: \mathfrak{u} \to \operatorname{End}(U(\mathcal{X}^*)/J)$ of the surjection $\mathfrak{u} \to \mathcal{X}^*$ which is dual to the inclusion $\mathcal{X} \subset \mathcal{M}^1$ and the faithful representation $\mathcal{X}^* \to \operatorname{End}(U(\mathcal{X}^*)/J)$. Since \mathcal{X} is a T-submodule of \mathcal{M}^1 , we can say that

$$(U(\mathcal{X}^*)/J, W_*, F^*, \Omega)$$

is a mixed Hodge (T, \mathfrak{u}) -representation. Hence, we obtain an \mathbb{R} -VMHS. More precisely, taking a basis x_{i1}, \ldots, x_{il_i} of each \mathcal{X}_i and the dual basis $\chi_{i1}, \ldots, \chi_{il_i}$ of \mathcal{X}_i^* , we have $\Omega = \sum_{ij} x_{ij} \otimes \chi_{ij}$ and so we can write $\Omega_{\phi} = \sum_{ij} \phi(x_{ij}) \otimes \chi_{ij}$.

Remark 8.2.2. Each $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n)$ is a *n*-step sub-structure of \mathcal{M}^* as in Definition 8.2.1. However, it is not finite-dimensional in general. By the construction, if \mathcal{V}_1 is finite-dimensional, then all \mathcal{V}_i are also finite-dimensional.

Since we have

$$\mathcal{V}_1 \cong H^1(A^*(M, \mathcal{O}_\rho)) \cong \bigoplus_{\alpha} H^1(M, \mathbf{E}_\alpha^*) \otimes V_\alpha,$$

if the group cohomology

$$\bigoplus_{\alpha} H^1(\pi_1(M,x), V_{\alpha}^*) \otimes V_{\alpha}.$$

is finite-dimensional, then \mathcal{V}_1 is finite-dimensional.

Proposition 8.2.3. We suppose the following conditions:

- $\operatorname{im}\rho$ is a co-compact discrete subgroup in T.
- $T \not\cong O(m,1)$ for any m.
- The group cohomology $H^1(\ker \rho, \mathbb{R})$ is finite-dimensional.

Then the group cohomology

$$\bigoplus_{\alpha} H^1(\pi_1(M,x),V_{\alpha}^*) \otimes V_{\alpha}$$

is finite-dimensional. Hence, in this case, $(\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n)$ is a finite-dimensional n-step sub-structure of \mathcal{M}^* and so we obtain the \mathbb{R} -VMHS associated with $(V_1 \oplus \cdots \oplus V_n)$ for each n.

Proof. Let T_0 be the identity component of T and $\Gamma = \operatorname{im} \rho \cap T_0$. Then Γ is a co-compact discrete subgroup of the connected semi-simple Lie group T_0 . We notice that Γ is a finite-index normal subgroup of $\operatorname{im} \rho$. For the extension

$$1 \longrightarrow \ker \rho \longrightarrow \pi_1(M, x) \longrightarrow \operatorname{im} \rho \longrightarrow 1 ,$$

we have the spectral sequence $E_r^{p,q}$ so that

$$E_2^{p,q} = \bigoplus_{\alpha} H^p(\operatorname{im}\rho, H^q(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*) \otimes V_{\alpha}$$

and it converges to $\bigoplus_{\alpha} H^{p+q}(\pi_1(M,x), V_{\alpha}^*) \otimes V_{\alpha}$. It is sufficient to show that $E_2^{1,0}$ and $E_2^{0,1}$ are finite-dimensional. By Raghunathan's result in [23], for non-trivial V_{α} , we have $H^1(\Gamma, V_{\alpha}) = 0$. Since Γ is a finite-index normal subgroup of $\operatorname{im} \rho$, we have $H^1(\operatorname{im} \rho, V_{\alpha}) = 0$. Thus

$$E_2^{1,0} = \bigoplus_{\alpha} H^1(\mathrm{im}\rho, H^0(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*) \otimes V_{\alpha}$$

is finite-dimensional. We have

$$E_2^{0,1} = \bigoplus_{\alpha} H^0(\operatorname{im}\rho, H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*) \otimes V_{\alpha} = \bigoplus_{\alpha} (H^1(\ker\rho, \mathbb{R}) \otimes V_{\alpha}^*)^{\operatorname{im}\rho} \otimes V_{\alpha}.$$

Since $\operatorname{im} \rho$ is Zariski-dense in T, we have

$$\bigoplus_{\alpha} (H^1(\ker \rho, \mathbb{R}) \otimes V_{\alpha}^*)^{\operatorname{im}\rho} \otimes V_{\alpha} = \bigoplus_{\alpha} (H^1(\ker \rho, \mathbb{R}) \otimes V_{\alpha}^*)^T \otimes V_{\alpha} \cong H^1(\ker \rho, \mathbb{R}).$$

Since $H^1(\ker \rho, \mathbb{R})$ is finite-dimensional, we can say that $E_2^{0,1}$ is finite-dimensional. Hence the proposition follows.

8.3. Lower step \mathbb{R} -VMHS. By using sub-structures of 1-minimal models, we can obtain explicit \mathbb{R} -VMHSs whose weight filtration is of length 1 or 2.

8.3.1. 1-step \mathbb{R} -VMHS. Let $\{V_{\alpha_1},\ldots,V_{\alpha_l}\}$ be a finite set of irreducible representations of T. Take $\mathcal{X}_1 = \bigoplus_i \mathcal{H}^1(M,\mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^*$. Then, regarding \mathcal{X}_1 as a subspace in \mathcal{V}_1 , obviously \mathcal{X}_1 is a finite-dimensional 1-step sub-structure of \mathcal{M}^* . In this case we have $U(\mathcal{X}_1^*)/J = \langle 1 \rangle \oplus \bigoplus_i \mathcal{H}^1(M,\mathbf{E}_{\alpha_i})^* \otimes V_{\alpha_i}$ Take a basis $x_1^{\alpha_i},\ldots x_{m_i}^{\alpha_i}$ of each $\mathcal{H}^1(M,\mathbf{E}_{\alpha_i}^*)$. We take the dual basis $\chi_1^{\alpha_i},\ldots \chi_{m_i}^{\alpha_i}$ of each $\mathcal{H}^1(M,\mathbf{E}_{\alpha_i})^*$. Then, the \mathbb{R} -VHMS as in subsection 8.2 is given by the flat connection $D + \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i}$ over the vector bundle $\mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M,\mathbf{E}_{\alpha_i}^*)^* \otimes \mathbf{E}_{\alpha_i}$. Precisely, for $(f,\sum \eta_{\alpha_i}) \in \mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M,\mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}$, we have

$$\left(D + \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i}\right) \left(f, \sum \eta_{\alpha_i}\right) = \left(df, \sum D_{\alpha_i} \eta_{\alpha_i} + f \sum x_j^{\alpha_i} \otimes \chi_j^{\alpha_i}\right)$$

The weight filtration \mathbf{W}_* is given by

$$\mathbf{W}_0 = \mathbb{R} \oplus \bigoplus_i H^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i},$$

$$\mathbf{W}_{-1} = \bigoplus_{i} H^{1}(M, \mathbf{E}_{\alpha_{i}})^{*} \otimes \mathbf{E}_{\alpha_{i}}$$

and $\mathbf{W}_{-2} = 0$. The Hodge filtration \mathbf{F}^* on $(\mathbb{R} \oplus \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^* \otimes \mathbf{E}_{\alpha_i}) \otimes \mathbb{C}$ is given by the dual Hodge structures on $\mathcal{H}^1(M, \mathbf{E}_{\alpha_i})^*$ and the \mathbb{R} -VHSs \mathbf{E}_{α_i} . Similar construction is given in [5, Example 2.2.2].

8.3.2. 2-step \mathbb{R} -VMHS. Take $\mathcal{X}_1 = \bigoplus_i \mathcal{H}^1(M, \mathbf{E}_{\alpha_i}) \otimes V_{\alpha_i}^*$. as the above argument. Let $\mathcal{X}_2 = d^{-1}(\mathcal{X}_1 \wedge \mathcal{X}_1) \subset \mathcal{V}_2$. As in Remark 6.7.2, we have $\mathcal{I}(\mathcal{X}_2 \otimes \mathbb{C}) = d^{-1}(\mathcal{I}(\mathcal{X}_1 \otimes \mathbb{C})) \wedge \mathcal{I}(\mathcal{X}_1 \otimes \mathbb{C})$. Hence, $\mathcal{X}_1 \oplus \mathcal{X}_2$ is a finite-dimensional sub-structure of \mathcal{M}^* and so we can construct the \mathbb{R} -VMHS associated with $\mathcal{X}_1 \oplus \mathcal{X}_2$. Thus, we obtain the \mathbb{R} -VMHS ($\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*$) associated with the mixed Hodge (T,\mathfrak{U})-module $\mathfrak{V} = (U(\mathcal{X}^*)/J, W_*, F^*, \Omega)$. Since the weight filtration W_* is of length 2, we can write the Hodge filtration $\mathbf{F}_{\mathfrak{V}}^*$ as in Example 7.3.3.

For this construction, it is necessary that the kernel of the cup product

$$\left(\bigoplus_{i} \mathcal{H}^{1}(M, \mathbf{E}_{\alpha_{i}}) \otimes V_{\alpha_{i}}^{*}\right) \wedge \left(\bigoplus_{i} \mathcal{H}^{1}(M, \mathbf{E}_{\alpha_{i}}) \otimes V_{\alpha_{i}}^{*}\right) \to H^{2}(M, \mathcal{O}_{\rho})$$

is non-trivial. On Example 6.3.2, by dim M=2, we have

$$H^2(M, \mathcal{O}_\rho) = \bigoplus_{k=0}^{\infty} H^2(M, S^k \mathbf{E}_0^*) \otimes S^k V_0 \cong \mathbb{R}.$$

By the Euler number $\chi(M) = 2 - 2g$ and dim $V_0 = 2$, we have dim $H^1(M, \mathbf{E}_0^*) \otimes V_0 = 8g - 8 \geq 8$. Thus the kernel of the cup product

$$(\mathcal{H}^1(M, \mathbf{E}_0^*) \otimes V_0) \wedge (\mathcal{H}^1(M, \mathbf{E}_0^*) \otimes V_0) \rightarrow H^2(M, \mathcal{O}_\rho)$$

is non-trivial.

9. Constructing R-VMHS from Deformation Theory

- 9.1. Functors of Artinian algebras. We briefly review the theory of Schlessinger's hull ([24]). Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . For a local \mathbb{K} -algebra R, we denote by \mathfrak{m}_R the maximal ideal of R. We define:
 - Art is the category of Artinian local K-algebras.
 - \overline{Art} is the category of complete Noetherian local K-algebra R so that $R/\mathfrak{m}_R^k \in Art$ for any k.
 - A functor F of Art is a covariant functor from Art to the category of sets so that $F(\mathbb{K})$ is a 1-point set.
 - For a functor F of Art, we define $t_F = F(\mathbb{K}[t]/(t^2))$.
 - For $R \in \overline{Art}$, we define the functor h_R of Art as $h_R(A) = Hom(R, A)$.
 - For a functor F of Art, $R \in \overline{Art}$ and $\in \xi \in F(R)$, we define the morphism $T_{\xi}: h_R \to F$ of functors such that $h_R(A) \ni u \mapsto F(u)(\xi) \in F(A)$ where we extend F to \overline{Art} .
 - A morphism $F \to G$ of functors of Art is smooth if for any surjection $B \to A$ in Art the map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective.

- A morphism $F \to G$ of functors of Art is étale if it is smooth and the induced map $t_F \to t_G$ is bijective.
- For a functor F of Art, $R \in \overline{Art}$ and $\xi \in F(R)$, a pair (R, ξ) is a hull if the morphism $T_{\xi}: h_R \to F$ is étale.

The following uniqueness is important.

Proposition 9.1.1 ([24]). For a functor F of Art, if two pairs (R_1, ξ_1) and (R_2, ξ_2) are hulls, then we have an isomorphism $u: R_1 \to R_2$ such that $F(u)(\xi_1) = \xi_2$.

9.2. **Deformation theory of DGLA.** We briefly review the Kuranishi theory of a DGLA (differential graded Lie algebra) ([17]). Let L^* be a DGLA over \mathbb{K} with a differential d. We define $MC(L^*) = \{x \in L^1 | dx + \frac{1}{2}[x,x] = 0\}$. If L^* is nilpotent, then for any $B \in L^0$, we can define the gauge transformation $\exp(B)*$ on L^1 by the exponential of the affine transformation $x \mapsto [B,x] - dB$. Moreover this action preserves $MC(L^*)$ and so we can define gauge equivalence on $MC(L^*)$ so that $x,y \in MC(L^*)$ are gauge equivalent if for some $B \in L^0$, $y = \exp(B)*(x)$.

For any DGLA L^* and $A \in Art$, the DGLA $L^* \otimes \mathfrak{m}_A$ is nilpotent and we define $Def_{L^*}(A)$ as the set of gauge equivalent classes of $MC(L^* \otimes \mathfrak{m}_A)$. Consider the functor $Def_{L^*}: A \mapsto Def_{L^*}(A)$ of Art. We notice that for $R \in \overline{Art}$ we can also define the gauge transformation $\exp(B)*$ of $B \in L^0 \otimes \mathfrak{m}_S$ by using formal power series and we can regard $Def_{L^*}(S)$ as the set of gauge equivalent classes of $MC(L^* \otimes \mathfrak{m}_S)$.

Let L^* be a DGLA such that the cohomologies $H^0(L^*)$, $H^1(L^*)$ and $H^2(L^*)$ are finite-dimensional. We will define the Kuranishi functor of L^* as in [17, Section 4]. For our application, we only consider the special case. A *special grading* is a grading $L^* = \bigoplus_{k>0} L_k^*$ on the vector space L^* so that:

• $dL_k^* \subset L_k^*$ and $[L_{k_1}^*, L_{k_2}^*] \subset L_{k_1+k_2}^*$.

- $\begin{array}{l} \bullet \ L_0^* = L^0. \\ \bullet \ L^1 = \bigoplus_{k \geq 1} L_k^1 \ \text{and } \ker d_{|L^1} = L_1^1. \\ \bullet \ L^2 = \bigoplus_{k \geq 2} L_k^2. \end{array}$
- $d: L_2^1 \to \ker d_{|L_2^2}$ is injective and $d: L_k^1 \to \ker d_{|L_k^2}$ is bijective for any $k \geq 3$.

This grading gives a special case of decomposition as in [17, Section 4].

We assume that L^* admits a special grading. For $A \in Art$, we define the set

$$Kur_{L^*}(A) = \{x_1 \in L^1 \otimes \mathfrak{m}_A | [x_1, x_1] \equiv 0 \in H^2(L^*) \otimes \mathfrak{m}_A \}$$

and the map $Kur_{L^*}(A) \ni x_1 \mapsto \sum_i x_i \in MC(L^* \otimes \mathfrak{m}_A)$ so that for $k \ge 2$,

$$dx_k = -\frac{1}{2} \sum_{i+j=k, i>0, j>0} [x_i, x_j].$$

By the assumptions, each x_k is uniquely determined by x_1 . We obtain the functor $Kur_{L^*}: A \mapsto Kur_{L^*}(A)$ and the morphism $Kur_{L^*} \to Def_{L^*}$ of functors so that $Kur_{L^*}(A) \ni x_1 \mapsto [\sum_i x_i] \in Def_{L^*}(A).$

Theorem 9.2.1. ([17, Theorem 4.7]) The morphism $Kur_{L^*} \to Def_{L^*}$ is étale.

We can easily check that $Kur_{L^*} = h_R$ so that $R = \mathbb{K}[[(L_1^1)^*]]/I$ where $\mathbb{K}[[(L_1^1)^*]]$ is the algebra of formal power series on $(L_1^1)^*$ and I is the ideal generated by the quadratic polynomials on $(L_1^1)^*$ associated with $L_1^1 \ni x \mapsto [x,x] \in H^2(L^*)$. Take $\xi_1 \in L^* \otimes R$ which is the extension of the identity map $I \in L^1_1 \otimes (L^1_1)^*$. Then, by elementary arguments, we have $[\xi_1, \xi_1] \equiv 0 \in H^2(L^*) \otimes \mathfrak{m}_R$ (see [6, Lemma 3.4]). Take the formal power series $\xi = \sum_{i=1}^{\infty} \xi_i \in MC(L^* \otimes \mathfrak{m}_R)$ so that

$$d\xi_k = -\frac{1}{2} \sum_{i+j=k, i>0, j>0} [\xi_i, \xi_j].$$

By Theorem 9.2.1, we have:

Corollary 9.2.2. For the functor Def_{L^*} of Art and the gauge equivalent class $[\xi] \in Def_{L^*}(R)$ of ξ , the pair $(R, [\xi])$ is a hull.

We notice that the definition of R is independent of the choice of a special grading $L^* = \bigoplus_{k \geq 0} L_k^*$ as above but ξ varies for the choice of a special grading. By Proposition 9.1.1, we have:

Corollary 9.2.3. For two Maurer-Cartan elements $\xi, \xi' \in MC(L^* \otimes \mathfrak{m}_R)$ constructed as above associated with two special gradings of L^* , there exists an automorphism $u: R \to R$ of R and $B \in L^0 \otimes \mathfrak{m}_R$ such that

$$\xi' = \exp(B) * u(\xi) = \exp(\operatorname{ad}_B) \circ u(\xi).$$

9.3. Mixed Hodge (T,\mathfrak{u}) -representation associated with deformation theory. We assume the same settings and use the same notations as in Section 6 and 7. Let U be a finite-dimensional rational T-representation with an irreducible decomposition $U = \bigoplus V_{\gamma_i}$. Now we have the polarized \mathbb{R} -Hodge structure on each V_{γ_i} induced by the polarized \mathbb{R} -Hodge structure on V_0 . We assume that all such \mathbb{R} -Hodge structures have same weight. Consider the polarized \mathbb{R} -VHS \mathbf{E}_{γ_i} corresponding to each irreducible representation γ_i . Then, we have the \mathbb{R} -VHS $\mathbf{E}_U = \bigoplus \mathbf{E}_{\gamma_i}$. We consider the DGLA $L^* = (\mathcal{M}^* \otimes \operatorname{End}(U))^T$ over \mathbb{R} . Then we have the grading

$$L^* = \bigoplus_k (\mathcal{M}_k^* \otimes \operatorname{End}(U))^T.$$

This grading is a special grading and $L_1^1 = (\mathcal{V}_1 \otimes \operatorname{End}(U))^T = \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)).$ We study $R = \mathbb{K}[[(L_1^1)^*]]/I$ as in the last subsection. Since the map $\phi: \mathcal{M}^* \to I$ $A^*(M,\mathcal{O}_{\rho})$ induces an injection on the second cohomology, the map $L^* \to (A^*(M,\mathcal{O}_{\rho}) \otimes$ $\operatorname{End}(U)^T \cong A^*(M,\operatorname{End}(\mathbf{E}_U))$ induces an injection $H^2(L^*) \to H^2(M,\operatorname{End}(\mathbf{E}_U))$. Thus I is the ideal generated by the quadratic polynomials on $\mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^*$ associated with $\mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \ni x \mapsto [x, x] \in H^2(M, \operatorname{End}(\mathbf{E}_U))$. Define $I_2 \subset$ $S^2\mathcal{H}^1(M,\operatorname{End}(\mathbf{E}_U))^*$ by the image of the dual

$$H^2(M, \operatorname{End}(\mathbf{E}_U))^* \to S^2 \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^*$$

of the cup bracket

$$[,]: \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \times \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \to H^2(M, \operatorname{End}(\mathbf{E}_U)).$$

We have

$$R/\mathfrak{m}_R^k \cong \bigoplus_{i=1}^{k-1} S^i \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^* / S^{i-2} \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^* \cdot I_2.$$

It is known that the cup bracket

$$[,]: \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \times \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \to H^2(M, \operatorname{End}(\mathbf{E}_U))$$

is a homomorphism of \mathbb{R} -Hodge structures for the \mathbb{R} -Hodge structures on $\mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))$ and $H^2(M, \operatorname{End}(\mathbf{E}_U)) \cong \mathcal{H}^2(M, \operatorname{End}(\mathbf{E}_U))$ as in Section 5. By this, we can say that

$$R/\mathfrak{m}_R \leftarrow R/\mathfrak{m}_R^2 \leftarrow R/\mathfrak{m}_R^3 \cdots$$

is a inverse system of \mathbb{R} -split \mathbb{R} -mixed Hodge structures (W_*, F_{sp}^*) such that $W_{-i}(R/\mathfrak{m}_R^k)=$

 $\mathfrak{m}_R^i/\mathfrak{m}_R^k$ and the multiplication on R is compatible with these structures. Take the formal power series $\xi = \sum_{i=1}^{\infty} \xi_i \in L^* \otimes \mathfrak{m}_R$ associated with the above special grading as in the last subsection. For the complexification $L^*_{\mathbb{C}}=(\mathcal{M}^*_{\mathbb{C}}\otimes$ $\operatorname{End}(U_{\mathbb{C}})^T$, we have another special grading

$$L_{\mathbb{C}}^* = \bigoplus_k \left(\bigoplus_{P+Q=k} \mathcal{I}^{-1} \left((\mathcal{N}^*)^{P,Q} \right) \otimes \operatorname{End}(U) \right)^T.$$

Take the formal power series $\xi' = \sum_{i=1}^{\infty} \xi'_i \in L^*_{\mathbb{C}} \otimes \mathfrak{m}_{R_{\mathbb{C}}}$ associated with this new special grading. Then, by Corollary 9.2.3, there exists an automorphism $u: R_{\mathbb{C}} \to \mathbb{C}$ $R_{\mathbb{C}}$ and $B \in L^0_{\mathbb{C}} \otimes \mathfrak{m}_{R_{\mathbb{C}}}$ such that

$$\xi' = \exp(\mathrm{ad}_B) \circ u(\xi).$$

We notice that u induces the identity map on $\mathfrak{m}_{R_{\mathbb{C}}}/\mathfrak{m}_{R_{\mathbb{C}}}^2$, since such induced map sends the identity map on $\mathcal{H}^1(M,\operatorname{End}(\mathbf{E}_{U_{\mathbb{C}}}))$ to itself by the constructions of ξ and ξ' . Since u is a ring homomorphism, u induces the identity map on $\mathfrak{m}_{R_{\mathbb{C}}}^k/\mathfrak{m}_{R_{\mathbb{C}}}^{k+1}$ for any k. We consider the map $\iota: R \to \operatorname{End}(R)$ associated with the multiplication on R. Let

$$\Omega = \iota(\xi) \in L^* \otimes \operatorname{End}(R) = (\mathcal{M}^* \otimes \operatorname{End}(U))^T \otimes \operatorname{End}(R)$$

and

$$\Omega' = \iota(\xi') \in L_{\mathbb{C}}^* \otimes \operatorname{End}(R_{\mathbb{C}}) = (\mathcal{M}_{\mathbb{C}}^* \otimes \operatorname{End}(U_{\mathbb{C}}))^T \otimes \operatorname{End}(R_{\mathbb{C}}).$$

Then we have

$$\Omega' = b^{-1}\Omega b.$$

where $b = u^{-1}e^{-B}$.

Consider each quotient $q_k: R \to R/\mathfrak{m}_R^k$. Take $\xi(k) = q_k(\xi)$, $\xi'(k) = q_k(\xi')$, $B_k = q_k(B)$ and the reduction $u_k: R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k \to R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k$ of $u: R_{\mathbb{C}} \to R_{\mathbb{C}}$. We have $\xi'(k) = \exp(\mathrm{ad}_{B_k}) \circ u_k(\xi(k))$. By the construction, we have $\xi(k) \equiv \sum_{i=1}^{k-1} \xi_i$ and $\xi'(k) \equiv \sum_{i=1}^{k-1} \xi_i'$. For the map $\iota_k: R/\mathfrak{m}_R^k \to \operatorname{End}(R/\mathfrak{m}_R^k)$ associated with the multiplication on R/\mathfrak{m}_R^k , let $\Omega_k = \iota_k(\xi(k))$ and $\Omega'_k = \iota_k(\xi'(k))$. We have

$$\Omega_k' = b_k^{-1} \Omega_k b_k.$$

where $b_k = u_k^{-1} e^{-B_k}$. Consider the \mathbb{R} -split \mathbb{R} -mixed Hodge structure (W_*, F_{sp}^*) on $U \otimes R/\mathfrak{m}_R^k$ induced by the \mathbb{R} -split \mathbb{R} -mixed Hodge structure on R/\mathfrak{m}_R^k as above and the \mathbb{R} -Hodge structure on $U = \bigoplus V_{\gamma_i}$. We regard Ω as a T-equivariant Lie algebra homomorphism $\mathfrak{u} \to \operatorname{End}(U) \otimes \operatorname{End}(R/\mathfrak{m}_R^k)$. By $b_k \in \operatorname{Aut}_1(U \otimes R/\mathfrak{m}_R^k, W)$, $(W_*, F^*) = (W_*, b_k^{-1} F_{sp}^*)$ is an \mathbb{R} -mixed Hodge structure. We have:

Proposition 9.3.1. $(U \otimes R/\mathfrak{m}_R^k, W_*, F^*, \Omega_k)$ is a mixed Hodge (T, \mathfrak{u}) -representation.

Proof. By the above arguments, it is sufficient to prove the following two claims.

- $\Omega_k : \mathfrak{u} \to \operatorname{End}(U) \otimes \operatorname{End}(R/\mathfrak{m}_R^k)$ is compatible with the weight filtrations W_* .
- $\Omega_k : \mathfrak{u}_{\mathbb{C}} \to \operatorname{End}(U_{\mathbb{C}}) \otimes \operatorname{End}(R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k)$ is compatible with the Hodge filtrations F^* .

Consider the sum $\xi(k) \equiv \sum_{i=1}^{k-1} \xi_i$. By the construction, we have $\xi_i \in \mathcal{V}_i \otimes \operatorname{End}(U) \otimes \mathfrak{m}_B^i$. Thus, we have

$$\xi(k)(\mathcal{V}_r^*)\cdot (U\otimes \mathfrak{m}_R^s/\mathfrak{m}_R^k)\subset U\otimes \mathfrak{m}_R^{r+s}/\mathfrak{m}_R^k.$$

This implies the first claim.

Consider the sum $\xi'(k) \equiv \sum_{i=1}^{k-1} \xi_i'$. By the splitting

$$R/\mathfrak{m}_R^k \cong \bigoplus_{i=1}^{k-1} S^i \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^* / S^{i-2} \mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U))^* \cdot I_2,$$

we took $\xi'_1 \in$ as an identity map on $\mathcal{H}^1(M, \operatorname{End}(\mathbf{E}_U)) \otimes \mathbb{C}$. Thus for the bigrading of the \mathbb{R} -split \mathbb{R} -mixed Hodge structure (W_*, F_{sp}^*) on R/\mathfrak{m}_R^k and the bigrading $\mathcal{M}_{\mathbb{C}}^* = \mathcal{I}^{-1}(\bigoplus (\mathcal{N}^*)^{P,Q}), \ \xi'_1 \in \mathcal{M}_{\mathbb{C}}^* \otimes \operatorname{End}(U) \otimes R/\mathfrak{m}_R^k$ is of type (0,0). Since we have $d\xi_l = -\frac{1}{2} \sum_{i+j=l, i>0, j>0} [\xi_i, \xi_j]$ for each l, we can say that each ξ_l is also of type

(0,0) inductively. Thus the sum $\xi'(k) = \sum_{i=1}^{k-1} \xi_i'$ is of type (0,0). By this, we can say

$$\begin{split} \Omega_k(F^r(\mathfrak{u}_{\mathbb{C}}))(F^s(U_{\mathbb{C}}\otimes R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k)) &= b^{-1}\Omega_k'(F_{sp}^s(U_{\mathbb{C}}\otimes R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k)) \\ &= b^{-1}\xi'(k)(F^r(\mathfrak{u}_{\mathbb{C}}))\cdot F_{sp}^s(U_{\mathbb{C}}\otimes R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k) \subset b^{-1}F_{sp}^{r+s}(U_{\mathbb{C}}\otimes R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k) \\ &= F^{r+s}(U_{\mathbb{C}}\otimes R_{\mathbb{C}}/\mathfrak{m}_{R_{\mathbb{C}}}^k) \end{split}$$

and hence the second claim follows. Thus the proposition follows.

Thus we can construct the \mathbb{R} -VMHSs associated with mixed Hodge (T, \mathfrak{u}) -representations $(U \otimes R/\mathfrak{m}_R^k, W_*, F^*, \Omega_k)$. The idea of this construction is inspired by Eyssidieux-Simpson's work in [6]. In [6, Theorem 3.15], Eyssidieux and Simpson construct \mathbb{R} -VMHSs starting from an \mathbb{R} -VHS \mathbf{E} , by using Goldman-Millson's theory in [9] and [10]. The construction of this section is very similar to Eyssidieux-Simpson's construction. They also use the \mathbb{R} -split \mathbb{R} -mixed Hodge structure on

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$$\bigoplus_{i=1}^{k-1} S^{i} \mathcal{H}^{1}(M, \operatorname{End}(\mathbf{E}))^{*} / S^{i-2} \mathcal{H}^{1}(M, \operatorname{End}(\mathbf{E}))^{*} \cdot I_{2}$$

(see [6, Subsection 2.3]). But they do not use 1-minimal model and it is not clear that the two constructions are same. This matter is left for future work.

10. Unipotent VMHS without base points

The main statement of this section is the following.

Theorem 10.0.1. On a compact Kähler manifold M, any unipotent \mathbb{R} -VMHS over M is isomorphic to the \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ associated with a mixed Hodge \mathfrak{u} -representation \mathfrak{V} as in Theorem (Prototype).

This result may be a counterpart of the construction of mixed Hodge representations of the fundamental group from unipotent VMHSs as in [14]. On the construction of Hain and Zucker in [14], the monodromy representation associated with a base point $x \in M$ play a central role. Certainly, the monodromy representation of the fiber at a base point is a very useful method for studying flat bundles. But, we consider alternative techniques for studying nilpotent flat bundles without using base points. In fact, for proving Theorem 10.0.1, we need "global trivializations" of unipotent \mathbb{R} -VMHSs but we never take fibers of them.

10.1. Algebraic model for graded nilpotent flat bundles. Let M be a connected manifold. Let \mathbf{E} be a flat bundle over M and \mathbf{W}_* a increasing filtration of the flat bundle \mathbf{E} such that each $Gr_k^{\mathbf{W}}(\mathbf{E})$ is the trivial local system. We fix a global flat frame of each $Gr_k^{\mathbf{W}}(\mathbf{E})$. By splittings of

$$0 \longrightarrow \mathbf{W}_{k-1}(\mathbf{E}) \longrightarrow \mathbf{W}_k(\mathbf{E}) \longrightarrow Gr_k^{\mathbf{W}}(\mathbf{E}) \longrightarrow 0 ,$$

we obtain a global \mathcal{C}^{∞} -frame $e_1^{m_0},\ldots,e_{i_{m_0}}^{m_0},e_1^{m_0+1},\ldots,e_{i_{m_0+1}}^{m_0+1},\ldots,e_1^{m_1},\ldots,e_{i_{m_1}}^{m_1}$ of \mathbf{E} such that $e_1^{m_0},\ldots,e_{i_k}^k$ is a global \mathcal{C}^{∞} -frame of $\mathbf{W}_k(\mathbf{E})$ and each $e_1^k,\ldots,e_{i_k}^k$ induces the fixed global flat frame of $Gr_k^{\mathbf{W}}(\mathbf{E})$. By this global \mathcal{C}^{∞} -frame, the flat connection on

E is represented by a connection form $\omega \in A^*(M) \otimes W_{-1}(\operatorname{End}(V))$ where $V = \langle e_i^k \rangle_{i,k}$ with a filtration $W_k(V) = \langle e_1^{m_0}, \dots, e_{i_k}^k \rangle$. (We do not regard V as any fiber of **E**.) For another such global frame $f_1^{m_0}, \dots, f_{i_{m_0}}^{m_0}, f_1^{m_0+1}, \dots, f_{i_{m_0+1}}^{m_0+1}, \dots, f_1^{m_1}, \dots, f_{i_{m_1}}^{m_1}$ of **E**, for the representation ω' , we have the gauge equivalence $a \in \operatorname{Id}_{|V|} + A^0(M) \otimes W_{-1}(\operatorname{End}(V))$ such that

$$a^{-1}da + a^{-1}\omega a = \omega'.$$

We will control these flat bundles by the following way.

Let A^* be a cohomologically connected DGA over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and (V, W_*) a finite-dimensional \mathbb{K} -vector space with an increasing filtration W_* . We define the category $F^{nil}(A^*, V, W_*)$ such that objects are $\omega \in A^1 \otimes W_{-1}(\operatorname{End}(V))$ satisfying the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

and for $\omega_1, \omega_2 \in \text{Ob}(F^{nil}(A^*, V, W_*))$, morphisms from ω_1 to ω_2 are $a \in A^0 \otimes W_0(\text{End}(V))$ satisfying

$$da + \omega_1 a - a\omega_2 = 0.$$

For $\omega_1, \omega_2 \in \text{Ob}(F^{nil}(A^*, V, W_*))$, we define the differential d_{ω_1, ω_2} on $A^* \otimes W_0(\text{End}(V))$ so that for $\eta \in A^r \otimes W_0(\text{End}(V))$,

$$d_{\omega_1,\omega_2}\eta = d\eta + \omega_1\eta - (-1)^r\eta\omega_2.$$

Then the set of morphisms from ω_1 to ω_2 is identified with the 0-th cohomology of the complex

$$(A^* \otimes W_0(\operatorname{End}(V)), d_{\omega_1, \omega_2}).$$

This complex admits the filtration

$$A^* \otimes W_0(\operatorname{End}(V)) \supset A^* \otimes W_{-1}(\operatorname{End}(V)) \supset A^* \otimes W_{-2}(\operatorname{End}(V)) \supset \dots$$

and the d_{ω_1,ω_2} induces the differential $d \otimes \mathrm{id}$ on

$$Gr_{-k}^W(A^* \otimes W_0(\text{End}(V))) = A^* \otimes Gr_{-k}^W(\text{End}(V)).$$

We also consider the category $F_{qr}^{nil}(A^*, V, W_*)$ such that

$$Ob(F_{gr}^{nil}(A^*, V, W_*)) = Ob(F^{nil}(A^*, V, W_*))$$

and for $\omega_1, \omega_2 \in \text{Ob}(F^{nil}_{gr}(A^*, V, W_*))$, morphisms from ω_1 to ω_2 are $a \in \text{Id}_{|V} + A^0 \otimes W_{-1}(\text{End}(V))$ satisfying

$$da + \omega_1 a - a\omega_2 = 0.$$

This category is a groupoid i.e. all morphisms are isomorphisms.

Let $\phi: A_1^* \to A_2^*$ be a morphism between DGAs. Then this induces the functors $F(\phi): F^{nil}(A_1^*, V, W_*) \to F^{nil}(A_2^*, V, W_*)$ and $F_{gr}(\phi): F^{nil}_{gr}(A_1^*, V, W_*) \to F^{nil}_{qr}(A_2^*, V, W_*)$.

Proposition 10.1.1. Suppose that $\phi: A_1^* \to A_2^*$ induces isomorphisms on 0th and first cohomologies. Then the functors $F(\phi): F^{nil}(A_1^*, V, W_*) \to F^{nil}(A_2^*, V, W_*)$ and $F_{gr}(\phi): F^{nil}_{gr}(A_1^*, V, W_*) \to F^{nil}_{gr}(A_2^*, V, W_*)$ are fully-faithful.

Proof. On the functor $F(\phi): F^{nil}(A_1^*, V, W_*) \to F^{nil}(A_2^*, V, W_*)$, it is sufficient to prove that the morphism

$$\phi: (A_1^* \otimes W_0(\operatorname{End}(V)), d_{\omega_1, \omega_2}) \to (A_2^* \otimes W_0(\operatorname{End}(V)), d_{\phi(\omega_1), \phi(\omega_2)})$$

induces an isomorphism on 0th cohomology. This is easily proved by the five-lemma on long-exact sequences of extensions $A_i^* \otimes W_{-k}(\operatorname{End}(V)) \supset A_i^* \otimes W_{-k-1}(\operatorname{End}(V))$ for i = 1, 2.

We can easily check that the map $F_{gr}(\phi)(\omega_1, \omega_2)$: $\operatorname{Hom}(\omega_1, \omega_2) \to \operatorname{Hom}(\phi(\omega_1), \phi(\omega_2))$ is the restriction of the above isomorphism and this is also an isomorphism.

Suppose that $\phi: A_1^* \to A_2^*$ induces isomorphisms on the 0th and first cohomologies and an injection on the second cohomology. We take a splitting $V = \bigoplus V_i$ of vector space such that $W_k(V) = \bigoplus_{i \leq k} V_i$. Corresponding to this splitting, we have the splitting $\operatorname{End}(V) = \bigoplus U_i$ and $U_iU_j \subset U_{i+j}$. For $\omega \in Ob(F^{nil}(A_2^*, V, W_*))$, we write $\omega = \sum \omega_i$ with $\omega_i \in A_2^1 \otimes U_i$. By the Maurer-Cartan equation, for each k, we have

$$d\omega_k = -\sum_{i+j=k} \omega_i \wedge \omega_j.$$

We denote $a_0 = \mathrm{Id} \in \mathrm{End}(V)$.

Lemma 10.1.2. For all positive integers i, there exist $\Omega_i \in A_1^1 \otimes U_i$ and $a_i \in A_2^0 \otimes U_i$ such that

$$d\Omega_k = -\sum_{i+j=k} \Omega_i \wedge \Omega_j$$

and

$$da_k = \sum_{i+j=k} (-\omega_i a_j + a_i \phi(\Omega_j)).$$

Proof. We prove the lemma inductively. Since $\phi: A_1^* \to A_2^*$ induces an isomorphism on the first cohomology, by $d\omega_1 = 0$, we have $\Omega_i \in A_1^1 \otimes U_1$ and $a_i \in A_2^0 \otimes U_1$ so that

$$da_1 = -\omega_1 + \phi(\Omega_1).$$

We suppose that for all $i \leq k-1$, we have $\Omega_i \in A_1^1 \otimes U_i$ and $a_i \in A_2^0 \otimes U_i$. Then by the equations on Ω_i and a_i for $i \leq k-1$ as in the statement, we can easily check

$$d\left(-\sum_{i+j=k}\omega_i a_j + \sum_{i+j=k, i \ge 1} a_i \phi(\Omega_j)\right) = \sum_{i+j=k} \phi(\Omega_i) \wedge \phi(\Omega_j)$$

and

$$d\left(\sum_{i+j=k}\Omega_i\wedge\Omega_j\right)=0.$$

Since $\phi: A_1^* \to A_2^*$ induces an injection on the second cohomology, we have $\Omega_k \in A_1^1 \otimes U_k$ so that

$$d\Omega_k = -\sum_{i+j=k} \Omega_i \wedge \Omega_j.$$

We obtain

$$d\left(-\sum_{i+j=k}\omega_i a_j + \sum_{i+j=k}a_i\phi(\Omega_j)\right) = d\left(-\sum_{i+j=k}\omega_i a_j + \sum_{i+j=k,i\geq 1}a_i\phi(\Omega_j)\right) + d\phi(\Omega_k) = 0.$$

Since $\phi: A_1^* \to A_2^*$ induces an isomorphism on the first cohomology, we can take Ω_k such that

$$da_k = -\sum_{i+j=k} \omega_i a_j + \sum_{i+j=k} a_i \phi(\Omega_j)$$

for some $a_k \in A_2^0 \otimes U_k$. Thus the lemma follows.

Let $\Omega = \sum \Omega_i \in A_1^1 \otimes W_{-1}(\operatorname{End}(V))$ and $a = \sum a_i \in \operatorname{Id} + A_2^0 \otimes W_{-1}(\operatorname{End}(V))$. We obtain the equations

$$d\Omega - \Omega \wedge \Omega = 0$$

and

$$da + \omega a - a\phi(\Omega) = 0.$$

Thus, by Proposition 10.1.1, we obtain the following statement.

Corollary 10.1.3. If $\phi: A_1^* \to A_2^*$ induces isomorphisms on 0th and first cohomologies and an injection on the second cohomology, then the functor $F_{gr}(\phi): F_{qr}^{nil}(A_1^*, V, W_*) \to F_{qr}^{nil}(A_2^*, V, W_*)$ is an equivalence.

Let M be a connected manifold. Let \mathcal{M} be the 1-minimal model of the DGA $A^*(M)$ and $\phi: \mathcal{M}^* \to A^*(M)$ a map which induces isomorphisms on 0th and first cohomologies and an injection on the second cohomology. Then, for the dual Lie algebra \mathfrak{u} of \mathcal{M} , each object in $F_{gr}^{nil}(\mathcal{M}^*, V, W_*)$ is regarded as a nilpotent representation $\mathfrak{u} \to W_{-1}(\operatorname{End}(V))$ by the Maurer-Cartan equation. By Lemma 10.1.2, for a connection form ω of a filtered nilpotent local system $(\mathbf{E}, \mathbf{W}_*)$ as above we obtain $\Omega \in \operatorname{Ob} F_{gr}^{nil}(\mathcal{M}^*, V, W_*)$ and hence a nilpotent representation $\mathfrak{u} \to W_{-1}(\operatorname{End}(V))$. This construction works on unipotent VMHSs like the monodromy of local systems in the work of Hain and Zucker ([14]).

10.2. **Unipotent VMHS.** Let $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ be an \mathbb{R} -VMHS. We assume that this is unipotent i.e. the \mathbb{R} -VHS on each $Gr_k^{\mathbf{W}}(\mathbf{E})$ is constant. Then $(\mathbf{E}, \mathbf{W}_*)$ is a local system as in the last subsection. We fix a global flat frame of $Gr_k^{\mathbf{W}}(\mathbf{E})$. We take a global \mathcal{C}^{∞} -frame $e_1^{m_0}, \ldots, e_{i_{m_0}}^{m_0}, e_1^{m_0+1}, \ldots, e_{i_{m_0+1}}^{m_0+1}, \ldots, e_{i_{m_1}}^{m_1}$ of \mathbf{E} such that $e_1^{m_0}, \ldots, e_{i_k}^k$ is a global \mathcal{C}^{∞} -frame of $\mathbf{W}_k(\mathbf{E})$ and each $e_1^k, \ldots, e_{i_k}^k$ induces the fixed global flat frame of $Gr_k^{\mathbf{W}}(\mathbf{E})$. Let $\omega \in A^*(M) \otimes W_{-1}(\mathrm{End}(V))$ be the connection form for this frame where $V = \langle e_i^k \rangle_{i,k}$.

We consider the complex \mathcal{C}^{∞} -vector bundle $\mathbf{E}_{\mathbb{C}}$ and take a splitting $\mathbf{E}_{\mathbb{C}} = \bigoplus \mathbf{E}^{k,r}$ such that

$$\mathbf{E}^{r,k} \cong Gr_r^{\mathbf{F}}Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathbb{C}})$$

as a \mathcal{C}^{∞} -vector bundle. By the assumption, each quotient

$$Gr_r^{\mathbf{F}}Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathbb{C}})$$

is a trivial flat bundle. Thus, we take a global \mathcal{C}^{∞} -frame $f_1^{r,k},\ldots,f_{i_{r,k}}^{r,k}$ of $\mathbf{E}^{k,r}$ which induces a global flat frame of $Gr_r^{\mathbf{F}}Gr_k^{\mathbf{W}}(\mathbf{E}_{\mathbb{C}})$. We take a global \mathcal{C}^{∞} -frame $f_1^{m_0},\ldots,f_{i_{m_0}}^{m_0},f_1^{m_0+1},\ldots,f_{i_{m_0+1}}^{m_0+1},\ldots,f_{i_{m_1}}^{m_1}$ of $\mathbf{E}_{\mathbb{C}}$ such that $f_1^{m_0},\ldots,f_{i_k}^k$ is a global \mathcal{C}^{∞} -frame of $\mathbf{W}_k(\mathbf{E})$, each $f_1^k,\ldots,f_{i_k}^k$ induces the fixed global flat frame of $Gr_k^{\mathbf{W}}(\mathbf{E})$ and each f_i^k is a linear combination of $\{f_i^{r,k}\}_{r,i}$. Let $V'=\langle f_i^k\rangle_{i,k}$ and $F^p(V')=\langle f_i^{r,k}\rangle_{r\geq p}$. Then, for the connection form $\omega'\in A^*(M)\otimes\mathbb{C}\otimes W_{-1}(\mathrm{End}(V'))$ for this frame, the condition of holomorphicity of \mathbf{F} and the Griffiths transversality is equivalent to the condition

$$\omega' \in F^0 \left(A^*(M) \otimes \mathbb{C} \otimes W_{-1}(\operatorname{End}(V')) \right).$$

By the identification $V_{\mathbb{C}} = V'$, we obtain an \mathbb{R} -mixed Hodge structure (V, W_*, F^*) . We remark that this \mathbb{R} -mixed Hodge structure (V, W_*, F^*) can not be considered as any fiber of $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ and can not be determined uniquely.

By these observations and the arguments in the last subsection, for considering unipotent \mathbb{R} -VMHSs without a base point, we would like to study the following category. Let V be a finite-dimensional \mathbb{R} -vector space with an \mathbb{R} -mixed Hodge structure (W_*, F^*) . We define the category $VMHS^{unip}_{\mathbb{R}}(M, V, W_*, F^*)$ by the following ways:

- Objects are (ω, ω', a) so that:
 - $-\omega \in \mathrm{Ob}(F^{nil}(A^*(M), V, W_*)).$
 - $-\omega' \in \mathrm{Ob}(F^{nil}(A^*(M) \otimes \mathbb{C}, V_{\mathbb{C}}, W_*))$ satisfying

$$\omega' \in F^0 \left(A^*(M) \otimes \mathbb{C} \otimes W_{-1}(\operatorname{End}(V_{\mathbb{C}})) \right).$$

- $-a \in \operatorname{Hom}(\omega, \omega')$ in the category $F_{gr}^{nil}(A^*(M) \otimes \mathbb{C}, V_{\mathbb{C}}, W_*)$.
- For $(\omega_1, \omega_1', a_1), (\omega_2, \omega_2', a_2) \in \text{Ob}(VMHS^{unip}_{\mathbb{R}}(M, V, W_*, F^*))$, morphisms from the first one to the second one are (b, b') so that
 - $-b \in \operatorname{Hom}(\omega_1, \omega_2).$
 - $-b' \in \text{Hom}(\omega_1', \omega_2')$ satisfying

$$b' \in F^0 (A^*(M) \otimes \mathbb{C} \otimes W_0(\text{End}(V)))$$
.

$$-b'\circ a_1=a_2\circ b.$$

For $(\omega, \omega', a) \in \text{Ob}(VMHS^{unip}_{\mathbb{R}}(M, V, W_*, F^*))$, define **E** by the trivial \mathcal{C}^{∞} -vector bundle $M \times V$ with the flat connection $d + \omega$, \mathbf{W}_* by the filtration of the vector bundle $M \times V$ induced by the weight filtration W_* on V and \mathbf{F}^* by the filtration of the vector bundle $M \times V$ induced by the Hodge filtration F^* on V. Then, $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ is in fact a unipotent \mathbb{R} -VMHS.

Suppose that M admits a Kähler metric g. We consider the canonical 1-minimal models $\phi: \mathcal{M}^* \to A^*(M)$ and $\varphi: \mathcal{N}^* \to A^*(M) \otimes \mathbb{C}$ with the trivial ρ and the isomorphism $\mathcal{I}: \mathcal{M}^*_{\mathbb{C}} \to \mathcal{N}$ with the homotopy H from $\varphi \circ \mathcal{I}$ to ϕ as in Section 6. Let $(\omega, \omega', a) \in \mathrm{Ob}(VMHS^{unip}_{\mathbb{R}}(M, V, W_*, F^*))$. Then, by Lemma 10.1.3, we can take $\Omega \in \mathrm{Ob}(F^{nil}(\mathcal{M}^*, V, W_*))$ and $b \in \mathrm{Id} + A^0(M) \otimes W_{-1}(\mathrm{End}(V))$ such that ω is isomorphic to $\Omega_{\phi} = \phi(\Omega)$ via b. We can also take $\Omega' \in \mathrm{Ob}(F^{nil}(\mathcal{N}^*, V, W_*))$ and $b' \in \mathrm{Id} + A^0(M) \otimes \mathbb{C} \otimes W_{-1}(\mathrm{End}(V_{\mathbb{C}}))$ such that ω' is isomorphic to $\Omega'_{\varphi} = \varphi(\Omega')$ via b'.

Lemma 10.2.1. We can choose

$$\Omega' \in F^0 \left(\mathcal{N}^* \otimes W_{-1}(\operatorname{End}(V_{\mathbb{C}})) \right)$$

and

$$b' \in F^0 (A^*(M) \otimes \mathbb{C} \otimes W_0(\text{End}(V)))$$
.

Proof. Consider the bigrading $V_{\mathbb{C}} = \bigoplus V^{p,q}$ of the \mathbb{R} -mixed Hodge structure (W_*, F^*) . Then, by the decoposition $V_{\mathbb{C}} = \bigoplus_k \left(\bigoplus_{p+q=k} V^{p,q} \right)$, taking $\omega' = \sum \omega'_k$ as in the last subsection, we can construct $\Omega' = \sum \Omega'_i$ and $b' = \sum b'_i$ as in Lemma 10.1.3. By

$$\omega' \in F^0 \left(A^1(M) \otimes \mathbb{C} \otimes W_{-1}(\operatorname{End}(V_{\mathbb{C}})) \right),$$

for each k, we have

$$\omega'_k \in F^0\left(A^1(M) \otimes \mathbb{C} \otimes W_{-k}(\operatorname{End}(V_{\mathbb{C}}))\right).$$

It is sufficient to prove that we can take

$$\Omega'_k \in F^0\left(\mathcal{N}^1 \otimes W_{-1}(\mathrm{End}(V_{\mathbb{C}}))\right)$$

and

$$b'_k \in F^0\left(A^0(M) \otimes \mathbb{C} \otimes W_{-k}(\mathrm{End}(V))\right).$$

We prove inductively. By the construction in the proof of Lemma 10.1.3, Ω'_1 is the harmonic representative of ω'_1 . Thus, by

$$\omega_1' \in F^0\left(A^*(M) \otimes \mathbb{C} \otimes W_{-1}(\mathrm{End}(V_{\mathbb{C}}))\right),$$

we have $\Omega_1' \in F^0(A^*(M) \otimes \mathbb{C} \otimes W_{-1}(\operatorname{End}(V_{\mathbb{C}})))$. By the standard argument of Hodge theory, we can take $b_1' \in F^0(A^0(M) \otimes \mathbb{C} \otimes W_{-1}(\operatorname{End}(V)))$ such that

$$db_1' = -\omega_1' + \varphi(\Omega_1').$$

We assume that for $i \leq k-1$ we have taken

$$\Omega_i' \in F^0 \left(\mathcal{N}^* \otimes W_{-i}(\operatorname{End}(V_{\mathbb{C}})) \right)$$

and

$$b_i' \in F^0(A^*(M) \otimes \mathbb{C} \otimes W_{-i}(\text{End}(V)))$$
.

By

$$d\Omega'_k = -\sum_{i+j=k} \Omega'_i \wedge \Omega'_j,$$

since the differential on \mathcal{N}^* is stricktly compatible with the filtration F^* , we obtain $\Omega'_k \in F^0(\mathcal{N}^* \otimes W_{-k}(\operatorname{End}(V_{\mathbb{C}})))$. Since $\varphi : \mathcal{N}^* \to A^*(M) \otimes \mathbb{C}$ is compatible with the filtration F^* ,

$$-\sum_{i+j=k}\omega_i'b_j'+\sum_{i+j=k}b_i'\phi(\Omega_j')\in F^0\left(A^0(M)\otimes\mathbb{C}\otimes W_{-k}(\mathrm{End}(V))\right)$$

and we can also take $b_k' \in F^0\left(A^0(M) \otimes \mathbb{C} \otimes W_{-k}(\mathrm{End}(V))\right)$ such that

$$db'_k = -\sum_{i+j=k} \omega'_i b'_j + \sum_{i+j=k} b'_i \phi(\Omega'_j).$$

Hence the lemma follows.

Define $\Omega_{\varphi} = \varphi(\mathcal{I}(\Omega))$. Then, as in Proposition 7.2.3, we obtain the isomorphism c from Ω_{ϕ} to Ω_{φ} in the category $F^{nil}_{gr}(A^*(M) \otimes \mathbb{C}, V_{\mathbb{C}}, W_*)$ which is determined by $H(\Omega)$. Now we have the isomorphism $b: \omega \to \Omega_{\phi}$ in the category $F^{nil}_{gr}(A^*(M), V, W_*)$ and the isomorphisms $a: \omega \to \omega', b': \omega' \to \Omega'_{\varphi}$ and $c: \Omega_{\phi} \to \Omega_{\varphi}$ in the category $F^{nil}_{gr}(A^*(M) \otimes \mathbb{C}, V_{\mathbb{C}}, W_*)$. Denote $c' = c^{-1}b^{-1}ab'$. Then, by Lemma 10.2.1, (ω, ω', a) is isomorphic to $(\Omega_{\phi}, \Omega'_{\varphi}, cc')$ in the category $VMHS^{unip}_{\mathbb{R}}(M, V, W_*, F^*)$ via (b, b').

By Proposition 10.1.1 and $\mathcal{N}^0 = \mathbb{C}$, we have $c' \in \mathrm{Id}_{|V_{\mathbb{C}}} + W_{-1}(\mathrm{End}(V_{\mathbb{C}}))$ and

$$c'^{-1}\mathcal{I}(\Omega)c' = \Omega'.$$

By Lemma 10.2.1, $\mathfrak{V} = (V, W_*, c'F^*, \Omega)$ is a mixed Hodge \mathfrak{u} -representation. We can easily check that the \mathbb{R} -VMHS $(\mathbf{E}_{\mathfrak{V}}, \mathbf{W}_{\mathfrak{V}*}, \mathbf{F}_{\mathfrak{V}}^*)$ associated with $\mathfrak{V} = (V, W_*, c'F^*, \Omega)$ as in Theorem (Prototype) corresponds to $(\Omega_{\phi}, \Omega_{\varphi}', cc') \in \mathrm{Ob}(VMHS_{\mathbb{R}}^{unip}(M, V, W_*, F^*))$. Thus we obtain Theorem 10.0.1.

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