NON-PARAMETRIC INVERSE CURVATURE FLOWS IN THE ADS-SCHWARZSCHILD MANIFOLD

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ABSTRACT. We consider the inverse curvature flows in the anti-de Sitter-Schwarzschild manifold with star-shaped initial hypersurface, driven by the 1-homogeneous curvature function. We show that the solutions exist for all time and the principle curvatures of the hypersurface converges to 1 exponentially fast.

Keywords: Inverse curvature flows, AdS-Schwarzschild manifold, homogeneous curvature function. **MSC**: Primary 58E20, Secondary 35J35.

1. Introduction

During the past decades, geometric flows have been studied intensively. Following the ground breaking work of Huisken [15], who considered the mean curvature flow, several authors started to investigate inverse, or expanding curvature flows of star-shaped closed hypersurfaces in ambient spaces of constant or asymptotically constant sectional curvature. Gerhardt [7] and Urbas [24] independently considered flows of the form

$$\frac{d}{dt}X = \frac{1}{F}\nu$$

in \mathbb{R}^{n+1} , where F is a curvature function homogeneous of degree 1, and proved that the flow exists for all time and converges to infinity. After a proper rescaling, the rescaled flow will converge to a sphere.

The equation (1.1) has the property that it is scale-invariant which seems to be the underlying reason why expanding curvature flows in Euclidean space do not develop singularities contrary to contracting curvature flows which will contract to a point in finite time (see [15]). Similar convergence results for inverse curvature flows in the hyperbolic space were estimated by Ding [1] and Gerhardt [8], and in the sphere by Gerhardt [11] and Makowski-Scheuer [18]. In [1], Ding also get similar results in rotationally symmetric spaces of Euclidean volume growth except the hyperbolic space. Compared with scale-invariant flows, there may be some difference for non-scale-invariant inverse curvature flows (see [25], [12] and [22]).

It is a natural question, whether one can prove long-time existence and the flow hypersurfaces become umbilic as in case of more general ambient spaces. Recently, Brendle-Hung-Wang [2] investigated the inverse mean curvature flow (IMCF for short) in anti-de Sitter-Schwarzschild manifold which is asymptotically hyperbolic at the infinity, and applied the convergence result to prove a sharp Minkowski inequality for strictly mean convex and star-shaped hypersurface in anti-de Sitter-Schwarzschild manifold. Similar applications can be found in the works [4] and [16], in which the IMCF was used to prove a Minkowski type inequality in the anti-de Sitter-Schwarzschild manifold and in the Schwarzschild manifold respectively. Other geometric inequalities, e.g., Aleksandrov-Fenchel inequalities in hyperbolic space as in [5, 6] have been

This research was supported in part by the National Natural Science Foundation of China (11201131,11401131) and Hubei Key Laboratory of Applied Mathematics (Hubei University).

proven also using inverse 1-homogeneous curvature flows [8] (also compare with [18], where additional isoperimetric type problems have been treated).

In the present work, we investigate the convergence of the flow (1.1) in some asymptotically hyperbolic space. More precisely, we consider the convergence of the flow (1.1) in anti-de Sitter-Schwarzschild manifold which is asymptotically hyperbolic at the infinity. Recently, Lu [17] considered the inverse hessian quotient curvature flow with star-shaped initial hypersurface in the anti-de Sitter-Schwarzschild manifold and proved that the solution exists for all time, and the second fundamental form converges to identity exponentially fast.

Let us first recall the definition of the anti-de Sitter-Schwarzschild manifold (see also [2]). Fixed a real number m > 0, and let s_0 denote the unique positive solution of the equation $1 + s_0 - ms_0^{1-n} = 0$. The anti-de Sitter-Schwarzschild manifold is an (n + 1)-dimensional manifold $M = [s_0, +\infty) \times \mathbb{S}^n$ equipped with the Riemannian metric

$$\overline{g} = \frac{1}{1 - ms^{1-n} + s^2} ds \otimes ds + s^2 g_{\mathbb{S}^n},$$

where $g_{\mathbb{S}^n}$ is the standard round metric on the unit sphere \mathbb{S}^n . Clearly, \overline{g} is asymptotically hyperbolic, since the sectional curvatures of (M, \overline{g}) approach -1 near infinity.

The anti-de Sitter-Schwarzschild manifold are examples of the static spaces. If we define $f = \sqrt{1 - ms^{1-n} + s^2}$, then it satisfies the equation

$$(\overline{\Delta}f)\overline{g} - \overline{\nabla}^2 f + fRic = 0.$$

In general, a Riemannian metric is called static if it satisfies (1.2) for some positive function f. The condition (1.2) guarantees the Lorentzian warped product $-f^2dt \otimes dt + \overline{g}$ is a solution of the Einstein equation.

In order to formulate the main result, we need a definition below (see also [22]).

Definition 1.1. Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone and $F \in C^{\infty}(\Gamma)$ be a symmetric function. A hypersurface Σ_0 in the anti-de Sitter-Schwarzschild manifold (M, \overline{g}) is called F-admissable, if at any point $x \in \Sigma_0$ the principal curvatures of Σ_0 , κ_1 , ..., κ_n , are contained in the cone Γ .

We mainly get the following result

Theorem 1.2. Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone that satisfies

$$\Gamma_+ = \{(\kappa_i) \in \mathbb{R}^n : \kappa_i > 0, \ \forall 1 \le i \le n\} \subset \Gamma$$

and $F \in C^{\infty}(\Gamma) \cap C^{0}(\overline{\Gamma})$ be a monotone, 1-homogeneous and concave curvature function, such that

$$F|_{\Gamma} > 0$$
 and $F|_{\partial \Gamma} = 0$.

We usually normalized F such that

$$F(1,...,1) = n.$$

Let Σ_0 be a smooth, star-shaped and F-admissable embedded closed hypersurface in AdS-Schwarzschild manifold (M, \overline{g}) , and Σ_0 can be written as a graph over a geodesic sphere \mathbb{S}^n ,

$$\Sigma_0 = \operatorname{graph} r(0,.).$$

Then

(1) There is a unique smooth curvature flow

$$X:[0,\infty)\times\Sigma\to M$$
,

which satisfies the flow equation

(1.3)
$$\begin{cases} \frac{d}{dt}X = \frac{1}{F}\nu, \\ X(0) = \Sigma_0. \end{cases}$$

where $\nu(t,\xi)$ is the outward normal to $\Sigma_t = X(t,\Sigma)$ at X(t,x), F is evaluated at the principle curvatures of Σ_t at $X(t,\xi)$ and the leaves Σ_t are graphs over \mathbb{S}^n ,

$$\Sigma_t = \text{graph } r(t,.).$$

(2) The leaves Σ_t become more and more umbilic, namely

$$\mid h_j^i - \delta_j^i \mid \le C e^{-\frac{2t}{n}}.$$

(3) Furthermore, the function

$$\widetilde{r}(t,\theta) = r(t,\theta) - \frac{t}{n}$$

converges to a well defined function $f(\theta) \in C^2(\mathbb{S}^n)$ in $C^{2,\alpha}$ as $t \to +\infty$, which implies that the limit of the rescaled induced metric of Σ_t is the conformal metric $e^{2f}g_{\mathbb{S}^n}$ on \mathbb{S}^n , where $g_{\mathbb{S}^n}$ is the round metric \mathbb{S}^n .

Remark 1.1. Similar to [14] and [19], in general, the function $f(\theta)$ in Theorem 1.2 may not be constant in the sense that the limit shape of the rescaled flow hypersurfaces does not have to be a round sphere.

The main techniques employed here were from [8] and later were developed by Scheuer in [22].

Acknowledgement: The authors would like to express gratitude to Professor Guofang Wang for some suggestive comments and they also thank Dr. Hengyu Zhou for pointing out the lost curvature terms in the Codazzi equation.

2. Graphic hypersurfaces in the AdS-Schwarzschild manifold and a reformulation of the problem

First, we state some general facts about the AdS-Schwarzschild manifold and the graphic hypersurfaces in it. We basically follow the description in [2, Section 2]. Denote the AdS-Schwarzschild manifold by (M, \overline{g}) and $\overline{\nabla}$ by the Levi-Civata connection with respect to the metric \overline{g} . By a change of variable, the AdS-Schwarzschild metric can be rewritten as

$$\overline{g} = dr \otimes dr + \lambda(r)^2 g_{\mathbb{S}^n},$$

where $\lambda(r)$ satisfies the ODE

(2.1)
$$\lambda'(r) = \sqrt{1 + \lambda^2 - m\lambda^{1-n}}$$

and the asymptotic expansion

(2.2)
$$\lambda(r) = \sinh(r) + \frac{m}{2(n+1)} \sinh^{-n}(r) + O(\sinh^{-n-2}(r)).$$

We can calculate the asymptotic expansion of Riemannian curvature tensors. Let e_{α} , $\alpha = 1, 2, ..., n + 1$, be an orthonormal frame and $\overline{R}_{\alpha\beta\gamma\mu}$ denote the Riemannian curvature tensor of the AdS-Schwarzschild metric. Then

(2.3)
$$\overline{R}_{\alpha\beta\gamma\mu} = -\delta_{\beta\mu}\delta_{\alpha\gamma} + \delta_{\beta\gamma}\beta_{\alpha\mu} + O(e^{-(n+1)r})$$

and

(2.4)
$$\overline{\nabla}_{\rho} \overline{R}_{\alpha\beta\gamma\mu} = O(e^{-(n+1)r}).$$

Since $\Sigma \subset M$ is a graphic hypersurface in M, it can be parametrized by

$$\Sigma = \{ (r(\theta), \theta) : \theta \in \mathbb{S}^n \}$$

for some smooth function r on \mathbb{S}^n . Let $\theta = \{\theta^i\}_{i=1,\dots,n}$ be a local coordinate system on \mathbb{S}^n and let ∂_i be the corresponding coordinate vector fields on \mathbb{S}^n and $\sigma_{ij} = g_{\mathbb{S}^n}(\partial_i, \partial_j)$. Let $\varphi_i = D_i \varphi$, $\varphi_{ij} = D_j D_i \varphi$ and $\varphi_{ijk} = D_k D_j D_i \varphi$ denote the covariant derivatives of φ with respect to the round metric $g_{\mathbb{S}^n}$ and ∇ be the Levi-Civata connection of Σ with respect to the induced metric g from (M, \overline{g}) . Set $X = (r(\theta), \theta)$, the tangential vectors on Σ take the form

$$X_i = \partial_i + D_i r \partial_r.$$

The induced metric on Σ is

$$g_{ij} = r_i r_j + \lambda^2 \sigma_{ij},$$

and the outward unit normal vector of Σ

$$\nu = \frac{1}{v} \left(\partial_r - \lambda^{-2} D^j r \partial_j \right).$$

Define a new function $\varphi: \mathbb{S}^n \to \mathbb{R}$ by

(2.5)
$$\varphi(\theta) = \int_{c}^{r(\theta)} \frac{1}{\lambda(s)} ds.$$

Then the induced metric on Σ takes the form

$$g_{ij} = \lambda^2 (\varphi_i \varphi_j + \sigma_{ij})$$

with the inverse

$$g^{ij} = \lambda^{-2} (\sigma^{ij} - \frac{\varphi^i \varphi^j}{2!^2}),$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$, $\varphi^i = \sigma^{ij}\varphi_j$ and

(2.6)
$$v^{2} = 1 + \sigma^{ij}\varphi_{i}\varphi_{j} \equiv 1 + |D\varphi|^{2} = 1 + \frac{|Dr|^{2}}{\lambda^{2}},$$

|.| is the norm corresponding to the metric $g_{\mathbb{S}^n}$. Let h_{ij} be the second fundamental form of $\Sigma \subset M$ in term of the coordinate θ^i . So

$$h_{ij} = \frac{\lambda}{v} \left(\lambda' (\varphi_i \varphi_j + \sigma_{ij}) - \varphi_{ij} \right)$$

and

(2.7)
$$h_j^i = \frac{1}{\lambda v} (\lambda' \delta_j^i - \widetilde{g}^{ik} \varphi_{kj}),$$

where $\widetilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$.

To calculate some curvature terms, we need the following result from Appendix A in [21].

Lemma 2.1.

$$\overline{R}(\partial_i, \partial_j, \partial_k, \partial_l) = \lambda^2 (1 - (\lambda')^2) (\sigma_{ik} \sigma_{jl} - \sigma_{il} \sigma_{jk})$$

$$\overline{R}(\partial_i, \partial_r, \partial_j, \partial_r) = -\lambda \lambda'' \sigma_{ij}$$

Then, we can calculate some curvature terms by using the above lemma.

$$(2.8) \overline{R}(X_i, \nu, X_j, \nu) \equiv \overline{R}_{i\nu j\nu} = -\frac{1}{v^2} \left[\lambda \lambda'' + ((\lambda')^2 - 1) \mid D\varphi \mid^2 \right] \sigma_{ij}$$

$$-\frac{1}{v^2} \left[\frac{2\lambda''}{\lambda} + \frac{1 - (\lambda')^2}{\lambda^2} + \frac{\lambda''}{\lambda} \mid D\varphi \mid^2 \right] r_i r_j$$

and

$$(2.9)\overline{R}(\nu, X_i, X_k, X_j) \equiv \overline{R}_{\nu i k j} = (-\lambda \lambda'' - (1 - (\lambda')^2)) \frac{r_k \sigma_{ij}}{v} + (\lambda \lambda'' + (1 - (\lambda')^2)) \frac{r_j \sigma_{ik}}{v}.$$

Thus,

(2.10)
$$v\overline{R}_{\nu ikj}r^{k} = \frac{1}{v^{2}\lambda^{2}}(\lambda\lambda'' + (1-(\lambda')^{2}))[r_{i}r_{j} - \lambda^{2} \mid D\varphi \mid^{2} \sigma_{ij}]$$

The geodesic spheres S_r in the AdS-Schwarzschild manifold (M, \overline{g}) are totally umbilic, their second fundamental form is given by

$$\overline{h}_{ij} = \overline{h}_{ij}(r) = \frac{\lambda'}{\lambda} \overline{g}_{ij},$$

where

$$\overline{g}_{ij} = \lambda^2 \sigma_{ij}$$
.

Thus $\overline{h}_j^i = \frac{\lambda'}{\lambda} \delta_j^i$, $\kappa_i = \frac{\lambda'}{\lambda}$ and the mean curvature \overline{H} of S_r is given by

$$\overline{H} = \overline{H}(r) = \frac{n\lambda'}{\lambda}.$$

For the evolution of graphic hypersurfaces, we can reform the equation (1.1). Let Σ_0 be a graphic hypersurface in AdS-Schwarzschild manifold (M, \overline{g}) which is given by an embedding

$$X_0: \mathbb{S}^n \to M$$
.

Let $X_t: \mathbb{S}^n \to M$, $t \in [0,T)$, be the solution of inverse curvature flow with initial data given by X_0 . In other word,

$$\frac{\partial X}{\partial t} = \frac{1}{F}\nu,$$

where ν is the outward unit normal vector and F is a monotone, 1-homogeneous and concave curvature function. We shall call (2.11) the parametric form of the flow. We can write the initial hypersurface Σ_0 as the graph of a function r_0 defined on the unit sphere:

$$\Sigma_0 = \{ (r_0(\theta), \theta) : \theta \in \mathbb{S}^n \}.$$

If each Σ_t is graphic, it can be parametrized as follows

$$\Sigma_t = \{ (r(\theta, t), \theta) : \theta \in \mathbb{S}^n \}.$$

Then the evolution equation (2.11) now yields

$$\frac{dr}{dt} = \frac{1}{Fv}$$
 and $\frac{d\theta^i}{dt} = -\frac{D^i r}{\lambda^2 F v}$,

from which we deduce

$$\frac{\partial r}{\partial t} = \frac{v}{F},$$

where v is given by (2.6). Therefore, as long as the solution of (1.3) exists and remain graphic, it is equivalent to a parabolic PDE (2.12) for r. The equation (2.12) is also referred as the non-parametric form of the inverse mean curvature flow. Notice that the velocity vector of (1.3) is always normal, while the velocity vector of (2.12) is in the direction of ∂_r . To go from one to the other, we take the difference which is a (time-dependent) tangential vector field and compose the flow of the reparametrization associated with the tangent vector field.

The proof of the short time existence of the flow (1.3) is standard, see Remark 3.4 in [22] and Remark 2.1 in [23]. For completeness, we describe it easily here. We can get the short time existence of the flow on a maximal interval $[0, T^*)$, $0 < T^* \le \infty$, and

$$X \in C^{\infty}([0, T^*) \times \Sigma, M).$$

Moreover, all the leaves M(t) = X(t, M), $0 \le t < T^*$, are admissable and can be written as graphs over \mathbb{S}^n . Furthermore, the flow X exists as long as the scalar flow (2.12) does, where

$$r: [0, T^*) \times \mathbb{S}^n \longrightarrow \mathbb{R}.$$

Thus, we will mainly investigate the long time existence of (1.3) in the following chapters.

3. THE LONG-TIME EXISTENCE

The proof of the long-time existence of (1.3) is standard which mainly relies on the following C^0 estimates, C^1 estimates and curvature estimates. Before proceeding, we give some notation. Covariant differentiation will usually be denoted by indices, e.g. r_{ij} for a function $r: \Sigma \to \mathbb{R}$, or, if ambiguities are possible, by a semicolon, e.g. $h_{ij;k}$. Usual partial derivatives will be denoted by a comma, e.g. $u_{i,j}$.

C^0 estimates

First, we recall the C^0 estimates whose proof is standard, see Lemma 3.1 in [8] and Section 4 in [1].

Lemma 3.1. The solution r of (2.12) satisfies

(3.1)
$$\lambda(\inf r(0,\cdot)) \le \lambda(r(t,\theta))e^{-\frac{t}{n}} \le \lambda(\sup r(0,\cdot)), \quad \forall \theta \in \mathbb{S}^n, t \in [0,T^*).$$

Remark 3.1. Noticing the asymptotic expansion (2.2) of $\lambda(r)$, we have from the above lemma

(3.2)
$$r(t,\theta) - \frac{t}{n} = o(t).$$

C^1 estimates

To get the C^1 estimate, we using the evolution equation of φ instead of r by noticing the relation (2.2). From (2.12), we get

(3.3)
$$\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda F(h_j^i)} = \frac{v}{F(\lambda h_j^i)} \equiv \frac{v}{F(\widetilde{h}_j^i)}.$$

Let
$$(\widetilde{g}_{ij}) = (\widetilde{g}^{ij})^{-1}$$
, clearly, $g_{ij} = \lambda^2 \widetilde{g}_{ij}$. Defining

$$\widetilde{h}_{ij} = \widetilde{g}_{ik}\widetilde{h}_i^k,$$

we see that in (3.3) we are considering the eigenvalues of h_{ij} with respect to \tilde{g}_{ij} and thus we define

$$F^{ij} = \frac{\partial F}{\partial \widetilde{h}_{ij}}$$
 and $F^i_j = \frac{\partial F}{\partial \widetilde{h}^j_i}$.

By a straightforward computation, it is easy to get the following relations.

Lemma 3.2.

$$\widetilde{h}_{k,i}^{l} = -\frac{v_{i}}{v}\widetilde{h}_{k}^{l} - v^{-1}(\widetilde{g}_{;i}^{lm}\varphi_{mk} + \widetilde{g}^{lm}\varphi_{mki} - \lambda\lambda''D_{i}\varphi\delta_{k}^{l}),$$

$$\widetilde{g}_{;i}^{kl} = \frac{2v_{i}\varphi^{k}\varphi^{l}}{v^{3}} - \frac{1}{v^{2}}\left(\varphi_{i}^{k}\varphi^{l} + \varphi^{k}\varphi_{i}^{l}\right),$$

$$v_i = v^{-1} \varphi_{ki} \varphi^k,$$

where the covariant derivatives as well as index raising are performed with respect to σ_{ij} .

Lemma 3.3. Let φ be a solution of (3.3), we have

$$(3.4) |D\varphi|^2 \le \sup_{\mathbb{S}^n} |D\varphi(0,\cdot)|^2.$$

Moreover, if F is bounded from above $F \leq C$, then there exists $0 < \mu = \mu(C)$ such that

$$(3.5) |D\varphi|^2 \le e^{-\mu t} \sup_{\mathbb{S}^n} |D\varphi(0,\cdot)|^2.$$

Proof. Let

$$w = \frac{1}{2}|D\varphi|^2.$$

By differentiating (3.3) with respect to the operator $D^k \varphi D_k$, we obtain

$$\frac{\partial}{\partial t}w = -\frac{v}{F^2}F_l^k \widetilde{h}_{k;i}^l \varphi^i + \frac{v_i \varphi^i}{F}.$$

Fix $0 < T < T^*$ and suppose

$$\sup_{[0,T]\times\mathbb{S}^n} w = w(t_0, \xi_0), \quad t_0 > 0.$$

Then at (t_0, ξ_0) , there holds

$$0 \leq \frac{\partial}{\partial t} w = -\frac{1}{F^2} \left(-\widetilde{g}^{lm}_{;i} \varphi_{mki} \varphi^i - \widetilde{g}^{lm} \varphi_{mki} + \lambda \lambda'' |D\varphi|^2 \delta_k^l \right) + \frac{2}{v^3} \varphi_{ki} \varphi^k \varphi^i$$
$$= -\frac{2}{F^2} \lambda \lambda'' F^{kl} \widetilde{g}_{kl} w + \frac{1}{F^2} F^{kl} \varphi_{kli} \varphi^i,$$

where we use Lemma 3.2 and the fact $\varphi_{ik}\varphi^i=0$, $\forall k$ at (t_0,ξ_0) . Then, we apply the rule for exchanging derivatives

$$\varphi_{kli} = \varphi_{ikl} + R_{ilkm}\varphi^m$$

and notice the fact on \mathbb{S}^n

$$R_{ilkm} = \sigma_{ik}\sigma_{lm} - \sigma_{im}\sigma_{lk}$$

we can obtain

$$0 \le \frac{\partial}{\partial t} w = \frac{1}{F^2} \left(-2\lambda \lambda'' F^{kl} \widetilde{g}_{kl} w + F^{kl} (\varphi_k \varphi_l - |D\varphi|^2 \sigma_{kl}) + F^{kl} w_{kl} - F^{kl} \varphi_{ik} \varphi_l^i \right) < 0,$$

where we use the assumption that F is a monotone, 1-homogeneous and concave curvature function and $F^{kl}w_{kl} \leq 0$ at (t_0, ξ_0) . Hence, the estimate (3.4) follows by the arbitrariness of T. To prove (3.5), we define

$$\widetilde{w} = we^{-\mu t}$$

where μ is a positive constant which will be chosen later. Then \widetilde{w} satisfies the same equation as w with an additional term $\mu\widetilde{w}$ at the right-hand side. Assume \widetilde{w} attains a positive maximum at a point (t_0, ξ_0) , $t_0 > 0$, by applying the maximum principle as before, there holds

$$(3.6) 0 \le -\frac{2}{F^2} \lambda \lambda'' F^{kl} \widetilde{g}_{kl} \widetilde{w} + \mu \widetilde{w}.$$

Then, since $\frac{\lambda''}{\lambda} = 1 + \frac{1}{2}m(n-1)\lambda^{1-n}$ is bounded by some constant C_1 from Lemma 3.1, $F(\tilde{h}_j^i)\lambda^{-1} = F(h_j^i)$ is bounded from above and $F^{kl}\tilde{g}_{kl} \geq n$, we can obtain

$$we^{\mu t} \le \sup_{\mathbb{S}^n} w(0)$$

for all

$$0 < \mu \le \frac{C_1 n}{C^2}.$$

Remark 3.2. In Theorem 3.13 below, we will estimate the optimal decay rate μ .

Curvature estimates

In this section, for convenience, we let $\Phi = \Phi(F) = -\frac{1}{F}$, $\Phi' = \frac{d\Phi}{dF}$ and

$$\chi = \langle \lambda \frac{\partial}{\partial r}, \nu \rangle = \frac{\lambda}{v}$$

Lemma 3.4. Under the flow (1.3), the following evolution equations hold true

(3.7)
$$\frac{\partial}{\partial t} \Phi - \Phi' F^{ij} \Phi_{ij} = \Phi' F^{ij} h_{ik} h_j^k \Phi + \Phi' F^{ij} \overline{R}_{\nu i \nu j} \Phi,$$

(3.8)
$$\frac{\partial}{\partial t}r - \Phi' F^{ij}r_{ij} = 2\Phi' F v^{-1} - \Phi' F^{ij} \overline{h}_{ij},$$

(3.9)
$$\frac{\partial}{\partial t} \chi - \Phi' F^{ij} \chi_{ij} = \Phi' F^{ij} h_i^k h_{kj} - \Phi' F^{ij} \overline{R} (\nu, X_i, (\lambda \partial_r)^T, X_j),$$

(3.10)
$$\frac{\partial}{\partial t}h_j^i = \Phi_j^i + \Phi h_k^i h_j^k + \Phi \overline{R}_{\nu j\nu k} g^{ki},$$

where $\partial_r = \frac{\partial X}{\partial r}$, $X_i = \frac{\partial X}{\partial \xi^i}$ and $(\lambda \partial_r)^T = \lambda \partial_r - \langle \lambda \partial_r, \nu \rangle \nu$.

Proof. This is a straightforward computation in any case by using the flow equation (1.3). For details, we can see the similar results in [8] for the flow in hyperbolic space.

Proposition 3.5. Let X be a solution of the inverse curvature flow (1.3). Then the curvature function is bounded from above, i.e. there exists $C = C(n, \Sigma_0)$ such that

(3.11)
$$F(t,\xi) \le C(n,\Sigma_0) < \infty \quad \forall (t,\xi) \in [0,T^*) \times \Sigma.$$

Proof. The proof proceeds similarly to that in Lemma 4.2 in [8]. Let

$$w = -\log(-\Phi) + \beta(r - \frac{t}{n}),$$

where β is supposed to be large. Fix $0 < T < T^*$ and suppose

$$\sup_{[0,T]\times\mathbb{S}^{n-1}} w = w(t_0, \xi_0), \quad t_0 > 0.$$

Then at (t_0, ξ_0) , there holds

$$0 = w_i = -\frac{\Phi_i}{\Phi} + cr_i$$

and

$$0 \leq \frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = -\Phi' F^{ij} h_{ik} h_j^k - \Phi' F^{ij} \overline{R}_{\nu i \nu j} - \Phi' F^{ij} \frac{\Phi_i \Phi_j}{\Phi^2} + 2\beta \Phi' F v^{-1} - \beta \Phi' F^{ij} \overline{h}_{ij} - \frac{1}{n}.$$

Thus, we have

$$0 \le \Phi' F^{ij} \left(-\overline{R}_{\nu i\nu j} - \beta^2 r_i r_j - \beta \frac{\lambda'}{\lambda} \lambda^2 \sigma_{ij} \right) + \beta \left(\frac{2}{Fv} - \frac{1}{n} \right).$$

It is easy to see from (2.1), (2.2) and Lemma 3.1

(3.12)
$$\frac{\lambda'}{\lambda} = 1 + O(e^{-\frac{n+1}{n}t}),$$

(3.13)
$$\frac{\lambda''}{\lambda} = 1 - \frac{1}{2}m(1-n)\lambda^{-n-2} = 1 + O(e^{-\frac{n+2}{n}t})$$

and

(3.14)
$$\frac{1 - (\lambda')^2}{\lambda^2} = -1 + m\lambda^{-n-1} = -1 + O(e^{-\frac{n+1}{n}t}).$$

Combing the above three estimates, as β is supposed to be large, we can get from (2.8)

$$\Phi' F^{ij} \left(-\overline{R}_{\nu i\nu j} - \beta^2 r_i r_j - \beta \frac{\lambda'}{\lambda} \lambda^2 \sigma_{ij} \right) \le 0.$$

Therefore, we can obtain

$$0 \le \beta \left(\frac{2}{Fv} - \frac{1}{n}\right).$$

Then,

$$F(t_0, \xi_0) \le C(n, \Sigma_0),$$

which leads to

$$w \leq C(n, \Sigma_0).$$

Therefore, the inequality

$$F < C(n, \Sigma_0)$$

holds.

Proposition 3.6. Let X be a solution of the inverse curvature flow (1.3). Then the curvature function is bounded from below, i.e., there exists $C = C(n, \Sigma_0)$ such that

$$(3.15) 0 < C(n, \Sigma_0) \le F(t, \xi), \forall (t, \xi) \in [0, T^*) \times \Sigma.$$

Proof. The proof proceeds similarly to that of [8, Lemma 4.1]. Let

$$w = \log(-\Phi) - \log(\chi e^{-\frac{t}{n}}).$$

Fix $0 < T < T^*$ and suppose

$$\sup_{[0,T]\times\mathbb{S}^n} w = w(t_0, \xi_0), \quad t_0 > 0.$$

Then at (t_0, ξ_0) , there holds

$$0 = w_i = \frac{\Phi_i}{\Phi} - \frac{\chi_i}{\chi},$$

which leads to

$$0 \le \frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = \Phi' \chi^{-1} F^{ij} \overline{R}(\nu, X_i, \lambda \partial_r, X_j) + \frac{1}{n}.$$

Then, we can have by using (2.8) and (2.10)

$$(3.16) \chi^{-1}F^{ij}\overline{R}(\nu, X_i, \lambda \partial_r, X_j) = F^{ij}\overline{R}(\nu, X_i, \nu, X_j) + vF^{ij}\overline{R}(\nu, X_i, X_k, X_j)r_lg^{kl}$$
$$= -\frac{\lambda''}{\lambda}F^{ij}g_{ij}.$$

Therefore,

$$0 \le \frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = -\frac{\lambda''}{\lambda} \Phi' F^{ij} g_{ij} + \frac{1}{n},$$

Since $F^{ij}g_{ij} \geq F(1, ., 1) = n$, we have from the estimate (3.13)

$$0 < C(n, \Sigma_0) \le F(t_0, \xi_0).$$

Thus,

$$w \le w(t_0, \xi_0) \le C(n, \Sigma_0).$$

From (3.1), we know there exists $C(n, \Sigma_0) > 0$ such that

$$C^{-1} \le \chi e^{-\frac{t}{n}} \le C.$$

Therefore, the inequality

$$0 < C(n, \Sigma_0) \le F$$

holds.

Now we begin to estimate the second fundamental form which is the most difficult part of the proof of the long-time existence. The proof is similar to that of [8, Lemma 4.4], but due to the non-vanishing term $\overline{\nabla}_i \overline{R}_{jklm}$ in non-constant curvature manifolds, our case is more complicated and needs a far more delicate treatment.

Proposition 3.7. Let X be a solution of the inverse curvature flow (1.3). Then, the principal curvatures of the flow hypersurfaces are uniformly bounded from above, i.e., there exists $C = C(n, \Sigma_0)$ such that

$$\kappa_i(t,\xi) \le C(n,\Sigma_0), \qquad \forall (t,\xi) \in [0,T^*) \times \Sigma.$$

Proof. First, we need the evolution equation of h_i^i . From (3.10) we can get

$$(3.17) \qquad \frac{\partial}{\partial t} h_j^i = \Phi' F^{kl} \nabla^i \nabla_j h_{kl} + \Phi'' F^i F_j + F^{kl,pq} h_{kl;}^{\ \ i} h_{pq;j} + \Phi h_k^i h_j^k + \Phi \overline{R}_{\nu j \nu k} g^{ki}.$$

Using Gauss equation and Codazzi equation, we have

$$(3.18) F^{kl}\nabla_{k}\nabla_{l}h_{ij} = F^{kl}\nabla_{i}\nabla_{j}h_{kl} + F^{kl}(\overline{R}_{kilp}h_{j}^{p} + \overline{R}_{kjlp}h_{i}^{p}) + 2F^{kl}\overline{R}_{kijp}h_{l}^{p}$$
$$-F^{kl}\overline{R}_{\nu ji\nu}h_{kl} - F^{kl}\overline{R}_{\nu k\nu l}h_{ij} + F^{kl}(\overline{\nabla}_{k}\overline{R}_{\nu ijl} + \overline{\nabla}_{i}\overline{R}_{\nu ljk})$$
$$+F^{kl}h_{kl}h_{i}^{p}h_{pj} - F^{kl}h_{il}h_{k}^{p}h_{pj} + F^{kl}h_{kj}h_{i}^{p}h_{pl} - F^{kl}h_{kp}h_{l}^{p}h_{ij}.$$

Then, we get the evolution equation of h_i^i by combing (3.17) and (3.18)

$$(3.19) \qquad \frac{\partial}{\partial t} h_{j}^{i} - F^{kl} \nabla_{k} \nabla_{l} h_{j}^{i}$$

$$= -\Phi' \left(F^{kl} (\overline{R}_{kqlp} h_{j}^{p} g^{qi} + \overline{R}_{kjlp} h_{q}^{p} g^{qi}) + 2F^{kl} \overline{R}_{kqjp} h_{l}^{p} g^{qi} \right)$$

$$-F^{kl} \overline{R}_{\nu jp\nu} h_{kl} g^{pi} - F^{kl} \overline{R}_{\nu k\nu l} h_{j}^{i} + F^{kl} (\overline{\nabla}_{k} \overline{R}_{\nu pjl} g^{pi} + g^{pi} \overline{\nabla}_{p} \overline{R}_{\nu ljk})$$

$$+F^{kl} h_{kl} h^{pi} h_{pj} - F^{kl} h_{l}^{i} h_{p}^{p} h_{pj} + F^{kl} h_{kj} h^{ip} h_{pl} - F^{kl} h_{kp} h_{l}^{p} h_{j}^{i} \right)$$

$$+\Phi''F^iF_j+F^{kl,pq}h_{kl}^ih_{pq;j}+\Phi h_k^ih_j^k+\Phi \overline{R}_{\nu j\nu k}g^{ki}$$

Using the estimates (3.1) and (3.4), there exists a constant $\vartheta > 0$ such that

$$2\vartheta \le \widetilde{\chi} \equiv \chi e^{-\frac{t}{n}}.$$

Setting

$$\rho = -\log(\widetilde{\chi} - \vartheta).$$

By using the equation (3.9), we get the evolution of ρ as follows

$$\frac{\partial}{\partial t}\rho - \Phi' F^{kl}\rho_{kl} = (\widetilde{\chi} - \vartheta)^{-1} \left(-\Phi' F^{kl} h_k^p h_{pl} \widetilde{\chi} + \frac{\widetilde{\chi}}{n} + \widetilde{\chi} \Phi' F^{ij} \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) \right) - \Phi' F^{kl} \frac{\widetilde{\chi}_k \widetilde{\chi}_l}{(\widetilde{\chi} - \vartheta)^2}.$$

Next, we define the functions

$$\zeta = \sup\{h_{ij}\eta_i\eta_j : g_{ij}\eta^i\eta^j = 1\}$$

and

$$w = \log \zeta + \rho + \beta (r - \frac{t}{n}),$$

where $\beta > 0$ is supposed to be large. We claim that w is bounded, if β is chosen sufficiently large. Fix $0 < T < T^*$, suppose w attains a maximal value at (t_0, ξ_0)

$$\sup_{[0,T]\times\mathbb{S}^n} w = w(t_0, \xi_0), \quad t_0 > 0.$$

Choose Riemannian normal coordinates at (t_0, ξ_0) such that at this point we have

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \le \kappa_2 \le \dots \le \kappa_n,$$

then

(3.20)
$$F^{kl,pq}\eta_{kl}\eta_{pq} \le \sum_{k \ne l} \frac{F^{kk} - F^{ll}}{\kappa_k - \kappa_l} (\eta_{kl})^2 \le \frac{2}{\kappa_n - \kappa_1} \sum_{i=1}^n (F^{nn} - F^{ii}) (\eta_{ni})^2$$

and

$$(3.21) F^{nn} \le \dots \le F^{11}.$$

For details, see, e.g., [10, Lemma 1.1] and [3, Lemma 2].

Since ζ is only continuous in general, we need to find a differential version instead. Set

$$\widetilde{\zeta} = \frac{h_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j},$$

where $\eta = (0, ..., 0, 1)$. There holds at (t_0, ξ_0) ,

$$h_{nn} = h_n^n = \kappa_n = \zeta = \widetilde{\zeta}$$

By a simple calculation, we find

$$\frac{\partial}{\partial t}\widetilde{\zeta} = \frac{(\frac{\partial}{\partial t}h_{ij})\eta^i\eta^j}{q_{ij}\eta^i\eta^j} - \frac{h_{ij}\eta^i\eta^j}{(q_{ij}\eta^i\eta^j)^2}(\frac{\partial}{\partial t}g_{ij})\eta^i\eta^j$$

and

$$\frac{\partial}{\partial t}h_n^n = \frac{\partial}{\partial t}(h_{nk}g^{kn}) = (\frac{\partial}{\partial t}h_{nk})g^{kn} - g^{ki}(\frac{\partial}{\partial t}g_{ij})g^{jn}h_{nk}.$$

Clearly, there holds in a neighborhood of (t_0, ξ_0)

$$\widetilde{\zeta} \leq \zeta$$

and we find at (t_0, ξ_0)

$$\frac{\partial}{\partial t}\widetilde{\zeta} = \frac{\partial}{\partial t}h_n^n$$

and the spatial derivatives do also coincide. This implies that $\tilde{\zeta}$ satisfies the same evolution (3.17) as h_n^n . Without loss of generality, we treat h_n^n like a scalar and pretend that w is defined by

$$w = \log h_n^n + \rho + \beta (r - \frac{t}{n}).$$

Using the asymptotic expansion of Riemannian curvature tensors (2.4), the non-vanishing terms $\overline{\nabla}_i \overline{R}_{jklm}$ which appear in (3.19) can be fortunately controlled by

$$|F^{kl}(\overline{\nabla}_k \overline{R}_{\nu pjl} g^{pi} + g^{pi} \overline{\nabla}_p \overline{R}_{\nu ljk})| \leq C F^{pq} g_{pq}.$$

Then, we get the evolution equation of h_n^n from (3.19)

$$(3.22) \qquad \frac{\partial}{\partial t} \log h_{n}^{n} - \Phi' F^{kl} \nabla_{k} \nabla_{l} \log h_{n}^{n}$$

$$= \frac{1}{\kappa_{n}} \left(\frac{\partial}{\partial t} h_{n}^{n} - \Phi' F^{kl} \nabla_{k} \nabla_{l} h_{n}^{n} \right) + \Phi' \frac{1}{\kappa_{n}^{2}} F^{kl} h_{n;k}^{n} h_{n;l}^{n}$$

$$\leq \frac{1}{\kappa_{n}} \Phi' \left(F^{kl} h_{kp} h_{l}^{p} \kappa_{n} - 2F \kappa_{n}^{2} - 2F^{kl} \overline{R}_{knln} \kappa_{n} - 2F^{kl} \overline{R}_{knnp} h_{l}^{p} \right)$$

$$+ F \overline{R}_{\nu n n \nu} + F^{kl} \overline{R}_{\nu k \nu l} \kappa_{n} + C F^{kl} g_{kl} - F \overline{R}_{\nu n \nu n} + \Phi' \frac{1}{\kappa_{n}^{2}} F^{kl} h_{n;k}^{n} h_{n;l}^{n}$$

$$+ F^{kl,pq} h_{kl} h_{nq;i} + \Phi'' F^{i} F_{i}.$$

Together with the evolution equations of ρ and r, we infer at (t_0, ξ_0) , the following inequality

$$0 \leq \Phi' F^{kl} h_{kp} h_l^p (1 - \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta}) - 2\Phi' F h_n^n + 2\beta \Phi' F v^{-1} - \beta \Phi' F^{ij} \overline{h}_{ij} - \frac{\beta}{n} + \frac{1}{n} \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta}$$

$$(3.23) \qquad + \Phi' F^{kl} (\log h_n^n)_k (\log h_n^n)_l - \Phi' F^{kl} \rho_k \rho_l + \frac{2}{\kappa_n - \kappa_1} \Phi' \sum_{i=1}^n (F^{nn} - F^{ii}) (h_{ni}^n)^2 (h_n^n)^{-1}$$

$$+ \frac{1}{\kappa_n} \Phi' \left(-2F^{kl} \overline{R}_{knln} \kappa_n - 2F^{kl} \overline{R}_{knnp} h_l^p + F^{kl} \overline{R}_{\nu k\nu l} \kappa_n + CF^{kl} g_{kl} - 2F \overline{R}_{\nu n\nu n} \right)$$

$$+ \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta} F^{ij} \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) + \Phi'' F^i F_j$$

holds. We can estimate the curvature terms by using (2.3)

$$|-2F^{kl}\overline{R}_{knln}\kappa_n - 2F^{kl}\overline{R}_{knnp}h_l^p - F^{kl}\overline{R}_{\nu k\nu l}\kappa_n + CF^{kl}g_{kl}| \le C(1+\kappa_n)F^{kl}g_{kl}$$

and

$$\mid F\overline{R}_{\nu n\nu n}\mid \leq CF.$$

Then, using the inequalities (3.20) and (3.21), $\Phi'' < 0$ and

$$(\log h_n^n)_i = -\rho_i - \beta r_i$$

at (t_0, ξ_0) , we can get from the above inequality

$$(3.24) \quad 0 \leq \Phi' F^{kl} h_{kp} h_l^p \frac{\vartheta}{\widetilde{\chi} - \vartheta} + \Phi' F^{kl} \left(C g_{kl} (1 + \kappa_n^{-1}) - \beta \overline{h}_{kl} \right) - 2 \Phi' F h_n^n$$

$$+ 2\beta \Phi' F v^{-1} - \frac{\beta}{n} + \left(\frac{1}{n} + F^{ij} \overline{R} (\nu, X_i, (\lambda \partial_r)^T, X_j) \right) \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta}$$

$$+\beta^{2}\Phi'F^{kl}r_{k}r_{l} - 2\beta\Phi'F^{kl}\rho_{k}r_{l} + \frac{2}{\kappa_{n} - \kappa_{1}}\Phi'\sum_{i=1}^{n}(F^{nn} - F^{ii})(h_{ni};^{n})^{2}(h_{n}^{n})^{-1} + C(\kappa_{n})^{-1}\Phi'F.$$

Now, we estimate the left curvature term in the above inequality. Clearly, we can get from (2.10)

$$(3.25) \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) = \lambda \overline{R}(\nu, X_i, X_k, X_j) r_l g^{kl}$$
$$= \frac{1}{v^3 \lambda} (\lambda \lambda'' + (1 - (\lambda')^2)) [r_i r_j - \lambda^2 \mid D\varphi \mid^2 \sigma_{ij}].$$

From (3.13) and (3.14), we can get

(3.26)
$$\frac{1}{\lambda}(\lambda \lambda'' + (1 - (\lambda')^2)) = (1 + \frac{n-1}{2\lambda})\frac{m}{\lambda^n},$$

which is clearly bounded. Therefore,

$$F^{ij}\overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) \leq CF^{ij}g_{ij}$$

Moreover, we know

$$F^{ij}\overline{h}_{ij} = \frac{\lambda'}{\lambda}F^{ij}\overline{g}_{ij} = \frac{\lambda'}{\lambda}F^{ij}(g_{ij} - r_i r_j) \ge \frac{\lambda'}{\lambda}F^{ij}g_{ij}(1 - g^{kl}r_k r_l)$$
$$= \frac{\lambda'}{\lambda}v^{-2}F^{ij}g_{ij} \ge C_0F^{ij}g_{ij},$$

where we use (3.4) and (3.12) in the last inequality. Furthermore, it is easy to check

$$v_i = v \frac{\overline{H}}{n} r_i - v^2 h_i^k r_k$$

(see (5.29) in [13]), and thus

$$|\nabla \rho| \leq C_2 |\nabla v| + C_2 |\nabla r| \leq C_2 |\kappa_n| |\nabla r| + C_2 |\nabla r|,$$

where $|\nabla \rho| = \sqrt{g^{ij} \nabla_i \rho \nabla_j \rho}$. We distinguish two cases.

Case 1. If $\kappa_1 < -\epsilon_1 \kappa_n$, $0 < \varepsilon_1 < 1$, then

$$F^{kl}h_{kp}h_l^p \geq \frac{1}{n}F^{kl}g_{kp}\epsilon_1^2\kappa_n^2.$$

Hence, after abandoning the negative term $-2\Phi'F\kappa_n$, (3.24) becomes

$$0 \leq \Phi' F^{kl} g_{kl} \left(-\frac{1}{n} \epsilon_1^2 \kappa_n^2 \frac{\vartheta}{\widetilde{\chi} - \vartheta} + C(1 + \kappa_n^{-1}) - \beta C_0 + C \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta} \right)$$

$$+ 2\beta C_2(\kappa_n + 1) |\nabla r|^2 + \beta^2 |\nabla r|^2$$

$$2\beta \Phi' F v^{-1} - \frac{\beta}{n} + \frac{1}{n} \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta} + C \kappa_n^{-1} \Phi' F.$$

Sine F bounded from above and below, $F^{ij}g_{ij} \geq F(1,...,1) = n$ and $|\nabla r| = \frac{|D\varphi|}{v} \leq C(n,\Sigma_0)$, the first line converges to $-\infty$ if $\kappa_n \to +\infty$. Moreover, the last line is uniformly bounded by some $C = C(n,\Sigma_0)$. Hence, in this case we conclude that

$$\kappa_n \leq C(n, \Sigma_0)$$

for any choice of β .

Case 2. If $\kappa_1 \geq -\epsilon_1 \kappa_n$, $0 < \varepsilon_1 < 1$, then

$$\frac{2}{\kappa_n - \kappa_1} \Phi' \sum_{i=1}^n (F^{nn} - F^{ii}) (h_{ni};^n)^2 (h_n^n)^{-1} \le \frac{2}{1 + \epsilon_1} \Phi' \sum_{i=1}^n (F^{nn} - F^{ii}) (\log h_n^n)_i^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2},$$

where we use $h_{ni;n} = h_{nn;i} + \overline{R}_{nni\nu}$ in view of the Codazzi equation and the boundedness of the curvature (1.1). Thus, the terms in (3.28) containing the derivatives of h_n^n can therefore be estimated from above by

$$\Phi' F^{ij} (\log h_n^n)_i (\log h_n^n)_j + \frac{2}{\kappa_n - \kappa_1} \Phi' \sum_{i=1}^n (F^{nn} - F^{ii}) (h_{ni};^n)^2 (h_n^n)^{-1}
\leq \frac{2}{1 + \epsilon_1} \Phi' \sum_{i=1}^n F^{nn} (\log h_n^n)_i^2 - \frac{1 - \epsilon_1}{1 + \epsilon_1} \Phi' \sum_{i=1}^n F^{ii} (\log h_n^n)_i^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
\leq \frac{2}{1 + \epsilon_1} \Phi' \sum_{i=1}^n F^{nn} (\log h_n^n)_i^2 - \frac{1 - \epsilon_1}{1 + \epsilon_1} \Phi' \sum_{i=1}^n F^{nn} (\log h_n^n)_i^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
= \Phi' F^{nn} |\nabla \rho + \beta \nabla r|^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
= \Phi' F^{nn} (|\nabla \rho|^2 + 2\beta \langle \nabla \rho, \nabla r \rangle + \beta^2 |\nabla r|^2) + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}.$$

Hence, taking the above inequality into the estimate (3.28) yields

$$0 \leq -\Phi' F^{nn} \kappa_n^2 \frac{\vartheta}{\widetilde{\chi} - \vartheta} + \Phi' F^{kl} g_{kl} \left(1 - \beta C_0 + C (1 + \kappa_n^{-1} + \kappa_n^{-2} + \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta}) \right)$$
$$-2\Phi' F \kappa_n + 2\beta \Phi' F v^{-1} - \frac{\beta}{n} + \frac{1}{n} \frac{\widetilde{\chi}}{\widetilde{\chi} - \vartheta} + \Phi' F^{nn} (2\beta |\nabla \rho| |\nabla r| + \beta^2 |\nabla r|^2)$$
$$+ C \kappa_n^{-1} \Phi' F < 0$$

for large κ_n if β is chosen large enough. Thus we obtain

$$\kappa_n(t_0, \xi_0) \le C(n, \Sigma_0).$$

Since ρ and \tilde{r} are bounded from above, we conclude our claim.

Corollary 3.8. Under the hypothesis of Proposition 3.7, there exists a compact set $K \subset \mathbb{R}^n$ such that

$$(\kappa_i) \subset K \subset \Gamma$$
.

Proof. Noticing that F is bounded from below and $F^{ij}h_{ij}=F$, Proposition 3.7 implies the result.

Theorem 3.9. Under the hypothesis of Theorem 1.2, we conclude

$$T^* = +\infty$$
.

Proof. Recalling that φ satisfies the equation (3.3)

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda F(h_j^i)} = G(x, \varphi, D\varphi, D^2\varphi).$$

By a simple calculation, we get

$$\frac{\partial G}{\partial \varphi_j^i} = \frac{1}{\lambda^2 F^2} F_k^j \widetilde{g}_i^k,$$

where \widetilde{g}_i^k and δ_i^k are equivalent norms, since $v \leq C$. Therefore, we can conclude the equation (3.3) is uniformly parabolic on finite intervals from Proposition 3.5, Proposition 3.6 and Corollary 3.8. Recalling that $h_j^i = \frac{1}{\lambda v} (\lambda' \delta_j^i - \widetilde{g}^{ik} \varphi_{kj})$, where $\widetilde{g}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$, we have

$$(3.27) |\varphi|_{C^2(\mathbb{S}^n)} \le C(n, \Sigma_0, T^*)$$

by using the estimate (3.4) and Corollary 3.8. Then by Krylov-Safonov estimate [20], we have

$$|\varphi|_{C^{2,\alpha}(\mathbb{S}^n)} \le C(n,\Sigma_0,T^*),$$

which implies the maximal time interval is unbounded, i.e., $T^* = +\infty$.

Optimal decay estimates

First, we recall [22, Lemma 4.2] which will be used in the next lemma.

Lemma 3.10. Let $f \in C^{0,1}(\mathbb{R}_+)$ and let D be the set of points of differentiability of f. Suppose that for all $\epsilon > 0$ there exist $T_{\epsilon} > 0$ and $\delta_{\epsilon} > 0$ such that

$$A_{\epsilon} = \{t \in [T, +\infty) \cap D : f(t) \ge \epsilon\} \subset \{t \in [T_{\epsilon}, +\infty) \cap D : f'(t) \ge -\delta_{\epsilon}\}.$$

Then there holds

$$\lim_{t \to \infty} \sup f(t) \le 0.$$

Lemma 3.11. Under the hypothesis of Theorem 1.2, the principle curvatures of the flow hypersurfaces converges to 1,

$$\sup_{\Sigma} |\kappa_i(t,\cdot) - 1| \to 0, \quad t \to \infty, \quad \forall 1 \le i \le n.$$

Proof. We use the method which first appears in [22]. Define the functions

$$\zeta = \sup\{h_{ij}\eta_i\eta_j : g_{ij}\eta^i\eta^j = 1\}$$

and

$$w = (\log \zeta - \log \widetilde{\chi} + \widetilde{r} - \log 2)t,$$

where $\tilde{\chi} = \chi e^{-\frac{t}{n}}$ and $\tilde{r} = r - \frac{t}{n}$. We claim that w is bounded. Fix $0 < T < +\infty$, suppose w attains a maximal value at (t_0, ξ_0) ,

$$\sup_{[0,T]\times\mathbb{S}^{n-1}} w = w(t_0,\xi_0), \quad t_0 > 0.$$

Choose Riemannian normal coordinates at (t_0, ξ_0) such that at this point we have

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \le \kappa_2 \le \dots \le \kappa_n.$$

Then it follows

$$w = (\log h_n^n - \log \widetilde{\chi} + \widetilde{r} - \log 2)t.$$

First, we claim that

$$(-\log \widetilde{\chi} + \widetilde{r} - \log 2)t = (\log v - \log \lambda + r - \log 2)t = (\log v - \log 2\lambda + r)t$$

is bounded. On the one hand, using the estimate (3.5),

$$t \log v = \log(1 + v - 1)^t \le \log(1 + Ce^{-\mu t})^t$$

is bounded. On the other hand, the asymptotic expansions (2.2) and (3.2) imply

$$e^{(-\log 2\lambda + r)t} = (1 - e^{-2r} + o(e^{-2r}))^{-t} < (1 - Ce^{-\frac{2t}{n}})^{-t}$$

is also bounded. Therefore, we prove our claim.

Using the evolution equations of h_n^n , $\widetilde{\chi}$ and \widetilde{r} , as (3.28), we can obtain the following evolution equation of w

$$\frac{\partial}{\partial t}w - \Phi'F^{ij}w_{ij}$$

$$= \left(-2\Phi'Fh_n^n + 2\Phi'Fv^{-1} - \Phi'F^{ij}\overline{h}_{ij} - \Phi'\kappa_nF^{kl}(\log h_n^n)_k(\log h_n^n)_l - \Phi'F^{kl}(\log \widetilde{\chi})_k(\log \widetilde{\chi})_l + \Phi'F^{kl,pq}h_{kl;n}h_{pq;}^n(h_n^n)^{-1} + \frac{1}{\kappa_n}\Phi'\left(-2F^{kl}\overline{R}_{knln}\kappa_n - 2F^{kl}\overline{R}_{knnp}h_l^p + F^{kl}\overline{R}_{\nu k\nu l}\kappa_n + F^{kl}(\overline{\nabla}_k\overline{R}_{\nu nnl} + \overline{\nabla}_n\overline{R}_{\nu lnk}) - 2F\overline{R}_{\nu n\nu n}\right) + F^{ij}\overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) + \Phi''F^iF_j\right)t_0 + \log h_n^n - \log \widetilde{\chi} + \widetilde{r} - \log 2.$$

Using the asymptotic expansion of Riemannian curvature tensors (2.4) and (2.3), we have

$$|F^{kl}(\overline{\nabla}_k \overline{R}_{\nu iml} g^{mj} + \overline{\nabla}^n \overline{R}_{\nu lnk})| \le Ce^{-\frac{n+1}{n}t}$$

and

$$-2F^{kl}\overline{R}_{knln}\kappa_n - 2F^{kl}\overline{R}_{knnp}h_l^p - F^{kl}\overline{R}_{\nu k\nu l}\kappa_n - 2F\overline{R}_{\nu n\nu n} = F^{kl}g_{kl}\kappa_n + O(e^{-\frac{n+1}{n}t}).$$

Moreover, we can get from (3.25) and (3.26)

$$|F^{ij}\overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_i)| \le Ce^{-t}$$
.

Therefore, we have

$$\frac{\partial}{\partial t}w - \Phi'F^{ij}w_{ij} \leq \left(\Phi'F^{kl}g_{kl} - 2\Phi'Fh_n^n + 2\Phi'Fv^{-1} - \Phi'F^{ij}\overline{h}_{ij} + \Phi''F_nF^n(h_n^n)^{-1} + \Phi'F^{kl}(\log h_n^n)_k(\log h_n^n)_l - \Phi'F^{kl}(\log \widetilde{\chi})_k(\log \widetilde{\chi})_l + \Phi'F^{kl,pq}h_{kl;n}h_{pq;}^n(h_n^n)^{-1}\right)t_0 + (\log h_n^n - \log \widetilde{\chi} + \widetilde{r} - \log 2) + O(1)$$

$$\leq \Phi(2h_n^n - 2v^{-1})t_0 + \Phi'F^{kl}(r_kr_l + (1 - \frac{\lambda'}{\lambda})\lambda^2\sigma_{ij})t_0 + \Phi'\left((\log h_n^n)_k(\log h_n^n)_l - (\log \widetilde{\chi})_k(\log \widetilde{\chi})_l\right)t_0 + O(1) + \log h_n^n - \log \widetilde{\chi} + \widetilde{r} - \log 2.$$

Using inequalities (3.20) and (3.21), $\Phi'' < 0$ and

$$(\log h_n^n)_i = -(\log \widetilde{\chi})_i - \widetilde{r}_i$$

at (t_0, ξ_0) , we can get from the above inequality

$$(3.28) 0 \leq \Phi(2h_n^n - 2v^{-1})t_0 + \Phi' F^{kl} r_k r_l t_0 + \Phi' F^{kl} (\log \widetilde{\chi})_k r_l t_0 + O(1) + \log h_n^n - \log \widetilde{\chi} + \widetilde{r} - \log 2.$$

From (2.7), we have

$$v_k = \frac{\varphi^j \varphi_{jk}}{v} = \lambda' v \varphi_k - \lambda v^2 h_k^i \varphi_i = \frac{\lambda'}{\lambda} v r_k - v^2 h_k^i r_i.$$

Then, we obtain

$$(\log \widetilde{\chi})_k = \frac{\chi_k}{\chi} = \frac{v}{\lambda} \frac{\lambda' r_k v - \lambda v_k}{v^2} = v h_k^i r_k.$$

Since the principal curvatures are bounded by Corollary 3.8 and F is also bounded by Propositions 3.5 and 3.6, the following two terms in (3.28) are controlled by

$$(3.29) \qquad \Phi' F^{kl} r_k r_l t_0 + \Phi' F^{kl} (\log \widetilde{\chi})_k r_l t_0 \le C g^{ij} r_i r_j t_0.$$

However, $g^{ij}r_ir_jt_0 = \frac{|D\varphi|^2}{1+|D\varphi|^2} \le Ce^{-\mu t_0}t_0 \le C(n,\Sigma_0)$ by Lemma 3.3. Therefore, from (3.28) at (t_0,ξ_0) , we get

$$0 \le \Phi(2h_n^n - 2v^{-1})t_0 + C$$

for some $C = C(n, \Sigma_0)$, which implies

$$h_n^n \le 1 + \frac{CF}{t_0}.$$

Thus, we have

$$w \le t_0 \log(1 + \frac{CF}{t_0}) + t_0(-\log \widetilde{\chi} + \widetilde{r} - \log 2) \le C(n, \Sigma_0),$$

which means w has a priori boundness. Hence,

(3.30)
$$\limsup_{t \to \infty} \sup_{M} \kappa_i(t, \cdot) \le 1, \qquad \forall 1 \le i \le n.$$

Now we define the function

$$\psi = \log(-\Phi) - \log \widetilde{\chi} + \widetilde{r} - \log 2 - \log \frac{1}{n}.$$

By a similar computation to that in the proofs of Propositions 3.5 and 3.6, we know that ψ satisfies

$$\frac{\partial}{\partial t}\psi - \Phi' F^{ij}\psi_{ij} = \Phi' F^{ij}(\log(-\Phi))_i(\log(-\Phi))_j - \Phi' F^{ij}(\log\widetilde{\chi})_i(\log\widetilde{\chi})_j + \frac{1}{\gamma}\overline{R}(\nu, \partial_i, \lambda\partial_r, \partial_j) + \frac{2}{Fv} - \Phi' F^{ij}\overline{h}_{ij}.$$

Then the Lipschitz function

$$\widetilde{\psi} = \sup_{\xi \in \Sigma} \psi(\cdot, \xi)$$

satisfies

$$\frac{\partial}{\partial t}\widetilde{\psi} \leq Ce^{-\mu t} - \Phi' F^{ij} g_{ij} (1 + O(e^{-\frac{n+2}{n}t})) + \frac{2}{Fv} - \Phi' F^{ij} \overline{h}_{ij}
\leq Ce^{-min\{\mu, \frac{n+2}{n}\}t} + \Phi'(\frac{2F}{v} - 2F^{ij} g_{ij}),$$

where we use a similar argument which has been done to (3.29) to get the first inequality by noticing (3.16) and (3.13). Setting

$$A_{\epsilon} = \{ t \in [T, +\infty) \cap D : \widetilde{\psi}(t) \ge \epsilon \},$$

where D is the set of points of differentiability of $\widetilde{\psi}$. Let $\epsilon > 0$ and choose T > 0 such that for all $(t, \xi) \in [T, \infty) \times \Sigma$,

$$-\log \widetilde{\chi} + \widetilde{r} - \log 2 < \frac{\epsilon}{2}.$$

Then we have

$$\left(\log(-\Phi) - \log\frac{1}{n}\right)(t,\xi_t) > \frac{\epsilon}{2}$$

for $t \in A_{\epsilon}$, where $\widetilde{\psi}(t) = \psi(t, \xi_t)$. Thus there exists $0 < \gamma = \gamma(\epsilon) = n(1 - e^{-\frac{\epsilon}{2}})$ such that

$$F(t, \xi_t) < n - \gamma$$

which implies

$$\Phi'(\frac{2F}{v} - 2F^{ij}g_{ij}) \le -\Phi'\frac{2n\gamma}{v}.$$

Therefore, if T is chosen large enough, we have

$$\frac{\partial}{\partial t}\widetilde{\psi} \le -\frac{1}{2}(\inf \Phi')\frac{2n\gamma}{v} \equiv \delta_{\epsilon}.$$

Now it follows from Lemma 3.10,

$$\lim_{t \to \infty} \sup \widetilde{\psi}(t) \le 0.$$

Hence, we have

$$\lim_{t\to\infty}\sup_{\Sigma}\log(-\Phi)-\log\frac{1}{n}\leq\lim_{t\to\infty}\sup_{\widetilde{\psi}}(t)+\lim_{t\to\infty}\sup_{\Sigma}(\log\widetilde{\chi}-\widetilde{r}+\log2)\leq0,$$

which leads to

$$\lim_{t \to \infty} \inf \inf_{M} F \ge n$$

Then, together with (3.30), we conclude that the following fact

$$\sup_{\Sigma} |\kappa_i(t,\cdot) - 1| \to 0, \quad t \to \infty, \quad \forall 1 \le i \le n$$

is true. \Box

Theorem 3.12. Under the assumptions of theorem 1.2, the principle curvatures of the flow hypersurfaces of (1.3) converge to 1 exponentially fast. There exists $C = C(n, \Sigma_0)$ such that for all $(t, \xi) \in [0, \infty) \times \Sigma$, the estimate

$$|h_j^i - \delta_j^i| \le Ce^{-\frac{2t}{n}}$$

holds.

Proof. Define the function

$$G(t,\xi) = \frac{1}{2} |h_j^i - \delta_j^i|^2(t,\xi), \forall (t,\xi) \in [T,\infty) \times \Sigma.$$

Using the evolution equation (3.22) of h_i^i , we can get the evolution equation of $G(t,\xi)$ as follows

$$\begin{split} \frac{\partial}{\partial t} G(t,\xi) - \Phi' F^{kl} \nabla_k \nabla_l G(t,\xi) &= (h^i_j - \delta^i_j) \bigg(\Phi' F^{kl} h_{kp} h^p_l h^i_j - 2 \Phi' F h^{ip} h_{pj} + \Phi'' F^i F_j \\ &+ \Phi' F^{kl,pq} h_{kl;j} h_{pq;}{}^i + \Phi' F^{kl} g_{kl} h^i_j + O(e^{-\frac{(n+1)}{n}t}) \bigg) \\ &- \Phi' F^{kl} h^i_{i:k} h^i_{j:l}. \end{split}$$

Set

$$G(t) = G(t, \xi_t) = \sup_{\xi \in \Sigma} G(t, \xi).$$

Since

$$F^{kl}h^i_{j;k}h^i_{j;l} \ge C|\nabla A|^2$$

and

$$|h_j^i - \delta_j^i| \to 0,$$

so for large t we can absorb the terms involving the derivatives of h_j^i by $\Phi' F^{kl} h_{j;k}^i h_{j;l}^i$. There holds the following identity

$$h_k^i h_i^k = (h_k^i - \delta_k^i)(h_i^k - \delta_i^k) + 2(h_i^i - \delta_i^i) + \delta_i^i.$$

Thus we have

$$\frac{\partial}{\partial t}G(t) = (h_{j}^{i} - \delta_{j}^{i}) \left(\Phi' F^{kl} h_{kp} h_{l}^{p} h_{j}^{i} - 2\Phi' F(h_{p}^{i} - \delta_{p}^{i}) (h_{j}^{p} - \delta_{j}^{p}) \right)
-4\Phi' F(h_{j}^{i} - \delta_{j}^{i}) - 2\Phi' F + \Phi' F^{kl} g_{kl} h_{j}^{i} + O(e^{-\frac{(n+1)}{n}t}) \right)
= (h_{j}^{i} - \delta_{j}^{i}) \left(\Phi' F^{kl} (h_{kp} h_{l}^{p} - 2h_{kl} + g_{kl}) h_{j}^{i} - 2\Phi' F(h_{p}^{i} - \delta_{p}^{i}) (h_{j}^{p} - \delta_{j}^{p}) \right)
-2\Phi' F(h_{j}^{i} - \delta_{j}^{i}) + O(e^{-\frac{(n+1)}{n}t}) \right)$$

Choose Riemannian normal coordinates at (t, ξ_t) such that at this point we have

$$g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \le \kappa_2 \le \dots \le \kappa_n.$$

For t large enough, we can find $\epsilon < \frac{4}{\sup_{\Sigma} F}$ such that

$$\frac{d}{dt}G(t) \leq \left(-\frac{4}{F} + 2\Phi' \sum_{j=1}^{n} |\kappa_{j}| |\kappa_{j} - 1| \sum_{k=1}^{n} F^{kk} + \frac{4}{F^{2}} \max_{1 \leq j \leq n} |\kappa_{j} - 1| \right) G(t) + \max_{1 \leq j \leq n} |\kappa_{j} - 1| O(e^{-\frac{n+1}{n}t})$$

$$\leq \left(-\frac{4}{F} + \epsilon\right) G(t) + \max_{1 \leq j \leq n} |\kappa_{j} - 1| O(e^{-\frac{n+1}{n}t}).$$

Therefore, we have

$$G(t) \leq Ce^{-\mu_1 t}$$

where $\mu_1 = \min\{\frac{4}{\sup_M F} - \epsilon, \frac{n+1}{n}\} > 0$. Thus,

Now we define

$$\overline{G} = \sup_{\Sigma} \frac{1}{2} |h_j^i - \delta_j^i|^2 e^{\frac{4}{n}t}.$$

Similar to the process of getting (3.32), we can obtain

$$\frac{d}{dt}\overline{G} \leq \left(-\frac{4}{F} + \frac{4}{n} + 2\Phi' \sum_{j=1}^{n} |\kappa_{j}| |\kappa_{j} - 1| \sum_{k=1}^{n} F^{kk} + \frac{4}{F^{2}} \max_{1 \leq j \leq n} |\kappa_{j} - 1| \right) \overline{G} + O(e^{-\frac{n+1}{n}t + \frac{4}{n}t - \frac{1}{2}\mu_{1}t})$$

$$\leq Ce^{-\frac{1}{2}\mu_{1}t}\overline{G} + O(e^{-\frac{n-3}{n}t - \frac{1}{2}\mu_{1}t}),$$

where we use (3.32) to get the last inequality. Thus,

$$\overline{G} \le C(n, \Sigma_0),$$

which implies our result.

Theorem 3.13. The estimate (3.5) in Lemma 3.3 is true for $\mu = \frac{2}{n}$.

Proof. Define

$$\widetilde{w} = \sup_{x \in \mathbb{S}^n} \frac{1}{2} |D\varphi(\cdot, x)|^2 e^{-\frac{2}{n}t}.$$

The same calculation as in (3.6) implies

$$\frac{d}{dt}\widetilde{w} \le -\frac{2}{F^2}\frac{\lambda''}{\lambda}F^{kl}\widetilde{g}_{kl}\widetilde{w} + \frac{2}{n}\widetilde{w} \le -\frac{2n}{F^2}\frac{\lambda''}{\lambda}\widetilde{w} + \frac{2}{n}\widetilde{w}.$$

Recalling the estimate (3.13)

$$\frac{\lambda''}{\lambda} = 1 - \frac{1}{2}m(1-n)\lambda^{-n-2} = 1 + O(e^{-\frac{n+2}{n}t}).$$

Together with (3.32), we have

$$\frac{d}{dt}\widetilde{w} \le Ce^{-\frac{1}{2}\mu_1 t}\widetilde{w},$$

which implies \widetilde{w} is bounded from above. Therefore, the theorem holds.

Theorem 3.14. Under the assumptions of Theorem 1.2. There exists a constant $C = C(n, \Sigma_0)$ such that

$$\left| D^2 \varphi \right| \le C e^{-\frac{t}{n}}.$$

Proof. Recalling (2.7), we have

$$\varphi_j^i = v^{-2} \varphi^i \varphi^k \varphi_{kj} + \lambda' \delta_j^i - v \lambda h_j^i.$$

From Lemma 3.1, we get

$$|\lambda' - \lambda| = \frac{1 - m\lambda^{1-n}}{\lambda\sqrt{1 + \lambda^2 - m\lambda^{1-n}}} \le \frac{1}{\lambda} \le Ce^{-\frac{1}{n}t}.$$

Together with Theorems 3.14 and 3.12, we obtain

$$\begin{split} |D^2\varphi| & \leq C|D\varphi|^2|D^2\varphi| + |\lambda'\delta^i_j - \lambda\delta^i_j| + |\lambda\delta^i_j - v\lambda\delta^i_j| + |v\lambda\delta^i_j - v\lambda h^i_j| \\ & \leq Ce^{-\frac{2}{n}t}|D^2\varphi| + Ce^{-\frac{1}{n}t}. \end{split}$$

Choosing T large enough $(Ce^{-\frac{2}{n}t} < \frac{1}{2})$, we know that the estimate

$$|D^2\varphi| \le Ce^{-\frac{1}{n}t}$$

holds for t > T.

Clearly, from Theorem 3.14, we can show that there exists a constant $C = C(n, \Sigma_0)$ such that

$$\parallel D^2r \parallel_{\mathbb{S}^n} \le C.$$

Then by Krylov-Safonov estimate [20], we have

$$\parallel r \parallel_{C^{2,\alpha}(\mathbb{S}^n)} \leq C(n,\Sigma_0),$$

which implies the following conclusion.

Theorem 3.15. Under the assumptions of theorem 1.2. The function

$$\widetilde{r}(t,\theta) = r(t,\theta) - \frac{t}{n}$$

converge to a well-defined C^2 function $f(\theta)$ in $C^{2,\alpha}$.

Proof. Because of the boundedness of $\widetilde{r} = r - \frac{t}{n}$ in $C^2(\mathbb{S}^n)$, we only have to show the pointwise limit

$$\lim_{t \to \infty} (r - \frac{t}{n})$$

exists for all $x \in \mathbb{S}^n$. We have

$$\frac{\partial}{\partial t}\widetilde{r} = \frac{v}{F} - \frac{1}{n} = \frac{v-1}{F} + \frac{n-F}{nF} \ge -C(n, \Sigma_0)e^{-\frac{t}{n}}.$$

Thus,

$$(\widetilde{r} - nCe^{-\frac{t}{n}})' \ge 0,$$

which implies the result.

Remark 3.3. Following the techniques in [8, Section 6] and [22, Section 5], we may also get estimates of high order for \tilde{r}

$$\|\widetilde{r}\|_{C^k(\mathbb{S}^n)} \le C(n, \Sigma_0), \quad \forall k \in \mathbb{N}.$$

Therefore, the C^{∞} convergence in the above theorem may be obtained.

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