REGULARITY OF PULLBACK ATTRACTORS AND EQUILIBRIUM FOR NON-AUTONOMOUS STOCHASTIC FITZHUGH-NAGUMO SYSTEM ON UNBOUNDED DOMAINS

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Abstract. A theory on bi-spatial random attractors developed recently by Li *et al.* is extended to study stochastic Fitzhugh-Nagumo system driven by a non-autonomous term as well as a general multiplicative noise. By using the so-called notions of uniform absorption and uniformly pullback asymptotic compactness, it is showed that every generated random cocycle has a pullback attractor in $L^l(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ with $l\in(2,p]$, and the family of obtained attractors is upper semi-continuous at any intensity of noise. Moreover, if some additional conditions are added, then the system possesses a unique equilibrium and is attracted by a single point.

1. Introduction

In this paper, we consider the random dynamics of solutions of the following non-autonomous FitzHugh-Nagumo system defined on $\mathbb{R}^N$ perturbed by a ε-multiplicative noise:

$$d\tilde{u}+(\lambda\tilde{u}-\Delta\tilde{u}+\alpha\tilde{v})dt=f(x,\tilde{u})dt+g(t,x)dt+\varepsilon\tilde{u}\circ d\omega(t), t>\tau, \tag{1.1}$$

$$d\tilde{v}+(\sigma\tilde{v}-\beta\tilde{u})dt=h(t,x)dt+\varepsilon\tilde{v}\circ d\omega(t), t>\tau, \tag{1.2}$$

with initial value numbers

$$\tilde{u}(x,\tau)=\tilde{u}_\tau(x), \qquad \tilde{v}(x,\tau)=\tilde{v}_\tau(x), \tag{1.3}$$

where the initial condition $(\tilde{u}_0,\tilde{v}_0)\in L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, the coefficients $\lambda,\alpha,\beta,\sigma$ are positive constants, the non-autonomous terms $g,h\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^N))$, the nonlinearity f is a smooth function satisfying some polynomial growth, ε is the intensity of noise with $\varepsilon\in[-a,a]\setminus\{0\}$, $a>0$, $\omega(t)$ is a Wiener process defined on a probability space $(\Omega,\mathcal{F},P)$, where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R});\omega(0)=0\}$, and $\mathcal{F}$ be the Borel σ-algebra induced by the compact-open topology of Ω and P be the corresponding Wiener measure on $(\Omega,\mathcal{F})$.

The deterministic FitzHugh-Nagumo system is an important mathematical model to describe the signal transmission across axons in neurobiology, see [22, 10, 15, 4] and references therein. It is well studied in the literature, see e.g. [20, 26, 21, 23]. In the random case, when g and h do not depend on the time, Wang [28] proved the existence and uniqueness of random attractors in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. For the general non-autonomous forcings g and h, under additive noises, Adili and Wang [2] obtained the pullback attractors in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, and Bao [5] developed

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this result and obtained the regularity in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. For our problem (1.1)-(1.3) , *i.e.*, under multiplicative noise, Adili and Wang [1] proved the existence and upper semi-continuity of attractors in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ recently. For the stochastic lattice FitzHugh-Nagumo system, the existences of random attractors are widely studied in [14, 12, 13]. However, to our knowledge, there are no literature to investigate the asymptotic high-order integrability of solutions to the FitzHugh-Nagumo system, even for the deterministic case.

In this paper, we strengthen these results offered by [1] and devote to obtain the asymptotic high-order integrability of solutions of problem (1.1)-(1.3). To this end, a theory on bi-spatial random attractors developed recently by Li *et al.* [16, 17] is extended to stochastic partial differential equations (SPDE) with both non-autonomous terms and random noises, see Theorem 2.9 and 2.10. It is showed that the *uniform absorption* and *uniformly pullback asymptotic compactness* are the appropriate notions to depict the existence and upper semi-continuity of attractor in both the initial space and terminate space [16]. As for the theory on the upper semi-continuity of pullback attractors in an initial space and its applications, we may refer to [29, 27, 31, 37] and references therein.

Then we apply the obtained theorems to prove that the problem (1.1)-(1.3) admits a unique pullback attractor in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, with the functions f, g and h satisfying almost the same conditions as in [1]. Furthermore, we derive the upper semi-continuity of pullback attractors of system in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ as the intensity ε approaches any $\varepsilon_0 \in \mathbb{R}$. These are achieved by checking the uniform absorption and uniformly pullback asymptotic compactness properties of random cocycles. The uniformly pullback asymptotic compactness in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ is completely proved by estimate of the L^2 and L^p-uniform boundedness and the L^p-truncation of solutions, see Lemma 4.4 and 4.5. it seems that the estimate of L^2-truncation is unnecessary, see [16, 17, 18, 34, 35, 19, 32, 36]. It is worth mention that an additional assumption on the non-autonomous terms (see [1]) is not used in our proof, see section 3.

The third goal is to study *stochastic fixed point* or *random equilibrium* of random dynamical system, see [6]. In this paper, we introduce the notion of equilibrium for SPDE with both non-autonomous terms and white noises. It is showed that if the parameters satisfy some additional conditions, then the system admits a unique equilibrium and is attracted by a single point.

This paper is organized as follows. In the next section, we introduce some concepts required for our further discussions and extend the results developed by [16, 17] to the general SPDE with non-autonomous forcing. In section 3, we give the assumptions on g, h and f, and define a family of continuous random cocycles for problem (1.1)-(1.3). In section 4, we prove the existence and upper semi-continuity of pullback attractors in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. The final section is concerned with the existence of equilibrium of the random cocycle.

2. Preliminaries and abstract results

In this section, we give the sufficient conditions for the existence and upper semi-continuity of pullback attractors in *the terminate space* for random dynamical systems over two parametric spaces, which are applicable to SPDE with both non-autonomous deterministic and random terms, and the structure of the pullback attractor is presented. This is an extension of the corresponding results just established by Li *et al.* [16], which is suitable for random systems with only one stochastic terms. The reader is also referred to [29, 30, 38] for the theory of pullback attractors and its applications in *the initial space* over two parametric spaces, and to [8, 25, 11, 9] for one parametric random attractors. The reader may also refer to [7, 24] for pullback attractor of deterministic dynamical systems.

2.1. **Preliminaries.** Let both $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be separable Banach spaces, where X is called *an initial space* which contains all initial data, and Y is called *a terminate space* which contains all values of solutions for a SPDE [16]. Both X and Y may not be embedded in any direction, but we assume that they are *limit-uniqueness* in the following sense:

(H1) *If $\{x_n\}_n \subset X \cap Y$ such that $x_n \to x$ in X and $x_n \to y$ in Y, respectively, then we have $x = y$.*

Let Q be a nonempty set and $(\Omega, \mathcal{F}, P)$ be a probability space. We assume that there are two groups $\{\sigma_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ over Q and Ω, respectively. Specifically, the mapping $\sigma : \mathbb{R} \times Q \mapsto Q$ satisfies that σ_0 is the identity on Q, and $\sigma_{s+t} = \sigma_s \circ \sigma_t$ for all $s, t \in \mathbb{R}$. Similarly, $\vartheta : \mathbb{R} \times \Omega \mapsto \Omega$ is a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping such that ϑ_0 is the identity on Ω, $\vartheta_{s+t} = \vartheta_s \circ \vartheta_t$ for all $s, t \in \mathbb{R}$ and $\vartheta_t P = P$ for all $t \in \mathbb{R}$. In particular, we call both $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ parametric dynamical systems. Let $\mathbb{R}^+ = \{x \in \mathbb{R}; x \geq 0\}$ and 2^X be the collection of all subsets of X.

Definition 2.1. *A measurable mapping $\varphi : \mathbb{R}^+ \times Q \times \Omega \times X \to X, (t, q, \omega, x) \mapsto \varphi(t, q, \omega, x)$ is called to be a random cocycle on X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ if for every $q \in Q, \omega \in \Omega$ and $s, t \in \mathbb{R}^+$ the following statements are satisfied:*

(i) $\varphi(0, q, \omega, .)$ is the identity on X, (ii) $\varphi(t + s, q, \omega, .) = \varphi(t, \sigma_s q, \vartheta_s\omega, .) \circ \varphi(s, q, \omega, .)$.

An random cocycle is said to be continuous in X iff each operator $\varphi(t, q, \omega, .)$ is continuous in X for each $q \in Q, \omega \in \Omega$ and $t \in \mathbb{R}^+$.

In particular, it is pointed out that in this paper, we need further to assume that the random cocycle φ acting on X takes its values in the terminate space Y for for all $t > 0$ (except that $t = 0$), *i.e.,*

(H2) *For every $t > 0, q \in Q,$ and $\omega \in \Omega$, $\varphi(t, q, \omega, .) : X \to Y$.*

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty subsets of X which is parameterized by $(q, \omega) \in (Q \times \Omega)$:

$$\mathcal{D} = \{D = \{\emptyset \neq D(q, \omega) \in 2^X; q \in Q, \omega \in \Omega\}; f_D \text{ satisfies some conditions}\}.$$

We further assume that $\mathcal{D}$ is inclusion closed, that is, for each $D \in \mathcal{D}$,

$$\{\tilde{D}(q, \omega) : \tilde{D}(q, \omega) \text{ is a nonempty subset of } D(q, \omega), \forall\ q \in Q, \omega \in \Omega\} \in \mathcal{D}.$$

Given $D_1, D_2 \in \mathcal{D}$, we say that $D_1 = D_2$ if and only if $D_1(q, \omega) = D_2(q, \omega)$ for each $q \in Q$ and $\omega \in \Omega$.

Throughout this paper, all assertions about ω are assumed to hold on a ϑ_t-invariant set of full measure (unless some exceptional cases needed).

Definition 2.2. *A set-valued mapping $K : Q \times \Omega \to 2^X$ is called measurable in X with respect to $\mathcal{F}$ in Ω if the mapping $\omega \in \Omega \mapsto dist_X(x, K(q,\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $q \in Q$, where $dist_X$ is the Haustorff semi-metric in X, i.e., for the two nonempty subsets $A, B \in 2^X$,*

$$dis_X(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_X.$$

Definition 2.3. *Suppose φ is a random cocycle on X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ and takes its value in Y. A set valued mappping $\mathcal{A} : Q \times \Omega \mapsto 2^{X\cap Y}$ is called a (X,Y)-pullback attractor for φ if*

(i) $\mathcal{A}$ is measurable in X (w.r.t the P-completion of $\mathcal{F}$ in Ω), and $\mathcal{A}(q,\omega)$ is compact in Y for all $q \in Q, \omega \in \Omega$,

(ii) $\mathcal{A}$ is invariant, that is, for every $q \in Q, \omega \in \Omega$,

$$\varphi(t, q, \omega, \mathcal{A}(q,\omega)) = \mathcal{A}(\sigma_t q, \vartheta_t \omega), \forall\ t \geq 0,$$

(iii) $\mathcal{A}$ attracts every element $D \in \mathcal{D}$ in Y, that is, for every $q \in Q, \omega \in \Omega$,

$$\lim_{t\to+\infty} dist_Y(\varphi(t, \sigma_{-t}q, \vartheta_{-t}\omega, D(\sigma_{-t}q, \vartheta_{-t}\omega)), \mathcal{A}(q,\omega)) = 0,$$

where $dist_Y$ is the Haustorff semi-distance in Y and the set $\varphi(t,q,\omega,D(q,\omega)) = \{\varphi(t,q,\omega,x); x \in D(q,\omega)\}$.

If $X = Y$, the above concept reduces to the well known notion of a $\mathcal{D}$-pullback attractors, which is first introduced in [30]. We also remark that the measurability of $\mathcal{A}$ is assumed in the initial space X.

Definition 2.4.(see[15].) *Let both Z and I be two metric spaces. A family $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ of sets in Z is said to be upper semi-continuous at α_0 if*

$$\lim_{\alpha\to\alpha_0} dist_Z(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) = 0.$$

A family $\mathcal{A}_\alpha$ of set-mappings over Q and Ω is called to be upper semi-continuous if $\mathcal{A}_\alpha(q,\omega)$ is upper semi-continuous for each $q \in Q$ and $\omega \in \Omega$.

In the sequel, we need to consider a family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ with $I = [-a,a] \setminus \{0\}$, where $a > 0$ and φ_0 is a deterministic cocycle over the parametric space $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$.

Definition 2.5.(see[16].) *A family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ is said to be convergent at point $\alpha = \alpha_0$ in X if for each $q \in Q, \omega \in \Omega$, and $x, x_0 \in X$,*

$$\varphi_\alpha(t,q,\omega,x) \to \varphi_{\alpha_0}(t,q,\omega,x_0) \text{ in } X,$$

whenever $\alpha \to \alpha_0$ and $x \to x_0$. A family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ is said to be convergent in X if it is convergent at any point α. We say a family of random cocycles $\alpha_\varepsilon(\alpha \in (0,a])$ converges to a deterministic cocycle φ_0 in X if for each $q \in Q$, and $x, x_0 \in X$,

$$\varphi_\alpha(t,q,\omega,x) \to \varphi_0(t,q,x_0) \text{ in } X,$$

whenever $\alpha \to \alpha_0$ and $x \to x_0$.

Definition 2.6.(see[16].) *A family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ is said to be uniformly absorbing in X if each φ_α has a closed and measurable pullback absorbing set K_α in X such that the closure $\overline{K} = \{\overline{\cup_{\alpha\in I}K_\alpha(q,\omega)}; q\in Q, \omega\in\Omega\}\in\mathcal{D}$ and for each $q\in Q,\omega\in\Omega$,*

$$\limsup_{\alpha\to 0}\|K_\alpha(q,\omega)\|_X \le c \quad \textit{for some deterministic constant } c>0.$$

Here a pullback absorbing set K_α means that for every $D\in\mathcal{D}$, there exists an absorbing time $T = T(D,q,\omega)>0$ such that for each $q\in Q,\omega\in\Omega$,

$$\varphi_\alpha(t,\sigma_{-t}q,\vartheta_{-t}\omega, D(\sigma_{-t}q,\vartheta_{-t}\omega))\subseteq K(q,\omega) \qquad \textit{for all } t\ge T.$$

Definition 2.7. *A family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ is said to be uniformly pullback asymptotically compact over I in X if for each $q\in Q,\omega\in\Omega$, $D\in\mathcal{D}$, the sequence*

$$\{\varphi_{\alpha_n}(t_n,\sigma_{-t_n}q,\vartheta_{-t_n}\omega,x_n)\} \textit{ has a convergence subsequence in } X, \tag{2.1}$$

whenever $\alpha_n\in I, t_n\to\infty$, and $x_n\in D(\sigma_{-t_n}q,\vartheta_{-t_n}\omega)$. A family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ is uniformly pullback asymptotically compact in Y if the convergence in (2.1) holds under Y-norm. A single cocycle φ_{α_0} is pullback asymptotically compact in X if (2.1) holds for a single point $\alpha=\alpha_0$.

Definition 2.8. *Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X, and φ be a random cocycle on X over $(Q,\{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega,\mathcal{F},P,\{\vartheta_t\}_{t\in\mathbb{R}})$. A mapping $\psi:\mathbb{R}\times Q\times\Omega\to X$ is called* a complete orbit *of φ if for each $\tau\in\mathbb{R}, t\in\mathbb{R}^+, q\in Q$ and $\omega\in\Omega$, there holds:*

$$\varphi(t,\sigma_\tau q,\sigma_\tau\omega,\psi(\tau,q,\omega)) = \psi(t+\tau,q,\omega).$$

If in addition, there exists $D=\{D(q,\omega);q\in Q,\omega\in\Omega\in\mathcal{D}\}$ such that $\psi(\tau,q,\omega)\in D(\sigma_\tau q,\sigma_\tau\omega)$, then ψ is called a $\mathcal{D}$-complete orbit of φ.

2.2. **Abstract results.** By slightly modifying the arguments of Theorem 3.1 in Li *et al.* [16], we can extend the corresponding theory to the case in which the random systems with non-autonomous term as well as random noises are considered.

Theorem 2.9. *Let (X,Y) be a pair of Banach spaces satisfying hypothesis* (H1)*, and φ a continuous random cocycle in X over $(Q,\{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega,\mathcal{F},P,\{\vartheta_t\}_{t\in\mathbb{R}})$ such that hypothesis* (H2) *holds. Assume further that*

(i) φ has a closed and measurable (w.r.t. the P-completion of $\mathcal{F}$) pullback absorbing set $K=\{K(q,\omega);q\in Q,\omega\in\Omega\}\in\mathcal{D}$ in X;
(ii) φ is pullback asymptotically compact in X;
(iii) φ is pullback asymptotically compact in Y.

Then the random cocycle φ admits a unique (X,Y)-pullback attractor $\mathcal{A}\in\mathcal{D}$, which is structured by

$$\begin{aligned}\mathcal{A}(q,\omega) &= \cap_{\tau>0}\overline{\cup_{t\ge\tau}(t,\sigma_{-t}q,\vartheta_{-t}\omega,K(\sigma_{-t}q,\vartheta_{-t}\omega))}^{Y}\\ &= \{\psi(0,q,\omega);\psi \textit{ is a } \mathcal{D}\textit{-complete orbit of the random cocycle } \varphi\}.\end{aligned} \tag{2.2}$$

Moreover, $\mathcal{A} = \mathcal{A}_X$*, where* $\mathcal{A}_X$ *is the* (X,X)*-pullback attractor.*

In the following, we will consider both the semi-continuity of a family of the bi-spatial pullback attractors and the existence problem. We give a unified result, where the concepts of uniform absorption, uniform convergence and uniformly pullback asymptotic compactness are used. To this end, we need to consider a family of random cocycles $\{\varphi_\alpha\}_{\alpha\in I}$ with $I = [-a,a]\setminus\{0\}$, where $a > 0$ and φ_0 is a deterministic cocycle over the parametric space $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$.

Then analogous to the proofs of Theorem 3.1 in Li *et al.* [17], or some small modifying the argument of Theorem 4.1 in Li *et al.* [16], we can extend their results about random attractors with single random noises to the random cocycle with non-autonomous term as well as random noises.

Theorem 2.10. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and (X, Y) a pair of Banach spaces satisfying hypothesis *(H1)*. Suppose that $\{\varphi_\alpha\}_{\alpha\in I}$ is a family of continuous random cocycles in X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ such that hypothesis *(H2)* holds, and φ_0 is a continuous deterministic cocycle in X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ satisfying $\varphi_0(t, q, .) : X \to Y$ for all $t > 0$ and $q \in Q$. Assume further that

(i) φ_α *is convergent in* X *at any* $\alpha \in [-a,a]$*;*
(ii) $\varphi_\alpha(\alpha \in I)$ *is uniformly absorbing in* X *;*
(iii) $\varphi_\alpha(\alpha \in I)$ *is uniformly pullback asymptotically compact in* X*;*
(iv) $\varphi_\alpha(\alpha \in I)$ *is uniformly pullback asymptotically compact in* Y*.*

Then each random cocycle φ_α *admits a unique (X,Y)-pullback attractor* $\mathcal{A}_\alpha \in \mathcal{D}$*, such that the family* $\{\varphi_\alpha\}_{\alpha\in I}$ *is upper semi-continuous at any* $\alpha \in I$ *in both* X *and* Y*. If in addition,* φ_0 *has an (X,Y)-attracting set* A_0*, then the* $\{\varphi_\alpha\}_{\alpha\in I}$ *is upper semi-continuous at* $\alpha = 0$ *in both* X *and* Y*.*

3. Non-autonomous FitzHugh-Nagumo system on $\mathbb{R}^N$ with multiplicative noise

For the non-autonomous FitzHugh-Nagumo system (1.1)-(1.3), the nonlinearity $f(x,s)$ satisfy almost the same assumptions as in [1], *i.e.*, for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$f(x,s)s \le -\alpha_1|s|^p + \psi_1(x), \tag{3.1}$$

$$|f(x,s)| \le \alpha_2|s|^{p-1} + \psi_2(x), \tag{3.2}$$

$$\frac{\partial f}{\partial s} f(x,s) \le \alpha_3, \tag{3.3}$$

$$\Big|\frac{\partial f}{\partial x} f(x,s)\Big| \le \psi_3(x). \tag{3.4}$$

where $p \ge 2$, $\alpha_i > 0 (i = 1,2,3)$ are determined constants, $\psi_1 \in L^1(\mathbb{R}^N) \cap L^{\frac{p}{2}}(\mathbb{R}^N)$, and $\psi_2, \psi_3 \in L^2(\mathbb{R}^N)$. The non-autonomous terms $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$ and $h \in$

$L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$ satisfy that for every $\tau \in \mathbb{R}$ and some $0 < \delta_0 < \delta = \min\{\lambda, \sigma\}$,

$$\int_{-\infty}^{\tau} e^{\delta_0 s}(\|g(s,.)\|^2_{L^2(\mathbb{R}^N)} + \|h(s,.)\|^2_{L^2(\mathbb{R}^N)})ds < +\infty, \tag{3.5}$$

where λ and δ are as in (1.1)-(1.3). The H^1-condition on the non-autonomous term h in (3.5) is required to prove the asymptotic compactness of solutions in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, see [1]:

$$\int_{-\infty}^{\tau} e^{\delta_0 s}\|h(s,.)\|^2_{H^1(\mathbb{R}^N)}ds < +\infty.$$

In order to model the random noises in system (1.1)-(1.3), we need to define a shift operator ϑ on Ω (which is defined in the introduction) by $\vartheta_t\omega(s) = \omega(s + t) - \omega(t)$ for every $\omega \in \Omega, t, s \in \mathbb{R}$. Then ϑ_t is a measure preserving transformation group on $(\Omega, \mathcal{F}, P)$, that is, $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system. By the law of the iterated logarithm (see [8]), there exists a ϑ_t-invarant set $\tilde{\Omega} \subset \Omega$ of full measure sucht that for $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0, \text{ as } |t| \to +\infty. \tag{3.6}$$

Put $Q = \mathbb{R}$. Define a family of shift operator $\{\sigma_t\}_{t\in\mathbb{R}}$ by $\sigma_t(\tau) = t + \tau$ for all $t, \tau \in \mathbb{R}$. Then both $\{\mathbb{R}, \{\sigma_t\}_{t\in\mathbb{R}}\}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ are parametric dynamical systems. We will define a continuous random cocycle for system (1.1)-(1.3) over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$.

Given $\omega \in \Omega$, put $z(t,\omega) = z_\varepsilon(t,\omega) = e^{-\varepsilon\omega(t)}$. Then we have $dz + \varepsilon z \circ d\omega(t) = 0$. Let $(\tilde{u}, \tilde{v})$ satisfy problem (1.1)-(1.3) and write

$$u(t,\tau,\omega,u_0) = z(t,\omega)\tilde{u}(t,\tau,\omega,\tilde{u}_0) \text{ and } v(t,\tau,\omega,v_0) = z(t,\omega)\tilde{v}(t,\tau,\omega,\tilde{v}_0). \tag{3.7}$$

Then (u, v) solves the follow system

$$\frac{du}{dt} + \lambda u - \Delta u + \alpha v = z(t,\omega)f(x, z^{-1}(t,\omega)u) + z(t,\omega)g(t,x), \tag{3.8}$$

$$\frac{dv}{dt} + \sigma v - \beta u = z(t,\omega)h(t,x), \tag{3.9}$$

with initial conditions $u_\tau = z(\tau,\omega)\tilde{u}_\tau$ and $v_\tau = z(\tau,\omega)\tilde{v}_\tau$.

It is known (see [1]) that for every $(u_\tau, v_\tau) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ the problem (1.1) possesses a unique solution (u, v) such that $u \in C([\tau, +\infty), L^2(\mathbb{R}^N)) \cap L^2(\tau, T, H^1(\mathbb{R}^N)) \cap L^p(\tau, T, L^p(\mathbb{R}^N))$ and $v \in C([0, +\infty), L^2(\mathbb{R}^N))$. In addition, the solution (u, v) is continuous in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with respect to the initial value (u_τ, v_τ) . Then formally $(\tilde{u}, \tilde{v}) = (z^{-1}(t,\omega)u, z^{-1}(t,\omega)v)$ is the solution to problem (1.1)-(1.3) with the initial value $\tilde{u}_\tau = z^{-1}(\tau,\omega)u_\tau$ and $\tilde{v}_\tau = z^{-1}(\tau,\omega)v_\tau$.

We are at the position to give the continuous random cocycle φ associated with problem (1.1)-(1.3) over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$. Define

$$\begin{aligned}\varphi(t,\tau,\omega,(\tilde{u}_\tau,\tilde{v}_\tau)) &= (\tilde{u}(t+\tau,\tau,\vartheta_{-\tau}\omega,\tilde{u}_\tau), \tilde{v}(t+\tau,\tau,\vartheta_{-\tau}\omega,\tilde{v}_\tau)) \\ &= (z^{-1}(t+\tau,\vartheta_{-\tau}\omega)u(t+\tau,\tau,\vartheta_{-\tau}\omega,u_\tau), z^{-1}(t+\tau,\vartheta_{-\tau}\omega)v(t+\tau,\tau,\vartheta_{-\tau}\omega,v_\tau)),\end{aligned} \tag{3.10}$$

where $u_\tau = z(\tau,\omega)\tilde{u}_\tau$ and $v_\tau = z(\tau,\omega)\tilde{v}_\tau$.

Suppose that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

$$\lim_{t\to+\infty} e^{-\delta_1 t}\|D(\tau - t, \vartheta_{-t}\omega)\|^2_{L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)} = 0, \tag{3.11}$$

where $0 < \delta_0 < \delta_1 < \delta = \min\{\lambda, \sigma\}$. Denote by $\mathcal{D}_\delta$ the collection of all families of nonempty subsets of $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that (3.11) holds. Then it is obvious that $\mathcal{D}_\delta$ is inclusion closed.

We emphasize that the choices of the constants δ_1, δ_0 in (3.11) and (3.5) respectively are different from the ones used in [1]. It makes us omit the additional assumption

$$\lim_{t\to-\infty} e^{\delta_1} \int_{-\infty}^{0} e^{\delta s}(\|g(s,.)\|^2_{L^2(\mathbb{R}^N)} + \|h(s,.)\|^2_{L^2(\mathbb{R}^N)})ds = 0, \tag{3.12}$$

which is intrinsically used in [1], see the detailed proof of Lemma 4.1 in the following section.

Note that Adili and Wang [1] established the existence and upper semi-continuous of pullback attractors for problem (1.1)-(1.3) in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. In this paper, we obtain an identical result in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, but we do not increase the restrictions (except that $\psi_1 \in L^{p/2}(\mathbb{R}^N)$ as in (3.1)) on the nonlinearity f. On the contrary, the restrictive assumption (3.12) on the non-autonomous terms g and h given in [1] is omitted. Furthermore, we construct a unique random equilibrium for this system when some additional assumptions on the physical parameters are added.

4. Existence and upper semi-continuous of attractors in $L^p \times L^2$

From now on, we assume without loss of generality that $\varepsilon \in (0, a]$ for any $a > 0$. Consider that $e^{-a|\omega(s)|} \le z_\varepsilon(s,\omega) = e^{-\varepsilon\omega(s)} \le e^{a|\omega(s)|}$ for $\varepsilon \in I$, and $\omega(.)$ is continuous on $[-2, 0]$. Then there exist two positive random constants $E = E(\omega)$ and $F = F(\omega)$ such that for each $\omega \in \Omega$,

$$E \le z_\varepsilon(s,\omega) \le F \quad \text{ for all } s \in [-2,0] \text{ and } \varepsilon \in (0,a]. \tag{4.1}$$

Hereafter, we denote by $\|.\|$ and $\|.\|_p$ the norms in $L^2(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)(p > 2)$, respectively. Throughout this paper, the number c is a generic positive constant independent of τ, ω, D and ε in any place, which may vary its values everywhere.

4.1. Uniform absorption and uniformly asymptotic compactness in $L^2 \times L^2$.

This subsection is concern with some uniform estimates of solutions on a certain compact interval $[\tau - 1, \tau]$ for $\tau \in \mathbb{R}$. The uniform absorption of the family of random cocycles φ_ε is proved. Note that the notations (u, v), $(\tilde{u}, \tilde{v})$, and φ are the abbreviations of $(u_\varepsilon, v_\varepsilon)$, $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ and φ_ε respectively, where the later implies the dependence of solutions on ε, omitting the subscript ε if there is no confusion.

Lemma 4.1. *Assume that (3.1)-(3.5) holds and $a > 0$. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\delta$, then there exists a constant $T = T(\tau,\omega,D) \ge 2$ such that for all $t \ge T$, $\varepsilon \in (0,a]$, and $(\tilde{u}_{\tau-t}, \tilde{v}_{\tau-t}) \in D(\tau - t, \vartheta_{-t}\omega)$, the solution $(u_\varepsilon, v_\varepsilon)$ of problem (3.8)-(3.9) satisfies*

$$\|(u_\varepsilon(\xi, \tau-t, \vartheta_{-\tau}\omega, u_{\tau-t}), v_\varepsilon(\xi, \tau-t, \vartheta_{-\tau}\omega, v_{\tau-t}))\|^2 \le ce^{2\varepsilon\omega(-\tau)}(1 + L_\varepsilon(\tau,\omega)), \xi \in [\tau-1, \tau], \tag{4.2}$$

$$\|(\tilde{u}_\varepsilon(\xi,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t}),\tilde{v}_\varepsilon(\xi,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_{\tau-t}))\|^2\leq c(1+L_\varepsilon(\tau,\omega)),\quad \xi\in[\tau-1,\tau], \tag{4.3}$$

and

$$\int_{\tau-t}^{\xi} e^{\delta(s-\tau)}\Big(\|v_\varepsilon(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2 + z_\varepsilon^{2-p}(s,\vartheta_{-\tau}\omega)\|u_\varepsilon(s,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\|_p^p\Big)ds\leq ce^{2\varepsilon\omega(-\tau)}(1+L_\varepsilon(\tau,\omega)), \tag{4.4}$$

where $(u_{\tau-t},v_{\tau-t})=z_\varepsilon(\tau-t,\vartheta_{-\tau}\omega)(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t})$*, and* $L_\varepsilon(\tau,\omega)$ *is given by*

$$L_\varepsilon(\tau,\omega)=\int_{-\infty}^{0} e^{\delta_{01}s+2\varepsilon|\omega(s)|}\Big(\|g(s+\tau,.)\|^2+\|h(s+\tau,.)\|^2+1\Big)ds, \tag{4.5}$$

such that $\varepsilon\to L_\varepsilon(\tau,\omega)$ is an increasing function on $(0,+\infty)$, where $0<\delta_0<\delta_{01}<\delta_1<\delta$.

In particular, the family of random cocycles $\varphi_\varepsilon(\varepsilon\in(0,a])$ *defined by (3.10) is uniformly absorbing on* $(0,a]$ *for any* $a>0$ *in the sense of Definition 2.6.*

Proof Taking the inner products of (3.8) and (3.9) with u and v, respectively, by using (3.1), we have

$$\frac{d}{dt}(\beta\|u\|^2+\alpha\|v\|^2)+\delta(\beta\|u\|^2+\alpha\|v\|^2) + 2\alpha_1\beta z^{2-p}(t,\omega)\|u\|_p^p\leq cz^2(t,\omega)(\|g(t,.)\|^2+\|h(t,.)\|^2+\|\psi_1\|_1). \tag{4.6}$$

By applying the Gronwall lemma over the interval $[\tau-t,\xi]$ with $\xi\in[\tau-1,\tau]$ and $t\geq 1$, along with ω replaced by $\vartheta_{-\tau}\omega$, we get from (3.7) that

$$\begin{aligned}
&\|u(\xi,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2+\|v(\xi,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2\\
&+\int_{\tau-t}^{\xi} e^{-\delta(\xi-s)}(\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2+z^{2-p}(s,\vartheta_{-\tau}\omega)\|u(s,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|_p^p)ds\\
&\leq ce^{-\delta(\xi-\tau+t)}(\|u_{\tau-t}\|^2+\|v_{\tau-t}\|^2)+c\int_{\tau-t}^{\xi} e^{-\delta(\xi-s)}z^2(s,\vartheta_{-\tau}\omega)(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds\\
&\leq ce^{2\varepsilon\omega(-\tau)}\Big(e^{-\delta t}z^2(-t,\omega)(\|\tilde{u}_{\tau-t}\|^2+\|\tilde{v}_{\tau-t}\|^2)+\int_{\tau-t}^{\tau} e^{-\delta(\tau-s)-2\varepsilon\omega(s-\tau)}(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds\Big)\\
&\leq ce^{2\varepsilon\omega(-\tau)}\Big(e^{-\delta t}z^2(-t,\omega)(\|\tilde{u}_{\tau-t}\|^2+\|\tilde{v}_{\tau-t}\|^2)+\int_{-t}^{0} e^{\delta_{01}s-2\varepsilon\omega(s)}(\|g(s+\tau,.)\|^2+\|h(s+\tau,.)\|^2+1)ds\Big),
\end{aligned} \tag{4.7}$$

where $\delta = \min\{\lambda, \sigma\}$ and $\delta_{01} < \delta$. By (3.6) we calculate that $\lim_{t\to+\infty} e^{-(\delta-\delta_1)t} z^2(-t,\omega) = 0$. Then from the property of $\mathcal{D}_\delta$ in (3.11) it follows that

$$\lim_{t\to+\infty} e^{-\delta t} z^2(-t,\omega)(\|\tilde{u}_{\tau-t}\|^2 + \|\tilde{v}_{\tau-t}\|^2) = 0. \tag{4.8}$$

Thus (4.7) and (4.8) together implies that there exists a random constant $T = T(\tau,\omega,D) \geq 1$ such that for each $\varepsilon \in (0,a]$ and all $t \geq T$, (4.2) and (4.4) hold. By (3.7) and (4.2) it is showed that (4.3) hold true for all $t \geq T$.

On the other hand, from (3.5) and (3.6) it follow that the integral in $L_\varepsilon(\tau,\omega)$ is meaningful and thus $L_\varepsilon(\tau,\omega)$ is finite. Further,

$$\begin{aligned} L_\varepsilon(\tau-t,\vartheta_{-t}\omega) &= \int_{-\infty}^{0} e^{\delta_{01}s+2\varepsilon|\vartheta_{-t}\omega(s)|}\Big(\|g(s+\tau-t,.)\|^2 + \|h(s+\tau-t,.)\|^2 + 1\Big)ds \\ \text{(letting } s-t=s'\text{)} &\leq e^{\delta_{01}t+2\varepsilon|\omega(-t)|} \int_{-\infty}^{-t} e^{\delta_{01}s+2\varepsilon|\omega(s)|}\Big(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1\Big). \end{aligned} \tag{4.9}$$

We see from (3.6) and the relation $\delta_0 < \delta_{01}$ that $\lim_{s\to-\infty} e^{-(\delta_{01}-\delta_0)s+2\varepsilon|\omega(s)|} = 0$, so there exists a positive variable $a(\omega)$ such that

$$0 < e^{-(\delta_{01}-\delta_0)s+2\varepsilon|\omega(s)|} \leq a(\omega), \quad s \in (-\infty, 0],$$

from which it follows that

$$L_\varepsilon(\tau-t,\vartheta_{-t}\omega) \leq a(\omega) e^{\delta_{01}t+2\varepsilon|\omega(-t)|} \int_{-\infty}^{-t} e^{\delta_0 s}\Big(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1\Big)ds. \tag{4.10}$$

Then from (3.11),(3.6), (3.5) and (4.10) we deduce that

$$K_\varepsilon = \{K_\varepsilon(\tau,\omega) = \{(\tilde{u},\tilde{v}) \in (L^2(\mathbb{R}))^2;\ \|(\tilde{u},\tilde{v})\|^2 \leq c(1+L_\varepsilon(\tau,\omega))\}; \tau\in\mathbb{R}, \omega\in\Omega\} \in \mathcal{D}_\delta,$$

and further the union $\overline{\cup_{\varepsilon\in(0,a]} K_\varepsilon(\tau,\omega)} \subset K_a(\tau,\omega)$. Thus $\overline{K} = \{\overline{\cup_{\varepsilon\in(0,a]} K_\varepsilon(\tau,\omega)}; \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\delta$. The measurability of the absorbing set $K_\varepsilon(\tau,\omega)$ follows from the measurability of the variable $L_\varepsilon(\tau,\omega)$. Finally since by (4.5) and (3.5) and the relation $\delta_0 < \delta_{01}$,

$$\limsup_{\varepsilon\to0} \|K_\varepsilon(\tau,\omega)\| \leq L_0(\tau,\omega) = \int_{-\infty}^{0} e^{\delta_{01}s}|\Big(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1\Big)ds < +\infty,$$

then we have showed the uniformly absorbing of $\varphi_\varepsilon(\varepsilon \in (0,a])$ for any $a > 0$. $\square$

The uniformly asymptotic compactness in $L^2 \times L^2$ has been proved by [1].

Lemma 4.2. *Assume that (3.1)-(3.5) hold. Then the family of random cocycles φ_ε defined by (3.10) is uniformly pullback asymptotically compact over $\varepsilon \in (0,a]$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.*

4.2. **Uniformly asymptotic compactness in $L^p \times L^2$.** In this subsection, we prove that the family of random cocycles $\varphi_\varepsilon(\varepsilon(0,a])$ is uniformly asymptotically compact in $L^p \times L^2$. We need to prove the L^p-uniform boundedness of the first component of solution u_ε as well as the uniform smallness of truncation of u_ε in L^p norm.

Lemma 4.3. *Assume that (3.1)-(3.5) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\delta$, then there exist some random constants $C = C(\tau,\omega)$ and $T = T(\tau,\omega,D) \geq 2$ such that for all $(\tilde{u}_{\tau-t}, \tilde{v}_{\tau-t}) \in D(\tau - t, \vartheta_{-t}\omega)$, the solution $(u_\varepsilon, v_\varepsilon)$ of problem (3.8)-(3.9) satisfies*

$$\sup_{t\geq T} \sup_{\xi\in[\tau-1,\tau]} \sup_{\varepsilon\in(0,a]} \|u_\varepsilon(\xi, \tau - t, \vartheta_{-\tau}\omega, u_{\tau-t})\|_p^p \leq C(\tau,\omega), \tag{4.11}$$

where $C(\tau,\omega)$ is independent ε.

Proof Multiplying (3.8) by $|u|^{p-2}u$ and then integrating over $\mathbb{R}^N$, we have

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + \lambda\|u\|_p^p \leq \alpha \int_{\mathbb{R}^N} v|u|^{p-1}dx$$
$$+ z(t,\omega) \int_{\mathbb{R}^N} f(x, z^{-1}(t,\omega)u)|u|^{p-2}udx + z(t,\omega) \int_{\mathbb{R}^N} g(t,x)|u|^{p-2}udx. \tag{4.12}$$

From (3.1) and $\psi \in L^{p/2}$, applying Young inequality, we obtain that

$$z(t,\omega) \int_{\mathbb{R}^N} f(x, z^{-1}(t,\omega)u)|u|^{p-2}udx \leq -\alpha_1 z^{2-p}(t,\omega)\|u\|_{2p-2}^{2p-2} + \frac{\lambda}{4}\|u\|_p^p + cz^p(t,\omega)\|\psi_1\|_{p/2}^{p/2}. \tag{4.13}$$

On the other hand,

$$\alpha \int_{\mathbb{R}^N} v|u|^{p-1}dx \leq \frac{1}{4}\alpha_1 z^{2-p}(t,\omega)\|u\|_{2p-2}^{2p-2} + cz^{p-2}(t,\omega)\|v\|^2, \tag{4.14}$$

and

$$z(t,\omega) \int_{\mathbb{R}^N} g(t,x)|u|^{p-2}udx \leq \frac{1}{4}\alpha_1 z^{2-p}(t,\omega)\|u\|_{2p-2}^{2p-2} + cz^p(t,\omega)\|g(t,.)\|^2. \tag{4.15}$$

Then combination (4.12)-(4.15), it give that

$$\frac{d}{dt}\|u\|_p^p + \delta\|u\|_p^p \leq cz^{p-2}(t,\omega)\|v\|^2 + cz^p(t,\omega)(\|g(t,.)\|^2 + \|\psi_1\|_{p/2}^{p/2}), \tag{4.16}$$

where $\delta = \min\{\lambda, \sigma\}$. Note that $\frac{1}{\xi-\tau+2} \leq 1$ for $\xi \in [\tau - 1, \tau]$. Applying Gronwall lemma (see also Lemma 5.1 in [33]) over the interval $[\tau-2, \xi]$, along with ω replaced by $\vartheta_{-\tau}\omega$, we deduce that

$$\|u(\xi, \tau - t, \vartheta_{-\tau}\omega, u_{\tau-t})\|_p^p \leq c \int_{\tau-2}^{\tau} e^{\delta(s-\tau)}\|u(s, \tau - t, \vartheta_{-\tau}\omega, u_{\tau-t})\|_p^p ds$$
$$+ c \int_{\tau-2}^{\tau} e^{\delta(s-\tau)}z^{p-2}(s, \vartheta_{-\tau})\|v(s, \tau - t, \vartheta_{-\tau}\omega, v_{\tau-t})\|^2 ds$$

$$+c\int_{\tau-2}^{\tau} e^{\delta(s-\tau)}z^p(s,\vartheta_{-\tau})(\|g(s,.)\|^2+1)ds. \tag{4.17}$$

We now estimate every term on the right hand side of (4.17). First from (4.1) it follows that for all $s\in[\tau-2,\tau]$ and $\varepsilon\in(0,a]$, $z^{2-p}(s,\vartheta_{-\tau}\omega)=e^{\varepsilon(2-p)\omega(-\tau)}z^{2-p}(s-\tau,\omega)\geq e^{\varepsilon(2-p)\omega(-\tau)}F^{2-p}$. Then from (4.4), there exists $T=T(\tau,\omega,D)\geq 2$, such that for all $\varepsilon\in(0,a]$ and $t\geq T$,

$$\int_{\tau-2}^{\tau} e^{\delta(s-\tau)}\|u(s,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\|_p^p ds\leq cF^{p-2}e^{ap|\omega(-\tau)|}(1+L_a(\tau,\omega)). \tag{4.18}$$

Noticing that $z^{p-2}(s,\vartheta_{-\tau})\leq e^{(p-2)\varepsilon\omega(-\tau)}F^{p-2}$ for $s\in[\tau-2,\tau]$, then from (4.4) again we see that

$$\int_{\tau-2}^{\tau} e^{\delta(s-\tau)}z^{p-2}(s,\vartheta_{-\tau})\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2 ds\leq cF^{p-2}e^{ap|\omega(-\tau)|}(1+L_a(\tau,\omega)). \tag{4.19}$$

On the other hand, by (3.5),

$$\int_{\tau-2}^{\tau} e^{\delta(s-\tau)}z^p(s,\vartheta_{-\tau})(\|g(s,.)\|^2+1)ds\leq F^p e^{ap|\omega(-\tau)|}\int_{-2}^{0} e^{\delta s}(\|g(s+\tau,.)\|^2+1)ds<+\infty. \tag{4.20}$$

Hence (4.17)-(4.20) implies the desired. $\square$

Let $M=M(\tau,\omega)>0$. Denote by $(u-M)_+$ the positive part of $u-M$, *i.e.*,

$$(u-M)_+=\begin{cases} u-M, & \text{if } u>M;\\ 0, & \text{if } u\leq M.\end{cases}$$

The next lemma will show that the unbounded part of the absolute value $|u|$ approaches zero in L^p-norm on the state domain $\mathbb{R}^N(|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|\geq M)$ for M large enough, where

$$\mathbb{R}^N(|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|\geq M)=\{x\in\mathbb{R}^N;|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|\geq M|\}.$$

Note that we need not to prove some auxiliary lemmas except Lemma 4.1 and Lemma 4.3, see [16, 17, 18, 19, 34, 35].

Lemma 4.4. Assume that (3.1)-(3.5) hold. Given $\tau\in\mathbb{R},\omega\in\Omega$ and $D=\{D(\tau,\omega);\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_\delta$, then for any $\eta>0$, there exist random constants $M=M(\tau,\omega,\eta,D)>1$ and $T=T(\tau,\omega,D)\geq 2$ such that for all $(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t})\in D(\tau-t,\vartheta_{-t}\omega)$, the first component $\tilde{u}_\varepsilon$ of solutions $(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)$ of problem (1.1)-(1.3) satisfies

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,a]}\int_{\mathbb{R}^N(|\tilde{u}_\varepsilon|\geq M)}|\tilde{u}_\varepsilon(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t})|^p dx\leq\eta,$$

where $\mathbb{R}^N(|\tilde{u}_\varepsilon|\geq M)=\mathbb{R}^N(|\tilde{u}_\varepsilon(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t})|\geq M)$, and M,T are independent of ε.

Proof Let $s \in [\tau-1,\tau]$ and $t \geq T \geq 2$, where T is determined by Lemma 4.1 and Lemma 4.2. Replacing ω by $\vartheta_{-\tau}\omega$ in (3.8)-(3.9), we see that $u = u(s,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t}), v = v(s,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})$, is a solution of the following system

$$\frac{du}{ds} + \lambda u - \Delta u + \alpha v = \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} f(x,\tilde{u}) + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} g(s,x), \tag{4.21}$$

$$\frac{dv}{ds} + \sigma v - \beta u = \frac{z(s-\tau,\omega)}{z(-\tau,\omega)} h(s,x). \tag{4.22}$$

For fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we assume that $M = M(\tau,\omega) > 1$. We multiply (4.21) by $(u-M)_+^{p-1}$ and integrate over $\mathbb{R}^N$ to yield

$$\begin{aligned}
&\frac{1}{p}\frac{d}{ds}\int_{\mathbb{R}^N}(u-M)_+^p dx + \lambda\int_{\mathbb{R}^N} u(u-M)_+^{p-1}dx - \int_{\mathbb{R}^N}\Delta u(u-M)_+^{p-1}dx \\
&\qquad = -\alpha\int_{\mathbb{R}^N} v(u-M)_+^{p-1}dx + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} f(x,\tilde{u})(u-M)_+^{p-1}dx \\
&\qquad\quad + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} g(s,x)(u-M)_+^{p-1}dx.
\end{aligned} \tag{4.23}$$

We now have to estimate every term in (4.23). First, it is obvious that

$$-\int_{\mathbb{R}^N}\Delta u(u-M)_+^{p-1}dx = (p-1)\int_{\mathbb{R}^N}(u-M)_+^{p-2}|\nabla u|^2 dx \geq 0, \tag{4.24}$$

$$\lambda\int_{\mathbb{R}^N} u(u-M)_+^{p-1}dx \geq \lambda\int_{\mathbb{R}^N}(u-M)_+^p dx. \tag{4.25}$$

The most involved work is to calculate the nonlinearity in (4.23). Consider that for $u(s) > M$ for $\in [\tau-1,\tau]$, we have $\tilde{u}(s) = z^{-1}(s,\vartheta_{-\tau}\omega)u(s) = \frac{z(-\tau,\omega)}{z(s-\tau,\omega)}u(s) > 0$, and thus by (3.1) and (4.1), we find that for every $s \in [\tau-1,\tau]$,

$$\begin{aligned}
f(x,\tilde{u}) &\leq -\alpha_1|\tilde{u}|^{p-1} + \frac{1}{\tilde{u}}\psi_1(x) \\
&= -\alpha_1\Big(\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\Big)^{1-p}|u|^{p-1} + \frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\frac{\psi_1(x)}{u} \\
&\leq -\frac{\alpha_1 z^{p-1}(-\tau,\omega)}{2F^{p-1}}M^{p-2}(u-M) - \frac{\alpha_1 z^{p-1}(-\tau,\omega)}{2F^{p-1}}(u-M)^{p-1} \\
&\quad + \frac{F}{z(-\tau,\omega)}|\psi_1(x)|(u-M)^{-1},
\end{aligned}$$

from which and (4.1) again it follows that

$$\begin{aligned}
&\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int_{\mathbb{R}^N} f(x,\tilde{u})(u-M)_+^{p-1}dx \\
&\leq -\frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{2F^{p-1}}M^{p-2}\int_{\mathbb{R}^N}(u-M)_+^p dx - \frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{2F^{p-1}}\int_{\mathbb{R}^N}(u-M)_+^{2p-2}dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{F^2}{z^2(-\tau,\omega)}\int\limits_{\mathbb{R}^N}|\psi_1(x)|(u-M)_+^{p-2}dx\\
\le&-\frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{2F^{p-1}}M^{p-2}\int\limits_{\mathbb{R}^N}(u-M)_+^p dx-\frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{2F^{p-1}}\int\limits_{\mathbb{R}^N}(u-M)_+^{2p-2}dx\\
&+\frac{1}{2}\lambda\int\limits_{\mathbb{R}^N}(u-M)_+^p dx+\frac{cF^p}{z^p(-\tau,\omega)}\int\limits_{\mathbb{R}^N(u\ge M)}|\psi_1(x)|^{p/2}dx, \qquad (4.26)
\end{aligned}
$$

in which we have used the Young inequality in the last term, and here $c=(\frac{2}{\lambda})^{p-2/2}$. On the other hand by using the Young inequality again, we get that for $s\in[\tau-1,\tau]$,

$$
\begin{aligned}
&\Big|\frac{z(s-\tau,\omega)}{z(-\tau,\omega)}\int\limits_{\mathbb{R}^N}g(s,x)(u-M)_+^{p-1}dx\Big|\le\frac{F}{z(-\tau,\omega)}\Big|\int\limits_{\mathbb{R}^N}g(s,x)(u(s)-M)_+^{p-1}dx\Big|\\
&\le\frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{4F^{p-1}}\int\limits_{\mathbb{R}^N}(u-M)_+^{2p-2}dx+\frac{F^{p+1}}{\alpha_1 E z^{p+1}(-\tau,\omega)}\int\limits_{\mathbb{R}^N(u(s)\ge M)}g^2(s,x)dx, \qquad (4.27)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|-\alpha\int\limits_{\mathbb{R}^N}v(u-M)_+^{p-1}dx\Big|\le&\frac{\alpha_1 E z^{p-1}(-\tau,\omega)}{4F^{p-1}}\int\limits_{\mathbb{R}^N}(u-M)_+^{2p-2}dx\\
&+\frac{\alpha^2F^{p-1}}{\alpha_1 E z^{p-1}(-\tau,\omega)}\int\limits_{\mathbb{R}^N(u(s)\ge M)}v^2dx. \qquad (4.28)
\end{aligned}
$$

For convenience of calculations, we introduce the following notations:

$$
k=k(\tau,\omega,M)=\frac{\alpha_1 E e^{-(p-1)a|\omega(-\tau)|}}{2F^{p-1}}M^{p-2}; \qquad (4.29)
$$

which is increasing to infinite in M for $p>2$, and

$$
G(\tau,\omega)=\max\Big\{\frac{F^{p+1}e^{a(p+1)|\omega(-\tau)|}}{\alpha_1 E};\frac{\alpha^2F^{p-1}e^{a(p-1)|\omega(-\tau)|}}{\alpha_1 E};(\frac{2}{\lambda})^{\frac{p-2}{2}}F^pz^{ap|\omega(-\tau)|}\Big\}, \qquad (4.30)
$$

which is a nonnegative random constant depending only on τ,ω. Combination (4.23)-(4.28) and using the notations (4.29)-(4.30), we obtain that

$$
\frac{d}{ds}\int\limits_{\mathbb{R}^N}(u(s)-M)_+^p dx+k\int\limits_{\mathbb{R}^N}(u(s)-M)_+^p dx\le G(\tau,\omega)(\|g(s,.)\|^2+\|v\|^2+1), \qquad (4.31)
$$

where $s\in[\tau-1,\tau]$. Applying Gronwall lemm (also see Lemma 5.1 in [33]) over $[\tau-1,\tau]$, by Lemma 4.1 and Lemma 4.3, we find that for all $t\ge T$ and $\varepsilon\in(0,a]$,

$$
\begin{aligned}
\int\limits_{\mathbb{R}^N}\Big(u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})-M\Big)_+^p dx\le&\int\limits_{\tau-1}^{\tau}e^{k(s-\tau)}\|u(s,\tau-t,\vartheta_{-\tau}\omega,u_0)\|_p^p ds\\
&+G(\tau,\omega)\Big(\int\limits_{\tau-1}^{\tau}e^{k(s-\tau)}\|v(s,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t})\|^2ds+\int\limits_{\tau-1}^{\tau}e^{k(s-\tau)}(\|g(s,.)\|^2+1)ds\Big)
\end{aligned}
$$

$$\leq \frac{C(\tau,\omega)+cG(\tau,\omega)e^{2a|\omega(-\tau)|}(1+L_a(\tau,\omega))+G(\tau,\omega)}{k}$$
$$+\,G(\tau,\omega)\int\limits_{\tau-1}^{\tau} e^{k(s-\tau)}\|g(s,.)\|^2ds. \tag{4.32}$$

For fixed $\tau\in\mathbb{R}$ and $\omega\in\Omega$, the first term in the last inequality of (4.32) varies only with the number k, but k is a large number as $M\to+\infty$. Therefore, we get that this term converges to zero when M goes to infinite. It remains to prove that the second term vanishes for M large enough. First, choosing a large M such that $k=k(\tau,\omega,M)>\delta_0$ (δ_0 is as in (3.5)) and taking $\varsigma\in(0,1)$, we have

$$\begin{aligned}\int\limits_{\tau-1}^{\tau} e^{k(s-\tau)}\|g(s,.)\|^2ds &= \int\limits_{\tau-1}^{\tau-\varsigma} e^{k(s-\tau)}\|g(s,.)\|^2ds+\int\limits_{\tau-\varsigma}^{\tau} e^{k(s-\tau)}\|g(s,.)\|^2ds\\ &= e^{-k\tau}\int\limits_{\tau-1}^{\tau-\varsigma} e^{(k-\delta_0)s}e^{\delta s}\|g(s,.)\|^2ds+e^{-k\tau}\int\limits_{\tau-\varsigma}^{\tau} e^{ks}\|g(s,.)\|^2ds\\ &\leq e^{-k\varsigma}e^{\delta_0(\varsigma-\tau)}\int\limits_{-\infty}^{\tau} e^{\delta_0 s}\|g(s,.)\|^2ds+\int\limits_{\tau-\varsigma}^{\tau}\|g(s,.)\|^2ds.\end{aligned} \tag{4.33}$$

By (3.5), the first term above vanishes as $k\to+\infty$, and by $g\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^N))$ we can choose ς small enough such that the second term in (4.33) is small. In terms of these arguments, from (4.32)and (4.33) we have proved that

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,a]}\int\limits_{\mathbb{R}^N}\Big(u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})-M\Big)_+^p dx\to 0, \tag{4.34}$$

when $M\to+\infty$. Therefore, for any $\eta>0$, there exists $M_1=M_1(\tau,\omega,\eta,D)>1$ large enough such that

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,a]}\int\limits_{\mathbb{R}^N}\Big(u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})-M_1\Big)_+^p dx\leq e^{-ap|\omega(-\tau)|}\frac{\eta}{2^{p+1}}. \tag{4.35}$$

If $u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\geq 2M_1$, then $u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})-M_1\geq\frac{u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})}{2}$, so by (4.35) it infer us that

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,a]}\int\limits_{\mathbb{R}^N(u(\tau)\geq 2M_1)}|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|^p dx\leq e^{-ap|\omega(-\tau)|}\frac{\eta}{2}. \tag{4.36}$$

We see from (3.7) that $\tilde u(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde u_{\tau-t})=z(-\tau,\omega)u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})$. Then in terms of the fact that $e^{-a|\omega(-\tau)|}\leq z(-\tau,\omega)=e^{-\varepsilon\omega(-\tau)}\leq e^{a|\omega(-\tau)|}$ for all $\varepsilon\in(0,a]$, it induce that $\mathbb{R}^N(\tilde u(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde u_{\tau-t})\geq 2M_1e^{a|\omega(-\tau)|})\subseteq\mathbb{R}^N(u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\geq 2M_1)$. This along with (4.36) implies that,

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,a]}\int\limits_{\mathbb{R}^N(\tilde u(\tau)\geq 2M_1e^{a|\omega(-\tau)|})}|\tilde u(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde u_{\tau-t})|^p dx$$

$$\leq \sup_{t\geq T}\sup_{\varepsilon\in(0,a]} e^{ap|\omega(-\tau)|}\int\limits_{\mathbb{R}^N(u(\tau)\geq 2M_1)}|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})|^p dx\leq\frac{\eta}{2}. \quad (4.37)$$

Similarly, we can deduce that there exists $M_2=M_2(\tau,\omega,\eta,D)>0$ large enough such that

$$\sup_{t\geq T}\sup_{\varepsilon\in(0,1]}\int\limits_{\mathbb{R}^N(\tilde{u}(\tau)\leq -2M_2e^{a|\omega(-\tau)|})}|\tilde{u}(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t})|^p dx\leq\frac{\eta}{2}. \quad (4.38)$$

Put $M=\max\{M_1,M_2\}\times e^{a|\omega(-\tau)|}$. Then (4.37) and (4.38) together imply the desired. $\square$

Lemma 4.5. *Assume that (3.1)-(3.5) hold. Then for every $\tau\in\mathbb{R},\omega\in\Omega$, $\{\tilde{u}_{\varepsilon_n}(\tau,\tau-t_n,\vartheta_{-t_n}\omega,\tilde{u}_{0,n})\}$ has a convergent subsequence in $L^p(\mathbb{R}^N)$ whenever $\varepsilon_n\in(0,a]$, $t_n\to+\infty$ and $(\tilde{u}_{0,n},\tilde{v}_{0,n})\in D(\tau-t_n,\vartheta_{-t_n}\omega)\in\mathcal{D}_\delta$.*

Proof. Denote by $\tilde{u}_n(\tau)=\tilde{u}_{\varepsilon_n}(\tau,\tau-t_n,\vartheta_{-\tau}\omega,\tilde{u}_{0,n})$. From Lemma 4.3, for any $\eta>0$, there exist random constants $M=M(\tau,\omega,\eta,D)>1$ and $\mathcal{Z}_1=\mathcal{Z}_1(\tau,\omega,D)\in\mathbb{Z}^+$ such that the solution $\tilde{u}_n(\tau)$ satisfies that for all $n\geq\mathcal{Z}_1$,

$$\sup_{\varepsilon_n\in(0,a]}\int\limits_{\mathbb{R}^N(|\tilde{u}_n(\tau)|\geq M)}|\tilde{u}_n(\tau)|^p dx\leq\frac{\eta^p}{2^{p+2}}. \quad (4.39)$$

On the other hand, Lemma 4.2 also implies that there exists a $\mathcal{Z}_2=\mathcal{Z}_1(\tau,\omega,B)\in\mathbb{Z}^+$ such that for all $n,m\geq\mathcal{Z}_2$,

$$\int\limits_{\mathbb{R}^N}|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|^2 dx\leq\frac{1}{(2M)^{p-2}}\frac{\eta^p}{4}, \quad (4.40)$$

whenever $\varepsilon_n,\varepsilon_m\in(0,a]$. Here M is as in (4.39). We then decompose the entire space $\mathbb{R}^N$ by $\mathbb{R}^N=\mathcal{O}_1\cup\mathcal{O}_2\cup\mathcal{O}_3\cup\mathcal{O}_4$, where

$$\mathcal{O}_1=\mathbb{R}^N(|\tilde{u}_n(\tau)|\leq M)\cap\mathcal{O}^N(|\tilde{u}_m(\tau)|\leq M);\quad \mathcal{O}_2=\mathbb{R}^N(|u_n(\tau)|\geq M)\cap\mathbb{R}^N(|\tilde{u}_m(\tau)|\leq M);$$

$$\mathcal{O}_3=\mathbb{R}^N(|\tilde{u}_n(\tau)|\leq M)\cap\mathbb{R}^N(|\tilde{u}_m(\tau)|\geq M);\quad \mathcal{O}_4=\mathbb{R}^N(|\tilde{u}_n(\tau)|\geq M)\cap\mathbb{R}^N(|\tilde{u}_m(\tau)|\geq M).$$

We now put $\mathcal{Z}=\max\{\mathcal{Z}_1,\mathcal{Z}_2\}$. Then for all $n,m\geq\mathcal{Z}$, (4.39) and (4.40) hold true. By (4.40), we have

$$\int\limits_{\mathcal{O}_1}|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|^p dx\leq\int\limits_{\mathbb{R}^N(|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|\leq 2M)}|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|^p dx$$
$$\leq(2M)^{p-2}\|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)\|^2\leq(2M)^{p-2}.(2M)^{2-p}(\frac{\eta^p}{4})=\frac{\eta^p}{4}. \quad (4.41)$$

On the other hand, according to (4.39),

$$\int\limits_{\mathcal{O}_2}|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|^p dx\leq 2^p\int\limits_{\mathbb{R}^N(|\tilde{u}_n(\tau)|\geq M)}|\tilde{u}_n(\tau)|^p dx\leq\frac{\eta^p}{4}, \quad (4.42)$$

$$\int\limits_{\mathcal{O}_3}|\tilde{u}_n(\tau)-\tilde{u}_m(\tau)|^p dx\leq 2^p\int\limits_{\mathbb{R}^N(|\tilde{u}_m(\tau)|\geq M)}|\tilde{u}_m(\tau)|^p dx\leq\frac{\eta^p}{4}, \quad (4.43)$$

$$\int\limits_{\mathcal{O}_4} |\tilde{u}_n(\tau) - \tilde{u}_m(\tau)|^p dx \leq 2^{p-1}\Big(\int\limits_{\mathbb{R}^N(|u(\tau)|\geq M)} |\tilde{u}_n(\tau)|^p dx + \int\limits_{\mathbb{R}^N(|\tilde{u}_m(\tau)|\geq M)} |\tilde{u}_m(\tau)|^p dx\Big) \leq \frac{\eta^p}{4}. \tag{4.44}$$

It follows form (4.41)-(4.44) that

$$\|\tilde{u}_n(\tau) - \tilde{u}_m(\tau)\|_p \leq \eta \quad \text{for all } n, m \geq \mathcal{Z},$$

whenever $\varepsilon_n, \varepsilon_m \in (0, a]$, which shows that $\{\tilde{u}_n(\tau)\}$ also has a convergent subsequence in $L^p(\mathbb{R}^N)$. Then the proof is concluded. $\square$

By Lemma 4.2 and Lemma 4.5 we immediately have

Lemma 4.6. *Assume that (3.1)-(3.5) hold. Then the family of random cocycles φ_ε defined by (3.10) is uniformly pullback asymptotically compact over $\varepsilon \in (0, a]$ in $L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. In particular, each φ_ε has a unique $(L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$-pullback attractor $\mathcal{A}_\varepsilon$*

Proof The uniformly pullback asymptotic compactness is followed from Lemma 4.2 and Lemma 4.4, and the existence and uniqueness of bi-spatial pullback attractor are from Theorem 2.9, or Theorem 2.10. $\square$

4.3. **Convergence of the family φ_ε on $(0, a]$ in $L^2 \times L^2$.** This subsection deal with the convergence of solutions at any intension ε of noise. The convergence at zero has been shown by [1]. Here we need to prove it also converges at any $\varepsilon > 0$. To this end, the following assumption on the nonlinearity f as in [2] is also required. That is, for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$\Big|\frac{\partial}{\partial s}(x, s)\Big| \leq \alpha_4 |s|^{p-2} + \psi_4(x), \tag{4.45}$$

where $\alpha_4 > 0$, $\psi_4 \in L^\infty(\mathbb{R}^N)$ if $p = 2$ and $\psi_4 \in L^{\frac{p}{p-2}}(\mathbb{R}^N)$ if $p > 2$. We need further to assume that $\psi_2 \in L^q(\mathbb{R}^N)$, where ψ_2 is as in (3.2) and $q = \frac{p}{p-1}$ is conjugation of p.

To begin with, from (4.6), it is very easy to derive the following inequality.

Lemma 4.7. *Assume that (3.1)-(3.5) hold. Then for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(\tilde{u}_\varepsilon(\tau), \tilde{v}_\varepsilon(\tau)) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, the solution (u, v) of problem (3.8) satisfies for all $t \geq \tau$,*

$$\begin{aligned}
&\|u_\varepsilon(t, \tau, \omega, u(\tau))\|^2 + \|v_\varepsilon(t, \tau, \omega, v(\tau))\|^2 \\
&\qquad + \int_\tau^t e^{\delta(s-t)}\Big(\|v_\varepsilon(s, \tau, \omega, v_\varepsilon(\tau))\|^2 + z_\varepsilon^2(s, \omega)\|\tilde{u}_\varepsilon(s, \tau, \omega, \tilde{u}_\varepsilon(\tau))\|_p^p\Big) ds \\
&\leq z_\varepsilon^2(\tau, \omega)(\|\tilde{u}_\varepsilon(\tau)\|^2 + \|\tilde{v}_\varepsilon(\tau)\|^2) + c\int_\tau^t z_\varepsilon^2(s, \omega)(\|g(s, .)\|^2 + \|h(s, .)\|^2 + 1) ds.
\end{aligned}$$

Then by applying Lemma 4.7 we have

Lemma 4.8. *Assume that (3.1)-(3.5) and (4.45) hold. Let $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ be the solution of problem (3.8)-(3.9) with initial data $(\tilde{u}_{\varepsilon,\tau}, \tilde{v}_{\varepsilon,\tau})$. Assume that $\varepsilon \to \varepsilon_0$ and $\|(\tilde{u}_{\varepsilon,\tau}, \tilde{v}_{\varepsilon,\tau}) - (\tilde{u}_{\varepsilon_0,\tau}, \tilde{v}_{\varepsilon_0,\tau})\| \to 0$ for $\varepsilon, \varepsilon_0 \in (0, a]$. Then for each $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$ and every $t \in [\tau, \tau + T]$,*

$$\lim_{\varepsilon\to\varepsilon_0} \|(\tilde{u}_\varepsilon(t,\tau,\omega,\tilde{u}_{\varepsilon,\tau}), \tilde{v}_\varepsilon(t,\tau,\omega,\tilde{v}_{\varepsilon,\tau})) - (\tilde{u}_{\varepsilon_0}(t,\tau,\omega,\tilde{u}_{\varepsilon_0,\tau}), \tilde{v}_{\varepsilon_0}(t,\tau,\omega,\tilde{v}_{\varepsilon_0,\tau}))\| = 0. \tag{4.46}$$

In particular, let $(\tilde{u}, \tilde{v})$ be the solution of problem (3.8)-(3.9) for $\varepsilon = 0$ with initial data $(\tilde{u}_\tau, \tilde{v}_\tau)$. Assume that $\varepsilon \to 0$ and $\|(\tilde{u}_{\varepsilon,\tau}, \tilde{v}_{\varepsilon,\tau})) - (\tilde{u}_\tau, \tilde{v}_\tau)\| \to 0$. Then for each $\tau \in \mathbb{R}$, $\omega \in \Omega, T > 0$

$$\lim_{\varepsilon\to 0} \|(\tilde{u}_\varepsilon(t,\tau,\omega,\tilde{u}_{\varepsilon,\tau}), \tilde{v}_\varepsilon(t,\tau,\omega,\tilde{v}_{\varepsilon,\tau})) - (\tilde{u}(t,\tau,\tilde{u}_\tau), \tilde{v}(t,\tau,\tilde{v}_\tau))\| = 0. \tag{4.47}$$

Proof Put $U = U(t) = u_\varepsilon(t,\tau,\omega,u_{\varepsilon,\tau}) - u_{\varepsilon_0}(t,\tau,\omega,u_{\varepsilon_0,\tau})$ and $V = V(t) = v_\varepsilon(t,\tau,\omega,v_{\varepsilon,\tau}) - v_{\varepsilon_0}(t,\tau,\omega,v_{\varepsilon_0,\tau})$. Then we get the following system:

$$\begin{cases} \frac{dU}{dt} + \lambda U - \Delta U + \alpha V = e^{-\varepsilon\omega(t)} f(x, e^{\varepsilon\omega(t)} u_\varepsilon) - e^{-\varepsilon_0\omega(t)} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0}) \\ \qquad\qquad\qquad\qquad + (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) g(t,x), \\ \frac{dV}{dt} + \sigma V - \beta U = (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) h(t,x), \end{cases} \tag{4.48}$$

where $u_\varepsilon = u_\varepsilon(t) = u_\varepsilon(t,\tau,\omega,u_{\varepsilon,\tau})$. Let η be a small positive number. Since ω is continuous on $\mathbb{R}$, then there exists a $\chi = \chi(\tau,\omega,\eta,T) > 0$ such that for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi) \subset (0, a]$ and $t \in [\tau, \tau + T]$,

$$|e^{\varepsilon\omega(t)} - e^{\varepsilon_0\omega(t)}| + |e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}| \le \eta. \tag{4.49}$$

By (4.48), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\beta\|U\|^2 + \alpha\|V\|^2) + \lambda\beta\|U\|^2 + \sigma\alpha\|V\|^2 \\ &\le \int_{\mathbb{R}^N} \Big(e^{-\varepsilon\omega(t)} f(x, e^{\varepsilon\omega(t)} u_\varepsilon) - e^{-\varepsilon_0\omega(t)} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0})\Big) U dx \\ &+ (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int_{\mathbb{R}^N} g(t,x) U dx + (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int_{\mathbb{R}^N} h(t,x) V dx. \end{aligned} \tag{4.50}$$

The first term on the right hand side of (4.50) is rewritten as

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big(e^{-\varepsilon\omega(t)} f(x, e^{\varepsilon\omega(t)} u_\varepsilon) - e^{-\varepsilon_0\omega(t)} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0})\Big) U dx \\ &= e^{-\varepsilon\omega(t)} \int_{\mathbb{R}^N} \Big(f(x, e^{\varepsilon\omega(t)} u_\varepsilon) - f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0})\Big) U dx + (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int_{\mathbb{R}^N} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0}) U dx \\ &\quad = e^{-\varepsilon\omega(t)} \int_{\mathbb{R}^N} \frac{\partial}{\partial s} f(x,s)(e^{\varepsilon\omega(t)} u_\varepsilon - e^{\varepsilon_0\omega(t)} u_{\varepsilon_0}) U dx + (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int_{\mathbb{R}^N} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0}) U dx \\ &\qquad = \int_{\mathbb{R}^N} \frac{\partial}{\partial s} f(x,s) U^2 dx + (e^{-\varepsilon_0\omega(t)} - e^{-\varepsilon\omega(t)}) \int_{\mathbb{R}^N} \frac{\partial}{\partial s} f(x,s) \tilde{u}_{\varepsilon_0} U dx \end{aligned}$$

$$+ (e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int\limits_{\mathbb{R}^N} f(x, e^{\varepsilon_0\omega(t)} u_{\varepsilon_0}) U dx. \tag{4.51}$$

By (4.45) and (4.49), the second term on the right hand side of (4.51) is bounded by

$$\begin{aligned}
&(e^{-\varepsilon_0\omega(t)} - e^{-\varepsilon\omega(t)}) \int\limits_{\mathbb{R}^N} \frac{\partial}{\partial s} f(x, s) \tilde{u}_{\varepsilon_0} U dx \\
&\qquad \le |e^{-\varepsilon_0\omega(t)} - e^{-\varepsilon\omega(t)}| \int\limits_{\mathbb{R}^N} \Big(\alpha_4 |\tilde{u}_\varepsilon + \tilde{u}_{\varepsilon_0}|^{p-2} |\tilde{u}_{\varepsilon_0}||U| + |\tilde{u}_{\varepsilon_0}||U|\psi_4|\Big) dx \\
&\qquad \le c\eta\Big(\|\tilde{u}_\varepsilon\|_p^p + \|\tilde{u}_{\varepsilon_0}\|_p^p + \|U\|_p^p + \|\psi_4\|_{\frac{p}{p-2}}^{\frac{p}{p-2}}\Big).
\end{aligned} \tag{4.52}$$

By (3.2) and (4.49),connection with $\psi_2 \in L^q$, the third term on the right hand side of (4.51) is bounded by

$$\begin{aligned}
(e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int\limits_{\mathbb{R}^N} f(x, \tilde{u}_{\varepsilon_0}) U dx &\le |e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}| \int\limits_{\mathbb{R}^N} (\varepsilon_2 |\tilde{u}_{\varepsilon_0}|^{p-1} + \psi_2)|U| dx \\
&\le c\eta\Big(\|\tilde{u}_{\varepsilon_0}\|_p^p + \|U\|_p^p + \|\psi_2\|_q^q\Big),
\end{aligned} \tag{4.53}$$

where $q = \frac{p}{p-1}$. Then combination (4.51)-(4.53), we find that for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi)$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \Big(e^{-\varepsilon\omega(t)} f(x, \tilde{u}_\varepsilon) - e^{-\varepsilon_0\omega(t)} f(x, \tilde{u}_{\varepsilon_0})\Big) U dx \\
&\qquad \le \alpha_3 \|U\|^2 + c\eta + c\eta\Big(\|\tilde{u}_\varepsilon\|_p^p + \|\tilde{u}_{\varepsilon_0}\|_p^p + \|U\|_p^p\Big) \\
&\qquad \le \alpha_3 \|U\|^2 + c_0\eta\Big(\|\tilde{u}_\varepsilon\|_p^p + \|\tilde{u}_{\varepsilon_0}\|_p^p + \|U\|_p^p\Big).
\end{aligned} \tag{4.54}$$

For the last two terms on the right hand side of (4.51), by (4.49), we have for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi)$ and $t \in [\tau, \tau + T]$,

$$(e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int\limits_{\mathbb{R}^N} g(t, x) U dx \le \eta \|U\|^2 + \eta \|g(t, .)\|^2, \tag{4.55}$$

$$(e^{-\varepsilon\omega(t)} - e^{-\varepsilon_0\omega(t)}) \int\limits_{\mathbb{R}^N} h(t, x) V dx \le \eta \|V\|^2 + \eta \|h(t, .)\|^2. \tag{4.56}$$

Then by (4.50) and (4.54)-(4.56), we get that for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi)$ and $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\frac{d}{dt}(\beta\|U\|^2 + \alpha\|V\|^2) \le c_1(\beta\|U\|^2 + \alpha\|V\|^2) \\
&\quad + c_2\eta\Big(\|\tilde{u}_\varepsilon\|_p^p + \|\tilde{u}_{\varepsilon_0}\|_p^p + \|u_{\varepsilon_0}\|_p^p + \||u_\varepsilon\|_p^p + \|g(t, .)\|^2 + \|h(t, .)\|^2\Big),
\end{aligned} \tag{4.57}$$

where c_1 and c_2 are positive constants independent of τ, ω and ε. By (4.57) we immediately have for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi)$ and $t \in [\tau, \tau + T]$,

$$\|U(t)\|^2 + \|V(t)\|^2 \le c_3 e^{c_1(t-\tau)}(\|U(\tau)\|^2 + \|V(\tau)\|^2)$$

$$+ c_4\eta e^{c_1(t-\tau)}\int_\tau^t \Big(\|\tilde{u}_\varepsilon(s)\|_p^p+\|\tilde{u}_{\varepsilon_0}(s)\|_p^p+\||u_\varepsilon(s)\|_p^p+\|u_{\varepsilon_0}(s)\|_p^p+\|g(s,.)\|^2+\|h(s,.)\|^2\Big)ds. \tag{4.58}$$

Since $e^{-\varepsilon\omega(s)}$ is continuous on $\mathbb{R}$, then for every fixed $\tau\in\mathbb{R}$ and $\omega\in\Omega$, and $s\in[\tau,\tau+T]$, there exist $\mu=\mu(\tau,\omega,T)$ and $\nu=\nu(\tau,\omega,T)$ such that for all $\varepsilon\in(0,a]$, $\mu\le z_\varepsilon(s,\omega)\le\nu$ for all $s\in[\tau,\tau+T]$. Therefor by Lemma 4.7, it follows that for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\int_\tau^t \|\tilde{u}_\varepsilon(s,\tau,\omega,u_\varepsilon(\tau))\|_p^p ds\\
&\le \mu^{-2}e^{\delta(t-\tau)}\int_\tau^t e^{\delta(s-t)}z_\varepsilon^2(s,\omega)\|\tilde{u}_\varepsilon(s,\tau,\omega,u_\varepsilon(\tau))\|_p^p ds\\
&\le \mu^{-2}e^{\delta(t-\tau)}\Big(z_\varepsilon^2(\tau,\omega)(\|\tilde{u}_\varepsilon(\tau)\|^2+\|\tilde{v}_\varepsilon(\tau)\|^2)+c\nu^2\int_\tau^t(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds\Big)\\
&\le \mu^{-2}e^{\delta(t-\tau)}\Big(e^{2a|\omega(\tau)|}(\|\tilde{u}_\varepsilon(\tau)\|^2+\|\tilde{v}_\varepsilon(\tau)\|^2)+c\nu^2\int_\tau^t(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds\Big).
\end{aligned} \tag{4.59}$$

By a similar technique we can calculate that for all $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\int_\tau^t \|u_\varepsilon(s,\tau,\omega,u_\varepsilon(\tau))\|_p^p ds\\
&\le \mu^{p-2}e^{\delta(t-\tau)}\Big(e^{2a|\omega(\tau)|}(\|\tilde{u}_\varepsilon(\tau)\|^2+\|\tilde{v}_\varepsilon(\tau)\|^2)+c\nu^2\int_\tau^t(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds\Big).
\end{aligned} \tag{4.60}$$

Then by (4.58-(4.60) it gives that

$$\begin{aligned}
&\|U(t)\|^2+\|V(t)\|^2\le c_3e^{c_1(t-\tau)}(\|U(\tau)\|^2+\|V(\tau)\|^2)\\
&+c_5\eta e^{c_6(t-\tau)}\Big(e^{2a|\omega(\tau)|}(\mu^{-2}+\mu^{p-2})(\|\tilde{u}_\varepsilon(\tau)\|^2+\|\tilde{u}_{\varepsilon_0,\tau}\|^2+\|\tilde{v}_\varepsilon(\tau)\|^2+\|\tilde{v}_{\varepsilon_0,\tau}\|^2)\\
&\quad+\nu^2(\mu^{-2}+\mu^{p-2})\int_\tau^t(\|g(s,.)\|^2+\|h(s,.)\|^2+1)ds+\int_\tau^t(\|g(s,.)\|^2+\|h(s,.)\|^2)ds\Big).
\end{aligned} \tag{4.61}$$

On the other hand, by (4.49) it follows that for every $\varepsilon\in(\varepsilon_0-\chi,\varepsilon_0+\chi)\subset(0,a]$,

$$\begin{aligned}
\|U(\tau)\|^2&=\|e^{-\varepsilon\omega(\tau)}\tilde{u}_\varepsilon(\tau)-e^{-\varepsilon_0\omega(\tau)}\tilde{u}_{\varepsilon_0,\tau}\|^2\\
&\le 2e^{-2\varepsilon\omega(\tau)}\|\tilde{u}_\varepsilon(\tau)-\tilde{u}_{\varepsilon_0,\tau}\|^2+2|e^{-\varepsilon\omega(\tau)}-e^{-\varepsilon_0\omega(\tau)}|^2\|\tilde{u}_{\varepsilon_0,\tau}\|^2\\
&\le 2e^{2a|\omega(\tau)|}\|\tilde{u}_\varepsilon(\tau)-\tilde{u}_{\varepsilon_0,\tau}\|^2+2\eta^2\|\tilde{u}_{\varepsilon_0,\tau}\|^2.
\end{aligned} \tag{4.62}$$

Similarly,

$$\|V(\tau)\|^2 \leq 2e^{2a|\omega(\tau)|}\|\tilde{v}_\varepsilon(\tau) - \tilde{v}_{\varepsilon_0,\tau}\|^2 + 2\eta^2\|\tilde{v}_{\varepsilon_0,\tau}\|^2. \tag{4.63}$$

We now let $\varepsilon \to \varepsilon_0$ and $\|u_{\varepsilon,\tau} - u_{\varepsilon_0,\tau}\| \to 0$. Then by (4.61)-(4.63) we obtain that for all $t \in [\tau, \tau + T]$,

$$\begin{aligned}\|U(t)\|^2 + \|V(t)\|^2 &= \|u_\varepsilon(t,\tau,\omega,u_{\varepsilon,\tau}) - u_{\varepsilon_0}(t,\tau,\omega,u_{\varepsilon_0,\tau})\|^2 \\ &\quad + \|v_\varepsilon(t,\tau,\omega,v_{\varepsilon,\tau}) - v_{\varepsilon_0}(t,\tau,\omega,v_{\varepsilon_0,\tau})\|^2 \to 0.\end{aligned} \tag{4.64}$$

Notice that by (4.49) we also have for every $\varepsilon \in (\varepsilon_0 - \chi, \varepsilon_0 + \chi)$ and $t \in [\tau, \tau + T]$,

$$\|\tilde{u}_\varepsilon(t,\tau,\omega,\tilde{u}_\varepsilon(\tau)) - \tilde{u}_{\varepsilon_0}(t,\tau,\omega,\tilde{u}_{\varepsilon_0,\tau})\|^2 \leq 2e^{2a|\omega(t)|}\|u_\varepsilon(t) - u_{\varepsilon_0}(t)\|^2 + 2\eta^2\|u_{\varepsilon_0}(t)\|^2, \tag{4.65}$$

$$\|\tilde{v}_\varepsilon(t,\tau,\omega,\tilde{v}_\varepsilon(\tau)) - \tilde{v}_{\varepsilon_0}(t,\tau,\omega,\tilde{v}_{\varepsilon_0,\tau})\|^2 \leq 2e^{2a|\omega(t)|}\|v_\varepsilon(t) - v_{\varepsilon_0}(t)\|^2 + 2\eta^2\|v_{\varepsilon_0}(t)\|^2. \tag{4.66}$$

Then by (4.64)-(4.66) we get (4.46). Repeating the same arguments we can derive (4.47). □

4.4. **Main results.** We are now at the point to present the main results in this paper.

Theorem 4.9. *Suppose $\varepsilon \in \mathbb{R}$ and (3.1)-(3.5) hold true. Then*

(i) Each random cocycle φ_ε generated by (1.1)-(1.3) has a unique pullback attractor $\mathcal{A}_\varepsilon$ and the corresponding deterministic cocycle φ_0 has a unique pullback attractor $\mathcal{A}_0$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, both $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ are $(L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N), L^l(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$-pullback attractors.

(ii) If further (4.45) holds true, then the family $\mathcal{A}_\varepsilon$ is upper semi-continuous under the Hausdorff semi-distance of $L^l(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ at any $\varepsilon_0 \in \mathbb{R}$. Here $l \in (2, p], p > 2$.

Proof Let $X = L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and $L^l(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Then it is known that the hypothesises (*H1*)) and (*H2*)) (see Lemma 2.7 in [34]) hold true. By the Sobolev interpolation and association with Lemma 4.2 and Lemma 4.6, we immediately obtain the uniformly pullback asymptotic compactness in $L^l(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ for any $2 < l < p$. Then along with uniform absorption (see Lemma 4.1) and convergence property (see Lemma 4.8), all conditions of Theorem 2.10 are satisfied. □

5. Existence of random equilibria for the generated random cocycle

Random equilibrium is a special case of omega-limit sets. We can refer to [3, 6] for the definitions and applications to monotone random dynamical system. The problem of the construction of equilibria for a general random dynamical system is rather complicate [6]. Recently, Gu [13] proved that the stochastic FitzHugh-Nagumo lattice equations driven by fractional Brownian motions possess a unique equilibrium. However, we here introduce the random equilibrium in the case of non-autonomous stochastic dynamical system. Specifically, we have

Definition 5.1. *Let $(Q, \{\sigma_t\}_{\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ be parametric dynamical systems. A random variable $u^* : Q \times \Omega \mapsto X$ is said to be an equilibrium (or fixed point, or stationary solution) of random cocycle φ if it is invariant under φ, i.e., if*

$$\varphi(t, q, \omega, u^*(q, \omega)) = u^*(\sigma_t q, \vartheta_t \omega) \quad for\ all\ t \geq 0,\ q \in Q,\ \omega \in \Omega.$$

In this section, the parametric dynamical systems $(Q, \{\sigma_t\}_{\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ are same as in section 3. We will prove the existence of equilibrium for problem (1.1)-(1.3) on the whole space $\mathbb{R}^N$. To this end, we need to assume that

$$\delta = \min\{\lambda, \sigma\} > \alpha_3, \quad \beta \geq 1, \tag{5.1}$$

where α_3 is as in (3.3) and λ, σ, β are as in the FitzHugh-Nagumo system (1.1)-(1.2).

For convenience, here we write $\varepsilon = 1$. First, we have

Lemma 5.2. *Suppose that g and h satisfy (3.5) and f satisfies (3.1) and (3.3) such that (5.1) holds. Then there exists a positive constant $0 < b_0 < \delta - \alpha_3$ such that the solutions of problem (1.1)-(1.3) with initial values $(\tilde{u}_{\tau-t_i}, \tilde{v}_{\tau-t_i})(i = 1, 2), t_1 < t_2$ satisfy the following decay property:*

$$\begin{aligned}
&\|\tilde{u}(\tau, \tau - t_1, \vartheta_{-\tau}\omega, \tilde{u}_{\tau-t_1}) - \tilde{u}(\tau, \tau - t_2, \vartheta_{-\tau}\omega, \tilde{u}_{\tau-t_2})\|^2 \\
&+ \|\tilde{v}(\tau, \tau - t_1, \vartheta_{-\tau}\omega, \tilde{v}_{\tau-t_1}) - \tilde{v}(\tau, \tau - t_2, \vartheta_{-\tau}\omega, \tilde{v}_{\tau-t_2})\|^2 \\
&\leq c\Big(e^{-b_0 t_1} z^2(-t_1, \omega)(\|\tilde{u}_{\tau-t_1}\|^2 + \|\tilde{v}_{\tau-t_1}\|^2) + e^{-b_0 t_2} z^2(-t_2, \omega)(\|\tilde{u}_{\tau-t_2}\|^2 + \|\tilde{v}_{\tau-t_2}\|^2)\Big) \\
&+ ce^{(b_0-b)t_1} \int_{-\infty}^{0} e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + \|h(s + \tau, .)\|^2 + 1)ds,
\end{aligned}$$

where c is a deterministic non-random constant.

Proof Put $\bar{u} = u(t, \tau - t_1, \vartheta_{-\tau}\omega, u_{\tau-t_1}) - u(t, \tau - t_2, \vartheta_{-\tau}\omega, u_{\tau-t_2})$ and

$$\bar{v} = v(t, \tau - t_1, \vartheta_{-\tau}\omega, v_{\tau-t_1}) - v(t, \tau - t_2, \vartheta_{-\tau}\omega, v_{\tau-t_2}).$$

Then from (3.8)-(3.9), along with (5.1), we have

$$\frac{d}{dt}(\beta\|\bar{u}\|^2 + \alpha\|\bar{v}\|^2) + b(\beta\|\bar{u}\|^2 + \alpha\|\bar{v}\|^2) \leq 0, \tag{5.2}$$

where $b = \delta - \alpha_3$. By applying Gronwall lemma to (5.2) over the interval $[\tau - t_1, \tau]$, we immediately get

$$\begin{aligned}
\|\bar{u}(\tau)\|^2 + \|\bar{v}(\tau)\|^2 &\leq ce^{-bt_1}(\|u(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, u_{\tau-t_2}) - u_{\tau-t_1}\|^2 \\
&\quad + \|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, v_{\tau-t_2}) - v_{\tau-t_1}\|^2) \\
&\leq ce^{-bt_1}(\|u(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, u_{\tau-t_2})\|^2 \\
&\quad + \|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, v_{\tau-t_2})\|^2) + ce^{-bt_1}(\|u_{\tau-t_1}\|^2 + \|v_{\tau-t_1}\|^2),
\end{aligned} \tag{5.3}$$

where $c = c(\alpha, \beta)$ is a positive deterministic constant. Write

$$0 < \delta_0 < b_0 < b = \delta - \alpha_3, \tag{5.4}$$

where δ_0 is as in (3.5). From (4.6) and using (5.4), we have

$$\frac{d}{dt}(\beta\|u\|^2 + \alpha\|v\|^2) + b_0(\beta\|u\|^2 + \alpha\|v\|^2) \leq cz^2(t, \omega)(\|g(t, .)\|^2 + \|h(t, .)\|^2 + 1). \tag{5.5}$$

Then by Gronwall lemma again over the interval $[\tau - t_2, \tau - t_1]$, we find that

$$\begin{aligned}
&\|u(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, u_{\tau-t_2})\|^2 + \|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau}\omega, v_{\tau-t_2})\|^2 \\
&\quad \le ce^{b_0(t_1-t_2)}(\|u_{\tau-t_2}\|^2 + \|v_{\tau-t_2}\|^2) \\
&\quad + c\int_{\tau-t_2}^{\tau-t_1} e^{-b_0(\tau-t_1-s)} z^2(s, \vartheta_{-\tau}\omega)(\|g(s,.)\|^2 + \|h(s,.)\|^2 + 1)ds \\
&\quad \le ce^{b_0(t_1-t_2)}(\|u_{\tau-t_2}\|^2 + \|v_{\tau-t_2}\|^2) \\
&\quad + ce^{b_0 t_1} e^{2\omega(-\tau)} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds,
\end{aligned} \tag{5.6}$$

where $c = c(\alpha, \beta)$ is a positive deterministic constant. Then combination (5.6) and (5.3) we have

$$\begin{aligned}
\|\bar u(\tau)\|^2 + \|\bar v(\tau)\|^2 &\le ce^{-bt_1}(\|u_{\tau-t_1}\|^2 + \|v_{\tau-t_1}\|^2) + ce^{(b_0-b)t_1} e^{-b_0 t_2}(\|u_{\tau-t_2}\|^2 + \|v_{\tau-t_2}\|^2) \\
&+ ce^{(b_0-b)t_1} e^{2\omega(-\tau)} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds \\
&\le ce^{-b_0 t_1}(\|u_{\tau-t_1}\|^2 + \|v_{\tau-t_1}\|^2) + e^{-b_0 t_2}(\|u_{\tau-t_2}\|^2 + \|v_{\tau-t_2}\|^2) \\
&+ ce^{(b_0-b)t_1} e^{2\omega(-\tau)} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds,
\end{aligned} \tag{5.7}$$

where we have used $e^{(b_0-b)t_1} \le 1$ for $b_0 < b$. In terms of the relation (3.7), we get

$$\begin{aligned}
\|\bar{\bar u}(\tau)\|^2 + \|\bar{\bar v}(\tau)\|^2 &\le e^{-2\omega(-\tau)}\|\bar u(\tau)\|^2 + \|\bar v(\tau)\|^2 \\
&\le ce^{-2\omega(-\tau)}\Big(e^{-b_0 t_1}(\|u_{\tau-t_1}\|^2 + \|v_{\tau-t_1}\|^2) + e^{-b_0 t_2}(\|u_{\tau-t_2}\|^2 + \|v_{\tau-t_2}\|^2)\Big) \\
&+ ce^{(b_0-b)t_1} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds \\
&= ce^{-2\omega(-\tau)}\Big(e^{-b_0 t_1} z^2(\tau - t_1, \vartheta_{-\tau}\omega)(\|\tilde u_{\tau-t_1}\|^2 + \|\tilde v_{\tau-t_1}\|^2) \\
&+ e^{-b_0 t_2} z^2(\tau - t_2, \vartheta_{-\tau}\omega)(\|\tilde u_{\tau-t_2}\|^2 + \|\tilde v_{\tau-t_2}\|^2)\Big) \\
&+ ce^{(b_0-b)t_1} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds \\
&= c\Big(e^{-b_0 t_1} z^2(-t_1,\omega)(\|\tilde u_{\tau-t_1}\|^2 + \|\tilde v_{\tau-t_1}\|^2) + e^{-b_0 t_2} z^2(-t_2,\omega)(\|\tilde u_{\tau-t_2}\|^2 + \|\tilde v_{\tau-t_2}\|^2)\Big) \\
&+ ce^{(b_0-b)t_1} \int_{-\infty}^{0} e^{b_0 s} z^2(s,\omega)(\|g(s+\tau,.)\|^2 + \|h(s+\tau,.)\|^2 + 1)ds,
\end{aligned}$$

which finishes the proof. $\square$

Lemma 5.3. *Suppose that g and h satisfy (3.5), f satisfies (3.1) and (3.3) such that (5.1) holds. Let $D = \{D(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_\delta$ where $\mathcal{D}_\delta$ is defined as in (3.11). Then for $\tau \in \mathbb{R}, \omega \in \Omega$, there exists a unique element $u^* = u^*(\tau,\omega) \in$*

$L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that

$$\lim_{t\to+\infty}(\tilde{u}(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t}),\tilde{u}(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t})) = u^*(\tau,\omega), \quad \text{in } L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N),$$

where $(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}) \in D(\tau-t,\vartheta_{-t}\omega)$. Furthermore, the convergence is uniform (w.r.t. $(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}) \in D(\tau-t,\vartheta_{-t}\omega)$).

Proof We choose the constant b_0 in Lemma 5.2 satisfying $b > b_0 > \delta_1$, where δ_1 is as in (3.11). If $(\tilde{u}_{\tau-t_i},\tilde{v}_{\tau-t_i}) \in D(\tau-t_i,\vartheta_{-t_i}\omega)$, $i=1,2$, then similar to (4.8) we have

$$\lim_{t_i\to+\infty} e^{-b_0 t_i} z^2(-t_i,\omega)\|(\tilde{u}_{\tau-t_i},\tilde{v}_{\tau-t_i})\|^2 = 0, i=1,2.$$

Thus the result is derived directly from Lemma 5.2. □

Lemma 5.4. *Suppose that g and h satisfies (3.5), f satisfies (3.1) and (3.3) such that (5.1) holds. Then for $\tau\in\mathbb{R},\omega\in\Omega$, the element $u^* = u^*(\tau,\omega)$ defined in Lemma 5.3 is a unique random equilibrium for the cocycle φ defined by (3.10) in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$, i.e.,*

$$\varphi(t,\tau,\omega,u^*(\tau,\omega)) = u^*(\tau+t,\vartheta_t\omega), \quad \text{for every } \ t\geq 0,\ \tau\in\mathbb{R},\ \omega\in\Omega.$$

Furthermore, the random equilibrium $\{u^*(\tau,\omega),\tau\in\mathbb{R},\ \omega\in\Omega\}$ is the unique element of the pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega);\tau\in\mathbb{R},\ \omega\in\Omega\}$ for the random cocycle φ, i.e., for every $\tau\in\mathbb{R},\omega\in\Omega$, $\mathcal{A}(\tau,\omega) = \{u^*(\tau,\omega)\}$.

Proof From the definition of random cocycle,

$$\varphi(t,\tau-t,\vartheta_{-t}\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t})) = (\tilde{u}(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{u}_{\tau-t}),\tilde{v}(\tau,\tau-t,\vartheta_{-\tau}\omega,\tilde{v}_{\tau-t})),$$

then for for every $\tau\in\mathbb{R},\omega\in\Omega$, we see from Lemma 5.3 that

$$u^*(\tau,\omega) = \lim_{t\to+\infty}\varphi(t,\tau-t,\vartheta_{-t}\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t})), \tag{5.8}$$

where $(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}) \in D(\tau-t,\vartheta_{-t}\omega)$. Thus by the continuity and the cocycle property of φ and (5.8), we find that for every $t\geq 0,\tau\in\mathbb{R},\omega\in\Omega$,

$$\begin{aligned}\varphi(t,\tau,\omega,u^*(\tau,\omega)) &= \varphi(t,\tau,\omega,.)\circ\lim_{s\to+\infty}\varphi(s,\tau-s,\vartheta_{-s}\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}))\\ &= \lim_{s\to+\infty}\varphi(t,\tau,\omega,.)\circ\varphi(s,\tau-s,\vartheta_{-s}\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}))\\ &= \lim_{s\to+\infty}\varphi(t+s,\tau-s,\vartheta_{-s}\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t}))\\ &= \lim_{s\to+\infty}\varphi(t+s,(\tau+t)-t-s,\vartheta_{-s-t}\vartheta_t\omega,(\tilde{u}_{\tau-t},\tilde{v}_{\tau-t})) = u^*(\tau+t,\vartheta_t\omega),\end{aligned}$$

which also implies the invariance of $\mathcal{A}$, that is, $\varphi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\vartheta_t\omega)$. The compactness of $\mathcal{A}(\tau,\omega)$ is obvious and the attracting property follows from (5.8). □

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