A global Torelli theorem for singular symplectic varieties

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ABSTRACT. We systematically study the moduli theory of symplectic varieties (in the sense of Beauville) which admit a resolution by an irreducible symplectic manifold. In particular, we prove an analog of Verbitsky's global Torelli theorem for the locally trivial deformations of such varieties. Verbitsky's work on ergodic complex structures replaces twistor lines as the essential global input. In so doing we extend many of the local deformation-theoretic results known in the smooth case to such (not-necessarily-projective) symplectic varieties. We deduce a number of applications to the birational geometry of symplectic manifolds, including some results on the classification of birational contractions of $K3^{[n]}$-type varieties.

CONTENTS

1. INTRODUCTION

The local and global deformation theories of irreducible holomorphic symplectic manifolds enjoy many beautiful properties. For example, unobstructedness of deformations [Bo78, Ti87, To89] and the local Torelli theorem [Be83] are at the origin of many results on symplectic manifolds. Among the highlights of the global theory are Huybrechts' surjectivity of the period map [Hu99] and Verbitsky's global Torelli theorem [Ve13] (see for example Markman's survey article [Ma11] and Huybrechts' Bourbaki talk [Hu11]). Verbitsky's result has since paved the way for many important developments, with applications to a wide variety of questons such as: birational boundedness [Ch16], lattice polarized mirror symmetry [Ca16], algebraic cycles [CP14], hyperbolicity questions [KLV14] and many more. Recent progress in MMP and interest

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in singular symplectic varieties [GKP11] has made it apparent that a global deformation theory of singular symplectic varieties would be equally valuable. As for smooth varieties, it is of utmost importance here to consider not-necessarily-projective Kähler varieties—even if one is mainly interested in moduli spaces of projective varieties, see e.g. Remark 5.21.

In the present article we initiate a systematic study of the moduli theory of compact Kähler singular symplectic varieties, beginning with those that admit symplectic resolutions by irreducible symplectic manifolds. We prove general results concerning their deformation theory and building on this we develop a global moduli theory for locally trivial families of such varieties. To fix ideas, recall that a *symplectic variety* X in the sense of Beauville [Be00, Definition 1.1] is a normal complex variety whose regular part admits a nondegenerate holomorphic 2-form that extends to some (hence any) resolution. Usually the extension will have zeroes, but if there is a resolution $\pi : Y \longrightarrow X$ by an (irreducible) holomorphic symplectic manifold Y, we call π an *(irreducible) symplectic resolution.*

Namikawa shows [Na01b, Theorem 2.2] that a projective variety X admitting a resolution $\pi : Y \longrightarrow X$ by a symplectic manifold Y has unobstructed deformations, and that there is a natural finite map $\mathrm{Def}(Y) \longrightarrow \mathrm{Def}(X)$. However, a generic deformation of π becomes an isomorphism, and therefore the only natural period map is that of the resolution Y. From a Hodge-theoretic perspective it is therefore more natural to consider the *locally trivial* deformations of X as in this case the pure weight two Hodge structure on $H^2(X, \mathbb{Z})$ varies in a local system, and the resulting theory is very closely analogous to the smooth situation. Our first result is the following

Theorem 1.1 (see Propositions 4.5 and 5.7). *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution of a symplectic variety X and $N = N_1(Y/X) \subset H_2(Y, \mathbb{Z})$ the group of 1-cycles contracted by π. The base space $\mathrm{Def}^{\mathrm{lt}}(X)$ of the universal locally trivial deformation is smooth, and there is a diagram*

$$
\begin{array}{ccc}
\mathscr{Y} & \longrightarrow & \mathscr{X} \\
\downarrow & & \downarrow \\
\mathrm{Def}(Y, N) & \xrightarrow{\;\cong\;} & \mathrm{Def}^{\mathrm{lt}}(X)
\end{array}
$$

where $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ is the universal locally trivial deformation of X, $\mathscr{Y} \longrightarrow \mathrm{Def}(Y, N) \subset \mathrm{Def}(Y)$ is the restriction of the universal deformation of Y to the closed subspace along which N remains algebraic, and $\mathscr{Y} \longrightarrow \mathscr{X}$ specializes to π.

The case of a divisorial contraction $\pi : Y \longrightarrow X$ of a projective symplectic variety was treated by Pacienza and the second author in [LP16, Proposition 2.3], where it was shown that locally trivial deformations of X correspond to deformations of Y such that all irreducible components of the exceptional divisor $\mathrm{Exc}(\pi)$ of π deform along. The space $\mathrm{Def}^{\mathrm{lt}}(X)$ of locally trivial deformations of X is smooth of dimension $h^{1,1}(X) = h^{1,1}(Y) - m$ where m is the number of irreducible components of $\mathrm{Exc}(\pi)$.

This description is equivalent to that of Theorem 1.1, since the Beauville–Bogomolov–Fujiki form q_Y on $H^2(Y, \mathbb{Z})$ yields an isomorphism $\widetilde{q}_Y : H^2(Y, \mathbb{Q}) \xrightarrow{\cong} H_2(Y, \mathbb{Q})$ identifying the subspace of $H^2(Y, \mathbb{Q})$ spanned by $\mathrm{Exc}(\pi)$ with $N_{\mathbb{Q}}$. In the case where π is a small contraction, X is not $\mathbb{Q}$-factorial, but we can still recast the description in Theorem 1.1 in terms of line bundles: under $\widetilde{q}_Y$, the Hodge classes in the orthogonal $N^\perp \subset H^2(Y, \mathbb{Q})$ are the $\mathbb{Q}$-line bundles on Y that vanish on N and can be pushed forward to $\mathbb{Q}$-line bundles on X (as the singularities are rational).

An important theorem of Huybrechts [Hu03, Theorem 2.5] shows that birational [1] irreducible holomorphic symplectic manifolds are deformation equivalent. Of course, birational singular symplectic varieties are not necessarily locally trivially deformation equivalent, and the correct analog of Huybrechts' theorem is the following.

Theorem 1.2 (See Theorem 4.9)**.** *Let $\pi : Y \longrightarrow X$ and $\pi' : Y' \longrightarrow X'$ be irreducible symplectic resolutions of symplectic varieties X, X'. Assume that there is a birational map $\phi : Y \dashrightarrow Y'$ such that the induced map $\phi^* : H^2(Y', \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$ sends $H^2(X', \mathbb{C})$ isomorphically to $H^2(X, \mathbb{C})$. Then X and X' are locally trivial deformations of one another.*

Let us now turn to global moduli theoretic questions. Fixing a lattice Λ, there is a natural locally trivial Λ-marked moduli space $\mathfrak{M}_\Lambda^{\mathrm{lt}}$ obtained by gluing the universal locally trivial deformation spaces together, and we also have an associated notion of parallel transport operator. Further, there is a period map $P : \mathfrak{M}_\Lambda^{\mathrm{lt}} \longrightarrow \Omega_\Lambda$ to the associated period domain Ω_Λ that is a local isomorphism.

The following is an analog of Verbitsky's global Torelli theorem [Ve13, Theorem 1.17], also encompassing analogs of Huybrechts' surjectivity of the period map [Hu99, Theorem 8.1], and Markman's description of the fibers of the period map in terms of the Kähler cone decomposition [Ma11, Theorem 5.16].

Theorem 1.3. *Let X be a symplectic variety with $b_2(X) > 4$ that admits an irreducible symplectic resolution. Let $\mathfrak{N}_\Lambda^{\mathrm{lt}} \subset \mathfrak{M}_\Lambda^{\mathrm{lt}}$ be a connected component containing (X, ν). Then*

(1) *The period map $P : \mathfrak{N}_\Lambda^{\mathrm{lt}} \longrightarrow \Omega_\Lambda$ is surjective, generically injective, and the points in any fiber are pairwise nonseparated. Moreover, varieties underlying points in the same fiber are birational.*

(2) *The points in the fiber containing (X, ν) are in bijective correspondence with the cones obtained by restricting the Kähler chambers of a resolution to $H^{1,1}(X, \mathbb{R})$.*

(3) *The locally trivial weight two monodromy group $\mathrm{Mon}^2(X)^{\mathrm{lt}}$ is a finite index subgroup of $\mathrm{O}(H^2(X, \mathbb{Z}))$.*

See Section 5 for more precise statements. The finiteness of the index of the monodromy group in the smooth case is proved by Verbitsky [Ve19, Theorem 2.6].

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[1] We will use the term birational instead of birational also for compact Kähler varieties.

The analogous results in the smooth case heavily rely on the existence of a hyperkähler metric, the theory of twistor lines, and deformations of complex structures. This presents a major difficulty for singular varieties as the aforementioned techniques are not available or much less understood as in the smooth case. We therefore deduce the largeness of the image of the period map by density results built on Ratner theory (as first explored in this context by Verbitsky in [Ve15], see e.g. Theorem 3.9 and Theorem 4.8 of that paper, also compare to Proposition 3.11 in Section 5 of the present paper), and this is responsible for the numerical conditions on $b_2(X)$. To the best of our knowledge this is the first general result in this direction which makes a statement about *large* deformations of singular symplectic varieties. Note that X with $b_2(X) = 3$ are locally trivially rigid in the sense that their only locally trivial deformation is what should be the (unique) twistor deformation, and unlike in the smooth case we can give examples (see Remark 5.10). These density results have many applications, and for instance we have the following application pointed out by Amerik–Verbitsky [AV19, Theorem 5.6].

Theorem 1.4 (See Corollary 5.9)**.** *Let Y be a smooth hyperkähler manifold with $b_2(Y) > 5$ and $\tau \subset H^{1,1}(Y, \mathbb{R})$ a face of the Kähler cone meeting the positive cone for which[2] $\dim \tau > 2$. Then there is a bimeromorphic contraction $\pi : Y \longrightarrow X$ contracting precisely $\tau^\perp$.*

Somewhat surprisingly, we also obtain a generalization to the non-projective setting of a result of Namikawa, see [Na01b, Theorem (2.2)].

Theorem 1.5 (See Proposition 5.23)**.** *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution with $b_2(X) > 4$. Then $\mathrm{Def}(X)$ is smooth of the same dimension as $\mathrm{Def}(Y)$ and the induced map $p : \mathrm{Def}(Y) \longrightarrow \mathrm{Def}(X)$ is finite.*

This is remarkable because one important ingredient in Namikawa's proof of the corresponding statement for projective varieties is a vanishing theorem by Steenbrink which we do not know to hold in the Kähler case. Our proof works by reduction to the projective case using the monodromy action.

Much more can be said when X admits a resolution deformation equivalent to a Hilbert scheme of points on a $K3$ surface. The possible projective contractions $M \to X$ of a moduli space of sheaves M on a $K3$ surface is completely described by wall-crossing in the space of Bridgeland stability conditions by work of Bayer–Macrì [BM14b]; Theorem 1.1 combined with density results then implies that no new singularities occur for general X:

Theorem 1.6 (See Proposition 6.6)**.** *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution where Y is a $K3^{[n]}$-type manifold. Provided $b_2(X) > 4$, X is locally trivially*

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[2]By this we mean the dimension of the real linear subspace generated by τ. Note that in this case we know the Kähler cone is locally polyhedral in the positive cone—see Remark 5.2.

deformation equivalent to a wall-crossing contraction of a moduli space of Bridgeland stable objects on a K3 surface.

Extremal contractions are particularly amenable to the lattice theory involved, and we for example have the following generalization of a beautiful result of Arbarello and Saccà [AS18, Theorem 1.1 part i)]:

Theorem 1.7 (See Proposition 6.8). *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution where Y is a $K3^{[n]}$-type manifold, and assume $\operatorname{rk} N_1(Y/X) = 1$. Then for any closed point $x \in X$ the analytic germ (X, x) is isomorphic to that of a Nakajima quiver variety.*

As another nice application, we are able to rule out the existence of certain contractions on $K3^{[n]}$-type varieties:

Theorem 1.8 (See Corollary 6.11). *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution where Y is a $K3^{[n]}$-type manifold, and assume $\operatorname{rk} N_1(Y/X) = 1$. Then for a generic closed point $x \in X^{\mathrm{sing}}$ the analytic germ (X, x) is isomorphic to $(S, 0) \times (\mathbb{C}^{2n-2}, 0)$ where $(S, 0)$ is an A_1-surface singularity.*

The surprising fact is that relative Picard rank one contractions whose generic singularity is transversally an A_2-surface singularity *do not occur* on irreducible symplectic manifolds while on other symplectic manifolds they do, see Corollary 6.11 and the discussion thereafter.

We expect most of our general results to apply to symplectic varieties not necessarily admitting a symplectic resolution by using a $\mathbb{Q}$-factorial terminalization in place of the resolution but we do not pursue that level of generality here. We do note however that this would fit very nicely with another result of Namikawa [Na06, Main Theorem, p. 97] that every flat deformation of a $\mathbb{Q}$-factorial terminal projective symplectic variety is locally trivial. As for the applications, we only restrict to $K3^{[n]}$-type varieties for simplicity, and our theory should yield similar results for the Kummer and O'Grady types. These directions are the topic of a forthcoming article.

Outline. In Sections 2 and 3 we explain some basic facts about symplectic varieties admitting irreducible symplectic resolutions and their Hodge structures. We recall that the Hodge structure on the second cohomology of such a symplectic variety is always pure and prove the degeneration of Hodge-de Rham spectral sequence for singular symplectic varieties in a suitable range. We also examine the Mumford–Tate group of a general irreducible symplectic manifold.

The locally trivial deformation theory of symplectic varieties admitting an irreducible symplectic resolution is studied in Section 4. This is one of the two technical centerpieces of the present paper. All our moduli theory builds on this in an essential way. The main results are the proof of smoothness of the Kuranishi space, the comparison of deformations of a symplectic variety admitting an irreducible symplectic

resolution and its resolution, the local Torelli theorem, and the analog of Huybrechts' theorem. Even though the deformations of a birational contraction can be characterized locally in the deformation space in terms of Noether-Lefschetz loci, it is—contrary to the surface case—nontrivial to show that a contraction deforms. We also include some remarks and applications to the question of existence of algebraically coisotropic subvarieties.

In Section 5 we introduce and investigate marked moduli spaces of locally trivial families of symplectic varieties admitting an irreducible symplectic resolution. We study the relation to moduli spaces of the resolution through compatibly marked moduli spaces of irreducible symplectic resolutions as well as various associated monodromy groups. The analysis of how these moduli spaces and their monodromy groups are connected is the second main ingredient of this work, and our analysis makes use of Verbitsky's ergodicity of complex structures (see [Ve15, Definition 1.12]) and Amerik-Verbitsky's concept of MBM classes (see [AV15, Definition 1.13]). It becomes clear at this point that it is not sufficient to study the moduli spaces of lattice polarized varieties as this perspective ignores the question of how the contraction deforms on two fronts: locally in terms of how the singularities smooth (and whether the contraction deforms at all) and globally in terms of how the monodromy group of the contraction differs from the isotropy group of the lattice polarization.

In Section 6 we give applications to $K3^{[n]}$-type varieties. We essentially make use of the description of the birational geometry of Bridgeland moduli spaces of stable objects by Bayer and Macrì. In principle, our methods allow to extend any result on singularities that can be proven for contractions of Bridgeland moduli spaces to arbitrary $K3^{[n]}$-type varieties. The above-mentioned generalization of Arbarello–Saccà's result is proven here.

Notation and Conventions. The term variety will denote an integral separated scheme of finite type over $\mathbb{C}$ in the algebraic setting or an irreducible and reduced Hausdorff complex space in the complex analytic setting.

2. Hodge theory of rational singularities

In this section we establish some basic facts about the Hodge structure on low degree cohomology groups of varieties with rational singularities. Recall that the Fujiki class $\mathscr{C}$

consists of all those compact complex varieties which are meromorphically dominated by a compact Kähler manifold, see [Fu78, §1]. This is equivalent to saying that there is a resolution of singularities by a compact Kähler manifold by Lemma 1.1 of op. cit. We will speak of a variety of Fujiki class for a variety in class $\mathscr{C}$. Let us also recall that by [Fu78, Corollary 1.7] for each $k \geq 0$ the singular cohomology of degree k of a smooth compact variety of Fujiki class carries a pure Hodge structure of weight k.

Let X be a variety carrying a pure Hodge structure on $H^{2k}(X, \mathbb{Z})$. We will write $H^{k,k}(X, \mathbb{Z})$ for the preimage of $H^{k,k}(X)$ under the map $H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{C})$. Recall that for $k = 1$ the *transcendental lattice* $H^2(X)_{\mathrm{tr}} \subset H^2(X, \mathbb{Z})$ is defined to be the smallest integral Hodge substructure such that $H^2(X, \mathbb{C})_{\mathrm{tr}} := H^2(X)_{\mathrm{tr}} \otimes \mathbb{C}$ contains $H^{2,0}(X)$.

The following lemma is well-known, we include it for the reader's convenience.

Lemma 2.1. *Let* $\pi : Y \rightarrow X$ *be a proper birational morphism where* X *is a complex variety with rational singularities. Then,* $\pi^* : H^1(X, \mathbb{Z}) \xrightarrow{\pi^*} H^1(Y, \mathbb{Z})$ *is an isomorphism and the sequence*

$$(2.1) \qquad 0 \longrightarrow H^2(X, \mathbb{Z}) \xrightarrow{\pi^*} H^2(Y, \mathbb{Z}) \longrightarrow H^0(X, R^2\pi_*\mathbb{Z})$$

is exact. In particular, if X *is compact and* Y *is a compact manifold of Fujiki class, then* $H^i(X, \mathbb{Z})$ *carries a pure Hodge structure for* $i = 1, 2$*. Moreover, the restriction* $\pi^* : H^2(X)_{\mathrm{tr}} \longrightarrow H^2(Y)_{\mathrm{tr}}$ *is an isomorphism, and* $\pi^* H^{1,1}(X, \mathbb{Z})$ *is the subspace of* $H^{1,1}(Y, \mathbb{Z})$ *of all classes that vanish on the classes of* π*-exceptional curves.*

Proof. The pushforward of the exponential sequence gives an exact sequence

$$\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow R^1\pi_*\mathbb{Z}_Y \longrightarrow R^1\pi_*\mathcal{O}_Y \longrightarrow \dots$$

where the map exp is again surjective. Thus, rationality of the singularities implies that $R^1\pi_*\mathbb{Z} = 0$. The Leray spectral sequence tells us now that π^* is an isomorphism on H^1 and that the sequence (2.1) is exact. As $H^i(Y, \mathbb{Z})$ carries a pure Hodge structure, also $H^i(X, \mathbb{Z})$ carries a pure Hodge structure for $i = 1, 2$ by strictness of morphisms of Hodge structures.

The last two statements follow from the proof and the statement of [KM92, (12.1.3) Theorem]. $\square$

For projective X the purity of the Hodge structure on the second cohomology was observed by Namikawa, see e.g. the proof of [Na06, Proposition 1]. See also [Sc16, Theorem 7] and [PR13, Lemma 3.1].

We will need to know to what extent the de Rham complex on either a resolution or the smooth part of a singular variety can be used to compute the singular cohomology and the Hodge decomposition on X. Note that there is an obvious morphism of complexes $\mathbb{C}_X \rightarrow \Omega_X^\bullet$, and for any resolution $\pi : Y \rightarrow X$ there is a natural pullback map of complexes $\Omega_X^\bullet \rightarrow \pi_*\Omega_Y^\bullet$. Likewise for an open embedding $j : U \rightarrow X$ whose complement has codimension ≥ 2, there is a pullback $\Omega_X^\bullet \rightarrow j_*\Omega_U^\bullet$. In the following

lemma, we will study the morphisms in cohomology coming from these morphisms of complexes.

Lemma 2.2. *Let X be a normal compact variety of Fujiki class with rational singularities and let us denote by $j : U = X^{\mathrm{reg}} \longrightarrow X$ the inclusion of the regular part. Then the following hold:*

(1) *For each $p \in \mathbb{N}_0$ the sheaf $j_* \Omega_U^p$ is a coherent and reflexive $\mathcal{O}_X$-module.*

(2) *For all $k \leq 2$ the canonical map $H^k(X, \mathbb{C}) \longrightarrow \mathbb{H}^k(X, j_* \Omega_U^\bullet)$ is an isomorphism and the Hodge-de Rham spectral sequence*

$$(2.2) \qquad E_1^{p,q} = H^q(j_* \Omega_U^p) \Rightarrow \mathbb{H}^{p+q}(X, j_* \Omega_U^\bullet)$$

degenerates on E_1 in the region where $p + q \leq 2$.

(3) *Let $\pi : Y \longrightarrow X$ be a resolution of singularities. Then for all $k \leq 2$ the canonical map $H^k(X, \mathbb{C}) \longrightarrow \mathbb{H}^k(X, \pi_* \Omega_Y^\bullet)$ is an isomorphism and the Hodge-de Rham spectral sequence*

$$(2.3) \qquad E_1^{p,q} = H^q(\pi_* \Omega_Y^p) \Rightarrow \mathbb{H}^{p+q}(X, \pi_* \Omega_Y^\bullet)$$

degenerates on E_1 in the region where $p + q \leq 2$.

Proof. Observe that Ω_U^p is locally free for each p, in particular torsion free. As X is normal, we infer that $j_* \Omega_U^p$ is coherent by Serre's result [Se66, Théorème 1]. Here we use that Ω_X is a coherent extension of Ω_U.

By [KS18, Corollary 1.8], we have $j_* \Omega_U^\bullet = \pi_* \Omega_Y^\bullet$ so that in particular $j_* \Omega_U^p$ is reflexive for every p and (1) follows. Moreover, (2) is a consequence of (3), and it suffices to prove the third statement. For every morphism $C_\bullet \to D_\bullet$ of complexes on Y the diagram

$$(2.4) \qquad \begin{array}{ccc} \pi_* C_\bullet & \longrightarrow & \pi_* D_\bullet \\ \downarrow & & \downarrow \\ R\pi_* C_\bullet & \longrightarrow & R\pi_* D_\bullet \end{array}$$

of complexes on X commutes. For $C_\bullet = \mathbb{C} \to \Omega_Y^\bullet = D_\bullet$ we obtain the commuting diagram

$$(2.5) \qquad \begin{array}{ccc} H^k(X, \mathbb{C}) & \longrightarrow & \mathbb{H}^k(\pi_* \Omega_Y^\bullet) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \psi} \\ H^k(Y, \mathbb{C}) & \longrightarrow & \mathbb{H}^k(\Omega_Y^\bullet) \end{array}$$

The lower horizontal map is an isomorphism by Grothendieck's theorem and π^* is injective for $k \leq 2$ by Lemma 2.1. We will show that for $k \leq 2$ the map ψ is injective and the codimension of its image is the same as that of π^*.

For the injectivity we compare the spectral sequences on X and Y. Let us show first that the E_1-level of the spectral sequence (2.3) embeds into the E_1-level of the spectral sequence of the complex $\Omega_Y^\bullet$, which degenerates on E_1 by Hodge theory. Note

that with X also Y is of Fujiki class and therefore it carries a Hodge structure on its cohomology and the Hodge-de Rham spectral sequence degenerates on E_1.

Indeed, we have $H^k(X, \mathcal{O}_X) \cong H^k(Y, \mathcal{O}_Y)$ by rationality of singularities for all $k \in \mathbb{N}_0$ and obviously $H^0(X, \pi_* \Omega_Y^k) \cong H^0(Y, \Omega_Y^k)$. The inclusion $H^1(X, \pi_* \Omega_Y) \subset H^1(Y, \Omega_Y)$ is deduced from the Leray spectral sequence.

Thus, degeneration of the spectral sequence and injectivity of ψ will follow once we show that $H^1(X, \pi_* \Omega_Y) \subset H^1(Y, \Omega_Y)$ has codimension equal to $m := \dim N_1(Y/X) = \dim H^{1,1}(Y) - \dim H^{1,1}(X)$, where $N_1(Y/X)$ is the kernel of the surjection $N_1(Y) \to N_1(X)$, see [KM92, (12.1.5)] and note that log terminal singularities are rational. But coker $\psi^{1,1}$ is the image of $H^1(\Omega_Y) \to H^0(R^1 \pi_* \Omega_Y)$. So let $C_1, \ldots, C_m$ be curves in Y contracted to a point under π such that their classes form a basis of $N_1(Y/X)$ and let $L_1, \ldots, L_m$ be line bundles on Y such that their Chern classes $\xi_i := c_1(L_i) \in H^1(\Omega_Y)$ define linearly independent functionals on $N_1(Y/X)$. Choose irreducible components $F_1, \ldots, F_m$ of fibers of π such that $C_i \subset F_i$ for all i. Take resolutions of singularities $\nu_i : \widetilde{F}_i \to F_i$ and curves $\widetilde{C}_i \subset \widetilde{F}_i$ such that $\nu_{i*} \widetilde{C}_i = C_i$ for all i. If we denote $F := \coprod \widetilde{F}_i$ and by $\nu : F \to Y$ the composition of the resolutions with the inclusion, then by the projection formula $\nu^* \xi_i . \widetilde{C}_j = \xi_i . C_j$ so that the $\nu^* \xi_i$ are still linearly independent. This implies that the ξ_i are mapped to an m-dimensional subspace of $H^1(F, \Omega_F)$ under the composition $H^1(\Omega_Y) \to H^0(R^1 \pi_* \Omega_Y) \to H^1(F, \Omega_F)$. In particular, rk $\left(H^1(\Omega_Y) \to H^0(R^1 \pi_* \Omega_Y) \right) \geq m$, which completes the proof of the lemma. $\quad\square$

From the proof of the preceding lemma we deduce

Corollary 2.3. *Let X be a normal compact variety of Fujiki class with rational singularities, let $\pi : Y \to X$ be a resolution of singularities, and denote by $j : U = X^{\mathrm{reg}} \to X$ the inclusion of the regular part. Then for $k, p + q \leq 2$ we have*

(1) $H^{p,q}(X) \cong H^q(X, \pi_* \Omega_Y^p) \cong H^q(X, j_* \Omega_U^p)$,

(2) $H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \pi_* \Omega_Y^\bullet) \cong \mathbb{H}^k(X, j_* \Omega_U^\bullet)$, *and*

(3) $F^p H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \pi_* \Omega_Y^{\geq p}) \cong \mathbb{H}^k(X, j_* \Omega_U^{\geq p})$.

Proof. The isomorphism $H^{p,q}(X) \cong H^q(X, \pi_* \Omega_Y^p)$ was shown in the proof of Lemma 2.2. There, we saw that $H^{p,q}(X) \subset H^q(X, \pi_* \Omega_Y^p)$ and both spaces were shown to have the same codimension in $H^p(Y, \Omega_Y^q)$. Therefore, they are equal. The statement about the pushforward of Ω_U^p is again deduced from [KS18, Corollary 1.8]. The second statement is contained in Lemma 2.2 and the third statement is a consequence of it together with the first statement of the corollary. $\quad\square$

Again, the second statement was shown by Schwald in [Sc16, Theorem 7] for projective varieties. Now we turn to the relative situation. The following result follows from the absolute case by homological algebra.

Lemma 2.4. *Let X_0 be a normal compact variety of Fujiki class with rational singularities, let $f : X \to S$ be a flat deformation of X_0 over a local Artinian base scheme S of finite type over $\mathbb{C}$, let $j_0 : U_0 \hookrightarrow X_0$ be the inclusion of the regular locus and let*

$U \to S$ be the induced deformation of U_0. Let us denote by $j : U \hookrightarrow X$ the inclusion and suppose that $j_*\Omega^\bullet_{U/S}$ is flat over S. Then the following hold:

(1) For each $p \in \mathbb{N}_0$ and for every closed subscheme $S' \hookrightarrow S$ we have $\left(j_*\Omega^p_{U/S}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} = j'_*\Omega^p_{U'/S'}$ where $j' : U' \hookrightarrow X'$ is the base change of j to S'. Moreover, the sheaf $j_*\Omega^p_{U/S}$ is a coherent $\mathcal{O}_X$-module.

(2) For all $k \le 2$ the canonical map $H^k(X, f^{-1}\mathcal{O}_S) \to \mathbb{H}^k(X, j_*\Omega^\bullet_{U/S})$ is an isomorphism and the Hodge-de Rham spectral sequence

(2.6)
$$E_1^{p,q} = H^q(j_*\Omega^p_{U/S}) \Rightarrow \mathbb{H}^{p+q}(X, j_*\Omega^\bullet_{U/S})$$

degenerates on E_1 in the region where $p + q \le 2$.

(3) The $\mathcal{O}_S$-modules $H^q(X, j_*\Omega^p_{U/S})$ are free for $p + q \le 2$ and compatible with arbitrary base change.

Proof. We put $R := \Gamma(S, \mathcal{O}_S)$ and denote by $\mathfrak{m} \subset R$ its maximal ideal. The case $R = \mathbb{C}$ was treated in Lemma 2.2.

For the proof of (1), we first note that $\left(j_*\Omega^p_{U/S}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} = \left(j_*\Omega^p_{U/S}\right) \otimes_{j_*\mathcal{O}_U} j'_*\mathcal{O}_{U'} = j'_*\Omega^p_{U'/S'}$ by normality of X_0. It remains to prove coherence. Let us assume $\mathfrak{m} \ne 0$ and denote by $n \in \mathbb{N}$ the unique natural number such that $\mathfrak{m}^n \ne 0$ but $\mathfrak{m}^{n+1} = 0$. We will argue by induction on n. From now on, let $S' \hookrightarrow S$ be the closed subscheme defined by $\mathfrak{m}^n$ and let $j' : U' \hookrightarrow X' \to S'$ be the base change of $j : U \hookrightarrow X$ to S'. By normality of X_0 we have that $\left(j_*\Omega^p_{U/S}\right) \otimes \mathcal{O}_{S'} = j'_*\Omega^p_{U'/S'}$ and thus by flatness for each p there is an exact sequence

(2.7)
$$0 \to j_{0*}\Omega^p_{U_0} \otimes \mathfrak{m}^n \to j_*\Omega^p_{U/S} \to j'_*\Omega^p_{U'/S'} \to 0$$

Note that $\mathfrak{m}^n$ is a $R/\mathfrak{m} = \mathbb{C}$-vector space. By the inductive hypothesis, the left and the right term in the sequence are coherent, thus the same holds for the middle term.

The strategy for (2) and (3) is basically the same as in the proof of [De68, Théorème 5.5]. The differentials on all modules $E_1^{p,q}$ with $p + q = k$ will be zero if and only if $\sum_{p+q=k} \lg_R E_1^{p,q} = \lg_R \mathbb{H}^k(X, j_*\Omega^\bullet_{U/S})$ where $\lg_R$ denotes the length as an R-module. Note that both sides are finite.

Flatness and coherence of $j_*\Omega^\bullet_{U/S}$ entail that there is a bounded below complex $L^\bullet$ of free R-modules of finite rank such that there is an isomorphism $H^q(X, j_*\Omega^p_{U/S} \otimes f^*M) \cong H^q(L \otimes_R M)$ which is functorial in the R-module M, see [BS77, Ch 3, Théorème 4.1]. This is where in the algebraic category, Deligne uses [EGA3, Théorème (6.10.5)] instead. By [De68, (3.5.1)], we obtain that $\lg_R H^q(X, j_*\Omega^p_{U/S}) \le \lg R \cdot \lg_R H^q(X_0, j_{0*}\Omega^p_{U_0})$ and equality holds if and only if $H^q(X, j_*\Omega^p_{U/S})$ is R-free.

For $k \le 2$ we have

(2.8)
$$\lg_R \mathbb{H}^k(X, j_*\Omega^\bullet_{U/S}) \le \sum_{p+q=k} \lg_R H^q(X, j_*\Omega^p_{U/S})$$
$$\le \lg R \cdot \sum_{p+q=k} \dim_{\mathbb{C}} H^q(X_0, j_{0*}\Omega^p_{U_0})$$
$$= \lg R \cdot H^k(X_0, \mathbb{C})$$

where the first inequality is the existence of the spectral sequence, the second one was explained just before and the equality is the degeneracy of the spectral sequence for X_0, see Lemma 2.2.

As in the proof of (1) we may assume $\mathfrak{m} \neq 0$, and we will show by induction on the minimal $n \in \mathbb{N}$ satisfying $\mathfrak{m}^{n+1} = 0$ that $\mathbb{H}^k(X, j_*\Omega^{\bullet}_{U/S}) = H^k(X, f^{-1}\mathcal{O}_S) = H^k(X, \underline{R}_X)$ for $k \leq 2$ where $\underline{R}_X$ denotes the constant sheaf R. The complexes in (2.7) have natural augmentations from $\underline{\mathbb{C}}_X \otimes \mathfrak{m}^n$, $\underline{R}_X$, and $\underline{R}'_X$ where $R' = R/\mathfrak{m}^n = \Gamma(S', \mathcal{O}_{S'})$. Applying cohomology we obtain the following diagram with exact rows. The upper row is exact by the universal coefficient theorem.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^k(X_0, \mathbb{C}) \otimes \mathfrak{m}^n & \longrightarrow & H^k(X_0, \underline{R}_X) & \longrightarrow & H^k(X_0, \underline{R}'_{X'}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathbb{H}^k(X_0, j_{0*}\Omega^{\bullet}_{U_0}) \otimes \mathfrak{m}^n & \xrightarrow{\beta_k} & \mathbb{H}^k(X, j_*\Omega^{\bullet}_{U/S}) & \xrightarrow{\alpha_k} & \mathbb{H}^k(X', j'_*\Omega^{\bullet}_{U'/S'}) & \longrightarrow & \cdots
\end{array}
$$

where the outer vertical morphisms are isomorphisms by induction. Thus, α_k is surjective for all $k \leq 2$. As the bottom row is part of the long exact sequence in cohomology, surjectivity of α_{k-1} implies injectivity of β_k. Thus, the middle vertical morphism is an isomorphism by the 5-lemma. Therefore, the inequality in (2.8) is an equality which entails all the freeness statements we wanted to prove.

The base change property follows from the local freeness by [BS77, Ch 3, Corollaire 3.10] (this is the analog of [EGA3, (7.8.5)] in the analytic case). $\qquad \square$

3. Symplectic varieties

Recall from [Be83, Proposition 4] that an *irreducible symplectic manifold* is a simply connected compact Kähler manifold Y such that $H^0(Y, \Omega^2_Y) = \mathbb{C}\sigma$ for a holomorphic symplectic 2-form σ. By [Be83, Théorème 5], there is a nondegenerate quadratic form $q_Y : H^2(Y, \mathbb{Z}) \to \mathbb{Z}$ of signature $(3, b_2(Y) - 3)$, the *Beauville–Bogomolov–Fujiki form*. Thus, the associated bilinear form gives an injection $\widetilde{q}_Y : H^2(Y, \mathbb{Z}) \hookrightarrow H_2(Y, \mathbb{Z})$ which becomes an isomorphism over $\mathbb{Q}$.

We will also need the notion of a Kähler complex space, due to Grauert, see [Gr62, §3, 3., p. 346]. Let X be a reduced complex space. A *Kähler form* on X is given by an open covering $X = \bigcup_{i \in I} U_i$ and smooth strictly plurisubharmonic functions $\varphi_i : U_i \to \mathbb{R}$ such that on $U_{ij} := U_i \cap U_j$ the function $\varphi_i|_{U_{ij}} - \varphi_j|_{U_{ij}}$ is pluriharmonic, i.e., locally the real part of a holomorphic function. Here, a *smooth function* on X is by definition just a function $f : X \to \mathbb{R}$ such that under a local holomorphic embedding of X into an open set $U \subset \mathbb{C}^n$, there is a smooth (i.e., C^∞) function on U (in the usual sense) that restricts to f on X. A *Kähler space* is then a reduced complex space admitting a Kähler form. Note that the form is not part of the structure.

We do not need many further details on Kähler spaces. Indeed, we will only use that compact Kähler spaces can be resolved by compact Kähler manifolds (see e.g. [Va89,

II, 1.3.1 Proposition]) and therefore come equipped with mixed Hodge structures on their cohomology groups.

Definition 3.1. We will use the term *symplectic variety* in the same sense as Beauville to mean a normal complex variety X together with a holomorphic symplectic 2-form σ on its regular part that extends holomorphically to one (and hence to any) resolution of singularities, see [Be00, Definition 1.1]. An *(irreducible) symplectic resolution* is a resolution $\pi : Y \longrightarrow X$ of a compact Kähler symplectic variety X by an (irreducible) symplectic manifold Y. Note that a symplectic resolution is not guaranteed to exist.

Remark 3.2. Different classes of singular symplectic varieties have been studied recently. The class of irreducible symplectic varieties defined by Greb–Kebekus–Peternell [GKP11, Definition 8.16] for example is the relevant class of symplectic varieties showing up in the singular version of the Beauville-Bogomolov decomposition. The decomposition theorem has been established by Höring–Peternell [HP19] building on work of Druel [Dru18] and Greb–Guenancia–Kebekus [GGK19].

Example 3.3.

(1) Recall that there are the following known deformation types of irreducible symplectic manifolds, all arising from moduli spaces of sheaves on $K3$ or abelian surfaces: for any $n > 1$, the Hilbert scheme $S^{[n]}$ of length n subschemes of a $K3$ surface S as described in [Be83, Section 6.] is an irreducible symplectic manifold, which by Lemme 2 of op. cit. satisfies $b_2 = 23$. For any $n > 1$, the generalized Kummer variety $K_n(A)$, i.e., the Hilbert scheme of zero-sum length $n + 1$ subschemes of an abelian surface A as described in [Be83, Section 7.] is irreducible symplectic and satisfies $b_2 = 7$ by Proposition 8 of op. cit. There are two more sporadic examples: in dimension 10, the symplectic resolution constructed in [OG99] of a singular moduli space of sheaves on a $K3$ surface S with a certain Mukai vector is an irreducible symplectic manifold. It has $b_2 = 24$ by [Ra08, Theorem 1.1]. The proof that this desingularization (denoted $\widetilde{\mathcal{M}_4}$ in op. cit.) is an irreducible symplectic manifold is spread over [OG99]: in (2.0.3) Proposition it is shown that $\widetilde{\mathcal{M}_4}$ is smooth, projective, and symplectic. Connectedness and uniqueness are proven in Section 3 through various propositions, see the summary right before the remark on the bottom of p. 95. Finally, simple-connectedness is shown in (4.1.9) Proposition together with Section 4.2. In dimension 6, the symplectic resolution of a singular moduli space of trivial-determinant sheaves on an abelian surface A is an irreducible symplectic manifold with $b_2 = 8$, see [OG03, (1.4) Theorem]. Note that the Betti numbers are different in all these examples; as of now these are all the known examples of irreducible symplectic manifolds up to deformation.

(2) Evidently, if $\pi : Y \longrightarrow X$ is a proper birational morphism from an (irreducible) symplectic manifold Y to a normal Kähler variety X, then X is a symplectic

variety and π is an (irreducible) symplectic resolution. Thus, symplectic resolutions can instead be viewed as birational contractions of symplectic manifolds.

As all known deformation types of irreducible symplectic manifolds arise from moduli spaces of sheaves, a rich source of symplectic resolutions is given by wall-crossing contractions associated to non-generic polarizations. We systematically investigate such examples in Section 6.

(3) Divisorial contractions yields $\mathbb{Q}$-factorial X, and these were studied in [LP16], see especially Section 2 there. The more general context dealt with in the current paper allows for small contractions, the easiest example of which is the contraction of a Lagrangian projective space to a point. See [Ba15, Examples 8-10] for explicit examples.

(4) Let $Y \subset \mathbb{P}^5$ be a (possibly singular) cubic fourfold not containing a plane. Then it has been shown in [Le18, Theorem 3.3] that the variety $M_1(Y)$ of lines on Y is a symplectic variety which is birational to the second punctual Hilbert scheme of an associated K3 surface. It follows that $M_1(Y)$ admits a crepant resolution by an irreducible symplectic manifold, see [Le18, Corollary 5.6]. A similar statement is deduced for the target space $Z(Y)$ of the MRC-fibration of the Hilbert scheme compactification of the space of twisted cubics on Y, see Theorem 1.1, Corollary 5.5, and Corollary 6.2 of op. cit.

Given a symplectic resolution $\pi : Y \longrightarrow X$, from [KM92, (12.1.3) Theorem] it follows that we have a short exact sequence

$$0 \longrightarrow H_2(Y/X, \mathbb{Q}) \longrightarrow H_2(Y, \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q}) \longrightarrow 0.$$

Recall that $N_1(Y/X)$ is the subspace of the space $N_1(Y)$ of (integral) 1-cycles on Y which are contracted under the map $Y \longrightarrow X$. Similarly, $H_2(Y/X, \mathbb{Q})$ denotes the subspace of $H_2(Y, \mathbb{Q})$ generated by all pushforwards of cohomology classes in the fibers of π. Moreover, by loc. cit. $H_2(Y/X, \mathbb{Q})$ is generated by algebraic cycles and therefore it coincides with $N_1(Y/X)_{\mathbb{Q}}$. Here, $N_1(Y/X) \subset N_1(Y)$ is defined as the kernel of the push forward map $\pi_* : N_1(Y) \longrightarrow N_1(X)$. As in the introduction, we denote $\widetilde{q}_Y : H^2(Y, \mathbb{Q}) \longrightarrow H_2(Y, \mathbb{Q})$ the isomorphism induced by the quadratic form q_Y.

Lemma 3.4. *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution. Then $\pi^* : H^2(X, \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$ is injective and the restriction of q_Y to $H^2(X, \mathbb{C})$ is nondegenerate. The q_Y-orthogonal complement to $H^2(X, \mathbb{C})$ in $H^2(Y, \mathbb{C})$ is $N \otimes \mathbb{C}$ where $N := (\widetilde{q}_Y)^{-1}(N_1(Y/X))$. In particular, the inclusion $H^2(X, \mathbb{C}) \subset H^2(Y, \mathbb{C})$ is an equality on the transcendental part $H^2(X, \mathbb{C})_{\mathrm{tr}} = H^2(Y, \mathbb{C})_{\mathrm{tr}}$. Moreover, N is negative definite with respect to q_Y.*

Proof. Injectivity follows from Lemma 2.1. To see that N is q_Y-orthogonal to $H^2(X, \mathbb{C})$ we argue as follows: as the bilinear form q_Y is compatible with the Hodge structures, $N \subset H^{1,1}(Y)$ is orthogonal to the $(2,0)$-part of $H^2(Y, \mathbb{C})$ and hence to $H^2(Y, \mathbb{C})_{\mathrm{tr}}$ (which is by the way isomorphic to $H^2(X, \mathbb{C})_{\mathrm{tr}}$ by Lemma 2.2). So it suffices to prove

that N is q_Y-orthogonal to $N^1(X) \subset H^2(X, \mathbb{Q}) \subset H^2(Y, \mathbb{Q})$. For this we only have to unravel the definition of N: let C be a curve in Y that is contracted to a point under π. Then we define the class $D_C = \widetilde{q}_Y^{-1}([C]) \in H^2(X, \mathbb{Q})$ to be the unique class such that $C.D' = q_Y(D_C, D')$ for all $D' \in H^2(X, \mathbb{Q})$. In particular, $q_Y(D_C, D') = 0$ for all $\mathbb{Q}$-line bundles D' on X. So $H^2(X, \mathbb{C}) \subset N^\perp$ and to conclude it is sufficient to exhibit a positive class in $H^2(X, \mathbb{R})$. As X is supposed to be Kähler, a Kähler class $h \in H^{1,1}(X, \mathbb{R})$ will do. Then $q_Y(h) > 0$ as h is big and nef on X, so the orthogonal to h in $H^{1,1}(Y)$ is negative definite and thus we see that $H^2(Y, \mathbb{C}) = H^2(X, \mathbb{C}) \oplus N$. $\square$

Proposition 3.5. *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution and let $j : U \longrightarrow X$ be the inclusion of the regular locus. Then, we have*

$$T_X = j_* \Omega_U = \pi_* \Omega_Y.$$

Proof. We have

$$T_X = j_* T_U = j_* \Omega_U = \pi_* \Omega_Y,$$

where the first equality comes from the reflexivity of T_X, the second from the symplectic form on the regular part, and the third is again [KS18, Corollary 1.8]. $\square$

Later on we will need that the Hodge structure of a very general symplectic variety admitting an irreducible symplectic resolution has no automorphisms different from $\pm \mathrm{id}$. As we were not able to find a reference, we include a short proof. In fact, we include a proof—which is basically taken from [vGV16, Lemma 9]—of a more general result about Mumford–Tate groups, which might be of independent interest, so let us recall this notion. Let H be a pure weight k integral Hodge structure with underlying finitely-generated free $\mathbb{Z}$-module $H_\mathbb{Z}$ and consider the action $\rho : \mathbb{S}^1 \to \mathrm{End}(H_\mathbb{C})$ of the unit circle $\mathbb{S}^1 \subset \mathbb{C}^\times$ on $H_\mathbb{C} = H \otimes \mathbb{C}$ defined by $\rho(z).h := z^{p-q}h$ for $h \in H^{p,q}$. One easily checks that this action is defined over $\mathbb{R}$.

Recall the following definition (see [CMP03, §15.2] for details):

Definition 3.6. The Mumford–Tate group of a pure Hodge structure H is the smallest algebraic subgroup $\mathrm{MT}(H) \subset \mathrm{GL}(H_\mathbb{Q})$ which is defined over $\mathbb{Q}$ and whose $\mathbb{R}$-points contains the image of ρ.

Definition 3.7. A *Hodge structure of hyperkähler type* is a weight two integral Hodge structure H with $h^{2,0} = h^{0,2} = 1$ together with a nondegenerate symmetric form $q : H \otimes H \to \mathbb{Z}(-2)$ which is a morphism of Hodge structures, of signature $(3, \mathrm{rk}(H_\mathbb{Z}) - 3)$, and for which the restriction of q to $(H^{2,0} \oplus H^{0,2})_\mathbb{R}$ is positive definite.

For a hyperkähler Hodge structure (H, q), we define $C(H) \subset H_\mathbb{R}$ and $C^{1,1}(H) \subset H_\mathbb{R}^{1,1}$ to be the positive cones. The real plane $(H^{2,0} \oplus H^{0,2})_\mathbb{R}$ is canonically oriented by $\mathrm{Re}(\sigma) \wedge \mathrm{Im}(\sigma)$ where $\sigma \in H^{2,0}$ is a generator. The cone $C^{1,1}(H)$ has two components, and a choice of component is equivalent to a choice of generator of $H^2(C(H), \mathbb{Z})$ (see for example [Ma11, §4]).

Hodge structures of hyperkähler type on H are parametrized by the period domain

$$(3.1) \qquad \Omega_{H_{\mathbb{Z}}} := \{\sigma \in \mathbb{P}(H_{\mathbb{C}}) \mid q(\sigma, \sigma) = 0, q(\sigma, \bar{\sigma}) > 0\}.$$

A Hodge structure of hyperkähler type is called *Mumford–Tate general* if it is not contained in any proper Hodge locus $\mathrm{Hdg}_\alpha \subset \Omega_{H_{\mathbb{Z}}}$ for some $\alpha \in H_{\mathbb{Z}}^s \otimes (H_{\mathbb{Z}}^*)^r$ for some $r, s \in \mathbb{N}_0$. As there are countably many such Hodge loci Hdg_α, it follows that the very general Hodge structure of hyperkähler type is Mumford–Tate general.

Proposition 3.8. *Let (H, q) be a Mumford–Tate general Hodge structure of hyperkähler type. Then $\mathrm{MT}(H) = \mathrm{SO}(H_{\mathbb{Q}})$, the special orthogonal group of $H_{\mathbb{Q}}$ with respect to q.*

Proof. The proof is basically identical to [vGV16, Lemma 9] so we only sketch it very briefly. It follows from the definition that $\mathrm{MT}(H) \subset \mathrm{SO}(H_{\mathbb{Q}})$. The main ingredient to show equality is that for every Hodge structure H_ω corresponding to some $\omega \in \Omega_H$ there is an inclusion of Mumford–Tate groups $\mathrm{MT}(H_\omega) \subset \mathrm{MT}(H)$. This is because the Mumford–Tate group may be characterized as the stabilizer group of all Hodge classes $\alpha \in H^s \otimes (H^*)^r$ for all $r, s \in \mathbb{N}_0$. Then one proceeds inductively: for $\alpha \in H_{\mathbb{Q}}$ and $\omega \in \alpha^\perp \subset \Omega_H$ which is Mumford–Tate general with respect to the Hodge structure on $\alpha^\perp$ we have that $\mathrm{MT}(H_\omega) = \mathrm{SO}(\alpha^\perp)$. The result follows as $\mathrm{SO}(H_{\mathbb{Q}})$ is the smallest algebraic subgroup containing all $\mathrm{SO}(\alpha^\perp)$. $\qquad \square$

Corollary 3.9. *A Mumford–Tate general Hodge structure of hyperkähler type only has $\pm\mathrm{id}$ as automorphisms.*

Proof. This is a consequence of Proposition 3.8 together with the fact that $\mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(H_{\mathbb{Q}}) = \mathrm{End}(H_{\mathbb{Q}})^{\mathrm{SO}(H_{\mathbb{Q}})} = \mathbb{Q}\mathrm{id}_H$. $\qquad \square$

We will also need Verbitsky's classification [Ve15, Theorem 4.8] of orbit closures of hyperkähler periods under the orthogonal group. As described above, the period domain Ω_Λ can alternatively be thought of as the space of oriented positive-definite planes in $\Lambda_{\mathbb{R}}$. We make the following definition:

Definition 3.10. Let H be a pure weight two integral Hodge structure with underlying $\mathbb{Z}$-module $H_{\mathbb{Z}}$. The *rational rank* of H is $\mathrm{rrk}(H) := \mathrm{rk}((H^{2,0} \oplus H^{0,2}) \cap H_{\mathbb{Z}})$. Note that $0 \leq \mathrm{rrk}(H) \leq 2 \cdot h^{2,0}$.

Fixing an oriented positive-definite plane $P_0 \in \Omega_\Lambda$ as a basepoint, we obtain an isomorphism

$$(3.2) \qquad \Omega_\Lambda \cong \mathrm{SO}(\Lambda_{\mathbb{R}})/\mathrm{SO}(P_0) \times \mathrm{SO}(P_0^\perp).$$

Let $\ell_0 = P_0 \cap \Lambda$ so that $\mathrm{rrk}(P_0) = \mathrm{rk}(\ell_0)$. Given an arithmetic lattice $\Gamma \subset \mathrm{SO}(\Lambda_{\mathbb{R}})$, by an application of Ratner's theorem (see [Mo15, §1.1.15(2)]) the orbit closure of $\Gamma \cdot \mathrm{id} \in \Gamma \backslash \mathrm{SO}(\Lambda)$ under $\mathrm{SO}(P_0^\perp)$ is the (closed) orbit under $\mathrm{SO}((\ell_0^\perp)_{\mathbb{R}})$ provided $\mathrm{rk}(\Lambda) \geq 5$ so that $\mathrm{SO}(P_0^\perp)$ is generated by unipotents. Indeed, $\mathrm{SO}((\ell_0^\perp)_{\mathbb{R}})$ is the smallest connected Lie subgroup of $\mathrm{SO}(\Lambda_{\mathbb{R}})$ containing $\mathrm{SO}(P_0^\perp)$ and which is defined over $\mathbb{Q}$, as the only

closed connected Lie subgroups of $\mathrm{SO}(3,n)$ containing $\mathrm{SO}(1,n)$ are $\mathrm{SO}(1,n)$, $\mathrm{SO}(2,n)$, and $\mathrm{SO}(3,n)$. It follows then that the orbit closure under $\mathrm{SO}(P_0) \times \mathrm{SO}(P_0^\perp)$ is either the orbit itself, $(\Gamma \cdot \mathrm{id})\, \mathrm{SO}((\ell_0^\perp)_\mathbb{R})\, \mathrm{SO}(P_0)$, or all of $\Gamma \backslash \mathrm{SO}(\Lambda_\mathbb{R})$ when $\mathrm{rk}(\ell_0) = 2, 1,$ or 0, respectively. Note that for the middle case, $(\Gamma \cdot \mathrm{id})\, \mathrm{SO}((\ell_0^\perp)_\mathbb{R})\, \mathrm{SO}(P_0)$ is closed (as $(\Gamma \cdot \mathrm{id})\, \mathrm{SO}((\ell_0^\perp)_\mathbb{R})$ is closed and $\mathrm{SO}(P_0)$ compact) and contained in the orbit closure, hence equal to it.

Proposition 3.11 (Theorem 2.5 in [Ve17]). *There are three possibilities for the orbit closure of $P_0 \in \Omega_\Lambda$ under Γ depending on $r = \mathrm{rrk}(P_0)$:*

($r = 2$) *the orbit is closed;*

($r = 1$) *the orbit closure is the union of $G_{\gamma \ell_0}$ for $\gamma \in \Gamma$, where G_ℓ is the subset of positive planes P containing ℓ;*

($r = 0$) *the orbit is dense.*

Remark 3.12. The $r = 1$ case was omitted in [Ve15, Theorem 4.8] (and corrected recently in [Ve17]). Note that in the $r = 1$ case, $G_\ell \subset \Omega_\Lambda$ is a real-analytic submanifold of real codimension $\mathrm{rk}\,\Lambda - 2$ in Ω_Λ.

Now, suppose we have a Γ-invariant collection $M \subset \Lambda - \{0\}$ of classes of bounded nonpositive square. For each $p \in \Omega_\Lambda$ we define a wall-and-chamber decomposition of $C^{1,1}(\Lambda_p)^+ \subset (\Lambda_\mathbb{R})_p^{1,1}$ whose open chambers are the connected components of $C^{1,1}(\Lambda_p)^+ - \bigcup_{\alpha \in M_p} \alpha^\perp$, where $M_p = M \cap \Lambda_p^{1,1}$. The index p refers to the decomposition into types for the Hodge structure corresponding to p. Let $\Omega_\Lambda^{\mathrm{class}}$ be the space of pairs (p, ω) for $p \in \Omega_\Lambda$ and $\omega \in C^{1,1}(\Lambda_p)^+$, which is naturally identified with $\mathrm{SO}(\Lambda)/\mathrm{SO}(P_0) \times \mathrm{SO}(\omega_0^\perp \cap P_0^\perp)$ for a choice of $P_0 \in \Omega_\Lambda$ and positive class $\omega_0 \in C^{1,1}(\Lambda_{P_0})^+$. Define the space $\Omega_\Lambda^{\mathrm{cone}}$ consisting of pairs (p, C) for $p \in \Omega_\Lambda$ and an open chamber C of the decomposition of $C^{1,1}(\Lambda_p)^+$, topologized as a quotient of $\Omega_\Lambda^{\mathrm{class}}$. The forgetful map $\pi : \Omega_\Lambda^{\mathrm{cone}} \longrightarrow \Omega_\Lambda$ is evidently continuous and moreover a local isomorphism.

We have the following formulation of result of Verbitsky:

Theorem 3.13 (Theorem 3.1 of [Ve17]). *Assume $\mathrm{rk}(\Lambda) > 4$. Then the forgetful map $\pi : \Omega_\Lambda^{\mathrm{cone}} \longrightarrow \Omega_\Lambda$ commutes with closures.*

Corollary 3.14. *For any $x = (p, C) \in \Omega_\Lambda^{\mathrm{cone}}$ of non-maximal Picard rank, we have $\overline{\Gamma x} = \pi^{-1}\left(\overline{\Gamma p}\right)$.*

Proof. Let $y \in \overline{\Gamma x}$, and suppose y' is inseparable from y. Let U' and U be open neighborhoods of y' and y, respectively, which we may assume have the same image in Ω_Λ. Then U' and U must meet at every point in U with Picard rank zero, and therefore meet at a point in $\overline{\Gamma x}$. $\qquad\square$

4. DEFORMATIONS

As usual in deformation theory, when we speak about the semi-universal deformation $\mathscr{Z} \to \mathrm{Def}(Z)$ of a complex space Z, then the complex space $\mathrm{Def}(Z)$ has a distinguished

point $0 \in \mathrm{Def}(Z)$ such that the fiber of $\mathscr{Z} \to \mathrm{Def}(Z)$ over 0 is Z and we should actually speak about the morphism of space germs $(\mathscr{Z}, Z) \to (\mathrm{Def}(Z), 0)$. All deformation theoretic statements have to be interpreted as statements about germs.

Let X be a normal compact complex variety with rational singularities and let $\pi : Y \to X$ be a resolution of singularities. Recall that by [KM92, Proposition 11.4] there is a morphism $p : \mathrm{Def}(Y) \to \mathrm{Def}(X)$ between the Kuranishi spaces of Y and X and also between the semi-universal families $\mathscr{Y} \to \mathrm{Def}(Y)$ and $\mathscr{X} \to \mathrm{Def}(X)$ fitting in a diagram

(4.1)
$$
\begin{array}{ccc}
\mathscr{Y} & \xrightarrow{\;P\;} & \mathscr{X} \\
\downarrow & & \downarrow \\
\mathrm{Def}(Y) & \xrightarrow{\;p\;} & \mathrm{Def}(X)
\end{array}
$$

Recall from [FK87, (0.3) Corollary] that there exists a closed complex subspace $\mathrm{Def}^{\mathrm{lt}}(X) \subset \mathrm{Def}(X)$ parametrizing locally trivial deformations of X. More precisely, the restriction of the semi-universal family to this subspace, which by abuse of notation we denote also by $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$, is a locally trivial deformation of X and is semi-universal for locally trivial deformations of X.

Let $\pi : Y \to X$ be an irreducible symplectic resolution with X of dimension $2n$. As $H^0(T_Y) = 0$, every semi-universal deformation of Y is universal. We also have $H^0(T_X) = H^0(\pi_* \Omega_Y) = 0$ by Proposition 3.5 so that every semi-universal deformation of X is universal. Let us fix universal deformations of X and Y and a diagram as (4.1). It is well-known that $\mathscr{Y} \to \mathrm{Def}(Y)$ is a family of irreducible symplectic manifolds, at least in the sense of germs, i.e., possibly after shrinking the representative of $\mathrm{Def}(Y)$. If X is projective, then also $\mathscr{X} \to \mathrm{Def}(X)$ is a family of symplectic varieties admitting irreducible symplectic resolutions by [Na01b, Theorem 2.2]. We will see in Proposition 5.23 that as an application of our results this statement also holds without the projectivity assumption. Our first goal is to address smoothness of the space of locally trivial deformations. Recall from the introduction and Lemma 3.4 that we have an orthogonal decomposition

(4.2)
$$
H^2(Y, \mathbb{Q}) = H^2(X, \mathbb{Q}) \oplus N
$$

where N corresponds under the isomorphism $\widetilde{q}_Y : H^2(Y, \mathbb{Q}) \to H_2(Y, \mathbb{Q})$ to the curves contracted by π and put $m := \dim N$.

Theorem 4.1. *Let X be a symplectic variety admitting an irreducible symplectic resolution. Then the space $\mathrm{Def}^{\mathrm{lt}}(X)$ of locally trivial deformations of X is smooth of dimension $h^{1,1}(X) = h^{1,1}(Y) - m$.*

Proof. Smoothness is shown using the T^1-lifting principle of Kawamata-Ran [Ra92, Ka92, Ka97], in particular Theorem 1 of [Ka92]. We refer to [GHJ, §14] or [Le11, VI.3.6] for more detailed introductions. The tangent space to $\mathrm{Def}^{\mathrm{lt}}(X)$ at the origin is $H^1(T_X)$ which thanks to the symplectic form can be identified with $H^1(j_* \Omega_U)$ where

$j : U = X^{\text{reg}} \rightarrow X$ is the inclusion. For the T^1-lifting property one has to show that for every infinitesimal locally trivial deformation $\mathcal{X} \rightarrow S$ of X over an Artinian base scheme S the space $H^1(T_{\mathcal{X}/S})$ is locally $\mathcal{O}_S$-free and compatible with arbitrary base change. We denote again by $j : \mathcal{U} \rightarrow \mathcal{X}$ the inclusion of the smooth locus of $\mathcal{X} \rightarrow S$. Take an extension $\sigma \in H^0(\mathcal{U}, j_*\Omega^2_{\mathcal{X}/S})$ of the symplectic form on $U \subset X$. It remains nondegenerate and hence yields an isomorphism $T_{\mathcal{X}/S} \rightarrow j_*\Omega_{\mathcal{X}/S}$, consequently also $H^1(T_{\mathcal{X}/S}) \cong H^1(\Omega_{\mathcal{X}/S})$ which is free by Lemma 2.4. Thus, it satisfies the T^1-lifting property and thus the space $\text{Def}^{\text{lt}}(X)$ is smooth. It follows from Corollary 2.3 that $\dim H^1(T_X) = h^{1,1}(X)$ which shows the dimension statement and completes the proof. $\qquad\square$

It is convenient to introduce the following terminology:

Definition 4.2. An irreducible symplectic resolution $\pi : Y \rightarrow X$ is *deformable* if $\text{Def}^{\text{lt}}(X)$ is contained in the image of the map $p : \text{Def}(Y) \rightarrow \text{Def}(X)$ considered in (4.1).

Proposition 4.3 below says that any irreducible symplectic resolution of a *projective* symplectic variety is deformable. We will in fact see in Proposition 5.7 that all symplectic resolutions are deformable.

The following is an easy consequence of Namikawa's work.

Proposition 4.3. *Let $\pi : Y \rightarrow X$ be a deformable irreducible symplectic resolution and consider the induced morphism $p : \text{Def}(Y) \rightarrow \text{Def}(X)$. Then, any small locally trivial deformation of X is again a compact Kähler symplectic variety admitting an irreducible symplectic resolution. Moreover, any irreducible symplectic resolution of a projective symplectic variety X is deformable.*

Proof. The variety X has rational singularities, hence $\mathcal{X} \rightarrow \text{Def}(X)$ is a family of Kähler varieties by [Na01a, Proposition 5]. As $\mathcal{Y} \rightarrow \text{Def}(X)$ is a family of irreducible symplectic manifolds, $\mathcal{Y} \rightarrow \mathcal{X}$ is fiberwise an irreducible symplectic resolution.

Suppose now that X is projective. Then by [Na01b, Theorem (2.2)], the spaces $\text{Def}(Y)$ and $\text{Def}(X)$ are smooth of the same dimension and the map p from diagram (4.1) is finite. In particular, p is surjective and $\text{Def}^{\text{lt}}(X)$ is contained in the image. $\qquad\square$

Corollary 4.4. *Let $\pi : Y \rightarrow X$ be a deformable irreducible symplectic resolution and $\mathcal{Y} \rightarrow \mathcal{X}$ the base-change of the top map of (4.1) to $p^{-1}(\text{Def}^{\text{lt}}(X))$. Then points $t \in p^{-1}(\text{Def}^{\text{lt}}(X))$ for which $\mathcal{Y}_t$ and $\mathcal{X}_t$ are both projective are dense in every positive dimensional subvariety of $\text{Def}^{\text{lt}}(X)$.*

Proof. By [GHJ, Proposition 26.6] the claim is true for $\mathcal{Y}_t$. Since $\mathcal{X}_t$ has rational singularities, see e.g. [Ki15, Theorem 3.3.3], and is Kähler, it is projective whenever $\mathcal{Y}_t$ is by Namikawa's result [Na02, Corollary 1.7]. $\qquad\square$

Next, we describe $p^{-1}(\text{Def}^{\text{lt}}(X)) \subset \text{Def}(Y)$. For $N \subset H^2(Y, \mathbb{Q})$ as in (4.2) let us denote by $\text{Def}(Y, N) \subset \text{Def}(Y)$ the subspace of those deformations of Y where all line

bundles on Y with first Chern class in N deform along. This is also the subspace where N remains of type $(1,1)$. It is a smooth submanifold of $\mathrm{Def}(Y)$ of codimension $m = \dim N$ by [Hu99, 1.14].

Proposition 4.5. *Let $\pi : Y \longrightarrow X$ be a deformable irreducible symplectic resolution. Let $\mathscr{Y} \longrightarrow \mathrm{Def}(Y,N)$ and $\mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ be the (restrictions of the) universal deformations. Then for the natural morphism $p : \mathrm{Def}(Y) \longrightarrow \mathrm{Def}(X)$ we have $p^{-1}(\mathrm{Def}^{\mathrm{lt}}(X)) = \mathrm{Def}(Y,N)$, and (4.1) restricts to a diagram*

(4.3)
$$
\begin{array}{ccc}
\mathscr{Y} & \xrightarrow{\;\;P\;\;} & \mathscr{X} \\
\downarrow & & \downarrow \\
\mathrm{Def}(Y,N) & \xrightarrow{\;\;p\;\;} & \mathrm{Def}^{\mathrm{lt}}(X).
\end{array}
$$

Moreover, $p : \mathrm{Def}(Y,N) \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ is an isomorphism.

Proof. By Proposition 4.3 we know that for each $t \in \mathrm{Def}(Y)$ mapping to $\mathrm{Def}^{\mathrm{lt}}(X)$ the morphism $P_t : \mathscr{Y}_t \longrightarrow \mathscr{X}_{p(t)}$ is an irreducible symplectic resolution. By Lemma 2.4 the second cohomology of locally trivial deformations of X form a vector bundle on $\mathrm{Def}^{\mathrm{lt}}(X)$, in particular, $h^{1,1}(\mathscr{X}_{p(t)}) = h^{1,1}(X)$. Thus, by the decomposition $H^2(Y,\mathbb{C}) = N \oplus H^2(X,\mathbb{C})$ from Lemma 3.4 we see that the space $N_1(\mathscr{Y}_t/\mathscr{X}_{p(t)})$ of curves contracted by P_t has dimension m for all $t \in p^{-1}(\mathrm{Def}^{\mathrm{lt}}(X))$. As N is the orthogonal complement of $H^2(X,\mathbb{C})$, it also varies in a local system. This shows the sought-for equality.

One shows as in [LP16] that p is an isomorphism, we only sketch this: It suffices to show that the differential $T_{p,0} : T_{\mathrm{Def}(Y,N),0} \longrightarrow T_{\mathrm{Def}^{\mathrm{lt}}(X),0} = H^1(T_X)$ is an isomorphism. We know from [Hu99, (1.8) and (1.14)] that $T_{\mathrm{Def}(Y,N),0} \subset H^1(T_Y)$ can be identified with the orthogonal complement to $N \subset H^{1,1}(Y)$ under the isomorphism $H^1(T_Y) \cong H^{1,1}(Y)$ induced by the symplectic form. In other words, $T_{\mathrm{Def}(Y,N),0} \cong H^{1,1}(X) \subset H^{1,1}(Y)$. That this is mapped to $H^1(T_X) \cong H^1(j_*\Omega_U)$ under the restriction of $T_{p,0} : H^1(T_Y) \longrightarrow \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ is easily verified. $\qquad\square$

Remark 4.6. Recall from [Fu87, Theorem 4.7] that for an irreducible symplectic manifold Y of dimension $2n$ there is a deformation invariant constant c_Y such that $q_Y(\alpha)^n = c_Y \cdot \int_Y \alpha^{2n}$ for any $\alpha \in H^2(Y,\mathbb{Z})$. Now if $\pi : Y \longrightarrow X$ is an irreducible symplectic resolution, it follows that the restriction $q_X(\beta) := q_Y(\pi^*\beta)$ only depends on X, since all symplectic resolutions are deformation-equivalent by [Hu03, Theorem 2.5] and $\int_Y(\pi^*\beta)^{2n} = \int_X \beta^{2n}$. Furthermore, from Lemma 3.4 we have that q_X is non-degenerate of signature $(3, b_2(X) - 3)$, and by Proposition 4.5 it is locally trivially deformation-invariant. We refer to q_X as the Beauville–Bogomolov–Fujiki form of X.

Using Beauville–Bogomolov–Fujiki form and Proposition 4.5, we obtain a singular version of the local Torelli theorem for locally trivial deformations as a direct corollary of the local Torelli theorem for a resolution [Be83, Théorème 5].

Proposition 4.7 (Local Torelli Theorem). *Let X be a symplectic variety admitting a deformable irreducible symplectic resolution, let q_X be its Beauville–Bogomolov–Fujiki form, and let*

$$(4.4) \qquad \Omega(X) := \{[\sigma] \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q_X(\sigma) = 0, q_X(\sigma, \bar{\sigma}) > 0\} \subset \mathbb{P}(H^2(X, \mathbb{C}))$$

be the period domain for X. If $f : \mathscr{X} \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X)$ denotes the universal locally trivial deformation of X and $X_t := f^{-1}(t)$, then the period map

$$\wp : \mathrm{Def}^{\mathrm{lt}}(X) \longrightarrow \Omega(X), \quad t \mapsto H^{2,0}(X_t)$$

is a local isomorphism. $\hfill\square$

It should be mentioned that Namikawa has proven a local Torelli theorem for certain singular projective symplectic varieties in [Na01a, Theorem 8] and this has been generalized by Kirschner [Ki15, Theorem 3.4.12] to a larger class of varieties. In particular, Kirschner has proven a local Torelli theorem in the context of symplectic compact Kähler spaces. Let us emphasize, however, that neither Namikawa's version nor Kirschner's is what we need as they do not make any statement about local triviality. Also observe that our version of local Torelli—unlike Namikawa's or Kirschner's—does not make any assumption on the codimension of the singular locus of the variety X.

Remark 4.8. In view of the local Torelli theorem 4.7, we will from now on identify the spaces $\mathrm{Def}(Y, N)$ and $\mathrm{Def}^{\mathrm{lt}}(X)$ via the morphism p if we are given a birational contraction.

Theorem 4.9. *Let $\pi : Y \longrightarrow X$ and $\pi' : Y' \longrightarrow X'$ be deformable irreducible symplectic resolutions. Assume that there is a birational map $\phi : Y \dashrightarrow Y'$ such that the induced map $\phi^* : H^2(Y', \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$ sends $H^2(X', \mathbb{C})$ isomorphically to $H^2(X, \mathbb{C})$. Then there is an isomorphism $\varphi : \mathrm{Def}^{\mathrm{lt}}(X) \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X')$ such that for each $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ we have a birational map $\phi_t : \mathscr{X}_t \dashrightarrow \mathscr{X}'_{\varphi(t)}$. In particular, for general $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ the map ϕ_t is an isomorphism, and X and X' are locally trivial deformations of one another.*

Note that as a birational map between K-trivial varieties, ϕ has to be an isomorphism in codimension one and therefore $\phi^* : H^2(Y', \mathbb{C}) \longrightarrow H^2(Y, \mathbb{C})$ has to be an isomorphism.

Proof. The birational map $\phi : Y \dashrightarrow Y'$ between irreducible symplectic manifolds induces an isomorphism between $H^2(Y, \mathbb{Z})$ and $H^2(Y', \mathbb{Z})$ compatible with the Beauville–Bogomolov–Fujiki forms and the local Torelli theorem gives an isomorphism $\mathrm{Def}(Y) \longrightarrow \mathrm{Def}(Y')$. Let us consider the orthogonal decompositions $H^2(Y, \mathbb{Q}) = H^2(X, \mathbb{Q}) \oplus N$ and $H^2(Y', \mathbb{Q}) = H^2(X', \mathbb{Q}) \oplus N'$ as in (4.2). By hypothesis, this decomposition is respected by ϕ^* from which we infer that the isomorphism $\mathrm{Def}(Y) \longrightarrow \mathrm{Def}(Y')$ restricts to an isomorphism $\mathrm{Def}(Y, N) \longrightarrow \mathrm{Def}(Y', N')$ and therefore yields an isomorphism $\varphi : \mathrm{Def}^{\mathrm{lt}}(X) \longrightarrow \mathrm{Def}^{\mathrm{lt}}(X')$ via Proposition 4.5. It remains to show the existence of

a birational map $\phi_t : \mathscr{X}_t \dashrightarrow \mathscr{X}'_{\varphi(t)}$ for $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ which is an isomorphism at the general point. We will identify the spaces

$$S := \mathrm{Def}^{\mathrm{lt}}(X) \cong \mathrm{Def}^{\mathrm{lt}}(X') \cong \mathrm{Def}(Y, N) \cong \mathrm{Def}(Y', N')$$

and consider the universal families

$$\mathscr{Y} \to \mathscr{X} \to S \leftarrow \mathscr{X}' \leftarrow \mathscr{Y}'$$

from Proposition 4.5. For any point $t \in S$, the fibers $\mathscr{Y}_t$ and $\mathscr{Y}'_t$ are deformation equivalent (by Huybrechts' theorem, see [Hu03, Theorem 2.5]) and have the same periods, hence they are birational by Verbitsky's global Torelli theorem [Ve13, Theorem 1.17]. By countability of components of the Douady space, there is a cycle $\Gamma \subset \mathscr{Y} \times_S \mathscr{Y}'$ such that over a Zariski open $U \subset S$ the fiber of Γ over S is the graph of a birational map. For each $t \in S$, the cycles Γ_t induce Hodge isometries $[\Gamma_t]_* : H^2(\mathscr{Y}_t, \mathbb{Q}) \to H^2(\mathscr{Y}'_t, \mathbb{Q})$ which form a morphism of local systems when t varies. Choosing $t \in S$ very general, we see that these cycles have to send $H^2(\mathscr{X}_t, \mathbb{Q})$ isomorphically to $H^2(\mathscr{X}'_t, \mathbb{Q})$ or equivalently, N to N'.

The image of Γ in $\mathscr{X} \times_S \mathscr{X}'$ is a cycle whose fiber for $t \in U$ is the graph of a birational map $\psi_t : \mathscr{X}_t \dashrightarrow \mathscr{X}'_{t'}$. As $[\Gamma_t]_*$ sends $H^2(\mathscr{X}_t, \mathbb{Q})$ isomorphically to $H^2(\mathscr{X}'_t, \mathbb{Q})$, we see that ψ_{t*} sends $\mathrm{Pic}(\mathscr{X}_t) \otimes \mathbb{Q}$ isomorphically to $\mathrm{Pic}(\mathscr{X}'_t) \otimes \mathbb{Q}$. If we choose $t \in U$ such that $\mathscr{X}_t$ and $\mathscr{X}'_t$ are projective of Picard number one, then ψ^* of the ample generator of $\mathrm{Pic}(\mathscr{X}'_t)$ is again an ample line bundle on $\mathscr{X}_t$. Therefore, ψ_t must be regular and hence an isomorphism. This completes the proof. $\square$

Theorem 4.10. *Let X and X' be projective symplectic varieties with irreducible symplectic resolutions $\pi : Y \to X$ and $\pi' : Y' \to X'$. Let $\phi : Y \dashrightarrow Y'$ be a birational map such that the induced map $\phi^* : H^2(Y', \mathbb{C}) \to H^2(Y, \mathbb{C})$ sends $H^2(X', \mathbb{C})$ isomorphically to $H^2(X, \mathbb{C})$. Then there are one parameter locally trivial deformations $f : \mathscr{X} \to \Delta$, $f' : \mathscr{X}' \to \Delta$ of X and X' such that $\mathscr{X}$ and $\mathscr{X}'$ are birational over Δ and such that $\mathscr{X}^* = f^{-1}(\Delta^\times) \cong (f')^{-1}(\Delta^\times) = (\mathscr{X}')^*$.*

Proof. The argument of Huybrechts works in this context almost literally, see [LP16, Theorem 1.1] for the necessary changes. $\square$

Following Voisin [Vo15, Definition 0.6], we call a subvariety $P \subset Y$ of an irreducible symplectic manifold an *algebraically coisotropic subvariety* if it is coisotropic and admits a rational map $\phi : P \dashrightarrow B$ onto a variety of dimension $\dim Y - 2 \cdot \mathrm{codim}\, P$ such that the restriction of the symplectic form to P satisfies $\sigma|_P = \phi^* \sigma_B$ for some 2-form σ_B on B.

Proposition 4.11. *Every irreducible component P of the exceptional locus of an irreducible symplectic resolution $Y \to X$ of a projective symplectic variety X is algebraically coisotropic and the coisotropic fibration of P is given by $\pi : P \to B := \pi(P)$. In particular, it is holomorphic. Moreover, the general fiber of $\pi|_P$ is rationally connected.*

Proof. Let F denote a resolution of singularities of the general fiber of $P \to B$. By [Ka06, Lemma 2.9] it follows that the pullback of the symplectic form σ of Y to F vanishes identically so that F is isotropic. It remains to show that $\dim B = 2(n - \dim F)$ where $\dim Y = 2n$. This is a consequence of [Wi03, Theorem 1.2].

To prove rational connectedness, we use that for every rational curve C on a symplectic variety admitting a symplectic resolution the morphism $\nu : \mathbb{P}^1 \to C \subset Y$ obtained by normalization and inclusion deforms in a family of dimension at least $2n - 2$, see [Ra95, Corollary 5.1] or [CP14, Proposition 3.1]. Being an exceptional locus, every fiber of $\pi : P \to B$ is rationally chain connected by [HM07, Corollary 1.5]. In particular, there are rational curves in P that are contracted by π. Let H be an ample divisor on Y and take an irreducible rational curve C on P contracted by π such that the intersection product $H.C$ is minimal among all such rational curves on P. In the Chow scheme of Y we look at an irreducible component Ch containing $[C]$ of the locus parametrizing rational curves. Let $U \subset \mathrm{Ch} \times Y$ be the graph of the universal family of cycles. As C is contracted by π, the same holds true for all curves in Ch (otherwise e.g. the intersection with a pullback of an ample divisor from X would change) and as U is irreducible, we have in fact $U \subset \mathrm{Ch} \times P$. By minimality of $H.C$ all points in U correspond to irreducible and reduced rational curves. Thus, $U \to \mathrm{Ch}$ is a family of curves in the fibers of $P \to B$.

By what we noted above, $\dim \mathrm{Ch} \geq 2n - 2$ and thus a simple dimension count and the fact that a positive dimensional family of rational curves with two basepoints has to have reducible or nonreduced members (Bend and Break, see e.g. [Ko96, Theorem (5.4.2)]) imply that through the general point of a general fiber F of $P \to B$ there is a family of rational curves without further basepoints of dimension $\dim(F) - 1$. Consequently, F is rationally connected. □

Remark 4.12. Rational connectedness of F would follow from [CMSB02, Theorem 9.1] the proof of which however seems to be incomplete. Instead, it might also be possible to use [CMSB02, Theorem 2.8 (2)] and the well-known fact that a variety is rationally connected if and only if it contains a very free rational curve, see e.g. [Ko96, Ch IV, 3.7 Theorem].

Remark 4.13. As mentioned in the proof of Proposition 4.11, rational chain connectedness follows from the much stronger result [HM07, Corollary 1.5]. This notion coincides for smooth varieties with rational connectedness, however, this is not the case for singular varieties. The cone over an elliptic curve is the easiest example of a variety which is rationally chain connected but not rationally connected.

Recall from [LP19, Theorem 4.6] that given an algebraically coisotropic subvariety P with almost holomorphic coisotropic fibration $\phi : P \dashrightarrow B$ whose generic fiber F is smooth, the subvariety F deforms all over its Hodge locus $\mathrm{Hdg}_F \subset \mathrm{Def}(Y)$. Moreover, if for $t \in \mathrm{Hdg}_F$ we denote by Y_t the corresponding deformation of Y, then the deformations of F inside Y_t cover an algebraically coisotropic subvariety $P_t \subset Y_t$

with F_t as a generic fiber of the coisotropic fibration. It seems, however, unclear how to relate the cycle class of P_t with that of P let alone to show that P_t is a flat deformation.

In the context of birational contractions of symplectic varieties (in dimension ≥ 4 at least) it is rather common that the generic fiber of the exceptional locus over its image is smooth. Thus, it is worthwhile to mention that the main result of [LP19] can be strengthened in this special situation. But first we need some notation.

Let $F \subset Y$ be a closed subvariety in an irreducible symplectic manifold. If $\mathscr{Y} \to \mathrm{Def}(Y)$ denotes the universal deformation we let $\mathscr{H} \to \mathrm{Def}(Y)$ be the union of all those components of the relative Hilbert scheme (or Douady space) of $\mathscr{Y}$ over $\mathrm{Def}(Y)$ which contain $[F]$. We define the closed subspace $\mathrm{Def}(Y, F) \subset \mathrm{Def}(Y)$ to be the scheme theoretic image of $\mathscr{H} \to \mathrm{Def}(Y)$; this is the space of deformations of Y that contain a deformation of F.

Theorem 4.14. *Let $\pi : Y \to X$ be an irreducible symplectic resolution of a projective symplectic variety X, let $P \subset Y$ be the exceptional locus of π, put $B := \pi(P)$, and let $\mathscr{Y} \to \mathscr{X}$ be the restriction of the universal deformation of $Y \to X$ over $\mathrm{Def}(Y, N)$. Suppose that P is irreducible and that a general fiber F of $\pi : P \to B$ is smooth. Then we have $\mathrm{Def}(Y, N) \subset \mathrm{Def}(Y, F)$ and the Hodge locus Hdg_P of P contains $\mathrm{Def}(Y, N)$.*

Proof. Let $\mathscr{F} \to \mathscr{H} \to \mathrm{Def}(Y, F)$ be the universal deformation of F over the closed subspace $\mathscr{H}$ of the relative Hilbert scheme of $\mathscr{Y} \to \mathrm{Def}(Y, N)$. By Proposition 4.11 the variety P is algebraically coisotropic and the fibers of $\pi : P \to B$ are rationally connected so that $H^1(F, \mathcal{O}_F) = 0$ and [LP19, Theorem 1.1] can be applied. We deduce that $\mathrm{Def}(Y, F)$ and $\mathscr{H} \to \mathrm{Def}(Y, F)$ are smooth at 0 respectively $[F]$. In particular, $\mathscr{H}$ is irreducible. Moreover, by [LP19, Corollary 1.2] the period map identifies $\mathrm{Def}(Y, F)$ with $Q \cap \mathbb{P}(K)$ where $Q \subset \mathbb{P}(H^2(Y, \mathbb{C}))$ is the period domain of the irreducible symplectic manifold Y and $K = \ker(H^2(Y, \mathbb{C}) \to H^2(F, \mathbb{C}))$. If $b \in B$ denotes the point with $F = \pi^{-1}(b)$, then by commutativity of

$$
\begin{array}{ccc}
F & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\{b\} & \longrightarrow & X
\end{array}
$$

it follows that we have $H^2(X, \mathbb{C}) \subset K$ and hence using $H^2(X, \mathbb{C})^\perp = N$ and the period map once more we obtain $\mathrm{Def}(Y, N) \subset \mathrm{Def}(Y, F)$.

In order to show that P remains a Hodge class all over $\mathrm{Def}(Y, N)$ we will construct flat families $\mathscr{P}_\Delta \to \Delta$ over curves $\Delta \subset \mathrm{Def}(Y, N)$ passing through the origin such that the cycle underlying the central fiber $\mathscr{P}_{\Delta,0}$ is a multiple of P. To this end we replace $\mathscr{F} \to \mathscr{H}$ as well as $\mathscr{Y} \to \mathscr{X}$ by their restrictions to a given smooth curve germ $\Delta \subset \mathrm{Def}(Y, N)$ and obtain morphisms $\mathscr{H} \longleftarrow \mathscr{F} \longrightarrow \mathscr{Y} \longrightarrow \mathscr{X}$ over Δ. The map in the middle is induced by the projection to the second factor of $\mathscr{F} \subset \mathscr{H} \times \mathscr{Y}$. As Δ is smooth and smoothness is stable under base change we may still assume that $\mathscr{H}$, $\mathscr{H} \to \Delta$ and $\mathscr{F} \to \mathscr{H}$ are smooth at $[F]$ respectively in a neighborhood of $F \subset \mathscr{F}$.

In particular, there is still a unique irreducible component of $\mathscr{H}$ passing through $[F]$ and by shrinking the representative of $\mathrm{Def}(Y, F)$ and throwing away components of $\mathscr{H}$ we may assume that $\mathscr{H}$ is irreducible.

On the other hand, as $\mathscr{X} \to \Delta$ is a locally trivial deformation, it induces a flat (even locally trivial) deformation of all components of its singular locus (with the reduced structure). Let $\mathscr{B} \to \mathrm{Def}(Y, N)$ be the so induced deformation of B. Then $\mathscr{B} \subset \mathscr{X}$ is an irreducible (and reduced) subspace. If we knew that also $\mathscr{Y} \to \mathscr{X}$ were a locally trivial deformation of $Y \to X$, the claim would follow immediately. As we cannot prove this so far, cf. Question 4.16, we have to argue differently.

We take the unique closed irreducible and reduced subspace $\mathscr{F}' \subset \mathscr{F}$ that coincides with $\mathscr{F}$ in a neighborhood of F. As $\rho : \mathscr{F}' \to \mathscr{H}$ is proper it therefore is surjective as well. If we take the Stein factorization, $\mathscr{F}' \to \widetilde{\mathscr{H}} \to \mathscr{H}$, then by the Rigidity Lemma (see e.g. [De01, Lemma 1.15]) there is a commutative diagram

$$
\begin{array}{ccc}
\mathscr{F} & \longrightarrow & \mathscr{Y} \\
\downarrow & & \downarrow \\
\widetilde{\mathscr{H}} & \longrightarrow & \mathscr{X}
\end{array}
$$

Note that $\widetilde{\mathscr{H}} \to \mathscr{H}$ is finite, birational, and an isomorphism over $[F] \in \mathscr{H}$ and that $\widetilde{\mathscr{H}}$ is irreducible. Moreover, the image of $\widetilde{\mathscr{H}} \to \mathscr{X}$ coincides in a neighborhood of $[f] \in \widetilde{\mathscr{H}}$ with the closed subvariety $\mathscr{B} \subset \mathscr{X}$ thanks to the smoothness of $\mathscr{F} \to \mathscr{H}$ in a neighborhood of $F \subset \mathscr{F}$. Invoking the irreducibility of $\widetilde{\mathscr{H}}$ and $\mathscr{B}$ we conclude that we have $\widetilde{\mathscr{H}} \twoheadrightarrow \mathscr{B} \subset \mathscr{X}$. We define $\mathscr{P} = \mathscr{P}_\Delta \subset \mathscr{Y}$ to be the image of $\mathscr{F}' \to \mathscr{Y}$. The variety $\mathscr{F}'$ being irreducible the same holds true for $\mathscr{P}$ and hence the induced map $\rho : \mathscr{P} \to \Delta$ is flat. It remains to show that $P \subset \mathscr{P}$ is the unique component of the central fiber $\mathscr{P}_0$ of ρ of dimension $\dim P$. This follows from the irreducibility of P by invoking the Rigidity Lemma once more. $\qquad\square$

Recall from [Vo15, Definition 1.5] that a cohomology class $p \in H^{2i}(Y, \mathbb{C})$ on a symplectic manifold is called *coisotropic* if it is a Hodge class and $[\sigma]^{n-i+1} \cup p = 0$ in $H^{2n+2}(Y, \mathbb{C})$ where σ is the symplectic form on Y. We refer to Huybrechts' article [Hu14, Definition 3.1] for the notion of a constant cycle subvariety. This is roughly speaking a subvariety of a given variety all of whose points have the same cycle class in the ambient variety.

Corollary 4.15. *In the situation of Theorem 4.14, the class of $[P]$ remains an effective coisotropic Hodge class all over $S = \mathrm{Def}(Y, N)$. Moreover, there are varieties $P_t \subset Y_t$ for each $t \in S$ representing (a multiple of) $[P]$ which are algebraically coisotropic with rationally connected fibers. In particular, the fibers are constant cycle subvarieties of Y_t.*

Proof. It follows from the preceding theorem that the class $[P_t]$ of the subvarieties $P_t \subset Y_t$ for a deformation Y_t of Y with $t \in \mathrm{Def}(Y, N)$ is (a multiple of) $[P]$. The

proof showed moreover that P_t is covered by deformations of the general fiber F of the coisotropic fibration of P. The claim follows as F was rationally connected and rational connectedness is known to be invariant under deformations (for smooth varieties). $\square$

Question 4.16. *Let X be a projective symplectic variety, $Y \to X$ an irreducible symplectic resolution, $N = \widetilde{q}^{-1}(N_1(Y/X))$ (cf. Lemma 3.4), and $\mathscr{Y} \to \mathscr{X}$ be the morphism between universal families over $S := \mathrm{Def}(Y, N) = \mathrm{Def}^{\mathrm{lt}}(X)$ from diagram (4.3). In this case we ask:*

$$\text{Is } \mathscr{Y} \to \mathscr{X} \text{ a locally trivial deformation of } Y \to X?$$

Note that if this were the case, the whole diagram

$$
\begin{array}{ccc}
\mathscr{E} & \longrightarrow & \mathscr{Y} \\
\downarrow & & \downarrow \\
\mathscr{S} & \longrightarrow & \mathscr{X}
\end{array}
$$

of complex spaces over S where $\mathscr{S} \to S$ is the singular locus of $\mathscr{X} \to S$ and $\mathscr{E} \to S$ is the exceptional locus of $\mathscr{Y} \to \mathscr{X}$ were a locally trivial (in particular flat) deformation over S of its central fiber.

5. Period maps and monodromy groups

In this section we first show that any symplectic resolution $\pi : Y \to X$ is deformable. We then use this to develop the global locally trivial deformation theory.

5.1. **Deformable resolutions.** We begin with some preliminary remarks regrading the Kähler cone of a smooth hyperkähler manifold.

In parallel to the notation in Definition 3.7, we define $C(X) \subset H^2(X, \mathbb{R})$ and $C^{1,1}(X) \subset H^{1,1}(X, \mathbb{R})$ for X a symplectic variety admitting an irreducible symplectic resolution. The cone $C^{1,1}(X)$ has two connected components, and we define $C^{1,1}(X)^+$ to be the component containing a Kähler class. As above, this induces a choice of generator of $H^2(C(X), \mathbb{Z})$ which is constant in locally trivial families. Under the pullback map, $C^{1,1}(X)^+$ maps to $C^{1,1}(Y)^+$.

Remark 5.2. For an irreducible holomorphic symplectic manifold Y, the cone $C^{1,1}(Y)^+$ further has a wall-and-chamber decomposition such that the complement of the walls are the images of the Kähler cones of birational models of Y under the action of monodromy operators that preserve the Hodge structure (see [Ma11, Definition 5.10] for details). Following Markman, the walls and chambers of this decomposition will be referred to as *Kähler-type walls* and *Kähler-type chambers*, respectively. Let Λ' be the abstract lattice underlying $H^2(Y, \mathbb{Z})$, and let $\mathfrak{M}_{\Lambda'}$ be the Λ'-marked moduli space of hyperkähler manifolds. By [Ma11, Theorem 5.16], for any connected component $\mathfrak{N}_{\Lambda'}$ of $\mathfrak{M}_{\Lambda'}$ and any $(Y, \mu) \in \mathfrak{N}_{\Lambda'}$ the points in the same fiber of $P : \mathfrak{N}_{\Lambda'} \to \Omega_{\Lambda'}$ as (Y, μ) are in natural correspondence with the Kähler-type chambers of $C^{1,1}(Y)^+$.

The Kähler-type walls of $C^{1,1}(Y)^+$ have been described by Amerik–Verbitsky in terms of monodromy birationally minimal (MBM) classes [AV15, Definition 1.13] (see also the wall divisors of [Mo15]). A nonzero class $\alpha \in H^{1,1}(Y, \mathbb{Z})$ with $q_Y(\alpha) < 0$ is MBM_Y if up to the action of the monodromy group $\alpha^\perp \subset H^{1,1}(Y, \mathbb{R})$ is a wall of the Kähler cone of a birational model of Y. By [AV15, Theorem 1.19] the Kähler-type walls of $C^{1,1}(Y)^+$ are precisely the hyperplanes $\alpha^\perp$ for α ranging over all MBM_Y classes. Note that MBM_Y classes are deformation invariant along deformations for which they remain of Hodge type $(1,1)$ by [AV15, Theorem 1.17], and that they have bounded square assuming $b_2(Y) > 5$ by [AV17, Theorem 5.3]. In particular, if $b_2(Y) > 5$, the Kähler cone is locally polyhedral in $C^{1,1}(Y)^+$ and if $\mathrm{Pic}(Y)$ is negative-definite there are only finitely many Kähler-type chambers. We say a class $\alpha \in H^2(Y, \mathbb{Z})$ is MBM if it becomes $MBM_{Y'}$ for some deformation Y' of Y, so that the MBM_Y classes are precisely the Hodge MBM classes.

Lemma 5.3. *Let* $\pi : Y \longrightarrow X$ *be an irreducible symplectic resolution with* $b_2(Y) > 5$. *Then* $N(Y/X) := (\pi^* H^2(X, \mathbb{Z}))^\perp \subset H^2(Y, \mathbb{Z})$ *is rationally generated by* MBM_Y *classes.*

Proof. We know that that $N(Y/X)^\perp$ in $H^{1,1}(Y, \mathbb{R})$ intersects the nef cone of Y in an extremal face τ because $N(Y/X)$ is generated by the duals of curves that are contracted. As τ contains a positive class ω (namely the pullback of a Kähler class of X) and the nef cone is polyhedral near ω and cut out by $\alpha^\perp$ for α MBM_Y, it follows that $N(Y/X)_{\mathbb{Q}}$ is generated by MBM_Y classes. $\square$

Before proceeding we introduce the following terminology.

Definition 5.4. Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution. The cones of the Kähler cone decomposition of $C^{1,1}(Y)^+$ at a very general deformation of Y along which $\pi^* H^2(Y, \mathbb{Z})$ remains algebraic will be called *resolution chambers* of π. Equivalently, these are the connected components of

$$C^{1,1}(Y)^+ - \bigcup_{\alpha \in MBM_Y \cap N(Y/X)} \alpha^\perp.$$

The chamber containing the Kähler cone of Y will be called the *resolution Kähler chamber* of π.

By the above, if $b_2(Y) > 5$ there are finitely many resolution chambers.

Definition 5.5. Provided $b_2(X) > 4$, for a choice of resolution $\pi : Y \longrightarrow X$ we define a Kähler-type wall-and-chamber decomposition of $C^{1,1}(X)^+$ whose open chambers are the connected components of

$$\pi^* C^{1,1}(X)^+ - \bigcup_{\substack{\alpha \in MBM_Y \\ \alpha \notin N(Y/X)}} \alpha^\perp.$$

As the Kähler cone of X pulls back to the nef cone of Y, there is a unique chamber containing the Kähler cone of X which we call the Kähler-type cone of X.

Remark 5.6. Note that the Kähler-type decomposition may depend on the choice of resolution. However, given another resolution $\pi' : Y' \to X$ a parallel transport operator $f : H^2(Y, \mathbb{Z}) \to H^2(Y', \mathbb{Z})$ sending $\pi^* H^2(X, \mathbb{Z})$ to $\pi'^* H^2(X, \mathbb{Z})$ maps Kähler-type chambers with respect to π isomorphically to Kähler-type chambers with respect to π'.

Proposition 5.7. *Every symplectic resolution* $\pi : Y \to X$ *with* $b_2(X) > 4$ *is deformable.*

Proof. Choose a marking $\mu : \Lambda' \xrightarrow{\cong} H^2(Y, \mathbb{Z})$ sending $\Lambda \subset \Lambda'$ to $\pi^* H^2(X, \mathbb{Z})$, and let $\mathfrak{N}_{\Lambda'}$ be the component of $\mathfrak{M}_{\Lambda'}$ containing (Y, μ). Consider the subspace $\mathfrak{N}_{\Lambda', \Lambda}$ of $\mathfrak{N}_{\Lambda'}$ for which $N = \Lambda^\perp$ is Hodge. Clearly $\mathfrak{N}_{\Lambda', \Lambda}$ is the preimage of $\Omega_\Lambda \subset \Omega_{\Lambda'}$ under the period map. Let $\Gamma_N \subset \mathrm{O}(\Lambda)$ be the image of the subgroup of the monodromy group of Y acting trivially on $N(Y/X)$. For a choice of resolution chamber τ of π (which we identify with its image in Λ'), let $\mathfrak{N}_{\Lambda', \Lambda}^\tau \subset \mathfrak{N}_{\Lambda', \Lambda}$ be the subspace of (Y, μ) whose Kähler cone is contained in τ and has Λ as a face. As Γ_N stabilizes Λ, each $\mathfrak{N}_{\Lambda', \Lambda}^\tau$ is an open Γ_N-invariant subspace of $\mathfrak{N}_{\Lambda', \Lambda}$.

Now, let $M \subset \Lambda$ be the (images) of orthogonal projections α' of MBM_Y classes α for which α is not contained in $N(Y/X)$ and α' is of nonpositive square. As N is negative definite, the classes in M have bounded nonpositive square, and moreover their denominators are bounded by the size of the discriminant group of Λ. Thus, as in the discussion after Proposition 3.11 we obtain an induced cone decomposition of $C^{1,1}(\Lambda_p)^+ \subset (\Lambda_\mathbb{R})_p^{1,1}$ for any $p \in \Omega_\Lambda$, and a space $\Omega_\Lambda^{\mathrm{cone}}$.

For each τ there is a natural map $\rho : \mathfrak{N}_{\Lambda', \Lambda}^\tau \to \Omega_\Lambda^{\mathrm{cone}}$ sending $y = (Y, \mu)$ to the pair $(P(y), C)$ where C is the interior of the intersection of the nef cone of Y with Λ.

Lemma 5.8. *The map* $\rho : \mathfrak{N}_{\Lambda', \Lambda}^\tau \to \Omega_\Lambda^{\mathrm{cone}}$ *is an isomorphism.*

Proof. By the local Torelli theorem, ρ is a local isomorphism, so it is enough to show that it is bijective. Consider a point $(Y, \mu) \in \mathfrak{N}_{\Lambda', \Lambda}^\tau$ with period $p \in \Omega_\Lambda$. As the cones of the Kähler cone decomposition of $C^{1,1}(Y)^+$ have disjoint interiors, any cone C in this decomposition for which $\overline{C} \cap \Lambda_\mathbb{R}$ has nonempty interior is determined by $\overline{C} \cap \Lambda_\mathbb{R}$, so the map is injective. Conversely, since the interior of the intersection of $\overline{\tau}$ with $\Lambda_\mathbb{R}$ is the entire positive cone $C^{1,1}(\Lambda_p)^+$, it follows that any $(C, p) \in \Omega_\Lambda^{\mathrm{cone}}$ can be lifted to a cone C' in the decomposition of $C^{1,1}(\Lambda_p')^+$ for which $\overline{C'} \cap \Lambda_\mathbb{R} = \overline{C}$. $\square$

By Proposition 4.3 we may assume X is not projective, and in particular not of maximal Picard rank. Now, Γ_N is a finite index subgroup of $\mathrm{O}(\Lambda)$, so by Corollary 3.14 we can choose a projective point $(Y_0, \mu_0) \in \mathfrak{N}_{\Lambda', \Lambda}$ in the orbit closure of (Y, μ) for which Λ is a face of the nef cone. By the basepoint free theorem there is a contraction $\pi_0 : Y_0 \to X_0$ to a primitive symplectic variety contracting the classes in N, and moreover by Proposition 4.3 this contraction deforms to $\mathcal{Y}_0 \to \mathcal{X}_0$ over $\mathrm{Def}(Y_0, N) \subset \mathfrak{N}_{\Lambda', \Lambda}^\tau$, with $\mathcal{X}_0$ deforming locally trivially. After changing the marking by an element of Γ_N, we have $(Y, \mu') \in \mathrm{Def}(Y_0, N)$, and therefore there is a deformable

contraction $\pi' : Y \longrightarrow X'$. As π' contracts exactly N, it follows that this is in fact the original contraction $\pi : Y \longrightarrow X$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We record the following corollary of the proof, which is was pointed out by Amerik–Verbitsky [AV19, Theorem 5.8]:

Corollary 5.9. *Let Y be a smooth hyperkähler manifold with $b_2(Y) > 4$ and $\tau \subset H^{1,1}(Y, \mathbb{R})$ a face of the Kähler cone meeting the positive cone $C^{1,1}(Y)^+$ for which $\dim \tau > 2$. Then there is a bimeromorphic contraction $\pi : Y \longrightarrow X$ contracting precisely $\tau^\perp$.*

Remark 5.10. It should be noted that the assumption $b_2(X) > 4$ that we used several times throughout this section is nontrivial. Unlike in the smooth case, we know that there are examples of symplectic varieties X with $b_2(X) = 3$ that admit an irreducible symplectic resolution: it may well happen that a birational contraction $Y \longrightarrow X$ of an irreducible symplectic manifold Y contracts a negative definite subspace of $H^2(Y, \mathbb{R})$ of maximal dimension. In the case of $K3$ surfaces for example, one may take X to be S/G where S is a $K3$ surface and G is a group of symplectic automorphisms with minimal invariant second cohomology lattice $H^2(S, \mathbb{Z})^G$ (i.e. of rank 3). Finite groups of symplectic automorphisms were classified by Nikulin [Ni76] and Mukai [Mu88], explicit examples of groups with the sought-for rank of $H^2(S, \mathbb{Z})^G$ may be found in [Xi96, Ha12]. An example of a different kind may be found in [OZ96]. The minimal resolution $Y \longrightarrow X$ then gives us an example of a contraction of relative Picard rank 19. The induced contraction $\mathrm{Hilb}^n(Y) \longrightarrow \mathrm{Sym}^n(Y) \longrightarrow \mathrm{Sym}^n(X)$ gives an example of a contraction of a $K3^{[n]}$-type variety of dimension $2n$ with relative Picard rank 20.

5.11. **Marked moduli spaces.** Recall that given a lattice Λ with quadratic form q of signature $(3, \mathrm{rk}(\Lambda) - 3)$, we define an analytic coarse moduli space $\mathfrak{M}_\Lambda$ of Λ-marked irreducible holomorphic symplectic manifolds by gluing together the Kuranishi families. Here, a *marking* of an ireducible holomorphic symplectic manifold Y is an isometry $\mu : (\Lambda, q) \xrightarrow{\cong} (H^2(Y, \mathbb{Z}), q_Y)$ and a *marked manifold* is a pair (Y, μ) where μ is a marking. Likewise, we define $\mathfrak{M}_\Lambda^{\mathrm{lt}}$ to be the analytic space obtained by gluing together the locally trivially Kuranishi spaces of Λ-marked symplectic varieties admitting irreducible symplectic resolutions. By Theorem 4.1, $\mathfrak{M}_\Lambda^{\mathrm{lt}}$ is a not-necessarily-Hausdorff complex manifold—see, for example, [Hu11, §4.2] and the references therein for more details on the construction of marked moduli spaces (in the smooth case).

With the period domain Ω_Λ defined as in (3.1), by the local Torelli theorem, see Corollary 4.7, there is a period map $P : \mathfrak{M}_\Lambda^{\mathrm{lt}} \longrightarrow \Omega_\Lambda$ that is a local isomorphism.

Let (X, ν) be a Λ-marked symplectic variety admitting an irreducible symplectic resolution. By the deformability property and [Hu03, Theorem 2.5], the deformation type of a symplectic resolution is constant along each connected component of $\mathfrak{M}_\Lambda^{\mathrm{lt}}$. Given a lattice Λ' with quadratic form q' of signature $(3, \mathrm{rk}(\Lambda') - 3)$ and a primitive embedding of lattices $\iota : \Lambda \hookrightarrow \Lambda'$, we define a *compatibly marked symplectic resolution* $\pi : (Y, \mu) \longrightarrow (X, \nu)$ to be a symplectic resolution π and a commutative diagram

$$
\begin{array}{ccc}
\Lambda' & \xrightarrow[\cong]{\ \mu\ } & H^2(Y,\mathbb{Z}) \\
\iota \big\uparrow & & \big\uparrow \pi^* \\
\Lambda & \xrightarrow[\nu]{\ \cong\ } & H^2(X,\mathbb{Z})
\end{array}
$$

compatible with q and q'. We define $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$ to be the set of compatibly marked irreducible symplectic resolutions $(Y,\mu) \to (X,\nu)$ modulo the following equivalence relation: we identify $\pi : (Y,\mu) \to (X,\nu)$ with $\pi' : (Y',\mu') \to (X',\nu')$ provided there is an isomorphism

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \cong\ } & Y' \\
\pi \big\downarrow & & \big\downarrow \pi' \\
X & \xrightarrow[\cong]{} & X'
\end{array}
$$

compatible with the markings. There are obvious forgetful maps that fit into a diagram

(5.1)

$$
\begin{array}{ccc}
& & \mathfrak{M}^{\mathrm{lt}}_{\Lambda} \xrightarrow{\ P\ } \Omega_{\Lambda} \\
& \nearrow & \big\downarrow \\
\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda} & & \big\downarrow \\
& \searrow & \\
& & \mathfrak{M}_{\Lambda'} \xrightarrow[\ P\ ]{} \Omega_{\Lambda'}
\end{array}
$$

where the vertical arrow is the embedding of Ω_{Λ} into $\Omega_{\Lambda'}$ as the Noether-Lefschetz locus $\Omega_{\Lambda'} \cap \mathbb{P}(\Lambda \otimes \mathbb{C})$. The set $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$ is given the structure of an analytic space by requiring the top diagonal map to be a local isomorphism, using Proposition 4.5.

For the following proposition, let $\mathfrak{M}_{\Lambda',\Lambda}$ be the inverse image under P of $\Omega_{\Lambda} \subset \Omega_{\Lambda'}$ in $\mathfrak{M}_{\Lambda'}$—that is, the locus where $\Lambda^\perp$ is Hodge. Recall from the proof of Proposition 5.7 that for a component $\mathfrak{N}_{\Lambda'} \subset \mathfrak{M}_{\Lambda'}$ and for a choice of resolution chamber τ the subset $\mathfrak{N}^\tau_{\Lambda',\Lambda}$ consists of those (Y,μ) for which the Kähler cone of Y is contained in τ and has Λ as a face. Note that $\mathfrak{N}^\tau_{\Lambda',\Lambda}$ surjects onto Ω_{Λ} via P and is connected since the fiber of P over a very general point is a single point and Ω_{Λ} is connected.

Proposition 5.12. *Assume* $\mathrm{rk}(\Lambda) > 4$. *For each choice of component* $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$ *of* $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$, *the diagonal map from* (5.1) *yields an isomorphism onto some* $\mathfrak{N}^\tau_{\Lambda',\Lambda} \subset \mathfrak{M}_{\Lambda',\Lambda}$ *(respectively some component* $\mathfrak{N}^{\mathrm{lt}}_{\Lambda'}$ *of* $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$*). Moreover, each such* $\mathfrak{N}^\tau_{\Lambda',\Lambda}$ *(resp.* $\mathfrak{N}^{\mathrm{lt}}_{\Lambda}$*) arises as the image of some component* $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$ *of* $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$.

Proof. Given the first claim, the second claim is obvious by Corollary 5.9 (resp. by taking a resolution). Both diagonal maps are local isomorphisms, so for the first claim we need only show bijectivity.

For the bottom diagonal map, $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$ lands in the union $\bigcup_\tau \mathfrak{N}^\tau_{\Lambda',\Lambda}$, and therefore in a single $\mathfrak{N}^\tau_{\Lambda',\Lambda}$. Corollary 5.9 implies that any path in $\mathfrak{N}^\tau_{\Lambda',\Lambda}$ can be locally lifted to

$\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$. The lift of a point is unique if it exists (as a contraction is unique if it exists and the marking is determined), so paths lift globally, and it follows that $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$ bijects onto $\mathfrak{N}^{\tau}_{\Lambda',\Lambda}$.

For the top diagonal map, suppose $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda}$ lands in a component $\mathfrak{N}^{\mathrm{lt}}_{\Lambda}$. Note that a very general point in $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$ has uniformly finitely many lifts to $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$, as the Kähler cone decomposition of a resolution Y has finitely many chambers and the marking extends in finitely many ways. By Proposition 5.7, for any path $\gamma \subset \mathfrak{N}^{\mathrm{lt}}_{\Lambda}$ and for each very general point $x \in \gamma$ we can find an open neighborhood in which the path lifts through every lift of x. As γ can be covered by finitely many such neighborhoods, paths can be lifted globally, and $\mathfrak{N}^{\mathrm{res}}_{\Lambda',\Lambda} \longrightarrow \mathfrak{N}^{\mathrm{lt}}_{\Lambda}$ is surjective.

The injectivity claim means the following: if we have two resolutions $\pi : Y \longrightarrow X$ and $\pi' : Y' \longrightarrow X$ together with a parallel transport operator $f : H^2(Y,\mathbb{Z}) \longrightarrow H^2(Y',\mathbb{Z})$ arising from a simultaneously resolved family connecting π to π' which is the identity on $H^2(X,\mathbb{Z})$, then f arises from an isomorphism $\phi : Y \longrightarrow Y'$ such that $\pi' \circ \phi = \pi$. By the statement of the proposition regarding the lower diagonal map, f must send the resolution Kähler chamber τ of π to the resolution Kähler chamber τ' of π'. Moreover, f fixes a Kähler class on X and therefore by Remark 5.6 both the image of the nef cone of Y and the nef cone of Y' have the same intersection with $\pi'^* H^2(X,\mathbb{R})$. It follows that f maps the Kähler cone of Y to the Kähler cone of Y' and the claim then follows. $\qquad\square$

Corollary 5.13. *Let $\pi : Y \longrightarrow X$ be an irreducible symplectic resolution with $b_2(X) > 4$ and let $\mathfrak{N}^{\mathrm{lt}}_{\Lambda}$ be a component of the locally trivial marked moduli space of X. The fibers of $P : \mathfrak{N}^{\mathrm{lt}}_{\Lambda} \longrightarrow \Omega_{\Lambda}$ are in bijective correspondence with the Kähler-type chambers of X with respect to π.*

Corollary 5.14. *Any locally trivial (respectively simultaneously resolved) parallel transport operator can be realized as parallel transport along a path in a locally trivial (respectively simultaneously resolved) family over a connected analytic base.*

Proof. It follows from the previous corollary that a Picard rank zero point of $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$ (respectively $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$) is a separated point (i.e. can be separated from any other point). Thus, any path in $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$ (respectively $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$) can be covered by finitely many open sets over which families exist, and these may be glued at very general points. $\qquad\square$

Corollary 5.15. *If (X,ν) and (X',ν') are inseparable in $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$, then X and X' are birational.*

Proof. (X,ν) and (X',ν') admit marked resolutions that are inseperable in moduli, hence birational. $\qquad\square$

5.16. **Monodromy groups.** Each of the above moduli spaces $\mathfrak{M}$ has a corresponding notion of parallel transport operator, which we call *locally trivial parallel transport operators* for $\mathfrak{M}^{\mathrm{lt}}_{\Lambda}$ and *simultaneously resolved parallel transport operators* for $\mathfrak{M}^{\mathrm{res}}_{\Lambda',\Lambda}$.

Precisely, for (X, ν) and (X', ν') in the same component of $\mathfrak{M}_\Lambda^{\mathrm{lt}}$, we define $\nu' \circ \nu^{-1} :$ $H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ to be a locally trivial parallel transport operator, and likewise for the simultaneously resolved parallel transport operators. A simultaneously resolved parallel transport operator from $\pi : Y \to X$ to $\pi' : Y' \to X'$ yields a diagram

$$
\begin{array}{ccc}
H^2(Y, \mathbb{Z}) & \xrightarrow{\;f\;} & H^2(Y', \mathbb{Z}) \\
{\scriptstyle \pi^*}\Big\uparrow & & \Big\uparrow {\scriptstyle \pi'^*} \\
H^2(X, \mathbb{Z}) & \xrightarrow[\;g\;]{} & H^2(X', \mathbb{Z})
\end{array}
$$

and f, g are evidently locally trivial parallel transport operators.

Proposition 5.12 allows us to describe the relationship between these three notions of parallel transport operators.

Corollary 5.17. *Suppose X admits an irreducible symplectic resolution and $b_2(X) > 4$.*

    (1) *For each choice of symplectic resolution $\pi : Y \to X$, a locally trivial parallel transport operator $g : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ lifts uniquely to a simultaneously resolved parallel transport operator.*

    (2) *Let $\pi : Y \to X$ be a symplectic resolution. A parallel transport operator $f : H^2(Y, \mathbb{Z}) \to H^2(Y', \mathbb{Z})$ for which $f(N(Y/X))$ is Hodge extends to a simultaneously resolved parallel transport operator if and only if the image of the resolution Kähler chamber of Y contains the Kähler cone of Y'.*

Let us now turn to the associated monodromy groups. We adopt the following notation:

    (1) $\mathrm{Mon}^2(X)^{\mathrm{lt}} \subset \mathrm{O}(H^2(X, \mathbb{Z}))$ will be the image of the monodromy representation associated to locally trivial families. Here, the orthogonal group is taken with respect to the Beauville–Bogomolov–Fujiki form q_X on $H^2(X, \mathbb{C})$, see Remark 4.6. Likewise we define $\mathrm{Mon}^2(Y)$ if Y is a smooth hyperkähler manifold.

    (2) Likewise for an irreducible symplectic resolution $\pi : Y \to X$ we define $\mathrm{Mon}^2(\pi) \subset$ $\mathrm{O}(H^2(Y, \mathbb{Z})) \times \mathrm{O}(H^2(X, \mathbb{Z}))$ to be the monodromy group associated to simultaneously resolved families. Note that we can in fact think of $\mathrm{Mon}^2(\pi) \subset$ $\mathrm{O}(H^2(Y, \mathbb{Z}))$.

Let $\pi : Y \to X$ be an irreducible symplectic resolution. The monodromy group $\mathrm{Mon}^2(Y)$ clearly acts on the resolution chambers of π.

Corollary 5.18. *Let $\pi : Y \to X$ be an irreducible symplectic resolution with $b_2(X) > 4$.*

    (1) $\mathrm{Mon}^2(\pi) \subset \mathrm{Mon}^2(Y)$ *is the stabilizer of the resolution Kähler chamber of π.*

    (2) $\mathrm{Mon}^2(X)^{\mathrm{lt}}$ *is the image of $\mathrm{Mon}^2(\pi)$ in $\mathrm{O}(H^2(X, \mathbb{Z}))$.*

Corollary 5.19. *With the above setup, $\mathrm{Mon}^2(\pi)$ (respectively $\mathrm{Mon}^2(X)^{\mathrm{lt}}$) is finite index in $\mathrm{O}(H^2(Y, \mathbb{Z}))$ (respectively $\mathrm{O}(H^2(X, \mathbb{Z}))$).*

5.20. **Global Torelli theorem.** We have now proved all the statements in our main Torelli theorem.

Proof of Theorem 1.3. The surjectivity and generic injectivity of the period map follow from Proposition 5.12, Verbitsky's global Torelli theorem [Ve13, Theorem 1.17], and the description of the fibers of the period map in the smooth case in terms of the Kähler-type chambers [Ma11, Theorem 5.16]. Fix a choice of resolution chamber τ for the component $\mathfrak{N}_\Lambda^{\mathrm{lt}}$, which thereby determines a preferred marked resolution at every point by Proposition 5.12. Points in a fiber are birational by Corollary 5.15 and are in bijective correspondence with the Kähler-type chambers with respect to τ. That $\mathrm{Mon}^2(X)^{\mathrm{lt}}$ is finite index in $\mathrm{O}(H^2(X, \mathbb{Z}))$ is Corollary 5.19. $\square$

Remark 5.21. The Hausdorff reduction of the moduli space of quasi-polarized locally trivial deformations of a given projective symplectic variety admitting an irreducible symplectic resolution is thus a locally symmetric variety of orthogonal type. Therefore, Mumford's theory of toroidal compactifications applies and enables one to use methods from algebraic geometry. The geometry of such varieties, and in particular their Kodaira dimensions, have been studied using modular forms by Gritsenko–Hulek–Sankaran in a series of papers, see for example [GHS07, GHS08, GHS10]. A rather general recent result in this direction has been obtained by Ma [Ma18]. The singularities of compactifications of such moduli spaces have studied by Giovenzana in [Gio19].

5.22. **On Namikawa's result.** We conclude this section by generalizing the result of Namikawa [Na01b, Theorem (2.2)], used in the proof of Proposition 4.3.

Proposition 5.23. *Let* $\pi : Y \longrightarrow X$ *be an irreducible symplectic resolution with* $b_2(X) > 4$. *Then* $\mathrm{Def}(X)$ *is smooth of the same dimension as* $\mathrm{Def}(Y)$ *and the induced map* $p : \mathrm{Def}(Y) \longrightarrow \mathrm{Def}(X)$ *is finite.*

Proof. Consider the analytic locus $Z \subset \mathrm{Def}^{\mathrm{lt}}(X)$ where the dimension of the tangent space $\dim \mathrm{Ext}^1(\Omega^1_{\mathscr{X}_t}, \mathcal{O}_{\mathscr{X}_t})$ to the full Kuranishi space $\mathrm{Def}(\mathscr{X}_t)$ is nongeneric, that is, stirctly greater than $\dim H^{1,1}(Y)$. The projective points in $\mathrm{Def}^{\mathrm{lt}}(X)$ (and in every subvariety) are dense by Corollary 4.4, so by [Na01b, Theorem (2.2)] we have that Z is at most the distinguished point $0 \in \mathrm{Def}^{\mathrm{lt}}(X)$ (possibly after shrinking $\mathrm{Def}^{\mathrm{lt}}(X)$). Now we may assume X is not projective (and in particular not of maximal Picard rank), so for any open neighborhood $0 \in U \subset \mathrm{Def}^{\mathrm{lt}}(X)$, by Corollary 3.14 and Lemma 5.8 there is another point $0 \neq t \in U$ corresponding to an isomorphic variety $\mathscr{X}_t \cong X$, and thus $\dim \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ has dimension $\dim H^{1,1}(Y)$.

We now claim $p : \mathrm{Def}(Y) \longrightarrow \mathrm{Def}(X)$ has no positive-dimensional fibers. If it did, then the projective points would be dense in such a fiber by [GHJ, Proposition 26.6], and by the same argument as in Corollary 4.4 therefore X would be projective, contradicting Namikawa's theorem [Na01b, Theorem (2.2)]. Thus,

$$\dim H^{1,1}(Y) = \mathrm{Def}(Y) \leq \dim \mathrm{Def}(X) \leq \dim \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$$

and by the above we have equality everywhere. This proves that $\mathrm{Def}(X)$ must be smooth at 0 as well. The finiteness claim follows from quasi-finiteness upon shrinking the representative $\mathrm{Def}(X)$ as in Namikawa's proof from [Fi87, 3.2 Lemma, p. 132]. $\qquad\square$

6. Applications to $K3^{[n]}$-type manifolds

Recall that a compact Kähler manifold Y is said to be of $K3^{[n]}$-type if it is deformation equivalent to a Hilbert scheme of n points on a $K3$ surface. The $K3^{[n]}$-type manifolds form one of the two known infinite families of irreducible holomorphic symplectic manifolds. We assume throughout that $n \geq 2$ (i.e. that $\dim Y \geq 4$).

By work of Markman [Ma11, Corollary 9.5] there is a canonical extension of weight 2 integral Hodge structures

$$(6.1) \qquad\qquad 0 \to H^2(Y, \mathbb{Z}) \to \widetilde{\Lambda}(Y, \mathbb{Z}) \to Q \to 0$$

where $Q \cong \mathbb{Z}(-1)$. The lattice underlying $\widetilde{\Lambda}(Y, \mathbb{Z})$ is the Mukai lattice $\widetilde{\Lambda}_{K3} = E_8(-1)^2 \oplus U^4$. We denote the primitive generator of the orthogonal to $H^2(Y, \mathbb{Z})$ in $\widetilde{\Lambda}(Y, \mathbb{Z})$ by $v = v(Y)$, which is determined up to sign and satisfies $v^2 = 2 - 2n$. Note that

$$H^2(Y, \mathbb{Z}) \cong \Lambda_{K3^{[n]}} := E_8(-1)^2 \oplus U^3 \oplus (2 - 2n).$$

Denote by $\mathrm{Mon}^2(K3^{[n]}) \subset O(\Lambda_{K3^{[n]}})$ the image of the weight two monodromy representation, which has been computed by Markman [Ma08] to be the subgroup $\widetilde{O}^+(\Lambda_{K3^{[n]}})$ preserving the orientation class and acting as ± 1 on the discriminant group $D(\Lambda_{K3^{[n]}}) := \Lambda^*_{K3^{[n]}}/\Lambda_{K3^{[n]}}$. We have the following well-known consequence of this computation and Verbitsky's global Torelli theorem: the extension (6.1) determines the birational class of Y. More precisely, for two symplectic manifolds Y, Y', there is a Hodge isometry $\phi : H^2(Y, \mathbb{Z}) \to H^2(Y', \mathbb{Z})$ lifting to a Hodge isometry $\widetilde{\phi} : \widetilde{\Lambda}(Y, \mathbb{Z}) \to \widetilde{\Lambda}(Y', \mathbb{Z})$ of the Markman Hodge structures if and only if Y is birational to Y' [Ma11, Corollary 9.8]. We therefore refer to (6.1) as the *extended period*.

6.1. **Bridgeland stability conditions.** Recall that for a $K3$ surface S, the total cohomology $H^*(S, \mathbb{Z})$ carries the so-called Mukai Hodge structure

$$\widetilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z})(-1) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})(1)$$

which comes equipped with the Mukai pairing defined by

$$(a_0 + a_2 + a_4, b_0 + b_2 + b_4) := (a_2, b_2)_S - a_0 b_4 - a_4 b_0$$

for $a_i, b_i \in H^i(S, \mathbb{Z})$. The Bridgeland stability condition and moduli space formalism we discuss below will also work in the larger category of twisted $K3$ surfaces. Recall that the cohomological Brauer group $\mathrm{Br}(S)$ of a $K3$ surface S can defined as the torsion part of the cohomology group $H^2(S, \mathcal{O}^*_S)$ in the analytic topology and is naturally the image of $H^2(S, \mathbb{Q}/\mathbb{Z})$ under the exponential map $e(-) = e^{2\pi i -}$. A twisted $K3$ surface (S, α) in the sense of [HS05, §1] is a $K3$ surface together with a Brauer class $\alpha \in \mathrm{Br}(S)$ and a choice of $\beta \in H^2(S, \mathbb{Q})$ with $e(\beta) = \alpha$, which exists for any α since $H^3(S, \mathbb{Z}) = 0$. A twisted $K3$ surface (S, α) likewise has a Mukai Hodge structure $\widetilde{H}(S, \alpha, \mathbb{Z})$ with

underlying lattice $H^2(S, \mathbb{Z})$, and we say (S, α) is projective if S is. We refer to [BM14b, §2] and the references therein for more details and for the theory of α-twisted sheaves and Bridgeland stability conditions on twisted $K3$ surfaces.

Throughout the following we will only consider primitive Mukai vectors $v \in \widetilde{H}(S, \alpha, \mathbb{Z})$. By work of Bayer–Macrì [BM14a, Theorem 1.3], for a generic Bridgeland stability condition σ on (S, α), the moduli space $Y = M_\sigma(v)$ of Bridgeland σ-stable objects on (S, α) of a Mukai vector $v \in \widetilde{H}(S, \alpha, \mathbb{Z})_{\mathrm{alg}}$ is a projective $K3^{[n]}$-type manifold, and we canonically have $\widetilde{\Lambda}(Y, \mathbb{Z}) = \widetilde{H}(S, \alpha, \mathbb{Z})$ with $v(Y) = v$. The identification $v^\perp \xrightarrow{\cong} H^2(Y, \mathbb{Z})$ is achieved by the Fourier–Mukai transform.

Note that by [BM14b, Theorem 1.2(c)], every symplectic birational model of a Bridgeland moduli space is a Bridgeland moduli space. We will need below the following Hodge-theoretic characterization of Bridgeland moduli spaces which follows from [Hu17, Lemma 2.5] and [Ad16, Proposition 4].

Proposition 6.2. *A projective $K3^{[n]}$-type manifold Y is isomorphic to a Bridgeland moduli space on a projective twisted $K3$ surface if and only if one of the following equivalent conditions holds:*

(1) $\widetilde{\Lambda}(Y, \mathbb{Q})_{\mathrm{alg}}$ *contains U as a sublattice;*
(2) *The rational transcendental lattice $H^2(Y, \mathbb{Q})_{\mathrm{tr}} \cong \widetilde{\Lambda}(Y, \mathbb{Q})_{\mathrm{tr}}$ is Hodge-isometric to the rational transcendental lattice of a projective $K3$ surface.*

Furthermore, the projective twisted $K3$ surface can be taken untwisted if and only if one of the following equivalent conditions holds:

(1′) $\widetilde{\Lambda}(Y, \mathbb{Z})_{\mathrm{alg}}$ *contains U as a primitive sublattice;*
(2′) *The transcendental lattice $H^2(Y, \mathbb{Z})_{\mathrm{tr}} \cong \widetilde{\Lambda}(Y, \mathbb{Z})_{\mathrm{tr}}$ is Hodge-isometric to the transcendental lattice of a projective $K3$ surface.*

6.3. $K3^{[n]}$-type contractions. We now turn to the singular case.

Definition 6.4. Let X be a symplectic variety and $\pi : Y \longrightarrow X$ a irreducible symplectic resolution. We say π is a $K3^{[n]}$-type contraction if in addition Y is a $K3^{[n]}$-type manifold. We will often abuse terminology and refer to X itself as a $K3^{[n]}$-type contraction as well.

Note that if X is a $K3^{[n]}$-type contraction, *every* (Kähler) symplectic resolution is a $K3^{[n]}$-type manifold by Huybrechts' theorem [Hu03, Theorem 2.5].

Example 6.5. Our main source of examples of $K3^{[n]}$-type contractions come from contractions of Bridgeland moduli spaces, which we call Bridgeland contractions. Bridgeland moduli spaces of untwisted $K3$ surfaces will be called untwisted Bridgeland contractions for emphasis. Their geometry is beautifully described via wall-crossing by Bayer–Macrì theory [BM14b]. Given a projective twisted $K3$ surface (S, α), a primitive Mukai vector $v \in \widetilde{H}(S, \alpha, \mathbb{Z})_{\mathrm{alg}}$, and an open chamber $\mathcal{C} \subset \mathrm{Stab}^\dagger(S)$ associated to v, then by [BM14b, Theorem 1.2(a)] any $\sigma_0 \in \partial\mathcal{C}$ yields a semiample class ℓ_{σ_0}

on $M_{\mathcal{C}}(v)$, and the associated morphism $\pi : M_{\mathcal{C}}(v) \to M$ is a $K3^{[n]}$-type contraction. Conversely, by [BM14b, Theorem 1.2(b)] any contraction arises from this construction. By [BM14a, Theorem 4.1] the morphism π contracts a curve if and only if two generic stable objects in the corresponding family are S-equivalent with respect to σ_0.

Much is known about the singularities of Bridgeland contractions, and Theorem 4.9 roughly says that arbitrary $K3^{[n]}$-type contractions exhibit no new singularities:

Proposition 6.6. *Any $K3^{[n]}$-type contraction is locally trivially deformation-equivalent to a Bridgeland contraction. Furthermore, if $b_2(X) \neq 4$, it is locally trivially deformation-equivalent to an untwisted Bridgeland contraction.*

Proof. Fix an identification $\Lambda_{K3^{[n]}} = H^2(Y, \mathbb{Z})$, and let $\Lambda = \pi^* H^2(X, \mathbb{Z})$. If we choose an arbitrary primitive embedding $U \subset \widetilde{\Lambda}_{K3}$ containing $\Lambda_{K3^{[n]}}^\perp$, then $U \cap \Lambda$ is of rank at most one and negative definite. Assuming $b_2(X) > 4$, then there is a rational-rank-zero projective period $\omega \in U^\perp \cap \Omega_\Lambda$, and we can find a point of its orbit under $\Gamma \subset \mathrm{O}(\Lambda)$ arbitrarily close to the period of X by Proposition 3.11. Here we take Γ to be the finite-index subgroup of isometries extending to $\widetilde{\Lambda}_{K3}$ and stabilizing $\Lambda_{K3^{[n]}}$ (and Λ). Thus, π has a (small) locally trivial deformation to a contraction $\pi' : Y' \to X'$ where Y' is an untwisted Bridgeland moduli space, by Proposition 6.2. π' is necessarily a Bridgeland contraction since every contraction of a Bridgeland moduli space is a Bridgeland contraction.

If $b_2(X) = 3$, then in fact $U \cap \Lambda = 0$, and we may simply apply Corollary 4.4. Finally, if $b_2(X) = 4$, let Π be the orthogonal to Λ in $\widetilde{\Lambda}_{K3}$. As $\mathrm{rk}(\Pi) > 4$, we can find $U \subset \Pi_{\mathbb{Q}}$ since Π has (many) isotropic vectors by a classical theorem of Meyer, and the claim again follows from Corollary 4.4. $\qquad\square$

The work of Bayer-Macrì [BM14b] in principle provides a complete description of the singularities of Bridgeland contractions as they are all realized by wall-crossing. For the following we refer specifically to [BM14b, §5]. A Bridgeland stability condition σ_0 on a projective twisted $K3$ surface (S, α) comes with a central charge $Z_0 : \widetilde{H}(S, \alpha, \mathbb{Z}) \to \mathbb{C}$, and we denote by $\mathcal{H}_{\sigma_0}(v) \subset \widetilde{H}(S, \alpha, \mathbb{Z})_{\mathrm{alg}}$ the primitive sublattice of algebraic vectors a such that $\mathrm{Im}\, \frac{Z(a)}{Z(v)} = 0$, *i.e.*, those vectors for which $Z(a)$ and $Z(v)$ are $\mathbb{R}$-linearly dependent. A nearby generic stability condition σ yields a contraction $\pi : M_\sigma(v) \to M$ which identifies σ-stable sheaves which are S-equivalent with respect to σ_0. The lattice $\mathcal{H}_{\sigma_0}(v)$ is of signature $(1, \rho)$, and

$$N_{\sigma_0}(v) := \mathcal{H}_{\sigma_0}(v) \cap H^2(M_\sigma(v), \mathbb{Z}) = \mathcal{H}_{\sigma_0}(v) \cap v^\perp$$

is negative definite. Denoting by $R_{\sigma_0}(v) \subset H_2(M, \mathbb{Z})$ the primitive sublattice corresponding to $N_{\sigma_0}(v)$ under the isomorphism $H^2(M_\sigma(v), \mathbb{Q}) \cong H_2(M_\sigma(v), \mathbb{Q})$ given by the Mukai pairing, we see that $R_{\sigma_0}(v)$ is identified with $N_1(M_\sigma(v)/M)$, and $N_{\sigma_0}(v)_{\mathbb{Q}}$ with the orthogonal to $\pi^* H^2(M, \mathbb{Q})$ in $H^2(M_\sigma(v), \mathbb{Q})$ (which we have called simply $N_{\mathbb{Q}}$ above).

For an arbitrary $K3^{[n]}$-type contraction $\pi : Y \to X$, we likewise denote by N and $\mathcal{H}$ the orthogonals to $\pi^* H^2(X, \mathbb{Z})$ in $H^2(Y, \mathbb{Z})$ and $\widetilde{\Lambda}(Y, \mathbb{Z})$, respectively. Relative Picard rank one contractions are particularly easy to analyze:

Lemma 6.7. *Let $\pi : Y \to X$ be a relative Picard rank one $K3^{[n]}$-type contraction. Then the locally trivial deformation type of X is uniquely determined by each of the following:*

(1) *the abstract isomorphism class of $(\mathcal{H}, v)$ as a pointed lattice;*

(2) $q_Y(\lambda)$ *and* $\mathrm{div}(\lambda) := [\mathbb{Z} : q_Y(\lambda, H^2(Y, \mathbb{Z}))]$, *where $\lambda \in N$ is a primitive generator.*

Proof. Suppose $\pi' : Y' \to X'$ is a second $K3^{[n]}$-type contraction with associated lattices N' and $\mathcal{H}'$. By Corollary 5.17, X and X' are locally trivially deformation equivalent if and only if there is a parallel transport operator $f : H^2(Y, \mathbb{Z}) \to H^2(Y', \mathbb{Z})$ sending N to N', which in this case is equivalent to there being an isometry $\widetilde{f} : \widetilde{\Lambda}(Y, \mathbb{Z}) \to \widetilde{\Lambda}(Y', \mathbb{Z})$ sending $(\mathcal{H}, v)$ to $(\mathcal{H}', v')$. Such a $\widetilde{f}$ exists if and only if $(\mathcal{H}, v) \cong (\mathcal{H}', v')$ by [Ni79, Corollary 1.5.2]. This proves (1).

For (2), $\mathrm{Mon}^2(K3^{[n]})$ orbits of primitive vectors λ are uniquely determined by λ^2 and $\mathrm{div}(\lambda)$, see [Ei74, Section 10]. $\qquad\qquad\square$

Proposition 6.8. *Let X be a relative Picard rank one $K3^{[n]}$-type contraction and $x \in X$ a (closed) point. The analytic germ (X, x) is isomorphic to that of a Nakajima quiver variety.*

Proof. By [AS18, Theorem 1.1], the statement is known for Gieseker moduli spaces $X = M_{H_0}(v_0)$ where v_0 is a primitive Mukai vector of a pure 1-dimensional sheaf on a $K3$ surface S with $v_0^2 \geq 2$ and H_0 is a nongeneric polarization. Let H be a generic polarization such that $M_H(v_0) \to M_{H_0}(v_0)$ is a symplectic resolution, and set $\mathcal{H}_0 = \mathcal{H}(M_H(v_0)/M_{H_0}(v_0))$.

By the previous lemma, we just need to show any pointed lattice $(\mathcal{H}, v)$ arising from a relative Picard rank one contraction $\pi : Y \to X$ is abstractly isomorphic to some $(\mathcal{H}_0, v_0)$. By Proposition 6.6 we may assume $\pi : Y \to X$ is a Bridgeland contraction, and then by [BM14b, Theorem 12.1] we must have that $\mathcal{H}$ contains a class $a \in \mathcal{H}$ with $0 \leq (v, a) \leq (v, v)/2$ and $(a, a) \geq -2$. Let S be a $K3$ surface such that
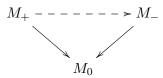
(1) $\mathrm{Pic}(S) \cong \mathcal{H}$. Let $D \in \mathrm{Pic}(S)$ correspond to v and A to a.

(2) $D - \epsilon A$ is ample.

Such a surface S exists since $\mathcal{H}$ embeds primitively into Λ_{K3} [Ni79, Theorem 1.1.2]. Then D, A and $D - A$ are effective by the conditions on a, so $|D|$ contains reducible curves. Choose an ample H_0 and $\delta, \alpha \in \mathbb{Z}$ nonzero such that

$$\frac{\delta}{H_0.D} = \frac{\alpha}{H_0.A}.$$

Then there are strictly H_0-semistable sheaves of Mukai vector $v_0 = (0, D, \delta)$. Set $M_\pm = M_{H^\pm}(v_0)$ for $H^\pm = H_0 \pm \epsilon D$, and $M_0 = M_{H_0}(v_0)$. We conclude by noticing

that the lattice $\mathcal{H}_0$ in $\widetilde{H}(S, \mathbb{Z}) = \widetilde{\Lambda}(M_+, \mathbb{Z})$ associated to the wall crossing

$$M_+ \; - - - - - - - \rightarrow \; M_-$$
$$M_0$$

is the saturation of $\langle (0, D, \delta), (0, A, \alpha) \rangle$, which is isomorphic to $\mathcal{H}$ and the isomorphism takes v_0 to v. $\qquad\square$

The proof of Proposition 6.8 gives explicit models for every relative Picard rank one $K3^{[n]}$-type contraction among compactified Jacobians of linear systems on $K3$ surfaces. Knutsen, Lelli-Chiesa, and Mongardi [KLCM19] have also used compactified Jacobians to construct contractible ruled subvarieties of $K3^{[n]}$-type manifolds, and analyze the geometry more closely. Models for such contractions have further been treated by Hassett–Tschinkel [HT16], where it is shown that every wall $\mathcal{H}$ can be realized on the Hilbert scheme of points for a projective $K3$ surface of Picard rank one.

The Bayer–Macrì picture strongly suggests that the answer to the following question is affirmative:

Question 6.9. *Let $\pi : Y \to X$ be a relative Picard rank one $K3^{[n]}$-type contraction, and let $E \subset Y$ be an irreducible component of the exceptional locus. Is the generic fiber of the map $E \to X$ isomorphic to $\mathbb{P}^{\mathrm{codim}\, E}$?*

Indeed, for a Bridgeland moduli space $Y = M_{\sigma_+}(v)$ and a contraction induced by a wall-crossing, the Harder–Narasimhan filtration of a generic point $[F] \in E$ with respect to a generic nearby stability condition σ_- on the other side of the wall is often of the form

(6.2)
$$0 \to A \to F \to B \to 0$$

for A, B σ_0-stable. All such extensions are σ_+-stable, and this yields a $\mathbb{P}^k = \mathbb{P}\operatorname{Ext}^1(B, A)$ fiber that is contracted. Moreover, setting $a = v(A)$ and $b = v(B)$,

$$\dim E = k + \dim M^{\mathrm{st}}_{\sigma_0}(a) + \dim M^{\mathrm{st}}_{\sigma_0}(b)$$
$$= ((a, b) - 1) + (a^2 + 2) + (b^2 + 2)$$
$$= (v^2 + 2) - k$$

so $k = \operatorname{codim} E$. Thus, in this case, we are done if the Harder–Narasimhan filtration of the general point of E has the form (6.2) for fixed[3] a and b. By Lemma 6.7 and [LP19], it would be sufficient to consider one model in each monodromy orbit, and many special cases have been established previously, see [HT16, KLCM19].

It is also not difficult to prove the following special case, since for an ADE singularities the singularity determines the exceptional divisor of the (unique) symplectic resolution:

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[3]Note the constancy of the Mukai vectors is automatic if a universal family exists over E, by the existence of Harder–Narasimhan filtrations in families.

Proposition 6.10. *Let $\pi : Y \longrightarrow X$ be a relative Picard rank one $K3^{[n]}$-type contraction, and let $E \subset Y$ be an irreducible divisorial component of the exceptional locus. Then the exceptional locus is irreducible and the generic fiber of the map $E \longrightarrow X$ is $\mathbb{P}^1$.*

Proof. In the notation of the proof of Proposition 6.8, we may assume the contraction is of the form $\pi : M_H(v_0) \longrightarrow M_{H_0}(v_0)$ where $v_0 = (0, D, \delta)$. Now a sheaf F is strictly H_0-semistable only if the support is reducible. $M_H(v_0)$ is stratified by the decomposition[4] of the support, with the generic stratum corresponding to the trivial decomposition $D = D$ (since the generic element of $|D|$ is integral, as D is big and nef), so the generic point of E has support $D = A + B$ with A and B smooth integral curves. Every point in the fiber of π through the generic point of E has the same Harder–Narasimhan factors, which are line bundles L_A, L_B supported on A, B respectively. There is a $\mathbb{P}^1$ fo such sheaves—namely, the extensions $\mathbb{P}\operatorname{Ext}^1(L_A, L_B)$ (or in the reverse direction), which must be a $\mathbb{P}^1$ as its necessarily a curve. $\square$

In [Ba15] the first author classified contractions at the other extreme, namely those for which the exceptional locus contains a Lagrangian $\mathbb{P}^n$.

As an application, assume that Y is a symplectic $2n$-fold which admits a divisorial contraction $\pi : Y \longrightarrow X$ of relative Picard rank one such that X has transversal A_2 singularities. We call this an A_2-contraction. Note that while ADE singularities admit unique symplectic resolutions, in the relative Picard rank one setting the monodromy action on the set of components of the general fiber of the exceptional locus yields a group of automorphisms of the ADE graph in question acting transitively on the nodes, so only A_1 and A_2 singularities remain as possibilities. We would like to know whether an A_2 contraction exists if Y is an irreducible symplectic manifold.

Corollary 6.11. *Let Y be a $K3^{[n]}$-type manifold. Then Y does not admit any A_2-contraction of relative Picard rank one.*

Note that there are however examples of A_2-contractions of relative Picard rank one of smooth and projective symplectic varieties. See e.g. [Wi03, §1.4, Example 2] for an explicit construction.

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