TIME-DEPENDENT PHOTON STATISTICS IN VARIABLE MEDIA

SERGEY I. KRYUCHKOV, ERWIN SUAZO, AND SERGEI K. SUSLOV

Abstract. We find explicit solutions of the Heisenberg equations of motion for a quadratic Hamiltonian, describing a generic model of variable media, in the case of multi-parameter squeezed input photon configuration. The corresponding probability amplitudes and photon statistics are also derived in the Schrödinger picture in an abstract operator setting of the quantum electrodynamics (QED). Their time evolution is given in terms of solutions of certain Ermakov-type system. The unitary transformation and an extension of the squeeze/evolution operator are introduced formally.

1. An Introduction

Although for most applications in optics, the electromagnetic field can be treated classically [38], [39], [86], a quantum description is required when quantum limits are approached and one is interested in the photon statistics of the field (see [29], [40], [44], [45] and the references therein). A fundamental effect of squeezing, or simultaneous (counter-)oscillations of variances of the electric and magnetic field operators, is one of the characteristic features of quantized light propagating in a variable medium [10]. Observations of squeezed states of light, generated by advanced methods of nonlinear optics, are discussed in [7], [14], [62], [65] (see also the references therein). Various aspects of the corresponding photon statistics and photon-counting were studied in detail [4], [5], [15], [16], [28], [37], [48], [55], [69], [71], [72], [73], [74], [75], [87], [88], [89], [92], [93], [96].

In spite of the considerable literature on squeezing and photon statistics, the most general case of time evolution of multi-parameter squeezed input photons propagating in a variable inhomogeneous medium, to the best of our knowledge, has never been thoroughly discussed with an exception of [4], [28], [55], [87], [89] and our recent article [3] on the degenerate parametric amplifiers [81], [82]. Moreover, traditionally, the interaction picture [77], [86], [94] is commonly used for these exactly solvable models in quantum nonlinear optics even though the statistics is postulated in the Schrödinger picture [50], [52]. This is why, in this article, we continue to study the statistical properties of output squeezed quanta in terms of explicit solutions of certain Ermakov-type system [61] (the initial data/constants of motion identify the squeezed states under consideration). In particular, the time-dependent probability amplitudes for a generic model of variable media are derived. Once again, we elaborate on time development of the corresponding photon states in Fock's space in the Schrödinger picture. The corresponding evolution operator and the transformation to Heisenberg's picture are also discussed. In order to achieve these goals, we utilize a unified approach to generalized harmonic oscillators discussed in detail in several recent publications [17], [19], [20], [56], [57], [61], [63], [83] (see also [25], [27], [58], [67] for the classical accounts).

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Dynamical features of the squeezed n-photon configurations, which are not usually discussed in cavity QED in detail, is our primarily motivation for writing of this comment. The article is organized as follows. To begin with, we briefly review the method of dynamical invariants for quantization of radiation field in order to make our presentation as self-contained as possible. Solutions of Heisenberg's equation of motion are found with the help of an evolution operator and then independently verified by a direct substitution. Finally, probability amplitudes with respect to the number states are re-derived in an operator form (for a generic model of variable medium). As a by-product, their time evolution is established, by making use of a general solution of the Ermakov-type system. Potential applications of these results to some recent experiments are mentioned in the end.

2. Quantization of Radiation Field in Variable Media

In this article, we consider the quantization of radiation field in a variable dielectric medium in the Schrödinger picture, as outlined in [8], [43] (in vacuum), by using the method of dynamical invariants originally developed in [25], [27], [67] and revisited in [18], [83], [90]. In summary, we separate spatial and time variables in the phenomenological Maxwell equations with factorized electric permittivity and magnetic permeability (tensors) [24] (see also Appendix in [57]). Under proper boundary conditions, solution of the spatial equations provides a complete set of eigenfunctions and eigenvalues for the corresponding field expansions. In the time domain, we follow the mathematical technique of the field quantization for a variable quadratic system in an abstract (Fock-)Hilbert space recently discussed in [57], primarily concentrating on a single mode of the radiation field. In this picture, the electric and magnetic fields are represented by certain time-independent operators and the time evolution is governed by the Schrödinger equation for the field state vector $|\psi(t)\rangle$:

$$i\frac{d}{dt}\left|\psi\left(t\right)\right\rangle = \widehat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle \tag{2.1}$$

with a variable quadratic Hamiltonian of the form

$$\widehat{H}\left(t\right) = a(t)\,\widehat{p}^{2} + b(t)\,\widehat{q}^{2} + c(t)\,\widehat{q}\,\widehat{p} - id(t) - f(t)\,\widehat{q} - g(t)\,\widehat{p}\,, \tag{2.2}$$

where a, b, c, d, f, and g are suitable real-valued functions of time only and the time-independent operators $\widehat{q}$ and $\widehat{p}$ obey the canonical commutation rule $[\widehat{q}\,,\widehat{p}] = i$ (in the units of $\hbar$).

The time-dependent annihilation $\widehat{b}(t)$ and creation $\widehat{b}^{\dagger}(t)$ operators (linear integrals of motion [25], [67], [83]) are described by Theorem 1 of [57],

$$\widehat{b}(t) = \frac{e^{-2i\gamma(t)}}{\sqrt{2}}\left(\beta\left(t\right)\widehat{q} + \varepsilon\left(t\right) + i\frac{\widehat{p} - 2\alpha\left(t\right)\widehat{q} - \delta\left(t\right)}{\beta\left(t\right)}\right), \tag{2.3}$$

$$\widehat{b}^{\dagger}(t) = \frac{e^{2i\gamma(t)}}{\sqrt{2}}\left(\beta\left(t\right)\widehat{q} + \varepsilon\left(t\right) - i\frac{\widehat{p} - 2\alpha\left(t\right)\widehat{q} - \delta\left(t\right)}{\beta\left(t\right)}\right),$$

in terms of solutions of the Ermakov-type system

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^{2} = a\beta^{4}, \tag{2.4}$$

$$\frac{d\beta}{dt} + \left(c + 4a\alpha\right)\beta = 0, \tag{2.5}$$

$$\frac{d\gamma}{dt} + a\beta^{2} = 0 \tag{2.6}$$

and

$$\frac{d\delta}{dt}+\left(c+4a\alpha\right)\delta-f-2g\alpha=2a\beta^{3}\varepsilon, \tag{2.7}$$

$$\frac{d\varepsilon}{dt}-\left(g-2a\delta\right)\beta=0, \tag{2.8}$$

$$\frac{d\kappa}{dt}-g\delta+a\delta^{2}=a\beta^{2}\varepsilon^{2} \tag{2.9}$$

subject to a given initial data. These dynamical invariants obey the canonical commutation rule $\widehat{b}(t)\widehat{b}^{\dagger}(t)-\widehat{b}^{\dagger}(t)\widehat{b}(t)=1$ at all times. The corresponding evolution (Heisenberg-type) equations take the form [57]:

$$i\frac{d}{dt}\widehat{b}\left(t\right)+\left[\widehat{b}\left(t\right),\ \widehat{H}\left(t\right)\right]=0,\qquad i\frac{d}{dt}\widehat{b}^{\dagger}\left(t\right)+\left[\widehat{b}^{\dagger},\ \widehat{H}\left(t\right)\right]=0. \tag{2.10}$$

Different forms of solutions of the Ermakov-type system have been found in [56], [57], [61], and [63] in general. Explicit special cases are discussed in [2], [3], [56], [64], [70].

The corresponding dynamical Fock states,

$$\left|\psi_{n}\left(t\right)\right\rangle=\frac{1}{\sqrt{n!}}\left(\widehat{b}^{\dagger}\left(t\right)\right)^{n}\left|\psi_{0}\left(t\right)\right\rangle,\qquad \widehat{b}\left(t\right)\left|\psi_{0}\left(t\right)\right\rangle=0, \tag{2.11}$$

where the phase $\kappa\left(t\right)$ finally shows up, can be obtained in a standard fashion [8], [33], [34] (see also [54], in particular, dialogues 8 and 9 and section 3.4, and (3.13)). Under a certain condition, they do satisfy the time-dependent Schrödinger equation (2.1) (see [32], Lemma 2 of Ref. [57] and (3.13) for more details). The state $\left|\psi_{n}\left(t\right)\right\rangle$ can be associated with the multi-parameter squeezed n-photon radiation field, whereas the initial data of Ermakov-type system provide the corresponding classical constants/integrals of motion, and we elaborate on it in the next section (see also [9], [41], [42], [50], [52]).

3. The Canonical Transformation and Evolution Operator

In the case of the standard time-independent annihilation and creation operators for a given harmonic mode ω,

$$\widehat{a}=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}+i\,\widehat{p}\right),\qquad \widehat{a}^{\dagger}=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}-i\,\widehat{p}\right),\qquad \left[\widehat{a},\ \widehat{a}^{\dagger}\right]=1, \tag{3.1}$$

one can formally get with the help of the Baker–Campbell–Hausdorff formula that

$$e^{i\left(\widehat{a}^{\dagger}\ \widehat{a}\right)\omega t}\,\widehat{a}\,e^{-i\left(\widehat{a}^{\dagger}\ \widehat{a}\right)\omega t}=\widehat{a}\,e^{-i\omega t}. \tag{3.2}$$

Two other familiar identities are valid:

$$e^{\xi^{*}\widehat{a}-\xi\widehat{a}^{\dagger}}\ \widehat{a}\,e^{-\xi^{*}\widehat{a}+\xi\widehat{a}^{\dagger}}=\widehat{a}+\xi, \tag{3.3}$$

$$\begin{aligned} &e^{\left(e^{2i\varphi}\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2}\,\widehat{a}\,e^{-\left(e^{2i\varphi}\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2}\\ &\qquad=\left(\cosh\tau\right)\widehat{a}+e^{-2i\varphi}\left(\sinh\tau\right)\widehat{a}^{\dagger}\end{aligned} \tag{3.4}$$

for the single-mode displacement [37] and squeeze [87], [88] operators, respectively (see, for example, [84], [85] for more details). With the aid of (3.2) and (3.3)–(3.4), we obtain the canonical transformation of the form:

$$\widehat{b}\left(t\right)=\boldsymbol{U}\left(t\right)\widehat{a}\,\boldsymbol{U}^{-1}\left(t\right)=\frac{e^{-2i\gamma}}{\sqrt{2}}\left(\beta\widehat{q}+\varepsilon+i\frac{\widehat{p}-2\alpha\widehat{q}-\delta}{\beta}\right) \tag{3.5}$$

for the following unitary operator:

$$\boldsymbol{U}(t)=e^{i\left(\widehat{a}^{\dagger}\ \widehat{a}\right)\theta}e^{\left(e^{2i\varphi}\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2}e^{\xi^{\ast}\widehat{a}-\xi\widehat{a}^{\dagger}}e^{2i\left(\widehat{a}^{\dagger}\ \widehat{a}\right)\gamma} \tag{3.6}$$

provided

$$\frac{1}{\sqrt{\omega}}\left(\beta-\frac{2i\alpha}{\beta}\right)+\frac{\sqrt{\omega}}{\beta}=2e^{-i\theta}\cosh\tau, \tag{3.7}$$

$$\frac{1}{\sqrt{\omega}}\left(\beta-\frac{2i\alpha}{\beta}\right)-\frac{\sqrt{\omega}}{\beta}=2e^{i(\theta-2\varphi)}\sinh\tau, \tag{3.8}$$

and

$$\xi\sqrt{2}=\varepsilon-i\frac{\delta}{\beta}. \tag{3.9}$$

As a result, the time-dependent parameters of our single-mode "multi-parameter squeeze/evolution operator" (3.6), namely, $\theta(t)$, $\tau(t)$, $\varphi(t)$ and $\xi(t)$, are determined in terms of solutions of the corresponding Ermakov-type system as follows

$$\tan\theta(t)=\frac{2\alpha}{\beta^{2}+\omega},\quad \tan 2\varphi(t)=\frac{4\alpha\beta^{2}}{\beta^{4}-4\alpha^{2}-\omega^{2}}, \tag{3.10}$$

$$4\left[\cosh\tau(t)\right]^{2}=\left(\frac{\beta}{\sqrt{\omega}}+\frac{\sqrt{\omega}}{\beta}\right)^{2}+\frac{4\alpha^{2}}{\omega\beta^{2}}, \tag{3.11}$$

$$4\left[\sinh\tau(t)\right]^{2}=\left(\frac{\beta}{\sqrt{\omega}}-\frac{\sqrt{\omega}}{\beta}\right)^{2}+\frac{4\alpha^{2}}{\omega\beta^{2}} \tag{3.12}$$

(see also (3.9)). To the best of our knowledge, these relations are omitted in the available literature and, therefore, may be considered as a main result of this article.

Assuming that the vacuum state is nondegenerate, one gets $\widehat{a}\left(\boldsymbol{U}^{-1}(t)\left|\psi_{0}(t)\right\rangle\right)=0$ and $\left|\psi_{0}(t)\right\rangle=\boldsymbol{U}(t)\left|0\right\rangle$. Then,

$$\left|\psi_{n}(t)\right\rangle=\frac{1}{\sqrt{n!}}\left(\widehat{b}^{\dagger}(t)\right)^{n}\left|\psi_{0}(t)\right\rangle=\boldsymbol{U}(t)\left(\frac{1}{\sqrt{n!}}\left(\widehat{a}^{\dagger}\right)^{n}\left|0\right\rangle\right)=\boldsymbol{U}(t)\left|n\right\rangle \tag{3.13}$$

in terms of standard Fock's number states $\left|n\right\rangle$. With the help of expansion, $\left|\psi\right\rangle_{0}=\sum_{n=0}^{\infty}c_{n}\ \left|n\right\rangle$, the action of operator $\boldsymbol{U}(t)$ can be extended, by completeness and linearity, to a certain class of arbitrary initial data,

$$\boldsymbol{U}(t)\left|\psi\right\rangle_{0}=\sum_{n=0}^{\infty}c_{n}\ \left|\psi_{n}(t)\right\rangle=\left|\psi(t)\right\rangle. \tag{3.14}$$

Therefore, our evolution operator $\boldsymbol{U}(t)$ satisfies, formally, the time-dependent Schrödinger equation (2.1), whereas $\boldsymbol{U}(0)\left|\psi\right\rangle_{0}=\left|\psi(0)\right\rangle$. Here, $\boldsymbol{U}(0)\neq\ id$, not the identity operator in general, but a composition involving the familiar time-independent displacement and squeeze operators.

As a result, one may conclude that the minimum-uncertainty squeezed states, which are important in most applications, occur when $\alpha(t_{\min})=0$ and $n=0$ (see, for example, Eq. (5.5) of Ref. [57]). Indeed, only at these instances, by (3.10) the following conditions hold, $\theta(t_{\min})=\varphi(t_{\min})=\alpha(t_{\min})=0$, and a traditional definition of single-mode squeeze operator from Refs. [84], [85], [87], [88] can be used, say, "stroboscopically"[89]. In general, the evolution operator in (3.6) and (3.13) should be applied for our initial "multi-parameter squeezed number states" given by $\left|\psi_{n}(0)\right\rangle=$

$\boldsymbol{U}(0)|n\rangle$ (see also [59], [63], [64], [70]). The traditional squeeze operator corresponds to the special initial data $\alpha(0)=\theta(0)=\varphi(0)=0$ and $\tau(0)=(1/2)\ln\left(\beta^2(0)/\omega\right).$

In summary, our mathematical analysis identifies(/supports a concept of) the multi-parameter squeezed n-photon (polarized) states of radiation field in the generic model of variable media [24]. Their time-evolution is described in terms of solutions of the Ermakov-type system.

4. Solving Heisenberg's Equations of Motion

In this article, we use the Schrödinger picture for investigation of the quantum statistics of the field.[1] On the contrary, in the Heisenberg picture, in view of (3.5) one gets

$$\widehat{q}(t)=\boldsymbol{U}^{-1}(t)\,\widehat{q}\,\boldsymbol{U}(t)=\frac{1}{\beta\sqrt{2}}\left(e^{2i\gamma}\,\widehat{a}+\widehat{a}^{\dagger}\,e^{-2i\gamma}\right)-\frac{\varepsilon}{\beta} \tag{4.1}$$

and

$$\begin{aligned}\widehat{p}(t)=\boldsymbol{U}^{-1}(t)\,\widehat{p}\,\boldsymbol{U}(t)=&\frac{e^{2i\gamma}}{\beta\sqrt{2}}\left(2\alpha-i\beta^2\right)\widehat{a}\\&+\frac{e^{-2i\gamma}}{\beta\sqrt{2}}\left(2\alpha+i\beta^2\right)\widehat{a}^{\dagger}+\delta-\frac{2\alpha\varepsilon}{\beta}.\end{aligned} \tag{4.2}$$

The Heisenberg equations of motion hold,

$$i\frac{d}{dt}\widehat{p}(t)=\left[\widehat{p}(t),\ \widehat{\mathcal{H}}(t)\right],\qquad i\frac{d}{dt}\widehat{q}(t)=\left[\widehat{q}(t),\ \widehat{\mathcal{H}}(t)\right], \tag{4.3}$$

where $\widehat{\mathcal{H}}(t)=\boldsymbol{U}^{-1}(t)\,\widehat{H}(t)\,\boldsymbol{U}(t)$. This can also be verified by a direct substitution with the help of a computer algebra system.[2]

Here, all information about the state of radiation field is encoded into our time-dependent operators (4.1)–(4.2) in the form of initial data/constants of motion of the Ermakov-type system (2.4)–(2.9). The corresponding field evolution is completely determined by explicit solutions of this system subject to given initial data. In particular, $\widehat{q}(0)=\widehat{q}$ and $\widehat{p}(0)=\widehat{p}$, when $\alpha(0)=\gamma(0)=\delta(0)=\varepsilon(0)=0$ and $\beta(0)=\sqrt{\omega}$. It is worth noting that our solutions correspond to the most general set of (known) quantum numbers for the state of radiation field under consideration. Moreover, Heisenberg's operators are given by

$$\begin{aligned}\widehat{a}(t)=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}(t)+i\,\widehat{p}(t)\right)=&\frac{e^{2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega+\beta^2+2i\alpha\right)\widehat{a}\\&+\frac{e^{-2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega-\beta^2+2i\alpha\right)\widehat{a}^{\dagger}-\frac{1}{\sqrt{2\omega}}\left[\frac{\omega\varepsilon}{\beta}-i\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)\right]\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\widehat{a}^{\dagger}(t)=\frac{1}{\sqrt{2\omega}}\left(\omega\widehat{q}(t)-i\,\widehat{p}(t)\right)=&\frac{e^{2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega-\beta^2-2i\alpha\right)\widehat{a}\\&+\frac{e^{-2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega+\beta^2-2i\alpha\right)\widehat{a}^{\dagger}-\frac{1}{\sqrt{2\omega}}\left[\frac{\omega\varepsilon}{\beta}+i\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)\right],\end{aligned} \tag{4.5}$$

[1] A detailed analysis of different representations of quantum mechanics and quantum optics is given in Refs. [50], [51], [52].

[2] See complementary *Mathematica* notebook: HeisenbergEquations.nb.

once again, in terms of solutions of the Ermakov-type system, which can be derived from (4.3) (see *Mathematica* notebook: HeisenbergEquations.nb). Thus

$$\left[\widehat{a}\left(t\right), \ \widehat{a}\left(t'\right)\right] = A\left(t\right)B\left(t'\right) - A\left(t'\right)B\left(t\right), \tag{4.6}$$

$$\left[\widehat{a}\left(t\right), \ \widehat{a}^{\dagger}\left(t'\right)\right] = A\left(t\right)A^{*}\left(t'\right) - B\left(t\right)B^{*}\left(t'\right), \tag{4.7}$$

$$\left[\widehat{a}^{\dagger}\left(t\right), \ \widehat{a}^{\dagger}\left(t'\right)\right] = B^{*}\left(t\right)A^{*}\left(t'\right) - A^{*}\left(t\right)B^{*}\left(t'\right). \tag{4.8}$$

Here, the asterisk represents the complex conjugate and, by definition,

$$A\left(t\right) = \frac{e^{2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega + \beta^{2} + 2i\alpha\right), \qquad B\left(t\right) = \frac{e^{-2i\gamma}}{2\beta\sqrt{\omega}}\left(\omega - \beta^{2} + 2i\alpha\right). \tag{4.9}$$

These relations reveal the fact of noncommutativity of the radiation field operators at different times.

The expectation values and variances of Heisenberg's operators $\widehat{q}\left(t\right)$ and $\widehat{p}\left(t\right)$ with respect to the standard Fock number states $|n\rangle$ coincide, of course, with those found in Ref. [57], section 5, in the Schrödinger picture.

Example. Special cases include the degenerate parametric amplifiers [2], [3], [78], [81], [82], [89], which provide a natural mechanism of creation of the multi-parameter squeezed states of light (see also [59], [70] and the references therein). For the Hamiltonian of the form

$$\widehat{H}\left(t\right) = \frac{\omega}{2}\left(\widehat{a}\,\widehat{a}^{\dagger} + \widehat{a}^{\dagger}\,\widehat{a}\right) + \frac{\lambda}{2i}\left(e^{2i\omega t}\ \ \widehat{a}^{2} - e^{-2i\omega t}\left(\widehat{a}^{\dagger}\right)^{2}\right), \tag{4.10}$$

our solutions of Heisenberg's equations (4.3) are given by

$$\begin{aligned} \widehat{a}\left(t\right) = \boldsymbol{U}^{-1}\left(t\right)\widehat{a}\,\boldsymbol{U}\left(t\right) = {} & \frac{\omega e^{2\lambda t} + \beta^{2}\left(0\right) + 2i\alpha\left(0\right)}{2\beta\left(0\right)\sqrt{\omega}}e^{-\left(\lambda + i\omega\right)t}\,\widehat{a} \\ & + \frac{\omega e^{2\lambda t} - \beta^{2}\left(0\right) + 2i\alpha\left(0\right)}{2\beta\left(0\right)\sqrt{\omega}}e^{-\left(\lambda + i\omega\right)t}\,\widehat{a}^{\dagger} \\ & + i\frac{\beta\left(0\right)\delta\left(0\right) - 2\alpha\left(0\right)\varepsilon\left(0\right) + i\omega e^{2\lambda t}\varepsilon\left(0\right)}{\beta\left(0\right)\sqrt{2\omega}}e^{-\left(\lambda + i\omega\right)t} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \widehat{a}^{\dagger}\left(t\right) = \boldsymbol{U}^{-1}\left(t\right)\widehat{a}^{\dagger}\,\boldsymbol{U}\left(t\right) = {} & \frac{\omega e^{2\lambda t} - \beta^{2}\left(0\right) - 2i\alpha\left(0\right)}{2\beta\left(0\right)\sqrt{\omega}}e^{\left(i\omega - \lambda\right)t}\,\widehat{a} \\ & + \frac{\omega e^{2\lambda t} + \beta^{2}\left(0\right) - 2i\alpha\left(0\right)}{2\beta\left(0\right)\sqrt{\omega}}e^{\left(i\omega - \lambda\right)t}\,\widehat{a}^{\dagger} \\ & + i\frac{2\alpha\left(0\right)\varepsilon\left(0\right) - \beta\left(0\right)\delta\left(0\right) + i\omega e^{2\lambda t}\varepsilon\left(0\right)}{\beta\left(0\right)\sqrt{2\omega}}e^{\left(i\omega - \lambda\right)t}. \end{aligned} \tag{4.12}$$

The particular solutions, corresponding to the initial data $\alpha\left(0\right) = \gamma\left(0\right) = \delta\left(0\right) = \varepsilon\left(0\right) = 0$ and $\beta\left(0\right) = \sqrt{\omega}$, are originally found in [78].

5. On Variable Photon Statistics

The explicit form of the unitary operator (3.6) allows us to (re-)evaluate the time-dependent photon amplitudes with respect to the Fock basis, namely,

$$\left|\psi_{n}\left(t\right)\right\rangle =\sum_{m=0}^{\infty}\left(e^{2i\gamma}\left[\left(\sum_{k=0}^{\infty}S_{mk}D_{kn}\right)\right]e^{im\theta}\right)\left|m\right\rangle , \tag{5.1}$$

in a pure algebraic fashion similar to Refs. [37], [72], [73], [92] (but for the generic variable quadratic Hamiltonian). Here, the matrix elements of the displacement operator are given by

$$\begin{aligned} D_{mn}\left(\xi\right) &= \left\langle m\left|e^{\xi^{*}\widehat{a}-\xi\widehat{a}^{\dagger}}\right|n\right\rangle \\ &= e^{-\left|\xi\right|^{2}/2}\frac{\left(-\xi\right)^{m}\left(\xi^{*}\right)^{n}}{\sqrt{m!n!}}\ {}_{2}F_{0}\left(-n,-m;-\frac{1}{\left|\xi\right|^{2}}\right), \end{aligned} \tag{5.2}$$

where parameter ξ is determined by (3.9) in terms of solutions of Ermakov-type system. (Familiar relations with the Charlier polynomials are discussed in [80].)

In a similar fashion, the matrix elements of the squeeze operator can be readily evaluated. By using an important factorization identity [84], [85],

$$\begin{aligned} e^{\left(e^{2i\varphi}\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2} &= e^{-(1/2)e^{-2i\varphi}\tanh\tau\left(\widehat{a}^{\dagger}\right)^{2}} \\ &\quad\times e^{-\ln\cosh\tau\left(\widehat{a}^{\dagger}\widehat{a}+1/2\right)}e^{(1/2)e^{2i\varphi}\tanh\tau\left(\widehat{a}\right)^{2}}, \end{aligned} \tag{5.3}$$

one can derive the following expression:

$$\begin{aligned} S_{mn}\left(\alpha,\beta\right) &= \left\langle m\left|e^{\left(e^{2i\varphi}\widehat{a}^{2}-e^{-2i\varphi}\left(\widehat{a}^{\dagger}\right)^{2}\right)\tau/2}\right|n\right\rangle \\ &= \sqrt{\frac{m!n!\pi}{2^{m+n}\cosh\tau}}\frac{\left(-e^{-2i\varphi}\sinh\tau\right)^{(m-n)/2}\left(\cosh\tau\right)^{-(m+n)/2}}{\Gamma\left(\frac{m-n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n+1}{2}+1\right)} \\ &\quad\times\ {}_{2}F_{1}\left(\begin{array}{c} (1-n)/2,\quad -n/2 \\ 1+\left(m-n\right)/2 \end{array};\ -\sinh^{2}\tau\right),\qquad m\geq n. \end{aligned} \tag{5.4}$$

With the help of familiar transformations of the terminating hypergeometric functions [6], we finally obtain the non-vanishing matrix elements as follows

$$\begin{aligned} S_{mn}\left(\alpha,\beta\right) &= \frac{\left(-1\right)^{m/2}e^{-i(m-n)\varphi}}{\sqrt{\cosh\tau}}\left[\frac{\left(1/2\right)_{m/2}\left(1/2\right)_{n/2}}{\left(m/2\right)!\left(n/2\right)!}\right]^{1/2} \\ &\quad\times\left(\tanh\tau\right)^{(m+n)/2}\ {}_{2}F_{1}\left(\begin{array}{c} -n/2,\quad -m/2 \\ 1/2 \end{array};\ -\frac{1}{\sinh^{2}\tau}\right), \end{aligned} \tag{5.5}$$

if $m,\ n$ are even and

$$S_{mn}\left(\alpha,\beta\right) = 2\frac{\left(-1\right)^{(m-1)/2}e^{-i(m-n)\varphi}}{\sinh\tau\sqrt{\cosh\tau}}\left[\frac{\left(3/2\right)_{\frac{m-1}{2}}\left(3/2\right)_{\frac{n-1}{2}}}{\left(\frac{m-1}{2}\right)!\left(\frac{n-1}{2}\right)!}\right]^{1/2} \tag{5.6}$$

$$\times (\tanh \tau)^{(m+n)/2} \ {}_2F_1 \left(\begin{array}{c} (1-n)/2, \quad (1-m)/2 \\ 3/2 \end{array} ; \quad -\frac{1}{\sinh^2 \tau} \right),$$

if $m,\ n$ are odd. In view of (3.10)–(3.12), the time evolution of these matrix elements is also found in terms of solutions of the corresponding Ermakov-type system for the generic model of a variable medium. (The latter hypergeometric functions are related to Meixner's polynomials [80].)

In some cases, the complex parametrization and quasi-invariants of the Ermakov-type system [57] may help to simplify the arguments of the hypergeometric functions. Examples are discussed in [3] and [59].

6. A Conclusion

Light propagation in a variable medium, say in the process of degenerate parametric amplification [3], [81], [82], [77], [89] in a nonlinear crystal, has became one of the standard ways of creation of the squeezed states of radiation field [14], [65]. The latter are expected to be utilized for detection of gravitational waves [1], [7], [15], [21], [46]. Nowadays, advanced experimental techniques allow one to measure photon correlation functions of input microwave signals [11], to perform quantum tomography on itinerant microwave photons [30], and to study squeezing of microwave fields [31], [68] (see also [13], [36] for experimental study of a single-mode thermal field using a microwave parametric amplifier). Connections with the experimentally observed dynamical Casimir effect [60], [95] are discussed in [22], [23], [35], [47], [79]. Moreover, nonclassical states of phonons are recently reviewed in [76]. In summary, coherent and, possibly, squeezed and entangled states of a crystal lattice can be created by using ultrashort laser pulses. In this article, we have discussed the time-evolution of photon (or phonon) statistics for the generic model of variable media which may be important in some of the aforementioned applications.

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School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: `sergeykryuchkov@yahoo.com`

School of Mathematical and Statistical Sciences, University of Texas of Rio Grande Valley, 1201 W. University Drive, Edinburg, TX 78541, U.S.A.

E-mail address: `erwin.suazo@utrgv.edu`

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: `sks@asu.edu`

URL: `http://hahn.la.asu.edu/~suslov/index.html`