VISCOSITY SOLUTIONS OF PATH-DEPENDENT INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We extend the notion of viscosity solutions for path-dependent PDEs introduced by Ekren et al. [Ann. Probab. 42 (2014), no. 1, 204-236] to path-dependent integro-differential equations and establish well-posedness, i.e., existence, uniqueness, and stability, for a class of semilinear path-dependent integro-differential equations with uniformly continuous data. Closely related are non-Markovian backward SDEs with jumps, which provide a probabilistic representation for solutions of our equations. The results are potentially useful for applications using non-Markovian jump-diffusion models.

1. Introduction

The goal of this paper is to extend the theory of viscosity solutions (in the sense of [17] and [18]) for path-dependent partial differential equations (PPDEs) to path-dependent integro-differential equations. In particular, we investigate semilinear path-dependent integro-differential equations of the form

(1.1)
$$\mathcal{L}u(t,\omega) - f_t(\omega, u(t,\omega), \partial_\omega u(t,\omega), \mathcal{I}u(t,\omega)) = 0,$$
$$(t,\omega) \in [0,T) \times \mathbb{D}([0,T], \mathbb{R}^d),$$

where $\mathbb{D}([0,T],\mathbb{R}^d)$ is the space of right-continuous functions with left limits from [0,T] to \mathbb{R}^d , \mathcal{L} is a linear integro-differential operator of the form

$$\begin{split} \mathcal{L}u(t,\omega) &= -\partial_t u(t,\omega) - \sum_{i=1}^d b_t^i(\omega) \partial_{\omega^i} u(t,\omega) - \frac{1}{2} \sum_{i,j=1}^d c_t^{ij}(\omega) \partial_{\omega^i \omega^j}^2 u(t,\omega) \\ &- \int_{\mathbb{R}^d} \left[u(t,\omega + z.\mathbf{1}_{[t,T]}) - u(t,\omega) - \sum_{i=1}^d z^i \, \partial_{\omega^i} u(t,\omega) \right] \, K_t(\omega,dz), \end{split}$$

Date: 12/29/2014.

2000 Mathematics Subject Classification. 45K05,35D40, 60H10, 60H30.

Key words and phrases. Path-dependent integro-differential equations, viscosity solutions, backward SDEs with jumps, Skorohod topologies, martingale problems.

The author would like to thank Remigijus Mikulevicius and Jianfeng Zhang for very valuable discussions.

and \mathcal{I} is an integral operator of the form

$$\mathcal{I}u(t,\omega) = \int_{\mathbb{R}^d} \left[u(t,\omega + z.\mathbf{1}_{[t,T]}) - u(t,\omega) \right] \, \eta_t(\omega,z) \, K_t(\omega,dz).$$

Well-posedness for semilinear PPDEs has been first established by Ekren, Keller, Touzi, and Zhang [17], where also the here used notion of viscosity solutions has been introduced. Subsequent work by Ekren, Touzi and Zhang deals with fully nonlinear PPDEs ([18] and [19]), by Pham and Zhang with path-dependent Isaacs equations ([35]), by Ekren with obstacle PPDEs ([16]), and by Ren with fully nonlinear elliptic PPDEs ([36]).

Initial motivation for this line of research came from Peng [34], who considered non-Markovian backward stochastic differential equations (BSDEs) as PPDEs analogously to the relationship between Markovian BSDEs and (standard) PDEs, from Dupire [15], who introduced new derivatives on $\mathbb{D}([0,T],\mathbb{R}^d)$ so that for smooth functionals on $[0,T)\times\mathbb{D}([0,T],\mathbb{R}^d)$ a functional counterpart to Itôs formula holds, and from Cont and Fournié (see [8], [9], and [10]), who extended Dupire's seminal work. However, fully nonlinear PPDEs of first order have been studied earlier by Lukoyanov (see, for example, [27], [29], [28]). He used derivatives introduced by Kim [26] and adapted first the notion of so-called minimax solution and then of viscosity solutions from PDEs to PPDEs. Minimax solutions for PDEs have been introduced by Subbotin (see, e.g., [39] and [40]) motivated by the study of differential games. In the case of PDEs of first order, minimax and viscosity solutions are equivalent (see [41]). Another approach for generalized solution for first-order PPDEs can be found in work by Aubin and Haddad [1], where so-called Clio derivatives for path-dependent functionals are introduced in order to study certain path-dependent Hamiltion-Jacobi-Bellman equations that occur in portfolio theory.

Possible applications of path-dependent integro-differential equations are non-Markovian problems in control, differential games, and financial mathematics that involve jump processes.

Some comments about differences between PDEs and PPDEs seem to be in order. Contrary to standard PDEs, even linear PPDEs have rarely classical solutions in most relevant situations. Hence, one needs to consider a weaker forms of solutions. In the case of PDEs, the notion of viscosity solutions introduced by Crandall and Lions [11] turned out to be extremely successful. The main difficulty in the path-dependent case compared to the standard PDE case is the lack of local compactness of the state space, e.g., $[0,T] \times \mathbb{D}$ vs. $[0,T] \times \mathbb{R}^d$. Local compactness is essential for proofs of uniqueness of viscosity solutions to PDEs, i.e., PDE standard methods can, in general, not easily adapted to the path-dependent case. The main contribution of [17] was to replace the pointwise supremum/ infimum occuring in the definition of viscosity solutions to PDEs via test functions by an optimal stopping problem. The lack of local compactness could be circumvented by the existence of an optimal stopping time. This is crucial in establishing

the comparison principle. In this paper, additional intricacies caused by the jumps have to be faced. For example, it turns out that in contrast to the PPDE case the uniform topology is not always appropriate. In order to prove the comparison principle, it seems necessary to equip $\mathbb D$ with one of Skorohod's nonuniform topologies.

The general methodology to establish well-posedness of viscosity solutions for (1.1) follows [17] and [18]. Existence will be proven by using a stochastic representation. An intermediate result is a so-called partial comparison principle, which is a comparison principle, where one the involved solutions is a viscosity subsolution (resp. a viscosity supersolution), and the other one a classical super- (resp. a classical subsolution). The partial comparison principle is essential to prove the comparison principle.

The rest of the paper is organized as follows. In Section 2, we introduce most of the notation and preliminaries. In Section 3, viscosity solutions for semilinear path-dependent integro-differential equation are defined and the main results are stated. In Section 4, we prove consistency of classical solutions with viscosity solutions as well as existence of viscosity solutions. In Section 5, the partial comparison principle and a stability result is proved. This section also contains some auxiliary results about backward SDEs and optimal stopping. In Section 6, the comparison principle is proved. Appendix A deals with conditional probability distributions and their applications to martingale problems. In Appendix B, Skorohod's topologies are defined. Appendix C contains additional auxiliary results.

2. Setup

2.1. **Notation and preliminaries.** For unexplained notation, we refer to [25] and [37].

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all strictly positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, \mathbb{Q} be the set of rational numbers, and \mathbb{R} be the set of real numbers. Given $d' \in \mathbb{N}$, we denote by $\mathbb{S}^{d'}$ the set of all symmetric real-valued $d' \times d'$ -matrices. For any matrix A, we denote by A^{\top} its transpose. Given a topological space E, let $\mathcal{B}(E)$ its Borel σ -field. We write $\mathbf{0}$ for zero vectors, zero matrices, constant functions attaining only the value 0, etc. The meaning should be clear from context. We write $\mathbf{1}$ for indicator functions. The expected value with respect to some probability measure \mathbb{P} is denoted by $\mathbb{E}^{\mathbb{P}}$. On $\mathbb{R}^{d'}$, $d' \in \mathbb{N}$, denote the ℓ^p -norms by $|\cdot|_p$, $p \in \mathbb{N} \cup \{\infty\}$. Also set $|\cdot| := |\cdot|_2$.

Fix T>0 and $d\in\mathbb{N}$. Let $\Omega:=\mathbb{D}([0,T],\mathbb{R}^d)$ be the canonical space, X the canonical process on Ω , i.e., $X_t(\omega)=\omega_t$, and $\mathbb{F}^0=\{\mathcal{F}^0_t\}_{t\in[0,T]}$ the (raw) filtration generated by X. Denote the right-limit of \mathbb{F}^0 by $\mathbb{F}^0_+=\{\mathcal{F}^0_{t+}\}_{t\in[0,T]}$. Given $t\in[0,T]$, let $\Lambda^t:=[t,T)\times\Omega$ and $\bar{\Lambda}^t:=[t,T]\times\Omega$. Also, put $\Lambda:=\Lambda^0$ and $\bar{\Lambda}:=\bar{\Lambda}^0$. Given random times $\tau_1,\,\tau_2:\Omega\to[0,\infty]$, put

$$[\![\tau_1,\tau_2]\!]:=\{(t,\omega)\in\bar\Lambda:\tau_1(\omega)\leq t\leq \tau_2(\omega)\}.$$

The other stochastic intervals $[\![\tau_1, \tau_2[\![]]]$, etc., are defined similarly. We equip Ω with the uniform norm $\|\cdot\|_{\infty}$ and $\bar{\Lambda}$ with the pseudometric \mathbf{d}_{∞} defined by

$$\mathbf{d}_{\infty}((t,\omega),(t',\omega')) := |t-t'| + ||\omega_{\cdot \wedge t} - \omega'_{\cdot \wedge t'}||_{\infty}.$$

Often, we consider a functional $u: \bar{\Lambda} \to \mathbb{R}$ as a stochastic process, in which case, we write u_t instead of u(t, X).

Definition 2.1. Let E_1 and E_2 be nonempty sets. Let A be a nonempty subset of $\bar{\Lambda} \times E_1$. Consider a mapping $u = u(t, \omega, x) : A \to E_2$. We call u non-anticipating if, for every $(t, \omega, x) \in A$,

$$u(t,\omega,x) = u(t,\omega_{\cdot \wedge t},x).$$

We call u deterministic if it does not depend on ω .

Given a nonempty subset A of $\bar{\Lambda}$ and a topological space E, we denote by C(A, E) the set of all functionals from A to E that are continuous under \mathbf{d}_{∞} . If $E = \mathbb{R}$, we just write C(A) instead.

Remark 2.2. Note that any $u \in C(\bar{\Lambda})$ satisfies the following:

- (i) u is non-anticipating. This follows immediately from the definition of \mathbf{d}_{∞} .
- (ii) The trajectories $t \mapsto u(t,\omega)$ are càdlàg and the trajectories $t \mapsto u(t,\omega_{\cdot\wedge t-})$ are left-continuous (Proposition 1 of [9]). Also, for fixed $t \in (0,T]$, the path $\tilde{\omega} := \omega_{\cdot\wedge t-}$ is continuous at t, which again, by Proposition 1 of [9], implies that

$$u_{t-}(\omega) = \lim_{s \uparrow t} u(s, \tilde{\omega}) = u(t, \tilde{\omega}) = u(t, \omega_{\cdot \land t-}).$$

That is, $u_{-} = (u(t, \omega_{\cdot \wedge t_{-}}))_{t}$. Moreover, considered as processes, u and u_{-} are \mathbb{F}^{0} -adapted (Theorem 2 of [9]).

(iii) X jumps whenever u jumps (Lemma C.1).

Often, we write $H \cdot S$ for stochastic integrals with respect to semimartingales, i.e., $H \cdot S_t = \int_s^t H_r \, dS_r$. The initial time s is usually clear from context. We also write sometimes $H \cdot t$ instead of $\int H_t \, dt$. Similarly, we write $W * \mu$ for stochastic integrals with respect to random measures (see Chapter II of [25]). Given a probability measure \mathbb{P} , denote by $\mathbb{F}^{\mathbb{P}}$ its induced filtration satisfying the usual conditions. If S is an $(\mathbb{F}^{\mathbb{P}}, \mathbb{P})$ -semimartingale, write $L^2_{\text{loc}}(S, \mathbb{P})$ for the set of all $\mathbb{F}^{\mathbb{P}}$ -predictable processes H such that $H^2 \cdot \langle X, X \rangle$ is locally integrable (cf. I.439 in [25]). Similarly, given a random measure μ , the set $G_{\text{loc}}(\mu, \mathbb{P})$ is defined (see Definition II.1.27 in [25]). Given a process S with left limits, define ΔS by $\Delta S_t := S_t - S_{t-}$. If S is a semimartingale under a probability measure \mathbb{P} , then we denote by $S^c = S^{c,\mathbb{P}}$ the continuous local martingale part of S (p. 45 in [25]) and by μ^S the random measure associated to the jumps of S (p. 69 in [25]). It is defined by

$$\mu^{S}(\cdot;dt,dx) := \sum_{s} \mathbf{1}_{\{\Delta S_s \neq 0\}} \, \delta_{(s,\Delta S_s)}(dt,dx),$$

where δ denotes the Dirac measure.

Given a nonempty domain D in $\mathbb{R}^{d'}$, $d' \in \mathbb{N}$, denote by $\|\cdot\|_{n+\alpha,D}$, $n \in \mathbb{N}_0$, $\alpha \in (0,1)$, the standard Hölder norms. Similarly, denote by $\|\cdot\|_{n+\alpha,Q}$, $Q = (t_1,t_2) \times D$, $t_1 < t_2$, the corresponding parabolic Hölder norms. We refer to [31] for the definition. The corresponding Hölder spaces are denoted by $C^{n+\alpha}(\bar{D})$ and $C^{n+\alpha}(\bar{Q})$, resp., and the corresponding local Hölder spaces by $C^{n+\alpha}(D)$ and $C^{n+\alpha}(Q)$, resp. Also, put $\|\cdot\|_D := \|\cdot\|_{0,D}$ and $\|\cdot\|_Q := \|\cdot\|_{0,Q}$.

2.2. **Standing assumptions.** The assumptions in this section are always in force unless explicitly stated otherwise.

Let $b=(b^i)_{i\leq d}$ be a d-dimensional, non-anticipating, and \mathbb{F}^0_+ -predictable process, $c=(c^{ij})_{i,j\leq d}$ a non-anticipating and \mathbb{F}^0_+ -predictable process with values in the set of nonnegative definite real $d\times d$ -matrices, and $K=K_t(dz)$ a non-anticipating and \mathbb{F}^0_+ -predictable process with values in the set of σ -finite measures on $\mathcal{B}(\mathbb{R}^d)$.

Assumption 2.3. Let (b, c, K) satisfy $c^{ij} = \sum_{k \leq d} \sigma^{ik} \sigma^{jk}$, $i, j \leq d$, and $K_t(A) = \int \mathbf{1}_{A\setminus \{\mathbf{0}\}}(\delta_t(z)) F(dz)$, $A \in \mathcal{B}(\mathbb{R}^d)$, where $\sigma = (\sigma^{i,j})_{i,j\leq d}$ is a non-anticipating and \mathbb{F}^0_+ -predictable process with values in the set of real $d \times d$ -matrices, $\delta = (\delta^i)_{i\leq d}$ is a d-dimensional, non-anticipating, and \mathbb{F}^0_+ -predictable random field on \mathbb{R}^d , and F is a nonnegative σ -finite measure on $\mathcal{B}(\mathbb{R}^d)$. Let b, σ , and δ be right-continuous in t. Let t and t be bounded by a common constant t and t be bounded by a common constant t be bounded by t by t be bounded by t by t be bounded by t by t be bounded by t by t be bounded by t by t

Let $\eta = \eta_t(\omega, z) : \bar{\Lambda} \times \mathbb{R}^d \to \mathbb{R}$ and $f = f_t(\omega, y, z, p) : \bar{\Lambda} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be functions that are non-anticipating in (t, ω) .

Assumption 2.4. Let ξ and f be bounded from above by C'_0 . Let ξ be uniformly continuous under $\|\cdot\|_{\infty}$ with modulus of continuity ρ_0 . Let η and f be uniformly continuous in (t,ω) under \mathbf{d}_{∞} uniformly in (y,z,p) with modulus of continuity ρ_0 . Also let, uniformly in (t,ω) ,

$$\left| f_t(\omega, y, z, p) - f_t(\omega, y', z', p') \right| \le L_0 \left[\left| y - y' \right| + \left| \sigma(t, \omega)^\top (z - z') \right|_1 + \left| p - p' \right| \right].$$

Let $0 \le \eta(t, \omega, z) \le C_0'(1 \land |z|).$

Remark 2.5. Note that our Lipschitz condition of f in z is the same as in [6].

To be able to use the comparison principle for backward SDEs with jumps, we also need the following assumption.

Assumption 2.6. Let f be nondecreasing in p.

2.3. Canonical setup. We introduce probability measures $\mathbb{P}_{s,\omega}$, which will be employed in the rest of this paper. To this end, let (B,C,ν) be a candidate for a characteristic triplet of X (see §III.2.a in [25]) such that

$$dB_t = b_t dt$$
, $dC_t = c_t dt$, $\nu(dt, dz) = K_t(z) dt$.

For every $s \in [0, T]$, define (cf. §III.2.d in [25])

$$p_sB: [s,T] \times \Omega \to \mathbb{R}^d, (p_sB)_t := B_t - B_s,$$

$$p_sC: [s,T] \times \Omega \to \mathbb{S}^d, (p_sC)_t := C_t - C_s,$$

$$p_s\mu: \mathcal{B}([s,T] \times \mathbb{R}^d) \to \mathbb{R}, (p_s\mu)((s,t] \times A) := \mu(((s,t] \times A).$$

Then, by Assumption 2.3, for every $(s,\omega) \in [0,T] \times \Omega$, the martingale problem for $(p_sB, p_sC, p_s\nu)$ starting at (s,ω) has a unique solution $\mathbb{P}_{s,\omega}$ (cf. Theorem III.2.26 in [25] for the Markovian case). That is, X is an $(\mathbb{F}^0_+, \mathbb{P}_{s,\omega})$ -semimartingale on [s,T] with characteristics $(p_sB, p_sC, p_s\nu)$ and $X.\mathbf{1}_{[0,s]} = \omega.\mathbf{1}_{[0,s]}, \mathbb{P}_{s,\omega}$ -a.s.

We also write $\mathbb{E}_{s,\omega}$ instead of $\mathbb{E}^{\mathbb{P}_{s,\omega}}$ and the continuous local martingal part of X under $\mathbb{P}_{s,\omega}$ on [s,T] is denoted by $X^{c,s,\omega}$.

Remark 2.7. By Theorem II.2.34 in [25], the canonical representation of X on [s, T] is given by

$$X = X_s + p_s B + X^{c,s,\omega} + z.\mathbf{1}_{\{|z| \le C_0'\}} * (\mu^X - \nu) + z.\mathbf{1}_{\{|z| > C_0'\}} * \mu^X, \quad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Also note that, since δ is bounded from above by C_0' , the random measure K assigns no mass to $\{z \in \mathbb{R}^d : |z| > C_0'\}$. Consequently, the jumps of X on [s,T] are bounded from above by C_0' , $\mathbb{P}_{s,\omega}$ -a.s., i.e., we have on [s,T],

$$X = X_s + p_s B + X^{c,s,\omega} + z * (\mu^X - \nu), \quad \mathbb{P}_{s,\omega}$$
-a.s.

and X is a special $(\mathbb{P}_{s,\omega}, \mathbb{F}^0_+)$ -semimartingale on [s,T].

We augment the raw filtration \mathbb{F}^0 similarly as in the theory of Markov processes (see, e.g., [37]). To this end, let $\mathcal{N}_{s,\omega}$ be the collection of all $\mathbb{P}_{s,\omega}$ -null sets in \mathcal{F}_T^0 and put, for every $t \in [0,T]$,

$$\mathcal{F}_t^{s,\omega} := \sigma(\mathcal{F}_{t+}^0, \mathcal{N}_{s,\omega}), \quad \mathcal{F}_t := \bigcap_{(s,\omega) \in \bar{\Lambda}} \mathcal{F}_t^{s,\omega}.$$

Now we can define the following filtrations:

$$\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}, \quad \mathbb{F}^{s,\omega} := \{\mathcal{F}_t^{s,\omega}\}_{t \in [0,T]}, \quad (s,\omega) \in \bar{\Lambda}.$$

Note that \mathbb{F} is right-continuous.

Next, we introduce several classes of stopping times.

Definition 2.8. Let $s \in [t,T]$. Given a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$ on Ω , denote by $\mathcal{T}_s(\mathbb{G})$ the set of all \mathbb{G} -stopping times τ such that $s \leq \tau$. Set $\mathcal{T}_s := \mathcal{T}_s(\mathbb{F})$. Let $\omega \in \Omega$. Denote by \mathcal{H}_s (resp. $\mathcal{H}_{s,\omega}$) the set of all $\tau \in \mathcal{T}_s$

for which there exist some $d' \in \mathbb{N}$, a right-continuous, non-anticipating, \mathbb{F} -adapted, d'-dimensional process $Y = (Y^i)_{i \leq d'}$, and a closed subset E of $\mathbb{R}^{d'}$ such that, for every $\tilde{\omega} \in \Omega$ (resp. for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$),

$$\tau(\tilde{\omega}) = \inf\{t \ge s : Y_t(\tilde{\omega}) \in E\} \wedge T.$$

Given a stopping time τ and a path $\omega \in \Omega$, we often write (τ, ω) instead of $(\tau(\omega), \omega)$ if there is no danger of confusion.

Lemma 2.9. Fix $(s, \omega) \in \Lambda$ and $t \in [s, T)$. Let $\tau \in \mathcal{H}_{s,\omega}$. If $\tau(\omega) > t$ and X coincides with ω on [0, t], $\mathbb{P}_{s,\omega}$ -a.s., then $\tau > t$, $\mathbb{P}_{s,\omega}$ -a.s.

Proof. Let Y be the corresponding process, E be the corresponding closed set, and Ω' the corresponding subset of Ω with $\mathbb{P}_{s,\omega}(\Omega') = 1$ in the definition of $\mathcal{H}_{s,\omega}$ such that, for every $\tilde{\omega} \in \Omega'$, $\tau(\tilde{\omega}) = \inf\{t \geq s : Y_t(\tilde{\omega}) \in E\} \wedge T$ and ω coincides with $\tilde{\omega}$ on [0,s]. Since $\tau(\omega) > t$, we have $Y_r(\tilde{\omega}) = Y_r(\omega) \in E^c$ for every $r \in [s,t]$. This yields $\tau(\tilde{\omega}) > t$ because Y is right-continuous and E is closed.

2.4. Path-dependent stochastic analysis. First, we introduce a new space of continuous functionals. The reason is that we want the trajectories $t \mapsto u(t, \omega + x.\mathbf{1}_{[t,T]})$ to be right-continuous, which, in general, is not the case if the functional u is only in $C(\bar{\Lambda})$ as Example 2.11 below demonstrates.

Definition 2.10. Let $s \in [0,T]$. Denote by $C^0(\bar{\Lambda}^s)$ the set of all $u \in C(\bar{\Lambda}^s)$ such that, for every $x \in \mathbb{R}^d$, the map $(t,\omega) \mapsto u(t,\omega+x\mathbf{1}_{[t,T]})$ is continuous under \mathbf{d}_{∞} . Denote by $C_b^0(\bar{\Lambda}^s)$ the set of all bounded functionals in $C^0(\bar{\Lambda}^s)$ and by $UC_b^0(\bar{\Lambda}^s)$ the set of all uniformly continuous functionals in $C_b^0(\bar{\Lambda}^s)$.

Example 2.11. Consider $u=u(t,\omega):=\sup_{0\leq s\leq t}|\omega_s|$. Fix t>0. Let $\omega=-2.\mathbf{1}_{[t,T]}$. Then $u(t,\omega+\mathbf{1}_{[t,T]})=1$ but $u(t+n^{-1},\omega+\mathbf{1}_{[t+n^{-1},T]})=2$ for every $n\in\mathbb{N}$.

Next, we give an implicit definition of our path-dependent derivatives.

Definition 2.12. Let $(s,\omega) \in \Lambda$ and let $H \in \mathcal{H}_{s,\omega}$ with H > s, $\mathbb{P}_{s,\omega}$ -a.s. Denote by $C_b^{1,2}(\llbracket s, \mathbf{H} \rrbracket)$ the set of all bounded functionals $u \in C(\bar{\Lambda}^s)$ for which there exist bounded, right-continuous, non-anticipating, $\mathbb{F}^{s,\omega}$ -adapted functionals $\partial_t u : \bar{\Lambda}^s \to \mathbb{R}$, $\partial_\omega u = (\partial_\omega i u)_{i \leq d} : \bar{\Lambda}^s \to \mathbb{R}^d$, and $\partial^2_{\omega\omega} u = (\partial_\omega i \omega^j u)_{i,j \leq d} : \bar{\Lambda}^s \to \mathbb{S}^d$ such that $\partial_t u \in C(\llbracket s, \mathbf{H} \rrbracket)$, $\partial_\omega u \in C(\llbracket s, \mathbf{H} \rrbracket, \mathbb{R}^d)$, $\partial^2_{\omega\omega} u \in C(\llbracket s, \mathbf{H} \rrbracket, \mathbb{S}^d)$, and that, for every $\tau \in \mathcal{T}_s$,

$$\begin{split} u_{\tau \wedge \mathbf{H}} &= u_s + \int_s^{\tau \wedge \mathbf{H}} \partial_t u_t \, dt + \sum_{i=1}^d \int_s^{\tau \wedge \mathbf{H}} \partial_{\omega^i} u_{t-} \, dX_t^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_s^{\tau \wedge \mathbf{H}} \partial_{\omega^i \omega^j}^2 u_{t-} \, d\langle X^{i,s,\omega,c}, X^{j,s,\omega,c} \rangle_t \\ &+ \int_s^{\tau \wedge \mathbf{H}} \int_{\mathbb{R}^d} \left[u_t(X_{\cdot \wedge t-} + z. \mathbf{1}_{[t,T]}) - u_{t-} - \sum_{i=1}^d z^i \partial_{\omega^i} u_{t-} \right] \mu^X(dt, dz), \, \mathbb{P}_{s,\omega}\text{-a.s.} \end{split}$$

Given $z \in \mathbb{R}^d$, we sometimes use the operator ∇_z^2 defined by

$$\nabla_z^2 u(t,\omega) := u(t,\omega + z.\mathbf{1}_{[t,T]}) - u(t,\omega) - \sum_{i=1}^d z^i \partial_{\omega^i} u(t,\omega).$$

Remark 2.13. If $u \in C_b^{1,2}(\llbracket s, H \rrbracket)$, then, for every $\tau \in \mathcal{T}_s$,

$$u_{\tau \wedge H} = u_s - \int_s^{\tau \wedge H} \mathcal{L}u_t \, dt + \text{local martingale part}, \, \mathbb{P}_{s,\omega}\text{-a.s.}$$

3. VISCOSITY SOLUTIONS AND MAIN RESULTS

In this section, we introduce the notion of viscosity solutions for equations of the form (1.1). A minimal requirement for those solutions is consistency with classical solutions. They are defined as follows:

Definition 3.1. If
$$u \in C_b^0(\bar{\Lambda}) \cap C_b^{1,2}(\Lambda)$$
 and $\mathcal{L}u - f(\cdot, u, \partial_{u}u, \mathcal{I}u) \leq (\text{resp.} \geq, =) 0$ in Λ ,

then u is a classical subsolution (resp. classical supersolution, classical solution) of (1.1).

To state the actual definition of viscosity solutions, we need first to introduce two nonlinear expectations and spaces of test functionals.

Fix $(s,\omega) \in \Lambda$ and $L \geq 0$. Given a process $H \in L^2_{loc}(X^{c,s,\omega}, \mathbb{P}_{s,\omega})$ and a random field $W \in G_{loc}(p_s\mu^X, \mathbb{P}_{s,\omega})$, denote by $\Gamma^{H,W}$ the solution to

$$\Gamma = 1 + (\Gamma_{-}H) \cdot X^{c,s,\omega} + (\Gamma_{-}W) * (\mu^{X} - \nu)$$

on [s,T] with $\Gamma=1$ on $[0,s), \mathbb{P}_{s,\omega}$ -a.s.

Definition 3.2. Let $L \geq 0$ and let $(s,\omega) \in \bar{\Lambda}$. Denote by $\mathcal{P}^L(s,\omega)$ the set of all probability measures \mathbb{P} on $(\Omega, \mathcal{F}_T^0)$ for which there exists a process $H \in L^2_{\text{loc}}(X^{c,s,\omega}, \mathbb{P}_{s,\omega})$ with $|\sigma^\top H|_{\infty} \leq L$ and a random field $W \in G_{\text{loc}}(p_s\mu^X, \mathbb{P}_{s,\omega})$ with $0 \leq W \leq L\eta$ such that, for every $A \in \mathcal{F}_T^0$,

$$\mathbb{P}(A) = \int_{A} \Gamma_{T}^{H,W}(\tilde{\omega}) d\mathbb{P}_{s,\omega}(\tilde{\omega}).$$

Now we can define the following nonlinear expectations:

$$\underline{\mathcal{E}}_{s,\omega}^L := \inf_{\mathbb{P} \in \mathcal{P}^L(s,\omega)} \mathbb{E}^{\mathbb{P}}, \qquad \overline{\mathcal{E}}_{s,\omega}^L := \sup_{\mathbb{P} \in \mathcal{P}^L(s,\omega)} \mathbb{E}^{\mathbb{P}}.$$

Definition 3.3. Let $u: \overline{\Lambda} \to \mathbb{R}$ be an \mathbb{F} -adapted process, let $L \geq 0$, and let $(s,\omega) \in \Lambda$. Denote by $\underline{\mathcal{A}}^L u(s,\omega)$ (resp. $\overline{\mathcal{A}}^L u(s,\omega)$) the set of all functionals $\varphi \in C_b^0(\overline{\Lambda}^s)$ for which there exists a hitting time $H \in \mathcal{H}_{s,\omega}$ with H > s, $\mathbb{P}_{s,\omega}$ -a.s., such that $\varphi \in C_b^{1,2}(\llbracket s, H \rrbracket)$ and that

$$0 = (\varphi - u)(s, \omega) = \inf_{\tau \in \mathcal{T}_s} \underline{\mathcal{E}}_{s, \omega}^L \left[(\varphi - u)_{\tau \wedge \mathbf{H}} \right] \text{ (resp.} = \sup_{\tau \in \mathcal{T}_s} \overline{\mathcal{E}}_{s, \omega}^L \left[(\varphi - u)_{\tau \wedge \mathbf{H}} \right] \text{)}.$$

Definition 3.4. Let u be a bounded, right-continuous, non-anticipating, \mathbb{F} -adapted process that is $\mathbb{P}_{s,\omega}$ -quasi-left-continuous on [s,T] for every $(s,\omega) \in \bar{\Lambda}$.

(i) Given $L \geq 0$, we say u is a viscosity L-subsolution (resp. viscosity L-supersolution) of (1.1) if, for every $(t,\omega) \in \Lambda$ and every $\varphi \in \underline{\mathcal{A}}^L u(t,\omega)$ (resp. $\overline{\mathcal{A}}^L u(t,\omega)$),

$$\mathcal{L}\varphi(t,\omega) - f_t(\omega,\varphi(t,\omega),\partial_\omega\varphi(t,\omega),\mathcal{I}\varphi(t,\omega)) \le \text{(resp. } \ge) 0.$$

- (ii) We say u is a viscosity subsolution (resp. viscosity supersolution) of (1.1) if it is a viscosity L-subsolution (resp. viscosity L-supersolution) of (1.1) for some $L \geq 0$.
- (iii) We say u is a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1).

Remark 3.5. If $u \in C(\bar{\Lambda})$, then u is $\mathbb{P}_{s,\omega}$ -quasi-left-continuous on [s,T] for every $(s,\omega) \in \Lambda$.

Indeed, since X is $\mathbb{P}_{s,\omega}$ -quasi-left-continuous on [s,T], there exists, by Proposition I.2.26 in [25], a sequence of totally inaccessible stopping times exhausting the jumps of X. By Remark 2.2 (iii), this sequence also exhausts the jumps of u. Hence, again by Proposition I.2.26 in [25], u is $\mathbb{P}_{s,\omega}$ -quasi-left-continuous.

Theorem 3.6 (Consistency with classical solutions). Let $u \in C^0(\bar{\Lambda}) \cap C^{1,2}(\Lambda)$. Then u is a classical subsolution (classical supersolution, classical solution) of (1.1) if and only if u is a viscosity subsolution (viscosity supersolution, viscosity solution) of (1.1).

Our semilinear path-dependent integro-differential equation is closely connected to a family of non-Markovian BSDEs with jumps. To introduce this family, fix first $(s,\omega) \in \bar{\Lambda}$. Denote by $(Y^{s,\omega},Z^{s,\omega},U^{s,\omega})$ the unique solution to the BSDE

$$Y_t^{s,\omega} = \xi + \int_t^T f_r \left(X, Y_r^{s,\omega}, Z_r^{s,\omega}, \int_{\mathbb{R}^d} U_r^{s,\omega}(z) \, \eta_r(z) \, K_r(dz) \right) dr$$
$$- \int_t^T Z_r^{s,\omega} \, dX_r^{c,s,\omega} - \int_t^T \int_{\mathbb{R}^d} U_r^{s,\omega}(z) \, (\mu^X - \nu)(dr, dz), \ t \in [s, T], \ \mathbb{P}_{s,\omega}\text{-a.s.}$$

Without loss of generality, we assume that $Y^{s,\omega}$ is right-continuous and \mathbb{F}^0_+ -adapted.

Remark 3.7. By Theorem III.4.29 in [25] every $(\mathbb{P}_{s,\omega}, \mathbb{F}^0_+)$ -local martingale has the representation property relative to X (see Definition III.4.22 in [25]). Therefore, one can prove well-posedness of the BSDE above by standard methods. For related results for BSDEs driven by càdlàg martingales see [20] and [6]. In [43], which deals with BSDEs driven by càdlàg martingales and random measures, a special case of our BSDE is covered. Moreover, BSDEs driven by random measures in a general setting are treated in [7].

Next, define a functional $u^0: \bar{\Lambda} \to \mathbb{R}$ by

$$u^0(t,\omega) := \mathbb{E}_{t,\omega}[Y_t^{t,\omega}].$$

It will turn out that under additional assumptions u^0 is the unique solution to (1.1) satisfying $u_T^0 = \xi$.

Theorem 3.8 (Existence). If (B, C, ν) and ν are deterministic, then u^0 is a viscosity solution of (1.1) and $u^0 \in UC_b(\bar{\Lambda})$.

Theorem 3.9 (Partial comparison I). Fix $(s, \omega) \in \Lambda$. Let u^1 be a viscosity subsolution of (1.1) on Λ^s and let u^2 be a classical supersolution of (1.1) on Λ^s . Suppose that $u_T^1 \leq u_T^2$, $\mathbb{P}_{s,\omega}$ -a.s. Then $u^1(s,\omega) \leq u^2(s,\omega)$.

Theorem 3.10 (Stability). For every $\varepsilon > 0$, let $(b^{\varepsilon}, c^{\varepsilon}, K^{\varepsilon})$ together with some process σ^{ε} , some random field δ^{ε} , and some σ -finite measure F^{ε} satisfy Assumption 2.3 in place of (b, c, K) together with σ , δ , and F, and denote the corresponding linear integro-differential operator by $\mathcal{L}^{\varepsilon}$ (see Section 3). Also, for every $\varepsilon > 0$, let $\eta^{\varepsilon} = \eta^{\varepsilon}_{t}(\omega, z)$ and $f^{\varepsilon} = f^{\varepsilon}_{t}(\omega, y, z, p)$ satisfy Assumption 2.4 and Assumption 2.6 in place of η and f, respectively, and denote the corresponding integral operator by $\mathcal{I}^{\varepsilon}$, where (η, K) is replaced with $(\eta^{\varepsilon}, K^{\varepsilon})$. Suppose that $b^{\varepsilon} \to b$, $c^{\varepsilon} \to c$, $K^{\varepsilon} \to K$, $\eta^{\varepsilon} \to \eta$, and $f^{\varepsilon} \to f$ uniformly as $\varepsilon \downarrow 0$. Fix L > 0. For every $\varepsilon > 0$, let $u^{\varepsilon} = u^{\varepsilon}(t, \omega)$ be a viscosity L-supersolution of (1.1) with \mathcal{L} replaced by $\mathcal{L}^{\varepsilon}$ and f replaced by f^{ε} . Suppose that u^{ε} converges to some functional $u = u(t, \omega)$ on Λ uniformly as $\varepsilon \downarrow 0$. Then u is a viscosity L-supersolution of (1.1).

For the comparison principle, we have to employ the subsequent set of assumptions.

Assumption 3.11. Let ξ be uniformly continuous with respect to the weak M_1 -topology, i.e., with respect to the metric d_p defined by $d_p(\omega, \tilde{\omega}) := \max_{i \leq d} d_{M_1}(\omega^i, \tilde{\omega}^i), \ \omega = (\omega^i)_{i \leq d}, \ \tilde{\omega} = (\tilde{\omega}^i)_{i \leq d} \in \Omega$ (see Theorem 12.5.2 in [42]).

Remark 3.12. If d = 1, then the weak M_1 -topology coincides with the M_1 -topology. For its definition, see Appendix B. For more details, we refer the reader to [42].

Assumption 3.13. The triple (b, c, K) is constant, the random field η is deterministic and does not depend on z, and there exist positive constants $(L_{\varepsilon})_{\varepsilon \in (0,1)}$ and $\check{\nu} \in (0,1)$ such that the following holds:

(i) For every $\zeta = (\zeta^i)_{i \leq d} \in \mathbb{R}^d$,

$$\check{\nu} \left| \zeta \right|^2 \leq \frac{1}{2} \sum_{i,j=1}^d c^{ij} \zeta^i \zeta^j \leq \check{\nu}^{-1} \left| \zeta \right|^2.$$

(ii) For every $\alpha \in (0,1) \cap \mathbb{Q}$, there exists a positive constant $L_2(\alpha)$ such that

$$|\eta|_{\alpha/2,[0,T]} \le L_2(\alpha).$$

(iii) For every $\varepsilon \in (0,1)$, there exist nonnegative σ -finite measures $K_{1,\varepsilon}$ and $K_{2,\varepsilon}$ on $\mathcal{B}(\mathbb{R}^d)$ such that

$$K(dz) \le K_{1,\varepsilon}(dz) + K_{2,\varepsilon}(dz),$$

$$\int_{\mathbb{R}^d} (|z|^2 + |z|) K_{1,\varepsilon}(dz) \le \varepsilon,$$

$$K_{2,\varepsilon}(\mathbb{R}^d \setminus \{\mathbf{0}\}) \le L_{\varepsilon}.$$

Assumption 3.14. Suppose that there exists a constant $c_0' \in (0, C_0')$ such that $K(\{z \in \mathbb{R}^d : |z| < c_0'\}) = 0$ and that $K(\mathbb{R}^d) < \infty$.

Assumption 3.15. For every $\omega \in \Omega$ and every $\alpha \in (0,1) \cap \mathbb{Q}$, there exists a positive constant $L_1(\omega, \alpha)$ such that the following holds:

- (i) For every $(y, z, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $|f_{\cdot}(\omega, y, z, p)|_{\alpha/2, [0, T]} \leq L_1(\omega, \alpha) \cdot [|(y, z, p)| + 1].$
- (ii) For every $(t,p) \in [0,T] \times \mathbb{R}$, we have $f_t(\omega,\cdot,p) \in C^{\infty}(\mathbb{R} \times \mathbb{R}^d)$.

The following sets are also used in the proof of the partial comparison principle and the comparison principle.

Definition 3.16 (The sets Π_{∞}^t and Π_i^t). Let $t \in [0,T]$ and $i \in \mathbb{N}$. Denote by Π_{∞}^t the set of all $\pi_{\infty} = (t_0, x_0; t_1, x_1; \dots; x_{\infty})$ such that

- (i) $t = t_0 \le t_1 \le \ldots \le T$,
- (ii) $t_i = T$ for all but finitely many $i \in \mathbb{N}_0$,
- (iii) $x_i \in \mathbb{R}^d$ for all $i \in \mathbb{N}_0 \cup \{\infty\}$.

Denote by Π_i^t the set of all $\pi_i = (s_0, y_0; \dots; s_{i-1}, y_{i-1})$ such that

- (i) $t = s_0 \le ... \le s_{i-1} \le T$, (ii) $y_j \in \mathbb{R}^d$ for all $j \in \{0, ..., i-1\}$.

The following assumption is only needed for measurability issues in the proof of the comparison principle.

Assumption 3.17. For every $(s, \omega) \in \Lambda$, $i \in \mathbb{N}$, and $\tilde{\omega} \in \Omega$, let the functions

$$\Pi_{i} \to \mathbb{R}, \pi_{i} = (s_{0}, y_{0}; \dots; s_{i-1}, y_{i-1}) \mapsto \xi \left(\omega. \mathbf{1}_{[0, s_{0})} + \sum_{j=0}^{i-1} y_{j}. \mathbf{1}_{[s_{j}, s_{j+1})} + \tilde{\omega}. \mathbf{1}_{[s_{i}, T]} \right),$$

$$\Pi_{i} \to \mathbb{R}, \pi_{i} \mapsto f_{t} \left(\omega. \mathbf{1}_{[0, s_{0})} + \sum_{j=0}^{i-1} y_{j}. \mathbf{1}_{[s_{j}, s_{j+1})} + \tilde{\omega}. \mathbf{1}_{[s_{i}, T]}, y, z, p \right)$$

be continuous uniformly in (t, y, z, p).

Theorem 3.18 (Comparison). Suppose that Assumptions 3.11, 3.13, 3.15, 3.14, and 3.17 are satisfied. If u^1 is a viscosity subsolution of (1.1), if u^2 is a viscosity supersolution of (1.1), and if $u_T^1 \leq u_T^2$, then $u^1 \leq u^2$.

Theorems 3.8 and 3.18 immediately yield our final main result.

Theorem 3.19 (Well-posedness). Suppose that Assumptions 3.11, 3.13, 3.15, 3.14, and 3.17 are satisfied. Then u^0 is the unique viscosity solution of (1.1) with $u_T^0 = \xi$.

4. Consistency and Existence

Proof of Theorem 3.6. Clearly, if u is a viscosity subsolution of (1.1), then it is a classical subsolution of (1.1).

Let us now assume that u is not a viscosity L_0 -subsolution of (1.1) but a classical subsolution of (1.1). Then there exist $(s_0, \omega) \in \Lambda$ and $\varphi \in \underline{\mathcal{A}}^{L_0}u(s_0, \omega)$ with corresponding hitting time $H \in \mathcal{H}_{s_0}$ (see the definition of $\underline{\mathcal{A}}^{L_0}$) such that

$$c' := \mathcal{L}\varphi(s_0, \omega) - f_{s_0}(\omega, \varphi(s_0, \omega), \partial_\omega \varphi(s_0, \omega), \mathcal{I}\varphi(s_0, \omega)) > 0.$$

Without loss of generality, $s_0 = 0$. Put

$$\tau := \inf \left\{ t \ge 0 : \mathcal{L}\varphi_t - f_t(X, \varphi_t, \partial_\omega \varphi_t, \mathcal{I}\varphi_t) \le \frac{c'}{2} \right\} \wedge T.$$

By right-continuity of the involved process, $\tau > t$, $\mathbb{P}_{0,\omega}$ -a.s. Let $H = (H^i)_{i \leq d}$ be a stochastic process with $|\sigma^\top H|_{\infty} \leq L_0$, let W be a random field with $0 \leq W \leq L_0 \eta$, and let $\Gamma = \Gamma^{H,W}$ (see Section 3). Then integration-by-parts (Lemma C.5) yields

$$\begin{aligned} & (4.1) \\ & \Gamma(\varphi - u) \\ & = \Gamma \Bigg[-\mathcal{L}\varphi + \mathcal{L}u + \sum_{k,l} H^k \partial_{\omega^l}(\varphi - u) c^{kl} + \int_{\mathbb{R}^d} W \nabla_z^2 (\varphi - u) K(dz) \Bigg] \bullet t \\ & + \text{ martingale.} \end{aligned}$$

Given a strictly positive stopping time $\tilde{\tau} \leq \tau \wedge H$ such that

$$|f_t(X, \varphi_t, \partial_\omega \varphi_t, \mathcal{I}\varphi_t) - f_t(X, u_t, \partial_\omega \varphi_t, \mathcal{I}\varphi_t)| \le \frac{c'}{4}$$

on $[0, \tilde{\tau}[$, taking expectations yields

$$\mathbb{E}_{0,\omega}\left[\Gamma_{\tilde{\tau}}(\varphi - u)_{\tilde{\tau}}\right] \leq \mathbb{E}_{0,\omega}\left[\int_{0}^{\tilde{\tau}} \Gamma_{t}\left[-\frac{c'}{2} - f_{t}(X,\varphi_{t},\partial_{\omega}\varphi_{t},\mathcal{I}\varphi_{t}) + f_{t}(X,u_{t},\partial_{\omega}u_{t},\mathcal{I}u_{t})\right] + \sum_{k,l} H_{t}^{k}\partial_{\omega^{l}}(\varphi - u)_{t}e^{kl} + \int_{\mathbb{R}^{d}} W_{t}\nabla_{z}^{2}(\varphi - u)_{t} K_{t}(dz)\right] dt \right] \leq \mathbb{E}_{0,\omega}\left[\int_{0}^{\tilde{\tau}} \Gamma_{t}\left[-\frac{c'}{4} - f_{t}(X,u_{t},\partial_{\omega}\varphi_{t},\mathcal{I}\varphi_{t}) + f_{t}(X,u_{t},\partial_{\omega}u_{t},\mathcal{I}u_{t})\right] + \sum_{k,l} H_{t}^{k}\partial_{\omega^{l}}(\varphi - u)_{t}e^{kl} + \int_{\mathbb{R}^{d}} W_{t}\nabla_{z}^{2}(\varphi - u)_{t} K_{t}(dz)\right] dt \right].$$

Define a function $\tilde{f}: \bar{\Lambda} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}_t(\tilde{\omega}, y, z, p) := f_t(\tilde{\omega}, y, (\sigma_t^{\top}(\tilde{\omega}))^{-1}z, p).$$

Then
$$f_t(\tilde{\omega}, y, z, p) = \tilde{f}_t(\tilde{\omega}, u, \sigma_t^{\top}(\tilde{\omega})z, p)$$
 and
$$\left| \tilde{f}_t(\tilde{\omega}, y, z, p) - \tilde{f}_t(\tilde{\omega}, y, z', p) \right| \leq L_0 \left| z - z' \right|_1.$$

Consequently,

$$\begin{split} &-f_t(X,u_t,\partial_\omega\varphi_t,\mathcal{I}\varphi_t)+f_t(X,u_t,\partial_\omega u_t,\mathcal{I}u_t)\\ &=\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega u_t,\mathcal{I}u_t)-\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega\varphi_t,\mathcal{I}\varphi_t)\\ &=[\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega u_t,\mathcal{I}u_t)\\ &-\tilde{f}_t(X,u_t,(\sigma_t^\top\partial_\omega u_t)^1,\dots,(\sigma_t^\top\partial_\omega u_t)^{d-1},(\sigma_t^\top\partial_\omega\varphi_t)^d),\mathcal{I}u_t)]\\ &+[\tilde{f}_t(X,u_t,(\sigma_t^\top\partial_\omega u_t)^1,\dots,(\sigma_t^\top\partial_\omega u_t)^{d-1},(\sigma_t^\top\partial_\omega\varphi_t)^d),\mathcal{I}u_t)\\ &-\tilde{f}_t(X,u_t,(\sigma_t^\top\partial_\omega u_t)^1,\dots,(\sigma_t^\top\partial_\omega u_t)^{d-2},(\sigma_t^\top\partial_\omega\varphi_t)^{d-1},(\sigma_t^\top\partial_\omega\varphi_t)^d),\mathcal{I}u_t)]\\ &+\cdots\\ &+\tilde{f}_t(X,u_t,(\sigma_t^\top\partial_\omega u_t)^1,(\sigma_t^\top\partial_\omega\varphi_t)^2,\dots,(\sigma_t^\top\partial_\omega\varphi_t)^d,\mathcal{I}u_t)\\ &-\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega\varphi_t,\mathcal{I}u_t)]\\ &+[\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega\varphi_t,\mathcal{I}u_t)-\tilde{f}_t(X,u_t,\sigma_t^\top\partial_\omega\varphi_t,\mathcal{I}\varphi_t)]\\ &=\kappa_t^d(\sigma_t^\top\partial_\omega(u-\varphi)_t)^d+\cdots\kappa_t^1(\sigma_t^\top\partial_\omega(u-\varphi)_t)^1\\ &+\int_{\mathbb{T}^d}\lambda_t\eta_t(z)\nabla_z^2(u-\varphi)_t\,K_t(dz), \end{split}$$

where, for every $i \in \{1, \ldots, d\}$,

$$\kappa_t^i := [(\sigma_t^\top \partial_\omega (u - \varphi)_t)^i]^{-1} \cdot \mathbf{1}_{\{(\sigma_t^\top \partial_\omega (u - \varphi)_t)^i \neq 0\}}$$

$$\cdot [\tilde{f}_t(X, u_t, (\sigma_t^\top \partial_\omega u_t)^1, \dots, (\sigma_t^\top \partial_\omega u_t)^i, (\sigma_t^\top \partial_\omega \varphi_t)^{i+1}, \dots, (\sigma_t^\top \partial_\omega \varphi_t)^d)$$

$$- \tilde{f}_t(X, u_t, (\sigma_t^\top \partial_\omega u_t)^1, \dots, (\sigma_t^\top \partial_\omega u_t)^{i-1}, (\sigma_t^\top \partial_\omega \varphi_t)^i, \dots, (\sigma_t^\top \partial_\omega \varphi_t)^d)]$$

and

$$\lambda_t := [\mathcal{I}(u_t - \varphi_t)]^{-1} \cdot \mathbf{1}_{\{\mathcal{I}(u_t - \varphi_t) \neq 0\}} \cdot [\tilde{f}_t(X, u_t, \sigma_t^\top \partial_\omega \varphi_t, \mathcal{I}u_t) - \tilde{f}_t(X, u_t, \sigma_t^\top \partial_\omega \varphi_t, \mathcal{I}\varphi_t)].$$

Note that, by Assumption 2.4, we have $|\kappa^i| \leq L_0$ and that, by Assumption 2.4 and Assumption 2.6, we have $0 \leq \lambda \leq L_0$. Our goal now is to establish $H^{\top}c\partial_{\omega}(\varphi - u) = \kappa^{\top}[\sigma^{\top}\partial_{\omega}(\varphi - u)]$, which, by putting $\tilde{H} = \sigma^{\top}H$ and $\tilde{Z} = \sigma^{\top}\partial_{\omega}(\varphi - u)$, is equivalent to $\tilde{H}^{\top}\tilde{Z} = \kappa^{\top}\tilde{Z}$. Hence, if H is given by $H = (\sigma^{\top})^{-1}\tilde{H}$ in the case d > 1 and by $H = \sigma^{-1}\tilde{H}.\mathbf{1}_{\{\sigma \neq 0\}}$ in the case d = 1, where $\tilde{H}^i = \kappa^i$, and if W is given by $W_t(z) = \lambda_t \eta_t(z)$, then $|\sigma^{\top}H|_{\infty} = |\kappa|_{\infty} \leq L_0$, $0 \leq W \leq L_0 \eta$, and, provided $\tilde{\tau}$ is sufficiently small, which is possible because Γ is right-continuous, we have, by (4.2),

$$\mathbb{E}_{0,\omega}[\Gamma_T(\varphi-u)_{\tilde{\tau}}] = \mathbb{E}_{0,\omega}[\Gamma_{\tilde{\tau}}(\varphi-u)_{\tilde{\tau}}] \leq \mathbb{E}_{0,\omega}\left[\int_0^{\tilde{\tau}} \Gamma_t \left[-\frac{c'}{4}\right] dt\right] \leq \mathbb{E}_{0,\omega}\left[-\frac{\tilde{\tau}c'}{8}\right] < 0,$$

i.e., $\underline{\mathcal{E}}_{0,\omega}^{L_0}[(\varphi-u)_{\tilde{\tau}}] < 0$, which contradicts $\varphi \in \underline{\mathcal{A}}^{L_0}u(0,\omega)$. Thus u is a viscosity L_0 -subsolution.

Similarly, one can show the corresponding statement for supersolutions.

Next, we prove regularity of u^0 . To this end, we need the following result.

Lemma 4.1. Suppose that (B, C, ν) and η are deterministic. Fix $(s, \omega) \in \bar{\Lambda}$. Define a process \tilde{Y} on [s, T] by

$$\tilde{Y}_t := Y_t^{s,\omega}(\omega.\mathbf{1}_{[0,s)} + (X + \omega_s).\mathbf{1}_{[s,T]}).$$

Then there exist processes \tilde{Z} and $\tilde{U}(z)$ in the appropriate spaces such that $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is the solution to the BSDE

(4.3)

$$\tilde{Y}_{t} = \xi(\omega.\mathbf{1}_{[0,s)} + (X + \omega_{s}).\mathbf{1}_{[s,T]})
+ \int_{t}^{T} f_{r}(\omega.\mathbf{1}_{[0,s)} + (X + \omega_{s}).\mathbf{1}_{[s,T]}, \tilde{Y}_{r}, \tilde{Z}_{r}, \int_{\mathbb{R}^{d}} \tilde{U}_{r}(z) \, \eta_{r}(z) \, K_{r}(dz)) \, dr
- \int_{t}^{T} \tilde{Z}_{r} \, dX_{r}^{c,s,\mathbf{0}} - \int_{t}^{T} \int_{\mathbb{R}^{d}} \tilde{U}_{r}(z) \, (\mu^{X} - \nu)(dr, dz), \, t \in [s, T], \, \mathbb{P}_{s,\mathbf{0}}\text{-}a.s.$$

Proof. Put

$$Z^{0} := Z^{s,\omega}(\omega.\mathbf{1}_{[0,s)} + (X + \omega_{s}).\mathbf{1}_{[s,T]}),$$

$$U^{0} := U^{s,\omega}(\omega.\mathbf{1}_{[0,s)} + (X + \omega_{s}).\mathbf{1}_{[s,T]}).$$

Since (B, C, ν) and η are deterministic, the process M on [s, T] defined by

$$M_t := \tilde{Y}_t - \tilde{Y}_s - \int_s^t f_r(\omega.\mathbf{1}_{[0,s)} + (X + \omega_s).\mathbf{1}_{[s,T]}, \tilde{Y}_r, Z_r^0, \int_{\mathbb{R}^d} U_r^0(z) \, \eta_r(z) \, K_r(dz))$$

is an $(\mathbb{F}^{s,0},\mathbb{P}_{s,0})$ -martingale. Thus, we can, for every $n\in\mathbb{N}_0$, define a pair (Z^{n+1},U^{n+1}) inductively by

$$\begin{split} \tilde{Y}_t &= \xi(\omega.\mathbf{1}_{[0,s)} + (X + \omega_s).\mathbf{1}_{[s,T]}) \\ &+ \int_t^T f_r(\omega.\mathbf{1}_{[0,s)} + (X + \omega_s).\mathbf{1}_{[s,T]}, \tilde{Y}_r, Z_r^n, \int_{\mathbb{R}^d} U_r^n(z) \, \eta_r(z) \, K_r(dz)) \, dr \\ &- \int_t^T Z_r^{n+1} \, dX_r^{c,s,\mathbf{0}} - \int_t^T \int_{\mathbb{R}^d} U_r^{n+1}(z) \, (\mu^X - \nu) (dr, dz), \, t \in [s,T], \, \mathbb{P}_{s,\mathbf{0}}\text{-a.s.} \end{split}$$

Since (Z^n, U^n) converges to some limit (\tilde{Z}, \tilde{U}) , the triple $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is a solution to (4.3).

Proposition 4.2. If (B, C, ν) and ν are deterministic, then $u^0 \in UC_b(\bar{\Lambda})$.

Proof. Let ρ_1 be an increasing modulus of continuity of ξ and of $(t, \tilde{\omega}) \mapsto f_t(\tilde{\omega}, y, z, p)$, uniformly in t, y, z, and p, with upper bound $\|\rho_1\|_{\infty} > 0$. Let $(s, \omega), (s', \omega') \in \bar{\Lambda}$ with $s \leq s'$. Then

$$|u^{0}(s,\omega) - u^{0}(s',\omega')| \le |u^{0}(s,\omega) - u^{0}(s',\omega_{\wedge s})| + |u^{0}(s',\omega_{\wedge s}) - u^{0}(s',\omega')|$$

=: $A_{1} + A_{2}$.

Let us start with estimating A_2 . To this end, put

$$\tilde{Y}_{t}^{1} := Y_{t}^{s',\omega,\wedge s}(\omega_{\cdot,\wedge s}.\mathbf{1}_{[0,s')} + (X + \omega_{s}).\mathbf{1}_{[s',T]}),
\tilde{\xi}^{1} := \xi(\omega_{\cdot,\wedge s}.\mathbf{1}_{[0,s')} + (X + \omega_{s}).\mathbf{1}_{[s',T]}),
\tilde{f}_{t}^{1}(X, y, z, p) := f_{t}(\omega_{\cdot,\wedge s}.\mathbf{1}_{[0,s')} + (X + \omega_{s}).\mathbf{1}_{[s',T]}, y, z, p),$$

and

$$\begin{split} \tilde{Y}_t^2 &:= Y_t^{s',\omega'}(\omega'_{\cdot \wedge s'}.\mathbf{1}_{[0,s')} + (X + \omega'_{s'}).\mathbf{1}_{[s',T]}), \\ \tilde{\xi}^2 &:= \xi(\omega'_{\cdot \wedge s'}.\mathbf{1}_{[0,s')} + (X + \omega'_{s'}).\mathbf{1}_{[s',T]}), \\ \tilde{f}_t^2(X,y,z,p) &:= f_t(\omega'_{\cdot \wedge s'}.\mathbf{1}_{[0,s')} + (X + \omega'_{s'}).\mathbf{1}_{[s',T]},y,z,p). \end{split}$$

Since (B, C, ν) and η are deterministic, there exists, by Lemma 4.1, for every $i \in \{1, 2\}$, a pair $(\tilde{Z}^i, \tilde{U}^i)$ such that the triple $(\tilde{Y}^i, \tilde{Z}^i, \tilde{U}^i)$ is the solution to

the BSDE

$$\begin{split} \tilde{Y}_{t}^{i} &= \tilde{\xi}^{i} + \int_{t}^{T} \tilde{f}_{r}^{i}(X, \tilde{Y}_{r}^{i}, \tilde{Z}_{r}^{i}, \int_{\mathbb{R}^{d}} \tilde{U}_{r}^{i}(z) \, \eta_{r}(z) \, K_{r}(dz)) \, dr \\ &- \int_{t}^{T} \tilde{Z}_{r}^{i} \, dX_{r}^{c,s',\mathbf{0}} - \int_{t}^{T} \int_{\mathbb{R}^{d}} \tilde{U}_{r}^{i}(z) \, (\mu^{X} - \nu)(dr, dz), \, t \in [s', T], \, \mathbb{P}_{s',\mathbf{0}}\text{-a.s.} \end{split}$$

Therefore and using again the fact that (B, C, ν) is deterministic, we have

(4.4)
$$A_2 = \left| \mathbb{E}_{s', \omega_{\cdot \wedge s}} [Y_{s'}^{s', \omega_{\cdot \wedge s}}] - \mathbb{E}_{s', \omega'} [Y_{s'}^{s', \omega'}] \right| = \left| \mathbb{E}_{s', \mathbf{0}} [\tilde{Y}_{s'}^1 - \tilde{Y}_{s'}^2] \right|.$$

Now, note that, for every $t \in [s', T]$,

$$\begin{split} \tilde{f}_{t}^{1}(X, \tilde{Y}_{t}^{1}, \tilde{Z}_{t}^{1}, \int_{\mathbb{R}^{d}} \tilde{U}_{t}^{1}(z) \, \eta_{t}(z) \, K_{t}(dz)) &- \tilde{f}_{t}^{2}(X, \tilde{Y}_{t}^{2}, \tilde{Z}_{t}^{2}, \int_{\mathbb{R}^{d}} \tilde{U}_{t}^{2}(z) \, \eta_{t}(z) \, K_{t}(dz)) \\ &= \gamma_{t} [\tilde{Y}_{t}^{1} - \tilde{Y}_{t}^{2}] + \sum_{j=1}^{d} \kappa_{t}^{j} \, [\sigma_{t}^{\top} (\tilde{Z}_{t}^{1} - \tilde{Z}_{t}^{2})]^{j} + \int_{\mathbb{R}^{d}} \lambda_{t} \, \eta_{t}(z) \, (\tilde{U}_{t}^{1}(z) - \tilde{U}_{t}^{2}(z)) \, K_{t}(dz) \\ &+ \tilde{f}_{t}^{1}(X, \tilde{Y}_{t}^{2}, \tilde{Z}_{t}^{2}, \int_{\mathbb{R}^{d}} \tilde{U}_{t}^{2}(z) \, \eta_{t}(z) \, K_{t}(dz)) \\ &- \tilde{f}_{t}^{2}(X, \tilde{Y}_{t}^{2}, \tilde{Z}_{t}^{2}, \int_{\mathbb{R}^{d}} \tilde{U}_{t}^{2}(z) \, \eta_{t}(z) \, K_{t}(dz)), \end{split}$$

where

$$\begin{split} \gamma_t := & [\tilde{Y}_t^1 - \tilde{Y}_t^2]^{-1}.\mathbf{1}_{\{\tilde{Y}_t^1 - \tilde{Y}_t^2 \neq 0\}} \\ & \cdot [\tilde{f}_t^1(X, \tilde{Y}_t^1, \tilde{Z}_t^1, \int_{\mathbb{R}^d} \tilde{U}_t^1(z) \, \eta_t(z) \, K_t(dz)) - \tilde{f}_t^1(X, \tilde{Y}_t^2, \tilde{Z}_t^1, \int_{\mathbb{R}^d} \tilde{U}_t^1(z) \, \eta_t(z) \, K_t(dz))], \end{split}$$

and the processes κ^j , $j=1,\ldots,d$, and λ are defined similarly (with the obvious changes) as in the proof of Theorem 3.6. Also, as in said proof, define a d-dimensional process H by $H^j:=[(\sigma^\top)^{-1}\kappa]^j$ in the case d>1 and by $H:=\sigma^{-1}\kappa$ in the case d=1, define a random field W by $W_t(z):=\lambda_t \eta_t(z)$, and consider the solution $\tilde{\Gamma}$ of

$$\tilde{\Gamma} = 1 + (\tilde{\Gamma}_{-}\gamma) \bullet t + (\tilde{\Gamma}_{-}H) \bullet X^{c,s',\mathbf{0}} + (\tilde{\Gamma}_{-}W) * (\mu^{X} - \nu)$$

on [s',T] with $\tilde{\Gamma}=1$ on [0,s'), $\mathbb{P}_{s',\mathbf{0}}\text{-a.s.}$ Integration-by-parts (Lemma C.5) yields

$$\begin{split} \tilde{\Gamma}(\tilde{Y}^1 - \tilde{Y}^2) &= (\tilde{Y}^1 - \tilde{Y}^2)_{s'} - \tilde{\Gamma}[\tilde{f}^1(X, \tilde{Y}^2, \tilde{Z}^2, \int_{\mathbb{R}^d} \tilde{U}^2(z) \, \eta(z) \, K(dz)) \\ &- \tilde{f}^2(X, \tilde{Y}^2, \tilde{Z}^2, \int_{\mathbb{R}^d} \tilde{U}^2(z) \, \eta(z) \, K(dz))] \bullet t \\ &+ \text{martingale}, \, \mathbb{P}_{s', \mathbf{0}}\text{-a.s.} \end{split}$$

Since ξ , f, and γ are bounded, we get, together with (4.4),

$$A_{2} \leq e^{(T-s')L_{0}} \mathbb{E}_{s',\mathbf{0}} \left[\left| \tilde{\xi}^{1} - \tilde{\xi}^{2} \right| \right] + \int_{s'}^{T} e^{(t-s')L_{0}} \rho_{1}((t,\omega_{\cdot \wedge s}),(t,\omega'_{\cdot \wedge s'})) dt$$

$$\leq C' \rho_{1}(\|\omega_{\cdot \wedge s} - \omega'_{\cdot \wedge s'}\|_{\infty}),$$

where C' does not depend on (s, ω) and (s', ω') .

To deal with A_1 , let $\varepsilon > 0$. Our goal is to find a $\delta' > 0$ such that

$$\mathbf{d}_{\infty}((s,\omega),(s',\omega'))<\delta'$$

implies $A_1 < \varepsilon$. In order to estimate A_1 , put for every $\tilde{\omega} \in \Omega$,

$$\begin{split} \tilde{Y}_t^{3,\tilde{\omega}} &:= Y_t^{s,\omega} \big(\tilde{\omega}.\mathbf{1}_{[0,s')} + (X + \tilde{\omega}_{s'}).\mathbf{1}_{[s',T]} \big), \\ \tilde{\xi}^{3,\tilde{\omega}} &:= \xi \big(\tilde{\omega}.\mathbf{1}_{[0,s')} + (X + \tilde{\omega}_{s'}).\mathbf{1}_{[s',T]} \big), \\ \tilde{f}_t^{3,\tilde{\omega}}(X,y,z,p) &:= f_t \big(\tilde{\omega}.\mathbf{1}_{[0,s')} + (X + \tilde{\omega}_{s'}).\mathbf{1}_{[s',T]},y,z,p \big). \end{split}$$

Since (B, C, ν) is deterministic and since, for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$Y_{s'}^{s,\omega}(\tilde{\omega}) = \mathbb{E}_{s,\omega}[Y_{s'}^{s,\omega}|\mathcal{F}_{s'+}^0](\tilde{\omega}) = \mathbb{E}_{s',\tilde{\omega}}[Y_{s'}^{s,\omega}],$$

we have

$$A_{1} = \left| \mathbb{E}_{s,\omega}[Y_{s}^{s,\omega}] - \mathbb{E}_{s',\omega_{\cdot,\wedge s}}[Y_{s'}^{s',\omega_{\cdot,\wedge s}}] \right|$$

$$\leq \left| \mathbb{E}_{s,\omega} \left[\int_{s}^{s'} f_{t} \left(X, Y_{t}^{s,\omega}, Z_{t}^{s,\omega}, \int_{\mathbb{R}^{d}} U_{t}^{s,\omega}(z) \, \eta_{t}(z) \, K_{t}(dz) \right) \, dt \right] \right|$$

$$+ \left| \mathbb{E}_{s,\omega}[Y_{s'}^{s,\omega}] - \mathbb{E}_{s',\omega_{\cdot,\wedge s}}[Y_{s'}^{s',\omega_{\cdot,\wedge s}}] \right|$$

$$\leq (s'-s)C'_{0} + \left| \int_{\Omega} \mathbb{E}_{s',\tilde{\omega}}[Y_{s'}^{s,\omega}] - \mathbb{E}_{s',\omega_{\cdot,\wedge s}}[Y_{s'}^{s',\omega_{\cdot,\wedge s}}] \, \mathbb{P}_{s,\omega}(d\tilde{\omega}) \right|$$

$$= (s'-s)C'_{0} + \left| \int_{\Omega} \mathbb{E}_{s',0}[\tilde{Y}_{s'}^{3,\tilde{\omega}} - \tilde{Y}_{s'}^{1}] \, \mathbb{P}_{s,\omega}(d\tilde{\omega}) \right|.$$

Similarly, as we estimated A_2 in (4), we get, for every $\tilde{\omega} \in \Omega$,

$$\left| \mathbb{E}_{s',\mathbf{0}} [\tilde{Y}_{s'}^{3,\tilde{\omega}} - \tilde{Y}_{s'}^{1}] \right| \leq C' \rho_1(\|\tilde{\omega}_{\cdot \wedge s'} - \omega_{\cdot \wedge s}\|_{\infty}),$$

where C'>0 does not depend on s, s', ω , and $\tilde{\omega}$. Note that, since ρ_1 is continuous at 0, there exists a δ'' such that $C'\rho_1(\delta'')<\varepsilon/2$. Thus, by (4.5) together with (4.6),

$$A_{1} \leq (s'-s)C'_{0} + C'\mathbb{E}_{s,\omega}[\rho_{1}(\|X_{\cdot \wedge s'} - \omega_{\cdot \wedge s}\|_{\infty}]$$

$$= (s'-s)C'_{0} + C'\mathbb{E}_{s,\omega}[\rho_{1}(\sup_{t \in [s,s']}|X_{t} - X_{s}|)$$

$$\cdot \mathbf{1}_{\{\sup_{t \in [s,s']}|X_{t} - X_{s}| < \delta''\}} + \mathbf{1}_{\{\sup_{t \in [s,s']}|X_{t} - X_{s}| \ge \delta''\}}]$$

$$\leq (s'-s)C'_{0} + \frac{\varepsilon}{2} + C'\|\rho_{1}\|_{\infty} \mathbb{P}_{s,\omega}\left(\sup_{t \in [s,s']}|X_{t} - X_{s}| \ge \delta''\right).$$

Recall that on [s, T],

$$X = X_s + p_s B + M,$$
 $\mathbb{P}_{s,\omega}$ -a.s.,

where $M := X^{c,s,\omega} + z * (\mu^X - \nu)$ is a $(\mathbb{P}_{s,\omega}, \mathbb{F}^0_+)$ -martingale on [s,T]. Without loss of generality, let

(4.8)
$$\delta' C_0' < \frac{\delta''}{2} \quad \text{and} \quad s' - s < \delta'.$$

Thus, since

$$\sup_{t \in [s,s']} |X_t - X_s| \ge \delta''$$

implies

$$\sup_{t \in [s,s']} |p_s B_t| + \sup_{t \in [s,s']} |M_t| \ge \delta''$$

but, by (4.8),

$$\sup_{t \in [s,s']} |p_s B_t| \le \int_s^{s'} |b_t| \ dt \le (s'-s)C_0' \le \frac{\delta''}{2},$$

we have, by Doob's inequality and Itô's lemma,

$$\mathbb{P}_{s,\omega} \left(\sup_{t \in [s,s']} |X_t - X_s| \ge \delta'' \right) \\
\leq \mathbb{P}_{s,\omega} \left(\sup_{t \in [s,s']} |M_t| \ge \frac{\delta''}{2} \right) \\
\leq \mathbb{P}_{s,\omega} \left(\sup_{t \in [s,s']} |M_t|^2 \ge \frac{|\delta''|^2}{4} \right) \\
\leq \frac{4}{|\delta''|^2} \mathbb{E}_{s,\omega} \left[|M_{s'}|^2 \right] \\
\leq \frac{4}{|\delta''|^2} \mathbb{E}_{s,\omega} \left[\sum_{i \le d} \int_s^{s'} c_t^{ii} dt + \int_s^{s'} \int_{\mathbb{R}^d} \left(|z|^2 \wedge C_0' \right) K_t(dz) dt \right] \\
\leq \frac{4}{|\delta''|^2} \cdot (s' - s) (dC_0' + C_0'').$$

Together with (4.7), we get

$$A_1 \le (s'-s)C'' + \frac{\varepsilon}{2}$$

for some constant C''>0 that does not depend on $s,\,s',\,\omega,$ and ω' provided that (4.8) holds. I.e., if

$$\mathbf{d}_{\infty}((s,\omega),(s',\omega')) < \frac{\varepsilon}{2C''} \wedge \frac{\delta''}{2C'_0},$$

then $A_1 < \varepsilon$.

Lemma 4.3. Fix $(s, \omega) \in \bar{\Lambda}$ and $t \in [s, T]$. For $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$, the processes $Y^{s,\omega}$ and $Y^{t,\tilde{\omega}}$ are $\mathbb{P}_{t,\tilde{\omega}}$ -indistinguishable on [t, T].

Proof. Let Ω' be the set of all $\omega' \in \Omega$ such that the process M on [s,T] defined by

$$M_r := Y_r^{s,\omega} - Y_t^{s,\omega} - \int_t^r f_{\theta}(X, Y_{\theta}^{s,\omega}, Z_{\theta}^{s,\omega}, \int_{\mathbb{R}^d} U_{\theta}^{s,\omega}(z) \eta_{\theta}(z) K_{\theta}(dz)) d\theta$$

is an $(\mathbb{F}^{t,\omega'}, \mathbb{P}_{t,\omega'})$ -martingale. By Proposition A.8, $\mathbb{P}_{s,\omega}(\Omega') = 1$. Now, let $\tilde{\omega} \in \Omega'$. Put $(Z^0, U^0) := (Z^{s,\omega}, U^{s,\omega})$. Since M is an $(\mathbb{F}^{t,\tilde{\omega}}, \mathbb{P}_{t,\tilde{\omega}})$ -martingale, we can, for every $n \in \mathbb{N}_0$, define (Z^{n+1}, U^{n+1}) inductively by

$$Y_r^{s,\omega} = \xi + \int_r^T f_{\theta}(X, Y_{\theta}^{s,\omega}, Z_{\theta}^n, \int_{\mathbb{R}^d} U_{\theta}^n(z) \eta_{\theta}(z) K_{\theta}(dz)) d\theta - \int_r^T Z_{\theta}^{n+1} dX_{\theta}^{c,t,\tilde{\omega}}$$
$$- \int_r^T \int_{\mathbb{R}^d} U_{\theta}^{n+1}(z) (\mu^X - \nu) (d\theta, dz), \quad r \in [t, T], \quad \mathbb{P}_{t,\tilde{\omega}}\text{-a.s.}$$

Note that (Z^n,U^n) converges to some limit (Z,U) and that $(Y^{s,\omega},Z,U)$ solves

$$Y_r^{s,\omega} = \xi + \int_r^T f_{\theta}(X, Y_{\theta}^{s,\omega}, Z_{\theta}, \int_{\mathbb{R}^d} U_{\theta}(z) \eta_{\theta}(z) K_{\theta}(dz)) d\theta - \int_r^T Z_{\theta} dX_{\theta}^{c,t,\tilde{\omega}} - \int_r^T \int_{\mathbb{R}^d} U_{\theta}(z) (\mu^X - \nu) (d\theta, dz), \quad r \in [t, T], \quad \mathbb{P}_{t,\tilde{\omega}}\text{-a.s.}$$

Uniqueness for BSDEs concludes the proof.

Remark 4.4. Fix $(s, \omega) \in \bar{\Lambda}$. By Lemma 4.3 and by Proposition A.7, for every $t \in [s, T]$, and for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$u^{0}(t,\tilde{\omega}) = \mathbb{E}_{t,\tilde{\omega}}[Y_{t}^{s,\omega}] = \mathbb{E}_{s,\omega}[Y_{t}^{s,\omega}|\mathcal{F}_{t+}^{0}](\tilde{\omega} = Y_{t}^{s,\omega}(\tilde{\omega}).$$

If $u^0 \in C(\bar{\Lambda})$, then u^0 and $Y^{s,\omega}$ are $\mathbb{P}_{s,\omega}$ -indistinguishable on [s,T] because both processes are right-continuous.

Proof of Theorem 3.8. By Proposition 4.2, $u^0 \in C_b(\bar{\Lambda})$. Thus u^0 is bounded, right-continuous, non-anticipating, and $\mathbb{P}_{s,\omega}$ -quasi-left-continuous for every $(s,\omega) \in \bar{\Lambda}$. Keeping Remark 4.4 in mind, one can easily show that u^0 is a viscosity subsolution. To do so, follow the lines of the corresponding part of the proof of Theorem 3.6 and replace in (4.1)

$$\mathcal{L}u$$
 by $f\left(X, u, Z^{0,\omega}, \int U^{0,\omega}(z)\eta(z) K(dz)\right)$

and everywhere

$$\partial_{\omega}u$$
 by $Z^{0,\omega}$, $\mathcal{I}u$ by $\int U^{0,\omega}(z)\eta(z)K(dz)$, and ∇_z^2u by $U^{0,\omega}(z)$.

Similarly, one can show that u^0 is a viscosity supersolution.

5. Partial Comparison and Stability

Before we begin to prove the partial comparison principle itself, we need to establish some auxiliary results about BSDEs, reflected BSDEs (RBSDEs), and optimal stopping.

5.1. **BSDEs with jumps and nonlinear expectations.** Given $(s, \omega) \in \bar{\Lambda}$, L > 0, $\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$, and an $\mathcal{F}_{\tau}^{s,\omega}$ -measurable random variable $\tilde{\xi} : \Omega \to \mathbb{R}$, denote by

$$(Y^{s,\omega}(L,\tau,\tilde{\xi}),Z^{s,\omega}(L,\tau,\tilde{\xi}),U^{s,\omega}(L,\tau,\tilde{\xi}))$$

the solution to the BSDE

$$Y_t = \tilde{\xi} + \int_t^T \mathbf{1}_{\{r < \tau\}} L \left[\left| \sigma_r^\top Z_r \right|_1 + \int_{\mathbb{R}^d} U_r(z)^+ \, \eta_r(z) \, K_r(dz) \right] dr$$
$$- \int_t^T Z_r \, dX_r^{c,s,\omega} - \int_t^T \int_{\mathbb{R}^d} U_r(z) \, (\mu^X - \nu) (dr, dz), \quad t \in [s, T], \quad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Remark 5.1. Note that in the driver of the BSDE above we use the positive part $U(z)^+$ instead of the absolute value |U(z)| in order for the comparison principle for BSDEs to hold (see [2]).

Lemma 5.2. We have

(5.1)
$$\overline{\mathcal{E}}_{s,\omega}^{L}[\tilde{\xi}] = \mathbb{E}_{s,\omega}[Y_s^{s,\omega}(L,\tau,\tilde{\xi})].$$

Proof. For the sake of readability, we omit to write ${}^{s,\omega}(L,\tau,\tilde{\xi})$ in this proof. Given a process H and a random field W, let $\Gamma=\Gamma^{H,W}$ be the solution to

$$\Gamma = 1 + (\Gamma_{-}H) \cdot X^{c,s,\omega} + (\Gamma_{-}W) * (\mu^{X} - \nu)$$

on [s,T] with $\Gamma=1$ on [0,s), $\mathbb{P}_{s,\omega}$ -a.s. Since $Z=\mathbf{1}_{\llbracket s,\tau \rrbracket}Z,\,U(z)=\mathbf{1}_{\llbracket s,\tau \rrbracket}U(z),$ and

$$Y = Y_s - \mathbf{1}_{\llbracket s,\tau \rrbracket} L \left[\left| \sigma^\top Z \right|_1 + \int_{\mathbb{R}^d} LU(z)^+ \eta K(dz) \right] \bullet t$$
$$+ \sum_i Z^i \bullet X^{i,c,s,\omega} + U * (\mu^X - \nu),$$

integration-by-parts (Lemma C.5) yields

(5.2)

$$\begin{split} \Gamma Y &= Y_s + \mathbf{1}_{\llbracket s,\tau \rrbracket} \Gamma \left[-L \left| \sigma^\top Z \right|_1 + \sum_{i,j} H^i Z^j c^{ij} \right] \bullet t \\ &+ \mathbf{1}_{\llbracket s,\tau \rrbracket} \Gamma \left[\int_{\mathbb{R}^d} -L U(z)^+ \, \eta(z) \, K(dz) + \int_{\mathbb{R}^d} W(z) U(z) \, K(dz) \right] \bullet t \\ &+ \text{martingale} \\ &=: Y_s + A^1 + A^2 + \text{martingale}. \end{split}$$

Our goal is to choose H and W so that the drift term $A^1 + A^2$ vanishes. Let us first deal with A^2 . If W is defined by

(5.3)
$$W(z) = L\eta(z).\mathbf{1}_{\{U(z)>0\}},$$

then $A^2 = 0$ and $0 \le W(z) \le L\eta(z)$. Next, we deal with A^1 . We need

(5.4)
$$H^{\top}cZ = (H^{\top}\sigma)(\sigma^{\top}Z) = L \left|\sigma^{\top}Z\right|_{1}$$

to hold. To this end, put $\tilde{Z} = \sigma^{\top} Z$ and $\tilde{H} = \sigma^{\top} H$. Then (5.4) is equivalent to

$$\sum_i \tilde{H}^i \tilde{Z}^i = L \sum_i \left| \tilde{Z}^i \right|.$$

Thus, if

$$(5.5) \quad \tilde{H}^i = L \frac{\left| \tilde{Z}^i \right|}{\tilde{Z}^i}.\mathbf{1}_{\{\tilde{Z}^i \neq 0\}} \quad \text{and} \quad H = \begin{cases} (\sigma^\top)^{-1} \tilde{H} & \text{if } d > 1, \\ \sigma^{-1} \tilde{H}.\mathbf{1}_{\{\sigma \neq 0\}} & \text{if } d = 1, \end{cases}$$

then we get (5.4), i.e., $A^1 = 0$, and, moreover, we have $\left|\sigma^{\top}H\right|_{\infty} = \left|\tilde{H}\right|_{\infty} \leq L$. Consequently, for H defined by (5.5) and W defined by (5.3), we have

(5.6)
$$\overline{\mathcal{E}}_{s,\omega}^{L}[\tilde{\xi}] \ge \mathbb{E}_{s,\omega} \left[\Gamma_{\tau}^{H,W} Y_{\tau} \right] = \mathbb{E}_{s,\omega}[Y_{s}].$$

On the other hand, for every process H with $|\sigma^{\top}H|_{\infty} \leq L$ and every random field W with $0 \leq W(z) \leq L\eta(z)$, we have

$$H^{\top}cZ \leq \left|\sigma^{\top}H\right|_{\infty} \left|\sigma^{\top}Z\right|_{1} \leq L\left|\sigma^{\top}Z\right|_{1} \quad \text{and} \quad W(z)U(z) \leq L\eta(z)U(z)^{+},$$

which, by (5.2), yields

$$\mathbb{E}_{s,\omega}\left[\Gamma_{\tau}^{H,W}Y_{\tau}\right] \leq \mathbb{E}_{s,\omega}[Y_{s}].$$

This, together with (5.6), establishes (5.1).

5.2. **RBSDEs with jumps.** Our proof of the partial comparison principle relies heavily on the theory of RBSDEs. See [21], [22], [12], and Chapter 14 of [14] for more details.

Fix a bounded, right-continuous, \mathbb{F} -adapted process $R: \bar{\Lambda} \to \mathbb{R}$ that is $\mathbb{P}_{s,\omega}$ -quasi-left-continuous on [s,T] for every $(s,\omega) \in \bar{\Lambda}$. Fix also $L \geq 0$.

For every $(s, \omega) \in \bar{\Lambda}$ and $H \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$, there exists, because of the martingale representation property (Theorem III.4.29 in [25]), a unique solution

$$(\bar{Y},\bar{Z},\bar{U},\bar{K})=(\bar{Y}^{s,\omega}(L,\mathbf{H},R),\bar{Z}^{s,\omega}(L,\mathbf{H},R),\bar{U}^{s,\omega}(L,\mathbf{H},\mathbb{R}),\bar{K}^{s,\omega}(L,\mathbf{H},\mathbb{R}))$$

with $\bar{Y} = \bar{Y}_{. \wedge H}$, $\bar{Z} = \mathbf{1}_{\{. \leq H\}} \bar{Z}$, $\bar{U} = \mathbf{1}_{\{. \leq H\}} \bar{U}$, and $\bar{K} = \bar{K}_{. \wedge H}$ to the following RBSDE with lower barrier R and random terminal time H (cf. Remark 2.4 in

[12]):

$$\begin{split} \bar{Y}_t &= R_{\mathrm{H}} + \int_t^T \mathbf{1}_{\{r \leq \mathrm{H}\}} \, L \left[\left| \sigma_r^\top \bar{Z}_r \right|_1 + \int_{\mathbb{R}^d} \bar{U}_r(z)^+ \, \eta_r(z) \, K_r(dz) \right] \, dr + \bar{K}_{\mathrm{H}} - \bar{K}_t \\ &- \int_t^T \bar{Z}_r \, dX_r^{c,s,\omega} - \int_t^T \int_{\mathbb{R}^d} \bar{U}_r(z) \, (\mu^X - \nu) (dr,dz), t \in [s,T], \, \, \mathbb{P}_{s,\omega}\text{-a.s.}, \\ \bar{Y}_t &\geq R_{t \wedge \mathrm{H}}, t \in [s,T], \, \, \mathbb{P}_{s,\omega}\text{-a.s.}, \\ \int_s^T (\bar{Y}_t - R_t) \, d\bar{K}_t = 0, \quad \bar{K}_s^{s,\omega} = 0, \quad \bar{K} \text{ is continuous and nondecreasing.} \end{split}$$

Lemma 5.3. Fix $\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$ and $H \in \mathcal{H}_{s,\omega}$ with $\tau \leq H$, $\mathbb{P}_{s,\omega}$ -a.s. Then, for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$, the processes $\bar{Y}^{s,\omega}(L,H,R)$ and $\bar{Y}^{\tau,\tilde{\omega}}(L,H,R)$ are $\mathbb{P}_{\tau,\tilde{\omega}}$ -indistinguishable on $[\tau(\tilde{\omega}),T]$ and $\bar{K}^{\tau,\tilde{\omega}}(L,H,R) = \bar{K}^{s,\omega}(L,H,R) - \bar{K}^{s,\omega}_{\tau(\tilde{\omega})}(L,H,R)$ on $[\tau(\tilde{\omega}),T]$.

Proof. Proceed as in the proof of Lemma 4.3 and note that Proposition A.8 also applies to stopping times. Utilizing uniqueness for RBSDEs will conclude the proof. \Box

Given $H \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$, consider the optimal stopping times

$$\begin{split} \tau^*_{s,\omega;\mathbf{H}} &:= \inf\{t \geq s : \bar{Y}^{s,\omega}_t(L,\mathbf{H},R) = R_{t \wedge \mathbf{H}}\}, \\ \tau^{**}_{s,\omega;\mathbf{H}} &:= \inf\{t \geq s : \bar{K}^{s,\omega}_t(L,\mathbf{H},R) > 0\}. \end{split}$$

Note that, since $\bar{K}^{s,\omega}(L,H,R)$ is continuous, we have $\tau_{s,\omega;H}^* \leq \tau_{s,\omega;H}^{**} \wedge H$.

Lemma 5.4. If $H \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$, then

(5.7)
$$\mathbb{E}_{s,\omega}[\bar{Y}_s^{s,\omega}(L,H,R)] = \sup_{\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})} \overline{\mathcal{E}}_{s,\omega}^L[R_{\tau \wedge H}].$$

A corresponding result for RBSDEs without jumps has been proven in [32]. We follow the approach in [4], where quadratic RBSDEs without jumps are studied.

Proof of Lemma 5.4. Provided there is no danger of confusion, we omit to write ${}^{s,\omega}(L,H,R)$ in this proof. Let us first fix a stopping time $\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$. Note that

$$\begin{split} \bar{Y}_t &= \bar{Y}_{\tau \wedge \mathbf{H}} + \int_t^T \mathbf{1}_{\{r \leq \tau \wedge \mathbf{H}\}} L \left[\left| \sigma_r^\top \bar{Z}_r \right|_1 + \int_{\mathbb{R}^d} \bar{U}_r(z)^+ \, \eta_r(z) \, K_r(dz) \right] \, dr \\ &+ \bar{K}_{\tau \wedge \mathbf{H}} - \bar{K}_t - \int_t^T \mathbf{1}_{\{r \leq \tau \wedge \mathbf{H}\}} \, \bar{Z}_r \, dX_r^{c,s,\omega} \\ &- \int_t^T \int_{\mathbb{R}^d} \mathbf{1}_{\{r \leq \tau \wedge \mathbf{H}\}} \, \bar{U}_r(z) \, (\mu^X - \nu) (dr, dz), \, t \in [s, T], \, \mathbb{P}_{s,\omega} \text{-a.s.} \end{split}$$

Since $\bar{Y}_{\tau \wedge H} \geq R_{\tau \wedge H}$ and $\bar{K}_{\tau \wedge H} - \bar{K}_t \geq 0$, the comparison principle for BSDEs with jumps (combine, e.g., the proofs of Theorem 4.2 of [12] and Theorem 5.1

of [4]) yields

$$\bar{Y}^{s,\omega}_t(L,\mathbf{H},R) \geq Y^{s,\omega}_t(L,\tau \wedge \mathbf{H},R_\tau), \quad s \leq t \leq T, \qquad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Consequently, by Lemma 5.2,

(5.8)
$$\mathbb{E}_{s,\omega}[\bar{Y}_s] \ge \sup_{\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})} \overline{\mathcal{E}}_{s,\omega}^L[R_{\tau \wedge \mathbf{H}}].$$

Next, consider the (optimal) stopping time $\tau^* := \tau_{s,\omega;H}^*$. Since $\bar{K} = 0$ on $[\![s,\tau^*]\!]$, $\tau^* \leq H$, and $\bar{Y}_{\tau^*} = R_{\tau^*}$, we have

$$\bar{Y}^{s,\omega}_{\cdot\wedge\tau^*}(L,\mathbf{H},R) = Y^{s,\omega}(L,\tau^*\wedge\mathbf{H},R_{\tau^*\wedge\mathbf{H}}).$$

Thus, by Lemma 5.2, $\mathbb{E}_{s,\omega}[\bar{Y}_s] = \overline{\mathcal{E}}_{s,\omega}^L[R_{\tau^* \wedge H}]$. Together with (5.8), we get (5.7).

Lemma 5.5. If $H \in \mathcal{H}_{s,\omega}$, then for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$\bar{Y}^{s,\omega}_{\tau^*_{s,\omega;\mathrm{H}}}(\tilde{\omega};L,\mathrm{H},R) = \sup_{\tau \in \mathcal{T}_{\tau^*_{s,\omega;\mathrm{H}}(\tilde{\omega})}(\mathbb{F}^{\tau^*_{s,\omega;\mathrm{H}},\tilde{\omega}})} \overline{\mathcal{E}}^L_{\tau^*_{s,\omega;\mathrm{H}}(\tilde{\omega})}[R_{\tau \wedge \mathrm{H}}].$$

Proof. Write τ^* instead of $\tau^*_{s,\omega;H}(\tilde{\omega})$. Let $\tilde{\tau}$ be a stopping time belonging to $\mathcal{T}_s(\mathbb{F}^0_+)$ such that $\tilde{\tau} = \tau^*$, $\mathbb{P}_{s,\omega}$ -a.s. Note that $\tilde{\tau} \leq H$, $\mathbb{P}_{s,\omega}$ -a.s. Then, for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$\begin{split} \bar{Y}^{s,\omega}_{\tau^*}(\tilde{\omega};L,\mathbf{H},R) &= \bar{Y}^{s,\omega}_{\tilde{\tau}}(\tilde{\omega};\mathbf{L},\mathbf{H},R) \\ &= \mathbb{E}_{s,\omega}[\bar{Y}^{s,\omega}_{\tilde{\tau}}(L,\mathbf{H},R)|\mathcal{F}^0_{\tilde{\tau}+}](\tilde{\omega}) \\ &= \mathbb{E}_{\tilde{\tau},\tilde{\omega}}[\bar{Y}^{s,\omega}_{\tilde{\tau}(\tilde{\omega})}(L,\mathbf{H},R)] \qquad \text{by (A.2) and Lemma A.5} \\ &= \mathbb{E}_{\tilde{\tau},\tilde{\omega}}[\bar{Y}^{\tilde{\tau},\tilde{\omega}}_{\tilde{\tau}(\tilde{\omega})}(L,\mathbf{H},R)] \qquad \text{by Lemma 5.3} \\ &= \sup_{\tau \in \mathcal{T}_{\tilde{\tau}(\tilde{\omega})}(\mathbb{F}^{\tilde{\tau},\tilde{\omega}})} \overline{\mathcal{E}}^L_{\tilde{\tau},\tilde{\omega}}[R_{\tau \wedge H}] \qquad \text{by Lemma 5.4} \\ &= \sup_{\tau \in \mathcal{T}_{\tau^*(\tilde{\omega})}(\mathbb{F}^{\tau^*,\tilde{\omega}})} \overline{\mathcal{E}}^L_{\tau^*,\tilde{\omega}}[R_{\tau \wedge H}]. \end{split}$$

This concludes the proof.

5.3. **Partial Comparison.** We will need the following modification of the partial comparison principle. Theorem 3.9 can be proven similarly. In order to formulate our result we need the following definition. It might be helpful to recall Definition 3.16.

Definition 5.6. Fix $t \in [0,T)$. The space $\bar{C}_b^{1,2}(\bar{\Lambda}^t)$ is the set of all universally measurable functionals $u: \bar{\Lambda}^t \to \mathbb{R}$ for which there exist a sequence $(\tau_n)_{n \in \mathbb{N}_0}$ of stopping times in \mathcal{H}_t and a collection of functions

$$(\vartheta_n(\pi_n;\cdot))_{n\in\mathbb{N},\pi_n\in\Pi_n^t}$$

on $[t,T] \times \mathbb{R}^d$ such that the following holds:

- (i) The sequence (τ_n) is nondecreasing, $\tau_0 = t$, $\tau_n < \tau_{n+1}$ if $\tau_n < T$, and, for every $\omega \in \Omega$, there exists an $m \in \mathbb{N}$ such that $\tau_m(\omega) = T$.
- (ii) For every $n \in \mathbb{N}$, $\vartheta_n = \vartheta_n(\pi_n; t, x)$ is universally measurable and, for every $\pi_n = (s_i, y_i)_{0 \le i \le n-1}$, the function $\vartheta_n(\pi_n; \cdot)$ is continuous on $[s_{n-1}, T] \times \mathbb{R}^d$ and belongs to $C_b^{1,2}([s_{n-1}, T) \times O_{\varepsilon}(y_{n-1}))$ for some $\varepsilon > 0$.
- (iii) For every $n \in \mathbb{N}$ and $\omega \in \Omega$.

$$\vartheta_n((\tau_i(\omega), X_{\tau_i}(\omega))_{0 \le i \le n-1}; \tau_n(\omega), X_{\tau_n}(\omega)) =$$

$$\vartheta_{n+1}((\tau_i(\omega), X_{\tau_i}(\omega))_{0 \le i \le n}; \tau_n(\omega), X_{\tau_n}(\omega)).\mathbf{1}_{\{\tau_n < T\}}(\omega) + u(T, \omega).\mathbf{1}_{\{\tau_n = T\}}(\omega)$$

(iv) We have the representation

$$u(s,\omega) = \sum_{n\geq 1} \vartheta_n((\tau_i(\omega), X_{\tau_i}(\omega))_{0\leq i\leq n-1}; s, \omega_s). \mathbf{1}_{\llbracket \tau_{n-1}, \tau_n \rrbracket}(s,\omega) + u(T,\omega). \mathbf{1}_{\{T\}}(s).$$

Theorem 5.7 (Partial Comparison II). Fix $(s, \omega) \in \Lambda$. Let u^1 be a viscosity subsolution of (1.1) on Λ^s . Let $u^2 \in \bar{C}_b^{1,2}(\bar{\Lambda}^s)$ with a corresponding sequence (τ_n) of stopping times and a corresponding collection (ϑ_n) of functionals such that, for every $n \in \mathbb{N}$ and every $(r, \tilde{\omega}) \in [\tau_{n-1}, \tau_n[]$, we have, with $\pi_n = (H_i^{t,\varepsilon}(\tilde{\omega}), X_{H_i^{t,\varepsilon}}(\tilde{\omega}))_{0 \le i \le n-1}$,

$$-\partial_{t}\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}) - \sum_{i=1}^{d}b_{r}^{i}(\tilde{\omega})\partial_{x^{i}}\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}) - \frac{1}{2}\sum_{i,j=1}^{d}c_{r}^{ij}(\tilde{\omega})\partial_{x^{i}x^{j}}^{2}\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r})$$

$$-\int_{\mathbb{R}^{d}}\left[\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}+z) - \vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}) - \sum_{i=1}^{d}z^{i}\partial_{x^{i}}\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r})\right]K_{r}(\tilde{\omega},dz)$$

$$-f_{r}\left(\tilde{\omega},\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}),\partial_{x}\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}),$$

$$\int_{\mathbb{R}^{d}}\left[\vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r}+z) - \vartheta_{n}(\pi_{n};r,\tilde{\omega}_{r})\right]\eta_{r}(\tilde{\omega})K_{r}(\tilde{\omega},dz)\right) \geq 0.$$

Suppose that $u_T^1 \leq u_T^2$, $\mathbb{P}_{s,\omega}$ -a.s. Then $u^1(s,\omega) \leq u^2(s,\omega)$.

Proof. Our proof follows the approach of Lemma 5.7 in [17]. However, due to our more general setting and different definition of $\bar{C}_b^{1,2}$, details are somewhat more involved and therefore we shall give a complete proof.

Without loss of generality, let s=0 and let f be nonincreasing in y (cf. Remark 3.9 in [17]). For the sake of a contradiction, assume that

$$c := (u^1 - u^2)(0, \omega) > 0.$$

Define a process $R: \bar{\Lambda} \to \mathbb{R}$ by

$$R_t := (u^1 - u^2)_t + \frac{ct}{2T}.$$

Note that R is $\mathbb{P}_{t,\tilde{\omega}}$ -quasi-left-continuous for every $(t,\tilde{\omega}) \in \bar{\Lambda}$, bounded, right-continuous, and \mathbb{F} -adapted. Thus the results about RBSDEs and optimal stopping in this section are applicable. Put

$$H := \inf\{t \ge 0 : (u^1 - u^2)_t \le 0\}.$$

Clearly, $H \leq T$, $\mathbb{P}_{0,\omega}$ -a.s. Put

$$\bar{Y} := \bar{Y}^{0,\omega}(L, H, R)$$
 and $\tau^* := \inf\{t \ge 0 : \bar{Y}_t = R_{t \land H}\}.$

Since $\bar{Y}_H = R_H$, we have $\tau^* \leq H$. Note that $\overline{\mathcal{E}}_{0,\omega}^L[R_H] \leq c/2$, but, by Lemma 5.4,

$$\overline{\mathcal{E}}_{0,\omega}^{L}[R_{\tau^*}] = \mathbb{E}_{0,\omega}[\bar{Y}_0] \le \mathbb{E}_{0,\omega}[R_0] = c.$$

Thus, $\mathbb{P}_{0,\omega}(\tau^* < H) > 0$. Consequently, there exists an $\omega^* \in \Omega$ such that $t^* := \tau^*(\omega^*) < H(\omega^*)$, that $(t^*, \omega^*) \in \llbracket \tau_{n-1}, \tau_n \llbracket$ for some $n \in \mathbb{N}$, and that, according to Lemma 5.5 and Lemma 2.9, from which we get the existence of a hitting time $\tilde{H} \in \mathcal{H}_{t^*}$ with $\tilde{H} > t^*$, $\tilde{H}(\tilde{\omega}) = H(\tilde{\omega})$, and $\tilde{H} = H$, $\mathbb{P}_{0,\omega}$ -a.s., we have

$$R_{t^*}(\omega^*) = \sup_{\tau \in \mathcal{T}_{t^*}(\mathbb{F}^{t^*,\omega^*})} \overline{\mathcal{E}}_{t^*,\omega^*}^L[R_{\tau \wedge \tilde{\mathbf{H}}}].$$

Next, define a process $\tilde{R}: \bar{\Lambda}^{t^*} \to \mathbb{R}$ by $\tilde{R}_t := R_t - R_{t^*}(\omega^*)$. Then

$$0 = \tilde{R}_{t^*}(\omega^*) = \sup_{\tau \in \mathcal{T}_{t^*}(\mathbb{F}^{t^*,\omega^*})} \overline{\mathcal{E}}_{t^*,\omega^*}^L[\tilde{R}_{\tau \wedge \tilde{\mathbf{H}}}].$$

Our goal is to establish a decomposition $\tilde{R} = u^1 - \varphi$ such that $\varphi \in C_b^0(\bar{\Lambda}^{t^*}) \cap C_b^{1,2}(\llbracket t^*, \tau \rrbracket)$ for some stopping time $\tau \geq t^*$. Since

$$\tilde{R}_t = u_t^1 - \left[u_t^2 + (u^1 - u^2)(t^*, \omega^*) - \frac{c(t - t^*)}{2T} \right],$$

we get such a decomposition with $\tau = \tau_n$ by setting $\varphi := u^1 - \tilde{R}$. Then $\varphi \in \underline{\mathcal{A}}^L u^1(t^*, \omega^*)$, and since $(u^1 - u^2)(t^*, \omega^*) \geq 0$ (, which follows from $t^* < \tilde{\mathbf{H}}(\omega^*)$), we have

$$0 \geq \mathcal{L}\varphi(t^{*}, \omega^{*}) - f_{t^{*}}(\omega^{*}, \varphi(t^{*}, \omega^{*}), \partial_{\omega}\varphi(t^{*}, \omega^{*}), \mathcal{I}\varphi(t^{*}, \omega^{*}))$$

$$= \mathcal{L}u^{2}(t^{*}, \omega^{*}) + \frac{c}{2T} - f_{t^{*}}(\omega^{*}, \varphi(t^{*}, \omega^{*}), \partial_{\omega}u^{2}(t^{*}, \omega^{*}), \mathcal{I}u^{2}(t^{*}, \omega^{*}))$$

$$> \mathcal{L}u^{2}(t^{*}, \omega^{*}) - f_{t^{*}}(\omega^{*}, u^{2}(t^{*}, \omega^{*}), \partial_{\omega}u^{2}(t^{*}, \omega^{*}), \mathcal{I}u^{2}(t^{*}, \omega^{*})).$$

But this is a contradiction to u^2 being a classical supersolution.

5.4. Stability.

Proof of Theorem 3.10. Let $\varepsilon' > 0$ be undetermined for the moment. Assume that u is not a viscosity L-supersolution of (1.1). Then there exist $(s_0, \omega) \in \Lambda$ and $\varphi \in \overline{\mathcal{A}}^L u(s_0, \omega)$ such that

$$c' := \mathcal{L}\varphi(s_0, \omega) - f_{s_0}(\omega, \varphi(s_0, \omega), \partial_\omega \varphi(s_0, \omega), \mathcal{I}\varphi(s_0, \omega)) < 0.$$

Without loss of generality, $s_0 = 0$. Next, define processes R, $R^{\varepsilon} : \bar{\Lambda} \to \mathbb{R}$, $\varepsilon > 0$, by

$$R_t := \varphi_t - u_t - \varepsilon' t, \quad R_t^{\varepsilon} := \varphi_t - u_t^{\varepsilon} - \varepsilon' t.$$

Also, put

$$\tau_1 := \inf \left\{ t \ge 0 : \mathcal{L}\varphi_t - f_t(X, \varphi_t, \partial_\omega \varphi_t, \mathcal{I}\varphi_t) \ge \frac{c'}{2} \right\}.$$

Note that $\tau_1 \in \mathcal{H}_0$ with $\tau_1 > 0$, $\mathbb{P}_{0,\omega}$ -a.s. Thus, since $\varphi \in \overline{\mathcal{A}}^L u(0,\omega)$, there exists an $\mathbf{H} \in \mathcal{H}_{0,\omega}$ with $\mathbf{H} > 0$, $\mathbb{P}_{0,\omega}$ -a.s., such that $0 = R_0 > \overline{\mathcal{E}}^L_{0,\omega}[R_{\tau_1 \wedge \mathbf{H}}]$ because we have $\overline{\mathcal{E}}^L_{0,\omega}[(-\varepsilon')(\tau_1 \wedge \mathbf{H})] < 0$ for otherwise $\overline{\mathcal{E}}^L_{0,\omega}[(-\varepsilon')(\tau_1 \wedge \mathbf{H})] = 0 = \overline{\mathcal{E}}^L_{0,\omega}[0]$ would, by the comparison principle for BSDEs with jumps (cf. Theorem 3.2.1 in [14]) together with Lemma 5.2, imply that $(-\varepsilon')(\tau_1 \wedge \mathbf{H}) = 0$, $\mathbb{P}_{0,\omega}$ -a.s., which is a contradiction. Now, let ε sufficiently small so that $R_0^{\varepsilon} > \overline{\mathcal{E}}^L_{0,\omega}[R_{\tau_1 \wedge \mathbf{H}}^{\varepsilon}]$. Put

$$\tau_2:=\tau_1\wedge \mathbf{H},\quad \bar{Y}^\varepsilon:=\bar{Y}^{0,\omega}(L,\tau_2,R^\varepsilon),\quad \tau^\varepsilon:=\inf\{t\geq 0: \bar{Y}^\varepsilon_t=R^\varepsilon_{t\wedge\tau_2}\},$$

where we used the notation of Subsection 5 for RBSDEs. Then, $\mathbb{P}_{0,\omega}(\tau^{\varepsilon} < \tau_2) > 0$ because otherwise, by Lemma 5.4, $R_0^{\varepsilon} \leq \bar{Y}_0^{\varepsilon} = \overline{\mathcal{E}}_{0,\omega}^L[R_{\tau_2}^{\varepsilon}] < R_0^{\varepsilon}$. That is, there exists an $\omega^{\varepsilon} \in \Omega$ such that $t^{\varepsilon} := \tau^{\varepsilon}(\omega^{\varepsilon}) < \tau_2(\omega^{\varepsilon})$, that $\tau_2 \in \mathcal{H}_{t^{\varepsilon}}$ with $\tau_2 > t^{\varepsilon}$, $\mathbb{P}_{t^{\varepsilon},\omega^{\varepsilon}}$ -a.s. (, which is possible by Lemma 2.9), and that

$$R_{t^{\varepsilon}}^{\varepsilon}(\omega^{\varepsilon}) = \sup_{\tau \in \mathcal{T}_{t^{\varepsilon}}(\mathbb{F}^{t^{\varepsilon},\omega^{\varepsilon}})} \overline{\mathcal{E}}_{0,\omega}^{L} \left[R_{\tau \wedge \tau_{2}}^{\varepsilon} \right].$$

Define $\tilde{R}^{\varepsilon}: \bar{\Lambda}^{t^{\varepsilon}} \to \mathbb{R}$ by $\tilde{R}^{\varepsilon}_t := R^{\varepsilon}_t - R_{t^{\varepsilon}}(\omega^{\varepsilon})$. Then,

$$\tilde{R}_t^{\varepsilon} = \varphi_t - \varepsilon'(t - t^{\varepsilon}) - u_t^{\varepsilon} - [\varphi(t^{\varepsilon}, \omega^{\varepsilon}) - u(t^{\varepsilon}, \omega^{\varepsilon})]$$

and, with $\varphi_t^{\varepsilon} := \varphi_t - \varepsilon'(t - t^{\varepsilon}) - (\varphi - u)(t^{\varepsilon}, \omega^{\varepsilon}), t \in [t^{\varepsilon}, T]$, we have

$$0 = (\varphi^{\varepsilon} - u^{\varepsilon})(t^{\varepsilon}, \omega^{\varepsilon}) = \sup_{\tau \in \mathcal{T}_{t^{\varepsilon}}(\mathbb{F}^{t^{\varepsilon}, \omega^{\varepsilon}})} \overline{\mathcal{E}}_{t^{\varepsilon}, \omega^{\varepsilon}}^{L} \left[(\varphi^{\varepsilon} - u^{\varepsilon})_{\tau \wedge \tau_{2}} \right].$$

That is, $\varphi^{\varepsilon} \in \overline{\mathcal{A}}^{L} u^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon})$, and thus

$$\begin{split} 0 & \leq \mathcal{L}^{\varepsilon} \varphi^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon}) - f_{t^{\varepsilon}}^{\varepsilon}(\omega^{\varepsilon}, \varphi^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I}^{\varepsilon} \varphi^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon})) \\ & = \mathcal{L} \varphi(t^{\varepsilon}, \omega^{\varepsilon}) + \varepsilon' - \left[f_{t^{\varepsilon}}^{\varepsilon}(\omega^{\varepsilon}, u^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I}^{\varepsilon} \varphi(t^{\varepsilon}, \omega^{\varepsilon})) \right. \\ & \qquad - f_{t^{\varepsilon}}(\omega^{\varepsilon}, u(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I} \varphi(t^{\varepsilon}, \omega^{\varepsilon})) \right] \\ & \qquad - f_{t^{\varepsilon}}(\omega^{\varepsilon}, u(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I} \varphi(t^{\varepsilon}, \omega^{\varepsilon})) \\ & \leq \frac{c'}{2} + \varepsilon' - \left[f_{t^{\varepsilon}}^{\varepsilon}(\omega^{\varepsilon}, u^{\varepsilon}(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I}^{\varepsilon} \varphi(t^{\varepsilon}, \omega^{\varepsilon})) \right. \\ & \qquad - f_{t^{\varepsilon}}(\omega^{\varepsilon}, u(t^{\varepsilon}, \omega^{\varepsilon}), \partial_{\omega} \varphi(t^{\varepsilon}, \omega^{\varepsilon}), \mathcal{I} \varphi(t^{\varepsilon}, \omega^{\varepsilon})) \right]. \end{split}$$

Hence, for $\varepsilon' = -c'/8$ and for sufficiently small ε , uniform convergence of f^{ε} , u^{ε} , η^{ε} , and K^{ε} yields $0 \le c'/4 < 0$, which is a contradiction.

6. Comparison

In this subsection, Assumption 3.11, Assumption 3.13, Assumption 3.15, Assumption 3.14, and Assumption 3.17 are in force. Furthermore, without loss of generality, assume that f is nonincreasing in y (cf. Remark 3.9 in [17]).

Given
$$\delta \in (0, \infty]$$
, $t \in (-\infty, T]$, and $y \in \mathbb{R}^d$, put
$$O_{\delta}(y) := \{x \in \mathbb{R}^d : |x - y| < \delta\},$$

$$\bar{O}_{\delta}(y) := \{x \in \mathbb{R}^d : |x - y| \le \delta\},$$

$$\partial O_{\delta}(y) := \{x \in \mathbb{R}^d : |x - y| = \delta\},$$

$$Q_{t,y}^{\delta} := (t, T) \times O_{\delta}(y),$$

$$\bar{Q}_{t,y}^{\delta} := [t, T] \times \bar{O}_{\delta}(y),$$

$$\partial Q_{t,y}^{\delta} := ((t, T] \times \partial O_{\delta}(y)) \cup (\{T\} \times O_{\delta}(y)),$$

$$\partial \bar{Q}_{t,y}^{\delta} := ([t, T] \times \partial O_{\delta}(y)) \cup (\{T\} \times O_{\delta}(y)),$$

$$Q_{t}^{\infty} := (t, T) \times \mathbb{R}^d,$$

$$\bar{Q}_{t}^{\infty} := [t, T] \times \mathbb{R}^d.$$

6.1. Hitting times. Given $\varepsilon > 0$, $t \in [0,T]$, and $y \in \mathbb{R}^d$, define hitting times

$$\begin{split} \mathrm{H}_0^{t,y,\varepsilon} &:= t, \\ \mathrm{H}_1^{t,y,\varepsilon} &:= \inf\{s \geq t : X_s \not\in O_\varepsilon(y)\} \wedge T, \\ \mathrm{H}_{i+1}^{t,y,\varepsilon} &:= \inf\{s \geq \mathrm{H}_i^{t,y,\varepsilon} : \left|X_s - X_{\mathrm{H}_i^{t,y,\varepsilon}}\right| \geq \varepsilon\} \wedge T. \end{split}$$
 Also put $\mathrm{H}_j^{t,\varepsilon} := \mathrm{H}_j^{t,X_t^t,\varepsilon}, \text{ that is,} \\ \mathrm{H}_0^{t,\varepsilon} &:= t, \\ \mathrm{H}_1^{t,\varepsilon} &= \inf\{s \geq t : |X_s - X_t| \geq \varepsilon\} \wedge T, \\ \mathrm{H}_1^{t,\varepsilon} &= \inf\{s \geq \mathrm{H}_j^{t,\varepsilon} : \left|X_s - X_{\mathrm{H}_i^t,\varepsilon}\right| \geq \varepsilon\} \wedge T. \end{split}$

Lemma 6.1. Let $(G_n)_n$ be an increasing sequence of non-empty open subsets of \mathbb{R}^d with $G = \bigcup_n G_n$. Let Y be a d-dimensional, càdlàg, and \mathbb{F}_+^0 -adapted process that is \mathbb{P} -quasi-left-continuous for some probability measure \mathbb{P} on $(\Omega, \mathcal{F}_0^1)$. Suppose that $Y_0 \in G_1$, \mathbb{P} -a.s. Consider the first-exit times

(6.1)
$$\tau_G := \inf\{t \ge 0 : Y_t \in G^c\} \wedge T,$$

(6.2)
$$\tau_{G_n} := \inf\{t \ge 0 : Y_t \in G_n^c\} \land T, \quad n \in \mathbb{N}.$$

Then $\lim_n \tau_{G_n} = \tau_G$, \mathbb{P} -a.s. Moreover, $\lim_n Y_{\tau_{G_n}} = Y_{\tau_G}$, \mathbb{P} -a.s.

Proof. Without loss of generality, we can assume that that τ_G and τ_{G_n} , $n \in \mathbb{N}$, are \mathbb{F}^0_+ -stopping times. Clearly, $(\tau_{G_n})_n$ is increasing and

$$\tau := \sup_{n} \tau_{G_n} \le \tau_G.$$

Consider the sets $A := \{ \tau < \tau_G \}$ and $A_n := \{ \tau_{G_n} < \tau_G \}, n \in \mathbb{N}$. We have to show that $\mathbb{P}(A) = 0$.

Let us first note that, for every $n \in \mathbb{N}$, we have $\tau_{G_n} < \tau$ on A because otherwise there exists an $\omega \in A$ and an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\tau_{G_n}(\omega) = \tau(\omega)$, which yields $Y_{\tau}(\omega) \in \cap_{n \geq n_0} G_n^c = G^c$, i.e., $\tau(\omega) \geq \tau_G(\omega)$ and thus $\omega \notin A$. Consequently, the stopping times

$$\tilde{\tau} := \tau \cdot \mathbf{1}_A + T \cdot \mathbf{1}_{A^c},$$

$$\tilde{\tau}_{G_n} := \left(\tau_{G_n} \cdot \mathbf{1}_{A_n} + T \cdot \mathbf{1}_{A_n^c}\right) \wedge [T(1 - n^{-1})], \quad n \in \mathbb{N},$$

satisfy the following: For every $n \in \mathbb{N}$, $\tilde{\tau}_{G_n} < \tilde{\tau}$, i.e., $\tilde{\tau}$ is \mathbb{F}^0_+ -predictable. Thus, using \mathbb{P} -quasi-left-continuity of Y, we get

$$(6.3) \qquad \mathbb{P}(\tilde{\tau} < T) = \mathbb{P}(\{\Delta Y_{\tilde{\tau}} = 0\} \cap \{\tilde{\tau} < T\}) = \mathbb{P}(\{\Delta Y_{\tau} = 0\} \cap A).$$

Next, note that $|\Delta Y_{\tau}| > 0$ on A because otherwise there exists an $\omega \in A$ such that $\Delta Y_{\tau}(\omega) = 0$ and then, since, for every $m \in \mathbb{N}$, $\{Y_{\tau_{G_n}}(\omega)\}_{n \geq m}$ takes values in G_m^c and G_m^c is closed, we get $Y_{\tau}(\omega) \in \cap_m G_m^c = G^c$, i.e., $\tau(\omega) \geq \tau_G(\omega)$ and thus $\omega \notin A$. Therefore, $\mathbb{P}(\{\Delta Y_{\tau} = 0\} \cap A) = 0$ and, together with (6.3), we get $\mathbb{P}(A) = 0$.

I.e., we have shown that $\lim_n \tau_{G_n} = \tau_G$, \mathbb{P} -a.s. Hence, by Proposition I.2.26 of [25], $\lim_n Y_{\tau_{G_n}} = Y_{\tau_G}$, \mathbb{P} -a.s.

In the following statement and its proof, the random times τ_G and τ_{G_n} , $n \in \mathbb{N}$, resp., are defined by (6.1) and (6.2), resp. Also, we do not need the full strength of Assumption 3.13, in particular, (B, C, ν) is allowed to be random; only Part (iii) of Assumption 3.13 is actually used.

Lemma 6.2. Let $(G_n)_n$ be an increasing sequence of open connected subsets of \mathbb{R}^d with $G = \bigcup_n G_n$. Let H be an open subset of \mathbb{R}^d with $G \subseteq H$. Put $Q_n := [0,T) \times G_n$, $n \in \mathbb{N}$, $Q := [0,T) \times G$, and $R := [0,T) \times H$. Suppose that there exists an $\varepsilon \in (0,1)$ such that, for all $n \in \mathbb{N}$,

$$\operatorname{dist}(G_{n+1}^c, G_n) \le \frac{\varepsilon}{n(n+1)}.$$

Let $v \in C(\bar{R}) \cap C^{1,2}(Q)$ and $x \in G_1$. Then there exists an $(\mathbb{F}^0_+, \mathbb{P}_{0,x})$ martingale M such that

$$\begin{split} v(\tau_G, X_{\tau_G}) - v(0, x) &= \int_0^{\tau_G} \left\{ \partial_t v(t, X_t) + \sum_{i=1}^d b_t^i \partial_{x^i} v(t, X_t) \right. \\ &+ \frac{1}{2} \sum_{i,j=1}^d c_t^{ij} \partial_{x^i x^j}^2 v(t, X_t) \\ &+ \int_{|z| \le C_0'} \left[v(t, X_t + z) - v(t, X_t) - \sum_{i=1}^d z^i \partial_{x^i} v(t, X_t) \right] \, K_t(dz) \right\} dt \\ &+ M_{\tau_G}, \quad \mathbb{P}_{0, x} \text{-} a.s. \end{split}$$

Proof. Let $(a_n)_n$ be a sequence of positive real numbers converging to 0. For every $n \in \mathbb{N}$, let $v^n \in C^{1,2}(\bar{R})$ such that $v = v^n$ on Q_n and $|v - v^n|_R \leq a_n$. By Lemma 6.1, $v(\tau_G, X_{\tau_G}) = \lim_n v^{n+1}(\tau_{G_n}, X_{\tau_{G_n}})$, $\mathbb{P}_{0,x}$ -a.s. Moreover, for every $n \in \mathbb{N}$, there exists, by Itô's formula an $(\mathbb{F}^0_+, \mathbb{P}_{0,x})$ -martingale M^n such that

$$\begin{split} v^{n+1}(\tau_{G_n}, X_{\tau_{G_n}}) - v(0, x) &= \int_0^{\tau_{G_n}} \left\{ \partial_t v(t, X_t) + \sum_{i=1}^d b_t^i \partial_{x^i} v(t, X_t) \right. \\ &+ \frac{1}{2} \sum_{i,j=1}^d c_t^{ij} \partial_{x^i x^j}^2 v(t, X_t) \\ &+ \int_{|z| \leq C_0'} \left[v(t, X_t + z) . \mathbf{1}_{\left\{ |z| \leq \frac{\varepsilon}{n(n+1)} \right\}} + v^{n+1}(t, X_t + z) . \mathbf{1}_{\left\{ |z| > \frac{\varepsilon}{n(n+1)} \right\}} \right. \\ &- v(t, X_t) - \sum_{i=1}^d z^i \partial_{x^i} v(t, X_t) \right] K_t(dz) \right\} dt \\ &+ M_{\tau_{G_n}}^n, \quad \mathbb{P}_{0, x}\text{-a.s.} \end{split}$$

Now, if

$$a_{n+1} = \frac{1}{(L_{1/n} \vee n)(n+1)}, \quad n \in \mathbb{N},$$

then, by Part (iii) of Assumption 3.13,

$$\begin{split} & \left| \int_{0}^{\tau_{G_{n}}} \int_{\frac{\varepsilon}{n(n+1)} < |z| \le C_{0}'} [v^{n+1}(t, X_{t} + z) - v(t, X_{t} + z)] K_{t}(dz) dt \right| \\ & \leq \int_{0}^{\tau_{G_{n}}} \int_{\frac{\varepsilon}{n(n+1)} < |z| \le C_{0}'} a_{n+1} K_{t}(dz) dt \\ & \leq T \left[\int_{\frac{\varepsilon}{n(n+1)} < |z| \le C_{0}'} a_{n+1} K_{t,1,1/n}(dz) + \int_{\frac{\varepsilon}{n(n+1)} < |z| \le C_{0}'} a_{n+1} K_{t,2,1/n}(dz) \right] \\ & \leq T \left[\frac{n(n+1)a_{n+1}}{\varepsilon} \right) \int_{\frac{\varepsilon}{n(n+1)} < |z| \le C_{0}'} |z| K_{t,1,1/n}(dz) + a_{n+1}(L_{1/n} \vee n) \right] \\ & \leq T \left[\frac{1}{(L_{1/n} \vee n)\varepsilon} + \frac{1}{n+1} \right] \to 0 \end{split}$$

as $n \to \infty$. This concludes the proof.

Remark 6.3. If (b, c, K) is constant, $t \in [0, T]$, and $x \in \mathbb{R}^d$, then

$$\mathbb{E}_{t,x}[\mathbf{H}_1^{t,\varepsilon}] = \mathbb{E}_{0,x}[t + [\mathbf{H}_1^{\varepsilon} \wedge (T-t)]].$$

Indeed, using $(a_1 \wedge a_2) - t = (a_1 - t) \wedge (a_2 - t)$, we get

$$\begin{split} \mathbb{E}_{t,x}[(\mathbf{H}_1^{t,\varepsilon}-t] &= \mathbb{E}_{t,x}[(\inf\{s\geq t: |X_s-X_t|\geq \varepsilon\}\wedge T)-t]\\ &= \mathbb{E}_{0,x}[(\inf\{s\geq t: |X_{s-t}-X_0|\geq \varepsilon\}\wedge T)-t]\\ &= \mathbb{E}_{0,x}[((t+\inf\{r\geq 0: |X_r-X_0|\geq \varepsilon\})\wedge T)-t]\\ &= \mathbb{E}_{0,x}[\inf\{r\geq 0: |X_r-X_0|\geq \varepsilon\}\wedge (T-t)]\\ &= \mathbb{E}_{0,x}[\mathbf{H}_1^\varepsilon\wedge (T-t)]. \end{split}$$

Proposition 6.4. Fix $(s, \omega) \in \bar{\Lambda}$ and $\varepsilon > 0$. Then the map $x \mapsto H_1^{s,x,\varepsilon}(\omega)$, $\mathbb{R}^d \to [s,T]$, is universally measurable.

Proof. Note that we can express $x \mapsto H_1^{s,x,\varepsilon}(\omega)$ as the debùt of the set

$$A := \{(t, x) \in [s, T] \times \mathbb{R}^d : |\omega_t - x| \ge \varepsilon\} \cup \{(T, x) : x \in \mathbb{R}^d\},\$$

i.e., $H_1^{s,x,\varepsilon}(\omega) = D_A(x)$, where $D_A : \mathbb{R}^d \mapsto [0,T]$ is defined by

$$D_A(x) := \inf\{t \in [s, T] : (t, x) \in A\}.$$

Since $A \in \mathcal{B}([s,T]) \otimes \mathcal{B}(\mathbb{R}^d)$ as inverse image of a Borel set under a Borel measurable map, we can deduce from Theorem III.44 in [13] that D_A is universally measurable.

Remark 6.5. If d=1, then $\operatorname{H}_1^{t,x,\varepsilon}$ can be written as infimum of first-passage times that are monotone in x and thus $x\mapsto \operatorname{H}_1^{s,x,\varepsilon}(\omega)$ is even Borel measurable.

Lemma 6.6 (Shifting of hitting times). Let $\varepsilon > 0$, $t \in [0,T]$, $y \in \mathbb{R}$, $\omega \in \Omega$, and $s = \mathrm{H}_1^{t,y,\varepsilon}(\omega)$. Then $\mathrm{H}_{i+1}^{t,y,\varepsilon} = \mathrm{H}_i^{s,\omega_s,\varepsilon}$, $i \in \mathbb{N}_0$, $\mathbb{P}_{s,\omega}$ -a.s.

Proof. The case i=0 follows from Remark A.6. Next, let i=1. Then, by Remark A.6,

$$\operatorname{H}_{2}^{t,y,\varepsilon} = \inf\{r \geq s : |X_{r} - X_{s}| \geq \varepsilon\} \wedge T = \operatorname{H}_{1}^{s,\omega_{s},\varepsilon}, \quad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Finally, assume that the statement is true for some $i \in \mathbb{N}$. Then

$$\operatorname{H}_{i+2}^{t,y,\varepsilon} = \inf\{r \geq \operatorname{H}_{i+1}^{s,\omega_s,\varepsilon} : \left| X_r - X_{\operatorname{H}_i^{t,\omega_s,\varepsilon}} \right| \geq \varepsilon\} \wedge T = \operatorname{H}_{i+1}^{s,\omega_s,\varepsilon}, \quad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Mathematical induction concludes the proof.

6.2. Regularity of path-frozen approximations. We are going to define candidate solutions for so-called path-frozen integro-differential equations by means stochastic representation. To this end fix $\varepsilon > 0$ and $(s^*, \omega^*) \in \Lambda$. Next, define a map $g^{s^*, \omega^*} = g : \Pi_{\infty}^{s^*} \to \mathbb{R}$ by

$$g(\pi_{\infty}) := \xi(\omega^*.\mathbf{1}_{[0,s^*)} + \sum_{i \in \mathbb{N}_0} x_i.\mathbf{1}_{[t_i,t_{i+1})} + x_{\infty}.\mathbf{1}_{\{T\}})$$

and a map $\tilde{f}^{s^*,\omega^*} = \tilde{f} : \mathbb{R} \times \Pi_{\infty}^{s^*} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}_t(\pi_{\infty}, y, z, p) := f_{(t \vee s^*) \wedge T}(\omega^*.\mathbf{1}_{[0, s^*)} + \sum_{i \in \mathbb{N}_0} x_i.\mathbf{1}_{[t_i, t_{i+1})} + x_{\infty}.\mathbf{1}_{\{T\}}, y, z, p).$$

Given $i \in \mathbb{N}$, $\pi_i = (s_0, y_0; \dots; s_{i-1}, y_{i-1}) \in \Pi_i^{s^*}$ with $s^* = s_0$, $s = s_{i-1}$, and $y = y_{i-1}$, denote, for every $(t, x) \in [s, T] \times \mathbb{R}^d$, by

$$\left(\tilde{Y}^{s^*,\omega^*,\varepsilon;\pi_i,t,x},\tilde{Z}^{s^*,\omega^*,\varepsilon;\pi_i,t,x},\tilde{U}^{s^*,\omega^*,\varepsilon;\pi_i,t,x}\right) = \left(\tilde{Y}^{\pi_i,t,x},\tilde{Z}^{\pi_i,t,x},\tilde{U}^{\pi_i,t,x}\right)$$

the solution of the BSDE

$$\begin{split} \tilde{Y}_{r}^{\pi_{i},t,x} &= g(\pi_{i}; (\mathbf{H}_{j}^{t,y-x,\varepsilon}, x + X_{\mathbf{H}_{j}^{t,y-x,\varepsilon}})_{j \in \mathbb{N}}); x + X_{T}) \\ &+ \int_{r}^{T} \tilde{f}_{\tilde{r}} \Big((\pi_{i}; (\mathbf{H}_{j}^{t,y-x,\varepsilon}, x + X_{\mathbf{H}_{j}^{t,y-x,\varepsilon}})_{j \in \mathbb{N}}); x + X_{T}), \\ &\qquad \qquad \tilde{Y}_{\tilde{r}}^{\pi_{i},t,x}, \tilde{Z}_{\tilde{r}}^{\pi_{i},t,x}, \int_{\mathbb{R}^{d}} \tilde{U}_{\tilde{r}}^{\pi_{i},t,x}(z) \, \eta_{\tilde{r}}(z) \, K_{\tilde{r}}(dz) \Big) \, d\tilde{r} \\ &- \int_{r}^{T} \tilde{Z}_{\tilde{r}}^{\pi_{i},t,x} \, dX_{\tilde{r}}^{c,t,0} - \int_{r}^{T} \int_{\mathbb{R}^{d}} \tilde{U}_{\tilde{r}}^{\pi_{i},t,x}(z) \, (\mu^{X} - \nu) (d\tilde{r}, dz), \, r \in [t,T], \, \mathbb{P}_{t,0}\text{-a.s.}, \\ \text{and define } \theta_{i}^{s^{*},\omega^{*},\varepsilon}(\pi_{i};\cdot) = \theta_{i}(\pi_{i};\cdot) : [s,T] \times \mathbb{R}^{d} \to \mathbb{R} \text{ by} \\ \theta_{i}(\pi_{i};t,x) := \mathbb{E}_{t} \, \mathbf{0}[\tilde{Y}_{t}^{\pi_{i};t,x}]. \end{split}$$

Lemma 6.7 (Dynamic programming). We have

$$\begin{split} \theta_i(\pi_i;t,x) &= \mathbb{E}_{t,\mathbf{0}} \Big[\theta_{i+1}(\pi_i;\mathbf{H}_1^{t,y-x,\varepsilon},x+X_{\mathbf{H}_1^{t,y-x,\varepsilon}};\mathbf{H}_1^{t,y-x,\varepsilon},x+X_{\mathbf{H}_1^{t,y-x,\varepsilon}}) \\ &+ \int_t^{\mathbf{H}_1^{t,y,\varepsilon}} \tilde{f}_r \Big((\pi_i;(\mathbf{H}_j^{t,y-x,\varepsilon},x+X_{\mathbf{H}_j^{t,y-x,\varepsilon}})_{j\in\mathbb{N}});x+X_T), \\ &\qquad \qquad \tilde{Y}_r^{\pi_i,t,x},\tilde{Z}_r^{\pi_i,t,x}, \int_{\mathbb{R}^d} \tilde{U}_r^{\pi_i,t,x}(z) \, \eta_r(z) \, K_r(dz) \Big) \, dr \Big]. \end{split}$$

If, additionally, $x \notin O_{\varepsilon}(y)$, then

$$\theta_i(\pi_i; t, x) = \theta_{i+1}(\pi_i; t, x; t, x).$$

Proof. Skip superscript ε . Put $\tau := H_1^{t,y-x}$. Since

$$\theta_{i}(\pi_{i};t,x) = \mathbb{E}_{t,0} \Big[\tilde{Y}_{\tau}^{\pi_{i},t,x} + \int_{t}^{\tau} \tilde{f}_{r} \Big((\pi_{i}; (\mathbf{H}_{j}^{t,y-x,\varepsilon}, x + X_{\mathbf{H}_{j}^{t,y-x,\varepsilon}})_{j \in \mathbb{N}}); x + X_{T}),$$

$$\tilde{Y}_{r}^{\pi_{i},t,x}, \tilde{Z}_{r}^{\pi_{i},t,x}, \int_{\mathbb{R}^{d}} \tilde{U}_{r}^{\pi_{i},t,x}(z) \, \eta_{r}(z) \, K_{r}(dz) \Big) \, dr \Big]$$

it suffices to show that

(6.4)
$$\mathbb{E}_{t,\mathbf{0}}[\tilde{Y}_{\tau}^{\pi_{i},t,x}] = \mathbb{E}_{t,\mathbf{0}}[\theta_{i+1}(\pi_{i};\mathbf{H}_{1}^{t,y-x,\varepsilon},x+X_{\mathbf{H}_{1}^{t,y-x,\varepsilon}};\mathbf{H}_{1}^{t,y,\varepsilon},x+X_{\mathbf{H}_{1}^{t,y,\varepsilon}})].$$

By Corollary A.7,

(6.5)
$$\mathbb{E}_{t,\mathbf{0}}[\tilde{Y}_{\tau}^{\pi_i,t,x}] = \int \int \tilde{Y}_{\tau(\omega)}^{\pi_i,t,x}(\tilde{\omega}) \, \mathbb{P}_{\tau,\omega}(d\tilde{\omega}) \, \mathbb{P}_{t,\mathbf{0}}(d\omega).$$

For every $\omega \in \Omega$, define a process Y^{ω} on $[\tau(\omega), T]$ by

(6.6)
$$Y_r^{\omega} := \tilde{Y}^{\pi_i, t, x}(\omega. \mathbf{1}_{[0, \tau(\omega))} + (X + \omega_{\tau(\omega)}). \mathbf{1}_{[\tau(\omega), T]}).$$

Note that, by Lemma 6.6, for $\mathbb{P}_{\tau,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$(\pi_i; (\mathbf{H}_j^{t,y-x,\varepsilon}(\tilde{\omega}), x + X_{\mathbf{H}_j^{t,y-x,\varepsilon}}(\tilde{\omega}))_{j \in \mathbb{N}}, x + \tilde{\omega}_T)$$

$$= (\pi_i; (\mathbf{H}_j^{\tau(\omega),\omega_{\tau(\omega)}}(\tilde{\omega}), x + X_{\mathbf{H}_j^{\tau(\omega),\omega_{\tau(\omega)}}}(\tilde{\omega}))_{j \in \mathbb{N}_0}, x + \tilde{\omega}_T).$$

Thus (cf. Lemma 4.1 and Lemma 4.3), for $\mathbb{P}_{t,\mathbf{0}}$ -a.e. $\omega \in \Omega$, there exists a pair (Z^{ω}, U^{ω}) such that $(Y^{\omega}, Z^{\omega}, U^{\omega})$ is the solution to the BSDE

$$Y_{r}^{\omega} = g\left(\pi_{i}; \left(\mathbf{H}_{j}^{\tau(\omega),\omega_{\tau(\omega)}}, x + \omega_{\tau(\omega)} + X_{\mathbf{H}_{j}^{\tau(\omega),\omega_{\tau(\omega)}}}\right)_{j \in \mathbb{N}_{0}}, x + \omega_{\tau(\omega)} + X_{T}\right)$$

$$+ \int_{r}^{T} \tilde{f}_{\tilde{r}}\left(\left(\pi_{i}; \left(\mathbf{H}_{j}^{\tau(\omega),\omega_{\tau(\omega)}}, x + \omega_{\tau(\omega)} + X_{\mathbf{H}_{j}^{\tau(\omega),\omega_{\tau(\omega)}}}\right)_{j \in \mathbb{N}_{0}}, x + \omega_{\tau(\omega)} + X_{T}\right),$$

$$Y_{\tilde{r}}^{\omega}, Z_{\tilde{r}}^{\omega}, \int_{\mathbb{R}^{d}} U_{\tilde{r}}^{\omega}(z) \, \eta_{\tilde{r}}(z) \, K_{\tilde{r}}(dz)\right) d\tilde{r}$$

$$+ \int_{r}^{T} Z_{\tilde{r}}^{\omega} \, dX^{c,\tau(\omega),\mathbf{0}} + \int_{r}^{T} \int_{\mathbb{R}^{d}} U_{\tilde{r}}^{\omega}(z) \, (\mu^{X} - \nu)(d\tilde{r}, dz), \, r \in [\tau(\omega), T], \, \mathbb{P}_{\tau(\omega),\mathbf{0}}\text{-a.s.}$$

Together with (6.5) and (6.6), we obtain

$$\mathbb{E}_{t,\mathbf{0}} \left[\tilde{Y}_{\tau}^{\pi_{i};t,x} \right] = \int \int Y_{\tau(\omega)}^{\omega}(\tilde{\omega}) \, \mathbb{P}_{\tau(\omega),\mathbf{0}}(d\tilde{\omega}) \, \mathbb{P}_{t,\mathbf{0}}(d\omega)
= \int \int Y_{\tau(\omega)}^{(\pi_{i};\tau(\omega),x+\omega_{\tau(\omega)}),\tau(\omega),x+\omega_{\tau(\omega)}}(\tilde{\omega}) \, \mathbb{P}_{\tau(\omega),\mathbf{0}}(d\tilde{\omega}) \, \mathbb{P}_{t,\mathbf{0}}(d\omega)
= \int \theta_{i+1}(\pi_{i};\tau(\omega),x+\omega_{\tau(\omega)};\tau(\omega),x+\omega_{\tau(\omega)}) \, \mathbb{P}_{t,\mathbf{0}}(d\omega).$$

Thus (6.4) has been established.

Fix $(s^*, \omega^*) \in \bar{\Lambda}$ and $\varepsilon \in (0, 1)$. We will write g instead of g^{s^*, ω^*} and \tilde{f} instead of $\tilde{f}^{s^*, \omega^*}$. The generic notation for an element of $\Pi_i^{s^*}$ is

$$\pi_i = (s_0, y_0; \dots; s_{i-1}, y_{i-1}), s = s_{i-1}, y = y_{i-1}.$$

Define a function $h_i^{s^*,\omega^*,\varepsilon} = h_i^{\varepsilon} : \Pi_i^{s^*} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h_i^{\varepsilon}(\pi_i;t,x) := \theta_{i+1}^{s^*,\omega^*,\varepsilon}(\pi_i;(s\vee t)\wedge T,x;(s\vee t)\wedge T,x).$$

The following result is needed for the approximation of the functions $\theta_{i+1}(\pi_i;\cdot)$ by smooth functions.

Lemma 6.8. Let $\pi_i \in \Pi_i^{s^*}$. Then $(t,x) \mapsto h_i^{\varepsilon}(\pi;t,x)$, $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, is continuous.

Remark 6.9. If ξ were only d_{J_1} -continuous, then, in contrast to corresponding mappings in [18] and [19], the mapping $(y;t,x) \mapsto \xi(y.\mathbf{1}_{[0,t)} + x.\mathbf{1}_{[t,T]})$ cannot be expected to be continuous. For example, assume that d=1 and let $(y^0;t^0,x^0):=(1;0,2)$ and $(y^n;t^n,x^n):=(1;1/n,2)$. Then $(y^n;t^n,x^n) \to (y^0;t^0,x^0)$ as $n\to\infty$, but

$$d_{J_1}(y^n.\mathbf{1}_{[0,t^n)} + x^n.\mathbf{1}_{[t^n,T]}, y^0.\mathbf{1}_{[0,t^0)} + x^0.\mathbf{1}_{[t^0,T]}) \ge 1.$$

Remark 6.10. If ξ is just d_{J_1} -continuous, then we cannot expect $(t,x) \mapsto \theta_i^{\varepsilon}(\pi_i;t,x)$ on $\overline{Q}_{s,y}^{2C'_0} \setminus Q_{s,y}^{\varepsilon}$ to be left-continuous in t. To see this, assume that d=1 and let $t^0=T$ and $t^n \uparrow t^0$ with $t^n < T$. Then, given $\omega \in \Omega$, we have, for sufficiently large n and with $y_j=0,\ j=0,\ \ldots,\ i-1$, (which we can assume without loss of generality,)

$$|g(\pi_{i}; (t^{n} + [\mathbf{H}_{j}^{\varepsilon}(\omega) \wedge (T - t^{n})], x + X_{\mathbf{H}_{j}^{\varepsilon} \wedge (T - t^{n})}(\omega))_{j \in \mathbb{N}_{0}}; x + X_{T - t^{n}}(\omega)) - g(\pi_{i}; (T, x + X_{0}(\omega))_{j \in \mathbb{N}_{0}}; x + X_{0}(\omega))|$$

$$= |\xi((x + \omega_{0}).\mathbf{1}_{[t^{n}, T]}) - \xi((x + \omega_{0}).\mathbf{1}_{\{T\}})|$$

but $d_{J_1}(z.\mathbf{1}_{[t^n,T]},z.\mathbf{1}_{\{T\}}) \ge 1$ for $z \ne 0$. However $d_{M_2}(z.\mathbf{1}_{[t^n,T]},z.\mathbf{1}_{\{T\}}) \to 0$ as $n \to \infty$.

Proof of Lemma 6.8. Let $(t^n, x_n) \to (t^0, x_0)$ in $\mathbb{R} \times \mathbb{R}^d$ as $n \to \infty$. For every $n \in \mathbb{N}$,

$$\left| h_i^{\varepsilon}(\pi_i; t^0, x_0) - h_i^{\varepsilon}(\pi_i; t^n, x_n) \right| \le \left| h_i^{\varepsilon}(\pi_i; t^0, x_0) - h_i^{\varepsilon}(\pi_i; t^0, x_n) \right| + \left| h_i^{\varepsilon}(\pi_i; t^0, x_n) - h_i^{\varepsilon}(\pi_i; t^n, x_n) \right|$$

$$=: A_n^x + A_n^t.$$

Note that, for every $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, with $t' := (t \vee s) \wedge T$, (6.7)

$$h_{i}^{\varepsilon}(\pi_{i};t,x) = \mathbb{E}_{t',\mathbf{0}} \left[g \left(\pi_{i};t',x; \left(\mathbf{H}_{j}^{t',\varepsilon}, x + X_{\mathbf{H}_{j}^{t',\varepsilon}} \right)_{j \in \mathbb{N}}; x + X_{T} \right) \right.$$

$$\left. + \int_{t'}^{T} \tilde{f}_{r} \left(\left(\pi_{i};t',x; \left(\mathbf{H}_{j}^{t',\varepsilon}, x + X_{\mathbf{H}_{j}^{t',\varepsilon}} \right)_{j \in \mathbb{N}}; x + X_{T} \right),$$

$$\tilde{Y}_{r}^{(\pi_{i};t',x);t',x}, \tilde{Z}_{r}^{(\pi_{i};t',x);t',x}, \int_{\mathbb{R}^{d}} \tilde{U}_{r}^{(\pi_{i};t',x);t',x}(z) \, \eta_{r}(z) \, K(dz) \right) dr \right]$$

and also (cf. Remark 6.3)

(6.8)

$$\mathbb{E}_{t',\mathbf{0}} \left[g \left(\pi_i; t', x; \left(\mathbf{H}_j^{t',\varepsilon}, x + X_{\mathbf{H}_j^{t',\varepsilon}} \right)_{j \in \mathbb{N}}; x + X_T \right) \right]$$

$$= \mathbb{E}_{0,\mathbf{0}} \left[g \left(\pi_i; \left(t' + \left[\mathbf{H}_j^{\varepsilon} \wedge (T - t') \right], x + X_{\mathbf{H}_j^{\varepsilon} \wedge (T - t')} \right)_{j \in \mathbb{N}_0}; x + X_{T - t'} \right) \right].$$

as well as

(6.9)

$$\mathbb{E}_{t',\mathbf{0}} \left[\int_{t'}^{T} \tilde{f}_r \left(\left(\pi_i; t', x; \left(\mathbf{H}_j^{t',\varepsilon}, x + X_{\mathbf{H}_j^{t',\varepsilon}} \right)_{j \in \mathbb{N}}; x + X_T \right), \right.$$

$$\tilde{Y}_r^{(\pi_i;t',x);t',x}, \tilde{Z}_r^{(\pi_i;t',x);t',x}, \int_{\mathbb{R}^d} \tilde{U}_r^{(\pi_i;t',x);t',x}(z) \, \eta_r(z) \, K(dz) \right) dr \right]$$

$$= \mathbb{E}_{0,\mathbf{0}} \left[\int_0^{T-t'} \tilde{f}_{r+t'} \left(\left(\pi_i; \left(t' + \left[\mathbf{H}_j^{\varepsilon} \wedge (T-t') \right], x + X_{\mathbf{H}_j^{\varepsilon} \wedge (T-t')} \right)_{j \in \mathbb{N}_0}; x + X_{T-t'} \right), \right.$$

$$\tilde{Y}_r^{t',x}, \hat{Z}_r^{t',x}, \int_{\mathbb{R}^d} \tilde{U}_r^{t',x}(z) \, \eta_r(z) \, K(dz) \right) dr \right],$$

where $(\hat{Y}^{t',x}, \hat{Z}^{t',x}, \hat{U}^{t',x})$ is the solution of the BSDE

(6.10)

$$\hat{Y}_{r}^{t',x} = g\left(\pi_{i}; \left(t' + [\mathbf{H}_{j}^{\varepsilon} \wedge (T - t')], x + X_{\mathbf{H}_{j}^{\varepsilon} \wedge (T - t')}\right)_{j \in \mathbb{N}_{0}}; x + X_{T - t'}\right) \\
+ \int_{r}^{T - t'} \tilde{f}_{\tilde{r} + t'} \left(\left(\pi_{i}; \left(t' + [\mathbf{H}_{j}^{\varepsilon} \wedge (T - t')], x + X_{\mathbf{H}_{j}^{\varepsilon} \wedge (T - t')}\right)_{j \in \mathbb{N}_{0}}; x + X_{T - t'}\right), \\
\hat{Y}_{\tilde{r}}^{t',x}, \hat{Z}_{\tilde{r}}^{t',x}, \int_{\mathbb{R}^{d}} \hat{U}_{\tilde{r}}^{t',x}(z) \, \eta_{\tilde{r}}(z) \, K(dz)\right) d\tilde{r} \\
- \int_{r}^{T - t'} \hat{Z}_{\tilde{r}}^{t',x} \, dX_{\tilde{r}}^{c,0,0} - \int_{r}^{T - t'} \int_{\mathbb{R}^{d}} \hat{U}_{\tilde{r}}^{t',x}(z) \, (\mu^{X} - \nu)(d\tilde{r}, dz), \, r \in [0, T - t'], \, \mathbb{P}_{0,0}\text{-a.s.}$$

Since ξ is uniformly continuous under d_U , since f is uniformly continuous under \mathbf{d}_{∞} in (t,ω) uniformly in (y,z,p), and since $d_{M_1} \leq d_U$, one can show using BSDE standard techniques (keeping (6.7) in mind) that there exists a constant $C' = C'(t^0) > 0$ such that

$$A_n^x \le C' \rho_0(|x_0 - x_n|).$$

Put $s^n:=(t^n\vee s)\wedge T.$ To show convergence of $A_n^t,$ let us initially fix $\omega\in\Omega.$ Set

$$r_j = r_j(\omega) := \operatorname{H}_j^{\varepsilon}(\omega), \ j \in \mathbb{N}_0,$$

 $\iota = \iota(\omega) := \max\{j \in \mathbb{N}_0 : s^0 + r_j \le T\}.$

We treat first the case that $t^n \ge t^0$, whence $s^n \ge s^0$. Since ω is fixed, we can and will assume that, without loss of generality,

$$s^n + r_t \leq T$$
.

Since $s^0 \leq T - r_\iota$, we have $s_n \in [s^0, T - r_\iota]$. Since, for every $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$,

$$g(\pi_{i}; (s^{n} + [r_{j} \wedge (T - s^{n})], \omega_{r_{j} \wedge (T - s^{n})})_{j \in \mathbb{N}_{0}}; x + \omega_{T - s^{n}})$$

$$= \xi \left(\sum_{j=0}^{i-2} y_{j} \cdot \mathbf{1}_{[s_{j}, s_{j+1})} + y \cdot \mathbf{1}_{[s, s^{n})} + \sum_{j=0}^{i-1} (x + \omega_{r_{j}}) \cdot \mathbf{1}_{[s^{n} + r_{j}, s^{n} + r_{j+1})} + (x + \omega_{r_{i}}) \cdot \mathbf{1}_{[s^{n} + r_{i}, T)} + (x + \omega_{T - s^{n}}) \cdot \mathbf{1}_{\{T\}} \right)$$

$$=: \xi(\tilde{\omega}(x, s^{n})).$$

we have, by Lemma C.3,

$$\sup_{x} \left| \xi(\tilde{\omega}(x, s^n)) - \xi(\tilde{\omega}(x, s^0)) \right| \le \sup_{x} \rho_0(d_{J_1}(\tilde{\omega}(x, s^n), \tilde{\omega}(x, s^0)))$$

$$\le \sup_{x} \rho_0(2(s^n - s^0) + |\omega_{T-s^n} - \omega_{T-s^0}| \to 0)$$

as $n \to \infty$ provided ω is left-continuous at T, which, however is the case for $\mathbb{P}_{0,\mathbf{0}}$ -a.e. ω because X is $\mathbb{P}_{0,\mathbf{0}}$ -quasi-left-continuous. A corresponding result can be shown for the driver $\tilde{f}(\cdots)$ of the BSDE (6.10). Thus, keeping (6.8) and (6.9) in mind, we can employ standard a-priori estimates for BSDEs (cf. Lemma 3.1.1 in [14]) to deduce that, for every $x \in \mathbb{R}^d$, $h_i^{\varepsilon}(\pi_i; t^n, x) \to h_i^{\varepsilon}(\pi_i; t^0, x)$ as $n \to \infty$ with $t^n \geq t$. Hence, by Lemma C.2, $(t^n, x_n) \to (t^0, x_0)$ as $n \to \infty$ with $t^n \geq t^0$ implies

$$A_n^t \le \sup_{x} \left| h_i^{\varepsilon}(\pi_i; t^n, x) - h_i^{\varepsilon}(\pi_i; t^0, x) \right| \to 0$$

as $n \to \infty$

Now we treat the case $t^n \leq t^0$. Again, fix $\omega = (\omega^k)_{k \leq d} \in \Omega$ and, in addition to the notation introduced in the previous paragraph, set

$$\iota^n = \iota^n(\omega) := \max\{j \in \mathbb{N}_0 : s^n + r_j \le T\}.$$

Note that $\iota \leq \iota^n$. Recall that

$$\tilde{\omega}(x, s^{0}) = (\tilde{\omega}(x, s^{0})^{k})_{k \leq d} = \sum_{j=0}^{i-2} y_{j} \cdot \mathbf{1}_{[s_{j}, s_{j+1})} + y \cdot \mathbf{1}_{[s, s^{0})}$$

$$+ \sum_{j=0}^{i-1} (x + \omega_{r_{j}}) \cdot \mathbf{1}_{[s^{0} + r_{j}, s^{0} + r_{j+1})} + (x + \omega_{r_{i}}) \cdot \mathbf{1}_{[s^{0} + r_{i}, T)} + (x + \omega_{T-s^{0}}) \cdot \mathbf{1}_{\{T\}}$$

For any $n \in \mathbb{N}$, let

$$\bar{\omega}(x,s^{n}) = (\bar{\omega}(x,s^{n})^{k})_{k \leq d} := \sum_{j=0}^{i-2} y_{j}.\mathbf{1}_{[s_{j},s_{j+1})} + y.\mathbf{1}_{[s,s^{n})}$$

$$+ \sum_{j=0}^{i-1} (x + \omega_{r_{j}}).\mathbf{1}_{[s^{n}+r_{j},s^{n}+r_{j+1})}$$

$$+ (x + \omega_{r_{i}}).\mathbf{1}_{[s^{n}+r_{i},(s^{n}+r_{i+1})\wedge T)} + \sum_{j=i+1}^{i^{n}-1} (x + \omega_{r_{j}}).\mathbf{1}_{[s^{n}+r_{j},s^{n}+r_{j+1})}$$

$$+ \mathbf{1}_{\{i\}^{c}}(\iota^{n}).(x + \omega_{r_{i}n}).\mathbf{1}_{[s^{n}+r_{i}n,T)} + (x + \omega_{T-s^{n}}).\mathbf{1}_{\{T\}}.$$

Right-continuity of ω yields $\omega_{r_j} = \omega_{r_j \wedge (T-s^n)} \to \omega_{T-s^0}$ as $n \to \infty$ for $j \le \iota^n$. Hence, since d_{M_1} is a metric, by the triangle inequality together with Lemma B.1,

$$d_p(\bar{\omega}(x,s^n),\tilde{\omega}(x,s^0)) \leq \max_{k \leq d} d_{M_1}(\bar{\omega}(x,s^n)^k,\tilde{\omega}(x,s^0)^k) \to 0$$

uniformly in x as $n \to \infty$. Thus, corresponding considerations as in the previous paragraph yield

$$\sup_{x} \left| \xi(\bar{\omega}(x,s^n)) - \xi(\tilde{\omega}(x,s^0)) \right| \le \sup_{x} \rho_0(d_p(\bar{\omega}(x,s^n),\tilde{\omega}(x,s^0)) \to 0$$

and

$$\sup_{x} \left| h_i^{\varepsilon}(\pi_i; t^n, x) - h_i^{\varepsilon}(\pi_i; t^0, x) \right| \to 0.$$

as $n \to \infty$.

This concludes the proof.

6.3. Path-frozen integro-differential equations. Let $\varepsilon \in (0, c_0')$. Given $(s,y) \in [s^*,T] \times \mathbb{R}^d$, let $K_{s,y}^{4C_0'} := [s,T] \times \prod_{i=1}^d [y^i - 4C_0',y^i + 4C_0']$. Then, by the Weierstrass approximation theorem, for any $\delta > 0$, there exists a polynomial $h_i^{\varepsilon,\delta}(\pi_i;\cdot)$ on $\mathbb{R} \times \mathbb{R}^d$ such that

$$\left| h_i^{\varepsilon,\delta}(\pi_i;\cdot) - h_i^{\varepsilon}(\pi_i;\cdot) \right|_{K_{s,n}^{4C_0'}} < \delta.$$

Since we can take multivariate Bernstein polynomials as approximating functions (see, e.g., Appendix B of [23]), we can and will, by Assumption 3.17,

assume that the mapping $(\pi_i; t, x) \mapsto h_i^{\varepsilon, \delta}(\pi_i; \cdot), \ \Pi_i^{s^*} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, is continuous. Put

$$\overline{h}_i^{\varepsilon,\delta} := h_i^{\varepsilon,\delta} + \delta.$$

Now, we proceed similarly as in the approach of the proof of Lemma 5.4 in the first arxiv version of [19] to define inductively (in three steps) a functional $\psi^{s^*,\omega^*,\varepsilon}\in \bar{C}_b^{1,2}(\bar{\Lambda}^{s^*})$, which will have properties that are needed in the proof of Theorem 3.18 below. To this end, let us, first of all, introduce some notation.

Definition 6.11. Let $0 < \delta < c_0' < C_0'$ and $2C_0' < \delta' < \infty$. (Recall that c_0' is a lower bound and C_0' is an upper bound of the jump size of X. See Remark 2.7 and Assumption 3.14.) Let $(t^*, y^*) \in (-\infty, T) \times \mathbb{R}^d$. Put $D := O_{\delta}(y^*), D' := O_{\delta'}(y^*), Q := (t^*, T) \times O_{\delta}(y^*), Q' := (t^*, T) \times O_{\delta'}(y^*),$ etc. Fix $\alpha \in (0, 1)$ and $h \in C^{\infty}(\bar{Q})$. Set

$$\mathcal{C}_{\alpha}(h) := \{ w : \bar{Q}' \to \mathbb{R} \quad \text{such that } w \in C^{2,\alpha}_{\text{loc}}(Q)$$
 with $|\partial_t w|_Q$, $|\partial_x w|_Q$, $|\partial^2_{xx} w|_Q$ being bounded and that $w = h$ in $(t^*, T) \times (D' \setminus D)$ and on $\{T\} \times D'\}$.

Let $\check{f} = \check{f}(t, y, z, p) : \bar{Q} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a function. Define $\bar{\eta} : (-\infty, T] \to \mathbb{R}$ by $\bar{\eta}(t) := \eta_{t \vee 0}$. Given $t \in (-\infty, T]$, define an operator $\mathbf{I}_t^h = \mathbf{I}_t$ on $\mathcal{C}_{\alpha}(h)$ by

$$\mathbf{I}_{t}w(t,x) := \int_{c'_{0} \leq |z| \leq C'_{0}} \left[h(t,x+z) \,\bar{\eta}(t) \right] \, K(dz) - w(t,x) \bar{\eta}(t) K(\mathbb{R}^{d}).$$

Define a mapping $\tilde{F} = \tilde{F}(t, x, y, z, w) : \bar{Q} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{C}_{\alpha}(h) \to \mathbb{R}$ by

$$\tilde{F}(t, x, y, z, w) := \check{f}(t, y, z, \mathbf{I}_t w(t, x)).$$

Given $v \in \mathcal{C}_{\alpha}(h)$, put

$$\tilde{F}[v](t,x) := \tilde{F}(t,x,v(t,x),\partial_x v(t,x),v(\cdot,\cdot)).$$

Define an operator $\mathbf{L}^h = \mathbf{L}$ on $\mathcal{C}_{\alpha}(h)$ by

$$\mathbf{L}w(t,x) := -\partial_t w(t,x) - \sum_{i=1}^d \bar{b}^i) \, \partial_{x^i} w(t,x) - \frac{1}{2} \sum_{i,j=1}^d \bar{c}^{i,j} \, \partial_{x^i x^j} w(t,x)$$
$$- \int_{\mathbb{R}^d} \left[h(t,x+z) - \sum_{i=1}^d z^i \, \partial_{x^i} w(t,x) \right] \, K(dz)$$
$$+ w(t,x) K(\mathbb{R}^d).$$

Remark 6.12. Given the context of the preceding definition, we have

$$\begin{split} \mathbf{I}_t^h w(t,x) &:= \int_{\mathbb{R}^d} \left[w(t,x+z) - w(t,x) \right] \bar{\eta}(t) \, \bar{K}(t,dz), \\ \mathbf{L}^h w(t,x) &:= -\partial_t w(t,x) - \sum_{i=1}^d b^i \, \partial_{x^i} w(t,x) - \frac{1}{2} \sum_{i,j=1}^d c^{i,j} \, \partial_{x^i x^j} w(t,x) \\ &- \int_{\mathbb{R}^d} \left[w(t,x+z) - w(t,x) - \sum_{i=1}^d z^i \, \partial_{x^i} w(t,x) \right] \, K(dz). \end{split}$$

Given $y \in \mathbb{R}^d$ and $h \in C^{\infty}(\mathbb{R} \times \mathbb{R}^d)$, put $D_y := O_{\varepsilon}(y)$ $D_y' := O_{2C_0'}(y)$, and let the function space $\mathcal{C}_{\alpha}(h,y)$ be defined as the space $\mathcal{C}_{\alpha}(h)$ with $\bar{Q}' = \bar{Q}_{-1,y}^{2C_0'}$ and $Q = \bar{Q}_{-1,y}^{\varepsilon}$.

Step 1. Let $\pi_1 = (s^*, y)$. Set $\delta_1 := \varepsilon/4$. Write $h = \overline{h}_1^{\varepsilon, \delta_1}(\pi_1; \cdot)$. Write $\hat{f}(t, \cdot) = \tilde{f}_t((\pi_1; (T, y)_{j \in \mathbb{N}}; y), \cdot)$ and let \tilde{F} as well as $\tilde{F}[\cdot]$ be defined as in Definition 6.11. By standard PDE theory, there exists a function $w_1^{s^*, \omega^*, \varepsilon}(\pi_1; \cdot) = w_1(\pi_1; \cdot) = w_1 \in \mathcal{C}_{\alpha}(h, y)$ such that

$$\mathbf{L}w_1 - \tilde{F}[w_1] = 0 \text{ in } Q_{-1}^{\varepsilon}, \ w_1 = h \text{ in } (-1, T) \times D'_y \setminus D_y, \ w_1 = h \text{ on } \{T\} \times D'_y.$$

Define a function $v_1^{s^*,\omega^*,\varepsilon}(\pi_1;\cdot) = v_1(\pi_1;\cdot) = v_1$ on $\bar{Q}_{-1,y}^{2C_0'}$ by

$$v_1(\pi_1; t, x) := w_1(\pi_1; t, x) - w_1(\pi_1; \pi_1) + \theta_1(\pi_1; \pi_1) + \frac{\varepsilon}{2}$$

Then $v_1(\pi_1;\cdot) \in \mathcal{C}_{\alpha}(h',y)$ for some $h' \in C^{\infty}(\mathbb{R} \times \mathbb{R}^d)$ and

(6.11)
$$\mathbf{L}v_1 - \tilde{F}[v_1] \ge 0 \text{ in } Q_{-1,y}^{\varepsilon},$$

(6.12)
$$v_1(\pi_1; \pi_1) = \theta_1(\pi_1; \pi_1) + \frac{\varepsilon}{2},$$

(6.13)
$$v_1 \ge h_1^{\varepsilon}(\pi_1; \cdot) \text{ in } [s^*, T) \times D_y' \setminus D_y \text{ and on } \{T\} \times D_y'.$$

To see that (6.13) is true, it suffices, by definition of v_1 , to show that $\varepsilon/2 + \theta_1(\pi_1; \pi_1) - w_1(\pi_1; \pi_1) \geq 0$. Indeed, noting that f is non-anticipating and using the dynamic programming principle (Lemma 6.7), we can employ the comparison principle for BSDEs with jumps together with constant-translatibility of sublinear expectations (see, e.g., [33]) to deduce that $h \leq h_i^{\varepsilon,\delta}(\pi_1;\cdot) + 2\delta_1$ implies $w_1(\pi_1;\pi_1) \leq \theta_1(\pi_1;\pi_1) + \varepsilon/2$. Note that Itô's formula together with quasi-left-continuity makes it possible to represent w_1 as BSDE (see Lemmas 6.1 and 6.2).

Define

$$\psi^{s^*,\omega^*,\varepsilon}(t,\omega) := v_1(s^*,\omega_{s^*};t,\omega_t) + \sum_{j=1}^{\infty} \delta_j, \quad s^* \le t \le \mathrm{H}_1^{s^*,\varepsilon}(\omega).$$

Then, by (6.12),

$$\theta_1^{s^*,\omega^*,\varepsilon}(s^*,\omega_{s^*};s^*,\omega_{s^*}) < \psi^{s^*,\omega^*,\varepsilon}(s^*,\omega) < \theta_1^{s^*,\omega^*,\varepsilon}(s^*,\omega_{s^*};s^*,\omega_{s^*}) + \varepsilon.$$

Universal measurability of $(t, \omega) \mapsto v_1(s^*, \omega_{s^*}; t, \omega_t)$ follows from Assumption 3.17.

Step 2. Let $\pi_2 = (s_0, y_0; s, y) \in \Pi_2^{s^*}$. Set $\delta_2 := \varepsilon/8$. Write $h = \bar{h}_2^{\varepsilon, \delta_2}(\pi_2; \cdot)$ and $\hat{f}(t, \cdot) = \tilde{f}_t((\pi_2; (T, y)_{j \in \mathbb{N}}; y), \cdot)$ and let \tilde{F} as well as $\tilde{F}[\cdot]$ be defined as in Definition 6.11. By standard PDE theory, there exists a function $w_2^{s^*, \omega^*, \varepsilon}(\pi_2; \cdot) = w_2(\pi_2; \cdot) = w_2 \in \mathcal{C}_{\alpha}(h, y)$ such that

$$\mathbf{L}w_2 - \tilde{F}[w_2] = 0 \text{ in } Q_{-1}^{\varepsilon}, \ w_2 = h \text{ in } (-1, T) \times D_y' \setminus D_y, \ w_2 = h \text{ on } \{T\} \times D_y'.$$

Define a function $v_2^{s^*,\omega^*,\varepsilon}(\pi_2;\cdot) = v_2(\pi_2;\cdot) = v_2$ on $\bar{Q}_{-1,y}^{2C_0'}$ by

$$v_2(\pi_2; t, x) := w_2(\pi_2; t, x) - w_2(\pi_2; s, y) + v_1(\pi_1; s, y) + \delta_1.$$

Then $v_2(\pi_2; \cdot) \in \mathcal{C}_{\alpha}(h', y)$ for some $h' \in C^{\infty}(R \times \mathbb{R}^d)$ and

(6.14)
$$\mathbf{L}v_2 - \tilde{F}[v_2] \ge 0 \text{ in } Q_{-1,y}^{\varepsilon},$$

(6.15)
$$v_2(\pi_2; s, y) = v_1(\pi_1; s, y) + \delta_1,$$

and, if $\varepsilon \leq |y - y_0| \leq 2C'_0$, then

(6.16)
$$v_2 \ge h_2^{\varepsilon}(\pi_2; \cdot) \text{ in } [s, T) \times D_y' \setminus D_y \text{ and on } \{T\} \times D_y'.$$

To see that (6.16) is true, let $(t, x) \in [s, T) \times D'_y \setminus D_y$ or $(t, x) \in \{T\} \times D'_y$. Then, by (6.13) in Step 1,

$$v_{2}(\pi_{2};t,x) = h(t,x) + v_{1}(\pi_{1};s,y) - w_{2}(\pi_{2};s,y) + \delta_{1}$$

$$\geq h_{2}^{\varepsilon}(\pi_{2};t,x) + v_{1}(\pi_{1};s,y) - w_{2}(\pi_{2};s,y) + \delta_{1}$$

$$= h_{2}^{\varepsilon}(\pi_{2};t,x)$$

$$+ \bar{h}_{1}^{\varepsilon,\delta_{1}}(s_{0},y_{0};s,y) + \theta_{1}(s^{*},y_{0};s^{*},y_{0}) - w_{1}(s^{*},y_{0};s^{*},y_{0}) + 2\delta_{1}$$

$$- w_{2}(\pi_{2};s,y) + \delta_{1}.$$

That is, we have to show that

(6.17)

$$w_2(\pi_2; s, y) \le \bar{h}_1^{\varepsilon, \delta_1}(s^*, y_0; s, y) + \theta_1(s^*, y_0; s^*, y_0) - w_1(s^*, y_0; s^*, y_0) + 3\delta_1.$$

Note that, similarly as in Step 1, one can show that

$$w_2(\pi_2; s, y) \le \theta_2(\pi_2; s, y) + 2\delta_2.$$

We also have $\theta_2(\pi_2; s, y) = h_1^{\varepsilon}(s^*, y_0; s, y)$ because $\varepsilon \leq |y - y_0|$. Thus

$$w_2(\pi_2; s, y) \le \bar{h}_1^{\varepsilon, \delta_1}(s^*, y_0; s, y) + 2\delta_2,$$

and together with $2\delta_1 \le \theta_1(s^*, y_0; s^*, y_0) - w_1(s^*, y_0; s^*, y_0)$ from Step 1 we get (6.17) and consequently (6.16).

Define

$$\psi^{s^*,\omega^*,\varepsilon}(t,\omega) := v_2^{s^*,\omega^*,\varepsilon}(s^*,\omega_{s^*}; \mathbf{H}_1^{s^*,\varepsilon}(\omega), X_{\mathbf{H}_1^{s^*,\varepsilon}}(\omega); t, \omega_t)$$

$$+ \sum_{j=2}^{\infty} \delta_j, \quad \mathbf{H}_1^{s^*,\varepsilon}(\omega) < t \le \mathbf{H}_2^{s^*,\varepsilon}(\omega).$$

Note that, by Step 1 and by definition of v_2 ,

$$\psi^{s^*,\omega^*,\varepsilon}(\mathbf{H}_1^{s^*,\varepsilon},\omega) = v_2^{s^*,\omega^*,\varepsilon}(s^*,\omega_{s^*};\mathbf{H}_1^{s^*,\varepsilon}(\omega),X_{\mathbf{H}_1^{s^*,\varepsilon}}(\omega);\mathbf{H}_1^{s^*,\varepsilon}(\omega),X_{\mathbf{H}_1^{s^*,\varepsilon}}(\omega)) + \sum_{j=2}^{\infty} \delta_j.$$

Universal measurability of

$$(t,\omega) \mapsto v_2(s^*,\omega_{s^*}; \mathbf{H}_1^{s^*,\varepsilon}(\omega), X_{\mathbf{H}_1^{s^*,\varepsilon}}(\omega); t,\omega_t),$$

follows from Assumption 3.17 and standard BSDE error estimates.

Step 3 $(i \to i+1)$. Let $i \in \mathbb{N}$. Set $\delta_j := \varepsilon/2^{j+1}$, $j \in \mathbb{N}$. For every $\pi_j = (s_0, y_0; \dots; s_{j-1}, y_{j-1}) \in \Pi_j^{s^*}$, $j \in \mathbb{N}$, there exists, by standard PDE theory, $w_j^{s^*, \omega^*, \varepsilon}(\pi_j; \cdot) = w_j(\pi_j; \cdot) = w_j \in \mathcal{C}_{\alpha}(\bar{h}_j^{\varepsilon, \delta_j}(\pi_j; \cdot), y_{j-1})$ such that

$$\mathbf{L}w_j - \tilde{f}_t((\pi_j; (T, y_{j-1})_{k \in \mathbb{N}}; y_{j-1}), w_j, \partial_x w_j, \mathbf{I}_t w_j) = 0 \text{ in } Q_{-1, y_{j-1}}^{\varepsilon},$$

$$w_j = \bar{h}_j^{\varepsilon,\delta_j}(\pi_j;\cdot) \text{ in } (-1,T) \times D_y' \setminus D_{y_{j-1}}, w_j = \bar{h}_j^{\varepsilon,\delta_j}(\pi_j;\cdot) \text{ on } \{T\} \times D_{y_{j-1}}'.$$

Define
$$v_j^{s^*,\omega^*,\varepsilon}(\pi_j;\cdot)=v_j(\pi_j;\cdot)$$
 on $\bar{Q}_{-1,y_{j-1}}^{2C_0'}$ recursively by

$$(6.20) v_j(\pi_j; t, x) := w_j(\pi_j; t, x) + v_{j-1}(\pi_j) - w_j(\pi_j; s_{j-1}, y_{j-1}) + \delta_{j-1}.$$

Suppose that the following induction hypothesis holds:

For every
$$j \in \{1, \ldots, i\}$$
,

$$\mathbf{L}v_j - \tilde{f}_t((\pi_j; (T, y_{j-1})_{k \in \mathbb{N}}; y_{j-1}), v_j, \partial_x v_j, \mathbf{I}_t v_j) \ge 0 \text{ in } Q_{-1, y_{j-1}}^{\varepsilon},$$

and, if
$$\varepsilon \leq |y_{k+1} - y_k| \leq 2$$
, for every $k \in \{0, \dots, j-1\}$, then

$$v_j(\pi_j;\cdot) \ge h_j^{\varepsilon}(\pi_j;\cdot)$$
 in $(s_{j-1},T) \times D'_{y_{j-1}} \setminus D_{y_{j-1}}$ and on $\{T\} \times D'_{y_{j-1}}$.

Let $i \geq 2$. Fix $\pi_{i+1} = (s_0, y_0; \dots; s_i, y_i) \in \Pi_{i+1}^{s^*}$ with $s = s_i$ and $y = y_i$. Let $\pi_j := (s_0, y_0; \dots; s_{j-1}, y_{j-1}), \ j = 1, \dots, \ i-2$. Then $v_{i+1}(\pi_{i+1}; \cdot) \in \mathcal{C}_{\alpha}(h', y)$ for some $h' \in C^{\infty}(\mathbb{R} \times \mathbb{R}^d)$ and

(6.21)
$$\mathbf{L}v_{i+1} - \tilde{f}_t((\pi_{i+1}; (T, y)_{k \in \mathbb{N}}; y), v_{i+1}, \partial_{\omega}v_{i+1}, \mathbf{I}_t v_{i+1}) \ge 0 \text{ in } Q_{-1, v}^{\varepsilon},$$

(6.22)
$$v_{i+1}(\pi_{i+1}; s, y) = v_i(\pi_i) + \delta_i,$$

and, if

(6.23)
$$\varepsilon \le |y_{j+1} - y_j| \le 2, \quad j = 0, \dots, i - 1,$$

then

$$(6.24) v_{i+1} \ge h_{i+1}^{\varepsilon}(\pi_{i+1};\cdot) \text{in } (s,T) \times D_y' \setminus D_y \text{ and on } \{T\} \times D_y'.$$

To see that (6.24) is true, let $(t,x) \in \bar{Q}_{s,y}^{2C_0'} \setminus Q_{s,y}^{\varepsilon}$. Then

$$\begin{aligned} & v_{i+1}(\pi_{i+1};t,x) \\ & \geq h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + v_{i}(\pi_{i};s,y) - w_{i+1}(\pi_{i+1};s,y) + \delta_{i} \end{aligned} \qquad \text{by } (6.23) \\ & \geq h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + \bar{h}_{i}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) + \boxed{v_{i-1}(\pi_{i})} \\ & - w_{i}(\pi_{i};s_{i-1},y_{i-1}) + \delta_{i-1} \\ & - w_{i+1}(\pi_{i+1};s,y) + \delta_{i} \end{aligned} \qquad \text{by } (6.19), (6.20), (6.23) \\ & \geq h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + \bar{h}_{i}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) \\ & + \boxed{h}_{i-1}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) \\ & + \boxed{h}_{i-1}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) + \delta_{i} \end{aligned} \qquad \text{by } (6.19), (6.20), (6.23) \\ & \geq h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + \bar{h}_{i}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) + \delta_{i} \end{aligned} \qquad \text{by } (6.19), (6.20), (6.23) \\ & = h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + \bar{h}_{i}^{\varepsilon,\delta_{i}}(\pi_{i};s,y) + \bar{h}_{i-1}^{\varepsilon,\delta_{i-1}}(\pi_{i}) \\ & + (\delta_{i} + \delta_{i-1} + \delta_{i-2}) \\ & - (w_{i-1}(\pi_{i-1};s_{i-2},y_{i-2}) + w_{i}(\pi_{i};s_{i-1};y_{i-1}) + w_{i+1}(\pi_{i+1};s,y)) \\ & + \boxed{v_{i-2}(\pi_{i-1})} \\ & \geq \cdots \\ & \geq h_{i+1}^{\varepsilon}(\pi_{i+1};t,x) \\ & + \left[\bar{h}_{i}^{\varepsilon,\delta_{i}}(\pi_{i+1}) + \cdots + \bar{h}_{1}^{\varepsilon,\delta_{1}}(\pi_{2}) + \theta_{1}(\pi_{1};\pi_{1})\right] \\ & + \left[(\delta_{i} + \cdots + \delta_{1}) + 2\delta_{1}\right] \\ & - \left[w_{i+1}(\pi_{i+1};s,y) + w_{i}(\pi_{i};s_{i-1},y_{i-1}) + \cdots + w_{1}(\pi_{1};s_{0};y_{0})\right]. \end{aligned}$$

I.e., we have to show that

$$[w_{i+1}(\pi_{i+1}; s, y) + w_i(\pi_i; s_{i-1}, y_{i-1}) + \dots + w_1(\pi_1; s_0; y_0)]$$

$$\leq \left[\bar{h}_i^{\varepsilon, \delta_i}(\pi_{i+1}) + \dots + \bar{h}_1^{\varepsilon, \delta_1}(\pi_2) + \theta_1(\pi_1; \pi_1)\right]$$

$$+ \left[(\delta_i + \dots + \delta_1) + 2\delta_1\right].$$

Again, similarly, as in Step 1, one can show that, for every $j \in \{2, \ldots, i+1\}$, we have $w_j(\pi_j; s_{j-1}, y_{j-1}) \leq \theta_j(\pi_j; s_{j-1}, y_{j-1}) + 2\delta_j$. Also, $\varepsilon \leq |y_{j-1} - y_j|$, $j = 0, \ldots, i-1$, implies that, for every $j \in \{2, \ldots, i+1\}$, we have $\theta_j(\pi_j; s_{j-1}, y_{j-1}) = h_{j-1}^{\varepsilon}(\pi_{j-1}; s_{j-1}, y_{j-1})$, which yields

$$w_j(\pi_j; s_{j-1}, y_{j-1}) \le h_{j-1}^{\varepsilon}(\pi_{j-1}; s_{j-1}, y_{j-1}) + 2\delta_j.$$

Together with $w_1(\pi_1; \pi_1) \leq \theta_1(\pi_1; \pi_1) + 2\delta_1$, from Step 1 and with

$$2\delta_{i+1} + \ldots + 2\delta_2 = (\delta_i + \ldots + \delta_1),$$

we get (6.25) and thus (6.24).

Define

$$\psi^{s^*,\omega^*,\varepsilon}(t,\omega) := v_{i+1}((\mathrm{H}_j^{s^*,\varepsilon}(\omega), X_{\mathrm{H}_j^{s^*,\varepsilon}}(\omega))_{0 \leq j \leq i}; t,\omega_t) + \sum_{j=i+1}^{\infty} \delta_j, \quad \mathrm{H}_i^{s^*,\varepsilon}(\omega) < t \leq \mathrm{H}_{i+1}^{s^*,\varepsilon}(\omega).$$

Note that, by the induction hypothesis by definition of v_{i+1} ,

$$\psi^{s^*,\omega^*,\varepsilon}(\mathbf{H}_i^{s^*,\varepsilon},\omega) = v_{i+1}((\mathbf{H}_j^{s^*,\varepsilon}(\omega), X_{\mathbf{H}_j^{s^*,\varepsilon}}(\omega))_{0 \le j \le i}; \mathbf{H}_i^{s^*,\varepsilon}(\omega), X_{\mathbf{H}_i^{s^*,\varepsilon}}(\omega)) + \sum_{j=i+1}^{\infty} \delta_j.$$

As in Step 2, universal measurability of

$$(t,\omega)\mapsto v_{i+1}((\mathbf{H}_{j}^{s^{*},\varepsilon}(\omega),X_{\mathbf{H}_{i}^{s^{*},\varepsilon}}(\omega))_{0\leq j\leq i};t,\omega_{t}),$$

follows from Assumption 3.17 and standard BSDE error estimates.

By mathematical induction, we obtain the following result.

Lemma 6.13. The mapping $\psi^{s^*,\omega^*,\varepsilon}: \bar{\Lambda} \to \mathbb{R}$ defined in Step 1, Step 2, and Step 3 belongs to $\bar{C}_h^{1,2}(\bar{\Lambda})$.

6.4. Proof of Comparison.

Definition 6.14. Let $(t, \omega) \in \Lambda$. Denote by $\overline{\mathcal{D}}(t, \omega)$ (resp. $\underline{\mathcal{D}}(t, \omega)$) the set of all $\varphi \in \overline{C}_b^{1,2}(\overline{\Lambda}^t)$ with corresponding sequences (τ_n) of stopping times and corresponding collections (ϑ_n) of functionals such that the following holds:

(i) For every $n \in \mathbb{N}$ and every $(r, \tilde{\omega}) \in [\![\tau_{n-1}, \tau_n[\![$, we have, with $\pi_n = (\mathrm{H}_i^{t,\varepsilon}(\tilde{\omega}), X_{\mathrm{H}_i^{t,\varepsilon}}(\tilde{\omega}))_{0 \leq i \leq n-1},$

$$\mathbf{L}\vartheta_{n}(\pi_{n}; r, \tilde{\omega}_{r}) - f_{r}(\tilde{\omega}, \vartheta_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \partial_{\omega}\vartheta_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \mathbf{I}_{r}\vartheta_{n}(\pi_{n}; r, \tilde{\omega}_{r})) \geq \text{ (resp. } \leq \text{) } 0.$$

(ii) For $\mathbb{P}_{t,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$\varphi(T, \tilde{\omega}) \ge \text{ (resp. } \le) \xi(\tilde{\omega}).$$

Proof of Theorem 3.18.

$$Put\overline{u}(t,\omega):=\inf\{\varphi(t,\omega):\varphi\in\overline{\mathcal{D}}(t,\omega)\},\quad \underline{u}(t,\omega):=\sup\{\varphi(t,\omega):\varphi\in\underline{\mathcal{D}}(t,\omega)\}.$$

We assert that $\overline{u}(t,\omega) \leq \underline{u}(t,\omega)$. To show this, we proceed nearly exactly as in the corresponding part of the proof of Proposition 7.5 in [18]. Define functionals $\overline{\psi}^{t,\omega,\varepsilon}$, $\psi^{t,\omega,\varepsilon}$ on $\bar{\Lambda}^t$ by

$$\overline{\psi}_r^{t,\omega,\varepsilon} := \psi_r^{t,\omega,\varepsilon} + \rho(2\varepsilon)[1+T-r], \quad \underline{\psi}_r^{t,\omega,\varepsilon} := \psi_r^{t,\omega,\varepsilon} - \rho(2\varepsilon)[1+T-r].$$

Note that $\overline{\psi}^{t,\omega,\varepsilon}$, $\underline{\psi}^{t,\omega,\varepsilon} \in \overline{C}^{1,2}(\overline{\Lambda}^t)$ and the corresponding sequences of stopping times are in both cases $(\mathbf{H}_n^{t,\varepsilon})$ and the corresponding collections of functionals are (\overline{v}_n) and (\underline{v}_n) , resp., defined by

$$\overline{v}_n(\cdot;r,\cdot) := v_n(\cdot;r,\cdot) + \rho(2\varepsilon)[1+T-r], \quad \underline{v}_n(\cdot;r,\cdot) := v_n(\cdot;r,\cdot) - \rho(2\varepsilon)[1+T-r].$$

Moreover, $\overline{\psi}^{t,\omega,\varepsilon} \in \overline{\mathcal{D}}(t,\omega)$ because, whenever $(r,\tilde{\omega}) \in [\![\mathbf{H}_{n-1}^{t,\varepsilon},\mathbf{H}_n^{t,\varepsilon}]\![$ for some $n \in \mathbb{N}$, we have, with $\pi_n = (\mathbf{H}_i^{t,\varepsilon}(\tilde{\omega}), X_{\mathbf{H}^{t,\varepsilon}}(\tilde{\omega}))_{0 \le i \le n-1}$,

$$\mathbf{L}\overline{v}_{n}(\pi_{n}; r, \tilde{\omega}_{r}) - f_{r}(\tilde{\omega}, \overline{v}_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \partial_{x}\overline{v}_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \mathbf{I}_{r}\overline{v}_{n}(\pi_{n}; r, \tilde{\omega}_{r}))$$

$$\geq \mathbf{L}v_{n}(\pi_{n}; r, \tilde{\omega}_{r}) + \rho_{0}(2\varepsilon) - f_{r}(\tilde{\omega}, v_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \partial_{x}v_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \mathbf{I}_{r}v_{n}(\pi_{n}; r, \tilde{\omega}_{r}))$$

$$\geq \mathbf{L}v_{n}(\pi_{n}; r, \tilde{\omega}_{r}) - \tilde{f}_{r}((\pi_{n}; (T, \tilde{\omega}_{r})_{k \in \mathbb{N}}; \tilde{\omega}_{r}), v_{n}(\pi_{n}; r, \tilde{\omega}_{r}), \mathbf{I}_{r}v_{n}(\pi_{n}; r, \tilde{\omega}_{r}))$$

 ≥ 0

and, similarly, $\overline{\psi}_T^{t,\omega,\varepsilon} \geq \xi$, $\mathbb{P}_{t,\omega}$ -a.s. Thus $\overline{u}(t,\omega) \leq \overline{\psi}^{t,\omega,\varepsilon}(t,\omega)$ and, similarly, one can show that $\psi^{t,\omega,\varepsilon}(t,\omega) \leq \underline{u}(t,\omega)$. Consequently, $\overline{u}(t,\omega) - \underline{u}(t,\omega) \leq \underline{u}(t,\omega)$ $2\rho_0(2\varepsilon)[1+T-t]$. Letting $\varepsilon \downarrow 0$ yields $\overline{u}(t,\omega) \leq \underline{u}(t,\omega)$.

Finally, by the partial comparison principle (Theorem 5.7), $u^1(t,\omega) \leq$ $\overline{u}(t,\omega)$ and $\underline{u}(t,\omega) \leq u^2(t,\omega)$. Our previous assertion yields then $u^1(t,\omega) \leq u^2(t,\omega)$ $u^2(t,\omega)$.

Appendix A. Martingale problems and regular conditioning

The results in this appendix are actually valid in a more general context than in our canonical setup and might be of independent interest. In particular, (B, C, ν) can be as general as in III.2a. of [25], in which case standard conventions of [25] are in force.

First, we recall the definitions of [38] for conditional probability distributions (c.p.d.) and regular conditional probability distributions (r.c.p.d). A c.p.d. of a probability measure \mathbb{P} on $(\Omega, \mathcal{F}_T^0)$ given a sub σ -field $\mathcal{F} \subseteq \mathcal{F}_T^0$ is a collection $\{\mathbb{P}_{\omega}\}_{{\omega}\in\Omega}$ of probability measures satisfying the following:

- (i) For every $A \in \mathcal{F}_T^0$, the map $\omega \mapsto \mathbb{P}_{\omega}(A)$ is \mathcal{F} -measurable. (ii) For every $A \in \mathcal{F}_T^0$ and every $B \in \mathcal{F}$,

$$\mathbb{P}(A \cap B) = \int_A \mathbb{P}_{\omega}(B) \, \mathbb{P}(d\omega).$$

If a c.p.d. $\{\mathbb{P}_{\omega}\}_{{\omega}\in\Omega}$ given \mathcal{F} satisfies $\mathbb{P}_{\omega}(A({\omega}))=1$ for \mathbb{P} -a.e ${\omega}\in\Omega$, where $A(\omega) := \bigcap \{A \in \mathcal{F} : x \in A\}, \text{ then we call } \{\mathbb{P}_{\omega}\}_{\omega \in \Omega} \text{ an r.c.p.d. given } \mathcal{F}.$

The following two results are straight-forward generalizations of Theorem 6.1.3 and Theorem 6.2.2 in [38]. For unexplained notation we refer to [25].

Proposition A.1. Let X be an $(\mathbb{F}^0_+, \mathbb{P})$ -semimartingale with characteristics (B,C,ν) after time $s \in [0,T], \ \tau \in \mathcal{T}_s(\mathbb{F}^0)$ and $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ be a c.p.d. of \mathbb{P} given \mathcal{F}_{τ}^{0} . Then, for \mathbb{P} -a.e. $\omega \in \Omega$, the process X is an $(\mathbb{F}_{+}^{0}, \mathbb{P}_{\omega})$ -semimartingale with characteristics $(p_{\tau(\omega)}B, p_{\tau(\omega)}C, p_{\tau(\omega)}\nu)$ after time $\tau(\omega)$.

Proof. By Theorem II.2.2.1 in [25], the processes $X(h)-B-X_s$, $M(h)^iM(h)^j \tilde{C}^{ij}$, $i, j \leq d$, and $g * (p_s \mu^X) - g * \nu$, $g \in \mathcal{C}^+(\mathbb{R}^d)$ (see [25] for the definition

of $\mathcal{C}^+(\mathbb{R}^d)$), are $(\mathbb{F}^0_+, \mathbb{P})$ -local martingales after time s. Hence, by Theorem 1.2.10 in [38] (,which, after localization, is applicable by the same argument as Lemma III.2.48 in [25] in the proof of Theorem III.2.40, p. 165, in [25]), there exists a \mathbb{P} -null set $N \subset \Omega$ such that, for every $\omega \in \Omega \setminus N$, the processes $X(h) - p_{\tau(\omega)}B - X_{\tau(\omega)}$, $M(h)^i M(h)^j - p_{\tau(\omega)} \tilde{C}^{ij} - M(h)^i_{\tau(\omega)} - M(h)^j_{\tau(\omega)}$, $i, j \leq d$, and $g * (p_{\tau(\omega)}\mu^X) - g * (p_{\tau(\omega)}\nu)$, $g \in \mathcal{C}^+(\mathbb{R}^d)$ are local martingales. Hence, since the canonical decomposition of X(h) after time $\tau(\omega)$, $\omega \in \Omega \setminus N$, is

$$X(h) = X_{\tau(\omega)} + M(h) - M(h)_{\tau(\omega)} + B(h) - B(h)_{\tau(\omega)},$$

Theorem II.2.21 in [25] concludes the proof.

Corollary A.2. Suppose that, for every $(s, \omega) \in [0, T] \times \Omega$, there exists a unique solution $\mathbb{P}_{s,\omega}$ of the martingale problem for $(p_s B, p_s C, p_s \nu)$ starting at (s, ω) . Then, for every $\tau \in \mathcal{T}_s(\mathbb{F}^0)$, the family $\{\mathbb{P}_{\tau(\tilde{\omega}),\tilde{\omega}}\}_{\tilde{\omega}\in\Omega}$ is an r.c.p.d. of $\mathbb{P}_{s,\omega}$ given \mathcal{F}_{τ}^0 .

Proof. Let $\{\mathbb{P}_{\tilde{\omega}}\}_{\tilde{\omega}\in\Omega}$ be an r.c.p.d. of $\mathbb{P}_{s,\omega}$ given \mathcal{F}_{τ}^{0} . By Proposition A.1, for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega}\in\Omega$, $\mathbb{P}_{\tilde{\omega}}$ is a solution of the martingale problem for

$$(p_{\tau(\tilde{\omega})}B, p_{\tau(\tilde{\omega})}C, p_{\tau(\tilde{\omega})}\nu)$$

starting at $(\tau(\tilde{\omega}), \tilde{\omega})$. By uniqueness, $\mathbb{P}_{\tilde{\omega}} = \mathbb{P}_{\tau(\tilde{\omega}), \tilde{\omega}}$.

The next result is crucial.

Theorem A.3 (Proof communicated by R. Mikulevicius). Let \mathbb{P} be a probability measure on $(\Omega, \mathcal{F}_T^0)$. Let $\tau \in \mathcal{T}(\mathbb{F}_+^0)$. Let $\{\mathbb{P}_{\omega}\}_{\omega \in \Omega}$ be a c.p.d. of \mathbb{P} given $\mathcal{F}_{\tau+}^0$. Then, for every $\omega \in \Omega$,

(A.1)
$$\mathbb{P}_{\omega}(X_t = \omega_t, \ 0 \le t \le \tau(\omega)) = 1.$$

Proof. Step 1. Fix a bounded $\mathcal{F}_T^0 \otimes \mathcal{F}_{\tau+}^0$ -measurable function $H: \Omega \times \Omega \to \mathbb{R}$ and put $\bar{H}(\tilde{\omega}) := H(\tilde{\omega}, \tilde{\omega})$. We claim that, for \mathbb{P} -a.e. $\omega \in \Omega$,

(A.2)
$$\mathbb{E}^{\mathbb{P}}[\bar{H}|\mathcal{F}_{\tau+}^{0}](\omega) = \int H(\tilde{\omega}, \omega) \, \mathbb{P}_{\omega}(d\tilde{\omega}).$$

If H is of the form $H(\tilde{\omega}, \omega) = G_1(\tilde{\omega})G_2(\omega)$, G_1 \mathcal{F}_T^0 -measurable, G_2 $\mathcal{F}_{\tau+}^0$ -measurable, then, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\mathbb{E}^{\mathbb{P}}[\bar{H}|\mathcal{F}_{\tau+}^{0}](\omega) = G_{2}(\omega)\mathbb{E}^{\mathbb{P}}[G_{1}|\mathcal{F}_{\tau+}^{0}](\omega)$$
$$= G_{2}(\omega)\int G_{1}(\tilde{\omega})\,\mathbb{P}_{\omega}(d\tilde{\omega}) = \int H(\omega,\tilde{\omega})\,\mathbb{P}_{\omega}(d\tilde{\omega}).$$

A monotone-class argument yields the claim.

Step 2. Fix $t \in [0,T]$ and define $H: \Omega \times \Omega \to \mathbb{R}$ by $H(\tilde{\omega},\omega) := \mathbf{1}_A(\tilde{\omega},\omega)$, where

$$A:=\{(\tilde{\omega},\omega)\in\Omega\times\Omega:\tilde{\omega}_{t\wedge\tau(\omega)}=\omega_{t\wedge\tau(\omega)}\}.$$

Since H is $\mathcal{F}_T^0 \otimes \mathcal{F}_{\tau+}^0$ -measurable and $\bar{H}(\tilde{\omega}) = 1$, Step 1 yields that, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$1 = \mathbb{E}^{\mathbb{P}}[\bar{H}|\mathcal{F}_{\tau+}^0] = \int \mathbf{1}_A(\tilde{\omega}, \omega) \, \mathbb{P}_{\omega}(d\tilde{\omega}) = \mathbb{P}_{\omega}(X_{t \wedge \tau(\omega)} = \omega_{t \wedge \tau(\omega)}).$$

Thus (A.1) holds up to a \mathbb{P} -null set and on this null set we can redefine \mathbb{P}_{ω} such that (A.1) holds there, too (cf. p. 34 in [38]). This concludes the proof.

Remark A.4. Note that the σ -field $\mathcal{F}_{\tau+}^0$ is not countably generated, which yields non-existence of r.c.p.d. (see [5], where r.c.p.d. in our sense are called proper r.c.p.d.). Hence we cannot rely on the corresponding proof in [38], when $\mathcal{F}_{\tau+}^0$ is replaced by \mathcal{F}_{τ}^0 and $\tau \in \mathcal{T}(\mathbb{F}^0)$.

The following result is an adaption of Lemma 2 in [30] to our setting. Again, for unexplained notation, see [25] and also [37].

Lemma A.5. Let $(s, \omega) \in \overline{\Lambda}$, \mathbb{P} be a solution of the martingale problem for $(p_s B, p_s C, p_s \nu)$ starting at (s, ω) , $\tau \in \mathcal{T}_s(\mathbb{F}^0_+)$, and $\{\mathbb{P}_{\tilde{\omega}}\}_{\tilde{\omega} \in \Omega}$ be a c.p.d. of \mathbb{P} given $\mathcal{F}^0_{\tau+}$. Then, for \mathbb{P} -a.e. $\tilde{\omega} \in \Omega$, the probability measure $\mathbb{P}_{\tilde{\omega}}$ is a solution of the martingale problem for $(p_{\tau(\tilde{\omega})}B, p_{\tau(\tilde{\omega})}C, p_{\tau(\tilde{\omega})}\nu)$ starting at $(\tau(\tilde{\omega}), \tilde{\omega})$.

Remark A.6. (i) Each \mathbb{F}^0 -stopping time τ satisfies $\mathbb{P}_{\tau,\omega}(\tau = \tau(\omega)) = 1$ for every ω . This follows easily from Galmarino's test (see [13]).

- (ii) Given a right-continuous \mathbb{F} -adapted process Y such that $Y_t(\omega) = Y_t(\omega_{\cdot \wedge t})$ (this is sometimes nearly impossible to verify) and a closed subset E of \mathbb{R} , the \mathbb{F} -stopping time $\tau := \inf\{t \geq 0 : Y_t \in E\} \wedge T$ satisfies $\mathbb{P}_{\tau,\omega}(\tau = \tau(\omega)) = 1$. To see this, let ω and $\tilde{\omega}$ two paths that coincide on $[0, \tau(\omega)]$. First note that $\tau(\omega) = T$ or, by right-continuity, $Y_{\tau}(\omega) = Y_{\tau(\omega)}(\tilde{\omega}) \in E$. Moreover, if $0 \leq t < \tau(\omega)$, then $Y_t(\omega) = Y_t(\tilde{\omega}) \not\in E$. Hence $\tau(\tilde{\omega}) = \tau(\omega)$.
- (iii) If we assume that the set E in the preceding paragraph is open instead of closed, then a corresponding result does not necessarily hold. For example, let $T=2, \ \tau=\inf\{t\geq 0: |X_t|>1\} \land T, \ \omega\in\Omega$ be defined by $\omega_t=t$, and $\tilde{\omega}_t:=t.\mathbf{1}_{[0,1]}+(2-t).\mathbf{1}_{(1,T]}$. Then $\tau(\omega)=1$ but $\tau(\tilde{\omega})=T$.

Proof of Lemma A.5. First, note that, by Theorem A.3, for \mathbb{P} -a.e. $\tilde{\omega} \in \Omega$, we have $X_t = \tilde{\omega}_t$, $0 \le t \le \tau(\tilde{\omega})$.

In the next two steps, let $M = (M_t)_{t \geq s}$ be one of following processes:

$$X(h) - p_s B - X_s,$$

$$M(h)^i M(h)^j - p_s \tilde{C}^{ij}, i, j \le d,$$

$$g * (p_s \mu^X) - g * (p_s \nu), g \in \mathcal{C}^+(\mathbb{R}^d).$$

Here, we can and will assume that $C^+(\mathbb{R}^d)$ is countable.

Step 1. By Theorem II.2.2.1 in [25], M is an $(\mathbb{F}_+^0, \mathbb{P})$ -local martingale. Moreover, M is \mathbb{F}^0 -adapted. Let $(\sigma_l)_l$ be a corresponding localizing sequence of \mathbb{F}^0 -stopping times (cf. the proof of Lemma III.2.48 in [25]). Without loss of generality, let us assume that M^{σ_l} is bounded. Then, for every $A \in \mathcal{F}_{\tau+}^0$,

 $l \in \mathbb{N}$, $r, r' \in [0, T]$ with $r \leq r'$, and $\eta \in b\mathcal{F}_r^0$, we can apply the claim in Step 1 of the proof of Theorem A.3 to get

$$\int_{A} \mathbb{E}^{\mathbb{P}_{\tilde{\omega}}} \left[\eta (M_{r' \vee \tau(\tilde{\omega})}^{\sigma_{l}} - M_{r \vee \tau(\tilde{\omega})}^{\sigma_{l}}) \right] \mathbb{P}(d\tilde{\omega})
= \int_{A} \mathbb{E}^{\mathbb{P}} \left[\eta \mathbb{E}^{\mathbb{P}} \left[M_{r' \vee \tau}^{\sigma_{l}} - M_{r \vee \tau_{k}}^{\sigma_{l}} | \mathcal{F}_{(r \vee \tau)+}^{0} \right] | \mathcal{F}_{\tau+}^{0} \right] \mathbb{P}(d\tilde{\omega}) = 0.$$

Step 2. For every $n \in \mathbb{N}$, fix a countable dense subset J_n of $C_0^{\infty}(\mathbb{R}^{dn})$ with respect to the locally uniform topology. For every $r \in [0,T] \cap (\mathbb{Q} \cup \{T\})$, denote by Ξ_r the set of all $\eta: \Omega \to \mathbb{R}$ of the form $\eta = f(X_{s_n}, \ldots, X_{s_1})$ for some $n \in \mathbb{N}$, $s_1, \ldots, s_n \in [0,r] \cap (\mathbb{Q} \cup \{r\})$ with $s_1 \leq \ldots \leq s_n$, and $f \in J_n$. Put $\Xi := \cap_r \Xi_r$. Since Ξ is countable, there exists, by Step 1, a set $\Omega_M \subset \Omega$ with $\mathbb{P}(\Omega_M) = 1$ such that, for every $l \in \mathbb{N}$, every $r, r' \in [0,T] \cap (\mathbb{Q} \cup \{T\})$ with $r \leq r'$, every $\eta \in \Xi_r$, and every $\tilde{\omega} \in \Omega_M$,

$$(A.3) \mathbb{E}^{\mathbb{P}_{\tilde{\omega}}}[\eta(M^{\sigma_l}_{r'\vee\tau(\tilde{\omega})}-M^{\sigma_l}_{r\vee\tau(\tilde{\omega})})]=0.$$

Since $\sigma(\Xi_r) = \mathcal{F}_r^0$, (A.3) holds also for every $\eta \in b\mathcal{F}_r^0$. Right-continuity of M implies that $M_{\tau(\tilde{\omega})\vee}^{\sigma_l}$ is an $(\mathbb{F}_+^0, \mathbb{P}_{\tilde{\omega}})$ -martingale after time $\tau(\tilde{\omega})$. Hence M is an $(\mathbb{F}_+^0, \mathbb{P}_{\tilde{\omega}})$ -local martingale after time $\tau(\tilde{\omega})$.

Step 3. Since $\mathbb{P}(\cap_M \Omega_M) = 1$, Step 2 and a second application of Theorem II.2.2.1 in [25] conclude the proof.

Proposition A.7. For every $(s,\omega) \in \bar{\Lambda}$, $\eta \in b\mathcal{F}_T^{s,\omega}$, and $\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$,

$$\mathbb{E}_{\tau,X}[\eta] = \mathbb{E}_{s,\omega}[\eta|\mathcal{F}^{s,\omega}_{\tau}], \quad \mathbb{P}_{s,\omega}$$
-a.s.

Proof. Note that there exists a $\tilde{\tau} \in \mathcal{T}_s(\mathbb{F}^0_+)$ and set $\Omega' \subset \Omega$ such that $\mathbb{P}_{s,\omega}(\Omega') = 1$ and $\tau(\tilde{\omega}) = \tilde{\tau}(\tilde{\omega})$ for every $\tilde{\omega} \in \Omega'$. Also note that there exists a c.p.d. $\{\mathbb{P}_{\tilde{\omega}}\}_{\tilde{\omega}\in\Omega}$ of $\mathbb{P}_{s,\omega}$ given $\mathcal{F}^0_{\tilde{\tau}+}$, which, by Theorem A.3, satisfies

$$\mathbb{P}_{\tilde{\omega}}(X_t = \tilde{\omega}_t, \, 0 \le t \le \tilde{\tau}(\tilde{\omega})) = 1.$$

Step 1. We will show that, for every $\eta \in b\mathcal{F}_T^0$, $\mathbb{E}\left[\eta \middle| \mathcal{F}_{\tau}^{s,\omega}\right] = \mathbb{E}_{\tilde{\tau},X}\left[\eta\right]$, \mathbb{P} -a.s. First, we show that $\mathcal{F}_{\tau}^{s,\omega} = \mathcal{F}_{\tilde{\tau}}^{s,\omega}$. To this end, let $A \in \mathcal{F}_{\tau}^{s,\omega}$ and $t \in [s,T]$. Then

$$A \cap \{\tilde{\tau} \leq t\} = [A \cap \{\tau \leq t\} \cap \{\tau = \tilde{\tau}\}] \cup [A \cap \{\tilde{\tau} \leq t\} \cap \{\tau \neq \tilde{\tau}\}] \in \mathcal{F}^{s,\omega}_t.$$

That is, $\mathcal{F}_{\tau}^{s,\omega} \subseteq \mathcal{F}_{\tilde{\tau}}^{s,\omega}$. The other inclusion can be shown in the same way. Consequently, using Lemma A.5, we have (A.4)

$$\mathbb{E}\left[\eta|\mathcal{F}^{s,\omega}_{\tau}\right](\tilde{\omega}) = \mathbb{E}\left[\eta|\mathcal{F}^{s,\omega}_{\tilde{\tau}}\right](\tilde{\omega}) = ^{(*)} \ \mathbb{E}\left[\eta|\mathcal{F}^{0}_{\tilde{\tau}+}\right](\tilde{\omega}) = \mathbb{E}^{\mathbb{P}_{\tilde{\omega}}}[\eta] = \mathbb{E}_{\tilde{\tau},\tilde{\omega}}[\eta]$$

for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$. Note that the equality (*) follows from the fact that first for every $A \in \mathcal{F}^{s,\omega}_{\tilde{\tau}}$ there exist, by Theorem II.75.3 in [37], sets $A' \in \mathcal{F}^0_{\tilde{\tau}+}$ and $N \in \mathcal{N}_{s,\omega}$ such that $A = A' \cup N$ and then

$$\mathbb{E}_{s,\omega} \left[\mathbf{1}_{A} \mathbb{E}_{s,\omega} \left[\eta | \mathcal{F}_{\tilde{\tau}}^{s,\omega} \right] \right] = \mathbb{E}_{s,\omega} \left[\mathbf{1}_{A} \eta \right] = \mathbb{E}_{s,\omega} \left[\mathbf{1}_{A'} \eta \right] = \mathbb{E}_{s,\omega} \left[\mathbf{1}_{A'} \mathbb{E}_{s,\omega} \left[\eta | \mathcal{F}_{\tilde{\tau}+}^{0} \right] \right]$$
$$= \mathbb{E}_{s,\omega} \left[\mathbf{1}_{A} \mathbb{E}_{s,\omega} \left[\eta | \mathcal{F}_{\tilde{\tau}+}^{0} \right] \right]$$

and that second $\mathbb{E}_{s,\omega}\left[\eta|\mathcal{F}_{\tilde{\tau}+}^{0}\right]$ is $\mathcal{F}_{\tilde{\tau}}^{s,\omega}$ -measurable. Finally,

$$\mathbb{E}_{\tau,X}[\eta] = \mathbf{1}_{\{\tau = \tilde{\tau}\}}.\mathbb{E}_{\tilde{\tau},X}[\eta] + \mathbf{1}_{\{\tau \neq \tilde{\tau}\}}.\mathbb{E}_{\tau,X}[\eta], \quad \mathbb{P}_{s,\omega}\text{-a.s.}$$

Thus, the map $\tilde{\omega} \mapsto \mathbb{E}_{\tau(\tilde{\omega}),\tilde{\omega}}[\eta]$ is $\mathcal{F}_{\tau}^{s,\omega}$ -measurable and $\mathbb{E}_{\tau,X}[\eta] = \mathbb{E}_{s,\omega}[\eta|\mathcal{F}_{\tau}^{s,\omega}]$, $\mathbb{P}_{s,\omega}$ -a.s.

Step 2. To finish the proof of the corollary, we can, without loss of generality because of Step 1, assume that $\eta = \mathbf{1}_A$ with $A \subset B \in \mathcal{F}_T^0$ and $\mathbb{P}_{s,\omega}(B) = 0$. Using Step 1, we have

$$0 \leq \mathbb{E}_{\tau(\tilde{\omega}),\tilde{\omega}} \mathbf{1}_A \leq \mathbb{E}_{\tau(\tilde{\omega}),\tilde{\omega}} \mathbf{1}_B = \mathbb{E}_{s,\omega} [\mathbf{1}_B | \mathcal{F}_{\tau}^{s,\omega}](\tilde{\omega}) = \mathbb{E}_{s,\omega} [\mathbf{1}_A | \mathcal{F}_{\tau}^{s,\omega}](\tilde{\omega}) = 0$$
 for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$.

Proposition A.8. Fix $(s, \omega) \in \bar{\Lambda}$. Let M be a bounded right-continuous \mathbb{F}^0_+ -adapted $(\mathbb{F}^{s,\omega}, \mathbb{P}_{s,\omega})$ -martingale. Let $\tau \in \mathcal{T}_s(\mathbb{F}^{s,\omega})$. Then, for $\mathbb{P}_{s,\omega}$ -a.e. $\tilde{\omega} \in \Omega$, M is an $(\mathbb{F}^{\tau,\tilde{\omega}}, \mathbb{P}_{\tau,\tilde{\omega}})$ -martingale after time $\tau(\tilde{\omega})$. A corresponding result holds for sub- and supermartingales.

Proof. Let $\tilde{\tau} \in \mathcal{T}_s(\mathbb{F}^0_+)$ satisfy $\tilde{\tau} = \tau$, $\mathbb{P}_{s,\omega}$ -a.s. For every $n \in \mathbb{N}$, fix a countable dense subset J_n of $C_b(\mathbb{R}^n)$ with respect to the locally uniform topology. For every $r \in [0,T] \cap (\mathbb{Q} \cup \{T\})$, denote by Ξ_r the set of all $\eta: \Omega \to \mathbb{R}$ of the form $\eta = f(X_{s_n}, \ldots, X_{s_1})$ for some $n \in \mathbb{N}$, $s_1, \ldots, s_n \in [0,r] \cap (\mathbb{Q} \cup \{r\})$ with $s_1 \leq \ldots \leq s_n$, and $f \in J_n$. Put $\Xi := \cap_r \Xi_r$. Since Ξ is countable, there exists, by Step 1 of the proof of Proposition A.7, a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for every $r, r' \in [0,T] \cap (\mathbb{Q} \cup \{T\})$ with $r \leq r'$, every $\eta \in \Xi_r$, and every $\tilde{\omega} \in \Omega'$,

$$\begin{split} \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{r' \vee \tau(\tilde{\omega})} \right] &= \mathbb{E}_{\tilde{\tau},\tilde{\omega}} \left[\eta M_{r' \vee \tilde{\tau}(\tilde{\omega})} \right] \\ &= \mathbb{E}_{s,\omega} \left[\eta M_{r' \vee \tilde{\tau}} | \mathcal{F}_{\tilde{\tau}}^{0} \right] \left(\tilde{\omega} \right) & \text{by (A.2)} \\ &= \mathbb{E}_{s,\omega} \left[\eta M_{r' \vee \tilde{\tau}} | \mathcal{F}_{\tilde{\tau}}^{s,\omega} \right] \left(\tilde{\omega} \right) & \text{by (A.4)} \\ &= \mathbb{E}_{s,\omega} \left[\eta \mathbb{E}_{s,\omega} \left[M_{r' \vee \tilde{\tau}} | \mathcal{F}_{r \vee \tilde{\tau}}^{s,\omega} \right] | \mathcal{F}_{\tilde{\tau}}^{s,\omega} \right] \left(\tilde{\omega} \right) & \text{since } \eta \in \mathbf{b} \mathcal{F}_{r}^{0} \subset \mathbf{b} \mathcal{F}_{r \vee \tilde{\tau}} \\ &= \mathbb{E}_{s,\omega} \left[\eta M_{r \vee \tilde{\tau}} | \mathcal{F}_{\tilde{\tau}}^{s,\omega} \right] \left(\tilde{\omega} \right) & \text{by (A.4)} \\ &= \mathbb{E}_{s,\omega} \left[\eta M_{r \vee \tilde{\tau}} | \mathcal{F}_{\tilde{\tau}}^{0} \right] \left(\tilde{\omega} \right) & \text{by (A.2)} \\ &= \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{r \vee \tilde{\tau}} (\tilde{\omega}) \right] & \text{by (A.2)} \end{split}$$

Since $\mathcal{F}_r^0 = \sigma(\Xi_r)$, $\mathbb{E}_{\tau,\tilde{\omega}}[\eta M_{r'\vee\tau(\tilde{\omega})}] = \mathbb{E}_{\tau,\tilde{\omega}}[\eta M_{r\vee\tau(\tilde{\omega})}]$ for every $\eta \in b\mathcal{F}_r^0$ hence, also for every $\eta \in b\mathcal{F}_r^{\tau,\tilde{\omega}}$, because, by Proposition III.4.32 in [25], we have $\mathcal{F}_r^0 \vee \mathcal{N}_{\tau,\tilde{\omega}} = \mathcal{F}_r^{\tau,\tilde{\omega}}$. Next, let $0 \leq \hat{s} \leq s' < T$, and let (r_n) and (r'_n) be sequences in $[0,T] \cap \mathbb{Q}$ with $\hat{s} = \inf_n r_n$, $s' = \inf_n r'_n$, and $r_n \leq r'_n$. Then, for every $k \in \mathbb{N}$, every $\tilde{\omega} \in \Omega'$, and every $\eta \in b\mathcal{F}_{\hat{s}}^{\tau,\tilde{\omega}}$, we have, by right-continuity of M and the dominated convergence theorem,

$$\begin{split} \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{s' \vee \tau(\tilde{\omega})} \right] &= \lim_{n} \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{r'_{n} \vee \tau(\tilde{\omega})} \right] \\ &= \lim_{n} \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{r_{n} \vee \tau(\tilde{\omega})} \right] = \mathbb{E}_{\tau,\tilde{\omega}} \left[\eta M_{\hat{s} \vee \tau(\tilde{\omega})} \right]. \end{split}$$

Since M is \mathbb{F}^0_+ -adapted, it is also $\mathbb{F}^{\tau,\tilde{\omega}}$ -adapted and thus $M_{\cdot\vee\tau(\tilde{\omega})}$ is an $(\mathbb{F}^{\tau,\tilde{\omega}},\mathbb{P}_{\tau,\tilde{\omega}})$ -martingale after time $\tau(\tilde{\omega})$.

APPENDIX B. SKOROHOD'S TOPOLOGIES

In this appendix, we recall some definitions and basic results from [42]. Put $\mathbb{D} := \mathbb{D}([0,T],\mathbb{R})$. The *completed graph* of a path $\omega \in \mathbb{D}$ is defined as the set

$$\Gamma_{\omega} := \{(t, x) \in [0, T] \times \mathbb{R} : \exists \alpha \in [0, 1] : x = \alpha \omega_{t-} + (1 - \alpha)\omega_t \},$$

where $\omega_{0-} := \omega_0$. We equip Γ_{ω} with a linear order \leq defined as follows: Given $(t,x), (t',x') \in \Gamma_{\omega}$, we write $(t,x) \leq (t',x')$ if either t < t' or both, t = t' as well as $|x - \omega_{t-}| \leq |x' - \omega_{t-}|$ hold. A parametric representation of ω is a mapping $(r,z) : [0,1] \to \Gamma_{\omega}$ that is continuous, nondecreasing, and surjective. The set of all parametric representations of ω is denoted by $\Pi(\omega)$. Define $d_{M_1} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}_+$ by

$$d_{M_1}(\omega,\omega') := \inf_{\substack{(r,z) \in \Pi(\omega) \\ (r',z') \in \Pi(\omega')}} ||r-r'||_{\infty} \vee ||z-z'||_{\infty}.$$

Note that d_{M_1} is a metric (Theorem 12.3.1 in [42]).

Lemma B.1. Let $0 \le t^n \le t^0 \le T$. Put $\omega^n = \mathbf{1}_{[t^n,T]}$, $\omega = \mathbf{1}_{[t^0,T]} \in \mathbb{D}$. Then $d_{M_1}(\omega,\omega^n) \le t^0 - t^n$.

Proof. Fix 0 < a < b < 1. We distinguish between the cases $t^0 < T$ and $t^0 = T$.

(i) Without loss of generality, let $T=2,\,t^0=1,\,$ and $t^n=1-n^{-1}.$ Then $\omega=\mathbf{1}_{[1,2]}$ and $\omega^n=\mathbf{1}_{[1-n^{-1},2]}.$ Define $(r,z)\in\Pi(\omega)$ and $(r^n,z^n)\in\Pi(\omega^n)$ by

$$r(t) := \frac{t}{a} \mathbf{1}_{[0,a]}(t) + \mathbf{1}_{(a,b]}(t) + \left(\frac{1-t}{1-b} \cdot 1 + \frac{t-b}{1-b} \cdot 2\right) \mathbf{1}_{(b,1]}(t),$$

$$r^{n}(t) := \frac{t}{a} (1-n^{-1}) \mathbf{1}_{[0,a]}(t) + (1-n^{-1}) \mathbf{1}_{(a,b]}(t)$$

$$+ \left(\frac{1-t}{1-b} \cdot (1-n^{-1}) + \frac{t-b}{1-b} \cdot 2\right) \mathbf{1}_{(b,1]}(t),$$

$$z(t) = z^{n}(t) := \frac{t-a}{b-a} \mathbf{1}_{[a,b]}(t) + \mathbf{1}_{(b,1]}(t).$$

Then $||r - r^n||_{\infty} = n^{-1}$ and $||z - z^n||_{\infty} = 0$. Thus $d_{M_1}(\omega, \omega^n) \le n^{-1}$.

(ii) Without loss of generality, let $T=t^0=1$ and $t^n=1-n^{-1}$. Then $\omega=\mathbf{1}_{\{T\}}$ and $\omega^n=\mathbf{1}_{[1-n^{-1},1]}$. Define $(r,z)\in\Pi(\omega)$ and $(r^n,z^n)\in\Pi(\omega^n)$ by

$$r(t) := \frac{t}{a} \mathbf{1}_{[0,a]}(t) + \mathbf{1}_{(a,1]}(t),$$

$$r^{n}(t) := \frac{t}{a} (1 - n^{-1}) \mathbf{1}_{[0,a]}(t) + (1 - n^{-1}) \mathbf{1}_{(a,b]}(t)$$

$$+ \left(\frac{1 - t}{1 - b} \cdot (1 - n^{-1}) + \frac{t - b}{1 - b} \cdot 1 \right) \mathbf{1}_{(b,1]}(t),$$

$$z(t) = z^{n}(t) := \frac{t - a}{b - a} \mathbf{1}_{[a,b]}(t) + \mathbf{1}_{(b,1]}(t).$$

Then $||r-r^n||_{\infty} = n^{-1}$, $||z-z^n||_{\infty} = 0$ and consequently $d_{M_1}(\omega,\omega^n) \leq n^{-1}$.

The d_{M_2} -metric on \mathbb{D} is defined by,

$$d_{M_2}(\omega, \tilde{\omega}) := m_H(\Gamma_\omega, \Gamma_{\tilde{\omega}}),$$

where m_H is the Hausdorff distance, that is, given closed sets $A, B \subseteq [0,T] \times \mathbb{R}$,

$$m_H(A,B) := \left[\sup_{a \in A} \inf_{b \in B} |a - b| \right] \vee \left[\sup_{b \in B} \inf_{a \in A} |b - a| \right].$$

The uniform topology on \mathbb{D} and the (usual) Skorohod J_1 -topology are induced by the following metrics:

$$d_{U}(\omega, \tilde{\omega}) := \|\omega - \tilde{\omega}\|_{\infty},$$

$$d_{J_{1}}(\omega, \tilde{\omega}) := \inf_{\lambda} \sup_{[s \in [0,T]} |\lambda(s) - s| \vee |\omega_{\lambda(s)} - \omega'_{s}|,$$

where λ runs through the set of all strictly increasing continuous functions from [0,T] to [0,T] satisfying $\lambda(0)=0$ and $\lambda(T)=T$.

Remark B.2. We have $d_{M_1} \leq d_{J_1} \leq d_U$ (Theorem 12.3.2 in [42]).

APPENDIX C. AUXILIARY RESULTS

Lemma C.1. Let $u \in C(\bar{\Lambda})$. Then $\Delta u_t > 0$ implies $\Delta X_t > 0$.

Proof. Fix $(t, \omega) \in \Lambda$ and let $c := |\Delta u(t, w)| > 0$. Then, there exists an m_0 such that, for every $m \ge m_0$,

$$\left| u(t,\omega) - u(t-m^{-1},\omega) \right| \ge c/2.$$

Since $u \in C^0(\Lambda)$, there exists a $\delta = \delta(t, \omega) > 0$ (independent from m) such that, for every $(t', \omega') \in \Lambda$,

$$\mathbf{d}_{\infty}((t,\omega),(t',\omega')) < \delta \Rightarrow |u(t,\omega) - u(t',\omega')| < c/2.$$

Let $t' = t - m^{-1}$, $\omega' = \omega$. Then

$$\mathbf{d}_{\infty}((t,\omega),(t',\omega')) = m^{-1} + \sup_{s \in [0,T]} \left| \omega_{s \wedge t} - \omega_{s \wedge (t-m^{-1})} \right|$$
$$= m^{-1} + \sup_{s \in [t-m^{-1},t]} \left| \omega_s - \omega_{t-m^{-1}} \right|$$
$$> \delta$$

if $m \ge m_0$. Let $m_1 \ge m_0$ be large enough so that $m^{-1} < \delta/2$ whenever $m \ge m_1$. Now, let $m \ge m_1$. Then

$$\sup_{s\in[t-m^{-1},t]}|\omega_s-\omega_{t-m^{-1}}|>\delta/2.$$

Letting $m \to \infty$ yields $|\Delta X_t(\omega)| \ge \delta/2 > 0$.

Lemma C.2. Let (S, \mathfrak{S}, μ) be a finite measure space, (P, \mathfrak{P}) be a measurable space, and $\{f_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded family of measurable functions from $S\times P$ to $[0,\infty)$ such that, for μ -a.e. $s\in S$,

$$\sup_{p \in P} |f_n(s, p) - f(s, p)| \to 0$$

as $n \to \infty$. Then

$$\sup_{p \in P} \int |f_n(s, p) - f(s, p)| \ \mu(ds) \to 0$$

as $n \to \infty$.

Proof. The proof follows the lines of a standard proof of the Dominated Convergence Theorem. By Fatou's lemma,

$$\limsup_{n \to \infty} \sup_{p \in P} \int |f_n(s, p) - f(s, p)| \ \mu(ds)$$

$$\leq \limsup_{n \to \infty} \int \operatorname{ess}_{p \in P}^{\mu} |f_n(s, p) - f(s, p)| \ \mu(ds)$$

$$\leq \int \limsup_{n \to \infty} \operatorname{ess}_{p \in P}^{\mu} |f_n(s, p) - f(s, p)| \ \mu(ds)$$

$$\leq \int \limsup_{n \to \infty} \sup_{p \in P} |f_n(s, p) - f(s, p)| \ \mu(ds)$$

$$\leq \int \limsup_{n \to \infty} \sup_{p \in P} |f_n(s, p) - f(s, p)| \ \mu(ds)$$

$$= 0.$$

This concludes the proof.

Lemma C.3. Let $t, t^n \in [0,T]$ with $t \leq t^n$. Let $\iota \in \mathbb{N}$, $0 = r_0 < r_1 < \ldots < r_\iota < T - t$ and $z_j \in \mathbb{R}^d$, $j = 0, \ldots, \iota$. Let $t^n + r_\iota < T$. Consider two paths

 $\omega, \omega^n \in \Omega$ defined by

$$\omega := \sum_{j=0}^{\iota-1} z_j . \mathbf{1}_{[t+r_j, t+r_{j+1})} + z_{\iota} . \mathbf{1}_{[t+r_{\iota}, T)},$$

$$\omega^n := \sum_{i=0}^{\iota-1} z_j . \mathbf{1}_{[t^n + r_j, t^n + r_{j+1})} + z_{\iota} . \mathbf{1}_{[t^n + r_{\iota}, T)}.$$

Then $d_{J_1}(\omega^n, \omega) \leq 2(t^n - t)$.

Proof. Let $\lambda_n: [0,T] \to [0,T]$ be a strictly increasing and continuous function with $\lambda_n(0) = 0$, $\lambda_n(t+r_j) = t^n + r_j$, $j = 0, \ldots, \iota$, $\lambda_n(T) = T$, and $\|\lambda_n - \mathrm{id}\|_{\infty} \le 2(t_n - t)$. Then, given $s \in [0,T)$, we have $\omega_s - \omega_{\lambda_n(s)}^n = 0$, given $s \in [t+r_j, t+r_{j+1})$, $j = 0, \ldots, \iota - 1$, we have

$$\omega_s - \omega_{\lambda_n(s)}^n = z_j - \omega_{\lambda_n(t+r_i)}^n = z_j - \omega_{t^n+r_i}^n = 0,$$

and, given $s \in [t + r_{\iota}, T)$, we have

$$\omega_s - \omega_{\lambda_n(s)}^n = z_\iota - \omega_{\lambda_n(t+r_\iota)}^n = z_\iota - \omega_{t^n+r_\iota}^n = 0.$$

This concludes the proof.

Lemma C.4. Fix $(t, \omega) \in \bar{\Lambda}$ and $s \in [t, T]$. For $\mathbb{P}_{t,\omega}$ -a.e. $\tilde{\omega} \in \Omega$,

$$dX_r^{c,s,\tilde{\omega}} = dX_r^{c,t,\omega}, \ s \le r \le T, \ \mathbb{P}_{s,\tilde{\omega}}\text{-}a.s.$$

Proof. Note that

$$X = X_t + (B - B_t) + X^{c,t,\omega} + z * (\mu - \nu)$$
 on $[t, T]$, $\mathbb{P}_{t,\omega}$ -a.s.,
 $X = X_s + (B - B_s) + X^{c,s,\tilde{\omega}} + z * (\mu - \nu)$ on $[s, T]$, $\mathbb{P}_{s,\tilde{\omega}}$ -a.s.

Define a process V on [t,T] by $V:=X-B-z*(\mu-\nu)$ and put

$$\Omega' := \{ \omega' \in \Omega : X_r^{c,t,\omega}(\omega') = (V - X_t + B_t)(\omega'), \ t \le r \le T \}.$$

Then $\mathbb{P}_{t,\omega}(\Omega') = 1$. Let $\tilde{\omega} \in \Omega'$. Put

$$\Omega'' := \{ \omega'' \in \Omega' : X_r^{c,s,\tilde{\omega}}(\omega'') = (V - X_s + B_s)(\omega''), s \le r \le T \}.$$

Then $\mathbb{P}_{s,\tilde{\omega}}(\Omega'') = 1$ and, for every $(r,\omega'') \in [s,T] \times \Omega''$,

$$X_r^{c,s,\tilde{\omega}}(\omega'') = (V_r - X_t + B_t)(\omega'') + (X_t - X_s + B_s - B_t)(\omega'')$$

= $X_r^{c,t,\omega}(\omega'') + (X_t - X_s + B_s - B_t)(\omega''),$

which concludes the proof.

Lemma C.5. Fix $(s, \omega) \in \bar{\Lambda}$. For i = 1, 2, let $S_s^i \in \mathbb{R}$, β^i be a predictable process on [s, T], $H^i \in L^2_{loc}(X^{c,s,\omega}, \mathbb{P}_{s,\omega})$, and $W^i \in G_{loc}(p_s\mu^X, \mathbb{P}_{s,\omega})$. Then we have the following:

(a) The process S^i defined by

$$S^i := S^i_s + \beta^i \bullet t + H^i \bullet X^{c,s,\omega} + W^i * (\mu^X - \nu)$$

is a special $(\mathbb{F}^0_+, \mathbb{P}^{s,\omega})$ -semimartingale on [s,T] with characteristics $(\tilde{B}^i, \tilde{C}^i, \tilde{\nu}^i)$, where $\tilde{B}^i = \beta^i \cdot t$, $\tilde{C}^i = \sum_{k,l=1}^d (H^{ik}H^{il}c^{kl}) \cdot t$, and $\tilde{\nu}^i$ is a random measure on $[s,T] \times \mathbb{R}$ defined by

$$\tilde{\nu}^i([s,t] \times A, \tilde{\omega}) := \nu(\{(r,z) \in [s,T] \times \mathbb{R}^d : (r,W^i(r,z,\tilde{\omega})) \in [s,t] \times (A \setminus \{\mathbf{0}\})\}, \tilde{\omega})$$
for every $(t,A,\tilde{\omega}) \in [s,T] \times \mathcal{B}(\mathbb{R}) \times \Omega$.

(b) The 2-dimensional process $S := (S^1, S^2)$ is a special $(\mathbb{F}^0_+, \mathbb{P}^{s,\omega})$ -semi-martingale on [s, T] with characteristics $(\tilde{B}, \tilde{C}, \tilde{\nu})$, where $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$,

$$\tilde{C}^{ij} = \sum_{k,l=1}^{d} (H^{ik}H^{jl}c^{kl}) \bullet t$$

for i, j = 1, 2, and $\tilde{\nu}$ is a random measure on $[s, T] \times \mathbb{R}^2$ defined by

$$\tilde{\nu}([s,t] \times A_1 \times A_2, \tilde{\omega}) := \nu(\{(r,z) \in [t,T] \times \mathbb{R}^d : (r,W^1(r,z,\tilde{\omega}),W^2(r,z,\tilde{\omega})) \in [s,t] \times (A_1 \setminus \{\mathbf{0}\}) \times (A_2 \setminus \{\mathbf{0}\})\}, \tilde{\omega})$$

for every $(t, A_1, A_2, \tilde{\omega}) \in [s, T] \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \Omega$.

$$S^{1}S^{2} = S_{s}^{1}S_{s}^{2} + \left[S^{2}\beta^{1} + S^{1}\beta^{2} + \sum_{k,l=1}^{d} H^{1,k}H^{2,l}c^{kl} + \int_{\mathbb{R}^{d}} W^{1}W^{2}K(dz)\right] \cdot t + \sum_{k=1}^{d} (S_{-}^{2}H^{1,k} + S_{-}^{1}H^{2,k}) \cdot X^{k,c,s,\omega} + (S_{-}^{1}x_{2} + S_{-}^{2}x_{1} + x_{1}x_{2}) * (\mu^{S} - \tilde{\nu}).$$

Proof. (a) Using Theorem I.4.40 of [25], we get

$$\left\langle \sum_{k=1}^{d} \gamma^{i,k} \bullet X^{k,c,s,\omega} \right\rangle = \sum_{k=1}^{d} \left\langle H^{ik} \bullet X^{k,c,s,\omega}, \sum_{l=1}^{d} H^{il} \bullet X^{l,c,s,\omega} \right\rangle$$
$$= \sum_{k,l=1}^{d} (H^{ik}H^{il}) \langle X^{k,c,s,\omega}, X^{l,c,s,\omega} \rangle$$
$$= \sum_{k,l=1}^{d} (H^{ik}H^{il}c^{kl}) \bullet t.$$

Now it suffices to show that $x*\tilde{\nu}^i = W^i*\nu$. Define a function $\pi_x: [s,T] \times \mathbb{R} \to \mathbb{R}$ by $\pi_x(t,x) := x$. Then, by Satz 19.1 in [3],

$$(x * \tilde{\nu}^i)_t = \int_s^t \int_{\mathbb{R}} \pi_x(r, x) \, \tilde{\nu}^i(dr, dx, \tilde{\omega})$$
$$= \int_s^t \int_{\mathbb{R}^d} \pi_x(r, W^i(r, z, \tilde{\omega})) \, \nu(dr, dz, \tilde{\omega})$$
$$= (W^i * \nu)_t.$$

(b) By Theorem I.4.40 in [25],

$$\begin{split} \langle S^{i,c}, S^{j,c} \rangle &= \left\langle \sum_{k=1}^d H^{ik} \bullet X^{k,c,s,\omega}, \sum_{l=1}^d H^{jl} \bullet X^{k,c,s,\omega} \right\rangle \\ &= \sum_{k,l=1}^d (H^{ik} H^{jl} c^{kl}) \bullet t. \end{split}$$

Next we show that $x*\tilde{\nu} = (W^1*\nu, W^2*\nu)$. Define functions $\pi_i : [s, T] \times \mathbb{R}^2 \to \mathbb{R}$ by $\pi(t, x_1, x_2) := x_i$, i = 1, 2. Then, again by Satz 19.1 in [3],

$$(x_i * \tilde{\nu})_t = \int_s^t \int_{\mathbb{R}^2} \pi_i(r, x_1, x_2) \, \tilde{\nu}(dr, dx, \tilde{\omega})$$
$$= \int_s^t \int_{\mathbb{R}^d} \pi_i(r, W^1(r, z, \tilde{\omega}), W^2(r, z, \tilde{\omega})) \, \nu(dr, dz, \tilde{\omega})$$
$$= (W^i * \nu)_t.$$

(c) By (b) and by Itô's formula based on local characteristics (see, e.g., Section 2.1.2 of [24]),

$$\begin{split} S^1S^2 &= S_s^1S_s^2 + S_-^2 \bullet \tilde{B}^1 + S_-^1 \bullet \tilde{B}^2 + \tilde{C}^{1,2} \\ &+ \left[(S_-^1 + x_1)(S_-^2 + x_2) - S_-^1S_-^2 - S_-^2x_1 - S_-^1x_2 \right] * \tilde{\nu} \\ &+ S_-^2 \bullet S^{1,c} + S_-^1 \bullet S^{2,c} \\ &+ \left[(S_-^1 + x_1)(S_-^2 + x_2) - S_-^1S_-^2 \right] * (\mu^S - \tilde{\nu}). \end{split}$$

To conclude, note that $S_-^j \bullet \tilde{B}^i = (S^j \beta^i) \bullet t$, $\tilde{C}^{1,2} = \sum_{k,l=1}^d (H^{1,k} H^{2,l} c^{kl}) \bullet t$,

$$\begin{split} & \left[(S_{-}^{1} + x_{1})(S_{-}^{2} + x_{2}) - S_{-}^{1}S_{-}^{2} - S_{-}^{2}x_{1} - S_{-}^{1}x_{2} \right] * \tilde{\nu} \\ &= x_{1}x_{2} * \tilde{\nu} \\ &= (W^{1}W^{2}) * \nu \qquad \qquad \text{by Satz 19.1 of [3]} \\ &= (W^{1}W^{2}) * (dr \otimes K(dz)), \end{split}$$

and
$$S_{-}^{j} \cdot S^{i,c} = \sum_{k=1}^{d} (S_{-}^{j} \gamma^{i,k}) \cdot X^{k,c,s,\omega}$$
.

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