LIE 2-ALGEBRAS OF VECTOR FIELDS

DANIEL BERWICK-EVANS AND EUGENE LERMAN

ABSTRACT. We show that the category of vector fields on a geometric stack has the structure of a Lie 2-algebra. This proves a conjecture of R. Hepworth. The construction uses a Lie groupoid that presents the geometric stack. We show that the category of vector fields on the Lie groupoid is equivalent to the category of vector fields on the stack. The category of vector fields on the Lie groupoid has a Lie 2-algebra structure built from known (ordinary) Lie brackets on multiplicative vector fields of Mackenzie and Xu and the global sections of the Lie algebroid of the Lie groupoid. After giving a precise formulation of Morita invariance of the construction, we verify that the Lie 2-algebra structure defined in this way is well-defined on the underlying stack.

CONTENTS

1. INTRODUCTION

Vector fields on a Lie groupoid G form a category [8]. We denote it by $\mathbb{X}(G)$. The *objects* of $\mathbb{X}(G)$ are the multiplicative vector fields of Mackenzie and Xu [12]. These are functors $v : G \to TG$ satisfying $\pi_G \circ v = \mathrm{id}_G$ where TG denotes the tangent groupoid and $\pi_G : TG \to G$ is the projection functor. A *morphism* $\alpha : v \Rightarrow v'$ in this category is a natural transformation α such that $\pi_G(\alpha(x)) = \mathrm{id}_x$ for every object x of the groupoid G. The first result of this paper is

Theorem 3.4. The category of vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a (strict) Lie 2-algebra. That is, $\mathbb{X}(G)$ is a category internal to the category of Lie algebras.

Remark 1.1. When a manifold M is regarded as a discrete Lie groupoid, $\mathbb{X}(M)$ is the usual Lie algebra of vector fields on M regarded as a discrete Lie 2-algebra.

To every Lie groupoid G there corresponds the stack $\mathbb{B}G$ of principal G-bundles, and Morita equivalent Lie groupoids G and H correspond to isomorphic stacks $\mathbb{B}G$ and $\mathbb{B}H$. It is natural to wonder if the Lie 2-algebra $\mathbb{X}(G)$ lives on the stack $\mathbb{B}G$ in some appropriate sense. To start, we can ask whether Morita equivalent Lie groupoids G and H have "Morita equivalent" Lie 2-algebras $\mathbb{X}(G)$ and $\mathbb{X}(H)$. More precisely we could ask for the existence of a (2-)functor $\mathbb{X}$ from the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles to an appropriate bicategory of Lie 2-algebras that sends Morita equivalences to Morita equivalences. It turns out that such a functor is too much to ask for but there is a functor from a sub-bicategory of Bi.

The reasons behind this fact can already be seen in the case of manifolds. Recall that there is no naturally defined functor from the category of manifolds to the category of Lie algebras that assigns to each manifold its Lie algebra of vector fields. However if we restrict ourselves to the category $\mathsf{Man}_{\mathrm{iso}}$ whose objects are manifolds and whose morphisms are diffeomorphisms then there is a perfectly well defined functor with the desired properties.

Getting back to Lie groupoids, recall that Bi is a localization of the strict 2-category of Lie groupoids, internal functors, and internal natural transformations at the essential equivalences. Lie 2-algebras, internal functors and internal natural transformations form the strict 2-category $\mathsf{Lie2Alg}_{strict}$, and localizing at the essential equivalences produces a bicategory $\mathsf{Lie2Alg}$ (see Subsections 2.b and 2.c below). Let $\mathsf{Bi}_{\mathrm{iso}}$ be the sub-bicategory of Bi whose objects are Lie groupoids, 1-morphisms are (weakly) *invertible* bibundles (i.e., Morita equivalences) and 2-morphisms are isomorphisms of bibundles.

Theorem 4.1. The map $G \mapsto \mathbb{X}(G)$ that assigns to each Lie groupoid its category of vector fields extends to a functor
$$\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}.$$
In particular, if $P : G \to H$ is a Morita equivalence of Lie groupoids then $\mathbb{X}(P) : \mathbb{X}(G) \to \mathbb{X}(H)$ is a (weakly) invertible 1-morphism of Lie 2-algebras in the bicategory $\mathsf{Lie2Alg}$.

Remark 1.2. In the Lie groupoid literature there are two standard constructions that associate a Lie algebra to a Lie groupoid: global sections of its Lie algebroid and Mackenzie and Xu's multiplicative vector fields. The Lie 2-algebra structure on $\mathbb{X}(G)$ is built out of this pair of Lie algebras. At first pass this might seem surprising: neither multiplicative vector fields nor sections of Lie algebroids are well-behaved under Morita equivalence of Lie groupoids. Theorem 4.1 shows that combining this pair of Lie algebras into a Lie 2-algebra gives us an object that *is* preserved by Morita equivalence.

How does the existence of the functor in Theorem 4.1 imply that the Lie 2-algebra $\mathbb{X}(G)$ "lives" on the stack $\mathbb{B}G$? To answer this, we need to recall the relationship between the bicategory Bi and the 2-category Stack of stacks over the site of smooth manifolds. The assignment $G \mapsto \mathbb{B}G$ extends to a fully faithful functor
$$\mathbb{B} : \mathsf{Bi} \to \mathsf{Stack}.$$
The essential image of the functor $\mathbb{B}$ is the 2-category $\mathsf{GeomStack}$ of geometric stacks. Restricting the functor $\mathbb{B}$ to the bicategory $\mathsf{Bi}_{\mathrm{iso}}$ of groupoids and Morita equivalences gives us an equivalence of bicategories
$$\mathbb{B} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{GeomStack}_{\mathrm{iso}},$$
where $\mathsf{GeomStack}_{\mathrm{iso}}$ is the 2-category of geometric stacks, isomorphisms of stacks (that is, weakly invertible 1-morphisms of stacks) and 2-morphisms. By inverting this equivalence $\mathbb{B}$ and composing it with the functor $\mathbb{X}$ we get a functor
$$\mathsf{GeomStack}_{\mathrm{iso}} \xrightarrow{\mathbb{B}^{-1}} \mathsf{Bi}_{\mathrm{iso}} \xrightarrow{\mathbb{X}} \mathsf{Lie2Alg}. \tag{1.1}$$
So in particular we get a functorial assignment of a Lie 2-algebra to every geometric stack, with isomorphic stacks being assigned "isomorphic" Lie 2-algebras.

In [8] Hepworth introduced a category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$. We introduce a category $\mathsf{Vect}'(\mathcal{A})$ equivalent to $\mathsf{Vect}(\mathcal{A})$ which is more convenient for our purposes. In particular, the assignment
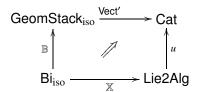$$\mathcal{A} \mapsto \mathsf{Vect}'(\mathcal{A})$$
easily extends to a functor
$$\mathsf{Vect}' : \mathsf{GeomStack}_{\mathrm{iso}} \to \mathsf{Cat}$$
where Cat is the 2-category of categories. We'll show that the functor Vect' is compatible with the functors $\mathbb{B} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{GeomStack}_{\mathrm{iso}}$ and $\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}$ in the following sense.

Theorem 6.1. The diagram of bicategories and functors

$$
\begin{array}{ccc}
\mathsf{GeomStack_{iso}} & \xrightarrow{\ \mathsf{Vect'}\ } & \mathsf{Cat} \\
\Big\uparrow{\mathbb{B}} & \nearrow\!\!\!\nearrow & \Big\uparrow{u} \\
\mathsf{Bi_{iso}} & \xrightarrow[\ \mathbb{X}\ ]{} & \mathsf{Lie2Alg}
\end{array}
$$

2-commutes. Here $u : \mathsf{Lie2Alg} \to \mathsf{Cat}$ denotes the functor that assigns to each Lie 2-algebra its underlying category. In particular for a geometric stack $\mathcal{A}$ the category underlying the Lie 2-algebra $(\mathbb{X} \circ \mathbb{B}^{-1})(\mathcal{A})$ is equivalent to Hepworth's category $\mathsf{Vect}(\mathcal{A})$ of vector fields on the stack.

Related work. The recent work of Cristian Ortiz and James Waldron [16] lies in a similar circle of ideas. Recall that an $\mathcal{LA}$-groupoid is a groupoid object in Lie algebroids. Given an $\mathcal{LA}$-groupoid, Ortiz and Waldron introduce its category of multiplicative sections and show that it carries a natural strict Lie 2-algebra structure in the language of crossed modules of Lie algebras (which affords an equivalent description of the category of strict Lie 2-algebras). They show that if two $\mathcal{LA}$-groupoids are Morita equivalent then the corresponding crossed modules of Lie algebras are connected by a zig-zag of equivalences. Furthermore, to every stack Ortiz and Waldron assign an ordinary Lie algebra and show that in the case of proper geometric stacks the set underlying this Lie algebra is in bijective correspondence with isomorphism classes of vector fields in Hepworth's definition.

Outline of the paper. In Section 2 we review some of the background material used in the paper. In particular we recall the strict 2-category LieGpd of Lie groupoids, smooth functors and natural transformations. We then briefly discuss the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles and the functor $\langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ that localizes the strict 2-category LieGpd at the class of the essential equivalences. We then discuss the localizations of bicategories in general and recall a criterion due to Pronk for a functor between bicategories to be a localization. We then review 2-vector spaces, strict Lie 2-algebras and crossed modules of Lie 2-algebras. We localize the strict 2-category $\mathsf{Lie2Alg_{strict}}$ of Lie 2-algebras, internal functors and natural transformations at essential equivalences and obtain a bicategory $\mathsf{Lie2Alg}$. Under the correspondence between Lie 2-algebras and crossed modules $\mathsf{Lie2Alg}$ corresponds to Noohi's bicategory of crossed-modules and butterflies [15]. We finish the section by discussing the extension of the tangent functor T on the category of manifolds to tangent functors on the 2-category LieGpd and the bicategory Bi, respectively.

In Section 3 we prove Theorem 3.4: the category of multiplicative vector fields on a Lie groupoid underlies a strict Lie 2-algebra. In Section 4 we prove that the assignment $G \mapsto \mathbb{X}(G)$ of the category of vector fields to a Lie groupoid extends to a functor $\mathbb{X} : \mathsf{Bi_{iso}} \to \mathsf{Lie2Alg}$ from the bicategory $\mathsf{Bi_{iso}}$ of Lie groupoids, invertible bibundles and isomorphisms of bibundles to the bicategory $\mathsf{Lie2Alg}$ of Lie 2-algebras. Hence, in particular, if $P : G \to H$ is a Morita equivalence of Lie groupoids then $\mathbb{X}(P) : \mathbb{X}(G) \to \mathbb{X}(H)$ is a (weakly) invertible 1-morphism of Lie 2-algebras in the bicategory $\mathsf{Lie2Alg}$. Along the way we introduce the category $\mathbb{X}_{\mathsf{gen}}(G)$ of *generalized* vector fields on a Lie groupoid G. The objects of $\mathbb{X}_{\mathsf{gen}}(G)$ are pairs (P, α_P) where $P : G \to TG$ is a bibundle and $\alpha_P : \langle\pi_G\rangle \circ P \Rightarrow \langle\mathrm{id}_G\rangle$ is an isomorphism of bibundles. Here and below $\langle\pi_G\rangle$ is the bibundle corresponding to the projection functor $\pi_G : TG \to G$ and $\langle\mathrm{id}_G\rangle$ is the identity bibundle on the Lie groupoid G.

In Section 5 we discuss Hepworth's category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$ and construct an equivalent category $\mathsf{Vect'}(\mathcal{A})$. In Section 6 we promote the assignment $\mathcal{A} \mapsto \mathsf{Vect'}(\mathcal{A})$ of a category of vector fields on a geometric stack to a functor $\mathsf{Vect'} : \mathsf{GeomStack_{iso}} \to \mathsf{Cat}$ from the 2-category of geometric stacks and isomorphisms to the category Cat of (small) categories and prove Theorem 6.1. In Section 7 we prove Theorem 4.11: for any Lie groupoid G the "inclusion" functor

$$
\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\mathsf{gen}}(G), \qquad v \mapsto (\langle v \rangle, \alpha_{\langle v \rangle} : \langle\pi_G\rangle \circ \langle v \rangle \Rightarrow \langle id_G \rangle).
$$

of the category of multiplicative vector fields into the category of generalized vector fields is fully faithful and essentially surjective. This generalizes a result of Hepworth for proper Lie groupoids. The result is technical but important for the purposes of this paper.

Acknowledgments. We thank Henrique Bursztyn for many helpful discussions. In particular this paper has partially originated from conversations of one of us (E.L.) with Henrique at Poisson 2014. We thank James Waldron for making us exercise more care with the equivalences of Lie 2-algebras.

We thank the anonymous referees for their helpful comments.

2. Background and notation

We assume that the reader is familiar with ordinary categories, strict 2-categories and bicategories (also known as weak 2-categories). We work exclusively with (2,1)-bicategories, that is with bicategories whose 2-morphisms are invertible. Standard references for bicategories are [3] and [5]. For the reader's convenience the definitions of a bicategory, (pseudo-)functors and natural transformations are summarized in Appendix A. We assume familiarity with Lie groupoids. Standard references are [11] and [14]. We also assume that the reader is comfortable with stacks over the site of manifolds. This said, sections 3, 4 and 7 do not use stacks and should be accessible to readers comfortable with Lie groupoids and bibundles. While there is no textbook covering stacks over manifolds, a number of references exist: [2], [4], [10], [13], [18] (this list is not exhaustive).

Given a category C we denote its collection of objects by C_0 and the collection of arrows/morphisms by C_1.[1] We usually denote the source and target maps of C by s and t, respectively. We write

$$C = \{C_1 \rightrightarrows C_0\}$$

and suppress the other structure maps of the category C. We denote the unit map by 1. Thus the map $1 : C_0 \to C_1$ assigns the identity arrow 1_x to each object x of the category C. The composition/multiplication in the category C is defined on the collection C_2 of pairs of composable arrows. Our convention is that

$$C_2 := \{(\gamma_2, \gamma_1) \in C_1 \times C_1 \mid s(\gamma_2) = t(\gamma_1)\} =: C_1 \times_{s,C_0,t} C_1.$$
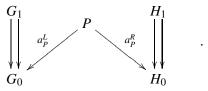
We denote the composition in the category C by m:

$$m : C_1 \times_{s,C_0,t} C_1 \to C_1, \qquad (\gamma_2, \gamma_1) \mapsto m(\gamma_2, \gamma_1) \equiv \gamma_2\gamma_1.$$

In particular, we write the composition from right to left: $\gamma_2\gamma_1$ means γ_1 followed by γ_2. If the category C is a groupoid we denote the inversion map by i:

$$i : C_1 \to C_1, \qquad i(\gamma) := \gamma^{-1}.$$

2.a. **Bicategories of Lie groupoids.** There are two natural bicategories whose objects are Lie groupoids:

- the strict 2-category LieGpd of Lie groupoids, (smooth) functors and (smooth) natural transformations, and
- the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles.

We write a 1-arrow in the category Bi as $P : G \to H$. We recall that P is a manifold with left and right anchor maps a^L and a^R,

together with commuting left and right actions, $G_1 \times_{G_0} P \to P$ and $P \times_{H_0} H_1 \to P$, respectively. We further require that the right H-action is principal.

---

[1] We use the words "arrow" and "morphism" interchangeably.

The composition of bibundles $P : G \to H$ and $Q : H \to K$ is the bibundle $Q \circ P$. It is defined to be the quotient of the fiber product:

$$Q \circ P = (P \times_{H_0} Q)/H. \tag{2.1}$$

Notation 2.1. In the bicategories LieGpd and Bi, we write the horizontal composition of 2-arrows as $\star$. Given a 1-morphism f and a 2-morphism α we abuse notation by writing $f \star \alpha$ for the horizontal composition (whiskering) $1_f \star \alpha$ where 1_f is the identity 2-arrow on the 1-morphism f. The vertical composition of 2-morphisms is denoted by $\circ$. When convenient, we also use arrow notation to denote morphisms in groupoids with specified source or target, e.g., $x \xleftarrow{g} y$ for a morphism g with target x and source y.

Remark 2.2. There is a functor (for example, see [10])

$$\langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi} \tag{2.2}$$

that is the identity on objects, and on 1-morphisms sends a functor $f : G \to H$ to the bibundle

$$\langle f \rangle := G_0 \times_{f,H_0,t} H_1 := \{(x,\gamma) \mid f(x) = t(\gamma)\} = \{(x, f(x) \xleftarrow{\gamma}) \mid x \in G_0, \gamma \in H_1\} \tag{2.3}$$

whose left and right anchor maps are given respectively by

$$a^L(x,\gamma) = x, \qquad a^R(x,\gamma) = s(\gamma).$$

Here, as before, $s : H_1 \to H_0$ is the source map. The left action of the groupoid G on the manifold $\langle f \rangle$ is given by

$$(g,(x,\gamma)) \mapsto (t(g), f(g)\gamma).$$

The right action of the groupoid H on $\langle f \rangle$ is given by

$$((x,\gamma),\nu) \mapsto (x, \gamma\nu).$$

Note that $a^L : \langle f \rangle \to G_0$ has a canonical section

$$x \mapsto (x, 1_{f(x)}).$$

Given a pair of functors $f, k : G \to H$ and a natural isomorphism $\alpha : f \Rightarrow k$, we get an isomorphism of bibundles

$$\langle \alpha \rangle : \langle f \rangle \Rightarrow \langle k \rangle.$$

The isomorphism $\langle \alpha \rangle$ is defined by

$$\langle \alpha \rangle(x, f(x) \xleftarrow{\gamma}) = (x, k(x) \xleftarrow{\alpha(x)\gamma}).$$

It is not hard to check that the map $\langle \alpha \rangle$ defined above is smooth, commutes with the left and right anchor maps and is equivariant with respect to the actions of G and H.

The functor $\langle\ \rangle$ takes vertical and horizontal composition of natural transformations to the composition of isomorphisms of bibundles and horizontal composition of isomorphisms, respectively.

Remark 2.3. By construction of the functor $\langle\ \rangle$ the total space of the bibundle $\langle id_G \rangle$ corresponding to the identity functor $id_G : G \to G$ on a Lie groupoid G is the fiber product $G_0 \times_{G_0} G_1$. This fiber product is diffeomorphic to G_1. We therefore define the manifold G_1 together with the actions of G by left and right multiplication to be the identity bibundle for a Lie groupoid G.

The functor $\langle\ \rangle$ is far from being an equivalence of 2-categories. The issue is that for almost all groupoids G and H the functor

$$\langle\ \rangle : \mathsf{Hom}_{\mathsf{LieGpd}}(G, H) \to \mathsf{Hom}_{\mathsf{Bi}}(G, H) \tag{2.4}$$

fails to be essentially surjective. The failure of essential surjectivity follows from the well-known fact:

Lemma 2.4. A bibundle $P : G \to H$ is isomorphic to a bibundle $\langle f \rangle$ for some functor $f : G \to H$ if and only if the left anchor $a_P^L : P \to G_0$ has a section.

5

Proof. For a functor $f : G \to H$, the left anchor map $a^L : \langle f \rangle = G_0 \times_{H_0} H_1 \to G_0$ has a canonical global section $x \mapsto (x, 1_{f(x)})$.

Conversely suppose $a^L : P \to G_0$ has a global section σ. We define the corresponding functor $f_\sigma : G \to H$ on objects by

$$f_\sigma(x) := (a^R \circ \sigma)(x).$$

Since $a_L : P \to G_0$ is a principal H-bundle for any arrow $y \xleftarrow{\gamma} x \in G_1$ there is a unique arrow τ in H_1, which depends smoothly on γ, so that

$$\gamma \cdot \sigma(x) = \sigma(y) \cdot \tau.$$

We set

$$f_\sigma(\gamma) := \tau.$$

It is easy to check that f is indeed a morphism of Lie groupoids. $\qquad\square$

In contrast to failure of $\langle\ \rangle$ to be surjective on 1-morphisms, for 2-morphisms the following result holds. The result must be known but we are not aware of a reference.

Theorem 2.5 (Folklore). For any pair of functors $f, k : G \to H$ of Lie groupoids the map

$$\langle\ \rangle : \mathsf{Hom}_{\mathsf{LieGpd}}(f, k) \to \mathsf{Hom}_{\mathsf{Bi}}(\langle f \rangle, \langle k \rangle), \qquad \alpha \mapsto \langle \alpha \rangle$$

is a bijection.

Sketch of proof. Let $\delta : \langle f \rangle \to \langle k \rangle$ be an isomorphism of bibundles. The left anchor $a^L_{\langle f \rangle} : \langle f \rangle \to G_0$ has a natural section σ_f. It is defined by

$$\sigma_f(x) = (x, f(x) \xleftarrow{1_{f(x)}}).$$

Similarly we have a natural section $\sigma_k : \langle k \rangle \to G_0$ of the left anchor $a^L_{\langle k \rangle} : \langle k \rangle \to G_0$. Since $a^L_{\langle k \rangle} : \langle k \rangle \to G_0$ is a principal H bundle, for any $x \in G_0$ there is a unique arrow $\bar{\delta}(x) \in H_1$ so that

$$\delta(\sigma_f(x)) = \sigma_k(x) \cdot \bar{\delta}(x)$$

for all $x \in G_0$. By equivariance of δ, the map

$$\bar{\delta} : G_0 \to H_1 \qquad x \mapsto \bar{\delta}(x)$$

is a natural isomorphism from f to k. $\qquad\square$

2.b. **Localizations of bicategories.** Suppose we are given a bicategory C and a class of 1-morphisms W in C. A localization of C at the class W (if it exists) is a bicategory $\mathsf{C}[W^{-1}]$ equipped with a functor $U : \mathsf{C} \to \mathsf{C}[W^{-1}]$ satisfying the following universal property: for any bicategory D the precomposition with U induces an equivalence of bicategories

$$\mathsf{Hom}(\mathsf{C}[W^{-1}], \mathsf{D}) \xrightarrow{-\circ U} \mathsf{Hom}_W(\mathsf{C}, \mathsf{D})$$

where $\mathsf{Hom}_W(\mathsf{C}, \mathsf{D})$ denotes the bicategory of functors sending elements of W to weakly invertible 1-morphisms in D. In particular given any functor $F : \mathsf{C} \to \mathsf{D}$ mapping elements of W to invertible morphisms of D there exists a functor $\tilde{F} : \mathsf{C}[W^{-1}] \to \mathsf{D}$ and a natural isomorphism

$$F \Rightarrow \tilde{F} \circ U.$$

The localization $\mathsf{C}[W^{-1}]$ is defined up to equivalence of bicategories, so it will be convenient to refer to any functor $F : \mathsf{C} \to \mathsf{C}'$ between bicategories as a localization of C at the class W if it has the same universal property as $U : \mathsf{C} \to \mathsf{C}[W^{-1}]$. Namely we ask that for any bicategory D the precomposition with F induces an equivalence of bicategories

$$\mathsf{Hom}(\mathsf{C}', \mathsf{D}) \xrightarrow{-\circ F} \mathsf{Hom}_W(\mathsf{C}, \mathsf{D}).$$

Pronk [17] gives a criterion for a functor $F : \mathsf{C} \to \mathsf{C}'$ to be "the" localization of C at the class W:

Proposition 2.6. ([17, Proposition 24]) A functor $F : \mathsf{C} \to \mathsf{C}'$ between bicategories is a localization of C at the class W if

(1) F sends the elements of W to (weakly) invertible 1-morphisms in C';
(2) F is essentially surjective on objects;
(3) for every 1-morphism f in C' there are 1-morphisms w in W and g in C with a 2-morphism $F(g) \Rightarrow f \circ F(w)$;
(4) F is fully faithful on 2-morphisms.

Example 2.7. The functor $\langle \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is the localization of the 2-categories of Lie groupoids, functors and natural transformations at the class of essential equivalences. Indeed the functor is surjective on objects and sends essential equivalences to invertible bibundles. Finally, for any bibundle P a choice of local sections of the left anchor leads to a factorization $P \Rightarrow \langle g \rangle \circ \langle w \rangle^{-1}$ where g is a functor and w is an essential equivalence.

Localizations of bicategories will come up several times in this paper. In the next subsection we will discuss the localization of the strict category $\mathsf{Lie2Alg}_{strict}$ of Lie 2-algebras at essential equivalences. In Section 4 we will need the fact that the bicategory Bi_{iso} of Lie groupoids, weakly invertible bibundles and isomorphisms of bibundles is a localization of a certain 2-category of embeddings of Lie groupoids.

2.c. **2-vector spaces and Lie 2-algebras.**

Definition 2.8. A 2-vector space (in the sense of Baez and Crans [1]) is a category V internal to the category of vector spaces. Hence $V = \{V_1 \rightrightarrows V_0\}$ where V_0 a vector space of objects, V_1 a vector space of morphisms, and all the structure maps (source, target, unit, and composition) are linear. All 2-vector spaces in this paper are defined over $\mathbb{R}$.

There is a 2-category $\mathsf{2Vect}$ whose objects are 2-vector spaces, 1-morphisms are (linear) functors and 2-morphism are (linear) natural transformations. There is a forgetful functor

$$\mathsf{2Vect} \to \mathsf{Cat} \tag{2.5}$$

from the 2-category of 2-vector spaces to the 2-category Cat of categories that forgets the linear structure.

Remark 2.9. There is an equivalence of categories of 2-vector spaces and of 2-term chain complexes of vector spaces. See, for example, [1]. (A similar result characterizing Picard categories was obtained much earlier by Deligne [6].) We remind the reader of how this equivalence is defined on objects. Given a 2-term complex $\partial : U \to W$ there is an action of the abelian group U on W given by

$$u \cdot w := \partial(u) + w \tag{2.6}$$

for all $u \in U, w \in W$. The corresponding action groupoid $\{U \times W \rightrightarrows W\}$ is a 2-vector space.

The converse is true as well: any 2-vector space $V = \{V_1 \rightrightarrows V_0\}$ is isomorphic to an action groupoid defined by the 2-term complex $\partial = t|_{\ker s} : \ker s \to V_0$. Here as before $s, t : V_1 \to V_0$ are the source and target map of the category V; see [1] for a proof.

Next we recall the definition of a strict Lie 2-algebra [1].

Definition 2.10. A strict Lie 2-algebra is a category internal to the category of Lie algebras (over the reals): the space of objects and morphisms of a Lie 2-algebra are ordinary Lie algebras and all the structure maps are maps of Lie algebras.

Notation 2.11. Categories internal to Lie algebras, internal functors and internal natural transformations form a strict 2-category which we denote by $\mathsf{Lie2Alg}_{strict}$.

Definition 2.12. (see, for example, [7, Definition 15]) A crossed module of Lie algebras consists of a Lie algebra homomorphism $\partial : \mathfrak{m} \to \mathfrak{n}$ together with a Lie algebra homomorphism

$$D : \mathfrak{n} \to \mathsf{Der}(\mathfrak{m})$$

from $\mathfrak{n}$ to the Lie algebra $\mathsf{Der}(\mathfrak{m})$ of derivations of $\mathfrak{m}$ so that for all $m, m' \in \mathfrak{m}, n \in \mathfrak{n}$

(i) $\partial(D(n)m) = [n, \partial(m)]$ and
(ii) $D(\partial(m))m' = [m, m']$.

A crossed module of Lie algebras determines a Lie 2-algebra: see, for example, the proof of Theorem 3 in [7]. A converse is true as well: any Lie 2-algebra canonically defines a crossed module of Lie 2-algebras. In fact more is true: crossed modules form a strict 2-category, and the 2-categories of Lie 2-algebras and of crossed modules are equivalent (see [7, Theorem 3] cited above). We won't need the full strength of this theorem in the present paper. We do, however, need the following result:

Lemma 2.13. Let $V = \{V_1 \rightrightarrows V_0\}$ be a 2-vector space. Suppose the corresponding 2-term complex $\partial = t|_{\ker s} : \ker s \to V_0$ is part of the data of a Lie algebra crossed module. That is, suppose that V_0, $\ker s$ are Lie algebras, ∂ is a Lie algebra map, and that there is an action $D : V_0 \to \mathsf{Der}(\ker s)$ of V_0 on $\ker s$ by derivations making $(\partial : \ker s \to V_0, D : V_0 \to \mathsf{Der}(\ker s))$ into a crossed module of Lie algebras. Then V is a Lie 2-algebra.

Sketch of proof. Since $1 \circ s = id_{V_0}$, $V_1 = \ker s \oplus V_0$. We define a bracket on $\ker s \oplus V_0$ by

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2] + D(y_1)x_2 - D(y_2)x_1, [y_1, y_2]). \tag{2.7}$$

for all $(x_1, y_1), (x_2, y_2) \in \ker s \oplus V_0$. That is, we define the Lie algebra V_1 to be the semi-direct product of V_0 and $\ker s$. Checking that source, target and unit maps of V are Lie algebra maps is easy. To check that the composition $m : V_1 \times_{V_0} V_1 \to V_1$ in the category V is a Lie algebra map we observe that m is given by
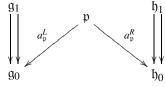
$$m((x_1, y_1), (x_2, y_2)) = (x_1 + x_2, y_2) \tag{2.8}$$

for all $(x_1, y_1), (x_2, y_2) \in \ker s \oplus V_0$ with $y_1 = t(x_2, y_2) = \partial x_2 + y_2$. This fact is not completely obvious. It lies in the heart of the correspondence between 2-vector spaces and 2-term chain complexes. See Remark 2.9 and [1]. A computation now shows that the map m is a Lie algebra map. □

There is a problem with the 2-category $\mathsf{Lie2Alg}_{strict}$ of Lie 2-algebras. Namely, suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie 2-algebras which is fully faithful and essentially surjective, that is, an essential equivalence. Then f has a weak inverse (as a functor), but there is no reason for that inverse to be a morphism of Lie 2-algebras. In fact it is easy to come up with examples where such morphism of Lie 2-algebras does not exist. Here is one. The $2n + 1$ dimensional Heisenberg Lie algebra $\mathfrak{h}$ is a central extension of a $2n$ dimensional abelian Lie algebra $\mathfrak{a}$ by the reals. Consequently we have a map φ of 2-term complexes of Lie algebras $(\mathbb{R} \to \mathfrak{h}) \to (0 \to \mathfrak{a})$, but the map $\phi_0 : \mathfrak{h} \to \mathfrak{a}$ has no Lie algebra sections. In fact φ is a map of crossed modules of Lie algebras. The morphism φ of crossed modules corresponds to an essential equivalence of Lie 2-algebras for which there is no inverse map in $\mathsf{Lie2Alg}_{strict}$.

Fortunately the problem has a universal solution: we localize the 2-category $\mathsf{Lie2Alg}_{strict}$ at the class of essential equivalences and obtain a bicategory $\mathsf{Lie2Alg}$ (see [17] and Subsection 2.b above). This localization has a simple and explicit description: we define a morphism between Lie 2-algebras as a "bibundle internal to the category of Lie algebras." The definition makes sense since fiber products exist in the category of Lie algebras. Here are the details.

Definition 2.14. A (left-principal) bibundle $\mathfrak{p} : \mathfrak{g} \to \mathfrak{h}$ from a Lie 2-algebra $\mathfrak{g}$ to a Lie 2-algebra $\mathfrak{h}$ is a Lie algebra $\mathfrak{p}$ with left and right anchor maps $a_{\mathfrak{p}}^L$ and $a_{\mathfrak{p}}^R$ (which are maps of Lie algebras),

along with a left action of the groupoid $\mathfrak{g}$ and right action of the groupoid $\mathfrak{h}$

$$\mathfrak{g}_1 \times_{s,\mathfrak{g}_0,a_{\mathfrak{p}}^L} \mathfrak{p} \to \mathfrak{p} \quad (g,p) \mapsto g \cdot p, \qquad \mathfrak{p} \times_{a_{\mathfrak{p}}^R,\mathfrak{h}_0,t} \mathfrak{h}_1 \to \mathfrak{p} \quad (p,h) \mapsto p \cdot h.$$

We require that the actions are maps of Lie algebras, commute with each other and satisfy associative and unital conditions. Finally, we require that the map

$$\mathfrak{p} \times_{a_{\mathfrak{p}}^R,\mathfrak{g}_0,t} \mathfrak{h}_1 \to \mathfrak{p} \times_{a_{\mathfrak{p}}^L,\mathfrak{g}_0,a_{\mathfrak{p}}^L} \mathfrak{p} \qquad (p,h) \mapsto (p,p\cdot h)$$

is an isomorphism of Lie algebras, i.e., that the $\mathfrak{h}$ action is principal.

Remark 2.15. The composition of bibundles between Lie 2-algebras is defined in the same way as in the case of bibundles between Lie groupoids. We will omit a proof that Lie 2-algebras, bibundles of Lie algebras and isomorphisms of bibundles form a bicategory. We denote this bicategory by Lie2Alg. We note that biprincipal bibundles are weakly invertible in this bicategory.

As in the case of Lie groupoids there is a functor $\langle\ \rangle :$ Lie2Alg$_{strict} \to$ Lie2Alg. It sends a strict map $f : \mathfrak{g} \to \mathfrak{h}$ of Lie 2-algebras to the bibundle

$$\langle f \rangle := \mathfrak{g}_0 \times_{f,\mathfrak{h}_0,t} \mathfrak{h}_1 := \{(x,\gamma) \mid f(x) = t(\gamma)\},$$

whose left and right anchor maps are given respectively by

$$a^L(x,\gamma) = x, \qquad a^R(x,\gamma) = s(\gamma).$$

The left action of $\mathfrak{g}$ on $\langle f \rangle$ is

$$(g,(x,\gamma)) \mapsto (t(g),f(g)\gamma),$$

and the right action of $\mathfrak{h}$ on $\langle f \rangle$ is

$$((x,\gamma),\nu) \mapsto (x,\gamma\nu).$$

In order to check that the functor $\langle\ \rangle :$ Lie2Alg$_{strict} \to$ Lie2Alg is the localization of the 2-category Lie2Alg$_{strict}$ at the class of essential equivalences we need

Lemma 2.16. Suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a strict map of Lie 2-algebras whose underlying functor is fully faithful and essentially surjective. Then the bibundle of Lie 2-algebras

$$\langle f \rangle : \mathfrak{g} \to \mathfrak{h}$$

is weakly invertible.

Proof. It is enough to show that $a^R : \langle f \rangle \to \mathfrak{h}$ is $\mathfrak{g}$-principal. That is, it's enough to show that a^R is surjective and that the map

$$\varphi : \mathfrak{p} \times_{a_{\mathfrak{p}}^R,\mathfrak{h}_0,t} \mathfrak{h}_1 \to \mathfrak{p} \times_{a_{\mathfrak{p}}^L,\mathfrak{g}_0,a_{\mathfrak{p}}^L} \mathfrak{p} \qquad \varphi(p,h) := (p,p\cdot h)$$

is an isomorphism of Lie algebras. Since $a^R(x,\gamma) = s(\gamma)$ the surjectivity of a^R is equivalent to the essential surjectivity of the functor f. The fullness of f translates into φ being onto and faithfulness of f translates into φ being 1-1. $\qquad\square$

We now apply Proposition 2.6 to conclude that $\langle\ \rangle :$ Lie2Alg$_{strict} \to$ Lie2Alg is the localization of Lie2Alg$_{strict}$ at the class of essential equivalences. See also Theorem 4.4 below for a similar argument.

Remark 2.17. A reader familiar with Noohi's butterflies (see [15] and reference therein) should not have much trouble showing that the bicategory Lie2Alg of Lie 2-algebras defined above is equivalent to the bicategory of crossed modules of Lie algebras, butterflies and isomorphisms of butterflies. Alternatively, this equivalence can be seen as an equivalence of localizations of equivalent 2-categories. Indeed, as recalled above the strict 2-category of Lie 2-algebras is equivalent to the 2-category of crossed modules of Lie algebras. Noohi's butterflies localize crossed modules at the class of 1-morphisms that correspond precisely to the class of essential equivalences of Lie 2-algebras.

2.d. **The tangent groupoid of a Lie groupoid.** The map T that assigns to each manifold M its tangent bundle TM and to each smooth map $f : M \to N$ between manifolds the differential $Tf : TM \to TN$ is a functor $T : \mathsf{Man} \to \mathsf{Man}$ from the category Man of manifolds to itself. The fact that T preserves the composition of maps is the chain rule. Moreover for any map $f : M \to N$ of manifolds the diagram

$$
\begin{array}{ccc}
TM & \xrightarrow{\;Tf\;} & TN \\
{\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_N} \\
M & \xrightarrow{\;f\;} & N
\end{array}
$$

commutes. Hence we have a natural transformation $\pi : T \Rightarrow id_{\mathsf{Man}}$.

The category Man of manifolds embeds into the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles. On objects the embedding is given by sending a manifold M to the groupoid $\{M \rightrightarrows M\}$. We argue that the functor $T : \mathsf{Man} \to \mathsf{Man}$ extends to a functor $T^{\mathsf{Bi}} : \mathsf{Bi} \to \mathsf{Bi}$. Indeed, an application of the functor T to a Lie groupoid $G = \{G_1 \rightrightarrows G_0\}$ gives us the tangent groupoid $TG := \{TG_1 \rightrightarrows TG_0\}$ and a functor $\pi_G : TG \to G$. Similarly for a given a bibundle $P : G \to H$ an application of the functor T gives us the bibundle $TP : TG \to TH$. If $\alpha : P \Rightarrow Q$ is a map between bibundles from G to H then its differential $T\alpha : TP \Rightarrow TQ$ is also a map between bibundles by functoriality of T. For any pair of composible bibundles

$$
G \xrightarrow{P} H \xrightarrow{Q} K
$$

the tangent bundle $T(Q \circ P)$ is isomorphic to the composition $TQ \circ TP$. This follows from the diffeomorphism

$$
T(P \times_{a_P^R, H_0, a_Q^L} Q) \simeq TP \times_{Ta_P^R, TH_0, Ta_Q^L} TQ
$$

(which is true for any transverse fiber product) and from the fact that for any groupoid H and any H-principal bundle $R \to B$

$$
TR/TH \simeq TB.
$$

Finally $T\langle id_G \rangle = TG_1 = \langle id_{TG} \rangle$ for any Lie groupoid G. We conclude that the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ extends to a functor $T^{\mathsf{Bi}} : \mathsf{Bi} \to \mathsf{Bi}$.

For any bibundle $P : G \to H$ the diagram in Bi
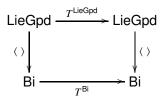
$$
\begin{array}{ccc}
TG & \xrightarrow{\;TP\;} & TH \\
{\scriptstyle \langle \pi_G \rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle \pi_H \rangle} \\
G & \xrightarrow{\;P\;} & H
\end{array}
$$

2-commutes. Hence the collection of functors $\{\pi_G : TG \to G\}_{G \in \mathsf{Bi}}$ gives rise to a natural transformation $\pi : T^{\mathsf{Bi}} \Rightarrow id_{\mathsf{Bi}}$.

We note in closing that the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ and the natural transformation $\pi : T \Rightarrow id_{\mathsf{Man}}$ also extend to the pair $(T^{\mathsf{LieGpd}} : \mathsf{LieGpd} \to \mathsf{LieGpd}, \pi : T^{\mathsf{LieGpd}} \Rightarrow id_{\mathsf{LieGpd}})$ consisting of a functor and a natural transformation on the 2-category LieGpd of Lie groupoids, smooth functors and natural transformations. Moreover by construction of the tangent functors and the functor $\langle\,\rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ the diagram

$$
\begin{array}{ccc}
\mathsf{LieGpd} & \xrightarrow{\;T^{\mathsf{LieGpd}}\;} & \mathsf{LieGpd} \\
{\scriptstyle \langle\,\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle\,\rangle} \\
\mathsf{Bi} & \xrightarrow{\;T^{\mathsf{Bi}}\;} & \mathsf{Bi}
\end{array}
$$

2-commutes.

3. The Lie 2-algebra $\mathbb{X}(G)$ of vector fields on a Lie groupoid G

In this section we prove Theorem 3.4: the category of multiplicative vector fields on a Lie groupoid underlies a strict Lie 2-algebra. We start by recalling the definition of the category of multiplicative vector fields.

As we saw in Subsection 2.d for any Lie groupoid G we have the tangent groupoid TG and a functor $\pi_G : TG \to G$.

Definition 3.1 ([8]). Consider a Lie groupoid G with its tangent groupoid $\pi_G : TG \to G$. The category $\mathbb{X}(G)$ of multiplicative vector fields is defined as follows. The set of objects of $\mathbb{X}(G)$ is

$$\mathbb{X}(G)_0 := \{v : G \to TG \mid v \text{ is a functor and } \pi_G \circ v = \mathrm{id}_G\}.$$

This is the set of multiplicative vector fields of Mackenzie and Xu [12]. A morphism in $\mathbb{X}(G)$ from a multiplicative vector field v to a multiplicative vector field w is a natural transformation $\alpha : v \Rightarrow w$ such that $\pi_G \star \alpha = 1_{\mathrm{id}_G}$. That is, for every point $x \in G_0$ we require that

$$\pi_G(\alpha_x) = 1_x. \tag{3.1}$$

The composition of morphisms is the vertical composition of natural transformations. Note that every morphism of $\mathbb{X}(G)$ is automatically invertible (since TG is a groupoid). Hence $\mathbb{X}(G)$ is a groupoid.

Notation 3.2. We denote the source and target maps in the category $\mathbb{X}(G)$ by $\mathfrak{s}$ and $\mathfrak{t}$, respectively. The unit map is denoted by $\mathbf{1}$, the inversion by $()^{-1}$ and the composition/multiplication of morphisms by $\circ$.

Lemma 3.3. The category of multiplicative vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a 2-vector space.

Proof. Mackenzie and Xu proved that the set $\mathbb{X}(G)_0$ of multiplicative vector fields is a real vector space [12].

We next argue that the set of morphisms $\mathbb{X}(G)_1$ of the category $\mathbb{X}(G)$ is a vector space as well. Suppose $\alpha_1 : v_1 \Rightarrow w_1$ and $\alpha_2 : v_2 \Rightarrow w_2$ are morphisms between multiplicative vector fields. Equation (3.1) says that α_1 and α_2 are both sections of the vector bundle

$$TG_1|_{G_0} \to G_0$$

where we have suppressed the unit map $1_G : G_0 \to G_1$. Clearly the linear combination $\lambda_1\alpha_1 + \lambda_2\alpha_2$ is again a section of the bundle $TG_1|_{G_0} \to G_0$ for any choice of scalars λ_1, λ_2. We need to check that it is actually a natural transformation from $\lambda_1 v_1 + \lambda_2 v_2$ to $\lambda_1 w_1 + \lambda_2 w_2$. That is, we need to check that for any arrow $y \overset{\gamma}{\leftarrow} x$ in the groupoid G

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2)_y \bullet ((\lambda_1 v_1 + \lambda_2 v_2)(\gamma)) = ((\lambda_1 w_1 + \lambda_2 w_2)(\gamma)) \bullet (\lambda_1\alpha_1 + \lambda_2\alpha_2)_x.$$

Here and below $\bullet : TG_1 \times_{TG_0} TG_1 \to TG_1$ denotes the multiplication in the Lie groupoid TG.

Since $\bullet$ is the derivative of the multiplication $m : G_1 \times_{G_0} G_1 \to G_1$ in the groupoid G, it is fiberwise linear: for any $(\gamma_2, \gamma_1) \in G_1 \times_{G_0} G_1 \to G_1$ and $(a_1, a_2), (b_1, b_2) \in T_{\gamma_2}G_1 \times_{TG_0} T_{\gamma_1}G_1 = T_{(\gamma_1,\gamma_2)}(TG_1 \times_{TG_0} TG_1)$ we have (in the prefix notation)

$$\bullet (\lambda(a_1, a_2) + \mu(b_1, b_2)) = \lambda(\bullet(a_2, a_1)) + \mu(\bullet(b_1, b_2)) \tag{3.2}$$

for all scalars λ, μ. In the infix notation (3.2) reads

$$(\lambda a_1 + \mu b_1) \bullet (\lambda a_2 + \mu b_2) = \lambda(a_1 \bullet a_2) + \mu(b_1 \bullet b_2). \tag{3.3}$$

Hence

$$\begin{aligned}
(\lambda_1\alpha_1 + \lambda_2\alpha_2)_y \bullet ((\lambda_1 v_1 + \lambda_2 v_2)(\gamma)) &= \lambda_1((\alpha_1)_y \bullet v_1(\gamma)) + \lambda_2((\alpha_2)_y \bullet v_2(\gamma)) \\
&= \lambda_1(w_1(\gamma) \bullet (\alpha_1)_x) + \lambda_2(w_2(\gamma) \bullet (\alpha_2)_x) \\
&= ((\lambda_1 w_1 + \lambda_2 w_2)(\gamma)) \bullet (\lambda_1\alpha_1 + \lambda_2\alpha_2)_x.
\end{aligned}$$

Here the first and third equalities hold by (3.3). In the second equality we used the fact that $\alpha_1 : v_1 \Rightarrow w_1$, and $\alpha_2 : v_2 \Rightarrow w_2$ are natural transformations. Therefore the space of morphisms $\mathbb{X}(G)_1$ is a vector space.

Moreover the computation above shows that for $\lambda_1, \lambda_2 \in \mathbb{R}$, $\alpha_1 : v_1 \Rightarrow w_1$, $\alpha_2 : v_2 \Rightarrow w_2 \in \mathbb{X}(G)_1$ the source of $\lambda_1\alpha_1 + \lambda_2\alpha_2$ is $\lambda_1 v_1 + \lambda_2 v_2$. That is, the source map $\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0$ of the category $\mathbb{X}(G)$ is linear. Similarly the target map $\mathfrak{t}$ is linear. It is also easy to see that the unit map $\mathbb{X}(G)_0 \to \mathbb{X}(G)_1$ is linear as well.

Finally we need to check that multiplication/composition $\circ$ in the category $\mathbb{X}(G)$, which is the vertical composition of natural transformations, is linear as a map from $\mathbb{X}(G)_1 \times_{\mathbb{X}(G)_0} \mathbb{X}(G)_1$ to $\mathbb{X}(G)_1$. That is, we need to check that

$$(\lambda\alpha_2 + \mu\beta_2) \circ (\lambda\alpha_1 + \mu\beta_1) = \lambda(\alpha_2 \circ \alpha_1) + \mu(\beta_2 \circ \beta_1) \tag{3.4}$$

for all $\lambda, \mu \in \mathbb{R}$, $(\alpha_2, \alpha_1), (\beta_2, \beta_1) \in \mathbb{X}(G)_1 \times_{\mathbb{X}(G)_0} \mathbb{X}(G)_1$. Recall that the vertical composition $\circ$ is computed pointwise: for any composible natural transformations δ_1, δ_1 and any point $x \in G_0$

$$(\delta_2 \circ \delta_1)_x = (\delta_2)_x \bullet (\delta_1)_x,$$

where as before $\bullet$ is the multiplication in TG. Since $\bullet$ is fiberwise linear (3.4) follows. This concludes our proof that the category $\mathbb{X}(G)$ of vector fields on a Lie groupoid G is internal to the category of vector spaces, that is, $\mathbb{X}(G)$ is a 2-vector space. $\qquad\square$

Theorem 3.4. The category of vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a (strict) Lie 2-algebra.

Proof. Recall the notation: $s : G_1 \to G_0$ is the source map for the groupoid G, its differential $Ts : TG_1 \to TG_0$ is the source map for the tangent groupoid TG. We use $\mathfrak{s}, \mathfrak{t}$ to denote the source and target maps of the groupoid $\mathbb{X}(G)$, respectively.

By Lemma 2.13 it is enough to: (i) give the vector spaces $\ker(\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0)$ and $\mathbb{X}(G)_0$ the structure of Lie algebras, (ii) check that $\partial := \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} \to \mathbb{X}(G)_0$ is a Lie algebra map, (iii) define an action $D : \mathbb{X}(G)_0 \to \mathsf{Der}(\ker \mathfrak{s}))$ on $\ker \mathfrak{s}$ by derivations, and (iv) check the compatibility of ∂ and D:

$$\partial(D(X)\alpha) = [X, \partial(\alpha)], \tag{3.5}$$
$$D(\partial\alpha_1)\alpha_2 = [\alpha_1, \alpha_2] \tag{3.6}$$

for all $\alpha, \alpha_1, \alpha_2 \in \ker \mathfrak{s}$ and all multiplicative vector fields X on the Lie groupoid G (compare with Definition 2.12).

The fact that the vector space $\mathbb{X}(G)_0$ of multiplicative vector fields carries a Lie bracket is due to Mackenzie and Xu [12]. We argue next that $\ker \mathfrak{s}$ is the space of sections of the Lie algebroid $A_G \to G_0$. By definition of the source map $\mathfrak{s}$,

$$\ker \mathfrak{s} = \{\alpha : X \Rightarrow Y \mid X = 0\}.$$

Therefore $\alpha \in \ker \mathfrak{s}$ if and only if there is a multiplicative vector field Y so that the diagram

$$
\begin{array}{ccc}
0_x & \xrightarrow{\;0_\gamma\;} & 0_y \\
{\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\
Y(x) & \xrightarrow[Y(\gamma)]{} & Y(y)
\end{array}
\tag{3.7}
$$

commutes for all arrows $y \xleftarrow{\gamma} x$ in G_1. Hence if $\alpha \in \ker \mathfrak{s}$ then $Ts(\alpha_x) = 0_x$ for all $x \in G_0$. That is, α is a section of $A_G \to G_0$. Conversely if $\alpha : G_0 \to A_G$ is a section of the Lie algebroid we can define a multiplicative vector field $Y : G \to TG$ so that (3.7) commutes. Namely on objects we define

$$Y(x) := Tt(\alpha_x) \qquad \text{for all } x \in G_0.$$

And for $y \xleftarrow{\gamma} x$ in G_1 we set

$$Y(\gamma) = \alpha_y \bullet 0_\gamma \bullet (\alpha_x)^{-1}. \tag{3.8}$$

Here as before $\bullet$ is the multiplication in TG and $()^{-1} = Ti : TG_1 \to TG_1$ is the inverse map, which is the derivative of the inverse map i of the groupoid G. We conclude that

$$\ker(\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0) = \Gamma(A_G)$$

and that

$$\partial := \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} \to \mathbb{X}(G)_0$$

is given by

$$(\partial \alpha)(\gamma) = \alpha_{t(\gamma)} \bullet 0_\gamma \bullet (\alpha_{s(\gamma)})^{-1} \tag{3.9}$$

for all $\gamma \in G_1$. Note that (3.9) can be written as

$$\partial \alpha = \overrightarrow{\alpha} + \overleftarrow{\alpha}. \tag{3.10}$$

where

$$\overrightarrow{\alpha}(\gamma) = TR_\gamma \, \alpha(t(\gamma))$$

and

$$\overleftarrow{\alpha}(\gamma) = T(L_\gamma \circ i) \, \alpha(t(\gamma))$$

for all $\gamma \in G_1$. Here R_γ and L_γ are right and left multiplications by γ, respectively. Recall that the bracket on the space of sections $\Gamma(A_G)$ of the Lie algebroid $A_G \to G_0$ is defined by requiring that the injective map

$$\overrightarrow{} : \Gamma(A_G) \to \Gamma(TG_1), \qquad \alpha \mapsto \overrightarrow{\alpha}.$$

is a map of Lie algebras. Consequently

$$\overleftarrow{} : \Gamma(A_G) \to \Gamma(TG_1), \qquad \alpha \mapsto \overleftarrow{\alpha}$$

is also a map of Lie algebras. We conclude that

$$\partial = \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} = \Gamma(A_G) \to \mathbb{X}(G)_0$$

is a Lie algebra map.

Following Mackenzie and Xu we define the map D from the space $\mathbb{X}(G)_0$ of multiplicative vector fields to $\mathsf{Hom}(\Gamma(TG_1|_{G_0}), \Gamma(TG_1|_{G_0}))$ by setting

$$D(X)\alpha := [X, \overrightarrow{\alpha}]|_{G_0}$$

for all multiplicative vector fields X and all sections $\alpha \in \Gamma(A_G)$. Mackenzie and Xu prove [12, Proposition 3.7] that $[X, \overrightarrow{\alpha}]$ is tangent to the fibers of s and is right invariant. Hence $[X, \overrightarrow{\alpha}]|_{G_0}$ is a section of the Lie algebroid $A_G \to G_0$. They furthermore show [12, Proposition 3.8] that $D(X)$ is a derivation of $\Gamma(A_G)$ and that $D : \mathbb{X}(G)_0 \to \mathsf{Der}(\Gamma(A_G))$ is a map of Lie algebras.

Since left- and right-invariant vector fields commute, for any $\alpha_1, \alpha_2 \in \Gamma(A_G)$ we have

$$[\partial \alpha_1, \overrightarrow{\alpha}_2] = [\overrightarrow{\alpha}_1 + \overleftarrow{\alpha}_1, \overrightarrow{\alpha}_2] = [\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2]$$

and (3.6) follows.

We end the proof by showing that (3.5) holds. On the right hand side we have

$$[X, \partial \alpha] = [X, \overrightarrow{\alpha} + \overleftarrow{\alpha}] = [X, \overrightarrow{\alpha}] + [X, \overleftarrow{\alpha}]$$

while on the left,

$$\partial (D(X)\alpha) = (D(X)\alpha)^{\rightarrow} + (D(X)\alpha)^{\leftarrow}.$$

By definition of D,

$$(D(X)\alpha)^{\rightarrow} = [X, \overrightarrow{\alpha}],$$

so it remains to prove that $[X, \overleftarrow{\alpha}] = (D(X)\alpha)^{\leftarrow}$. Since X is a functor,

$$Ti \circ X = X \circ i.$$

The inversion map i relates right- and left-invariant vector fields. That is,

$$Ti \circ \overrightarrow{\alpha} = \overleftarrow{\alpha} \circ i$$

for all α. Consequently

$$(D(X)\alpha)^{\leftarrow}(g) = T(L_g \circ i)(D(X)\alpha)(1_{s(g)}) = T(L_g)Ti([X, \overrightarrow{\alpha}](1_{s(g)}) = TL_g[X, \overleftarrow{\alpha}](i(1_{s(g)})).$$

Since $[X, \overleftarrow{\alpha}]$ is left-invariant, $TL_g[X, \overleftarrow{\alpha}](i(1_{s(g)})) = [X, \overleftarrow{\alpha}](g)$. Therefore,

$$(D(X)\alpha)^{\leftarrow}(g) = [X, \overleftarrow{\alpha}](g)$$

for all $g \in G_1$ and we are done.

$\square$

4. Morita invariance of the Lie 2-algebra of vector fields

The goal of this section is to prove

Theorem 4.1. The assignment $G \mapsto \mathbb{X}(G)$ of the category of vector fields to a Lie groupoid extends to a functor

$$\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg} \tag{4.1}$$

from the bicategory $\mathsf{Bi}_{\mathrm{iso}}$ of Lie groupoids, invertible bibundles and isomorphisms of bibundles to the bicategory $\mathsf{Lie2Alg}$ of Lie 2-algebras. Hence, in particular, if $P : G \to H$ is a Morita equivalence of Lie groupoids then $\mathbb{X}(P) : \mathbb{X}(G) \to \mathbb{X}(H)$ is a (weakly) invertible 1-morphism of Lie 2-algebras in the bicategory $\mathsf{Lie2Alg}$.

Our strategy for constructing the functor $\mathbb{X}$ is to first construct it on a simpler category.

Definition 4.2. An **essentially surjective open embedding** of Lie groupoids is a functor $f : U \to G$ so that

   (1) The maps on objects $f_0 : U_0 \to G_0$ and on morphisms $f_1 : U_1 \to G_1$ are open embeddings and
   (2) the functor f is a weak equivalence, i.e., the corresponding bibundle

$$\langle f \rangle := U_0 \times_{f_0, G_0, t} G_1 : U \to G$$

     is weakly invertible.

It is clear that the identity functors are essentially surjective open embeddings. Moreover the composition of essentially surjective open embeddings is again an essentially surjective open embedding. Consequently Lie groupoids, essentially surjective open embeddings and natural transformations form a 2-category.

Notation 4.3. We denote the 2-category of Lie groupoids, essentially surjective open embeddings and natural transformations by $\mathscr{E}mb$.

The relevance of the 2-category $\mathscr{E}mb$ is that it provides a different geometric description of $\mathsf{Bi}_{\mathrm{iso}}$, as follows.

Theorem 4.4. The localization of the bicategory $\mathscr{E}mb$ at the class W of all 1-morphisms is the bicategory $\mathsf{Bi}_{\mathrm{iso}}$ of bicategory of Lie groupoids, invertible bibundles and isomorphisms of bibundles.

Proof. We apply Proposition 2.6. Consider the localization functor $\langle \ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ introduced in Remark 2.2. By definition of the 2-category $\mathscr{E}mb$ the restriction of the functor $\langle \ \rangle$ to $\mathscr{E}mb$ sends every 1-morphism $w : U \to G$ of $\mathscr{E}mb$ to an invertible bibundle $\langle w \rangle$ (and a 2-morphism to an isomorphism of bibundles). This gives us a functor

$$\langle \ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathrm{iso}}, \qquad (G \xrightarrow{w} H) \mapsto (G \xrightarrow{\langle w \rangle} H). \tag{4.2}$$

The functor is surjective on objects. By Theorem 2.5 the functor is fully faithful on 2-morphisms.

It remains to check that given an invertible bibundle $P : G \to H$ there exist essentially surjective open embeddings w_G, w_H so that $\langle w_H \rangle \circ P$ is isomorphic to $\langle w_G \rangle$. Since the bibundle P is weakly invertible, it gives rise to the *linking groupoid* [19, Proposition 4.3], denoted $G *_P H$ and recalled presently. The manifold of objects $(G *_P H)_0$ is the disjoint union $G_0 \sqcup H_0$ of the objects of the groupoids G and H. The manifold of arrows $(G *_P H)_1$ is the disjoint union $G_1 \sqcup P \sqcup P^{-1} \sqcup H_1$. We think of the manifold P as the space of arrows

from the points of H_0 to the points of G_0. We think of the elements of P^{-1} as the inverses of the elements of P. The multiplication in $G *_P H$ comes from the multiplications in the groupoids G and H and the actions of G and H on P and on P^{-1}. The inclusion $w_G : G \to G *_P H$ is given by the open embeddings

$$G_0 \hookrightarrow G_0 \sqcup H_0, \qquad G_1 \hookrightarrow G_1 \sqcup P \sqcup P^{-1} \sqcup H_1.$$

It is easy to see that w_G is an essential equivalence, i.e., that the bibundle $\langle w_G \rangle$ is biprincipal, hence weakly invertible. Similarly we have the essentially surjective open embedding

$$w_H : H \hookrightarrow G *_P H.$$

A computation shows that the bibundles $\langle w_H \rangle \circ P$ and $\langle w_G \rangle$ are isomorphic. $\qquad\square$

Proposition 4.5. The assignment

$$G \mapsto \mathbb{X}(G)$$

of the Lie 2-algebra of vector fields to a Lie groupoid extends to a contravariant functor

$$(\mathscr{E}mb)^{op} \to \mathsf{Lie2Alg}_{strict}, \quad (G \xrightarrow{w} H) \mapsto (\mathbb{X}(H) \xrightarrow{w^*} \mathbb{X}(G)) \tag{4.3}$$

from the bicategory $\mathscr{E}mb$ of Lie groupoids, essentially surjective open embeddings and natural isomorphism to the strict 2-category $\mathsf{Lie2Alg}_{strict}$ of Lie 2-algebras.

Proof. Consider an essentially surjective open embedding $w : G \to H$. Then $w(G) \subset H$ is an open Lie subgroupoid and $w : G \to w(G)$ is an isomorphism of Lie groupoids. We now assume without any loss of generality that G is an open subgroupoid of H. Then the tangent bundle TG is an open subgroupoid of TH. Moreover, any multiplicative vector field $v : H \to TH$ restricts to a multiplicative vector field $v|_G : G \to TG$. Similarly, a morphism $\alpha : v \Rightarrow u$ of multiplicative vector fields restricts to a morphism $\alpha|_G : v|_G \Rightarrow u|_G$. This gives us a functor

$$w^* : \mathbb{X}(H) \to \mathbb{X}(G), \qquad w^*(\alpha : v \Rightarrow u) = (\alpha|_G : v|_G \Rightarrow u|_G). \tag{4.4}$$

The restriction to an open subgroupoid is a map of 2-vector spaces and preserves the brackets. Hence (4.4) is a map of Lie 2-algebras. $\qquad\square$

We next observe that the notion of a category of vector fields makes sense in many bicategories $\mathcal{B}$ with a tangent functor. For simplicity we will assume that all 2-arrows of $\mathcal{B}$ are invertible, that is, $\mathcal{B}$ is (2,1)-bicategory.

Definition 4.6. A (2,1)-bicategory $\mathcal{B}$ is a **bicategory with a tangent functor** if there a functor $T : \mathcal{B} \to \mathcal{B}$ and a natural transformation $\pi : T \Rightarrow \mathrm{id}_{\mathcal{B}}$. We will refer to T as the tangent functor and to π as the projection.

Example 4.7. We care about three bicategories with tangent functors: (1) the strict 2-category LieGpd of Lie groupoids, smooth functors and smooth natural transformations, (2) the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundle and (3) the strict 2-category $\mathsf{GeomStack}$ of geometric stacks. The tangent functor on on the 2-category $\mathsf{GeomStack}$ is discussed in Section 5 below.

We defined the bicategories with tangent functors in order to extend Hepworth's definition of the category of vector fields on a stack (Definition 5.1) to other bicategories of interest such as the bicategory Bi.

Definition 4.8. Let $\mathcal{B}$ be a (2,1)-bicategory with a tangent functor $T : \mathcal{B} \to \mathcal{B}$ and a projection $\pi : T \Rightarrow \mathrm{id}_{\mathcal{B}}$. Let B be an object of $\mathcal{B}$. The category of (generalized) **vector fields** $\mathbb{X}(B)$ on B is defined as follows. The objects of $\mathbb{X}(B)$ are pairs $(v : B \to TB, \alpha_v : \pi_B \circ v \Rightarrow \mathrm{id}_v)$. A morphism β in $\mathbb{X}(B)$ from (v, α_v) to (u, α_u) is a 2-morphism $\beta : v \Rightarrow u$ so that

$$\alpha_u = \alpha_v \circ (\pi_B \star \beta).$$

Here as before $\star$ denotes whiskering in $\mathcal{B}$, and $\circ$ is the vertical composition of 2-morphisms.

Example 4.9 (The category $\mathbb{X}_{\text{gen}}(G)$ of generalized vector fields on a Lie groupoid G)**.** Suppose the bicategory $\mathcal{B}$ is the bicategory Bi of Lie groupoids and bibundles. Recall that there is a tangent functor $T^{\text{Bi}} : \text{Bi} \to \text{Bi}$ and a projection $\pi : T \Rightarrow id_{\text{Bi}}$ (see Subsection 2.d). An object of Bi is a Lie groupoid G. We denote corresponding category of vector fields by $\mathbb{X}_{\text{gen}}(G)$ (to distinguish it from the category $\mathbb{X}(G)$ of multiplicative vector fields).

The objects of $\mathbb{X}_{\text{gen}}(G)$ are pairs $(P : G \to TG, \alpha_P : \langle \pi_G \rangle \circ P \Rightarrow \langle id_G \rangle)$ where P is a bibundle and α_P is an isomorphism of bibundles. A morphism β in $\mathbb{X}_{\text{gen}}(G)$ from (P, α_P) to (Q, α_Q) is a map of bibundles $\beta : P \Rightarrow Q$ so that

$$\alpha_Q = \alpha_P \circ (\langle \pi \rangle \star \beta).$$

Here as before $\star$ denotes whiskering in Bi, and $\circ$ is the composition of isomorphisms of bibundles.

Lemma 4.10. Let $\mathcal{B}$ be a bicategory with a tangent functor T and a projection π. A weakly invertible 1-morphism $P : G \to H$ between objects of $\mathcal{B}$ induces an equivalence of categories

$$P_* : \mathbb{X}(G) \to \mathbb{X}(H)$$
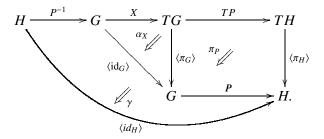
between the corresponding categories of vector fields.

Proof. Since the 1-morphism P is (weakly) invertible, there is 2-morphism

$$\gamma : (P \circ id_G) \circ P^{-1} \Rightarrow id_H.$$

Given an object (X, α_X) of $\mathbb{X}(G)$ we define

$$P_* X := TP \circ (X \circ P^{-1})$$

The 2-morphism $\alpha_{P_* X} : \pi_H \circ P_* X \Rightarrow id_H$ comes from the 2-commutative diagram

(The 2-morphism π_P is part of the data of the natural transformation $\pi : T \Rightarrow id_{\text{Bi}}$; see Appendix A.) We set

$$\alpha_{P_* X} := \gamma \circ (\alpha_X \star P^{-1}) \circ (\pi_P \star (X \circ P^{-1})).$$

Given a morphism $\beta : (X, \alpha_X) \to (Y, \alpha_Y)$ in the category $\mathbb{X}(G)$ we define

$$P_* \beta := TP \star \beta \star P^{-1}.$$

A diagram chase ensures that $P_* \beta$ *is* a morphism in $\mathbb{X}(H)$ from $(P_* X, \alpha_{P_* X})$ to $(P_* Y, \alpha_{P_* Y})$.

Finally one checks that the functor $(P^{-1})_* : \mathbb{X}(H) \to \mathbb{X}(G)$ is a weak inverse of P_*. Hence P_* is an equivalence of categories as claimed. $\square$

Theorem 4.11. For any Lie groupoid G the evident "inclusion" functor

$$\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\text{gen}}(G), \qquad v \mapsto (\langle v \rangle, \alpha_{\langle v \rangle} : \langle \pi_G \rangle \circ \langle v \rangle \Rightarrow \langle \pi_G \circ v \rangle = \langle id_G \rangle).$$

is an equivalence of categories.

Remark 4.12. In the case where the groupoid G is proper Theorem 4.11 can be deduced from [8, Theorem 15]; see Remark 5.4 below.

The proof of Theorem 4.11 is technical; we carry it out in section 7 below.

Lemma 4.13. Let $w : G \to G'$ be an essentially surjective open embedding, $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$ the pull-back/restriction functor of Proposition 4.5 and $(\langle w \rangle)_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathbb{X}_{\mathsf{gen}}(G')$ the push-forward along the bibundle $\langle w \rangle$ constructed in Lemma 4.10. Then the diagram
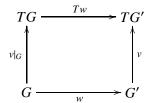
$$
\begin{array}{ccc}
\mathbb{X}(G) & \xrightarrow{\;\iota_G\;} & \mathbb{X}_{\mathsf{gen}}(G) \\[2pt]
\Big\uparrow{\scriptstyle w^*} & & \Big\downarrow{\scriptstyle (\langle w \rangle)_*} \\[2pt]
\mathbb{X}(G') & \xrightarrow{\;\iota_{G'}\;} & \mathbb{X}_{\mathsf{gen}}(G')
\end{array}
\tag{4.5}
$$

2-commutes in the 2-category Cat of categories.

Proof. Again, we may assume that $G \subset G'$ and $w^* : \mathbb{X}(G) \to \mathbb{X}(G')$ is given by $(\beta : v \Rightarrow v') \mapsto (\beta|_G : v|_G \Rightarrow v'|_G)$. To prove 2-commutativity of (4.5) we need to check that for any multiplicative vector field $v : G' \to TG'$, the generalized vector field $(\langle w \rangle)_*(\langle v|_G \rangle, \alpha_{v|_G})$ is isomorphic to the vector field $(\langle v \rangle, \alpha_{v_G})$.

Now given a multiplicative vector field $v : G' \to TG'$ the diagram

$$
\begin{array}{ccc}
TG & \xrightarrow{\;Tw\;} & TG' \\[2pt]
\Big\uparrow{\scriptstyle v|_G} & & \Big\uparrow{\scriptstyle v} \\[2pt]
G & \xrightarrow{\;w\;} & G'
\end{array}
$$

commutes. That is,

$$
v \circ w = Tw \circ (v|_G). \tag{4.6}
$$

Since $\langle \, \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is a functor,

$$
\langle v \rangle \circ \langle w \rangle \Rightarrow \langle Tw \rangle \circ \langle v|_G \rangle,
$$

where $\Rightarrow$ here and below denotes an unspecified isomorphism of bibundles. Note that the bibundle $\langle Tw \rangle$ is isomorphic to the bibundle $T\langle w \rangle$. Hence $\langle v \rangle \circ \langle w \rangle \Rightarrow T\langle w \rangle \circ \langle v|_G \rangle$. Now multiply both sides of the equation by $\langle w \rangle^{-1}$ and move the brackets around. We get

$$
\langle v \rangle \Rightarrow T\langle w \rangle \circ (\langle v|_G \rangle \circ \langle w \rangle^{-1}).
$$

The fact that $(\langle w \rangle)_*(\alpha_{v|_G})$ equals α_v follows from the fact that $\langle \, \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is a functor that intertwines the tangent functors: $\langle \, \rangle \circ T^{\mathsf{LieGpd}} \Rightarrow T^{\mathsf{Bi}} \circ \langle \, \rangle$ (see end of Section 2). $\qquad \square$

Lemma 4.14. The functor (4.3) of Proposition 4.5 takes every essentially surjective open embedding to a weakly invertible 1-morphism of Lie 2-algebras.

Proof. The diagram (4.5) 2-commutes by Lemma 4.13. By Theorem Theorem 4.11 the functors ι_G and $\iota_{G'}$ are equivalences of categories. Since the functor $\langle w \rangle$ is weakly invertible, the functor $(\langle w \rangle)_*$ is an equivalence of categories by Lemma 4.10. Hence the functor w^* is an equivalence of categories as well, hence an essential equivalence. Finally, the localization functor $\langle \, \rangle : \mathsf{Lie2Alg}_{strict} \to \mathsf{Lie2Alg}$ takes all essential equivalences to weakly invertible 1- morphisms. $\qquad \square$

We are now in position to extend the assignment $G \mapsto \mathbb{X}(G)$ to a (covariant) functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$.

Definition 4.15. We define the functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$ on objects to be the assignment

$$
G \mapsto \mathbb{X}(G).
$$

Given an essentially surjective open embedding $G \xrightarrow{w} G'$, the bibundle $\langle w^* \rangle$ is weakly invertible in $\mathsf{Lie2Alg}$ by Lemma 4.14. We set
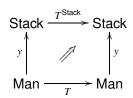
$$
\mathbb{X}(w) := (\langle w^* \rangle)^{-1}.
$$

Proof of Theorem 4.1. Since $\langle\ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathsf{iso}}$ is a localization of the bicategory $\mathscr{E}mb$ at the class $\mathscr{E}mb_1$ of all 1-morphisms and since the functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$ sends every 1-morphism of $\mathscr{E}mb$ to an invertible morphism there exists by Proposition 2.6 functor

$$\tilde{\mathbb{X}} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$$

(which is unique up to isomorphism) and an isomorphism $\tilde{\mathbb{X}} \circ \langle\ \rangle \overset{\sim}{\Leftrightarrow} \mathbb{X}$ of functors. It is no loss of generality to assume that $\tilde{\mathbb{X}}(G) = \mathbb{X}(G)$ for every Lie groupoid G. We now drop the $\tilde{}$ and obtain the desired functor $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$. $\qquad\square$

5. Categories of vector fields on stacks and Lie 2-algebras

Now we recall Hepworth's construction [8] of the category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$. The first step is to extend the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ on the category of manifolds to a functor $T^{\mathsf{Stack}} : \mathsf{Stack} \to \mathsf{Stack}$ on the 2-category of stacks over manifolds along the Yoneda embedding $y : \mathsf{Man} \to \mathsf{Stack}$. This results in a 2-commuting diagram

and there is a natural transformation $\pi : T^{\mathsf{Stack}} \Rightarrow id_{\mathsf{Stack}}$.

Definition 5.1 (Hepworth). The objects of the category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$ are pairs (v, α_v) where $v : \mathcal{A} \to T^{\mathsf{Stack}}\mathcal{A}$ is a 1-morphism of stacks and $\alpha_v : \pi_{\mathcal{A}} \circ v \Rightarrow id_{\mathcal{A}}$ is a 2-morphism. A morphism in $\mathsf{Vect}(\mathcal{A})$ from (v, α_v) to (u, α_u) is a 2-morphism $\beta : v \Rightarrow u$ so that

$$\alpha_u \circ (\pi_{\mathcal{A}} \star \beta) = \alpha_v.$$

Here $\circ$ is the vertical composition and $\star$ is whiskering.

Next recall that for any Lie groupoid G there is a stack $\mathbb{B}G$ of principal G-bundles. The assignment

$$G \mapsto \mathbb{B}G$$

can be promoted to a functor $\mathbb{B}$ in different ways depending on which source 2-category one chooses. Hepworth takes the source to be the 2-category LieGpd of Lie groupoids, smooth functors and natural isomorphisms and considers the functor

$$\mathbb{B} : \mathsf{LieGpd} \to \mathsf{Stack}. \tag{5.1}$$

The essential image of this functor consists of the 2-category $\mathsf{GeomStack}$ of geometric stacks. The functor $\mathbb{B}$ is faithful but not full. In particular the functor $\mathbb{B}$ maps essential equivalences of Lie groupoids (which need not be invertible in LieGpd, even weakly) to isomorphisms of stacks.[2] The tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ is easily extended to a functor $T^{\mathsf{LieGpd}} : \mathsf{LieGpd} \to \mathsf{LieGpd}$. We have a natural transformation $\pi^{\mathsf{LieGpd}} : T^{\mathsf{LieGpd}} \Rightarrow id_{\mathsf{LieGpd}}$.

Hepworth proves [8, Theorem 3.11] that there is a natural isomorphism

$$\mathbb{B} \circ T^{\mathsf{LieGpd}} \Leftrightarrow T^{\mathsf{Stack}} \circ \mathbb{B}. \tag{5.2}$$

Consequently given a vector field $v : G \to TG$ on a Lie groupoid G we get a map of stacks

$$\mathbb{B}v : \mathbb{B}G \to \mathbb{B}TG.$$

---

[2]Recall that by tradition a weakly invertible 1-morphism of stacks is called an *isomorphism.*

Composing v with the isomorphism $\mathbb{B}TG \to T^{\mathsf{Stack}}(\mathbb{B}G)$ gives us a functor that we again denoted by $\mathbb{B}v$: $\mathbb{B}G \to T^{\mathsf{Stack}}(\mathbb{B}G)$. This determines an object $(\mathbb{B}v, a_{Bv})$ in the category $\mathsf{Vect}(\mathbb{B}G)$ of vector fields on the stack $\mathbb{B}G$. Hepworth shows that the assignment

$$v \mapsto (\mathbb{B}v, a_{Bv})$$

can be promoted to a functor

$$\mathbb{X}(G) \to \mathsf{Vect}(\mathbb{B}G). \tag{5.3}$$

Here as before $\mathbb{X}(G)$ denotes the category of vector fields on a Lie groupoid G (see Definition 3.1). He proves in [8, Theorem 4.15] that if the groupoid G is proper then the functor (5.3) is an equivalence of categories.[3]

Another important consequence of the existence of the isomorphism (5.2) is that for any geometric stack $\mathcal{A}$ the tangent stack $T^{\mathsf{Stack}}\mathcal{A}$ is geometric as well.

We can promote the assignment $G \to \mathbb{B}G$ to a functor out of a different bicategory, which at a slight risk of confusion we will again denote by $\mathbb{B}$. Namely we can choose as our source the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles. The advantage is that the functor

$$\mathbb{B} : \mathsf{Bi} \to \mathsf{Stack}$$

is fully faithful: for Lie groupoids G and H, the functor

$$\mathbb{B} : \mathsf{Hom}_{\mathsf{Bi}}(G, H) \to \mathsf{Hom}_{\mathsf{Stack}}(\mathbb{B}G, \mathbb{B}H)$$

is an equivalence of categories. Consequently the functor

$$\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$$

is an equivalence of bicategories [4]. It is not hard to adapt [8, Theorem 3.11] to this setting: the diagram

2-commutes. It will be convenient for us to choose a weak inverse $\mathbb{B}^{-1} : \mathsf{GeomStack} \to \mathsf{Bi}$ and consider the functor

$$T^{\mathsf{GeomStack}} : \mathsf{GeomStack} \to \mathsf{GeomStack}, \qquad T^{\mathsf{GeomStack}} := \mathbb{B} \circ T^{\mathsf{Bi}} \circ \mathbb{B}^{-1},$$

which by construction is isomorphic to Hepworth's functor T^{Stack} restricted to geometric stacks. As in the case of T^{Stack} we have a transformation $\pi^{GS} : T^{\mathsf{GeomStack}} \Rightarrow id_{\mathsf{GeomStack}}$.

Given a geometric stack $\mathcal{A}$ we now define a category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on $\mathcal{A}$ as follows (compare with Definition 5.1).

Definition 5.2. The category of vector fields on a geometric stack $\mathcal{A}$, denoted $\mathsf{Vect}'(\mathcal{A})$, has as objects pairs (X, α_X) where $X : \mathcal{A} \to T^{\mathsf{GeomStack}}\mathcal{A}$ is a 1-morphism of stacks and $\alpha_X : \pi_{\mathcal{A}} \circ X \Rightarrow id_{\mathcal{A}}$ is a 2-morphism. A morphism from (X, α_X) to (Y, α_Y) in $\mathsf{Vect}'(\mathcal{A})$ is a 2-morphism $\beta : X \Rightarrow Y$ so that

$$\alpha_Y \circ (\pi_X \star \beta) = \alpha_X.$$

___

[3]The hypothesis that the groupoid G is proper is not explicit in the statement of [8, Theorem 4.15]. However the proof depends on several lemmas: 4.11, 4.12, 2.11, 2.12. In particular the proof uses the existence of partitions of unity and Weinstein-Zung linearization, both of which require properness.

In other words we apply Definition 4.8 to the strict 2-category $\mathsf{GeomStack}$ with the tangent functor $T^{\mathsf{GeomStack}} : \mathsf{GeomStack} \to \mathsf{GeomStack}$ rather than to Hepworth's tangent functor $T^{\mathsf{Stack}} : \mathsf{Stack} \to \mathsf{Stack}$. It is easy to see that for a geometric stack $\mathcal{A}$ the categories $\mathsf{Vect}(\mathcal{A})$ and $\mathsf{Vect}'(\mathcal{A})$ are equivalent (and even isomorphic). For us there are several advantages in working with $\mathsf{Vect}'(\mathcal{A})$. First of all, the functor Vect' is more explicit than T^{Stack}: the latter involves 2-limits and stackification. Additionally the following result is easy to prove:

Lemma 5.3. For a Lie groupoid G the classifying stack functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$ induces an equivalence of categories

$$\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G),$$

where the category $\mathbb{X}_{\mathsf{gen}}(G)$ of generalized vector fields is defined in Example 4.9.

Proof. Consider a generalized vector field (P, α_P) on the Lie groupoid G. By definition we have an isomorphism $\alpha_P : \langle \pi_G \rangle \circ P \Rightarrow \langle \mathrm{id}_G \rangle$ of bibundles. Apply the classifying stack functor $\mathbb{B}$ to the 2-morphism α_P. We get the 2-morphism of stacks

$$\mathbb{B}\alpha_P : \mathbb{B}(\langle \pi_G \rangle \circ P) \Rightarrow \mathbb{B}\langle \mathrm{id}_G \rangle.$$

Since $\mathbb{B}$ is functor between bicategories, we have canonical 2-arrows $\mathbb{B}\langle \mathrm{id}_G \rangle \Rightarrow \mathrm{id}_{\mathbb{B}G}$ and $\mathbb{B}\langle \pi_G \rangle \circ \mathbb{B}P \Rightarrow \mathbb{B}(\langle \pi_G \rangle \circ P)$. Note that these 2-morphisms are 2-isomorphisms since all 2-arrows in the 2-category of stacks are invertible. Composing the three 2-arrows we get a 2-arrow

$$\mathbb{B}\langle \pi_G \rangle \circ \mathbb{B}P \Rightarrow \mathrm{id}_{\mathbb{B}G}$$

which we denote by $\alpha_{\mathbb{B}P}$. By definition the pair $(\mathbb{B}P, \alpha_{\mathbb{B}P})$ is an object of $\mathsf{Vect}'(\mathbb{B}G)$.

Similarly a morphism $\beta : (P, \alpha_P) \to (Q, \alpha_Q)$ in $\mathbb{X}_{\mathsf{gen}}(G)$ gives rise to a morphism $\mathbb{B}\beta : \mathbb{B}P \Rightarrow \mathbb{B}Q$. One checks that

$$\alpha_{\mathbb{B}Q} \circ (\pi_{\mathbb{B}G} \star \mathbb{B}\beta) = \alpha_{\mathbb{B}Q}.$$

Consequently $\mathbb{B}\beta$ is a morphism in $\mathsf{Vect}'(\mathbb{B}G)$ from $(\mathbb{B}P, \alpha_{\mathbb{B}P})$ to $(\mathbb{B}Q, \alpha_{\mathbb{B}Q})$. We therefore get a functor

$$\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G).$$

A weak inverse $\mathbb{B}^{-1} : \mathsf{GeomStack} \to \mathsf{Bi}$ gives rise to the functor

$$(\mathbb{B}^{-1})_* : \mathsf{Vect}'(\mathbb{B}G) \to \mathbb{X}_{\mathsf{gen}}(G)$$

in the other direction. The induced functors $\mathbb{B}_*$ and $(\mathbb{B}^{-1})_*$ are weak inverses of each other. $\qquad\square$

Remark 5.4. Suppose G is a Lie groupoid. Tracing carefully through the definitions: of the map $\mathbb{X}(G) \to \mathsf{Vect}(\mathbb{B}G)$ (this map is defined by Hepworth), of the equivalence $\mathsf{Vect}'(\mathbb{B}G) \to \mathsf{Vect}(\mathbb{B}G)$, of the inclusion $\mathbb{X}(G) \hookrightarrow \mathbb{X}_{\mathsf{gen}}(G)$ and of the map $\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G)$ one can show that the diagram

2-commutes. By Lemma 5.3 the functor $\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G)$ is an equivalence of categories.

If the Lie groupoid G is proper, the functor $\mathbb{X}(G) \to \mathsf{Vect}(\mathbb{B}G)$ is an equivalence of categories by [8, Theorem 4.15]. Consequently the functor $\mathbb{X}(G) \to \mathbb{X}_{\mathsf{gen}}(G)$ has to be an equivalence of categories in this case.

In general Theorem 4.11 tells us that the functor $\mathbb{X}(G) \to \mathbb{X}_{\mathsf{gen}}(G)$ is an equivalence of categories for any Lie groupoid G. Consequently the functor $\mathbb{X}(G) \to \mathsf{Vect}(\mathbb{B}G)$ is always an equivalence of categories, regardless of whether the Lie groupoid G is proper or not. Thus Theorem 4.11 generalizes [8, Theorem 4.15].

We now address the issue of giving the category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on a geometric stack $\mathcal{A}$ the structure of a Lie 2-algebra. We may proceed as follows. Choose an atlas $G_0 \to \mathcal{A}$ on the stack $\mathcal{A}$. The atlas induces an isomorphism of stacks $\mathcal{A} \xrightarrow{p} \mathbb{B}G$, where G is the Lie groupoid defined by the atlas. The isomorphism p induces an equivalence of categories

$$p_* : \mathsf{Vect}'(\mathbb{B}G) \to \mathsf{Vect}'(\mathcal{A}).$$

By Lemma 5.3 the classifying stack functor induces an equivalence of categories

$$\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G).$$

By Theorem 4.11 the inclusion

$$\iota_G : \mathbb{X}(G) \to \mathbb{X}_{\mathsf{gen}}(G)$$

is an equivalence of categories. Consequently the composite functor $\phi_G : \mathbb{X}(G) \to \mathsf{Vect}'(\mathcal{A})$,

$$\phi_G := p_* \circ \mathbb{B}_* \circ \iota_G$$

is an equivalence of categories. By Lemma 3.4 the category $\mathbb{X}(G)$ of multiplicative vector fields has a natural structure of a strict Lie 2-algebra. We may view the functor ϕ_G as a kind of a "Lie 2-algebra atlas" on the category $\mathsf{Vect}'(\mathcal{A})$.

What happens if we choose a different atlas $q : H_o \to \mathcal{A}$ on the stack $\mathcal{A}$? By the same argument as above we get an equivalence of categories $\phi_H : \mathbb{X}(H) \to \mathsf{Vect}'(\mathcal{A})$, which we may view as a different "Lie 2-algebra atlas" on the category $\mathsf{Vect}'(\mathcal{A})$. We would like the two "atlases" to be compatible. In particular we would like to make sure that the functor

$$\phi_G^{-1} \circ \phi_H : \mathbb{X}(H) \to \mathbb{X}(G)$$

underlies a Morita equivalence of Lie 2-algebras. At the very least we would like the Lie 2-algebras $\mathbb{X}(G)$ and $\mathbb{X}(H)$ to be Morita equivalent in general. That is, we would like there to exist a weakly invertible 1-morphism in the bicategory $\mathsf{Lie2Alg}$ from the Lie 2-algebra $\mathbb{X}(H)$ to the Lie 2-algebra $\mathbb{X}(G)$.

To address this issue we study the functoriality of the assignment $\mathcal{A} \mapsto \mathbb{X}(G)$ of a Lie 2-algebra of vector fields to a geometric stack by a choice of an atlas $G_0 \to \mathcal{A}$. Consider the 2-category $\mathsf{GeomStack}_{\mathsf{iso}}$ of geometric stacks, *isomorphisms* of stacks and 2-morphisms of stacks. The classifying stack functor

$$\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$$

restricts to an equivalence of bicategories

$$\mathbb{B} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{GeomStack}_{\mathsf{iso}}.$$

A choice of a weak inverse $\mathbb{B}^{-1}$ of $\mathbb{B}$ amounts to choosing an atlas for each geometric stack. Once the inverse $\mathbb{B}^{-1}$ is chosen, we have the composite functor

$$\mathsf{GeomStack}_{\mathsf{iso}} \xrightarrow{\mathbb{B}^{-1}} \mathsf{Bi}_{\mathsf{iso}} \xrightarrow{\mathbb{X}} \mathsf{Lie2Alg}.$$

By construction, for a stack $\mathcal{A}$ the Lie 2-algebra $\mathbb{X}(\mathbb{B}^{-1}\mathcal{A})$ is the Lie 2-algebra of vector fields on the Lie groupoid $G = \mathbb{B}^{-1}\mathcal{A}$. By the discussion above the category underlying the Lie 2-algebra $\mathbb{X}(\mathbb{B}^{-1}\mathcal{A})$ is equivalent to the category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on the stack $\mathcal{A}$.

A different choice of a weak inverse $(\mathbb{B}^{-1})'$ of $\mathbb{B}$ amounts to choosing a possibly different atlas for each geometric stack. Once $(\mathbb{B}^{-1})'$ is chosen we have a natural isomorphism $\alpha : \mathbb{B}^{-1} \Rightarrow (\mathbb{B}^{-1})'$. For each geometric stack $\mathcal{A}$ the component $\alpha_{\mathcal{A}}$ of the natural transformation α is an invertible bibundle

$$\alpha_{\mathcal{A}} : \mathbb{B}^{-1}\mathcal{A} \to (\mathbb{B}^{-1})'\mathcal{A}.$$

Applying the functor $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$ to $\alpha_{\mathcal{A}}$ we get an invertible bibundle

$$\mathbb{X}(\alpha_{\mathcal{A}}) : \mathbb{X}(\mathbb{B}^{-1}\mathcal{A}) \to \mathbb{X}((\mathbb{B}^{-1})'\mathcal{A})$$

in the bicategory $\mathsf{Lie2Alg}$.

One can be fairly explicit as to what the bibundle $\mathbb{X}(\alpha_{\mathcal{A}})$ actually is. Namely let $G_0 \to \mathcal{A}$ be the atlas giving rise to the Lie groupoid $G = \mathbb{B}^{-1}(\mathcal{A})$ and $H_0 \to \mathcal{A}$ be the atlas giving rise to $H = (\mathbb{B}^{-1})'(\mathcal{A})$. Then the total space of the bibundle $\alpha_{\mathcal{A}} : G \to H = (\mathbb{B}^{-1})'(\mathcal{A})$ represents the fiber product $G_0 \times_{\mathcal{A}} H_0$. The linking groupoid $G *_{\alpha_{\mathcal{A}}} H$ is the groupoid corresponding to the atlas $G_0 \sqcup H_0 \to \mathcal{A}$. The linking groupoid comes with two canonical essentially surjective open embeddings

$$ i_G : G \hookrightarrow G *_{\alpha_{\mathcal{A}}} H \quad \text{and} \quad i_H : H \hookrightarrow G *_{\alpha_{\mathcal{A}}} H. $$

By (the proof of) Proposition 4.5 the pullback/restriction functors

$$ i_G^* : \mathbb{X}(G *_{\alpha_{\mathcal{A}}} H) \to \mathbb{X}(G), \qquad i_H^* : \mathbb{X}(G *_{\alpha_{\mathcal{A}}} H) \to \mathbb{X}(G) $$

are 1-morphisms of Lie 2-algebras that are fully faithful and essentially surjective. Hence the bibundle $\langle i_G^* \rangle$ is invertible in the bicategory Lie2Alg. On the other hand, as was noted in the proof of Theorem 4.4, the bibundles $\langle i_H \rangle \circ \alpha_{\mathcal{A}}$ and $\langle i_G \rangle$ are isomorphic. Hence

$$ \mathbb{X}(\langle i_H \rangle) \circ \mathbb{X}(\alpha_{\mathcal{A}}) \simeq \mathbb{X}(\langle i_G \rangle). $$

By construction of the functor $\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}$ we have

$$ \mathbb{X}(\langle i_G \rangle) = \langle i_G^* \rangle^{-1} \qquad \text{and} \qquad \mathbb{X}(\langle i_H \rangle) = \langle i_H^* \rangle^{-1}. $$

Hence

$$ \mathbb{X}(\alpha_{\mathcal{A}}) \simeq \langle i_H^* \rangle \circ \langle i_G^* \rangle^{-1}. $$

6. Lie 2-algebras of vector fields on stacks and their underlying categories

In the previous section we constructed a functor

$$ \mathbb{X} \circ \mathbb{B}^{-1} : \mathsf{GeomStack}_{\mathrm{iso}} \to \mathsf{Lie2Alg}. $$

Recall that there is a forgetful functor $u : \mathsf{Lie2Alg} \to \mathsf{Cat}$ that assigns to a Lie 2-algebra its underlying category. Therefor for every geometric stack $\mathcal{A}$ we have the category $(u \circ \mathbb{X} \circ \mathbb{B}^{-1})(\mathcal{A})$. We should make sure that this category is equivalent to the category of vector fields $\mathsf{Vect}'(\mathcal{A})$ (and hence to Hepworth's category $\mathsf{Vect}(\mathcal{A})$ of vector fields on the stack $\mathcal{A}$).

We start by promoting the assignment $\mathcal{A} \mapsto \mathsf{Vect}'(\mathcal{A})$ to a functor

$$ \mathsf{Vect}' : \mathsf{GeomStack}_{\mathrm{iso}} \to \mathsf{Cat} $$

whose source is the 2-category of geometric stacks and isomorphisms and whose target is the 2-category Cat of (small) categories. We then prove the following theorem:

Theorem 6.1. The diagram of bicategories and functors

$$
\begin{array}{ccc}
\mathsf{GeomStack}_{\mathrm{iso}} & \xrightarrow{\ \mathsf{Vect}'\ } & \mathsf{Cat} \\
\mathbb{B} \Big\uparrow & \nearrow & \Big\uparrow u \\
\mathsf{Bi}_{\mathrm{iso}} & \xrightarrow[\ \mathbb{X}\ ]{} & \mathsf{Lie2Alg}
\end{array}
$$

2-commutes. Here $u : \mathsf{Lie2Alg} \to \mathsf{Cat}$ denotes the functor that on objects sends a Lie 2-algebra to its underlying category (for the value of u on 1- and 2-morphisms see the proof of Lemma 6.3). In particular for every geometric stack $\mathcal{A}$ the category underlying the Lie 2-algebra $\mathbb{X} \circ \mathbb{B}^{-1}(\mathcal{A})$ is equivalent to the category $\mathsf{Vect}'(\mathcal{A})$ of vector fields on the stack $\mathcal{A}$.

We now construct the (2-)functor $\mathsf{Vect}' : \mathsf{GeomStack} \to \mathsf{Cat}$ (see Appendix A for a definition of a 2-functor). Recall that by Lemma 4.10 an isomorphism of stacks $f : \mathcal{A}_1 \to \mathcal{A}_2$ induces an equivalence of categories

$$ f_* : \mathsf{Vect}'(\mathcal{A}_1) \to \mathsf{Vect}'(\mathcal{A}_2). $$

Note that if $f = \text{id}_{\mathcal{A}}$ we may take $f_* = \text{id}_{\text{Vect}'(\mathcal{A})}$.

Given isomorphisms $f : \mathcal{A}_1 \to \mathcal{A}_2$ and $g : \mathcal{A}_2 \to \mathcal{A}_3$ of stacks we get equivalences of categories: $(g \circ f)_*$ and $g_* \circ f_*$. We need to produce a natural transformation $\mu_{gf} : g_* \circ f_* \Rightarrow (g \circ f)_*$. So given an object (v, a_v) of $\text{Vect}'(\mathcal{A}_1)$ we need to produce a 2-cell

$$(\mu_{gf})_{(v,a_v)} : g_*(f_*(v, a_v)) \Rightarrow (g \circ f)_*(v, a_v)$$

in the category $\text{Vect}'(\mathcal{A}_3)$. By the proof of Lemma 4.10

$$g_*(f_*(v)) = T^{\text{GeomStack}}g \circ (T^{\text{GeomStack}}f \circ v \circ f^{-1}) \circ g^{-1}.$$

Since $T^{\text{GeomStack}}$ is a (pseudo-) functor, there is a natural isomorphism

$$T^{\text{GeomStack}}g \circ T^{\text{GeomStack}}f \Rightarrow T^{\text{GeomStack}}(g \circ f).$$

Consequently there is an isomorphism

$$T^{\text{GeomStack}}g \circ (T^{\text{GeomStack}}f \circ v \circ f^{-1}) \circ g^{-1} \Rightarrow T^{\text{GeomStack}}(g \circ f) \circ v \circ (g \circ f)^{-1}.$$

This isomorphism is the desired 2-cell $(\mu_{gf})_{(v,a_v)}$. We are now ready to describe the functor Vect'. To a geometric stack $\mathcal{A}$ it assigns the category $\text{Vect}'(\mathcal{A})$. To an arrow $f : \mathcal{A}_1 \to \mathcal{A}_2$ it assigns the equivalence of categories $\text{Vect}'(f) := f_*$. Additionally for each pair (g, f) we have a natural isomorphism $\mu_{gf} : g_* \circ f_* \to (g \circ f)_*$ constructed above.

Proceeding similarly (and keeping track of the coherence data) we can promote the assignment

$$\text{Bi}_{\text{iso}} \ni (G \xrightarrow{P} H) \mapsto (\mathbb{X}_{\text{gen}}(G) \xrightarrow{P_*} \mathbb{X}_{\text{gen}}(H))$$

to a functor

$$\mathbb{X}_{\text{gen}} : \text{Bi}_{\text{iso}} \to \text{Cat}.$$

Lemma 5.3 now generalizes as follows:

Lemma 6.2. The equivalences of categories

$$(\mathbb{B}_*)_G = \mathbb{B}_* : \mathbb{X}_{\text{gen}}(G) \to \text{Vect}'(\mathbb{B}G)$$

(one for each Lie groupoid G) assemble into a natural isomorphism

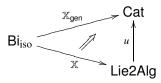$$\mathbb{B}_* : \mathbb{X}_{\text{gen}} \Rightarrow \mathbb{B} \circ \text{Vect}'.$$

That is, the diagram

2-commutes. Moreover the components of the natural transformation are equivalences of categories.

Next we prove

Lemma 6.3. The diagram

2-commutes.

Proof. We have the underlying category functor $u_{\text{strict}} : \text{Lie2Alg}_{\text{strict}} \to \text{Cat}$ which sends Lie 2-algebras to their underlying categories and morphisms of Lie 2-algebras to the underlying functors. The functor u_{strict} sends essential equivalences of Lie 2-algebras to weakly invertible functors. By the universal property of the localization $\langle\,\rangle : \text{Lie2Alg}_{\text{strict}} \to \text{Lie2Alg}$ we get the underlying category functor $u : \text{Lie2Alg} \to \text{Cat}$ with $u(\langle f \rangle)$ isomorphic to $u_{\text{strict}}(f)$ for every essential equivalence of Lie 2-algebras. It follows that for any essential equivalence f in $\text{Lie2Alg}_{\text{strict}}$ the functor $u(\langle f \rangle^{-1})$ is a weak inverse of $u_{\text{strict}}(f)$. We proved that for any essentially surjective open embedding $w : G \to G'$ of Lie groupoids the pullback functor $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$ is an essential equivalence. We defined $\mathbb{X}(w) = \langle w^* \rangle^{-1}$. It follows that $u(\mathbb{X}(w))$ is a weak inverse of $u_{\text{strict}}(w^*)$.

By Lemma 4.13 the diagram (4.5) 2-commutes in Cat for any 1-morphism $w : G \to G'$ in $\mathscr{E}mb$. Hence the diagram

$$
\begin{array}{ccc}
\mathbb{X}(G) & \xrightarrow{\;\;\iota_G\;\;} & \mathbb{X}_{\text{gen}}(G) \\
{\scriptstyle u(\mathbb{X}(w))}\Big\downarrow & & \Big\downarrow{\scriptstyle \langle w \rangle_* = \mathbb{X}_{\text{gen}}(w)} \\
\mathbb{X}(G') & \xrightarrow[\;\;\iota_{G'}\;\;]{} & \mathbb{X}_{\text{gen}}(G')
\end{array}
$$

2-commutes as well. It follows that the functors

$$u \circ \mathbb{X}, \mathbb{X}_{\text{gen}} \circ \langle\,\rangle \in \text{Hom}_W(\mathscr{E}mb, \text{Cat})$$

are isomorphic. Here $\text{Hom}_W(\mathscr{E}mb, \text{Cat})$ denotes the category of functors that send the collection W of all 1-cells in $\mathscr{E}mb$ to weakly invertible functors.

By the definition of the functor $\mathbb{X} : \text{Bi}_{\text{iso}} \to \text{Lie2Alg}$ its precomposition with the localization functor $\langle\,\rangle : \mathscr{E}mb \to \text{Bi}_{\text{iso}}$ is isomorphic to $\mathbb{X} : \mathscr{E}mb \to \text{Lie2Alg}$. It follows that the functors $u \circ \mathbb{X} \circ \langle\,\rangle$ and $\mathbb{X}_{\text{gen}} \circ \langle\,\rangle$ are isomorphic in $\text{Hom}_W(\mathscr{E}mb, \text{Cat})$. By the universal property of the localization $\langle\,\rangle : \mathscr{E}mb \to \text{Bi}_{\text{iso}}$, the functors $u \circ \mathbb{X}$ and $\mathbb{X}_{\text{gen}}$ are isomorphic in $\text{Hom}(\text{Bi}_{\text{iso}}, \text{Cat})$. $\qquad\square$

Proof of Theorem 6.1. This now follows directly from Lemmas 6.2 and 6.3. $\qquad\square$

7. Generalized vector fields on a Lie groupoid versus multiplicative vector fields

In this section we prove Theorem 4.11: for any Lie groupoid G the inclusion

$$\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\text{gen}}(G), \qquad v \mapsto (\langle v \rangle, \alpha_{\langle v \rangle} : \langle \pi_G \rangle \circ \langle v \rangle \Rightarrow \langle id_G \rangle).$$

of the category of multiplicative vector fields into the category of generalized vector fields is fully faithful and essentially surjective.

Remark 7.1. In the case of *proper* Lie groupoids Theorem 4.11 follows from [8, Theorem 4.15] — see Remark 5.4.

The fact that ι_G is fully faithful is an easy consequence of Theorem 2.5. We now address essential surjectivity. We first prove:

Lemma 7.2. Let $V = \{V_1 \rightrightarrows V_0\}$ be a 2-vector space, $v_1, \ldots, v_s \in V_0$ a finite collection of objects and $\{v_i \xleftarrow{w_{ij}} v_j\}_{i,j=1}^s$ a collection of morphisms satisfying the cocycle conditions:

- $w_{ii} = 1_{v_i}$ for all i;
- $w_{ji} = w_{ij}^{-1}$ for all i, j;
- $w_{ij} w_{jk} = w_{ik}$ for all i, j, k.

Then for any $\lambda_1, \ldots, \lambda_s \in [0, 1]$ with $\sum \lambda_k = 1$ there are morphisms $v_i \xleftarrow{z_i} \sum \lambda_k v_k$ $(i = 1, \ldots, s)$ with

$$w_{ij} = z_i z_j^{-1}$$

for all i, j.

Proof. By Remark 2.9 the category V is isomorphic to the action groupoid $\{U \times V_0 \rightrightarrows V_0\}$ where $U_0 = \ker(s : V_1 \to V_0)$, $\partial : U \to V_0$ is $t|_U$ and the action of U on V_0 is given by

$$u \cdot v := v + \partial(u).$$

Note that the multiplication/composition in $\{U \times V_0 \rightrightarrows V_0\}$ is given by

$$(u', v + \partial(u))(u, v) = (u' + u, v)$$

for all $v \in V_0$, $u, u' \in U$. Consequently

$$(u, v)^{-1} = (-u, v + \partial(u)).$$

The isomorphism $f : V \to \{U \times V_0 \rightrightarrows V_0\}$ is given on morphisms by

$$f(w) = (w - 1_{s(w)}, s(w)) \in U \times V_0 \qquad \text{for all } w \in V_1.$$

The isomorphism f followed by the projection onto U sends the morphisms w_{ij} to vectors $u_{ij} \in U$. It is easy to see that the cocycle conditions translate into:

- $u_{ii} = 0$ for all i;
- $u_{ji} = -u_{ij}$ for all i, j;
- $u_{ik} - u_{jk} = u_{ij}$ for all i, j, k.

Moreover

$$\partial(u_{ij}) = v_i - v_j \qquad \text{for all } i, j.$$

Now consider

$$y_i = (\sum \lambda_k u_{ik}, \sum \lambda_k v_k) \in U \times V_0$$

and set

$$z_i := f^{-1}(y_i) \in V_1.$$

We now verify that the z_i's are the desired morphisms. By definition the source of y_i is $\sum \lambda_k v_k$. The target of y_i is

$$\partial(\sum_k \lambda_k u_{ik}) + \sum_k \lambda_k v_k = \sum_k \lambda_k \partial(u_{ik}) + \sum_k \lambda_k v_k$$

$$= \sum_k \lambda_k (v_i - v_k) + \sum_k \lambda_k v_k = \sum_k \lambda_k v_i = v_i.$$
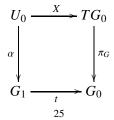
Hence z_i is an arrow from $\sum \lambda_k v_k$ to v_i. Finally

$$y_i y_j^{-1} = (\sum_k \lambda_k u_{ik}, \sum_k \lambda_k v_k)(-\sum_k \lambda_k u_{jk}, v_j)$$

$$= (\sum_k \lambda_k (u_{ik} - u_{jk}), v_j) = (\sum_k \lambda_k u_{ij}, v_j) = (u_{ij}, v_j),$$

and so $z_i z_j^{-1} = w_{ij}$ as desired. $\qquad \square$

Proposition 7.3. Let $G = \{G_1 \rightrightarrows G_0\}$ be a Lie groupoid, $U_0 \subset G_0$ an open submanifold and $U = \{U_1 \rightrightarrows U_0\}$ the restriction of G to U_0 (that is, U_1 consists of arrows of G with source and target in U_0). Given a functor $X : U \to TG$ together with a natural isomorphism $\alpha : (i : U \hookrightarrow G) \Rightarrow \pi_G \circ X$ there exists a functor $Y : U \to TU$ so that $\pi_U \circ Y = \mathrm{id}_U$ and a natural isomorphisms $\beta : Ti \circ Y \Rightarrow X$.

Proof. By definition of α the diagram

25

commutes. Hence there is a smooth map
$$(\alpha, X) : U_0 \to G_1 \times_{t, G_0, \pi} TG_0 = t^*TG_0.$$

Since the target map $t : G_1 \to G_0$ is a submersion, its differential
$$Tt_\gamma : T_\gamma G_1 \to T_{t(\gamma)}G_0$$

is a surjective linear map for each $\gamma \in G_1$. Consequently the map
$$\Phi : TG_1 \to t^*TG_0, \qquad \Phi(\gamma, v) = (\gamma, Tt_\gamma v)$$

is a surjective map of vector bundles over G_1. Choose a smooth section $\sigma : t^*TG_0 \to TG_1$ of Tt of Φ and consider the composite
$$\beta := \sigma \circ (\alpha, X) : U_0 \to TG_1.$$

By construction of β
$$\beta(x) \in T_{\alpha(x)}G_1 \qquad \text{and} \qquad Tt_{\alpha(x)}\beta(x) = X(x)$$

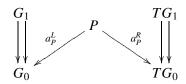for any $x \in U_0$. We now define a functor $Y : U \to TU$. On objects we set
$$Y(x) = Ts(\beta(x)).$$

For an arrow $x \xrightarrow{\gamma} y \in U_1$ we set
$$Y(\gamma) = \beta(y)^{-1}X(\gamma)\beta(x).$$

It is easy to check that Y is indeed functor, $\beta : Ti \circ Y \Rightarrow X$ is a natural transformation and $\pi_U \circ Y = \text{id}_U$. $\square$

Proposition 7.4. Let G be a Lie groupoid and

be a bibundle from G to the tangent groupoid TG such that the composite $\langle \pi \rangle \circ P$ is isomorphic to $\langle id_G \rangle$ by way of a bibundle isomorphism
$$\mathbf{a} : \langle \pi \rangle \circ P \Rightarrow \langle \text{id}_G \rangle.$$

Then the left anchor $a_P^L : P \to G_0$ has a global section $\tau : G_0 \to P$. Moreover we may choose τ so that the corresponding functor $X_\tau : G \to TG$ is a multiplicative vector field (i.e., $\pi_G \circ X_\tau = \text{id}_G$). Consequently the functor $\iota_G : \mathbb{X}(G) \to \mathbb{X}_{\text{gen}}(G)$ of Theorem 4.11 is essentially surjective.

Proof. Since $a_P^L : P \to G_0$ is a surjective submersion, it has local sections. Choose a collection of local sections $\{\sigma_i : U_0^{(i)} \to P\}$ of a_P^L so that $\{U_0^{(i)}\}$ is an open cover of G_0. It is no loss of generality to assume that the cover is locally finite. Denote the restriction of the groupoid G to $U_0^{(i)}$ by $U^{(i)}$. That is, the manifold of objects of $U^{(i)}$ is $U_0^{(i)}$ and the manifold of arrows $U_1^{(i)}$ consists of all arrows of G with source and target in $U_0^{(i)}$, so $U_1^{(i)} := s^{-1}(U_0^{(i)}) \cap t^{-1}(U_0^{(i)})$.

For each section σ_i we get a functor $X_i : U^{(i)} \to TG$ whose value on objects is
$$X_i(x) = a_P^R(\sigma_i(x)).$$

The value of X_i on an arrow $y \xleftarrow{\gamma} x \in U_1^{(i)}$ is uniquely defined by the equation
$$\gamma \cdot \sigma_i(x) = \sigma_i(y) \cdot X_i(\gamma)$$

(see Lemma 2.4). We next observe that the isomorphism $\mathbf{a} : \langle \pi_G \rangle \circ P \to \langle \text{id}_G \rangle$ gives rise to natural isomorphisms $\alpha_j : \pi_G \circ X_j \Rightarrow (\iota_j : U^{(j)} \hookrightarrow G)$ where $\iota_j : U^{(j)} \hookrightarrow G$ is the inclusion functor. This can be seen as follows.

Recall that the composite $Q \circ P$ of bibundles $P : K \to L$ and $Q : L \to M$ is the quotient of the fiber product $P \times_{a_P^R, L_0, a_Q^L} Q$ by the action of L. We denote by $[p, q]$ the orbit of $(p, q) \in P \times_{a_P^R, L_0, a_Q^L} Q$ in $Q \circ P = (P \times_{a_P^R, L_0, a_Q^L} Q)/L$. The bibundle $\langle \pi_G \rangle$ is the fiber product $TG_0 \times_{\pi_G, G_0, t} G_1$ with the anchor maps $a_{\langle \pi_G \rangle}^R(v, \gamma) = v$, $a_{\langle \pi_G \rangle}^L(v, \gamma) = s(\gamma)$. Consequently in our case

$$\langle \pi_G \rangle \circ P = (P \times_{a_p^R, TG_0, a_{\langle \pi_G \rangle}^L} (TG_0 \times_{\pi_G, G_0, s} G_1))/TG.$$

It is convenient to identify $P \times_{a_p^R, TG_0, a_{\langle \pi_G \rangle}^L} (TG_0 \times_{\pi_G, G_0, s} G_1)$ with $P \times_{\pi_G \circ a_p^R, G_0, s} G_1$ by way of the TG-equivariant isomorphism

$$(p, (\pi_G \circ a_P^R)(p), \gamma) \mapsto (p, \gamma).$$

We then have a $G \times G$ equivariant diffeomorphism

$$\mathbf{a} : (P \times_{\pi_G \circ a_p^R, G_0, s} G_1)/TG \to G_1, \qquad [p, \gamma] \mapsto \mathbf{a}([p, \gamma])$$

with

$$s(\mathbf{a}([p, \gamma])) = s(\gamma) \quad \text{and} \quad t(\mathbf{a}([p, \gamma])) = a_L^P(p).$$

A local section $\sigma_i : U_0^{(i)} \to P$ also defines a local section

$$\bar{\sigma}_i : U_0^{(i)} \to (P \times_{G_0} G_1)/TG$$

of $a_{\langle \pi_G \rangle \circ P}^L : (P \times_{G_0} G_1)/TG \to G_0$. It is given by

$$\bar{\sigma}_i(x) = [\sigma_i(x), 1_{(\pi_G \circ a_p^R \circ \sigma_i)(x)}](= [\sigma_i(x), 1_{\pi_G \circ X_i (x)}]).$$

The arrow $\mathbf{a}(\bar{\sigma}_i(x)) \in G_1 = \langle \mathrm{id}_G \rangle$ is an arrow with the target $a_P^L(\sigma_i(x)) = x$ and the source $s(1_{\pi_G \circ X_i (x)}) = \pi_G \circ X_i (x)$. We define the desired natural isomorphism α_i by setting

$$\alpha_i(x) = (\mathbf{a}(\bar{\sigma}_i(x)))^{-1} .$$

By Proposition 7.3 there are smooth maps $\beta_i : U_0^{(i)} \to TG_1$ so that

$$\pi_G \circ \beta_i = \alpha_i$$

and

$$Tt \circ \beta_i = X_i.$$

Moreover the functors $Y_i : U^{(i)} \to TG$ given by

$$Y_i = Ts \circ \beta_i$$

define multiplicative vector fields on each groupoid $U^{(i)}$. This is because their images land in $TU^{(i)} \subset TG$. In particular $\pi_G(Y_i(x)) = x$ for all $x \in U_0^{(i)}$.

Define the local sections $\nu_i : U_0^{(i)} \to P$ of a_P^L by

$$\nu_i(x) := \sigma_i(x) \cdot \beta_i(x)$$

for all $x \in U_0^{(i)}$. Then by definition

$$a^R(\nu_i(x)) = Y_i(x)$$

and

$$\gamma \cdot \nu_i(x) = \nu_i(y) \cdot Y_i(x)$$

for all arrows $y \xleftarrow{\gamma} x$. For all i and all $x \in U_0^{(i)}$

$$\mathbf{a}([\nu_i(x), 1_{\pi_G \circ a^R \circ \nu_i(x))}]) = \mathbf{a}([\sigma_i(x)\beta_i(x), 1_{\pi_G \circ Y_i(x)}]) = \mathbf{a}([\sigma_i(x), \pi_G(\beta(x))]$$
$$= \mathbf{a}([\sigma_i(x), 1_{\pi_G \circ X_i(x)}])\pi_G(\beta(x)) = \mathbf{a}(\bar{\sigma}_i(x))\alpha_i(x) = 1_x.$$

Hence

$$\mathbf{a}([\nu_i(x), 1_{\pi_G \circ Y_i(x)}] = 1_x.$$

Finally, we construct a global section $\tau : G_0 \to P$ of a_P^L and the corresponding global multiplicative vector fields $X_\tau : G \to TG$ using a partition of unity argument. Choose a partition of unity $\{\lambda_i\}$ on G_0 subordinate to the cover $\{U_0^{(i)}\}$. Since the cover is locally finite it is no loss of generality to assume that the cover is in fact finite.

Consider a point $x \in U_0^{(i)} \cap U_0^{(j)}$. Then

$$\pi_G \circ a_P^R \circ v_i(x) = x = \pi_G \circ a_P^R \circ v_j(x).$$

Moreover

$$\mathbf{a}([v_i(x), 1_x]) = \mathbf{a}([v_i(x), 1_{\pi_G \circ a_R \circ v_i(x)}]) = 1_x.$$

Similarly

$$\mathbf{a}([v_j(x), 1_x]) = 1_x.$$

Since $\mathbf{a}$ is a diffeomorphism it follows that

$$[v_j(x), 1_x] = [v_i(x), 1_x]$$

in the orbit space $(P \times_{G_0} G_1)/TG$. Therefore there is an arrow $w_{ij} \in TG_1$ so that

$$(v_i(x)w_{ij}(x), 1_x) = (v_j(x), \pi_G(w_{ij}(x))1_x).$$

Consequently

$$v_i(x)w_{ij}(x) = v_j(x) \qquad \text{and} \qquad \pi_G(w_{ij}(x)) = 1_x,$$

that is, $w_{ij}(x) \in T_{1_x}G_1$. Moreover since $a_P^L : P \to G_0$ is a principal TG_1 bundle, the arrow $w_{ij}(x)$ with this property is unique and depends smoothly on x. Note that the source of w_{ij} is $Y_j(x)$ and the target is $Y_i(x)$. The uniqueness of the $w_{ij}(x)$'s implies that the collection $\{w_{ij}(x)\}$ satisfies the cocycle conditions of Lemma 7.2. Therefore there exist arrows $Y_i(x) \xleftarrow{z_i(x)} \sum_k \lambda_k Y_k(x)$ with $z_i(x)z_j(x)^{-1} = w_{ij}(x)$. A quick look at the proof of Lemma 7.2 should convince the reader that $z_i(x)$'s depend smoothly on x.

For $x \in U_0^{(i)}$ we set $\tau(x) = v_i(x) \cdot z_i(x)$. Note that for $x \in U_0^{(i)} \cap U_0^{(j)}$

$$v_j(x) = v_i(x)w_{ij}(x) = v_i(x)z_i(x)z_j(x)^{-1}.$$

Therefore

$$v_j(x) \cdot z_j(x) = v_i(x) \cdot z_i(x).$$

It follows that τ is a globally defined section of $a_P^L : P \to G_0$. It remains to show that the corresponding functor $X_\tau : G \to TG$ is a multiplicative vector field. By construction for each index i we have a natural isomorphism $z_i : Y_i \Rightarrow X_\tau|_{U^{(i)}}$. Since $z_i(x) \in T_{1_x}G_1$ and since Y_i is a multiplicative vector field, the restriction $X_\tau|_{U^{(i)}}$ is also a multiplicative vector field. We conclude that X_τ is a multiplicative vector field globally. $\square$

APPENDIX A. BICATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

In this section for the reader's convenience we record the definitions of a bicategories, (2,1)-bicategories, (pseudo-)functors and natural transformations. Our presentation closely follows [9].

Definition A.1. A *bicategory* $\mathcal{B}$ consists of the following data subject to the following axioms:

DATA

- A collection $\mathcal{B}_0$ of 0-cells (or objects).
- For every pair A, B of 0-cells a *category* $\mathsf{Hom}_\mathcal{B}(A, B)$ of morphisms from A to B. The objects of $\mathsf{Hom}_\mathcal{B}(A, B)$ are called 1-cells (or 1-morphisms) and are written $f : A \to B$ (small Latin letters). The morphisms $\mathsf{Hom}_\mathcal{B}(A, B)$ are called 2-cells (or 2-morphisms) and are written $\alpha : f \Rightarrow g$ (Greek letters). Note that for every 1-cell f we have the 2-cell $\mathrm{id}_f : f \Rightarrow f$.

  We refer to the composition of 2-cells in $\mathsf{Hom}_\mathcal{B}(A, B)$ as a vertical composition and write it as $\circ$ or as blank: $\beta \circ \alpha \equiv \beta\alpha$.

28

- For every triple of 0-cells A, B, C a *composition* functor

$$c_{ABC} = c : \mathsf{Hom}_{\mathscr{B}}(B, C) \times \mathsf{Hom}_{\mathscr{B}}(A, B) \quad \to \quad \mathsf{Hom}_{\mathscr{B}}(A, C)$$

$$c_{ABC}(g, f) \quad = \quad g \circ f \equiv gf, \qquad \text{for all 1-cells } g, f$$

$$c_{ABC}(\beta, \alpha) \quad = \quad \beta \star \alpha. \qquad \text{for all 2-cells } \beta, \alpha$$

We refer to the composition $\star$ of 2-cells as the horizontal composition.
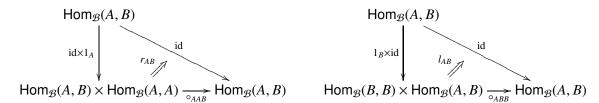- For every object A of $\mathscr{B}$ a 1-cell $1_A \in \mathsf{Hom}_{\mathscr{B}}(A, A)$.
- Natural isomorphisms $a_{ABCD} : c_{ABD} \circ (c_{BCD} \times \mathrm{id}) \Rightarrow c_{ACD} \circ (\mathrm{id} \times c_{ABC})$ called associators for every quadruple of objects A, B, C, D:

$$
\begin{array}{ccc}
\mathsf{Hom}_{\mathscr{B}}(C, D) \times \mathsf{Hom}_{\mathscr{B}}(B, C) \times \mathsf{Hom}_{\mathscr{B}}(A, B) & \xrightarrow{\ \mathrm{id} \times c_{ABC}\ } & \mathsf{Hom}_{\mathscr{B}}(C, D) \times \mathsf{Hom}_{\mathscr{B}}(A, C) \\
\Big\downarrow{\scriptstyle c_{BCD} \times \mathrm{id}} & \nearrow{\scriptstyle a_{ABCD}} & \Big\downarrow{\scriptstyle c_{ACD}} \\
\mathsf{Hom}_{\mathscr{B}}(B, D) \times \mathsf{Hom}_{\mathscr{B}}(A, B) & \xrightarrow[\ c_{ABD}\ ]{} & \mathsf{Hom}_{\mathscr{B}}(A, D)
\end{array}
$$

and in particular invertible 2-cells

$$a_{hgf} : (hg)f \Rightarrow h(gf)$$

for every triple of 1-cells $(h, g, f) \in \mathsf{Hom}_{\mathscr{B}}(C, D) \times \mathsf{Hom}_{\mathscr{B}}(B, C) \times \mathsf{Hom}_{\mathscr{B}}(A, B)$.
- Natural isomorphisms $r_{AB} : c_{AAB} \circ (\mathrm{id} \times 1_A) \Rightarrow \mathrm{id}$ and $l_{AB} : c_{ABB} \circ (1_B \times \mathrm{id}) \Rightarrow \mathrm{id}$ called right and left unitors for every pair of objects A, B of $\mathscr{B}$:

$$
\begin{array}{cc}
\mathsf{Hom}_{\mathscr{B}}(A, B) & \qquad \mathsf{Hom}_{\mathscr{B}}(A, B) \\
\Big\downarrow{\scriptstyle \mathrm{id} \times 1_A} \ \searrow{\scriptstyle \mathrm{id}} & \qquad \Big\downarrow{\scriptstyle 1_B \times \mathrm{id}} \ \searrow{\scriptstyle \mathrm{id}} \\
\scriptstyle r_{AB} \nearrow & \qquad \scriptstyle l_{AB} \nearrow \\
\mathsf{Hom}_{\mathscr{B}}(A, B) \times \mathsf{Hom}_{\mathscr{B}}(A, A) \xrightarrow[\circ_{AAB}]{} \mathsf{Hom}_{\mathscr{B}}(A, B) & \qquad \mathsf{Hom}_{\mathscr{B}}(B, B) \times \mathsf{Hom}_{\mathscr{B}}(A, B) \xrightarrow[\circ_{ABB}]{} \mathsf{Hom}_{\mathscr{B}}(A, B)
\end{array}
$$

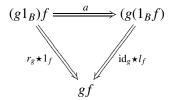where $\mathrm{id} : \mathsf{Hom}_{\mathscr{B}}(A, B) \to \mathsf{Hom}_{\mathscr{B}}(A, B)$ denotes the identity functor. By abuse of notation $1_A : \mathsf{Hom}_{\mathscr{B}}(A, B) \to \mathsf{Hom}_{\mathscr{B}}(A, A)$ denotes the functor that takes every 2-cell to the identity 2-cell $\mathrm{id}_{1_A} : 1_A \Rightarrow 1_A$. The functor 1_B is defined similarly. Thus for every 1-cell $f \in \mathsf{Hom}_{\mathscr{B}}(A, B)$ we have *invertible* 2-cells

$$r_f : f \circ 1_A \Rightarrow f \quad \text{and} \quad l_f : 1_B \circ f \Rightarrow f.$$
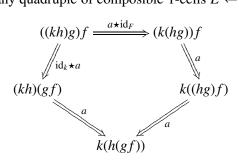
CONDITIONS ON THE DATA

- (Triangle identity) For any pair of composable 1-cells $C \xleftarrow{g} B \xleftarrow{f} A$ the diagram of 2-cells

$$
\begin{array}{ccc}
(g 1_B)f & \xRightarrow{\ a\ } & (g(1_B f) \\
& & \\
\scriptstyle r_g \star 1_f \searrow & & \swarrow \scriptstyle \mathrm{id}_g \star l_f \\
& gf &
\end{array}
$$

commutes.

- (Pentagon identity) For any quadruple of composible 1-cells $E \xleftarrow{k} D \xleftarrow{h} C \xleftarrow{g} B \xleftarrow{f} A$ the diagram

$$
\begin{array}{ccc}
((kh)g)f & \xRightarrow{\;a\star\mathrm{id}_F\;} & (k(hg))f \\
\Downarrow{\scriptstyle \mathrm{id}_k\star a} & & \Downarrow{\scriptstyle a} \\
(kh)(gf) & & k((hg)f) \\
& {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle a} & \\
& k(h(gf)) &
\end{array}
$$

commutes.

Remark A.2.   • If the natural isomorphisms a, r and l of a bicategory $\mathcal{B}$ are identities so that $(hg)f = h(gf)$ and $1_B \circ f = f = f \circ 1_A$ for all 1-cells $h, g, f : A \rightarrow B$, and similarly for the horizontal composition of 2-cells, then the bicategory $\mathcal{B}$ is a (strict) **2-category**.
- If all the 2-cells in a bicategory $\mathcal{B}$ are invertible, that is, for any 2-cell $\alpha : f \Rightarrow g$ there is $\beta : g \Rightarrow f$ with $\beta\alpha = \mathrm{id}_f$ and $\alpha\beta = \mathrm{id}_g$ then $\mathcal{B}$ is a **(2,1)-bicategory**.

The following definitions are special cases of more general definitions that are specialized to (2,1)-bicategories.

Definition A.3. A (pseudo-)**functor** from a (2,1)-bicategory $\mathcal{B}$ to a (2,1)-bicategory $\mathcal{B}'$ consists of the follow data:

- A function $F = F_0 : \mathcal{B}_0 \rightarrow \mathcal{B}'_0$ on objects.
- For any pair of objects $A, B \in \mathcal{B}_0$ a functor

$$F_{AB} : \mathsf{Hom}_{\mathcal{B}}(A, B) \rightarrow \mathsf{Hom}_{\mathcal{B}'}(FA, FB)$$

- For every triple of objects A, B, C of $\mathcal{B}$ a natural isomorphism

$$\mu_{ABC} : (c'_{FA,FB,FC}) \circ (F_{BC} \times F_{AB}) \Rightarrow F_{AC} \circ (c_{ABC}) :$$

$$
\begin{array}{ccc}
\mathsf{Hom}_{\mathcal{B}}(B, C) \times \mathsf{Hom}_{\mathcal{B}}(A, B) & \xrightarrow{\quad c_{ABC} \quad} & \mathsf{Hom}_{\mathcal{B}}(A, C) \\
{\scriptstyle F_{BC}\times F_{AB}}\Big\downarrow & {\scriptstyle \mu_{ABC}}\nearrow & \Big\downarrow{\scriptstyle F_{AC}} \\
\mathsf{Hom}_{\mathcal{B}'}(FB, FC) \times \mathsf{Hom}_{\mathcal{B}'}(FA, FB) & \xrightarrow[\quad c'_{FA,FB,FC} \quad]{} & \mathsf{Hom}_{\mathcal{B}'}(FA, FC)
\end{array}
$$

thus invertible 2-cells $\mu_{gf} : Fg \circ Ff \Rightarrow F(g \circ f)$ for every pair of composible 1-cells $C \xleftarrow{g} B \xleftarrow{f} A$. Here $c'_{FA,FB,FC}$ is the composition functor in the target bicategory $\mathcal{B}'$.
- 2-cells $\mu_A : 1_{FA} \Rightarrow F(1_A)$ for every object A of $\mathcal{B}$.

conditions on the data
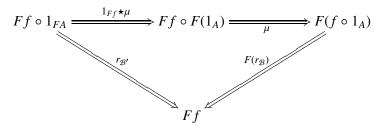
For any triple $D \xleftarrow{h} C \xleftarrow{g} B \xleftarrow{f} A$ of composible 1-cells the following diagrams of 2-cells commute:

$$
\begin{array}{ccccc}
(Fh \circ Fg) \circ Ff & \xRightarrow{\;\mu\star 1_{Ff}\;} F(h \circ g) \circ Ff \xRightarrow{\;\mu\;} & F((h \circ g) \circ f) \\
\Big\Downarrow{\scriptstyle a_{\mathcal{B}'}} & & \Big\Downarrow{\scriptstyle F(a_{\mathcal{B}})} \\
Fh \circ (Fg \circ Ff) & \xRightarrow[\;1_{Fh}\star\mu\;]{} Fh \circ (F(g \circ f)) \xRightarrow[\;\mu\;]{} & F(h \circ (g \circ f))
\end{array}
$$
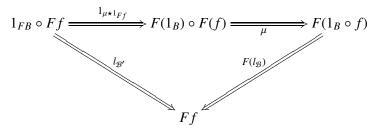
(here $a_{\mathcal{B}'}$ denotes the associator in the bicategory $\mathcal{B}'$),

$$Ff \circ 1_{FA} \xrightarrow{\;1_{Ff} \star \mu\;} Ff \circ F(1_A) \xrightarrow{\quad \mu \quad} F(f \circ 1_A)$$

with $r_{\mathcal{B}'}$ and $F(r_{\mathcal{B}})$ to Ff.

and

$$1_{FB} \circ Ff \xrightarrow{\;1_\mu \star 1_{Ff}\;} F(1_B) \circ F(f) \xrightarrow{\quad \mu \quad} F(1_B \circ f)$$

with $l_{\mathcal{B}'}$ and $F(l_{\mathcal{B}})$ to Ff.

Here $r_{\mathcal{B}}$ and $r_{\mathcal{B}'}$ denote the right unitors in $\mathcal{B}$ and $\mathcal{B}'$ respectively. The meaning of $l_{\mathcal{B}'}$ and $l_{\mathcal{B}}$ is similar.

Definition A.4. Let (F, μ), (G, τ) be (pseudo-) functors from a (2,1)-bicategory $\mathcal{B}$ to a (2,1)-bicategory $\mathcal{B}'$. A natural transformation $\sigma : (F, \mu) \Rightarrow (G, \tau)$ consists of the following data:

- 1-cells $\sigma_A : FA \to GA$ for every object A of $\mathcal{B}$;
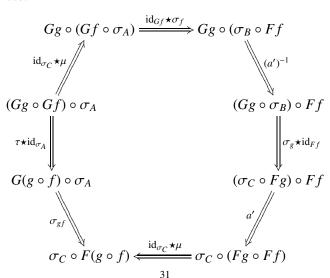- natural isomorphisms

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(A, B) & \xrightarrow{\;F_{AB}\;} & \mathrm{Hom}_{\mathcal{B}'}(FA, FB) \\[2pt]
\scriptstyle{G_{AB}} \Big\downarrow & \overset{\sigma_{AB}}{\nearrow} & \Big\downarrow \scriptstyle{\sigma_B \circ -} \\[2pt]
\mathrm{Hom}_{\mathcal{B}'}(GA, GB) & \xrightarrow[\; - \,\circ\, \sigma_A \;]{} & \mathrm{Hom}_{\mathcal{B}'}(FA, GB)
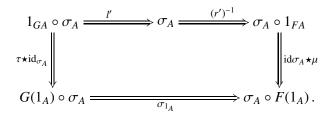\end{array}
$$

and thus 2-cells $\sigma_f : Gf \circ \sigma_A \Rightarrow \sigma_B \circ Ff$.

The data are subject to the following compatibility conditions — for any pair of composable 1-cells

$$C \xleftarrow{\;g\;} B \xleftarrow{\;f\;} A$$

the diagrams below commute:

$$
\begin{array}{ccc}
Gg \circ (Gf \circ \sigma_A) & \xRightarrow{\;\mathrm{id}_{Gf} \star \sigma_f\;} & Gg \circ (\sigma_B \circ Ff) \\[4pt]
\scriptstyle{\mathrm{id}_{\sigma_C} \star \mu} \Big\Uparrow & & \Big\Downarrow \scriptstyle{(a')^{-1}} \\[4pt]
(Gg \circ Gf) \circ \sigma_A & & (Gg \circ \sigma_B) \circ Ff \\[4pt]
\scriptstyle{\tau \star \mathrm{id}_{\sigma_A}} \Big\Downarrow & & \Big\Downarrow \scriptstyle{\sigma_g \star \mathrm{id}_{Ff}} \\[4pt]
G(g \circ f) \circ \sigma_A & & (\sigma_C \circ Fg) \circ Ff \\[4pt]
\scriptstyle{\sigma_{gf}} \searrow & & \swarrow \scriptstyle{a'} \\[4pt]
\sigma_C \circ F(g \circ f) & \xLeftarrow[\;\mathrm{id}_{\sigma_C} \star \mu\;]{} & \sigma_C \circ (Fg \circ Ff)
\end{array}
$$

and

$$1_{GA} \circ \sigma_A \xrightarrow{\quad l' \quad} \sigma_A \xrightarrow{\quad (r')^{-1} \quad} \sigma_A \circ 1_{FA}$$

$$\tau \star \mathrm{id}_{\sigma_A} \Big\Downarrow \qquad\qquad\qquad\qquad\qquad \Big\Downarrow \mathrm{id}\sigma_A \star \mu$$

$$G(1_A) \circ \sigma_A \xRightarrow{\qquad \sigma_{1_A} \qquad} \sigma_A \circ F(1_A).$$

References

[1] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004) 492–538, `arXiv:math.QA/0307263`.

[2] K. Behrend and P. Xu, Differentiable stacks and gerbes, *Journal of Symplectic Geometry*,**9**(3) (2011), 285–341.

[3] J. Bénabou, Introduction to bicategories, in *Reports of the Midwest Category Seminar* pp. 1–77, Springer, Berlin, 1967.

[4] C. Blohmann, Stacky Lie groups, *Int. Math. Res. Notices*, 2008, rnn082.

[5] F. Borceux *Handbook of categorical algebra*, volume 50 of Encyclopedia of Mathematics and its Applications, Cambridge Univ. Press, 1994.

[6] P. Deligne, La formule de dualité globale in *Théorie des topos et cohomologie étale des schémas* Tome 3, Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973. vi+640 pp.

[7] Y. Fregier and F. Wagemann, *On Hopf 2-algebras*, IMRN, vol. 2011, no. 15, 3471–3501

[8] R. Hepworth, Vector fields and flows on differentiable stacks, *Theory Appl. Categ.* **22** (2009), 542–587.

[9] T. Leinster, Basic bicategories, `https://arxiv.org/abs/math/9810017v1`

[10] E. Lerman, Orbifolds as Stacks?. *L'Enseignment Mathématique* **56** (2010), 315–363.

[11] K.C.H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series **213**, Cambridge University Press, Cambridge, 2005. xxxviii+501 pp.

[12] K.C.H. Mackenzie and P. Xu, Classical lifting properties and multiplicative vector fields, *Quart. J. Math. Oxford(2),* **49** (1998), 59–85.

[13] D. Metzler, Topological and smooth stacks, arXiv:math/0306176 [math.DG] (2003).

[14] I. Moerdijk and J. Mrcun, *Introduction to Foliations and Lie groupoids*, Cambridge University Press, Cambridge, 2003. ix+173 pp.

[15] B. Noohi, Integrating morphisms of Lie 2-algebras, *Compositio Math.* **149** (2013), 264–294. [10.1112/S0010437X1200067X]

[16] C. Ortiz and J. Waldron, Lie 2-algebra of vector fields on Lie groupoids, arXiv:1703.09791 [math.DG] (2017).

[17] D.A. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Mathematica* **102**, no. 3 (1996), pp. 243–303.

[18] C. Schommer-Pries, Central extensions of smooth 2-groups and a finite-dimensional string 2-group, *Geometry and Topology*, **15** (2011), pp. 609-676.

[19] A. Weinstein, The Volume of a Differentiable Stack, *Lett. Math. Phys.* **90** (2009), 353–371.

E-mail address: `danbe@illinois.edu`

E-mail address: `lerman@illinois.edu`

Department of Mathematics, University of Illinois, Urbana, IL 61801