# A strengthened inequality of Alon-Babai-Suzuki's conjecture on set systems with restricted intersections modulo $p$

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## Abstract

Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be disjoint subsets of $\{0, 1, \ldots, p-1\}$, where $p$ is a prime and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for $i \neq j$. In 1991, Alon, Babai and Suzuki conjectured that if $n \geq s + \max_{1 \leq i \leq r} k_i$, then $|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. In 2000, Qian and Ray-Chaudhuri proved the conjecture under the condition $n \geq 2s - r$. In 2015, Hwang and Kim verified the conjecture of Alon, Babai and Suzuki.

In this paper, we will prove that if $n \geq 2s - 2r + 1$ or $n \geq s + \max_{1 \leq i \leq r} k_i$, then

$$|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This result strengthens the upper bound of Alon, Babai and Suzuki's conjecture when $n \geq 2s - 2$.

## 1 Introduction

A family $\mathcal{A}$ of subsets of $[n]$ is called *intersecting* if every pair of distinct subsets $A_i, A_j \in \mathcal{A}$ have a nonempty intersection. Let $L$ be a set of $s$ nonnegative integers. A family $\mathcal{A}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ is *L-intersecting* if $|A_i \cap A_j| \in L$ for every pair of distinct subsets $A_i, A_j \in \mathcal{A}$. A family $\mathcal{A}$ is *k-uniform* if it is a collection of $k$-subsets of $[n]$. Thus, a $k$-uniform intersecting family is $L$-intersecting for $L = \{1, 2, \ldots, k-1\}$.

The following is an intersection theorem of de Bruijin and Erdös [4].

**Theorem 1.1** (de Bruijin and Erdös, 1948 [4]). *If $\mathcal{A}$ is a family of subsets of $[n]$ satisfying $|A_i \cap A_i| = 1$ for every pair of distinct subsets $A_i, A_j \in \mathcal{A}$, then $|\mathcal{A}| \leq n$.*

A year later, Bose [2] obtained the following more general intersection theorem which requires the intersections to have exactly $\lambda$ elements.

**Theorem 1.2** (Bose, 1949 [2]). *If $\mathcal{A}$ is a family of subsets of $[n]$ satisfying $|A_i \cap A_i| = \lambda$ for every pair of distinct subsets $A_i, A_j \in \mathcal{A}$, then $|\mathcal{A}| \leq n$.*

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In 1961, Erdös, Ko and Rado [5] proved the following classical result on $k$-uniform intersecting families.

**Theorem 1.3** (Erdös, Ko and Rado, 1961 [5])**.** *Let $n \geq 2k$ and let $\mathcal{A}$ be a $k$-uniform intersecting family of subsets of $[n]$. Then $|\mathcal{A}| \leq \binom{n-1}{k-1}$ with equality only when $\mathcal{A}$ consists of all $k$-subsets containing a common element.*

In 1975, Ray-Chaudhuri and Wilson [11] made a major progress by deriving the following upper bound for a $k$-uniform $L$-intersecting family.

**Theorem 1.4** (Ray-Chaudhuri and Wilson, 1975 [11])**.** *If $\mathcal{A}$ is a $k$-uniform $L$-intersecting family of subsets of $[n]$, then $|\mathcal{A}| \leq \binom{n}{s}$.*

In terms of parameters $n$ and $s$, this inequality is best possible, as shown by the set of all $s$-subsets of $[n]$ with $L = \{0, 1, \ldots, s-1\}$.

In 1981, Frankl and Wilson [6] obtained the following celebrated theorem which extends Theorem 1.4 by allowing different subset sizes.

**Theorem 1.5** (Frankl and Wilson, 1981 [6])**.** *If $\mathcal{A}$ is an $L$-intersecting family of subsets of $[n]$, then $\mathcal{A} \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$.*

The upper bound in Theorem 1.5 is best possible, as demonstrated by the set of all subsets of size at most $s$ of $[n]$.

In the same paper, a modular version of Theorem 1.4 was also proved.

**Theorem 1.6** (Frankl and Wilson, 1981 [6])**.** *If $\mathcal{A}$ is a $k$-uniform family of subsets of $[n]$ such that $k \pmod p \notin L$ and $|A_i \cap A_j| \pmod p \in L$ for all $i \neq j$, then $|\mathcal{A}| \leq \binom{n}{s}$.*

In 1991, Alon, Babai and Suzuki [1] proved the following theorem, which is a generalization of Theorem 1.6 by replacing the condition of uniformity with the condition that the members of $\mathcal{A}$ have $r$ different sizes.

**Theorem 1.7** (Alon, Babai and Suzuki, 1991 [1])**.** *Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$, where $p$ is a prime, and let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A_i| \pmod p \in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j| \pmod p \in L$ for $i \neq j$. If $r(s - r + 1) \leq p - 1$ and $n \geq s + \max_{1 \leq i \leq r} k_i$, then $|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

In the proof of Theorem 1.7, Alon, Babai and Suzuki used a very elegant linear algebra method together with their Lemma 3.6 which needs the condition $r(s - r + 1) \leq p - 1$ and $n \geq s + \max_{1 \leq i \leq r} k_i$. They conjectured that the condition $r(s - r + 1) \leq p - 1$ in the statement of their theorem can be dropped off. However, their approach cannot work for this stronger claim. In an effort to prove the Alon-Babai-Suzuki's conjecture, Snevily [12] obtained the following result.

**Theorem 1.8** (Snevily, 1994 [12])**.** *Let $p$ be a prime and $K, L$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Let $|L| = s$ and let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A_i| \pmod p \in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j| \pmod p \in L$ for $i \neq j$. Then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.*

Since $\binom{n-1}{s} + \binom{n-1}{s-1} = \binom{n}{s}$ and $\binom{n}{s-1} > \sum_{i=0}^{s-2} \binom{n-1}{i}$ when $n$ is sufficiently large, Theorem 1.8 not only confirms the conjecture of Alon, Babai and Suzuki in many cases but also strengthens the upper bound of their theorem when $n$ is sufficiently large.

In 2000, Qian and Ray-Chaudhuri [10] developed a new linear algebra approach and proved the next theorem which shows that the same conclusion in Theorem 1.7 holds if the two conditions $r(s - r + 1) \leq p - 1$ and $n \geq s + \max_{1 \leq i \leq r} k_i$ are replaced by a single more relaxed condition $n \geq 2s - r$.

**Theorem 1.9** (Qian and Ray-Chaudhuri, 2000 [10])**.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $n \geq 2s - r$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. Then $|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

Recently, Hwang and Kim [8] verified the conjecture of Alon, Babai and Suzuki.

**Theorem 1.10** (Hwang and Kim, 2015 [8])**.** *Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$, where $p$ is a prime, and let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for $i \neq j$. If $n \geq s + \max_{1 \leq i \leq r} k_i$, then $|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

We note here that in some instances Alon, Babai and Suzuki's condition holds but Qian and Ray-Chaudhuri's condition does not, while in some other instances the later condition holds but the former condition does not.

In [3], Chen and Liu strengthened the upper bounds of Theorem 1.8 under the condition $\min\{k_i\} > \max\{l_i\}$.

**Theorem 1.11** (Chen and Liu, 2009 [3])**.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $\min\{k_i\} > \max\{l_i\}$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. Then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

In [9], Liu and Yang generalized Theorem 1.11 under a relaxed condition $k_i > s - r$ for every $i$.

**Theorem 1.12** (Liu and Yang, 2014 [3])**.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $k_i > s - r$ for every $i$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. Then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

In the same paper, they also obtained the same bound under the condition of Theorem 1.7.

**Theorem 1.13** (Liu and Yang, 2014 [3])**.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$ such that $r(s - r + 1) \leq p - 1$ and $n \geq s + \max_{1 \leq i \leq r} k_i$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. Then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

In this paper, we show that Theorem 1.13 still holds under the Alon, Babai and Suzuki's condition; that is to say, we can drop the condition $r(s - r + 1) \leq p - 1$ in Theorem 1.13.

**Theorem 1.14.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. If $n \geq s + \max_{1 \leq i \leq r} k_i$, then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

Note that $\binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1} = \binom{n}{s} + \binom{n}{s-2} + \cdots + \binom{n}{s-2(r-1)}$ and $\binom{n}{s-2i} < \binom{n}{s-i}$ for $1 \leq i \leq r - 1$ when $n \geq 2s - 2$. Our result strengthens the upper bound of Alon-Babai-Suzuki's conjecture (Theorems 1.10) when $n \geq 2s - 2$.

In the proof of Theorem 1.14, we first prove that the bound holds under the condition $n \geq 2s - 2r + 1$, which relaxes the condition $n \geq 2s - r$ in the theorem of Qian and Ray-Chaudhuri.

**Theorem 1.15.** *Let $p$ be a prime and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A_i|$ (mod $p$) $\in K$ for all $A_i \in \mathcal{A}$ and $|A_i \cap A_j|$ (mod $p$) $\in L$ for every $i \neq j$. If $n \geq 2s - 2r + 1$, then $|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.*

Theorems 1.7, 1.9, 1.12 and 1.13 have been extended to $k$-wise $L$-intersecting families in [7, 9]. With a similar idea, our results can also be extended to the $k$-wise case.

## 2 Proof of Theorem 1.15

In this section we prove Theorem 1.15, which will be helpful in the proof of Theorem 1.14.

Throughout this section, let $X = [n-1] = \{1, 2, \ldots, n-1\}$ be an $(n-1)$-element set, $p$ be a prime, and let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of $[n]$ such that (1) $|A_i|$ (mod $p$) $\in K$ for every $1 \leq i \leq m$, (2) $|A_i \cap A_j|$ (mod $p$) $\in L$ for $i \neq j$. Without loss of generality, assume that there exists a positive integer $t$ such that $n \notin A_i$ for $1 \leq i \leq t$ and $n \in A_i$ for $i \geq t+1$. Denote

$$\mathbb{P}_i(X) = \{S | S \subset X \text{ and } |S| = i\}.$$

We associate a variable $x_i$ for each $A_i \in \mathcal{A}$ and set $x = (x_1, x_2, \ldots, x_m)$. For each $I \subset X$, define

$$L_I = \sum_{i:I \subset A_i \in \mathcal{A}} x_i.$$

Consider the system of linear equation over the field $\mathbb{F}_p$:

$$\{L_I = 0, \quad \text{where } I \text{ runs through } \cup_{i=0}^{s} \mathbb{P}_i(X)\}. \tag{1}$$

**Proposition 2.1.** *Assume that $L \cap K = \emptyset$. If $\mathcal{A}$ is a mod $p$ $L$-intersecting family with $|A_i|$ (mod $p$) $\in K$ for every $i$, then the only solution of the above system of linear equations is the trivial solution.*

*Proof.* Let $v = (v_1, v_2, \ldots, v_m)$ be a solution to the system (1). We will show that $v$ is the zero solution over the field $\mathbb{F}_p$. Define

$$g(x) = \prod_{j=1}^{s} (x - l_j),$$

and

$$h(x) = g(x+1) = \prod_{j=1}^{s} (x + 1 - l_j).$$

Since $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{s}$ form a basis for the vector space spanned by all the polynomials in $\mathbb{F}_p[x]$ of degree at most $s$, there exist $a_0, a_1, \ldots, a_s \in \mathbb{F}_p$ and $b_0, b_1, \ldots, b_s \in \mathbb{F}_p$ such that

$$g(x) = \sum_{i=0}^{s} a_i \binom{x}{i},$$

and

$$h(x) = \sum_{i=0}^{s} b_i \binom{x}{i}.$$

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Let $A_{i_0}$ be an element in $\mathcal{A}$ with $v_{i_0} \neq 0$. Next we prove the following identities:

If $n \notin A_{i_0}$, then

$$\sum_{i=0}^{s} a_i \sum_{I \in \mathbb{P}_i(X), I \subset A_{i_0}} L_I = \sum_{A_i \in \mathcal{A}} g(|A_i \cap A_{i_0}|)x_i; \qquad (2)$$

if $n \in A_{i_0}$, then

$$\sum_{i=0}^{s} b_i \sum_{I \in \mathbb{P}_i(X), I \subset A_{i_0}} L_I = \sum_{i=1}^{t} h(|A_i \cap A_{i_0}|)x_i + \sum_{i \geq t+1} h(|A_i \cap A_{i_0}| - 1)x_i. \qquad (3)$$

We prove them by comparing the coefficients of both sides. For any $A_i \in \mathcal{A}$, the coefficient of $x_i$ in the left hand side of (2) is

$$\sum_{i=0}^{s} a_i |\{I \in \mathbb{P}_i(X) : I \subset A_{i_0}, I \subset A_i\}| = \sum_{i=0}^{s} a_i \binom{|A_i \cap A_{i_0}|}{i},$$

which is equal to $g(|A_i \cap A_{i_0}|)$ by the definition of $a_i$. This proves the identity (2).

For any $i \leq t$, the coefficient of $x_i$ in the left hand side of (3) is

$$\sum_{i=0}^{s} b_i |\{I \in \mathbb{P}_i(X) : I \subset A_{i_0}, I \subset A_i\}| = \sum_{i=0}^{s} b_i \binom{|A_i \cap A_{i_0}|}{i},$$

for any $i \geq t+1$, the coefficient of $x_i$ in the left hand side of (3) is

$$\sum_{i=0}^{s} b_i |\{I \in \mathbb{P}_i(X) : I \subset A_{i_0}, I \subset A_i\}| = \sum_{i=0}^{s} b_i \binom{|A_i \cap A_{i_0}| - 1}{i}.$$

This proves the identity (3).

If $n \notin A_{i_0}$, substituting $x_i$ with $v_i$ for all $i$ in the identity (2), we have

$$\sum_{i=0}^{s} a_i \sum_{I \in \mathbb{P}_i(X), I \subset A_{i_0}} L_I(v) = \sum_{A_i \in \mathcal{A}} g(|A_i \cap A_{i_0}|)v_i.$$

It is clear that the left hand side is 0 since $v$ is a solution to (1). For $A_i \in \mathcal{A}$ with $i \neq i_0$, $|A_i \cap A_{i_0}|$ (mod $p$) $\in L$ and so $g(|A_i \cap A_{i_0}|) = 0$. Thus the right hand side of the above identity is equal to $g(|A_{i_0}|)v_{i_0}$. So $g(|A_{i_0}|)v_{i_0} = 0$. Since $L \cap K = \emptyset$, we have $g(|A_{i_0}|) \neq 0$ and so $v_{i_0} = 0$. This is a contradiction to the definition of $v$.

If $n \in A_{i_0}$, substituting $x_i$ with $v_i$ for all $i$ in the identity (3), we have

$$\sum_{i=0}^{s} b_i \sum_{I \in \mathbb{P}_i(X), I \subset A_{i_0}} L_I(v) = \sum_{i=1}^{t} h(|A_i \cap A_{i_0}|)v_i + \sum_{i \geq t+1} h(|A_i \cap A_{i_0}| - 1)v_i$$

$$= \sum_{i \geq t+1} h(|A_i \cap A_{i_0}| - 1)v_i \quad \text{since } v_i = 0 \text{ for all } i \leq t.$$

Since $h(|A_i \cap A_{i_0}| - 1) = g(|A_i \cap A_{i_0}|)$, with a similar argument to the above case, we can deduce the same contradiction. Then the proposition follows. $\qquad\square$

As a result of this proposition, we have:

$$|\mathcal{A}| \leq \dim(\{L_I : I \in \cup_{i=0}^{s}\mathbb{P}_i(X)\}),$$

where $\dim(\{L_I : I \in \cup_{i=0}^{s}\mathbb{P}_i(X)\})$ is defined to be the dimension of the space spanned by $\{L_I : I \in \cup_{i=0}^{s}\mathbb{P}_i(X)\}$. In the remaining of this section, we make efforts to give an upper bound on this dimension.

**Lemma 2.2.** *For any $i \in \{0, 1, \ldots, s - 2r + 1\}$ and every $I \in \mathbb{P}_i(X)$, the linear form*

$$\sum_{H\in\mathbb{P}_{i+2r}(X), I\subset H} L_H$$

*is linearly dependent on the set of linear forms $\{L_H : i \leq |H| \leq i + 2r - 1, H \subset X\}$ over $\mathbb{F}_p$.*

*Proof.* Define

$$f(x) = \left(\prod_{j=1}^{r}(x - (k_j - i))\right) \times \left(\prod_{j=1}^{r}(x - (k_j - 1 - i))\right).$$

We distinguish two cases.

(a) $i \pmod p \notin K$ and $i + 1 \pmod p \notin K$ for all $i$. In this case $\forall\, k_j \in K$, $k_j - i \neq 0$ and $k_j - i - 1 \neq 0$ in $\mathbb{F}_p$ and so $c = (k_1 - i)(k_2 - i)\cdots(k_r - i)(k_1 - i - 1)\cdots(k_r - i - 1) \neq 0$ in $\mathbb{F}_p$. It is clear that there exist $a_1, a_2, \ldots, a_{2r-1} \in \mathbb{F}_p$, $a_{2r} = (2r)! \in \mathbb{F}_p - \{0\}$ such that

$$a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots + a_{2r}\binom{x}{2r} = f(x) - c,$$

since the polynomial in the right hand side has constant term equal to 0.

Next we show that

$$\sum_{j=1}^{2r} a_j \sum_{H\in\mathbb{P}_{i+j}(X), I\subset H} L_H = -cL_I. \tag{4}$$

In fact both sides are linear forms in $x_A$, for $A \in \mathcal{A}$. The coefficient of $x_A$ in the left hand side is $\sum_{j=1}^{2r} a_j|\{H|I \subset H \subset A, n \notin H, |H| = i + j\}|$. So it is equal to

$$\begin{cases} 0, & \text{if } I \not\subset A; \\ a_1\binom{|A|-i}{1} + a_2\binom{|A|-i}{2} + \cdots + a_{2r}\binom{|A|-i}{2r}, & \text{if } I \subset A \text{ and } n \notin A; \\ a_1\binom{|A|-i-1}{1} + a_2\binom{|A|-i-1}{2} + \cdots + a_{2r}\binom{|A|-i-1}{2r}, & \text{if } I \subset A \text{ and } n \in A. \end{cases}$$

By the above polynomial identity,

$$\sum_{j=1}^{2r} a_j \binom{|A|-i}{j} = f(|A| - i) - c = -c \text{ since } |A| \pmod p \in K;$$

$$\sum_{j=1}^{2r} a_j \binom{|A|-i-1}{j} = f(|A| - i - 1) - c = -c \text{ since } |A| \pmod p \in K.$$

The coefficient of $x_A$ in the right hand side is obviously the same. This proves (4).

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Writing (4) in a different way, we have

$$\sum_{H\in\mathbb{P}_{i+2r}(X),I\subset H} L_H = -\frac{1}{(2r)!}(cL_I + \sum_{j=1}^{2r-1} a_j \sum_{H\in\mathbb{P}_{i+j}(X),I\subset H} L_H).$$

This proves the lemma in case (a).

(b) $i\ (\mathrm{mod}\ p)\in K$ or $i+1\ (\mathrm{mod}\ p)\in K$ for some $i$. In this case, the constant term of $(x-(k_1-i))(x-(k_2-i))\cdots(x-(k_r-i))(x-(k_1-i-1))\cdots(x-(k_r-i-1))$ is $0\in\mathbb{F}_p$. So there exists $a_1,a_2,\ldots,a_{2r-1}\in\mathbb{F}_p$, $a_{2r}=(2r)!\in\mathbb{F}_p-\{0\}$ such that

$$a_1\binom{x}{1}+a_2\binom{x}{2}+\cdots+a_{2r}\binom{x}{2r}=f(x)$$

As a consequence we have

$$\sum_{j=1}^{2r} a_j \sum_{H\in\mathbb{P}_{i+j}(X),I\subset H} L_H = 0 \ \ \forall I\in\mathbb{P}_i(X),$$

i.e. we have

$$\sum_{H\in\mathbb{P}_{i+2r}(X),I\subset H} L_H = -\frac{1}{(2r)!}(\sum_{j=1}^{2r-1} a_j \sum_{H\in\mathbb{P}_{i+j}(X),I\subset H} L_H).$$

This finishes the proof of this lemma.

$\square$

**Corollary 2.3.** *With the same condition as in Lemma 2.2, we have*

$$\langle L_H : H \in \cup_{j=i}^{i+2r-1}\mathbb{P}_j(X)\rangle$$

$$= \langle L_H : H \in \cup_{j=i}^{i+2r-1}\mathbb{P}_j(X)\rangle + \left\langle \sum_{H\in\mathbb{P}_{i+2r}(X),I\subset H} L_H : I\in\mathbb{P}_i(X)\right\rangle$$

*Here $\langle L_H : H \in \cup_{j=i}^{i+2r-1}\mathbb{P}_j(X)\rangle$ is the vector space spanned by $\{L_H : H \in \cup_{j=i}^{i+2r-1}\mathbb{P}_j(X)\}$.*

The rest of the proof is similar to the proof of Theorem 1.9 given by Qian and Ray-Chaudhuri [10]. The next lemma is a restatement of [10, Lemma 2], and is used to prove Lemma 2.5.

**Lemma 2.4.** *For any positive integers $u, v$ with $u < v < p$ and $u + v \leq n - 1$, we have*

$$\mathrm{dim}\left(\frac{\langle L_J : J\in\mathbb{P}_v(X)\rangle}{\langle\sum_{J\in\mathbb{P}_v(X),I\subset J} L_J : I\in\mathbb{P}_u(X)\rangle}\right) \leq \binom{n-1}{v} - \binom{n-1}{u}.$$

*Here $\frac{A}{B}$ is the quotient space of two vector spaces $A$ and $B$ with $B\leq A$.*

**Lemma 2.5.** *For any $i\in\{0,1,\ldots,s-2r+1\}$,*

$$\binom{n-1}{i}+\binom{n-1}{i+1}+\cdots+\binom{n-1}{i+2r-1}+\mathrm{dim}\left(\frac{\langle L_H : H\in\cup_{j=i}^s\mathbb{P}_j(X)\rangle}{\langle L_H : H\in\cup_{j=i}^{i+2r-1}\mathbb{P}_j(X)\rangle}\right)$$

$$\leq \binom{n-1}{s-2r+1}+\binom{n-1}{s-2r+2}+\cdots+\binom{n-1}{s}.$$

*Proof.* We induct on $s - 2r + 1 - i$. It is clearly true when $s - 2r + 1 - i = 0$. Suppose the lemma holds for $s - 2r + 1 - i < l$ for some positive integer $l$. Now we want to show that it holds for $s - 2r + 1 - i = l$.

We observe that $i + i + 2r \leq (s - 2r) + (s - 2r) + 2r \leq n - 1$ by the condition in the theorem. By Corollary 2.3 and Lemma 2.4, we have

$$\dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{i+2r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle} \right)$$

$$= \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle + \langle L_H : H \in \mathbb{P}_{i+2r}(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle + \langle \sum_{H \in \mathbb{P}_{i+2r}(X), I \subset H} L_H : I \in \mathbb{P}_i(X) \rangle} \right)$$

$$\leq \dim \left( \frac{L_H : H \in \mathbb{P}_{i+2r}(X)}{\sum_{H \in \mathbb{P}_{i+2r}(X), I \subset H} L_H : I \in \mathbb{P}_i(X)} \right)$$

$$\leq \binom{n-1}{i+2r} - \binom{n-1}{i}.$$

Now we are ready to prove the lemma.

$$\binom{n-1}{i} + \binom{n-1}{i+1} + \cdots + \binom{n-1}{i+2r-1} + \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle} \right)$$

$$= \binom{n-1}{i} + \binom{n-1}{i+1} + \cdots + \binom{n-1}{i+2r-1} + \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{i+2r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle} \right)$$

$$+ \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r} \mathbb{P}_j(X) \rangle} \right)$$

$$= \binom{n-1}{i} + \binom{n-1}{i+1} + \cdots + \binom{n-1}{i+2r-1} + \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{i+2r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle} \right)$$

$$+ \dim \left( \frac{\langle L_H : H \in \mathbb{P}_i(X) \rangle + \langle L_H : H \in \cup_{j=i+1}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \mathbb{P}_i(X) \rangle + \langle L_H : H \in \cup_{j=i+1}^{i+2r} \mathbb{P}_j(X) \rangle} \right)$$

$$\leq \binom{n-1}{i} + \binom{n-1}{i+1} + \cdots + \binom{n-1}{i+2r-1} + \dim \left( \frac{\langle L_H : H \in \cup_{j=i}^{i+2r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i}^{i+2r-1} \mathbb{P}_j(X) \rangle} \right)$$

$$+ \dim \left( \frac{\langle L_H : H \in \cup_{j=i+1}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i+1}^{i+2r} \mathbb{P}_j(X) \rangle} \right)$$

$$\leq \binom{n-1}{i} + \binom{n-1}{i+1} + \cdots + \binom{n-1}{i+2r-1} + \binom{n-1}{i+2r} - \binom{n-1}{i}$$

$$+ \dim \left( \frac{\langle L_H : H \in \cup_{j=i+1}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i+1}^{i+2r} \mathbb{P}_j(X) \rangle} \right)$$

$$= \binom{n-1}{i+1} + \cdots + \binom{n}{i+2r} + \dim \left( \frac{\langle L_H : H \in \cup_{j=i+1}^{s} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \cup_{j=i+1}^{i+2r} \mathbb{P}_j(X) \rangle} \right)$$

$$\leq \binom{n-1}{s-2r+1} + \cdots + \binom{n-1}{s},$$

where the last step follows from the induction hypothesis since $s - 2r + 1 - (i+1) < l$. $\square$

8

We are now turning to the proof of Theorem 1.15.

*Proof.*

$$
\begin{aligned}
|\mathcal{A}| \leq\; & \dim(\langle L_H : H \in \cup_{i=0}^{s} \mathbb{P}_i(X)\rangle) \\
\leq\; & \dim(\langle L_H : H \in \cup_{i=0}^{2r-1}\mathbb{P}_i(X)\rangle) + \dim\left(\frac{\langle L_H : H \in \cup_{i=0}^{s}\mathbb{P}_j(X)\rangle}{\langle L_H : H \in \cup_{i=0}^{2r-1}\mathbb{P}_j(X)\rangle}\right) \\
\leq\; & \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{2r-1} + \dim\left(\frac{\langle L_H : H \in \cup_{i=0}^{s}\mathbb{P}_j(X)\rangle}{\langle L_H : H \in \cup_{i=0}^{2r-1}\mathbb{P}_j(X)\rangle}\right) \\
\leq\; & \binom{n-1}{s-2r+1} + \binom{n-1}{s-2r+2} + \cdots + \binom{n-1}{s} \quad \text{by taking } i=0 \text{ in Lemma 2.5,}
\end{aligned}
$$

which completes the proof of the theorem. $\qquad\square$

# 3   Proof of Theorem 1.14

Throughout this section, we let $p$ be a prime and we will use $x = (x_1, x_2, \ldots, x_n)$ to denote a vector of $n$ variables with each variable $x_i$ taking values 0 or 1. A polynomial $f(x)$ in $n$ variables $x_i$, for $1 \leq i \leq n$, is called *multilinear* if the power of each variable $x_i$ in each term is at most one. Clearly, if each variable $x_i$ only takes the values 0 or 1, then any polynomial in variable $x$ can be regarded as multilinear. For a subset $A$ of $[n]$, we define the incidence vector $v_A$ of $A$ to be the vector $v = (v_1, v_2, \ldots, v_n)$ with $v_i = 1$ if $i \in A$ and $v_i = 0$ otherwise.

Let $L = \{l_1, l_2, \ldots, l_s\}$ and $K = \{k_1, k_2, \ldots, k_r\}$ be two disjoint subsets of $\{0, 1, \ldots, p-1\}$, where the elements of $K$ are arranged in increasing order. Suppose that $\mathcal{A} = \{A_1, \ldots, A_m\}$ is the family of subsets of $[n]$ satisfying the conditions in Theorem 1.14. Without loss of generality, we may assume that $n \in A_j$ for $j \geq t+1$ and $n \notin A_j$ for $1 \leq j \leq t$.

For each $A_j \in \mathcal{A}$, define

$$
f_{A_j}(x) = \prod_{i=1}^{s}(v_{A_j} x - l_i),
$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector of $n$ variables with each variable $x_i$ taking values 0 or 1. Then each $f_{A_j}(x)$ is a multilinear polynomial of degree at most $s$.

Let $Q$ be the family of subsets of $[n-1]$ with sizes at most $s-1$. Then $|Q| = \sum_{i=0}^{s-1}\binom{n-1}{i}$. For each $L \in Q$, define

$$
q_L(x) = (1 - x_n)\prod_{i\in L} x_i.
$$

Then each $q_L(x)$ is a multilinear polynomial of degree at most $s$.

Denote $K - 1 = \{k_i - 1 | k_i \in K\}$. Then $|K \cup (K-1)| \leq 2r$. Set

$$
g(x) = \prod_{h\in K\cup(K-1)}\left(\sum_{i=1}^{n-1} x_i - h\right).
$$

Let $W$ be the family of subsets of $[n-1]$ with sizes at most $s-2r$. Then $|W| = \sum_{i=0}^{s-2r}\binom{n-1}{i}$. For each $I \in W$, define

$$
g_I(x) = g(x)\prod_{i\in I} x_i.
$$

Then each $g_I(x)$ is a multilinear polynomial of degree at most $s$.

9

We want to show that the polynomials in

$$\{f_{A_i(x)}|1 \le i \le m\} \cup \{q_L(x)|L \in Q\} \cup \{g_I(x)|I \in W\}$$

are linearly independent over the field $\mathbb{F}_p$. Suppose that we have a linear combination of these polynomials that equals 0:

$$\sum_{i=1}^{m} a_i f_{A_i}(x) + \sum_{L \in Q} b_L q_L(x) + \sum_{I \in W} u_I g_I(x) = 0, \tag{5}$$

with all coefficients $a_i, b_L$ and $u_I$ being in $\mathbb{F}_p$.

**Claim 1.** $a_i = 0$ for each $i$ with $n \in A_i$.

Suppose, to the contrary, that $i_0$ is a subscript such that $n \in A_{i_0}$ and $a_{i_0} \ne 0$. Since $n \in A_{i_0}$, $q_L(v_{A_{i_0}}) = 0$ for every $L \in Q$. Recall that $f_{A_j}(v_{i_0}) = 0$ for $j \ne i_0$ and $g(v_{i_0}) = 0$. By evaluating (5) with $x = v_{A_{i_0}}$, we obtain that $a_{i_0} f_{A_{i_0}}(v_{A_{i_0}}) = 0 \pmod{p}$. Since $f_{A_{i_0}}(v_{A_{i_0}}) \ne 0$, we have $a_{i_0} = 0$, a contradiction. Thus, Claim 1 holds.

**Claim 2.** $a_i = 0$ for each $i$ with $n \notin A_i$. Applying Claim 1, we get

$$\sum_{i=1}^{t} a_i f_{A_i}(x) + \sum_{L \in Q} b_L q_L(x) + \sum_{I \in W} u_I g_I(x) = 0. \tag{6}$$

Suppose, to the contrary, that $i_0$ is a subscript such that $n \notin A_{i_0}$ and $a_{i_0} \ne 0$. Let $v'_{i_0} = v_{i_0} + (0, 0, \ldots, 0, 1)$. Then $q_L(v'_{i_0}) = 0$ for every $L \in Q$. Note that $f_{A_j}(v'_{i_0}) = f_{A_j}(v_{i_0})$ for each $j$ with $n \notin A_j$ and $g(v'_{i_0}) = 0$. By evaluating (6) with $x = v'_{i_0}$, we obtain $a_{i_0} f_{A_{i_0}}(v'_{i_0}) = a_{i_0} f_{A_{i_0}}(v_{i_0}) = 0 \pmod{p}$ which implies $a_{i_0} = 0$, a contradiction. Thus, the claim is verified.

**Claim 3.** $b_L = 0$ for each $L \in Q$.

By Claims 1 and 2, we obtain

$$\sum_{L \in Q} b_L q_L(x) + \sum_{I \in W} u_I g_I(x) = 0. \tag{7}$$

Set $x_n = 0$ in (7), then

$$\sum_{L \in Q} b_L \prod_{i \in L} x_i + \sum_{I \in W} u_I g_I(x) = 0.$$

Subtracting the above equality from (7), we get

$$\sum_{L \in Q} b_L \left( x_n \prod_{i \in L} x_i \right) = 0.$$

Setting $x_n = 1$, we obtain

$$\sum_{L \in Q} b_L \prod_{i \in L} x_i = 0.$$

It is not difficult to see that the polynomials $\prod_{i \in L} x_i$, $L \in Q$, are linearly independent. Therefore, we conclude that $b_L = 0$ for each $L \in Q$.

By Claims 1-3, we now have

$$\sum_{I \in W} u_I g_I(x) = 0.$$

Thus it is sufficient to prove $g_I$'s are linearly independent.

Let $N$ be a positive integer and $H = \{h_1, h_2, \ldots, h_u\}$ be a subset of $[N]$ with all the elements being arranged in increasing order. We say $H$ *has a gap* of size $\geq g$ if either $h_1 \geq g-1$, $N - h_u \geq g - 1$, or $h_{i+1} - h_i \geq g$ for some $i$ $(1 \leq i \leq u - 1)$. The following result obtained by Alon, Babai and Suzuki [1] is critical to our proof.

**Lemma 3.1.** *Let $H$ be a subset of $\{0, 1, \ldots, p-1\}$. Let $p(x)$ denote the polynomial function defined by $p(x) = \prod_{h \in H}(x_1 + x_2 + \cdots + x_N - h)$. If the set $(H + p\mathbb{Z}) \cap [N]$ has a gap $\geq g + 1$, where $g$ is a positive integer, then the set of polynomials $\{p_I(x) : |I| \leq g - 1, I \in N\}$ is linearly independent over $\mathbb{F}_p$, where $p_I(x) = p(x) \prod_{i \in I} x_i$.*

To apply Lemma 3.1, we define the set $H$ as follows: $H = (K \cup (K-1) + p\mathbb{Z}) \cap [n-1]$. We can divide $n - 1$ into the the following four cases:

1. $s + k_r - 1 \leq n - 1 < p + k_1 - 1$;

2. $s + k_r - 1 < p + k_1 - 1 \leq n - 1$;

3. $(s - 2r + 1) + k_r < p + k_1 - 1 \leq s + k_r - 1 \leq n - 1$;

4. $p + k_1 - 1 \leq (s - 2r + 1) + k_r \leq s + k_r - 1 \leq n - 1$.

**Case 1:** $s + k_r - 1 \leq n - 1 < p + k_1 - 1$.

Since $n - 1 < p + k_1 - 1$, the set $H$ consists of only $\{k_1 - 1, k_1, \ldots, k_r\}$. From $s + k_r - 1 \leq n - 1$, we obtain $n - 1 - k_r \geq s - 1 \geq s - 2r + 1$. By the definition of the gap, $H$ has a gap $\geq s - 2r + 2$.

**Case 2:** $s + k_r - 1 < p + k_1 - 1 \leq n - 1$.

Since $n - 1 \geq p + k_1 - 1$, the set $H$ contains at least the following elements $\{k_1 - 1, k_1, \ldots, k_r, p + k_1 - 1\}$. From $s + k_r - 1 < p + k_1 - 1$, we derive $(p + k_1 - 1) - k_r \geq s \geq s - 2r + 2$. Thus, $H$ has a gap $\geq s - 2r + 2$.

**Case 3:** $(s - 2r + 1) + k_r < p + k_1 - 1 \leq s + k_r - 1 \leq n - 1$.

Since $n - 1 \geq p + k_1 - 1$, $H$ contains at least the following elements $\{k_1 - 1, k_1, \ldots, k_r, p + k_1 - 1\}$. Since $(s - 2r + 1) + k_r < p + k_1 - 1$, we have $(p + k_1 - 1) - k_r > s - 2r + 1$. Then $H$ has a gap $\geq s - 2r + 2$.

By applying Lemma 3.1, we conclude that the set of polynomials $\{g_I(x) : I \in W\}$ is linearly independent over $\mathbb{F}_p$, and so $u_I = 0$ for each $I \in W$.

In summary, for the Cases 1–3, we have shown that the polynomials in

$$\{f_{A_i(x)} | 1 \leq i \leq m\} \cup \{q_L(x) | L \in Q\} \cup \{g_I(x) | I \in W\}$$

are linearly independent over the field $\mathbb{F}_p$. Since the set of all monomials in variables $x_1, x_2, \ldots, x_n$ of degree at most $s$ forms a basis for the vector space of multilinear polynomials of degree at most $s$, it follows that

$$|\mathcal{A}| + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-2r} \binom{n-1}{i} \leq \sum_{i=0}^{s} \binom{n}{i},$$

which implies that

$$|\mathcal{A}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}.$$

This completes the proof of the theorem for the Cases 1–3.

Since Theorem 1.15 has shown that the statement of Theorem 1.14 remains true under the condition $n \geq 2s - 2r + 1$, we just consider $n \leq 2s - 2r$ for the Case 4. The following argument is similar to the technique Hwang and Kim used for the proof of Alon-Babai-Suzuki's conjecture.

Since $p + k_1 - 1 \le (s - 2r + 1) + k_r \le s + k_r - 1 \le n - 1 \le 2s - 2r - 1$, we obtain $k_r \le s - 2r$. Thus, we have $r + s \le p \le s - 2r + 2 + k_r - k_1 \le 2s - 4r + 1$. This implies $s \ge 5r - 1$. Since $n \le 2s - 2r < 2p$, we have $|A_i| \in (K + p\mathbb{Z}) \cap [n] = \{k_1, k_2, \ldots, k_r, p + k_1, \ldots, p + k_c\}$ for some $1 \le c \le r$. This gives

$$|\mathcal{A}| \le \binom{n}{k_1} + \binom{n}{k_2} + \cdots + \binom{n}{k_r} + \binom{n}{p + k_1} + \cdots + \binom{n}{p + k_c}.$$

We will show that the right hand side of the above inequality is less than or equal to $\binom{n-1}{s} + \binom{n-1}{s-1} + \ldots + \binom{n-1}{s-2r+1} = \binom{n}{s} + \binom{n}{s-2} + \ldots + \binom{n}{s-2r+2}$. Since $s + r + k_1 - 1 \le p + k_1 - 1 \le (s - 2r + 1) + k_r$, we have $k_r \ge 3r - 2 + k_1$. Let $n = 2s - 2r - \delta$ for integer $\delta$, where $0 \le \delta \le s - 5r + 1$, since $2s - 2r \ge n \ge s + k_r \ge s + 3r - 2 + k_1$. Since the sequence $\{\binom{n}{k}\}$ is unimodal and symmetric around $n/2$, we have $|s - n/2| = r + \delta/2 > r - \delta/2 - 2 = |n/2 - (s - 2r + 2)|$.

Therefore we have

$$\min\left[\binom{n}{s}, \binom{n}{s-2}, \ldots, \binom{n}{s-2r+2}\right] = \binom{n}{s}. \tag{8}$$

Since $n = 2s - 2r - \delta \ge p + k_c \ge r + s + k_c$, we have $k_c \le s - 3r - \delta$. For $1 \le i \le c$, $k_i$ can be written as $k_i = s - 3r - \delta - a_i$, where $0 < a_i \le s - 3r - \delta$. Thus, we have $p + k_i \ge r + s + k_i = 2s - 2r - \delta - a_i$ where $1 \le i \le c$. Since $2s - 2r - \delta - a_i \ge s + r > n/2$, we have

$$\sum_{i=1}^{c}\left(\binom{n}{k_i} + \binom{n}{p + k_i}\right) \le \sum_{i=1}^{c}\left(\binom{n}{s - 3r - \delta - a_i} + \binom{n}{2s - 2r - \delta - a_i}\right).$$

For $c + 1 \le i \le r$, we derive $k_i \le k_r < s - 2r - \delta < n/2$. Noting that $|s - n/2| = r + \delta/2 = |n/2 - (s - 2r - \delta)|$, we have $\binom{n}{k_i} \le \binom{n}{s}$ for all $c + 1 \le i \le r$. Then

$$|\mathcal{A}| \le \sum_{i=1}^{c}\left(\binom{n}{k_i} + \binom{n}{p + k_i}\right) + \sum_{i=c+1}^{r}\binom{n}{k_i}$$

$$\le \sum_{i=1}^{c}\left(\binom{n}{s - 3r - \delta - a_i} + \binom{n}{2s - 2r - \delta - a_i}\right) + (r - c)\binom{n}{s}.$$

With the help of the next lemma, we can complete our proof.

**Lemma 3.2.** [8] *For all $0 \le c < k \le n/2$, we have*

$$\binom{n}{k - 1 - c} + \binom{n}{c} \le \binom{n}{k}.$$

Let $k = n - s = s - 2r - \delta < n/2$, apply Lemma 3.2. For every $0 \le a \le s - 3r - \delta < k$, we have

$$\binom{n}{s - 3r - \delta - a} + \binom{n}{2s - 2r - \delta - a}$$

$$= \binom{n}{n - s - r - a} + \binom{n}{n - a}$$

$$= \binom{n}{k - r - a} + \binom{n}{a}$$

$$\le \binom{n}{k - 1 - a} + \binom{n}{a}$$

$$\le \binom{n}{k} = \binom{n}{s}.$$

We now finish the proof of Theorem 1.14 for the Case 4.

$$|\mathcal{A}| \leq \sum_{i=1}^{c} \left( \binom{n}{s - 3r - \delta - a_i} + \binom{n}{2s - 2r - \delta - a_i} \right) + (r - c)\binom{n}{s} \leq r\binom{n}{s}.$$

By (8), we have

$$|\mathcal{A}| \leq \binom{n}{s} + \binom{n}{s - 2} + \cdots + \binom{n}{s - 2r + 2} = \binom{n - 1}{s} + \binom{n - 1}{s - 1} + \cdots + \binom{n - 1}{s - 2r + 1}.$$

## Acknowledgements

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