VECTOR BUNDLES INDUCED FROM JET SCHEMES

BAILIN SONG

ABSTRACT. A family of holomorphic vector bundles is constructed on a complex manifold X. The space of the holomorphic sections of these bundles are calculated in certain cases. As an application, if X is an N-dimensional compact Kähler manifold with holonomy group SU(N), the space of holomorphic vector fields on its jet scheme $J_m(X)$ is calculated. We also prove that the space of the global sections of the chiral de Rham complex of a K3 surface is the simple N=4 superconformal vertex algebra with central charge 6.

1. Introduction

On a Ricci flat compact Kähler manifold X, the holomorphic sections of the bundle given by tensors of tangent and cotangent bundles are exactly the parallel sections (page 142 of [11]). So if the holonomy group of X is G, the space of the holomorphic sections of the bundle is isomorphic to the space of the G invariants of the fibre. In this paper, on a complex manifold X, given holomorphic vector bundles E and E, we construct a family of holomorphic vector bundles E and E are sums of copies of the holomorphic tangent and cotangent bundles, we show that the space of the holomorphic sections of E0 is isomorphic to the space of the E1 is isomorphic to the space of the E2 is invariants of the fibre of the bundle. Here E3 is isomorphic to the space of the E4 is scheme of E5 invariants of the fibre of the bundle. Here E4 is is the E5 is the E6 invariants of the fibre of the bundle. Here E6 is the E7 is the E8 is the E9 invariants of the fibre of the bundle. Here E9 is the E9 invariants of the fibre of the bundle. Here E9 is the E9 is the E9 is the E9 invariants of the fibre of the bundle. Here E9 is the E9 is the E9 is the E9 invariants of the fibre of the bundle. Here E9 is the E9 is the E9 is the E9 invariants of the fibre of the bundle.

The idea of the construction of $A_m(E, F)$ is from the jet scheme. The mth jet scheme [1] $J_m(X)$ of an algebraic scheme X over $\mathbb C$ is determined by its functor of points: for every $\mathbb C$ -algebra A, we have a bijection

$$\operatorname{Hom}(\operatorname{Spec}(A), J_m(X)) \cong \operatorname{Hom}(\operatorname{Spec}(A[t]/(t^{m+1})), X).$$

There is a canonical projection $\pi_m: J_m(X) \to X$. If X is a projective complex manifold, E is a vector bundle over X, then $J_m(E)$ is a vector bundle over $J_m(X)$. If X is Ricci flat, we expect that sections of the bundle given by tensors of $J_m(TX)$ and $J_m(T^*X)$ are $J_m(SL(N,\mathbb{C}))$ invariants of the fibre. $A_m(E,F)$ is the vector bundle over X, such that the sheaf of sections of $A_m(E,F)$ is the push forward of the sheaf of sections of $\mathrm{Sym}^* J_m(E) \otimes \wedge^* J_m(F)$ through π_m .

There are two applications of Theorem 5.6. The first application is that we can calculate the holomorphic vector fields on $J_m(X)$,

Theorem 1.1. If X is an N-dimensional compact Kähler manifold with holonomy group SU(N), then the space of holomorphic vector fields on $J_m(X)$ has dimension m.

The second application is that we can calculate the global sections of the chiral de Rham complex on any K3 surface.

Theorem 1.2. If X is a K3 surface, the space of global sections of the chiral de Rham complex of X is the simple N=4 superconformal vertex algebra with central charge 6, which is strongly generated by eight sections

$$Q(z), L(z), J(z), G(z), B(z), D(z), C(z), E(z).$$

This generalizes our previous result in [9], which calculates the global sections of the chiral de Rham complex on a Kummer surface. According to [4], on any Calabi-Yau manifold, the cohomology of the chiral de Rham complex can be identified with the infinite-volume limit of the half-twisted sigma model defined by E. Witten. This calculation may help to understand the half-twisted sigma model on K3 surfaces.

The paper is organized as follows. In section 2, we review some basic facts of complex geometry. In section 3, we construct two holomorphic vector bundles $A_m(E,F)$ and $B_m(E,F)$. In section 4, we compare the holomorphic structures of $A_m(E,F)$ and $B_m(E,F)$, and calculate the mean curvature of $A_m(E,F)$ when the mean curvatures of E,F and E,F and E,F vanish. In section 5, we give some results of the holomorphic sections of E,F in Finally, in section 6, we calculate the global sections of the chiral de Rham complex on a K3 surface.

2. THE CHERN CONNECTIONS AND CURVATURES

For a complex manifold X, let $\Omega_X^{k,l}$ be the space of smooth forms on X of type (k,l). Similarly, for a holomorphic vector bundle E on X, let $\Omega_X^{k,l}(E)$ be the space of all smooth forms of type (k,l) with values in E. Let $h_E=(-,-)_E$ be a Hermitian metric on E and $\nabla_E=\nabla_E^{1,0}+\bar{\partial}$ be the associated Chern connection with $\nabla_E^{1,0}:\Omega^{0,0}(E)\to\Omega^{1,0}(E)$. Then its curvature is

$$\Theta^E = [\bar{\partial}, \nabla_E] = [\bar{\partial}, \nabla_E^{1,0}] \in \Omega^{1,1}(\operatorname{Hom}(E, E)).$$

Locally, if $e = (e_1, \dots, e_p)$ is a holomorphic frame of E over $U \subset X$, the connection one form θ^E of ∇_E is given by

$$\nabla_E e_i = \sum \theta_{ij}^E e_j.$$

If we define the matrix $H^E = (H^E_{ij})$,

$$H_{ij}^E = (e_i, e_j)_E,$$

then

$$\theta_{ij}^E = \partial H_{il}^E H_E^{lj}.$$

Here (H_E^{lj}) is the inverse matrix of H^E . The curvature form of ∇_E is

$$\Theta^E = \bar{\partial}\theta^E \in \Omega^{1,1}(\operatorname{Hom}(E, E)).$$

Covariant derivatives of the curvature. According to [3], the tensor fields $F_k = F_k^E$, $k \ge 2$ as higher covariant derivatives of the curvature can be defined by induction,

$$F_{2} = \Theta^{E} \in \Omega^{1,1}(\text{Hom}(E, E)) = \Omega^{0,1}(\text{Hom}(T \otimes E, E)),$$

$$F_{k+1} = \nabla^{1,0}_{E} F_{k} \in \Omega^{0,1}(\text{Hom}(T^{\otimes (k-1)} \otimes E, E)), \quad k \geq 3.$$

Since ∇_E is a Chern connection,

$$[\nabla_E^{1,0}, \nabla_E^{1,0}] = 0.$$

So we have a lemma from [3],

Lemma 2.1.

$$F_n \in \Omega^{0,1}(\operatorname{Hom}(\operatorname{Sym}^{n-1} T \otimes E, E)).$$

Locally, if $y=(y_1,\cdots,y_N)$ is a holomorphic coordinate system and $e=(e_1,\cdots,e_l)$ is a holomorphic frame of E on $U\subset X$, F_{k,i_1,\cdots,i_n}^l is denoted by

$$F_{k,i_1\cdots,i_n}^l e_l = F_{n+1}(\frac{\partial}{\partial y_{i_1}},\cdots,\frac{\partial}{\partial y_{i_n}})e_k \in \Omega^{0,1}(E).$$

If X is an Hermitian manifold with the Hermitian metric h=(-,-), let $M=M^E$ be the mean curvature of E, i,e,

$$M = \Theta^E(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j}) H_{ij} \in \text{Hom}(E, E).$$

Here $H_{ij} = (dy_i, dy_i)$. We define the tensor field $M_n = M_n^E$ by

$$M_2 = M$$
, $M_n = \nabla^{1,0} M_{n-1}$, $n \ge 3$.

Then

$$M_n = F_n(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j}) H_{ij} \in \operatorname{Hom}(\operatorname{Sym}^{n-2} T \otimes E, E).$$

If $M = C_0 Id$ for a constant C_0 , then h is an Einstein-Hermitian metric. We have the following lemma obviously,

Lemma 2.2. If h_E is an Einstein-Hermitian metric, then $M_n^E=0$ for $n\geq 3$. If the mean curvature of h_E is zero, then $M_n^E=0$ for $n\geq 2$. In particular, if the Ricci curvature of X is zero, then $M_n^T=0$ for $n\geq 2$.

If X is Kähler, then for the holomorphic tangent bundle T of X,

$$\Theta^T \in \Omega^{0,1}(\operatorname{Hom}(\operatorname{Sym}^2 T, T)),$$

we have

Lemma 2.3.

$$F_n^T \in \Omega^{0,1}(\operatorname{Hom}(\operatorname{Sym}^n T, T)).$$

The Chern connections of the vector bundle with different holomorphic structures. Assume E has another holomorphic structure given by $\bar{\partial}'$ and $\tilde{\nabla}$ is the Chern connection corresponding to $\bar{\partial}'$. Let $K = \bar{\partial} - \bar{\partial}' \in \Omega^{0,1}(\mathrm{Hom}(E,E))$. Let $K^* \in \Omega^{1,0}(\mathrm{Hom}(E,E))$ be the dual of K, i.e. K^* is determined by

$$(K(\frac{\partial}{\partial \bar{y}_i})a, b) = (a, K^*(\frac{\partial}{\partial y_i})b).$$

We have the following relation between the two connections,

Lemma 2.4.

$$\nabla = \tilde{\nabla} + K - K^*$$

Proof. This is because that the Chern connections ∇ and $\tilde{\nabla}$ can be determined by

$$\nabla = \nabla^{1,0} + \bar{\partial}, \quad \tilde{\nabla} = \tilde{\nabla}^{1,0} + \bar{\partial}',$$

and for any smooth sections a, b of E,

$$d(a,b) = (\nabla a, b) + (a, \nabla b) = (\tilde{\nabla} a, b) + (a, \tilde{\nabla} b).$$

Now

$$((\tilde{\nabla} + K - K^*)a, b) + (a, (\tilde{\nabla} + K - K^*)b) = (\tilde{\nabla} a, b) + (a, \tilde{\nabla} b) = d(a, b)$$

and

$$\tilde{\nabla} + K - K^* = \tilde{\nabla}^{1,0} + \bar{\partial}' + K - K^* = \tilde{\nabla}^{1,0} - K^* + \bar{\partial}$$

Since $K^* \in \Omega^{1,0}(\operatorname{Hom}(E,E), \tilde{\nabla}^{1,0} - K^*)$ takes values in $\Omega^{1,0}(E)$. So $\tilde{\nabla} + K - K^*$ is the Chern connection on E corresponding to the holomorphic structure given by $\bar{\partial}$. Thus by the uniqueness of the Chern connection, $\nabla = \tilde{\nabla} - K + K^*$.

Let E^{\vee} be the dual complex vector bundle of E, then E^{\vee} has two holomorphic structures induced from the two holomorphic structures on E. The two holomorphic structures are given by $\bar{\partial}$ and $\bar{\partial}'$, which are determined by

$$\bar{\partial}(a^\vee(b)) = (\bar{\partial}a^\vee)(b) + a^\vee(\bar{\partial}b) \text{ and } \bar{\partial}(a^\vee(b)) = (\bar{\partial}'a^\vee)(b) + a^\vee(\bar{\partial}'b)$$

for any section a^{\vee} of E^{\vee} and b of E. Let $\bar{\partial} = \bar{\partial}' + K^{\vee}$, then

$$\bar{\partial} a^{\vee}(b) = ((\bar{\partial} + K^{\vee})a^{\vee})(b) + a^{\vee}((\bar{\partial} + K)b).$$

Thus K^{\vee} is determined by

$$(K^{\vee}a^{\vee})(b) = -a^{\vee}(Kb).$$

Weitzenböck formulas. We list the following Weitzenböck formulas for the compact Hermitian manifold here:

(1) If α is a smooth section of E,

(2.1)
$$\Delta_{\bar{\partial}}\alpha = \nabla^* \nabla \alpha - M\alpha.$$

(2) If *X* is Kähler, and α is a smooth section of E = TX, then the Ricci curvature

$$\operatorname{Ric} \alpha = M\alpha$$
.

And

$$\Delta_{\bar{\partial}}\alpha = \nabla^* \nabla \alpha - \operatorname{Ric} \alpha.$$

3. VECTOR BUNDLES INDUCED FROM JET SCHEMES

Given holomorphic bundles E and F on a complex manifold X, a family of holomorphic vector bundles $A_m(E,F)$, $m \in \mathbb{Z}_{\geq 0}$ will be constructed in this section.

The algebra R_m . For $m \in \mathbb{Z}_{\geq 0}$, $y = (y_1, \dots, y_N)$, $e = (e_1, \dots, e_p)$ and $f = (f_1, \dots, f_q)$, let $R_m = R_m(y, e, f)$ be the algebra

$$\mathbb{C}[y_1^{(1)}, \cdots, y_i^{(j)}, \cdots, y_N^{(m)}] \otimes \mathbb{C}[e_1^{(0)}, \cdots, e_i^{(j)}, \cdots, e_p^{(m)}] \otimes \wedge_{\mathbb{C}}[f_1^{(0)}, \cdots, f_i^{(j)}, \cdots, f_q^{(m)}].$$

Here $y_i^{(j)}, e_i^{(j)}, f_i^{(j)}$ are new variables and $\wedge_{\mathbb{C}}[f_1^{(0)}, \cdots, f_i^{(j)}, \cdots, f_q^{(m)}]$ is the exterior algebra generated by $f_1^{(0)}, \cdots, f_i^{(j)}, \cdots, f_q^{(m)}$ over \mathbb{C} . Let

$$R_{\infty}(y, e, f) = \underset{\longrightarrow}{\lim} R_m(y, e, f).$$

Let

$$L = \sum_{i,j} j \, y_i^{(j)} \frac{\partial}{\partial y_i^{(j)}} + \sum_{i,j} j \, e_i^{(j)} \frac{\partial}{\partial e_i^{(j)}} + \sum_{i,j} j \, f_i^{(j)} \frac{\partial}{\partial f_i^{(j)}},$$

$$L_e = \sum_{i,j} e_i^{(j)} \frac{\partial}{\partial e_i^{(j)}}, \quad L_f = \sum_{i,j} f_i^{(j)} \frac{\partial}{\partial f_i^{(j)}}.$$

 L_e, L_f, L gives a $\mathbb{Z}^3_{\geq 0}$ grading of $R_m(y, e, f)$, i.e.

$$R_m(y, e, f) = \bigoplus_{j,k,l} R_m(y, e, f)[j, k, l]$$

with

$$L_e(a) = ja$$
, $L_f(a) = kf$, $L(a) = la$ for $a \in R_m(y, e, f)[j, k, l]$.

It is easy to see that $R_m(y, e, f)[j, k, l]$ is a finite dimensional complex vector space. Let

$$R_m(y, e, f)[j, k] = \bigoplus_{l} R_m(y, e, f)[j, k, l].$$

Let \tilde{D} be the derivation on $R_{\infty}[y, e, f]$ given by

$$\tilde{D}y_i^{(j)} = y_i^{(j+1)}, \quad \tilde{D}e_i^{(j)} = e_i^{(j+1)}, \quad \tilde{D}f_i^{(j)} = f_i^{(j+1)}.$$

 \tilde{D} maps $R_m(y, e, f)[j, k, l]$ to $R_{m+1}(y, e, f)[j, k, l+1]$.

Construction of the vector bundles $A_m(E,F)$. Let \mathcal{U} be the set which consists of (U,y,e,f) such that $y=(y_1,\cdots,y_N)$ is a coordinate system, $e=(e_1,\cdots,e_p)$ is a holomorphic frame of E and $f=(f_1,\cdots,f_q)$ is a holomorphic frame of F on U. Let $\mathcal{O}(U)$ be the space of holomorphic functions on U. For $m\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$, let

$$\mathcal{A}_m(U, y, e, f) = \mathcal{O}(U) \otimes R_m(y, e, f)$$

be the algebra of holomorphic maps from U to $R_m(y,e,f)$. For any $a \in \mathcal{A}_m(U,y,e,f)$, let $a(x) \in R_m(y,e,f)$ be the image of a at $x \in U$.

Let

(3.1)
$$D = \sum_{i} \frac{\partial}{\partial y_i} \otimes y_i^{(1)} + 1 \otimes \tilde{D}.$$

It is a derivation on $\mathcal{A}_{\infty}(U, y, e, f)$.

Now if $(U_{\alpha}, y_{\alpha}, e_{\alpha}, f_{\alpha}), (U_{\beta}, y_{\beta}, e_{\beta}, f_{\beta}) \in \mathcal{U}$ with

$$y_{\alpha,i} = f_i(y_\beta), \quad e_{\alpha,i} = g_{ij}e_{\beta,j}, \quad f_{\alpha,i} = h_{ij}f_{\beta,j} \quad \text{on } U_\alpha \cap U_\beta,$$

there is a unique $\mathcal{O}(U_{\alpha} \cap U_{\beta})$ algebra isomorphism

$$r_{\alpha\beta}: \mathcal{A}_{\infty}(U_{\alpha} \cap U_{\beta}, y_{\alpha}, e_{\alpha}, f_{\alpha}) \to \mathcal{A}_{\infty}(U_{\alpha} \cap U_{\beta}, y_{\beta}, e_{\beta}, f_{\beta})$$

with

$$r_{\alpha\beta}(y_{\alpha,i}^{(l)}) = D^l f(y_{\beta}), \quad r_{\alpha\beta}(\varphi) = \varphi, \quad \text{for } \varphi \in \mathcal{O}(U_{\alpha} \cap U_{\beta}),$$
$$r_{\alpha\beta}(e_{\alpha,i}^{(l)}) = D^l (g_{ij}e_{\beta,j}^{(0)}), \quad r_{\alpha\beta}(f_{\alpha,i}^{(l)}) = D^l (h_{ij}f_{\beta,j}^{(0)}).$$

It is easy to see

Lemma 3.1. For $a \in \mathcal{A}_{\infty}(U_{\alpha} \cap U_{\beta}, y_{\alpha}, e_{\alpha}, f_{\alpha})$,

$$r_{\alpha\beta}(Da) = D \, r_{\alpha\beta}(a).$$

For any $x \in U_{\alpha} \cap U_{\beta}$, we get an isomorphism of \mathbb{C} -algebras

$$r_{\alpha\beta}^x: R_{\infty}(y_{\alpha}, e_{\alpha}, f_{\alpha}) \to R_{\infty}(y_{\beta}, e_{\beta}, f_{\beta})$$

with

(3.2)
$$r_{\alpha\beta}^{x}(y_{\alpha,i}^{(k)}) = D^{k}f_{i}(y_{\beta})(x), \quad r_{\alpha\beta}(e_{\alpha,i}^{(k)}) = D^{k}(g_{ij}e_{\beta,j}^{(0)})(x), \\ r_{\alpha\beta}(f_{\alpha,i}^{(k)}) = D^{k}(h_{ij}f_{\beta,j}^{(0)})(x).$$

 $r_{\alpha\beta}$ satisfies

$$\begin{split} r_{\alpha\beta}^x \circ r_{\beta\alpha}^x &= Id, \quad \text{for } x \in U_\alpha \cap U_\beta, \\ r_{\beta\gamma}^x \circ r_{\alpha\beta}^x &= r_{\alpha\gamma}^x, \quad \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{split}$$

Thus the open cover $\{U_{\alpha} \times R(y_{\alpha}, e_{\alpha}, f_{\alpha}) : (U_{\alpha}, y_{\alpha}, e_{\alpha}, f_{\alpha}) \in \mathcal{U}\}$ and the transition functions $r_{\alpha\beta}^{x}$ define an algebra bundle $A_{\infty}(E, F)$ on X.

By (3.2), the transition functions $r_{\alpha\beta}^x$ map $R_m(y_\alpha,e_\alpha,f_\alpha)[j,k,l]$ to $R_m(y_\beta,e_\beta,f_\beta)[j,k,l]$. Let $A_m(E,F)[j,k,l]$, $A_m(E,F)[j,k]$ and $A_m(E,F)$ be the subbundles of $A_\infty[E,F]$ with fibres $R_m[j,k,l]$, $R_m[j,k]$ and R_m , respectively. We have

$$A_m(E,F) = \bigoplus_{j,k,l} A_m(E,F)[j,k,l],$$

$$A_m(E,F)[j,k] = \bigoplus_l A_m(E,F)[j,k,l].$$

 $A_m(E,F)[j,k,l]$ are finite dimensional holomorphic vector bundles. Let O_m be the algebra bundle $A_m(E,F)[0,0]=A_m(0,0)$ and $O_m[l]=A_m(E,F)[0,0,l]$.

Let $\mathcal{A}_m(E,F)$, $\mathcal{A}_m(E,F)[j,k]$ and \mathcal{O}_m be the sheaf of holomorphic sections of $A_m(E,F)$, $A_m(E,F)[j,k]$ and O_m , respectively. $\mathcal{A}_m(E,F)$ are algebras of \mathcal{O}_m and $\mathcal{A}_m(E,F)[j,k]$ are modules of \mathcal{O}_m .

Remark 3.2. If X is a complex projective manifold, then $A_m(E,F)[j,k]$ can be induced from the mth jet scheme of X. For each integer $m \geq 0$, the sheaf \mathcal{O}_m is the push forward of the structure sheaf of $J_m(X)$ through π_m . $J_m(E)$ and $J_m(F)$ are vector bundles over $J_m(X)$. $A_m(E,F)[j,k]$ is the push forward of the sheaf of sections of $\operatorname{Sym}^j J_m(E) \otimes \wedge^k J_m(F)$ through π_m .

By Lemma 3.1, $r_{\alpha\beta}$ commutes with the derivation D. So D is a derivation of the sheaf of algebras $\mathcal{A}_{\infty}(E,F)$. According to (3.1), D maps $\mathcal{A}_{m}(E,F)[j,k,l]$ to $\mathcal{A}_{m+1}(E,F)[j,k,l+1]$. Thus D is a derivation of the \mathcal{O}_{∞} module $\mathcal{A}_{\infty}(E,F)[j,k]$.

D can be extended to a derivation of $\Omega_U^{0,*}(A_\infty(E,F))$ by assuming $D\bar{d}y_i=0$ and $Df=\sum_i \frac{\partial f}{\partial y_i}y_i^{(1)}$, for any smooth function f on U.

Vector bundles $B_m(E,F)$. Let $B_m(E,F)$ be the algebra bundle constructed from the open cover $\{U_\alpha \times R(y_\alpha,e_\alpha,f_\alpha): (U_\alpha,y_\alpha,e_\alpha,f_\alpha) \in \mathcal{U}\}$ and the transition functions

$$\tilde{r}_{\alpha\beta}^x: R_{\infty}(y_{\alpha}, e_{\alpha}, f_{\alpha}) \to R_{\infty}(y_{\beta}, e_{\beta}, f_{\beta}).$$

For $x \in U_{\alpha} \cap U_{\beta}$, $\tilde{r}_{\alpha\beta}^x$ is the isomorphism of algebras given by

$$(3.3) \quad \tilde{r}_{\alpha\beta}^{x}(y_{\alpha,i}^{(k)}) = \frac{\partial f_{i}(y_{\beta})}{\partial y_{\beta,j}}(x)y_{\beta,j}^{(k)}, r_{\alpha\beta}(e_{\alpha,i}^{(k)}) = g_{ij}(x)e_{\beta,j}^{(k)}(x), \ r_{\alpha\beta}(f_{\alpha,i}^{(k)}) = h_{ij}(x)f_{\beta,j}^{(k)}.$$

 $\tilde{r}_{\alpha\beta}^x$ maps $R_m(y_\alpha,e_\alpha,f_\alpha)[j,k,l]$ to $R_m(y_\beta,e_\beta,f_\beta)[j,k,l]$. Let $B_m(E,F)[j,k,l]$, $B_m(E,F)[j,k]$ and $B_m(E,F)$ be the subbundles of $B_\infty[E,F]$ with fibres $R_m[j,k,l]$, $R_m[j,k]$ and R_m respectively. Then $B_m(E,F)[j,k,l]$ are finite dimensional holomorphic vector bundles. We have

$$(3.4) B_m(E,F) = \bigoplus_{j,k,l} B_m(E,F)[j,k,l]$$

$$= \operatorname{Sym}^*(\bigoplus_{1 \le l \le m} T^*) \otimes \operatorname{Sym}^*(\bigoplus_{0 \le l \le m} E) \otimes \wedge^*(\bigoplus_{0 \le l \le m} F),$$

$$(3.5) B_m(E,F)[j,k] = \bigoplus_{l} B_m(E,F)[j,k,l]$$
$$= \operatorname{Sym}^*(\bigoplus_{1 < l < m} T^*) \otimes \operatorname{Sym}^j(\bigoplus_{0 < l < m} E) \otimes \wedge^k(\bigoplus_{0 < l < m} F).$$

It is easy to see that \tilde{D} commute with $r_{\alpha\beta}^x$. So \tilde{D} is a derivation of the algebra bundle $B_m(E,F)$.

Let $O_{m,1} = \bigoplus_{k>0} O_m[k]$ and $O_{m,k} = (O_{m,1})^k$, the sub bundle of O_m generated by product of k elements of O_m with positive weights. Thus

$$\cdots O_{m,k} \subset \cdots \subset O_{m,2} \subset O_{m,1} \subset O_{m,0} = \mathcal{O}_m.$$

$$\bigoplus_{k>0} O_{m,k}/O_{m,k+1} \cong \bigoplus_{k>0} \operatorname{Sym}^k(\bigoplus_{1\leq l\leq m} T_l^*).$$

In general we have

$$\bigoplus_{k \ge 0} A(E, F)[j, k] O_{m,l} / A(E, F)[j, k] O_{m,l+1} = B_m(E, F)[j, k].$$

4. Holomorphic structures

Let $h_{T^*} = (-, -)_{T^*}$ be an Hermitian metric of the cotangent bundle T^* of X, and $h_E = (-, -)_E$ and $h_F = (-, -)_F$ be Hermitian metrics of E and F respectively. Let ∇_{T^*} , ∇_E and ∇_F be the Chern connections of T^* , E and F respectively. For $(U, y, e, f) \in \mathcal{U}$,

$$\nabla_T dy_i = \sum \theta_{ij}^{T^*} dy_j, \nabla_E e_i = \sum \theta_{ij}^E e_j, \nabla_F f_i = \sum \theta_{ij}^F f_j.$$

Let Θ^{T^*} , Θ^E and Θ^F be their curvatures.

Chern connection on $B_m(E,F)$. By (3.4), there is a canonical Hermitian metric h=(-,-) on $B_m(E,F)$ induced from h_{T^*} , h_E and h_F . Let $\tilde{\nabla}=\tilde{\nabla}^{1,0}+\bar{\partial}'$ be the Chern connection associated with h. For $(U,y,e,f)\in\mathcal{U}$, $\tilde{\nabla}$ satisfies

(4.1)
$$\tilde{\nabla} y_i^{(l)} = \sum_{ij} \theta_{ij}^{T^*} y_j^{(l)}, \quad \tilde{\nabla} e_i^{(l)} = \sum_{ij} \theta_{ij}^E e_j^{(l)}, \quad \tilde{\nabla} f_i^{(l)} = \sum_{ij} \theta_{ij}^F f_j^{(l)}, \\ \tilde{\nabla} (ab) = (\tilde{\nabla} a)b + a(\tilde{\nabla} b), \quad \text{for } a, b \in \Omega_U^{0,0}(B_m(E, F)).$$

So its curvature Θ satisfies

$$(4.2) \qquad \tilde{\Theta}y_i^{(l)} = \sum_{ij} \Theta_{ij}^{T^*} y_j^{(l)}, \quad \tilde{\Theta}e_i^{(l)} = \sum_{ij} \Theta_{ij}^E e_j^{(l)}, \quad \tilde{\Theta}f_i^{(l)} = \sum_{ij} \Theta_{ij}^F f_j^{(l)}, \\ \tilde{\Theta}(ab) = (\tilde{\Theta}a)b + a(\tilde{\Theta}b), \quad \text{for } a, b \in \Omega_U^{0,0}(B_m(E, F)).$$

A canonical isomorphism. $A_m(E,F)$ and $B_m(E,F)$ are different holomorphic vector bundles. But as complex vector bundles, they are isomorphic. Here we construct a canonical isomorphism from $A_m(E,F)$ to $B_m(E,F)$.

For $(U, y, e, f) \in \mathcal{U}$, let

$$H_{ij} = (dy_i, dy_j)_{T^*}, \quad H_{ij}^E = (e_i, e_j)_E, \quad H_{ij}^F = (f_i, f_j)_F,$$

and let (H^{ij}) , (H^{ij}_E) and (H^{ij}_F) be the inverse matrices of (H_{ij}) , H^E_{ij}) and (H^F_{ij}) respectively. For k > 0, let

$$(4.3) Y_l^{(k+1)} = H_{lj} D^k (H^{ji} y_i^{(1)}), \quad E_l^{(k)} = H_{lj}^E D^k (H_E^{ji} E_i^{(0)}), \quad F_l^{(k)} = H_{lj}^F D^k (H_F^{ji} F_i^{(0)}).$$

Proposition 4.1. There is an isomorphism from $\Phi_m: A_m(E,F) \to B_m(E,F)$ of smooth complex vector bundles, which locally maps $Y_l^{(k)}$, $E_l^{(k)}$ and $F_l^{(k)}$ to $y_l^{(k)}$, $e_l^{(k)}$ and $f_l^{(k)}$ respectively. This isomorphism maps $A_m(E,F)[j,k,l]$ to $B_m(E,F)[j,k,l]$ and preserves the algebra structures.

Proof. For any point $x \in X$, let $(U_{\alpha}, y_{\alpha}, e_{\alpha}, f_{\alpha}) \in \mathcal{U}$ with $x \in U_{\alpha}$. By the definition of $Y_l^{(k+1)}$, $E_l^{(k)}$ and $F_l^{(k)}$ in (4.3),

$$Y_{\alpha,l}^{(k+1)} \equiv y_{\alpha,l}^{(k+1)} \mod O_{m,2};$$

$$E_{\alpha,l}^{(k)} \equiv e_{\alpha,l}^{(k)}, \quad F_{\alpha,l}^{(k)} \equiv f_{\alpha,l}^{(k)} \mod A_m(E,F)O_{m,1}.$$

So $R_m(Y_\alpha, E_\alpha, F_\alpha)$ are the fibres of $A_m(E, F)$ at x.

Let $\Phi_{m,x}: A_m(E,F)|_x \to B_m(E,F)|_x$ be the isomorphism of \mathbb{C} -algebras given by

$$\Phi_{m,x}(Y_{\alpha,l}^{(k+1)}) = y_{\alpha,l}^{(k+1)}, \quad \Phi_{m,x}(E_{\alpha,l}^{(k)}) = e_{\alpha,l}^{(k)}, \quad \Phi_{m,x}(F_{\alpha,l}^{(k)}) = f_{\alpha,l}^{(k)}.$$

 $Y_{\alpha,l}^{(k+1)}$, $E_{\alpha,l}^{(k)}$ and $F_{\alpha,l}^{(k)}$ have the same grades as $y_{\alpha,l}^{(k+1)}$, $e_{\alpha,l}^{(k)}$ and $f_{\alpha,l}^{(k)}$ respectively. So $\Phi_{m,x}$ preserves the $\mathbb{Z}^3_{>0}$ grading of each fibre.

If $(U_{\beta}, y_{\beta}, e_{\beta}, f_{\beta}) \in \mathcal{U}$ with $x \in U_{\beta}$, $y_{\alpha,i} = f_i(y_{\beta})$, $e_{\alpha,i} = g_{ij}e_{\beta,j}$, $f_{\alpha,i} = h_{ij}f_{\beta,j}$ on $U_{\alpha} \cap U_{\beta}$.

$$H_{\beta,ij} = (dy_{\beta,i}, dy_{\beta,j}) = \frac{\partial y_{\beta,i}}{\partial y_{\alpha,k}} H_{\alpha,kl} \frac{\overline{\partial y_{\beta,j}}}{\partial y_{\alpha,l}}.$$

So

$$(4.4) \gamma_{\alpha\beta}(Y_{\alpha,l}^{(k+1)}) = H_{\alpha,lj}D^{k}(H_{\alpha}^{ji}\gamma_{\alpha\beta}(y_{\alpha,i}^{(1)}))$$

$$= (\frac{\partial y_{\alpha,l}}{\partial y_{\beta,s}}H_{\beta,st}\frac{\overline{\partial y_{\alpha,j}}}{\partial y_{\beta,t}})D^{k}(\frac{\overline{\partial y_{\beta,u}}}{\partial y_{\alpha,j}}H_{\beta}^{uv}\frac{\partial y_{\beta,v}}{\partial y_{\alpha,i}}\gamma_{\alpha\beta}(y_{\alpha,i}^{(1)}))$$

$$= (\frac{\partial y_{\alpha,l}}{\partial y_{\beta,s}}H_{\beta,st}\frac{\overline{\partial y_{\alpha,j}}}{\partial y_{\beta,t}}\frac{\overline{\partial y_{\beta,u}}}{\partial y_{\alpha,j}})D^{k}(H_{\beta}^{uv}\frac{\partial y_{\beta,v}}{\partial y_{\alpha,i}}\gamma_{\alpha\beta}(y_{\alpha,i}^{(1)}))$$

$$= \frac{\partial y_{\alpha,l}}{\partial y_{\beta,s}}H_{\beta,su}D^{k}(H_{\beta}^{uv}y_{\beta,v}^{(1)})$$

$$= \frac{\partial y_{\alpha,l}}{\partial y_{\beta,s}}Y_{\beta,s}^{(k+1)}.$$

Similarly,

(4.5)
$$\gamma_{\alpha\beta}(E_{\alpha,l}^{(k)}) = g_{ls}E_{\beta,s}^{(k)}, \quad \gamma_{\alpha\beta}(F_{\alpha,l}^{(k)}) = h_{ls}F_{\beta,s}^{(k)}.$$

Thus $\Phi_{m,x} \circ \gamma_{\alpha\beta}^x = \tilde{\gamma}_{\alpha\beta}^x \circ \Phi_{m,x}$. $\Phi_{m,x}$ is independent of the choice of $(U_\alpha, y_\alpha, e_\alpha, f_\alpha) \in \mathcal{U}$.

 $\Phi_{m,x}$ smoothly depends on x, so it gives an isomorphism $\Phi_m: A_m(E,F) \to B_m(E,F)$ of smooth complex vector bundles. $\Phi_{m,x}$ preserves the grading, so Φ_m maps $A_m(E,F)[j,k,l]$ to $B_m(E,F)[j,k,l]$. $\Phi_{m,x}$ is an isomorphism of \mathbb{C} -algebra, so Φ_m preserves the algebra structure.

Holomorphic structures on $A_m(E,F)$. Through the isomorphism Φ_m , $B_m(E,F)$ can be regarded as the same smooth complex vector bundle as $A_m(E,F)$ with a different holomorphic structure. Now the underlying complex vector bundle of $A_m(E,F)$ has a Hermitian metric $(-,-)_m$ (from $B_m(E,F)$) and two holomorphic structures. One is from $A_m(E,F)$, which is determined by $\bar{\partial}$ with

(4.6)
$$\bar{\partial}y_i^{(l)} = 0, \quad \bar{\partial}e_i^{(l)} = 0, \quad \bar{\partial}f_i^{(l)} = 0, \\ \bar{\partial}ab = \bar{\partial}ab + a\bar{\partial}b, \quad a, b \in \Omega^{0,0}(A_m(E, F)).$$

The other is induced from $B_m(E, F)$, which is determined by $\bar{\partial}'$ with

(4.7)
$$\bar{\partial}' Y_i^{(l)} = 0, \quad \bar{\partial}' E_i^{(l)} = 0, \quad \bar{\partial}' F_i^{(l)} = 0, \\ \bar{\partial}' ab = \bar{\partial}' ab + a\bar{\partial}' b, \quad a, b \in \Omega^{0,0}(A_m(E, F)).$$

Under the fixed Hermitian metric $(-,-)_m$ on $A_m(E,F)$, let $\nabla=\nabla^{1,0}+\bar{\partial}$ and $\tilde{\nabla}=\tilde{\nabla}^{1,0}+\bar{\partial}'$ be the Chern connections associated with $\bar{\partial}$ and $\bar{\partial}'$ respectively. By (4.1) and (4.2) and Proposition 4.1, for $(U,y,e,f)\in\mathcal{U}$, $\tilde{\nabla}$ satisfies

(4.8)
$$\tilde{\nabla}Y_{i}^{(l)} = \sum \theta_{ij}^{T^{*}}Y_{j}^{(l)}, \quad \tilde{\nabla}E_{i}^{(l)} = \sum \theta_{ij}^{E}E_{j}^{(l)}, \quad \tilde{\nabla}F_{i}^{(l)} = \sum \theta_{ij}^{F}F_{j}^{(l)}, \\ \tilde{\nabla}(ab) = (\tilde{\nabla}a)b + a(\tilde{\nabla}b), \quad \text{for } a, b \in \Omega_{U}^{0,0}(A_{m}(E, F)).$$

Its curvature $\tilde{\Theta}$ satisfies

$$(4.9) \qquad \tilde{\Theta}Y_i^{(l)} = \sum_{i} \Theta_{ij}^{T^*} Y_j^{(l)}, \quad \tilde{\Theta}E_i^{(l)} = \sum_{i} \Theta_{ij}^E E_j^{(l)}, \quad \tilde{\Theta}F_i^{(l)} = \sum_{i} \Theta_{ij}^F F_j^{(l)},$$

$$\tilde{\Theta}(ab) = (\tilde{\Theta}a)b + a(\tilde{\Theta}b), \quad \text{for } a, b \in \Omega_U^{0,0}(A_m(E, F)).$$

There is also a derivation $\tilde{D}: A_{\infty}(E,F) \to A_{\infty}(E,F)$ induced from \tilde{D} on $B_m(E,F)$ through Φ_{∞} . By Proposition 4.1, for $(U,y,e,f) \in \mathcal{U}$, \tilde{D} satisfies

$$\tilde{D}Y_i^{(j)} = Y_i^{(j+1)}, \quad \tilde{D}E_i^{(j)} = E_i^{(j+1)}, \quad \tilde{D}F_i^{(j)} = F_i^{(j+1)}.$$

Lemma 4.2.

$$D = \tilde{D} + Y_t^{(1)} \tilde{\nabla}_{\frac{\partial}{\partial y_t}}.$$

Proof. D, \tilde{D} and $\sum_t Y_t^{(1)} \tilde{\nabla}_{\frac{\partial}{\partial y_t}}$ are globally defined operators. To prove the equation, we only need to prove it locally. For $(U, y, e, f) \in \mathcal{U}$,

$$\begin{split} DY_{l}^{(k+1)} &= D(H_{lj}D^{k}(H^{ji}y_{i}^{(1)})) \\ &= (DH_{lj})D^{k}(H^{ji}y_{i}^{(1)}) + H_{lj}D^{k+1}(H^{ji}y_{i}^{(1)}) \\ &= Y_{t}^{(1)}\frac{\partial H_{lj}}{\partial y_{t}}H^{js}H_{sm}(D^{k}(H^{mi}y_{i}^{(1)})) + Y_{l}^{(k+2)} \\ &= Y_{l}^{(k+2)} + Y_{t}^{(1)}\theta_{ls}^{T^{*}}(\frac{\partial}{\partial y_{t}})Y_{s}^{(k+1)}. \\ &= \tilde{D}Y_{l}^{k+1} + Y_{t}^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_{t}}}Y_{l}^{k+1} \end{split}$$

Similarly,

$$DE_l^{(k)} = \tilde{D}E_l^k + Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}E_l^k$$

$$DF_l^{(k)} = \tilde{D}F_l^k + Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}F_l^k$$

For a smooth function f on U,

$$\tilde{D}f + Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}f = 0 + Y_t^{(1)}\frac{\partial f}{\partial y_t} = Df.$$

Since both D and $\tilde{D}+Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}$ are derivations on $\Omega_U^{0,0}(A_\infty(E,F))$ and $Y_l^{(k+1)}$, $E_l^{(k)}$, $F_l^{(k)}$ and smooth functions on U generate $\Omega_U^{0,0}(A_\infty(E,F))$, D is equal to $\tilde{D}+Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}$.

We can find the relation between $\bar{\partial}$ and $\bar{\partial}'$ by the above lemma and the following fact.

Lemma 4.3.

$$[D, \bar{\partial}] = 0, \quad [\tilde{D}, \bar{\partial}'] = 0$$

Proof. $[D, \bar{\partial}]$ and $[\tilde{D}, \bar{\partial}']$ are derivations on the sheaf of smooth sections of $A_{\infty}(E, F)$. Locally, for $(U, y, e, f) \in \mathcal{U}$ and a smooth function f on U,

$$\begin{split} [D,\bar{\partial}]y_l^{(k+1)} &= [D,\bar{\partial}]e_l^{(k)} = [D,\bar{\partial}]f_l^{(k)} = [D,\bar{\partial}]f = 0.\\ [\tilde{D},\bar{\partial}']Y_l^{(k+1)} &= [\tilde{D},\bar{\partial}']E_l^{(k)} = [\tilde{D},\bar{\partial}']F_l^{(k)} = [\tilde{D},\bar{\partial}']f = 0. \end{split}$$

Let $K = \bar{\partial} - \bar{\partial}'$. For any smooth function f on X, $Kf = \bar{\partial}f - \bar{\partial}'f = 0$. By (4.6) and (4.7), K(ab) = (Ka)b + a(Kb), for $a, b \in \Omega^{0,0}(A_m(E,F))$.

So K is determined if we know $KY_l^{(k+1)}$, $KE_l^{(k)}$ and $KF_l^{(k)}$. Using the following lemma, we can calculate them out.

Lemma 4.4.

$$[K, \tilde{D}] = Y_t^{(1)} [\tilde{\nabla}_{\frac{\partial}{\partial y_t}}, \bar{\partial}] = -Y_t^{(1)} \tilde{\Theta}(\frac{\partial}{\partial y_t}) + Y_t^{(1)} [\tilde{\nabla}_{\frac{\partial}{\partial y_t}}, K].$$

Proof. $\bar{\partial} Y_l^{(1)} = \bar{\partial} y_l^{(1)} = 0.$

$$\begin{split} [K,\tilde{D}] &= [\bar{\partial} - \bar{\partial}',\tilde{D}] \\ &= [\bar{\partial},\tilde{D}] \qquad \text{(by Lemma 4.3)} \\ &= [\bar{\partial},D - Y_t^{(1)}\tilde{\nabla}_{\frac{\partial}{\partial y_t}}] \qquad \text{(by Lemma 4.2)} \\ &= Y_t^{(1)}[\tilde{\nabla}_{\frac{\partial}{\partial y_t}},\bar{\partial}] \qquad \text{(by Lemma 4.3)} \\ &= Y_t^{(1)}[\tilde{\nabla}_{\frac{\partial}{\partial y_t}},\bar{\partial}' + K] \\ &= -Y_t^{(1)}\tilde{\Theta}(\frac{\partial}{\partial y_t}) + Y_t^{(1)}[\tilde{\nabla}_{\frac{\partial}{\partial y_t}},K]. \end{split}$$

Let $C_{j_1,\cdots,j_a} = -\frac{1}{(a-1)!} \frac{(j_1+\cdots+j_a)!}{j_1!j_2!\cdots j_a!}$, if $j_1 \geq 0$ and $j_l \geq 1$ for $l \geq 2$, otherwise $C_{j_1,\cdots,j_a} = 0$.

Lemma 4.5. For $k \ge 0$, If $P_l^{(k)}$ is $Y_l^{(k+1)}$, $E_l^{(k)}$ or $F_l^{(k)}$, and F is F^{T^*} , F^E or F^F respectively, then

$$KP_l^{(k)} = \sum_{a=2}^k \sum_{j_1 + \dots + j_a = k} C_{j_1, \dots, j_a} F_{i_1, \dots, i_a}^l P_{i_1}^{(j_1)} Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}.$$

Proof. Here gives the proof for $P_l^{(k)} = Y_l^{(k+1)}$. The proofs are similar for $P_l^{(k)} = E_l^{(k)}$ and $P_l^{(k)} = F_l^{(k)}$.

We prove it by induction on k. When n=0, $Y_l^{(1)}=y_l^{(1)}$. So $\bar{\partial}Y_l^{(1)}=\bar{\partial}y_l^{(1)}=0$. Since $\bar{\partial}'Y_l^{(1)}=0$, $KY_l^{(1)}=0$, the lemma is true. Assume the lemma is true for n=k. If n=k+1, by Lemma 4.4,

$$\begin{split} KY_l^{(k+2)} &= K\tilde{D}Y_l^{(k+1)} \\ &= \tilde{D}KY_l^{(k+1)} - Y_t^{(1)}\tilde{\Theta}(\frac{\partial}{\partial y_t})Y_l^{(k+1)} + Y_t^{(1)}[\tilde{\nabla}_{\frac{\partial}{\partial y_t}},K]Y_l^{(k+1)} \quad \text{(By Lemma4.4)} \\ &= \tilde{D}(\sum_{a=2}^k \sum_{j_1+\dots+j_a=k} C_{j_1,\dots,j_a}F_{i_1,\dots,i_a}^lY_{i_1}^{(j_1+1)}Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}) \\ &- Y_t^{(1)}\Theta_{ls}^{T^*}(\frac{\partial}{\partial y_t})Y_s^{(k+1)} \\ &+ Y_t^{(1)}\sum_{a=2}^k \sum_{j_1+\dots+j_a=k} C_{j_1,\dots,j_a}(\nabla_{\frac{\partial}{\partial y_t}}^{T^*}F_a)_{i_1,\dots,i_a}^lY_{i_1}^{(j_1+1)}Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)} \\ &= \sum_{a=2}^k \sum_{j_1+\dots+j_a=k+1} (\sum_{l=1}^a C_{j_1,\dots,j_{l-1},\dots,j_a})F_{i_1,\dots,i_a}^lY_{i_1}^{(j_1+1)}Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)} \\ &- F_{i_1,i_2}^2Y_{i_1}^{(k+1)}Y_{i_2}^{(1)} \\ &+ \sum_{a=2}^k \sum_{j_1+\dots+j_a=k} \frac{a}{k+1}C_{j_1,\dots,j_a,1}F_{i_1,\dots,i_a,i_{a+1}}^lY_{i_1}^{(j_1+1)}Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}Y_{i_{a+1}}^{(1)} \\ &= \sum_{a=2}^{k+1} \sum_{j_1+\dots+j_a=k+1} C_{j_1,\dots,j_a}F_{i_1,\dots,i_a}^lY_{i_1}^{(j_1+1)}Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}. \end{split}$$

So the lemma is true for n = k + 1. Thus the lemma is true for any $k \ge 0$.

By Lemma 4.5, we immediately have

Lemma 4.6. For $k \geq 0$, If $P_l^{(k)}$ is $Y_l^{(k+1)}$, $E_l^{(k)}$ or $F_l^{(k)}$, and F is F^{T^*} , F^E or F^F respectively, then

$$[\tilde{\nabla}^{1,0}, K]P_l^{(k)} = \sum_{a=2}^k \sum_{j_1+\dots+j_a=k} C_{j_1,\dots,j_a} dy_{i_{a+1}} \wedge F_{i_1,\dots,i_a,i_{a+1}}^l P_{i_1}^{(j_1)} Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}.$$

The mean curvature of $A_m(E,F)$. By Lemma 2.4, the connections on $A_m(E,F)$ satisfy

$$\nabla = \tilde{\nabla} + K - K^*.$$

So their curvatures satisfy

$$(4.10) \qquad \Theta = [\bar{\partial}, \nabla] = [\bar{\partial}' + K, \tilde{\nabla}^{1,0} - K^*] = \tilde{\Theta} + [K, \tilde{\nabla}^{1,0}] - [\bar{\partial}', K^*] - [K, K^*].$$

Now $\tilde{\nabla}^{1,0}$ and K are derivations on the sheaf of smooth sections of $A_m(E,F)$, so $[\tilde{\nabla}^{1,0},K]$ is a derivation on the sheaf of smooth sections of $A_m(E,F)$.

Lemma 4.7. If h_{T^*} , h_E and h_F are Einstein-Hermitian metrics then

$$[\tilde{\nabla}^{1,0}, K](\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_i})H_{ij} = 0.$$

Proof. $\tilde{\nabla}^{1,0}$ and K are derivations on the sheaf of smooth sections of $A_m(E,F)$, so $[\tilde{\nabla}^{1,0},K]$ and $[\tilde{\nabla}^{1,0},K](\frac{\partial}{\partial y_i},\frac{\partial}{\partial \bar{y}_j})H_{ij}$ are derivations on the sheaf of smooth sections of $A_m(E,F)$. From Lemma 4.6, if $P_l^{(k)}$ is $Y_l^{(k+1)}$, $E_l^{(k)}$ or $F_l^{(k)}$, and F is F^{T^*} , F^E or F^F respectively,

$$[\tilde{\nabla}^{1,0}, K] P_l^{(k)} (\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j}) H_{ij} = \sum_{a=2}^k \sum_{j_1 + \dots + j_a = k} C_{j_1, \dots, j_a} F_{i_1, \dots, i_a, i}^l (\frac{\partial}{\partial \bar{y}_j}) H_{ij} P_{i_1}^{(j_1)} Y_{i_2}^{(j_1)} \cdots Y_{i_a}^{(j_a)}.$$

By Lemma 2.2, if h_{T^*} , h_E and h_F are Einstein-Hermitian metrics, then for $n \geq 3$, $M_n^{T^*}$, M_n^E and M_n^F all vanish. So

$$F_{i_1,\dots,i_a,i}^l(\frac{\partial}{\partial \bar{y}_i})H_{ij}=0.$$

Thus $[\tilde{\nabla}^{1,0},K]P_l^{(k)}(\frac{\partial}{\partial y_i},\frac{\partial}{\partial \bar{y}_j})H_{ij}=0$. For any smooth function f on X,

$$[\tilde{\nabla}^{1,0},K]f = \tilde{\nabla}^{1,0}Kf + K\tilde{\nabla}^{1,0}f = K\partial f = 0.$$

Since $[\tilde{\nabla}^{1,0},K](\frac{\partial}{\partial y_i},\frac{\partial}{\partial \bar{y}_j})H_{ij}$ is a derivation on the sheaf of smooth sections of $A_m(E,F)$, it is zero.

We have the following theorem for the mean curvature of the connection ∇ on $A_m(E, F)$.

Theorem 4.8. If h_{T^*} , h_E and h_F are Einstein-Hermitian metrics then the mean curvature of ∇ is

$$\Theta(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j}) H_{ij} = \tilde{\Theta}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j} H_{ij} + K(\frac{\partial}{\partial \bar{y}_j}) K^*(\frac{\partial}{\partial y_i}) H_{ij} - K^*(\frac{\partial}{\partial y_i}) K(\frac{\partial}{\partial \bar{y}_j}) H_{ij}.$$

In particular, if the mean curvatures of h_{T^*} , h_E and h_F are zero,

$$\Theta(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j}) H_{ij} = K(\frac{\partial}{\partial \bar{y}_j}) K^*(\frac{\partial}{\partial y_i}) H_{ij} - K^*(\frac{\partial}{\partial y_i}) K(\frac{\partial}{\partial \bar{y}_j}) H_{ij}.$$

Proof. By Lemma 4.7, $[K, \tilde{\nabla}^{1,0}](\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij} = 0$, $-[\bar{\partial}', K^*](\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij}$ also vanishes since it is the conjugacy of $[K, \tilde{\nabla}^{1,0}](\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij}$. This proves the first equation by (4.10). By (4.9), $\tilde{\Theta}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij}$ vanishes if the mean curvatures of h_{T^*} , h_E and h_F vanish. The second equation follows.

5. Holomorphic sections

In this section, we assume X is a compact Hermitian manifold. We calculate the space of holomorphic sections of $A_m(E, F)$ if the mean curvatures of h_{T^*} , h_E and h_F are zero.

Lemma 5.1. Let E be a holomorphic vector bundle with a Hermitian metric h=(-,-) and its Chern connection is $\nabla=\nabla^{1,0}+\bar{\partial}$. Assume E has another holomorphic holomorphic structure determined by $\bar{\partial}'=\bar{\partial}-K$ with its Chern connection $\tilde{\nabla}=\tilde{\nabla}^{1,0}+\bar{\partial}'$, such that under this holomorphic structure,

- (1) $E = \bigoplus_{k \in \mathbb{Z}} E^k$, E^k are holomorphic vector bundles;
- (2) E^k is perpendicular to E^l for $k \neq l$;
- (3) the mean curvature of $\tilde{\nabla}$ is zero.
- (4) K maps $\Omega^{0,0}(E_l)$ to $\Omega^{0,1}(\bigoplus_{k=l+1}^{\infty} E_k)$.
- (5) $[\tilde{\nabla}^{1,0}, K](\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij} = 0.$

Then $a \in \Omega_X^{0,0}(E)$ is holomorphic if and only if $\tilde{\nabla} a = 0$ and Ka = 0.

Proof. By condition (1), any smooth section a of E can be uniquely written as a finite sum of smooth sections a_k of E^k , that is $a = \sum_k a_k$ where only a finite number of a_k not zero. By condition (2), any smooth sections a and b of E, $(a,b) = \sum_k (a_k,b_k)$. For any $\lambda \geq 0$, let $h^{\lambda} = (-,-)^{\lambda}$ be the Hermitian metric of E given by

$$(a,b)^{\lambda} = \sum_{k} \lambda^{k} (a_{k}, b_{k}).$$

Under this Hermitian metric, Let ∇^{λ} and $\tilde{\nabla}^{\lambda}$ are the Chern connections corresponding to $\bar{\partial}$ and $\bar{\partial}'$ respectively. Since on E^k , h^{λ} is a rescale of h, $\tilde{\nabla}^{\lambda} = \tilde{\nabla}$.

Let $K_{k,l} \in \Omega^{0,1}(\operatorname{Hom}(E,E))$ with $K_{k,l}a = (Ka_k)_l$. By condition (4), $K = \sum_{k < l} K_{k,l}$. Let $K_{k,l}^* \in \Omega^{1,0}(\operatorname{Hom}(E,E))$ be the dual of $K_{k,l}$, so

$$(K_{k,l}(\frac{\partial}{\partial \bar{y}_i})a,b)^{\lambda} = \lambda^l(K_{k,l}(\frac{\partial}{\partial \bar{y}_i})a_k,b_l) = \lambda^l(a_k,K_{k,l}^*(\frac{\partial}{\partial y_i})b_l) = \lambda^{l-k}(a,K_{k,l}^*(\frac{\partial}{\partial y_i})b)^{\lambda}.$$

Let

$$K_{\lambda}^* = \sum_{k < l} \lambda^{l-k} K_{k,l}^*,$$

we have $(K(\frac{\partial}{\partial \bar{y}_i})a,b)^{\lambda}=(a,K^*_{\lambda}(\frac{\partial}{\partial y_i})b)^{\lambda}$. By Lemma 2.4, $\nabla^{\lambda}=\tilde{\nabla}+K-K^*_{\lambda}$, so its curvature is

$$\Theta^{\lambda} = [\bar{\partial}, \nabla] = [\bar{\partial}' + K, \tilde{\nabla}^{1,0} - K_{\lambda}^*] = [\bar{\partial}', \tilde{\nabla}] + [K, \tilde{\nabla}^{1,0}] - [\bar{\partial}', K_{\lambda}^*] - [K, K_{\lambda}^*].$$

By conditions (3) and (5) and the proof of Theorem 4.8, the mean curvature of ∇^{λ} is

$$M = \Theta^{\lambda}(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \bar{y}_j})H_{ij} = K(\frac{\partial}{\partial \bar{y}_j})K_{\lambda}^*(\frac{\partial}{\partial y_i})H_{ij} - K_{\lambda}^*(\frac{\partial}{\partial y_i})K(\frac{\partial}{\partial \bar{y}_j})H_{ij}.$$

Apply the Weitzenböck formula 2.1 to a smooth section $a \in \Omega_X^{0,0}(E)$. We have

$$\int_{X} (\Delta_{\bar{\partial}} a^{\lambda}, a)^{\lambda} = \int_{X} (\nabla^{\lambda *} \nabla^{\lambda} a - Ma, a)^{\lambda}
= \int_{X} (\nabla^{\lambda} a, \nabla^{\lambda} a)^{\lambda} - (K_{\lambda}^{*} a, K_{\lambda}^{*} a)^{\lambda} + (Ka, Ka)^{\lambda}
= \int_{X} (\nabla^{\lambda 1, 0} a, \nabla^{\lambda 1, 0} a)^{\lambda} + (\bar{\partial} a, \bar{\partial} a)^{\lambda} - (K_{\lambda}^{*} a, K_{\lambda}^{*} a)^{\lambda} + (Ka, Ka)^{\lambda}.$$

Now

(5.1)
$$\int_{X} (\bar{\partial}a, \bar{\partial}a)^{\lambda} \ge 0, \quad \int_{X} (Ka, Ka)^{\lambda} \ge 0.$$

Let

$$P(\lambda) = \int_X (\nabla^{\lambda 1,0} a, \nabla^{\lambda 1,0} a)^{\lambda} - (K_{\lambda}^* a, K_{\lambda}^* a)^{\lambda}.$$

Assume $a_k = 0$ for k < l or k > m, $a_l \neq 0$ and $a_m \neq 0$.

$$\begin{split} &(\nabla^{\lambda 1,0}a,\nabla^{\lambda 1,0}a)^{\lambda} - (K_{\lambda}^{*}a,K_{\lambda}^{*}a)^{\lambda} \\ &= ((\tilde{\nabla}^{1,0} - K_{\lambda}^{*})a,(\tilde{\nabla}^{1,0} - K_{\lambda}^{*})a)^{\lambda} - (K_{\lambda}^{*}a,K_{\lambda}^{*}a)^{\lambda} \\ &= (\tilde{\nabla}^{1,0}a,\tilde{\nabla}^{1,0}a)^{\lambda} - (\tilde{\nabla}^{1,0}a,K_{\lambda}^{*}a)^{\lambda} - (K_{\lambda}^{*}a,\tilde{\nabla}^{1,0}a)^{\lambda}. \\ &= \sum_{k=l}^{m} \lambda^{l}(\tilde{\nabla}^{1,0}a_{l},\tilde{\nabla}^{1,0}a_{l}) - \sum_{m \geq k' > k \geq l} \lambda^{k'}(\tilde{\nabla}^{1,0}a_{k},K_{k,k'}^{*}a_{k'}) - \sum_{m \geq k' > k \geq l} (K_{k,k'}^{*}a_{k'},\tilde{\nabla}^{1,0}a_{k}). \end{split}$$

 $\lambda^{-l}P(\lambda)$ is a polynomial of λ with constant term

$$\int_X (\tilde{\nabla}^{1,0} a_l, \tilde{\nabla}^{1,0} a_l) \ge 0.$$

If a is a holomorphic section, we have $\int_X (\Delta_{\bar{\partial}}^{\lambda} a, a)^{\lambda} = 0$ for any $\lambda > 0$. By (5.1), $P(\lambda) \leq 0$. We must have

$$\int_{V} (\tilde{\nabla}^{1,0} a_l, \tilde{\nabla}^{1,0} a_l) = 0.$$

So

$$\tilde{\nabla}^{1,0}a_l = 0.$$

 $K_{\lambda}^* a_l \in \Omega_X^{1,0}(\bigoplus_{k < l} E^k)$, which is perpendicular to $\tilde{\nabla}^{1,0} a_k$, $k \ge l$. Thus

$$P(\lambda) = \int_X (\sum_{k=l+1}^m \lambda^l (\tilde{\nabla}^{1,0} a_l, \tilde{\nabla}^{1,0} a_l) - \sum_{m \ge k' > k \ge l+1} \lambda^{k'} (\tilde{\nabla}^{1,0} a_k, K_{k,k'}^* a_{k'}) - \sum_{m \ge k' > k \ge l+1} (K_{k,k'}^* a_{k'}, \tilde{\nabla}^{1,0} a_k)).$$

Induction on l, we can show that $\tilde{\nabla}^{1,0}a = 0$ and $P(\lambda) = 0$.

Thus to keep $\int_X (\Delta_{\bar{\partial}} a, a)^{\lambda} = 0$, the left side of the inequation (5.1) is zero. So

$$Ka = 0$$
, and $\bar{\partial}a = 0$.

$$\bar{\partial}'a = \bar{\partial}a - Ka = 0$$
 and $\tilde{\nabla}a = \tilde{\nabla}^{1,0}a + \bar{\partial}'a = 0$.

On the other hand, If $\tilde{\nabla} a = 0$ and Ka = 0. Since the mean curvature of $\tilde{\nabla}$ is zero, by the Weitzenböck formula (2.1), $\bar{\partial}' a = 0$. So $\bar{\partial} a = \bar{\partial}' a + Ka = 0$. a is a holomorphic section of E.

Let

$$(5.2) \quad E^n = \Phi_m^{-1}(\bigoplus_{j+k+l=n} \operatorname{Sym}^j(\bigoplus_{1 \le l \le m} T^*) \otimes \operatorname{Sym}^k(\bigoplus_{0 \le l \le m} E^*) \otimes \wedge^l(\bigoplus_{0 \le l \le m} F^*)).$$

 $A_m(E,F) = \bigoplus_{k=0}^{\infty} E^k$. Let $A_m^*(E,F)[i,j,k]$ be the dual vector bundle of $A_m^*(E,F)[i,j,k]$. Let

$$A_m^*(E,F) = \bigoplus_{i,j,k} A_m^*(E,F)[i,j,k].$$

We have the following theorem:

Theorem 5.2. If X is a compact Hermitian manifold and the mean curvatures of h_{T^*} , h_E and h_F are zero, then a smooth section a of $A_m(E,F)$ is holomorphic if and only if $\tilde{\nabla} a = 0$ and Ka = 0. A smooth section a of $A_m(E,F)^*$ is holomorphic if and only if $\tilde{\nabla} a = 0$ and $K^{\vee} a = 0$.

Proof. We use Lemma 5.1 to prove the theorem. $A_m(E,F)$ is a holomorphic bundle with the Hermitian metric h and its Chern connection is ∇ . $A_m(E,F)$ has another holomorphic structure coming from $B_m(E,F)$ by Φ_m . Its Chern connection is $\tilde{\nabla}$. E^k are holomorphic under the holomorphic structure from $B_m(E,F)$. E^k is perpendicular to E^l if $k \neq l$. $A_m(E,F) = \bigoplus_{k=0}^{\infty} E^k$. Let $E^{-k} = E^{k*}$, then $A_m^*(E,F) = \bigoplus_{-\infty}^{0} E^k$. So $A_m(E,F)$ and $A_m^*(E,F)$ satisfies condition (1) and (2). Since the mean curvature of h_{T*} , h_E and h_F vanish, by (4.9), $A_m^*(E,F)$ satisfies condition (3). By Lemma 4.5 and Lemma 4.7, $A_m^*(E,F)$ satisfies conditions (4) and (5). Since E^{-k} is the dual of E^k , it is easy to see that $A_m^*(E,F)$ satisfies conditions (3), (4) and (5). So by Lemma 5.1, the theorem is true.

When the Hermitian metric of E, F and T^* are flat, then K = 0 and a smooth section b of $A_m(E, F)^*$ is holomorphic if and only if $\tilde{\nabla} a = 0$. In particular

Corollary 5.3. If X is a flat Kähler torus, E and F are sums of some copies of tangent and cotangent bundles of X, then A smooth section a of $A_m(E,F)$ is holomorphic if and only if $\tilde{\nabla} a = 0$. So the space of holomorphic sections of $A_m(E,F)$ is isomorphic to its fibre by restriction.

 $\mathfrak{g}[t]$ invariants. Let \mathfrak{g} be a Lie algebra. Let

$$\mathfrak{g}[t] = \bigoplus_{n \ge 0} \mathfrak{g}t^n$$

be the Lie algebra given by

$$[g_i t^i, g_j t^j] = [g_i, g_j] t^{i+j},$$
for $g_i, g_j \in \mathfrak{g}.$

Let $\mathfrak{g}_m = \mathfrak{g}[t]/(t^{m+1})$. The following lemma is obvious.

Lemma 5.4. if \mathfrak{g} is simple, then \mathfrak{g}_m is generated by \mathfrak{g} and an element K of $\mathfrak{g}t$. So if R is a representation of \mathfrak{g}_m , then the invariant subspace of R is

$$R^{\mathfrak{g}_m} = \{ r \in R^{\mathfrak{g}} | Kr = 0 \}.$$

If \mathfrak{g} is the Lie algebra of a connected algebraic group G, then \mathfrak{g}_m is the Lie algebra of the Lie group $J_m(G)$. $R^{\mathfrak{g}_m} = R^{J_m(G)}$.

Let $\mathfrak{g}=\mathfrak{sl}(N,\mathbb{C})$ (or $\mathfrak{sp}(2N,\mathbb{C})$), $V=\mathbb{C}^N$ (or \mathbb{C}^{2N}) be the fundamental representation of \mathfrak{g} . By the Weyl's dimension formula for the finite irreducible representation of \mathfrak{g} , we have the following lemma about the representation of $\mathfrak{g}=\mathfrak{sl}(N,\mathbb{C})$ (or $\mathfrak{sp}(2N,\mathbb{C})$):

Lemma 5.5. Let V be the fundamental representation of \mathfrak{g} . Regarding \mathfrak{g} as a subspace of $V^* \otimes V$, $V^* \otimes \mathfrak{g} \cap \operatorname{Sym}^2 V^* \otimes V$ is an irreducible representation of \mathfrak{g} .

Now we assume X is an N (or 2N) dimensional compact Kähler manifold with the holonomy group G=SU(N) (or G=Sp(N)). Then the mean curvature of h_{T^*} is zero. Let E and F be sums of some copies of the holomorphic tangent and cotangent bundles of X. Then the mean curvatures of h_E and h_F are zeros. The holonomy group of $A_m(E,F)$ for $\tilde{\nabla}$ is G. By Theorem 5.2, a is a holomorphic section of $A_m(E,F)$ if and only if $\tilde{\nabla} a=0$ and Ka=0. $\tilde{\nabla} a=0$ means that a is a parallel section of $\tilde{\nabla}$ in $A_m(E,F)$. The space of the parallel sections of $\tilde{\nabla}$ in $A_m(E,F)$ is isomorphic to $(A_m(E,F)|_x)^G$, for any point $x\in X$. The isomorphism is given by the restriction. So the restriction

$$r_x: \Gamma(A_m(E,F)) \to (A_m(E,F)|_x)^G$$

is injective. Let $R=A_m(E,F)|_x$ be the fibre of $A_m(E,F)$ at x. Let $\mathfrak g$ be the Lie algebra $\mathfrak{sl}(N,\mathbb C)$ (or $\mathfrak{sp}(2N,\mathbb C)$). The action of G on R induces the action of $\mathfrak g$ on R, then we have the action of $\mathfrak g_m$ on R given by

$$gt^k(ab) = (gt^ka)b + a(gt^kb), \text{ for } a, b \in R$$

and if $P_i^{(j)}$ is $Y_i^{(j+1)}$, $E_i^{(j)}$ or $F_i^{(j)}$,

$$gt^k(P_i^{(j)}) = 0$$
, for $j < k$

$$gt^k(P_i^{(j)}) = \frac{j!}{(j-k)!}g(P_i^{(j-k)}), \text{ for } j \ge k.$$

We have the following theorem on the holomorphic sections of $A_m(E, F)$.

Theorem 5.6. Let X be an N (or 2N) dimensional compact Kähler manifold with holonomy group G = SU(N) (or G = Sp(N)). Let E and F be sums of some copies of the holomorphic tangent and cotangent bundles of X. Then the image of $r_x : \Gamma(A_m(E,F)) \to (A_m(E,F)|_x)^G$ is $(A_m(E,F)|_x)^{\mathfrak{g}_m}$. So r_x induces an isomorphism from $\Gamma(A_m(E,F))$ to $(A_m(E,F)|_x)^{\mathfrak{g}_m}$.

Proof. If a is a parallel section of $\tilde{\nabla}$ with $r_x(a) \in (A_m(E,F)|_x)^{\mathfrak{g}_m}$ for a point $x \in X$. For any point $x' \in X$, let's prove that $r_{x'}(a) \in (A_m(E,F)|_{x'})^{\mathfrak{g}_m}$. Let γ be a path from x to x'. Through the parallel transformation along the path γ under the connection $\tilde{\nabla}$, we get an isomorphism of G representation from $A_m(E,F)|_x$ to $A_m(E,F)|_{x'}$. From the definition of the action \mathfrak{g}_m , we actually get an isomorphism of \mathfrak{g}_m representation. Now since a is a parallel section, the isomorphism maps $a|_x$ to $a|_{x'}$. Thus $a|_x$ is \mathfrak{g}_m invariant if and only if $a|_{x'}$ is \mathfrak{g}_m invariant.

The holonomy group of $A_m(E,F)$ for $\tilde{\nabla}$ is G. Through the action of G on $A_m(E,F)|_x$, we can regard $\mathfrak g$ as a subspace of $\operatorname{End}(A_m(E,F)|_x,(A_m(E,F)|_x)$. We have $\tilde{\Theta}(\frac{\partial}{\partial y_i},\frac{\partial}{\partial \bar{y}_j})|_x\in \mathfrak g$. Let $\tilde{F}_n=F_n^{A_m(E,F)}$ be the higher covariant derivative of $\tilde{\Theta}$. Then

$$g_{i_2\cdots i_a,j} = \tilde{F}_n(\frac{\partial}{\partial y_{i_2}}, \cdots, \frac{\partial}{\partial y_{i_n}}, \frac{\partial}{\partial \bar{y}_j})|_x \in \mathfrak{g}.$$

By the expression of K in Lemma 4.5,

$$K(\frac{\partial}{\partial \bar{y}_{j}})P_{l}^{(k)}|_{x} = \sum_{a=2}^{k} \sum_{j_{1}+\dots+j_{a}=k} C_{j_{1},\dots,j_{a}}(\tilde{F}_{n}(\frac{\partial}{\partial y_{i_{2}}},\dots,\frac{\partial}{\partial y_{i_{a}}},\frac{\partial}{\partial \bar{y}_{j}})|_{x}P_{l}^{(k-j_{1})})Y_{i_{2}}^{(j_{2})}\dots Y_{i_{a}}^{(j_{a})}$$

$$= \sum_{a=2}^{k} \sum_{j_{1}+\dots+j_{a}=k} \frac{1}{(a-1)!j_{2}!\dots j_{a}!} (g_{i_{2}\dots i_{a},j}t^{k-j_{1}}P_{l}^{(k)})Y_{i_{2}}^{(j_{2})}\dots Y_{i_{a}}^{(j_{a})}$$

So

(5.3)
$$K(\frac{\partial}{\partial \bar{y}_i})|_x = \sum_{a=2}^k \sum_{j_1+\dots+j_a=k} \frac{1}{(a-1)!j_2!\cdots j_a!} Y_{i_2}^{(j_2)}\cdots Y_{i_a}^{(j_a)} g_{i_2\cdots i_a,j} t^{k-j_1}.$$

Since $a|_x$ is \mathfrak{g}_m invariant for any $x \in X$, so K(a) = 0. By Theorem 5.2, $a \in \Gamma(A_m(E, F))$. On the other hand, if $a \in \Gamma(A_m(E, F))$, $a = \sum_{k=1}^n a_k$ with $a_k \in \Omega^{0,0}(E^k)$. Let

$$\tilde{K}_i = \sum_{j_2 \ge 1} \frac{Y_{i_2}^{(j_2)}}{j_2!} \tilde{\Theta}(\frac{\partial}{\partial y_{i_2}}, \frac{\partial}{\partial \bar{y}_i}) t^{j_2}.$$

Then

$$K(\frac{\partial}{\partial \bar{y}_i})a|_x = \tilde{K}_i a_l|_x + b, \quad b \in \bigoplus_{k>l} E^k|_x.$$

So Ka = 0 implies $\tilde{K}_i a_l|_x = 0$.

Since $\tilde{\nabla}$ preserves E^k , the action of the holomomy group G on the fibre of $A_m(E,F)$ preserves the grading. So $a_l|_x\in (E^l|_x)^G$. Since the holomomy group of X is G, the curvature R of T is not equals to zero. There is some point $x\in X$, and some i_0 , $R(\frac{\partial}{\partial \bar{y}_{i_0}})|_p\neq 0$.

The action of \mathfrak{g} on K_{i_0} forms a representation W of \mathfrak{g} . Every element of W is a derivation of $\Omega^{0,0}(E_m)$ and it vanishes on $a_l|_p$.

Now for any $g \in \mathfrak{g}$.

$$[g, \tilde{K}_{i_0}] = \sum_{j_2 \ge 1} \left((g \frac{Y_{i_2}^{(j_2)}}{j_2!}) (\tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}}, \frac{\partial}{\partial y_{i_2}}) t^{j_2}) + \frac{Y_{i_2}^{(j_2)}}{j_2!} ([g, \tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}}, \frac{\partial}{\partial y_{i_2}})] t^{j_2}) \right)$$

$$= \sum_{j_2 \ge 1} \left(-\frac{Y_{i_2}^{(j_2)}}{j_2!} (\tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}}, g \frac{\partial}{\partial y_{i_2}}) t^{j_2}) + \frac{Y_{i_2}^{(j_2)}}{j_2!} ([g, \tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}}, \frac{\partial}{\partial y_{i_2}})] t^{j_2}) \right)$$

Thus the action of \mathfrak{g} on K_{i_0} is the same as the action of \mathfrak{g} on $\tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}})$. Since E and F are direct sums of copies of holomorphic tangent and cotangent bundles. By (4.9), the action of \mathfrak{g} on $\tilde{\Theta}(\frac{\partial}{\partial \bar{y}_{i_0}})$ is the same as its action on $R(\frac{\partial}{\partial \bar{y}_{i_0}}) \in \operatorname{Sym}^2 T_x^* \otimes T_x \cap T_x^* \otimes \mathfrak{g}$. By Lemma 5.5, $\operatorname{Sym}^2 T_x^* \otimes T_x \cap T_x^* \otimes \mathfrak{g}$ is an irreducible representation of \mathfrak{g} . So

$$W \cong \operatorname{Sym}^2 T_x^* \otimes T_x \bigcap T_x^* \otimes \mathfrak{g}.$$

Let $g_1, g_2 \in \mathfrak{g}$, which are determined by

$$g_1(\frac{\partial}{\partial y_i}) = \delta_i^1 \frac{\partial}{\partial y_s}, \quad g_2(\frac{\partial}{\partial y_i}) = \delta_i^1 \frac{\partial}{\partial y_1} - \delta_i^s \frac{\partial}{\partial y_s}.$$

here s=2 if $g=\mathfrak{sl}(N,\mathbb{C})$ and s=N+1 if $g=\mathfrak{sp}(2N,\mathbb{C})$. We have $[g_2,g_1]=-2g_1$. Then the two derivations

$$K_1 = \sum_{j \ge 1} \frac{Y_1^{(j)}}{j!} g_1 t^j, \quad K_2 = -\sum_{j \ge 1} \frac{Y_s^{(j)}}{j!} g_1 t^j + \sum_{j \ge 1} \frac{Y_1^{(j)}}{j!} g_2 t^j$$

are in W. We have $K_1a_l|_x=0$ and $K_2a_l|_x=0$. By Lemma 5.7, $a_l|_x$ is g_m invariant. Then we have $Ka_l=0$. Now $K(a-a_l)=0$. By induction on l, we can show a_x is \mathfrak{g}_m invariant. This proves the theorem.

Lemma 5.7. if $a \in A_m(E, F)|_x$ is \mathfrak{g} invariant and $K_1a = 0$, $K_2a = 0$, then a is \mathfrak{g}_m invariant.

Proof. Let \mathbb{P}_n be a derivation on $A_m(E,F)|_x$ defined inductively by

$$\mathbb{P}_1 = K_1, \quad \mathbb{P}_n = [\mathbb{P}_{n-1}, K_2].$$

We have $\mathbb{P}_n a = 0$. Now

$$\left[\frac{Y_1^{(j)}}{j!}, K_2\right] = \frac{1}{j} \sum_{s=1}^{j-1} \frac{Y_1^{(j-s)}}{(j-s-1)!} \frac{Y_1^{(s)}}{s!},$$

$$[g_1t^j, K_2] = 2\sum_{s=1}^{j-1} \frac{Y_1^{(s)}}{s!} g_1t^{j+s}.$$

By the above two equations, we can show inductively that

$$\mathbb{P}_n = \sum_{j_i > 1} c_{j_1, \dots, j_a} Y_1^{(j_1)} \cdots Y_1^{(j_n)} g_1 t^{j_1 + \dots + j_n}$$

with $c_{j_1,\dots,j_n} > 0$. a has finite weight, when k large enough $gt^k a = 0$ for any $g \in \mathfrak{g}$. Let L be the largest number such that a is not $\mathfrak{g}t^L a$ invariant. if $L \geq 1$,

$$\mathbb{P}_{L}a = c_{1,\dots,1}(Y_1^{(1)})^{L}g_1t^{L}a.$$

So $g_1t^La=0$. Since a is \mathfrak{g} invariant and $\mathfrak{g}t^L$ is an irreducible representation of \mathfrak{g} , a is $\mathfrak{g}t^L$ invariant. But we assume that a is not $\mathfrak{g}t^L$ invariant. So $L\leq 0$. we conclude that a is \mathfrak{g}_m invariant.

Now we can prove Theorem 1.1.

Proof of theorem 1.1. Let $\mathfrak{g}=\mathfrak{sl}(N,\mathbb{C})$. $TJ_m(X)$ is isomorphic to $J_m(TX)$. The sheaf of sections of $A_m(T,0)[1,0]$ over X is the sheaf of the push forward of the sheaf of sections of $J_m(TX)$ through $\pi_m:J_m(X)\to X$. Thus the space of holomorphic vector fields of $J_m(X)$ is isomorphic to $\Gamma(A_m(T,0)[1,0])$. By Theorem 5.6, $\Gamma(A_m(T,0))$ is isomorphic to $(A_m(T,0)|_x)^{\mathfrak{g}_m}$.

$$A_m(T,0)|_x \cong R(y,y^*,0) = \mathbb{C}[y_1^{(1)},\cdots,y_i^{(j)},\cdots,y_N^{(m)},y_1^{*(0)},\cdots,y_i^{*(j)},\cdots,y_N^{*(m)}],$$

So $\Gamma(A_m(T,0)[1,0])$ is isomorphic to the subspace of $(A_m(T,0)|_x)^{\mathfrak{g}_m}$ with the degree of y^* is one. By Theorem 4.3 of [5], $(A_m(T,0)|_x)^{\mathfrak{g}_m}$ is generated by $\mathbb{C}[y_1^{(1)},\cdots,y_N^{(1)},y_1^{*(0)},\cdots,y_N^{*(0)}]^{\mathfrak{g}}$, which is generated by $v=\sum y_i^{(1)}y_i^{*(0)}$. So

$$\Gamma(A_m(T,0)[1,0]) \cong \bigoplus_{k=0}^{m-1} \mathbb{C}\tilde{D}^k v,$$

has dimension m.

Theorem 5.6 can be genelezed to the case when *X* is a compact Ricci-flat Kähler manifold.

Theorem 5.8. Let X be compact Ricci-flat Kähler manifold with holonomy group G. Let $\mathfrak g$ be the complexified Lie algebra of G. Let E and F be sums of some copies of the holomorphic tangent and cotangent bundles of X. Then the image of $r_x: \Gamma(A_m(E,F)) \to (A_m(E,F)|_x)^G$ is $((A_m(E,F)|_x)^{\mathfrak g_m})^G$. So r_x induces an isomorphism from $\Gamma(A_m(E,F))$ to $((A_m(E,F)|_x)^{\mathfrak g_m})^G$.

Proof. It is well known that if X is a compact Ricci-flat Kähler manifold, then it admits a finite cover $p: \tilde{X} \to X$,

$$\tilde{X} = T \times X_1 \times \cdots \times X_k,$$

where T is a flat Kähler torus, and X_i has holonomy group $SU(m_i)$ or $Sp(\frac{m_i}{2})$ with $\dim X_i = m_i$. (See for example page 124 of [2].) If E and F are sums of some copies of the holomorphic tangent and cotangent bundles of X, then the pullback p^*E and p^*F are sums of some copies of the holomorphic tangent and cotangent bundles of \tilde{X} . We have

$$p^*A_m(E,F) = A_m(p^*E, p^*F).$$

By Theorem 5.6, Corollary 5.3 and the fact that holonomoy group of T is trivial, $r_{\tilde{x}}$ induces an isomorphism from $\Gamma(A_m(p^*E, p^*F))$ to $(A_m(p^*E, p^*F)|_{\tilde{x}})^{\mathfrak{g}_m}$.

The pullback gives an imbedding $p^*: \Gamma(X,A(E,F)) \to \Gamma(\tilde{X},A(p^*E,p^*F))$. By Theorem 5.2, any element $\tilde{a} \in \Gamma(\tilde{X},A(p^*E,p^*F))$ satisfies $\tilde{\nabla}a=0$. So $\tilde{a}=p^*a$ for some smooth section a of A(E,F) if and only if $a|_{\tilde{x}}$ is G invariant. Since \tilde{a} is holomorphic, a must be holomorphic. Thus $r_{\tilde{x}} \circ p^*$ induces an isomorphism from $\Gamma(X,A(E,F))$ to $((A_m(p^*E,p^*F)|_{\tilde{x}})^{\mathfrak{g}_m})^G$, which means r_x induces an isomorphism from $\Gamma(A_m(E,F))$ to $((A_m(E,F)|_x)^{\mathfrak{g}_m})^G$.

G has a connected component G_0 which contain the identity. G_0 is a normal subgroup of G and it acts trivially on $(A_m(E,F)|_x)^{\mathfrak{g}_m}$. So $((A_m(E,F)|_x)^{\mathfrak{g}_m})^G = ((A_m(E,F)|_x)^{\mathfrak{g}_m})^{G/G_0}$

6. The global sections of chiral de Rham complex on K3 surfaces

Chiral de Rham complex. The chiral de Rham complex [7, 8] is a sheaf of vertex algebras Ω^{ch}_X defined on any smooth manifold X in either the algebraic, complex analytic, or C^{∞} categories. In this paper we work exclusively in the complex analytic setting. Let Ω_N be the tensor product of N copies of the $\beta\gamma-bc$ system. It has 2N even generators $\beta^1(z),\cdots,\beta^N(z),\gamma^1(z),\cdots,\gamma^N(z)$ and 2N odd generators $b^1(z),\cdots,b^N(z),c^1(z),\cdots,c^N(z)$. Their nontrivial OPEs are

$$\beta^i(z)\gamma^j(w) \sim \frac{\delta^i_j}{z-w}, \quad b^i(z)c^j(w) \sim \frac{\delta^i_j}{z-w}.$$

Given a coordinate system $(U, \gamma^1, \cdots \gamma^N)$ of X, $\mathbb{C}[\gamma^1, \cdots \gamma^N] \subset \mathcal{O}(U)$ can be regarded as a subspace of Ω_N by identifying γ^i with $\gamma^i(z) \in \Omega_N$. As a linear space, Ω_N has a $\mathbb{C}[\gamma^1, \cdots \gamma^N]$ module structure. $\Omega_X^{ch}(U)$ is the localization of Ω_N on U,

$$\Omega_X^{ch}(U) = \Omega_N \otimes_{\mathbb{C}[\gamma^1, \dots \gamma^N]} \mathcal{O}(U).$$

Then $\Omega_X^{ch}(U)$ is the vertex algebra generated by $\beta^i(z), b^i(z), c^i(z)$ and $f(z), f \in \mathcal{O}(U)$. These generators satisfy the nontrivial OPEs

$$\beta^{i}(z)f(w) \sim \frac{\frac{\partial f}{\partial \gamma^{i}}(z)}{z-w}, \quad b^{i}(z)c^{j}(w) \sim \frac{\delta^{i}_{j}}{z-w},$$

as well as the normally ordered relations

$$: f(z)g(z) := fg(z), \text{ for } f, g \in \mathcal{O}(U).$$

 $\Omega_X^{ch}(U)$ is spanned by the elements (6.1)

(6.1)

$$: \partial^{k_1} \beta^{i_1}(z) \cdots \partial^{k_s} \beta^{i_s}(z) \partial^{l_1} b^{j_1}(z) \cdots \partial^{l_t} b^{j_t} \partial^{m_1} c^{r_1}(z) \cdots \partial^{n_1} \gamma^{s_1}(z) \cdots f(\gamma)(z) :, \quad f(\gamma) \in \mathcal{O}(U).$$

Let $\tilde{\gamma}^1, \dots \tilde{\gamma}^N$ be another set of coordinates on U, with

$$\tilde{\gamma}^i = f^i(\gamma^1, \cdots \gamma^N), \quad \gamma^i = g^i(\tilde{\gamma}^1, \cdots \tilde{\gamma}^N).$$

The coordinate transformation equations for the generators are

$$\partial \tilde{\gamma}^{i}(z) =: \frac{\partial f^{i}}{\partial \gamma^{j}}(z)\partial \gamma^{j}(z):,$$

$$\tilde{b}^{i}(z) =: \frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}}(g(\gamma))b^{j}:,$$

$$\tilde{c}^{i}(z) =: \frac{\partial f^{i}}{\partial \gamma^{j}}(z)c^{j}(z):,$$

$$\tilde{\beta}^{i}(z) =: \frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}}(g(\gamma))(z)\beta^{j}(z): + :: \frac{\partial}{\partial \gamma}(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}}(g(\gamma)))(z)c^{k}(z): b^{j}(z):.$$

From [9], Ω_X^{ch} has an increasing filtration Q_n , $n \in \mathbb{Z}_{\geq 0}$ and its associated graded sheaf is

$$\operatorname{gr}(\mathcal{Q}) = \bigoplus_n \mathcal{Q}_n/\mathcal{Q}_{n-1}.$$

Locally, $Q_n(U)$ is spanned by the elements with only at most n copies of β and b, i.e. the elements in equation (6.1) with $s + t \le n$.

Then the associated graded object

$$(\operatorname{gr} \mathcal{Q})(U) = \bigoplus_{n} \mathcal{Q}_{n}(U)/\mathcal{Q}_{n-1}(U)$$

is a ∂ -ring. A $\mathbb{Z}_{\geq 0}$ graded, associative, super-commutative algebra equipped with a derivation ∂ of degree zero is called ∂ -ring. On $(\operatorname{gr} \mathcal{Q})(U)$, the product and the derivation ∂ are induced from the wick product and ∂ from $\Omega^{ch}_X(U)$, respectively.

For each $n \ge 0$, let

$$\phi_n: \mathcal{Q}_n(U) \to \mathcal{Q}_n(U)/\mathcal{Q}_{n-1}(U).$$

be the projection. As a ring with a derivation, gr(Q(U)) is generated by

$$\beta^{i} = \phi_{1}(\beta^{i}(z)), \quad b^{i} = \phi_{1}(b^{i}(z)), \quad c^{i} = \phi_{0}(c^{i}(z)), \quad \phi_{0}(f(z)), f \in \mathcal{O}(U).$$

 $Q_n(U)/Q_{n-1}(U)$ is spanned by

$$(6.3) \ a = \partial^{k_1} \beta^{i_1} \cdots \partial^{k_s} \beta^{i_s} \partial^{k_{s+1}} b^{i_{s+1}} \cdots \partial^{k_n} b^{i_n} \partial^{m_1} c^{r_1} \cdots \partial^{n_1} \gamma^{s_1} \cdots \partial^{n_t} \gamma^{s_t} f(\gamma), \quad f(\gamma) \in \mathcal{O}(U).$$

For the sheaf gr(Q), the coordinate transformation equations of β , γ , b, c are

$$\partial^{n} \tilde{\gamma}^{i} = \partial^{n-1} \left(\frac{\partial f^{i}}{\partial \gamma^{j}} \partial \gamma^{j} \right),$$

$$\partial^{n} \tilde{b}^{i} = \partial^{n} \left(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}} (g(\gamma)) b^{j} \right),$$

$$\partial^{n} \tilde{c}^{i} = \partial^{n} \left(\frac{\partial f^{i}}{\partial \gamma^{j}} c^{j} \right),$$

$$\partial^{n} \tilde{\beta}^{i} = \partial^{n} \left(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}} (g(\gamma)) \beta^{j} \right) + \partial^{n} \left(\frac{\partial}{\partial \gamma} \left(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}} (g(\gamma)) c^{k} b^{j} \right).$$

The only difference between these equations and those of the chiral de Rham sheaf is that the Wick product is replaced by the ordinary product in an associated super commutative algebra.

gr(Q) has an increasing filtration $gr(Q)_{n,s}$, $0 \le s \le n$ and its associated graded sheaf is

$$\operatorname{gr}^2(\mathcal{Q}) = \bigoplus_{n,s} \operatorname{gr}(\mathcal{Q})_{n,s}/\operatorname{gr}(\mathcal{Q})_{n,s-1}.$$

Locally, $\operatorname{gr}(\mathcal{Q})_{n,s}(U)$ is spanned by all elements $a \in \mathcal{Q}_n(U)/\mathcal{Q}_{n-1}(U)$ of the form (6.3) with the number of β less or equal than s.

The associated graded object

$$\operatorname{gr}^2(\mathcal{Q})(U) = \bigoplus_{n,s} \operatorname{gr}(\mathcal{Q})_{n,s}(U)/\operatorname{gr}(\mathcal{Q})_{n,s-1}(U)$$

is a ∂ -ring. The product and the derivation ∂ are induced from the product and derivation of $gr(\mathcal{Q})(U)$.

Let

$$\psi_{n,s}: \operatorname{gr}(\mathcal{Q})_{n,s}(U) \to \operatorname{gr}(\mathcal{Q})_{n,s}(U)/\operatorname{gr}(\mathcal{Q})_{n,s-1}(U)$$

be the projection as a ring with a derivation, $gr^2(Q(U))$ is generated by

$$\psi_{1,1}(\beta^i), \psi_{1,0}(b^i), \psi_{0,0}(c^i)$$
 and $\psi_{0,0}(f(\gamma)), f(\gamma) \in \mathcal{O}(U)$.

When no confusion arise, the symbols $\beta^i, \gamma^i, b^i, c^i$ in $gr(\mathcal{Q})(U)$ will also be used to denote the corresponding elements $\psi_{1,1}(\beta^i), \psi_{0,0}(\gamma^i), \psi_{1,0}(b^i), \psi_{0,0}(c^i)$ in $gr^2(\mathcal{Q})(U)$.

For the sheaf $gr^2(Q)$, the relations of β , γ , b, c under the coordinate transformation are

$$\partial^{n}\tilde{\gamma}^{i} = \partial^{n-1}\left(\frac{\partial f^{i}}{\partial \gamma^{j}}\partial \gamma^{j}\right),$$

$$\partial^{n}\tilde{b}^{i} = \partial^{n}\left(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}}(g(\gamma))b^{j}\right),$$

$$\partial^{n}\tilde{c}^{i} = \partial^{n}\left(\frac{\partial f^{i}}{\partial \gamma^{j}}c^{j}\right),$$

$$\partial^{n}\tilde{\beta}^{i} = \partial^{n}\left(\frac{\partial g^{i}}{\partial \tilde{\gamma}^{j}}(g(\gamma))\beta^{j}\right).$$

By these coordinate transformation equations, we have

Proposition 6.1. $gr^2(\mathcal{Q})$ is exactly the sheaf $\mathcal{A}_{\infty}(T, T \oplus T^*)$.

From [9], we have the following reconstruction properties for the holomorphic sections of Ω_X^{ch} .

Lemma 6.2. If $a_i \in \mathcal{Q}_{n_i}(X)$, for $1 \leq i \leq l$, such that $\phi_{n_i}(a_i)$ generate $gr(\mathcal{Q})(X)$ as a ∂ -ring, then ϕ_n is surjective, and it therefore induces an isomorphism

$$Q_n(M)/Q_{n-1}(X) \cong Q_n/Q_{n-1}(X).$$

Furthermore, $\{a_i | 1 \le i \le l\}$ strongly generates the vertex algebra $\Omega_X^{ch}(M)$.

Lemma 6.3. If $a_i \in gr(\mathcal{Q})_{n_i,s_i}$ for $1 \leq i \leq l$, such that $\psi_{n_i,s_i}(a_i)$ generate $gr(\mathcal{Q})(X)$ as a ∂ -ring, then $\psi_{n,s}$ is surjective, and it induces an isomorphism

$$gr(\mathcal{Q})_{n,s}(X)/gr(\mathcal{Q})_{n,s-1}(X) \cong gr(\mathcal{Q})_{n,s}/gr(\mathcal{Q})_{n,s-1}(X).$$

Furthermore, $\{a_i | 1 \le i \le l\}$ generate gr(Q)(X) as a ∂ -ring.

Holomorphic sections of the chiral de Rham complex. If X is a Calabi-Yau manifold, then $\Omega_X^{ch}(X)$ is a topological vertex algebra. There are four global sections Q(z), L(z), J(z), G(z). Locally, they can be represented by

$$Q(z) = \sum_{i=1}^{N} : \beta^{i}(z)c^{i}(z) :, \qquad L(z) = \sum_{i=1}^{N} (:\beta^{i}(z)\partial\gamma^{i}(z) : - :b^{i}(z)\partial c^{i}(z) :),$$

$$J(z) = -\sum_{i=1}^{N} :b^{i}(z)c^{i}(z) :, \qquad G(z) = \sum_{i=1}^{N} :b^{i}(z)\partial\gamma^{i}(z) :,$$

If ω is a nowhere vanishing holomorphic N form of X, there are two global sections D(z) and E(Z) of Ω_X^{ch} can be constructed. Locally, if (U,γ) are coordinate of U such that $\omega = d\gamma^1 \wedge \cdots \wedge d\gamma^N$, D(z) and E(Z) can be represented by

$$D(z) =: b^1(z)b^2(z)\cdots b^N(z):, \quad E(z) =: c^1(z)c^2(z)\cdots c^N(z):.$$

Let $B(z) = Q(z)_{(0)}D(z)$, $C(z) = G(z)_{(0)}E(z)$. Let \mathcal{V}_N be the vertex algebra generated by these eight sections. When N=2, \mathcal{V} is an N=4 superconformal vertex algebra with central charge 6. (Here the Virasolo field is $L(z) - \frac{1}{2}J(z)$.) We will show that if X is a K3 surface, these eight sections strongly generate $\Omega_X^{ch}(X)$ as a vertex algebra.

Let $\bar{\Omega}_N$ be the subalgebra of the Ω_N , which is generated by $\beta^i(z), \partial \gamma^i(z), b^i(z), c^i(z)$. It is a tensor product of a system of free bosons and a system of free fermions. \mathcal{V}_0 is a subalgebra of $\bar{\Omega}_N$. On $\bar{\Omega}_N$ there is a positive definite Hermitian form (-,-) with the following property:

(6.6)
$$(\beta_{(n)}^{x_i} A, B) = (A, \alpha_{(-n)}^{x_i'} B), \quad \text{for any } n \in \mathbb{Z}, n \neq 0, \forall A, B \in \bar{\Omega}_N;$$

$$(b_{(n)}^{x_i} A, B) = (A, c_{(-n-1)}^{x_i'} B), \quad \text{for any } n \in \mathbb{Z}, \forall A, B \in \bar{\Omega}_N.$$

It is easy to check

(6.7)
$$Q_{(n)}^* = G_{(-n+1)}, \qquad J_{(n)}^* = J_{(-n)},$$

$$L_{(n)}^* = L_{(-n+2)} - (n-1)J_{(-n+1)}, \qquad D_{(n)}^* = (-1)^{\frac{N(N-1)}{2}} E_{(N-2-n)},$$

Let $\tilde{L}=L-\frac{1}{2}\partial J$. \tilde{L} is a virasoro field with central charge 3N. $\tilde{L}_{(n)}^*=\tilde{L}_{(-n+2)}$. We can conclude

Lemma 6.4. $\bar{\Omega}_N$ is a unitary representation of the Lie algebra generated by

$${Q_{(n)}, G_{(n)}, J_{(n)}, L_{(n)}, D_{(n)}, E_{(n)}, B_{(n)}, C_{(n)}}_{n \in \mathbb{Z}}$$

and V_N is a simple conformal vertex algebra with central charge 3N.

Assume X is an N dimensional complex manifold with holonomy group SU(N). By Theorem 5.6 and Proposition 6.1, $\operatorname{gr}^2(\mathcal{Q})(X)$ is isomorphic to $\operatorname{gr}^2(\mathcal{Q})|_x^{\mathfrak{sl}(N,\mathbb{C})[t]}$. $\operatorname{gr}^2(\mathcal{Q})|_x$ is a ∂ -ring, which is isomorphic to

$$R = \mathbb{C}[\partial^k \beta^i, \partial^{k+1} \gamma^i, \partial^k b^i, \partial^k c^i], \quad k \geq 0, 1 \leq i \leq N.$$

So to calculate the holomorphic sections of $\operatorname{gr}^2(\mathcal{Q})$ is to calculate $R^{\mathfrak{sl}_N[t]}$. It is easy to see that the following eight elements of R are $\mathfrak{sl}_N[t]$ invariant.

$$\sum_{i=1}^{N} \beta^{i} c^{i},$$

$$\sum_{i=1}^{N} \beta^{i} \partial \gamma^{i},$$

$$\sum_{i=1}^{N} b^{i} \partial \gamma^{i},$$

$$\sum_{i=1}^{N} b^{i} \partial \gamma^{i},$$

$$\sum_{i=1}^{N} b^{i} \partial \gamma^{i},$$

$$\sum_{i=1}^{N} c^{1} c^{2} \cdots c^{N},$$

$$\sum_{i=1}^{N} (-1)^{i-1} b^{1} \cdots b^{i-1} \beta^{i} b^{i+1} \cdots b^{N},$$

$$\sum_{i=1}^{N} (-1)^{i-1} c^{1} \cdots c^{i-1} \partial \gamma^{i} c^{i+1} \cdots c^{N}.$$

We have the following theorem.

Theorem 6.5. If the holonomy group of the N-dimensional Kähler manifold X is SU(N) and $R^{\mathfrak{sl}(N,\mathbb{C})[t]}$ is generated by above eight elements as a ∂ -ring, then the space of global sections of the chiral de Rham complex of X is a simple vertex algebra strongly generated by

$$Q(z),L(z),J(z),G(z),B(z),D(z),C(z),E(z). \\$$

Proof. The images of the above eight sections under the map of ϕ_* and $\psi_{*,*}$ locally are

$$\psi_{1,1}(\phi_{1}(Q(z))) = \sum_{i=1}^{N} \beta^{i} c^{i}, \qquad \psi_{1,1}(\phi_{1}(L(z))) = \sum_{i=1}^{N} \beta^{i} \partial \gamma^{i},
\psi_{1,0}(\phi_{1}(J(z))) = -\sum_{i=1}^{N} b^{i} c^{i}, \qquad \psi_{1,0}(\phi_{1}(G(z))) = \sum_{i=1}^{N} b^{i} \partial \gamma^{i},
(6.9)
\psi_{N,0}(\phi_{N}(D(z))) = \frac{1}{f} b^{1} b^{2} \cdots b^{N}, \qquad \psi_{0,0}(\phi_{0}(E(z))) = f c^{1} c^{2} \cdots c^{N},
\psi_{N,1}(\phi_{N}(B(z))) = \sum_{i=1}^{N} (-1)^{i-1} \frac{1}{f} b^{1} \cdots b^{i-1} \beta^{i} b^{i+1} \cdots b^{N},
\psi_{0,0}(\phi_{0}(C(z))) = \sum_{i=1}^{N} (-1)^{i-1} f c^{1} \cdots c^{i-1} \partial \gamma^{i} c^{i+1} \cdots c^{N}.$$

They are global sections of $\operatorname{gr}^2(Q)$. By assumption, the eight elements of (6.8) generated $\operatorname{gr}^2(Q)|_p^{\mathfrak{sl}_N[t]}$ as a ∂ -ring. Compare (6.8) and (6.9). By Theorem 5.6 the eight global

sections in (6.9) generate $\operatorname{gr}^2(\mathcal{Q})(X)$ as a ∂ -ring. By Lemma 6.2 and 6.3, the eight sections Q(z), L(z), J(z), G(z), B(z), D(z), C(z), E(z) strongly generate the vertex algebra $\Omega_X^{ch}(X)$.

From [6] and the classical invariant theory [10], when N = 2, $R^{\mathfrak{sl}_N[t]}$ is generated by the eight elements of (6.8), so Theorem 1.2 is a corollary of Theorem 6.5.

Let X be an Enriques surface. Then X is the quotient of a K3 surface \tilde{X} by a fixed point-free involution σ . σ induce an automorphism of $\Omega^{ch}_{\tilde{X}}(\tilde{X})$ which maps Q(z), L(z), J(z), G(z), B(z), D(z), C(z), E(z) to Q(z), L(z), J(z), G(z), -B(z), -D(z), -C(z), -E(z), respectively. Obviously, $\Omega^{ch}_{\tilde{X}}(X)$ is isomorphic to the subalgebra of $\Omega^{ch}_{\tilde{X}}(\tilde{X})^{\sigma}$ consisting the invariant elements of σ .

Theorem 6.6.

$$\Omega^{ch}_X(X) = \Omega^{ch}_{\tilde{X}}(\tilde{X})^{\sigma}.$$

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KEY LABORATORY OF WU WEN-TSUN MATHEMATICS, CHINESE ACADEMY OF SCIENCES, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, ANHUI 230026, P.R. CHINA

E-mail address: bailinso@ustc.edu.cn