## Singularities of the wave trace for the Friedlander model

Yves Colin de Verdière \*
Victor Guillemin †
David Jerison ‡

January 21, 2014

#### Abstract

In a recent preprint, [CGJ], we showed that for the Dirichlet Laplacian  $\Delta$  on the unit disk, the wave trace  $\text{Tr}(e^{it\sqrt{\Delta}})$ , which has complicated singularities on  $2\pi - \varepsilon < t < 2\pi$ , is, on the interval  $2\pi < t < 2\pi + \varepsilon$ , the restriction to this interval of a  $C^{\infty}$  function on its closure. In this paper we prove the analogue of this somewhat counter-intuitive result for the Friedlander model. The proof for the Friedlander model is simpler and more transparent than in the case of the unit disk.

### Introduction

Let  $\Omega$  be a smooth bounded strictly convex region in the plane, and let  $0 < \lambda_1 < \lambda_2 \leq \ldots$  be the eigenvalues, with multiplicity, of the Dirichlet problem for the Laplacian on  $\Omega$ :

$$\Delta u_j = \lambda_j u_j$$
 on  $\Omega$ ;  $u_j = 0$  on  $\partial \Omega$ ,  $(\Delta = -(\partial_{x_1}^2 + \partial_{x_2}^2))$ 

<sup>\*</sup>Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

<sup>†</sup>MIT; supported in part by NSF grant DMS-1005696

<sup>&</sup>lt;sup>‡</sup>MIT; supported in part by NSF grant DMS-1069225 and the Stefan Bergman Trust.

In the spirit of [CdV0, Ch, DG, AM, GM]) we will define the wave trace of the Dirichlet Laplacian  $\Delta$  to be the sum

$$Z(t) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}.$$
 (1)

By a theorem of Andersson and Melrose [AM], Re Z(t) is a tempered distribution whose singular support in t > 0 is the closure of the set

$$\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}_k \tag{2}$$

where  $\mathcal{L}_k$  is the set of critical values of the length function given by

$$|x_1 - x_2| + |x_2 - x_3| + \dots + |x_k - x_1|,$$
 (3)

in which  $x_1 \dots x_k$  are distinct boundary points.

Each critical point of the length function (3) defines a polygonal trajectory consisting of the line segments joining  $x_i$  to  $x_{i+1}$  and  $x_k$  to  $x_1$ . These are, by definition, the closed geodesics on  $\Omega$ , and  $\mathcal{L}$  is the set of lengths of these geodesics, i.e. the length spectrum of  $\Omega$ . Each set  $\mathcal{L}_k$  is closed, but there are limiting geodesics as  $k \to \infty$ . These limits are closed curves, called gliding rays, that lie entirely in  $\partial\Omega$ .

Near T in  $\mathcal{L}$ , the nature of the singularity of Z(t) is fairly well understood in generic cases (see [GM]); but, at least at present, this is not so for T in the closure of the length spectrum. For example, if T is the perimeter of  $\Omega$ , then there are trajectories forming a convex polygon with k sides whose perimeters tend to T as  $k \to \infty$ . In general, the limiting lengths T are integer multiples of the perimeter of  $\Omega$ , and the set  $(T - \varepsilon, T) \cap \mathcal{L}$  is infinite for every  $\varepsilon > 0$ . The cumulative effect of infinitely many singularities could be quite complicated and has not been analyzed completely even in simple model examples. However to the right of T one doesn't encounter this problem.  $(T, T + \varepsilon) \cap \mathcal{L} = \emptyset$  for sufficiently small  $\varepsilon > 0$ , so that  $\operatorname{Re} Z(t)$  is  $C^{\infty}$  for  $t \in (T, T + \varepsilon)$ . Therefore, several years ago the second author posed the question, what is the asymptotic behavior of Z(t) as  $t \to T$  from the right?

In one simple case, the case of the unit disk, we showed in [CGJ] that Z(t) is  $C^{\infty}$  to the right, i.e., for some  $\epsilon > 0$ , is  $C^{\infty}$  on  $[T, T + \epsilon)$ . We will publish details of that result elsewhere; however in this article we will discuss another example for which we have been able to show that this assertion is

true, and, in fact, for which the verification is simpler and more transparent than in the case of the disk. This is the operator L introduced by F. G. Friedlander in [F], defined by

$$L = -\partial_x^2 - (1+x)\partial_y^2, \quad x \in [0, \infty), \quad y \in \mathbb{R}/2\pi\mathbb{Z}$$
 (4)

on the manifold  $M = [0, \infty) \times (R/2\pi\mathbb{Z})$  with Dirichlet boundary conditions at x = 0. It represents the simplest approximation to the Laplace operator near the boundary of a convex domain.

The operator L has continuous spectrum on the space of functions that depend on x alone. On the orthogonal complement of this space, the spectrum is discrete, and the eigenfunctions of L are described in terms of the Airy function Ai as follows (see Appendix A (28)).

**Proposition 1.** The Dirichlet eigenfunctions of L on

$$\{\varphi \in L^2(M): \int_{\mathbb{R}/2\pi\mathbb{Z}} \varphi(x,y) \, dy = 0\}$$

are the functions

$$\varphi_{m,n}(x,y) = \operatorname{Ai}(n^{2/3}x - t_m)e^{iny}, \quad m = 1, 2, \dots, \quad n = \pm 1, \pm 2, \dots; \quad (5)$$

and their eigenvalues are

$$\lambda(m,n) = n^2 + n^{4/3}t_m,\tag{6}$$

in which  $0 < t_1 < t_2 < \cdots$  are the roots of Ai(-t) = 0.

We will define the trace of the wave operator associated to L by

$$Z_F(t) = \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{it\sqrt{\lambda(m,n)}} . \tag{7}$$

Our main result is as follows.

**Theorem 1.** Let  $\mathcal{L}_F$  be the Friedlander length spectrum, as computed in Section 1.

- a) The limit points of  $\mathcal{L}_F$  are  $\bar{\mathcal{L}}_F \setminus \mathcal{L}_F = 2\pi \mathbb{N}$ .
- b)  $Z_F$  is  $C^{\infty}$  in the complement of  $\bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}$
- c) For each  $\ell \in \mathbb{N}$ , there is  $\varepsilon_{\ell} > 0$  such that  $Z_F$  extends to a  $C^{\infty}$  function on  $[2\pi\ell, 2\pi\ell + \varepsilon_{\ell})$ .

The main issue in the proof is to identify appropriate non-classical symbol properties for the function  $\lambda(m,n)$ . The Friedlander model is significantly easier to treat than other integrable models such as the disk because the asymptotics of the spectral function  $\lambda(m,n)$  follow easily from well known asymptotics of the Airy function.

The paper is organized as follows. In Section 1, we find the length spectrum of L. In Section 2, we show how the Poisson summation formula is used to deduce smoothness of the trace when the spectral function is a classical symbol. Next, we show, in Section 3, that the Friedlander spectral function  $\lambda(m,n)$  belongs to an appropriate non-classical symbol class of product type. In Section 4, we prove the smoothness properties of the Friedlander wave trace  $Z_F$  (Theorem 1) by dividing the (m,n) lattice into sectors and using Poisson summation methods adapted in each sector to the symbol properties of  $\lambda(m,n)$ . We conclude, in Section 5, by computing the action coordinates of the completely integrable system corresponding to the operator L and show that they capture the first order properties of eigenvalues and eigenfunctions of L in sectors where  $\lambda(m,n)$  behaves like a classical symbol. This final computation reveals more details about which parts of the spectrum are linked to which aspects of the geodesic flow.

In Appendix A, we recall the properties of the Airy function and prove Proposition 1 characterizing the spectrum of L. In Appendix B we review the semiclassical analysis of joint eigenfunctions of commuting operators in a general integrable system, showing how to find the parametrization of the eigenfunctions by a discrete lattice and how to carry out the computations of Section 5 linking the classical and quantum systems in a more general context.

### 1 Closed geodesics

Our first task is to calculate the length spectrum  $\mathcal{L}_F$  for the Friedlander model. Define the Hamiltonian function H as the square root of the symbol of L,

$$H(x, y; \xi, \eta) = \sqrt{\xi^2 + (x+1)\eta^2}$$

The integral curves of the Hamiltonian vector field of H restricted to the hypersurface H=1 are defined as solutions to

$$\dot{x} = \xi, \quad \dot{y} = (1+x)\eta, \quad \dot{\xi} = -\eta^2/2, \quad \dot{\eta} = 0$$
 (8)

with the constraint

$$\xi^2 + (1+x)\eta^2 = 1\tag{9}$$

We begin by analyzing a single trajectory between boundary points.

**Proposition 2.** Consider an integral curve of the Hamiltonian system (8)-(9) on  $0 \le t \le T$ , with

$$x(0) = 0$$
,  $x(t) > 0$  for  $0 < t < T$ ,  $x(T) = 0$ 

If the initial velocity is  $\xi(0) = \xi_0 > 0$ ,  $\eta(0) = \eta_0$ , then final velocity is given by

$$\dot{x}(T) = -\xi_0, \quad \dot{y}(T) = \eta_0,$$

and

$$T = 4\xi_0/\eta_0^2$$
;  $y(T) - y(0) = 4\xi_0/\eta_0 + (8/3)\xi_0^3/\eta_0^3$ 

*Proof.* Solving the equations, we get

$$\eta(t) = \eta_0; \quad \xi(t) = \xi_0 - \eta_0^2 t/2, \quad x(t) = \xi_0 t - \eta_0^2 t^2/4$$

By translation invariance in y, we may as well assume y(0) = 0. Then

$$y(t) = \eta_0 t + \eta_0 \xi_0 t^2 / 2 - \eta_0^3 t^3 / 12$$

and

$$x(T) = 0 \implies \xi_0 T - \eta_0^2 t^2 / 4 = 0 \implies T = 4\xi_0 / \eta_0^2$$

Substituting for T we obtain

$$\dot{x}(T) = \xi_0 - \eta_0^2 T/2 = -\xi_0,$$

and

$$y(T) - y(0) = y(T) = \eta_0 T + \eta_0 \xi_0 T^2 / 2 - \eta_0^3 T^3 / 12 = 4\xi_0 / \eta_0 + (8/3)\xi_0^3 / \eta_0^3$$

Finally, x(T) = 0 and  $\eta(T) = \eta_0$  imply

$$\dot{y}(T) = (1 + x(T))\eta(T) = \eta_0$$

which concludes the proof.

We call a continuous curve  $(x,y):[a,b]\to M$  a geodesic if there are finitely many points  $t_0< t_1< \cdots t_k$  for which the curve meets the boundary (i. e.  $x(t_j)=0$ ) and for  $t_j< t< t_{j+1}$ , the curve is the projection onto (x,y) of a solution to (8) and (9), and, finally, at each  $t_j$  the curve undergoes a reflection in the boundary:

$$\lim_{t \to t_j^+} \dot{x}(t) = -\lim_{t \to t_j^-} \dot{x}(t); \quad \lim_{t \to t_j^+} \dot{y}(t) = \lim_{t \to t_j^-} \dot{y}(t). \tag{10}$$

The Hamiltonian equations are translation invariant in t and y. Proposition 2 shows that a geodesic with starting velocity  $(\dot{x}(t_j^+),\dot{y}(t_j^+))=(\xi_0,\eta_0)$  has final velocity  $(\dot{x}(t_{j+1}^-),\dot{y}(t_{j+1}^-))=(-\xi_0,\eta_0)$ . By (10), the reflected velocity  $(\dot{x}(t_{j+1}^+),\dot{y}(t_{j+1}^+))=(\xi_0,\eta_0)$ , the same as the starting velocity. Consequently, each arc of a geodesic between encounters with the boundary is a translate of the others. This is analogous to equilateral polygonal trajectories in the disk. The figure below depicts a trajectory reflected at the boundary x=0. Note also that it is tangent to the caustic,  $x=\xi_0^2/\eta_0^2$ , depicted by a dotted line.



**Proposition 3.** A closed geodesic with k = 1, 2, ... reflections and winding number  $\ell = \pm 1, \pm 2, ...$  in the y variable, has length  $L_{k,\ell} > 0$  determined by the equations

$$L_{k,\ell} \frac{\eta_0^2}{\sqrt{1 - \eta_0^2}} = 4k, \quad L_{k,\ell} \left(\frac{1}{3}\eta_0 + \frac{2}{3\eta_0}\right) = 2\pi\ell$$
 (11)

*Proof.* Proposition 2 shows that the length of a closed geodesic with k reflections is kT, with T given as in the proposition. Hence, using  $\xi_0^2 + \eta_0^2 = 1$ ,

$$L_{k,\ell} = kT = 4k\xi_0/\eta_0^2 \implies L_{k,\ell} \frac{\eta_0^2}{\sqrt{1-\eta_0^2}} = 4k.$$

Moreover, if the winding number is  $\ell$ , then if y(t) is as in Proposition 2 with y(0) = 0,

$$ky(T) = 2\pi\ell \implies k(4\xi_0/\eta_0 + (8/3)\xi_0^3/\eta_0^3) = 2\pi\ell$$

Substituting  $L_{k,\ell} = 4k\xi_0/\eta_0^2$  into the last equation, it can be written

$$L_{k,\ell}(\eta_0 + (2/3)\xi_0^2/\eta_0) = 2\pi\ell \iff L_{k,\ell}\left(\frac{1}{3}\eta_0 + \frac{2}{3\eta_0}\right) = 2\pi\ell$$

The Friedlander length spectrum is defined by

$$\mathcal{L}_F = \{ L_{k,\ell} : k \in \mathbb{N}, \ \ell \in \mathbb{Z} - \{0\} \}$$

where  $L_{k,\ell}$  solve the equations of Proposition 3 for some  $\eta_0$ ,  $-1 < \eta_0 < 1$ . Note that if  $\eta_0$  is replaced by  $-\eta_0$ , then  $\ell$  is replaced by  $-\ell$  and  $L_{k,-\ell} = L_{k,\ell} > 0$ . Thus we may confine our attention to  $(k,\ell) \in \mathbb{N} \times \mathbb{N}$ .

The implicit definition of  $L_{k,\ell}$  can be restated as follows. Let

$$H: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$$

be the homogeneous function of degree 1 that is equal to 1 on the curve

$$u \to \left(\frac{\pi}{2} \frac{u^2}{\sqrt{1-u^2}}, \left(\frac{u}{3} + \frac{2}{3u}\right)\right) \quad 0 < u < 1.$$
 (12)

Then  $L_{k,\ell} = H(2\pi k, 2\pi \ell)$ .

**Proposition 4.** The Friedlander length spectrum  $\mathcal{L}_F$  can be written

$$\mathcal{L}_F = H(2\pi\mathbb{N} \times 2\pi\mathbb{N}) \tag{13}$$

Furthermore,

- a)  $\bar{\mathcal{L}}_F = \mathcal{L}_F \cup 2\pi \mathbb{N}$ .
- b) For each  $\ell \in \mathbb{N}$ , there is  $\varepsilon_{\ell} > 0$  such that  $(2\pi\ell, 2\pi\ell + \varepsilon_{\ell}) \cap \mathcal{L}_F = \emptyset$ .
- c)  $\mathcal{L}_F \cap 2\pi \mathbb{N} = \emptyset$

*Proof.* Because the curve (12) is the graph of a decreasing function,  $L_{k,\ell} < L_{k+1,\ell}$ . Furthermore,  $L_{1,\ell} \to \infty$  as  $\ell \to \infty$ . Therefore, every sequence of distinct values of  $L_{k,\ell}$  tends to infinity unless  $\ell$  remains bounded. It follows from this and the fact that the curve (12) is asymptotic to the horizontal line of height 1 that all the limit points are given by

$$\lim_{k \to \infty} L_{k,\ell} = 2\pi \ell \lim_{k \to \infty} H(k/\ell, 1) = 2\pi \ell$$

The sequence  $L_{k,\ell}$  is increasing in k for fixed  $\ell$ , so (a) and (b) are established. Finally, to confirm (c), note that if  $2\pi m = L_{k,\ell}$  for some  $(k,\ell)$ , then the equations (11) imply that  $\pi$  is algebraic, a contradiction.

Proposition 4 (a) and (c) imply Theorem 1 (a).

### 2 Poisson summation formula with classical symbols

The symbol class  $S^m(\mathbb{R}^n)$  is the set of functions  $a \in C^\infty(\mathbb{R}^n)$  satisfying

$$|\partial_{\omega}^{\alpha} a(\omega)| \le C_{\alpha} (1 + |\omega|)^{m - \alpha_1 - \alpha_2 - \dots - \alpha_n}, \quad \omega \in \mathbb{R}^n$$
 (14)

for all multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  and some constants  $C_{\alpha}$ .  $S_{cl}^m(\mathbb{R}^n)$  consists of the symbols in  $S^m(\mathbb{R}^n)$  with an asymptotic expansion, that is, there are homogeneous functions  $a_j$  of degree m-j, smooth except at the origin, and a cut-off function  $\chi_N \in C^{\infty}(\mathbb{R}^n)$  such that  $\chi_N = 1$  on  $|\omega| \geq C_N$ , and

$$\left(a - \sum_{j=0}^{N} a_j\right) \chi_N \in S^{m-N-1}(\mathbb{R}^2)$$
(15)

The function  $a_0$  is the principal symbol of a. Functions defined on a cone  $\Gamma \subset \mathbb{R}^n$  belong to the symbol class  $S^m(\Gamma)$   $(S_{cl}^m(\Gamma))$  if they are restrictions of symbols in the class  $S^m(\mathbb{R}^n)$   $(S_{cl}^m(\mathbb{R}^n))$ .

Let F be a real-valued symbol of first order,  $F \in S^1_{cl}(\mathbb{R}^2)$  with positive principal symbol. Let  $\mathbb{T}^2$  be the 2-torus,  $\mathbb{R}^2/2\pi\mathbb{Z}^2$ , and let  $P: C^{\infty}(\mathbb{T}^2) \to C^{\infty}(\mathbb{T}^2)$  be the constant coefficient self-adjoint first order elliptic pseudodifferential operator given by

$$P(e^{2\pi i(mx+ny)}) = F(m,n)e^{2\pi i(mx+ny)}$$

The eigenvalues of P are F(m,n),  $(m,n) \in \mathbb{Z}^2$ , and hence its wave trace is

$$Z(t) = \sum_{(m,n)\in\mathbb{Z}^2} e^{itF(m,n)}. \tag{16}$$

Since the summands are smooth functions of t, the singularities of Z(t) depend only on the asymptotic behavior of F(m,n) as  $(m,n) \to \infty$ . For simplicity we will assume that  $F = F_1$  on  $\xi^2 + \eta^2 \ge 1/2$ , where  $F_1$  is homogeneous of degree 1 and positive. (With small modifications the proof we will give applies in general.)

To identify the length spectrum, consider the Hamiltonian flow on the cotangent bundle of  $\mathbb{T}^2$  associated with the symbol,  $F_1$ , defined by the ordinary differential equations

$$\dot{x} = \partial_{\xi} F_1, \ \dot{y} = \partial_{\eta} F_1, \quad \dot{\xi} = \dot{\eta} = 0, \tag{17}$$

Its periodic trajectories are curves of the form,  $(x, y) + t\nabla F_1(\xi, \eta)$ ,  $0 \le t \le T$ , satisfying

$$T\nabla F_1(\xi, \eta) = 2\pi(p, q) \tag{18}$$

for some T > 0 and some  $(p,q) \in \mathbb{Z}^2$ . Let  $T_{p,q}$  be the set of T for which (18) has a solution for some  $(\xi,\eta) \neq (0,0)$ . (Note that  $\nabla F_1$  is homogeneous of degree 0, so that we may as well assume  $\xi^2 + \eta^2 = 1$ .) The length spectrum is given by

$$\mathcal{L} = \bigcup_{(p,q) \in \mathbb{Z}^2} T_{p,q}$$

Note that because  $\nabla F_1$  is nonzero, bounded, and continuous on  $\xi^2 + \eta^2 = 1$ , the length spectrum is closed and bounded away from 0. Our goal is to prove that the function Z(t) in (16) is  $C^{\infty}$  in t > 0 on the complement of the length spectrum.

To do so we will apply the Poisson summation formula to the function,  $e^{itF(\xi,\eta)}$ . The Fourier transform of this function is

$$\int_{\mathbb{R}^2} e^{i(tF(\xi,\eta) - 2\pi(x\xi + y\eta))} d\xi d\eta$$

so by the Poisson summation formula,

$$Z(t) = \sum_{(p,q)\in\mathbb{Z}^2} Z_{p,q}(t) \tag{19}$$

where

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} e^{i(tF(\xi,\eta) - 2\pi(p\xi + q\eta))} d\xi d\eta.$$
 (20)

We will first show that on the complement of the length spectrum these individual summands are  $C^{\infty}$ . If  $t_0 > 0$  and  $t_0 \notin \mathcal{L}$  then there exists an interval,  $t_0 - \epsilon \le t \le t_0 + \epsilon$ , such that for all  $|(\xi, \eta)| \ge 1$ , either  $t \partial_{\xi} F(\xi, \eta) - 2\pi p$  or  $t \partial_{\eta} F - 2\pi q$  is non-zero. Hence for t on this interval  $Z_{p,q}(t)$  can be written as a sum

$$\sum_{r=1}^{3} \int_{\mathbb{R}^2} e^{i(tF - 2\pi(p\xi + q, n))} \rho_r(\xi, \eta) \, d\xi \, d\eta \tag{21}$$

where  $\rho_1$  and  $\rho_2$  are smooth homogeneous functions of degree zero on  $|(\xi, \eta)| \ge 1$  and the first of the functions above is non-zero on supp  $\rho_1$  and the second non-zero on supp  $\rho_2$ . Finally  $\rho_3$  is supported in a compact neighborhood of the origin. The third integrand is compactly supported, and the corresponding integral converges along with all derivatives with respect to t, regardless of the range of t. Let

$$a_0(\xi, \eta, p, t) = i (t \partial_{\xi} F - 2\pi p)^{-1}$$
 (22)

Then

$$e^{i(tF-2\pi(p\xi+q\eta))}\rho_1 = a_0\partial_\xi \left(e^{i(tF-2\pi(p\xi+q\eta))}\right)\rho_1$$

hence by integration by parts the first summand of (21) can be written as

$$\int_{\mathbb{R}^2} e^{i(tF - 2\pi(p\xi + q\eta))} a_1 \, d\xi d\eta$$

or inductively as

$$\int_{\mathbb{R}^2} e^{i(tF - 2\pi(p\xi + q\eta))} a_N \, d\xi d\eta$$

where

$$a_1 = -\partial_{\xi}(a_0 \rho_1); \quad a_N(\xi, \eta, p, t) = -\partial_{\xi}(a_0 a_{N-1}).$$
 (23)

Since  $a_0$  is a homogeneous function of degree zero in  $(\xi, \eta)$  outside a small neighborhood of the origin, it follows by induction from (23) that  $a_N$  is a homogeneous function of degree -N, and hence

$$|a_N(\xi, \eta, p, t)| \le C_N(1 + |\xi|^2 + |\eta|^2)^{-N/2}$$
(24)

uniformly on the interval  $-\epsilon + t_0 \le t \le \epsilon + t_0$ . Differentiation k times with respect to t introduces the factor  $F^k$  in the integrand, which is dominated by  $|(\xi,\eta)|^k$ ; hence the integral converges. In all, the first of the summand in (21) is a  $C^{\infty}$  function of t in the range specified. The proof for the term  $\rho_2$  is similar, replacing  $a_0$  by  $b_0 = i (t \partial_{\eta} F - 2\pi q)^{-1}$ ,  $a_1$  by  $b_1 = -\partial_{\eta} (b_0 \rho_2)$  and  $a_N$  by  $b_N = -\partial_{\eta} (a_0 b_{N-1})$  one concludes by the same argument that the second of these summands is  $C^{\infty}$  using integration by parts in  $\eta$ .

For |p| large,  $t\partial F/\partial \xi - 2\pi p$  is non-vanishing for all  $(\xi, \eta)$  and all t in  $t_0 - \epsilon \le t \le t_0 + \epsilon$ , and integration by parts with respect to  $\xi$  yields

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} e^{i(tF - 2\pi(p\xi + q\eta))} \tilde{a}_N \, d\xi d\eta, \quad N = 1, 2, \dots$$

with  $\tilde{a}_0 = 1$  and

$$\tilde{a}_N = -\partial_{\xi}(a_0 \tilde{a}_{N-1})$$

The estimate (24) is replaced by

$$|\tilde{a}_N(\xi, \eta, p, t)| \le C_N (1 + |\xi|^2 + |\eta|^2)^{-N/2} |p|^{-(N+1)}, \quad N \ge 1$$
 (25)

Similarly, for |q| large, integration by parts with respect to  $\eta$  yields an analogous formula with better decay as  $q \to \infty$ . Hence if we break the sum  $\sum Z_{p,q}$  into subsums,  $\sum Z_{p,q}$ , |p| < |q|, and  $\sum Z_{p,q}$ ,  $|p| \ge |q|$ , and make use of these estimates we conclude that not only are the individual summands of (19)  $C^{\infty}$  but that the sum itself is  $C^{\infty}$  on the complement of the length spectrum  $\mathcal{L}$ .

### 3 Non-classical symbols

Let  $\Gamma$  be a cone (sector with vertex at the origin) in  $\mathbb{R}^2$ .

**Definition 1.** A smooth function  $a: \Gamma \to \mathbb{C}$  belongs to  $\Sigma^{\alpha,\beta}(\Gamma)$  if there exists constants  $C_{i,k}$  so that for all  $(\xi,\eta) \in \Gamma$ , the following upper bounds hold

$$\forall j, k \ge 0, \ |\partial_{\xi}^{j} \partial_{\eta}^{k} a(\xi, \eta)| \le C_{j,k} (1 + |\xi|)^{\alpha - j} (1 + |\eta|)^{\beta - k}.$$

A typical example is  $b(\xi)c(\eta) \in \Sigma^{\alpha,\beta}(\mathbb{R}^2)$  if  $b \in S^{\alpha}(\mathbb{R})$  and  $c \in S^{\beta}(\mathbb{R})$ .

**Proposition 5.** The following properties hold.

- a)  $S^0(\Gamma) \subset \Sigma^{0,0}(\Gamma)$ .
- b) If  $a \in \Sigma^{\alpha,\beta}(\Gamma)$  and  $b \in \Sigma^{\alpha',\beta'}(\Gamma)$ , the product ab belongs to  $\Sigma^{\alpha+\alpha',\beta+\beta'}(\Gamma)$ .
- c) If  $a \in \Sigma^{\alpha,\beta}(\Gamma)$  and  $a \ge C(1+|\xi|)^{\alpha}(1+|\eta|)^{\beta}$  with C > 0 in  $\Gamma$  (a is said to be elliptic positive), then  $a^{-1} \in \Sigma^{-\alpha,-\beta}(\Gamma)$ . Moreover,  $(a+c)^{-1} \in \Sigma^{-\alpha,-\beta}(\Gamma)$  uniformly for all constants  $c \ge 0$ .

To establish the symbol properties of the phase  $\sqrt{\lambda(m,n)}$  in the formula (7) for the Friedlander wave trace  $Z_F$ , we will need a well known asymptotic formula for the roots of the Airy function. (A proof is given in Appendix A; see also [AS] page 450.)

**Proposition 6.** There is a symbol  $\tau \in S_{cl}^{2/3}(\mathbb{R}_+)$  such that  $\tau(\xi) > 0$  for all  $\xi > 1/2$ ,

$$t_m = \tau(m), \quad m = 1, 2, \dots$$

are the roots of  $\operatorname{Ai}(-t)=0$ , and the principal symbol of  $\tau(\xi)$  is  $(3\pi\xi/2)^{2/3}$   $(\xi\to\infty)$ .

Denote

$$F(\xi, \eta) = (\eta^2 + \eta^{4/3} \tau(\xi))^{1/2}$$

Then  $\sqrt{\lambda(m,n)} = F(m,|n|)$  and the sum (7) can be written

$$Z_F(t) = 2 \sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} e^{itF(m,n)}$$

Let  $0 < \kappa_1 < \kappa_2 < \infty$ . The first quadrant is covered by the union of three overlapping sectors

$$\Gamma_{1} = \{(\xi, \eta) : |\xi| < \kappa_{1}\eta\};$$

$$\Gamma_{2} = \{(\xi, \eta) : (\kappa_{1}/2)\eta < \xi < 2\kappa_{2}\eta\}$$

$$\Gamma_{3} = \{(\xi, \eta) : \kappa_{2}|\eta| < \xi\}$$

One can deduce easily from Proposition 6 the following.

**Proposition 7.** Let  $G(\xi, \eta) = F(\xi, \eta) - \eta$ . Then

- a)  $G \in \Sigma^{2/3,1/3}(\Gamma_1)$ , and  $\partial_{\eta}G$  is positive, elliptic in  $\Sigma^{2/3,-2/3}(\Gamma_1)$ .
- b)  $F \in S^1_{cl}(\Gamma_2)$  with principal symbol  $F_0$  given by

$$F_0(\xi,\eta) = \sqrt{\eta^2 + (3\pi\xi/2)^{2/3}\eta^{4/3}}$$

c)  $F \in \Sigma^{1/3,2/3}(\Gamma_3)$ , and  $\partial_{\xi}F$  is positive elliptic in  $\Sigma^{-2/3,2/3}(\Gamma_3)$ .

### 4 Poisson summation formula with non-classical symbols

We are now ready to finish the proof of Theorem 1. First, we divide the sum representing  $Z_F$  into three regions corresponding to different behaviors of the symbol  $F(\xi, \eta)$ . Let  $\chi_i \in S^0(\mathbb{R}^2)$  be supported in  $\Gamma_i$  and such that

$$\sum_{j=1}^{3} \chi_j(\xi, \eta) = 1 \quad \text{for all } \xi > 0, \ \eta > 0, \ \xi^2 + \eta^2 \ge 1/4$$

Let  $\psi \in \mathbb{C}^{\infty}(\mathbb{R})$  satisfy  $\psi(\xi) = 1$  for all  $\xi \geq 1$  and  $\psi(\xi) = 0$  for all  $\xi \leq 1/2$ . Then

$$Z_F(t) = 2\sum_{j=1}^3 I_j(t), \quad I_j(t) = \sum_{(m,n)\in\mathbb{Z}^2} e^{itF(m,n)} \psi(m)\psi(n)\chi_j(m,n)$$

The method of Section 2 applies in the sector  $\Gamma_2$ . The times t at which singularities can occur are the stationary points  $T_{p,q}$  of the phases of the representation using the Poisson summation formula, namely, solutions to

$$T_{p,q}\nabla F_0(\xi,\eta) = 2\pi(p,q) \tag{26}$$

Furthermore, on  $F_0(\xi, \eta) = 1$ ,

$$\nabla F_0(\xi, \eta) = \left(\frac{\pi}{2} \frac{\eta^2}{\sqrt{1 - \eta^2}}, \frac{1}{3} \eta + \frac{2}{3} \eta^{-1}\right)$$

Thus if  $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ , then  $T_{k,\ell} = L_{k,\ell}$ , exactly the points of the length spectrum  $\mathcal{L}_F$ . The set of  $T_{p,q}$  is

$$-\mathcal{L}_F \cup \{0\} \cup \mathcal{L}_F$$

(Evidently,  $T_{0,0} = 0$  is a stationary point; there are no roots  $T_{0,q}$ ,  $q \neq 0$  and no roots  $T_{p,0}$ ,  $p \neq 0$  of (26). The rest of the range  $(p,q) \in \mathbb{Z} \times \mathbb{Z}$  is covered by the symmetries  $T_{k,-\ell} = T_{k,\ell}$  and  $T_{-k,-\ell} = -T_{k,\ell}$ .)

The restriction to  $\Gamma_2$  gives a further limitation on the set of stationary points. Define a smaller collection of stationary points by

$$\mathcal{L}_F(M) = \{ L_{k,\ell} : (k,\ell) \in \mathbb{N}^2, \quad 1 \le k \le M\ell \}$$

 $\mathcal{L}_F(M)$  is a closed, discrete set in  $\mathbb{R}$ . For M sufficiently large depending on  $\kappa_1$ , all the stationary values that occur in the the range  $\xi \geq \kappa_1 |\eta|/2$  are in  $\mathcal{L}_F(M)$ . Because  $\mathcal{L}_F(M)$  has no limit points, and the symbol of the phase has classical behavior in  $\Gamma_2$ , the method of Section 2 applies and yields the following.

**Proposition 8.** For M sufficiently large depending on  $\kappa_1$ ,  $I_2$  is smooth in the complement of

$$\mathcal{L}_F(M) \cup (-\mathcal{L}_F(M)) \cup \{0\}$$

In particular, for any  $\ell \in \mathbb{N}$  and any  $\kappa_1 > 0$  there is  $\varepsilon > 0$  such that

$$I_2 \in C^{\infty}((2\pi\ell - \varepsilon, 2\pi\ell + \varepsilon))$$

All the remaining singularities in  $\bar{\mathcal{L}}_F - \mathcal{L}_F(M)$  are accounted for in  $\Gamma_1$ , associated with  $I_1$ . The main issue is the limit points at  $t = 2\pi\ell$ . Our theorem in  $\Gamma_1$  is stated in sufficient generality to apply as well to the case of the disk.

**Theorem 2.** Let  $\psi(\xi)$  be a classical symbol of degree 0 in one variable supported in  $\xi \geq 0$  and let  $\chi(\xi, \eta)$  be a classical symbol of degree 0 supported in a cone  $\Gamma_1$  given by  $|\xi| \leq \kappa_1 \eta$ . Suppose that the distribution Z is defined by

$$Z(t) = \sum_{(m,n)\in\mathbb{Z}^2} \psi(m)\chi(m,n)e^{it(n+G(m,n))} ,$$

where G belongs to  $\Sigma^{2/3,1/3}$ , with  $\partial_{\eta}G > 0$  is positive elliptic in  $\Sigma^{2/3,-2/3}$  on the support of  $\psi(m)\chi(m,n)$ . For any T > 0 there exist  $\varepsilon > 0$  and  $\kappa_1 > 0$  sufficiently small depending on T and the constant in the symbol upper bound on  $\partial_{\eta}G$  such that

- a) if  $T = 2\pi \ell$ ,  $\ell \in \mathbb{N}$ , then Z is smooth on  $[T, T + \varepsilon)$ ;
- b) if  $T \notin 2\pi \mathbb{N}$ , then Z is smooth on  $(T \varepsilon, T + \varepsilon)$ .

*Proof.* In the proof of part (a) we will assume for simplicity that  $\ell = 1$ . The other cases are similar. Let  $F(\xi, \eta) = \eta + G(\xi, \eta)$ . The Poisson summation formula says

$$Z(t) = \sum_{(p,q)\in\mathbb{Z}^2} Z_{p,q}(t)$$

with

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} \psi(\xi) \chi(\xi, \eta) e^{i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))} d\xi d\eta ,$$

Denote  $Q = i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))$  and define two differential operators in order to integrate by parts as follows.

$$Af = e^{itF(\xi,\eta)} \frac{i}{2\pi} \partial_{\xi} (e^{-itF(\xi,\eta)} f) = \frac{i}{2\pi} \partial_{\xi} f + \frac{t}{2\pi} (\partial_{\xi} F) f.$$
$$Bf = \frac{1}{i(t\partial_{\eta} F(\xi,\eta) - 2\pi q)} \partial_{\eta} f$$

Then 
$$\frac{1}{p}A(e^Q) = B(e^Q) = e^Q$$
.

Let us start with the most singular case q=1. Because  $G\in \Sigma^{2/3,1/3}$ , we have

$$\partial_{\xi} F = \partial_{\xi} G \in \Sigma^{-1/3,1/3}, \quad \partial_{\eta} G \in \Sigma^{2/3,-2/3}.$$

The main point is that by Proposition 5,

$$\frac{\psi(\xi)\chi(\xi,\eta)}{t\partial_{\eta}F - 2\pi} = \frac{\psi(\xi)\chi(\xi,\eta)}{t - 2\pi + \partial_{\eta}G} \in \Sigma^{-2/3,2/3}$$

uniformly in  $t > 2\pi$ . Let A' and B' denote the adjoint operators to A and B. Then Proposition 5 also implies that

$$A'(\Sigma^{\alpha,\beta}) \subset \Sigma^{\alpha-1/3,\beta+1/3}; \quad B'(\Sigma^{\alpha,\beta}) \subset \partial_{\eta} \Sigma^{\alpha-2/3,\beta+2/3} \subset \Sigma^{\alpha-2/3,\beta-1/3}$$

with constants in the estimates independent of p.

$$Z_{p,1}(t) = p^{-M} \int e^{Q} (A')^{M} (B')^{N} (\psi(\xi)\chi(\xi,\eta)) d\xi d\eta$$
.

Since  $\psi(\xi)\chi(\xi,\eta) \in \Sigma^{0,0}$ , the symbol  $p^{-M}(A')^M(B')^N(\psi(\xi)\chi(\xi,\eta))$  belongs to  $\Sigma^{-(M+2N)/3,(M-N)/3}$  with constants  $O(|p|^{-M})$ . Note that symbols in  $\Sigma^{\alpha,\beta}$  supported in the cone  $0 \le \xi \le \eta$  are integrable if  $\max(\alpha,0) + \beta < -1$ .

Therefore it suffices to take  $N \ge M+5$  to get a convergent integral. In case p=0, take M=0 and N large. Otherwise, take  $M\ge 2$  so that the sum over p converges. So far we have shown that that sum over (p,1) is bounded. To prove smoothness in t, differentiate k times. This yields an extra factor  $F^k \in \Sigma^{2k/3,k}$  in the integrand. Therefore,

$$\sum_{p\in\mathbb{Z}} (d/dt)^k Z_{p,1}(t)$$

is represented by sum of integrals of symbols  $\Sigma^{2k/3-M/3-2N/3,k+M/3-N/3}$  with constants  $O(|p|^{-M})$ . The integrals converge for N large, for instance  $N \ge M+3k+10$ . Again, take M=0 when p=0 and  $M\ge 2$  when  $p\ne 0$ .

Next, consider  $q \neq 1$ . The same estimates as before hold for A'. Recall that we may assume that  $\chi$  is zero on a fixed compact subset. Provided t is close enough to  $2\pi$ , and  $\kappa_1$  sufficiently small, and  $|\xi| + |\eta|$  sufficiently large on the support of  $\chi$ , we have

$$|\partial_{\eta}G| \le C((1+|\xi|)/(1+|\eta|))^{2/3} \le C\kappa_1 << 1$$

and hence  $|t\partial_{\eta}F - 2\pi q| \sim |q-1|$  and

$$\psi(\xi)\chi(\xi,\eta)/(t\partial_n F - 2\pi q) \in \Sigma^{0,0}$$
, with constants  $O(1/|q-1|)$ 

Hence,

$$B': \Sigma^{\alpha,\beta} \to \Sigma^{\alpha,\beta-1}$$

with constants O(1/|q-1|). So the symbol  $p^{-M}(A')^M(B')^N(\psi(\xi)\chi(\xi,\eta))$  belongs to  $\Sigma^{-M/3,-N}$  with constants  $O(|p|^{-M}|q-1|^{-N})$ . The integral converges provided  $N \geq 2$ . When p=0, take M=0; when  $p\neq 0$  take  $M\geq 2$ . Then the sum over (p,q), with  $q\neq 1$  also converges. Lastly, if we apply  $(d/dt)^k$ , we obtain integrands that belong to  $\Sigma^{2k/3-M/3,k-N}$  with constants  $O(|p|^{-M}|q-1|^{-N})$ , so the symbols are integrable if  $N\geq 5k/3+2$ . To sum the series, for  $p\neq 0, q\neq 1$ , let  $M\geq 2$ . For  $p=0, q\neq 1$ , use M=0. This concludes the proof of part (a) of Theorem 2.

For part (b), once again for any  $\delta > 0$  we can choose  $\kappa_1 > 0$  sufficiently small that on the support of  $\chi$  (where  $|\xi| + |\eta|$  is sufficiently large)

$$|\partial_{\eta}G| \le C((1+|\xi|/(1+|\eta|)^{2/3} \le C\kappa_1^{2/3} < \delta$$

Let  $q_0$  be the integer such that  $|T - 2\pi q_0|$  is smallest, and choose both  $\delta > 0$  and  $\varepsilon > 0$  less than  $|T - 2\pi q_0|/4$ . Then for  $t \in (T - \varepsilon, T + \varepsilon)$ ,  $|t\partial_{\eta} F - 2\pi q_0| \ge |T - 2\pi q_0|/4$  on the support of  $\chi$ . Hence,

$$\psi(\xi)\chi(\xi,\eta)/(t\partial_{\eta}F - 2\pi q) \in \Sigma^{0,0}$$
, with constants  $O(1/(1+|q-q_0|))$ 

and the rest of the proof proceeds as in the case  $q \neq 1$  above. This concludes the proof of Theorem 2.

The next proposition will take care the integral  $I_3$ .

**Proposition 9.** Let  $\psi \in C^{\infty}(\mathbb{R})$  satisfy  $\psi(\eta) = 1$  for  $\eta$  sufficiently large and  $\psi(\eta) = 0$  for  $\eta \leq 0$ . Suppose that where F belongs to  $\Sigma^{1/3,2/3}$  and  $\partial_{\xi}F$  is positive elliptic in  $\Sigma^{-2/3,2/3}$  on the cone  $0 < \eta < \xi$ . For each  $T < \infty$  there exists  $\kappa_2$  such that if  $\chi(\xi,\eta)$  is a classical symbol of degree 0 supported in a cone  $\Gamma$  given by  $\xi > \kappa_2 |\eta|$ , then the distribution

$$Z(t) = \sum_{(m,n)\in\mathbb{Z}^2} \psi(n)\chi(m,n)e^{itF(m,n)} ,$$

satisfies  $Z \in C^{\infty}((0,T))$ .

*Proof.* As usual, the Poisson summation formula implies

$$Z(t) = \sum_{(p,q)\in\mathbb{Z}^2} Z_{p,q}(t)$$

with

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} \psi(\eta) \chi(\xi, \eta) e^{i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))} d\xi d\eta ,$$

Denote  $Q = i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))$ .

Case 1. p = 0. Then  $Q = i(tF - 2\pi q\eta)$  and

$$Z_{0,q}(t) = \int \int e^{Q} \psi(\eta) \chi(\xi, \eta) \, d\xi d\eta = -\int \int e^{Q} \partial_{\xi} (\chi/\partial_{\xi} Q) \psi(\eta) \, d\xi d\eta$$
$$= \dots = \int \int e^{Q} a_{M}(\xi, \eta) \psi(\eta) \, d\xi d\eta$$

with  $a_0 = \chi$ ,  $a_{M+1} = -\partial_{\xi}(a_M/\partial_{\xi}Q)$ . Since  $\partial_{\xi}F$  is positive elliptic in  $\Sigma^{-2/3,2/3}(\Gamma)$ ,

$$1/\partial_{\xi}Q = 1/it\partial_{\xi}F$$

belongs to  $\Sigma^{2/3,-2/3}$  with support on  $\Gamma$  and bounds depending on 1/t. By induction,  $a_M \in \Sigma^{-M/3,-2M/3}$  with support in  $\Gamma_3$ . Thus  $Z_{0,q}$  is represented by a convergent integral. Moreover, if q=0, we may differentiate k times with respect to t to obtain an integrand dominated by  $|a_M F^k|$  which is integrable if k < M/3 - 2. For  $q \neq 0$ , use

$$e^{2\pi i q\eta} = \frac{1}{2\pi i q} \partial_{\eta} e^{2\pi i q\eta}$$

to integrate by parts N times to obtain

$$(d/dt)^{j} Z_{0,q}(t) = \frac{\pm 1}{(2\pi iq)^{N}} \int \int e^{Q} \partial_{\eta}^{N} \left[ (iF)^{j} e^{itF} \psi(\eta) a_{M}(\xi, \eta) \right] d\xi d\eta$$

The integrand is majorized by  $(1+|\xi|)^{j+(N-M)/3}$  which is convergent in  $\Gamma$  for sufficiently large M depending on j and N. (Note that because  $\chi$  is supported in  $\xi < |\eta|, \ \psi'(\eta)\chi(\xi,\eta)$  has compact support, and every term in which a derivative falls on  $\psi$  is convergent.) The factor  $1/q^N$  makes the sum over q convergent as well.

Case 2.  $p \neq 0$ . We use the same integration by parts as in Case 1. Note that the set of  $(m,n) \in \mathbb{Z}^2$  where  $\psi(n)\chi(m,n) > 0$  and  $m < \xi_0$  is finite. Any finite sum of exponentials is smooth in t, so we may assume without loss of generality that  $\chi(\xi,\eta)$  is supported on  $\xi > \xi_0$  for some large  $\xi_0$ . It follows that

$$\partial_{\xi} F(\xi, \eta) \le C \xi^{-2/3} \eta^{2/3}$$

on the support of  $\psi(\eta)\chi(\xi,\eta)$ . Furthermore, if  $\kappa_2$  is sufficiently large, depending on T,

$$0 \le t \partial_{\xi} F \le CT \kappa_2^{-2/3} \le \pi$$
, for all  $0 \le t \le T$ 

Thus

$$\chi/\partial_{\xi}Q = 1/i(t\partial_{\xi}F - 2\pi p) \in \Sigma^{0,0}(\Gamma)$$

with bounds O(1/|p|). By induction,

$$a_M \in \Sigma^{-M,0}$$

with bounds  $O(1/|p|^M)$ . The rest of the proof proceeds as in Case 1. If q = 0, then no more integrations by parts are needed. If  $q \neq 0$ , then one uses integration by parts in  $\eta$  as in Case 1 to introduce a factor  $|q|^{-N}$  so as to be able to sum in both p and q.

Finally, we assemble these ingredients to prove Theorem 1. Part (a) was already proved at the end of Section 1. For part(b), choose  $T \notin \bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}$ , and choose  $\varepsilon$  so that

$$[T-\varepsilon, T+\varepsilon] \cap (\bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}) = \emptyset$$

Choose  $\kappa_1$  sufficiently small depending on the symbol bounds and the distance from  $[T - \varepsilon, T + \varepsilon]$  to  $2\pi\mathbb{Z}$  and  $\kappa_2$  sufficiently large depending on the size of T, we may apply Theorem 2 (b), Proposition 8 and Proposition 9, respectively, to  $I_1$ ,  $I_2$  and  $I_3$  to obtain smoothness in  $(T - \varepsilon, T + \varepsilon)$ . Note that Proposition 7 implies that the corresponding phases satisfy the appropriate symbol estimates. For part (c), apply Theorem 2 (a) to  $I_1$ , to obtain smoothness in  $[2\pi\ell, 2\pi\ell + \varepsilon)$ . By Propositions 8 and 9  $I_2$  and  $I_3$  are smooth near  $2\pi\ell$  on both sides. This concludes the proof of Theorem 1.

### 5 Action variables

The classical mechanical system associated to the Friedlander model has two commuting conserved quantities,

$$H(x, y, \xi, \eta) = \sqrt{\xi^2 + (1+x)\eta^2}; \quad J(x, y, \xi, \eta) = \eta.$$

In this section we will show how the action variables associated to this system determine the first order asymptotics of the eigenvalues and eigenfunctions of the Friedlander operator L. (See Appendix B for the general framework of this calculation.)

Let  $H_0$  and  $J_0$  be real numbers satisfying  $0 < |J_0| < H_0$ . The Lagrangian submanifold

$$H = H_0, \quad J = J_0 \tag{27}$$

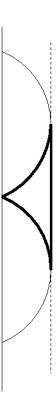
is a torus foliated by trajectories with initial velocity  $(\xi_0, \eta_0)$  on x = 0, with  $\xi_0 > 0$ ,  $\xi_0^2 + \eta_0^2 = H_0$  and  $J_0 = \eta_0$ . These trajectories are tangent to the caustic

$$x = \frac{\xi_0^2}{\eta_0^2} = \frac{H_0^2 - J_0^2}{J_0^2},$$

depicted as a dotted line in the figure below and in Section 1.

Fix H=1, so that  $\xi_0^2+\eta_0^2=1$ . Denote by  $\gamma(t)=(x(t),y(t),\xi(t),\eta(t))$  the trajectory given by Proposition 2 on the interval  $-T/2 \le t \le T/2$ . Define by  $\gamma_1$  the loop (whose projection onto  $\mathbb{R}^+ \times S^1$  is pictured in bold in the figure

below) that follows  $\gamma$  on its two curved portions and then the segment of the caustic  $x = \xi_0^2/\eta_0^2$ , with  $\xi = 0$ ,  $\eta = \eta_0$  (oriented by  $\dot{y} < 0$ ) going from  $\gamma(T/2)$  to  $\gamma(-T/2)$ .



Define by  $\gamma_2$  the loop that follows the caustic  $x = \xi_0^2/\eta_0^2$ ,  $0 \le y \le 2\pi$ ,  $\xi = 0$ , and  $\eta = \eta_0$ , with orientation  $\dot{y} > 0$ .

The loops  $\gamma_1$  and  $\gamma_2$  are a basis for the homology of the torus H=1,  $J=\eta_0,$  and the action coordinates are defined by

$$I_1 = \int_{\gamma_1} \xi \, dx + \eta \, dy, \quad I_2 = \int_{\gamma_2} \xi \, dx + \eta \, dy.$$

Since  $\eta(t) = \eta_0$  is constant,  $T = 4\xi_0/\eta_0^2$ , and  $H = H_0 = \xi_0^2 + \eta_0^2$ , we have

$$I_1 = \int_{\gamma_1} \xi \, dx = \int_{\gamma} \xi \, dx$$
$$= 2 \int_0^{T/2} (\xi_0 - \eta_0^2 t/2)^2 \, dt$$
$$= \frac{4\xi_0^3}{3\eta_0^2} = \frac{4}{3} (H_0^2 - \eta_0^2)^{3/2} \eta_0^{-2}$$

Moreover,

$$I_2 = \int_{\gamma_2} \xi \, dx + \eta \, dy = \int_{\gamma_2} \eta \, dy = 2\pi \eta_0$$

More generally, by homogeneity,

$$I_1 = \frac{4}{3}(H^2 - J^2)^{3/2}J^{-2}; \quad I_2 = 2\pi J.$$

Thus H and J can be written as functions of the action variables,

$$H^2 = J^2 + \left(\frac{3}{4}I_1J^2\right)^{2/3} = \frac{I_2^2}{4\pi^2} + \left(\frac{3I_1I_2^2}{16\pi^2}\right)^{2/3}; \quad J = \frac{I_2}{2\pi}.$$

Define  $\Lambda(m,n)$  as the values of  $H^2$  when the action variables belong to the integer lattice  $(I_1,I_2)=2\pi(m,n), (m,n)\in\mathbb{Z}^2$ . (Since  $I_1>0$ , we consider only m>0.) Then

$$\Lambda(m,n) = n^2 + (3\pi/2)^{2/3} m^{2/3} n^{4/3}.$$

The Bohr-Sommerfeld energy levels of H are defined as the values of H on the lattice, namely  $\sqrt{\Lambda(m,n)}$ .

**Proposition 10.** In any sector of the form  $0 < c_1 \le |n|/m \le c_2$ , the eigenvalues  $\lambda(m,n)$  of the Friedlander operator satisfy

$$\sqrt{\lambda(m,n)} = \sqrt{\Lambda(m,n)} + O(1), \quad (m,n) \to \infty$$

The proposition is an immediate consequence of the formula for  $\mathcal{L}(m,n)$  and the fact that

$$\lambda(m,n) = n^2 + n^{4/3}\tau(m),$$

with  $\tau(m) = (3\pi m/2)^{2/3} + O(1)$ , m = 1, 2, ... (see Proposition 6).

Next, we show that the phases of the eigenfunctions are equal, asymptotically, to the generating function of the canonical relation of the Lagrangian submanifold (27). Indeed, if  $\rho(x,y)$  is the generating function, then by definition, the Lagrangian is the graph of  $(x,y) \mapsto (\rho_x, \rho_y)$ :

$$\rho_x^2 + (1+x)\rho_y^2 = H_0^2; \quad \rho_y = J_0.$$

Thus,

$$\rho_x = \pm \sqrt{H_0^2 - J_0^2 - xJ_0^2}; \quad 0 \le x \le \frac{H_0^2 - J_0^2}{J_0^2},$$

and

$$\rho(x,y) = \pm \frac{2}{3} \frac{(H_0^2 - J_0^2 - xJ_0^2)^{3/2}}{J_0^2} + J_0 y$$

Notice that the upper limit of the range of x is exactly the caustic

$$x = \frac{H_0^2 - J_0^2}{J_0^2},$$

so that the graph over this interval in x (and all y) covers the whole Lagrangian submanifold.

Now let the submanifold be given by  $(I_1, I_2) = 2\pi(m, n)$ . Then

$$H_0^2 = n^2 + \left(\frac{3}{2}\pi mn^2\right)^{3/2}; \quad J_0 = n,$$

and we have two generating functions

$$\rho_{\pm}(x,y) = \pm \frac{2}{3} \left( \left( \frac{3\pi}{2} m \right)^{2/3} - n^{2/3} x \right)^{3/2} + ny.$$

To compare this with the phase of  $\varphi_{m,n}$ , recall that for  $t_m > n^{2/3}x$ ,

$$\varphi_{m,n}(x,y) = \operatorname{Ai}(n^{2/3}x - t_m)e^{iny}$$

$$\sim \frac{1}{2}\pi^{-1/2}|n^{2/3}x - t_m|^{-1/4}\sin\left(\frac{2}{3}(t_m - n^{2/3}x)^{3/2} + \pi/4\right)e^{iny}.$$

(See Proposition 1 and Appendix A.) Thus  $\varphi_{m,n}$  is a sum of two terms with phases

$$\pm \left[ \frac{2}{3} \left( t_m - n^{2/3} x \right)^{3/2} + \pi/4 \right] - \pi/2 + ny = \rho_{\pm}(x, y) + O(1),$$

as  $(m, n) \to \infty$  in sectors of the form  $0 < c_1 < |n|/m < c_2$  (since  $t_m = (3\pi m/2)^{2/3}(1 + O(1/m))$ ).

### Appendix A: Airy functions

An Airy function is a solution y(t) of the differential equation y'' = ty. The classical properties of Airy functions are treated in [AS], Section 10.4 and [H], Section 7.6, and reviewed briefly here.

The Airy function (or Airy integral), denoted Ai, was introduced in 1838 by G.B. Airy in the remarkable paper [Ai] in order to describe the intensity of the light near a caustic. This allowed him to propose a good theory for the rainbow. He defined the function Ai by the oscillatory integral

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(sx+x^3/3)} dx$$
 (28)

In other words, Ai is the inverse Fourier transform of  $x \mapsto e^{ix^3/3}$ .

The Airy function Ai(t) is positive for  $t \geq 0$  and decays exponentially as  $t \to +\infty$ ; it is oscillating for t < 0 with an expansion, given by the method of stationary phase, of the form

$$\operatorname{Ai}(t) = \pi^{-1/2} |t|^{-1/4} \left[ \sin \left( \zeta + \frac{\pi}{4} \right) + \operatorname{Re} \left( e^{i\zeta} \alpha(\zeta) \right) \right], \quad t < 0,$$

with  $\zeta = \frac{2}{3}|t|^{3/2}$  and  $\alpha$  a classical symbol of order -1. Differentiating, we find

$$\operatorname{Ai}'(t) = -\pi^{-\frac{1}{2}} |t|^{1/4} \left[ \cos \left( \zeta + \frac{\pi}{4} \right) + \operatorname{Re} \left( e^{i\zeta} \beta(\zeta) \right) \right], \quad t < 0,$$

with  $\beta$  a classical symbol of order -1.

The function Ai has infinitely many real negative zeros

$$\cdots < -t_m < \cdots < -t_1 < 0.$$

To derive the asymptotic formula for  $t_m$ , denote

$$f(s) = is^{1/3} \text{Ai}(-s^{2/3}) - \text{Ai}'(-s^{2/3}), \quad s > 0.$$

The foregoing asymptotic expansions can be written

$$f(s) = \frac{1}{\sqrt{\pi}} s^{1/6} e^{i(2s/3 + \pi/4) + c(s)},$$

where c is a complex valued classical symbol of order -1. Define

$$\theta(s) = 2s/3 + \pi/4 + \text{Im } c(s) = \arg f(s),$$

Im f(s) > 0 for all  $0 < s < t_1^{3/2}$ , and specify the branch of  $\theta$  by  $0 < \theta(s) < \pi$  in that range. Thus  $\theta$  is a classical symbol of order 1 with principal part 2s/3 as  $s \to \infty$ , and

$$\tan \theta(s) = -\frac{s^{1/3} \operatorname{Ai}(-s^{2/3})}{\operatorname{Ai}'(-s^{2/3})}, \quad s > 0.$$
 (29)

**Proposition 11.** Let  $\theta$  be as above. Then  $\theta'(s) > 0$  for all s > 1/4, and there is an absolute constant  $c_1 < 1$  such that

$$\tau(\xi) = [\theta^{-1}(\pi\xi)]^{2/3} > 0, \text{ for } \xi \ge c_1,$$

is well defined;  $\tau \in S_{cl}^{2/3}(\mathbb{R}_+)$  with principal term  $(3\pi\xi/2)^{2/3}$ . Finally,

$$t_m = \tau(m), \quad m = 1, 2, \dots,$$

where  $0 > -t_1 > -t_2 > \cdots$  are the zeros of Ai.

Proof.

$$\theta'(s) = (1/3|f'|^2)[2s^{2/3}a^2 + 2b^2 - s^{-2/3}ab], \quad a = Ai(-s^{2/3}), \quad b = Ai'(-s^{2/3}).$$

Therefore,  $\theta'(s) > 0$  for s > 1/4. Because  $\theta$  is increasing, the zeros of Ai are given by

$$\theta(t_m^{3/2}) = m\pi, \quad m = 1, 2, \dots$$

Let  $\theta(1/4) = \theta_0$ .  $0 < \theta_0 < \theta(t_1^{3/2}) = \pi$  because  $t_1^{3/2} \approx (2.33)^{3/2} > 1/4$ . The inverse function  $\sigma$  of  $\theta$  is a classical symbol of order 1 on  $[\theta_0, \infty)$ . Then  $\tau(\xi) = \sigma(\pi\xi)^{2/3}$  is defined for all  $\xi \geq \theta_0/\pi$  and satisfies all the properties of the proposition.

Now we turn to proof of Proposition 1 describing the spectrum of the Friedlander operator L. Begin by noting that if L is restricted to functions f(x) of x alone, it becomes

$$Lf(x) = -f''(x),$$

which has completely continuous spectrum on  $L^2([0,\infty))$ . Next, consider  $\varphi$  in the orthogonal complement,

$$\{\varphi \in L^2(M): \int_{\mathbb{R}/2\pi\mathbb{Z}} \varphi(x,y) \, dy = 0\}$$

By separation of variables, the eigenfunctions are of the form  $f(x)e^{iny}$ , where n is a non-zero integer, and f satisfies

$$L_n f := (-\partial_x^2 + n^2(1+x))f = -\lambda f \tag{30}$$

For  $n \neq 0$ , the operator  $L_n$  acting on functions satisfying f(0) = 0 is self-adjoint and has compact resolvent, so a complete system of eigenfunctions is obtained by solving (30) for  $\lambda$  and f such that f(0) = 0 and  $f \in L^2([0, \infty))$ . Making the change of variables

$$f(x) = A(n^{2/3}x - t),$$

equation (30) becomes

$$n^{4/3}(-A''(s) + sA(s)) = (\lambda - n^{4/3}t - n^2)A(s) .$$

Choose t so that  $\lambda = n^2 + n^{4/3}t$ , then the equation is the Airy equation

$$-A''(s) + sA(s) = 0 (31)$$

Since f is in  $L^2([0,\infty))$ , the same is true of A. Up to a constant multiple, there is a unique such function A, the Airy function Ai. The Dirichlet condition f(0) = 0 implies  $\operatorname{Ai}(-t) = 0$ , and the roots of this equation are  $0 < t_1 < t_2 < \cdots$ . Hence the formulas in the proposition for the eigenfunctions and eigenvalues are confirmed.

# Appendix B: Semiclassical properties of eigenfunctions and eigenvalues for completely integrable systems

Let X be an n-dimensional manifold and  $P_1, \ldots, P_n$  be commuting self-adjoint zero order semiclassical pseudodifferential operators with leading symbols  $p_1, \ldots, p_n$ . In Section 5 we considered the example  $X = \mathbb{R}^+ \times S^1$ ,

$$P_1 = \hbar^2 (\partial_x^2 + (1+x)\partial_y^2); \quad P_2 = i\hbar \partial_y.$$

(These operators are of order zero with respect to  $\partial_x$ ,  $\partial_y$  and multiplication by  $\hbar$ .)

We seek the joint eigenfunctions  $\varphi(x)$ ,  $x \in X$ , of  $P_j$  in the form

$$\varphi(x) = a(x,h)e^{i\rho(x)/h}, \quad a(x,h) = a_0(x) + ha_1(x) + h^2a_2(x) + \cdots, \quad (32)$$

solving asymptotically as  $\hbar \to 0$ 

$$P_i \varphi = \lambda_i(h) \varphi, \quad j = 1, \dots, n, \qquad \lambda_i(h) = \lambda_i(0) + h \lambda_{i,1} + h^2 \lambda_{i,2} + \dots$$

Let  $\Lambda_{\rho}$  be the graph of  $d\rho$  in  $T^*X$ .  $\Lambda_{\rho}$  is a Lagrangian submanifold and determines  $\rho$  up to an additive constant;  $\rho$  is known as the *generating function* of  $\Lambda_{\rho}$ . Setting  $\hbar = 0$ , one finds the *eikonal equation*, the first of the heirarchy of equations for a,  $\rho$ , and  $\lambda_{i}(h)$ :

$$p_j|_{\Lambda_\rho} = \lambda_j(0) \,. \tag{33}$$

Let us confine ourselves to the sequence  $\hbar = 1/N$  for integers  $N \to \infty$ . Then (32) defines  $\varphi$  provided  $\rho$  is well-defined modulo  $2\pi\mathbb{Z}$ . To see what this entails, denote by  $\kappa$  the projection,  $\Lambda_{\rho} \to X$ , and by  $\iota$  the inclusion,  $\Lambda_{\rho} \to T^*X$ , and denote

$$\alpha = \sum_{j=1}^{n} \xi_j dx_j \,,$$

the canonical 1-form on  $T^*X$ . The fact that  $\Lambda_{\rho}$  is the graph of  $d\varphi$  implies

$$d\kappa^* \rho = \iota^* \alpha \,. \tag{34}$$

If  $(\lambda_1(0), \ldots, \lambda_n(0))$  is a regular value of  $(p_1, \ldots, p_n)$  and  $\Lambda_{\rho}$  is compact, then  $\Lambda_{\rho}$  is a torus of dimension n. Thus if  $\gamma_1, \ldots, \gamma_n$  is a basis of the first homology group of  $\Lambda_{\rho}$ , then (34) implies

**Proposition 12.**  $\rho$  is well-defined modulo  $2\pi\mathbb{Z}$  if and only if

$$I_j := \int_{\gamma_j} \alpha \in 2\pi \mathbb{Z}, \quad j = 1, 2, \dots, n.$$

The functions  $I_j$  are known as action coordinates on  $T^*X$ . Let  $\mathbf{m} \in \mathbb{Z}^n$ . In the generic case in which

$$(I_1,\ldots,I_n)=2\pi\mathbf{m}$$

defines an *n*-dimensional Lagrangian torus in  $T^*X$ , the torus is known as a Bohr-Sommerfeld surface, and we will denote it by  $S(\mathbf{m})$ .

Proposition 12 says that the generating function  $\rho_{\mathbf{m}}$  of  $S(\mathbf{m})$  is well-defined modulo  $2\pi\mathbb{Z}$ . Thus one can follow the ansatz (32) to seek a joint eigenfunction  $\varphi$  whose phase function  $\rho$  equals  $\rho_{\mathbf{m}}$  to first order as  $\hbar \to 0$  and whose eigenvalue  $\lambda_j$  with respect  $P_j$  is given to first order by  $p_j$  evaluated on  $S(\mathbf{m})$  as in (33). Under suitable further hypotheses, all but finitely many joint eigenfunctions are obtained in this way. Typically,  $\varphi$  and  $S(\mathbf{m})$  exist for a cone of values of  $\mathbf{m} \in \mathbb{Z}^n$  (see [CdV1]).

#### References

- [AS] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematics Series 55, 10th edition (1972).
- [Ai] G.B. Airy, On the Intensity of Light in the neighborhood of a Caustic. Transactions of the Cambridge Philosophical Society 6:379–402 (1838).
- [AM] K. G. Andersson and R. B. Melrose, The propagation of singularities along gliding rays, *Invent. Math.* **41** (1977) no. 3, 97–232.
- [Ch] J. Chazarain, Formule de Poisson pour les variétés riemanniennes. *Inventiones Math.* **24**: 65–82 (1974).
- [CdV0] Y. Colin de Verdière, Spectre du laplacien et longueurs des géodésiques périodiques. *Compositio Math.* **27**:159–184 (1973).
- [CdV1] Y. Colin de Verdière, Spectre joint d'opérateurs pseudo-différentiels qui commutent II. Le cas intégrable. *Math. Zeitschrift* **171**:51–73 (1980).
- [CGJ] Y. Colin de Verdière, V. Guillemin and D. Jerison, Singularities of the wave trace near cluster points of the length spectrum. arXiv:1101.0099v1 (2011).
- [DG] H. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic geodesics. *Inventiones Math.* **29**:39–79 (1975).

- [F] F.G. Friedlander, The wave front set of the solution of a simple initial boundary value problem with glancing rays. *Math. Proc. Cambridge Phil. Soc.* **79**:145–159 (1976).
- [GM] V. Guillemin and R. Melrose, The Poisson summation formula for manifolds with boundary. *Advances Math.* **32**:204–232 (1979).
- [H] L. Hörmander, The Analysis of Partial Differential Operators I, 2nd ed., Springer Verlag (1990).