A GEOMETRIC INSTABILITY OF THE LAMINAR AXISYMMETRIC EULER FLOWS WITH OSCILLATING FLUX

TSUYOSHI YONEDA

ABSTRACT. The dynamics along the particle trajectories for the 3D axisymmetric Euler equations in an infinite cylinder are considered. It is shown that if the inflow-outflow is highly oscillating in time, the corresponding Euler flow cannot keep the uniformly smooth laminar profile provided that the swirling component is not zero. In the proof, Frenet-Serret formulas and orthonormal moving frame are essentially used.

1. INTRODUCTION

We study the dynamics along the particle trajectories for the 3D axisymmetric Euler equations. Such Lagrangian dynamics have already been studied in mathematics (see [1, 2, 3]). For example, in [2], Chae considered a blow-up problem for the axisymmetric 3D incompressible Euler equations with swirl. More precisely, he showed that under some assumption of local minima for the pressure on the axis of symmetry with respect to the radial variations along some particle trajectory, the solution blows up in finite time. Although the blowup problem of 3D Euler equation is still an outstanding open problem, in this paper, we focus on a different problem in physics, especially, the cardiovascular system [5]. If the blood flow is in large and medium sized vessels, the flow is governed by the usual incompressible Navier-Stokes equations. In this paper we focus on behavior of the interior flow, thus it is reasonable to use a simpler model: the 3D axisymmetric Euler flow in an infinite cylinder $\Omega := \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} < 1, \ x \in \mathbb{R}\}$. The configuration of the boundary is not important anymore, thus the setting Ω is just for simplicity. The incompressible Euler equations are expresses as follows:

$$(1.1) \quad \partial_t u + (u \cdot \nabla)u = -\nabla \pi, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega,$$
$$u|_{t=0} = u_0, \quad u \cdot n = 0 \quad \text{on} \quad \partial\Omega, \quad u(x,t) \to (0,0,g(t)) \quad (x_3 \to \pm\infty)$$

with $u = u(x,t) = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t))$, $\pi = \pi(x,t)$ and an uniform inflow-outflow condition $g = g(t)$ (the uniform setting is just for simplicity, we can easily generalize it), where n is a unit normal vector on the boundary. We show a geometric instability of the laminar profile (more precisely, "non-uniformly smooth laminar profile" which will be defined rigorously later) when the uniform inflow-outflow is highly oscillating in time, more precisely,

$$(1.2) \qquad g(t) = 2 + (1-t)^{\beta_1} \sin \left((1-t)^{-\beta_2} \right) \quad (\beta_1, \beta_2 > 0).$$

Note that $u = (0, 0, g)$ in $\Omega \times [0, 1)$ is one of the solution to (1.1). Throughout this paper we assume existence of a unique smooth solution to (1.1) in $t \in [0, 1)$. If there is no unique smooth solution in $t \in [0, 1)$ for some initial data, then we can regard it as one of the instability. The above inflow-outflow has the following property: for any $\epsilon > 0$, there is a time interval I such that

$$(1.3) \qquad 1 < \frac{\epsilon g'(t)^2}{g(t)} < \epsilon^2 g''(t) \quad \text{for} \quad t \in I \subset [1 - \epsilon, 1).$$

Throughout this paper we always focus on the flow behavior in I to extract a dominant term. In this case, notations "$\approx$" and "$\lesssim$" are convenient. The notation "$a \approx b$" means there is a positive constant $C > 0$ such that

$$C^{-1} a \leq b \leq Cb,$$

and "$a \lesssim 1$" means that there is a positive constant $C > 0$ such that

$$0 \leq a \leq C.$$

Remark 1.1. If $g(t) = (1 - t)^{-\beta}$ $(\beta > 0)$, then it does not satisfy (1.3). Thus the oscillating flux setting might be essential.

This flow setting arises from a reduced cardiovascular 1D model [5, Section 10]. To obtain the reduced model, we need to assume the flow is always unilateral laminar flow, especially, for $D := \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} < 1\}$, the axis direction of the flow u_3 is assumed to satisfy

$$(1.4) \qquad \int_D u_3(x_1, x_2, x_3, t)^2 dx_1 dx_2 = \alpha \left(\int_D u_3(x_1, x_2, x_3, t) dx_1 dx_2 \right)^2$$

for some positive constant $\alpha > 0$ (see [5, (10.18)]). However, in this setting, it is not clear whether or not such condition (1.4) is always valid. For example, if the flow is not unilateral, containing the reverse flow, then α may become infinity. In this paper we show all axisymmetric Euler flows with swirl component have non-uniformly smooth laminar profile (possibly, turbulent transition) when these corresponding inflow-outflow are highly oscillating in time. These non-uniformly smooth laminar profiles suggest us that we may need to construct more suitable cardiovascular 1D model with which such non-uniformly smooth laminar profiles are more involved.

Since we consider the axisymmetric Euler flow, we can simplify the Euler equations (1.1). Let $e_r := x_h/|x_h|$, $e_\theta := x_h^\perp/|x_h|$ and $e_z = (0, 0, 1)$ with $x_h = (x_1, x_2, 0)$, $x_h^\perp = (-x_2, x_1, 0)$. The vector valued function u can be rewritten as $u = v_r e_r + v_\theta e_\theta + v_z e_z$, where $v_r = v_r(r, z, t)$, $v_\theta = v_\theta(r, z, t)$ and $v_z = v_z(r, z, t)$ with $r = |x_h|$, $z = x_3$. Then the axisymmetric Euler equations can be expressed as follows:

$$(1.5) \qquad \partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r - \frac{v_\theta^2}{r} + \partial_r p = 0,$$

$$(1.6) \qquad \partial_t v_\theta + v_r \partial_r v_\theta + v_z \partial_z v_\theta + \frac{v_r v_\theta}{r} = 0,$$

$$(1.7) \qquad \partial_t v_z + v_r \partial_r v_z + v_z \partial_z v_z + \partial_z p = 0,$$

$$(1.8) \qquad \frac{\partial_r(r v_r)}{r} + \partial_z v_z = 0.$$

In order to state "uniformly smooth laminar profiles" rigorously, first we need to give several definitions.

Definition 1.2. We call "unilateral flow" iff $v_z = u \cdot e_z > 0$ in Ω.

Definition 1.3. (Axis-length streamline in z.) For a unilateral flow, we can define an axis-length streamline $\gamma(z)$. Let t be fixed, and let $\gamma(z)$ be such that

$$\gamma(\bar{r}_0, z, t) = \gamma(z) := (\bar{R}(z) \cos \bar{\Theta}(z), \bar{R}(z) \sin \bar{\Theta}(z), z)$$

with $\bar{R}(z) = \bar{R}(\bar{r}_0, z, t)$, $\bar{R}(\bar{r}_0, 0, t) = \bar{r}_0$, $\bar{\Theta}(z) = \bar{\Theta}(z, t)$ and we choose $\bar{R}$ and $\bar{\Theta}$ in order to satisfy

$$\partial_z \gamma(z) = \left(\frac{u}{u \cdot e_z} \right) (\gamma(z), t).$$

We easily see

$$\partial_z \gamma \cdot e_z = 1, \quad \partial_z \gamma \cdot e_r = \partial_z \bar{R} = \frac{u_r}{u_z} \quad \text{and} \quad \partial_z \gamma \cdot e_\theta = \bar{R} \partial_z \Theta = \frac{u_\theta}{u_z}.$$

Since $\partial_{\bar{r}_0} \bar{R} > 0$ (otherwise uniqueness does not hold), we have its inverse $r_0 = \bar{R}^{-1}(r, z, t)$. In order to define "uniformly smooth laminar profiles", we use the regularity of $\bar{R}$ and $\bar{R}^{-1}$ up to three derivatives. Note that the definition of "laminar profile" should come from geometry, thus it seems $\bar{R}$ and $\bar{R}^{-1}$ are the suitable concepts rather than the velocity.

Definition 1.4. (uniformly smooth laminar profile.) Let $\partial = \partial_z$ or $\partial_{\bar{r}_0}$, and let $\bar{\partial} = \partial_z$ or ∂_r. We call " uniformly smooth laminar profile" if and only if $\bar{R}$ and $\bar{R}^{-1}$ satisfy the following

$$\partial_{\bar{r}_0} \bar{R} \approx 1, \quad |\partial^\ell \bar{R}|, \ |\bar{\partial}^\ell \bar{R}^{-1}|, \ |\partial_t \bar{R}^{-1}|, \ |\partial_t \partial_{\bar{r}_0} \bar{R}| \lesssim 1$$

for $t \in [0, 1)$, $\ell = 1, 2, 3$. Later we deal with the curvature and torsion of the particle trajectory, thus it is natural to see up to three derivatives.

Remark 1.5. As we remarked that $u = (0, 0, g)$ in $\Omega \times [0, 1)$ is one of the solution to (1.1). This flow is the typical laminar flow. In this case

$$\partial_{\bar{r}_0} \bar{R} = \partial_r \bar{R}^{-1} = 1, \quad \partial^\ell \partial_z \bar{R} = \bar{\partial}^\ell \partial_z \bar{R}^{-1} = \partial_t \bar{R}^{-1} = \partial_t \partial_{\bar{r}_0} \bar{R} = 0$$

for $t \in [0, 1)$, $\ell = 0, 1, 2$.

Now we define the particle trajectory. The associated Lagrangian flow $\eta(t)$ is a solution of the initial value problem

$$\text{(1.9)} \qquad\qquad \frac{d}{dt} \eta(x, t) = u(\eta(x, t), t),$$

$$\text{(1.10)} \qquad\qquad \eta(x, 0) = x.$$

Now we give the main theorem.

Theorem 1.6. *Let $X(0)$ be such that*

$$X(0) := \{ x \in \Omega : u_0(x) \cdot e_\theta \neq 0 \}$$

and $X(t)$ be

$$X(t) := \{ \eta(x, t) \in \Omega : x \in X(0) \}.$$

Assume there is a unique smooth solution to the Euler equations (1.1) *in $\Omega \times [0, 1)$. For any $x \in \Omega(0)$, then at least, either of the following two cases must happen:*

- *Its corresponding laminar profile at $\eta(x, t) \in \Omega(t)$ is not uniformly smooth,*
- *The particle $\eta(x, t) \in \Omega(t)$ touches the axis in $t \in (0, 1]$.*

In the next section, we prove the main theorem.

2. PROOF OF THE MAIN THEOREM.

In order to prove the main theorem, we define the Lagrangian flow along r,z-direction. Let

$$\text{(2.1)} \qquad \frac{d}{dt} Z(t) = v_z(R(t), Z(t), t),$$
$$Z(0) = z_0$$

and

$$\text{(2.2)} \qquad \frac{d}{dt} R(t) = v_r(R(t), Z(t), t),$$
$$R(0) = r_0$$

with $Z(t) = Z(r_0, z_0, t)$ and $R(t) = R(r_0, z_0, t)$. Assume the axisymmetric smooth Euler flow has "uniformly smooth laminar profile" and "particles never touch the axis". Rigorously, "particles never touch the axis" means that R satisfies the following: For any $\tilde{r}_0 > 0$, there is $C > 0$ such that

$$R(t) > C \quad \text{for} \quad r_0 > \tilde{r}_0 \quad \text{and} \quad t \in [0, 1).$$

First we express v_z and v_r by using $\bar{R}$ and $\bar{R}^{-1}$. To do so, we define the cross section of the stream-tube (annulus). Let $B_{-\infty}(\bar{r}_0) = \{x \in \mathbb{R}^3 : |x_h| < \bar{r}_0, \ x_3 = -\infty\}$ and let

$$A(\bar{r}_0, z, \epsilon, t) := \bigcup_{x \in B_{-\infty}(\bar{r}_0 + \epsilon) \backslash B_{-\infty}(\bar{r}_0)} \gamma(x, z, t).$$

We see that its measure is

$$|A(\bar{r}_0, z, \epsilon, t)| = \pi \left(\bar{R}(\bar{r}_0, \epsilon, z, t)^2 - \bar{R}(\bar{r}_0, z, t)^2 \right).$$

Definition 2.1. (Inflow propagation.) Let ρ be such that

$$\rho(\bar{r}_0, z, t) := \lim_{\epsilon \to 0} \frac{|A(\bar{r}_0, -\infty, \epsilon, t)|}{|A(\bar{r}_0, z, \epsilon, t)|}.$$

We see that

$$\rho(\bar{r}_0, z, t) = \frac{\partial_{\bar{r}_0} \bar{R}(\bar{r}_0, -\infty, t) \bar{R}(\bar{r}_0, -\infty, t)}{\partial_{\bar{r}_0} \bar{R}(\bar{r}_0, z, t) \bar{R}(\bar{r}_0, z, t)} = \frac{\bar{r}_0}{\partial_{\bar{r}_0} \bar{R}(\bar{r}_0, z, t) \bar{R}(\bar{r}_0, z, t)} = \frac{2\bar{r}_0}{\partial_{\bar{r}_0} \bar{R}(\bar{r}_0, z, t)^2}.$$

Remark 2.2. If the laminar profile is uniformly smooth, then we have the estimates of the inflow propagation ρ:

$$\rho \approx 1, \ |\partial_z \rho|, |\partial_z^2 \rho|, |\partial_{\bar{r}_0} \rho|, |\partial_{\bar{r}_0}^2 \rho| \lesssim 1 \quad \text{for} \quad t \in [0, 1).$$

Since

$$2\pi \int_{\bar{R}(\bar{r}_0, z, t)}^{\bar{R}(\bar{r}_0 + \epsilon, z, t)} u_z(r', z, t) r' dr' = 2\pi \int_{\bar{r}_0}^{\bar{r}_0 + \epsilon} u_z(r', -\infty, t) r' dr'$$

by divergence free and Gauss's divergence theorem, we can figure out v_z by using the inflow propagation ρ,

$$
\begin{aligned}
v_z(r, z, t) &= \lim_{\epsilon \to 0} \frac{2\pi}{|A(\bar{r}_0, z, \epsilon, t)|} \int_{\bar{R}(\bar{r}_0, z, t)}^{\bar{R}(\bar{r}_0 + \epsilon, z, t)} v_z(r', z, t) r' dr' \\
&= \lim_{\epsilon \to 0} \frac{|A(\bar{r}_0, -\infty, \epsilon, t)|}{|A(\bar{r}_0, z, \epsilon, t)|} \frac{2\pi}{|A(\bar{r}_0, -\infty, \epsilon, t)|} \int_{\bar{r}_0}^{\bar{r}_0 + \epsilon} v_z(r', -\infty, t) r' dr' \\
&= \rho(\bar{r}_0, z, t) u_z(\bar{r}_0, -\infty, t).
\end{aligned}
$$

Thus we have the following proposition.

Proposition 2.3. *We have the following formula of v_z and v_r:*

(2.3) $\qquad v_z(r,z,t) = \rho(\bar{R}^{-1}(r,z,t),z,t)u_z(\bar{R}^{-1}(r,z,t),0,t) = \rho(\bar{R}^{-1},z,t)g(t)$

and

(2.4) $\qquad\qquad v_r(r,z,t) = (\partial_z \bar{R})(\bar{R}^{-1}(r,z,t),z,t)u_z(r,z,t).$

By the above proposition, we see

$$|v_z|,\ |v_r| \lesssim 1.$$

Since $v_z > 0$, then we can define the inverse of Z in t: $t = Z_t^{-1}(z,r_0,z_0)$. In this case we can estimate $\partial_z Z_t^{-1} = 1/\partial_t Z = 1/v_z \approx 1/g(t)$ and $\partial_z^2 Z_t^{-1} = -\frac{\partial_z v_z}{v_z^2}$. First we show the following estimates.

Lemma 2.4. *For $t \in I$, we have the following estimates along the axis-length trajectory:*

(2.5) $\qquad \partial_z v_z(R(Z_t^{-1}(z)),z,Z_t^{-1}(z)) \approx g'(t)/g(t),$
$\qquad\qquad \partial_z^2 Z_t^{-1} \approx -g'(t)/g(t)^3,$
$\qquad\qquad \partial_z^2 v_z(R(Z_t^{-1}(z)),z,Z_t^{-1}(z)) \approx g''(t)/g(t)^2,$
$\qquad\qquad |\partial_z v_r(R(Z_t^{-1}(z)),z,Z_t^{-1}(z))| \lesssim g'(t)/g(t),$
$\qquad\qquad |\partial_z^2 v_r(R(Z_t^{-1}(z)),z,Z_t^{-1}(z))| \lesssim g''(t)/g(t)^2 \quad \text{with} \quad t = Z_t^{-1}(z).$

Moreover, we have

(2.6) $\qquad\qquad v_\theta(R(Z_t^{-1}(z)),z,Z_t^{-1}(z)) \approx 1,$

(it is reasonable to assume $v_\theta(r_0,z_0,0) > 0$)

(2.7) $\qquad |\partial_z v_\theta(R(Z_t^{-1}(z)),z,Z_t^{-1}(z))| \lesssim 1,$
$\qquad\qquad |\partial_z^2 v_\theta(R(Z_t^{-1}(z)),z,Z_t^{-1}(z))| \lesssim g'(t)/g(t) \quad \text{with} \quad t = Z_t^{-1}(z)$

and

$$\partial_t |u(\eta(x,t),t)| \approx g'(t).$$

Proof. First we consider a non-incompressible 2D-flow composed by R and Z. Let us denote $\eta_{2D} = \eta_{2D}(t) = (R(t),Z(t))$ and $D\eta_{2D}$ be its Lagrangian deformation:

$$D\eta_{2D} = \begin{pmatrix} \partial_{r_0}R & \partial_{z_0}R \\ \partial_{r_0}Z & \partial_{z_0}Z \end{pmatrix}.$$

We see $\det(D\eta_{2D}) = \partial_{r_0}R\partial_{z_0}Z - \partial_{z_0}R\partial_{r_0}Z$ and thus we have

$$D(\eta_{2D}^{-1}) = (D\eta_{2D})^{-1} = \frac{1}{\det D\eta_{2D}}\begin{pmatrix} \partial_{z_0}Z & -\partial_{z_0}R \\ -\partial_{r_0}Z & \partial_{r_0}R \end{pmatrix}.$$

A direct calculation with (1.8), (2.1) and (2.2) yields

$$\frac{d}{dt}(\det D\eta_{2D}) = (\partial_r v_r + \partial_z v_z)(\det D\eta_{2D}) = -\frac{v_r}{R(t)}(\det D\eta_{2D}).$$

Thus

$$\det D\eta_{2D}(t) = \det D\eta_{2D}(0)\exp\left\{-\int_0^t \frac{v_r(R(\tau),Z(\tau),\tau)}{R(\tau)}d\tau\right\}.$$

Since $|v_r| \lesssim 1$ and $R(t) \geq C$, we have

$$\det D\eta_{2D} \approx 1$$

by Gronwall's equality. Recall the inflow propagation $v_z(\bar{R}^{-1}(R(t), Z(t), t), -\infty, t) = g(t)$. Then we see

$$(2.8) \qquad v_z(R(t), Z(t), t) = \rho(\bar{R}^{-1}(R(t), Z(t), t), Z(t), t)g(t) = \partial_t Z(t)$$

and

$$(2.9) \qquad (\partial_z \bar{R})(\bar{R}^{-1}(R, Z, t), Z, t)\rho(\bar{R}^{-1}(R, Z, t), Z, t)g(t) = v_r(R, Z, t) = \partial_t R.$$

Since we have already controlled $\det D\eta_{2D}$, here we estimate $\partial_{r_0} R$, $\partial_{r_0} Z$, $\partial_{z_0} R$ and $\partial_{z_0} Z$ respectively. We see the following estimates of Lagrangian deformation:

$$\partial_t \partial_{z_0} Z(t) = \left[\partial_{z_0} R \partial_{\bar{r}_0} \rho \partial_r \bar{R}^{-1} + \partial_{z_0} Z \partial_{\bar{r}_0} \rho \partial_z \bar{R}^{-1} + \partial_{z_0} Z \partial_z \rho \right] g(t)$$

$$\partial_t \partial_{z_0} R(t) = \left[\partial_{z_0} \partial_z \bar{R} \partial_r \bar{R}^{-1} \partial_{z_0} R + \partial_{r_0} \partial_z \bar{R} \partial_z \bar{R}^{-1} \partial_{z_0} Z + \partial_z^2 \bar{R} \partial_{z_0} Z \right] g(t) + (v_z \text{ part}).$$

Then we can construct a Gronwall's inequality of $|\partial_{z_0} Z| + |\partial_{z_0} R|$ and then we can control each $|\partial_{z_0} Z|$ and $|\partial_{z_0} R|$ (here we use "uniformly smooth laminar profile"). By the same calculation, we can also control $|\partial_{r_0} Z|$ and $|\partial_{r_0} R|$. This means

$$(2.10) \qquad |\partial_{z_0} Z|, \ |\partial_{z_0} R|, \ |\partial_{r_0} Z|, \ |\partial_{r_0} R| \lesssim 1.$$

By the same argument again, we have

$$(2.11) \qquad |\partial^2 Z|, \ |\partial^2 R| \lesssim 1$$

with $\partial = \partial_{r_0}$ or ∂_{z_0}. By (2.10) and (2.11), we can estimate v_r and v_z along the particle trajectory in z-valuable (note that the dominant terms are always g with derivatives). Thus we obtain (2.5). Now we control v_θ by using (2.5). By (1.6) we see that

$$\partial_t v_\theta(R(t), Z(t), t) = v_\theta(r_0, z_0, 0) - \frac{v_r(R(t), Z(t), t)v_\theta(R(t), Z(t), t)}{R(t)}.$$

Applying the Gronwall equality, we see
(2.12)

$$v_\theta(R^{-1}(Z_t^{-1}(z)), z, Z_t^{-1}(z)) = v_\theta(r_0, z_0, 0) \exp\left\{ -\int_0^{Z_t^{-1}(z)} \frac{v_r(R(\tau), Z(\tau), \tau)}{R(\tau)} d\tau \right\}.$$

Since $|v_r| \lesssim 1$, we have (2.6). Just taking derivatives to (2.12) in z-valuable, then we also have (2.7). Now we estimate $\partial_t |u(\eta, t)|$. Recall the usual trajectory $\eta(x, t) = (R(t) \cos \Theta(t), R(t) \sin \Theta(t), Z(t))$,
$e_\theta = (-\sin \Theta(t), \cos \Theta(t), 0)$ and $e_r = (\cos \Theta(t), \sin \Theta(t), 0)$. Then, by a direct calculation with $u = v_r e_r + v_\theta e_\theta + v_z e_z$, we see that

$$\frac{1}{2}\partial_t |u(\eta(x, t), t)|^2 = \partial_t u \cdot u = \partial_t v_r v_r + \partial_t v_\theta v_\theta + \partial_t v_z v_z$$

along the trajectory. Just take a time derivative to v_z along the trajectory, then we have

$$\begin{aligned} \partial_t(v_z(R(t), Z(t), t)) \ &= \ \partial_{\bar{r}_0} \rho \partial_t \bar{R}^{-1} g(t) + \partial_{\bar{r}_0} \rho \partial_z \bar{R}^{-1} \partial_t Z g + \partial_{\bar{r}_0} \rho \partial_r \bar{R}^{-1} \partial_t R g \\ &\quad + \partial_z \rho \partial_t Z \rho + \partial_t \rho g + \rho g'. \end{aligned}$$

Thus

$$\partial_t v_z v_z \approx \rho^2 g'(t)g(t) \quad \text{for} \quad t \in I.$$

By the similar calculation,

$$\partial_t v_r v_r \lesssim (\partial_z \bar{R})^2 \rho^2 g'(t) g(t) \quad \text{for} \quad t \in I.$$

Clearly $\partial_t v_\theta v_\theta$ is not large anymore. Thus we have

$$\partial_t |u(\eta(x,t),t)|^2 \approx g'(t) g(t) \quad \text{for} \quad t \in I$$

and then

$$\partial_t |u(\eta(x,t),t)| \approx g'(t) \quad \text{for} \quad t \in I.$$

$\square$

Now we define the axis-length trajectory $\tilde{\eta}$ in z.

Definition 2.5. (Axis-length trajectory.) Let $\tilde{\eta}$ be such that

$$\tilde{\eta}(z) := (r(z)\cos\theta(z), r(z)\sin\theta(z), z)$$

and we choose $r(z)$ and $\theta(z)$ in order to satisfy $\tilde{\eta}(z) = \eta(x, Z_t^{-1}(z))$.

For $t \in I$, we see

$$\partial_z \tilde{\eta} \cdot e_\theta = \frac{\partial_t \eta \cdot e_\theta}{v_z} = r\theta' = \frac{v_\theta(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))}{v_z(R(Z_t^{-1}(z)), z, Z_t^{-1}(z))} \approx 1/g(t),$$

$$\partial_z \tilde{\eta} \cdot e_r = \frac{v_r}{v_z} = r', \quad |r'| \lesssim 1, \quad |r''| \lesssim g'/g$$

with $t = Z_t^{-1}(z)$. We need the estimates θ'' and θ''' to specify the geometry of the trajectory, in particular, its curvature and torsion. By Lemma 2.4, we can immediately obtain the following proposition.

Proposition 2.6. *For $t \in I$, by Lemma 2.4 we have (just see the highest order term)*

$$\theta''(z) \approx -\frac{v_\theta \partial_z v_z}{v_z^2} \approx -g'(t)/g(t)^3$$

and

$$\theta'''(z) \approx -\frac{v_\theta \partial_z^2 v_z}{v_z^2} + \frac{2v_\theta(\partial_z v_z)^2}{v_z^3} \approx -g''(t)/g(t)^4$$

with $t = Z_t^{-1}(z)$.

From the trajectory $\eta(x,t)$, we define the arc-length trajectory $\eta^*(s) = \eta^*(x,s)$.

Definition 2.7. (Arc-length trajectory.) Let η^* be such that

$$\eta^*(s) := \eta(x, t(s)) \quad \text{and} \quad \eta^*(x,0) = \eta(x,0)$$

with $\partial_s t(s) = |u|^{-1}$.

In this case we see $|\partial_s \eta^*(s)| = 1$. We define the unit tangent vector τ as

$$\tau(s) = \partial_s \eta^*(x,s),$$

the unit curvature vector n as $\kappa n = \partial_s \tau$ with a curvature function $\kappa(s) > 0$, the unit torsion vector b as : $b(s) := \pm\tau(s) \times n(s)$ ($\times$ is an exterior product) with a torsion function to be positive $T(s) > 0$ (once we restrict T to be positive, then the direction of b can be determined), that is,

$$Tb := \partial_s n + \kappa\tau, \quad |b| = 1$$

due to the Frenet-Serret formula.

By the estimates of θ'' and θ''' in Proposition 2.6, we obtain the following lemma.

Lemma 2.8. *For $t \in I$, we have $n \cdot e_\theta \to -1$ $(t \to 1)$, $\partial_s \kappa \approx g''(t)/g(t)^4$ (with $t = t(s)$) and $\partial_s \kappa \gg |\kappa Tb \cdot e_\theta|$.*

Proof. Recall the arc-length trajectory $(z = z(s))$:

$$\eta^*(x,s) = \tilde{\eta}(x,z) = (r(z)\cos\theta(z), r(z)\sin\theta(z), z) \quad \text{with} \quad \theta' > 0.$$

Thus τ and κn are expressed as

$$\tau = (\partial_z \tilde{\eta})z', \quad \kappa n = \partial_s^2 \eta^* = \partial_z^2 \tilde{\eta}(z')^2 + \partial_z \tilde{\eta} z''.$$

We recall that

$$\partial_z \tilde{\eta} \cdot e_z = 1, \quad \partial_z \tilde{\eta} \cdot e_\theta = r\theta' = \frac{u_\theta}{u_z}, \quad \partial_z \tilde{\eta} \cdot e_r = r' = \frac{u_r}{u_z}.$$

Clearly, $r' = \partial_z \bar{R}$ (this essentially links the streamline and the trajectory for fixed t). Near the possible blowup time, we easily see that θ' is small enough and positive. We also see that

$$
\begin{aligned}
\partial_z \tilde{\eta}(x,z) &= (-r\theta'\sin\theta, r\theta'\cos\theta, 1) + (r'\cos\theta, r'\sin\theta, 0) \approx (r'\cos\theta, r'\sin\theta, 1), \\
\partial_z^2 \tilde{\eta}(x,z) &= -r(\theta')^2(\cos\theta, \sin\theta, 0) + (-r\theta''\sin\theta, r\theta''\cos\theta, 0) \\
&\quad + r''(\cos\theta, \sin\theta, 0) + 2r'\theta'(-\sin\theta, \cos\theta, 0) \\
&= r\theta''(-\sin\theta, \cos\theta, 0) + r''(\cos\theta, \sin\theta, 0) + \text{remainder} \\
z'(s) &= (1 + (r')^2 + (r\theta')^2)^{-1/2} = (1 + (r')^2)^{-1/2} + \text{remainder} \\
z''(s) &= -(1 + (r')^2 + (r\theta')^2)^{-2}(r'r'' + r\theta'(r'\theta' + r\theta'')) \\
&= -(1 + (r')^2)^{-2}(r'r'' + r^2\theta'\theta'') + \text{remainder}.
\end{aligned}
$$

r', r'' and r''' can be controlled due to the uniformly smooth laminar profile, and recall that $\theta'' \approx -g'(t)/g(t)^3$. Thus

$$
\begin{aligned}
\kappa^2 &= |\kappa n|^2 = |\partial_z^2 \eta|^2(z')^4 + 2(\partial_z \eta \cdot \partial_z^2 \eta)(z')^2 z'' + |\partial_z \eta|^2(z'')^2 \\
&= \left((r'')^2 + (r\theta'')^2\right)(1 + (r')^2)^{-2} - 2r'r''(1 + (r')^2)^{-3}r^2\theta'\theta'' \\
&\quad + (1 + (r')^2)^{-4}r^4(\theta'\theta'')^2(r')^2 + \text{remainder} \\
&= (r\theta'')^2(1 + (r')^2)^{-2} + \text{remainder} \\
&\approx g'(t)^2/g(t)^6.
\end{aligned}
$$

Also recall that $\theta''' \approx -g''(t)/g(t)^4$. The dominant term of $\partial_s(\kappa^2)$ is composed by θ'' and θ''', more precisely,

$$\partial_s(\kappa^2) = 2(\partial_s \kappa)\kappa = 2r\theta''(r\theta''')(1 + (r')^2)^{-5/2} + \text{remainder} \approx g'(t)g''(t)/g(t)^7.$$

We also see that

$$
\begin{aligned}
\kappa &= |r\theta''|(1 + (r')^2)^{-1} + \text{remainder} \approx g'(t)/g(t)^3, \\
\kappa n \cdot e_\theta &= r\theta''(1 + (r')^2)^{-1} + \text{remainder} \approx -g'(t)/g(t)^3, \\
\partial_s(\kappa n) \cdot e_\theta &= \partial_s^3 \tilde{\eta} \cdot e_\theta = r\theta'''(1 + (r')^2)^{-3/2} + \text{remainder} \approx -g''(t)/g(t)^4, \\
\partial_s \kappa &= \frac{r\theta''(r\theta''')(1 + (r')^2)^{-5/2}}{\kappa} + \text{remainder} \\
&= -r\theta'''(1 + (r')^2)^{-3/2} + \text{remainder} \approx g''(t)/g(t)^4
\end{aligned}
$$

near the blowup time. Thus

$$n \cdot e_\theta = \frac{\kappa n \cdot e_\theta}{\kappa} \to -1 \quad (t \to 1).$$

By the Frenet-Serret formula,

$$Tb = \partial_s n + \kappa\tau,$$

we see that

$$\kappa Tb \cdot e_\theta = \partial_s(\kappa n) \cdot e_\theta - (\partial_s\kappa)n \cdot e_\theta + \kappa^2\tau \cdot e_\theta.$$

Thus, by the direct calculation, we can find a cancellation on the (candidate) highest order term $\theta''' \approx \partial_s\kappa$, namely,

$$|\kappa Tb \cdot e_\theta| \ll |\partial_s\kappa|$$

for $t \in I$. $\qquad\square$

In what follows, we use a differential geometric idea. See Chan-Czubak-Y [4, Section 2.5], more originally, see Ma-Wang [6, (3.7)]. They considered 2D separation phenomena using fundamental differential geometry. The key idea here is "local pressure estimate" on a normal coordinate in $\bar\theta$, $\bar r$ and $\bar z$ valuables. Two derivatives to the scalar function p on the normal coordinate is commutative, namely, $\partial_{\bar r}\partial_{\bar\theta}p(\bar\theta,\bar r,\bar z) - \partial_{\bar\theta}\partial_{\bar r}p(\bar\theta,\bar r,\bar z) = 0$ (Lie bracket). This fundamental observation is the key to extract the local effect of the pressure. For any point $x \in \mathbb{R}^3$ near the arc-length trajectory η^* is uniquely expressed as $x = \eta^*(\bar\theta) + \bar r n(\bar\theta) + \bar z b(\bar\theta)$ with $(\bar\theta,\bar r,\bar z) \in \mathbb{R}^3$ (the meaning of the parameters s and $\bar\theta$ are the same along the arc-length trajectory). Thus we have that

$$
\begin{aligned}
\partial_{\bar\theta}x &= \tau + \bar r(Tb - \kappa\tau) + \bar z\kappa n, \\
\partial_{\bar r}x &= n, \\
\partial_{\bar z}x &= b.
\end{aligned}
$$

This means that

$$
\begin{pmatrix} \partial_{\bar\theta} \\ \partial_{\bar r} \\ \partial_{\bar z} \end{pmatrix} = \begin{pmatrix} 1 - \kappa\bar r & \bar z\kappa & \bar rT \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ n \\ b \end{pmatrix}.
$$

Remark 2.9. For any smooth scalar function f, we have

$$\partial_{\bar\theta}f(x) = \nabla f \cdot \partial_{\bar\theta}x.$$

∇f itself is essentially independent of any coordinates. Thus we can regard a partial derivative as a vector.

By the fundamental calculation, we have the following inverse matrix:

$$
\begin{pmatrix} \tau \\ n \\ b \end{pmatrix} = \begin{pmatrix} (1 - \kappa\bar r)^{-1} & -\bar zT(1 - \kappa\bar r)^{-1} & -\bar rT(1 - \kappa\bar r)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_{\bar\theta} \\ \partial_{\bar r} \\ \partial_{\bar z} \end{pmatrix}.
$$

Therefore we have the following orthonormal moving frame: $\partial_{\bar r} = n$, $\partial_{\bar z} = b$ and

$$(1 - \kappa\bar r)^{-1}\partial_{\bar\theta} - \bar zT(1 - \kappa\bar r)^{-1}\partial_{\bar r} - \bar rT(1 - \kappa\bar r)^{-1}\partial_{\bar z} = \tau.$$

In order to abbreviate the complicated indexes, we re-define the absolute value of the velocity along the trajectory. Let (the indexes are x and t respectively)

$$|u| := |u(\eta(x',t),t)| \quad \text{with} \quad x' = \eta^{-1}(x,t)$$

and

$$\partial_t|u| := \partial_{t'}|u(\eta(x',t'),t')|\Big|_{t'=t} \quad \text{with} \quad x' = \eta^{-1}(x,t).$$

Lemma 2.10. *We see $\nabla p \cdot \tau = \partial_t |u|$ along the trajectory.*

Proof. Let us define a unit tangent vector $\tilde{\tau}$ (in time t') as follows:

$$\tilde{\tau}_{x,t}(t') := \frac{u}{|u|}(\eta(x',t'),t') \quad \text{with} \quad x' = \eta^{-1}(x,t).$$

Note that there is a re-parametrize factor $s(t')$ such that

$$\tau(s(t')) = \tilde{\tau}(t').$$

Since $u \cdot \partial_s \tau = 0$, we see that

$$
\begin{aligned}
\partial_{t'}|u(\eta(x',t'),t')| &= \partial_{t'}(u(\eta(x',t'),t') \cdot \tilde{\tau}_{x,t}(t')) \\
&= \partial_{t'}(u(\eta(x',t'),t')) \cdot \tilde{\tau}_{x,t}(t') + u(\eta(x',t'),t') \cdot \partial_s \tau \partial_{t'} s \\
&= \partial_{t'}(u(\eta(x',t'),t')) \cdot \tilde{\tau}_{x,t}(t').
\end{aligned}
$$

By the above calculation we have

$$\nabla p \cdot \tau = \partial_{t'}(u(\eta(x',t'),t') \cdot \tau|_{t'=t} = \partial_{t'}(u(\eta(x',t'),t') \cdot \tilde{\tau}|_{t'=t} = \partial_{t'}|u|\Big|_{t'=t}.$$

$\square$

Lemma 2.11. *Along the arc-length trajectory, we have*

$$3\kappa \partial_t |u| + \partial_s \kappa |u|^2 = \partial_{\bar{r}} \partial_t |u|$$

and

$$T\kappa |u|^2 = \partial_{\bar{z}} \partial_t |u|.$$

Proof. By using the orthonormal moving frame, we have the following gradient of the pressure,

$$\nabla p = (\partial_\tau p)\tau + (\partial_n p)n + (\partial_b p)b.$$

By the unit tangent vector, we see

$$\partial_s \eta^*(s) = \partial_t \eta \partial_s t = \tau$$

and thus

$$\partial_s t = |u|^{-1}.$$

By the unit normal vector with the curvature constant, we see

$$\partial_s^2 \eta^* = \partial_s(\partial_t \eta \partial_s t) = \partial_t^2 \eta(\partial_s t)^2 + \partial_t \eta \partial_s^2 t = \kappa n.$$

Thus we have

$$
\begin{aligned}
-(\nabla p \cdot n) &= (\partial_t^2 \eta \cdot n) = \kappa |u|^2, \\
-\partial_s(\nabla p \cdot n) &= \partial_s(\kappa(\partial_s t)^{-2}) = \partial_s \kappa(\partial_s t)^{-2} - 2\kappa(\partial_s t)^{-3}(\partial_s^2 t), \\
-\nabla p \cdot \tau &= -|u|^3 \partial_s^2 t, \\
-\nabla p \cdot b &= 0.
\end{aligned}
$$

Recall that

$$\partial_\tau = (1-\kappa\bar{r})^{-1}\partial_{\bar{\theta}} - \bar{z}T(1-\kappa\bar{r})^{-1}\partial_{\bar{r}} - \bar{r}T(1-\kappa\bar{r})^{-1}\partial_{\bar{z}}.$$

Along the arc-length trajectory, we have

$$
\begin{aligned}
-\partial_{\bar{r}}(\nabla p \cdot \tau) &= -\partial_{\bar{r}}\partial_{\tau}p \\
&= -\kappa\partial_{\bar{\theta}}p - \partial_{\bar{r}}\partial_{\bar{\theta}}p - T\partial_{\bar{z}}p \\
(\text{commute } \partial_{\bar{r}} \text{ and } \partial_{\bar{\theta}}) \quad &= -\kappa(\nabla p \cdot \tau) - \partial_{\bar{\theta}}(\nabla p \cdot n) - T(\nabla p \cdot b) \\
&= -\kappa|u|^3\partial_s^2 t + \partial_s\kappa(\partial_s t)^{-2} - 2\kappa(\partial_s t)^{-3}(\partial_s^2 t) \\
&= 3\kappa\partial_t|u| + \partial_s\kappa|u|^2.
\end{aligned}
$$

Since $\nabla p \cdot b = \partial_{\bar{z}}p \equiv 0$ along the trajectory, then

$$
-\partial_{\bar{z}}(\nabla p \cdot \tau)|_{\bar{r},\bar{z}=0} = -\partial_{\bar{z}}\partial_{\bar{\theta}}p - T\partial_{\bar{r}}p = -T(\nabla p \cdot n) = T\kappa|u|^2.
$$

By Lemma 2.10 along the arc-length trajectory η^*, we have

$$
3\kappa\partial_t|u| + \partial_s\kappa|u|^2 = -\partial_{\bar{r}}(\nabla p \cdot \tau)|_{\bar{r},\bar{z}=0} = \partial_{\bar{r}}\partial_t|u|
$$

and

$$
T\kappa|u|^2 = -\partial_{\bar{z}}(\nabla p \cdot \tau)|_{\bar{r},\bar{z}=0} = \partial_{\bar{z}}\partial_t|u|.
$$

$\square$

By using the above lemma we can finally prove the main theorem. Since

$$
\partial_\theta = (e_\theta \cdot n)\partial_{\bar{r}} + (e_\theta \cdot b)\partial_{\bar{z}}
$$

and the axisymmetric flow is rotation invariant,

$$
\begin{aligned}
0 &= \partial_\theta\partial_t|u| = (e_\theta \cdot n)\partial_{\bar{r}}\partial_t|u| + (e_\theta \cdot b)\partial_{\bar{z}}\partial_t|u| \\
&= 3(e_\theta \cdot n)\left(\kappa\partial_t|u| + \partial_s\kappa|u|^2\right) + (e_\theta \cdot b)T\kappa|u|^2.
\end{aligned}
$$

However, $(e_\theta \cdot n)\partial_s\kappa|u|^2 \approx -g''(t)/g(t)^2$ is the dominant term, and it is in contradiction to (1.3), since

$$
(e_\theta \cdot n)\kappa\partial_t|u| \approx -g'(t)^2/g(t)^3 \quad \text{and} \quad |T\kappa| \ll |\partial_s\kappa|.
$$

References

1. D. Chae, *On the Lagrangian dynamics for the 3D incompressible Euler equations*, Comm. Math. Phys. **269**, (2007), 557-569.
2. D. Chae, *On the blow-up problem for the axisymmetric 3D Euler equations*, Nonlinearity, **21**, (2008), 2053-2060.
3. D. Chae, *On the Lagrangian dynamics of the axisymmetric 3D Euler equations*, J. Diff. Eq., **249** (2010), 571-577.
4. C-H. Chan, M. Czubak and T. Yoneda, *An ODE for boundary layer separation on a sphere and a hyperbolic space.* Physica D, **282** (2014), 34-38.
5. L. Formaggia, A. Quarteroni and A. Veneziani, *Cardiovascular mathematics, modeling and simulation of the circulatory system*, Springer-Verlag, Italia, Milano, 2009.
6. T. Ma and S. Wang, *Boundary layer separation and structural bifurcation for 2-D incompressible fluid flows. Partial differential equations and applications.* Discrete Contin. Dyn. Syst. 10 (2004), 459–472.

Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1 Meguro, Tokyo 153-8914, Japan

E-mail address: yoneda@ms.u-tokyo.ac.jp