A FOURTH-ORDER DISPERSIVE FLOW EQUATION FOR CLOSED CURVES ON COMPACT RIEMANN SURFACES

EIJI ONODERA

ABSTRACT. A fourth-order dispersive flow equation for closed curves on the canonical two-dimensional unit sphere arises in some contexts in physics and fluid mechanics. In this paper, a geometric generalization of the sphere-valued model is considered, where the solutions are supposed to take values in compact Riemann surfaces. As a main results, time-local existence and the uniqueness of a solution to the initial value problem is established under the assumption that the sectional curvature of the Riemann surface is constant. The analytic difficulty comes from the so-called loss of derivatives and the absence of the local smoothing effect. The proof is based on the geometric energy method combined with a kind of gauge transformation to eliminate the loss of derivatives. Specifically, to show the uniqueness of the solution, the detailed geometric analysis of the solvable structure for the equation is presented.

1. INTRODUCTION

Dispersive partial differential equations have been extensively studied in mathematical research. Many studies have paid attention to real or complex-valued functions as solutions to these equations. However, some nonlinear dispersive partial differential equations in contexts in classical mechanics and fluid mechanics require their solutions to take values in a (curved) Riemannian manifold. In general, their nonlinear structures depend on the geometric setting of the manifold. Therefore, concerning how to solve their initial value problem, geometric analysis of the relationship between their solvable structure and the geometric setting of the manifold plays an essential role.

In this field, after the pioneering work of Koiso [11], the method of geometric analysis for the so-called one-dimensional Schrödinger flow equation, the higher-dimensional generalization and a third-order analogue has been developed extensively. Many results on how to solve their initial value problem have been established mainly from the following three points of view: analysis of the solvable structure of dispersive partial differential equations(systems), an application of Riemannian geometry, and analysis of nonlinear partial differential equations with physical backgrounds. See, e.g., [2, 3, 4, 5, 10, 11, 13, 14, 15, 16, 17], and references therein. In this paper, we study a fourth-order analogue whose solutions are required to take values in a compact Riemann surface. This is a continuation of [6, 18] and presents the answer to the problem suggested in [4].

The setting of our problem is stated as follows: Given a compact Riemann surface N with the complex structure J and with a hermitian metric g, consider the following initial value problem

$$u_t = a\, J_u \nabla_x^3 u_x + \{\lambda + b\, g(u_x, u_x)\} J_u \nabla_x u_x + c\, g(\nabla_x u_x, u_x) J_u u_x \quad \text{in} \quad \mathbb{R} \times \mathbb{T}, \qquad (1.1)$$
$$u(0, x) = u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \mathbb{T}. \qquad (1.2)$$

Here $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional flat torus, $u = u(t, x) : \mathbb{R} \times \mathbb{T} \to N$ is the unknown map describing the deformation of closed curves lying on N parameterized by t, $u_0 = u_0(x)$:

$\mathbb{T} \to N$ is the given initial map, $u_t = du(\frac{\partial}{\partial t})$, $u_x = du(\frac{\partial}{\partial x})$, du is the differential of the map u, ∇_x is the covariant derivative along u in x, $J_u : T_u N \to T_u N$ is the complex structure at $u \in N$, a, b, c, and λ are real constants. If $a, b, c = 0$ and $\lambda = 1$, (1.1) is reduced to the second-order dispersive equation of the form

$$u_t = J_u \nabla_x u_x, \tag{1.3}$$

which is called a one-dimensional Schrödinger flow equation. As a fourth-order analogue of (1.3), we call (1.1) with $a \neq 0$ a fourth-order dispersive flow equation. Hereafter it is assumed that $a \neq 0$.

An example of (1.1) with $a \neq 0$ arises in two areas of physics, where N is supposed to be the canonical two-dimensional unit sphere $\mathbb{S}^2$. Indeed, if $N = \mathbb{S}^2$ equipped with the complex structure acting as $\pi/2$-degree rotation on each tangent plane and with the canonical metric induced from the Euclidean metric in $\mathbb{R}^3$, (1.1) is described by

$$u_t = u \wedge \left[a\, \partial_x^3 u_x + \{\lambda + (a+b)\,(u_x, u_x)\}\partial_x u_x + (5a+c)\,(\partial_x u_x, u_x)u_x \right], \tag{1.4}$$

where $u : \mathbb{R} \times \mathbb{T} \to \mathbb{S}^2 \subset \mathbb{R}^3$, ∂_x is the partial differential operator in x acing on $\mathbb{R}^3$-valued functions, $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^3$, and $\wedge$ is the exterior product in $\mathbb{R}^3$. In particular, the $\mathbb{S}^2$-valued model (1.4) with $3a - 2b + c = 0$ and $\lambda = 1$ models the continuum limit of the Heisenberg spin chain systems with biquadratic exchange interactions([12]), where each of a, b, c is decided by two independent physical constants. Interestingly, the same equation can be derived from an equation modelling the motion of a vortex filament in an incompressible perfect fluid in $\mathbb{R}^3$ by taking into account of the elliptical deformation effect of the core due to the self-induced strain ([7, 8]).

For the Schrödinger flow equation (1.3) and the higher-dimensional generalization, almost all results on the existence of solutions have been established assuming essentially that (N, J, g) is a compact Kähler manifold. See, e,g, [2, 10, 11, 13, 14, 19] and references therein. Under the assumption, the classical energy method combined with geometric analysis works to show the local existence results. On the other hand, if (N, J, g) is a compact almost hermitian manifold without the Kähler condition, the classical energy method breaks down, since the so-called loss of derivatives occurs from the covariant derivative of the almost complex structure. However, Chihara in [3] overcame the difficulty by the geometric energy method combined with a kind of the gauge transformation acting on the pull-buck bundle. Indeed, he established a local existence and uniqueness result for maps from a compact Riemannian manifold into a compact almost hermitian manifold. After that, he and the author obtained similar results in [5, 16, 17, 18] for a third-order dispersive flow equation for maps from $\mathbb{R}$ or $\mathbb{T}$ into a compact almost hermitian manifold.

In contrast, for our fourth-order dispersive flow equation (1.1), we face with the difficulty due to loss of derivatives even if (N, J, g) satisfies the Kähler condition, which is also the case for the $\mathbb{S}^2$-valued physical model (1.4). If the spacial domain is the real line $\mathbb{R}$ instead of $\mathbb{T}$, the difficulty can be overcome by making use of the local dispersive smoothing effect of the equation in some sense. Besides, there is much room for the solvable structure. Indeed, in [6], the local existence and the uniqueness of a solution to the problem on $\mathbb{R}$ were established and were extended to compact Kähler manifolds as N. Unfortunately, however, the local smoothing effect is absent in our problem since the spacial domain $\mathbb{T}$ is compact. In other words, the method of the proof in [6] is not applicable to our problem. Thus the obstruction coming from the loss of derivatives is expected to be avoided by finding out a kind of special nice solvable structure of the equation.

The previous studies of (1.1) on $\mathbb{T}$ are limited as follows: Guo, Zeng, and Su in [9] investigated the $\mathbb{S}^2$-valued physical model (1.4) with $3a - 2b + c = 0$ and $\lambda = 1$ imposing an additional assumption $c = 0$. Under the assumption, (1.4) is completely integrable, and they made use of

some conservation laws of (1.4) to show the local existence of a weak solution to the initial value problem, though the uniqueness was unsolved. Chihara in [4] investigated fourth-order dispersive systems for $\mathbb{C}^2$-valued functions including a system which is reduced from (1.1) by the generalized Hasimoto transformation, and pointed out that the assumption that the sectional curvature of N is constant provides the solvable structure of the initial value problem. To the present author's knowledge, though the insights seems to grasp the solvable structure of (1.1)-(1.2) essentially, it is nontrivial whether we can recover the solution to (1.1)-(1.2) from the solution to the reduced dispersive system.

Motivated by them, the present author tried to solve directly (1.1)-(1.2) imposing that the sectional curvature on N is constant, without using the generalized Hasimoto transformation. Recently, he in [18] succeeded to show the local existence of a unique solution to the initial value problem for the $\mathbb{S}^2$-valued model (1.4) without any assumption on a, b, c, λ (except for $a \neq 0$), where u_0 is taken so that $u_{0x} \in H^k(\mathbb{T}; \mathbb{R}^3)$ with $k \geqslant 6$. This is proved by the energy method based on the standard Sobolev norm for $\mathbb{R}^3$-valued functions, combined with a kind of gauge transformation.

The purpose of the present paper is to extend the results obtained in [18] for $\mathbb{S}^2$-valued model (1.4), that is, to establish the time-local existence and uniqueness theorem for (1.1)-(1.2) under the assumption that $k \geqslant 6$ and the sectional curvature on (N, g) is constant. More precisely, our main results is stated as follows:

Theorem 1.1. *Suppose that (N, J, g) is a compact Riemann surface whose sectional curvature is constant. Let k be an integer satisfying $k \geqslant 6$. Then for any $u_0 \in C(\mathbb{T}; N)$ satisfying $u_{0x} \in H^k(\mathbb{T}; TN)$, there exists $T = T(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 0$ such that (1.1)-(1.2) has a unique solution $u \in C([-T, T] \times \mathbb{T}; N)$ satisfying $u_x \in C([-T, T]; H^k(\mathbb{T}; TN))$.*

Notation. For $\phi : \mathbb{T} \to N$, we denote by $\Gamma(\phi^{-1}TN)$ the set of all vector fields along ϕ. Let $V \in \Gamma(\phi^{-1}TN)$ and let m be nonnegative integer. Then we say $V \in H^m(\mathbb{T}; TN)$ if

$$\|V\|_{H^m(\mathbb{T};TN)} := \sum_{\ell=0}^{m} \int_{\mathbb{T}} g(\nabla_x^\ell V(x), \nabla_x^\ell V(x)) \, dx < \infty.$$

In particular, if $m = 0$, we replace $H^0(\mathbb{T}; TN)$ with $L^2(\mathbb{T}; TN)$.

Remark 1.2. Precisely speaking, the existence time T of the solution in Theorem 1.1 depends on a, b, c, λ, and the constant sectional curvature of (N, g) as well as $\|u_{0x}\|_{H^4(\mathbb{T};TN)}$.

Remark 1.3. The local existence of the solution in Theorem 1.1 holds if $k \geqslant 4$. The assumption $k \geqslant 6$ comes from the requirement to show the uniqueness.

Remark 1.4. Let w be an isometric embedding of (N, g) into some Euclidean space $\mathbb{R}^d$ so that N is considered as a submanifold of $\mathbb{R}^d$. By the Gagliardo-Nirenberg inequality, it is found for u_0 in Theorem 1.1 that $u_{0x} \in H^k(\mathbb{T}; TN)$ if and only if $(w \circ u_0)_x \in H^k(\mathbb{T}; \mathbb{R}^d)$, where $H^k(\mathbb{T}; \mathbb{R}^d)$ denotes the standard k-th order Sobolev space for $\mathbb{R}^d$-valued functions on $\mathbb{T}$. By the equivalence, Theorem 1.1 actually extends the results obtained in [18].

Remark 1.5. We can extend Theorem 1.1 to the case where (N, J, g) is a compact Kähler manifold with non-zero constant sectional curvature. Indeed, the argument using (2.12) and (3.28) in the proof can be replaced by that using (2.9) if the curvature is not zero. This seems a little bit artificial and the proof is not so different. Thus we do not pursue that.

Remark 1.6. It is unlikely that we can remove the assumption on the curvature of (N, g) in general. To see this, let (N, g) be a Riemann surface whose sectional curvature is not necessarily

constant. In view of [4, Section 4], if we can construct a sufficiently smooth solution u to (1.1)-(1.2), the following necessary condition

$$\int_{\mathbb{T}} \frac{\partial}{\partial x} \{S(u(t,x))\} \, g(u_x(t,x), u_x(t,x)) \, dx = 0 \tag{1.5}$$

is expected to be satisfied for all existence time, where $S(u(t,x))$ denotes the sectional curvature of (N, g) at $u(t,x) \in N$. This requires at least that the left hand side of (1.5) is a conserved quantity in time. Even if (1.5) is true, the initial map u_0 is required to satisfy

$$\int_{\mathbb{T}} \frac{\partial}{\partial x} \{S(u_0(x))\} \, g(u_{0x}(x), u_{0x}(x)) \, dx = 0. \tag{1.6}$$

On the other hand, (1.5) and (1.6) are obviously satisfied if the sectional curvature of (N, g) is constant.

   The idea of the proof of the local existence comes from the following formal observation. Suppose that u solves (1.1)-(1.2). If $k \geqslant 4$, $\nabla_x^k u_x$ satisfies

$$(\nabla_t - a \, J_u \nabla_x^4 - c_1 \, P_1 \nabla_x^2 - c_2 \, P_2 \nabla_x) \nabla_x^k u_x = \mathcal{O}\left(\sum_{m=0}^{k+2} |\nabla_x^m u_x|_g \right) \tag{1.7}$$

where $| \cdot |_g = \{g(\cdot, \cdot)\}^{1/2}$, c_1 and c_2 are real constants depending on a, b, c, k and the sectional curvature on (N, g), and P_1 and P_2 are defined by

$$P_1 Y = g(Y, u_x) J_u u_x, \quad P_2 Y = g(\nabla_x u_x, u_x) J_u Y$$

for any $Y \in \Gamma(u^{-1}TN)$. It is found that (1.7) leads to the classical energy estimate for $\|\nabla_x^k u_x\|_{L^2(\mathbb{T};TN)}^2$ with loss of derivatives coming only from $c_1 \, P_1 \nabla_x^2$ and $c_2 \, P_2 \nabla_x$. Though the right hand side of (1.7) includes $\nabla_x^2(\nabla_x^k u_x)$ and $\nabla_x(\nabla_x^k u_x)$, no loss of derivatives occur thanks to the curvature condition and the Kähler condition on (N, J, g). To eliminate the loss of derivatives coming from $c_1 \, P_1 \nabla_x^2$ and $c_2 \, P_2 \nabla_x$, we introduce the so-called gauged function V_k defined by

$$V_k = \nabla_x^k u_x - \frac{d_1}{2a} g(\nabla_x^{k-2} u_x, J_u u_x) J_u u_x + \frac{d_2}{8a} g(u_x, u_x) \nabla_x^{k-2} u_x, \tag{1.8}$$

where d_1 and d_2 are constants decided later. Here V_k is formally expressed by $V_k = (I_d + \Phi_1 \nabla_x^{-2} + \Phi_2 \nabla_x^{-2}) \nabla_x^k u_x$, where I_d is the identity on $\Gamma(u^{-1}TN)$ and

$$\Phi_1 Y = -\frac{d_1}{2a} g(Y, J_u u_x) J_u u_x, \quad \Phi_2 = \frac{d_2}{8a} g(u_x, u_x) Y$$

for any $Y \in \Gamma(u^{-1}TN)$. Noting that J_u commutes with Φ_2 and not with Φ_1, we see

$$\left[a \, J_u \nabla_x^4, \Phi_1 \nabla_x^{-2} \right] \nabla_x^k u_x = (d_1 \, P_1 \nabla_x^2 - d_1 \, P_2 \nabla_x) \nabla_x^k u_x + \text{harmless terms}, \tag{1.9}$$

$$\left[a \, J_u \nabla_x^4, \Phi_2 \nabla_x^{-2} \right] \nabla_x^k u_x = d_2 \, P_2 \nabla_x \nabla_x^k u_x + \text{harmless terms}. \tag{1.10}$$

Therefore, if we set $d_1 = c_1$ and $d_2 = c_1 + c_2$, the above two commutators eliminate $c_1 \, P_1 \nabla_x^2 + c_2 \, P_2 \nabla_x$ in the partial differential equation satisfied by V_k, and hence the energy estimate for $\|V_k\|_{L^2(\mathbb{T};TN)}^2$ works. The nice choice of the above gauged function is inspired by [4].

   The strategy for the proof of the local existence of a solution is as follows: First, we construct a family of fourth-order parabolic regularized solutions $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$. Second, we obtain ε-independent uniform estimates for $\|u_x^\varepsilon\|_{H^{k-1}(\mathbb{T};TN)}^2 + \|V_k^\varepsilon\|_{L^2(\mathbb{T};TN)}^2$ and the lower bound $T > 0$ of existence time of $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$, where V_k^ε is defined by (1.8) replacing u with u^ε. Finally, the standard compactness argument concludes the existence of $u \in C([0,T] \times \mathbb{T}; N)$ so that

$u_x \in L^\infty(0,T; H^k(\mathbb{T}; TN)) \cap C([0,T]; H^{k-1}(\mathbb{T}; TN))$ and u solves (1.1)-(1.2). The two commutators (1.9) and (1.10) in the above formal observation will be generated essentially in the computation of the second and the third term of the right hand side of (3.102). One can refer to [10, 11, 13] for tools of computation and [4, 5, 6] for the method of the gauged energy employed in the proof.

The strategy for the proof of the uniqueness of the solution is stated as follows: Suppose that $u, v \in C([0,T] \times \mathbb{T}; N)$ are solutions to (1.1)-(1.2) satisfying $u_x, v_x \in L^\infty(0,T; H^6(\mathbb{T}; TN)) \cap C([0,T]; H^5(\mathbb{T}; TN))$ with same initial data u_0. Their existence is ensured by the above local existence results. To estimate the difference between u and v, we regard u and v as functions with values in some Euclidean space $\mathbb{R}^d$. Indeed, letting w be an isometric embedding of (N, g) into $\mathbb{R}^d$, we consider $\mathbb{R}^d$-valued functions defined as follows:

$$U := w \circ u, \quad V := w \circ v, \quad Z := U - V,$$
$$\mathcal{U} := dw_u(\nabla_x u_x), \quad \mathcal{V} := dw_v(\nabla_x v_x), \quad \mathcal{W} := \mathcal{U} - \mathcal{V},$$

where $dw_p : T_p N \to T_{w \circ p}\mathbb{R}^d \cong \mathbb{R}^d$ is the differential of w at $p \in N$. To complete the proof of the uniqueness, it suffices to show $Z = 0$. First, as shown in (3.94), we obtain the classical energy estimate for $\|Z\|_{L^2}^2 + \|Z_x\|_{L^2}^2 + \|\mathcal{W}\|_{L^2}^2$ with the loss of derivatives, where $\|\cdot\|_{L^2}$ expresses the standard L^2-norm for $\mathbb{R}^d$-valued functions on $\mathbb{T}$. The loss of derivatives has similar form as that eliminated by the method of the gauge transformation in the proof of the local existence of a solution. Observing the analogy, we can easily find $\widetilde{\mathcal{W}} = \mathcal{W} + \widetilde{\Lambda}$ as a gauged function of $\mathcal{W}$ so that the energy estimate for $\|Z\|_{L^2}^2 + \|Z_x\|_{L^2}^2 + \|\widetilde{\mathcal{W}}\|_{L^2}^2$ can be closed. This shows $Z = 0$. The precise form of $\widetilde{\Lambda}$ will be given in (3.97).

In the proof of the uniqueness, we face with another difficulty, which does not appear in the proof of the local existence. On one hand, the proof of the local existence seems clear, thanks to the nice matching between the geometric formulation of (1.1) and the geometric L^2-norm $\|\cdot\|_{L^2(\mathbb{T};TN)}$. On the other hand, the proof of the uniqueness requires lengthier computations, due to the worse matching between the form of the equation satisfied by U and the standard L^2-norm $\|\cdot\|_{L^2(\mathbb{T};\mathbb{R}^d)}$. More concretely, the most crucial part of the proof of the uniqueness is how to derive the energy estimate for $\mathcal{W}$ of the form (3.94). To derive this, the partial differential equation satisfied by $\mathcal{W}$ and the energy estimate in $L^2(\mathbb{T};\mathbb{R}^d)$ are required. However, the analysis of the structure of lower order terms in the equation becomes complicated, since many terms related to the second fundamental form on N and the derivatives appear to describe the equation satisfied by U or V. As (1.1) is higher-order equation than the Schrödinger flow equation or the third-order dispersive flow equation previously studied, the situation becomes worse. Fortunately, however, we can successfully formulate the Kähler condition and the curvature condition on (N, J, g) to be applicable to our problem, and demonstrate that only weak loss of derivatives is allowed to appear in the energy estimate for $\|\mathcal{W}\|_{L^2(\mathbb{T};\mathbb{R}^d)}^2$. In addition, it is to be noted that we does not choose $\partial_x Z_x$ but choose $\mathcal{W}$ in the energy estimate. The choice also plays an important role (See,e.g.,Lemma 3.1) in our proof, as well as the choice of $\widetilde{\Lambda}$.

By the way, the geometric formulation of (1.1) was originally proposed by [15]. Independently, Anco and Myrzakulov in [1] derived the equation, named a fourth-order Schrödinger map equation, for $u : \mathbb{R} \times \mathbb{R} \to N$ or $u : \mathbb{R} \times \mathbb{T} \to N$ of the form

$$-u_t = J_u \nabla_x^3 u_x + \frac{1}{2}\nabla_x\{g(u_x, u_x)J_u u_x\} - \frac{1}{2}g(J_u u_x, \nabla_x u_x)u_x. \tag{1.11}$$

Interestingly, if N is a Riemann surface, (1.11) is identical with (1.1) with $a = -1$, $b = -1$, $c = -1/2$, and $\lambda = 0$. Therefore, we immediately find that Theorem 1.1 is valid for the initial value problem also for (1.11).

The organization of the present paper is as follows: In Section 2, a time-local solution to (1.1)-(1.2) is constructed. In Section 3, the proof of Theorem 1.1 is completed.

2. PROOF OF THE EXISTENCE OF A TIME-LOCAL SOLUTION

This section is devoted to the construction of a time-local solution to (1.1)-(1.2). More concretely, the goal of this section is to show the following.

Theorem 2.1. *Suppose that the sectional curvature of (N, g) is constant. Let k be an integer satisfying $k \geqslant 4$. Then for any $u_0 \in C(\mathbb{T}; N)$ satisfying $u_{0x} \in H^k(\mathbb{T}; TN)$, there exists $T = T(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 0$ such that (1.1)-(1.2) has a solution $u \in C([-T, T] \times \mathbb{T}; N)$ satisfying $u_x \in L^\infty(-T, T; H^k(\mathbb{T}; TN)) \cap C([-T, T]; H^{k-1}(\mathbb{T}; TN))$.*

Proof of Theorem 2.1. Let $k \geqslant 4$ be fixed. It suffices to solve the problem in the positive direction in time. We first assume that $u_0 \in C^\infty(\mathbb{T}; N)$ and construct a local solution.

As a beginning, we consider the initial value problem of the form

$$u_t = (-\varepsilon + a J_u)\nabla_x^3 u_x$$
$$+ b g(u_x, u_x) J_u \nabla_x u_x + c g(\nabla_x u_x, u_x) J_u u_x + \lambda J_u \nabla_x u_x \quad \text{in} \quad (0, \infty) \times \mathbb{T}, \quad (2.1)$$
$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}, \quad (2.2)$$

where $\varepsilon \in (0, 1]$ is a small positive parameter. Thanks to the added term $-\varepsilon \nabla_x^3 u_x$, (2.1) is a fourth-order quasilinear parabolic system, and (2.1)-(2.2) has a unique local smooth solution which we will denote u^ε.

Lemma 2.2. *For each $\varepsilon \in (0, 1]$, there exists a positive constant T_ε depending on ε and $\|u_{0x}\|_{H^4(\mathbb{T};TN)}$ such that (2.1)-(2.2) possesses a unique solution $u^\varepsilon \in C^\infty([0, T_\varepsilon] \times \mathbb{T}; N)$.*

We can show Lemma 2.2 by the mix of a sixth-order parabolic regularization and a geometric classical energy method without the constant curvature condition on (N, g). The proof almost falls into the scope of that of [6, Lemma 3.1] by replacing $\mathbb{R}$ with $\mathbb{T}$ and by restricting to a compact Riemann surface as N. Thought a slight modification is required in the proof, the difference is not essential and thus we omit the detail of the proof.

In the next step, letting $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ be a family of solutions to (2.1)-(2.2) constructed in Lemma 2.2, we obtain ε-independent energy estimates for $\{u_x^\varepsilon\}_{\varepsilon \in (0,1]}$. Precisely speaking, we obtain a uniform lower bound T of $\{T_\varepsilon\}_{\varepsilon \in (0,1]}$ and show that $\{u_x^\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^k(\mathbb{T}; TN))$. However, the classical energy estimate for $\|u_x^\varepsilon\|_{H^k(\mathbb{T};TN)}$ causes loss of derivatives. To overcome the difficulty, we introduce a gauged function V_k^ε defined by

$$V_k^\varepsilon = \nabla_x^k u_x^\varepsilon + \Lambda^\varepsilon = \nabla_x^k u_x^\varepsilon + \Lambda_1^\varepsilon + \Lambda_2^\varepsilon, \quad (2.3)$$

where

$$\Lambda_1^\varepsilon = -\frac{d_1}{2a} g(\nabla_x^{k-2} u_x^\varepsilon, J_u u_x^\varepsilon) J_u u_x^\varepsilon, \quad \Lambda_2^\varepsilon = \frac{d_2}{8a} g(u_x^\varepsilon, u_x^\varepsilon) \nabla_x^{k-2} u_x^\varepsilon,$$

and $d_1, d_2 \in \mathbb{R}$ are real constants which will be decided later depending only on a, b, c, k and the constant sectional curvature of (N, g). Furthermore, we introduce the associated gauged energy $N_k(u^\varepsilon(t))$ defined by

$$N_k(u^\varepsilon(t)) = \sqrt{\|u_x^\varepsilon(t)\|_{H^{k-1}(\mathbb{T};TN)}^2 + \|V_k^\varepsilon(t)\|_{L^2(\mathbb{T};TN)}^2}. \quad (2.4)$$

We restrict the time interval on $[0, T_\varepsilon^\star]$ with $T_\varepsilon^\star$ defined by

$$T_\varepsilon^\star = \sup\{T > 0 \mid N_4(u^\varepsilon(t)) \leqslant 2N_4(u_0) \quad \text{for all} \quad t \in [0, T]\}.$$

By the Sobolev embedding, we immediately find that there holds

$$\frac{1}{C}N_k(u^\varepsilon(t)) \leqslant \|u_x^\varepsilon(t)\|_{H^k(\mathbb{T};TN)} \leqslant C\,N_k(u^\varepsilon(t)) \quad \text{for any} \quad t \in [0, T_\varepsilon^\star], \qquad (2.5)$$

with $C = C(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 1$ being an ε-independent constant. We shall show that there exists a constant $T = T(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 0$ which is independent of $\varepsilon \in (0,1]$ and k such that $T_\varepsilon^\star \geqslant T$ uniformly in $\varepsilon \in (0,1]$ and that $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0,T)$. If it is true, this together with (2.5) implies that $\{u_x^\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0,T;H^k(\mathbb{T};TN))$.

Having them in mind, let us focus on the uniform energy estimate for $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$. We set $u = u^\varepsilon$, $V_k = V_k^\varepsilon$, $\Lambda = \Lambda^\varepsilon$, $\Lambda_1 = \Lambda_1^\varepsilon$, $\Lambda_2 = \Lambda_2^\varepsilon$, $\|\cdot\|_{H^0(\mathbb{T};TN)} = \|\cdot\|_{L^2(\mathbb{T};TN)} = \|\cdot\|_{L^2}$, $\|\cdot\|_{H^m(\mathbb{T};TN)} = \|\cdot\|_{H^m}$ for $m = 1, \ldots, k$, and $\sqrt{g(\cdot,\cdot)} = |\cdot|_g$, for ease of notation. Since g is a hermitian metric, $g(J_u Y_1, J_u Y_2) = g(Y_1, Y_2)$ holds for any $Y_1, Y_2 \in \Gamma(u^{-1}TN)$. Since Riemann surfaces with hermitian metric are Kähler manifolds, $\nabla_x J_u = J_u \nabla_x$ and $\nabla_t J_u = J_u \nabla_t$ hold. We denote the sectional curvature of (N,g) by S which is constant. Any positive constant which depends on a, b, c, λ, k, S, $\|u_{0x}\|_{H^4}$ and not on $\varepsilon \in (0,1]$ will be denoted by the same C. Note that $k \geqslant 4$ and the Sobolev embedding $H^1(\mathbb{T}) \subset C(\mathbb{T})$ yield $\|\nabla_x^4 u_x\|_{L^\infty(0,T_\varepsilon^\star;L^2)} \leqslant C$ and $\|\nabla_x^m u_x\|_{L^\infty((0,T_\varepsilon^\star)\times\mathbb{T})} \leqslant C$ for $m = 0,1,\ldots,3$. These properties will be used without any comment in this section.

We now investigate the energy estimate for $\|V_k\|_{L^2}^2$. It follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|V_k\|_{L^2}^2 &= \int_\mathbb{T} g(\nabla_t V_k, V_k)dx \\ &= \int_\mathbb{T} g(\nabla_t(\nabla_x^k u_x), V_k)dx + \int_\mathbb{T} g(\nabla_t \Lambda, V_k)dx \\ &= \int_\mathbb{T} g(\nabla_t(\nabla_x^k u_x), \nabla_x^k u_x)dx + \int_\mathbb{T} g(\nabla_t(\nabla_x^k u_x), \Lambda)dx + \int_\mathbb{T} g(\nabla_t \Lambda, V_k)dx. \quad (2.6) \end{aligned}$$

To evaluate the right hand side (denoted by RHS hereafter for short) of (2.6), we compute the partial differential equation satisfied by $\nabla_x^k u_x$. Recalling that $\nabla_x u_t = \nabla_t u_x$ and $(\nabla_x \nabla_t - \nabla_t \nabla_x)Y = R(u_x, u_t)Y$ for any $Y \in \Gamma(u^{-1}TN)$ where $R = R(\cdot,\cdot)$ denotes the Riemann curvature tensor on (N,g), we have

$$\nabla_t(\nabla_x^k u_x) = \nabla_x^{k+1} u_t + \sum_{m=0}^{k-1} \nabla_x^{k-1-m}\left\{R(u_t, u_x)\nabla_x^m u_x\right\} =: \nabla_x^{k+1} u_t + Q. \qquad (2.7)$$

First, we use (2.1) to compute the second term of the RHS of the above, which becomes

$$\begin{aligned} Q = &-\varepsilon \sum_{m=0}^{k-1} \nabla_x^{k-1-m}\left\{R(\nabla_x^3 u_x, u_x)\nabla_x^m u_x\right\} \\ &+ a \sum_{m=0}^{k-1} \nabla_x^{k-1-m}\left\{R(J_u \nabla_x^3 u_x, u_x)\nabla_x^m u_x\right\} \\ &+ \lambda \sum_{m=0}^{k-1} \nabla_x^{k-1-m}\left\{R(J_u \nabla_x u_x, u_x)\nabla_x^m u_x\right\} \\ &+ b \sum_{m=0}^{k-1} \nabla_x^{k-1-m}\left\{g(u_x, u_x)R(J_u \nabla_x u_x, u_x)\nabla_x^m u_x\right\} \end{aligned}$$

$$+ c \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \left\{ g(\nabla_x u_x, u_x) R(J_u u_x, u_x) \nabla_x^m u_x \right\}.$$

Thus, by using the Sobolev embedding and the Gagliardo-Nirenberg inequality, we obtain

$$Q = \varepsilon \, \mathcal{O}(|\nabla_x^{k+2} u_x|_g) + a \, Q_0 + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g \right), \tag{2.8}$$

where

$$Q_0 = \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \left\{ R(J_u \nabla_x^3 u_x, u_x) \nabla_x^m u_x \right\}.$$

Since S is the constant sectional curvature of (N, g),

$$R(Y_1, Y_2)Y_3 = S \left\{ g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2 \right\} \tag{2.9}$$

holds for any $Y_1, Y_2, Y_3 \in \Gamma(u^{-1}TN)$. Using the formula, Q_0 is expressed as follows.

$$\begin{aligned}
Q_0 &= S \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \left\{ g(\nabla_x^m u_x, u_x) J_u \nabla_x^3 u_x - g(\nabla_x^m u_x, J_u \nabla_x^3 u_x) u_x \right\} \\
&= S \left(Q_{0,1} + Q_{0,2} + Q_{0,3} \right), \tag{2.10}
\end{aligned}$$

where

$$\begin{aligned}
Q_{0,1} &= \nabla_x^{k-1} \left\{ g(u_x, u_x) J_u \nabla_x^3 u_x - g(u_x, J_u \nabla_x^3 u_x) u_x \right\}, \\
Q_{0,2} &= \nabla_x^{k-2} \left\{ g(\nabla_x u_x, u_x) J_u \nabla_x^3 u_x - g(\nabla_x u_x, J_u \nabla_x^3 u_x) u_x \right\}, \\
Q_{0,3} &= \sum_{m=2}^{k-1} \nabla_x^{k-1-m} \left\{ g(\nabla_x^m u_x, u_x) J_u \nabla_x^3 u_x - g(\nabla_x^m u_x, J_u \nabla_x^3 u_x) u_x \right\}.
\end{aligned}$$

For $Q_{0,1}$, the product formula implies

$$\begin{aligned}
Q_{0,1} &= \sum_{\mu+\nu=0}^{k-1} \frac{(k-1)!}{\mu! \nu! (k-1-\mu-\nu)!} \, g(\nabla_x^\mu u_x, \nabla_x^\nu u_x) J_u \nabla_x^{k+2-\mu-\nu} u_x \\
&\quad - \sum_{\mu+\nu=0}^{k-1} \frac{(k-1)!}{\mu! \nu! (k-1-\mu-\nu)!} \, g(\nabla_x^\mu u_x, J_u \nabla_x^{\nu+3} u_x) \nabla_x^{k-1-\mu-\nu} u_x \\
&= g(u_x, u_x) J_u \nabla_x^{k+2} u_x + 2(k-1) g(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x - g(u_x, J_u \nabla_x^{k+2} u_x) u_x \\
&\quad - (k-1) g(\nabla_x u_x, J_u \nabla_x^{k+1} u_x) u_x - (k-1) g(u_x, J_u \nabla_x^{k+1} u_x) \nabla_x u_x \\
&\quad + \sum_{\mu+\nu=2}^{k-1} \frac{(k-1)!}{\mu! \nu! (k-1-\mu-\nu)!} \, g(\nabla_x^\mu u_x, \nabla_x^\nu u_x) J_u \nabla_x^{k+2-\mu-\nu} u_x \\
&\quad - \sum_{\substack{\mu+\nu=0, \\ \nu \leqslant k-3}} \frac{(k-1)!}{\mu! \nu! (k-1-\mu-\nu)!} \, g(\nabla_x^\mu u_x, J_u \nabla_x^{\nu+3} u_x) \nabla_x^{k-1-\mu-\nu} u_x \\
&= g(u_x, u_x) J_u \nabla_x^{k+2} u_x + 2(k-1) g(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x \\
&\quad - g(u_x, J_u \nabla_x^{k+2} u_x) u_x - (k-1) g(\nabla_x u_x, J_u \nabla_x^{k+1} u_x) u_x \\
&\quad - (k-1) g(u_x, J_u \nabla_x^{k+1} u_x) \nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g \right). \tag{2.11}
\end{aligned}$$

Here it is to be emphasized that

$$g(Y, u_x)u_x + g(Y, J_u u_x)J_u u_x = g(u_x, u_x)Y \tag{2.12}$$

holds for any $Y \in \Gamma(u^{-1}TN)$, since N is a two-dimensional real manifold. Using (2.12) with $Y = J_u \nabla_x^{k+2} u_x$, we rewrite the third term of the RHS of (2.11) to have

$$-g(u_x, J_u \nabla_x^{k+2} u_x)u_x = -g(u_x, u_x)J_u \nabla_x^{k+2} u_x + g(J_u u_x, J_u \nabla_x^{k+2} u_x)J_u u_x$$
$$= -g(u_x, u_x)J_u \nabla_x^{k+2} u_x + g(\nabla_x^{k+2} u_x, u_x)J_u u_x. \tag{2.13}$$

Substituting (2.13) into the RHS of (2.11), we obtain

$$Q_{0,1} = 2(k-1)g(\nabla_x u_x, u_x)J_u \nabla_x^{k+1} u_x + g(\nabla_x^{k+2} u_x, u_x)J_u u_x$$
$$+ (k-1)g(\nabla_x^{k+1} u_x, J_u \nabla_x u_x)u_x + (k-1)g(\nabla_x^{k+1} u_x, J_u u_x)\nabla_x u_x$$
$$+ \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.14}$$

For $Q_{0,2}$, in the same way as that for $Q_{0,1}$, we deduce

$$Q_{0,2} = \sum_{\mu+\nu=0}^{k-2} \frac{(k-2)!}{\mu!\nu!(k-2-\mu-\nu)!} g(\nabla_x^{\mu+1} u_x, \nabla_x^\nu u_x)J_u \nabla_x^{k+1-\mu-\nu} u_x$$
$$- \sum_{\mu+\nu=0}^{k-2} \frac{(k-2)!}{\mu!\nu!(k-2-\mu-\nu)!} g(\nabla_x^{\mu+1} u_x, J_u \nabla_x^{\nu+3} u_x)\nabla_x^{k-2-\mu-\nu} u_x$$
$$= g(\nabla_x u_x, u_x)J_u \nabla_x^{k+1} u_x - g(\nabla_x u_x, J_u \nabla_x^{k+1} u_x)u_x$$
$$+ \sum_{\mu+\nu=1}^{k-2} \frac{(k-2)!}{\mu!\nu!(k-2-\mu-\nu)!} g(\nabla_x^{\mu+1} u_x, \nabla_x^\nu u_x)J_u \nabla_x^{k+1-\mu-\nu} u_x$$
$$- \sum_{\substack{\mu+\nu=0, \\ \nu\leqslant k-3}}^{k-2} \frac{(k-2)!}{\mu!\nu!(k-2-\mu-\nu)!} g(\nabla_x^{\mu+1} u_x, J_u \nabla_x^{\nu+3} u_x)\nabla_x^{k-2-\mu-\nu} u_x$$
$$= g(\nabla_x u_x, u_x)J_u \nabla_x^{k+1} u_x + g(\nabla_x^{k+1} u_x, J_u \nabla_x u_x)u_x + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.15}$$

For $Q_{0,3}$, the Sobolev embedding and the Gagliardo-Nirenberg inequality imply

$$Q_{0,3} = \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.16}$$

Collecting (2.8), (2.10), (2.14), (2.15), and (2.16), we obtain

$$Q = \varepsilon\, \mathcal{O}\left(|\nabla_x^{k+2} u_x|_g\right) + aS\, g(\nabla_x^2(\nabla_x^k u_x), u_x)J_u u_x$$
$$+ aS(2k-1)\, g(\nabla_x u_x, u_x)J_u \nabla_x(\nabla_x^k u_x) + aSk\, g(\nabla_x(\nabla_x^k u_x), J_u \nabla_x u_x)u_x$$
$$+ aS(k-1)\, g(\nabla_x(\nabla_x^k u_x), J_u u_x)\nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.17}$$

Second, we use (2.1) to compute the first term of the RHS of (2.7). A simple computation shows

$$\nabla_x^{k+1} u_t = -\varepsilon \nabla_x^4(\nabla_x^k u_x) + a\, J_u \nabla_x^4(\nabla_x^k u_x) + \lambda\, J_u \nabla_x^2(\nabla_x^k u_x) + b\, Q_{1,1} + c\, Q_{1,2}, \tag{2.18}$$

where

$$Q_{1,1} = \nabla_x^{k+1} \{g(u_x, u_x)J_u\nabla_x u_x\}$$

$$= \sum_{\mu+\nu=0}^{k+1} \frac{(k+1)!}{\mu!\nu!(k+1-\mu-\nu)!}\, g(\nabla_x^\mu u_x, \nabla_x^\nu u_x)J_u\nabla_x^{k+2-\mu-\nu}u_x$$

$$= g(u_x, u_x)J_u\nabla_x^{k+2}u_x + 2(k+1)g(\nabla_x u_x, u_x)J_u\nabla_x^{k+1}u_x$$

$$+ 2g(\nabla_x^{k+1}u_x, u_x)J_u\nabla_x u_x$$

$$+ \sum_{\substack{\mu+\nu=2, \\ \mu,\nu\leqslant k}}^{k+1} \frac{(k+1)!}{\mu!\nu!(k+1-\mu-\nu)!}\, g(\nabla_x^\mu u_x, \nabla_x^\nu u_x)J_u\nabla_x^{k+2-\mu-\nu}u_x$$

$$= \nabla_x\left\{g(u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)\right\} + 2k\, g(\nabla_x u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)$$

$$+ 2\, g(\nabla_x(\nabla_x^k u_x), u_x)J_u\nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_g\right), \qquad (2.19)$$

and

$$Q_{1,2} = \nabla_x^{k+1}\{g(\nabla_x u_x, u_x)J_u u_x\}$$

$$= \sum_{\mu+\nu=0}^{k+1} \frac{(k+1)!}{\mu!\nu!(k+1-\mu-\nu)!}\, g(\nabla_x^{\mu+1}u_x, \nabla_x^\nu u_x)J_u\nabla_x^{k+1-\mu-\nu}u_x$$

$$= g(\nabla_x u_x, u_x)J_u\nabla_x^{k+1}u_x + g(\nabla_x^{k+2}u_x, u_x)J_u u_x$$

$$+ (k+1)g(\nabla_x^{k+1}u_x, \nabla_x u_x)J_u u_x + g(\nabla_x u_x, \nabla_x^{k+1}u_x)J_u u_x$$

$$+ (k+1)g(\nabla_x^{k+1}u_x, u_x)J_u\nabla_x u_x$$

$$+ \sum_{\substack{\mu+\nu=1, \\ \mu\leqslant k-1, \\ \nu\leqslant k}}^{k+1} \frac{(k+1)!}{\mu!\nu!(k+1-\mu-\nu)!}\, g(\nabla_x^{\mu+1}u_x, \nabla_x^\nu u_x)J_u\nabla_x^{k+1-\mu-\nu}u_x$$

$$= g(\nabla_x^2(\nabla_x^k u_x), u_x)J_u u_x + g(\nabla_x u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)$$

$$+ (k+2)g(\nabla_x(\nabla_x^k u_x), \nabla_x u_x)J_u u_x + (k+1)g(\nabla_x(\nabla_x^k u_x), u_x)J_u\nabla_x u_x$$

$$+ \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_g\right). \qquad (2.20)$$

By collecting (2.17) and (2.18) with (2.19) and with (2.20), we have

$$\nabla_t(\nabla_x^k u_x) = -\varepsilon\nabla_x^4(\nabla_x^k u_x) + \varepsilon\,\mathcal{O}\left(|\nabla_x^{k+2}u_x|_g\right)$$

$$+ a\,J_u\nabla_x^4(\nabla_x^k u_x) + \lambda\,J_u\nabla_x^2(\nabla_x^k u_x) + b\,\nabla_x\left\{g(u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)\right\}$$

$$+ (aS+c)\,g(\nabla_x^2(\nabla_x^k u_x), u_x)J_u u_x$$

$$+ \{aS(2k-1) + 2kb + c\}\,g(\nabla_x u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)$$

$$+ \{2b + (k+1)c\}\,g(\nabla_x(\nabla_x^k u_x), u_x)J_u\nabla_x u_x$$

$$+ (k+2)c\,g(\nabla_x(\nabla_x^k u_x), \nabla_x u_x)J_u u_x$$

$$+ aSk\,g(\nabla_x(\nabla_x^k u_x), J_u\nabla_x u_x)u_x$$

$$+ aS(k-1)\, g(\nabla_x(\nabla_x^k u_x), J_u u_x)\nabla_x u_x$$

$$+ \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.21}$$

Furthermore, we modify the expression of some terms including $\nabla_x(\nabla_x^k u_x)$ to detect their essential structure. Let $Y \in \Gamma(u^{-1}TN)$ be fixed. We first use (2.12) to see

$$g(u_x, u_x)J_u Y = g(J_u Y, u_x)u_x + g(J_u Y, J_u u_x)J_u u_x$$
$$= g(Y, u_x)J_u u_x - g(Y, J_u u_x)u_x.$$

Acting ∇_x to both sides of the above, we have

$$2\, g(\nabla_x u_x, u_x)J_u Y = g(Y, \nabla_x u_x)J_u u_x + g(Y, u_x)J_u \nabla_x u_x$$
$$- g(Y, J_u \nabla_x u_x)u_x - g(Y, J_u u_x)\nabla_x u_x. \tag{2.22}$$

We next introduce the following expression:

$$A_1 Y = g(Y, \nabla_x u_x)J_u u_x + g(Y, u_x)J_u \nabla_x u_x$$
$$+ g(Y, J_u \nabla_x u_x)u_x + g(Y, J_u u_x)\nabla_x u_x,$$
$$A_2 Y = g(Y, J_u u_x)\nabla_x u_x - g(Y, J_u \nabla_x u_x)u_x.$$

We find ${}^t A_1 = A_1$ and ${}^t A_2 = A_2$ in $T_u N$. More precisely we can show the following.

Proposition 2.3. *Let $Y_1, Y_2 \in \Gamma(u^{-1}TN)$. Then*

$$g(A_i Y_1, Y_2) = g(Y_1, A_i Y_2) \tag{2.23}$$

holds for each $(t, x) \in [0, T_\varepsilon^*] \times \mathbb{T}$ with $i = 1, 2$.

Proof of Proposition 2.3. If $i = 1$, (2.23) immediately follows from the definition of A_1. If $i = 2$, (2.23) follows from

$$\{g(u_x, u_x)\}^2 \{g(A_2 Y_1, Y_2) - g(Y_1, A_2 Y_2)\} = 0, \tag{2.24}$$

since both sides of (2.23) vanish at the point (t, x) with $u_x(t, x) = 0$. Indeed we can show (2.24) by the following computations. We first write

$$g(u_x, u_x)A_2 Y_1 = g(u_x, u_x)\{g(Y_1, J_u u_x)\nabla_x u_x - g(Y_1, J_u \nabla_x u_x)u_x\}$$
$$= g(g(u_x, u_x)Y_1, J_u u_x)\nabla_x u_x - g(g(u_x, u_x)Y_1, J_u \nabla_x u_x)u_x,$$

and we use (2.12) with $Y = Y_1$ to see

$$g(u_x, u_x)A_2 Y_1 = g(g(Y_1, u_x)u_x + g(Y_1, J_u u_x)J_u u_x, J_u u_x)\nabla_x u_x$$
$$- g(g(Y_1, u_x)u_x + g(Y_1, J_u u_x)J_u u_x, J_u \nabla_x u_x)u_x$$
$$= g(u_x, u_x)g(Y_1, J_u u_x)\nabla_x u_x - g(u_x, J_u \nabla_x u_x)g(Y_1, u_x)u_x$$
$$- g(u_x, \nabla_x u_x)g(Y_1, J_u u_x)u_x.$$

This implies

$$\{g(u_x, u_x)\}^2 g(A_2 Y_1, Y_2) = g(g(u_x, u_x)A_2 Y_1, g(u_x, u_x)Y_2)$$
$$= g(u_x, u_x)g(Y_1, J_u u_x)g(\nabla_x u_x, g(u_x, u_x)Y_2)$$
$$- g(u_x, J_u \nabla_x u_x)g(Y_1, u_x)g(u_x, g(u_x, u_x)Y_2)$$
$$- g(u_x, \nabla_x u_x)g(Y_1, J_u u_x)g(u_x, g(u_x, u_x)Y_2). \tag{2.25}$$

Using (2.12) again with $Y = Y_2$, we see

$$g(u_x, g(u_x, u_x)Y_2) = g(u_x, u_x)g(Y_2, u_x),$$

$$g(\nabla_x u_x, g(u_x, u_x)Y_2) = g(\nabla_x u_x, u_x)g(Y_2, u_x) + g(\nabla_x u_x, J_u u_x)g(Y_2, J_u u_x).$$

Substituting them into (2.25), we have

$$\{g(u_x, u_x)\}^2 \, g(A_2 Y_1, Y_2)$$
$$= g(u_x, u_x)g(\nabla_x u_x, J_u u_x) \{g(Y_1, J_u u_x)g(Y_2, J_u u_x) + g(Y_1, u_x)g(Y_2, u_x)\} .$$

As the form of the RHS is symmetric with respect to Y_1 and Y_2, we immediately conclude that the desired property (2.24) holds. $\qquad\square$

Using (2.22) and the definition of A_1 and A_2, we have

$$g(Y, J_u u_x)\nabla_x u_x$$
$$= \frac{1}{4}\left\{g(Y, \nabla_x u_x)J_u u_x + g(Y, u_x)J_u\nabla_x u_x + g(Y, J_u\nabla_x u_x)u_x + g(Y, J_u u_x)\nabla_x u_x\right\}$$
$$- \frac{1}{4}\left\{g(Y, \nabla_x u_x)J_u u_x + g(Y, u_x)J_u\nabla_x u_x - g(Y, J_u\nabla_x u_x)u_x - g(Y, J_u u_x)\nabla_x u_x\right\}$$
$$+ \frac{1}{2}\left\{g(Y, J_u u_x)\nabla_x u_x - g(Y, J_u\nabla_x u_x)u_x\right\}$$
$$= -\frac{1}{2}\, g(\nabla_x u_x, u_x)J_u Y + \frac{1}{4}A_1 Y + \frac{1}{2}A_2 Y. \tag{2.26}$$

In the same way, we have

$$g(Y, J_u\nabla_x u_x)u_x = -\frac{1}{2}\, g(\nabla_x u_x, u_x)J_u Y + \frac{1}{4}A_1 Y - \frac{1}{2}A_2 Y. \tag{2.27}$$

Using ${}^t J_u = -J_u$ in $T_u N$, (2.23), and (2.27), we deduce

$$g(Y, u_x)J_u\nabla_x u_x = {}^t\left(g(\cdot, J_u\nabla_x u_x)u_x\right)Y$$
$$= -\frac{1}{2}\, g(\nabla_x u_x, u_x)\, {}^t J_u Y + \frac{1}{4}\, {}^t A_1 Y - \frac{1}{2}\, {}^t A_2 Y$$
$$= \frac{1}{2}\, g(\nabla_x u_x, u_x)J_u Y + \frac{1}{4}A_1 Y - \frac{1}{2}A_2 Y, \tag{2.28}$$

and

$$g(Y, \nabla_x u_x)J_u u_x = {}^t\left(g(\cdot, J_u u_x)\nabla_x u_x\right)Y$$
$$= \frac{1}{2}\, g(\nabla_x u_x, u_x)J_u Y + \frac{1}{4}A_1 Y + \frac{1}{2}A_2 Y. \tag{2.29}$$

Applying (2.26), (2.27), (2.28), and (2.29) to the RHS of (2.21), we derive

$$\nabla_t(\nabla_x^k u_x) = -\varepsilon \nabla_x^4(\nabla_x^k u_x) + \varepsilon \, \mathcal{O}\left(|\nabla_x^{k+2} u_x|_g\right)$$
$$+ a\, J_u \nabla_x^4(\nabla_x^k u_x) + \lambda\, J_u \nabla_x^2(\nabla_x^k u_x) + b\, \nabla_x\left\{g(u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)\right\}$$
$$+ c_1\, g(\nabla_x^2(\nabla_x^k u_x), u_x)J_u u_x + c_2\, g(\nabla_x u_x, u_x)J_u\nabla_x(\nabla_x^k u_x)$$
$$+ c_3\, A_1\nabla_x(\nabla_x^k u_x) + c_4\, A_2\nabla_x(\nabla_x^k u_x)$$
$$+ \mathcal{O}\left(\sum_{m=0}^{k}|\nabla_x^m u_x|_g\right), \tag{2.30}$$

where $c_1, \ldots, c_4$ are constants given by a, b, c and S. More concretely,

$$c_1 = aS + c, \tag{2.31}$$

$$c_2 = \{aS(2k-1) + 2kb + c\} + \frac{1}{2}\{2b + (k+1)c + (k+2)c - aSk - aS(k-1)\}$$

$$= \left(k - \frac{1}{2} \right) aS + (2k+1)b + \left(k + \frac{5}{2} \right) c. \tag{2.32}$$

We omit to describe the explicit form of c_3 and c_4, as they will not be used later.

We are now in position to evaluate the first term of the RHS of (2.6). Using (2.30), we have

$$\int_{\mathbb{T}} g(\nabla_t(\nabla_x^k u_x), \nabla_x^k u_x)dx$$

$$= -\varepsilon \int_{\mathbb{T}} g(\nabla_x^4(\nabla_x^k u_x), \nabla_x^k u_x)dx + \varepsilon \int_{\mathbb{T}} g(\mathcal{O}\left(|\nabla_x^{k+2} u_x|_g\right), \nabla_x^k u_x)dx$$

$$+ a \int_{\mathbb{T}} g(J_u \nabla_x^4(\nabla_x^k u_x), \nabla_x^k u_x)dx + \lambda \int_{\mathbb{T}} g(J_u \nabla_x^2(\nabla_x^k u_x), \nabla_x^k u_x)dx$$

$$+ b \int_{\mathbb{T}} g(\nabla_x \left\{ g(u_x, u_x) J_u \nabla_x(\nabla_x^k u_x) \right\}, \nabla_x^k u_x)dx$$

$$+ c_1 \int_{\mathbb{T}} g(g(\nabla_x^2(\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x)dx$$

$$+ c_2 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)dx$$

$$+ c_3 \int_{\mathbb{T}} g(A_1 \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)dx + c_4 \int_{\mathbb{T}} g(A_2 \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)dx$$

$$+ \int_{\mathbb{T}} g(\mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g \right), \nabla_x^k u_x)dx.$$

We compute each term of the above separately. By integrating by parts, we obtain

$$a \int_{\mathbb{T}} g(J_u \nabla_x^4(\nabla_x^k u_x), \nabla_x^k u_x)dx = a \int_{\mathbb{T}} g(J_u \nabla_x^2(\nabla_x^k u_x), \nabla_x^2(\nabla_x^k u_x))dx = 0,$$

$$\lambda \int_{\mathbb{T}} g(J_u \nabla_x^2(\nabla_x^k u_x), \nabla_x^k u_x)dx = -\lambda \int_{\mathbb{T}} g(J_u \nabla_x(\nabla_x^k u_x), \nabla_x(\nabla_x^k u_x))dx = 0,$$

$$b \int_{\mathbb{T}} g(\nabla_x \left\{ g(u_x, u_x) J_u \nabla_x(\nabla_x^k u_x) \right\}, \nabla_x^k u_x)dx$$

$$= -b \int_{\mathbb{T}} g(g(u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x(\nabla_x^k u_x))dx = 0.$$

By using the Cauchy-Schwartz inequality, we have

$$\int_{\mathbb{T}} g(\mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g \right), \nabla_x^k u_x)dx \leqslant C\|u_x\|_{H^k}\|\nabla_x^k u_x\|_{L^2} \leqslant C\|u_x\|_{H^k}^2. \tag{2.33}$$

Using the integration by parts, the Young inequality $AB \leqslant A^2/2 + B^2/2$ for any $A, B \geqslant 0$, and $\varepsilon \leqslant 1$, we deduce

$$-\varepsilon \int_{\mathbb{T}} g(\nabla_x^4(\nabla_x^k u_x), \nabla_x^k u_x)dx + \varepsilon \int_{\mathbb{T}} g(\mathcal{O}\left(|\nabla_x^{k+2} u_x|_g\right), \nabla_x^k u_x)dx$$

$$\leqslant -\varepsilon\|\nabla_x^2(\nabla_x^k u_x))\|_{L^2}^2 + \varepsilon C\|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}\|\nabla_x^k u_x\|_{L^2}$$

$$\leqslant -\varepsilon\|\nabla_x^2(\nabla_x^k u_x))\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 + \frac{\varepsilon C^2}{2}\|\nabla_x^k u_x\|_{L^2}^2$$

$$\leqslant -\frac{\varepsilon}{2}\|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 + \frac{C^2}{2}\|u_x\|_{H^k}^2.$$

By integrating by parts and by using (2.23), we have

$$c_3 \int_{\mathbb{T}} g(A_1 \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x) dx + c_4 \int_{\mathbb{T}} g(A_2 \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x) dx$$

$$= -\frac{c_3}{2} g\left(\nabla_x(A_1)\nabla_x^k u_x, \nabla_x^k u_x\right) dx - \frac{c_4}{2} g\left(\nabla_x(A_2)\nabla_x^k u_x, \nabla_x^k u_x\right) dx$$

$$\leqslant C \|u_x\|_{H^k}^2.$$

Collecting them and noting that $\|u_x\|_{H^k} \leqslant C N_k(u)$ follows from (2.5), we derive

$$\int_{\mathbb{T}} g(\nabla_t(\nabla_x^k u_x), \nabla_x^k u_x) dx$$

$$\leqslant -\frac{\varepsilon}{2} \|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 + c_1 \int_{\mathbb{T}} g(g(\nabla_x^2(\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x) dx$$

$$+ c_2 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x) dx + C (N_k(u))^2. \tag{2.34}$$

We next evaluate the second term of the RHS of (2.6). In the computation, it is to be noted that $\Lambda = \mathcal{O}(|\nabla_x^{k-2} u_x|_g)$ and

$$\nabla_t(\nabla_x^k u_x) = -\varepsilon \nabla_x^4(\nabla_x^k u_x) + a J_u \nabla_x^4(\nabla_x^k u_x) + \mathcal{O}\left(\sum_{m=0}^{k+2} |\nabla_x^m u_x|_g\right).$$

By noting them and by integrating by parts, we obtain

$$\int_{\mathbb{T}} g(\nabla_t(\nabla_x^k u_x), \Lambda) dx$$

$$\leqslant -\varepsilon \int_{\mathbb{T}} g(\nabla_x^4(\nabla_x^k u_x), \Lambda) dx + a \int_{\mathbb{T}} g(J_u \nabla_x^4(\nabla_x^k u_x), \Lambda) dx + C \|u_x\|_{H^k}^2. \tag{2.35}$$

For the first term of the RHS of (2.35), by using $\varepsilon \leqslant 1$, the integration by parts, the Young inequality $AB \leqslant A^2/8 + 2B^2$ for any $A, B \geqslant 0$, and $\Lambda = \mathcal{O}(|\nabla_x^{k-2} u_x|_g)$, we have

$$-\varepsilon \int_{\mathbb{T}} g(\nabla_x^4(\nabla_x^k u_x), \Lambda) dx = -\varepsilon \int_{\mathbb{T}} g(\nabla_x^2(\nabla_x^k u_x), \nabla_x^2(\Lambda)) dx$$

$$\leqslant \varepsilon \|\nabla_x^2(\nabla_x^k u_x)\|_{L^2} \|\nabla_x^2(\Lambda))\|_{L^2}$$

$$\leqslant \frac{\varepsilon}{8} \|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 + 2\varepsilon \|\nabla_x^2(\Lambda))\|_{L^2}^2$$

$$\leqslant \frac{\varepsilon}{8} \|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 + C \|u_x\|_{H^k}^2. \tag{2.36}$$

For the second term of the RHS of (2.35), we compute $\nabla_x^2 \Lambda$ to see

$$\nabla_x^2 \Lambda = -\frac{d_1}{2a} \nabla_x^2 \left\{ g(\nabla_x^{k-2} u_x, J_u u_x) J_u u_x \right\} + \frac{d_2}{8a} \nabla_x^2 \left\{ g(u_x, u_x) \nabla_x^{k-2} u_x \right\}$$

$$= -\frac{d_1}{2a} g(\nabla_x^k u_x, J_u u_x) J_u u_x + \frac{d_2}{8a} g(u_x, u_x) \nabla_x^k u_x$$

$$- \frac{d_1}{a} g(\nabla_x^{k-1} u_x, J_u \nabla_x u_x) J_u u_x - \frac{d_1}{a} g(\nabla_x^{k-1} u_x, J_u u_x) J_u \nabla_x u_x$$

$$+ \frac{d_2}{2a} g(\nabla_x u_x, u_x) \nabla_x^{k-1} u_x + \mathcal{O}\left(\sum_{m=0}^{k-2} |\nabla_x^m u_x|_g\right).$$

Thus, by integrating by parts and by substituting the above, we obtain

$$a \int_{\mathbb{T}} g(J_u \nabla_x^4 (\nabla_x^k u_x), \Lambda) dx$$

$$= a \int_{\mathbb{T}} g(J_u \nabla_x^2 (\nabla_x^k u_x), \nabla_x^2 \Lambda) dx$$

$$= -\frac{d_1}{2} Q_{2,1} + \frac{d_2}{8} Q_{2,2} - d_1 Q_{2,3} - d_1 Q_{2,4} + \frac{d_2}{2} Q_{2,5} + Q_{2,6}, \qquad (2.37)$$

where

$$Q_{2,1} = \int_{\mathbb{T}} g(\nabla_x^k u_x, J_u u_x) g(J_u \nabla_x^2 (\nabla_x^k u_x), J_u u_x) \, dx,$$

$$Q_{2,2} = \int_{\mathbb{T}} g(u_x, u_x) g(J_u \nabla_x^2 (\nabla_x^k u_x), \nabla_x^k u_x) \, dx,$$

$$Q_{2,3} = \int_{\mathbb{T}} g(\nabla_x^{k-1} u_x, J_u \nabla_x u_x) g(J_u \nabla_x^2 (\nabla_x^k u_x), J_u u_x) \, dx,$$

$$Q_{2,4} = \int_{\mathbb{T}} g(\nabla_x^{k-1} u_x, J_u u_x) g(J_u \nabla_x^2 (\nabla_x^k u_x), J_u \nabla_x u_x) \, dx,$$

$$Q_{2,5} = \int_{\mathbb{T}} g(\nabla_x u_x, u_x) g(J_u \nabla_x^2 (\nabla_x^k u_x), \nabla_x^{k-1} u_x) \, dx,$$

$$Q_{2,6} = \int_{\mathbb{T}} g\left(J_u \nabla_x^2 (\nabla_x^k u_x), \mathcal{O}\left(\sum_{m=0}^{k-2} |\nabla_x^m u_x|_g\right)\right) dx.$$

We compute $Q_{2,1}, \ldots, Q_{2,6}$ separately. By the integration by parts and the property of hermitian metric g, we deduce

$$Q_{2,1} = \int_{\mathbb{T}} g(\nabla_x^k u_x, J_u u_x) g(\nabla_x^2 (\nabla_x^k u_x), u_x) \, dx,$$

$$= \int_{\mathbb{T}} g(g(\nabla_x^2 (\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x) \, dx,$$

$$Q_{2,2} = \int_{\mathbb{T}} g(\nabla_x \left\{g(u_x, u_x) J_u \nabla_x (\nabla_x^k u_x)\right\}, \nabla_x^k u_x) \, dx$$

$$\qquad - 2 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx$$

$$= -2 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx,$$

$$Q_{2,3} = \int_{\mathbb{T}} g(\nabla_x^{k-1} u_x, J_u \nabla_x u_x) g(\nabla_x^2 (\nabla_x^k u_x), u_x) \, dx$$

$$\leqslant - \int_{\mathbb{T}} g(\nabla_x^k u_x, J_u \nabla_x u_x) g(\nabla_x (\nabla_x^k u_x), u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$= - \int_{\mathbb{T}} g(g(\nabla_x (\nabla_x^k u_x), u_x) J_u \nabla_x u_x, \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2,$$

$$Q_{2,4} = \int_{\mathbb{T}} g(\nabla_x^{k-1} u_x, J_u u_x) g(\nabla_x^2 (\nabla_x^k u_x), \nabla_x u_x) \, dx$$

$$\leqslant - \int_{\mathbb{T}} g(\nabla_x^k u_x, J_u u_x) g(\nabla_x (\nabla_x^k u_x), \nabla_x u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$= -\int_{\mathbb{T}} g(g(\nabla_x(\nabla_x^k u_x), \nabla_x u_x) J_u u_x, \nabla_x^k u_x)\, dx + C\|u_x\|_{H^k}^2,$$

$$Q_{2,5} \leqslant -\int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx + C\|u_x\|_{H^k}^2,$$

$$Q_{2,6} \leqslant C\|u_x\|_{H^k}^2.$$

Applying them to (2.37), we obtain

$$a\int_{\mathbb{T}} g(J_u \nabla_x^4(\nabla_x^k u_x), \Lambda) dx \leqslant -\frac{d_1}{2}\int_{\mathbb{T}} g(g(\nabla_x^2(\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x)\, dx$$

$$-\frac{3d_2}{4}\int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx$$

$$+ d_1 \int_{\mathbb{T}} g(g(\nabla_x(\nabla_x^k u_x), u_x) J_u \nabla_x u_x, \nabla_x^k u_x)\, dx$$

$$+ d_1 \int_{\mathbb{T}} g(g(\nabla_x(\nabla_x^k u_x), \nabla_x u_x) J_u u_x, \nabla_x^k u_x)\, dx$$

$$+ C\|u_x\|_{H^k}^2. \tag{2.38}$$

Here we rewrite the sum of the third and the fourth term of the RHS by using (2.28) and (2.29), and use the integration by parts and (2.23) to find

$$d_1 \int_{\mathbb{T}} g(g(\nabla_x(\nabla_x^k u_x), u_x) J_u \nabla_x u_x, \nabla_x^k u_x)\, dx$$

$$+ d_1 \int_{\mathbb{T}} g(g(\nabla_x(\nabla_x^k u_x), \nabla_x u_x) J_u u_x, \nabla_x^k u_x)\, dx$$

$$= d_1 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx + \frac{d_1}{2}\int_{\mathbb{T}} g(A_1 \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx$$

$$\leqslant d_1 \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx + C\|u_x\|_{H^k}^2. \tag{2.39}$$

Combining (2.38) and (2.39), we have

$$a\int_{\mathbb{T}} g(J_u \nabla_x^4(\nabla_x^k u_x), \Lambda) dx \leqslant -\frac{d_1}{2}\int_{\mathbb{T}} g(g(\nabla_x^2(\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x)\, dx$$

$$+ \left(d_1 - \frac{3d_2}{4}\right)\int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx$$

$$+ C\|u_x\|_{H^k}^2. \tag{2.40}$$

Therefore, from (2.35), (2.36), and (2.40), it follows that

$$\int_{\mathbb{T}} g(\nabla_t(\nabla_x^k u_x), \Lambda)\, dx$$

$$\leqslant \frac{\varepsilon}{8}\|\nabla_x^2(\nabla_x^k u_x)\|_{L^2}^2 - \frac{d_1}{2}\int_{\mathbb{T}} g(g(\nabla_x^2(\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x)\, dx$$

$$+ \left(d_1 - \frac{3d_2}{4}\right)\int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x(\nabla_x^k u_x), \nabla_x^k u_x)\, dx + C\|u_x\|_{H^k}^2. \tag{2.41}$$

We next evaluate the third term of the RHS of (2.6). For this purpose, we compute $\nabla_t \Lambda$. Using the product formula and noting $\nabla_t u_x = \nabla_x u_t = \mathcal{O}\left(\sum_{m=0}^{4} |\nabla_x^m u_x|_g\right)$, we have

$$
\begin{aligned}
\nabla_t \Lambda &= -\frac{d_1}{2a} g(\nabla_t \nabla_x^{k-2} u_x, J_u u_x) J_u u_x - \frac{d_1}{2a} g(\nabla_x^{k-2} u_x, J_u \nabla_t u_x) J_u u_x \\
&\quad - \frac{d_1}{2a} g(\nabla_x^{k-2} u_x, J_u u_x) J_u \nabla_t u_x + \frac{d_2}{8a} g(u_x, u_x) \nabla_t \nabla_x^{k-2} u_x \\
&\quad + \frac{d_2}{4a} g(\nabla_x u_t, u_x) \nabla_x^{k-2} u_x \\
&= -\frac{d_1}{2a} g(\nabla_t \nabla_x^{k-2} u_x, J_u u_x) J_u u_x + \frac{d_2}{8a} g(u_x, u_x) \nabla_t \nabla_x^{k-2} u_x \\
&\quad + \mathcal{O}\left(|\nabla_x^{k-2} u_x|_g \sum_{m=0}^{4} |\nabla_x^m u_x|_g\right).
\end{aligned}
$$

Thus, we have

$$
\int_{\mathbb{T}} g(\nabla_t \Lambda, V_k)\, dx = Q_{3,1} + Q_{3,2} + Q_{3,2},
$$

where,

$$
\begin{aligned}
Q_{3,1} &= -\frac{d_1}{2a} \int_{\mathbb{T}} g(g(\nabla_t \nabla_x^{k-2} u_x, J_u u_x) J_u u_x, V_k)\, dx, \\
Q_{3,2} &= \frac{d_2}{8a} \int_{\mathbb{T}} g(g(u_x, u_x) \nabla_t \nabla_x^{k-2} u_x, V_k)\, dx, \\
Q_{3,3} &= \int_{\mathbb{T}} g(\mathcal{O}\left(|\nabla_x^{k-2} u_x|_g \sum_{m=0}^{4} |\nabla_x^m u_x|_g\right), V_k)\, dx.
\end{aligned}
$$

For $Q_{3,3}$, since $k \geqslant 4$, we use the Sobolev embedding and the Cauchy-Schwartz inequality to obtain

$$
Q_{3,3} \leqslant C \|u_x\|_{H^k}^2. \tag{2.42}
$$

For $Q_{3,1}$ and $Q_{3,2}$, we need to compute $\nabla_t \nabla_x^{k-2} u_x$. Indeed, by the same computation as that we obtain $\nabla_t(\nabla_x^k u_x)$, we find

$$
\begin{aligned}
\nabla_t \nabla_x^{k-2} u_x &= -\varepsilon \nabla_x^4 (\nabla_x^{k-2} u_x) + a\, J_u \nabla_x^4 (\nabla_x^{k-2} u_x) + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right) \\
&= \varepsilon\, \mathcal{O}(|\nabla_x^{k+2} u_x|_g) + a\, J_u \nabla_x^2 (\nabla_x^k u_x) + \mathcal{O}\left(\sum_{m=0}^{k} |\nabla_x^m u_x|_g\right). \tag{2.43}
\end{aligned}
$$

Applying (2.43), we deduce

$$
\begin{aligned}
Q_{3,1} &= -\frac{d_1}{2a} \int_{\mathbb{T}} g(g(\nabla_t \nabla_x^{k-2} u_x, J_u u_x) J_u u_x, \nabla_x^k u_x + \Lambda)\, dx \\
&\leqslant -\frac{d_1}{2a} \int_{\mathbb{T}} g(g(\nabla_t \nabla_x^{k-2} u_x, J_u u_x) J_u u_x, \nabla_x^k u_x)\, dx + C \|u_x\|_{H^k}^2 \\
&\leqslant \varepsilon \int_{\mathbb{T}} g(\mathcal{O}(|\nabla_x^{k+2} u_x|_g), \nabla_x^k u_x)\, dx
\end{aligned}
$$

$$- \frac{d_1}{2} \int_{\mathbb{T}} g(g(J_u \nabla_x^2 (\nabla_x^k u_x), J_u u_x) J_u u_x, \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$\leqslant \frac{\varepsilon}{8} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 - \frac{d_1}{2} \int_{\mathbb{T}} g(g(\nabla_x^2 (\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2. \quad (2.44)$$

In the same way, applying (2.43), we deduce

$$Q_{3,2} = \frac{d_2}{8a} \int_{\mathbb{T}} g(g(u_x, u_x) \nabla_t \nabla_x^{k-2} u_x, \nabla_x^k u_x + \Lambda) \, dx$$

$$\leqslant \frac{d_2}{8a} \int_{\mathbb{T}} g(g(u_x, u_x) \nabla_t \nabla_x^{k-2} u_x, \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$\leqslant \varepsilon \int_{\mathbb{T}} g(\mathcal{O}(|\nabla_x^{k+2} u_x|_g), \nabla_x^k u_x) \, dx$$

$$+ \frac{d_2}{8} \int_{\mathbb{T}} g(g(u_x, u_x) J_u \nabla_x^2 (\nabla_x^k u_x), \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$\leqslant \frac{\varepsilon}{8} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 + \frac{d_2}{8} \int_{\mathbb{T}} g(\nabla_x \left\{ g(u_x, u_x) J_u \nabla_x (\nabla_x^k u_x) \right\}, \nabla_x^k u_x) \, dx$$

$$- \frac{d_2}{4} \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2$$

$$= \frac{\varepsilon}{8} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 - \frac{d_2}{4} \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx$$

$$+ C \|u_x\|_{H^k}^2. \quad (2.45)$$

Collecting (2.42), (2.44), and (2.45), we obtain

$$\int_{\mathbb{T}} g(\nabla_t \Lambda, V_k) \, dx$$

$$\leqslant \frac{\varepsilon}{4} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 - \frac{d_1}{2} \int_{\mathbb{T}} g(g(\nabla_x^2 (\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x) \, dx$$

$$- \frac{d_2}{4} \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx + C \|u_x\|_{H^k}^2. \quad (2.46)$$

Consequently, collecting the information (2.6), (2.34), (2.41), and (2.46), we derive

$$\frac{1}{2} \frac{d}{dt} \|V_k\|_{L^2}^2 \leqslant -\frac{\varepsilon}{8} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 + (c_1 - d_1) \int_{\mathbb{T}} g(g(\nabla_x^2 (\nabla_x^k u_x), u_x) J_u u_x, \nabla_x^k u_x) \, dx$$

$$+ (c_2 + d_1 - d_2) \int_{\mathbb{T}} g(g(\nabla_x u_x, u_x) J_u \nabla_x (\nabla_x^k u_x), \nabla_x^k u_x) \, dx$$

$$+ C \|u_x\|_{H^k}^2 + C(N_k(u))^2,$$

where c_1 and c_2 are given by (2.31) and (2.32). To cancel the second and the third term of the RHS of above, we set d_1 and d_2 so that

$$d_1 = c_1 = aS + c,$$

$$d_2 = c_2 + d_1 = \left(k + \frac{1}{2} \right) aS + (2k + 1)b + \left(k + \frac{7}{2} \right) c.$$

Therefore, using $\|u_x\|_{H^k} \leqslant C N_k(u)$, we conclude that

$$\frac{1}{2} \frac{d}{dt} \|V_k\|_{L^2}^2 \leqslant -\frac{\varepsilon}{8} \|\nabla_x^2 (\nabla_x^k u_x)\|_{L^2}^2 + C(N_k(u))^2 \quad (2.47)$$

holds on $[0, T_\varepsilon^\star]$

Let us now go back to the original purpose to derive the uniform estimate for $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$. To achieve this, it remains to consider the energy estimate for $\|u_x^\varepsilon\|_{H^{k-1}}^2$. However, by using the integration by parts, the Sobolev embedding, and the Cauchy-Schwartz inequality repeatedly, we can easily show that

$$\frac{1}{2}\frac{d}{dt}\|u_x^\varepsilon\|_{H^{k-1}}^2 \leqslant -\frac{\varepsilon}{2}\sum_{m=0}^{k-1}\|\nabla_x^{m+2}u_x^\varepsilon\|_{L^2}^2 + C\left(N_k(u^\varepsilon)\right)^2. \tag{2.48}$$

Therefore, from (2.47) and (2.48), we conclude that there exits a positive constant C depending on $a, b, c, k, \lambda, S, \|u_{0x}\|_{H^4}$ and not on ε such that

$$\frac{d}{dt}(N_k(u^\varepsilon))^2 = \frac{d}{dt}\left(\|u_x^\varepsilon\|_{H^{k-1}}^2 + \|V_k^\varepsilon\|_{L^2}^2\right) \leqslant C(N_k(u^\varepsilon))^2$$

on the time-interval $[0, T_\varepsilon^\star]$. This implies $(N_k(u^\varepsilon(t)))^2 \leqslant (N_k(u_0))^2 e^{Ct}$ for any $t \in [0, T_\varepsilon^\star]$. Thus, by the definition of $T_\varepsilon^\star$, there holds

$$4(N_4(u_0))^2 = (N_4(u^\varepsilon(T_\varepsilon^\star)))^2 \leqslant (N_4(u_0))^2 e^{C_4 T_\varepsilon^\star}$$

with $C_4 > 0$ which depends on $a, b, c, \lambda, S, \|u_{0x}\|_{H^4}$ and not on ε. This shows $e^{C_4 T_\varepsilon^\star} \geqslant 4$ and hence $T_\varepsilon^\star \geqslant \log 4/C_4$ holds. Therefore, if we set $T = \log 4/C_4$, it follows that $T_\varepsilon^\star \geqslant T$ for any $\varepsilon \in (0, 1]$ and $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T)$.

As stated before, this shows that $\{u_x\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^k(\mathbb{T}; TN))$. Hence the standard compactness argument and the compactness of N show the existence of a map $u \in C([0, T] \times \mathbb{T}; N)$ and a subsequence $\{u^{\varepsilon(j)}\}_{j=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ that satisfy

$$u_x^{\varepsilon(j)} \to u_x \quad \text{in} \quad C([0, T]; H^{k-1}(\mathbb{T}; TN)),$$
$$u_x^{\varepsilon(j)} \to u_x \quad \text{in} \quad L^\infty(0, T; H^k(\mathbb{T}; TN)) \quad \text{weakly star}$$

as $j \to \infty$, and this u is smooth and solves (1.1)-(1.2).

Finally, in the general case where $u_0 \in C(\mathbb{T}; N)$ and $u_{0x} \in H^k(\mathbb{T}; TN)$, we modify the above argument as follows: We take a sequence $\{u_0^i\}_{i=1}^\infty \subset C^\infty(\mathbb{T}; N)$ such that

$$u_{0x}^i \to u_{0x} \quad \text{in} \quad H^k(\mathbb{T}; TN) \tag{2.49}$$

as $i \to \infty$. There exist $T_i = T(\|u_{0x}^i\|_{H^4}) > 0$ and $u^i \in C^\infty([0, T_i] \times \mathbb{T}; N)$ which satisfies (1.1) and $u^i(0, x) = u^i(x)$ for each $i = 1, 2, \ldots$, since $u_0^i \in C^\infty(\mathbb{T}; N)$. Recalling the above argument, it is not difficult to show the estimate $T_i^\star \geqslant \log 4/C_{4,i}$ where

$$T_i^\star = \sup\left\{T > 0 \mid N_4(u^i(t)) \leqslant 2N_4(u_0^i) \quad \text{for all} \quad t \in [0, T]\right\},$$

and $C_{4,i} > 0$ depends on $a, b, c, \lambda, S, \|u_{0x}^i\|_{H^4}$. Note that $C_{4,i}$ depends on $\|u_{0x}^i\|_{H^4}$ continuously. This together with (2.49) shows that there exists $C_4' > 0$ depending on $a, b, c, \lambda, S, \|u_{0x}\|_{H^4}$ and not on i such that $T_i^\star \geqslant \log 4/C_4'$ for sufficient large i. Therefore, if we set $T = \log 4/C_4'$, there exists a sufficiently large $i_0 \in \mathbb{N}$ such that $T_i^\star \geqslant T$ for any $i \geqslant i_0$ and $\{N_k(u^i)\}_{i=i_0}^\infty$ is bounded in $L^\infty(0, T)$. Therefore, by applying the compactness argument again, we can construct the desired solution to (1.1)-(1.2). This completes the proof. $\qquad\square$

3. PROOF OF THEOREM 1.1

The goal of this section is to complete the proof of Theorem 1.1. Throughout this section, it is assumed that $k \geqslant 6$.

Proof of Theorem 1.1. Since $k \geqslant 6 \geqslant 4$, Theorem 2.1 established in Section 2 guarantees the existence of $T = T(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 0$ and a map $u \in C([0,T] \times \mathbb{T}; N)$ so that $u_x \in L^\infty(0,T; H^k(\mathbb{T};TN)) \cap C([0,T]; H^{k-1}(\mathbb{T};TN))$ and u solves (1.1)-(1.2) on the time-interval $[0,T]$. In what follows, we shall concentrate on the proof of the uniqueness of the solution. Once the uniqueness is established, we can easily prove the time-continuity of $\nabla_x^k u_x$ in L^2 by the standard argument, which implies $u_x \in C([0,T]; H^k(\mathbb{T};TN))$. In this way, the proof of Theorem 1.1 is completed.

Let u,v be solutions constructed in Theorem 2.1. Then u and v solve (1.1)-(1.2) and satisfy $u_x, v_x \in L^\infty(0,T; H^6(\mathbb{T};TN)) \cap C([0,T]; H^5(\mathbb{T};TN))$. We shall show $u = v$. For this purpose, fix w as an isometric embedding of (N,g) into some Euclidean space $\mathbb{R}^d$ so that N is considered as a submanifold of $\mathbb{R}^d$. We set $U = w{\circ}u$, $V = w{\circ}v$, $Z = U - V$, $\mathcal{U} = dw_u(\nabla_x u_x)$, $\mathcal{V} = dw_v(\nabla_x v_x)$, and $\mathcal{W} = \mathcal{U} - \mathcal{V}$. To prove $u = v$, it suffices to show $Z = 0$. The proof of $Z = 0$ consists of the following four steps:

1. Notations and tools of computations used below.
2. Analysis of the partial differential equation satisfied by $\mathcal{U}$.
3. Classical energy estimates for $\|\mathcal{W}\|_{L^2(\mathbb{T};\mathbb{R}^d)}$ with the loss of derivatives.
4. Energy estimates for $\|\widetilde{\mathcal{W}}\|_{L^2(\mathbb{T};\mathbb{R}^d)}$ (defined later) to eliminate the loss of derivatives

1. Notations and tools of computations used below.
We state some notations and gather tools of computations which will be used below.

The inner product and the norm in $\mathbb{R}^d$ is expressed by $(\cdot,\cdot)$ and $|\cdot|$ respectively. The inner product and the norm in L^2 for $\mathbb{R}^d$-valued functions on $\mathbb{T}$ is expressed by $\langle\cdot\rangle$ and $\|\cdot\|_{L^2}$ respectively. That is, for $\phi,\psi : \mathbb{T} \to \mathbb{R}^d$, $\langle\phi,\psi\rangle$ and $\|\phi\|_{L^2}$ is given by $\langle\phi,\psi\rangle = \int_{\mathbb{T}}(\phi(x),\psi(x))\,dx$ and $\|\phi\|_{L^2} = \sqrt{\langle\phi,\phi\rangle}$ respectively.

Let $p \in N$ be a fixed point. We consider the orthogonal decomposition $\mathbb{R}^d = dw_p(T_pN) \oplus (dw_p(T_pN))^\perp$, where $dw_p : T_pN \to T_{w{\circ}p}\mathbb{R}^d \cong \mathbb{R}^d$ is the differential of $w : N \to \mathbb{R}^d$ at $p \in N$ and $(dw_p(T_pN))^\perp$ is the orthogonal complement of $dw_p(T_pN)$ in $\mathbb{R}^d$. We denote the orthogonal projection mapping of $\mathbb{R}^d$ onto $dw_p(T_pN)$ by $P(w{\circ}p)$ and define $N(w{\circ}p)$ by $N(w{\circ}p) = I_d - P(w{\circ}p)$, where I_d is the identity mapping on $\mathbb{R}^d$. Moreover, we define $J(w{\circ}p)$ as an action on $\mathbb{R}^d$ by first projecting onto $dw_p(T_pN)$ and then applying the complex structure at $p \in N$. More precisely, we define $J(w{\circ}p)$ by

$$J(w{\circ}p) = dw_p \circ J_p \circ dw_{w{\circ}p}^{-1} \circ P(w{\circ}p). \tag{3.1}$$

We can extend $P(\cdot)$, $N(\cdot)$, and $J(\cdot)$ to a smooth linear operator on $\mathbb{R}^d$ so that $P(q)$, $N(q)$, and $J(q)$ make sense for all $q \in \mathbb{R}^d$ following the argument in e.g. [14, pp.17]. Though $J(q)$ is not skew-symmetric and the square is not the minus identity in general, similar properties hold if q is restricted to $w(N)$. Indeed, from the definition of $P(\omega{\circ}p)$ and $J(\omega{\circ}p)$, it follows that

$$(P(w{\circ}p)Y_1, Y_2) = (Y_1, P(w{\circ}p)Y_2), \tag{3.2}$$

$$(J(w{\circ}p)Y_1, Y_2) = -(Y_1, J(w{\circ}p)Y_2), \tag{3.3}$$

$$(J(w{\circ}p))^2 Y_3 = -P(w{\circ}p)Y_3, \tag{3.4}$$

for any $p \in N$ and $Y_1, Y_2 \in \mathbb{R}^d$.

Let $Y \in \Gamma(u^{-1}TN)$ be fixed. For $(t,x) \in [0,T] \times \mathbb{T}$, let $\{\nu_3,\ldots,\nu_d\}$ denote a smooth local orthonormal frame field for the normal bundle $(dw(TN))^\perp$ near $U(t,x) = w{\circ}u(t,x) \in w(N)$. Recalling that $dw_u(\nabla_x Y)$ is the $dw_u(T_u N)$-component of $\partial_x(dw_u(Y))$, we see

$$dw_u(\nabla_x Y) = \partial_x(dw_u(Y)) - \sum_{k=3}^{d}(\partial_x(dw_u(Y)), \nu_k(U))\nu_k(U)$$

$$= \partial_x \left(dw_u(Y) \right) + \sum_{k=3}^{d} (dw_u(Y), \partial_x \left(\nu_k(U) \right)) \nu_k(U)$$

$$= \partial_x \left(dw_u(Y) \right) + \sum_{k=3}^{d} (dw_u(Y), D_k(U)U_x) \nu_k(U)$$

$$= \partial_x \left(dw_u(Y) \right) + A(U)(dw_u(Y), U_x), \tag{3.5}$$

where $D_k = \operatorname{grad} \nu_k$ for $k = 3, \ldots, d$ and $A(q)(\cdot, \cdot) = \sum_{k=3}^{d} (\cdot, D_k(q)\cdot) \nu_k(q)$ is the second fundamental form at $q \in w(N)$. In the same way, only by replacing x with t, we see

$$dw_u(\nabla_t Y) = \partial_t \left(dw_u(Y) \right) + \sum_{k=3}^{d} (dw_u(Y), D_k(U)U_t) \nu_k(U). \tag{3.6}$$

The Sobolev embedding and the Gagliardo-Nirenberg inequality lead to the equivalence between $U_x, V_x \in L^\infty(0, T; H^6(\mathbb{T}; \mathbb{R}^d))$ and $u_x, v_x \in L^\infty(0, T; H^6(\mathbb{T}; TN))$. In particular, from the Sobolev embedding $H^1(\mathbb{T})$ into $C(\mathbb{T})$, it follows that $\partial_x^k U_x, \partial_x^k V_x \in L^\infty((0, T) \times \mathbb{T}; \mathbb{R}^d)$ for $k = 0, 1, \ldots, 5$, which will be used below without any comments.

We next observe some properties related to ν_k and D_k for $k = 3 \ldots, d$.

Lemma 3.1. *For each $(t, x) \in [0, T] \times \mathbb{T}$, the following properties hold.*

$$J(U)\nu_k(U) = 0, \tag{3.7}$$

$$(\nu_k(U), \mathcal{W}) = -(\nu_k(U) - \nu_k(V), \mathcal{V}) = \mathcal{O}(|Z|), \tag{3.8}$$

$$(\nu_k(U), \partial_x \mathcal{W}) = -(D_k(U)U_x, \mathcal{W}) - (D_k(U)Z_x, \mathcal{V}) + \mathcal{O}(|Z|), \tag{3.9}$$

$$(\nu_k(U), \partial_x^2 \mathcal{W}) = -2 \left(D_k(U)U_x, \partial_x \mathcal{W} \right) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|), \tag{3.10}$$

$$(D_k(U)Y_1, Y_2) = (Y_1, D_k(U)Y_2) \quad \text{for any } Y_1, Y_2 : [0, T] \times \mathbb{T} \to \mathbb{R}^d. \tag{3.11}$$

Remark 3.2. In particular, in view of (3.8), we find (see the argument to show (3.66) in the third step) that the term including $\partial_x^2 \mathcal{W}$ or $\partial_x \mathcal{W}$ can be handled as a harmless term if the vector part is described by $\nu_k(U)$ with some $k = 3, \ldots, d$. This is related to the reason why we choose $dw_u(\nabla_x u_x) - dw_v(\nabla_x v_x)$ as $\mathcal{W}$.

Proof of Lemma 3.1. First, (3.7) is a direct consequence of the definition of $J(U)$ and the orthogonality $\nu_k(U) \perp dw_u(T_u N)$. Next, in view of $(\nu_k(U), \mathcal{U}) = (\nu_k(V), \mathcal{V}) = 0$, we have

$$(\nu_k(U), \mathcal{W}) = (\nu_k(U), \mathcal{U} - \mathcal{V}) = -(\nu_k(U), \mathcal{V}) = -(\nu_k(U) - \nu_k(V), \mathcal{V}),$$

which shows (3.8). Moreover, by taking the derivative of (3.8) in x, we have

$$(\nu_k(U), \partial_x \mathcal{W})$$
$$= \partial_x \left\{ (\nu_k(U), \mathcal{W}) \right\} - (\partial_x \left\{ \nu_k(U) \right\}, \mathcal{W})$$
$$= -(\partial_x \left\{ \nu_k(U) - \nu_k(V) \right\}, \mathcal{V}) - (\nu_k(U) - \nu_k(V), \partial_x \mathcal{V}) - (D_k(U)U_x, \mathcal{W})$$
$$= -(D_k(U)U_x - D_k(V)V_x, \mathcal{V}) - (\nu_k(U) - \nu_k(V), \partial_x \mathcal{V}) - (D_k(U)U_x, \mathcal{W})$$
$$= -(D_k(U)Z_x - (D_k(U) - D_k(V))V_x, \mathcal{V}) - (\nu_k(U) - \nu_k(V), \partial_x \mathcal{V}) - (D_k(U)U_x, \mathcal{W})$$
$$= -(D_k(U)U_x, \mathcal{W}) - (D_k(U)Z_x, \mathcal{V}) + \mathcal{O}(|Z|),$$

which shows (3.9). We obtain (3.10), by taking the derivative of (3.9) in x. We omit the proof of (3.11), since it has been proved in [16, pp.912]. $\qquad\square$

The following lemma comes from the Kähler condition on (N, J, g).

Lemma 3.3. *(i): For any* $Y \in \Gamma(u^{-1}TN)$,

$$\partial_x(J(U))dw_u(Y) = \sum_{k=3}^{d} (dw_u(Y), J(U)D_k(U)U_x)\, \nu_k(U). \tag{3.12}$$

(ii): For any $Y : [0, T] \times \mathbb{T} \to \mathbb{R}^d$,

$$\partial_x(J(U))Y = \sum_{k=3}^{d} (Y, J(U)D_k(U)U_x)\, \nu_k(U) - \sum_{k=3}^{d} (Y, \nu_k(U))J(U)D_k(U)U_x. \tag{3.13}$$

Remark 3.4. Using (3.13) combined with (3.8), we can handle the term $\partial_x(J(U))\partial_x W$ as a harmless term in the energy estimate for $\mathcal{W}$ in L^2.

Proof of Lemma 3.3. First we show (i). For $Y \in \Gamma(u^{-1}TN)$, the Kähler condition on (N, J, g) implies $\nabla_x J_u Y = J_u \nabla_x Y$. Hence there holds

$$dw_u(\nabla_x J_u Y) = dw_u(J_u \nabla_x Y). \tag{3.14}$$

From (3.5) and (3.7), the RHS of (3.14) satisfies

$$dw_u(J_u \nabla_x Y) = J(U)dw_u(\nabla_x Y) = J(U)\partial_x(dw_u(Y)). \tag{3.15}$$

On the other hand, from (3.5), the left hand side of (3.14) satisfies

$$dw_u(\nabla_x J_u Y) = \partial_x \left\{ dw_u(J_u Y) \right\} + \sum_{k=3}^{d} (dw_u(J_u Y), D_k(U)U_x)\, \nu_k(U)$$

$$= \partial_x \left\{ J(U)dw_u(Y) \right\} + \sum_{k=3}^{d} (J(U)dw_u(Y), D_k(U)U_x)\, \nu_k(U)$$

$$= \partial_x(J(U))dw_u(Y) + J(U)\partial_x(dw_u(Y))$$

$$+ \sum_{k=3}^{d} (J(U)dw_u(Y), D_k(U)U_x)\, \nu_k(U). \tag{3.16}$$

By substituting (3.15) and (3.16) into (3.14), and by using (3.3), we have

$$\partial_x(J(U))dw_u(Y) = - \sum_{k=3}^{d} (J(U)dw_u(Y), D_k(U)U_x)\, \nu_k(U)$$

$$= \sum_{k=3}^{d} (dw_u(Y), J(U)D_k(U)U_x)\, \nu_k(U),$$

which shows (3.12). Next we show (ii). Decomposing $Y = P(U)Y + N(U)Y$ where $P(U)Y \in dw(T_u N)$ and $N(U)Y \in (dw(T_u N))^{\perp}$ for each (t, x), we have

$$\partial_x(J(U))Y = \partial_x(J(U))P(U)Y + \partial_x(J(U))N(U)Y. \tag{3.17}$$

By using (3.12) and by noting that $N(U)Y$ is perpendicular to $J(U)D_k(U)U_x$, we find that the first term of the RHS of (3.17) satisfies

$$\partial_x(J(U))P(U)Y = \sum_{k=3}^{d} (P(U)Y, J(U)D_k(U)U_x)\, \nu_k(U)$$

$$= \sum_{k=3}^{d} (Y, J(U)D_k(U)U_x)\, \nu_k(U). \tag{3.18}$$

Moreover, since

$$\partial_x(J(U))\nu_k(U) = \partial_x(J(U)\nu_k(U)) - J(U)\partial_x(\nu_k(U)) = -J(U)D_k(U)U_x \tag{3.19}$$

follows from (3.7), the second term of the RHS of (3.17) satisfies

$$\partial_x(J(U))N(U)Y = \sum_{k=3}^d (Y, \nu_k(U))\partial_x(J(U))\nu_k(U)$$

$$= -\sum_{k=3}^d (Y, \nu_k(U))J(U)D_k(U)U_x. \tag{3.20}$$

Substituting (3.18) and (3.20) into (3.17), we obtain (3.13). $\qquad\square$

As in the proof of Theorem 2.1, we denote the sectional curvature on (N, g) by S which is supposed to be a constant. Recall that the Riemann curvature tensor R is expressed by

$$R(Y_1, Y_2)Y_3 = S\left\{g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2\right\} \tag{3.21}$$

for any $Y_1, Y_2, Y_3 \in \Gamma(u^{-1}TN)$. The next lemma comes from (3.21).

Lemma 3.5. *For any $Y_1, Y_2, Y_3 \in \Gamma(u^{-1}TN)$,*

$$dw_u\left(R(Y_1, Y_2)Y_3\right) = \sum_k \left(dw_u(Y_3), D_k(U)dw_u(Y_2)\right) P(U)D_k(U)dw_u(Y_1)$$

$$- \sum_k \left(dw_u(Y_3), D_k(U)dw_u(Y_1)\right) P(U)D_k(U)dw_u(Y_2), \tag{3.22}$$

$$\sum_k \left(dw_u(Y_3), D_k(U)dw_u(Y_2)\right) P(U)D_k(U)dw_u(Y_1)$$

$$- \sum_k \left(dw_u(Y_3), D_k(U)dw_u(Y_1)\right) P(U)D_k(U)dw_u(Y_2)$$

$$= S\left\{(dw_u(Y_3), dw_u(Y_2))dw_u(Y_1) - (dw_u(Y_3), dw_u(Y_1))dw_u(Y_2)\right\} \tag{3.23}$$

Proof of Lemma 3.5. We can understand that (3.22) is a kind of the expression of the Gauss-Codazzi formula in Riemannian geometry. Fix $(t, x) \in [0, T] \times \mathbb{T}$. We take a two-parameterized smooth map $\gamma = \gamma(s, \sigma) : (-\delta, \delta) \times (-\delta, \delta) \to N$ with sufficiently small $\delta > 0$, and a $Y_4 \in \Gamma(\gamma^{-1}TN)$ so that $\gamma(0, 0) = u(t, x)$, $\gamma_s(0, 0) = Y_1(t, x)$, $\gamma_\sigma(0, 0) = Y_2(t, x)$, and $Y_4(0, 0) = Y_3(t, x)$. Since $R(\gamma_s, \gamma_\sigma)Y_4 = \nabla_s\nabla_\sigma Y_4 - \nabla_\sigma\nabla_s Y_4$, we deduce

$$dw_\gamma\left(R(\gamma_s, \gamma_\sigma)Y_4\right) = dw_\gamma\left(\nabla_s\nabla_\sigma Y_4\right) - dw_\gamma\left(\nabla_\sigma\nabla_s Y_4\right)$$

$$= P(w\circ\gamma)\partial_s\left(dw_\gamma(\nabla_\sigma Y_4)\right) - P(w\circ\gamma)\partial_\sigma\left(dw_\gamma(\nabla_s Y_4)\right)$$

$$= P(w\circ\gamma)\left\{\partial_s\left(dw_\gamma(\nabla_\sigma Y_4)\right) - \partial_\sigma\left(dw_\gamma(\nabla_s Y_4)\right)\right\}. \tag{3.24}$$

Similarly to (3.5) or (3.6), the definition of the covariant derivatives yields

$$\partial_s\left(dw_\gamma(\nabla_\sigma Y_4)\right) = \partial_s\left(\partial_\sigma(dw_\gamma(Y_4)) + \sum_{k=3}^d \left(dw_\gamma(Y_4), D_k(w\circ\gamma)\partial_\sigma(w\circ\gamma)\right)\nu_k(w\circ\gamma)\right)$$

$$= \partial_s\partial_\sigma(dw_\gamma(Y_4)) + \sum_{k=3}^d \partial_s\left\{(dw_\gamma(Y_4), D_k(w\circ\gamma)\partial_\sigma(w\circ\gamma))\right\}\nu_k(w\circ\gamma)$$

$$+ \sum_{k=3}^d \left(dw_\gamma(Y_4), D_k(w\circ\gamma)\partial_\sigma(w\circ\gamma)\right) D_k(w\circ\gamma)\partial_s(w\circ\gamma), \tag{3.25}$$

and

$$\partial_\sigma \left(dw_\gamma(\nabla_s Y_4) \right) = \partial_\sigma \partial_s (dw_\gamma(Y_4)) + \sum_{k=3}^{d} \partial_\sigma \left\{ (dw_\gamma(Y_4), D_k(w{\circ}\gamma)\partial_s(w{\circ}\gamma)) \right\} \nu_k(w{\circ}\gamma)$$

$$+ \sum_{k=3}^{d} (dw_\gamma(Y_4), D_k(w{\circ}\gamma)\partial_s(w{\circ}\gamma)) \, D_k(w{\circ}\gamma)\partial_\sigma(w{\circ}\gamma). \tag{3.26}$$

By substituting (3.25) and (3.26) into (3.24), and by noting $P(w{\circ}\gamma)\nu_k(w{\circ}\gamma) = 0$, we have

$$dw_\gamma \left(R(\gamma_s, \gamma_\sigma)Y_4 \right) = \sum_{k=3}^{d} (dw_\gamma(Y_4), D_k(w{\circ}\gamma)\partial_\sigma(w{\circ}\gamma)) \, P(w{\circ}\gamma)D_k(w{\circ}\gamma)\partial_s(w{\circ}\gamma)$$

$$- \sum_{k=3}^{d} (dw_\gamma(Y_4), D_k(w{\circ}\gamma)\partial_s(w{\circ}\gamma)) \, P(w{\circ}\gamma)D_k(w{\circ}\gamma)\partial_\sigma(w{\circ}\gamma).$$

Thus, by taking the limit $(s, \sigma) \to (0,0)$, we obtain

$$dw_u \left(R(Y_1, Y_2)Y_3 \right) = \sum_{k=3}^{d} (dw_u(Y_3), D_k(w{\circ}u)dw_u(Y_2)) \, P(w{\circ}u)D_k(w{\circ}u)dw_u(Y_1)$$

$$- \sum_{k=3}^{d} (dw_u(Y_3), D_k(w{\circ}u)dw_u(Y_1)) \, P(w{\circ}u)D_k(w{\circ}u)dw_u(Y_2)$$

for each (t,x). This implies (3.22). Noting that $w : (N, g) \to (\mathbb{R}^d, (\cdot, \cdot))$ is isometric, we see that (3.23) follows from (3.21) and (3.22). $\qquad\square$

Next properties come from the assumption that N is a two-dimensional real manifold. Noting that $\left\{ \frac{U_x}{|U_x|}, \frac{J(U)U_x}{|U_x|}, \nu_3(U), \ldots, \nu_d(U) \right\}$ forms an orthonormal basis of $\mathbb{R}^d$ for each (t,x) where $U_x(t,x) \neq 0$, we see

$$|U_x|^2 Y = (Y, U_x)U_x + (Y, J(U)U_x)J(U)U_x + \sum_{k=3}^{d}(|U_x|^2 Y, \nu_k(U))\nu_k(U) \tag{3.27}$$

holds for every (t,x). Note also that (3.27) is valid also for (t,x) where $U_x(t,x) = 0$, as each of both sides of (3.27) vanishes. Using (3.27) with $J(U)Y$ instead of Y, we have

$$|U_x|^2 J(U)Y = (J(U)Y, U_x)U_x + (J(U)Y, J(U)U_x)J(U)U_x$$

$$+ \sum_{k=3}^{d}(|U_x|^2 J(U)Y, \nu_k(U))\nu_k(U)$$

$$= -(Y, J(U)U_x)U_x + (Y, U_x)J(U)U_x. \tag{3.28}$$

Moreover, we introduce $T_2(U), \ldots, T_5(U) : [0,T] \times \mathbb{T} \to \mathbb{R}^d$ defined by the following.

Definition 3.6. For any $Y : [0,T] \times \mathbb{T} \to \mathbb{R}^d$,

$$T_2(U)Y = \frac{1}{2}|U_x|^2 J(U)Y, \tag{3.29}$$

$$T_3(U)Y = \frac{1}{2}\left\{ (Y, \partial_x U_x)J(U)U_x + (Y, U_x)J(U)\partial_x U_x \right.$$

$$\left. + (Y, J(U)\partial_x U_x)U_x + (Y, J(U)U_x)\partial_x U_x \right\}, \tag{3.30}$$

$$T_4(U)Y = \left(Y, \partial_x U_x + \sum_{k=3}^{d}(U_x, D_k(U)U_x)\nu_k(U)\right)J(U)U_x - (Y, U_x)J(U)\partial_x U_x, \quad (3.31)$$

$$T_5(U)Y = \frac{1}{2}\bigg\{(Y, \partial_x U_x)J(U)U_x + (Y, U_x)J(U)\partial_x U_x$$

$$- (Y, J(U)\partial_x U_x)U_x - (Y, J(U)U_x)\partial_x U_x\bigg\}. \quad (3.32)$$

We use (3.27) or (3.28) to show the following.

Lemma 3.7. *For any $Y, Y_1, Y_2 : [0, T] \times \mathbb{T} \to \mathbb{R}^d$, it follows that*

$$T_2(U)Y = \frac{1}{2}\{(Y, U_x)J(U)U_x - (Y, J(U)U_x)U_x\}, \quad (3.33)$$

$$\partial_x(T_2(U))Y = (\partial_x U_x, U_x)J(U)Y + \frac{1}{2}|U_x|^2\partial_x(J(U))Y, \quad (3.34)$$

$$\partial_x(T_2(U))Y = T_5(U)Y - \frac{1}{2}(Y, U_x)\sum_{k=3}^{d}(J(U)U_x, D_k(U)U_x)\nu_k(U)$$

$$+ \frac{1}{2}\sum_{k=3}^{d}(Y, \nu_k(U))(J(U)U_x, D_k(U)U_x)U_x, \quad (3.35)$$

$$(T_3(U)Y_1, Y_2) = (Y_1, T_3(U)Y_2), \quad (3.36)$$

$$(T_4(U)Y_1, Y_2) = (Y_1, T_4(U)Y_2). \quad (3.37)$$

Proof of Lemma 3.7. First, (3.33) is a direct consequence of (3.28). Second, (3.34) follows from substituting (3.29) into $\partial_x(T_2(U))Y = \partial_x\{T_2(U)Y\} - T_2(U)\partial_x Y$. Third, by substituting (3.33) into $\partial_x(T_2(U))Y = \partial_x\{T_2(U)Y\} - T_2(U)\partial_x Y$ and by using (3.32), we have

$$\partial_x(T_2(U))Y = T_5(U)Y + \frac{1}{2}(Y, U_x)\partial_x(J(U))U_x - \frac{1}{2}(Y, \partial_x(J(U))U_x)U_x. \quad (3.38)$$

Recall here that (3.13) yields $\partial_x(J(U))U_x = \sum_{k=3}^{d}(U_x, J(U)D_k(U)U_x)\nu_k(U)$. Substituting this into the RHS of (3.38), we get (3.35). Next, in view of (3.30), it is immediate that (3.36) holds. Finally we show (3.37). The proof is reduced to that of (2.23) with $i = 2$. Noting that there exists $\Xi_i \in \Gamma(u^{-1}TN)$ such that $dw_u(\Xi_i) = P(U)Y_i$ for each $i = 1, 2$, we have

$$T_4(U)Y_1 = (Y_1, dw_u(\nabla_x u_x))dw_u(J_u u_x) - (Y_1, dw_u(u_x))dw_u(J_u \nabla_x u_x)$$

$$= (P(U)Y_1, dw_u(\nabla_x u_x))dw_u(J_u u_x) - (P(U)Y_1, dw_u(u_x))dw_u(J_u \nabla_x u_x)$$

$$= (dw_u(\Xi_1), dw_u(\nabla_x u_x))dw_u(J_u u_x) - (dw_u(\Xi_1), dw_u(u_x))dw_u(J_u \nabla_x u_x).$$

Since w is an isometric, this shows

$$T_4(U)Y_1 = dw_u\{g(\Xi_1, \nabla_x u_x)J_u u_x - g(\Xi_1, u_x)J_u \nabla_x u_x\},$$

and thus we obtain

$$(T_4(U)Y_1, Y_2) = (dw_u\{g(\Xi_1, \nabla_x u_x)J_u u_x - g(\Xi_1, u_x)J_u \nabla_x u_x\}, P(U)Y_2)$$

$$= (dw_u\{g(\Xi_1, \nabla_x u_x)J_u u_x - g(\Xi_1, u_x)J_u \nabla_x u_x\}, dw_u(\Xi_2))$$

$$= g(g(\Xi_1, \nabla_x u_x)J_u u_x - g(\Xi_1, u_x)J_u \nabla_x u_x, \Xi_2)$$

$$= g(A_2\Xi_1, \Xi_2).$$

Since $g(A_2\Xi_1, \Xi_2) = g(\Xi_1, A_2\Xi_2)$ follows from (2.23), we conclude that (3.37) holds. $\square$

In what follows, for simplicity, we sometimes write dw instead of dw_u or dw_v and use the expression $\sum_k$ and $\sum_{k,\ell}$ instead of $\sum_{k=3}^{d}$ and $\sum_{k,\ell=3}^{d}$ respectively. Any confusion will not occur.

2. Analysis of the partial differential equation satisfied by $\mathcal{U}$.
We compute the PDE satisfied by $\mathcal{U}$.

First, we start by the computation of the PDE satisfied by U. Since u satisfies (1.1),

$$
\begin{aligned}
U_t &= dw(u_t)\\
&= a\,dw(\nabla_x J_u \nabla_x^2 u_x) + \lambda\,dw(J_u \nabla_x u_x)\\
&\quad + b\,dw(g(u_x,u_x)J_u\nabla_x u_x) + c\,dw(g(\nabla_x u_x,u_x)J_u u_x)\\
&= a\,dw(\nabla_x J_u \nabla_x^2 u_x) + \lambda\,J(U)dw(\nabla_x u_x)\\
&\quad + b\,(dw(u_x),dw(u_x))J(U)dw(\nabla_x u_x) + c\,(dw(\nabla_x u_x),dw(u_x))J(U)dw(u_x)\\
&= a\,dw(\nabla_x J_u \nabla_x^2 u_x) + \left\{\lambda + b|U_x|^2\right\}J(U)\mathcal{U} + c\,(\mathcal{U},U_x)J(U)U_x. \qquad (3.39)
\end{aligned}
$$

Using (3.5) and (3.7) repeatedly, we have

$$
\begin{aligned}
dw(\nabla_x^2 u_x) &= \partial_x\left(dw(\nabla_x u_x)\right) + \sum_\ell (dw(\nabla_x u_x),D_\ell(U)U_x)\nu_\ell(U)\\
&= \partial_x \mathcal{U} + \sum_\ell (\mathcal{U},D_\ell(U)U_x)\nu_\ell(U), \qquad (3.40)
\end{aligned}
$$

$$
J(U)dw(\nabla_x^2 u_x) = J(U)\partial_x\mathcal{U} + \sum_\ell (\mathcal{U},D_\ell(U)U_x)J(U)\nu_\ell(U) = J(U)\partial_x\mathcal{U}, \qquad (3.41)
$$

$$
\begin{aligned}
dw(\nabla_x J_u \nabla_x^2 u_x) &= \partial_x\left(dw(J_u\nabla_x^2 u_x)\right) + \sum_k (dw(J_u\nabla_x^2 u_x),D_k(U)U_x)\nu_k(U)\\
&= \partial_x\left(J(U)dw(\nabla_x^2 u_x)\right) + \sum_k (J(U)dw(\nabla_x^2 u_x),D_k(U)U_x)\nu_k(U)\\
&= \partial_x\left(J(U)\partial_x\mathcal{U}\right) + \sum_k (J(U)\partial_x\mathcal{U},D_k(U)U_x)\nu_k(U). \qquad (3.42)
\end{aligned}
$$

From (3.39) and (3.42), we have

$$
U_t = a\,\partial_x\left(J(U)\partial_x\mathcal{U}\right) + a\sum_k (J(U)\partial_x\mathcal{U},D_k(U)U_x)\nu_k(U) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \quad (3.43)
$$

Next, we compute the PDE satisfied by $\mathcal{U}$. From (1.1), (3.21), and (3.6), it follows that

$$
\begin{aligned}
\partial_t\mathcal{U} &= \partial_t(dw(\nabla_x u_x))\\
&= dw(\nabla_t\nabla_x u_x) - \sum_k (dw(\nabla_x u_x),D_k(U)U_t)\nu_k(U)\\
&= dw(\nabla_x^2 u_t + R(u_t,u_x)u_x) - \sum_k (\mathcal{U},D_k(U)U_t)\nu_k(U)\\
&= dw(\nabla_x^2 u_t + S\left\{g(u_x,u_x)u_t - g(u_t,u_x)u_x\right\}) - \sum_k (\mathcal{U},D_k(U)U_t)\nu_k(U)\\
&= dw(\nabla_x^2 u_t) + S\left\{(dw(u_x),dw(u_x))dw(u_t) - (dw(u_t),dw(u_x))dw(u_x)\right\}\\
&\quad - \sum_k (\mathcal{U},D_k(U)U_t)\nu_k(U)
\end{aligned}
$$

$$= dw(\nabla_x^2 u_t) + S|U_x|^2 U_t - S(U_x, U_t)U_x - \sum_k (\mathcal{U}, D_k(U)U_t)\nu_k(U)$$

$$=: I + II + III + IV. \tag{3.44}$$

We compute II, III, IV, and I in order.

Applying (3.43), we have

$$II = aS \, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{U} \right\} - 2aS \, (\partial_x U_x, U_x)J(U)\partial_x \mathcal{U}$$
$$+ aS \, |U_x|^2 \sum_k (J(U)\partial_x \mathcal{U}, D_k(U)U_x)\nu_k(U) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.45}$$

In the same way, by noting $(U_x, \nu_k(U)) = 0$ and (3.2), we obtain

$$III = -aS \, (U_x, \partial_x (J(U)\partial_x \mathcal{U})) \, U_x + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$
$$= -aS \, (J(U)\partial_x^2 \mathcal{U}, U_x)U_x - aS \, (\partial_x(J(U))\partial_x \mathcal{U}, U_x)U_x + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$
$$= aS \, (\partial_x^2 \mathcal{U}, J(U)U_x)U_x - aS \, (\partial_x(J(U))\partial_x \mathcal{U}, U_x)U_x + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.46}$$

Furthermore, by substituting (3.28) with $Y = \partial_x^2 \mathcal{U}$ into the first term of the RHS of (3.46),

$$III = aS \, (\partial_x^2 \mathcal{U}, U_x)J(U)U_x - aS \, |U_x|^2 J(U)\partial_x^2 \mathcal{U} - aS \, (\partial_x(J(U))\partial_x \mathcal{U}, U_x)U_x$$
$$+ \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$
$$= aS \, (\partial_x^2 \mathcal{U}, U_x)J(U)U_x - aS \, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{U} \right\}$$
$$+ 2aS \, (\partial_x U_x, U_x)J(U)\partial_x \mathcal{U} + aS \, |U_x|^2 \partial_x(J(U))\partial_x \mathcal{U}$$
$$- aS \, (\partial_x(J(U))\partial_x \mathcal{U}, U_x)U_x + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.47}$$

In the same way, by applying (3.43),

$$IV = -a \sum_k (\mathcal{U}, D_k(U)\partial_x (J(U)\partial_x \mathcal{U})) \, \nu_k(U)$$
$$- a \sum_k \left(\mathcal{U}, D_k(U) \sum_\ell (J(U)\partial_x \mathcal{U}, D_\ell(U)U_x)\nu_\ell(U) \right) \nu_k(U)$$
$$+ \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$
$$= -a \sum_k (\mathcal{U}, D_k(U)\partial_x (J(U)\partial_x \mathcal{U})) \, \nu_k(U)$$
$$- a \sum_{k,\ell} (\mathcal{U}, D_k(U)\nu_\ell(U))(J(U)\partial_x \mathcal{U}, D_\ell(U)U_x)\nu_k(U)$$
$$+ \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.48}$$

Let us now move on to the computation of I. We start with

$$I = dw(\nabla_x^2 u_t)$$
$$= a \, dw(\nabla_x^4 J_u \nabla_x u_x) + \lambda \, dw(\nabla_x^2 J_u \nabla_x u_x)$$
$$+ b \, dw \left(\nabla_x^2 \{ g(u_x, u_x) J_u \nabla_x u_x \} \right) + c \, dw \left(\nabla_x^2 \{ g(\nabla_x u_x, u_x) J_u u_x \} \right)$$
$$=: I_1 + I_2 + I_3 + I_4. \tag{3.49}$$

We compute I_2, I_3, I_4, and I_1 in order.

For I_2, we have

$$I_2 = \lambda \, dw \left(\nabla_x (\nabla_x J_u \nabla_x u_x) \right)$$

$$= \lambda \, \partial_x \left(dw(\nabla_x J_u \nabla_x u_x) \right) + \lambda \sum_\ell \left(dw(\nabla_x J_u \nabla_x u_x), D_\ell(U) U_x \right) \nu_\ell(U).$$

Since

$$dw(\nabla_x J_u \nabla_x u_x) = \partial_x \left(dw(J_u \nabla_x u_x) \right) + \sum_k \left(dw(J_u \nabla_x u_x), D_k(U) U_x \right) \nu_k(U)$$

$$= \partial_x \left(J(U)\mathcal{U} \right) + \sum_k \left(J(U)\mathcal{U}, D_k(U) U_x \right) \nu_k(U)$$

$$= J(U)\partial_x \mathcal{U} + \partial_x(J(U))\mathcal{U} + \sum_k \left(J(U)\mathcal{U}, D_k(U) U_x \right) \nu_k(U), \qquad (3.50)$$

we obtain

$$I_2 = \lambda \, \partial_x \left\{ J(U)\partial_x \mathcal{U} \right\} + \lambda \, \partial_x(J(U))\partial_x \mathcal{U} + 2\lambda \sum_k \left(J(U)\partial_x \mathcal{U}, D_k(U) U_x \right) \nu_k(U)$$

$$+ \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \qquad (3.51)$$

For I_3, we have

$$I_3 = b \, dw \left\{ g(u_x, u_x) \nabla_x^2 J_u \nabla_x u_x \right\} + b \, dw \left\{ 2\nabla_x(g(u_x, u_x)) \nabla_x J_u \nabla_x u_x \right\}$$

$$+ b \, dw \left\{ \nabla_x^2(g(u_x, u_x)) J_u \nabla_x u_x \right\}$$

$$= b \, g(u_x, u_x) dw(\nabla_x^2 J_u \nabla_x u_x) + 4b \, g(\nabla_x u_x, u_x) dw(\nabla_x J_u \nabla_x u_x)$$

$$+ 2b \, g(\nabla_x^2 u_x, u_x) dw(J_u \nabla_x u_x) + 2b \, g(\nabla_x u_x, \nabla_x u_x) dw(J_u \nabla_x u_x)$$

$$= b \, |U_x|^2 dw(\nabla_x^2 J_u \nabla_x u_x) + 4b \, (dw(\nabla_x u_x), U_x) dw(\nabla_x J_u \nabla_x u_x)$$

$$+ 2b \, (dw(\nabla_x^2 u_x), U_x) dw(J_u \nabla_x u_x) + 2b \, |dw(\nabla_x u_x)|^2 dw(J_u \nabla_x u_x)$$

$$= b \, |U_x|^2 dw(\nabla_x^2 J_u \nabla_x u_x) + 4b \, (\mathcal{U}, U_x) dw(\nabla_x J_u \nabla_x u_x)$$

$$+ 2b \, (dw(\nabla_x^2 u_x), U_x) J(U)\mathcal{U} + 2b \, |\mathcal{U}|^2 J(U)\mathcal{U}.$$

Here, we recall the Kähler condition on (N, J, g) to see $dw(\nabla_x^2 J_u \nabla_x u_x) = dw(\nabla_x J_u \nabla_x^2 u_x)$. Hence, substituting (3.42), (3.50), and (3.40), we deduce

$$I_3 = b \, |U_x|^2 \partial_x \left\{ J(U)\partial_x \mathcal{U} \right\} + b \, |U_x|^2 \sum_k (J(U)\partial_x \mathcal{U}, D_k(U) U_x) \nu_k(U)$$

$$+ 4b \, (\mathcal{U}, U_x) J(U)\partial_x \mathcal{U} + 2b \, (\partial_x \mathcal{U}, U_x) J(U)\mathcal{U} + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$

$$= b \, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{U} \right\} - 2b \, (\partial_x U_x, U_x) J(U)\partial_x \mathcal{U}$$

$$+ b \, |U_x|^2 \sum_k (J(U)\partial_x \mathcal{U}, D_k(U) U_x) \nu_k(U)$$

$$+ 4b \, (\mathcal{U}, U_x) J(U)\partial_x \mathcal{U} + 2b \, (\partial_x \mathcal{U}, U_x) J(U)\mathcal{U} + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \qquad (3.52)$$

Furthermore, by noting $\mathcal{U} = dw(\nabla_x u_x) = \partial_x U_x + \sum_k (U_x, D_k(U) U_x) \nu_k(U)$, we see

$$(\mathcal{U}, U_x) = (\partial_x U_x, U_x) + \sum_k (U_x, D_k(U) U_x)(\nu_k(U), U_x) = (\partial_x U_x, U_x), \qquad (3.53)$$

$$J(U)\mathcal{U} = J(U) \left(\partial_x U_x + \sum_k (U_x, D_k(U) U_x) \nu_k(U) \right) = J(U)\partial_x U_x. \qquad (3.54)$$

Collecting the information (3.52), (3.53), and (3.54), we obtain

$$I_3 = b \, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{U} \right\} + 2b \, (\partial_x U_x, U_x) J(U)\partial_x \mathcal{U} + 2b \, (\partial_x \mathcal{U}, U_x) J(U)\partial_x U_x$$

$$+ b\,|U_x|^2 \sum_k (J(U)\partial_x \mathcal{U}, D_k(U)U_x)\nu_k(U) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.55}$$

For I_4, we have

$$I_4 = c\,dw\left(g(\nabla_x^3 u_x, u_x)J_u u_x\right) + 3c\,dw\left(g(\nabla_x^2 u_x, \nabla_x u_x)J_u u_x\right)$$
$$+ 2c\,dw\left(g(\nabla_x^2 u_x, u_x)J_u \nabla_x u_x\right) + 2c\,dw\left(g(\nabla_x u_x, \nabla_x u_x)J_u \nabla_x u_x\right)$$
$$+ c\,dw\left(g(\nabla_x u_x, u_x)J_u \nabla_x^2 u_x.\right)$$
$$= c\left(dw(\nabla_x^3 u_x), U_x\right)J(U)U_x + 3c\left(dw(\nabla_x^2 u_x), \mathcal{U}\right)J(U)U_x$$
$$+ 2c\left(dw(\nabla_x^2 u_x), U_x\right)J(U)\mathcal{U} + 2c\,|\mathcal{U}|^2 J(U)\mathcal{U} + c\,(\mathcal{U}, U_x)J(U)dw(\nabla_x^2 u_x).$$

From (3.40), it follows that

$$dw(\nabla_x^3 u_x) = \partial_x\left\{dw(\nabla_x^2 u_x)\right\} + \sum_\ell \left(dw(\nabla_x^2 u_x), D_\ell(U)U_x\right)\nu_\ell(U)$$

$$= \partial_x^2 \mathcal{U} + 2\sum_k (\partial_x \mathcal{U}, D_k(U)U_x)\nu_k(U) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|).$$

This together with $(\nu_k(U), U_x) = 0$ yields

$$(dw(\nabla_x^3 u_x), U_x) = (\partial_x^2 \mathcal{U}, U_x) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.56}$$

By using (3.40), (3.53), (3.54), and (3.56), we obtain

$$I_4 = c\,(\partial_x^2 \mathcal{U}, U_x)J(U)U_x + 3c\,(\partial_x \mathcal{U}, \mathcal{U})J(U)U_x$$
$$+ 2c\,(\partial_x \mathcal{U}, U_x)J(U)\mathcal{U} + c\,(\mathcal{U}, U_x)J(U)\partial_x \mathcal{U} + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$$
$$= c\,(\partial_x^2 \mathcal{U}, U_x)J(U)U_x + 3c\,(\partial_x \mathcal{U}, \partial_x U_x)J(U)U_x$$
$$+ 2c\,(\partial_x \mathcal{U}, U_x)J(U)\partial_x U_x + c\,(\partial_x U_x, U_x)J(U)\partial_x \mathcal{U} + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|). \tag{3.57}$$

For $I_1 = a\,dw(\nabla_x \nabla_x J_u \nabla_x^2 \nabla_x u_x)$, we start with

$$dw(\nabla_x \nabla_x J_u \nabla_x^2 \nabla_x u_x)$$
$$= \partial_x\left\{dw(\nabla_x J_u \nabla_x^2 \nabla_x u_x)\right\} + \sum_k \left(dw(\nabla_x J_u \nabla_x^2 \nabla_x u_x), D_k(U)U_x\right)\nu_k(U), \tag{3.58}$$

and

$$dw(\nabla_x J_u \nabla_x^2 \nabla_x u_x)$$
$$= \partial_x\left\{dw(J_u \nabla_x^2 \nabla_x u_x)\right\} + \sum_\ell \left(dw(J_u \nabla_x^2 \nabla_x u_x), D_\ell(U)U_x\right)\nu_\ell(U). \tag{3.59}$$

From (3.3), (3.13) with $Y = \partial_x \mathcal{U}$, the Kähler condition on (N, J, g), and (3.42), it follows that

$$dw(J_u \nabla_x^2 \nabla_x u_x) = J(U)\partial_x^2 \mathcal{U} + \partial_x(J(U))\partial_x \mathcal{U} + \sum_k (J(U)\partial_x \mathcal{U}, D_k(U)U_x)\nu_k(U)$$

$$= J(U)\partial_x^2 \mathcal{U} + \sum_k (\partial_x \mathcal{U}, J(U)D_k(U)U_x)\,\nu_k(U)$$

$$- \sum_k (\partial_x \mathcal{U}, \nu_k(U))J(U)D_k(U)U_x - \sum_k (\partial_x \mathcal{U}, J(U)D_k(U)U_x)\nu_k(U)$$

$$= J(U)\partial_x^2 \mathcal{U} - \sum_k (\partial_x \mathcal{U}, \nu_k(U))J(U)D_k(U)U_x.$$

Here note that $(\mathcal{U}, \nu_k(U)) = 0$ holds. By taking the derivative of both sides in x, we see

$$(\partial_x \mathcal{U}, \nu_k(U)) = -(\mathcal{U}, \partial_x(\nu_k(U))) = -(\mathcal{U}, D_k(U)U_x).$$

Using this, we obtain

$$dw(J_u\nabla_x^2\nabla_x u_x) = J(U)\partial_x^2\mathcal{U} + \sum_k(\mathcal{U}, D_k(U)U_x)J(U)D_k(U)U_x. \tag{3.60}$$

Furthermore, by substituting (3.60) into (3.59), we have

$$dw(\nabla_x J_u\nabla_x^2\nabla_x u_x)$$

$$= \partial_x\left\{J(U)\partial_x^2\mathcal{U} + \sum_n(\mathcal{U}, D_n(U)U_x)J(U)D_n(U)U_x\right\}$$

$$+ \sum_\ell\left(J(U)\partial_x^2\mathcal{U} + \sum_n(\mathcal{U}, D_n(U)U_x)J(U)D_n(U)U_x, D_\ell(U)U_x\right)\nu_\ell(U)$$

$$= \partial_x\left\{J(U)\partial_x^2\mathcal{U}\right\} - \sum_\ell\left(\partial_x^2\mathcal{U}, J(U)D_\ell(U)U_x\right)\nu_\ell(U)$$

$$+ \sum_n(\partial_x\mathcal{U}, D_n(U)U_x)J(U)D_n(U)U_x$$

$$+ \sum_n(\mathcal{U}, \partial_x\{D_n(U)U_x\})J(U)D_n(U)U_x$$

$$+ \sum_n(\mathcal{U}, D_n(U)U_x)\partial_x\{J(U)D_n(U)U_x\}$$

$$+ \sum_{\ell,n}(\mathcal{U}, D_n(U)U_x)(J(U)D_n(U)U_x, D_\ell(U)U_x)\nu_\ell(U). \tag{3.61}$$

Therefore, by substituting (3.61) into (3.58), and by using $\partial_x^2 U_x = \partial_x\mathcal{U} + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|)$, we deduce

$$I_1 = a\,\partial_x^2\left\{J(U)\partial_x^2\mathcal{U}\right\} - a\sum_\ell\left(\partial_x^3\mathcal{U}, J(U)D_\ell(U)U_x\right)\nu_\ell(U)$$

$$- a\sum_\ell\left(\partial_x^2\mathcal{U}, \partial_x\{J(U)D_\ell(U)U_x\}\right)\nu_\ell(U)$$

$$- a\sum_\ell\left(\partial_x^2\mathcal{U}, J(U)D_\ell(U)U_x\right)D_\ell(U)U_x$$

$$+ a\sum_n(\partial_x^2\mathcal{U}, D_n(U)U_x)J(U)D_n(U)U_x$$

$$+ a\sum_n(\partial_x\mathcal{U}, \partial_x\{D_n(U)U_x\})J(U)D_n(U)U_x$$

$$+ a\sum_n(\partial_x\mathcal{U}, D_n(U)U_x)\partial_x\{J(U)D_n(U)U_x\}$$

$$+ a\sum_n(\partial_x\mathcal{U}, \partial_x\{D_n(U)U_x\})J(U)D_n(U)U_x$$

$$+ a\sum_n\left(\mathcal{U}, D_n(U)\partial_x^2 U_x\right)J(U)D_n(U)U_x$$

$$+ a\sum_n(\partial_x\mathcal{U}, D_n(U)U_x)\partial_x\{J(U)D_n(U)U_x\}$$

$$+ a\sum_n(\mathcal{U}, D_n(U)U_x)J(U)D_n(U)\partial_x^2 U_x$$

$$+ a \sum_{\ell,n} (\partial_x \mathcal{U}, D_n(U)U_x) \left(J(U)D_n(U)U_x, D_\ell(U)U_x \right) \nu_\ell(U)$$

$$+ a \sum_{k} \left(\partial_x \left\{ J(U)\partial_x^2 \mathcal{U} \right\}, D_k(U)U_x \right) \nu_k(U)$$

$$- a \sum_{k,\ell} \left(\partial_x^2 \mathcal{U}, J(U)D_\ell(U)U_x \right) \left(\nu_\ell(U), D_k(U)U_x \right) \nu_k(U)$$

$$+ a \sum_{k,n} (\partial_x \mathcal{U}, D_n(U)U_x) \left(J(U)D_n(U)U_x, D_k(U)U_x \right) \nu_k(U)$$

$$+ \mathcal{O}(|U|, |U_x|, |\mathcal{U}|)$$

$$= a \, \partial_x^2 \left\{ J(U)\partial_x^2 \mathcal{U} \right\} - 2a \, F_1(\partial_x^3 \mathcal{U}) - a \, F_2(\partial_x^2 \mathcal{U}) + a \, F_3(\partial_x^2 \mathcal{U})$$

$$+ 2a \, F_4(\partial_x \mathcal{U}) + 2a \, F_5(\partial_x \mathcal{U}) + a \, F_6(\partial_x \mathcal{U}) + a \, F_7(\partial_x \mathcal{U})$$

$$+ \sum_{k} \mathcal{O} \left(|U| + |U_x| + |\mathcal{U}| + |\partial_x \mathcal{U}| + |\partial_x^2 \mathcal{U}| \right) \nu_k(U)$$

$$+ \mathcal{O}(|U| + |U_x| + |\mathcal{U}|), \tag{3.62}$$

where for any $Y : [0, T] \times \mathbb{T} \to \mathbb{R}^d$,

$$F_1(Y) = \sum_{k} (Y, J(U)D_k(U)U_x) \, \nu_k(U),$$

$$F_2(Y) = \sum_{k} (Y, J(U)D_k(U)U_x) \, D_k(U)U_x,$$

$$F_3(Y) = \sum_{k} (Y, D_k(U)U_x) J(U)D_k(U)U_x,$$

$$F_4(Y) = \sum_{k} (Y, \partial_x \left\{ D_k(U)U_x \right\}) J(U)D_k(U)U_x,$$

$$F_5(Y) = \sum_{k} (Y, D_k(U)U_x)\partial_x \left\{ J(U)D_k(U)U_x \right\},$$

$$F_6(Y) = \sum_{k} (\mathcal{U}, D_k(U)Y) \, J(U)D_k(U)U_x,$$

$$F_7(Y) = \sum_{k} (\mathcal{U}, D_k(U)U_x) J(U)D_k(U)Y.$$

Combining (3.45), (3.47), (3.48), (3.51), (3.55), (3.57), and (3.62), we derive

$$\partial_x \mathcal{U} = I_1 + I_2 + I_3 + I_4 + II + III + IV$$

$$= a \, \partial_x^2 \left\{ J(U)\partial_x^2 \mathcal{U} \right\} - 2a \, F_1(\partial_x^3 \mathcal{U}) + \lambda \, \partial_x \left\{ J(U)\partial_x \mathcal{U} \right\}$$

$$+ (b + aS - aS) \, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{U} \right\} + (c + aS) \, (\partial_x^2 \mathcal{U}, U_x) J(U)U_x$$

$$- a \, F_2(\partial_x^2 \mathcal{U}) + a \, F_3(\partial_x^2 \mathcal{U}) + 2a \, F_4(\partial_x \mathcal{U}) + 2a \, F_5(\partial_x \mathcal{U})$$

$$+ a \, F_6(\partial_x \mathcal{U}) + a \, F_7(\partial_x \mathcal{U}) + (2b + c - 2aS + 2aS) \, (\partial_x U_x, U_x) J(U)\partial_x \mathcal{U}$$

$$+ (2b + 2c) \, (\partial_x \mathcal{U}, U_x) J(U)\partial_x U_x + 3c \, (\partial_x \mathcal{U}, \partial_x U_x) J(U)U_x$$

$$- aS \, (\partial_x(J(U))\partial_x \mathcal{U}, U_x) U_x + aS \, |U_x|^2 \partial_x(J(U))\partial_x \mathcal{U}$$

$$+ \lambda \, \partial_x(J(U))\partial_x \mathcal{U} + r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) + \mathcal{O}(|U| + |U_x| + |\mathcal{U}|), \tag{3.63}$$

where

$$r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) = \sum_k \mathcal{O}\left(|U| + |U_x| + |\mathcal{U}| + |\partial_x \mathcal{U}| + |\partial_x^2 \mathcal{U}|\right) \nu_k(U).$$

3. Classical energy estimates for $\|\mathcal{W}\|_{L^2(\mathbb{T};\mathbb{R}^d)}$ with the loss of derivatives

We compute $\partial_t \mathcal{W} = \partial_t \mathcal{U} - \partial_t \mathcal{V}$ and next evaluate the classical energy estimate for $\mathcal{W}$ in L^2. Obviously, $\mathcal{V}$ also satisfies (3.63) replacing $\mathcal{U}$ with $\mathcal{V}$. Hence, by using the mean value formula, we obtain

$$\begin{aligned}
\partial_t \mathcal{W} = {}& a\, \partial_x^2 \left\{ J(U)\partial_x^2 \mathcal{W} \right\} - 2a\, F_1(\partial_x^3 \mathcal{W}) + \lambda\, \partial_x \left\{ J(U)\partial_x \mathcal{W} \right\} \\
& + b\, \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{W} \right\} + (c + aS)\,(\partial_x^2 \mathcal{W}, U_x)J(U)U_x \\
& - a\, F_2(\partial_x^2 \mathcal{W}) + a\, F_3(\partial_x^2 \mathcal{W}) + 2a\, F_4(\partial_x \mathcal{W}) + 2a\, F_5(\partial_x \mathcal{W}) \\
& + a\, F_6(\partial_x \mathcal{W}) + a\, F_7(\partial_x \mathcal{W}) + (2b + c)\,(\partial_x U_x, U_x)J(U)\partial_x \mathcal{W} \\
& + (2b + 2c)\,(\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x + 3c\,(\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x \\
& - aS\,(\partial_x(J(U))\partial_x \mathcal{W}, U_x)U_x + aS\,|U_x|^2 \partial_x(J(U))\partial_x \mathcal{W} + \lambda\, \partial_x(J(U))\partial_x \mathcal{W} \\
& + r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) - r(V, V_x, \mathcal{V}, \partial_x \mathcal{V}, \partial_x^2 \mathcal{V}) \\
& + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.64}
\end{aligned}$$

Note that $F_1(\cdot), \ldots, F_7(\cdot)$ should be expressed globally not by using local orthonormal frame. It is possible by using the second fundamental form on $w(N)$ and the derivatives, or by following the argument in [18] to use the partition of unity on $w(N)$. However, for simplicity and for better understandings, we will continue to use the local expression without loss of generality.

We move on to the classical energy estimate for $\|\mathcal{W}\|_{L^2}^2$. Since $k \geqslant 6$, $\mathcal{W} \in L^\infty(0, T; H^5) \cap C([0, T]; H^4) \cap C^1([0, T]; L^2)$. This together with (3.64) implies

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\mathcal{W}\|_{L^2}^2 \\
& = \langle \partial_t \mathcal{W}, \mathcal{W} \rangle \\
& = a\, \langle \partial_x^2 \left\{ J(U)\partial_x^2 \mathcal{W} \right\}, \mathcal{W} \rangle - 2a\, \langle F_1(\partial_x^3 \mathcal{W}), \mathcal{W} \rangle + \lambda\, \langle \partial_x \left\{ J(U)\partial_x \mathcal{W} \right\}, \mathcal{W} \rangle \\
& \quad + b\, \langle \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{W} \right\}, \mathcal{W} \rangle + (c + aS)\, \langle (\partial_x^2 \mathcal{W}, U_x)J(U)U_x, \mathcal{W} \rangle \\
& \quad - a\, \langle F_2(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle + a\, \langle F_3(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle + 2a\, \langle F_4(\partial_x \mathcal{W}), \mathcal{W} \rangle \\
& \quad + 2a\, \langle F_5(\partial_x \mathcal{W}), \mathcal{W} \rangle + a\, \langle F_6(\partial_x \mathcal{W}), \mathcal{W} \rangle + a\, \langle F_7(\partial_x \mathcal{W}), \mathcal{W} \rangle \\
& \quad + (2b + c)\, \langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \rangle + (2b + 2c)\, \langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \rangle \\
& \quad + 3c\, \langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \rangle - aS\, \langle (\partial_x(J(U))\partial_x \mathcal{W}, U_x)U_x, \mathcal{W} \rangle \\
& \quad + aS\, \langle |U_x|^2 \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W} \rangle + \lambda\, \langle \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W} \rangle \\
& \quad + \langle r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) - r(V, V_x, \mathcal{V}, \partial_x \mathcal{V}, \partial_x^2 \mathcal{V}), \mathcal{W} \rangle \\
& \quad + \langle \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|), \mathcal{W} \rangle. \tag{3.65}
\end{aligned}$$

Let us compute the RHS of the above term by term. First, by integrating by parts, it is immediate to see

$$\begin{aligned}
a\, \langle \partial_x^2 \left\{ J(U)\partial_x^2 \mathcal{W} \right\}, \mathcal{W} \rangle &= a\, \langle J(U)\partial_x^2 \mathcal{W}, \partial_x^2 \mathcal{W} \rangle = 0, \\
\lambda\, \langle \partial_x \left\{ J(U)\partial_x \mathcal{W} \right\}, \mathcal{W} \rangle &= -\lambda\, \langle J(U)\partial_x \mathcal{W}, \partial_x \mathcal{W} \rangle = 0, \\
b\, \langle \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{W} \right\}, \mathcal{W} \rangle &= -b\, \langle |U_x|^2 J(U)\partial_x \mathcal{W}, \partial_x \mathcal{W} \rangle = 0.
\end{aligned}$$

Next, by the Cauchy-Schwartz inequality, there holds

$$\langle \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|), \mathcal{W} \rangle \leqslant \|\mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)\|_{L^2} \|\mathcal{W}\|_{L^2}$$
$$\leqslant C \left\{ \|Z\|_{L^2}^2 + \|Z_x\|_{L^2}^2 + \|\mathcal{W}\|_{L^2}^2 \right\}$$

for some $C > 0$. Here and hereafter, various positive constants depending on $\|u_x\|_{L^\infty(0,T;H^6)}$ and $\|v_x\|_{L^\infty(0,T;H^6)}$ will be denoted by the same C without any comments. Besides, we use the notation $D(t)$ so that the square is defined by

$$D(t)^2 = \|Z\|_{L^2}^2 + \|Z_x\|_{L^2}^2 + \|\mathcal{W}\|_{L^2}^2.$$

Next, by noting

$$r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) - r(V, V_x, \mathcal{V}, \partial_x \mathcal{V}, \partial_x^2 \mathcal{V})$$
$$= \sum_k \mathcal{O}\left(|Z| + |Z_x| + |\mathcal{W}| + |\partial_x \mathcal{W}| + |\partial_x^2 \mathcal{W}|\right) \nu_k(U)$$
$$+ \sum_k \mathcal{O}\left(|U| + |U_x| + |\mathcal{U}| + |\partial_x \mathcal{U}| + |\partial_x^2 \mathcal{U}|\right) (\nu_k(U) - \nu_k(V)),$$

we use (3.8) obtained in Lemma 3.1, $\partial_x Z_x = \mathcal{W} + \mathcal{O}(|Z| + |Z_x|)$, and the integration by parts, to obtain

$$\langle r(U, U_x, \mathcal{U}, \partial_x \mathcal{U}, \partial_x^2 \mathcal{U}) - r(V, V_x, \mathcal{V}, \partial_x \mathcal{V}, \partial_x^2 \mathcal{V}), \mathcal{W} \rangle$$
$$\leqslant \left\langle \sum_k \mathcal{O}\left(|Z| + |Z_x| + |\mathcal{W}| + |\partial_x \mathcal{W}| + |\partial_x^2 \mathcal{W}|\right) \nu_k(U), \mathcal{W} \right\rangle + C\, D(t)^2$$
$$= \int_{\mathbb{T}} \sum_k \mathcal{O}\left(|Z| + |Z_x| + |\mathcal{W}| + |\partial_x \mathcal{W}| + |\partial_x^2 \mathcal{W}|\right) \mathcal{O}(|Z|)\, dx + C\, D(t)^2$$
$$= \int_{\mathbb{T}} \sum_k \mathcal{O}\left(|Z| + |Z_x| + |\mathcal{W}|\right) \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)\, dx + C\, D(t)^2$$
$$\leqslant C\, D(t)^2. \tag{3.66}$$

In the next computation, the Kähler condition on (N, J, g) plays the crucial parts. Indeed, we apply (3.13) with $Y = \partial_x \mathcal{W}$ and use (3.9) to obtain

$$\partial_x (J(U)) \partial_x \mathcal{W} = \sum_k (\partial_x \mathcal{W}, J(U) D_k(U) U_x)\, \nu_k(U) - \sum_k (\partial_x \mathcal{W}, \nu_k(U)) J(U) D_k(U) U_x$$
$$= \sum_k (\partial_x \mathcal{W}, J(U) D_k(U) U_x)\, \nu_k(U) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.67}$$

By using (3.67) and (3.8), we see

$$(\partial_x (J(U)) \partial_x \mathcal{W}, U_x) = \sum_k (\partial_x \mathcal{W}, J(U) D_k(U) U_x)\, (\nu_k(U), U_x) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)$$
$$= \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|), \tag{3.68}$$
$$(\partial_x (J(U)) \partial_x \mathcal{W}, \mathcal{W}) = \sum_k (\partial_x \mathcal{W}, J(U) D_k(U) U_x)\, (\nu_k(U), \mathcal{W}) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)$$
$$= \sum_k (\partial_x \mathcal{W}, J(U) D_k(U) U_x)\, \mathcal{O}(|Z|) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.69}$$

Thus, by using (3.68) and the Cauchy-Schwartz inequality, we have

$$-aS\, \langle (\partial_x (J(U)) \partial_x \mathcal{W}, U_x) U_x, \mathcal{W} \rangle \leqslant C\, D(t)^2.$$

In the same manner, by using (3.69), the Cauchy-Schwartz inequality, together with the integration by parts, we deduce

$$aS \left\langle |U_x|^2 \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W} \right\rangle + \lambda \left\langle \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W} \right\rangle \leqslant C\, D(t)^2.$$

Collecting them, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{W}\|_{L^2}^2$$
$$\leqslant (c+aS)\left\langle (\partial_x^2 \mathcal{W}, U_x)J(U)U_x, \mathcal{W} \right\rangle + (2b+c)\left\langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \right\rangle$$
$$+ (2b+2c)\left\langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \right\rangle + 3c\left\langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \right\rangle$$
$$+ \sum_{i=1}^{7} R_i + C\, D(t)^2, \tag{3.70}$$

where

$$R_1 = -2a \left\langle F_1(\partial_x^3 \mathcal{W}), \mathcal{W} \right\rangle, R_2 = -a \left\langle F_2(\partial_x^2 \mathcal{W}), \mathcal{W} \right\rangle, R_3 = a \left\langle F_3(\partial_x^2 \mathcal{W}), \mathcal{W} \right\rangle,$$
$$R_4 = 2a \left\langle F_4(\partial_x \mathcal{W}), \mathcal{W} \right\rangle, R_5 = 2a \left\langle F_5(\partial_x \mathcal{W}), \mathcal{W} \right\rangle, R_6 = a \left\langle F_6(\partial_x \mathcal{W}), \mathcal{W} \right\rangle,$$
$$R_7 = a \left\langle F_7(\partial_x \mathcal{W}), \mathcal{W} \right\rangle.$$

In what follows, we need to compute more carefully. Let us consider R_1. We start by integrating by parts to see

$$R_1 = 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, \partial_x \left\{ J(U)D_k(U)U_x \right\} \right) \nu_k(U), \mathcal{W} \right\rangle$$
$$+ 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) D_k(U)U_x, \mathcal{W} \right\rangle$$
$$+ 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) \nu_k(U), \partial_x \mathcal{W} \right\rangle.$$

By applying (3.8) to the first term of the RHS of the above and by applying (3.9) to the third term of the RHS of the above, we have

$$R_1 = 2a \int_{\mathbb{T}} \sum_k \left(\partial_x^2 \mathcal{W}, \partial_x \left\{ J(U)D_k(U)U_x \right\} \right) \mathcal{O}(|Z|)\, dx$$
$$+ 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) D_k(U)U_x, \mathcal{W} \right\rangle$$
$$- 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) D_k(U)U_x, \mathcal{W} \right\rangle$$
$$- 2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) D_k(U)Z_x, \mathcal{V} \right\rangle$$
$$- 2a \int_{\mathbb{T}} \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x \right) \mathcal{O}(|Z|)\, dx$$
$$= 2a \int_{\mathbb{T}} \sum_k \left(\partial_x^2 \mathcal{W}, \partial_x \left\{ J(U)D_k(U)U_x \right\} \right) \mathcal{O}(|Z|)\, dx$$

$$-2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)Z_x, \mathcal{V} \right\rangle$$

$$-2a \int_{\mathbb{T}} \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x\right) \mathcal{O}(|Z|) \, dx.$$

Here, by integrating by parts, the first and the third term are bounded by $C\, D(t)^2$. Therefore we obtain

$$R_1 \leqslant -2a \left\langle \sum_k \left(\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)Z_x, \mathcal{V} \right\rangle + C\, D(t)^2.$$

Furthermore, by using integration by parts, $\partial_x Z_x = \mathcal{W} + \mathcal{O}(|Z| + |Z_x|)$, (3.3), and (3.11), we deduce

$$R_1 \leqslant 2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)\partial_x Z_x, \mathcal{V} \right\rangle + C\, D(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)(\mathcal{W} + \mathcal{O}(|Z| + |Z_x|)), \mathcal{V} \right\rangle + C\, D(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)\mathcal{W}, \mathcal{V} \right\rangle + C\, D(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)\mathcal{W}, \mathcal{U} \right\rangle$$

$$-2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)\mathcal{W}, \mathcal{W} \right\rangle + C\, D(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(\partial_x \mathcal{W}, J(U)D_k(U)U_x\right) D_k(U)\mathcal{W}, \mathcal{U} \right\rangle + C\, D(t)^2$$

$$= -2a \left\langle \sum_k \left(J(U)\partial_x \mathcal{W}, D_k(U)U_x\right) D_k(U)\mathcal{U}, \mathcal{W} \right\rangle + C\, D(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(J(U)\mathcal{W}, D_k(U)U_x\right) D_k(U)\mathcal{U}, \partial_x \mathcal{W} \right\rangle + C\, D(t)^2$$

$$=: R_{11} + R_{12} + C\, D(t)^2, \tag{3.71}$$

where

$$R_{11} = 2a \left\langle \sum_k \left(J(U)\mathcal{W}, D_k(U)U_x\right) P(U)D_k(U)\mathcal{U}, \partial_x \mathcal{W} \right\rangle,$$

$$R_{12} = 2a \left\langle \sum_k \left(J(U)\mathcal{W}, D_k(U)U_x\right) N(U)D_k(U)\mathcal{U}, \partial_x \mathcal{W} \right\rangle.$$

For R_{12}, recall (3.9) to see

$$(N(U)D_k(U)\mathcal{U}, \partial_x \mathcal{W}) = \sum_\ell (D_k(U)\mathcal{U}, \nu_\ell(U))(\nu_\ell(U), \partial_x \mathcal{W}) = \mathcal{O}\left(|Z| + |Z_x| + |\mathcal{W}|\right).$$

This shows $R_{12} \leqslant C\,D(t)^2$. For R_{11}, by using (3.2) and (3.11), we see

$$R_{11} = 2a\left\langle \sum_k (J(U)\mathcal{W}, D_k(U)U_x)\,D_k(U)P(U)\partial_x\mathcal{W}, \mathcal{U}\right\rangle.$$

Since $(N(U)D_k(U)P(U)\partial_x\mathcal{W}, \mathcal{U}) = 0$, we have

$$R_{11} = 2a\left\langle \sum_k (J(U)\mathcal{W}, D_k(U)U_x)\,P(U)D_k(U)P(U)\partial_x\mathcal{W}, \mathcal{U}\right\rangle.$$

Applying (3.23) in Lemma 3.5 with $dw_u(Y_1) = P(U)\partial_x\mathcal{W}$, $dw_u(Y_2) = U_x$, and with $dw_u(Y_3) = J(U)\mathcal{W}$, we obtain

$$R_{11} = 2a\left\langle \sum_k (J(U)\mathcal{W}, D_k(U)P(U)\partial_x\mathcal{W})\,P(U)D_k(U)U_x, \mathcal{U}\right\rangle$$
$$+ 2aS\left\langle (J(U)\mathcal{W}, U_x)P(U)\partial_x\mathcal{W}, \mathcal{U}\right\rangle - 2aS\left\langle (J(U)\mathcal{W}, P(U)\partial_x\mathcal{W})U_x, \mathcal{U}\right\rangle$$
$$=: R_{111} + R_{112} + R_{113}.$$

Here we recall (3.9) to see

$$N(U)\partial_x\mathcal{W} = \sum_k (\partial_x\mathcal{W}, \nu_k(U))\nu_k(U) = \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.72}$$

This implies $P(U)\partial_x\mathcal{W} = \partial_x\mathcal{W} + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)$. Using this (3.2), (3.3) and $P(U)\mathcal{U} = \mathcal{U}$, we obtain

$$R_{111} = -2a\left\langle \sum_k (\mathcal{W}, J(U)D_k(U)P(U)\partial_x\mathcal{W})\,D_k(U)U_x, P(U)\mathcal{U}\right\rangle$$
$$\leqslant -2a\left\langle \sum_k (\mathcal{W}, J(U)D_k(U)\partial_x\mathcal{W})\,D_k(U)U_x, \mathcal{U}\right\rangle + C\,D(t)^2$$
$$= -2a\left\langle \sum_k (\mathcal{U}, D_k(U)U_x)J(U)D_k(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle + C\,D(t)^2.$$

In the same way, using (3.2), (3.3) and $\mathcal{U} = \partial_x U_x + \sum_\ell (U_x, D_\ell(U)U_x)\nu_\ell(U)$, we obtain

$$R_{112} \leqslant -2aS\left\langle (\partial_x\mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W}\right\rangle + C\,D(t)^2,$$
$$R_{113} \leqslant 2aS\left\langle (\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle + C\,D(t)^2.$$

Collecting them, we obtain

$$R_1 = R_{111} + R_{112} + R_{113} + R_{12} + C\,D(t)^2$$
$$\leqslant -2a\left\langle \sum_k (\mathcal{U}, D_k(U)U_x)J(U)D_k(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$
$$- 2aS\left\langle (\partial_x\mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W}\right\rangle + 2aS\left\langle (\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$
$$+ C\,D(t)^2. \tag{3.73}$$

The first term of the RHS of (3.73) is cancelled with the same term appearing from the computation of $R_6 + R_7$.

We compute $R_6 + R_7 = a \ \langle F_6(\partial_x \mathcal{W}), \mathcal{W} \rangle + a \ \langle F_7(\partial_x \mathcal{W}), \mathcal{W} \rangle$. By noting $J(U) = J(U)P(U)$ and by applying (3.23) with $dw_u(Y_1) = U_x$, $dw_u(Y_2) = P(U)\partial_x \mathcal{W}$ and with $dw_u(Y_3) = \mathcal{U}$, we obtain

$$
\begin{aligned}
F_6(\partial_x \mathcal{W}) &= \sum_k (\mathcal{U}, D_k(U)\partial_x \mathcal{W}) \, J(U) D_k(U) U_x \\
&= J(U) \sum_k (\mathcal{U}, D_k(U)\partial_x \mathcal{W}) \, P(U) D_k(U) U_x \\
&= J(U) \sum_k (\mathcal{U}, D_k(U)P(U)\partial_x \mathcal{W}) \, P(U) D_k(U) U_x \\
&\quad + J(U) \sum_k (\mathcal{U}, D_k(U)N(U)\partial_x \mathcal{W}) \, P(U) D_k(U) U_x \\
&= J(U) \sum_k (\mathcal{U}, D_k(U)U_x) \, P(U) D_k(U) P(U) \partial_x \mathcal{W} \\
&\quad + S \, J(U) \left\{ (\mathcal{U}, P(U)\partial_x \mathcal{W}) U_x - (\mathcal{U}, U_x)P(U)\partial_x \mathcal{W} \right\} \\
&\quad + J(U) \sum_k (\mathcal{U}, D_k(U)N(U)\partial_x \mathcal{W}) \, P(U) D_k(U) U_x.
\end{aligned}
$$

Furthermore, we use $J(U) = J(U)P(U)$ and (3.72) to obtain

$$
\begin{aligned}
F_6(\partial_x \mathcal{W}) &= \sum_k (\mathcal{U}, D_k(U)U_x) \, J(U) D_k(U) P(U) \partial_x \mathcal{W} \\
&\quad + S \, (\partial_x \mathcal{W}, \mathcal{U}) J(U) U_x - S \, (\mathcal{U}, U_x) J(U) \partial_x \mathcal{W} + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|) \\
&= \sum_k (\mathcal{U}, D_k(U)U_x) \, J(U) D_k(U) \partial_x \mathcal{W} \\
&\quad + S \, (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x - S \, (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W} \\
&\quad + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|) \\
&= F_7(\partial_x \mathcal{W}) + S \, (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x - S \, (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W} \\
&\quad + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
R_6 + R_7 \leqslant 2a \ & \left\langle \sum_k (\mathcal{U}, D_k(U)U_x) \, J(U) D_k(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle \\
&+ aS \ \langle (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x, \mathcal{W} \rangle \\
&- aS \ \langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \rangle + C \, D(t)^2.
\end{aligned} \tag{3.74}
$$

Combining (3.73) and (3.74), and using (3.11), we have

$$
\begin{aligned}
R_1 &+ R_6 + R_7 \\
&\leqslant -aS \ \langle (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x, \mathcal{W} \rangle + aS \ \langle (U_x, \partial_x U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \rangle + C \, D(t)^2. \tag{3.75}
\end{aligned}
$$

Next, we compute $R_2 + R_3 = -a \langle F_2(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle + a \langle F_3(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle$. As in deriving (3.73), we use (3.3), (3.11), and (3.23) with $dw_u(Y_1) = U_x$, $dw_u(Y_2) = J(U)\partial_x^2 \mathcal{W}$ and with $dw_u(Y_3) = U_x$, to deduce

$$
F_2(\partial_x^2 \mathcal{W}) = \sum_k (\partial_x^2 \mathcal{W}, J(U) D_k(U) U_x) D_k(U) U_x
$$

$$\begin{aligned}
&= \sum_k (\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x)N(U)D_k(U)U_x \\
&\quad + \sum_k (\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x)P(U)D_k(U)U_x \\
&= \sum_k (\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x)N(U)D_k(U)U_x \\
&\quad - \sum_k (U_x, D_k(U)J(U)\partial_x^2\mathcal{W})P(U)D_k(U)U_x \\
&= \sum_k (\partial_x^2 \mathcal{W}, J(U)D_k(U)U_x)N(U)D_k(U)U_x \\
&\quad - \sum_k (U_x, D_k(U)U_x)P(U)D_k(U)J(U)\partial_x^2\mathcal{W} \\
&\quad - S\left\{ (U_x, J(U)\partial_x^2\mathcal{W})U_x - (U_x, U_x)J(U)\partial_x^2\mathcal{W} \right\} \\
&= -\sum_k (U_x, D_k(U)U_x)D_k(U)J(U)\partial_x^2\mathcal{W} \\
&\quad + S(\partial_x^2\mathcal{W}, J(U)U_x)U_x + S|U_x|^2 J(U)\partial_x^2\mathcal{W} \\
&\quad + \sum_k (U_x, D_k(U)U_x)N(U)D_k(U)J(U)\partial_x^2\mathcal{W} \\
&\quad + \sum_k (\partial_x^2\mathcal{W}, J(U)D_k(U)U_x)N(U)D_k(U)U_x \\
&= -\sum_k (U_x, D_k(U)U_x)D_k(U)J(U)\partial_x^2\mathcal{W} \\
&\quad + S(\partial_x^2\mathcal{W}, J(U)U_x)U_x + S|U_x|^2 J(U)\partial_x^2\mathcal{W} \\
&\quad + \sum_\ell \mathcal{O}\left(|\partial_x^2\mathcal{W}|\right)\nu_\ell(U), \tag{3.76}
\end{aligned}$$

and in the same way we use (3.3), (3.11), and (3.23) with $dw_u(Y_1) = U_x$, $dw_u(Y_2) = \partial_x^2\mathcal{W}$ and with $dw_u(Y_3) = U_x$, to deduce

$$\begin{aligned}
F_3(\partial_x^2\mathcal{W}) &= \sum_k (\partial_x^2\mathcal{W}, D_k(U)U_x)J(U)D_k(U)U_x \\
&= J(U)\sum_k (U_x, D_k(U)\partial_x^2\mathcal{W})P(U)D_k(U)U_x \\
&= J(U)\sum_k (U_x, D_k(U)N(U)\partial_x^2\mathcal{W})P(U)D_k(U)U_x \\
&\quad + J(U)\sum_k (U_x, D_k(U)P(U)\partial_x^2\mathcal{W})P(U)D_k(U)U_x \\
&= J(U)\sum_k (U_x, D_k(U)N(U)\partial_x^2\mathcal{W})P(U)D_k(U)U_x \\
&\quad + J(U)\sum_k (U_x, D_k(U)U_x)P(U)D_k(U)P(U)\partial_x^2\mathcal{W} \\
&\quad + S J(U)\left\{ (U_x, P(U)\partial_x^2\mathcal{W})U_x - (U_x, U_x)P(U)\partial_x^2\mathcal{W} \right\}
\end{aligned}$$

$$= \sum_k (U_x, D_k(U)N(U)\partial_x^2\mathcal{W})J(U)D_k(U)U_x$$

$$+ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)P(U)\partial_x^2\mathcal{W}$$

$$+ S\left(\partial_x^2\mathcal{W}, U_x\right)J(U)U_x - S\left|U_x\right|^2 J(U)\partial_x^2\mathcal{W}$$

$$= \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x^2\mathcal{W}$$

$$+ S\left(\partial_x^2\mathcal{W}, U_x\right)J(U)U_x - S\left|U_x\right|^2 J(U)\partial_x^2\mathcal{W}$$

$$+ \sum_k (U_x, D_k(U)N(U)\partial_x^2\mathcal{W})J(U)D_k(U)U_x$$

$$- \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)N(U)\partial_x^2\mathcal{W}. \tag{3.77}$$

Here, we use (3.10) to see

$$N(U)\partial_x^2\mathcal{W} = \sum_\ell (\partial_x^2\mathcal{W}, \nu_\ell(U))\nu_\ell(U)$$

$$= -2\sum_\ell (\partial_x\mathcal{W}, D_\ell(U)U_x)\nu_\ell(U) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.78}$$

By substituting (3.78) into the fourth and fifth term of the RHS of (3.77), we have

$$F_3(\partial_x^2\mathcal{W}) = \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x^2\mathcal{W}$$

$$+ S\left(\partial_x^2\mathcal{W}, U_x\right)J(U)U_x - S\left|U_x\right|^2 J(U)\partial_x^2\mathcal{W}$$

$$- 2\sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)\nu_\ell(U))J(U)D_k(U)U_x$$

$$+ 2\sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U)$$

$$+ \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.79}$$

Thus, from (3.76), (3.79), and (3.28), it follows that

$$- F_2(\partial_x^2\mathcal{W}) + F_3(\partial_x^2\mathcal{W})$$

$$= \sum_k (U_x, D_k(U)U_x)D_k(U)J(U)\partial_x^2\mathcal{W} + \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x^2\mathcal{W}$$

$$- S\left(\partial_x^2\mathcal{W}, J(U)U_x\right)U_x + S\left(\partial_x^2\mathcal{W}, U_x\right)J(U)U_x - 2S\left|U_x\right|^2 J(U)\partial_x^2\mathcal{W}$$

$$- 2\sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)\nu_\ell(U))J(U)D_k(U)U_x$$

$$+ 2\sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U)$$

$$+ \sum_\ell \mathcal{O}\left(|\partial_x^2\mathcal{W}|\right)\nu_\ell(U) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)$$

$$= \sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U))\partial_x^2\mathcal{W}$$

$$- S\left|U_x\right|^2 J(U)\partial_x^2\mathcal{W}$$

$$- 2 \sum_{k,\ell} (\partial_x \mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)\nu_\ell(U))J(U)D_k(U)U_x$$

$$+ 2 \sum_{k,\ell} (\partial_x \mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U)$$

$$+ \sum_{\ell} \mathcal{O}\left(|\partial_x^2 \mathcal{W}|\right) \nu_\ell(U) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.80}$$

Therefore, from (3.80) and

$$|U_x|^2 J(U)\partial_x^2 \mathcal{W} = \partial_x \left\{|U_x|^2 J(U)\partial_x \mathcal{W}\right\} - 2(\partial_x U_x, U_x)J(U)\partial_x \mathcal{W} - |U_x|^2 \partial_x(J(U))\partial_x \mathcal{W},$$

we see that $R_2 + R_3 = -a \langle F_2(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle + a \langle F_3(\partial_x^2 \mathcal{W}), \mathcal{W} \rangle$ is evaluated as follows:

$$R_2 + R_3 \leqslant a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U))\partial_x \mathcal{W} \right\}, \mathcal{W} \right\rangle$$

$$- a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U)) \right\} \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$- aS \left\langle \partial_x \left\{ |U_x|^2 J(U)\partial_x \mathcal{W} \right\}, \mathcal{W} \right\rangle + 2aS \left\langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$+ aS \left\langle |U_x|^2 \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$- 2a \left\langle \sum_{k,\ell} (\partial_x \mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)\nu_\ell(U))J(U)D_k(U)U_x, \mathcal{W} \right\rangle$$

$$+ 2a \left\langle \sum_{k,\ell} (\partial_x \mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U), \mathcal{W} \right\rangle$$

$$+ \left\langle \sum_{\ell} \mathcal{O}\left(|\partial_x^2 \mathcal{W}|\right) \nu_\ell(U), \mathcal{W} \right\rangle$$

$$+ C\, D(t)^2. \tag{3.81}$$

Note here that

$$((J(U)D_k(U) + D_k(U)J(U))Y_1, Y_2) = -(Y_1, (J(U)D_k(U) + D_k(U)J(U))Y_2)$$

holds for any $Y_1, Y_2 : [0, T] \times \mathbb{T} \to \mathbb{R}^d$. This implies that the first term of the RHS of (3.81) vanishes. Indeed, the integration by parts yields

$$a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U))\partial_x \mathcal{W} \right\}, \mathcal{W} \right\rangle$$

$$= -a \left\langle \sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U))\partial_x \mathcal{W}, \partial_x \mathcal{W} \right\rangle$$

$$= 0.$$

In addition, the third term of the RHS of (3.81) vanishes by integrating by parts. Beside, due to the presence of $N(U)$, we can bound the eighth term of the RHS of (3.81) by $C\, D(t)^2$ using the

same argument to show (3.66). Consequently, we derive

$$
\begin{aligned}
R_2 + R_3 \leqslant -a &\left\langle \partial_x\left\{\sum_k (U_x, D_k(U)U_x)(D_k(U)J(U) + J(U)D_k(U))\right\}\partial_x\mathcal{W}, \mathcal{W}\right\rangle \\
&+ 2aS \left\langle (\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle \\
&- 2a \left\langle \sum_{k,\ell}(\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)\nu_\ell(U))J(U)D_k(U)U_x, \mathcal{W}\right\rangle \\
&+ 2a \left\langle \sum_{k,\ell}(\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U), \mathcal{W}\right\rangle \\
&+ C\, D(t)^2.
\end{aligned} \tag{3.82}
$$

The third and fourth term and the bad part of the first term of the RHS of (3.82) will be cancelled with the same term appearing in the computation of $R_4 + R_5$.

Let us next compute $R_4 + R_5$. To begin with, we introduce $T_1(U)$ which is defined by

$$
T_1(U)Y = \sum_k (Y, D_k(U)U_x)J(U)D_k(U)U_x
$$

for any $Y : [0,T]\times\mathbb{T} \to \mathbb{R}^d$. Substituting this with $Y = \partial_x^2\mathcal{W}$ into the RHS of $\partial_x(T_1(U))\partial_x\mathcal{W} = \partial_x\{T_1(U)\partial_x\mathcal{W}\} - T_1(U)\partial_x^2\mathcal{W}$, we can write

$$
R_4 + R_5 = 2a \left\langle \partial_x(T_1(U))\partial_x\mathcal{W}, \mathcal{W}\right\rangle. \tag{3.83}
$$

On the other hand, using (3.23) with $dw_u(Y_1) = U_x$, $dw_u(Y_2) = P(U)Y$ and with $dw_u(Y_3) = U_x$, we find that $T_1(U)Y$ has the following another expression.

$$
\begin{aligned}
T_1(U)Y &= J(U)\sum_k (Y, D_k(U)U_x)P(U)D_k(U)U_x \\
&= J(U)\sum_k (P(U)Y, D_k(U)U_x)P(U)D_k(U)U_x \\
&\quad + J(U)\sum_k (N(U)Y, D_k(U)U_x)P(U)D_k(U)U_x \\
&= J(U)\sum_k (U_x, D_k(U)P(U)Y)P(U)D_k(U)U_x \\
&\quad + J(U)\sum_k (N(U)Y, D_k(U)U_x)P(U)D_k(U)U_x \\
&= J(U)\sum_k (U_x, D_k(U)U_x)P(U)D_k(U)P(U)Y \\
&\quad + S\, J(U)\left\{(U_x, P(U)Y)U_x - |U_x|^2 P(U)Y\right\} \\
&\quad + J(U)\sum_k (N(U)Y, D_k(U)U_x)P(U)D_k(U)U_x \\
&= \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)P(U)Y \\
&\quad + S\,(U_x, P(U)Y)J(U)U_x - S|U_x|^2 J(U)Y \\
&\quad + \sum_k (N(U)Y, D_k(U)U_x)J(U)D_k(U)U_x
\end{aligned}
$$

$$= \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)Y$$

$$- \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)N(U)Y$$

$$+ S\,(Y, U_x)J(U)U_x - S|U_x|^2 J(U)Y$$

$$+ \sum_k (N(U)Y, D_k(U)U_x)J(U)D_k(U)U_x$$

for any $Y : [0, T] \times \mathbb{T} \to \mathbb{R}^d$. If we adopt this formulation, we have

$$\partial_x(T_1(U))Y = \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U) \right\} Y$$

$$- \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U) \right\} N(U)Y$$

$$- \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x(N(U))Y$$

$$+ S\,(Y, \partial_x U_x)J(U)U_x + S\,(Y, U_x)\partial_x(J(U))U_x + S\,(Y, U_x)J(U)\partial_x U_x$$

$$- 2S(\partial_x U_x, U_x)J(U)Y - S|U_x|^2 \partial_x(J(U))Y$$

$$+ \sum_k (\partial_x(N(U))Y, D_k(U)U_x)J(U)D_k(U)U_x$$

$$+ \sum_k (N(U)Y, \partial_x\{D_k(U)U_x\})J(U)D_k(U)U_x$$

$$+ \sum_k (N(U)Y, D_k(U)U_x)\partial_x\{J(U)D_k(U)U_x\}. \tag{3.84}$$

By substituting (3.84) into (3.83), we have

$$R_4 + R_5 = 2a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U) \right\} \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$- 2a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U) \right\} N(U)\partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$- 2a \left\langle \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x(N(U))\partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$+ 2aS\,\langle(\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W}\rangle + 2aS\,\langle(\partial_x \mathcal{W}, U_x)\partial_x(J(U))U_x, \mathcal{W}\rangle$$

$$+ 2aS\,\langle(\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W}\rangle - 4aS\,\langle(\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W}\rangle$$

$$- 2aS\,\langle|U_x|^2 \partial_x(J(U))\partial_x \mathcal{W}, \mathcal{W}\rangle$$

$$+ 2a \left\langle \sum_k (\partial_x(N(U))\partial_x \mathcal{W}, D_k(U)U_x)J(U)D_k(U)U_x, \mathcal{W} \right\rangle$$

$$+ 2a \left\langle \sum_k (N(U)\partial_x \mathcal{W}, \partial_x\{D_k(U)U_x\})J(U)D_k(U)U_x, \mathcal{W} \right\rangle$$

$$+ 2a \left\langle \sum_k (N(U)\partial_x \mathcal{W}, D_k(U)U_x)\partial_x \{J(U)D_k(U)U_x\}, \mathcal{W} \right\rangle. \qquad (3.85)$$

The second, the tenth, and the eleventh term of the RHS of (3.85) are bounded by $C\,D(t)^2$, in view of (3.72). For the fifth term of the RHS of (3.85), we use (3.13) and $(U_x, \nu_k(U)) = 0$ to see

$$\partial_x(J(U))U_x = \sum_k (U_x, J(U)D_k(U)U_x)\,\nu_k(U), \qquad (3.86)$$

which combined with (3.8) implies $(\partial_x(J(U))U_x, \mathcal{W}) = \mathcal{O}(|Z|)$. Therefore, by the integration by parts, the fifth term of the RHS of (3.85) is bounded by $C\,D(t)^2$. In the same way, we use (3.13), (3.8), and (3.9), to have

$$(\partial_x(J(U))\partial_x\mathcal{W}, \mathcal{W})$$
$$= \sum_k (\partial_x\mathcal{W}, J(U)D_k(U)U_x)\,(\nu_k(U), \mathcal{W}) - \sum_k (\partial_x\mathcal{W}, \nu_k(U))(J(U)D_k(U)U_x, \mathcal{W})$$
$$= \sum_k (\partial_x\mathcal{W}, J(U)D_k(U)U_x)\,\mathcal{O}(|Z|) + \mathcal{O}\left((|Z| + |Z_x| + |\mathcal{W}|)|\mathcal{W}|\right).$$

Thus the integration by parts shows that the eighth term of the RHS of (3.85) is bounded by $C\,D(t)^2$. For, the third and the ninth term of the RHS of (3.85), in view of (3.72), we have

$$\partial_x(N(U))\partial_x\mathcal{W} = \sum_\ell (\partial_x\mathcal{W}, D_\ell(U)U_x)\nu_\ell(U) + \sum_\ell (\partial_x\mathcal{W}, \nu_\ell(U))D_\ell(U)U_x$$
$$= \sum_\ell (\partial_x\mathcal{W}, D_\ell(U)U_x)\nu_\ell(U) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|),$$

which implies

$$-2a \left\langle \sum_k (U_x, D_k(U)U_x)J(U)D_k(U)\partial_x(N(U))\partial_x\mathcal{W}, \mathcal{W} \right\rangle$$
$$\leqslant -2a \left\langle \sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U), \mathcal{W} \right\rangle + C\,D(t)^2,$$

and

$$2a \left\langle \sum_k (\partial_x(N(U))\partial_x\mathcal{W}, D_k(U)U_x)J(U)D_k(U)U_x, \mathcal{W} \right\rangle$$
$$\leqslant 2a \left\langle \sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(\nu_\ell(U), D_k(U)U_x)J(U)D_k(U)U_x, \mathcal{W} \right\rangle + C\,D(t)^2.$$

Collecting them, we derive

$$R_4 + R_5 \leqslant 2a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)J(U)D_k(U) \right\} \partial_x\mathcal{W}, \mathcal{W} \right\rangle$$
$$- 2a \left\langle \sum_{k,\ell} (\partial_x\mathcal{W}, D_\ell(U)U_x)(U_x, D_k(U)U_x)J(U)D_k(U)\nu_\ell(U), \mathcal{W} \right\rangle$$

$$+ 2a \left\langle \sum_{k,\ell} (\partial_x \mathcal{W}, D_\ell(U)U_x)(\nu_\ell(U), D_k(U)U_x)J(U)D_k(U)U_x, \mathcal{W} \right\rangle$$

$$+ 2aS \langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \rangle + 2aS \langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \rangle$$

$$- 4aS \langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \rangle + C\,D(t)^2. \tag{3.87}$$

Therefore, by (3.82) and (3.87), we obtain

$$R_2 + R_3 + R_4 + R_5$$

$$\leqslant a \left\langle \partial_x \left\{ \sum_k (U_x, D_k(U)U_x)(J(U)D_k(U) - D_k(U)J(U)) \right\} \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$

$$+ 2aS \langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \rangle + 2aS \langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \rangle$$

$$- 2aS \langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \rangle + C\,D(t)^2. \tag{3.88}$$

Note here that

$$\left(\sum_k (\mathcal{U}, D_k(U)U_x)(J(U)D_k(U) - D_k(U)J(U))Y_1, Y_2 \right)$$

$$= \left(Y_1, \sum_k (\mathcal{U}, D_k(U)U_x)(J(U)D_k(U) - D_k(U)J(U))Y_2 \right) \tag{3.89}$$

holds for any $Y_1, Y_2 : [0, T] \times \mathbb{T} \to \mathbb{R}^d$. This follows immediately from (3.3) and (3.11). By taking the derivative of both sides of (3.89) in x, we see that

$$\left(\partial_x \left\{ \sum_k (\mathcal{U}, D_k(U)U_x)(J(U)D_k(U) - D_k(U)J(U)) \right\} Y_1, Y_2 \right)$$

$$= \left(Y_1, \partial_x \left\{ \sum_k (\mathcal{U}, D_k(U)U_x)(J(U)D_k(U) - D_k(U)J(U)) \right\} Y_2 \right)$$

holds for any $Y_1, Y_2 : [0, T] \times \mathbb{T} \to \mathbb{R}^d$. Hence, by integrating by parts, we see that the first term of the RHS of (3.88) is bounded by $C\,D(t)^2$. Therefore we obtain

$$R_2 + R_3 + R_4 + R_5$$

$$\leqslant 2aS \langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \rangle + 2aS \langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \rangle$$

$$- 2aS \langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \rangle + C\,D(t)^2. \tag{3.90}$$

Gathering the information (3.70), (3.75), and (3.90), we derive

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{W}\|_{L^2}^2 \leqslant (c + aS) \langle (\partial_x^2 \mathcal{W}, U_x)J(U)U_x, \mathcal{W} \rangle$$

$$+ (-aS + 2b + c) \langle (\partial_x U_x, U_x)J(U)\partial_x \mathcal{W}, \mathcal{W} \rangle$$

$$+ (2aS + 2b + 2c) \langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \rangle$$

$$+ (aS + 3c) \langle (\partial_x \mathcal{W}, \partial_x U_x)J(U)U_x, \mathcal{W} \rangle + C\,D(t)^2. \tag{3.91}$$

Furthermore we rewrite the third and fourth term of the RHS of (3.91) recalling Definition 3.6 and Lemma 3.7. From Definition 3.6, it follows that

$$(\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x$$

$$= \frac{1}{2} (T_3(U) - T_4(U) + T_5(U)) \partial_x \mathcal{W} + \frac{1}{2} \sum_k (U_x, D_k(U)U_x)(\partial_x \mathcal{W}, \nu_k(U))J(U)U_x.$$

Using (3.34) and (3.35) with $Y = \partial_x \mathcal{W}$, we see

$$T_5(U)\partial_x\mathcal{W} = (\partial_x U_x, U_x)J(U)\partial_x\mathcal{W} + \frac{1}{2}|U_x|^2\partial_x(J(U))\partial_x\mathcal{W}$$
$$+ \frac{1}{2}(\partial_x\mathcal{W}, U_x)\sum_k(J(U)U_x, D_k(U)U_x)\nu_k(U)$$
$$- \frac{1}{2}\sum_k(\partial_x\mathcal{W}, \nu_k(U))(J(U)U_x, D_k(U)U_x)U_x.$$

Substituting this and using (3.9), we obtain

$$(\partial_x\mathcal{W}, U_x)J(U)\partial_x U_x$$
$$= \frac{1}{2}(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W} + \frac{1}{2}\left(T_3(U) - T_4(U)\right)\partial_x\mathcal{W}$$
$$+ \frac{1}{4}|U_x|^2\partial_x(J(U))\partial_x\mathcal{W} + \frac{1}{4}(\partial_x\mathcal{W}, U_x)\sum_k(J(U)U_x, D_k(U)U_x)\nu_k(U)$$
$$+ \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.92}$$

In the same way, by using Definition 3.6 and Lemma 3.7, we obtain

$$(\partial_x\mathcal{W}, \partial_x U_x)J(U)U_x$$
$$= \frac{1}{2}(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W} + \frac{1}{2}\left(T_3(U) + T_4(U)\right)\partial_x\mathcal{W}$$
$$+ \frac{1}{4}|U_x|^2\partial_x(J(U))\partial_x\mathcal{W} + \frac{1}{4}(\partial_x\mathcal{W}, U_x)\sum_k(J(U)U_x, D_k(U)U_x)\nu_k(U)$$
$$+ \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|). \tag{3.93}$$

Thanks to (3.36) and (3.37), we can easily show $\langle T_i(U)\partial_x\mathcal{W}, \mathcal{W}\rangle \leqslant C\,D(t)^2$ with $i = 3, 4$, by integrating by parts. Besides, it is now immediate to see

$$\left\langle |U_x|^2\partial_x(J(U))\partial_x\mathcal{W}, \mathcal{W}\right\rangle \leqslant C\,D(t)^2,$$
$$\left\langle (\partial_x\mathcal{W}, U_x)\sum_k(J(U)U_x, D_k(U)U_x)\nu_k(U)\mathcal{W}\right\rangle \leqslant C\,D(t)^2$$

by the argument using (3.13) and (3.8), Therefore, we substitute (3.92) and (3.93) into (3.91) to derive

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{W}\|_{L^2}^2$$
$$\leqslant (c + aS)\left\langle(\partial_x^2\mathcal{W}, U_x)J(U)U_x, \mathcal{W}\right\rangle + (-aS + 2b + c)\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$
$$+ (aS + b + c)\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$
$$+ \frac{aS + 3c}{2}\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle + C\,D(t)^2.$$
$$= (c + aS)\left\langle(\partial_x^2\mathcal{W}, U_x)J(U)U_x, \mathcal{W}\right\rangle$$
$$+ \frac{aS + 6b + 7c}{2}\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle + C\,D(t)^2. \tag{3.94}$$

Even if we use the integration parts, the first and the second term of the RHS of (3.94) cannot be bounded by $C\,D(t)^2$. Fortunately, however, we will find in the next step that the two terms can be eliminated essentially by introducing a gauged function.

4. Energy estimates for $\|\widetilde{\mathcal{W}}\|_{L^2(\mathbb{T};\mathbb{R}^d)}$ to eliminate the loss of derivatives.

We introduce the function $\widetilde{\mathcal{W}}$ which is defined by

$$\widetilde{\mathcal{W}} = \mathcal{W} + \widetilde{\Lambda}, \tag{3.95}$$

where

$$\widetilde{\Lambda} = -\frac{e_1}{2a}(Z, J(U)U_x)J(U)U_x + \frac{e_2}{8a}|U_x|^2 Z, \tag{3.96}$$

$$e_1 = aS + c, \quad e_2 = e_1 + \frac{aS + 6b + 7c}{2}. \tag{3.97}$$

Moreover, we introduce the energy $\widetilde{D}(t)$ whose square is defined by

$$\widetilde{D}(t)^2 = \|Z(t)\|_{L^2}^2 + \|Z_x(t)\|_{L^2}^2 + \|\widetilde{\mathcal{W}}(t)\|_{L^2}^2. \tag{3.98}$$

Since u and v satisfy the same initial value, $\widetilde{D}(0) = 0$ holds. We shall show that there exists a positive constant C such that

$$\frac{1}{2}\frac{d}{dt}\widetilde{D}(t)^2 \leqslant C\,\widetilde{D}(t)^2 \tag{3.99}$$

for all $t \in (0, T)$. If it is true, (3.99) together with $\widetilde{D}(0) = 0$ shows $\widetilde{D}(t) \equiv 0$. This implies $Z = 0$.

In the proof of (3.99), by integrating by parts repeatedly, it is now not difficult to obtain the following estimate permitting the loss of derivatives of order one:

$$\frac{1}{2}\frac{d}{dt}\left\{\|Z(t)\|_{L^2}^2 + \|Z_x(t)\|_{L^2}^2\right\} \leqslant C\,\widetilde{D}(t)^2. \tag{3.100}$$

Having them in mind, we hereafter concentrate on how to derive the estimate of the form

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathcal{W}}(t)\|_{L^2}^2 \leqslant C\,\widetilde{D}(t)^2. \tag{3.101}$$

For this purpose, we begin with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathcal{W}}\|_{L^2}^2 &= \left\langle \partial_t\widetilde{\mathcal{W}}, \widetilde{\mathcal{W}}\right\rangle \\ &= \left\langle \partial_t\mathcal{W}, \widetilde{\mathcal{W}}\right\rangle + \left\langle \partial_t\widetilde{\Lambda}, \widetilde{\mathcal{W}}\right\rangle \\ &= \langle \partial_t\mathcal{W}, \mathcal{W}\rangle + \left\langle \partial_t\mathcal{W}, \widetilde{\Lambda}\right\rangle + \left\langle \partial_t\widetilde{\Lambda}, \widetilde{\mathcal{W}}\right\rangle. \end{aligned} \tag{3.102}$$

The first term of the RHS of (3.102) has already been investigated to satisfy (3.94). Hence we compute the second and the third term of the RHS of (3.102) below. Observing $\widetilde{\Lambda} = \mathcal{O}(|Z|)$, we see $\widetilde{\mathcal{W}} = \mathcal{W} + \mathcal{O}(|Z|)$, $\partial_x\widetilde{\mathcal{W}} = \partial_x\mathcal{W} + \mathcal{O}(|Z| + |Z_x|)$, and $\partial_x^2\widetilde{\mathcal{W}} = \partial_x^2\mathcal{W} + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|)$, which will be often used without comments.

We start the computation of $\left\langle \partial_t\widetilde{\Lambda}, \widetilde{\mathcal{W}}\right\rangle$ by investigating $\partial_t\widetilde{\Lambda}$. A simple computation shows

$$\partial_t\widetilde{\Lambda} = -\frac{e_1}{2a}(Z_t, J(U)U_x)J(U)U_x + \frac{e_2}{8a}|U_x|^2 Z_t + \mathcal{O}(|Z|). \tag{3.103}$$

Recalling (3.43), we see

$$Z_t = a\,\partial_x\left(J(U)\partial_x\mathcal{W}\right) + a\sum_k (J(U)\partial_x\mathcal{W}, D_k(U)U_x)\nu_k(U) + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|) \tag{3.104}$$

$$= a\,J(U)\partial_x^2\mathcal{W} + a\,\partial_x(J(U))\partial_x\mathcal{W} + a\sum_k (J(U)\partial_x\mathcal{W}, D_k(U)U_x)\nu_k(U)$$

$$+ \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|). \tag{3.105}$$

By using (3.105), we see

$$-\frac{e_1}{2a}(Z_t, J(U)U_x)J(U)U_x$$
$$= -\frac{e_1}{2}(J(U)\partial_x^2\mathcal{W}, J(U)U_x)J(U)U_x - \frac{e_1}{2}(\partial_x(J(U))\partial_x\mathcal{W}, J(U)U_x)J(U)U_x$$
$$- \frac{e_1}{2}\sum_k(J(U)\partial_x\mathcal{W}, D_k(U)U_x)(\nu_k(U), J(U)U_x)J(U)U_x + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|).$$

The third term of the RHS vanishes, since $(\nu_k(U), J(U)U_x) = 0$. By noting (3.67), we see that the second term of the RHS is $\mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|)$. Thus we have

$$-\frac{e_1}{2a}(Z_t, J(U)U_x)J(U)U_x = -\frac{e_1}{2}(\partial_x^2\mathcal{W}, U_x)J(U)U_x + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|). \tag{3.106}$$

On the other hand, by using (3.104), we obtain

$$\frac{e_2}{8a}|U_x|^2 Z_t = \frac{e_2}{8}|U_x|^2\partial_x(J(U)\partial_x\mathcal{W}) + \frac{e_2}{8}|U_x|^2\sum_k(J(U)\partial_x\mathcal{W}, D_k(U)U_x)\nu_k(U)$$
$$+ \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|)$$
$$= \frac{e_2}{8}\partial_x\left\{|U_x|^2 J(U)\partial_x\mathcal{W}\right\} - \frac{e_2}{4}(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}$$
$$+ \sum_k\mathcal{O}(|\partial_x\mathcal{W}|)\nu_k(U) + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|). \tag{3.107}$$

By substituting (3.106) and (3.107) into (3.103), we obtain

$$\partial_t\widetilde{\Lambda} = -\frac{e_1}{2}(\partial_x^2\mathcal{W}, U_x)J(U)U_x + \frac{e_2}{8}\partial_x\left\{|U_x|^2 J(U)\partial_x\mathcal{W}\right\} - \frac{e_2}{4}(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}$$
$$+ \sum_k\mathcal{O}(|\partial_x\mathcal{W}|)\nu_k(U) + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|).$$

This shows that

$$\left\langle\partial_t\widetilde{\Lambda}, \widetilde{\mathcal{W}}\right\rangle = -\frac{e_1}{2}\left\langle(\partial_x^2\mathcal{W}, U_x)J(U)U_x, \mathcal{W} + \mathcal{O}(|Z|)\right\rangle$$
$$+ \frac{e_2}{8}\left\langle\partial_x\left\{|U_x|^2 J(U)\partial_x\mathcal{W}\right\}, \mathcal{W} + \mathcal{O}(|Z|)\right\rangle$$
$$- \frac{e_2}{4}\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W} + \mathcal{O}(|Z|)\right\rangle$$
$$+ \left\langle\sum_k\mathcal{O}(|\partial_x\mathcal{W}|)\nu_k(U), \mathcal{W} + \mathcal{O}(|Z|)\right\rangle$$
$$+ \left\langle\mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|), \mathcal{W} + \mathcal{O}(|Z|)\right\rangle$$
$$\leqslant -\frac{e_1}{2}\left\langle(\partial_x^2\mathcal{W}, U_x)J(U)U_x, \mathcal{W}\right\rangle + \frac{e_2}{8}\left\langle\partial_x\left\{|U_x|^2 J(U)\partial_x\mathcal{W}\right\}, \mathcal{W}\right\rangle$$
$$- \frac{e_2}{4}\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$
$$+ \left\langle\sum_k\mathcal{O}(|\partial_x\mathcal{W}|)\nu_k(U), \mathcal{W}\right\rangle + C\,\widetilde{D}(t)^2$$
$$\leqslant -\frac{e_1}{2}\left\langle(\partial_x^2\mathcal{W}, U_x)J(U)U_x, \mathcal{W}\right\rangle - \frac{e_2}{4}\left\langle(\partial_x U_x, U_x)J(U)\partial_x\mathcal{W}, \mathcal{W}\right\rangle$$

$$+ C\,\widetilde{D}(t)^2. \tag{3.108}$$

We next compute $\left\langle \partial_t \mathcal{W}, \widetilde{\Lambda} \right\rangle$. Observing (3.64), we see

$$\partial_t \mathcal{W} = a\, \partial_x^2 \left\{ J(U) \partial_x^2 \mathcal{W} \right\} - 2a \sum_k \left(\partial_x^3 \mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U)$$

$$+ \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}| + |\partial_x \mathcal{W}| + |\partial_x^2 \mathcal{W}|).$$

By using this and by noting $\widetilde{\Lambda} = \mathcal{O}(|Z|)$, we integrate by parts to obtain

$$\left\langle \partial_t \mathcal{W}, \widetilde{\Lambda} \right\rangle \leqslant R_8 + R_9 + C\,\widetilde{D}(t)^2, \tag{3.109}$$

where

$$R_8 = a \left\langle \partial_x^2 \left\{ J(U) \partial_x^2 \mathcal{W} \right\}, \widetilde{\Lambda} \right\rangle,$$

$$R_9 = -2a \left\langle \sum_k \left(\partial_x^3 \mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U), \widetilde{\Lambda} \right\rangle.$$

For R_9, noting $\widetilde{\Lambda} = \mathcal{O}(|Z|)$, we use the integration by parts and $(\nu_k(U), J(U) U_x) = 0$ to obtain

$$R_9 \leqslant -2a\,(-1)^3 \left\langle \sum_k \left(\mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U), \partial_x^3 \widetilde{\Lambda} \right\rangle + C\,\widetilde{D}(t)^2$$

$$\leqslant 2a \left\langle \sum_k \left(\mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U), -\frac{e_1}{2a} (\partial_x^3 Z, J(U) U_x) J(U) U_x + \frac{e_2}{8a} |U_x|^2 \partial_x^3 Z \right\rangle$$

$$+ C\,\widetilde{D}(t)^2$$

$$= \frac{e_2}{4} \left\langle \sum_k \left(\mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U), |U_x|^2 \partial_x^3 Z \right\rangle + C\,\widetilde{D}(t)^2.$$

Furthermore, since $\partial_x^3 Z = \partial_x^2 Z_x = \partial_x \mathcal{W} + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|) = \partial_x \mathcal{W} + \mathcal{O}(|Z| + |Z_x| + |\widetilde{\mathcal{W}}|)$ and $(\nu_k(U), \partial_x \mathcal{W}) = \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|)$, we have

$$R_9 \leqslant \frac{e_2}{4} \left\langle \sum_k \left(\mathcal{W}, J(U) D_k(U) U_x \right) \nu_k(U), |U_x|^2 \partial_x \mathcal{W} \right\rangle + C\,\widetilde{D}(t)^2 \leqslant C\,\widetilde{D}(t)^2. \tag{3.110}$$

For R_8, we begin with

$$R_8 = -\frac{e_1}{2} \left\langle \partial_x^2 \left\{ J(U) \partial_x^2 \mathcal{W} \right\}, (Z, J(U) U_x) J(U) U_x \right\rangle + \frac{e_2}{8} \left\langle \partial_x^2 \left\{ J(U) \partial_x^2 \mathcal{W} \right\}, |U_x|^2 Z \right\rangle$$

$$=: R_{81} + R_{82}.$$

The integration by parts implies

$$R_{81} = -\frac{e_1}{2} \left\langle J(U) \partial_x^2 \mathcal{W}, \partial_x^2 \left\{ (Z, J(U) U_x) J(U) U_x \right\} \right\rangle$$

$$\leqslant -\frac{e_1}{2} \left\langle J(U) \partial_x^2 \mathcal{W}, (\partial_x Z_x, J(U) U_x) J(U) U_x \right\rangle$$

$$- e_1 \left\langle J(U) \partial_x^2 \mathcal{W}, (Z_x, \partial_x \left\{ J(U) U_x \right\}) J(U) U_x \right\rangle$$

$$- e_1 \left\langle J(U) \partial_x^2 \mathcal{W}, (Z_x, J(U) U_x) \partial_x \left\{ J(U) U_x \right\} \right\rangle + C\,\widetilde{D}(t)^2$$

$$\leqslant -\frac{e_1}{2} \left\langle J(U) \partial_x^2 \mathcal{W}, (\partial_x Z_x, J(U) U_x) J(U) U_x \right\rangle$$

$$+ e_1 \left\langle J(U) \partial_x \mathcal{W}, (\partial_x Z_x, \partial_x \left\{ J(U) U_x \right\}) J(U) U_x \right\rangle$$

$$+ e_1 \left\langle J(U)\partial_x \mathcal{W}, (\partial_x Z_x, J(U)U_x)\partial_x \left\{ J(U)U_x \right\} \right\rangle + C\,\widetilde{D}(t)^2$$

$$=: R_{83} + R_{84} + R_{85} + C\,\widetilde{D}(t)^2. \tag{3.111}$$

Since $\mathcal{U} = \partial_x U_x + \sum_k (U_x, D_k(U)U_x)\nu_k(U)$ and $\mathcal{V} = \partial_x V_x + \sum_k (V_x, D_k(V)V_x)\nu_k(V)$, we see

$$\partial_x Z_x = \mathcal{W} - \sum_k (Z_x, D_k(U)U_x)\nu_k(U) - \sum_k (V_x, D_k(U)Z_x)\nu_k(U) + \mathcal{O}(|Z|),$$

and thus $(\partial_x Z_x, J(U)U_x) = (\mathcal{W}, J(U)U_x) + \mathcal{O}(|Z|)$. Substituting this, using (3.3) and (3.4), and integrating by parts, we have

$$
\begin{aligned}
R_{83} &= -\frac{e_1}{2}\left\langle \partial_x^2 \mathcal{W}, (\partial_x Z_x, J(U)U_x)U_x \right\rangle \\
&\leqslant -\frac{e_1}{2}\left\langle \partial_x^2 \mathcal{W}, (\mathcal{W}, J(U)U_x)U_x \right\rangle - \frac{e_1}{2}\left\langle \partial_x^2 \mathcal{W}, \mathcal{O}(|Z|) \right\rangle + C\,\widetilde{D}(t)^2 \\
&\leqslant -\frac{e_1}{2}\left\langle (\partial_x^2 \mathcal{W}, U_x)J(U)U_x, \mathcal{W} \right\rangle + C\,\widetilde{D}(t)^2.
\end{aligned} \tag{3.112}
$$

From (3.86), it follows that

$$\partial_x \left\{ J(U)U_x \right\} = J(U)\partial_x U_x + \sum_k (U_x, J(U)D_k(U)U_x)\nu_k(U).$$

Using this, $\partial_x Z_x = \mathcal{W} + \mathcal{O}(|Z| + |Z_x|)$, and (3.8), we see

$$
\begin{aligned}
&(\partial_x Z_x, \partial_x \left\{ J(U)U_x \right\}) \\
&= (\partial_x Z_x, J(U)\partial_x U_x) + \sum_k (U_x, J(U)D_k(U)U_x)(\nu_k(U), \partial_x Z_x) + \mathcal{O}(|Z| + |Z_x|) \\
&= (\mathcal{W}, J(U)\partial_x U_x) + \sum_k (U_x, J(U)D_k(U)U_x)(\nu_k(U), \mathcal{W}) + \mathcal{O}(|Z| + |Z_x|) \\
&= (\mathcal{W}, J(U)\partial_x U_x) + \mathcal{O}(|Z| + |Z_x|).
\end{aligned}
$$

This implies

$$
\begin{aligned}
R_{84} &= e_1 \left\langle \partial_x \mathcal{W}, (\partial_x Z_x, \partial_x \left\{ J(U)U_x \right\})U_x \right\rangle \\
&\leqslant e_1 \left\langle \partial_x \mathcal{W}, (\mathcal{W}, J(U)\partial_x U_x)U_x \right\rangle + C\,\widetilde{D}(t)^2 \\
&= e_1 \left\langle (\partial_x \mathcal{W}, U_x)J(U)\partial_x U_x, \mathcal{W} \right\rangle + C\,\widetilde{D}(t)^2.
\end{aligned} \tag{3.113}
$$

In the same way, we use (3.4) to see

$$
\begin{aligned}
&(J(U)\partial_x \mathcal{W}, \partial_x \left\{ J(U)U_x \right\}) \\
&= (J(U)\partial_x \mathcal{W}, J(U)\partial_x U_x) + \sum_k (U_x, J(U)D_k(U)U_x)(J(U)\partial_x \mathcal{W}, \nu_k(U)) \\
&= (\partial_x \mathcal{W}, P(U)\partial_x U_x) \\
&= (\partial_x \mathcal{W}, \partial_x U_x + \sum_k (U_x, D_k(U)U_x)\nu_k(U)) \\
&= (\partial_x \mathcal{W}, \partial_x U_x) + \mathcal{O}(|Z| + |Z_x| + |\mathcal{W}|).
\end{aligned}
$$

Substituting this, we obtain

$$
\begin{aligned}
R_{85} &= e_1 \left\langle J(U)\partial_x \mathcal{W}, (\partial_x Z_x, J(U)U_x)\partial_x \left\{ J(U)U_x \right\} \right\rangle \\
&\leqslant e_1 \left\langle \partial_x \mathcal{W}, (\partial_x Z_x, J(U)U_x)\partial_x U_x \right\rangle + C\,\widetilde{D}(t)^2 \\
&\leqslant e_1 \left\langle \partial_x \mathcal{W}, (\mathcal{W}, J(U)U_x)\partial_x U_x \right\rangle + C\,\widetilde{D}(t)^2
\end{aligned}
$$

$$= e_1 \left\langle (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2. \tag{3.114}$$

Substituting (3.112), (3.113), and (3.114) into (3.111), we see $R_{81} = R_{83} + R_{84} + R_{85} + C \, \widetilde{D}(t)^2$ is bounded as follows:

$$R_{81} \leqslant -\frac{e_1}{2} \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle + e_1 \left\langle (\partial_x \mathcal{W}, U_x) J(U) \partial_x U_x, \mathcal{W} \right\rangle$$
$$+ e_1 \left\langle (\partial_x \mathcal{W}, \partial_x U_x) J(U) U_x, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2.$$

Furthermore, by applying (3.92) and (3.93) to the second and the third term of the RHS of above, and by using $\langle T_3(U) \partial_x \mathcal{W}, \mathcal{W} \rangle \leqslant C \, \widetilde{D}(t)^2$ again, we deduce

$$R_{81} \leqslant -\frac{e_1}{2} \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle + e_1 \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$
$$+ e_1 \left\langle T_3(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2$$
$$\leqslant -\frac{e_1}{2} \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle + e_1 \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2. \tag{3.115}$$

For R_{82}, the integration by parts and the same argument as above lead to

$$R_{82} = \frac{e_2}{8} \left\langle J(U) \partial_x^2 \mathcal{W}, \partial_x^2 \left\{ |U_x|^2 Z \right\} \right\rangle$$
$$= \frac{e_2}{8} \left\langle J(U) \partial_x^2 \mathcal{W}, |U_x|^2 \partial_x Z_x + 4(\partial_x U_x, U_x) Z_x + \mathcal{O}(|Z|) \right\rangle$$
$$\leqslant \frac{e_2}{8} \left\langle |U_x|^2 J(U) \partial_x^2 \mathcal{W}, \mathcal{W} \right\rangle + \frac{e_2}{2} \left\langle J(U) \partial_x^2 \mathcal{W}, (\partial_x U_x, U_x) Z_x \right\rangle + C \, \widetilde{D}(t)^2$$
$$\leqslant \frac{e_2}{8} \left\langle \partial_x \left\{ |U_x|^2 J(U) \partial_x \mathcal{W} \right\}, \mathcal{W} \right\rangle - \frac{e_2}{4} \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$
$$- \frac{e_2}{8} \left\langle |U_x|^2 \partial_x (J(U)) \partial_x \mathcal{W}, \mathcal{W} \right\rangle - \frac{e_2}{2} \left\langle J(U) \partial_x \mathcal{W}, (\partial_x U_x, U_x) \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2$$
$$\leqslant -\frac{3e_2}{4} \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2. \tag{3.116}$$

Therefore, from (3.115), and (3.116), it follows that $R_8 = R_{81} + R_{82}$ is bounded as follows:

$$R_8 \leqslant -\frac{e_1}{2} \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle + \left(e_1 - \frac{3e_2}{4} \right) \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle$$
$$+ C \, \widetilde{D}(t)^2. \tag{3.117}$$

Consequently, by substituting (3.110) and (3.117) into (3.109), we have

$$\left\langle \partial_t \mathcal{W}, \widetilde{\Lambda} \right\rangle \leqslant -\frac{e_1}{2} \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle$$
$$+ \left(e_1 - \frac{3e_2}{4} \right) \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2. \tag{3.118}$$

Collecting the information (3.102), (3.94), (3.108), (3.118), and (3.97), we conclude

$$\frac{1}{2} \frac{d}{dt} \|\widetilde{\mathcal{W}}\|_{L^2}^2 \leqslant (c + aS - e_1) \left\langle (\partial_x^2 \mathcal{W}, U_x) J(U) U_x, \mathcal{W} \right\rangle$$
$$+ \left(\frac{aS + 6b + 7c}{2} + e_1 - e_2 \right) \left\langle (\partial_x U_x, U_x) J(U) \partial_x \mathcal{W}, \mathcal{W} \right\rangle + C \, \widetilde{D}(t)^2$$
$$= C \, \widetilde{D}(t)^2,$$

which is the desired result (3.101). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Acknowledgements.

REFERENCES

[1] S.C. Anco and R. Myrzakulov, *Integrable generalizations of Schrödinger maps and Heisenberg spin models from Hamiltonian flows of curves and surfaces.* J. Geom. Phys. **60** (2010), 1576–1603.

[2] N.-H. Chang, J. Shatah, and K. Uhlenbeck, *Schrödinger maps.* Comm. Pure Appl. Math. **53** (2000), 590–602.

[3] H. Chihara, *Schrödinger flow into almost Hermitian manifolds.* Bull. Lond. Math. Soc. **45** (2013), 37–51.

[4] H. Chihara, *Fourth-order dispersive systems on the one-dimensional torus.* J. Pseudo-Differ. Oper. Appl. **6** (2015), 237–263.

[5] H. Chihara and E. Onodera, *A third order dispersive flow for closed curves into almost Hermitian manifolds*, J. Funct. Anal. **257**, 388–404, (2009).

[6] H. Chihara and E. Onodera, *A fourth-order dispersive flow into Kähler manifolds.* Z. Anal. Anwend. **34** (2015), 221–249.

[7] Y. Fukumoto, *Three-dimensional motion of a vortex filament and its relation to the localized induction hierarchy.* Eur. Phys. J. B **29** (2002), 167–171.

[8] Y. Fukumoto and H. K. Moffatt, *Motion and expansion of a viscous vortex ring. Part I. A higher-order asymptotic formula for the velocity.* J. Fluid. Mech. **417** (2000), 1–45.

[9] B. Guo, M. Zeng, and F. Su, *Periodic weak solutions for a classical one-dimensional isotropic biquadratic Heisenberg spin chain.* J. Math. Anal. Appl. **330** (2007), 729–739.

[10] C. Kenig, T. Lamm, D. Pollack, G. Staffilani, and T. Toro, *The Cauchy problem for Schrödinger flows.* Discrete Contin. Dyn. Syst. **27** (2010), 389–439.

[11] N. Koiso, *The vortex filament equation and a semilinear Schrödinger equation in a Hermitian symmetric space.* Osaka J. Math. **34** (1997), 199–214.

[12] M. Lakshmanan, K. Porsezian, and M. Daniel, *Effect of discreteness on the continuum limit of the Heisenberg spin chain.* Phys. Lett. A. **133** (1988), 483–488.

[13] H. McGahagan, *An approximation scheme for Schrödinger maps.* Comm. Partial Differential Equations **32** (2007), 375–400.

[14] A. Nahmod, J. Shatah, L. Vega, and C. Zeng, *Schrödinger maps and their associated frame systems.* Int. Math. Res. Not. IMRN 2007, no. 21, Art. ID rnm088, 29 pp.

[15] E. Onodera, *Generalized Hasimoto transform of one-dimensional dispersive flows into compact Riemann surfaces.* SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), article No. 044, 10 pages.

[16] E. Onodera, *A third-order dispersive flow for closed curves into Kähler manifolds.* J. Geom. Anal. **18** (2008), 889–918.

[17] E. Onodera, *A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces.* Comm. Partial Differential Equations **35** (2010), 1130–1144.

[18] E. Onodera, *The initial value problem for a fourth-order dispersive closed curve flow on the two-sphere.* to appear in Proc. Roy. Soc. Edinburgh Sect. A.

[19] P.L. Sulem, C. Sulem, and C.Bardos, *On the continuous limit for a system of classical spins.* Comm. Math. Phys. **107** (1986), 431–454.

(Eiji Onodera) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOCHI UNIVERSITY, KOCHI 780-8520, JAPAN

E-mail address: onodera@kochi-u.ac.jp