Subregular subalgebras and invariant generalized complex structures on Lie groups

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Abstract

We introduce the notion of a subregular subalgebra, which we believe is useful for classification of subalgebras of Lie algebras. We use it to construct a non-regular invariant generalized complex structure on a Lie group. As an illustration of the study of invariant generalized complex structures, we compute them all for the real forms of G_2 .

1 Subregular subalgebras

Let \mathfrak{g} be a finite-dimensional complex Lie algebra, $k \geq 0$ an integer.

Definition 1.1. A subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is called subregular in codimension k if \mathfrak{s} is normalized by a codimension k subalgebra of a Cartan subalgebra of \mathfrak{g} .

If $k \ge 1$, then $\mathfrak{s} \subset \mathfrak{g}$ is called subregular strictly in codimension k if $\mathfrak{s} \subset \mathfrak{g}$ is subregular in codimension k, but is not subregular in codimension k-1.

Note that any subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is subregular (strictly) in codimension k for some $k \in \{0, 1, \ldots, \operatorname{rank}(\mathfrak{g})\}$. Regular subalgebras, as defined in [2], [4], are precisely those which are subregular in codimension 0.

This notion may be useful for an explicit classification of subalgebras of Lie algebras as in [10]. In this note, we demonstrate how it can be applied to construction of invariant generalized complex structures on Lie groups.

2 Invariant generalized complex structures

Invariant generalized complex structures on homogeneous spaces were studied in [11] and [1]. In particular, Alekseevsky, David and Milburn classified invariant generalized complex structures on Lie groups in terms of the so-called admissible pairs. We will review a part of their classification.

Throughout this section, G_0 denotes a finite-dimensional connected real Lie group, \mathfrak{g}_0 the (real) Lie algebra of G_0 , $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ its complexification and $\tau \colon \mathfrak{g} \to \mathfrak{g}$ the corresponding antiinvolution. If \mathfrak{g} is semisimple and $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is a Cartan subalgebra, then $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{g}$ denotes its complexification and $\Phi \subset \mathfrak{h}^*$ the root system of \mathfrak{g} with respect to the Cartan subalgebra \mathfrak{h} .

Definition 2.1 (Alekseevsky-David [1], Milburn [11]). A \mathfrak{g}_0 -admissible pair is a pair (\mathfrak{s}, ω) , where $\mathfrak{s} \subset \mathfrak{g}$ is a complex subalgebra and $\omega \in \bigwedge^2 \mathfrak{s}^*$ is a closed 2-form such that:

- $\mathfrak{s} + \tau(\mathfrak{s}) = \mathfrak{g}$, and
- $\operatorname{Im}(\omega \mid_{\mathfrak{g}_0 \cap \mathfrak{s}})$ is non-degenerate.

Theorem 2.2 (Akelseevsky-David [1], Milburn [11]). There is a one-to-one correspondence between the invariant generalized complex structures on G_0 and the \mathfrak{g}_0 -admissible pairs (\mathfrak{s},ω) .

The following notion was introduced by Alekseevsky and David [1].

Definition 2.3 (Akelseevsky-David [1]). An invariant generalized complex structure on G_0 is called regular if the associated subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is normalized by a Cartan subalgebra of \mathfrak{g}_0 .

The following theorem strengthens [1], Theorem 15 and completes the classification of invariant generalized complex structures on finite-dimensional compact connected real semisimple Lie groups.

Theorem 2.4. If G_0 is a finite-dimensional compact connected real semisimple Lie group, then any invariant generalized complex structure on G_0 is regular.

Proof. Let $\mathfrak{s} \subset \mathfrak{g}$ be the complex subalgebra associated by Theorem 2.2 to an invariant generalized complex structure on G_0 . Let $N(\mathfrak{s}) \subset \mathfrak{g}$ be its normalizer.

By [9], Theorem 13, $N(\mathfrak{s}) \cap \mathfrak{g}_0$ generates a closed subgroup of G_0 . The same argument as in [1], Theorem 15, using [14], implies that $N(\mathfrak{s})$ is normalized by a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$, i.e.

$$N(\mathfrak{s}) = L \oplus \bigoplus_{\alpha \in R} \mathbb{C}X_{\alpha},$$

where $R \subset \Phi$ is a closed subset, X_{α} , $\alpha \in \Phi$, are root vectors of \mathfrak{g} with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$, and $L \subset \mathfrak{h}$ is the solution set of a system of equations of the form $\alpha - \beta = 0$, $\alpha, \beta \in \Phi$.

Since τ is the conjugation with respect to a compact real form \mathfrak{g}_0 of \mathfrak{g} and $\tau(\mathfrak{h}) = \mathfrak{h}$,

$$\tau \mid_{\mathfrak{h}(\mathbb{R})} = -\operatorname{Id}_{\mathfrak{h}(\mathbb{R})}, \quad \tau(\mathbb{C}X_{\alpha}) = \mathbb{C}X_{-\alpha},$$

where $\mathfrak{h}(\mathbb{R})$ is the real span in \mathfrak{h} of the coroots of \mathfrak{g} with respect to \mathfrak{h} [6].

Since
$$N(\mathfrak{s}) + \tau(N(\mathfrak{s})) = \mathfrak{g}$$
, $L + \tau(L) = \mathfrak{h}$, which is possible only if $L = \mathfrak{h}$. Hence $\mathfrak{h}_0 \subset N(\mathfrak{s})$ normalizes \mathfrak{s} .

In general, not all invariant generalized complex structures on real semisimple Lie groups are regular. Let G_0 be a finite-dimensional connected real Lie group, $k \ge 0$ an integer.

Definition 2.5. An invariant generalized complex structure \mathcal{J} on G_0 is called subregular in codimension k if the associated subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is normalized by a codimension k subalgebra of a Cartan subalgebra of \mathfrak{g}_0 . If $k \geq 1$, then \mathcal{J} is called subregular strictly in codimension k if \mathcal{J} is subregular in codimension k, but is not subregular in codimension k-1.

We illustrate this notion with an example of a non-regular invariant generalized complex structure on $SO_0(2n-1,1)$, $n \ge 3$ even.

3 A non-regular invariant generalized complex structure on $SO_0(2n-1,1)$

Let $G_0 = SO_0(2n-1,1)$, $n \ge 3$. Then $\mathfrak{g}_0 = \mathfrak{so}(2n-1,1)$ is a noncompact real form of $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$. We interpret \mathfrak{g} as the Lie algebra of $2n \times 2n$ skew symmetric complex matrices. Then

$$\tau \colon \mathfrak{g} \to \mathfrak{g}, \ A \mapsto J \cdot \bar{A} \cdot J, \quad J = \operatorname{diag}(\underbrace{1 \ 1 \cdots 1}_{2n-1} (-1)),$$

is the conjugation with respect to \mathfrak{g}_0 , where bar denotes the usual complex conjugation.

Let E_{ij} , $1 \le i, j \le 2n$, be a $2n \times 2n$ matrix with 1 in the $(i, j)^{\text{th}}$ place and 0 elsewhere. Following [6], define

$$\begin{split} H_k &= \sqrt{-1} \cdot \left(E_{2k-1,2k} - E_{2k,2k-1} \right), \ 1 \leq k \leq n, \\ G_{jk}^+ &= E_{2j-1,2k-1} - E_{2k-1,2j-1} + E_{2j,2k} - E_{2k,2j} + \sqrt{-1} \cdot \left(E_{2j-1,2k} - E_{2j,2k-1} - E_{2k,2j-1} + E_{2k-1,2j} \right), \\ G_{kj}^+ &= -\overline{G_{jk}^+}, \ 1 \leq j < k \leq n, \\ G_{jk}^- &= E_{2j-1,2k-1} - E_{2k-1,2j-1} - E_{2j,2k} + E_{2k,2j} + \sqrt{-1} \cdot \left(E_{2j-1,2k} + E_{2j,2k-1} - E_{2k,2j-1} - E_{2k-1,2j} \right), \\ G_{kj}^- &= -\overline{G_{jk}^-}, \ 1 \leq j < k \leq n. \end{split}$$

Then $\mathfrak{h} = \bigoplus_{k=1}^n \mathbb{C}H_k$ is a Cartan subalgebra of \mathfrak{g} . Let $\epsilon_k \in \mathfrak{h}^*$, $1 \leq k \leq n$, be such that $\epsilon_k(H_j) = 1$ if j = k and 0 otherwise. Then

$$\Phi = \{ \epsilon_j - \epsilon_k \mid 1 \le j \ne k \le n \} \cup \{ \pm (\epsilon_j + \epsilon_k) \mid 1 \le j < k \le n \}$$

is the root system of $(\mathfrak{g},\mathfrak{h})$. Let us choose the following root vectors:

$$X_{jk} = X_{\epsilon_j - \epsilon_k} = G_{jk}^+, \ 1 \le j \ne k \le n,$$
 $Y_{jk} = X_{\epsilon_j + \epsilon_k} = G_{kj}^-, \ 1 \le j < k \le n,$ $Z_{jk} = X_{-(\epsilon_j + \epsilon_k)} = G_{jk}^-, \ 1 \le j < k \le n.$

Note that

$$[Y_{jk}, Z_{jk}] = 4 \cdot (H_j + H_k), \ [X_{jk}, X_{kj}] = 4 \cdot (H_j - H_k), \ 1 \le j < k \le n.$$

Let $\mathfrak{h}_1 \subset \mathfrak{h}$ be a hyperplane cut out by the equation $\epsilon_{n-1} - \epsilon_n = 0$, $L \subsetneq \mathfrak{h}_1$ a vector subspace containing $H_{n-1} + H_n$, and $H \in \mathfrak{h}_1 \setminus L$. Define

$$\begin{split} \mathfrak{s} &= L \oplus \mathbb{C}(H + X_{n-1,n}) \oplus \bigoplus_{\substack{1 \leq j < k \leq n \\ (j,k) \neq (n-1,n)}} \mathbb{C}X_{jk} \ \oplus \bigoplus_{1 \leq j < k \leq n} \mathbb{C}Y_{jk} \ \oplus \ \mathbb{C}Z_{n-1,n} \\ &\subset \mathfrak{g} = \mathfrak{h} \ \oplus \bigoplus_{1 \leq j \neq k \leq n} \mathbb{C}X_{jk} \ \oplus \bigoplus_{1 \leq j < k \leq n} \left(\mathbb{C}Y_{jk} \oplus \mathbb{C}Z_{jk} \right). \end{split}$$

Lemma 3.1. The subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is subregular strictly in codimension 1.

Proof. By construction, \mathfrak{s} is normalized by a codimension 1 subalgebra $\mathfrak{h}_1 \subset \mathfrak{h}$. At the same time, \mathfrak{s} is not regular, because, for a suitable $l \in L$, $l + H + X_{n-1,n}$ lies in the radical of \mathfrak{s} but its nilpotent component $X_{n-1,n}$ does not.

Note that $s + \tau(s) = \mathfrak{g}$ if and only if $L + \mathbb{C}H + \tau(L + \mathbb{C}H) = \mathfrak{h}$.

To illustrate the general idea, let us assume for simplicity that $H = H_1$, $L = \bigoplus_{k=2}^{n-2} \mathbb{C}H_k \oplus \mathbb{C}(H_{n-1} + H_n)$. Then

$$\mathfrak{s} \cap \mathfrak{g}_0 = \{ \sqrt{-1} \cdot b_1 \cdot (H_1 + X_{n-1,n} - Z_{n-1,n}) + \sum_{j=2}^{n-2} \sqrt{-1} \cdot b_j \cdot H_j \mid b_j \in \mathbb{R} \}$$

is a real abelian Lie algebra of dimension n-2. If n is even, $\mathfrak{s} \cap \mathfrak{g}_0$ carries a symplectic form ω_0 , which may be any non-degenerate 2-form on the real vector space $\mathfrak{s} \cap \mathfrak{g}_0 = \mathbb{R}\sqrt{-1}(H_1+X_{n-1,n}-Z_{n-1,n}) \oplus \bigoplus_{j=2}^{n-2} \mathbb{R}\sqrt{-1}H_j \cong \mathbb{R}^{n-2}$. One can extend $\sqrt{-1}\omega_0$ to a closed 2-form $\omega \in \bigwedge^2 \mathfrak{s}^*$. Assume that ϵ_j , $1 \leq j \leq n$, vanish on the root vectors of \mathfrak{g} . This proves

Theorem 3.2. Let $n \geq 4$ be even, $\mathfrak{s} \subset \mathfrak{g}$ the complex subalgebra defined above, $H = H_1$, $L = \bigoplus_{k=2}^{n-2} \mathbb{C}H_k \oplus \mathbb{C}(H_{n-1} + H_n)$, and $\omega \in \bigwedge^2 \mathfrak{s}^*$ a closed 2-form such that

$$\omega\mid_{\mathfrak{s}\cap\mathfrak{g}_0}=\sqrt{-1}\cdot\sum_{j=1}^{\frac{n}{2}-1}\epsilon_{2j-1}\wedge\epsilon_{2j}.$$

Then (\mathfrak{s},ω) is a \mathfrak{g}_0 -admissible pair and defines a non-regular invariant generalized complex structure on $SO_0(2n-1,1)$.

4 Invariant generalized complex structures on real forms of G_2

In this section, G_0 denotes a connected real Lie group whose Lie algebra \mathfrak{g}_0 is a real form of the complex simple Lie algebra \mathfrak{g} of type G_2 , i.e. \mathfrak{g}_0 is either the compact real form G_2^c or the normal real form G_2^n of $\mathfrak{g} = G_2$. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the conjugation with respect to \mathfrak{g}_0 .

Recall that G_2^n has 4 conjugacy classes of Cartan subalgebras \mathfrak{l} : the maximally noncompact, the maximally compact, the one with a single short real root and the one with a single long real root [13]. The conjugation $\tau_n \colon \mathfrak{g} \to \mathfrak{g}$ with respect to G_2^n acts on the root system of $(\mathfrak{g}, \mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C})$ as Id, – Id, a reflection through a short root and a reflection through a long root respectively.

Let (\mathfrak{s}, ω) be a \mathfrak{g}_0 -admissible pair corresponding to an invariant generalized complex structure on G_0 . The subalgebras of the complex simple Lie algebra of type G_2 were classified in [10]. We will use the notation of [4] and [10]. Since $\mathfrak{s} + \tau(\mathfrak{s}) = \mathfrak{g}$, $\dim(\mathfrak{s}) \geq \dim(G_2)/2 = 7$ and $\mathfrak{s} \subset \mathfrak{g}$ is regular.

Lemma 4.1. The subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is normalized by a Cartan subalgebra of \mathfrak{g}_0 and is not isomorphic to $\mathfrak{sl}_3(\mathbb{C})$.

Proof. By [4], up to conjugacy either $\mathfrak{s} = \mathfrak{g}$ or $\mathfrak{s} = A_2$ or $\mathfrak{s} \subset G_2[\beta]$ or $\mathfrak{s} \subset G_2[\alpha]$.

Suppose $\mathfrak{s}=A_2$. Since \mathfrak{s} is semisimple, the 2-dimensional subalgebra $\mathfrak{s}\cap\tau(\mathfrak{s})$ contains semisimple and nilpotent components of its elements. Since $H^2(\mathfrak{s},\mathbb{C})=0$, ω is exact. Thus, if $\mathfrak{s}\cap\tau(\mathfrak{s})$ is abelian, $\omega\mid_{\mathfrak{s}\cap\mathfrak{g}_0}=0$, a contradiction. Hence $\mathfrak{s}\cap\tau(\mathfrak{s})$ is not abelian, and so every element of $\mathfrak{s}\cap\tau(\mathfrak{s})$ is either semisimple or nilpotent. Then we can choose a basis x_0,x_1 of $\mathfrak{s}\cap\tau(\mathfrak{s})$ such that $[x_0,x_1]=2\cdot x_1$, where x_0 is semisimple and x_1 is nilpotent. Since $\tau(x_1)\in\mathbb{C}x_1$, we may assume that $\tau(x_0)=x_0,\tau(x_1)=x_1$.

The proof of the Jacobson-Morozov theorem in [3] goes through and provides $x_2 \in \mathfrak{s}$ such that x_0, x_1, x_2 span an $\mathfrak{sl}_2(\mathbb{C})$ subalgebra of \mathfrak{s} . Since $\tau(x_2) = x_2$, we obtain a contradiction.

Suppose $\mathfrak{s} \subset G_2[\beta]$ or $\mathfrak{s} \subset G_2[\alpha]$. By [10], Table 1, \mathfrak{s} either is solvable and contains a Cartan subalgebra of \mathfrak{g} or is normalized by a Borel subalgebra of \mathfrak{g} or is the subalgebra

$$\mathfrak{s}_3 = \mathfrak{h}_1 \oplus \mathbb{C}Y_\beta \oplus \mathbb{C}Y_{-\beta} \oplus \mathbb{C}Y_{2\alpha+\beta} \oplus \mathbb{C}Y_{3\alpha+\beta} \oplus \mathbb{C}Y_{3\alpha+2\beta},$$

where $\mathfrak{h}_1 \subset \mathfrak{g}$ is a Cartan subalgebra, $\Phi = \{\pm \alpha, \pm \beta, \pm (\alpha + \beta), \pm (2\alpha + \beta), \pm (3\alpha + \beta), \pm (3\alpha + 2\beta)\}$ is the root system of $(\mathfrak{g}, \mathfrak{h}_1), Y_{\gamma}, \gamma \in \Phi$, are root vectors.

Note that any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ [15]. Let $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

If $\mathfrak{s} \subset \mathfrak{b}$ contains a Cartan subalgebra of \mathfrak{g} , then \mathfrak{h}_0 is maximally compact and $\mathfrak{h} \cap \mathfrak{s} \neq 0$. This implies that either \mathfrak{h} normalizes \mathfrak{s} or $\mathfrak{h} \cap \mathfrak{s} \subset \mathfrak{s} \cap \tau(\mathfrak{s}) = 0$, a contradiction.

Suppose $\mathfrak{s} = \mathfrak{s}_3$. Let $\mathfrak{b} \subset G_2[\alpha]$ be the Borel subalgebra of $(\mathfrak{g}, \mathfrak{h}_1)$ containing Y_{α} and Y_{β} , $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Since $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{s}$, we may write

$$\mathfrak{s} = \mathfrak{h}_1 \oplus \mathbb{C} x_3 \oplus \mathbb{C} x_4 \oplus \mathbb{C} X_{2\alpha+\beta} \oplus \mathbb{C} X_{3\alpha+\beta} \oplus \mathbb{C} X_{3\alpha+2\beta},$$

$$x_3 = a_0 \cdot X_\alpha + a_1 \cdot X_{\alpha+\beta} + X_\beta$$
, $x_4 = X_{-\beta} + b_0 \cdot X_\alpha + b_1 \cdot X_{\alpha+\beta}$,

where X_{γ} , $\gamma \in \Phi$, are root vectors of $(\mathfrak{g}, \mathfrak{h})$, \mathfrak{n} contains X_{α} and X_{β} .

We may assume that $\mathfrak{h}_1 = \mathbb{C}x_1 \oplus \mathbb{C}x_2$, where

$$x_1=z_1+X_\rho+\sum_{\gamma\succ\rho}u_\gamma\cdot X_\gamma,\quad x_2=z_2+\sum_{\gamma\succ\rho}v_\gamma\cdot X_\gamma,\quad z_1,z_2\in\mathfrak{h},\quad \rho(z_2)=0,$$

for some $\rho \in \{\alpha, \beta, \alpha + \beta\}$.

Since $\mathfrak{s} \ni [x_2, x_3] = \alpha(z_2) \cdot a_0 \cdot X_\alpha + \beta(z_2) \cdot X_\beta + x_{23}, \ x_{23} \in [\mathfrak{n}, \mathfrak{n}], \ \mathrm{and} \ (\alpha - \beta)(z_2) \neq 0, \ a_0 = 0.$ If $\rho \neq \alpha$, then also

$$\mathfrak{s} \ni (\alpha + \beta)(z_2) \cdot a_1 \cdot X_{\alpha + \beta} + \beta(z_2) \cdot X_{\beta},$$

and so $a_1 = 0$ in this case.

If $x_3 = X_\beta$, then $[x_3, x_4] = H_\beta + b_0 \cdot X_{\alpha+\beta}$. Hence $\rho \neq \alpha$, and so $[x_2, x_4] \in \mathfrak{s}$ implies that $b_1 = 0$. Then

$$\mathfrak{s} = \mathbb{C}x_1' \oplus \mathbb{C}x_2' \oplus \mathbb{C}x_4 \oplus \mathbb{C}X_\beta \oplus \mathbb{C}X_{2\alpha+\beta} \oplus \mathbb{C}X_{3\alpha+\beta} \oplus \mathbb{C}X_{3\alpha+2\beta},$$

where $x_1' = z_1' + u \cdot X_{\alpha+\beta}$, $x_2' = z_2' + v \cdot X_{\alpha+\beta}$, $z_1', z_2' \in \mathfrak{h}$. We may assume that u = 1, v = 0, and so $(\alpha + \beta)(z_2') = 0$.

In this case, τ acts on the roots either as – Id or as a reflection through β . Hence either z_2' or X_{β} is contained in $\mathfrak{s} \cap \tau(\mathfrak{s}) = 0$, a contradiction.

Hence we may assume that x_3 is not proportional to a root vector, and so $\rho = \alpha$. In this case, \mathfrak{s} contains $x_1' = z_1 + X_{\alpha} + u \cdot X_{\alpha+\beta}$ and $x_2' = z_2 + v \cdot X_{\alpha+\beta}$.

If $v \neq 0$, we may assume that v = 1 and u = 0. Since $[x'_1, x'_2] \in \mathfrak{s}$, $(\alpha + \beta)(z_1) = 0$.

Since $[x'_1, x_4] \in \mathfrak{s}$, $-\beta(z_1) \cdot X_{-\beta} + \alpha(z_1)b_0 \cdot X_{\alpha} \in \mathfrak{s}$, and so $b_1 = 0$.

Since $[x'_2, x_4] \in \mathfrak{s}$, $-\beta(z_2) \cdot X_{-\beta} - X_{\alpha} \in \mathfrak{s}$, and so $b_0 = 1/\beta(z_2)$.

Since $[x_1', x_3] \in \mathfrak{s}$, $\beta(z_1) \cdot X_{\beta} - X_{\alpha+\beta} \in \mathfrak{s}$, and so $a_1 = -1/\beta(z_1)$. Hence

$$\mathfrak{s} = \mathbb{C}x_1' \oplus \mathbb{C}x_2' \oplus \mathbb{C}x_3' \oplus \mathbb{C}x_4' \oplus \mathbb{C}X_{2\alpha+\beta} \oplus \mathbb{C}X_{3\alpha+\beta} \oplus \mathbb{C}X_{3\alpha+2\beta},$$

where $x'_1 = r_1 \cdot H_{3\alpha+\beta} - r_2 \cdot X_{-\beta}$, $x'_2 = r_2 \cdot H_{3\alpha+2\beta} - r_1 \cdot X_{\beta}$, $x'_3 = X_{\alpha+\beta} + r_1 \cdot X_{\beta}$, $x'_4 = X_{\alpha} + r_2 \cdot X_{-\beta}$.

Since τ acts on the roots either as - Id or as a reflection through β , $\mathfrak{h} \oplus \mathbb{C}X_{\beta} \oplus \mathbb{C}X_{-\beta}$ is spanned by x'_1 , x'_2 , $\tau(x'_1)$, $\tau(x'_2)$. We can choose the root vectors such that $\tau(X_{\gamma}) = \pm X_{\tau(\gamma)}$, $\gamma \in \Phi$.

If τ acts on the roots as $-\operatorname{Id}$, then $\tau(x_1') = -\overline{r_1} \cdot H_{3\alpha+\beta} \mp \overline{r_2} \cdot X_{\beta}$, $\tau(x_2') = -\overline{r_2} \cdot H_{3\alpha+2\beta} \mp \overline{r_1} \cdot X_{-\beta}$. Hence $\mathbb{C}X_{\beta} \oplus \mathbb{C}X_{-\beta}$ is spanned by a single element $(r_2/r_1) \cdot X_{-\beta} \pm \overline{(r_2/r_1)} \cdot X_{\beta}$, a contradiction.

If τ acts on the roots as a reflection through β , then $\tau(x_1') = \pm \overline{r_1} \cdot H_{3\alpha+2\beta} + \overline{r_2} \cdot X_{-\beta}$, $\tau(x_2') = \pm \overline{r_2} \cdot H_{3\alpha+\beta} + \overline{r_1} \cdot X_{\beta}$. Hence \mathfrak{h} is spanned by a single element $(r_1/r_2) \cdot H_{3\alpha+\beta} \pm \overline{(r_1/r_2)} \cdot H_{3\alpha+2\beta}$, a contradiction.

If v = 0, then u = 0. Since $[x'_2, x_4] \in \mathfrak{s}$, $X_{-\beta} - b_1 \cdot X_{\alpha+\beta} \in \mathfrak{s}$, and so $b_0 = b_1 = 0$. Since $\mathfrak{s} \cap \tau(\mathfrak{s}) = 0$ and $\alpha(z_2) = 0$, $\mathfrak{g}_0 = G_2^n$ and $\tau = \tau_n$ acts on the roots as a reflection through β . Hence $X_{-\beta} \in \mathfrak{s} \cap \tau(\mathfrak{s})$, a contradiction.

Corollary 4.2. The subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is normalized by a maximally compact Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$. Moreover, either $\mathfrak{s} = L \oplus \mathfrak{n}$ or $\mathfrak{s} = \mathfrak{b}$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$, $L \subset \mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Proof. Let $\mathfrak{h}_0 \subset \mathfrak{g}_0$ be a Cartan subalgebra normalizing \mathfrak{s} , $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$, X_{γ} , $\gamma \in \Phi$, root vectors of $(\mathfrak{g}, \mathfrak{h})$.

By Lemma 4.1 and [10], Table 1, the subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is one of the following:

$$L \oplus \mathfrak{n}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathbb{C}H_{\alpha} \oplus \mathbb{C}X_{-\alpha} \oplus \mathfrak{n}, \quad \mathbb{C}H_{\beta} \oplus \mathbb{C}X_{-\beta} \oplus \mathfrak{n},$$

$$G_2[\beta] = \mathbb{C}X_{-\alpha} \oplus \mathfrak{b}, \quad G_2[\alpha] = \mathbb{C}X_{-\beta} \oplus \mathfrak{b}, \quad \mathfrak{g},$$

where $L \subset \mathfrak{h}$, $\mathfrak{n} = \bigoplus_{\gamma \in \Phi^+} \mathbb{C} X_{\gamma}$ and Φ is suitably ordered so that $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\} \subset \Phi$ is the subset of positive roots.

By [1], Lemma 7, only two of these subalgebras can form a \mathfrak{g}_0 -admissible pair:

$$\mathfrak{s} = L \oplus \mathfrak{n}$$
 or $\mathfrak{s} = \mathfrak{b}$.

In both cases, τ acts on the roots as $-\operatorname{Id}$.

Let $\phi: G_0 \to G$ be the universal complexification [8], i.e. G is the connected complex simple Lie group of type G_2 , with Lie algebra \mathfrak{g} , $\ker(\phi)$ is the center of G_0 , and the differential of ϕ is the embedding $\mathfrak{g}_0 \subset \mathfrak{g}$.

Let $B \subset G$ be a fixed Borel subgroup, with Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing a maximally compact Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$. By [15], Theorem 5.4, $H_0 = B \cap G_0$ is connected and so is generated by $\mathfrak{h}_0 = \mathfrak{b} \cap \mathfrak{g}_0$.

Consider the flag manifold G/B parametrizing the Borel subalgebras of \mathfrak{g} . Let \mathcal{N}^* be the holomorphic homogeneous vector bundle over G/B corresponding to the isotropy representation $B \to GL(\operatorname{Hom}_{\mathbb{C}}([\mathfrak{b},\mathfrak{b}],\mathbb{C}))$ coming from the adjoint action of B on \mathfrak{b} .

Let $\mathcal{B} \subset G/B$ be the union of the open orbits of G_0 . By [15], Theorem 4.5, \mathcal{B} parametrizes the Borel subalgebras containing a maximally compact Cartan subalgebra of \mathfrak{g}_0 . If G_0 is compact, then $\mathcal{B} = G/B$. Otherwise, \mathcal{B} consists of exactly three open orbits of G_0 on G/B, corresponding to the three Weyl chambers of G_2 contained in a Weyl chamber of $A_1 + \tilde{A}_1$, [15], Theorem 4.7.

Let

$$\mathcal{I} = \mathcal{B} \times GL(2, \mathbb{R})/GL(1, \mathbb{C}), \quad \Sigma = \mathcal{B} \times \Sigma_0, \quad \text{where } \Sigma_0 = \{\sigma \in \bigwedge^2(\mathbb{C}^2)^* \mid \operatorname{Im}(\sigma|_{\mathbb{R}^2}) \text{ is symplectic}\},$$

be the trivial bundles over \mathcal{B} parametrizing the complex structures and certain extensions of symplectic structures on the fibers of $\mathcal{B} \times \mathfrak{h}_0 \to \mathcal{B}$ respectively, \mathfrak{h}_0 identified with \mathbb{R}^2 .

Remark 4.3. As sets, $GL(2,\mathbb{R})/GL(1,\mathbb{C}) \cong \{z \in \mathbb{C} \mid \text{Im}(z) \neq 0\} \cong \Sigma_0$.

Now we state the main result of this section.

Theorem 4.4. Any invariant generalized complex structure on G_0 , a real Lie group of type G_2 and real dimension 14, is regular. The set of invariant generalized complex structures on G_0 is parametrized by the disjoint union

$$\mathcal{C} \cup \mathcal{S}$$
,

where $C = \mathcal{I} \times_{\mathcal{B}} \mathcal{N}^* \cong \mathcal{N}^*|_{\mathcal{B}} \times GL(2,\mathbb{R})/GL(1,\mathbb{C})$ and $S = \Sigma \times_{\mathcal{B}} \mathcal{N}^* \cong \mathcal{N}^*|_{\mathcal{B}} \times \Sigma_0$.

Proof. We use the notation of Corollary 4.2.

Suppose $\mathfrak{s} = L \oplus \mathfrak{n}$, $\dim(L) = 1$. Then $\mathfrak{s} + \tau(\mathfrak{s}) = \mathfrak{g}$ if and only if $L \subset \mathfrak{h}$ is the holomorphic subspace of a complex structure on \mathfrak{h}_0 . Since $\mathfrak{s} \cap \tau(\mathfrak{s}) = 0$, any closed 2-form $\omega \in \bigwedge^2 \mathfrak{s}^*$ gives a \mathfrak{g}_0 -admissible pair (\mathfrak{s}, ω) .

The Chevalley-Eilenberg resolution gives $H^2(\mathfrak{s}, \mathbb{C}) = 0$. Hence $\omega = d\xi$ for a uniquely determined linear map $\xi : [\mathfrak{b}, \mathfrak{b}] \to \mathbb{C}$.

Thus, the \mathfrak{g}_0 -admissible pairs (\mathfrak{s}, ω) with $\mathfrak{s} = L \oplus \mathfrak{n}$ are parametrized by the triples $(\mathfrak{b}, \xi, \lambda)$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra containing a maximally compact Cartan subalgebra of $\mathfrak{g}_0, \xi \in \operatorname{Hom}_{\mathbb{C}}([\mathfrak{b}, \mathfrak{b}], \mathbb{C})$ and $\lambda \in GL(2, \mathbb{R})/GL(1, \mathbb{C})$ is a complex structure on the real vector space $\mathfrak{h}_0 = \mathfrak{b} \cap \mathfrak{g}_0 \cong \mathbb{R}^2$. Cf. [12].

Suppose $\mathfrak{s} = \mathfrak{b}$. In this case, $H^2(\mathfrak{s}, \mathbb{C}) = \mathbb{C} \cdot \omega_0$, where $0 \neq \omega_0 \in \bigwedge^2 \mathfrak{h}^*$ is extended by zero to a 2-form on \mathfrak{s} . Since $\mathfrak{b} \cap \mathfrak{g}_0 = \mathfrak{h}_0$, a 2-form $\omega \in \bigwedge^2 \mathfrak{b}^*$ gives a \mathfrak{g}_0 -admissible pair (\mathfrak{b}, ω) if and only if $\omega = c \cdot \omega_0 + d \xi$ for a uniquely determined linear map $\xi \colon [\mathfrak{b}, \mathfrak{b}] \to \mathbb{C}$, where $\mathrm{Im}(c \cdot \omega_0|_{\mathfrak{h}_0})$ is non-degenerate.

Thus, the \mathfrak{g}_0 -admissible pairs (\mathfrak{s}, ω) with $\mathfrak{s} = \mathfrak{b}$ are parametrized by the triples $(\mathfrak{b}, \xi, \sigma)$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra containing a maximally compact Cartan subalgebra of \mathfrak{g}_0 , $\xi \in \text{Hom}_{\mathbb{C}}([\mathfrak{b}, \mathfrak{b}], \mathbb{C})$ and $\sigma \in \Sigma_0$.

As we recalled above, \mathcal{B} consists of one or three orbits of G_0 [15].

Corollary 4.5. The set of invariant generalized complex structures on G_0 , up to conjugacy by G_0 , is parametrized by r copies of the disjoint union

$$N_0^* \times GL(2,\mathbb{R})/GL(1,\mathbb{C}) \cup N_0^* \times \Sigma_0,$$

where $N_0^* = \operatorname{Hom}_{\mathbb{C}}([\mathfrak{b}, \mathfrak{b}], \mathbb{C})/H_0$, r = 1 if G_0 is compact and 3 otherwise.

The following remark is an immediate consequence of Milburn's study of invariant generalized complex structures on homogeneous spaces [11].

Remark 4.6. There is no SO(2n+1)-invariant generalized complex structure on the 2n-dimensional sphere $S^{2n} = SO(2n+1)/SO(2n), n \geq 2$, and no G_2 -invariant generalized complex structure on $S^6 = G_2^c/SU(3)$. The SO(3)-invariant generalized complex structures on $S^2 = SO(3)/SO(2)$ are two biholomorphic complex structures \mathbb{CP}^1 and \mathbb{CP}^1 , and a family of invariant generalized complex structures with holomorphic subbundles of the form $L(\mathfrak{so}_3(\mathbb{C}), \omega_c), c \in \mathbb{C}$, $\mathrm{Im}(c) \neq 0$, which are B-transforms of the symplectic structures (up to symplectomorphism) on S^2 . Notation is from [5], [11], $\omega_c \in \bigwedge^2 \mathfrak{so}_3(\mathbb{C})^*$ is defined by

$$\omega_c(X,Y) = \operatorname{Trace}\begin{pmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot [X,Y])$$

for 3×3 skew symmetric complex matrices $X, Y \in \mathfrak{so}_3(\mathbb{C})$. See also [7].

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