On semi-classical limits of ground states of a nonlinear Maxwell-Dirac system

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Abstract

We study the semi-classical ground states of the nonlinear Maxwell-Dirac system:

$$\begin{cases} \alpha \cdot (i\hbar \nabla + q(x)\mathbf{A}(x))w - a\beta w - \omega w - q(x)\phi(x)w = P(x)g(|w|)w \\ -\Delta \phi = q(x)|w|^2 \\ -\Delta A_k = q(x)(\alpha_k w) \cdot \bar{w} \quad k = 1, 2, 3 \end{cases}$$

for $x \in \mathbb{R}^3$, where **A** is the magnetic field, ϕ is the electron field and q describes the changing pointwise charge distribution. We develop a variational method to establish the existence of least energy solutions for \hbar small. We also describe the concentration behavior of the solutions as $\hbar \to 0$.

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1 Introduction and main result

The Maxwell-Dirac system, which has been widely considered in literature (see [1], [14], [19], [25], [28], [29], [33] etc. and references therein), is fundamental in the relativistic description of spin 1/2 particles. It represents the time-evolution of fast (relativistic) electrons and positrons within external and self-consistent generated electromagnetic field. The system can be written as follows:

(1.1)
$$\begin{cases} i\hbar\partial_t\psi + \alpha \cdot (ic\hbar\nabla + q\mathbf{A})\,\psi - q\phi\psi - mc^2\beta\psi = 0\\ \partial_t\phi + c\sum_{k=1}^3 \partial_kA_k = 0\,, \quad \partial_t^2\phi - \Delta\phi = \frac{4\pi}{c}q\,|\psi|^2 & \text{in } \mathbb{R} \times \mathbb{R}^3\\ \partial_t^2A_k - \Delta A_k = \frac{4\pi}{c}q(\alpha_k\psi)\bar{\psi} \quad k = 1, 2, 3 \end{cases}$$

where $\psi(t,x) \in \mathbb{C}^4$, c is the speed of light, q is the charge of the particle, m > 0 is the mass of the electron, \hbar is the Planck's constant, and $u\bar{v}$ denotes the inner product of $u,v \in \mathbb{C}^4$. Furthermore, α_1 , α_2 , α_3 and β are 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

 $\mathbf{A} = (A_1, A_2, A_3) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \ \phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \text{ and we have used } \alpha = (\alpha_1, \alpha_2, \alpha_3), \ \alpha \cdot \nabla = \sum_{k=1}^3 \alpha_k \partial_k, \text{ and } \alpha \cdot \mathbf{V} = \sum_{k=1}^3 \alpha_k V_k \text{ for any vector } \mathbf{V} \in \mathbb{C}^3.$

The above system has been studied for a long time and results are available concerning the Cauchy problem (see [7], [8], [18], [20], [23], [31] etc. and references therein). The first result on the local existence and uniqueness of solutions of (1.1) was obtained by L. Gross in [23]. For later développements, we mention, e.g., that Sparber and Markowich [31] studied the existence and asymptotic description of the solution of Cauchy problem for Maxwell-Dirac system as $\hbar \to 0$, and obtained the asymptotic approximation up to order $O(\sqrt{\hbar})$.

In this paper, we are interested in finding stationary waves of (1.1) which have the form

$$\begin{cases} \psi(t,x) = w(x)e^{i\theta t/\hbar}, & \theta \in \mathbb{R}, \ w : \mathbb{R}^3 \to \mathbb{C}^4, \\ \mathbf{A} = \mathbf{A}(x), & \phi = \phi(x) \text{ in } \mathbb{R}^3. \end{cases}$$

For notation convenience, one shall denote $A_0 = \phi$. If (ψ, \mathbf{A}, A_0) is a stationary solution of (1.1), then (w, \mathbf{A}, A_0) is a solution of

(1.2)
$$\begin{cases} \alpha \cdot (i\hbar \nabla + Q\mathbf{A}) w - a\beta w - \omega w - QA_0 w = 0, \\ -\Delta A_k = 4\pi Q(\alpha_k w) \bar{w}, \quad k = 0, 1, 2, 3, \end{cases}$$

where a = mc > 0, $\omega \in \mathbb{R}$, Q = q/c and $\alpha_0 := I$.

The existence of stationary solution of the system has been an open problem for a long time, see [22]. Using variational methods, Esteban, Georgiev and Séré [15] proved the existence of regular solutions of the form $\psi(t,x) = w(x)e^{i\omega t}$ with $\omega \in (0,a)$, leaving open the question of existence of solutions for $\omega \leq 0$. On the other hand, in [25], Garrett Lisi gave numerical evidence of the existence of bounded states for $\omega \in (-a,a)$ by using an axially symmetric ansatz. After that, Abenda in [1] obtained the existence result of solitary wave solutions for $\omega \in (-a,a)$.

We emphasize that the works mentioned above mainly concerned with the autonomous system with null self-coupling. Besides, limited work has been done in the semi-classical approximation. For small \hbar , the solitary waves are referred to as semi-classical states. To describe the transition from quantum to classical mechanics, the existence of solutions w_{\hbar} , \hbar small, possesses an important physical interest. The idea to consider a nonlinear self-coupling, in Quantum electrodynamics, gives the description of models of self-interacting spinor fields (see [16], [17], [26] etc. and references therein). Due to the special physical importance, in the present paper, we are devoted to the existence and concentration phenomenon of stationary semi-classical solutions to the system with

- the varying pointwise charge distribution Q(x) including the constant q as a special one;
- general subcritical self-coupling nonlinearity.

More precisely, we consider the system, writing $\varepsilon = \hbar$,

(1.3)
$$\begin{cases} \alpha \cdot (i\varepsilon \nabla + Q(x)\mathbf{A}) w - a\beta w - \omega w - Q(x)A_0w = P(x)g(|w|)w, \\ -\Delta A_k = 4\pi Q(x)(\alpha_k w)\bar{w} & k = 0, 1, 2, 3. \end{cases}$$

Writing $G(|w|) := \int_0^{|w|} g(s) s ds$, we make the following hypotheses:

- (g_1) g(0) = 0, $g \in C^1(0, \infty)$, g'(s) > 0 for s > 0, and there exist $p \in (2, 3)$, $c_1 > 0$ such that $g(s) \le c_1(1 + s^{p-2})$ for $s \ge 0$;
- (g₂) there exist $\sigma > 2$, $\theta > 2$ and $c_0 > 0$ such that $c_0 s^{\sigma} \leq G(s) \leq \frac{1}{\theta} g(s) s^2$ for all s > 0.

A typical example is the power function $g(s) = s^{\sigma-2}$. For describing the charge distribution and external fields we always assume that Q(x) and P(x) verify, respectively

- $(Q_0) \ \ Q \in C^{0,1}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3) \text{ with } Q(x) \geq 0 \text{ a.e. on } \mathbb{R}^3;$
- (P_0) $P \in C^{0,1}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$ with $\inf P > 0$ and $\limsup_{|x| \to \infty} P(x) < \max P(x)$.

For showing the concentration phenomena, we set $m := \max_{x \in \mathbb{R}^3} P(x)$ and

$$\mathscr{P} := \{ x \in \mathbb{R}^3 : P(x) = m \}.$$

Our result reads as

Theorem 1.1. Assume that $\omega \in (-a, a)$, (g_1) - (g_2) , (Q_0) and (P_0) are satisfied. Then for all $\varepsilon > 0$ small,

(i) The system (1.3) has at least one least energy solution $w_{\varepsilon} \in W^{1,q}$ for all $q \geq 2$. In addition, if $P, Q \in C^{1,1}(\mathbb{R}^3)$ the solutions will be in C^1 class.

- (ii) The set of all least energy solutions is compact in $W^{1,q}$ for all $q \ge 2$;
- (iii) There is a maximum point x_{ε} of $|w_{\varepsilon}|$ with $\lim_{\varepsilon \to 0} \operatorname{dist}(x_{\varepsilon}, \mathscr{P}) = 0$ such that $u_{\varepsilon}(x) := w_{\varepsilon}(\varepsilon x + x_{\varepsilon})$ converges uniformly to a least energy solution of (the limit equation)

$$(1.4) i\alpha \cdot \nabla u - a\beta u - \omega u = mq(|u|)u.$$

(iv)
$$|w_{\varepsilon}(x)| \leq C \exp\left(-\frac{c}{\varepsilon}|x-x_{\varepsilon}|\right)$$
 for some $C, c > 0$.

It is standard that (1.3) is equivalent to, letting $u(x) = w(\varepsilon x)$,

(1.5)
$$\begin{cases} \alpha \cdot (i\nabla + Q_{\varepsilon} \mathbf{A}_{\varepsilon}) u - a\beta u - \omega u - Q_{\varepsilon} A_{\varepsilon,0} u = P_{\varepsilon} g(|u|) u, \\ -\Delta A_{\varepsilon,k} = \varepsilon^2 4\pi Q_{\varepsilon} J_k \quad k = 0, 1, 2, 3, \end{cases}$$

where
$$Q_{\varepsilon}(x) = Q(\varepsilon x)$$
, $P_{\varepsilon}(x) = P(\varepsilon x)$, $\mathbf{A}_{\varepsilon}(x) = \mathbf{A}(\varepsilon x)$, $A_{\varepsilon,k}(x) = A_k(\varepsilon x)$, $k = 0, 1, 2, 3$, and

$$J_k = (\alpha_k u)\bar{u}$$
 for $k = 0, 1, 2, 3$.

In fact, using variational methods, we are going to focus on studying the semiclassical solutions that are obtained as critical points of an energy functional Φ_{ε} associated to the equivalent problem (1.5).

There have been a large number of works on existence and concentration phenomenon of semi-classical states of nonlinear Schrödinger-Poisson systems arising in the non-relativistic quantum mechanics, see, for example, [3, 4, 5] and their references. It is quite natural to ask if certain similar results can be obtain for nonlinear Maxwell-Dirac systems arising in the relativistic quantum mechanics. Mathematically, the two systems possess different variational structures, the Mountain-Pass and the Linking structures respectively. The problems in Maxwell-Dirac systems are difficult because they are strongly indefinite in the sense that both the negative and positive parts of the spectrum of Dirac operator are unbounded and consist of essential spectrums. As far as the authors known there have been no results on the existence and concentration phenomena of semiclassical solutions to nonlinear Maxwell-Dirac systems.

Very recently, one of the authors, jointly with co-authors, developed an argument to obtain some results on existence and concentration of semi-classical solutions for nonlinear Dirac equations but not for Maxwell-Dirac system, see [10, 11, 12]. Compared with the papers, difficulty arises in the Maxwell-Dirac system because of the presence of nonlocal terms $A_{\varepsilon,k}$, k=0,1,2,3. In order to overcome this obstacle, we develop a cut-off arguments. Roughly speaking, an accurate uniform boundness estimates on $(C)_c$ (Cerami) sequences of the associate energy functional Φ_{ε} enables us to introduce a new functional $\widetilde{\Phi}_{\varepsilon}$ by virtue of the cut-off technique so that $\widetilde{\Phi}_{\varepsilon}$ has the same least energy solutions as Φ_{ε} and can be dealt with more

easily, in particular, the influence of these nonlocal terms can be reduced as $\varepsilon \to 0$. In addition, for obtaining the exponential decay, since the Kato's inequality seems not work well in the present situation, we handle, instead of considering $\Delta |u|$ as in [10], the square of |u|, that is $\Delta |u|^2$, with the help of identity (4.10), and then describe the decay at infinity in a subtle way.

2 The variational framework

2.1 The functional setting and notations

In this subsection we discuss the variational setting for the equivalent system (1.5). Throughout the paper we assume $0 \in \mathscr{P}$ without loss of generality, and the conditions (g_1) - (g_2) , (P_0) and (Q_0) are satisfied.

In the sequel, by $|\cdot|_q$ we denote the usual L^q -norm, and $(\cdot, \cdot)_2$ the usual L^2 -inner product. Let $H_0 = i\alpha \cdot \nabla - a\beta$ denote the self-adjoint operator on $L^2 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = H^1 \equiv H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is well known that $\sigma(H_0) = \sigma_c(H_0) = \mathbb{R} \setminus (-a, a)$ where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and the continuous spectrum. Thus the space L^2 possesses the orthogonal decomposition:

(2.1)
$$L^2 = L^+ \oplus L^-, \quad u = u^+ + u^-$$

so that H_0 is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $E := \mathcal{D}(|H_0|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$\langle u, v \rangle = \Re(|H_0|^{1/2} u, |H_0|^{1/2} v)_2$$

and the induced norm $||u|| = \langle u, u \rangle^{1/2}$, where $|H_0|$ and $|H_0|^{1/2}$ denote respectively the absolute value of H_0 and the square root of $|H_0|$. Since $\sigma(H_0) = \mathbb{R} \setminus (-a, a)$, one has

(2.2)
$$a|u|_2^2 \le ||u||^2$$
 for all $u \in E$.

Note that this norm is equivalent to the usual $H^{1/2}$ -norm, hence E embeds continuously into L^q for all $q \in [2,3]$ and compactly into L^q_{loc} for all $q \in [1,3)$. It is clear that E possesses the following decomposition

$$(2.3) E = E^+ \oplus E^- \text{ with } E^{\pm} = E \cap L^{\pm},$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$ inner products. This decomposition induces also a natural decomposition of L^p , hence there is $d_p > 0$ such that

(2.4)
$$d_p \left| u^{\pm} \right|_p^p \le |u|_p^p \text{ for all } u \in E.$$

Let $\mathcal{D}^{1,2} \equiv \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R})$ be the completion of $C_c^{\infty}(\mathbb{R}^3, \mathbb{R})$ with respect the Dirichlet norm

 $||u||_{\mathcal{D}}^2 = \int |\nabla u|^2 \, dx.$

Then (1.5) can be reduced to a single equation with a non-local term. Actually, since Q is bounded and $u \in L^q$ for all $q \in [2,3]$, one has $Q_{\varepsilon} |u|^2 \in L^{6/5}$ for all $u \in E$, and there holds, for all $v \in \mathcal{D}^{1,2}$,

(2.5)
$$\left| \int Q_{\varepsilon}(x) J_k \cdot v dx \right| \leq \left(\int \left| Q_{\varepsilon}(x) \left| u \right|^2 \right|^{6/5} dx \right)^{5/6} \left(\int \left| v \right|^6 \right)^{1/6}$$
$$\leq S^{-1/2} \left| Q_{\varepsilon} \left| u \right|^2 \right|_{6/5} \left\| v \right\|_{\mathcal{D}},$$

where S is the Sobolev embedding constant: $S|u|_6^2 \leq ||u||_{\mathcal{D}}^2$ for all $u \in \mathcal{D}^{1,2}$. Hence there exists a unique $A_{\varepsilon,u}^k \in \mathcal{D}^{1,2}$ for k = 0, 1, 2, 3 such that

(2.6)
$$\int \nabla A_{\varepsilon,u}^k \nabla v dx = \varepsilon^2 4\pi \int Q_{\varepsilon}(x) J_k v \, dx$$

for all $v \in \mathcal{D}^{1,2}$. It follows that $A_{\varepsilon,u}^k$ satisfies the Poisson equation

$$-\Delta A_{\varepsilon,u}^k = \varepsilon^2 4\pi Q_{\varepsilon}(x) J_k$$

and there holds

(2.7)
$$A_{\varepsilon,u}^k(x) = \varepsilon^2 \int \frac{Q_{\varepsilon}(y)J_k(y)}{|x-y|} dy = \frac{\varepsilon^2}{|x|} * (Q_{\varepsilon}J_k).$$

Substituting $A_{\varepsilon,u}^k$, k=0,1,2,3, in (1.5), we are led to the equation

(2.8)
$$H_0 u - \omega u - Q_{\varepsilon}(x) A_{\varepsilon,u}^0 u + \sum_{k=1}^3 Q_{\varepsilon}(x) \alpha_k A_{\varepsilon,u}^k u = P_{\varepsilon}(x) g(|u|) u.$$

On E we define the functional

$$\Phi_{\varepsilon}(u) = \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \Gamma_{\varepsilon}(u) - \Psi_{\varepsilon}(u)$$

for $u = u^+ + u^-$, where

$$\Gamma_{\varepsilon}(u) = \frac{1}{4} \int Q_{\varepsilon}(x) A_{\varepsilon,u}^{0}(x) J_{0} dx - \frac{1}{4} \sum_{k=1}^{3} \int Q_{\varepsilon}(x) A_{\varepsilon,u}^{k} J_{k} dx$$

and

$$\Psi_{\varepsilon}(u) = \int P_{\varepsilon}(x)G(|u|)dx.$$

2.2 Technical results

In this subsection, we shall introduce some lemmas that related to the functional Φ_{ε} .

Lemma 2.1. Under the hypotheses (g_1) - (g_2) , one has $\Phi_{\varepsilon} \in C^2(E, \mathbb{R})$ and any critical point of Φ_{ε} is a solution of (1.5).

Proof. Clearly, $\Psi_{\varepsilon} \in C^2(E, \mathbb{R})$. It remains to check that $\Gamma_{\varepsilon} \in C^2(E, \mathbb{R})$. It suffices to show that, for any $u, v \in E$,

$$(2.9) |\Gamma_{\varepsilon}(u)| \le \varepsilon^2 C_1 |Q|_{\infty}^2 ||u||^4,$$

$$\left|\Gamma'_{\varepsilon}(u)v\right| \leq \varepsilon^{2} C_{2} \left|Q\right|_{\infty}^{2} \left\|u\right\|^{3} \left\|v\right\|,$$

$$|\Gamma_{\varepsilon}''(u)[v,v]| \le \varepsilon^2 C_3 |Q|_{\infty}^2 ||u||^2 ||v||^2.$$

Observe that one has, by (2.5) and (2.6) with $v = A_{\varepsilon,u}^k$

$$(2.12) |A_{\varepsilon,u}^k|_6 \le S^{-1/2} ||A_{\varepsilon,u}^k||_{\mathcal{D}} \le \varepsilon^2 C_1 |Q|_{\infty} ||u||^2.$$

This, together with the Hölder inequality (with r=6, r'=6/5), implies (2.9). Note that $\Gamma'_{\varepsilon}(u)v = \frac{d}{dt}\Gamma_{\varepsilon}(u+tv)\big|_{t=0}$, so

$$\Gamma_{\varepsilon}'(u)v = \frac{\varepsilon^{2}}{2} \iint \frac{Q_{\varepsilon}(x)Q_{\varepsilon}(y)}{|x-y|} \Big(J_{0}(x)\Re[\alpha_{0}u\overline{v}(y)] + J_{0}(y)\Re[\alpha_{0}u\overline{v}(x)]$$

$$-\sum_{k=1}^{3} \Big(J_{k}(x)\Re[\alpha_{k}u\overline{v}(y)] + J_{k}(y)\Re[\alpha_{k}u\overline{v}(x)] \Big) \Big) dydx$$

$$= \int \Big(Q_{\varepsilon}A_{\varepsilon,u}^{0}\Re[\alpha_{0}u\overline{v}] - \sum_{k=1}^{3} Q_{\varepsilon}A_{\varepsilon,u}^{k}\Re[\alpha_{k}u\overline{v}] \Big) dx$$

which, together with the Hölder's inequality and (2.12), shows (2.10). Similarly,

$$\begin{split} \Gamma_{\varepsilon}^{''}(u)[v,v] &= \int Q_{\varepsilon} \Big(A_{\varepsilon,u}^{0} J_{k}^{v} - \sum_{k=1}^{3} A_{\varepsilon,u}^{k} J_{k}^{v} \Big) dx \\ &+ 2 \, \varepsilon^{2} \int \int \frac{Q_{\varepsilon}(x) Q_{\varepsilon}(y)}{|x-y|} \Big[\big(\Re[\alpha_{0} u \overline{v}(x)] \big) \big(\Re[\alpha_{0} u \overline{v}(y)] \big) \\ &- \sum_{k=1}^{3} \big(\Re[\alpha_{k} u \overline{v}(x)] \big) \big(\Re[\alpha_{k} u \overline{v}(y)] \big) \Big] dx dy \end{split}$$

where $J_k^u = \alpha_k u \overline{u}$ and $J_k^v = \alpha_k v \overline{v}$, and one gets (2.11).

Now it is a standard to verify that critical points of Φ_{ε} are solutions of (1.5).

We show further the following:

Lemma 2.2. For every $\varepsilon > 0$, Γ_{ε} is nonnegative and weakly sequentially lower semi-continuous.

Proof. Firstly, let us recall some technical results in [15]: For any $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ and $u \in \mathbb{C}^4$, we have

$$\begin{aligned} \left| \xi_{0}(\beta u, u) + \sum_{k=1}^{3} \xi_{k}(\alpha_{k} u, u) \right|^{2} \\ &= \left| \left(\beta u, \left[\xi_{0} + \sum_{k=1}^{3} \xi_{k} \pi_{k} \right] u \right) \right|^{2} \\ &\leq \left| \beta u \right|_{\mathbb{C}^{4}}^{2} \left(u, \left(\xi_{0} - \sum_{k=1}^{3} \xi_{k} \pi_{k} \right) \left(\xi_{0} + \sum_{k=1}^{3} \xi_{k} \pi_{k} \right) u \right) \\ &= \left| \xi \right|_{\mathbb{R}^{4}}^{2} \left| u \right|_{\mathbb{C}^{4}}^{4}. \end{aligned}$$

Here, we have used the formulas $(u,v)=u\bar{v}$ for all $u,v\in\mathbb{C}^4$, $\pi_k=\beta\cdot\alpha_k$, $\beta^*=\beta$, $\pi_k^*=-\pi_k$ and $\pi_i\pi_j+\pi_j\pi_j=-2\delta_{ij}$, $1\leq i,j\leq 3$. As a consequence, we find

(2.15)
$$(\beta u, u)^2 + \sum_{k=1}^{3} (\beta u, \pi_k u)^2 \le |u|_{\mathbb{C}^4}^4.$$

So, taking $u(x) \in E$, $x, y \in \mathbb{R}^3$, $\xi_0 = \pm(\beta u, u)(y)$, $\xi_k = (\beta u, \pi_k u)(y)$, we get from (2.14) and (2.15) that

$$(2.16) \pm (\beta u, u)(y)(\beta u, u)(x) + \sum_{k=1}^{3} (\beta u, \pi_{k} u)(y)(\beta u, \pi_{k} u)(x)$$

$$= \pm (\beta u, u)(y)(\beta u, u)(x) + \sum_{k=1}^{3} (\alpha_{k} u, u)(y)(\alpha_{k} u, u)(x)$$

$$\leq |\xi|_{\mathbb{R}^{4}} |u(x)|_{\mathbb{C}^{4}}^{2} \leq |u(y)|_{\mathbb{C}^{4}}^{2} |u(x)|_{\mathbb{C}^{4}}^{2}.$$

It is not difficult to see from (2.16) that

(2.17)
$$J_0(x)J_0(y) - \sum_{k=1}^3 J_k(x)J_k(y) \ge 0.$$

And hence (see (2.7))

$$\Gamma_{\varepsilon}(u) = \frac{\varepsilon^2}{4} \iint \frac{Q_{\varepsilon}(x)Q_{\varepsilon}(y) \left(J_0(x)J_0(y) - \sum_{k=1}^3 J_k(x)J_k(y)\right)}{|x - y|} dx dy \ge 0.$$

And if $u_n \rightharpoonup u$ in E, then $u_n \to u$ a.e.. Therefore (2.17) and Fatou's lemma yield

$$\Gamma_{\varepsilon}(u) \leq \liminf_{n \to \infty} \Gamma_{\varepsilon}(u_n)$$

as claimed.

Set, for
$$r > 0$$
, $B_r = \{u \in E : ||u|| \le r\}$, and for $e \in E^+$
$$E_e := E^- \oplus \mathbb{R}^+ e$$

with $\mathbb{R}^+ = [0, +\infty)$. In virtue of the assumptions (g_1) - (g_2) , for any $\delta > 0$, there exist $r_{\delta} > 0$, $c_{\delta} > 0$ and $c'_{\delta} > 0$ such that

(2.18)
$$\begin{cases} g(s) < \delta & \text{for all } 0 \le s \le r_{\delta}; \\ G(s) \ge c_{\delta} s^{\theta} - \delta s^{2} & \text{for all } s \ge 0; \\ G(s) \le \delta s^{2} + c_{\delta}' s^{p} & \text{for all } s \ge 0 \end{cases}$$

and

(2.19)
$$\widehat{G}(s) := \frac{1}{2}g(s)s^2 - G(s) \ge \frac{\theta - 2}{2\theta}g(s)s^2 \ge \frac{\theta - 2}{2}G(s) \ge c_{\theta}s^{\sigma}$$

for all $s \ge 0$, where $c_{\theta} = c_0(\theta - 2)/2$.

Lemma 2.3. For all $\varepsilon \in (0,1]$, Φ_{ε} possess the linking structure:

1) There are r > 0 and $\tau > 0$, both independent of ε , such that $\Phi_{\varepsilon}|_{B_r^+} \ge 0$ and $\Phi_{\varepsilon}|_{S_r^+} \ge \tau$, where

$$B_r^+ = B_r \cap E^+ = \{ u \in E^+ : ||u|| \le r \},$$

 $S_r^+ = \partial B_r^+ = \{ u \in E^+ : ||u|| = r \}.$

2) For any $e \in E^+ \setminus \{0\}$, there exist $R = R_e > 0$ and $C = C_e > 0$, both independent of ε , such that, for all $\varepsilon > 0$, there hold $\Phi_{\varepsilon}(u) < 0$ for all $u \in E_e \setminus B_R$ and $\max \Phi_{\varepsilon}(E_e) \leq C$.

Proof. Recall that $|u|_p^p \leq C_p ||u||^p$ for all $u \in E$ by Sobolev's embedding theorem. 1) follows easily because, for $u \in E^+$ and $\delta > 0$ small enough

$$\begin{split} \Phi_{\varepsilon}(u) &= \frac{1}{2} \|u\|^2 - \frac{\omega}{2} |u|_2^2 - \Gamma_{\varepsilon}(u) - \Psi_{\varepsilon}(u) \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\omega}{2} |u|_2^2 - \varepsilon^2 C_1 |Q|_{\infty}^2 \|u\|^4 - |P|_{\infty} \left(\delta |u|_2^2 + c_{\delta}' |u|_p^p\right) \end{split}$$

with C_1 , C_p independent of u and p > 2 (see (2.9) and (2.18)).

For checking 2), take $e \in E^+ \setminus \{0\}$. In virtue of (2.4) and (2.18), one gets, for $u = se + v \in E_e$,

(2.20)
$$\Phi_{\varepsilon}(u) = \frac{1}{2} \|se\|^2 - \frac{1}{2} \|v\|^2 - \frac{\omega}{2} |u|_2^2 - \Gamma_{\varepsilon}(u) - \Psi_{\varepsilon}(u)$$

$$\leq \frac{1}{2} s^2 \|e\|^2 - \frac{1}{2} \|v\|^2 - c_{\delta} d_{\theta} \inf P \cdot s^{\theta} |e|_{\theta}^{\theta}$$

proving the conclusion.

Recall that a sequence $\{u_n\} \subset E$ is called to be a $(PS)_c$ -sequence for functional $\Phi \in C^1(E,\mathbb{R})$ if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, and is called to be $(C)_c$ -sequence for Φ if $\Phi(u_n) \to c$ and $(1 + ||u_n||)\Phi'(u_n) \to 0$. It is clear that if $\{u_n\}$ is a $(PS)_c$ -sequence with $\{||u_n||\}$ bounded then it is also a $(C)_c$ -sequence. Below we are going to study $(C)_c$ -sequences for Φ_{ε} but firstly we observe the following

Lemma 2.4. Let $\{u_n\} \subset E \setminus \{0\}$ be bounded in $L^{\sigma}(\mathbb{R}^3)$, where $\sigma > 0$ is the constant in (g_2) . Then $\left\{\frac{A_{\varepsilon,u_n}^k}{\|u_n\|}\right\}$ is bounded in $L^6(\mathbb{R}^3)$ uniformly in $\varepsilon \in (0,1]$, for k = 0, 1, 2, 3.

Proof. Set $v_n = \frac{u_n}{\|u_n\|}$. Notice that A_{ε,u_n}^k satisfies the equation

$$-\Delta A_{\varepsilon,u_n}^k = \varepsilon^2 4\pi Q_{\varepsilon}(x) (\alpha_k u_n) \overline{u_n},$$

hence,

$$-\Delta \frac{A_{\varepsilon,u_n}^k}{\|u_n\|} = \varepsilon^2 4\pi Q_{\varepsilon}(x) (\alpha_k u_n) \overline{v_n}.$$

Observe that $||v_n|| = 1$, E embeds continuously into L^q for $q \in [2,3]$, and

$$\left| \int Q_{\varepsilon}(x)(\alpha_k u_n) \overline{v_n} \cdot \psi dx \right| \leq |Q|_{\infty} |u_n|_{\sigma} |v_n|_{q} |\psi|_{6}$$

$$\leq S^{-1/2} |Q|_{\infty} |u_n|_{\sigma} |v_n|_{q} ||\psi||_{\mathcal{D}}$$

for any $\psi \in \mathcal{D}^{1,2}(\mathbb{R}^3,\mathbb{C}^4)$ and $\frac{1}{\sigma} + \frac{1}{q} + \frac{1}{6} = 1$. We infer

$$\left\| \frac{A_{\varepsilon,u_n}^k}{\|u_n\|} \right\|_{\mathcal{D}} \le \varepsilon^2 \tilde{C} |Q|_{\infty} |u_n|_{\sigma},$$

which yields the conclusion.

We now turn to an estimate on boundness of $(C)_c$ -sequences which is the key ingredient in the sequel. Recall that, by (g_1) , there exist $r_1 > 0$ and $a_1 > 0$ such that

(2.21)
$$g(s) \le \frac{a - |\omega|}{2|P|_{\infty}} \quad \text{for all } s \le r_1,$$

and, for $s \ge r_1$, $g(s) \le a_1 s^{p-2}$, so $g(s)^{\sigma_0 - 1} \le a_2 s^2$ with

$$\sigma_0 := \frac{p}{p-2} > 3$$

which, jointly with (g_2) , yields (see (2.19))

(2.22)
$$g(s)^{\sigma_0} \le a_2 g(s) s^2 \le a_3 \widehat{G}(s)$$
 for all $s \ge r_1$.

Lemma 2.5. For any $\lambda > 0$, denoting $I = [0, \lambda]$, there is $\Lambda > 0$ independent of ε such that, for all $\varepsilon \in (0, 1]$, any $(C)_c$ -sequence $\{u_n^{\varepsilon}\}$ of Φ_{ε} with $c \in I$, there holds (up to a subsequence)

$$||u_n^{\varepsilon}|| \leq \Lambda$$

for all $n \in \mathbb{N}$.

Proof. Let $\{u_n^{\varepsilon}\}$ be a $(C)_c$ -sequence of Φ_{ε} with $c \in I$: $\Phi_{\varepsilon}(u_n^{\varepsilon}) \to c$ and $(1 + \|u_n^{\varepsilon}\|)\|\Phi'_{\varepsilon}(u_n^{\varepsilon})\| \to 0$. Without loss of generality we may assume that $\|u_n^{\varepsilon}\| \geq 1$. The form of Φ_{ε} and the representation (2.13) $(\Gamma'_{\varepsilon}(u)u = 4\Gamma_{\varepsilon}(u))$ implies that

$$(2.23) \quad 2\lambda > c + o(1) = \Phi_{\varepsilon}(u_n^{\varepsilon}) - \frac{1}{2}\Phi_{\varepsilon}'(u_n^{\varepsilon})u_n^{\varepsilon} = \Gamma_{\varepsilon}(u_n^{\varepsilon}) + \int P_{\varepsilon}(x)\widehat{G}(|u_n^{\varepsilon}|)$$

and

(2.24)
$$o(1) = \Phi_{\varepsilon}'(u_n^{\varphi})(u_n^{\varepsilon+} - u_n^{\varepsilon-})$$

$$= ||u_n^{\varepsilon}||^2 - \omega(|u_n^{\varepsilon+}|_2^2 - |u_n^{\varepsilon-}|_2^2) - \Gamma_{\varepsilon}'(u_n^{\varepsilon})(u_n^{\varepsilon+} - u_n^{\varepsilon-})$$

$$- \Re \int P_{\varepsilon}(x)g(|u_n^{\varepsilon}|)u_n^{\varepsilon} \cdot \overline{u_n^{\varepsilon+} - u_n^{\varepsilon-}}.$$

By Lemma 2.2, (2.19) and (2.23), $\{u_n^{\varepsilon}\}$ is bounded in L^{σ} uniformly in ε with the upper bound, denoted by C_1 , depending on λ , σ , θ and inf P. It follows from (2.24) that

$$o(1) + \frac{a - |\omega|}{a} ||u_n^{\varepsilon}||^2$$

$$\leq \Gamma_{\varepsilon}'(u_n^{\varepsilon})(u_n^{\varepsilon +} - u_n^{\varepsilon -}) + \Re \int P_{\varepsilon}(x)g(|u_n^{\varepsilon}|)u_n^{\varepsilon} \cdot \overline{u_n^{\varepsilon +} - u_n^{\varepsilon -}}.$$

This, together with (2.21) and (2.2), shows

(2.25)
$$o(1) + \frac{a - |\omega|}{2a} ||u_n^{\varepsilon}||^2 \\ \leq \Gamma_{\varepsilon}'(u_n^{\varepsilon})(u_n^{\varepsilon +} - u_n^{\varepsilon -}) + \Re \int_{|u_n^{\varepsilon}| \geq r_1} P_{\varepsilon}(x) g(|u_n^{\varepsilon}|) u_n^{\varepsilon} \cdot \overline{u_n^{\varepsilon +} - u_n^{\varepsilon -}}.$$

Recall that (g_1) and (g_2) imply $2 < \sigma \le p$. Setting $t = \frac{p\sigma}{2\sigma - p}$, one sees

$$2 < t < p$$
, $\frac{1}{\sigma_0} + \frac{1}{\sigma} + \frac{1}{t} = 1$.

By Hölder's inequality, the fact $\Gamma_{\varepsilon}(u_n^{\varepsilon}) \geq 0$, (2.22), (2.23), the boundedness of $\{|u_n^{\varepsilon}|_{\sigma}\}$ uniformly in ε , and the embedding of E into L^t , we have

$$\int_{|u_{n}^{\varepsilon}| \geq r_{1}} P_{\varepsilon}(x) g(|u_{n}^{\varepsilon}|) |u_{n}^{\varepsilon}| \left| u_{n}^{\varepsilon+} - u_{n}^{\varepsilon-} \right|
(2.26) \qquad \leq |P|_{\infty} \left(\int_{|u_{n}^{\varepsilon}| \geq r_{1}} g(|u_{n}^{\varepsilon}|)^{\sigma_{0}} \right)^{1/\sigma_{0}} \left(\int |u_{n}^{\varepsilon}|^{\sigma} \right)^{1/\sigma} \left(|u_{n}^{\varepsilon+} - u_{n}^{\varepsilon-}|^{t} \right)^{1/t}
\leq C_{2} ||u_{n}^{\varepsilon}||$$

with
$$C_2$$
 independent of ε .
Let $q = \frac{6\sigma}{5\sigma - 6}$. Then $2 < q < 3$ and $\frac{1}{\sigma} + \frac{1}{q} + \frac{1}{6} = 1$. Set

$$\zeta = \begin{cases} 0 & \text{if } q = \sigma; \\ \frac{2(\sigma - q)}{q(\sigma - 2)} & \text{if } q < \sigma; \\ \frac{3(q - \sigma)}{q(3 - \sigma)} & \text{if } q > \sigma \end{cases}$$

and note that

$$|u|_q \le \begin{cases} |u|_2^{\zeta} \cdot |u|_{\sigma}^{1-\zeta} & \text{if } 2 < q \le \sigma \\ |u|_3^{\zeta} \cdot |u|_{\sigma}^{1-\zeta} & \text{if } \sigma < q < 3. \end{cases}$$

By virtue of the Hölder inequality, Lemma 2.2, the boundedness of $\{|u_n^{\varepsilon}|_{\sigma}\}$, and the embedding of E to L^2 and L^3 , we obtain that

$$\begin{split} &\left|\Re\int Q_{\varepsilon}(x)A_{\varepsilon,u_{n}^{\varepsilon}}^{k}(\alpha_{k}u_{n}^{\varepsilon})\cdot\overline{u_{n}^{\varepsilon+}-u_{n}^{\varepsilon-}}\right| \\ &=\left|\left\|u_{n}^{\varepsilon}\right\|\Re\int Q_{\varepsilon}(x)\frac{A_{\varepsilon,u_{n}^{\varepsilon}}^{k}}{\left\|u_{n}^{\varepsilon}\right\|}(\alpha_{k}u_{n}^{\varepsilon})\cdot\overline{u_{n}^{\varepsilon+}-u_{n}^{\varepsilon-}}\right| \\ &\leq\left|Q\right|_{\infty}\left\|u_{n}^{\varepsilon}\right\|\left|\frac{A_{\varepsilon,u_{n}^{\varepsilon}}^{k}}{\left\|u_{n}^{\varepsilon}\right\|}\right|_{6}\left|u_{n}^{\varepsilon}\right|_{\sigma}\left|u_{n}^{\varepsilon+}-u_{n}^{\varepsilon-}\right|_{q} \\ &\leq\varepsilon^{2}C_{3}\left\|u_{n}^{\varepsilon}\right\|\left|u_{n}^{\varepsilon}\right|_{q}\leq\varepsilon^{2}C_{4}\left\|u_{n}^{\varepsilon}\right\|^{1+\zeta} \end{split}$$

with C_4 independent of ε . This, together with the representation of (2.13), implies that

$$(2.27) |\Gamma_{\varepsilon}'(u_n^{\varepsilon})(u_n^{\varepsilon+} - u_n^{\varepsilon-})| \le C_5 ||u_n^{\varepsilon}||^{1+\zeta}$$

with C_5 independent of ε .

Now the combination of (2.25), (2.26) and (2.27) shows that

(2.28)
$$||u_n^{\varepsilon}||^2 \le M_1 ||u_n^{\varepsilon}|| + M_2 ||u_n^{\varepsilon}||^{1+\zeta}$$

with M_1 and M_2 being independent of $\varepsilon \leq 1$. Therefore, either $||u_n^{\varepsilon}|| \leq 1$ or there is $\Lambda \geq 1$ independent of ε such that

$$||u_n^{\varepsilon}|| \le \Lambda$$

as desired.

Finally, for the later aim we define the operator $\mathcal{A}_{\varepsilon,k}: E \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ by $\mathcal{A}_{\varepsilon,k}(u) = A_{\varepsilon,u}^k$. We have

Lemma 2.6. For k = 0, 1, 2, 3,

- (1) $\mathcal{A}_{\varepsilon,k}$ maps bounded sets into bounded sets;
- (2) $\mathcal{A}_{\varepsilon,k}$ is continuous;

Proof. Clearly, (1) is a straight consequence of (2.12). (2) follows easily because, for $u, v \in E$, one sees that $A_{\varepsilon,u}^j - A_{\varepsilon,v}^j$ satisfies

$$-\Delta (A_{\varepsilon,u}^j - A_{\varepsilon,v}^j) = \varepsilon^2 4\pi Q_{\varepsilon}(x) \left[(\alpha_j u) \bar{u} - (\alpha_j v) \bar{v} \right].$$

Hence

$$\begin{aligned} \|A_{\varepsilon,u}^{j} - A_{\varepsilon,v}^{j}\|_{\mathcal{D}^{1,2}} &\leq \varepsilon^{2} C |Q|_{\infty} \left| (\alpha_{j} u) \bar{u} - (\alpha_{j} v) \bar{v} \right|_{6/5} \\ &\leq \varepsilon^{2} C |Q|_{\infty} \left(|u - v|_{12/5} |u|_{12/5} + |u - v|_{12/5} |v|_{12/5} \right) \\ &\leq \varepsilon^{2} C_{1} |Q|_{\infty} \left(\|u - v\| \cdot \|u\| + \|u - v\| \cdot \|v\| \right), \end{aligned}$$

and this implies the desired conclusion.

3 Preliminary results

Observe that the non-local term Γ_{ε} is rather complex. The main purpose of this section is, by cut-off arguments, to introduce an auxiliary functional which will simplify our proofs.

3.1 The limit equation

In order to prove our main result, we will make use of the limit equation. For any $\mu > 0$, consider the equation

$$i\alpha \cdot \nabla u - a\beta u - \omega u = \mu q(|u|)u$$
.

Its solutions are critical points of the functional

$$\mathscr{T}_{\mu}(u) := \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \mu \int G(|u|)$$
$$= \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \mathscr{G}_{\mu}(u).$$

defined for $u = u^+ + u^- \in E = E^+ \oplus E^-$. Denote the critical set, the least energy and the set of least energy solutions of \mathscr{T}_{μ} as follows

$$\begin{split} \mathscr{K}_{\mu} &:= \{ u \in E : \ \mathscr{T}'_{\mu}(u) = 0 \}, \\ \gamma_{\mu} &:= \inf \{ \mathscr{T}_{\mu}(u) : \ u \in \mathscr{K}_{\mu} \setminus \{ 0 \} \}, \\ \mathscr{R}_{\mu} &:= \{ u \in \mathscr{K}_{\mu} : \ \mathscr{T}_{\mu}(u) = \gamma_{\mu}, \ |u(0)| = |u|_{\infty} \}. \end{split}$$

The following lemma is from [9] (see also [13])

Lemma 3.1. There hold the following:

- i) $\mathscr{K}_{\mu} \neq \emptyset$, $\gamma_{\mu} > 0$ and $\mathscr{K}_{\mu} \subset \cap_{q \geq 2} W^{1,q}$,
- ii) γ_{μ} is attained and \mathscr{R}_{μ} is compact in $H^{1}(\mathbb{R}^{3},\mathbb{C}^{4})$,
- iii) there exist C, c > 0 such that

$$|u(x)| \le C \exp\left(-c|x|\right)$$

for all $x \in \mathbb{R}^3$ and $u \in \mathscr{R}_{\mu}$.

Motivated by Ackermann [2] (also see [10, 11, 13]), for a fixed $u \in E^+$, let $\varphi_u : E^- \to \mathbb{R}$ defined by $\varphi_u(v) = \mathscr{T}_{\mu}(u+v)$. We have, for any $v, w \in E^-$,

$$\varphi_u''(v)[w,w] = -\|w\|^2 - \omega |w|_2^2 - \mathcal{G}_{\mu}''(u+v)[w,w] \le -\|w\|^2.$$

In addition

$$\varphi_u(v) \le \frac{a + |\omega|}{2a} \|u\|^2 - \frac{a - |\omega|}{2a} \|v\|^2.$$

Therefore, there exists a unique $\mathscr{J}_{\mu}: E^+ \to E^-$ such that

$$\mathscr{T}_{\mu}(u + \mathscr{J}_{\mu}(u)) = \max_{v \in E^{-}} \mathscr{T}_{\mu}(u + v).$$

Define

$$J_{\mu}: E^+ \to \mathbb{R}, \quad J_{\mu}(u) = \mathscr{T}_{\mu}(u + \mathscr{J}_{\mu}(u)),$$

$$\mathscr{M}_{\mu} := \{ u \in E^+ \setminus \{0\} : \ J'_{\mu}(u)u = 0 \}.$$

Plainly, critical points of J_{μ} and \mathscr{T}_{μ} are in one-to-one correspondence via the injective map $u \mapsto u + \mathscr{J}_{\mu}(u)$ from E^+ into E. For any $u \in E^+$ and $v \in E^-$, setting $z = v - \mathscr{J}_{\mu}(u)$ and $l(t) = \mathscr{T}_{\mu}(u + \mathscr{J}_{\mu}(u) + tz)$, one has $l(1) = \mathscr{T}_{\mu}(u + v)$, $l(0) = \mathscr{T}_{\mu}(u + \mathscr{J}_{\mu}(u))$ and l'(0) = 0. Thus $l(1) - l(0) = \int_0^1 (1-t)l''(t)dt$. This implies that

$$\begin{split} & \mathscr{T}_{\mu}(u+v) - \mathscr{T}_{\mu}(u+\mathscr{J}_{\mu}(u)) \\ & = \int_{0}^{1} (1-t)\mathscr{T}_{\mu}''(u+\mathscr{J}_{\mu}(u)-tz) \, [z,z] dt \\ & = -\int_{0}^{1} (1-t) \left(\|z\|^{2} + \omega \, |z|_{2}^{2} \right) dt - \int_{0}^{1} (1-t)\mathscr{G}_{\mu}''(u+\mathscr{J}_{\mu}(u)-tz) [z,z] dt, \end{split}$$

hence

(3.1)
$$\int_{0}^{1} (1-t)\mathscr{G}''_{\mu}(u+\mathscr{J}_{\mu}(u)-tz)[z,z]dt + \frac{1}{2}\|z\|^{2} + \frac{\omega}{2}|z|_{2}^{2} = \mathscr{T}_{\mu}(u+\mathscr{J}_{\mu}(u)) - \mathscr{T}_{\mu}(u+v).$$

It is not difficult to see that, for each $u \in E^+ \setminus \{0\}$ there is a unique t = t(u) > 0 such that $tu \in \mathcal{M}_{\mu}$ and

$$\gamma_{\mu} = \inf\{J_{\mu}(u): u \in \mathscr{M}_{\mu}\} = \inf_{e \in E^{+}\setminus\{0\}} \max_{u \in E_{e}} \mathscr{T}_{\mu}(u)$$

(see [13], [10]). The following lemma is from [10].

Lemma 3.2. There hold:

- 1). Let $u \in \mathscr{M}_{\mu}$ be such that $J_{\mu}(u) = \gamma_{\mu}$ and set $E_{u} = E^{-} \oplus \mathbb{R}^{+}u$. Then $\max_{w \in E_{u}} \mathscr{T}_{\mu}(w) = J_{\mu}(u).$
- 2). If $\mu_1 < \mu_2$, then $\gamma_{\mu_1} > \gamma_{\mu_2}$.

3.2 Auxiliary functionals

In order to make the reduction method work for Φ_{ε} as ε small, we circumvent by cutting off the nonlocal terms. We find our current framework is more delicate, since the solutions we look for are at the least energy level and Γ_{ε} is not convex (even for u with ||u|| large). By cutting off the nonlocal terms, and using the reduction method, we are able to find a critical point via an appropriate min-max scheme. The critical point will eventually be shown to be a least energy solution of the original equation when ε is sufficiently small.

By virtue of (P_0) , set $\mu = b := \inf P(x) > 0$, take $e_0 \in \mathcal{M}_b$ such that $J_b(e_0) = \gamma_b$, and set $E_{e_0} = E^- \oplus \mathbb{R}^+ e_0$. One has

Lemma 3.3. For all
$$\varepsilon > 0$$
, $\max_{v \in E_{e_0}} \Phi_{\varepsilon}(v) \le \gamma_b$.

Proof. It is clear that $\Phi_{\varepsilon}(u) \leq \mathscr{T}_b(u)$ for all $u \in E$, hence, by Lemma 3.2

$$\max_{v \in E_{e_0}} \Phi_{\varepsilon}(v) \leq \max_{v \in E_{e_0}} \mathscr{T}_b(v) = J_b(e_0) = \gamma_b$$

as claimed. \Box

To introduce the modified functional, by virtue of Lemma 2.5, for $\lambda = \gamma_b$ and $I = [0, \gamma_b]$, let $\Lambda \geq 1$ be the associated constant (independent of ε). Denote $T := (\Lambda + 1)^2$ and let $\eta : [0, \infty) \to [0, 1]$ be a smooth function with $\eta(t) = 1$ if $0 \leq t \leq T$, $\eta(t) = 0$ if $t \geq T + 1$, $\max |\eta'(t)| \leq c_1$ and $\max |\eta''(t)| \leq c_2$. Define

$$\widetilde{\Phi}_{\varepsilon}(u) = \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \eta(\|u\|^{2}) \Gamma_{\varepsilon}(u) - \Psi_{\varepsilon}(u)$$

$$= \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \mathscr{F}_{\varepsilon}(u) - \Psi_{\varepsilon}(u).$$

By definition, $\Phi_{\varepsilon}|_{B_T} = \widetilde{\Phi}_{\varepsilon}|_{B_T}$. It is easy to see that $0 \leq \mathscr{F}_{\varepsilon}(u) \leq \Gamma_{\varepsilon}(u)$ and

$$\left|\mathscr{F}'_{\varepsilon}(u)v\right| \leq \left|2\eta'(\left\|u\right\|^{2})\Gamma_{\varepsilon}(u)\left\langle u,v\right\rangle\right| + \left|\Gamma'_{\varepsilon}(u)v\right|$$

for $u, v \in E$.

Lemma 3.4. There exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \leq \varepsilon_1$, if $\{u_n^{\varepsilon}\}$ is a $(C)_c$ sequence of $\widetilde{\Phi}_{\varepsilon}$ with $c \in I$ then $\|u_n^{\varepsilon}\| \leq \Lambda + \frac{1}{2}$, and consequently $\widetilde{\Phi}_{\varepsilon}(u_n^{\varepsilon}) = \Phi_{\varepsilon}(u_n^{\varepsilon})$.

Proof. We repeat the arguments of Lemma 2.5. Let $\{u_n^{\varepsilon}\}$ be a $(C)_c$ -sequence of $\widetilde{\Phi}_{\varepsilon}$ with $c \in I$. If $\|u_n^{\varepsilon}\|^2 \geq T+1$ then $\widetilde{\Phi}_{\varepsilon}(u_n^{\varepsilon}) = \Phi_{\varepsilon}(u_n^{\varepsilon})$ so, by Lemma 2.5, one has $\|u_n^{\varepsilon}\| \leq \Lambda$, a contradiction. Thus we assume that $\|u_n^{\varepsilon}\|^2 \leq T+1$. Then, using (2.9), $|\eta'(\|u_n^{\varepsilon}\|^2)\|u_n^{\varepsilon}\|^2\Gamma_{\varepsilon}(u_n^{\varepsilon})| \leq \varepsilon^2 d_1$ (here and in the following, by d_j we denote positive constants independent of ε). Similarly to (2.23),

$$2\gamma_b > c + o(1) \ge \left(\eta(\|u_n^{\varepsilon}\|^2) + 2\eta'(\|u_n^{\varepsilon}\|^2)\|u_n^{\varepsilon}\|^2\right)\Gamma_{\varepsilon}(u_n^{\varepsilon}) + \int P_{\varepsilon}(x)\widehat{G}(|u_n^{\varepsilon}|)$$

which yields

$$2\gamma_b + \varepsilon^2 d_1 > \eta(\|u_n^{\varepsilon}\|^2) \Gamma_{\varepsilon}(u_n^{\varepsilon}) + \int P_{\varepsilon}(x) \widehat{G}(|u_n^{\varepsilon}|),$$

consequently $|u_n^{\varepsilon}|_{\sigma} \leq d_2$. Similarly to (2.25) we get that

$$\frac{a - |\omega|}{2a} \|u_n^{\varepsilon}\|^2 \le \varepsilon^2 d_3 + \eta(\|u_n^{\varepsilon}\|^2) \Gamma_{\varepsilon}'(u_n^{\varepsilon}) (u_n^{\varepsilon +} - u_n^{\varepsilon -})$$
$$+ \Re \int_{|u_{\varepsilon}| > r_1} P_{\varepsilon}(x) g(|u_n^{\varepsilon}|) u_n^{\varepsilon} \cdot \overline{u_n^{\varepsilon +} - u_n^{\varepsilon -}}$$

which, together with (2.26) and (2.27), implies either $||u_n^{\varepsilon}|| \leq 1$ or as (2.28)

$$||u_n^{\varepsilon}||^2 \le \varepsilon^2 d_4 + M_1 ||u_n^{\varepsilon}|| + M_2 ||u_n^{\varepsilon}||^{1+\zeta},$$

thus

$$||u_n^{\varepsilon}|| \le \varepsilon^2 d_5 + \Lambda.$$

The proof is complete.

Based on this lemma, to prove Theorem 1.1 it suffices to study $\widetilde{\Phi}_{\varepsilon}$ and get its critical points with critical values in $[0, \gamma_b]$. This will be done via a series of arguments. The first is to introduce the minimax values of $\widetilde{\Phi}_{\varepsilon}$. It is easy to verify the following lemma.

Lemma 3.5. $\widetilde{\Phi}_{\varepsilon}$ possesses a linking structure and we can replace Φ_{ε} by $\widetilde{\Phi}_{\varepsilon}$ in Lemma 2.3. In addition,

$$\max_{v \in E_{e_0}} \widetilde{\Phi}_{\varepsilon}(v) \le \gamma_b,$$

where $e_0 \in \mathcal{M}_b$ such that $J_b(e_0) = \gamma_b$ and $E_{e_0} = E^- \oplus \mathbb{R}^+ e_0$

Proof. One can follow the proofs of Lemmas 2.3 and Lemma 3.3 with minor changes. \Box

Define (see [6, 32])

$$c_{\varepsilon} := \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \widetilde{\Phi}_{\varepsilon}(u).$$

As a consequence of Lemma 3.5 we have

Lemma 3.6. $\tau \leq c_{\varepsilon} \leq \gamma_b$.

We now describe further the minimax value c_{ε} . As before, for a fixed $u \in E^+$ we define $\phi_u : E^- \to \mathbb{R}$ by

$$\phi_u(v) = \widetilde{\Phi}_{\varepsilon}(u+v).$$

A direct computation gives, for any $v, z \in E^-$,

$$\phi_{u}''(v)[z,z] = -\|z\|^{2} - \omega |z|_{2}^{2} - \mathscr{F}_{\varepsilon}''(u+v)[z,z] - \Psi_{\varepsilon}''(u+v)[z,z],$$

$$\leq -\frac{(a-|\omega|)}{a} \|z\|^{2} - \mathscr{F}_{\varepsilon}''(u+v)[z,z],$$

and

$$\mathcal{F}''_{\varepsilon}(u+v)[z,z] = \left(4\eta''(\|u+v\|^2) |\langle u+v,z\rangle|^2 + 2\eta'(\|u+v\|^2) \|z\|^2\right) \Gamma_{\varepsilon}(u+v) + 4\eta'(\|u+v\|^2) \langle u+v,z\rangle \Gamma'_{\varepsilon}(u+v)z + \eta(\|u+v\|^2) \Gamma''_{\varepsilon}(u+v)[z,z].$$

Combining (2.9)-(2.11) yields that there exists $\varepsilon_0 \in (0, \varepsilon_1]$ such that

$$\phi_u''(v)[z,z] \le -\frac{a-|\omega|}{2a} \|z\|^2$$
 if $0 < \varepsilon \le \varepsilon_0$.

Since

$$\phi_u(v) \le \frac{a + |\omega|}{2a} \|u\|^2 - \frac{a - |\omega|}{2a} \|v\|^2,$$

there is $h_{\varepsilon}: E^+ \to E^-$, uniquely defined, such that

$$\phi_u(h_{\varepsilon}(u)) = \max_{v \in E^-} \phi_u(v)$$

and

$$v \neq h_{\varepsilon}(u) \Leftrightarrow \widetilde{\Phi}_{\varepsilon}(u+v) < \widetilde{\Phi}_{\varepsilon}(u+h_{\varepsilon}(u)).$$

It is clear that, for all $v \in E^-$, $0 = \phi'_u(h_\varepsilon(u))v$. Observe that, similarly to (3.1), we have for $u \in E^+$ and $v \in E^-$

$$(3.2) \widetilde{\Phi}_{\varepsilon}(u+h_{\varepsilon}(u)) - \widetilde{\Phi}_{\varepsilon}(u+v)$$

$$= \int_{0}^{1} (1-t) \Big[\mathscr{F}_{\varepsilon}''(u+h_{\varepsilon}(u)+t(v-h_{\varepsilon}(u)))[v-h_{\varepsilon}(u),v-h_{\varepsilon}(u)] + \Psi_{\varepsilon}''(u+h_{\varepsilon}(u)+t(v-h_{\varepsilon}(u)))[v-h_{\varepsilon}(u),v-h_{\varepsilon}(u)] \Big] dt$$

$$+ \frac{1}{2} \|v-h_{\varepsilon}(u)\|^{2} + \frac{\omega}{2} |v-h_{\varepsilon}(u)|^{2}_{2}.$$

Define $I_{\varepsilon}: E^+ \to \mathbb{R}$ by

$$I_{\varepsilon}(u) = \widetilde{\Phi}_{\varepsilon}(u + h_{\varepsilon}(u)),$$

and set

$$\mathcal{N}_{\varepsilon} := \{ u \in E^+ \setminus \{0\} : I_{\varepsilon}'(u)u = 0 \}.$$

Lemma 3.7. For any $u \in E^+ \setminus \{0\}$, there is a unique t = t(u) > 0 such that $tu \in \mathscr{N}_{\varepsilon}$.

Proof. This proof is quite technical, for details we refer [2, 13]. We only give a sketch of the proof. Firstly, we observe that for any $u \in E \setminus \{0\}$ and $v \in E$,

$$\begin{aligned} & \left(\Gamma_{\varepsilon}''(u)[u,u] - \Gamma_{\varepsilon}'(u)u\right) + 2\left(\Gamma_{\varepsilon}''(u)[u,v] - \Gamma_{\varepsilon}'(u)v\right) + \Gamma_{\varepsilon}''(u)[v,v] \\ &= 2\,\varepsilon^2 \iint \frac{Q_{\varepsilon}(x)Q_{\varepsilon}(y)}{|x-y|} \left[J_0(x)[\alpha_0(u+v)\overline{(u+v)}](y) \right] \\ & - \sum_{k=1}^3 J_k(x)[\alpha_k(u+v)\overline{(u+v)}](y) dxdy \\ & + 2\,\varepsilon^2 \iint \frac{Q_{\varepsilon}(x)Q_{\varepsilon}(y)}{|x-y|} \left[\left(\Re[\alpha_0 u\overline{v}(x)]\right) \left(\Re[\alpha_0 u\overline{v}(y)]\right) \right] \\ & - \sum_{k=1}^3 \left(\Re[\alpha_k u\overline{v}(x)]\right) \left(\Re[\alpha_k u\overline{v}(y)]\right) dxdy \\ & \geq O(\varepsilon^2) \|u\|^2 \|v\|^2. \end{aligned}$$

Here we used the formula

$$\pm (\beta z, z)(y)(\beta u, u)(x) + \sum_{k=1}^{3} (\beta z, \pi_k z)(y)(\beta u, \pi_k u)(x)$$

$$= \pm (\beta z, z)(y)(\beta u, u)(x) + \sum_{k=1}^{3} (\alpha_k z, z)(y)(\alpha_k u, u)(x)$$

$$\leq |z(y)|_{\mathbb{C}^4}^2 |u(x)|_{\mathbb{C}^4}^2$$

which follows from (2.14) and (2.15) with $z = u + v \in E$. Consequently, we deduce that

$$\left(\mathscr{F}_{\varepsilon}''(u)[u,u] - \mathscr{F}_{\varepsilon}'(u)u\right) + 2\left(\mathscr{F}_{\varepsilon}''(u)[u,v] - \mathscr{F}_{\varepsilon}'(u)v\right) + \mathscr{F}_{\varepsilon}''(u)[v,v] \ge o(\varepsilon)\|v\|^2 + o(\varepsilon).$$

Invoking the arguments in [2], if $z \in E^+ \setminus \{0\}$ with $I'_{\varepsilon}(z)z = 0$, we see by a delicate calculation that, for ε sufficiently small,

$$(3.3) I_{\varepsilon}''(z)[z,z] < 0.$$

Now for a fixed $u \in E^+ \setminus \{0\}$, we set $f(t) = I_{\varepsilon}(tu)$. From Lemma 3.5, we see that f(0) = 0, f(t) > 0 for t > 0 sufficiently small, and $f(t) \to -\infty$ as $t \to \infty$. Thus there exists t(u) > 0 such that

$$I_{\varepsilon}(t(u)u) = \sup_{t \ge 0} I_{\varepsilon}(tu).$$

It is clear that

$$\frac{dI_{\varepsilon}(tu)}{dt}\bigg|_{t=t(u)} = I'_{\varepsilon}(t(u)u)u = \frac{1}{t(u)}I'_{\varepsilon}(t(u)u)t(u)u = 0,$$

and consequently by (3.3)

$$I_{\varepsilon}''(t(u)u)[t(u)u,t(u)u] < 0.$$

Therefore, one sees that such t(u) > 0 is unique.

Lemma 3.8. $c_{\varepsilon} = \inf_{u \in \mathscr{N}_{\varepsilon}} I_{\varepsilon}(u)$.

Proof. Indeed, denoting $\hat{c}_{\varepsilon} = \inf_{u \in \mathscr{N}_{\varepsilon}} I_{\varepsilon}(u)$, given $e \in E^+$, if $u = v + se \in E_e$ with $\widetilde{\Phi}_{\varepsilon}(u) = \max_{z \in E_e} \widetilde{\Phi}_{\varepsilon}(z)$ then the restriction $\widetilde{\Phi}_{\varepsilon}|_{E_e}$ of $\widetilde{\Phi}_{\varepsilon}$ on E_e satisfies $(\widetilde{\Phi}_{\varepsilon}|_{E_e})'(u) = 0$ which implies $v = h_{\varepsilon}(se)$ and $I'_{\varepsilon}(se)(se) = 0$, i.e. $se \in \mathscr{N}_{\varepsilon}$. Thus $\hat{c}_{\varepsilon} \leq c_{\varepsilon}$. On the other hand, if $w \in \mathscr{N}_{\varepsilon}$ then $(\widetilde{\Phi}_{\varepsilon}|_{E_w})'(w + h_{\varepsilon}(w)) = 0$, hence, $c_{\varepsilon} \leq \max_{u \in E_w} \widetilde{\Phi}_{\varepsilon}(u) = I_{\varepsilon}(w)$. Thus $\hat{c}_{\varepsilon} \geq c_{\varepsilon}$.

Lemma 3.9. For any $e \in E^+ \setminus \{0\}$, there is $T_e > 0$ independent of ε such that $t_{\varepsilon} \leq T_e$ for $t_{\varepsilon} > 0$ satisfying $t_{\varepsilon}e \in \mathscr{N}_{\varepsilon}$.

Proof. Since $I'_{\varepsilon}(t_{\varepsilon}e)(t_{\varepsilon}e) = 0$, one gets

$$\widetilde{\Phi}_{\varepsilon}(t_{\varepsilon}e + h_{\varepsilon}(t_{\varepsilon}e)) = \max_{w \in E_{\varepsilon}} \widetilde{\Phi}_{\varepsilon}(w) \ge \tau.$$

This, together with Lemma 3.5, shows the assertion.

Let $\mathscr{K}_{\varepsilon} := \{u \in E : \widetilde{\Phi}'_{\varepsilon}(u) = 0\}$ be the critical set of $\widetilde{\Phi}_{\varepsilon}$. Since critical points of I_{ε} and $\widetilde{\Phi}_{\varepsilon}$ are in one-to-one correspondence via the injective map $u \mapsto u + h_{\varepsilon}(u)$ from E^+ into E, let us denoted by

$$\mathscr{C}_{\varepsilon} := \{ u \in \mathscr{K}_{\varepsilon} : \ \widetilde{\Phi}_{\varepsilon}(u) = c_{\varepsilon} \},$$

from Lemma 3.8, one easily sees that if $\mathscr{C}_{\varepsilon} \neq \emptyset$ then $c_{\varepsilon} = \inf \{ \widetilde{\Phi}_{\varepsilon}(u) : u \in \mathscr{K}_{\varepsilon} \setminus \{0\} \}$. Next we estimate the regularity of critical points of $\widetilde{\Phi}_{\varepsilon}$. By using the same iterative argument of [14] one obtains easily the following

Lemma 3.10. If $u \in \mathscr{K}_{\varepsilon}$ with $|\widetilde{\Phi}_{\varepsilon}(u)| \leq C_1$, then, for any $q \in [2, +\infty)$, $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ with $||u||_{W^{1,q}} \leq \Lambda_q$ where Λ_q depends only on C_1 and q.

Proof. See [14]. We outline the proof as follows. From (2.8), we write

$$u = H_0^{-1} \Big(\omega u + Q_{\varepsilon}(x) A_{\varepsilon,u}^0 u - \sum_{k=1}^3 Q_{\varepsilon}(x) \alpha_k A_{\varepsilon,u}^k u + P_{\varepsilon}(x) g(|u|) u \Big).$$

For the later use, let $\rho:[0,\infty)\to [0,1]$ be a smooth function satisfying $\rho(s)=1$ if $s\in [0,1]$ and $\rho(s)=0$ if $s\in [2,\infty)$. Then we have

$$g(s) := g_1(s) + g_2(s)$$

= $\rho(s)g(s) + (1 - \rho(s))g(s)$.

Consequently, $u = u_1 + u_2 + u_3$ with

$$u_{1} = H_{0}^{-1} (\omega u + P_{\varepsilon} \cdot g_{1}(|u|)u),$$

$$u_{2} = H_{0}^{-1} (Q_{\varepsilon} \cdot A_{\varepsilon,u}^{0} u - \sum Q_{\varepsilon} \cdot \alpha_{k} A_{\varepsilon,u}^{k} u),$$

$$u_{3} = H_{0}^{-1} (P_{\varepsilon} \cdot g_{2}(|u|)u).$$

Noting that, by Hölder's inequality, for $q \geq 2$

$$\left|Q_{\varepsilon}\alpha_{k}A_{\varepsilon,u}^{k}u\right|_{s}\leq\left|Q\right|_{\infty}\left|A_{\varepsilon,u}^{k}\right|_{6}\left|u\right|_{q}$$

with $\frac{1}{s} = \frac{1}{6} + \frac{1}{q}$ and since

$$|P_{\varepsilon}(x)|u|^{p-2}u|_{t} \leq |P|_{\infty}|u|_{t(p-1)}^{p-1}$$

one has

$$u_1 \in W^{1,2} \cap W^{1,3}, \ u_2 \in W^{1,s}, \ u_3 \in W^{1,t}.$$

Then, denoting $s^* = \frac{3s}{3-s}$ and $t^* = \frac{3t}{3-t}$, $u \in W^{1,q}$ with $q = \min\{s^*, t^*\}$. A standard bootstrap argument shows that $u \in \cap_{q \ge 2} L^q$, $u_1 \in \cap_{q \ge 2} W^{1,q}$, $u_2 \in \cap_{6 > q \ge 2} W^{1,q}$ and $u_3 \in \cap_{q \ge 2} W^{1,q}$.

By the Sobolev's embedding theorems, $u \in C^{0,\gamma}$ for some $\gamma \in (0,1)$. This, together with elliptic regularity (see [21]), shows $A_{\varepsilon,u}^k \in W_{loc}^{2,6} \cap L^6$ for k = 0, 1, 2, 3 and

$$||A_{\varepsilon,u}^{k}||_{W^{2,6}(B_{1}(x))} \le C_{2}(\varepsilon^{2}|Q|_{\infty}|u|_{L^{12}(B_{2}(x))}^{2} + |A_{\varepsilon,u}^{k}|_{L^{6}(B_{2}(x))})$$

for all $x \in \mathbb{R}^3$, with C_2 independent of x and ε , where $B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}$ for r > 0. Since $W^{2,6}(B_1(x)) \hookrightarrow C^1(B_1(x))$, we have

for all $x \in \mathbb{R}^3$ with C_3 independent of x and ε . Consequently $A_{\varepsilon,u}^k \in L^{\infty}$, and that yields

$$\left|Q_{\varepsilon}\alpha_{k}A_{\varepsilon,u}^{k}u\right|_{s}\leq\left|Q\right|_{\infty}\left|A_{\varepsilon,u}^{k}\right|_{\infty}\left|u\right|_{s}.$$

Thus $u_2 \in \cap_{q \geq 2} W^{1,q}$, and combining with $u_1, u_3 \in \cap_{q \geq 2} W^{1,q}$ the conclusion is obtained.

Remark 3.11. Let $\mathscr{L}_{\varepsilon}$ denote the set of all least energy solutions of $\widetilde{\Phi}_{\varepsilon}$. If $u \in \mathscr{L}_{\varepsilon}$, $\widetilde{\Phi}_{\varepsilon}(u) = c_{\varepsilon} \leq \gamma_b$. Recall that $\mathscr{L}_{\varepsilon}$ is bounded in E with an upper bound Λ independent of ε . Therefore, as a consequence of Lemma 3.10 we see that, for each $q \in [2, +\infty)$ there is $C_q >$ independent of ε such that

(3.5)
$$||u||_{W^{1,q}} \le C_q \quad \text{for all } u \in \mathscr{L}_{\varepsilon}.$$

This, together with the Sobolev embedding theorem, implies that there is $C_{\infty} > 0$ independent of ε with

(3.6)
$$||u||_{\infty} \leq C_{\infty}$$
 for all $u \in \mathscr{L}_{\varepsilon}$.

4 Proof of the main result

Throughout this section we suppose $\omega \in (-a, a)$ and that (g_1) - (g_2) , (Q_0) , (P_0) are satisfied, and recall that we always assume $0 \in \mathscr{P}$. The main theorem will be carried in three parts: Existence, Concentration, and Exponential decay.

Part 1. Existence

Keeping the notation of Section 3 we now turn to the existence result of the main theorem. The proof is carried out in three lemmas. The modified problem gives us an access to Lemma 4.1, which is the key ingredient for Lemma 4.2.

Recall that γ_m denotes the least energy of \mathscr{T}_m (see the subsection 3.1), where $\mu = m := \max_{x \in \mathbb{R}^3} P(x)$, and J_m denotes the associated reduction functional on E^+ . We have

Lemma 4.1. $c_{\varepsilon} \to \gamma_m$ as $\varepsilon \to 0$.

Lemma 4.2. c_{ε} is attained for all small $\varepsilon > 0$.

Lemma 4.3. $\mathcal{L}_{\varepsilon}$ is compact in $W^{1,q}$ for each $q \geq 2$, for all small $\varepsilon > 0$.

Proof of Lemma 4.1. Firstly we show that

$$\liminf_{\varepsilon \to 0} c_{\varepsilon} \ge \gamma_m.$$

Arguing indirectly, assume that $\liminf_{\varepsilon\to 0} c_{\varepsilon} < \gamma_m$. By the definition of c_{ε} and Lemma 3.8 we can choose an $e_j \in \mathscr{N}_{\varepsilon}$ and $\delta > 0$ such that

$$\max_{u \in E_{e_j}} \widetilde{\Phi}_{\varepsilon_j}(u) \le \gamma_m - \delta$$

as $\varepsilon_j \to 0$. Since $P_{\varepsilon}(x) \leq m$ and $\mathscr{F}(u) = o(1)$ as $\varepsilon \to 0$ uniformly in u (by (2.9) and the definition of η), the representations of $\widetilde{\Phi}_{\varepsilon}$ and \mathscr{T}_m imply

that $\widetilde{\Phi}_{\varepsilon}(u) \geq \mathscr{T}_m(u) - \delta/2$ for all $u \in E$ and ε small. Note also that $\gamma_m \leq J_m(e_j) \leq \max_{u \in E_{e_j}} \mathscr{T}_m(u)$. Therefore we get, for all ε_j small,

$$\gamma_m - \delta \ge \max_{u \in E_{e_j}} \widetilde{\Phi}_{\varepsilon_j}(u) \ge \max_{u \in E_{e_j}} \mathscr{T}_m(u) - \frac{\delta}{2} \ge \gamma_m - \frac{\delta}{2},$$

a contradiction.

We now turn to prove the desired conclusion. Set $P^0(x) = m - P(x)$ and $P^0_{\varepsilon}(x) = P^0(\varepsilon x)$. Then

(4.2)
$$\widetilde{\Phi}_{\varepsilon}(u) = \mathscr{T}_{m}(u) - \mathscr{F}_{\varepsilon}(u) + \int P_{\varepsilon}^{0}(x)G(|u|).$$

In virtue of Lemma 3.1, let $u=u^++u^-\in \mathcal{R}_m$, be a least energy solution of the limit equation with $\mu=m$, and set $e=u^+$. Clearly, $e\in \mathcal{M}_m$, $\mathcal{J}_m(e)=u^-$ and $J_m(e)=\gamma_m$. There is a unique $t_{\varepsilon}>0$ such that $t_{\varepsilon}e\in \mathcal{N}_{\varepsilon}$ and one has

$$(4.3) c_{\varepsilon} \leq I_{\varepsilon}(t_{\varepsilon}e).$$

By Lemma 3.9 t_{ε} is bounded. Hence, without loss of generality we can assume $t_{\varepsilon} \to t_0$ as $\varepsilon \to 0$. Using (3.1) and (3.2), we infer

$$\frac{1}{2} \|v_{\varepsilon}\|^{2} + (I) = \widetilde{\Phi}_{\varepsilon}(w_{\varepsilon}) - \widetilde{\Phi}_{\varepsilon}(u_{\varepsilon})$$

$$= \mathscr{T}_{m}(w_{\varepsilon}) - \mathscr{T}_{m}(u_{\varepsilon}) - \mathscr{F}_{\varepsilon}(w_{\varepsilon}) + \mathscr{F}_{\varepsilon}(u_{\varepsilon})$$

$$+ \int P_{\varepsilon}^{0}(x)G(|w_{\varepsilon}|) - \int P_{\varepsilon}^{0}(x)G(|u_{\varepsilon}|)$$

where, setting

$$u_{\varepsilon} = t_{\varepsilon}e + \mathscr{J}_m(t_{\varepsilon}e), \ w_{\varepsilon} = t_{\varepsilon}e + h_{\varepsilon}(t_{\varepsilon}e), \ v_{\varepsilon} = u_{\varepsilon} - w_{\varepsilon},$$

$$(I) := \frac{\omega}{2} \left| v_{\varepsilon} \right|_{2}^{2} + \int_{0}^{1} (1-s) \big(\mathscr{F}_{\varepsilon}''(w_{\varepsilon} + sv_{\varepsilon})[v_{\varepsilon}, v_{\varepsilon}] + \Psi_{\varepsilon}''(w_{\varepsilon} + sv_{\varepsilon})[v_{\varepsilon}, v_{\varepsilon}] \big) dt.$$

Taking into account that

$$\mathscr{F}_{\varepsilon}(u_{\varepsilon}) - \mathscr{F}_{\varepsilon}(w_{\varepsilon}) = \mathscr{F}'_{\varepsilon}(w_{\varepsilon})v_{\varepsilon} + \int_{0}^{1} (1-s)\mathscr{F}''_{\varepsilon}(w_{\varepsilon} + sv_{\varepsilon})[v_{\varepsilon}, v_{\varepsilon}]dt$$

and

$$\int P_{\varepsilon}^{0}(x) (G(|w_{\varepsilon}|) - G(|u_{\varepsilon}|))$$

$$= -\int P_{\varepsilon}^{0}(x) g(|u_{\varepsilon}|) u_{\varepsilon} \cdot \overline{v_{\varepsilon}} + \int_{0}^{1} (1 - s) \mathscr{G}''_{m}(u_{\varepsilon} - sv_{\varepsilon}) [v_{\varepsilon}, v_{\varepsilon}] dt$$

$$-\int_{0}^{1} (1 - s) \Psi''_{\varepsilon}(u_{\varepsilon} - sv_{\varepsilon}) [v_{\varepsilon}, v_{\varepsilon}] dt,$$

setting

$$(II) := \int_0^1 (1-s)\Psi_{\varepsilon}''(u_{\varepsilon} - sv_{\varepsilon})[v_{\varepsilon}, v_{\varepsilon}]dt,$$

one has

$$\frac{1}{2} \|v_{\varepsilon}\|^{2} + (I) + (II)$$

$$\leq \mathscr{F}'_{\varepsilon}(w_{\varepsilon})v_{\varepsilon} + \int_{0}^{1} (1-s)\mathscr{F}''_{\varepsilon}(w_{\varepsilon} + sv_{\varepsilon})[v_{\varepsilon}, v_{\varepsilon}] - \int P_{\varepsilon}^{0}(x)g(|u_{\varepsilon}|)u_{\varepsilon} \cdot \overline{v_{\varepsilon}}.$$

So we deduce, noticing that $0 \le P_{\varepsilon}^0(x) \le m$,

(4.4)
$$\frac{1}{2} \|v_{\varepsilon}\|^{2} + \frac{\omega}{2} |v_{\varepsilon}|_{2}^{2} + \int_{0}^{1} (1 - s) \Psi_{\varepsilon}''(w_{\varepsilon} + sv_{\varepsilon}) [v_{\varepsilon}, v_{\varepsilon}]$$

$$\leq |\mathscr{F}_{\varepsilon}'(w_{\varepsilon})v_{\varepsilon}| + \int P_{\varepsilon}^{0}(x) g(|u_{\varepsilon}|) |u_{\varepsilon}| \cdot |v_{\varepsilon}|.$$

Since $t_{\varepsilon} \to t_0$, it is clear that $\{u_{\varepsilon}\}, \{w_{\varepsilon}\}$ and $\{v_{\varepsilon}\}$ are bounded, hence, by the definitions and (2.9), (2.10),

$$\mathscr{F}_{\varepsilon}(z_{\varepsilon}) = o(1), \quad \|\mathscr{F}'_{\varepsilon}(z_{\varepsilon})\| = o(1)$$

as $\varepsilon \to 0$ for $z_{\varepsilon} = u_{\varepsilon}, w_{\varepsilon}, v_{\varepsilon}$. In addition, by noting that for $q \in [2, 3]$

$$\limsup_{r \to \infty} \int_{|x| > r} |u_{\varepsilon}|^q = 0,$$

using the assumption $0 \in \mathcal{P}$ one deduces

$$\int \left(P_{\varepsilon}^{0}(x)\right)^{q/(q-1)} |u_{\varepsilon}|^{q}$$

$$= \left(\int_{|x| \le r} + \int_{|x| > r}\right) P_{\varepsilon}^{0}(x)^{q/(q-1)} |u_{\varepsilon}|^{q}$$

$$\leq \int_{|x| \le r} \left(P_{\varepsilon}^{0}(x)\right)^{q/(q-1)} |u_{\varepsilon}|^{q} + m^{q/(q-1)} \int_{|x| > r} |u_{\varepsilon}|^{q}$$

$$= o(1)$$

as $\varepsilon \to 0$. Thus by (4.4) one has $||v_{\varepsilon}||^2 \to 0$, that is, $h_{\varepsilon}(t_{\varepsilon}e) \to \mathscr{J}_m(t_0e)$. Consequently,

$$\int P_{\varepsilon}^{0}(x)G(|w_{\varepsilon}|) \to 0$$

as $\varepsilon \to 0$. This, jointly with (4.2), shows

$$\widetilde{\Phi}_{\varepsilon}(w_{\varepsilon}) = \mathscr{T}_m(w_{\varepsilon}) + o(1) = \mathscr{T}_m(u_{\varepsilon}) + o(1),$$

that is,

$$I_{\varepsilon}(t_{\varepsilon}e) = J_m(t_0e) + o(1)$$

as $\varepsilon \to 0$. Then, since

$$J_m(t_0e) \le \max_{v \in E_e} \mathscr{T}_m(v) = J_m(e) = \gamma_m,$$

we obtain by using (4.1) and (4.3)

$$\gamma_m \le \lim_{\varepsilon \to 0} c_\varepsilon \le \lim_{\varepsilon \to 0} I_\varepsilon(t_\varepsilon e) = J_m(t_0 e) \le \gamma_m,$$

hence, $c_{\varepsilon} \to \gamma_m$.

Proof of Lemma 4.2. Given $\varepsilon > 0$, let $\{u_n\} \subset \mathscr{N}_{\varepsilon}$ be a minimizing sequence: $I_{\varepsilon}(u_n) \to c_{\varepsilon}$. By the Ekeland variational principle we can assume that $\{u_n\}$ is in fact a $(PS)_{c_{\varepsilon}}$ -sequence for I_{ε} on E^+ (see [27, 34]). Then $w_n = u_n + h_{\varepsilon}(u_n)$ is a $(PS)_{c_{\varepsilon}}$ -sequence for $\widetilde{\Phi}_{\varepsilon}$ on E. It is clear that $\{w_n\}$ is bounded, hence is a $(C)_{c_{\varepsilon}}$ -sequence. We can assume without loss of generality that $w_n \rightharpoonup w_{\varepsilon} = w_{\varepsilon}^+ + w_{\varepsilon}^- \in \mathscr{K}_{\varepsilon}$ in E. If $w_{\varepsilon} \neq 0$ then $\widetilde{\Phi}_{\varepsilon}(w_{\varepsilon}) = c_{\varepsilon}$. So we are going to show that $w_{\varepsilon} \neq 0$ for all small $\varepsilon > 0$.

For this end, take $\limsup_{|x|\to\infty} P(x) < \kappa < m$ and define

$$P^{\kappa}(x) = \min\{\kappa, P(x)\}.$$

Consider the functional

$$\widetilde{\Phi}_{\varepsilon}^{\kappa}(u) = \frac{1}{2} \left(\|u^{+}\|^{2} - \|u^{-}\|^{2} - \omega |u|_{2}^{2} \right) - \mathscr{F}_{\varepsilon}(u) - \int P_{\varepsilon}^{\kappa}(x) G(|u|)$$

and as before define correspondingly $h_{\varepsilon}^{\kappa}: E^{+} \to E^{-}$, $I_{\varepsilon}^{\kappa}: E^{+} \to \mathbb{R}$, $\mathscr{N}_{\varepsilon}^{\kappa}$, c_{ε}^{κ} and so on. Following the proof of Lemma 4.1, one finds

$$\lim_{\varepsilon \to 0} c_{\varepsilon}^{\kappa} = \gamma_{\kappa}.$$

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in $E, u_n \to 0$ in L^q_{loc} for $q \in [1,3)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$. Let $t_n > 0$ be such that $t_n u_n \in \mathcal{N}^{\kappa}_{\varepsilon_j}$. Since $u_n \in \mathcal{N}_{\varepsilon}$, it is not difficult to see that $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. By (P_0) , the set $A_{\varepsilon} := \{x \in \mathbb{R}^3 : P_{\varepsilon}(x) > \kappa\}$ is bounded. Remark that $h^{\kappa}_{\varepsilon_j}(t_n u_n) \rightharpoonup 0$ in E and $h^{\kappa}_{\varepsilon_j}(t_n u_n) \to 0$ in L^q_{loc} for $q \in [1,3)$ as $n \to \infty$ (see [2]). Moreover, by virtue of Lemma 3.2, $\widetilde{\Phi}_{\varepsilon_j}(t_n u_n + h^{\kappa}_{\varepsilon_j}(t_n u_n)) \le I_{\varepsilon_j}(u_n)$. We obtain

$$\begin{split} c_{\varepsilon_{j}}^{\kappa} &\leq I_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n}) = \widetilde{\Phi}_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n} + h_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n})) \\ &= \widetilde{\Phi}_{\varepsilon_{j}}(t_{n}u_{n} + h_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n})) + \int \left(P_{\varepsilon_{j}}(x) - P_{\varepsilon_{j}}^{\kappa}(x)\right) G(|t_{n}u_{n} + h_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n})|) \\ &\leq I_{\varepsilon_{j}}(u_{n}) + \int_{A_{\varepsilon_{j}}} \left(P_{\varepsilon_{j}}(x) - P_{\varepsilon_{j}}^{\kappa}(x)\right) G(|t_{n}u_{n} + h_{\varepsilon_{j}}^{\kappa}(t_{n}u_{n})|) \\ &= c_{\varepsilon_{j}} + o(1) \end{split}$$

as $n \to \infty$. Hence $c_{\varepsilon_j}^{\kappa} \leq c_{\varepsilon_j}$. By (4.5), letting $j \to \infty$ yields

$$\gamma_{\kappa} \leq \gamma_m$$

which contradiction with $\gamma_m < \gamma_{\kappa}$.

Proof of Lemma 4.3. Since $\mathscr{L}_{\varepsilon} \subset B_{\Lambda}$ for all small $\varepsilon > 0$, assume by contradiction that, for some $\varepsilon_j \to 0$, $\mathscr{L}_{\varepsilon_j}$ is not compact in E. Then we can choose $u_n^j \in \mathscr{L}_{\varepsilon_j}$ be such that $u_n^j \to 0$ as $n \to \infty$, as done for proving the Lemma 4.2, we gets a contradiction.

Now let $\{u_n\} \subset \mathscr{L}_{\varepsilon}$ such that $u_n \to u$ in E, and recall $H_0 = i\alpha \cdot \nabla - a\beta$, by

$$H_0 u = \omega u + Q_{\varepsilon}(x) A_{\varepsilon, u}^0 u - \sum_{k=1}^3 Q_{\varepsilon}(x) \alpha_k A_{\varepsilon, u}^k u + P_{\varepsilon}(x) g(|u|) u$$

one has

$$|H_{0}(u_{n}-u)|_{2} \leq \omega |u_{n}-u|_{2} + \left|Q_{\varepsilon}(x)\left(A_{\varepsilon,u_{n}}^{0}u_{n}-A_{\varepsilon,u}^{0}u\right)\right|_{2}$$

$$+ \sum_{k=1}^{3} \left|Q_{\varepsilon}(x)\alpha_{k}\left(A_{\varepsilon,u_{n}}^{k}u_{n}-A_{\varepsilon,u}^{k}u\right)\right|_{2}$$

$$+ |P_{\varepsilon}(x)\left(g(|u_{n}|)u_{n}-g(|u|)u\right)|_{2}$$

A standard calculus shows that

$$\left| Q_{\varepsilon} \cdot \alpha_{k} \left(A_{\varepsilon, u_{n}}^{k} u_{n} - A_{\varepsilon, u}^{k} u \right) \right|_{2} \leq |Q|_{\infty} |u_{n}|_{\infty}^{1/6} \left| A_{\varepsilon, u_{n}}^{k} - A_{\varepsilon, u}^{k} \right|_{6} |u_{n}|_{5/2}^{5/6} \\
+ |Q|_{\infty} |u_{n} - u|_{\infty}^{1/6} \left| A_{\varepsilon, u}^{k} \right|_{6} |u_{n} - u|_{5/2}^{5/6}$$

and

$$\begin{aligned} &|P_{\varepsilon} \cdot (g(|u_n|)u_n - g(|u|)u)|_2\\ &\leq &|P|_{\infty} |g(|u_n|) - g(|u|)|_{\infty}^{\frac{1}{2}} \left| (g(|u_n|) - g(|u|))^{1/2} u_n \right|_2\\ &+ &|P|_{\infty} |g(|u|)|_{\infty} |u_n - u|_2 \,. \end{aligned}$$

By Lemma 2.6 and the fact that $u_n \to u$ in $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [2, 3]$, one gets $|H_0(u_n - u)|_2 \to 0$, so $u_n \to u$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$. With Lemma 3.10, $u_n \to u$ in $W^{1,q}$ for all $q \in [2, \infty)$.

Part 2. Concentration

The proof relies on the following lemma. To prove it, it suffices to show that for any sequence $\varepsilon_j \to 0$ the corresponding sequence of solutions $u_j \in \mathscr{L}_{\varepsilon_j}$ converges, up to a shift of x-variable, to a least energy solution of the limit problem (1.4).

Lemma 4.4. There is a maximum point x_{ε} of $|u_{\varepsilon}|$ such that $\operatorname{dist}(y_{\varepsilon}, \mathscr{P}) \to 0$ where $y_{\varepsilon} = \varepsilon x_{\varepsilon}$, and for any such x_{ε} , $v_{\varepsilon}(x) := u_{\varepsilon}(x + x_{\varepsilon})$ converges to a least energy solution of (1.4) in $W^{1,q}$ as $\varepsilon \to 0$ for all $q \geq 2$.

Proof. Let $\varepsilon_j \to 0$, $u_j \in \mathscr{L}_j$, where $\mathscr{L}_j = \mathscr{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument (see [24]) shows that there exist a sequence $\{x_j\} \subset \mathbb{R}^3$ and constant R > 0, $\delta > 0$ such that

$$\liminf_{j \to \infty} \int_{B(x_j, R)} |u_j|^2 \ge \delta.$$

Set

$$v_i = u_i(x + x_i).$$

Then v_j solves, denoting $\hat{Q}_j(x) = Q(\varepsilon_j(x+x_j))$, $\hat{A}_{\varepsilon,u_j}^k(x) = A_{\varepsilon,u_j}^k(x+x_j)$ and $\hat{P}_j(x) = P(\varepsilon_j(x+x_j))$,

$$(4.7) H_0 v_j - \omega v_j - \hat{Q}_j \hat{A}^0_{\varepsilon, u_j} v_j + \sum_{k=1}^3 \hat{Q}_j \alpha_k \hat{A}^k_{\varepsilon, u_j} v_j = \hat{P}_j \cdot g(|v_j|) v_j,$$

with energy

$$S(v_j) := \frac{1}{2} (\|v_j^+\|^2 - \|v_j^-\|^2 - \omega |v_j|_2^2) - \hat{\Gamma}_j(v_j) - \int \hat{P}_j(x) G(|v_j|)$$

$$= \widetilde{\Phi}_j(v_j) = \Phi_j(v_j) = \hat{\Gamma}_j(v_j) + \int \hat{P}_j(x) \widehat{G}(|v_j|)$$

$$= c_{\varepsilon_j}.$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \rightarrow v$ in L^q_{loc} for $q \in [1,3)$.

We now turn to prove that $\{\varepsilon_j x_j\}$ is bounded. Arguing indirectly we assume $\varepsilon_j |x_j| \to \infty$ and get a contradiction.

Without loss of generality assume $P(\varepsilon_j x_j) \to P_{\infty}$. Clearly, $m > P_{\infty}$ by (P_0) . Since for any $\psi \in C_c^{\infty}$

$$0 = \lim_{j \to \infty} \int \left(H_0 v_j - \omega v_j - \hat{Q}_j \hat{A}_{\varepsilon, u_j}^0 v_j + \sum_{k=1}^3 \hat{Q}_j \alpha_k \hat{A}_{\varepsilon, u_j}^k v_j - \hat{P}_j g(|v_j|) v_j \right) \bar{\psi}$$
$$= \lim_{j \to \infty} \int \left(H_0 v - \omega v - P_\infty g(|v|) v \right) \bar{\psi},$$

hence v solves

$$i\alpha \cdot \nabla v - a\beta v - \omega v = P_{\infty}g(|v|)v.$$

Therefore,

$$S_{\infty}(v) := \frac{1}{2} \left(\|v^{+}\|^{2} - \|v^{-}\|^{2} - \omega |v|_{2}^{2} \right) - \int P_{\infty}G(|v|) \ge \gamma_{P_{\infty}}.$$

It follows from $m > P_{\infty}$, by Lemma 3.2, one has $\gamma_m < \gamma_{P_{\infty}}$. Moreover, by Fatou's lemma,

$$\lim_{j \to \infty} \int \hat{P}_j(x) \widehat{G}(|v_j|) \ge \int P_{\infty} \widehat{G}(|v|) = S_{\infty}(v).$$

Consequently, noting that $\hat{\Gamma}_j(v_j) = o(1)$ as $j \to \infty$,

$$\gamma_m < \gamma_{P_\infty} \le S_\infty(v) \le \lim_{j \to \infty} c_{\varepsilon_j} = \gamma_m,$$

a contradiction.

Thus $\{\varepsilon_j x_j\}$ is bounded. Hence, we can assume $y_j = \varepsilon_j x_j \to y_0$. Then v solves

(4.8)
$$i\alpha \cdot \nabla v - a\beta v - \omega v = P(y_0)g(|v|)v.$$

Since $P(y_0) \leq m$, we obtain

$$S_0(v) := \frac{1}{2} (\|v^+\|^2 - \|v^-\|^2 - \omega |v|_2^2) - \int P(y_0) G(|v|) \ge \gamma_{P(y_0)} \ge \gamma_m.$$

Again, by Fatou's lemma, we have

$$S_0(v) = \int P(y_0)\widehat{G}(|v|) \le \lim_{j \to \infty} c_{\varepsilon_j} = \gamma_m.$$

Therefore, $\gamma_{P(y_0)} = \gamma_m$, which implies $y_0 \in \mathscr{P}$ by Lemma 3.2. By virtue of Lemma 3.10 and (3.6) it is clear that one may assume that $x_j \in \mathbb{R}^3$ is a maximum point of $|u_j|$. Moreover, from the above argument we readily see that any sequence of such points satisfies $y_j = \varepsilon_j x_j$, converging to some point in \mathscr{P} as $j \to \infty$.

In order to prove $v_i \to v$ in E, recall that as the argument shows

$$\lim_{j \to \infty} \int \hat{P}_j(x) \widehat{G}(|v_j|) = \int P(y_0) \widehat{G}(|v|).$$

By (g_2) and the exponential decay of v, using the Brezis-Lieb lemma, one obtains $|v_j - v|_{\sigma} \to 0$, then $|v_j^{\pm} - v^{\pm}|_{\sigma} \to 0$ by (2.4). Denote $z_j = v_j - v$. Remark that $\{z_j\}$ is bounded in E and $z_j \to 0$ in L^{σ} , therefore $z_j \to 0$ in L^q for all $q \in (2,3)$. The scalar product of (4.7) with z_j^+ yields

$$\langle v_j^+, z_j^+ \rangle = o(1).$$

Similarly, using the exponential decay of v together with the fact that $z_j^{\pm} \to 0$ in L_{loc}^q for $q \in [1,3)$, it follows from (4.8) that

$$\langle v^+, z_j^+ \rangle = o(1).$$

Thus

$$||z_i^+|| = o(1),$$

and the same arguments show

$$||z_i^-|| = o(1),$$

we then get $v_j \to v$ in E, and the arguments in Lemma 4.3 shows that $v_j \to v$ in $W^{1,q}$ for all $q \ge 2$.

Part 3. Exponential decay

See the following Lemma 4.6. For the later use denote $D = i\alpha \cdot \nabla$ and, for $u \in \mathscr{L}_{\varepsilon}$, write (2.8) as

$$Du = a\beta u + \omega u + Q_{\varepsilon}(x)A_{\varepsilon,u}^{0}u - \sum_{k=1}^{3} Q_{\varepsilon}(x)\alpha_{k}A_{\varepsilon,u}^{k}u + P_{\varepsilon}(x)g(|u|)u.$$

Applying the operator D on both sides and noting that $D^2 = -\Delta$, we get

$$\Delta u = a^{2}u - \left(\omega + Q_{\varepsilon}(x)A_{\varepsilon,u}^{0}(x) + P_{\varepsilon}(x)g(|u|)\right)^{2}u$$

$$- D\left(P_{\varepsilon}(x)g(|u|)\right)u - D\left(Q_{\varepsilon}A_{\varepsilon,u}^{0}\right)u$$

$$+ \sum_{k=1}^{3} \left(Q_{\varepsilon}A_{\varepsilon,u}^{k}\right)^{2}u + \sum_{k=1}^{3} D\left(Q_{\varepsilon}A_{\varepsilon,u}^{k}\right)\alpha_{k}u$$

$$+ 2i\sum_{k=1}^{3} Q_{\varepsilon}A_{\varepsilon,u}^{k}\partial_{k}u.$$

With the fact that

(4.10)
$$\Delta |u|^2 = \bar{u}\Delta u + u\Delta \bar{u} + 2|\nabla u|^2$$

and $\overline{\alpha_k u} \cdot u = \alpha_k u \cdot \overline{u}$ one deduces

$$\begin{split} \Delta \left|u\right|^2 &= 2a^2 \left|u\right|^2 - 2\left(\omega + Q_{\varepsilon}(x)A_{\varepsilon,u}^0(x) + P_{\varepsilon}(x)g(\left|u\right|)\right)^2 \left|u\right|^2 \\ &+ 2\sum_{k=1}^3 \left(Q_{\varepsilon}A_{\varepsilon,u}^k\right)^2 \left|u\right|^2 + 2i\sum_{k=1}^3 \sum_{\substack{1 \leq j \leq 3 \\ j \neq k}} \partial_j \left(Q_{\varepsilon}A_{\varepsilon,u}^k\right) \left(\alpha_j \alpha_k u\right) \cdot \bar{u} \\ &+ 4\Im\sum_{k=1}^3 Q_{\varepsilon}A_{\varepsilon,u}^k \partial_k u \cdot \bar{u} + 2\left|\nabla u\right|^2. \end{split}$$

In addition, setting

$$f_{\varepsilon}^{0}(x) := \max \left\{ \left| Q_{\varepsilon}(x) A_{\varepsilon,u}^{k}(x) \right| : k = 0, 1, 2, 3 \right\},$$

$$f_{\varepsilon}^{1}(x) := \max \left\{ \left| \nabla \left(Q_{\varepsilon}(x) A_{\varepsilon,u}^{k}(x) \right) \right| : k = 0, 1, 2, 3 \right\},$$

one has

$$\left| 2i \sum_{k=1}^{3} \sum_{\substack{1 \le j \le 3 \\ j \ne k}} \partial_{j} \left(Q_{\varepsilon} A_{\varepsilon, u}^{k} \right) \left(\alpha_{j} \alpha_{k} u \right) \cdot \bar{u} \right| \le c_{1} f_{\varepsilon}^{1}(x) \left| u \right|^{2}$$

and

$$\left| 4\Im \sum_{k=1}^{3} Q_{\varepsilon} A_{\varepsilon,u}^{k} \partial_{k} u \cdot \bar{u} \right| \leq c_{2} f_{\varepsilon}^{0}(x) \left(\left| \nabla u \right|^{2} + \left| u \right|^{2} \right).$$

Hence

$$\Delta |u|^{2} \ge \left(2a^{2} - 2\left(\omega + Q_{\varepsilon}A_{\varepsilon,u}^{0} + P_{\varepsilon}(x)g(|u|)\right)^{2} + 2\sum_{k=1}^{3} \left(Q_{\varepsilon}A_{\varepsilon,u}^{k}\right)^{2}\right) |u|^{2} - c_{1}f_{\varepsilon}^{1}(x) |u|^{2} - c_{2}f_{\varepsilon}^{0}(x) \left(|\nabla u|^{2} + |u|^{2}\right) + 2|\nabla u|^{2}.$$

Observe that for $\varepsilon > 0$ sufficiently small, by (3.4), we get

$$c_2 \left| f_{\varepsilon}^0 \right| < 2,$$

hence

$$(4.11) \quad \Delta |u|^{2} \geq 2 \left(a^{2} - \left(\omega + Q_{\varepsilon} A_{\varepsilon, u}^{0} + P_{\varepsilon} \cdot g(|u|) \right)^{2} + \sum_{k=1}^{3} \left(Q_{\varepsilon} A_{\varepsilon, u}^{k} \right)^{2} \right) |u|^{2} - c_{1} f_{\varepsilon}^{1}(x) |u|^{2} - c_{2} f_{\varepsilon}^{0}(x) |u|^{2}.$$

This together with the regularity results for u implies there is M>0 satisfying

$$\Delta |u|^2 \ge -M |u|^2.$$

By the sub-solution estimate [21, 30], one has

$$|u(x)| \le C_0 \left(\int_{B_1(x)} |u(y)|^2 \, dy \right)^{1/2}$$

with C_0 independent of x and $u \in \mathscr{L}_{\varepsilon}$, $\varepsilon > 0$ small.

Lemma 4.5. Let v_{ε} and $\hat{A}^k_{\varepsilon,u_{\varepsilon}}$ for k=0,1,2,3 be given in the proof of Lemma 4.4. Then $|v_{\varepsilon}(x)| \to 0$ and $|\hat{A}^k_{\varepsilon,u_{\varepsilon}}(x)| \to 0$ as $|x| \to \infty$ uniformly in $\varepsilon > 0$ small.

Proof. Arguing indirectly, if the conclusion of the lemma is not held, then by (4.12), there exist $\delta > 0$ and $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$ such that

$$\delta \le |v_j(x_j)| \le C_0 \Big(\int_{B_1(x_j)} |v_j|^2 \Big)^{1/2},$$

where $\varepsilon_j \to 0$ and $v_j = v_{\varepsilon_j}$. Since $v_j \to v$ in E, we obtain

$$\delta \le C_0 \left(\int_{B_1(x_j)} |v_j|^2 \right)^{1/2}$$

$$\le C_0 \left(\int |v_j - v|^2 \right)^{1/2} + C_0 \left(\int_{B_1(x_j)} |v|^2 \right)^{1/2} \to 0,$$

a contradiction. Now, jointly with (3.4), one sees also $\left|\hat{A}_{\varepsilon,u_{\varepsilon}}^{k}(x)\right| \to 0$ as $|x| \to \infty$ uniformly in $\varepsilon > 0$ small.

Lemma 4.6. There exist C > 0 such that for all $\varepsilon > 0$ small

$$|u_{\varepsilon}(x)| \le Ce^{-\frac{c_0}{2}|x-x_{\varepsilon}|}$$

where $c_0 = \sqrt{(a^2 - \omega^2)}$.

Proof. The conclusions of Lemma 4.5 with (4.11) allow us to take R>0 sufficiently large such that

$$\Delta |v_{\varepsilon}|^2 \ge (a^2 - \omega^2) |v_{\varepsilon}|^2$$

for all $|x| \geq R$ and $\varepsilon > 0$ small. Let $\Gamma(y) = \Gamma(y,0)$ be a fundamental solution to $-\Delta + (a^2 - \omega^2)$. Using the uniform boundedness, we may choose that $|v_{\varepsilon}(y)|^2 \leq (a^2 - \omega^2) \Gamma(y)$ holds on |y| = R for all $\varepsilon > 0$ small. Let $z_{\varepsilon} = |v_{\varepsilon}|^2 - (a^2 - \omega^2) \Gamma$. Then

$$\Delta z_{\varepsilon} = \Delta |v_{\varepsilon}|^2 - (a^2 - \omega^2) \Delta \Gamma$$

$$\geq (a^2 - \omega^2) (|v_{\varepsilon}|^2 - (a^2 - \omega^2) \Gamma) = (a^2 - \omega^2) z_{\varepsilon}.$$

By the maximum principle we can conclude that $z_{\varepsilon}(y) \leq 0$ on $|y| \geq R$. It is well known that there is C' > 0 such that $\Gamma(y) \leq C' \exp(-c_0 |y|)$ on $|y| \geq 1$, we see that

$$|v_{\varepsilon}(y)|^2 \le C'' e^{-c_0|y|}$$

for all $y \in \mathbb{R}^3$ and all $\varepsilon > 0$ small, that is

$$|u_{\varepsilon}(x)| \le Ce^{-\frac{c_0}{2}|x-x_{\varepsilon}|}$$

as claimed. \Box

Now, with the above arguments, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Going back to system (1.3) with the variable substitution: $x \mapsto x/\varepsilon$, Lemma 4.2 jointly with Lemma 3.10, shows that, for all $\varepsilon > 0$ small, Eq.(1.3) has at least one least energy solution $w_{\varepsilon} \in W^{1,q}$ for all $q \geq 2$. In addition, if $P, Q \in C^{1,1}(\mathbb{R}^3)$, with (4.9) and the elliptic regularity (see [21]) one obtains a classical solution, that is, the conclusion (i) of Theorem 1.1. And Lemma 4.3 is nothing but the conclusion (ii). Finally, the conclusion (iii) and (iv) follow from Lemma 4.4 and Lemma 4.6 respectively.

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