Carleman estimate and unique continuation for a structured population model

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Abstract

We consider a time-dependent structured population model equation and establish a Carleman estimate. We apply the Carleman estimate to prove the unique continution which means that Cauchy data on any lateral boundary determine the solution uniquely.

1 Introduction

Structured population models describe the change of distribution of individuals in a population. In these models, individuals are described by using several parameters– for example, age, size and so on– and a population density is considered as a function of not only time and spatial position but these individual parameters. In this meaning, we can say that structured population models describe "the detail structure of population." These models originated in Sharpe and Lotka [9] and McKendrik [8] and have been widely studied in the mathematical biology.

In this paper, we consider one of structured population models stated in Webb [11] in which age and size are considered as individual parameters. The model is described as follows: Let $\Omega \subset \mathbb{R}^n$ be an open set which represents an inhabited area and a_1, τ_1, τ_2, T

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be positive real constants. Henceforth $g > 0$ on $[\tau_1, \tau_2]$ and $g \in C^1[\tau_1, \tau_2]$, and we set

$$Kv(x,t,a,\tau) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j v(x,t,a,\tau) - \sum_{k=1}^{n} b_k(x,t,a,\tau)\partial_k v - c(x,t,a,\tau)v, \quad x \in \Omega, \, 0 < t < T$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $1 \leq i, j \leq n$, $b_k, c \in L^\infty(\Omega \times (0,T) \times (0,a_1) \times (\tau_1, \tau_2))$, and there exists a constant $\sigma_1 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sigma_1 \sum_{i=1}^{n} \xi_i^2$$

for $x \in \overline{\Omega}$ and $\xi_1, ..., \xi_n \in \mathbb{R}$.

Then our model equation is

$$\partial_t u(x,t,a,\tau) + \partial_a u(x,t,a,\tau) + \partial_\tau(g(\tau)u(x,t,a,\tau)) = Ku(x,t,a,\tau),$$

$$(x,t,a,\tau) \in \Omega \times (0,T) \times (0,a_1) \times (\tau_1, \tau_2), \tag{1.1}$$

with initial and boundary conditions

$$u(x,t,0,\tau) = \int_0^{a_1} \int_{\tau_1}^{\tau_2} \beta(x,a,\tau,\widetilde{\tau})\, u(x,t,a,\widetilde{\tau})d\widetilde{\tau}da,$$

$$(x,t,\tau) \in (0,T) \times \Omega \times (\tau_1, \tau_2), \tag{1.2}$$

$$u(x,t,a,\tau_1) = 0, \quad (x,t,a) \in \Omega \times (0,T) \times (0,a_1), \tag{1.3}$$

$$u(x,0,a,\tau) = p(x,a,\tau), \quad (x,a,\tau) \in \Omega \times (0,a_1) \times (\tau_1, \tau_2), \tag{1.4}$$

and

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega \times (0,T) \times (0,a_1) \times (\tau_1, \tau_2). \tag{1.5}$$

We can interpret equation (1.1) as follows. The variable a is the age of individual and τ the size, $u(x,t,a,\tau)$ can be interpreted as the population density at time t, position x, age a and size τ. Moreover

- $\partial_a u(x,t,a,\tau)$ represents *aging effect*. The coefficient is always exactly 1 because age increases exactly 1 per a year.

- $\partial_\tau(g(\tau)u(t,x,a,\tau))$ represents *growth effect* with $g(\tau)$ a *growth modulus*, that is, $\int_\tau^{\tau'} 1/g(\sigma)d\sigma$ is a spending time to grow the individual from size τ to size τ').

- The ellipic part Ku represents diffusion and taxis. In particular, $c(x,t,a,\tau)$ denotes a linearized mortality rate.

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- The condition (1.2) represents birth with birth rate β.

For details, see Webb [11] which also proves the existence of the solution of the system (1.1) - (1.5) by the semigroup theory.

In this paper, we discuss

Unique continuation. Let $\Gamma \subset \partial\Omega$ be an arbitrary subboundary. Then determine u in $\Omega \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2)$ by lateral Cauchy data $u, \nabla u$ on $\Gamma \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2)$.

For the unique continuation for elliptic, parabolic and hyperbolic equations, there are many works and see for example Bellassoued and Yamamoto [1], Hörmander [3], Isakov [6], Lavrent'ev, Romanov and Shishat·skiĭ[7] and the references therein. Here we do not intend any comprehensive lists of references. However as for the unique continuation for equation (1.1), to the authors' best knowledge, there are no results published.

We state our main result:

Theorem 1 (unique continuation). Let $\Gamma \subset \partial\Omega$ be an arbitrary subboundary. Let $u \in H^1(\Omega \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2))$ satisfy $\partial_i \partial_j u \in L^2(\Omega \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2))$ and (1.1). Then $u = |\nabla u| = 0$ on $\Gamma \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2)$ yields $u = 0$ in $\Omega \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2)$.

The proof is based on a Carleman estimate for (1.1), which may be interesting itself. In the case of $g \equiv 0$, we refer to Traore [10], which proves a Carleman estimate with a weight function in the form of $\exp\left(2s \frac{V(x)}{at(T-t)}\right)$ with some function $V(x)$. The weight function in [10] is inspired by [2] and [4], and [10] discusses the controllability, but it it very difficult to derive the unique continuation by the Carleman estimate in [10].

The paper is composed of three sections. In Section 2, we prove the key Carleman estimate for (1.1) and in Section 3, the proof of Theorem 1 is completed.

2   Carleman estimate

We set
$$L_0 u := \partial_t u + \partial_a u + \partial_\tau(g(\tau)u).$$

Let $D \subset \Omega \times (0,T) \times (0,a_1) \times (\tau_1, \tau_2)$ be a subdomain.

Lemma 1. Let $d \in C^2(\overline{\Omega})$ satisfy $|\nabla d(x)| \neq 0$ on $\overline{\Omega}$. We fix $t_0 \in (0,T)$, $a_0 \in (0,a_1)$ and $\tau_0 \in (\tau_1, \tau_2)$ arbitrarily and set

$$\psi(x,t,a,\tau) = d(x) - \beta(|t - t_0|^2 + |a - a_0|^2 + |\tau - \tau_0|^2)$$

and

$$\varphi(x,t,a,\tau) = e^{\lambda \psi(x,t,a,\tau)}.$$

Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exists a constant $C = C(s_0, \lambda_0) > 0$ such that

$$\int_D \left\{ \frac{1}{s\varphi}|L_0 u|^2 + s\lambda^2 \varphi |\nabla u|^2 + s^3 \lambda^4 \varphi^3 u^2 \right\} e^{2s\varphi} dx dt da d\tau \leq C \int_D |(L_0 - K)u|^2 e^{2s\varphi} dx dt da d\tau \tag{2.1}$$

for all $s > s_0$ and all $u \in H^1(D)$ satisfying $\partial_i \partial_j u \in L^2(D)$ and $\operatorname{supp} u \in D$.

The constant $C > 0$ in (2.1) depends continuously on

$$\max_{1 \leq i,j \leq n} \|a_{ij}\|_{C^1(\overline{D})}, \quad \|b_i\|_{L^\infty(D)}, \quad \|c\|_{L^\infty(D)}.$$

Remark. We further assume that for each $\widetilde{t}, \widetilde{a}, \widetilde{\tau}$, the cross section $\{x; (x, \widetilde{t}, \widetilde{a}, \widetilde{\tau}) \in D\}$ is composed of a finite number of smooth surfaces. Then similarly to Theorem 3.1 in Yamamoto [12], we can improve (2.1) as

$$\int_D \left\{ \frac{1}{s\varphi} \left(|L_0 u|^2 + \sum_{i,j=1}^{n} |\partial_i \partial_j u|^2 \right) + s\lambda^2 \varphi |\nabla u|^2 + s^3 \lambda^4 \varphi^3 u^2 \right\} e^{2s\varphi} dx dt da d\tau$$
$$\leq C \int_D |(L_0 - K)u|^2 e^{2s\varphi} dx dt da d\tau.$$

Proof of Lemma 1. We set

$$\widetilde{L}_0 u := \partial_t u + \partial_a u + g(\tau)\partial_\tau u.$$

Then $L_0 u = \widetilde{L}_0 u + g'(\tau)u$. Thanks to the large parameters s and λ, it is sufficient to prove the Carleman estimate for

$$Lu := \widetilde{L}_0 u - \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j u.$$

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By a usual density argument (i.e., the approximation of any u satisfying the condition in Lemma 1 by a sequence $u_m \in C_0^\infty(D)$), it suffices to prove the Carleman estimate for $u \in C_0^\infty(D)$.

We further set

$$\sigma(x) = \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)(x)(\partial_j d)(x), \quad x \in \overline{\Omega}$$

and

$$w(x,t) = e^{s\varphi(x,t)}u(x,t)$$

and

$$Pw(x,t) = e^{s\varphi}L(e^{-s\varphi}w) = e^{s\varphi}Lu. \tag{2.2}$$

The proof is very similar to the proof of Theorem 3.1 in Yamamoto [12], which is a Carleman estimate for a parabolic equation. More precisely,

(1) the decomposition of P into the part P_1 and P_2, where P_1 is composed of second-order and zeroth-order terms in x, and P_2 is composed of first-order terms in t and first-order terms in x.

(2) Estimation of $\int_D(|P_2 w|^2 + 2(P_1 w)(P_2 w))dxdtdad\tau$ from the below.

(3) Another estimate for

$$\int_D Pw \times [\text{the term } u \text{ with second highest order of } s, \lambda, \varphi \text{ among } Pw].$$

Direct calculation of (2.2) gives

$$Pw = \widetilde{L}_0 w - \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)\partial_j w$$

$$-s^2\lambda^2\varphi^2\sigma w + s\lambda^2\varphi\sigma w + s\lambda\varphi w \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d - s\lambda\varphi w(\widetilde{L}_0\psi) \text{ in } D. \tag{2.3}$$

Here we note that we have specified all the dependency of coefficients on s, λ and φ. We set

$$A_1 = s\lambda^2\varphi\sigma + s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d - s\lambda\varphi(\widetilde{L}_0\psi) =: s\lambda^2\varphi a_1(x,t,a,\tau;s,\lambda).$$

Then

$$Pw = \widetilde{L}_0 w - \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)\partial_j w$$

$$-s^2\lambda^2\varphi^2\sigma w + A_1 w = fe^{s\varphi} \quad \text{in } D.$$

We note that a_1 depends on s and λ but

$$|a_1(x,t,a,\tau;s,\lambda)| \le C$$

for $(x,t,a,\tau) \in \overline{D}$ and all sufficiently large $\lambda > 0$ and $s > 0$. Here and henceforth by C, C_1, etc., we denote generic constants which are independent of s, λ and φ but may change line by line.

Then taking into consideration the orders of (s,λ,φ), we divide Pw as follows:

$$P_1 w = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j w - s^2\lambda^2\varphi^2 w\sigma(x,t) + A_1 w \tag{2.4}$$

and

$$P_2 w = \widetilde{L}_0 w + 2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)\partial_j w. \tag{2.5}$$

By $\|fe^{s\varphi}\|^2_{L^2(D)} = \|P_1 w + P_2 w\|^2_{L^2(D)}$, we have

$$2\int_D (P_1 w)(P_2 w)dxdtdad\tau + \|P_2 w\|^2_{L^2(D)} \le \int_D f^2 e^{2s\varphi}dxdtdad\tau. \tag{2.6}$$

We estimate:

$$\int_D (P_1 w)(P_2 w)dxdtdad\tau = -\sum_{i,j=1}^{n}\int_D a_{ij}(\partial_i\partial_j w)(\widetilde{L}_0 w)dxdtdad\tau$$

$$-\sum_{i,j=1}^{n}\int_D a_{ij}(\partial_i\partial_j w)2s\lambda\varphi \sum_{k,\ell=1}^{n} a_{k\ell}(\partial_k d)(\partial_\ell w)dxdtdad\tau$$

$$-\int_D s^2\lambda^2\varphi^2\sigma w(\widetilde{L}_0 w)dxdtdad\tau - \int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)dxdtdad\tau$$

$$+\int_D (A_1 w)(\widetilde{L}_0 w)dxdtdad\tau + \int_D (A_1 w)2s\lambda\varphi \sum_{i,j=1}^{n} a_{ij}(\partial_i d)(\partial_j w)dxdtdad\tau$$

$$=: \sum_{k=1}^{6} J_k. \tag{2.7}$$

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Now, applying the integration by parts, $a_{ij} = a_{ji}$ and $u \in C_0^\infty(D)$ and assuming that $\lambda > 1$ and $s > 1$ are sufficiently large, we reduce all the derivatives of w to $w, \partial_i w, \widetilde{L}_0 w$. We note that

$$\int_D u(\widetilde{L}_0 v) dx dt da d\tau = - \int_D (L_0 u) v dx dt da d\tau, \quad u, v \in C_0^1(D).$$

We continue the estimation of J_k, $1 \le k \le 6$.

$$|J_1| = \left| - \sum_{i,j=1}^n \int_D a_{ij} (\partial_i \partial_j w)(\widetilde{L}_0 w) dx dt da d\tau \right|$$

$$= \left| \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_j w)(\widetilde{L}_0 w) dx dt da d\tau + \sum_{i,j=1}^n \int_D a_{ij}(\partial_j w)\partial_i(\widetilde{L}_0 w) dx dt da d\tau \right|$$

$$= \left| \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_j w)(\widetilde{L}_0 w) dx dt da d\tau \right.$$

$$+ \left(\sum_{i>j} \int_D a_{ij}((\partial_j w)\partial_i(\widetilde{L}_0 w) + (\partial_i w)\partial_j(\widetilde{L}_0 w)) dx dt da d\tau \right.$$

$$+ \left. \left. \int_D \sum_{i=1}^n a_{ii}(\partial_i w)\partial_i(\widetilde{L}_0 w) dx dt da d\tau \right) \right|$$

$$\le C \int_D |\nabla w||\widetilde{L}_0 w| dx dt da d\tau. \tag{2.8}$$

Here we used

$$\widetilde{L}_0((\partial_i w)\partial_j w) = (\widetilde{L}_0(\partial_i w))\partial_j w + (\partial_i w)\widetilde{L}_0(\partial_j w),$$

and

$$\left(\sum_{i>j} \int_D a_{ij}((\partial_j w)(\partial_i \widetilde{L}_0 w) + (\partial_i w)(\partial_j \widetilde{L}_0 w)) dx dt da d\tau \right.$$

$$+ \left. \int_D \sum_{i=1}^n a_{ii}(\partial_i w)(\partial_i \widetilde{L}_0 w) dx dt da d\tau \right) = \frac{1}{2} \sum_{i,j=1}^n \int_D a_{ij} \widetilde{L}_0((\partial_j w)(\partial_i w)) dx dt da d\tau$$

$$= -\frac{1}{2} \sum_{i,j=1}^n \int_D L_0(a_{ij})(\partial_j w)(\partial_i w) dx dt da d\tau = 0,$$

because a_{ij} are independent of t, a, τ.

Next

$$J_2 = - \sum_{i,j=1}^n \sum_{k,\ell=1}^n \int_D 2s\lambda \varphi a_{ij} a_{k\ell} (\partial_k d)(\partial_\ell w)(\partial_i \partial_j w) dx dt da d\tau$$

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$$= 2s\lambda \int_D \sum_{i,j=1}^n \sum_{k,\ell=1}^n \lambda(\partial_i d)\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_\ell w)(\partial_j w)dxdtdad\tau$$

$$+ 2s\lambda \int_D \sum_{i,j=1}^n \sum_{k,\ell=1}^n \varphi\partial_i(a_{ij}a_{k\ell}\partial_k d)(\partial_\ell w)(\partial_i w)dxdtdad\tau$$

$$+ 2s\lambda \int_D \sum_{i,j=1}^n \sum_{k,\ell=1}^n \varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_i\partial_\ell w)(\partial_j w)dxdtdad\tau.$$

We have

$$[\text{first term}] = 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) \right|^2 dxdtdad\tau \geq 0,$$

and similarly to J_1, we can estimate

$$[\text{third term}] = s\lambda \sum_{i,j=1}^n \sum_{k,\ell=1}^n \int_D \varphi a_{ij}a_{k\ell}(\partial_k d)\partial_\ell((\partial_i w)(\partial_j w))$$

$$= -s\lambda^2 \int_D \varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau - s\lambda \int_D \varphi \sum_{i,j=1}^n \sum_{k,\ell=1}^n \partial_\ell(a_{ij}a_{k\ell}\partial_k d)(\partial_i w)(\partial_j w)dxdtdad\tau.$$

Hence

$$J_2 \geq -\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau$$

$$-C \int_D s\lambda\varphi|\nabla w|^2 dxdtdad\tau + 2s\lambda^2 \int_D \varphi \left| \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) \right|^2 dxdtdad\tau$$

$$\geq -\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau - C \int_D s\lambda\varphi|\nabla w|^2 dxdtdad\tau. \qquad (2.9)$$

$$|J_3| = \left| -\int_D \frac{1}{2}s^2\lambda^2\varphi^2\sigma\widetilde{L}_0(w^2)dxdtdad\tau \right| = \frac{1}{2}\left| \int_D L_0(s^2\lambda^2\varphi^2\sigma)w^2 dxdtdad\tau \right|$$

$$\leq C\int_D s^2\lambda^3\varphi^2 w^2 dxdtdad\tau. \qquad (2.10)$$

$$J_4 = -\int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dxdtdad\tau$$

$$= -\int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^n \sigma a_{ij}(\partial_i d)\partial_j(w^2)dxdtdad\tau$$

$$= \int_D s^3\lambda^3 \sum_{i,j=1}^n 3\varphi^2\{\lambda(\partial_j d)\varphi\}\sigma a_{ij}(\partial_i d)w^2 dxdtdad\tau$$

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$$+ \int_D s^3 \lambda^3 \varphi^3 \sum_{i,j=1}^n \partial_j(\sigma a_{ij} \partial_i d) w^2 dx dt da d\tau$$

$$\geq \int_D 3s^3 \lambda^4 \varphi^3 \sigma^2 w^2 dx dt da d\tau - C \int_D s^3 \lambda^3 \varphi^3 w^2 dx dt da d\tau. \qquad (2.11)$$

$$|J_5| = \left| \int_D (A_1 w)(\widetilde{L}_0 w) dx dt da d\tau \right| = \left| \int_D s\lambda^2 \varphi a_1 w (\widetilde{L}_0 w) dx dt da d\tau \right|$$

$$= \frac{1}{2} \left| \int_D s\lambda^2 \varphi a_1 \widetilde{L}_0(w^2) dx dt da d\tau \right| = \frac{1}{2} \left| \int_D s\lambda^2 L_0(\varphi a_1) w^2 dx dt da d\tau \right|$$

$$\leq C \int_D s\lambda^3 \varphi w^2 dx dt da d\tau. \qquad (2.12)$$

$$|J_6| = \left| \int_D s\lambda^2 \varphi a_1 \times 2s\lambda \varphi w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) dx dt da d\tau \right|$$

$$= \left| \int_D 2a_1 s^2 \lambda^3 \varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d) w (\partial_j w) dx dt da d\tau \right|$$

$$= \left| \int_D a_1 s^2 \lambda^3 \varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d) \partial_j(w^2) dx dt da d\tau \right|$$

$$= \left| - \int_D \sum_{i,j=1}^n \partial_j(a_1 s^2 \lambda^3 \varphi^2 a_{ij}(\partial_i d)) w^2 dx dt da d\tau \right|$$

$$\leq C \int_D s^2 \lambda^4 \varphi^2 w^2 dx dt da d\tau. \qquad (2.13)$$

Hence, choosing $s > 0$ and $\lambda > 0$ large, by (2.7) - (2.13) we obtain

$$\int_D (P_1 w)(P_2 w) dx dt da d\tau \geq 3 \int_D s^3 \lambda^4 \varphi^3 \sigma^2 w^2 dx dt da d\tau - \int_D s\lambda^2 \varphi \sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dx dt da d\tau$$

$$- C \int_D s\lambda \varphi |\nabla w|^2 dx dt da d\tau - C \int_D (s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^2) w^2 dx dt da d\tau - C \int_D |\nabla w||\widetilde{L}_0 w| dx dt da d\tau.$$

Consequently

$$3 \int_D s^3 \lambda^4 \varphi^3 \sigma^2 w^2 dx dt da d\tau - \int_D s\lambda^2 \varphi \sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dx dt da d\tau$$

$$\leq \int_D (P_1 w)(P_2 w) dx dt da d\tau + C \int_D s\lambda \varphi |\nabla w|^2 dx dt da d\tau$$

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$$+C\int_D(s^3\lambda^3\varphi^3+s^2\lambda^4\varphi^2)w^2dxdtdad\tau+C\int_D|\nabla w||\widetilde{L}_0w|dxdtdad\tau. \tag{2.14}$$

Moreover for all large $s>0$, by the definition (2.5) of P_2 and an inequality: $|\alpha+\beta|^2\geq\frac{1}{2}|\alpha|^2-|\beta|^2$, we obtain

$$\int_D|P_2w|^2dxdtdad\tau\geq\int_D\frac{1}{s\varphi}|P_2w|^2dxdtdad\tau$$

$$=\int_D\frac{1}{s\varphi}\left|\widetilde{L}_0w+2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_id)(\partial_jw)\right|^2dxdtdad\tau$$

$$\geq\frac{1}{2}\int_D\frac{1}{s\varphi}|\widetilde{L}_0w|^2dxdtdad\tau-C\int_D s\lambda^2\varphi\left|\sum_{i,j=1}^n a_{ij}(\partial_id)(\partial_jw)\right|^2dxdtdad\tau,$$

that is,

$$\varepsilon\int_D\frac{1}{s\varphi}|\widetilde{L}_0w|^2dxdtdad\tau\leq C\varepsilon\int_D|P_2w|^2dxdtdad\tau+C\varepsilon\int_D s\lambda^2\varphi|\nabla w|^2dxdtdad\tau$$

for any $\varepsilon>0$. Hence by (2.14) and (2.6), we have

$$3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2dxdtdad\tau-\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_iw)(\partial_jw)dxdtdad\tau$$

$$+\varepsilon\int_D\frac{1}{s\varphi}|\widetilde{L}_0w|^2dxdtdad\tau$$

$$\leq C\int_D f^2e^{2s\varphi}dxdtdad\tau+C\int_D s\lambda\varphi|\nabla w|^2dxdtdad\tau+C\varepsilon\int_D s\lambda^2\varphi|\nabla w|^2dxdtdad\tau$$

$$+C\int_D(s^3\lambda^3\varphi^3+s^2\lambda^4\varphi^2)w^2dxdtdad\tau+C\int_D|\nabla w||\widetilde{L}_0w|dxdtdad\tau.$$

Now we note that the factor with the maximal order in s,λ,φ of w^2 is $s^3\lambda^4\varphi^3\sigma^2$, the maximal factor of $|\nabla w|^2$ is $s\lambda^2\varphi\sigma$, and the maximal order of $|\widetilde{L}_0w|^2$ is $\frac{1}{s\varphi}$. For example, since we can choose s,λ large, the term $(s^3\lambda^3\varphi^3+s^2\lambda^4\varphi^2)w^2$ is of lower order.

Here, since the Cauchy-Schwarz inequality implies

$$|\widetilde{L}_0w||\nabla w|=s^{-\frac{1}{2}}\varphi^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}|\widetilde{L}_0w|s^{\frac{1}{2}}\varphi^{\frac{1}{2}}\lambda^{\frac{1}{2}}|\nabla w|$$

$$\leq\frac{1}{2}\frac{1}{s\lambda\varphi}|\widetilde{L}_0w|^2+\frac{1}{2}s\lambda\varphi|\nabla w|^2,$$

we have

$$3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2dxdtdad\tau-\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_iw)(\partial_jw)dxdtdad\tau$$

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$$+ \left(\varepsilon - \frac{C}{\lambda} \right) \int_D \frac{1}{s\varphi} |\widetilde{L}_0 w|^2 dx dt da d\tau$$

$$\leq C \int_D f^2 e^{2s\varphi} dx dt da d\tau + C \int_D s\lambda\varphi |\nabla w|^2 dx dt da d\tau + C\varepsilon \int_D s\lambda^2\varphi |\nabla w|^2 dx dt da d\tau$$

$$+ C \int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2) w^2 dx dt da d\tau. \tag{2.15}$$

The first and the second terms on the left-hand side of (2.15) have different signs and so we need another estimate. Thus we will execute another estimation for

$$\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w) dx dt da d\tau$$

by means of

$$\int_D (P_1 w + P_2 w) \times (s\lambda^2\varphi\sigma w) dx dt da d\tau.$$

Here we have chosen the factor $s\lambda^2\varphi\sigma w$ for obtaining the term of $|\nabla w|^2$ with desirable (s, λ, φ)-factor $s\lambda^2\varphi$. That is, multiplying

$$\widetilde{L}_0 w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) - \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w = fe^{s\varphi}$$

with $s\lambda^2\varphi\sigma w$, we have

$$\int_D (\widetilde{L}_0 w)(s\lambda^2\varphi\sigma w) dx dt da d\tau + \int_D 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) s\lambda^2\varphi\sigma w dx dt da d\tau$$

$$- \int_D \left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w \right) s\lambda^2\varphi\sigma w dx dt da d\tau - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dx dt da d\tau$$

$$+ \int_D (A_1 w)(s\lambda^2\varphi\sigma w) dx dt da d\tau$$

$$=: \sum_{k=1}^5 I_k = \int_D fe^{s\varphi}s\lambda^2\varphi\sigma w dx dt da d\tau. \tag{2.16}$$

Now, in terms of the integration by parts and $w \in C_0^2(D)$, noting that $|\widetilde{L}_0\varphi| = |\lambda(\widetilde{L}_0\psi)\varphi| \leq C\lambda\varphi$ and $\partial_i\varphi = \lambda(\partial_i d)\varphi$, etc., we estimate the terms.

$$|I_1| = \left| \int_D \frac{1}{2}s\lambda^2\varphi\sigma\widetilde{L}_0(w^2) dx dt da d\tau \right| \leq C \int_D s\lambda^3\varphi w^2 dx dt da d\tau. \tag{2.17}$$

$$|I_2| = \left| \int_D s^2\lambda^3\varphi^2\sigma \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2) dx dt da d\tau \right|$$

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$$= \left| -\int_D s^2\lambda^3 \sum_{i,j=1}^n \partial_j(\varphi^2\sigma a_{ij}\partial_i d)w^2 dxdtdad\tau \right|$$

$$\leq C\int_D s^2\lambda^4\varphi^2 w^2 dxdtdad\tau. \tag{2.18}$$

$$I_3 = -\int_D s\lambda^2 \sum_{i,j=1}^n \varphi\sigma a_{ij}w(\partial_i\partial_j w)dxdtdad\tau$$

$$= \int_D s\lambda^2 \sum_{i,j=1}^n \varphi\sigma a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau + \int_D s\lambda^2 \sum_{i,j=1}^n \partial_i(\varphi\sigma a_{ij})w(\partial_j w)dxdtdad\tau$$

$$\geq \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau - C\int_D s\lambda^3\varphi|\nabla w||w|dxdtdad\tau. \tag{2.19}$$

$$I_4 = -\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdtdad\tau. \tag{2.20}$$

$$|I_5| \leq C\left|\int_D s\lambda^2\varphi \times s\lambda^2\varphi\sigma w^2 dxdtdad\tau\right| \leq C\int_D s^2\lambda^4\varphi^2 w^2 dxdtdad\tau. \tag{2.21}$$

Hence, choosing $s > 0$ and $\lambda > 0$ large, by (2.16) - (2.21) we obtain

$$\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dxdtdad\tau - \int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dxdtdad\tau$$

$$\leq C\int_D |fe^{s\varphi}s\lambda^2\varphi\sigma w|dxdtdad\tau + C\int_D s^2\lambda^4\varphi^2 w^2 dxdtdad\tau + C\int_D s\lambda^3\varphi|\nabla w||w|dxdtdad\tau$$

$$\leq C\int_D f^2 e^{2s\varphi}dxdtdad\tau + C\int_D s^2\lambda^4\varphi^2 w^2 dxdtdad\tau + C\int_D \lambda^2|\nabla w|^2 dxdtdad\tau. \tag{2.22}$$

At the last inequality, we argue as follows: By

$$s\lambda^3\varphi|\nabla w||w| = (s\lambda^2\varphi|w|)(\lambda|\nabla w|) \leq \frac{1}{2}s^2\lambda^4\varphi^2 w^2 + \frac{1}{2}\lambda^2|\nabla w|^2,$$

we have

$$\int_D s\lambda^3\varphi|\nabla w||w|dxdtdad\tau \leq \frac{1}{2}\int_D (s^2\lambda^4\varphi^2 w^2 + \lambda^2|\nabla w|^2)dxdtdad\tau.$$

Furthermore

$$|fe^{s\varphi}s\lambda^2\varphi\sigma w|$$

$$\leq \frac{1}{2}f^2 e^{2s\varphi} + \frac{1}{2}s^2\lambda^4\varphi^2\sigma^2 w^2 \leq \frac{1}{2}f^2 e^{2s\varphi} + Cs^2\lambda^4\varphi^2 w^2.$$

Finally we consider $2 \times (2.22) + (2.15)$. Using the uniform ellipticity and $\sigma_0 \equiv \inf_{x \in \Omega} \sigma(x) > 0$, we obtain

$$\int_D s^3\lambda^4\varphi^3\sigma_0^2 w^2 dxdtdad\tau + (\sigma_0\sigma_1 - C\varepsilon)\int_D s\lambda^2\varphi|\nabla w|^2 dxdtdad\tau$$

$$+ \left(\varepsilon - \frac{C}{\lambda}\right)\int_D \frac{1}{s\varphi}|\widetilde{L}_0 w|^2 dxdtdad\tau$$

$$\leq C\int_D f^2 e^{2s\varphi} dxdtdad\tau$$

$$+ C\int_D (s\lambda\varphi + \lambda^2)|\nabla w|^2 dxdtdad\tau + C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 dxdtdad\tau. \quad (2.23)$$

Therefore, first choosing $\varepsilon > 0$ sufficiently small such that $\sigma_0\sigma_1 - C\varepsilon > 0$ and then taking $\lambda > 0$ sufficiently large such that $\varepsilon - \frac{C}{\lambda} > 0$, we can absorb the second and the third terms on the right-hand side of (2.23) into the left-hand side and we obtain

$$\int_D s^3\lambda^4\varphi^3 w^2 dxdtdad\tau + \int_D s\lambda^2\varphi|\nabla w|^2 dxdtdad\tau + \int_D \frac{1}{s\varphi}|\widetilde{L}_0 w|^2 dxdtdad\tau$$

$$\leq C\int_D f^2 e^{2s\varphi} dxdtdad\tau. \quad (2.24)$$

Noting $w = ue^{s\varphi}$, we have

$$\int_D \left(\frac{1}{s\varphi}|\widetilde{L}_0 u|^2 + s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3 u^2\right)e^{2s\varphi} dxdtdad\tau$$

$$\leq C\int_D f^2 e^{2s\varphi} dxdtdad\tau.$$

Thus the proof of Lemma 1 is completed.

3 Proof of Theorem 1

We need a special weight function. The existence of such a function is proved in Fursikov and Imanuvilov [2], Imanuvilov [4], Imanuvilov, Puel and Yamamoto [5].

Lemma 2. Let ω be an arbitrarily fixed sub-domain of Ω. Then there exists a fucntion $d \in C^2(\overline{\Omega})$ such that

$$d(x) > 0 \quad x \in \Omega, \quad d|_{\partial\Omega} = 0, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega \setminus \omega}.$$

Example: Let $\Omega = \{x; |x| < 1\}$ and let $0 \in \omega$. Then $d(x) = 1 - |x|^2$ satisfies the conditions in Lemma 2.

Henceforth we set
$$B(\mathbf{p}, r) := \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x} - \mathbf{p}| < r\}$$
with $\mathbf{p} \in \mathbb{R}^3$ and $r > 0$, and
$$\|u\|_{H^{1,0}(D)} = (\|\nabla u\|_{L^2(D)}^2 + \|u\|_{L^2(D)}^2)^{\frac{1}{2}}.$$

Now we proceed to the proof of Theorem 1, which is similar to Theorem 5.1 in [12].

Let Ω_0 be an arbitrary subdomain of Ω such that $\overline{\Omega}_0 \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ is a non-empty open subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$, According to the geometry of Ω_0 and Γ, we have to choose a suitable weight function φ, that is, $d(x)$. For this, we first choose a bouned domain Ω_1 with smooth boundary such that
$$\Omega \subsetneqq \Omega_1, \quad \Gamma = \overline{\partial\Omega \cap \Omega_1}, \quad \partial\Omega \setminus \Gamma \subset \partial\Omega_1. \tag{3.1}$$

We note that Ω_1 is constructed by taking a union of Ω and a domain $\widetilde{\Omega}$ such that $\partial\widetilde{\Omega} \cap \overline{\Omega} = \Gamma$ and that $\Omega_1 \setminus \overline{\Omega}$ contains some non-empty open set. Choosing $\overline{\omega} \subset \Omega_1 \setminus \overline{\Omega}$, we apply Lemma 2 to obtain $d \in C^2(\overline{\Omega}_1)$ satisfying
$$d(x) > 0, \quad x \in \Omega_1, \quad d(x) = 0, \quad x \in \partial\Omega_1, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}_1 \cap \overline{\Omega}. \tag{3.2}$$

Then, since $\overline{\Omega}_0 \subset \Omega_1$, we can choose a sufficiently large $N > 1$ such that
$$\{x \in \Omega_1; d(x) > \frac{4}{N}\|d\|_{C(\overline{\Omega}_1)}\} \cap \overline{\Omega} \supset \Omega_0. \tag{3.3}$$

Let $\varepsilon > 0$ be an arbitrarily small number. Moreover we choose $\beta > 0$ such that
$$2\beta\varepsilon^2 > \|d\|_{C(\overline{\Omega}_1)} > \beta\varepsilon^2. \tag{3.4}$$

We fix $t_0 \in [\sqrt{2}\varepsilon, T - \sqrt{2}\varepsilon]$, $a_0 \in [\sqrt{2}\varepsilon, a_1 - \sqrt{2}\varepsilon]$ and $\tau_0 \in [\tau_1 + \sqrt{2}\varepsilon, \tau_2 - \sqrt{2}\varepsilon]$ arbitrarily. We set $\mathbf{p} = (t_0, a_0, \tau_0)$, and $\varphi(x, t, a, \tau) = e^{\lambda\psi(x,t,a,\tau)}$ with fixed large parameter $\lambda > 0$ and
$$\psi(x, t, a, \tau) = d(x) - \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2)$$
and $\mu_k = \exp\left(\lambda\left(\frac{k}{N}\|d\|_{C(\overline{\Omega}_1)} - \frac{\beta\varepsilon^2}{N}\right)\right)$, $k = 1, 2, 3, 4$, and
$$D = \{(x, t, a, \tau); x \in \overline{\Omega}, \quad \varphi(x, t, a, \tau) > \mu_1\}.$$

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Then we can verify that

$$\Omega_0 \times B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right) \subset D \subset \overline{\Omega} \times B(\mathbf{p}, \sqrt{2}\varepsilon). \tag{3.5}$$

In fact, let $(x, t, a, \tau) \in \Omega_0 \times B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right)$. Then, by (3.3) we have $x \in \overline{\Omega}$ and $d(x) > \frac{4}{N}\|d\|_{C(\overline{\Omega_1})}$, so that

$$d(x) - \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2) > \frac{4}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N},$$

that is, $\varphi(x, t, a, \tau) > \mu_4$, which implies that $(x, t, a, \tau) \in D$ by the definition. Next let $(x, t, a, \tau) \in D$. Then $d(x) - \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2) > \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N}$. Therefore

$$\|d\|_{C(\overline{\Omega_1})} - \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} + \frac{\beta\varepsilon^2}{N} > \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2).$$

Applying (3.4), we have $2\left(1 - \frac{1}{N}\right)\beta\varepsilon^2 + \frac{\beta\varepsilon^2}{N} > \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2)$, that is, $2\beta\varepsilon^2 > \beta((t - t_0)^2 + (a - a_0)^2 + (\tau - \tau_0)^2)$. The verification of (3.5) is completed.

Next we have

$$\partial D \subset \Sigma_1 \cup \Sigma_2, \quad \text{where } \Sigma_1 \subset \Gamma \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2),$$

$$\Sigma_2 = \{(x, t, a, \tau); \ x \in \Omega, \ \varphi(x, t, a, \tau) = \mu_1\}. \tag{3.6}$$

In fact, let $(x, t, a, \tau) \in \partial D$. Then $x \in \overline{\Omega}$ and $\varphi(x, t, a, \tau) \geq \mu_1$. We separately consider the cases $x \in \Omega$ and $x \in \partial\Omega$. First let $x \in \Omega$. If $\varphi(x, t, a, \tau) > \mu_1$, then (x, t, a, τ) is an interior point of D. This is impossible. Therefore if $x \in \Omega$, then $\varphi(x, t, a, \tau) = \mu_1$ must hold. Next let $x \in \partial\Omega$. Let $x \in \partial\Omega \setminus \Gamma$. Then $x \in \partial\Omega_1$ by the third condition in (3.1), and $d(x) = 0$ by the second condition in (3.2). On the other hand, $\varphi(x, t, a, \tau) \geq \mu_1$ yields that

$$d(x) - \beta((t-t_0)^2 + (a-a_0)^2 + (\tau-\tau_0)^2)) = -\beta((t-t_0)^2 + (a-a_0)^2 + (\tau-\tau_0)^2) \geq \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N},$$

that is, $0 \leq \beta((t-t_0)^2 + (a-a_0)^2 + (\tau-\tau_0)^2) \leq \frac{1}{N}(-\|d\|_{C(\overline{\Omega_1})} + \beta\varepsilon^2)$, which is impossible by (3.4). Therefore $x \in \Gamma$. In terms of (3.5), the verification of (3.6) is completed.

By replacing the coefficient $c(x, t, a, \tau)$ by $c(x, t, a, \tau) - g'(\tau)$, it is sufficient to consider $\widetilde{L}_0 u - Ku = 0$ in place of $L_0 u - Ku = 0$. We apply Lemma 1 in D with suitably fixed $\lambda > 0$. Henceforth $C > 0$ denotes generic constants depending on λ,

but independent of s. For it, we need a cut-off function because we have no data on $\partial D \setminus (\Gamma \times (0, T) \times (0, a_1) \times (\tau_1, \tau_2))$. Let $\chi \in C^\infty(\mathbb{R}^{n+3})$ such that $0 \le \chi \le 1$ and

$$\chi(x, t, a, \tau) = \begin{cases} 1, & \varphi(x, t, a, \tau) > \mu_3, \\ 0, & \varphi(x, t, a, \tau) < \mu_2. \end{cases} \tag{3.7}$$

We set $v = \chi u$, and have

$$Lv = (\widetilde{L}_0 \chi) u - 2 \sum_{i,j=1}^n a_{ij} (\partial_i \chi) \partial_j u$$

$$- \left(\sum_{i,j=1}^n a_{ij} \partial_i \partial_j \chi \right) u - \left(\sum_{i=1}^n b_i \partial_i \chi \right) u \quad \text{in } D.$$

Here we recall that $\widetilde{L}_0 \chi = \partial_t \chi + \partial_a \chi + g(\tau) \partial_\tau \chi$. By (3.6) and (3.7), we see that

$$v = |\nabla_{x,t,a,\tau} v| = 0 \qquad \text{on } \partial D.$$

Hence Lemma 1 yields

$$\int_D s^3 |v|^2 e^{2s\varphi} dx dt da d\tau$$

$$\le C \int_D \left| (\widetilde{L}_0 \chi) u - 2 \sum_{i,j=1}^n a_{ij} (\partial_i \chi) \partial_j u - \left(\sum_{i,j=1}^n a_{ij} \partial_i \partial_j \chi \right) u - \left(\sum_{i=1}^n b_i \partial_i \chi \right) u \right|^2 e^{2s\varphi} dx dt da d\tau \tag{3.8}$$

for all $s \ge s_0$. By (3.7), the second integral on the right-hand side does not vanish only if $\mu_2 \le \varphi(x, t, a, \tau) \le \mu_3$ and so

$$\int_D \left| (\widetilde{L}_0 \chi) u - 2 \sum_{i,j=1}^n a_{ij} (\partial_i \chi) \partial_j u - \left(\sum_{i,j=1}^n a_{ij} \partial_i \partial_j \chi \right) u - \left(\sum_{i=1}^n b_i \partial_i \chi \right) u \right|^2 e^{2s\varphi} dx dt da d\tau$$

$$\le C e^{2s\mu_3} \|u\|_{H^{1,0}(D)}^2.$$

By (3.3) and the definition of D, we can directly verify that if $(x, t, a, \tau) \in \Omega_0 \times B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right)$, then $\varphi(x, t, a, \tau) > \mu_4$. Therefore, noting (3.5) and (3.7), we see that

$$[\text{the left-hand side of (3.8)}]$$

$$\ge \int_{B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right)} \int_{\Omega_0} s^3 |v|^2 e^{2s\varphi} dx dt da d\tau \ge e^{2s\mu_4} \int_{B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right)} \int_{\Omega_0} s^3 |u|^2 dx dt da d\tau.$$

Hence (3.8) yields

$$e^{2s\mu_4} \int_{B\left(\mathbf{p}, \frac{\varepsilon}{\sqrt{N}}\right)} \int_{\Omega_0} s^3 |u|^2 dx dt da d\tau \le C e^{2s\mu_3} \|u\|_{H^{1,0}(D)}^2.$$

Therefore

$$\int_{B\left(\mathbf{p},\frac{\varepsilon}{\sqrt{N}}\right)} \int_{\Omega_0} s^3 |u|^2 dx dt da d\tau \le C e^{-2s(\mu_4-\mu_3)} \|u\|^2_{H^{1,0}(D)}$$

for all $s \ge s_0$. Letting $s \to \infty$, we obtain

$$u(x,t,a,\tau) = 0, \quad x \in \Omega_0, \ |t-t_0|^2 + |a-a_0|^2 + |\tau-\tau_0|^2 < \frac{\varepsilon^2}{N}. \qquad (3.9)$$

Since $(t_0, a_0, \tau_0) \in [\sqrt{2}\varepsilon, T - \sqrt{2}\varepsilon] \times [\sqrt{2}\varepsilon, a_1 - \sqrt{2}\varepsilon] \times [\tau_1 + \sqrt{2}\varepsilon, \tau_2 - \sqrt{2}\varepsilon]$ and $\Omega_0 \subset \Omega$ are chosen arbitrary provided that $\overline{\Omega}_0 \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ is a non-empty subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$, equality (3.9) yields $u = 0$ in $\Omega \times (0,T) \times (0,a_1) \times (\tau_1,\tau_2)$. Thus the proof of Theorem 1 is completed.

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