EXISTENCE OF COUPLED KÄHLER-EINSTEIN METRICS USING THE CONTINUITY METHOD

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ABSTRACT. In this paper we prove the existence of coupled Kähler-Einstein metrics on complex manifolds whose canonical bundle is ample. These metrics were introduced and their existence in the said case was proven by Hultgren and Nyström using calculus of variations. We prove the result using the method of continuity. In the process of proving estimates, akin to the usual Kähler-Einstein metrics, we reduce existence in the Fano case to a C^0 estimate.

1. Introduction

Let (X, ω_0) be a compact Kähler manifold which is either Fano $(c_1(X) > 0)$ or anti-Fano $(c_1(X) < 0)$. Consider the following equations (the "coupled Kähler-Einstein equations") on X, originally introduced in [8].

(1.1)
$$\operatorname{Ric}(\omega_1) = \operatorname{Ric}(\omega_2) = \dots = \pm \sum \omega_i,$$

where ω_i are Kähler metrics to be solved for in given Kähler classes $[\theta_i]$ satisfying $\pm \sum_i [\theta_i] = c_1(X)$. These equations seem vaguely reminiscent of

the bimetric theories of gravity (see [7] and the references therein).

It can easily be shown that 1.1 is equivalent to the following system of Monge-Ampère PDE if ω_0 satisfies $\text{Ric}(\omega_0) = \pm \sum_i \theta_i$. (This can be arranged using Yau's solution of the Calabi conjecture [12].)

(1.2)
$$(\theta_i + \sqrt{-1}\partial\bar{\partial}\phi_i)^n = C_i e^{\mp \sum_i \phi_i} \omega_0^n$$

for smooth functions ϕ_i satisfying $\sup \phi_2 = \sup \phi_3 = \ldots = \sup \phi_n = 0$ where $C_i = \frac{\int \theta_i^n}{\int \omega_0^n}$. In [8] the following existence result was proven for anti-Fano X.

Theorem 1.1 (Hultgren-Nyström). Let (X, ω_0) be a compact Kähler manifold which is anti-Fano. Let $[\theta_i]$ be Kähler classes such that $\sum_i [\theta_i] = -c_1(X)$. Then there exist unique Kähler metrics $\omega_i \in [\theta_i]$ such that

(1.3)
$$\operatorname{Ric}(\omega_1) = \operatorname{Ric}(\omega_2) = \dots = -\sum_i \omega_i$$

Hultegren and Nyström proved theorem 1.1 using calculus of variations. In this paper we prove this theorem using the method of continuity. To do this we establish the following *a priori* estimates.

Theorem 1.2. Let (X, ω_0) be a compact Kähler manifold that is either Fano or anti-Fano such that ω_0 satisfies $\mathrm{Ric}(\omega_0) = \pm \sum_i \theta_i$ where θ_i are Kähler forms such that $\pm \sum_i [\theta_i] = c_1(X)$ Let ϕ_i be a smooth solution of the following system of

such that $\pm \sum_{i} [\theta_i] = c_1(X)$. Let ϕ_i be a smooth solution of the following system of equations.

(1.4)
$$(\theta_i + \sqrt{-1}\partial\bar{\partial}\phi_i)^n = C_i e^{\mp \sum_i t_i \phi_i} \omega_0^n$$

where
$$C_i = \frac{\int \theta_i^n}{\int \omega_0^n}$$
 and $0 \le t_i \le 1$.

- (1) If X is anti-Fano then $\|\phi_i\|_{C^{2,\alpha}} \leq C$ where C is bounded uniformly.
- (2) If X is Fano then $\|\phi_i\|_{C^{2,\alpha}} \leq C$ where C depends on $\|\phi_1\|_{C^0}$.

Note that at $t_i = 0 \ \forall i$, the functions $\phi_i = 0$ solve the equations. By theorem 1.2 the set of t_i for which there exists a solution is closed for anti-Fano manifolds. Theorem 1.1 follows from the following openness result.

Theorem 1.3. The set of $0 \le t < 1$ for which there exists a unique smooth solution to the following system is open.

(1.5)
$$\operatorname{Ric}(\theta_{1\phi_1}) = \operatorname{Ric}(\theta_{2\phi_2}) = \dots = \pm \left(\sum t\theta_{i\phi_i} + \sum (1-t)\theta_i\right)$$

Notice that theorems 1.2 and 1.3 reduce the problem for Fano manifolds to the C^0 estimate just as in the usual Kähler-Einstein case. In [8] an obstruction to solving the equation akin to K-stability was discovered for Fano manifolds. It is interesting to see if the corresponding C^0 estimate can be proven along this continuity path using techniques of [2, 3, 4, 6].

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2. A priori estimates on solutions to equation 1.2

As is often the case in fully nonlinear PDE, we prove lower order estimates and improve upon them. In what follows unless clarity demands otherwise, we denote arbitrary uniform (in the time parameters in the method of continuity) constants by *C*.

We first prove a C^0 estimate in the anti-Fano case.

Lemma 2.1. If $c_1(X) < 0$ then any smooth solution ϕ_i satisfying $\sup \phi_2 = \sup \phi_3 = \ldots = 0$ of the system

(2.1)
$$(\theta_i + \sqrt{-1}\partial\bar{\partial}\phi_i)^n = C_i e^{-i} t_i \phi_i$$

where $C_i = \frac{\int \theta_i^n}{\int \omega_0^n}$ and $0 \le t_i \le 1$ satisfies $||\phi_i||_{C^0} \le C$.

Proof. If $|\phi_1|_{C^0} \le C$ then by the assumption that $\phi_i \le 0 \ \forall i \ge 2$, and either the Alexandrov-Bakelmann-Pucci (ABP) maximum principle [1] or L^p stability for p > 1 [9] we can see that $||\phi_i||_{C^0} \le C$ for all $1 \le i \le n$. In addition, the maximum principle shows that $\phi_1 \ge -C$. So we just need to prove that $\phi_1 \le C$.

Choosing a positive Green's function G for the Laplacian of ω_0 , we see using the representation formula (page 49 in [10] for instance) that

$$(2.2) u(x) - C \le \frac{\int u\omega_0^n}{V},$$

for every u satisfying $\sqrt{-1}\partial\bar{\partial}u \geq -C\omega_0$, where V is the volume of ω_0 . Taking $u=\sum_i t_i\phi_i$ and using Jensen's inequality we get

(2.3)
$$\sum_{i} t_{i} \phi_{i}(x) - C \leq \ln \left(\int e^{\sum t_{i} \phi_{i}} \omega_{0}^{n} \right)$$
$$\Rightarrow \sum_{i} t_{i} \phi_{i}(x) \leq C.$$

Therefore, $||e^{\sum t_i \phi_i}||_{L^p} \le C_p$ for all p > 1. Thus by the ABP estimate as before we see that $-C \le \phi_i \le C$.

We proceed to prove a bound on the Laplacian in both, the Fano, and the anti-Fano cases.

Lemma 2.2. Any smooth solution ϕ_i of the system 1.2 satisfies $||\Delta \phi_i|| \le C$.

Proof. Let $u_i = e^{-\lambda \phi_i}(n + \Delta_{\theta_i}\phi_i)$. We shall assume that $||\phi_i||_{C^0} \leq C$ in what follows. Just as in Yau's proof [12] we write the following inequality (inequality 2.3 from [5] for instance) for solutions of $\omega_v^n = (\omega + \sqrt{-1}\partial \bar{\partial}v)^n = e^{F-\lambda v}\omega^n$

$$\Delta_{\omega_{v}}(\exp(-C_{1}v)(n+\Delta v)) \ge \exp(-C_{1}v)\left[\Delta F - C_{2} - C_{1}(n+\Delta v)\right] + \exp\left(-C_{3}v - \frac{F}{n-1}\right)C_{4}(n+\Delta v)^{n/(n-1)}.$$

Replacing C_1 by λ , F by $F + a \sum_{i \neq j} t_j \phi_j$, ω by θ_i , and v by ϕ_i in the above inequality we get (after a couple of easy estimates) the following. Note that

a = 1 or a = -1 depending on whether the manifold is anti-Fano or Fano respectively.

$$\Delta_{\theta_{\phi_{i}}} u_{i} \geq -C + \tilde{C} u_{i}^{n/(n-1)} + a \sum_{i \neq j} t_{j} e^{-\lambda \phi_{i}} \Delta_{\theta_{i}} \phi_{j}$$

$$= -C + \tilde{C} u_{i}^{n/(n-1)} + a \sum_{i \neq j} e^{-\lambda \phi_{i}} t_{j} \frac{\theta_{i}^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \phi_{j}}{\theta_{i}^{n}}.$$
(2.5)

At this point we analyse the two cases $a = \pm 1$ separately.

(1) a = 1. In this case we may continue inequality 2.5 further as follows.

$$\Delta_{\theta_{\phi_i}} u_i \ge -C + \tilde{C} u_i^{n/(n-1)} - \sum_{i \ne j} t_j e^{-\lambda \phi} \frac{\theta_i^{n-1} \theta_j}{\theta_i^n}.$$

Therefore by the maximum principle $u_i \le C \ \forall \ 1 \le i \le n$.

(2) In this case we have the following consequence of inequality 2.5.

$$\Delta_{\theta_{\phi_{i}}} u_{i} \geq -C + \tilde{C} u_{i}^{n/(n-1)} - C \sum_{i \neq j} e^{-\lambda \phi_{i}} t_{j} \frac{\theta_{j}^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \phi_{j}}{\theta_{j}^{n}}$$

$$\geq -C + \tilde{C} u_{i}^{n/(n-1)} - C \sum_{i \neq j} t_{j} u_{j}.$$
(2.6)

Let $\max_{X} u_i = M_i$. By the maximum principle and inequality 2.6 we see that for every i we have the following inequality.

$$(2.7) C\left(1+\sum t_j M_j\right) \ge M_i^{n/(n-1)}.$$

Summing 2.7 over all i and using Young's inequality $a \le \epsilon a^{n/(n-1)} + C(n, \epsilon)$ we get (upon choosing a small enough ϵ),

(2.8)
$$C\left(1+\epsilon\sum_{j}M_{j}^{n/(n-1)}+\sum_{j}C(n,\epsilon)\right)\geq\sum_{i}M_{i}^{n/(n-1)}$$

$$\Rightarrow M_{i}\leq C\ \forall\ 1\leq i\leq n.$$

Finally, we need a $C^{2,\alpha}$ estimate in order to complete the proof of theorem 1.2. Indeed, theorem 1.1 of [11] implies the desired $C^{2,\alpha}$ estimate provided $\|\phi_i\|_{C^1} \leq C$. The latter inequality is true because of the Laplacian bound and $W^{2,p}$ elliptic regularity. We also note that standard elliptic theory (Schauder estimates) and bootstrapping imply that $\|\phi_i\|_{C^{k,\alpha}} \leq C$ for any k.

3. Uniqueness in the anti-Fano case and openness along the continuity $$\operatorname{\textsc{path}}$$

The uniqueness part of theorem 1.1 was proven in [8] but we prove it again for the convenience of the reader.

Proposition 3.1. Let X be an anti-Fano manifold. If a solution to the coupled Kähler-Einstein equations exists, then it is unique.

Proof. If $\phi'_i = \phi_i + u^{(i)}$ is another solution of 1.2 then upon subtraction we get

(3.1)
$$L_j^{ab} u_{a\bar{b}}^{(j)} = e^{\sum \phi_i} (e^{\sum u^{(i)}} - 1),$$

where $L_j^{ab}u_{a\bar{b}}^{(j)}=\frac{(\theta_j+\sqrt{-1}\partial\bar{\partial}\phi_j+\sqrt{-1}\partial\bar{\partial}u^{(j)})^n-(\theta_j+\sqrt{-1}\partial\bar{\partial}\phi_j)^n}{\omega_0^n}$. Note that L_j is a positive-definite matrix. Multiplying 3.1 by $u^{(j)}$, integrating-by-parts, and summing over j we see that

(3.2)
$$(\sum_{i} u^{(j)}) e^{\sum \phi_i} (e^{\sum u^{(i)}} - 1) \le 0.$$

This means that $\sum u^{(j)} = 0$ and $\partial u^{(j)} = \bar{\partial} u^{(j)} = 0$. Therefore $u^{(j)} = 0 \ \forall \ j$.

Now we proceed to prove openness, i.e., theorem 1.3.

Proof of theorem 1.3 : Suppose we know that $\omega_i \in [\theta_i]$ solve the system 1.5 for t. Then we need to prove that for $t + \delta$ where δ is in a small open interval, the system can still be solved. We shall in fact consider t_i to be potentially different for different i until the very end of this proof. This is because for the anti-Fano case, one can prove a slightly more general result than the one stated in theorem 1.3. To this end define the following Banach manifolds.

Definition. Let \mathcal{B}_1^i be the open subset of $C^{4,\alpha}$ functions ψ_i satisfying

$$\int_X \psi_i \omega_i^n = 0$$

and

$$\omega_i + \sqrt{-1}\partial\bar{\partial}\psi_i > 0.$$

Let \mathcal{B}_2 be the subspace of $C^{0,\alpha}$ real (1, 1)-forms η of the form

$$\eta = \sqrt{-1}\partial\bar{\partial}f$$

where f is a $C^{2,\alpha}$ function satisfying

$$\int_X f\omega_0^n = 0.$$

Notice that we have the map $T: U = \prod_{i=1}^k (\mathcal{B}_1^i \times [0,1]) \to V = \mathcal{B}_2^k$ given by

$$T(\psi_1,t_1,\psi_2,t_2,\ldots) = \left(\operatorname{Ric}(\omega_{1\psi_1}) + a\left(\sum t_i\omega_{i\psi_i} + \sum (1-t_i)\theta_i\right),\ldots\right),\,$$

where $a = \pm 1$ depending on the sign of $-c_1(X)$. Suppose we take a point $p = (0, t_1, 0, t_2, ...)$ such that T(p) = 0. The implicit function theorem states that if $DT_p(v_1, 0, v_2, 0, ...)$ is an isomorphism from TU to TV, then ψ_i can be

locally solved for in terms of t_j and therefore the set of t_j for which T=0 is open. The derivative DT_p is

$$(3.3) DT_p(v_1,0,v_2,0,\ldots) = (-\sqrt{-1}\partial\bar{\partial}\Delta_{\omega_i}v_1 + a\sum_i t_i\sqrt{-1}\partial\bar{\partial}v_i,\ldots).$$

For it to be surjective we need to solve

$$(-\sqrt{-1}\partial\bar{\partial}\Delta_{\omega_{i}}v_{1} + a\sum_{i} t_{i}\sqrt{-1}\partial\bar{\partial}v_{i}, \ldots) = (\sqrt{-1}\partial\bar{\partial}f_{1}, \sqrt{-1}\partial\bar{\partial}f_{2}, \ldots)$$

$$(3.4) \qquad \Rightarrow L(v_{1}, v_{2}, \ldots) = (-\Delta_{\omega_{i}}v_{1} + a\sum_{i} t_{i}v_{i}, \ldots) = (f_{1}, f_{2}, \ldots).$$

By the Fredholm alternative we simply need to prove that the kernel of *L* is trivial. The kernel consists of functions such that

$$\Delta_{\omega_1} v_1 = a \sum_{i} t_i v_i$$

$$\Delta_{\omega_2} v_2 = a \sum_{i} t_i v_i$$
(3.5)

Note that at $t_i = 0 \ \forall i$ we see that the kernel is obviously trivial and thus openness holds for small t_i . Therefore we may assume without loss of generality that $t_i > 0 \ \forall i$. We observe that the normalised volume forms $\frac{\omega_i^n}{\int \omega_i^n}$ are all equal (to some form dvol) because the Ricci curvatures of ω_i are equal. Multiplying the j^{th} equation of 3.5 by $t_i v_i dvol$ and integrating the left-hand side by parts

$$(3.6) - \int_X t_i \langle \nabla_j v_j, \nabla_j v_i \rangle_j dvol = a \int_X t_i v_i \sum_k t_k v_k dvol.$$

Taking i = j and summing over all j we get

$$(3.7) - \int_X \sum_j t_j |\nabla_j v_j|_j^2 dvol = a \int_X \left(\sum_k t_k v_k\right)^2 dvol.$$

There are two cases to consider.

- (1) *X* is anti-Fano, i.e. a = 1: Equation 3.7 implies that $\sum t_i v_i = 0$. Therefore by equations 3.5 all the v_i are constants and in fact equal to 0 (because $\int v_i \theta_i^n = 0$).
- (2) *X* is Fano, i.e., a = -1: A Weitzenböck identity (see page 65 of [10] for instance) that

(3.8)
$$\int_{X} (\Delta_{\omega_{i}} v_{i})^{2} dvol \geq \int_{X} \operatorname{Ric}(\omega_{i}) (\partial v_{i}, \bar{\partial} v_{i}) dvol$$
$$\geq \int_{X} \sum_{i} t_{j} |\nabla_{i} v_{i}|_{j}^{2} dvol.$$

Assume without loss of generality that none of the v_i are constant. Indeed, if let's say v_1 is a constant, then $\Delta_1 v_1 = \Delta_i v_i = 0$ which by

the maximum principle means that all the v_i are constant and in fact 0 by normalisation. Note that 3.6 implies that

(3.9)
$$\int_{X} \langle \nabla_{j} v_{j}, \nabla_{j} v_{i} \rangle_{j} dvol = \int_{X} |\nabla_{i} v_{i}|_{i}^{2} dvol.$$

Choose normal coordinates for ω_i at a point p. Further, assume that ω_j is diagonal at p with eigenvalues λ_μ . Writing the integrand of the left hand side of 3.9 at p in the said coordinates we get the following.

$$\langle \nabla_{j} v_{j}, \nabla_{j} v_{i} \rangle_{j}(p) = \sum_{\mu} \frac{\partial_{\mu} v_{j} \bar{\partial}_{\mu} v_{i}}{\lambda_{\mu}}$$

$$\leq \sqrt{\sum_{\mu} \frac{|\partial_{\mu} v_{j}|^{2}}{\lambda_{\mu}^{2}}} \sqrt{\sum_{\mu} |\partial_{\mu} v_{i}|^{2}}$$

$$= |\nabla_{j} v_{j}|_{i} |\nabla_{i} v_{i}|_{i}.$$
(3.10)

Thus using 3.9, 3.10, and the Cauchy-Schwarz inequality we get

At this point we put $t_i = t_j = t$ in 3.8, summing over i and j, and using 3.5 and 3.7 we get,

$$(3.12) \sum_{i,j} \int_{X} \left(|\nabla_{j} v_{j}|_{j}^{2} - |\nabla_{i} v_{i}|_{j}^{2} \right) dvol \ge 0$$

$$\Rightarrow \sum_{i < j} \int_{X} \left(|\nabla_{j} v_{j}|_{j}^{2} + |\nabla_{i} v_{i}|_{i}^{2} - |\nabla_{i} v_{i}|_{j}^{2} - |\nabla_{j} v_{j}|_{i}^{2} \right) dvol \ge 0.$$

Equation 3.12 in conjunction with 3.11 implied that equality holds in all the inequalities above. Therefore all the v_i are constants and in fact 0 by normalisation. Note that this may not be true for t=1 because equality holding in the inequalities would merely mean that $\nabla_i v_i$ are holomorphic vector fields proportional to each other.

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