# On the geometric structure of certain real algebraic surfaces

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#### Abstract

In this paper we study the affine geometric structure of the graph of a polynomial  $f \in \mathbb{R}[x,y]$ . We provide certain criteria to determine when the parabolic curve is compact and when the unbounded component of its complement is hyperbolic or elliptic. We analyse the extension to the real projective plane of both fields of asymptotic lines and the Poincaré index at its singular points at infinity. We exhibit an index formula for the field of asymptotic lines involving the number of connected components of the projective Hessian curve of f and the number of godrons. As an application of this investigation, we obtain upper bounds, respectively, for the number of godrons having an interior tangency and when they have an exterior tangency.

Keywords: parabolic curve, asymptotic fields of lines, real algebraic surfaces, quadratic differential forms.

MS classification: 53A15, 53A05, 14P05, 14N10, 34K32, 34G20

### 1 Introduction

There is a well known classification of the points of a smooth surface immersed in the three-dimensional real affine (projective or Euclidean) space. Any point belongs to one of the following types: elliptic, parabolic or hyperbolic. On generic smooth surfaces, parabolic points appear along a smooth

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curve (it may be empty) called the parabolic curve of the surface, whose complement is constituted by the elliptic and hyperbolic domains. The configuration of these sets, invariant under the action of the affine group (or projective group) on 3-space, is the basic affine geometric structure of the surface. One of the goals in projective and affine differential geometry has been the study of this basic geometric structure for smooth and also for algebraic surfaces, see for example [2, 3, 5, 12, 19, 25].

In this paper, we focus on the analysis of the basic geometric structure of generic algebraic surfaces in  $\mathbb{R}^3$  that are the graph of a real polynomial  $f \in \mathbb{R}[x,y]$ . When the parabolic curve of such a surface  $S_f$  is compact, there is one unbounded component  $C_u$  in the complement of this curve that plays a relevant role in the determination of this structure. The class of this component can be either elliptic or hyperbolic and when it is known we can specify, in the generic case, the class of the other connected components that are on the complement of the parabolic curve.

In section 3, we study the distribution of the elliptic and hyperbolic domains: we provide in Theorem 3.5 conditions on the homogeneous part of the highest degree of f that guarantee the parabolic curve is compact and indicate the class of the component  $C_u$ . At each hyperbolic point, there are two lines tangent to the surface that have a contact of order, at least three with the surface. These lines are called *asymptotic lines*. A parabolic point has exactly one asymptotic line.

When f is a differentiable function defined on the plane  $\mathbb{R}^2$ , it is usual to consider a projection of the geometric structure of  $S_f$  into the plane. The image of the parabolic curve under such projection is a plane curve called *Hessian curve of* f that is defined by the equation  $\operatorname{Hess} f(x,y) = 0$ . The images of the two fields of asymptotic lines are described by the second fundamental form of  $S_f$ ,

$$\Pi_f(dx, dy) = f_{xx}(x, y)dx^2 + 2f_{xy}(x, y)dxdy + f_{yy}(x, y)dy^2.$$

In [9], V. Guíñez considers positive quadratic differential equations on the plane  $\mathbb{R}^2$  of the form

$$a(x,y) dx^{2} + b(x,y) dxdy + c(x,y) dy^{2} = 0,$$
 (1)

where  $a, b, c \in \mathbb{R}[x, y]$  are polynomials of degree at most n, the function  $b^2 - 4ac$  is nonnegative at every point of the xy-plane and  $b^2 - 4ac$  vanishes at a point p if and only if a, b, c vanish simultaneously at p. He extends the foliations determined by equation (1) to the line at infinity and proves, among other things, that the topological behaviour of these foliations in a

neighbourhood of a singular point at infinity, is one of the types shown in Fig. 1 (see [9], Remark 2.9).



Fig. 1: Topological types at a singular point at infinity.

When  $f \in \mathbb{R}[x, y]$  is a polynomial, the second fundamental form  $\Pi_f$  is a polynomial quadratic differential form that, in general, is not positive: there are disjoint open sets on the plane where the discriminant of this form is negative.

Through the projection of Poincaré from a plane into the unitary sphere, we give, in Proposition 4.2, an analytic extension on the sphere of the two fields of asymptotic lines. The fields of lines obtained through such extension are tangent to the sphere and have the same singular points. If the surface  $S_f$  is generic, the singular points of these fields appear on the equator of the sphere and they will be referred to as singular points at infinity. In Theorem 4.6, we characterise these points. We prove that the Poincaré index at a singular point at infinity is equal to  $\frac{1}{2}$ , Theorem 4.8. As a consequence, we obtain an upper bound for the sum over all Poincaré indices of an extended field at its singular points. This analysis allowed us to itemize the Poincaré index at a singular point at infinity of a field of asymptotic lines when it is extended to the real projective plane, Remark 4.11.

The projective Hessian curve of f is, in general, a nonsingular algebraic curve in  $\mathbb{RP}^2$  of even degree. On this plane we define two surfaces,  $B^{\pm}$ , whose boundary is the projective Hessian curve of f. Among parabolic points of a generic surface,  $S_f$ , a godron is distinguished because its unique asymptotic line is tangent to the parabolic curve at such point. The problem of determining the lowest upper bound for the number of godrons of an algebraic surface in terms of the degree of the polynomial that defines it has been an interesting subject of research [13, 24, 1]. The tangency of the asymptotic direction with the parabolic curve at such a point may be interior or exterior [4]. When the surface  $B^{\pm}$  is hyperbolic we give, in Theorem 5.1, a formula that relates the following three values: the Euler characteristic of  $B^{\pm}$ , the number of godrons having either an interior or an exterior tangency and the Poincaré indices at the singular points of the extension to the real projective plane of a field of asymptotic lines. Derived from this result, upper bounds

for the number of interior and exterior tangencies are given in Corollary 5.2. Another consequence given in Theorem 5.3, is the determination of an upper bound for the number of godrons when the projective Hessian curve of f is convex and it is comprised only of exterior ovals. We conclude the paper with the proof of Theorem 4.8, section 6.

### 2 Preliminaries

### Classification of points on a generic surface

A point of a generic smooth surface in  $\mathbb{R}^3$  can be classified in terms of the maximum order of contact of the tangent lines at this point with the surface [24, 14, 22]. We say that a point p is *elliptic* if all straight lines tangent to the surface at p have a contact of order two with the surface at that point.

An asymptotic line at a point p is a straight line tangent to the surface at p that has a contact of order greater than two with the surface. A hyperbolic point has exactly two transversal asymptotic lines while a parabolic point has one (double) asymptotic line.

The sets of elliptic and hyperbolic points are open subsets on the surface called *elliptic* and *hyperbolic domains*, respectively. These two domains share a common boundary called the parabolic curve which is a smooth curve constituted by the parabolic points. The unique asymptotic line at a parabolic point is transversal to the parabolic curve except at some isolated points called *godrons* (other authors call them cusps of Gauss or special parabolic points). The order of contact of the asymptotic line at each parabolic point is three while at a godron, is four. The set of asymptotic directions makes up, globally, two continuous fields of directions tangent to the surface [26] (this property is proved locally in [6, 7]). The integral curves of these fields are known as asymptotic curves. A hyperbolic point p is called a point of inflexion if the order of contact of an asymptotic line with the surface at p is at least four. This property implies that an asymptotic curve passing through p has an inflexion point at such point. The set of points of inflexion is called *flecnodal curve*. The closure of the flecnodal curve is a curve which is tangent to the parabolic curve at the godrons.

In order to understand the geometric structure of  $S_f$ , when the surface  $S_f$  is the graph of a differentiable function f on the plane, it is usual to consider the projection of the elements constituting such structure under the map  $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ ,  $(x,y,z) \mapsto (x,y)$ . The image of the parabolic curve on the xy-plane under  $\pi$  is the zero locus of the Hessian function  $\operatorname{Hess} f = f_{xx} f_{yy} - f_{xy}^2$ . This curve will be called the Hessian curve of f. The

hyperbolic and elliptic domains are projected, respectively, on H and E, where the Hessian function of f is negative and positive, respectively. The projection of both fields of asymptotic directions over the xy-plane yields two fields of lines that are described by the quadratic differential equation:

$$f_{xx}(x,y) dx^2 + 2f_{xy}(x,y) + f_{yy}(x,y) dy^2 = 0.$$
 (2)

The quadratic form on the left will be referred to as the second fundamental form of f and will be denoted by  $\Pi_f$ . For sake of simplicity, we identify the solutions of this quadratic form with the asymptotic directions and they will be referred to as the fields of asymptotic directions of f. A point on the xy-plane is a flat point of  $\Pi_f$  if the coefficients of this form,  $f_{xx}$ ,  $f_{xy}$  and  $f_{yy}$ , vanish at this point.

We are interested in the particular case when  $f \in \mathbb{R}[x, y]$  is a polynomial. If the degree of f is n, its Hessian curve is a real plane algebraic curve of degree, at most 2n-4. Moreover, if we consider the homogeneous decomposition of f,  $f = \sum_{i=r}^{n} f_i$ , where  $f_i \in \mathbb{R}[x, y]$  is a homogeneous polynomial of degree i, then

$$\operatorname{Hess} f(x,y) = \sum_{j=2r-4}^{2n-4} h_j(x,y)$$
, where  $h_{2r-4} = \operatorname{Hess} f_r$  and  $h_{2n-4} = \operatorname{Hess} f_n$ .

**Definition 2.1** The projective Hessian curve of f is the zero locus of the homogeneous polynomial  $H_f \in \mathbb{R}[x,y,z]$  which is the homogenization of the polynomial Hess f(x,y).

It follows, from the homogeneous decomposition of f, that  $H_f$  has the expression:  $H_f(x,y,z) = \sum_{j=2r-4}^{2n-4} z^{2n-4} h_j\left(\frac{x}{z},\frac{y}{z}\right)$ . Therefore, the restriction of  $H_f$  to the line at infinity z=0 is

$$H_f(x, y, 0) = \operatorname{Hess} f_n(x, y).$$

When the degree of  $H_f$  is even it allows us to label the points at infinity. A point on the line at infinity z = 0 of  $\mathbb{RP}^2$  is called *elliptic*, *parabolic* or *hyperbolic* if the sign of the homogeneous polynomial  $H_f$  at this point is positive, zero or negative, respectively.

Now, we shall introduce the concept of generic surface for  $S_f$  but before we give some definitions.

Let S be a smooth surface in  $\mathbb{RP}^3$  and p a point on S. Two function germs of S at p are *equivalent* if one is transformed into the other under the diffeomorphism group action. In paper [22], O. A. Platonova proves that

"In the space of compact smooth surfaces in  $\mathbb{RP}^3$  there is an open everywhere dense set of surfaces of which the germs at each point are equivalent to the germs that have the p-jets in Table 1".

Notation	Normal form	Restrictions	p	$\operatorname{cod}$
$\Pi_2$	$x^2 + y^2$	_	2	0
$\Pi_{3,1}$	$xy + sx^3 + y^3$	$s \neq 0$	3	0
$\Pi_{3,2}$	$y^2 + x^3$	_	3	1
$\Pi_{4,1}$	$xy + y^3 + x^4 + hx^3y$	_	4	1
$\Pi_{4,2}$	$y^2 + x^2y + \nu x^4$	$\nu \neq 0, 1/4$	4	2
$\Pi_{4,3}$	$xy + x^4 + s_1x^3y + s_2xy^3 + s_3y^4$	$s_3 \neq 0$	4	2
$\Pi_5$	$xy + y^3 \pm x^3y \sum d_i x^{5-i} y^i$	$d_0 \neq 0$	5	2

**Table 1:** Normal Forms

**Definition 2.2** A smooth surface in  $\mathbb{RP}^3$  is *generic* if it belongs to the open everywhere dense set defined by Platonova.

When f is a polynomial, its graph  $S_f$  is an algebraic surface in  $\mathbb{R}^3$  and we will say that  $S_f$  is generic if the p-jet of the function germ at each point of  $S_f$  is equivalent to a normal form of Table 1 and if the projective Hessian curve of f is nonsingular.

### Real algebraic curves in $\mathbb{RP}^2$

A real algebraic curve in  $\mathbb{RP}^2$  of degree m is, up to nonzero constant factors, a homogeneous polynomial  $F \in \mathbb{R}[x,y,z]$  of degree m. The polynomial equation F(x,y,z) = 0 determines the set of real points of the curve in  $\mathbb{RP}^2$ . From now on, we shall also call this set a real algebraic curve in  $\mathbb{RP}^2$ .

Each connected component of a nonsingular algebraic curve in  $\mathbb{RP}^2$  is homeomorphic to a circle. There are two ways up to isotopy to embed a circle into the real projective plane which are called the two-sidedly and the one-sidedly [27]. In the two-sidedly case, the complement in  $\mathbb{RP}^2$  of the image L of the circle has two connected components, one of which is homeomorphic to an open disc and called the inside component of L while the other is homeomorphic to a Möbius strip and is known as the outside component of L. Under these conditions, the image of the circle is called oval. We say that an oval is an outer oval if it is not in the inside component of any other oval. In the one-sidedly case, the complement in  $\mathbb{RP}^2$  of the image of the circle is connected and homeomorphic to a disc. In this situation, the

image of the circle is called a pseudo-line. While all connected components of a nonempty nonsingular real algebraic curve in  $\mathbb{RP}^2$  of even degree are ovals, each nonsingular algebraic curve of odd degree is constituted by ovals (if there are any) and exactly one pseudo-line.

The complement of a nonsingular curve F of even degree in  $\mathbb{RP}^2$  is the union of two disjoint open subsets, say  $b^+$  and  $b^-$  (Fig. 2). The set  $b^+$  is an orientable smooth surface at which the sign of F does not change while the open set  $b^-$  is a nonorientable smooth surface at which F takes the other sign. The closure of  $b^+$  and  $b^-$  will be denoted by  $B^+$  and  $B^-$ , respectively.

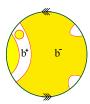


Fig. 2: Open sets  $b^+$  and  $b^-$ .

**Definition 2.3** An oval of a real algebraic curve in  $\mathbb{RP}^2$  of even degree m is called  $even\ (odd)$  if it is contained in an even (odd) number of ovals of the same curve. The number of even ovals is denoted by P and the number of odd ovals by N.

The numbers P and N contain information about the topology of the surfaces  $B^+$  and  $B^-$ . Indeed, the surface  $B^+$  has P connected components and the surface  $B^-$  has N+1 connected components. In 1906, Virginia Ragsdale proves that the Euler characteristics of these surfaces are  $\chi(B^+) = P - N$  and  $\chi(B^-) = N - P + 1$ , [23]. Three decades later, I. Petrowsky shows in [21] the following

**Theorem 2.4** Any nonsingular real projective algebraic curve of even degree m = 2k satisfies

$$-\frac{3}{2}k(k-1) \le P - N \le \frac{3}{2}k(k-1) + 1.$$

# 3 Determination of the elliptic and hyperbolic domains

In this paragraph, we analyse the geometric behaviour of the sets E and H.

**Definition 3.1** A homogeneous polynomial on  $\mathbb{R}[x,y]$  is called *hyperbolic* (*elliptic*) if its Hessian polynomial has no real linear factors and if it is nonpositive (nonnegative) at any point.

**Example 3.2** [8] If  $f \in \mathbb{R}[x,y]$  is a homogeneous polynomial of degree  $n \geq 2$  with n real linear factors that are distinct up to nonzero constant factors, then it is hyperbolic.

Let us denote by  $H^n[x,y] \subset \mathbb{R}[x,y]$  the set of real homogeneous polynomials of degree n. The set constituted by hyperbolic homogeneous polynomials of degree n is a topological subspace of  $H^n[x,y]$ , denoted by Hyp(n). The connectedness of this space has been studied as part of the subject known as the Hessian Topology introduced in [1, 2, 20] and named by V. I. Arnold in [3]. In fact, in reference [3] it is shown that this topological property of Hyp(n) depends on the degree of the polynomials that constitute it. For example, Hyp(3) and Hyp(4) are connected subspaces whereas Hyp(6) is a disconnected one. According to this, V.I. Arnold stated the following conjecture [3], p.1067:

"The number of connected components of the space of hyperbolic homogeneous polynomials of degree n increases as n increases (at least as a linear function of n)."

**Lemma 3.3** i) A hyperbolic homogeneous polynomial in  $\mathbb{R}[x,y]$  has at least one real linear factor. Moreover, every real linear factor of a hyperbolic polynomial has multiplicity one.

ii) An elliptic homogeneous polynomial has no real linear factors.

**Proof.** We firstly note that when a real linear factor of a homogeneous polynomial f has multiplicity greater than one, the Hessian curve of f is unbounded.

i) Let suppose that n is even and that f has no real linear factors. On the one hand, V.I. Arnold proves in [3] (p.1035) that the index of a field of asymptotic lines at the origin on the xy- plane is

$$\operatorname{ind}_{0}(\operatorname{cross}(\gamma)) = 1 - \frac{1}{4} \# \{ \theta \in [0, 2\pi) : F(\theta) = 0 \},$$
 (3)

where  $F(\theta)$  is the expression of the hyperbolic homogeneous polynomial in polar coordinates and  $\gamma$  is a parametrization of the unitary circle centred at the origin of the xy-plane.

According to (3) and considering the fact that f has no real linear factors,  $\operatorname{ind}_0(\operatorname{cross}(\gamma)) = 1$ . On the other hand, Arnold shows in the same paper

(p.1038) that if f is a hyperbolic homogeneous polynomial of degree even, then

$$\operatorname{ind}_0(\operatorname{cross}(\gamma)) \le 0.$$
 (4)

It is a contradiction to the first assertion.

ii) Let f be an elliptic homogeneous polynomial. In this case, its Hessian curve is compact, in fact, it is the origin. Let us suppose that f has a real linear factor l(x,y). Thus, l=0 is an asymptotic curve because it has an infinite order of contact with  $S_f$  and the multiplicity of l is one. This is a contradiction.

**Example 3.4** ([16], p.60) For each  $\mu \in (0, 1)$  and  $\alpha = \pm 1$ , the homogeneous polynomial  $\alpha (x^4 + 6\mu x^2y^2 + y^4)$  is elliptic. Therefore, any element of its orbit is elliptic, by considering the action of  $GL(2, \mathbb{R})$  on  $H^4[x, y]$ .

In Theorem 3.5, we can appreciate how  $f_n$  determines the geometric structure of the surface  $S_f$  when such homogeneous polynomial is hyperbolic or elliptic. In other circumstances, it can be untrue as shown by the following examples. Consider the polynomials  $f(x,y) = x^4 + 6x^2y^2 - y^4 + 3x^2y - 3xy^2 + 10y^2 - 10x^2$  and  $g(x,y) = x^4 + 6x^2y^2 - y^4 + 3x^2y - 3xy^2 + 10y^2 + 10x^2$ . While they only differ by the quadratic homogeneous part, its geometric structure is different because in the first case H is contained in  $B^-$ , and in the second case, E is contained in  $B^-$ . Indeed, the Hessian polynomial of f is

$$\operatorname{Hess} f(x,y) = 144x^4 - 576x^2y^2 - 144y^4 - 72x^3 - 216x^2y + 216xy^2 - 72y^3 - 36x^2 + 36xy + 444y^2 + 120x + 120y - 400,$$

and its restriction to the straight line x=0 is a one-variable polynomial without real roots. On the other hand, the Hessian polynomial of g is

$$\operatorname{Hess} g(x,y) = 144x^4 - 576x^2y^2 - 144y^4 - 72x^3 - 216x^2y + 216xy^2 - 72y^3 + 228x^2 + 36xy + 180y^2 - 12x + 120y + 40,$$

and its restriction to the line y=0 is a one-variable polynomial without real roots. We remark that the intersection of both projective Hessian curves with the line at infinity are the points  $P^{\pm}=[\pm\sqrt{\sqrt{10}-3}:1:0]$ . Therefore, the Hessian curve of f has two unbounded connected components: one of them is located in both quadrants, first and fourth, while the other component is located in the two complementary quadrants. This implies that H is contained in  $B^-$ . In an analogous way, the Hessian curve of g has two unbounded connected components: one of them is located in the first

and second quadrants while the other is located in the two complementary quadrants. So, the set E is contained in  $B^-$ .

The conclusions of the next result are proved in [11] (Theorem 2) by considering the compactification of a plane with the point at infinity. They show that the fields of asymptotic directions are extended up to the point at infinity by means of a polynomial binary differential form  $\widetilde{\Pi}_f$ . They assume the extra hypothesis: the associated form  $\widetilde{\Pi}_f$  (of  $\Pi_f$ ) at the point at infinity has good multiplicity, that is, the homogeneous part of the lowest degree of  $\widetilde{\Pi}_f$  is determined only by the form  $\Pi_{f_n}$ .

**Theorem 3.5** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree  $n \geq 3$ . If  $f_n$  is hyperbolic or elliptic, then the Hessian curve of f is compact. Moreover, the set  $b^- \cap \mathbb{R}^2$  is hyperbolic or elliptic providing that  $f_n$  is hyperbolic or elliptic, respectively.

**Proof.** Suppose that  $f_n$  is a hyperbolic polynomial. The elliptic case is similar. Since the polynomial  $\operatorname{Hess} f_n(x,y)$  has no real linear factors and  $h_{2n-4} = \operatorname{Hess} f_n$ , the projective Hessian curve does not intersect to the line at infinity. Accordingly, the curve  $\operatorname{Hess} f(x,y) = 0$  is compact in  $\mathbb{R}^2$  and the line at infinity is contained in  $B^-$ . To show that the set  $b^- \cap \mathbb{R}^2$  is hyperbolic, it will be enough to prove that any point on the line at infinity is a hyperbolic point. By taking p = [1:0:0] we have that  $H_f(p)$  is negative because  $H_f(p) = \operatorname{Hess} f_n(1,0)$ .

# 4 Projection into the Poincaré sphere

A good approach to studying the behaviour of the asymptotic curves of (2) "at infinity" is to use the so-called *Poincaré sphere* [18]. Let  $\mathbb{S}^2 = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}$  be the unit sphere centred at the origin O in  $\mathbb{R}^3$  and identify its tangent plane  $T_N\mathbb{S}^2$  at the north pole N=(0,0,1) with the xy-plane. Given a point  $\mathbf{x}=(x,y,1) \in T_N\mathbb{S}^2$ , the line through  $\mathbf{x}$  and O intersects  $\mathbb{S}^2$  at the following two points:

$$s_1(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{1 + x^2 + y^2}}, \quad s_2(\mathbf{x}) = -\frac{\mathbf{x}}{\sqrt{1 + x^2 + y^2}}.$$

The maps  $s_i : \mathbb{R}^2 \to \mathbb{S}^2$ , i = 1, 2, are called the projections to the Poincaré sphere.

Now, suppose that  $f \in \mathbb{R}[x, y]$  is a polynomial of degree n and consider on the xy-plane the two fields of asymptotic directions,  $\mathbb{X}_1$  and  $\mathbb{X}_2$ , defined by equation (2).

**Remark 4.1** The images of the two fields  $X_1$  and  $X_2$ , under the Poincaré projection, over both upper and lower hemispheres, are the zero loci of the induced quadratic differential forms,  $s_1^*(II_f)$  and  $s_2^*(II_f)$ , which are defined on the complement of the equator of  $S^2$ . Moreover, the images of both fields over each open hemisphere consist of two fields of lines diffeomorphic to  $X_1$  and  $X_2$ .

Similarly, as V. Guíñez does in [9], we shall prove that the induced quadratic differential forms  $s_1^*(II_f)$  and  $s_2^*(II_f)$  can be extended to an analytical quadratic differential form defined on the sphere.

**Proposition 4.2** The induced differential forms  $s_1^*(\Pi_f)$  and  $s_2^*(\Pi_f)$  are extended to this analytical quadratic differential form

$$\begin{pmatrix}
du & dv & d\omega
\end{pmatrix}
\begin{pmatrix}
\omega^{2} F_{uu} (u, v, \omega) & \omega^{2} F_{uv} (u, v, \omega) & \omega A (u, v, \omega) \\
\omega^{2} F_{uv} (u, v, \omega) & \omega^{2} F_{vv} (u, v, \omega) & \omega B (u, v, \omega) \\
\omega A (u, v, \omega) & \omega B (u, v, \omega) & S (u, v, \omega)
\end{pmatrix}
\begin{pmatrix}
du \\
dv \\
d\omega
\end{pmatrix} (5)$$

defined on the sphere with the property that the equator is an integral curve of the fields defined by this form. In such case

$$F(u,v,\omega) = \sum_{i=0}^{n} \omega^{n-i} f_i(u,v), \quad F_{uu} = \frac{\partial^2 F}{\partial u^2}, \quad F_{uv} = \frac{\partial^2 F}{\partial u \partial v}, \quad F_{vv} = \frac{\partial^2 F}{\partial v^2},$$

$$A(u,v,\omega) = -uF_{uu}(u,v,\omega) - vF_{uv}(u,v,\omega),$$

$$B(u,v,\omega) = -uF_{uv}(u,v,\omega) - vF_{vv}(u,v,\omega),$$

$$S(u,v,\omega) = u^2 F_{uu}(u,v,\omega) + 2uvF_{uv}(u,v,\omega) + v^2 F_{vv}(u,v,\omega).$$

We denote by  $\mathbb{Y}_1$ ,  $\mathbb{Y}_2$  the two fields of lines defined by the form (5). It is worth mentioning that the fields  $\mathbb{Y}_k$ , k = 1, 2, are not defined, in general, on the whole sphere.

**Proof.** Consider the map  $\varrho: \mathbb{R}^3 \setminus \{\omega=0\} \to \mathbb{R}^2$ ,  $(u,v,\omega) \mapsto (x,y)$  where  $x=\frac{u}{\omega}, \ y=\frac{v}{\omega}$ . The images under this map of a pair of antipodal points on the sphere  $\mathbb{S}^2$  are the same. We proceed to obtain the pullback  $\varrho^*(\mathrm{II}_f)$  of the second fundamental form  $\mathrm{II}_f$ . Replacing

$$(dx \quad dy) = \begin{pmatrix} \frac{\omega du - u d\omega}{\omega^2} & \frac{\omega dv - v d\omega}{\omega^2} \end{pmatrix} = \frac{1}{\omega^2} \begin{pmatrix} du & dv & d\omega \end{pmatrix} \begin{pmatrix} \omega & 0\\ 0 & \omega\\ -u & -v \end{pmatrix}$$

in the expression  $\Pi_f(dx,dy) = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} f_{xx}\left(x,y\right) & f_{xx}\left(x,y\right) \\ f_{xy}\left(x,y\right) & f_{yy}\left(x,y\right) \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$ , we have that  $\varrho^*\left(\Pi_f\right)$  is

$$\frac{1}{\omega^4} \begin{pmatrix} du & dv & d\omega \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & \omega \\ -u & -v \end{pmatrix} \begin{pmatrix} f_{xx} \begin{pmatrix} \underline{u} & \underline{v} \\ \omega & \underline{v} \end{pmatrix} & f_{xy} \begin{pmatrix} \underline{u} \\ \omega & \underline{v} \end{pmatrix} \\ f_{xy} \begin{pmatrix} \underline{u} \\ \omega & \underline{v} \end{pmatrix} & f_{yy} \begin{pmatrix} \underline{u} \\ \omega & \underline{v} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \omega & 0 & -u \\ 0 & \omega & -v \end{pmatrix} \begin{pmatrix} du \\ dv \\ d\omega \end{pmatrix}.$$

After multiplication by  $\omega^{n+2}$  we obtain the desired quadratic form (5).

Now, we shall prove that the equator is an integral curve of the fields  $\mathbb{Y}_k, k=1,2$ . Consider the chart u=1 in  $\mathbb{R}^3$ . In this chart, the projections of  $\mathbb{Y}_1$  and  $\mathbb{Y}_2$  restricted to the set  $\{(u,v,\omega)\in\mathbb{S}^2|u>0\}$  are described by the quadratic equation

$$(dv \quad d\omega) \begin{pmatrix} \omega^2 F_{vv} (1, v, \omega) & \omega B (1, v, \omega) \\ \omega B (1, v, \omega) & S (1, v, \omega) \end{pmatrix} \begin{pmatrix} dv \\ d\omega \end{pmatrix} = 0.$$
 (6)

If the origin of the  $v\omega$ -plane is not a solution of  $S(1, v, \omega)$ , the following two vector fields are tangent to  $\mathbb{Y}_1$  and  $\mathbb{Y}_2$  in a neighbourhood of the origin,

$$\frac{d\omega}{dv} = \frac{-\omega B \pm \sqrt{\omega^2 (B^2 - F_{vv} S)}}{S}.$$

So, the v-axis is locally an integral curve of  $\mathbb{Y}_k$ , k = 1, 2.

**Lemma 4.3** The polynomial S of (5) is equal to the expression  $S(u, v, \omega) = \sum_{k=2}^{n} k(k-1) \omega^{n-k} f_k(u, v)$ .

**Proof.** By definition  $F(u, v, \omega) = \sum_{k=0}^{n} \omega^{n-k} f_k(u, v)$ . Thus,

$$S(u, v, \omega) = \sum_{k=0}^{n} \omega^{n-k} \left( u^2 \frac{\partial^2}{\partial u^2} f_k(u, v) + 2uv \frac{\partial^2}{\partial u \partial v} f_k(u, v) + v^2 \frac{\partial^2}{\partial v^2} f_k(u, v) \right).$$

By considering the well known Euler's formula for a homogeneous polynomial  $P \in \mathbb{R}[x,y]$  of degree m:  $mP(x,y) = xP_x(x,y) + yP_y(x,y)$ , it follows the relation

$$m(m-1) P(x,y) = x^{2} P_{xx}(x,y) + 2xy P_{xy}(x,y) + y^{2} P_{yy}(x,y).$$

We obtain the desired equality by taking  $P = f_k$  and m = k.

**Definition 4.4** A singular point of  $\mathbb{Y}_k$  is called *singular point at infinity* if it is on the equator of  $\mathbb{S}^2$ .

We remark that if  $S_f$  is generic, every singular point of  $\mathbb{Y}_k$  is a singular point at infinity. We say that a point  $p \in \mathbb{S}^2$  is a flat point of the quadratic form (5) if the coefficients of this form vanish at this point.

**Remark 4.5** (i) A point  $(u_0, v_0, \omega_0) \in \mathbb{S}^2$ , with  $\omega_0 \neq 0$  is a flat point of (5) if and only if the point  $(x_0, y_0) = \left(\frac{u_0}{\omega_0}, \frac{v_0}{\omega_0}\right)$  is a flat point of the fundamental form  $\Pi_f$ .

(ii) By Lemma 4.3: a flat point of (5) lies in the equator if and only if the polynomial  $S(u, v, 0) = k(k-1)f_n(u, v)$  vanishes at that point. Therefore, the form (5) has a finite number of flat points on the equator.

**Theorem 4.6** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree  $n \geq 3$ . If p is a point on the equator of  $\mathbb{S}^2$ , then p is a flat point of (5) if and only if p is a singular point at infinity of  $\mathbb{Y}_k$ , k = 1, 2. Moreover, if p is a singular point at infinity of  $\mathbb{Y}_k$ , and  $f_n$  has no repeated factors, then  $H_f(p) < 0$ .

**Proof.** Let us suppose that p = (1,0,0). By taking the chart u = 1, the fields  $\mathbb{Y}_k$ , k = 1, 2, restricted to the set  $\{(u, v, \omega) \in \mathbb{S}^2 | u > 0\}$ , are described by the quadratic equation

$$\omega^2 F_{vv}(1, v, \omega) dv^2 + 2\omega B(1, v, \omega) dv d\omega + S(1, v, \omega) d\omega^2 = 0.$$
 (7)

The discriminant of the left-side form of (7) is  $\Delta = -\omega^2 \left( F_{vv}S - B^2 \right) |_{(1,v,\omega)}$ . A straightforward calculation shows that  $\Delta = -\omega^2 H_f(1,v,\omega)$ . By Proposition 4.2, the v-axis is an integral curve of  $\mathbb{Y}_k$ . Moreover, since S(1,v,0) has a finite number of solutions, the fields of (7) are described, in a neighbourhood of the origin, by

$$R_k(v,\omega) dv + 2S(1,v,\omega) d\omega = 0, \quad k = 1, 2, \tag{8}$$

where  $R_k(v,\omega) = -2\omega B(1,v,\omega) + 2(-1)^k \sqrt{-\omega^2 H_f(1,v,\omega)}$ . Suppose that p is a flat point of the form (5). So, the origin of the  $v\omega$ -plane is a singular point of the fields (8) since it is a zero of  $S(1,v,\omega)$ . Conversely, let us suppose that the origin of the  $v\omega$ -plane is a singular point of the fields defined by (8). Thus, S(1,0,0) = 0, that is, the point p is a flat point of the form (5).

In order to prove the second part, we suppose that S(1,0,0) = 0. By Lemma 4.3,  $S(1,0,0) = f_n(1,0) = 0$ . It implies that the polynomial y is a factor of  $f_n$ . Thus, by hypothesis the multiplicity of y is one. In this case  $H_f(1,y,0) = -(g_x(1,y))^2$ , where  $f_n(x,y) = yg(x,y)$ . Thus,  $H_f(p) < 0$ .

Moreover, the discriminant  $\Delta$  is locally positive in the complement of the  $\omega$ -axis.

In the next result, we prove that the number of singular points at infinity of the field  $\mathbb{Y}_k$ , k = 1, 2, is twice the number of distinct real linear factors of the homogeneous polynomial  $f_n$ . Its proof follows from Lemma 4.3 and Theorem 4.6.

**Corollary 4.7** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree  $n \geq 3$ . Then, the set of singular points at infinity of  $\mathbb{Y}_k$ , k = 1, 2, is

$$\{(u, v, 0) \in \mathbb{S}^2 \mid f_n(u, v) = 0\}.$$

The singular points at infinity of  $\mathbb{Y}_k$  that do not belong to the boundary of  $B^{\pm}$  are characterised in the following

**Theorem 4.8** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree  $n \geq 3$  such that its homogeneous part  $f_n$  has no repeated factors. Then, the Poincaré index of  $\mathbb{Y}_k$ , k = 1, 2, at a singular point at infinity is equal to  $\frac{1}{2}$ . Moreover, its topological type is the one shown in Fig. 3.



**Fig. 3:** Topological behaviour of  $\mathbb{Y}_k$  at a singular point at infinity.

The proof of Theorem 4.8 is given in section 6.

For k = 1, 2, the expression  $\operatorname{Sing}(\mathbb{Y}_k)$  denotes the set of singular points of the field  $\mathbb{Y}_k$ .

**Corollary 4.9** Let  $f \in \mathbb{R}[x,y]$  be a polynomial such that  $f_n$  has no repeated factors.

i) If the projective Hessian curve of f has a nonempty transversal intersection with the line at infinity, then

$$0 \le \sum_{\xi \in \operatorname{Sing}(\mathbb{Y}_k)} \operatorname{Ind}_{\xi} (\mathbb{Y}_k) \le n - 2, \text{ for } k = 1, 2.$$

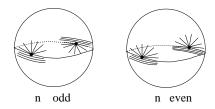
ii) If the projective Hessian curve of f does not intersect to the line at infinity, then

$$0 \le \sum_{\xi \in \operatorname{Sing}(\mathbb{Y}_k)} \operatorname{Ind}_{\xi}(\mathbb{Y}_k) \le n, \text{ for } k = 1, 2.$$

**Proof.** On the one hand, when the polynomial  $f_n$  has exactly n generic real linear factors, it is hyperbolic (Example 3.2). In such a case, the Hessian curve of f is compact by Theorem 3.5, and the field  $\mathbb{Y}_k$  has 2n singular points at infinity. In this case, in according to Theorem 4.8, the field  $\mathbb{Y}_k$  reaches the upper bound of ii). On the other hand, if the Hessian curve of f is unbounded,  $f_n$  has at most n-2 real linear factors, and by Lemma 4.7, the maximum number of singular points at infinity is 2(n-2). Inequalities of i) and ii) follow from Theorem 4.8.

### **Remark 4.10** The fields $\mathbb{Y}_1$ and $\mathbb{Y}_2$ behave qualitatively as:

- When n is even, if  $\mathbb{Y}_1$  is the projection on the upper hemisphere of  $\mathbb{X}_1$ , then, on the lower hemisphere,  $\mathbb{Y}_1$  is the projection of  $\mathbb{X}_2$ . Analogously for  $\mathbb{Y}_2$ . See Fig. 4.
- If n is odd, the field  $Y_i$  is the projection over both hemispheres of either  $X_1$  or  $X_2$  (Fig. 4).

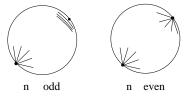


**Fig. 4:** Behaviour of  $\mathbb{Y}_k$  at antipodal singular points at infinity.

The restriction of the field  $\mathbb{Y}_k$ ,  $k \in \{1, 2\}$ , to the closure of a hemisphere of  $\mathbb{S}^2$  will be called a projective extension of the field of asymptotic directions  $\mathbb{X}_k$  and it will be denoted by  $\widetilde{\mathbb{X}}_k$ . Let us suppose that p is a singular point at infinity of  $\mathbb{Y}_k$ . When n is odd, a picture of the local qualitative behaviour of any projective extension at the points, p and -p, is shown in Fig. 5. We will say that  $[p] = \{p, -p\}$  is a singular point at infinity of the projective

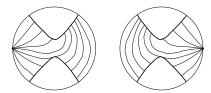
extension. Now, let n be even and choose a projective extension  $\widetilde{\mathbb{X}}_k$ . If the local qualitative behaviour of  $\widetilde{\mathbb{X}}_k$  at points, p and -p, is as shown in Fig. 5, we will say that  $[p] = \{p, -p\}$  is a singular point at infinity of  $\widetilde{\mathbb{X}}_k$ . Thus, from Remark 4.10 we have the following

**Remark 4.11** The Poincaré index of a projective extension  $\widetilde{\mathbb{X}}_k$  at a singular point at infinity is equal to  $\frac{1}{2}$  if n is odd, and it is 1 when n is even. Their topological types are shown in Fig. 5.



**Fig. 5:** Behaviour of  $\widetilde{\mathbb{X}}_k$  at a singular point at infinity.

**Example 4.12** Consider the cubic polynomial  $q(x,y) = x^2 + y^2 + y(x^2 + y^2)$ . In [10], it is proved that the Hessian curve of q is a hyperbola, it contains one godron and the flecnodal curve of q is the straight line y = 0. Moreover, the convex domain is elliptic while the concave is hyperbolic. By Corollary 4.7, each field  $\mathbb{Y}_k$  has two singular points at infinity. In Fig. 6, we draw the foliation of  $\mathbb{Y}_k$  in both closed hemispheres. We remark that this qualitative behaviour is the same for any nonhomogeneous cubic polynomial such that  $q_3$  has exactly one real linear factor.



**Fig. 6:** Foliation by asymptotic curves for the cubic polynomial q(x, y).

**Example 4.13** The product  $\Pi_{k=1}^n l_k$  of n linear polynomials on  $\mathbb{R}[x,y]$  is called a factorisable polynomial of degree n if (i) the intersection of each

pair of straight lines  $l_i = 0$ ,  $l_j = 0$ ,  $i \neq j, i, j \in \{1, ..., n\}$  is nonempty and (ii) for each i = 1, ..., n, the straight line  $l_i = 0$  has no critical points of the function  $\Pi_{j \neq i} l_j$ .

The geometrical structure of a factorisable polynomial of degree  $n \geq 3$  is described in Theorem 1 of [17]. When n=4, such geometrical composition is as follows: the parabolic curve of f is a quartic smooth and compact curve with three connected components. The unbounded component  $C_u$  is hyperbolic and the graph of f has eight godrons, all of index -1. Moreover, the flechodal curve is only constituted by the straight lines  $l_i(x,y) = 0, i = 1, \ldots, 4$ .







Fig. 7: Hessian and flecnodal curves for a factorisable polynomial.

**Fig. 8:** Foliation of  $\mathbb{Y}_k$  for a quartic factorisable polynomial.

In Fig. 7, we show the affine geometrical structure of the quartic factorisable polynomial g(x,y) = y(x+3)(x-y)(y+x-3). By Remark 4.11, each field  $\widetilde{\mathbb{X}}_k$  associated to this example has two singular points at infinity while each  $\mathbb{Y}_k$  has eight singular points at infinity. We conclude this example by offering in Fig. 8 a picture of the qualitative behaviour of  $\mathbb{Y}_k$  in the two hemispheres.

# 5 Upper bounds for the number of godrons

In the first part of this chapter, we prove a formula that relates the Euler characteristic of the surface  $B^{\pm}$  with the Poincaré indices at singularities of  $\widetilde{\mathbb{X}}$  when such field is defined on  $B^{\pm}$ , respectively. In the second part, as an application, we give an upper bound for the number of godrons when the projective Hessian curve of f is only constituted by exterior ovals. Before stating our results, we introduce some definitions.

The tangency of the asymptotic line with the Hessian curve of f at a godron is either, exterior or interior [4]. In the first case, we say that such godron has an interior tangency and in the second case, an exterior tangency.

**Theorem 5.1** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree n whose graph  $S_f$  is generic and its projective Hessian curve is not tangent to the line at

infinity. Assume that  $f_n$  has no repeated factors and that  $\widetilde{\mathbb{X}}$  is a projective extension of a field of asymptotic directions. Then

$$\sum_{\xi \in \operatorname{Sing}(\widetilde{\mathbb{X}})} \operatorname{Ind}_{\xi} \left( \widetilde{\mathbb{X}} \right) = \chi \left( B^{\epsilon} \right) + \frac{P_{i} - P_{e}}{2},$$

where  $\epsilon$  is either + or - and  $\widetilde{\mathbb{X}}$  is defined on  $B^{\epsilon}$ . In both cases,  $P_i$  denotes the number of godrons with an interior tangency and  $P_e$ , with an exterior tangency.

**Proof.** Since the projective Hessian curve is transversal to the line at infinity z = 0, all tangencies of  $\widetilde{\mathbb{X}}$  with the projective Hessian curve occur in the Hessian curve of f, Hess f(x,y) = 0.

Suppose that  $\mathbb{X}$  is defined on the smooth surface  $B^-$ . This surface is composed by a finite number of orientable connected components denoted by  $D_1, \ldots, D_s$  and a connected component  $\mathbb{M}$  homeomorphic to a closed Möbius strip with a finite number of open discs removed. For  $l=1,\ldots,s$ , we denote by  $P_i^{D_l}$  and  $P_e^{D_l}$  the number of godrons having an interior and exterior tangency on the boundary of  $D_l$ , respectively. Poincaré-Hopf's Theorem for surfaces with boundary implies

$$\sum_{l=1}^{s} \left( \sum_{\substack{\xi \in \operatorname{Sing}(\widetilde{\mathbb{X}}) \\ \xi \in D_{l}}} \operatorname{Ind}_{\xi}(\widetilde{\mathbb{X}}) \right) = \sum_{l=1}^{s} \chi\left(D_{l}\right) + \sum_{l=1}^{s} \frac{P_{i}^{D_{l}} - P_{e}^{D_{l}}}{2}. \tag{9}$$

Now, we shall prove a version of Poincaré-Hopf's Theorem for the nonorientable surface  $\mathbb{M}$ . The projective extension  $\widetilde{\mathbb{X}}$  is the restriction of a field  $\mathbb{Y}$  defined by (5) to a hemisphere. Such field  $\mathbb{Y}$  is defined on an orientable surface  $\mathbb{DM} \subset \mathbb{S}^2$  which is a double covering of  $\mathbb{M}$ . So,  $\chi(\mathbb{DM}) = 2\chi(\mathbb{M})$ . By considering the Poincaré-Hopf Theorem for the field  $\mathbb{Y}$ ,

$$\sum_{\substack{\xi \in \operatorname{Sing}(\mathbb{Y}) \\ \xi \in \mathbb{DM}}} \operatorname{Ind}_{\xi}(\mathbb{Y}) = \chi(\mathbb{DM}) + \frac{P_i^{\mathbb{DM}} - P_e^{\mathbb{DM}}}{2}.$$
 (10)

Since the number of tangencies that the field  $\mathbb Y$  has with the boundary of  $\mathbb D\mathbb M$  is twice the number of tangencies of  $\widetilde{\mathbb X}$  with the boundary of  $\mathbb M$ , we obtain the relation  $P_i^{\mathbb D\mathbb M} - P_e^{\mathbb D\mathbb M} = 2(P_i^{\mathbb M} - P_e^{\mathbb M})$ . Moreover, by Remark 4.10,  $\sum_{\xi \in \operatorname{Sing}(\mathbb Y)} \operatorname{Ind}_{\xi}(\mathbb Y) = 2 \sum_{\xi \in \operatorname{Sing}(\widetilde{\mathbb X})} \operatorname{Ind}_{\xi}(\widetilde{\mathbb X})$ . By replacing these expressions

in (10) we obtain the desired equality

$$\sum_{\substack{\xi \in \operatorname{Sing}(\widetilde{\mathbb{X}})\\\xi \in \mathbb{M}}} \operatorname{Ind}_{\xi} \left( \widetilde{\mathbb{X}} \right) = \chi \left( \mathbb{M} \right) + \frac{P_i^{\mathbb{M}} - P_e^{\mathbb{M}}}{2}. \tag{11}$$

The proof concludes from (9) and (11) since  $B^- = \mathbb{M} \coprod D_1 \coprod \cdots \coprod D_s$ .  $\square$ 

Points on a generic algebraic surface in  $\mathbb{CP}^3$  are also classified in terms of the maximum order of contact of the tangent lines at them with the surface. George Salmon proves in [24] that such a surface of degree n has 2n(n-2)(11n-24) godrons (points at which the asymptotic line is tangent to the parabolic curve [15]). This number is an upper bound for the number of godrons on a generic algebraic surface in  $\mathbb{RP}^3$ . When the graph of a polynomial  $f \in \mathbb{R}[x,y]$  of degree n is generic, an upper bound for the number of godrons is given ([10], Theorem 5), namely,

$$\# \{ \text{Godrons in } S_f \} \le (n-2)(5n-12).$$
 (12)

**Corollary 5.2** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree n such that  $S_f$  is generic. Suppose that the projective Hessian curve is not tangent to the line at infinity. If the polynomial  $f_n$  has k distinct real linear factors, then

$$P_i \le \frac{(n-2)(8n-21)+k}{2}$$
 and  $P_e \le 1 + \frac{(n-2)(8n-21)-k}{2}$ .

**Proof.** Since  $S_f$  is generic, the projective Hessian curve of f is an algebraic curve of degree 2n-4. By Theorem 2.4, the value  $\chi(B^+)$  satisfies:

$$-\frac{3(n-2)(n-3)}{2} \le \chi\left(B^+\right) \le 1 + \frac{3(n-2)(n-3)}{2}.$$

Because  $\chi(B^+)=1-\chi(B^-)$ , we have that  $\chi(B^-)$  satisfies the inequalities  $-\frac{3}{2}(n-2)(n-3) \leq \chi(B^-) \leq 1+\frac{3}{2}(n-2)(n-3)$ . In conclusion, we obtain

$$-\frac{3(n-2)(n-3)}{2} - 1 \le -\chi \left(B^{\pm}\right) \le \frac{3(n-2)(n-3)}{2}.$$
 (13)

If the set  $b^{\pm} \cap \mathbb{R}^2$  is hyperbolic, then, by Theorem 5.1

$$\frac{P_i - P_e}{2} = \sum_{\xi \in \operatorname{Sing}(\widetilde{\mathbb{X}})} \operatorname{Ind}_{\xi} \left( \widetilde{\mathbb{X}} \right) - \chi \left( B^{\pm} \right). \tag{14}$$

According to Corollary 4.7 and Theorem 4.8,  $\sum_{\xi \in \text{Sing}(\widetilde{\mathbb{X}})} \text{Ind}_{\xi}(\widetilde{\mathbb{X}}) = \frac{k}{2}$ . Therefore, by adding  $\frac{k}{2}$  to the inequalities (13) and using (14), we get

$$-3(n-2)(n-3) - 2 + k \le P_i - P_e \le 3(n-2)(n-3) + k. \tag{15}$$

The proof follows from inequalities (12) and (15).

When the Hessian curve of f is a convex compact curve and the set  $b^- \cap \mathbb{R}^2$  is hyperbolic, the second author of this paper joined to L.I. Hernández-Martínez and F. Sánchez-Bringas to prove that n(3n-14)+18 is an upper bound for the number of godrons lying on the boundary of the unbounded connected component  $C_u$  ([11], Theorem 10). In the following result, we improve such bound under different assumptions and we analyse the unbounded case: we give an upper bound for the number of godrons that are on the boundary of  $\mathbb{M}$ .

**Theorem 5.3** Let  $f \in \mathbb{R}[x,y]$  be a polynomial of degree n whose graph  $S_f$  is generic. Suppose that the projective Hessian curve of f, constituted only by exterior ovals, is convex and it is not tangent to the line at infinity. If H is contained in  $B^-$  and the polynomial  $f_n$  has k distinct real linear factors, then the maximal number of godrons is 3(n-2)(n-3)+k.

**Proof.** On the one hand, since any projective extension  $\widetilde{\mathbb{X}}$  of a field of asymptotic lines is defined on  $B^-$  the expression  $P_i - P_e$  satisfies the second inequality of (15), that is,  $P_i - P_e \leq 3 (n-2) (n-3) + k$ . On the other hand, all godrons have an interior tangency because the projective Hessian curve of f is convex and the set  $b^- \cap \mathbb{R}^2$  is hyperbolic. Therefore,  $P_e = 0$  and  $P_i$  equals the total number of godrons.

# 6 Appendix

**Proof of Theorem 4.8**. Let  $p \in \mathbb{S}^2 \subset \mathbb{R}^3 = \{(u, v, \omega)\}$  be a singular point at infinity of the field  $\mathbb{Y}_k, k = 1, 2$ . According to Corollary 4.7, a real linear factor l of  $f_n$  defines the point p and by hypothesis, the multiplicity of l is one. After a suitable linear change of coordinates on the xy-plane, we have that l(x, y) = y, p = (1, 0, 0) and

$$f_n(x,y) = y\left(\sum_{i=0}^{n-1} a_{i,n-i} x^i y^{n-1-i}\right), \text{ with } a_{n-1,1} \neq 0.$$
 (16)

In the chart u = 1 the fields  $\mathbb{Y}_k$ , k = 1, 2, restricted to the set  $\{(u, v, \omega) \in \mathbb{S}^2 | u > 0\}$  are described by the quadratic equation

$$\omega^{2} F_{vv}(1, v, \omega) (dv)^{2} + 2\omega B(1, v, \omega) dv d\omega + S(1, v, \omega) (d\omega)^{2} = 0.$$
 (17)

By Theorem 4.6, S(1,0,0) = 0 and  $H_f(p) < 0$ . Moreover, in a neighbourhood of the origin on the  $v\omega$ -plane the two fields of directions defined by (17) are described by (see equation (8))

$$R_k(v,\omega) dv + 2S(1,v,\omega) d\omega = 0, k = 1, 2.$$
 (18)

We denote by  $\mathcal{G}_1$  and  $\mathcal{G}_2$  the foliations of these fields. The proof is based on the following geometric idea. Consider the sets  $W_U = \{(v, \omega) \in \mathbb{R}^2 | \omega > 0\}$  and  $W_L = \{(v, \omega) \in \mathbb{R}^2 | \omega < 0\}$ . The key point is to prove that there exists a neighbourhood of the origin, denoted by  $W \subset \mathbb{R}^2$ , at which one of the two foliations,  $\mathcal{G}_1$  for example, is tangent, in  $W \cap W_U$ , to a vector field having a node at the the origin and, in  $W \cap W_L$ , is tangent to a nonsingular vector field. Simultaneously, we will have that the foliation  $\mathcal{G}_2$  is tangent to the same vector fields, but in this case in the sets  $W \cap W_L$  and  $W \cap W_U$ , respectively.

From the expression of the fields described in (18) and setting  $\tilde{S}(v,\omega) = S(1,v,\omega)$ , we define the following vector fields on the  $v\omega$ -plane which have similar qualitative behaviours.

$$Y_{k}\left(v,\omega\right) = \left(\tilde{S}\left(v,\omega\right),\,\omega\,T_{k}\left(v,\omega\right)\right),\quad k = 1,2,$$

where 
$$T_k(v, \omega) = -2B(1, v, \omega) + 2(-1)^k \sqrt{-H_f(1, v, \omega)}$$
.

It is clear that in a punctured neighbourhood of the origin the foliation  $\mathcal{G}_1$  is tangent to the vector field  $Y_1$  if  $\omega > 0$ , and tangent to the vector field  $Y_2$  if  $\omega < 0$ . Respectively, the foliation  $\mathcal{G}_2$  is tangent to the vector field  $Y_2$  when  $\omega > 0$ , and it is tangent to the vector field  $Y_1$  for  $\omega < 0$ .

Now, we shall describe the topological type of the origin. A straightforward calculation shows the equality

$$T_1(v,\omega) T_2(v,\omega) = 4 \tilde{S}(v,\omega) F_{vv}(1,v,\omega).$$
(19)

Because

$$B(1,0,0) = -\frac{\partial^2 f_n}{\partial u \partial v}|_{(1,0)} = -(n-1)a_{n-1,1} \quad \text{and} \quad H_f(1,0,0) = Hess f_n(1,0) = -(n-1)^2 a_{n-1,1}^2,$$

we assert that

$$T_k(0,0) = 2(n-1)a_{n-1,1} + 2(-1)^k(n-1)|a_{n-1,1}|$$

Therefore, if  $a_{n-1,1} > 0$ , then  $T_1(0,0) = 0$  and  $T_2(0,0) = 4(n-1)a_{n-1,1}$ . In case  $a_{n-1,1} < 0$ ,  $T_2(0,0) = 0$  and  $T_1(0,0) = 4(n-1)a_{n-1,1}$ . It allows us to analyse the linear part of the vector field  $Y_k$  at the origin.  $DY_k(0,0) =$ 

$$= \begin{pmatrix} 2\frac{\partial}{\partial v}\tilde{S}\left(v,\omega\right) & 2\frac{\partial}{\partial\omega}\tilde{S}\left(v,\omega\right) \\ \omega\frac{\partial}{\partial v}T_{k}\left(v,\omega\right) & \omega\frac{\partial}{\partial\omega}T_{k}(v,\omega) + T_{k}\left(v,\omega\right) \end{pmatrix}\Big|_{(0,0)}$$

$$= \begin{pmatrix} 2n(n-1)\frac{\partial}{\partial v}\sum_{i=0}^{n}\omega^{n-i}f_{i}\left(1,v\right) & 2n(n-1)\frac{\partial}{\partial\omega}\sum_{i=0}^{n}\omega^{n-i}f_{i}(1,v) \\ \omega\frac{\partial}{\partial v}T_{k}(v,\omega) & \omega\frac{\partial}{\partial\omega}T_{k}(v,\omega) + T_{k}(v,\omega) \end{pmatrix}\Big|_{(0,0)}$$

$$= \begin{pmatrix} 2n(n-1)a_{n-1,1} & 2n(n-1)\frac{\partial}{\partial\omega}f_{n-1}\left(1,v\right)\Big|_{v=0} \\ 0 & 2(n-1)a_{n-1,1} + 2(n-1)\left(-1\right)^{k}|a_{n-1,1}| \end{pmatrix}.$$

The matrix  $DY_1(0,0)$   $(DY_2(0,0))$  has two nonzero real eigenvalues with the same sign if  $a_{n-1,1} < 0$  (if  $a_{n-1,1} > 0$  respectively). In conclusion, if  $a_{n-1,1}$  is positive (negative) then the origin is a singular point of type node of the vector field  $Y_2$  (respectively  $Y_1$ ).

Now, suppose  $a_{n-1,1} > 0$  (the negative case is analogous) and consider the vector field

$$Z_{1}(v,\omega) = (T_{2}(v,\omega), 4\omega F_{vv}(1,v,\omega)).$$

Since  $T_2(0,0) \neq 0$ , the origin is a nonsingular point of this vector field. Moreover, by (19) this field satisfies the equality

$$T_2(v,\omega) Y_1(v,\omega) = \tilde{S}(v,\omega) Z_1(v,\omega).$$

This relation implies that  $Z_1$  is tangent to the foliation of  $Y_1$ .

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