# A Simple Bijection Between 231-Avoiding and 312-Avoiding Placements

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## Abstract

Stankova and West proved in 2002 that the patterns 231 and 312 are shape-Wilf-equivalent. Their proof was nonbijective. We give a new characterization of 231 and 312 avoiding full rook placements and use this to give a simple bijection that demonstrates the shape-Wilf-equivalence.

## 1. Introduction

For any pattern  $\tau \in S_k$ , let  $S_n(\tau)$  denote the set of permutations in  $S_n$  that avoid  $\tau$ , in the sense that they have no subsequence order-isomorphic to  $\tau$ . For any Ferrers board F, let  $S_F(\tau)$  denote the set of all full rook placements on F that avoid  $\tau$ . We say that two patterns  $\tau$  and  $\sigma$  are Wilf-equivalent, and write  $\tau \sim \sigma$ , if  $|S_n(\tau)| = |S_n(\sigma)|$  for all n > 0. We say that  $\tau$  and  $\sigma$  are shape-Wilf-equivalent, and write  $\tau \sim_s \sigma$ , if  $|S_F(\tau)| = |S_F(\sigma)|$  for all F. So shape-Wilf-equivalence implies Wilf-equivalence, as we see by considering square Ferrers boards. (The relevant definitions will be reviewed in Section 2.)

The concept of shape-Wilf-equivalence was introduced in [1], as a means for obtaining results about Wilf-equivalence. Since shape-Wilf-equivalence is stronger than Wilf-equivelence, positive results about it are rare. The only "general result" was obtained in [1], where it was shown that the patterns  $k \cdots 321$  and  $123 \cdots k$  are shape-Wilf-equivalent for every positive k. Later, in [4], Stankova and West proved that the patterns 231 and 312 are shape-Wilf-equivalent, and the motivation for our paper comes from their result. Their proof that  $|S_F(231)| = |S_F(312)|$  was nonbijective, and somewhat complicated. Our purpose here it to give a simple bijection between  $S_F(231)$  and  $S_F(312)$ .

We will do so by associating a sequence of nonnegative integers to each full rook placement on F, and characterizing those sequences that arise from 231-avoiding or 312-avoiding placements. We will give a simple way to transform a sequence arising from a 231-avoiding placement into a sequence arising from a 312-avoiding placement, and vice-versa.

In Section 2, we will review the needed definitions, define our bijection, and state the Theorems needed to verify that it is indeed a bijection. In Sections 3 and 4 we will prove these theorems.

Vit Jelínek has pointed out to us that a bijective proof of the shape-Wilf-equivalence of the patterns 231 and 312 can also be obtained from his work on pattern—avoidance in matchings. See [3], where a bijection is obtained by establishing an isomorphism between generating trees. Examples on small Ferrers boards show that Jelínek's bijection differs from ours.

# 2. The bijection

**Definitions.** Let  $\mathcal{A}$  be an  $n \times n$  array of unit squares and coordinatize it by placing the bottom left corner of  $\mathcal{A}$  at the origin in the xy-plane. We refer to the corners of all the squares in  $\mathcal{A}$  as vertices and reference them by their cardinal position. For example, the upper right corner will be called the NE corner. For any vertex V = (a, b) we define R(V) to be the rectangular array of squares bounded by the lines x = 0, x = a, y = 0, and y = b.

A Ferrers board is any subset F of A with the property that  $R(V) \subseteq F$  for each vertex in F. We define the right/up border of F to be the border of F excluding the vertical left hand side and horizontal bottom.

Next we need to define the generalization of a permutation for the context of Ferrers boards.

**Definitions.** A rook placement on a Ferrers board F is a subset of F that contains at most one square from each column of F and at most one square from each row of F. We indicate these squares by putting markers in them. Likewise a full rook placement is a rook placement such that each row and each column has exactly one marker in it. We say a rook placement P on a Ferrers board F avoids  $\tau$  if and only if for every vertex V on the right/up border the permutation that is order-isomorphic to the restriction of P to R(V) avoids  $\tau$  in the usual sense.

**Definition.** For any rook placement P on F and any vertex V of F, we denote by S(P,V) the maximal length of an increasing sequence of P in R(V).

To define our bijection from  $S_F(231)$  to  $S_F(312)$ , we first associate to each full rook placement P on F a sequence S(P, F).

**Notation**. For any full rook placement on a Ferrers board F, S(P, F) denotes the sequence of nonnegative integers obtained by taking S(P, V) for all V on the right/up border of F, starting with the vertex at the top left corner of F.

**Theorem 1.** If P is in  $S_F(231)$  or  $S_F(312)$ , then S(P,F) and F determine P.

We will prove Theorem 1 in Section 3 by giving a "reverse algorithm" for the map  $P \to S(P, F)$  consequently establishing injectivity.

Readers familiar with Fomin's growth diagram algorithm will note that the values of S(P, F) are the first entries of the partitions in the oscillating tableaux produced by the algorithm. Theorem 1 may be restated by saying that for P

in  $S_F(231)$  or P in  $S_F(312)$  the first entries in the partitions determine the oscillating tableaux.

To define our bijection, we will need to characterize those sequences that arise from  $P \in S_F(231)$  or  $P \in S_F(312)$ .

**Definition.** If F is a Ferrers board, then an F-sequence is a sequence of nonnegative integers assigned to the vertices on the right/up border of F, starting with the vertex at the top left corner.

**Definition** (the 231-conditions). If F is a Ferrers board and S is an F-sequence, then the 231-conditions for the pair (F, S) are the following three conditions:

- (i) (monotonicity conditions) If  $V_1$  and  $V_2$  are vertices on the right/up border and  $V_1$  is either directly to the left of  $V_2$  or directly below  $V_2$  then  $S(V_1) \leq S(V_2) \leq S(V_1) + 1$ .
- (ii) (0-conditions) The first and last values of S are 0, and there do not exist consecutive vertices  $V_1$  and  $V_2$  such that  $S(V_1) = 0 = S(V_2)$ .
- (iii) (diagonal condition) If  $V_1$  and  $V_2$  are vertices on the right/up border that are at the left and right ends of a diagonal with slope -1 that lies entirely within F, then  $S(V_1) < S(V_2)$ .

**Definition** (the 312-conditions). With S as in the preceding definition, the 312-conditions for the pair (F,S) are the same as the 231-conditions, except that we reverse the inequalities in the diagonal condition.

The following definition is often useful when dealing with the diagonal condition.

**Definition.** We refer to a pair of vertices  $V_1, V_2$  as diagonal vertices or F-diagonal vertices if they are on the right/up border of F and are at the left and right ends of a diagonal with slope -1 that lies entirely within F.

**Theorem 2.** If F is a Ferrers board whose longest row and longest column have the same length, and S is an F-sequence, then there exists  $P \in S_F(231)$  (respectively,  $P \in S_F(312)$ ) such that S(P,F) = S if and only if (F,S) satisfies the 231-conditions (respectively, the 312-conditions).

Theorem 2 will be proved in Section 4.

To obtain our bijection, we need a way to take  $P \in S_F(231)$  (respectively,  $P \in S_F(312)$  and transform S(P, F) into a sequence satisfying the 312-conditions (respectively, the 231-conditions). To do this we need our first lemma.

**Lemma 1.** For any Ferrers board F and vertex V on its right/up border, there exists an integer N(F, V) such that for every full rook placement P on F, there are exactly N(F, V) markers of P in R(V).

*Proof.* Take any full rook placement P on F. We proceed inductively, starting with the vertex V at the top left corner. Clearly, P has no markers in R(V). If  $V_1, V_2$  are vertices on the right/up border such that  $V_1$  is either directly to the left of  $V_2$  or directly below it, then the number of markers of P in  $R(V_2)$  is one greater than the number in  $R(V_1)$ .

**Definition.** If P is a full rook placement on a Ferrers board F, and S = S(P, F), then we define another F-sequence  $S^+$  by letting  $S^+(V) = 0$  if S(V) = 0, and  $S^+(V) = N(F, V) + 1 - S(V)$  otherwise.

It is clear that  $S^+$  is an F-sequence, because  $S(V) \leq N(F, V)$ .

**Lemma 2.** Let P be a full rook placement on F and let S = S(P, F). Then if (F, S) satisfies the 231-conditions (respectively, the 312-conditions),  $(F, S^+)$  satisfies the 312-conditions (respectively, the 231-conditions).

*Proof.* We give the proof when (F, S) satisfies the 231-conditions. The proof of the other case is nearly identical.

To verify the monotonicity conditions for  $(F, S^+)$ , first let  $V_1, V_2$  be vertices on the right/up border of F such that  $V_1$  is directly to the left of  $V_2$ . In the case that  $S(V_1) = 0$  then  $S(V_2) = 1$  by the 231-conditions. Observe that  $N(F, V_2) = 1$  as well, which implies that  $S^+(V_1) = 0$  and  $S^+(V_2) = 1$ . In the case that  $S(V_1) \neq 0$  then we know that  $S(V_1) \leq S(V_2) \leq S(V_1) + 1$ . Since  $N(F, V_1) + 1 = N(F, V_2)$  then we get

$$N(F, V_1) + 2 - S(V_1) \ge N(F, V_2) + 1 - S(V_2) \ge N(F, V_1) + 1 - S(V_1)$$

and hence  $1 + S^+(V_1) \ge S^+(V_2) \ge S^+(V_1)$ . The proof is the same if  $V_1$  is directly below  $V_2$  and therefore monotonicity holds.

The 0-conditions hold for  $(F, S^+)$  because  $S^+(V) = 0$  if and only if S(V) = 0. To verify the 312-diagonal condition for  $(F, S^+)$ , let  $V_1, V_2$  be F-diagonal vertices. We note that  $N(F, V_1) = N(F, V_2)$ , because N(F, V) increases by one each time we move to the right on the right/up border, and decreases by one each time we move downward, and the number of rightward steps between  $V_1$  and  $V_2$  equals the number of downward steps. By the 231-diagonal condition for (F, S), we have  $S(V_1) \leq S(V_2)$ . If  $S(V_1) \neq 0$ , then since  $N(F, V_1) = N(F, V_2)$ , we have  $S^+(V_1) \geq S^+(V_2)$ . If  $S(V_1) = 0$  then  $N(F, V_1) = 0$  so  $N(F, V_2) = 0$  and thus  $S(V_2) = 0$ .

**Definitions.** Let  $P \in S_F(231)$  and let S = S(P, F). By Theorems 1 and 2, let  $\alpha(P)$  denote the unique element of  $S_F(312)$  such that  $S(\alpha(P), F) = S^+$ . For  $P \in S_F(312)$ , define  $\beta(P) \in S_F(231)$  analogously.

**Theorem 3.** The maps  $\alpha: S_F(231) \to S_F(312)$  and  $\beta: S_F(312) \to S_F(231)$  are inverses, and therefore both are bijections.

*Proof.* This follows from the fact that if S = S(P, F) for P in either  $S_F(231)$  or  $S_F(312)$ , then  $S^{++} = S$ .

**Remark 1.** Although our proofs depend on the fact that we are working with full rook placements it follows from Theorem 3 that for any Ferrers board F the number of 231-avoiding rook placements on F is equal to the number of 312-avoiding rook placements on F. The idea is as follows. For any rook placement P on F we have the set C of column numbers corresponding to columns that contain a marker. Similarly we get the set C of row numbers. Now we may consider the set of squares

$$F_P = \{(c, r) | c \in \mathcal{C}, r \in \mathcal{R}\}.$$

Observe that we may view  $F_P$  as a Ferrers board by sliding all the squares down and then left. Likewise, P may be viewed as a full rook placement on  $F_P$ . We may now define an equivalence relation  $\sim$  on rook placements by saying two placements P and Q are related if and only if  $F_P = F_Q$ . Now let A (respectively, B) be the partition under  $\sim$  of the set of 231-avoiding (respectively, 312-avoiding) rook placements on F. Clearly |A| = |B| and if  $\overline{P} \in A$  then Theorem 3 implies that  $|\overline{P}| = |\alpha(P)|$  proving our claim.

# 3. The reverse algorithm

We will prove Theorem 1 by developing an "reverse algorithm" for the map  $P \to S(P, F)$ . To do this, we must first establish some properties of S(P, V).

**Lemma 3.** Let P be a rook placement on Ferrers board F, and let  $V_1$  and  $V_2$  be vertices of F. Then if  $V_1$  is directly to the left of  $V_2$ , or directly below  $V_2$ , we have

$$S(P, V_1) \le S(P, V_2) \le S(P, V_1) + 1.$$

*Proof.* This follows immediately from the definition of S(P, V).

**Lemma 4.** Suppose P is a rook placement on a Ferrers board F, and A, B, C are the vertices at the NW, NE, and SE corners, respectively, of a square  $\mathcal{B}$  in F. Let a, b, c be the values of S(P, V) at V = A, B, C, respectively. Then if P has no marker in  $\mathcal{B}$ , we have  $b = \max(a, c)$ . And P has a marker in  $\mathcal{B}$  if and only if b = a + 1 = c + 1.

*Proof.* First suppose P has no marker in  $\mathcal{B}$ . Consider an increasing sequence I of length b in R(B). If I is contained in R(C), then  $b \leq c$ . If I is not contained in R(C), then I must include a marker in the top row of R(B), so I terminates at this marker, which is to the left of  $\mathcal{B}$ , and therefore I is contained in R(A), yielding  $b \leq a$ . In either case,  $b \leq \max(a, c)$ . Since the reverse inequality follows from Lemma 3, we have  $b = \max(a, c)$ .

It follows that if P has no marker in  $\mathcal{B}$  then we cannot have b=a+1=c+1. It is clear that if P has a marker in  $\mathcal{B}$  then b=a+1=c+1.

**Lemma 5.** Suppose  $P \in S_F(231)$  and  $V_1, V_2$  are vertices of F such that  $V_1$  is directly below  $V_2$ . Suppose P has a marker X in the top row of  $R(V_2)$ , and another marker Y in  $R(V_2)$  that is to the right of X. Then  $S(P, V_1) = S(P, V_2)$ .

Proof. Since  $P \in S_F(231)$ , P has no 231-patterns in  $R(V_2)$ . If R is the set of markers of P in  $R(V_2)$  that are to the right of X, and L is the set of markers of P in  $R(V_2)$  that are to the left of X, it follows that all elements of R are in higher rows than all elements of L. Since  $R \neq \emptyset$  because of the presence of Y, both  $S(P, V_1)$  and  $S(P, V_2)$  are the sum of the maximal length of an increasing sequence in L and the maximal length of an increasing sequence in R. This proves the lemma.

Proof of Theorem 1. It will suffice to prove the result for  $P \in S_F(231)$ , for then by considering the inverse placement P' on the conjugate board F', we obtain the result for  $P \in S_F(312)$ .

So let  $P \in S_F(231)$ . Suppose the bottom row of F contains exactly n squares, and the right-hand column of F contains exactly r squares. Let the values of S(P,V) on the line x=n be  $b_r,\ldots,b_0$ , from top to bottom, and let the values on the line x=n-1 be  $a_r,\ldots,a_0$ , again from top to bottom. The values  $a_r,b_r,b_{r-1},\ldots,b_0$  are included in S(P,F), and we will show that from these values we can determine the location of the marker of P in the right-hand column, and the values of  $a_{r-1},\ldots,a_0$ .

Choose j as large as possible such that  $b_j > b_{j-1}$ . Then the markers  $X_r, \ldots, X_{j+1}$  of P in rows  $r, \ldots, j+1$  are not in the right-hand column, and since there is a marker Y in the right-hand column and there are no 231-patterns in R((n,r)), the markers  $X_r, \ldots, X_{j+1}$  must form a decreasing sequence. Applying Lemma 5 repeatedly, with the X of that lemma being  $X_r, \ldots, X_{j+1}$  in turn, we conclude that  $a_r = \cdots = a_{j+1}$ . The marker  $X_j$  in row j must be Y, for else, using  $X_j, Y$  and Lemma 5, we would have  $b_j = b_{j-1}$ . It follows that  $a_j = b_{j-1}$  and  $a_i = b_i$  for  $i \leq j-1$ .

We have determined the placement of the marker Y in the right-hand column and the values of  $a_{r-1}, \ldots, a_0$ . If we delete the right-hand column and the row containing the marker Y we obtain a smaller board  $F^*$  and a placement  $P^* \in S_{F^*}(231)$  such that the sequence of values  $S(P^*, F^*)$  is S(P, F) with the terminal r+1 values  $b_r, \ldots, b_0$  replaced by the r-1 values  $a_{r-1}, \ldots, a_j, a_{j-2}, \ldots, a_0$ . By iterating the above argument we can proceed to determine the positions of all the markers in P, from right to left.

# 4. The proof of Theorem 2

To prove Theorem 2, it will suffice to prove the assertion about  $P \in S_F(231)$ , for the assertion about  $P \in S_F(312)$  then follows by considering the inverse placement P' on the conjugate board F', with  $P' \in S_{F'}(231)$ .

**Notation**. Let F be a Ferrers board whose longest row and longest column each contain exactly n squares. Let  $\mathcal{B}$  be the square at the top of the right-hand column of F, and suppose  $\mathcal{B}$  is in row r. Let A, B, C be the vertices at the NW, NE, and SE corners of  $\mathcal{B}$ .

We will first prove the necessity of the 231-conditions, then the sufficiency.

Proof of Necessity.

The monotonicity conditions are clear by Lemma 3, and it is also clear that S(P,F) starts and ends with the value 0. If the values of S(P,F) at two successive vertices were both 0, then if one of these vertices were below (respectively, to the left of) the other, F would have a row (respectively, a column) with no marker in it, contradicting the fact that P is a full rook placement.

We now prove the diagonal condition by induction on the number of squares in F. For a board with one square, it is obvious that the 231-diagonal condition holds for the only possible placement. Assume now that  $P \in S_F(231)$  and the result holds for all boards with fewer squares than F.

## Case 1: $\mathcal{B}$ contains a marker.

Let  $V_0, \ldots V_{2n}$  be the sequence of vertices on the right/up border of F starting at the top left corner of F, and let  $B = V_k$ . Since  $S(P, V_{k-1}) = S(P, V_{k+1})$  by Lemma 4, it will suffice to check the diagonal condition for all diagonal vertices not containing  $V_{k+1}$ . To this end denote by  $F^*$  and  $P^*$  the board and placement obtained by deleting the row and column of  $\mathcal{B}$  from F. Now let  $V_i^*$  be the vertex directly under  $V_i$  for  $0 \le i \le k-1$  and the vertex directly to the left of  $V_i$  for  $k+2 \le i \le 2n$ . Observe that the sequence

$$V_0^*, \dots, V_{k-1}^*, V_{k+2}^*, \dots, V_{2n}^*$$

is precisely the sequence of vertices on the right/up border of  $F^*$ . Fix  $i, j \notin \{k, k+1\}$ . It is clear that

$$S(P, V_i) = S(P^*, V_i^*)$$
(1)

and that

$$V_i, V_j$$
 are  $F$ -diagonal vertices iff  $V_i^*, V_j^*$  are  $F^*$ -diagonal vertices. (2)

By induction  $S(P^*, F^*)$  satisfies the diagonal condition. Therefore (1) and (2) directly imply that S(P, F) also satisfies the diagonal condition.

## Case 2: B does not contain a marker.

Note that in this case we must have  $r \geq 2$ , and consider the smaller board  $F^* = F \setminus \mathcal{B}$ . By the induction hypothesis the pair  $(F^*, S(P, F^*))$  satisfies the diagonal condition. So we only need to show that  $S(P, A) \leq S(P, C)$ . Since  $\mathcal{B}$  contains no marker, Lemma 5 implies that S(P, C) = S(P, B). By monotonicity we must have  $S(P, A) \leq S(P, B)$  which concludes this case.  $\square$ 

## Proof of Sufficiency.

We prove the sufficiency of the 231-conditions by again using induction on the number of squares in F. Let S be an F-sequence such that (F, S) satisfies the 231-conditions, and let a, b, c denote S(A), S(B), S(C), respectively.

Case 1:  $b \neq a$  and  $b \neq c$ .

First note that in this case we must have a+1=b=c+1 by monotonicity of S. Define  $V_0, \ldots, V_{2n}$  as in the proof of necessity, with  $B=V_k$ . Also define  $F^*$ ,  $P^*$ , and vertices  $V_i^*$  as in that proof, so that the sequence

$$V_0^*, \dots, V_{k-1}^*, V_{k+2}^*, \dots, V_{2n}^*$$

is precisely the sequence of vertices on the right/up border of  $F^*$ .

Now define  $S^*(V_i^*) = S(V_i)$  for  $i \notin \{k, k+1\}$ . We claim that  $(F^*, S^*)$  satisfies the 231-conditions. Since a = c it is clear that  $(F^*, S^*)$  satisfies the monotonicity conditions. To see that  $(F^*, S^*)$  satisfies the 0-conditions it suffices to show that  $a \neq 0$  when  $r \geq 2$ . So suppose  $r \geq 2$  and a = 0. Draw the diagonal  $\ell$  extending NW from A and let  $V_1$  be the first vertex on the right/up border where  $\ell$  passes outside of F. Since A is above the diagonal from upper left to lower right, there must be a vertex  $V_2$  on the right/up border directly to the left of  $V_1$ . Since a = 0, the diagonal and monotonicity conditions for (F, S) yield  $S(V_1) = 0$  and  $S(V_2) = 0$ , contradicting the 0-conditions for (F, S).

To verify the diagonal condition for  $(F^*, S^*)$ , note that:  $V_i, V_j$  are F-diagonal vertices if and only if  $V_i^*, V_j^*$  are  $F^*$ -diagonal vertices. Therefore since (F, S) satisfies the diagonal condition so must  $(F^*, S^*)$ .

Since  $(F^*, S^*)$  satisfies the 231-conditions, there exists, by the induction hypothesis,  $P^* \in S_{F^*}(231)$  such that  $S(P^*, F^*) = S^*$ . Now restore the row and column we removed from F and place a marker X in square  $\mathcal{B}$  to obtain a placement P on F. It is clear that  $P \in S_F(231)$ , because of the position of X. Lastly we show that S(P, F) = S. Note that for  $V_i \neq B$  or C

$$S(P, V_i) = S(P^*, V_i^*) = S^*(V_i^*) = S(V_i).$$
(3)

Since P has a marker in  $\mathcal{B}$ , we have S(P,A)+1=S(P,C)+1=S(P,B). By (3) we know that S(P,A)=a. Putting these together we have that S(P,B)=a+1=b and S(P,C)=a=c.

Case 2: b = a or b = c.

Note that in this case we cannot have r=1, because if r=1 then c=0, so by the diagonal condition for (F,S), a=0 and thus b=0, violating the 0-conditions for (F,S). So we can let D be the vertex directly below C. Let E be the vertex at the SW corner of  $\mathcal{B}$ , and let d denote S(D). Denote by  $F^*$  the Ferrers board  $F \setminus \mathcal{B}$ .

Now consider the function  $S^*$  defined by

$$S^*(V) = \begin{cases} S(V) & \text{if } V \neq E \\ \min(a, d) & \text{if } V = E \end{cases}$$

where V is a vertex on the right/up border of  $F^*$ .

In order to apply the induction hypothesis to the smaller pair  $(F^*, S^*)$  we need to know that  $(F^*, S^*)$  satisfies the 231-conditions. Since  $r \geq 2$ , we have  $a \neq 0$  as in Case 1, and  $a \leq c$ . Since (F, S) satisfies both the monotonicity and 0-conditions it easily follows that  $(F^*, S^*)$  satisfies these two conditions as well.

So it only remains to show that  $(F^*, S^*)$  satisfies the diagonal condition. Now for the diagonal extending SE from E we have  $S^*(E) = \min(a, d) \le d = S^*(D)$ . Next consider the diagonal extending NW from E and let its right-most intersection point with the right/up border be  $E_0$ . (Note that  $E_0$  exists since  $r \ge 2$ .) Call the vertex to  $E_0$ 's immediate right  $A_0$  and note that  $A_0$  must be on the right/up border. Our choice of  $E_0$  implies that  $E_0$  and  $E_0$  are diagonal vertices. Now if  $E_0$  must be our definitions we have

$$S^*(E_0) \le S^*(A_0) = S(A_0) \le S(A) = S^*(E).$$

If on the other hand  $\min(a, d) = d$  then clearly  $S^*(E_0) \leq S^*(E)$  since  $E_0$  and D are diagonal vertices in F. Therefore  $(F^*, S^*)$  satisfies the diagonal condition.

Since the pair  $(F^*, S^*)$  satisfies the 231-conditions then by the induction hypothesis there exists a 231-avoiding full rook placement P on  $F^*$  such that  $S^* = S(P, F^*)$ . We claim that P is also a 231-avoiding full rook placement on F such that S = S(P, F).

To see that S = S(P, F) let V be any vertex on the right/up border of F. If  $V \neq B$  then we have  $S(P, V) = S^*(V) = S(V)$ . If V = B then since  $\mathcal{B}$  does not contain a marker we have  $S(P, B) = \max(a, c) = b$  where the last equality holds because  $\max(a, c) \leq b$  by the monotonicity of S and  $b \leq \max(a, c)$  since b = a or b = c in this case.

Lastly we need to show that P is a 231-avoiding placement on F. Assume it is not and let XYZ be a 231-pattern in F. Let marker Y be in square  $\mathcal{B}_1$  and Z be in square  $\mathcal{B}_2$ . Note that square  $\mathcal{B}_1$  must be in row r, along with square  $\mathcal{B}$ . Likewise, note that  $\mathcal{B}_2$  must be in the right-hand column. Since P in  $F^*$  has no 231-patterns then all the markers in the columns strictly between  $\mathcal{B}_1$  and  $\mathcal{B}$  must be above row r. For if not then some marker W is either in a row below X's row in which case XYW is a 231-pattern in  $F^*$ , or W is in a row between X's row and Y's row resulting in the 231-pattern XWZ. So if  $\overline{A}$  and  $\overline{E}$  denote the vertices in the NE and SE corners of  $\mathcal{B}_1$  respectively then it follows that, letting  $e = \min{(a,d)}$ ,

$$S(P, \overline{A}) = S(P, A) = a$$
 and  $S(P, \overline{E}) = S(P, E) = e$ .

If we could show that a=e, it would follow from Lemma 4 that  $\mathcal{B}_1$  could not contain a marker. But this would be a contradiction as  $\mathcal{B}_1$  contains the marker Y, and we would be done. To show that a=e, note that Z cannot be in row r-1, because XYZ is a 231-pattern. Since row r-1 must contain a marker, Lemma 5 implies that c=d and therefore  $e=\min(a,d)=a$  since  $a\leq c$  by the diagonal condition.  $\square$ 

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