DIMENSIONS OF MONOMIAL VARIETIES

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ABSTRACT. The dimensions of certain varieties defined by monomials are computed using only high school algebra.

1. Krull dimension and varieties

In this paper, a ring R is a commutative ring with a multiplicative identity, and a field \mathbf{F} is an infinite field of any characteristic.

Let S be a nonempty set of polynomials in $\mathbf{F}[t_1, \ldots, t_n]$. The variety (also called the algebraic set) V determined by S is the set of points in \mathbf{F}^n that are common zeros of the polynomials in S, that is,

$$V = V(S) = \{(x_1, \dots, x_n) \in \mathbf{F}^n : f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}.$$

The vanishing ideal $\mathfrak{I}(V)$ is the set of polynomials that vanish on the variety V, that is,

$$\mathfrak{I}(V) = \{ f \in \mathbf{F}[t_1, \dots, t_n] : f(x_1, \dots, x_n) = 0 \text{ for all } (x_1, \dots, x_n) \in V \}.$$

We have $S \subseteq \mathfrak{I}(V)$, and so $\mathfrak{I}(V)$ contains the ideal generated by S. The quotient ring

$$\mathbf{F}(V) = \mathbf{F}[t_1, \dots, t_n] / \mathfrak{I}(V)$$

is called the *coordinate ring* of V.

Note that $\Im(V)$ contains S, and so $\Im(V)$ contains the ideal generated by S.

A prime ideal chain of length n in the ring R is a strictly increasing sequence of n+1 prime ideals of R. The Krull dimension of R is the supremum of the lengths of prime ideal chains in R. We define the dimension of the variety V as the Krull dimension of its coordinate ring $\mathbf{F}(V)$.

It is a basic theorem in commutative algebra that the polynomial ring $\mathbf{F}[t_1,\ldots,t_n]$ has Krull dimension n. (Nathanson [4] gives an elementary proof. Other references are Atiyah and Macdonald [1, Chapter 11], Cox, Little, and O'Shea [2, Chapter 9], and Kunz [3, Chapter 2]). If $S = \{0\} \subseteq \mathbf{F}[t_1,\ldots,t_n]$ is the set whose only element is the zero polynomial, then $V = V(\{0\}) = \mathbf{F}^n$, and the vanishing ideal of V is $\mathcal{I}(V) = \mathcal{I}(\mathbf{F}^n) = \{0\}$. We obtain the coordinate ring

$$\mathbf{F}(V) = \mathbf{F}[t_1, \dots, t_n]/\Im(V) \cong \mathbf{F}[t_1, \dots, t_n]$$

and so the variety \mathbf{F}^n has dimension n.

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We adopt standard polynomial notation. Let \mathbf{N}_0 denote the set of nonnegative integers. Associated to every n-tuple $I = (i_1, \dots, i_n) \in \mathbf{N}_0^n$ is the monomial

$$t^I = t_1^{i_1} \cdots t_n^{i_n}.$$

Every polynomial $f \in R[t_1, \ldots, t_n]$ can be represented uniquely in the form

$$f = \sum_{I \in \mathbf{N}_0^n} c_I t^I$$

where $c_I \in R$ and $c_I \neq 0$ for only finitely many $I \in \mathbf{N}_0^n$.

In this paper, two results about polynomials from high school algebra will enable us to compute the dimensions of certain varieties defined by monomials. The first result is a factorization formula, and the second result follows from the fact that a polynomial of degree d has at most d roots in a field.

Lemma 1. For every nonnegative integer i, there is the polynomial identity

(1)
$$u^{i} - v^{i} = (u - v)\Delta_{i}(u, v).$$

where

(2)
$$\Delta_i(u,v) = \sum_{i=0}^{i-1} u^{i-1-j} v^j.$$

Proof. We have

$$\begin{split} (u-v)\sum_{j=0}^{i-1}u^{i-1-j}v^j &= \sum_{j=0}^{i-1}u^{i-j}v^j - \sum_{j=0}^{i-1}u^{i-1-j}v^{j+1} \\ &= \sum_{j=0}^{i-1}u^{i-j}v^j - \sum_{j=1}^{i}u^{i-j}v^j \\ &= u^i - v^i. \end{split}$$

Lemma 2. Let \mathbf{F} be an infinite field. A polynomial $f \in \mathbf{F}[t_1, \dots, t_n]$ satisfies $f(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \mathbf{F}^n$ if and only if f = 0.

Proof. The proof is by induction on n. Let n = 1. A nonzero polynomial $f \in \mathbf{F}[t_1]$ of degree d has at most d roots in \mathbf{F} , and so $f(x_1) \neq 0$ for some $x_1 \in \mathbf{F}$. Thus, if $f(x_1) = 0$ for all $x_1 \in \mathbf{F}$, then f = 0.

Let $n \geq 1$, and assume that the Lemma holds for polynomials in n variables. Let $f \in \mathbf{F}[t_1, \ldots, t_n, t_{n+1}]$ have degree d in the variable t_{n+1} . There exist polynomials $f_i \in \mathbf{F}[t_1, \ldots, t_n]$ such that

$$f = f(t_1, \dots, t_n, t_{n+1}) = \sum_{i=0}^{d} f_i(t_1, \dots, t_n) t_{n+1}^i$$

For all $(x_1, \ldots, x_n) \in \mathbf{F}^n$, the polynomial

$$g(t_{n+1}) = f(x_1, \dots, x_n, t_{n+1}) = \sum_{i=0}^{d} f_i(x_1, \dots, x_n) t_{n+1}^i \in \mathbf{F}[t_{n+1}]$$

satisfies $g(x_{n+1}) = 0$ for all $x_{n+1} \in \mathbf{F}$, and so g = 0. Therefore, $f_i(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \mathbf{F}^n$ and $i = 0, 1, \dots, d$. By the induction hypothesis, $f_i = 0$ for all $i = 0, 1, \dots, d$, and so f = 0.

2. An example of a plane curve

A hypersurface is a variety that is the set of zeros of one nonzero polynomial. A plane algebraic curve is a hypersurface in \mathbf{F}^2 . In this section we compute the dimension of a hypersurface V in \mathbf{F}^{m+1} defined by a polynomial of the form

$$f^* = t_{m+1} - \lambda t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m}$$

where $\lambda \in \mathbf{F}$ and $(a_1, \ldots, a_m) \in \mathbf{N}_0^m$. We shall prove that the monomial hypersurface

$$V = \{(x_1, \dots, x_m, x_{m+1}) \in \mathbf{F}^{m+1} : f^*(x_1, \dots, x_m, x_{m+1}) = 0\}$$
$$= \{(x_1, \dots, x_m, x_{m+1}) \in \mathbf{F}^{m+1} : x_{m+1} = \lambda x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}\}$$

has dimension m.

We begin with an example. Consider the monomial $4t_1^3$ and the curve in ${\bf F}^2$ defined by the polynomial

$$f^* = t_2 - 4t_1^3 \in \mathbf{F}[t_1, t_2].$$

Let

$$V = \{(x_1, x_2) \in \mathbf{F}^2 : f^*(x_1, x_2) = 0\} = \{(x_1, x_2) \in \mathbf{F}^2 : x_2 = 4x_1^3\}.$$

We shall prove that the vanishing ideal $\mathfrak{I}(V)$ is the principal ideal generated by f^* . Because $f^* \in \mathfrak{I}(V)$, it suffices to show that every polynomial in $\mathfrak{I}(V)$ is divisible by f^* .

For
$$I = (i_1, i_2) \in \mathbb{N}_0^2$$
 and $t^I = t_1^{i_1} t_2^{i_2}$, let

$$b_1 = i_1 + 3i_2$$

and let Δ_{i_2} be the polynomial defined by (2) in Lemma 1. We have

$$t^{I} - 4^{i_{2}}t_{1}^{b_{1}} = t_{1}^{i_{1}}t_{2}^{i_{2}} - 4^{i_{2}}t_{1}^{i_{1}+3i_{2}} = t_{1}^{i_{1}}\left(t_{2}^{i_{2}} - 4^{i_{2}}t_{1}^{3i_{2}}\right)$$

$$= t_{1}^{i_{1}}\left(t_{2}^{i_{2}} - \left(4t_{1}^{3}\right)^{i_{2}}\right) = t_{1}^{i_{1}}\Delta_{i_{2}}(t_{2}, 4t_{1}^{3})\left(t_{2} - 4t_{1}^{3}\right)$$

$$= g_{I}f^{*}$$

where $g_I = t_1^{i_1} \Delta_{i_2}(t_2, 4t_1^3) \in \mathbf{F}[t_1, t_2].$

Every polynomial $f \in \mathbf{F}[t_1, t_2]$ can be represented uniquely in the form

$$f = \sum_{I=(i_1,t_2)\in\mathbf{N}_0^2} c_I t_1^{i_1} t_2^{i_2} = \sum_{\substack{b_1\in\mathbf{N}_0\\i_1+3i_2=b_1}} \sum_{I=(i_1,i_2)\in\mathbf{N}_0^2\\i_1+3i_2=b_1} c_I t_1^{i_1} t^{i_2}$$

A polynomial $f \in \mathbf{F}[t_1, t_2]$ is in the vanishing ideal $\mathfrak{I}(V)$ if and only if, for all $x_1 \in \mathbf{F}$,

$$0 = f(x_1, 4x_1^3) = \sum_{\substack{b_1 \in \mathbf{N}_0 \\ i_1 + 3i_2 = b_1}} \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1}} c_I x_1^{i_1} \left(4x_1^3\right)^{i_2}$$

$$= \sum_{\substack{b_1 \in \mathbf{N}_0 \\ i_1 + 3i_2 = b_1}} \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1}} c_I 4^{i_2} x_1^{i_1 + 3i_{m+1}}$$

$$= \sum_{\substack{b_1 \in \mathbf{N}_0 \\ i_1 + 3i_2 = b_1}} \left(\sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1}} c_I 4^{i_2}\right) x_1^{b_1}.$$

By Lemma 2, because ${\bf F}$ is an infinite field, the coefficients of this polynomial are zero, and so

$$\sum_{\substack{I=(i_1,i_2)\in \mathbf{N}_0^2\\i_1+3i_2=b_1}} c_I 4^{i_2} = 0$$

for all $b_1 \in \mathbf{N}_0$. The ordered pair $I = (b_1, 0)$ is one of the terms in this sum, and so

$$-c_{(b_1,0)} = \sum_{\substack{I=(i_1,i_1)\in\mathbf{N}_0^2,\\i_\ell+3i_{m+1}=b_1\\I\neq(b_1,0)}} c_I 4^{i_2}.$$

Therefore, $f \in \mathfrak{I}(V)$ implies

$$f = \sum_{b_1 \in \mathbf{N}_0} \left(c_{(b_1,0)} t_1^{b_1} + \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1 \\ I \neq (b_1,0)}} c_I t_1^{i_1} t_2^{i_2} \right)$$

$$= \sum_{b_1 \in \mathbf{N}_0} \left(\sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1 \\ I \neq (b_1,0)}} c_I t_1^{i_1} t_2^{i_2} - \sum_{\substack{I = (i_1, i_1) \in \mathbf{N}_0^2 \\ i_\ell + 3i_{m+1} = b_1 \\ I \neq (b_1,0)}} c_I 4^{i_2} t_1^{b_1} \right)$$

$$= \sum_{b_1 \in \mathbf{N}_0} \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1 \\ I \neq (b_1,0)}} c_I \left(t_1^{i_1} t_2^{i_2} - 4^{i_2} t_1^{b_1} \right)$$

$$= \sum_{b_1 \in \mathbf{N}_0} \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1 \\ I \neq (b_1,0)}} c_I g^I f^*$$

$$= \sum_{b_1 \in \mathbf{N}_0} \sum_{\substack{I = (i_1, i_2) \in \mathbf{N}_0^2 \\ i_1 + 3i_2 = b_1 \\ I \neq (b_1,0)}} c_I g^I f^*$$

and so f^* divides f. Thus, every polynomial $f \in \mathfrak{I}(V)$ is contained in the principal ideal generated by f^* .

The function

$$\varphi: \mathbf{F}[t_1, t_2] \to \mathbf{F}[t_1]$$

defined by

$$\varphi(t_1) = t_1$$

and

$$\varphi(t_2) = 4t_1^3$$

is a surjective ring homomorphism with

$$\operatorname{kernel}(\varphi) = \{ f \in \mathbf{F}[t_1, t_2] : f(t_1, 4^{i_2}t_1^3) = 0 \} = \mathfrak{I}(V).$$

Therefore,

$$\mathbf{F}[V] = \mathbf{F}[t_1, t_2] / \mathcal{I}(V) \cong \mathbf{F}[t_1].$$

The polynomial ring $\mathbf{F}[t_1]$ has Krull dimension 1, and so the coordinate ring $\mathbf{F}(V)$ of the curve has Krull dimension 1 and the curve has dimension 1.

3. Dimension of a monomial hypersurface

We shall prove that every monomial hypersurface in \mathbf{F}^{m+1} has dimension m. The proof is elementary, like the proof in Section 2, but a bit more technical.

Lemma 3. For $\lambda \in \mathbf{F}$ and $(a_1, \ldots, a_m) \in \mathbf{N}_0^m$, consider the polynomial

$$f^* = t_{m+1} - \lambda t_1^{a_1} \cdots t_m^{a_m} \in \mathbf{F}[t_1, \dots, t_{m+1}].$$

For
$$I = (i_1, \dots, i_m, i_{m+1}) \in \mathbf{N}_0^{m+1}$$
, let

$$b_{\ell} = i_{\ell} + a_{\ell} i_{m+1}$$
 for $\ell = 1, \dots, m$.

There exists a polynomial $g_I \in \mathbf{F}[t_1, \dots, t_{m+1}]$ such that

$$t^{I} - \lambda^{i_{m+1}} t_1^{b_1} \cdots t_m^{b_m} = g_I f^*.$$

Proof. Let $\Delta_i(u,v)$ be the polynomial defined by (2). We have

$$\begin{split} t^I - \lambda^{i_{m+1}} t_1^{b_1} & \cdots t_m^{b_m} \\ &= t_1^{i_1} \cdots t_m^{i_m} t_{m+1}^{i_{m+1}} - \lambda^{i_{m+1}} t_1^{i_1 + a_1 i_{m+1}} \cdots t_m^{i_m + a_m i_{m+1}} \\ &= t_1^{i_1} \cdots t_m^{i_m} \left(t_{m+1}^{i_{m+1}} - \lambda^{i_{m+1}} t_1^{a_1 i_{m+1}} \cdots t_m^{a_m i_{m+1}} \right) \\ &= t_1^{i_1} \cdots t_m^{i_m} \left(t_{m+1}^{i_{m+1}} - (\lambda t_1^{a_1} \cdots t_m^{a_m})^{i_{m+1}} \right) \\ &= t_1^{i_1} \cdots t_m^{i_m} \Delta_{i_{m+1}} (t_{m+1}, \lambda t_1^{a_1} \cdots t_m^{a_m}) \left(t_{m+1} - \lambda t_1^{a_1} \cdots t_m^{a_m} \right) \\ &= a_I f^* \end{split}$$

where

$$g_I = t_1^{i_1} \cdots t_m^{i_m} \Delta_{i_{m+1}}(t_{m+1}, \lambda t_1^{a_1} \cdots t_m^{a_m}) \in \mathbf{F}[t_1, \dots, t_{m+1}].$$

This completes the proof.

Theorem 1. Let **F** be an infinite field. For $\lambda \in \mathbf{F}$ and $(a_1, \ldots, a_m) \in \mathbf{N}_0^m$, consider the polynomial

$$f^* = t_{m+1} - \lambda t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \in \mathbf{F}[t_1, \dots, t_m, t_{m+1}]$$

and the associated hypersurface

$$V = \{(x_1, \dots, x_m, x_{m+1}) \in \mathbf{F}^{m+1} : f^*(x_1, \dots, x_m, x_{m+1}) = 0\}.$$

= \{(x_1, \dots, x_m, \lambda x_1^{a_1} x_2^{a_2} \dots x_m^{a_n}) \in \mathbf{F}^{m+1} : (x_1, \dots, x_m) \in \mathbf{F}^m\}.

The vanishing ideal $\Im(V)$ is the principal ideal generated by f^* .

Proof. The vanishing ideal $\mathfrak{I}(V)$ contains f^* , and so $\mathfrak{I}(V)$ contains the principal ideal generated by f^* . Therefore, it suffices to prove that $\mathfrak{I}(V)$ is contained in the principal ideal generated by f^* .

For every (m+1)-tuple $I=(i_1,\ldots,i_m,i_{m+1})\in \mathbf{N}_0^{m+1}$, there is a unique m-tuple $(b_1,\ldots,b_m)\in \mathbf{N}_0^m$ such that

$$i_{\ell} + a_{\ell} i_{m+1} = b_{\ell}$$

for $\ell = 1, ..., m$. Thus, every polynomial $f \in \mathbf{F}[t_1, ..., t_m, t_{m+1}]$ can be represented uniquely in the form

$$f = \sum_{I \in \mathbf{N}_0^{m+1}} c_I t^I$$

$$= \sum_{\substack{(b_1, \dots, b_m) \in \mathbf{N}_0^m \\ (b_1, \dots, b_m) \in \mathbf{N}_0^m \\ \text{for } \ell = 1, \dots, m}} \sum_{I = (i_1, \dots, i_m, i_{m+1}) \in \mathbf{N}_0^{m+1}} c_I t_1^{i_1} \cdots t_m^{i_m} t_{m+1}^{i_{m+1}}.$$

A polynomial $f \in \mathbf{F}[t_1, \dots, t_m, t_{m+1}]$ is in the vanishing ideal $\mathfrak{I}(V)$ if and only if, for all $(x_1, \dots, x_m) \in \mathbf{F}^m$,

$$0 = f(x_{1}, \dots, x_{m}, \lambda x_{1}^{a_{1}} x_{2}^{a_{2}} \cdots x_{m}^{a_{m}})$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbf{N}_{0}^{m}} \sum_{\substack{I = (i_{1}, \dots, i_{m}, i_{m+1}) \in \mathbf{N}_{0}^{m+1} \\ i_{\ell} + a_{\ell} i_{m+1} = b_{\ell} \\ \text{for } \ell = 1, \dots, m}} c_{I} x_{1}^{i_{1}} \cdots x_{m}^{i_{m}} (\lambda x_{1}^{a_{1}} x_{2}^{a_{2}} \cdots x_{m}^{a_{m}})^{i_{m+1}}$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbf{N}_{0}^{m}} \sum_{\substack{I = (i_{1}, \dots, i_{m}, i_{m+1}) \in \mathbf{N}_{0}^{m+1} \\ i_{\ell} + a_{\ell} i_{m+1} = b_{\ell} \\ \text{for } \ell = 1, \dots, m}} c_{I} \lambda^{i_{m+1}} x_{1}^{i_{1} + a_{1} i_{m+1}} \cdots x_{m}^{i_{m} + a_{m} i_{m+1}}$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbf{N}_{0}^{m}} \left(\sum_{\substack{I = (i_{1}, \dots, i_{m}, i_{m+1}) \in \mathbf{N}_{0}^{m+1} \\ i_{\ell} + a_{\ell} i_{m+1} = b_{\ell} \\ \text{for } \ell = 1, \dots, m}} c_{I} \lambda^{i_{m+1}} \right) x_{1}^{b_{1}} \cdots x_{m}^{b_{m}}.$$

By Lemma 2, the coefficients of this polynomial are zero, and so

(3)
$$\sum_{\substack{I=(i_1,\dots,i_m,i_{m+1})\in\mathbf{N}_0^{m+1}\\i_\ell+a_\ell i_{m+1}=b_\ell\\\text{for }\ell=1,\dots,m}} c_I \lambda^{i_{m+1}}=0$$

for all $(b_1, \ldots, b_m) \in \mathbf{N}_0^m$. The (m+1)-tuple $I = (b_1, \ldots, b_m, 0)$ is one of the terms in the sum (3), and so

$$-c_{(b_1,\dots,b_m,0)} = \sum_{\substack{I = (i_1,\dots,i_m,i_{m+1}) \in \mathbf{N}_0^{m+1}, \\ i_\ell + a_\ell i_{m+1} = b_\ell \\ \text{for } \ell = 1,\dots,m, \\ I \neq (b_1,\dots,b_m,0)}} c_I \lambda^{i_{m+1}}.$$

Therefore, $f \in \mathfrak{I}(V)$ implies

by Lemma 3, and so f is in the principal ideal generated by f^* . This completes the proof.

Theorem 2. For $\lambda \in \mathbf{F}$ and $(a_1, \ldots, a_m) \in \mathbf{N}_0^m$, the hypersurface

$$V = \left\{ (x_1, \dots, x_m, \lambda x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}) \in \mathbf{F}^{m+1} : (x_1, \dots, x_m) \in \mathbf{F}^m \right\}$$

has dimension m.

Proof. The function

$$\varphi: \mathbf{F}[t_1,\ldots,t_{m+1}] \to \mathbf{F}[t_1,\ldots,t_m]$$

defined by

$$\varphi(t_{\ell}) = t_{\ell}$$
 for $\ell = 1, \dots, m$

and

$$\varphi(t_{m+1}) = \lambda t_1^{a_1} \cdots t_m^{a_m}$$

is a surjective ring homomorphism with

$$kernel(\varphi) = \{ f \in \mathbf{F}[t_1, \dots, t_m] : f(t_1, \dots, t_m, \lambda^{i_{m+1}} t_1^{a_1} \dots t_m^{a_m} = 0 \} = \Im(V).$$

Therefore,

$$\mathbf{F}[V] = \mathbf{F}[t_1, \dots, t_m, t_{m+1}] / \mathcal{I}(V) \cong \mathbf{F}[t_1, \dots, t_m].$$

The polynomial ring $\mathbf{F}[t_1,\ldots,t_m]$ has Krull dimension m, and so the coordinate ring $\mathbf{F}[V]$ has Krull dimension m and the hypersurface V has dimension m. This completes the proof.

4. Varieties defined by several monomials

Let m and k be positive integers, and let n=m+k. For $j=1,2,\ldots,k$, let $\lambda_j \in \mathbf{F}$ and $(a_{1,j},a_{2,j},\ldots,a_{m,j}) \in \mathbf{N}_0^m$. Consider the polynomials

(4)
$$f_j^* = t_{m+j} - \lambda_j t_1^{a_{1,j}} t_2^{a_{2,j}} \cdots t_m^{a_{m,j}} \in \mathbf{F}[t_1, \dots, t_n].$$

Let V be the variety in \mathbf{F}^n determined by the set of polynomials

$$S = \{f_i^* : j = 1, \dots, k\}$$

and let $\mathfrak{I}(V)$ be the vanishing ideal of V. We shall prove that the coordinate ring $\mathbf{F}[V] = \mathbf{F}[t_1, \dots, t_n]/\mathfrak{I}(V)$ is isomorphic to the polynomial ring $\mathbf{F}[t_1, \dots, t_m]$, and so V has dimension m.

Lemma 4. Let R be a ring. For $j=1,2,\ldots,k$, let $\lambda_j \in \mathbf{F}$ and $(a_{1,j},a_{2,j},\ldots,a_{m,j}) \in \mathbf{N}_0^m$. Define the polynomial f_j^* by (4). For $I=(i_1,\ldots,i_m,i_{m+1},\ldots,i_{m+k}) \in \mathbf{N}_0^{m+k}$, let

$$b_{\ell} = i_{\ell} + \sum_{j=1}^{k} a_{\ell,j} i_{m+j}$$
 for $\ell = 1, \dots, m$.

There exist polynomials $g_{I,1}, \ldots, g_{I,k} \in \mathbf{F}[t_1, \ldots, t_{m+k}]$ such that

(5)
$$t^{I} - \prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} t_{1}^{b_{1}} \cdots t_{m}^{b_{m}} = \sum_{j=1}^{k} g_{I,j} f_{j}^{*}.$$

Proof. The proof is by induction on k. The case k=1 is Lemma 3. Assume that Lemma 4 is true for the positive integer k. We shall prove the Lemma for k+1. For

$$I = (i_1, \dots, i_{m+k}) \in \mathbf{N}_0^{m+k}$$

and

$$I' = (i_1, \dots, i_{m+k+1}) \in \mathbf{N}_0^{m+k+1}$$

we have

$$t^{I} = \prod_{\ell=1}^{m} t_{\ell}^{i_{\ell}} \prod_{j=1}^{k} t_{m+j}^{i_{m+j}}$$

and

$$t^{I'} = \prod_{\ell=1}^{m} t_{\ell}^{i_{\ell}} \prod_{i=1}^{k+1} t_{m+j}^{i_{m+j}} = t^{I} t_{m+k+1}^{i_{m+k+1}}.$$

For $\ell = 1, \ldots, m$, define

$$b_{\ell} = i_{\ell} + \sum_{j=1}^{k} a_{\ell,j} i_{m+j}$$

and

$$b'_{\ell} = i_{\ell} + \sum_{j=1}^{k+1} a_{\ell,j} i_{m+j} = b_{\ell} + a_{\ell,k+1} i_{m+k+1}.$$

We have

$$t^{I'} - \prod_{j=1}^{k+1} \lambda_j^{i_{m+j}} t_1^{b'_1} \cdots t_m^{b'_m}$$

$$= t_{m+k+1}^{i_{m+k+1}} \left(t^I - \prod_{j=1}^k \lambda_j^{i_{m+j}} t_1^{b_1} \cdots t_m^{b_m} \right)$$

$$+ t_{m+k+1}^{i_{m+k+1}} \left(\prod_{j=1}^k \lambda_j^{i_{m+j}} t_1^{b_1} \cdots t_m^{b_m} \right) - \prod_{j=1}^{k+1} \lambda_j^{i_{m+j}} t_1^{b'_1} \cdots t_m^{b'_m}$$

By the induction hypothesis, there exist polynomials $g_{I,1}, \ldots, g_{I,k} \in \mathbf{F}[t_1, \ldots, t_{m+k}]$ that satisfy (5), and so

$$t_{m+k+1}^{i_{m+k+1}}\left(t^{I}-\prod_{j=1}^{k}\lambda_{j}^{i_{m+j}}\ t_{1}^{b_{1}}\cdots t_{m}^{b_{m}}\right)=t_{m+k+1}^{i_{m+k+1}}\sum_{j=1}^{k}g_{I,j}\ f_{j}^{*}.$$

Applying the factorization formula (1), we obtain

$$\begin{split} t_{m+k+1}^{i_{m+k+1}} \left(\prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} \ t_{1}^{b_{1}} \cdots t_{m}^{b_{m}} \right) &- \prod_{j=1}^{k+1} \lambda_{j}^{i_{m+j}} \ t_{1}^{b'_{1}} \cdots t_{m}^{b'_{m}} \\ &= \left(\prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} \ t_{1}^{b_{1}} \cdots t_{m}^{b_{m}} \right) \left(t_{m+k+1}^{i_{m+k+1}} - \lambda_{k+1}^{i_{m+k+1}} \prod_{\ell=1}^{m} t_{\ell}^{a_{\ell,k+1}i_{m+k+1}} \right) \\ &= \left(\prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} \ t_{1}^{b_{1}} \cdots t_{m}^{b_{m}} \right) \left(t_{m+k+1}^{i_{m+k+1}} - \left(\lambda_{k+1} \prod_{\ell=1}^{m} t_{\ell}^{a_{\ell,k+1}} \right)^{i_{m+k+1}} \right) \\ &= \left(\prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} \ t_{1}^{b_{1}} \cdots t_{m}^{b_{m}} \right) \Delta_{i_{m+k+1}} \left(t_{m+k+1}, \ \lambda_{k+1} \prod_{\ell=1}^{m} t_{\ell}^{a_{\ell,k+1}} \right) \left(t_{m+k+1} - \lambda_{k+1} \prod_{\ell=1}^{m} t_{\ell}^{a_{\ell,k+1}} \right) \\ &= g_{I,k+1} \ f_{k+1}^{*} \end{split}$$

where

$$g_{I,k+1} = \left(\prod_{j=1}^k \lambda_j^{i_{m+j}} \ t_1^{b_1} \cdots t_m^{b_m}\right) \Delta_{i_{m+k+1}} \left(t_{m+k+1}, \ \lambda_{k+1} \prod_{\ell=1}^m t_\ell^{a_{\ell,k+1}}\right).$$

This completes the proof.

Theorem 3. Let n = m + k. Let \mathbf{F} be an infinite field. For j = 1, 2, ..., k, let $\lambda_j \in \mathbf{F}$ and $(a_{1,j}, a_{2,j}, ..., a_{m,j}) \in \mathbf{N}_0^m$, and let f_j^* be the polynomial defined by (4). Let $V \subseteq \mathbf{F}^n$ be the variety determined by the set $S = \{f_1^*, ..., f_k^*\} \subseteq \mathbf{F}[t_1, ..., t_n]$. The vanishing ideal $\mathfrak{I}(V)$ is the ideal generated by S.

Proof. The ideal $\mathfrak{I}(V)$ contains S, and so $\mathfrak{I}(V)$ contains the ideal generated by S. Thus, it suffices to prove that ideal generated by S contains every polynomial in $\mathfrak{I}(V)$.

The variety determined by S is

$$V = \{ (x_1, \dots, x_m, \lambda_1 x_1^{a_{1,1}} \cdots x_m^{a_{m,1}}, \dots, \lambda_k x_1^{a_{1,k}} \cdots x_m^{a_{m,k}}) : (x_1, \dots, x_m) \in \mathbf{F}^m \}.$$

For every (m+k)-tuple $I=(i_1,\ldots,i_m,i_{m+1},\ldots,i_{m+k})\in \mathbf{N}_0^{m+k}$, there is a unique m-tuple $(b_1,\ldots,b_m)\in \mathbf{N}_0^m$ such that

$$i_{\ell} + \sum_{j=1}^{k} a_{\ell,j} i_{m+j} = b_{\ell}$$

for $\ell = 1, \ldots, m$. Let

$$\sum_{I(b_1,\dots,b_m)} = \sum_{\substack{I=(i_1,\dots,i_m,i_{m+1},\dots,i_{m+k}) \in \mathbf{N}_0^n \\ i_\ell + \sum_{j=1}^k a_{\ell,j} i_{m+j} = b_\ell \\ \text{for } \ell = 1,\dots,m}}.$$

Every polynomial $f \in \mathbf{F}[t_1, \dots, t_n]$ has a unique representation in the form

$$f = \sum_{I \in \mathbf{N}_0^n} c_I t^I = \sum_{(b_1, \dots, b_m) \in \mathbf{N}_0^m} \sum_{I(b_1, \dots, b_m)} c_I t^I$$

where $c_I \in \mathbf{F}$ and $c_I \neq 0$ for only finitely many n-tuples I. If $f \in \mathfrak{I}(V)$, then

$$0 = f\left(x_{1}, \dots, x_{m}, \lambda_{1}x_{1}^{a_{1,1}} \cdots x_{m}^{a_{m,1}}, \dots, \lambda_{k}x_{1}^{a_{1,k}} \cdots x_{m}^{a_{m,k}}\right)$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbb{N}_{0}^{m}} \sum_{I(b_{1}, \dots, b_{m})} c_{I}x_{1}^{i_{1}} \cdots x_{m}^{i_{m}} \left(\lambda_{1}x_{1}^{a_{1,1}} \cdots x_{m}^{a_{m,1}}\right)^{i_{m+1}} \cdots \left(\lambda_{k}x_{1}^{a_{1,k}} \cdots x_{m}^{a_{m,k}}\right)^{i_{m+k}}$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbb{N}_{0}^{m}} \sum_{I(b_{1}, \dots, b_{m})} c_{I} \prod_{j=1}^{k} \lambda_{j}^{i_{m+j}} \prod_{\ell=1}^{m} x_{\ell}^{i_{\ell} + \sum_{j=1}^{k} a_{\ell, j} i_{m+j}}$$

$$= \sum_{(b_{1}, \dots, b_{m}) \in \mathbb{N}_{m}^{m}} \left(\sum_{I(b_{1}, \dots, b_{m}) \in \mathbb{N}_{m}^{m}} c_{I} \prod_{j=1}^{k} \lambda_{j}^{i_{m+j}}\right) x_{1}^{b_{1}} \cdots x_{m}^{b_{m}}$$

for all $(x_1, \ldots, x_m) \in \mathbf{F}^m$. It follows from Lemma 2 that

$$0 = \sum_{I_{(b_1, \dots, b_m)}} c_I \prod_{j=1}^k \lambda_j^{i_{m+j}} = c_{(b_1, \dots, b_m, 0, \dots, 0)} + \sum_{\substack{I_{(b_1, \dots, b_m)} \\ I \neq (b_1, \dots, b_m, 0, \dots, 0)}} c_I \prod_{j=1}^k \lambda_j^{i_{m+j}}$$

for all $(b_1, \ldots, b_m) \in \mathbf{N}_0^m$, and so

$$f = \sum_{(b_1, \dots, b_m) \in \mathbf{N}_0^m} \sum_{I(b_1, \dots, b_m)} c_I t^I$$

$$= \sum_{(b_1, \dots, b_m) \in \mathbf{N}_0^m} \left(\sum_{\substack{I_{(b_1, \dots, b_m)} \\ I \neq (b_1, \dots, b_m, 0, \dots, 0)}} c_I t^I + c_{(b_1, \dots, b_m, 0, \dots, 0)} t^{b_1} \cdots t^{b_m} \right)$$

$$= \sum_{(b_1, \dots, b_m) \in \mathbf{N}_0^m} \left(\sum_{\substack{I_{(b_1, \dots, b_m)} \\ I \neq (b_1, \dots, b_m, 0, \dots, 0)}} c_I t^I - \sum_{\substack{I_{(b_1, \dots, b_m)} \\ I \neq (b_1, \dots, b_m, 0, \dots, 0)}} c_I \prod_{j=1}^k \lambda_j^{i_{m+j}} t_1^{b_1} \cdots t_m^{b_m} \right)$$

$$= \sum_{(b_1, \dots, b_m) \in \mathbf{N}_0^m} \sum_{\substack{I_{(b_1, \dots, b_m)} \\ I \neq (b_1, \dots, b_m, 0, \dots, 0)}} c_I \left(t^I - \prod_{j=1}^k \lambda_j^{i_{m+j}} t_1^{b_1} \cdots t_m^{b_m} \right).$$

Lemma 4 immediately implies that f is in the ideal generated by S. This completes the proof.

Theorem 4. The variety V has dimension m.

Proof. The function

$$\varphi: \mathbf{F}[t_1,\ldots,t_{m+1},\ldots,t_{m+k}] \to \mathbf{F}[t_1,\ldots,t_m]$$

defined by

$$\varphi(t_{\ell}) = t_{\ell}$$
 for $\ell = 1, \dots, m$

and

$$\varphi(t_{m+j}) = \lambda_j t_1^{a_{1,j}} \cdots t_m^{a_{m,j}} \quad \text{for } j = 1, \dots, k$$

is a surjective ring homomorphism with

 $kernel(\varphi)$

$$= \left\{ f \in \mathbf{F}[t_1, \dots, t_n] : f\left(t_1, \dots, t_m, \lambda_1 t_1^{a_{1,1}} \cdots t_m^{a_{m,1}}, \dots, \lambda_k t_1^{a_{1,k}} \cdots t_m^{a_{m,k}} \right) = 0 \right\}$$

= $\mathfrak{I}(V)$.

Therefore,

$$\mathbf{F}[V] = \mathbf{F}[t_1, \dots, t_n] / \mathcal{I}(V) \cong \mathbf{F}[t_1, \dots, t_m]$$

and the coordinate ring of $\Im(V)$ has Krull dimension m. This completes the proof.

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