INITIAL BOUNDARY VALUE PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITION

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Abstract. In this paper we consider an initial boundary value problem for a semilinear parabolic equation with nonlinear nonlocal boundary condition. We prove comparison principle, the existence theorem of a local solution and study the problem of uniqueness and nonuniqueness.

1. Introduction

In this paper we consider the initial boundary value problem for the following semilinear parabolic equation

$$u_t = \Delta u + c(x,t)u^p,\ x \in \Omega,\ t > 0, \tag{1.1}$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial datum

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{1.3}$$

where $p > 0$, $l > 0$, Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $c(x,t)$, $k(x,y,t)$ and $u_0(x)$ satisfy the following conditions:

$$c(x,t) \in C^\alpha_{loc}(\overline{\Omega} \times [0,+\infty)),\ 0 < \alpha < 1,\ c(x,t) \geq 0;$$

$$k(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)),\ k(x,y,t) \geq 0;$$

$$u_0(x) \in C^1(\overline{\Omega}),\ u_0(x) \geq 0 \text{ in } \Omega,\ \frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x,y,0)u_0^l(y)\,dy \text{ on } \partial\Omega.$$

A lot of articles have been devoted to the investigation of initial boundary value problems for parabolic equations and systems of parabolic equations with nonlinear nonlocal Dirichlet boundary condition (see, for example, [7, 12, 13, 14, 15, 16, 19, 20, 21] and the references therein). In particular, the initial boundary value problem for equation (1.1) with nonlocal boundary condition

$$u(x,t) = \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0,$$

was considered for $c(x,t) \leq 0$ and $c(x,t) \geq 0$ in [12, 13] and [14, 15] respectively.

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We note that for $p < 1$ and $l < 1$ the nonlinearities in equation (1.1) and boundary condition (1.2) are non-Lipschitzian. The problem of uniqueness and nonuniqueness for different parabolic nonlinear equations with non-Lipschitzian data in bounded domain has been addressed by several authors (see, for example, [3, 5, 6, 8, 9, 13, 15, 18] and the references therein).

The aim of this paper is to study problem (1.1)–(1.3) for any $p > 0$ and $l > 0$. We prove existence of a local solution and establish some uniqueness and nonuniqueness results.

This paper is organized as follows. In the next section we prove comparison principle. The existence theorem of a local solution is established in section 3. The problem of uniqueness and nonuniqueness for (1.1)–(1.3) is investigated in section 4.

2. Comparison principle

In this section the theorem of positiveness of solution and comparison principle for (1.1)–(1.3) will be proved. We begin with definitions of supersolution, subsolution and maximal solution of (1.1)–(1.3).

Let be $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Definition 2.1. We say that a nonnegative function $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of (1.1)–(1.3) in Q_T if

$$u_t \geq \Delta u + c(x,t)u^p, \ (x,t) \in Q_T, \tag{2.1}$$

$$\frac{\partial u(x,t)}{\partial \nu} \geq \int_\Omega k(x,y,t)u^l(y,t)\,dy, \ (x,t) \in S_T, \tag{2.2}$$

$$u(x,0) \geq u_0(x), \ x \in \Omega, \tag{2.3}$$

and $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of (1.1)–(1.3) in Q_T if $u \geq 0$ and it satisfies (2.1)–(2.3) in the reverse order. We say that $u(x,t)$ is a solution of problem (1.1)–(1.3) in Q_T if $u(x,t)$ is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T.

Definition 2.2. We say that a solution $u(x,t)$ of (1.1)–(1.3) in Q_T is maximal solution if for any other solution $v(x,t)$ of (1.1)–(1.3) in Q_T the inequality $v(x,t) \leq u(x,t)$ is satisfied for $(x,t) \in Q_T \cup \Gamma_T$.

Theorem 2.3. *Suppose that $u_0 \not\equiv 0$ in Ω and $u(x,t)$ is a solution of (1.1)–(1.3) in Q_T. Then $u(x,t) > 0$ in $Q_T \cup S_T$.*

Proof. Since $u_0(x) \not\equiv 0$ in Ω and $u_t - \Delta u = c(x,t)u^p \geq 0$ in Q_T, by the strong maximum principle $u(x,t) > 0$ in Q_T. Let $u(x_0,t_0) = 0$ in some point $(x_0,t_0) \in S_T$. Then according to Theorem 3.6 of [10] it yields $\partial u(x_0,t_0)/\partial\nu < 0$, which contradicts (1.2). □

Theorem 2.4. *Let $u(x,t)$ and $v(x,t)$ be a supersolution and a subsolution of problem (1.1)–(1.3) in Q_T, respectively, with $u(x,0) \geq v(x,0)$ in Ω. Suppose that $u(x,t) > 0$ or $v(x,t) > 0$ in $Q_T \cup \Gamma_T$ if $\min(p,l) < 1$. Then $u(x,t) \geq v(x,t)$ in $Q_T \cup \Gamma_T$.*

Proof. Let $\varphi(x,\tau) \in C^{2,1}(\overline{Q}_t)$ $(0 < t < T)$ be a nonnegative function which satisfies homogeneous Neumann boundary condition. Multiplying (2.1) by φ and then integrating over Q_t, we obtain

$$\int_\Omega u(x,t)\varphi(x,t)\,dx \geq \int_\Omega u(x,0)\varphi(x,0)\,dx \tag{2.4}$$

$$+\int_0^t \int_\Omega (u(x,\tau)\varphi_\tau(x,\tau) + u(x,\tau)\Delta\varphi(x,\tau) + c(x,\tau)u^p(x,\tau)\varphi(x,\tau))\,dx\,d\tau$$
$$+\int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau)u^l(y,\tau)\,dy\,dS_x\,d\tau.$$

On the other hand, the subsolution $v(x,t)$ satisfies (2.4) with reversed inequality

$$\int_\Omega v(x,t)\varphi(x,t)\,dx \le \int_\Omega v(x,0)\varphi(x,0)\,dx \tag{2.5}$$
$$+\int_0^t \int_\Omega (v(x,\tau)\varphi_\tau(x,\tau) + v(x,\tau)\Delta\varphi(x,\tau) + c(x,\tau)v^p(x,\tau)\varphi(x,\tau))\,dx\,d\tau$$
$$+\int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega k(x,y,\tau)v^l(y,\tau)\,dy\,dS_x\,d\tau.$$

Put $M = \max(\max_{\overline{Q}_t} u, \max_{\overline{Q}_t} v)$ and $w(x,t) = v(x,t) - u(x,t)$. Subtracting (2.4) from (2.5) and using mean value theorem, we get

$$\int_\Omega w(x,t)\varphi(x,t)\,dx \le \int_\Omega w(x,0)\varphi(x,0)\,dx \tag{2.6}$$
$$+\int_0^t \int_\Omega w(x,\tau)\left(\varphi_\tau(x,\tau) + \Delta\varphi(x,\tau) + p\theta_1^{p-1}(x,\tau)c(x,\tau)\varphi(x,\tau)\right)dx\,d\tau$$
$$+l\int_0^t \int_{\partial\Omega} \varphi(x,\tau) \int_\Omega \theta_2^{l-1}(y,\tau)k(x,y,\tau)w(y,\tau)\,dy\,dS_x\,d\tau,$$

where $\theta_i(x,\tau)$ $(i=1,2)$ are some positive continuous functions in $\overline{Q}_t$ if $\min(p,l) < 1$ and some nonnegative continuous functions in $\overline{Q}_t$ otherwise.

The function $\varphi(x,\tau)$ is defined as a solution of the following problem

$$\varphi_\tau + \Delta\varphi + p\theta_1^{p-1}(x,\tau)c(x,\tau)\varphi = 0,\ (x,\tau) \in Q_t,$$
$$\frac{\partial\varphi(x,\tau)}{\partial\nu} = 0,\ (x,\tau) \in S_t,$$
$$\varphi(x,t) = \psi(x),\ x \in \Omega,$$

where $\psi(x) \in C_0^\infty(\Omega)$, $0 \le \psi \le 1$. By virtue of the comparison principle for linear parabolic equations the solution $\varphi(x,\tau)$ of this problem is nonnegative and bounded. By (2.6) and $w(x,0) \le 0$ we have

$$\int_\Omega w(x,t)\psi(x)\,dx \le m\int_0^t \int_\Omega w_+(x,\tau)\,dx\,d\tau, \tag{2.7}$$

where $s_+ = \max(0,s)$, $m = l|\partial\Omega| \sup_{\partial\Omega\times Q_t} k(x,y,\tau) \sup_{Q_t} \theta_2^{l-1}(x,\tau) \sup_{S_t} \varphi(x,\tau)$, $|\partial\Omega|$ is the Lebesgue measure of $\partial\Omega$. Since the inequality holds for every function $\psi(x)$, we can choose a sequence $\psi_n(x) \in C_0^\infty(\Omega)$ converging in $L^1(\Omega)$ to the function

$$\gamma(x) = \begin{cases} 1,\ w(x,t) > 0, \\ 0,\ w(x,t) \le 0. \end{cases}$$

Substituting $\psi_n(x)$ instead of $\psi(x)$ in (2.7) and letting $n \to \infty$, we get

$$\int_\Omega w_+(x,t)\,dx \le m\int_0^t \int_\Omega w_+(x,\tau)\,dx\,d\tau.$$

By the Gronwall inequality we obtain $w_+(x,t) \le 0$. $\square$

From Theorem 2.4 the following assertion is easily deduced.

Theorem 2.5. *Suppose that problem (1.1)–(1.3) has a solution in* Q_T *with any nonnegative initial data for* $\min(p,l) \geq 1$ *and with positive initial data otherwise. Then the solution of (1.1)–(1.3) is unique in* Q_T.

3. Local existence

In this section we establish the local existence of solution for (1.1)–(1.3) using representation formula and the contraction mapping argument.

Let $\{\varepsilon_m\}$ be decreasing to 0 sequence such that $0 < \varepsilon_m < 1$. For $\varepsilon = \varepsilon_m$ let $u_{0\varepsilon}(x)$ be the functions with the following properties:

$$u_{0\varepsilon}(x) \in C^1(\overline{\Omega}),\ u_{0\varepsilon}(x) \geq \varepsilon,\ u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x),\ \varepsilon_i \geq \varepsilon_j,$$
$$u_{0\varepsilon}(x) \to u_0(x) \text{ as } \varepsilon \to 0,\ \frac{\partial u_{0\varepsilon}(x)}{\partial \nu} = \int_\Omega k(x,y,0) u_{0\varepsilon}^l(y)\,dy \text{ for } x \in \partial\Omega.$$

Since the nonlinearities in (1.1) and (1.2), the Lipschitz condition is not satisfied if $\min(p,l) < 1$, and thus we need to consider the auxiliary problem for equation (1.1) with boundary condition (1.2) and the initial datum

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x),\ x \in \Omega. \tag{3.1}$$

Theorem 3.1. *For some values of* T *problem (1.1), (1.2), (3.1) has a unique solution in* Q_T.

Proof. Let $G_N(x,y;t-\tau)$ be the Green function for the heat equation with homogeneous Neumann boundary condition. We note that the function $G_N(x,y;t-\tau)$ has the following properties (see, for example, [17]):

$$G_N(x,y;t-\tau) \geq 0,\ x,y \in \Omega,\ 0 \leq \tau < t < T, \tag{3.2}$$

$$\int_\Omega G_N(x,y;t-\tau)\,dy = 1,\ x \in \Omega,\ 0 \leq \tau < t < T. \tag{3.3}$$

It is well known that problem (1.1), (1.2), (3.1) in Q_T is equivalent to the equation

$$\begin{aligned} u_\varepsilon(x,t) &= \int_\Omega G_N(x,y;t) u_{0\varepsilon}(y)\,dy + \int_0^t \int_\Omega G_N(x,y;t-\tau) c(y,\tau) u_\varepsilon^p(y,\tau)\,dy\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau) u_\varepsilon^l(y,\tau)\,dy\,dS_\xi\,d\tau \equiv Lu_\varepsilon(x,t). \end{aligned} \tag{3.4}$$

To show that (3.4) is solvable for small T we use the contraction mapping argument. To this end we define a sequence of functions $\{u_{\varepsilon,n}(x,t)\}$, $n = 1,2,\ldots,$ in the following way:

$$u_{\varepsilon,1}(x,t) \equiv \varepsilon,\ (x,t) \in \overline{Q}_T, \tag{3.5}$$

and

$$u_{\varepsilon,n+1}(x,t) = Lu_{\varepsilon,n}(x,t),\ (x,t) \in \overline{Q}_T,\ n = 1,2,\ldots. \tag{3.6}$$

Set

$$M_{0\varepsilon} = \sup_{x\in\Omega} u_{0\varepsilon}(x).$$

Using the method of mathematical induction we prove that the inequalities

$$\sup_{Q_{T_1}} u_{\varepsilon,n}(x,t) \leq M,\ n = 1,2,\ldots, \tag{3.7}$$

hold for some constants $T_1 > 0$ and $M > \max(\varepsilon, M_{0\varepsilon})$. For $n = 1$ the validity of (3.7) is obvious. Supposing that (3.7) is true for $n = m$, we shall prove it for $n = m + 1$. Indeed, by (3.2)–(3.4) and (3.6) we have

$$\begin{aligned} u_{\varepsilon,m+1}(x,t) &= \int_\Omega G_N(x,y;t)u_{0\varepsilon}(y)\,dy \\ &+ \int_0^t \int_\Omega G_N(x,y;t-\tau)c(y,\tau)u^p_{\varepsilon,m}(y,\tau)\,dy\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)u^l_{\varepsilon,m}(y,\tau)\,dy\,dS_\xi\,d\tau \\ &\le M_{0\varepsilon} + M^p\nu(t) + M^l\mu(t), \end{aligned} \tag{3.8}$$

where

$$\nu(t) = \sup_{x\in\Omega}\int_0^t \int_\Omega G_N(x,y;t-\tau)c(y,\tau)\,dy\,d\tau,$$

$$\mu(t) = \sup_{x\in\Omega}\int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)\,dy\;dS_\xi\,d\tau.$$

We note that (see [11]) there exist positive constants δ_1 and a_1 such that

$$\mu(t) \le a_1\sqrt{t} \text{ for } t \le \delta_1. \tag{3.9}$$

Due to (3.2), (3.3) we have

$$\nu(t) \le a_2 t \text{ for } t \le \delta_2, \tag{3.10}$$

where δ_2 and a_2 are some positive constants. We choose $0 < T_1 < \min(\delta_1, \delta_2)$ such that

$$\sup_{0<t<T_1}(M^p\nu(t) + M^l\mu(t)) \le M - M_{0\varepsilon}. \tag{3.11}$$

By virtue of (3.8) and (3.11) we have (3.7) with $n = m+1$. By (3.2)–(3.6) and the properties of $u_{0\varepsilon}(x)$ we get

$$u_{\varepsilon,n}(x,t) \ge \varepsilon,\ (x,t) \in \overline{Q}_{T_1},\ n = 1,2,\dots\,. \tag{3.12}$$

Using mean value theorem we obtain for $n = 2,3,\dots$

$$\begin{aligned} &\sup_{Q_{T_1}}|u_{\varepsilon,n+1}(x,t) - u_{\varepsilon,n}(x,t)| \\ &= \sup_{Q_{T_1}}\Big|\int_0^t \int_\Omega G_N(x,\xi;t-\tau)c(y,\tau)(u^p_{\varepsilon,n}(\xi,\tau) - u^p_{\varepsilon,n-1}(\xi,\tau))\,d\xi\,d\tau \\ &+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau)\int_\Omega k(\xi,y,\tau)(u^l_{\varepsilon,n}(y,\tau) - u^l_{\varepsilon,n-1}(y,\tau))\,dy\,dS_\xi\,d\tau\Big| \\ &\le \sup_{Q_{T_1}}\left(p\theta^{p-1}_{1,n}(x,t)\nu(t) + l\theta^{l-1}_{2,n}(x,t)\mu(t)\right)\sup_{Q_{T_1}}|u_{\varepsilon,n}(x,t) - u_{\varepsilon,n-1}(x,t)| \\ &\le \sup_{(0,T_1)}\rho(t)\sup_{Q_{T_1}}|u_{\varepsilon,n}(x,t) - u_{\varepsilon,n-1}(x,t)| \le (M+\varepsilon)\left(\sup_{(0,T_1)}\rho(t)\right)^{n-1}, \end{aligned}$$

where $\theta_{i,n}(x,t)$ $(i = 1,2)$ are continuous functions in $\overline{Q}_{T_1}$ such that $\alpha_1 \le \theta_{i,n}(x,t) \le M_1$ for $(x,t) \in \overline{Q}_{T_1}$, $\rho(t) = p(\alpha_1^{p-1} + M_1^{p-1})\nu(t) + l(\alpha_1^{l-1} + M_1^{l-1})\mu(t)$ for $t \in [0,T_1]$.

We note that positive constants α_1 and M_1 do not depend on n. By (3.9) and (3.10) there exists a constant $T \in (0, T_1)$ such that

$$\sup_{(0,T)} \rho(t) < 1.$$

Hence, the sequence $\{u_{\varepsilon,n}(x,t)\}$ converges uniformly in $\overline{Q}_T$ as $n \to \infty$. We denote

$$u_\varepsilon(x,t) = \lim_{n\to\infty} u_{\varepsilon,n}(x,t).$$

By virtue of (3.7), (3.12) we have

$$\varepsilon \le u_\varepsilon(x,t) \le M,\ (x,t) \in \overline{Q}_T.$$

Passing to the limit as $n \to \infty$ in (3.6) by dominated convergence theorem we obtain that the function $u_\varepsilon(x,t)$ satisfies (3.4). Hence, $u_\varepsilon(x,t)$ solves problem (1.1), (1.2), (3.1) in Q_T.

By contradiction we shall prove uniqueness of the solution of (1.1), (1.2), (3.1) in Q_T for small values of T. Let problem (1.1), (1.2), (3.1) have at least two solutions $u_\varepsilon(x,t)$ and $v_\varepsilon(x,t)$ in Q_T. Arguing as above we get

$$\begin{aligned}
&\sup_{Q_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)| \\
&= \sup_{Q_T} \Big| \int_0^t \int_\Omega G_N(x,\xi;t-\tau) c(y,\tau)(u_\varepsilon^p(\xi,\tau) - v_\varepsilon^p(\xi,\tau))\, d\xi\, d\tau \\
&+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)(u_\varepsilon^l(y,\tau) - v_\varepsilon^l(y,\tau))\, dy\, dS_\xi\, d\tau \Big| \\
&\le \sup_{Q_T} \Big(p\theta_1^{p-1}(x,t)\nu(t) + l\theta_2^{l-1}(x,t)\mu(t) \Big) \sup_{Q_T} |u_\varepsilon(x,t) - v_\varepsilon(x,t)| \\
&\le \alpha \sup_{Q_T} |u_\varepsilon(x,t) - u_\varepsilon(x,t)|,
\end{aligned}$$

where $\theta_i(x,t)$ $(i = 1,2)$ are some positive continuous functions in $\overline{Q}_T$ and $\alpha < 1$ for small values of T. Obviously, $u_\varepsilon(x,t) = v_\varepsilon(x,t)$ in Q_T. □

Theorem 3.2. *For some values of T problem (1.1)–(1.3) has maximal solution in Q_T.*

Proof. Let u_ε be a solution of (1.1), (1.2), (3.1). It is easy to see that u_ε is a supersolution of (1.1)–(1.3). By Theorem 2.4 for $\varepsilon_1 \le \varepsilon_2$ we have $u_{\varepsilon_1} \le u_{\varepsilon_2}$. According to the Dini theorem (see [2]) for some $T > 0$ the sequence $\{u_\varepsilon(x,t)\}$ converges as $\varepsilon \to 0$ uniformly in $\overline{Q}_T$ to some function $u(x,t)$. Passing to the limit as $\varepsilon \to 0$ in (3.4) and using dominated convergence theorem we obtain that the function $u(x,t)$ satisfies in Q_T the following equation

$$\begin{aligned}
u(x,t) &= \int_\Omega G_N(x,y;t) u_0(y)\, dy + \int_0^t \int_\Omega G_N(x,y;t-\tau) c(y,\tau) u^p(y,\tau)\, dy\, d\tau \\
&+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau) u^l(y,\tau)\, dy\, dS_\xi\, d\tau.
\end{aligned}$$

Hence, $u(x,t)$ solves problem (1.1)–(1.3) in Q_T. It is easy to prove that $u(x,t)$ is maximal solution of (1.1)–(1.3) in Q_T. □

4. Uniqueness and nonuniqueness

In this section we shall use some arguments of [8] and [15].

Theorem 4.1. *Let $u_0(x) \equiv 0$ and $u(x,t)$ be maximal solution of (1.1)–(1.3) in Q_T. Suppose that for some $t_0 \in [0,T)$ at least one from the following conditions is fulfilled:*

$$c(x_0,t_0) > 0 \text{ for some } x_0 \in \Omega \text{ and } 0 < p < 1 \tag{4.1}$$

or

$$k(x,y_0,t_0) > 0 \text{ for any } x \in \partial\Omega \text{ and some } y_0 \in \partial\Omega \text{ and } 0 < l < 1. \tag{4.2}$$

Then maximal solution $u(x,t)$ of problem (1.1)–(1.3) is nontrivial in Q_T.

Proof. At first we suppose that (4.1) is true. By the continuity of the function $c(x,t)$ there exist a neighborhood $U(x_0) \subset \Omega$ of x_0 in Ω and a constant $T_1 < T$ such that $c(x,t) \geq c_0 > 0$ for $x \in U(x_0)$ and $t \in [t_0,T_1]$. We introduce the auxiliary problem

$$\begin{cases} u_t = \Delta u + c(x,t)u^p, \ x \in U(x_0), \ t_0 < t < T_1, \\ u(x,t) = 0, \ x \in \partial U(x_0), \ t_0 < t < T_1, \\ u(x,t_0) = 0, \ x \in U(x_0). \end{cases} \tag{4.3}$$

We shall construct a subsolution of problem (4.3). Put $\underline{u}(x,t) = C(t-t_0)^{\frac{1}{1-p}} w(x,t)$, where C is some positive constant and $w(x,t)$ is a solution of the following problem

$$\begin{cases} w_t = \Delta w, \ x \in U(x_0), \ t_0 < t < T_1, \\ w(x,t) = 0, \ x \in \partial U(x_0), \ t_0 < t < T_1, \\ w(x,t_0) = w_0(x), \ x \in U(x_0). \end{cases} \tag{4.4}$$

Here $w_0(x)$ is nontrivial nonnegative continuous function in $\overline{U(x_0)}$ which satisfies boundary condition. We note that $\underline{u}(x,t) = 0$ if $t = t_0$ or $x \in \partial U(x_0)$. By the strong maximum principle we get $0 < w(x,t) < M_0 = \sup_{x \in U(x_0)} w_0(x)$ for $x \in U(x_0)$ and $t_0 < t < T_1$.

For all $(x,t) \in U(x_0) \times (t_0,T_1)$ we have

$$\underline{u}_t - \Delta \underline{u} - c(x,t)\underline{u}^p = \frac{C}{1-p}(t-t_0)^{\frac{p}{1-p}} w - c(x,t)C^p(t-t_0)^{\frac{p}{1-p}} w^p \leq 0,$$

where $C \leq M_0^{-1}[c_0(1-p)]^{1/(1-p)}$.

Let $u(x,t)$ be maximal solution of (1.1)–(1.3) in Q_T with trivial initial datum. According to Theorem 3.2 $u(x,t) = \lim\limits_{\varepsilon \to 0} u_\varepsilon(x,t)$, where $u_\varepsilon(x,t)$ is positive supersolution of (1.1)–(1.3) in Q_T. It is easy to see that $u_\varepsilon(x,t)$ is a supersolution of (4.3). By the comparison principle for problem (4.3) we have $u_\varepsilon(x,t) \geq \underline{u}(x,t)$ for $(x,t) \in \overline{U(x_0)} \times [t_0,T_1)$. Passing to the limit as $\varepsilon \to 0$ we get $u(x,t) \geq \underline{u}(x,t)$ for $(x,t) \in \overline{U(x_0)} \times [t_0,T_1)$. By (1.2) and the strong maximum principle we obtain that maximal solution $u(x,t) > 0$ for all $x \in \overline{\Omega}$ and $t_0 < t < T_1$.

Now we suppose that (4.2) is realized. Then there exist a neighborhood $V(y_0) \subset \overline{\Omega}$ of y_0 and a constant $T_2 \in (t_0,T)$ such that $k(x,y,t) > 0$ for $t_0 \leq t \leq T_2, \ x \in \partial\Omega$ and $y \in V(y_0)$.

We use the change of variables in a neighborhood of $\partial\Omega$ as in [4]. Let $\overline{x}$ be a point in $\partial\Omega$ and $\widehat{n}(\overline{x})$ be the unit inner normal to $\partial\Omega$ at the point $\overline{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0,\delta] \to \mathbb{R}^n$ given by $\psi(\overline{x},s) = \overline{x} + s\widehat{n}(\overline{x})$ defines new coordinates $(\overline{x},s)$ in a neighborhood of $\partial\Omega$

in $\overline{\Omega}$. A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\overline{x}, s) = g(s)$, which is independent of the variable $\overline{x}$, evaluated at a point $(\overline{x}, s)$ is given by

$$\Delta g(\overline{x}, s) = \frac{\partial^2 g}{\partial s^2}(\overline{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \frac{\partial g}{\partial s}(\overline{x}, s), \tag{4.5}$$

where $H_j(\overline{x})$ for $j = 1, \dots, n-1$, denote the principal curvatures of $\partial\Omega$ at $\overline{x}$.

Let $\alpha > 1/(1-l)$, $0 < \xi_0 \le 1$ and $t_0 < T_3 \le \min(T_2, t_0 + \delta^2)$. For points in $Q_{\delta,T_3} = \partial\Omega \times [0,\delta] \times (t_0, T_3)$ of coordinates $(\overline{x}, s, t)$ we define

$$\underline{u}(\overline{x}, s, t) = (t - t_0)^\alpha \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^3,$$

and for points in $\overline{\Omega} \times [t_0, T_3) \setminus Q_{\delta,T_3}$ we put $\underline{u}(\overline{x}, s, t) \equiv 0$. We shall prove that $\underline{u}(\overline{x}, s, t)$ is subsolution of (1.1)–(1.3) in $\Omega \times (t_0, T_3)$. Indeed, using (4.5), we get

$$\begin{aligned}
&\underline{u}_t(\overline{x}, s, t) - \Delta \underline{u}(\overline{x}, s, t) - c(x,t)\underline{u}^p(\overline{x}, s, t) = \alpha (t-t_0)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t-t_0}} \right)_+^3 \\
+ \quad & \frac{3}{2} s (t-t_0)^{\alpha - 3/2} \left(\xi_0 - \frac{s}{\sqrt{t-t_0}} \right)_+^2 - 6(t-t_0)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t-t_0}} \right)_+ \\
- \quad & 3(t-t_0)^{\alpha-1/2} \left(\xi_0 - \frac{s}{\sqrt{t-t_0}} \right)_+^2 \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{(1 - sH_j(\overline{x}))} - c(x,t)\underline{u}^p(\overline{x}, s, t) \le 0
\end{aligned}$$

for sufficiently small values of ξ_0.

It is obvious,

$$\frac{\partial \underline{u}}{\partial \nu}(\overline{x}, 0, t) = -\frac{\partial \underline{u}}{\partial s}(\overline{x}, 0, t) = 3(t-t_0)^{\alpha - \frac{1}{2}} \xi_0^2.$$

For sufficiently small values of $t - t_0$ we get

$$\begin{aligned}
&\tfrac{\partial \underline{u}}{\partial \nu}(x,t) - \textstyle\int_\Omega k(x,y,t) \underline{u}^l(y,t)\, dy = 3(t-t_0)^{\alpha-\frac{1}{2}} \xi_0^2 \\
&-(t-t_0)^{\alpha l} \textstyle\int_{\partial\Omega \times [0,\delta]} k(x, (\overline{y}, s), t) |J(\overline{y}, s)| \left(\xi_0 - \frac{s}{\sqrt{t-t_0}} \right)_+^{3l} d\overline{y}\, ds \le 3(t-t_0)^{\alpha-\frac{1}{2}} \xi_0^2 \\
&-(t-t_0)^{\alpha l + \frac{1}{2}} \textstyle\int_{\partial\Omega} d\overline{y} \int_0^{\xi_0} k(x, (\overline{y}, z\sqrt{t-t_0}), t) |J(\overline{y}, z\sqrt{t-t_0})| \, (\xi_0 - z)_+^{3l}\, dz \\
&\le 3(t-t_0)^{\alpha - \frac{1}{2}} \xi_0^2 - C(t-t_0)^{\alpha l + \frac{1}{2}} \le 0,
\end{aligned}$$

where $J(\overline{y}, s)$ is Jacobian of the change of variables, and the constant C does not depend on t. Completion of the proof is the same as in the first part of the theorem. $\square$

Suppose that

$$c(x,t) \not\equiv 0 \text{ in } Q_\tau \text{ for any } \tau > 0 \text{ and } 0 < p < 1 \tag{4.6}$$

and there exist sequences $\{t_k\}$ and $\{y_k\}$, $k \in N$, such that

$$t_k > 0, \lim_{k\to\infty} t_k = 0,\ y_k \in \partial\Omega,\ k(x, y_k, t_k) > 0 \text{ for any } x \in \partial\Omega \text{ and } 0 < l < 1. \tag{4.7}$$

Remark 4.2. Let the assumptions of Theorem 4.1 hold but only at least one condition (4.6) or (4.7) is fulfilled instead of (4.1), (4.2). Then maximal solution of (1.1)–(1.3) is positive in $Q_T \cup S_T$.

Corollary 4.3. *Let the assumptions of Theorem 4.1 hold but only at least one condition (4.6) or (4.7) is fulfilled instead of (4.1), (4.2). Suppose that there exists* $\overline{t} \in (0,T)$ *such that*

$$c(x,t) \text{ and } k(x,y,t) \text{ are nondecreasing with respect to } t \in [0,\overline{t}]. \tag{4.8}$$

Then there exists exactly one solution of (1.1)–(1.3) which is positive in $Q_T \cup S_T$.

Proof. Denote $u(x,t)$ maximal solution of (1.1)–(1.3) with $u_0(x) \equiv 0$. Due to Remark 4.2 $u(x,t) > 0$ in $Q_T \cup S_T$. Suppose, for a contradiction, that there exists another solution $v(x,t)$ of (1.1)–(1.3) with trivial initial datum which is positive in $Q_T \cup S_T$. By virtue of (4.8) $v(x,t+\tau)$ is positive supersolution of (1.1)–(1.3) in $Q_{\overline{t}-\tau}$ for any $\tau \in (0,\overline{t})$. By Theorem 2.4 then we get $u(x,t) \le v(x,t+\tau)$ for $(x,t) \in Q_{\overline{t}-\tau} \cup \Gamma_{\overline{t}-\tau}$. Passing to the limit as $\tau \to 0$ we have $u(x,t) \le v(x,t)$ for $(x,t) \in Q_{\overline{t}} \cup \Gamma_{\overline{t}}$. By Definition 2.2 and Theorem 2.5 we obtain $v(x,t) = u(x,t)$ for all $t \in (0,T)$. □

Theorem 4.4. *Let* $\min(p,l) < 1,\ u_0 \not\equiv 0$ *and (4.8) be satisfied. Suppose that (4.6) or (4.7) hold. Then the solution of (1.1)–(1.3) is unique.*

Proof. In order to prove uniqueness we show that if v is any solution of (1.1)–(1.3) then

$$u \le v \text{ in } Q_{T_1}, \tag{4.9}$$

where u is maximal solution of (1.1)–(1.3).

We shall consider three cases: $0 < l < 1$ and $0 < p \le 1$, $0 < l < 1$ and $p > 1$, $0 < p < 1$ and $l \ge 1$.

Let $0 < l < 1,\ 0 < p \le 1$. Put $z = u - v$. Then z satisfies the problem

$$\begin{cases} z_t \le \Delta z + c(x,t) z^p,\ (x,t) \in Q_{T_1}, \\ \frac{\partial z(x,t)}{\partial \nu} \le \int_\Omega k(x,y,t) z^l(y,t)\, dy,\ (x,t) \in S_{T_1}, \\ z(x,0) \equiv 0,\ x \in \Omega. \end{cases} \tag{4.10}$$

By Corollary 4.3 there exists unique solution $h(x,t)$ of the following problem

$$\begin{cases} h_t = \Delta h + c(x,t) h^p,\ (x,t) \in Q_{T_2}, \\ \frac{\partial h(x,t)}{\partial \nu} = \int_\Omega k(x,y,t) h^l(y,t)\, dy,\ (x,t) \in S_{T_2}, \\ h(x,0) \equiv 0,\ x \in \Omega, \end{cases}$$

such that $h(x,t) > 0$ for $x \in \overline{\Omega}$ and $0 < t < T_2$. Let $T_3 = \min(T_1, T_2)$. In a similar way as in Corollary 4.3 and Theorem 2.4 it can be shown that $h \ge z$ and $u \ge h$. Put $a = h - z$ and use the following inequality (see, for example, [1])

$$h^q - u^q + v^q \ge (h - u + v)^q,$$

where $0 < q \le 1$ and $\max\{h,v\} \le u \le h + v$. Then we get

$$\begin{cases} a_t \ge \Delta a + c(x,t) a^p,\ (x,t) \in Q_{T_3}, \\ \frac{\partial a(x,t)}{\partial \nu} \ge \int_\Omega k(x,y,t) a^l(y,t)\, dy,\ (x,t) \in S_{T_3}, \\ a(x,0) \equiv 0,\ x \in \Omega. \end{cases}$$

We claim that $a > 0$ in Q_{T_3}. Indeed, otherwise by Theorem 2.3 there exists $\bar{t} \in (0,T_3)$ such that $a(x,t) \equiv 0$ for $(x,t) \in Q_{\bar{t}}$. Then we obtain

$$\int_\Omega k(x,y,t)(h^l(y,t) + v^l(y,t))\, dy = \frac{\partial h(x,t)}{\partial \nu} + \frac{\partial v(x,t)}{\partial \nu} = \frac{\partial z(x,t)}{\partial \nu} + \frac{\partial v(x,t)}{\partial \nu}$$

$$
\begin{aligned}
&= \frac{\partial u(x,t)}{\partial \nu} = \int_\Omega k(x,y,t)u^l(y,t)\,dy = \int_\Omega k(x,y,t)(z(y,t)+v(y,t))^l\,dy \\
&= \int_\Omega k(x,y,t)(h(y,t)+v(y,t))^l\,dy
\end{aligned}
$$

for all $x \in \partial\Omega$ and $0 < t < \bar{t}$. This is a contradiction, if (4.7) is satisfied, since $h > 0,\ v > 0$ in $Q_{\bar{t}}$, $k(x, y_k, t_k) > 0$ for all $x \in \partial\Omega$ and some $y_k \in \partial\Omega$, $0 < t_k < \bar{t}$ and $0 < l < 1$.

If (4.6) is fulfilled, we can get a contradiction in another way. Really,

$$
\begin{aligned}
&c(x,t)(h+v)^p = c(x,t)(z+v)^p = c(x,t)u^p = u_t - \Delta u = (z+v)_t - \Delta(z+v) \\
&= (h+v)_t - \Delta(h+v) = c(x,t)(h^p + v^p)
\end{aligned}
$$

for all $x \in \Omega$ and $0 < t < \bar{t}$. This is a contradiction since $h > 0,\ v > 0$ in $Q_{\bar{t}}$, $c(x_1, t_1) > 0$ for some $x_1 \in \Omega,\ t_1 \in (0, \bar{t})$, and $0 < p < 1$.

Since $a > 0$ in $Q_{\bar{t}}$ by Corollary 4.3 and Theorem 2.4 we conclude that $a \geq h$ in $Q_{\bar{t}} \cup \Gamma_{\bar{t}}$. This implies (4.9) for the case $0 < l < 1$ and $0 < p \leq 1$.

We consider the second case $0 < l < 1$ and $p > 1$. It is easy to see that there exists a constant $M > 0$ such that

$$
u^p(x,t) - v^p(x,t) \leq M(u(x,t) - v(x,t)),\ (x,t) \in Q_{T_4},
$$

where $T_4 < T_2$. Put $z = u - v$. Then the function $z(x,t)$ satisfies problem (4.10) with $p = 1$, and $Mc(x,t)$ instead of $c(x,t)$. Further the proof is the same as in the first case with $p = 1$.

In the third case $0 < p < 1$ and $l \geq 1$ either $0 < p < 1$ and $c(x,t) \equiv 0$ in Q_σ for some $\sigma > 0$ or (4.6) is true. If (4.6) is fulfilled, we can argue as in previous cases, otherwise, the solution of (1.1)–(1.3) is unique by Theorem 2.3 and Theorem 2.5. □

Remark 4.5. As we can see from the proof of Theorem 4.4, the solution of (1.1)–(1.3) is unique if $u_0 \not\equiv 0$, (4.8) hold and $k(x,y,t) \equiv 0$ in $\partial\Omega \times Q_\tau$ for some $\tau > 0$.

References

[1] J. Aguirre and M. Escobedo, *A Cauchy problem for $u_t - \Delta u = u^p$: Asymptotic behavior of solutions*, Ann. Fac. Sci. Toulouse **8** (1986-87), 175–203.

[2] R. Bartle and D. Sherbert, *Introduction to real analysis*, John Wiley & Sons, Inc. (2011), 402 pp.

[3] P. Bokes,*A uniqueness result for a semilinear parabolic system*, J. Math. Anal. Appl. **331** (2007), 567–584.

[4] C. Cortazar, M. del Pino and M. Elgueta, *On the short-time behaviour of the free boundary of a porous medium equation*, Duke Math. J. **87** (1997), no. 1, 133–149.

[5] C. Cortazar, M. Elgueta and J. D. Rossi, *Uniqueness and non-uniqueness for a system of heat equations with non-linear coupling at the boundary*, Nonlinear Anal. **37** (1999), no. 2, 257–267.

[6] C. Cortazar, M. Elgueta and J. D. Rossi, *Uniqueness and nonuniqueness for the porous medium equation with non linear boundary condition*, Diff. Int. Equat. **16** (2003), no. 10, 1215–1222.

[7] K. Deng, *Comparison principle for some nonlocal problems*, Quart. Appl. Math. **50** (1992), no. 3, 517–522.

[8] M. Escobedo and M. A. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. **CLXV** (1993), 315–336.

[9] H. Fujita and S. Watanabe, *On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), no. 6, 631–652.

[10] B. Hu, *Blow-up theories for semilinear parabolic equations*, Lecture Notes in Mathematics **2018** (2011), 127 p.
[11] B. Hu and H.-M. Yin, *Critical exponents for a system of heat equations coupled in a nonlinear boundary condition*, Math. Methods Appl. Sci. **19** (1996), no. 14, 1099–1120.
[12] A. Gladkov and M. Guedda, *Blow-up problem for semilinear heat equation with absorption and a nonlocal boundary condition*, Nonlinear Anal. **74** (2011), no. 13, 4573–4580.
[13] A. Gladkov and M. Guedda, *Semilinear heat equation with absorption and a nonlocal boundary condition*, Appl. Anal. **91** (2012), no. 12, 2267–2276.
[14] A. Gladkov and K. Ik Kim, *Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition*, J. Math. Anal. Appl. **338** (2008), 264–273.
[15] A. Gladkov and K. Ik Kim, *Uniqueness and nonuniqueness for reaction-diffusion equation with nonlocal boundary condition*, Adv. Math. Sci. Appl. **19** (2009), no. 1, 39–49.
[16] A. Gladkov and A. Nikitin, *A reaction-diffusion system with nonlinear nonlocal boundary conditions*, Int. J. Partial Differential Equations **2014** (2014), 1–10.
[17] C. S. Kahane, *On the asymptotic behavior of solutions of parabolic equations*, Czechoslovak Math. J. **33** (1983), no. 108, 262–285.
[18] M. Kordoš, *On uniqueness for a semilinear parabolic system coupled in an equation and a boundary condition*, J. Math. Anal. Appl. **298** (2004), 655–666.
[19] D. Liu and C. Mu, *Blowup properties for a semilinear reaction-diffusion system with nonlinear nonlocal boundary conditions*, Abs. Appl. Anal. **2010** (2010), 1–17.
[20] C. V. Pao, *Asimptotic behavior of solutions of solutions of reaction-diffusion equations with nonlocal boundary conditions*, J. Comput. Appl. Math. **88** (1998), 225–238.
[21] Y. Yin, *On nonlinear parabolic equations with nonlocal boundary condition*, J. Math. Anal. Appl. **185** (1994), no. 1, 161–174.

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