A Note on Total and Paired Domination of Cartesian Product Graphs

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Abstract

A dominating set D for a graph G is a subset of V(G) such that any vertex not in D has at least one neighbor in D. The domination number $\gamma(G)$ is the size of a minimum dominating set in G. Vizing's conjecture from 1968 states that for the Cartesian product of graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G\Box H)$, and Clark and Suen (2000) proved that $\gamma(G)\gamma(H) \leq 2\gamma(G\Box H)$. In this paper, we modify the approach of Clark and Suen to prove a variety of similar bounds related to total and paired domination, and also extend these bounds to the n-Cartesian product of graphs A^1 through A^n .

1 Introduction

We consider simple undirected graphs G = (V, E) with vertex set V and edge set E. The open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and closed neighborhood by $N_G[v]$. A dominating set D of a graph G is a subset of V(G) such that for all v, $N_G[v] \cap D \neq \emptyset$. A γ -set of G is a minimum dominating set for G, and its size is denoted $\gamma(G)$. A total dominating set D of a graph G is a subset of V(G) such that for all v, $N_G(v) \cap D \neq \emptyset$. A γ_t -set of G is a minimum total dominating set for G, and its size is denoted $\gamma_t(G)$. A paired dominating set D for a graph G is a dominating set such that the subgraph of G induced by D (denoted G[D]) has a perfect matching. A γ_{pr} -set of G

is a minimum paired dominating set for G, and its size is denoted $\gamma_{pr}(G)$. In general, for a graph containing no isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$.

The Cartesian product graph, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$, where vertices gh and g'h' are adjacent whenever g = g' and $(h, h') \in E(H)$, or h = h' and $(g, g') \in E(G)$. Just as the Cartesian product of graphs G and H is denoted $G \square H$, the n-product of graphs A^1, A^2, \ldots, A^n is denoted as $A^1 \square A^2 \square \cdots \square A^n$, and has vertex set $V(A^1) \times V(A^2) \times \cdots \times V(A^n)$, where vertices $u^1 \cdots u^n$ and $v^1 \cdots v^n$ are adjacent if and only if for some i, $(u^i, v^i) \in E(A^i)$, and $u^j = v^j$ for all other indices $j \neq i$.

Vizing's conjecture from 1968 states that $\gamma(G)\gamma(H) \leq \gamma(G\square H)$. For a thorough review of the activity on this famous open problem, see [1] and references therein. In 2000, Clark and Suen [2] proved that $\gamma(G)\gamma(H) \leq 2\gamma(G\square H)$ by a sophisticated double-counting argument which involved projecting a γ -set of the product graph $G\square H$ down onto the graph G. In this paper, we slightly modify the Clark and Suen double-counting approach and instead project subsets of $G\square H$ down onto both graphs G and G, which allow us to prove five theorems relating to total and paired domination. In this section, we state the results, and in Section 2, we prove the results.

Theorem 1. Given graphs G and H containing no isolated vertices,

$$max\{\gamma(G)\gamma_t(H), \gamma_t(G)\gamma(H)\} \le 2\gamma(G\square H)$$
.

In 2008, Ho [3] proved an inequality for total domination analogous to the Clark and Suen inequality for domination. In particular, Ho proved $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G\Box H)$. We provide a slightly different proof of Ho's inequality, and then extend the result to the n-product case.

Theorem 2 (Ho [3]). Given graphs G and H containing no isolated vertices,

$$\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G\square H)$$
.

Theorem 3. Given graphs A^1, A^2, \ldots, A^n containing no isolated vertices,

$$\prod_{i=1}^{n} \gamma_t(A^i) \le n\gamma_t(A^1 \square A^2 \square \cdots \square A^n) .$$

In 2010, Hou and Jiang [4] proved that $\gamma_{pr}(G)\gamma_{pr}(H) \leq 7\gamma_{pr}(G\square H)$, for graphs G and H containing no isolated vertices. We provide an improvement to this result, and extend the result to the n-product graph.

Theorem 4. Given graphs G and H containing no isolated vertices,

$$\gamma_{pr}(G)\gamma_{pr}(H) \leq 6\gamma_{pr}(G\Box H)$$
.

Theorem 5. Given graphs A^1, \ldots, A^n containing no isolated vertices,

$$\prod_{i=1}^{n} \gamma_{pr}(A_i) \le 2^{n-1} (2n-1) \gamma_{pr}(A_1 \square \cdots \square A_n) .$$

2 Main Results

We begin by introducing some notation which will be utilized throughout the proofs in this section. Given $S \subseteq V(G \square H)$, the projection of S onto graphs G and H is defined as

$$\Phi_G(S) = \{ g \in V(G) \mid \exists h \in V(H) \text{ with } gh \in S \},
\Phi_H(S) = \{ h \in V(H) \mid \exists g \in V(G) \text{ with } gh \in S \}.$$

In the case of the *n*-product graph $A^1 \square \cdots \square A^n$, we project a set of vertices in $V(A^1 \square \cdots \square A^n)$ down to a particular graph A_i . Therefore, given $S \subseteq V(A^1 \square \cdots \square A^n)$, we define

$$\Phi_{A^i}(S) = \{ a \in V(A^i) \mid \exists u^1 \cdots u^n \in S \text{ with } a = u^i \}$$
.

For $gh \in V(G \square H)$, the G-neighborhood and H-neighborhood of gh are defined as follows:

$$N_{\underline{\mathbf{G}} \square H}(gh) = \{ g'h \in V(G \square H) \mid g' \in N_G(g) \} ,$$

$$N_{G \square \mathbf{H}}(gh) = \{ gh' \in V(G \square H) \mid h' \in N_H(h) \} .$$

Thus, $N_{\underline{\mathbf{G}} \square H}(gh)$ and $N_{G \square \underline{\mathbf{H}}}(gh)$ are both subsets of $V(G \square H)$. Additionally, $E(G \square H)$ can be partitioned into two sets, \mathbf{G} -edges and \mathbf{H} -edges, where

G-edges =
$$\{(gh, g'h) \in E(G \square H) \mid h \in V(H) \text{ and } (g, g') \in E(G)\}$$
,
H-edges = $\{(gh, gh') \in E(G \square H) \mid g \in V(G) \text{ and } (h, h') \in E(H)\}$.

In the case of the *n*-product graph $A^1 \square \cdots \square A^n$, we identify the *i*-neighborhood of a particular vertex, and partition the set of edges $E(A^1 \square \cdots \square A^n)$ into *n* sets. Thus, we define E_i to be

$$E_i = \left\{ \left(u^1 \cdots u^n, v^1 \cdots v^n \right) \mid (u^i, v^i) \in E(A^i), \text{ and } u_j = v_j, \text{ for all other indices } j \neq i \right\},$$

and for a vertex $u \in V(A^1 \square \cdots \square A^n)$, we define

$$N_{\Box A^i}(u) = \left\{ v \in V(A^1 \Box \cdots \Box A^n) \mid v \text{ and } u \text{ are connected by } E_i\text{-edge} \right\}$$
.

Finally, we need two elementary propositions about matrices that will be utilized throughout the proofs.

Proposition 1. Let M be a binary matrix. Then either

- (a) each column contains a 1, or
- (b) each row contains a θ .

Prop. 1 refers only to $d_1 \times d_2$ binary matrices. Prop. 2 is a generalization of Prop. 1 for $d_1 \times d_2 \times \cdots \times d_n$ n-ary matrices.

Proposition 2. Let M be a $d_1 \times d_2 \times \cdots \times d_n$, n-ary matrix (n-ary in this case signifies that M contains entries only in the range $\{1, \ldots, n\}$). Then there exists a $j \in \{1, \ldots, n\}$ (not necessarily unique), such that each of the $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$ submatrices of M contains an entry with value j. Such a matrix M is called a j-matrix.

Note that, given any $d_1 \times d_2 \times \cdots \times d_n$ matrix, there are d_j submatrices of the form $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$. We will denote such a submatrix as $M[:, i_j, :]$ with $1 \leq i_j \leq d_j$.

Proof. Let M be a $d_1 \times d_2 \times \cdots \times d_n$ n-ary matrix which is not a j-matrix for $1 \leq j \leq n-1$. We will show that M is an n-matrix.

Consider j=1. Since M is not a 1-matrix, there exists at least one $1\times d_2\times d_3\times\cdots\times d_n$ submatrix that does not contain a 1. Without loss of generality, let $M[i_1,:]$ with $1\leq i_1\leq d_1$ be such a matrix. Next, consider j=2. Since M is also not a 2-matrix, let $M[:,i_2,:]$ with $1\leq i_2\leq d_2$ be a $d_1\times 1\times d_3\times\cdots\times d_n$ submatrix that does not contain a 2. Therefore, $M[i_1,i_2,:]$ is a $1\times 1\times d_3\times\cdots\times d_n$ submatrix that contains neither a 1 nor a 2. We continue this pattern for $1\leq j\leq n-1$. Since M is not a j-matrix for $1\leq j\leq n-1$, let $M[i_1,\ldots,i_{n-1},:]$ be the $1\times\cdots 1\times d_n$ submatrix containing no elements in the set $\{1,\cdots,n-1\}$. Therefore, for all $1\leq x\leq d_n$, $M[i_1,\ldots,i_{n-1},x]=n$, and all of the $d_1\times\cdots\times d_{n-1}\times 1$ submatrices of M contains an entry with value n. Thus, M is an n-matrix.

Now, we present the proofs of Theorems 1 through 5.

2.1 Proof of Theorem 1

Proof. Let $\{u_1, \ldots, u_{\gamma_t(G)}\}$ be a γ_t -set of G. Partition V(G) into sets $D_1, \ldots, D_{\gamma_t(G)}$, such that $D_i \subseteq N_G(u_i)$. Let $\{\overline{u}_1, \ldots, \overline{u}_{\gamma(H)}\}$ be a γ -set of H. Partition V(H) into sets $\overline{D}_1, \ldots, \overline{D}_{\gamma(H)}$, such that $\overline{u}_j \in \overline{D}_j$ and $\overline{D}_j \subseteq N_H[\overline{u}_j]$. We note that $\{D_1, \ldots, D_{\gamma_t(G)}\} \times \{\overline{D}_1, \ldots, \overline{D}_{\gamma(H)}\}$ is a partition of $V(G \square H)$. Let D be a γ -set of $G \square H$. Then, for each $gh \notin D$, either $N_{\underline{\mathbf{G}} \square H}(gh) \cap D$ or $N_{G \square \underline{\mathbf{H}}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F such that:

$$F(g,h) = \begin{cases} 1 & \text{if } gh \in D \text{ or } N_{G \square \underline{\mathbf{H}}}(gh) \cap D \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Since F is a $|V(G)| \times |V(H)|$ matrix, each of the $D_i \times \overline{D}_j$ subsets of $V(G \square H)$ determines a submatrix of F.

For
$$i = 1, ..., \gamma_t(G)$$
, let $Z_i = D \cap (D_i \times V(H))$, and let

$$S_i = \{\overline{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \overline{D}_x \text{ satisfies Prop. 1a,}$$

with $x \in \{1, \dots, \gamma(H)\}\}$.

For $j=1,\ldots,\gamma(H)$, let $\overline{Z}_j=D\cap (V(G)\times \overline{D}_j)$, and let $\overline{S}_j=\left\{D_x\mid \text{the submatrix of }F\text{ determined by }D_x\times \overline{D}_j\text{ satisfies Prop. 1b,} \right.$ with $x\in\{1,\ldots,\gamma_t(G)\}$.

Let $d_H = \sum_{i=1}^{\gamma_t(G)} |S_i|$, and $d_G = \sum_{j=1}^{\gamma(H)} |\overline{S}_j|$. Since the partition of $V(G \square H)$ composed of elements $D_i \times \overline{D}_j$ contains $\gamma_t(G)\gamma(H)$ components, and since every $D_i \times \overline{D}_j$ submatrix of F satisfies either conditions (a) or (b) of Prop. 1 (possibly both), $\gamma_t(G)\gamma(H) \leq d_H + d_G$. We will now prove two subclaims which will allow us to bound the size of our various sets.

Claim 1. If the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1a, then \overline{D}_j is dominated by $\Phi_H(Z_i)$.

Proof. Let $h \in \overline{D}_j$. We must show that either $h \in \Phi_H(Z_i)$, or h is adjacent to a vertex h' in $\Phi_H(Z_i)$. If $(D_i \times \{h\}) \cap D \neq \emptyset$, there exists a $g \in D_i$ such that $gh \in D$. Thus, $h \in \Phi_H(Z_i)$.

If $(D_i \times \{h\}) \cap D = \emptyset$, then recall that the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1a. Therefore, there is a 1 in every column of the submatrix. This implies there exists a $g \in D_i$ such that F(g,h) = 1. Since $gh \notin D$, there exists an $h' \in V(H)$ such that $gh' \in N_{G \square \underline{\mathbf{H}}}(gh) \cap D$. Therefore, (gh', gh) is an \mathbf{H} -edge, implying $(h, h') \in E(H)$ and h is adjacent to h'. Therefore, \overline{D}_i is dominated by $\Phi_H(Z_i)$.

Claim 2. If the submatrix of F determined by $D_i \times \overline{D}_j$ satisfies Prop. 1b, then D_i is dominated by $\Phi_G(\overline{Z}_j)$. Additionally, $\forall g \in D_i \cap \Phi_G(\overline{Z}_j)$, there exists a vertex $g' \in \Phi_G(\overline{Z}_j)$ such that $(g, g') \in E(G)$.

We note that this claim does not imply that $\Phi_G(\overline{Z}_j)$ is a total dominating set, but the claim is a slightly stronger condition on domination. When applying this condition, we will say that the set D_i is non-self dominated by $\Phi_G(\overline{Z}_i)$.

Proof. The argument for proving that $\Phi_G(\overline{Z}_j)$ dominates D_i is almost identical to the proof of Claim 1. The only difference is that the $D_i \times \overline{D}_j$ submatrix of F satisfies Prop. 1b. Thus, every row contains a 0. But since every vertex in $V(G \square H)$ is dominated by D, this implies that every vertex $g \in D_i$ is dominated by some other (not itself) vertex $g' \in \Phi_G(\overline{Z}_j)$. Thus, D_i is dominated by $\Phi_G(\overline{Z}_j)$, with the slightly stronger condition that every vertex in D_i (even those vertices in $D_i \cap \Phi_G(\overline{Z}_j)$) is adjacent to another vertex in $\Phi_G(\overline{Z}_j)$.

Claim 3. For $i = 1, ..., \gamma_t(G)$, $|S_i| \leq |Z_i|$. Similarly, for $j = 1, ..., \gamma(H)$, $|\overline{S}_j| \leq |\overline{Z}_j|$.

Proof. Let $S_i = \{\overline{D}_{j_1}, \overline{D}_{j_2}, \dots, \overline{D}_{j_k}\}$, and let $A = \Phi_H(Z_i)$. Note that $|A| \leq |Z_i|$. By Claim 1, A dominates $\bigcup_{x=1}^k \overline{D}_{j_x}$. Therefore, $A \cup \{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}$ is a dominating set of H, and, since the sets A and $\{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}$ are disjoint, then

$$|A \cup \{\overline{u}_j \mid j \notin \{j_1, j_2, \dots, j_k\}\}| = |A| + (\gamma(H) - k) \ge \gamma(H)$$
.

Hence, $k = |S_i| \le |A| \le |Z_i|$.

For the proof of second part, let $\overline{S}_j = \{D_{i_1}, D_{i_2}, \dots, D_{i_k}\}$, and let $A = \Phi_G(\overline{Z}_j)$. Again, note that $|A| \leq |\overline{Z}_j|$. Then by Claim 2, A dominates $\bigcup_{x=1}^k D_{i_x}$, with the stronger condition that $\forall g \in D_{i_x} \cap A$, there exists a vertex $g' \in A$ such that $(g, g') \in E(G)$. Now we consider $A \cap \{u_i \mid i \notin \{i_1, i_2, \dots, i_k\}\}$. If this intersection is non-empty, let $A \cap \{u_i \mid i \notin \{i_1, i_2, \dots, i_k\}\}$ = $\{u_{i_{k+1}}, \dots, u_{i_l}\}$. Then, A dominates $\bigcup_{x=1}^l D_{i_x}$ with the same stronger condition. Moreover, the sets A and $\{u_i \mid i \notin \{i_1, i_2, \dots, i_k, \dots, i_l\}\}$ are disjoint.

We claim that $A \cup \{u_i \mid i \notin \{i_1, \ldots, i_l\}\}$ is a total dominating set of G. To see this, consider any vertex $g \in V(G)$. If $g \in D_x$ with $x \in \{i_1, i_2, \ldots, i_k\}$, then by the stronger condition on domination associated with Claim 2, g is adjacent to another vertex in A. If $g \in D_x$ with $x \notin \{i_1, \ldots, i_k\}$, then $u_x \in \{u_i \mid i \notin \{i_1, \ldots, i_k\}\}$, and g is adjacent to u_x , since u_x dominates D_x . We note that u_x is either in A (if $k+1 \le x \le l$) or in $\{u_i \mid i \notin \{i_1, \ldots, i_l\}\}$. In either case, $A \cup \{u_i \mid i \notin \{i_1, \ldots, i_l\}\}$ is a total dominating set of G, and

$$|A \cup \{u_i \mid i \notin \{i_1, i_2, \dots, i_l\}\}| = |A| + (\gamma_t(G) - l) \ge \gamma_t(G)$$
.

Hence, as before, $k = |\overline{S}_i| \le l \le |A| \le |\overline{Z}_i|$.

To conclude the proof, we observe that

$$d_{H} = \sum_{i=1}^{\gamma_{t}(G)} |S_{i}| \le \sum_{i=1}^{\gamma_{t}(G)} |Z_{i}| \le |D|,$$

$$d_{G} = \sum_{j=1}^{\gamma(H)} |\overline{S}_{j}| \le \sum_{j=1}^{\gamma(H)} |\overline{Z}_{j}| \le |D|.$$

Hence, $\gamma_t(G)\gamma(H) \leq d_H + d_G \leq 2|D| \leq 2\gamma(G\square H)$. Moreover, we can similarly prove that $\gamma(G)\gamma_t(H) \leq 2\gamma(G\square H)$. Therefore, $\max\{\gamma(G)\gamma_t(H), \gamma_t(G)\gamma(H)\} \leq 2\gamma(G\square H)$.

2.2 Proof of Theorem 2

Proof. Let $\{u_1, \ldots, u_{\gamma_t(G)}\}$ be a γ_t -set of G. Partition V(G) into sets $D_1, \ldots, D_{\gamma_t(G)}$, such that if $u \in D_i$ then $u \in N_G(u_i)$ for all $i = 1, \ldots, \gamma_t(G)$. Similarly, let $\{\overline{u}_1, \ldots, \overline{u}_{\gamma_t(H)}\}$ be a γ_t -set of H and $\overline{D}_1, \ldots, \overline{D}_{\gamma_t(H)}$ be the corresponding partitions. Then, $\{D_1, \ldots, D_{\gamma_t(G)}\} \times \{\overline{D}_1, \ldots, \overline{D}_{\gamma_t(H)}\}$ forms a partition of $V(G \square H)$.

Let D be a γ_t -set of $G \square H$. Then, for each $gh \in V(G \square H)$, either the set $N_{\underline{\mathbf{G}} \square H}(gh) \cap D$ or the set $N_{G \square \underline{\mathbf{H}}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F:

$$F(g,h) = \begin{cases} 1 & \text{if } N_{G \square \underline{\mathbf{H}}}(gh) \cap D \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

For $i = 1, ..., \gamma_t(G)$, let $Z_i = D \cap (D_i \times V(H))$, and let

 $S_i = \{\overline{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \overline{D}_x \text{ satisfies Prop. 1a,}$ with $x \in \{1, \dots, \gamma_t(H)\}\}$.

For $j = 1, ..., \gamma_t(H)$, let $\overline{Z}_j = D \cap (V(G) \times \overline{D}_j)$, and let

 $\overline{S}_j = \{D_x \mid \text{the submatrix of } F \text{ determined by } D_x \times \overline{D}_j \text{ satisfies Prop. 1b,}$ with $x \in \{1, \dots, \gamma_t(G)\}\}$.

Let $d_H = \sum_{i=1}^{\gamma_t(G)} |S_i|$, and $d_G = \sum_{j=1}^{\gamma_t(H)} |\overline{S}_j|$. Since the partition of $V(G \square H)$ composed of elements $D_i \times \overline{D}_j$ contains $\gamma_t(G)\gamma_t(H)$ components, and since every submatrix of F determined by $D_i \times \overline{D}_j$ satisfies either Prop. 1a or 1b (or possibly both), then $\gamma_t(G)\gamma_t(H) \leq d_H + d_G$.

Furthermore, by similar arguments given in the proof of Theorem 1 (specifically, Claims 1 and 2), we can conclude, as before, that for $i = 1, ..., \gamma_t(G)$, $|S_i| \leq |Z_i|$ and, for $j = 1, ..., \gamma_t(H)$, $|\overline{S}_j| \leq |\overline{Z}_j|$. Finally,

$$d_{H} = \sum_{i=1}^{\gamma_{t}(G)} |S_{i}| \leq \sum_{i=1}^{\gamma_{t}(G)} |Z_{i}| = |D| = \gamma_{t}(G \square H) ,$$

$$d_{G} = \sum_{j=1}^{\gamma_{t}(H)} |\overline{S}_{j}| \leq \sum_{j=1}^{\gamma_{t}(H)} |\overline{Z}_{j}| = |D| = \gamma_{t}(G \square H) .$$

Summing these two equations, we see $d_H + d_G \leq 2\gamma_t(G\Box H)$, which implies $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G\Box H)$.

2.3 Proof of Theorem 3

Proof. For i = 1, ..., n, let $\{u_1^i, ..., u_{\gamma_t(A^i)}^i\}$ be a γ_t -set of A^i , and $D_1^i, ..., D_{\gamma_t(A^i)}^i$ be the corresponding partitions (as defined in the proof of Theorem 2).

Let $Q = \{D_1^1, \dots, D_{\gamma_t(A^1)}^1\} \times \dots \times \{D_1^n, \dots, D_{\gamma_t(A^n)}^n\}$. Then Q forms a partition of $V(A^1 \square \dots \square A^n)$ with $|Q| = \prod_{i=1}^n \gamma_t(A^i)$.

Let D be a γ_t -set of $A^1 \square \cdots \square A^n$. Then, for each $u \in V(A^1 \square \cdots \square A^n)$, there exists an i such that $N_{\square A^i}(u) \cap D$ is non-empty. Based on this observation (as in the 2-dimensional case), we define an n-ary $|V(A^1)| \times \cdots \times |V(A^n)|$ matrix F such that:

$$F(u_1, \dots, u_n) = \min\{i \mid N_{\square A^i}(u_1 \cdots u_n) \cap D \neq \emptyset\}.$$

For j = 1, ..., n, let $d_j \subseteq Q$ be the set of the elements in Q which are j-matrices. By Prop. 2, each element of Q belongs to at least one d_j -set. Then, $\prod_{i=1}^n \gamma_t(A^i) \leq \sum_{j=1}^n |d_j|$.

Claim 4. For $j = 1, ..., n, |d_j| \le |D|$.

Proof. We prove here that $|d_n| \leq |D|$, but a similar proof can be performed for any other j. Similar to Q, let $B = \{D_1^1, \ldots, D_{\gamma_t(A^1)}^1\} \times \cdots \times \{D_1^{n-1}, \ldots, D_{\gamma_t(A^{n-1})}^{n-1}\}$. For convenience,

we denote *B* as
$$\{B_1, ..., B_{|B|}\}$$
, where $|B| = \prod_{i=1}^{(n-1)} \gamma_t(A^i)$.

For
$$p = 1, ..., |B|$$
, let $Z_p = D \cap (B_p \times A^n)$, and

$$S_p = \{D_x^n \mid \text{the submatrix of } F \text{ determined by } B_p \times D_x^n \text{ is an } n\text{-matrix},$$

with $x \in \{1, \dots, \gamma_t(A^n)\}\}$.

Note that if $q \in Q$ is a *n*-matrix, then the projection of q on A^n is non-self-dominated by the projection of D on A^n (the same condition used in Claim 2). Moreover, if q is written as $B_p \times D_x^n$ for some $p \in \{1, \ldots, |B|\}$ and $x \in \{1, \ldots, \gamma_t(A^n)\}$, then D_x^n is non-self-dominated by the projection of Z_p on A^n .

We now claim that for $p=1,\ldots,|B|,\ |S_p|\leq |Z_p|.$ We prove this claim in a manner very similar to the proof of Claim 2. Let $S_p=\{D_{i_1}^n,D_{i_2}^n,\ldots,D_{i_t}^n\}$ and let $\varPhi_{A^n}(Z_p)$ be the projection of Z_p on A^n . As in Claim 2, $\varPhi_{A^n}(Z_p)$ dominates $\cup_{x=1}^t D_{i_x}^n$, and if $\varPhi_{A^n}(Z_p)\cap \{u_i^n\mid i\notin \{i_1,i_2,\ldots,i_t\}\}$ is non-empty, let $\varPhi_{A^n}(Z_p)\cap \{u_i^n\mid i\notin \{i_1,i_2,\ldots,i_t\}\}$ is a total dominating set of A^n , and the sets $\varPhi_{A^n}(Z_p)\cup \{u_i^n\mid i\notin \{i_1,i_2,\ldots,i_t,\ldots,i_l\}\}$ are disjoint. Therefore, $|\varPhi_{A^n}(Z_p)\cup \{u_i^n\mid i\notin \{i_1,i_2,\ldots,i_l\}\}|=|\varPhi_{A^n}(Z_p)|+(\gamma_t(A^n)-l)\geq \gamma_t(A^n)$. Hence, $t=|S_p|\leq l\leq |\varPhi_{A^n}(Z_p)|\leq |Z_p|$.

Now,
$$|d_n| = \sum_{p=1}^{|B|} |S_p| \le \sum_{p=1}^{|B|} |Z_p| \le |D|$$
.

To conclude the proof,
$$\prod_{i=1}^{n} \gamma_t(A^i) \leq \sum_{j=1}^{n} |d_j| \leq n|D| = n\gamma_t(A^1 \square \cdots \square A^n).$$

2.4 Proof of Theorem 4

Proof. Let $\{x_1, y_1, \ldots, x_k, y_k\}$ be a γ_{pr} -set of G, where for each $i, (x_i, y_i) \in E(G)$. Thus, $\gamma_{pr}(G) = 2k$. Partition V(G) into sets D_1, \ldots, D_k , such that $\{x_i, y_i\} \subseteq D_i \subseteq N_G[x_i, y_i]$ for $1 \leq i \leq k$. Similarly, let $\{\overline{x}_1, \overline{y}_1, \ldots, \overline{x}_l, \overline{y}_l\}$ be a γ_{pr} -set of H, where for each j, $(\overline{x}_j, \overline{y}_j) \in E(H)$. Thus, $\gamma_{pr}(H) = 2l$. Partition V(H) into sets $\overline{D}_1, \ldots, \overline{D}_l$, such that $\{\overline{x}_j, \overline{y}_j\} \subseteq \overline{D}_j \subseteq N_H[\overline{x}_j, \overline{y}_j]$ for $1 \leq j \leq l$. Now, $\{D_1, \ldots, D_k\} \times \{\overline{D}_1, \ldots, \overline{D}_l\}$ forms a partition of $V(G \square H)$.

Let D be a γ_{pr} -set of $G \square H$. Then, for each $gh \notin D$, either $N_{\underline{\mathbf{G}} \square H}(gh) \cap D$ or $N_{G \square \underline{\mathbf{H}}}(gh) \cap D$ is non-empty. Based on this observation, we define the binary $|V(G)| \times |V(H)|$ matrix F such that:

$$F(g,h) = \begin{cases} 1 & \text{if } gh \in D \text{ or } N_{G \square \underline{\mathbf{H}}}(gh) \cap D \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Since D is a γ_{pr} -set, the subgraph of $G \square H$ induced by D has a perfect matching. Thus, D can be written as the disjoint union of

$$D_G = \{gh \in D \mid \text{the matching edge incident to } gh \text{ is a } \mathbf{G}\text{-edge}\}, \text{ and } D_H = \{gh \in D \mid \text{the matching edge incident to } gh \text{ is an } \mathbf{H}\text{-edge}\}.$$

For $i=1,\ldots,k$, let $Z_{G_i}=D_G\cap (D_i\times V(H))$, and $Z_{H_i}=D_H\cap (D_i\times V(H))$. For $j=1,\ldots,l$, let $\overline{Z}_{G_j}=D_G\cap (V(G)\times \overline{D}_j)$, and let $\overline{Z}_{H_j}=D_H\cap (V(G)\times \overline{D}_j)$. By Claims 1 and 2, if the submatrix of F determined by $D_i\times \overline{D}_j$ satisfies Prop. 1a, then \overline{D}_j is dominated by $\Phi_H(Z_{G_i}\cup Z_{H_i})$, and if the submatrix of F determined by $D_i\times \overline{D}_j$ satisfies Prop. 1b, then D_i is dominated by $\Phi_G(\overline{Z}_{G_i}\cup \overline{Z}_{H_i})$.

For i = 1, ..., k, and j = 1, ..., l, let

$$S_i = \{\overline{D}_x \mid \text{the submatrix of } F \text{ determined by } D_i \times \overline{D}_x \text{ satisfies Prop. 1a,}$$

with $x \in \{1, \dots, l\}\}$,

$$\overline{S}_j = \{D_x \mid \text{the submatrix of } F \text{ determined by } D_x \times \overline{D}_j \text{ satisfies Prop. 1b,}$$

with $x \in \{1, \dots, k\}\}$.

Finally, let $d_H = \sum_{i=1}^k |S_i|$, and $d_G = \sum_{j=1}^l |\overline{S}_j|$. Then, as before, $kl \leq d_H + d_G$, since each of the kl submatrices of F determined by $D_i \times \overline{D}_j$ satisfies one (or both) of the conditions of Prop. 1. We now prove a claim that will allow us to bound the sizes of our various sets and conclude the proof.

Claim 5. For
$$i = 1..., k, 2|S_i| \le 2|Z_{G_i}| + |Z_{H_i}|$$
.

Proof. Let
$$S_i = \{\overline{D}_{j_1}, \overline{D}_{j_2}, \dots, \overline{D}_{j_t}\}$$
. Let $A = \Phi_H(Z_{G_i})$, $B = \Phi_H(Z_{H_i})$, and $C = \{\overline{x}_j \mid j \notin \{j_1, j_2, \dots, j_t\}\}$.

Let M be the matching on $B \cup C$ formed by taking all of the $\{\overline{x}_j, \overline{y}_j\}$ edges induced by the vertices in C, and then adding the edges from a maximal matching on the remaining unmatched vertices in B. Then, $E = A \cup B \cup C$ is a dominating set of H with M as a matching. Let $M_1 = V(M)$ and $M_2 = (B \cup C) \setminus M_1$. We note that M_1 consists of all the vertices in C plus the matched vertices from B, and M_2 contains only the unmatched vertices from B. Therefore, $|M_1| + 2|M_2| \leq |C| + |Z_{H_i}|$. To see this more clearly, consider a vertex $gh \in Z_{H_i}$ that is matched by an H-edge to a vertex gh' such that $h \notin V(M)$. This implies that either h' coincides with a vertex of C, or h' coincides with the projection of some other vertex of Z_{H_i} (because otherwise h would be matched with h'). Therefore, $2|M_2|$ is equivalent to counting h', and we see that $|M_1| + 2|M_2| \leq |C| + |Z_{H_i}|$.

In order to obtain a perfect matching of E, we recursively modify E by choosing an unmatched vertex h in E (a vertex in either A or B, since all vertices in C are automatically matched), and then either matching it with an appropriate vertex, or removing it from E. Specifically, if $N_H(h)\backslash V(M)$ is non-empty, there exists a vertex $h' \in N_H(h)\backslash V(M)$ such that we can add h' to E and (h, h') to the matching M. Otherwise, h is incident on only matched vertices, and we can remove h from E without altering the fact that E is a dominating set.

Our recursively modified E (denoted by $E_{\rm rec}$) is now a paired dominating set of H. Furthermore, in the worst case, we have doubled the unmatched vertices from B, and also doubled the vertices in A. Thus,

$$2l \le |E_{\text{rec}}| \le 2|A| + |M_1| + 2|M_2|$$
.

Since $|M_1| + 2|M_2| \le |C| + |Z_{H_i}|$, this implies that $2l - |C| \le 2|A| + |Z_{H_i}|$. Furthermore, since $2l - |C| = 2|S_i|$, we see that $2|S_i| \le 2|Z_{G_i}| + |Z_{H_i}|$.

Similarly, for $j=1,\ldots,l$, we can show that $2|\overline{Z}_j| \leq |\overline{Z}_{G_j}| + 2|\overline{Z}_{H_j}|$. We now see

$$2\sum_{i=1}^{k} |S_{i}| + 2\sum_{j=1}^{l} |\overline{S}_{j}| \leq 2\sum_{i=1}^{k} |Z_{G_{i}}| + \sum_{i=1}^{k} |Z_{H_{i}}| + \sum_{j=1}^{l} |\overline{Z}_{G_{j}}| + 2\sum_{j=1}^{l} |\overline{Z}_{H_{j}}|,$$

$$\leq \sum_{i=1}^{k} |Z_{G_{i}}| + \sum_{i=1}^{k} |Z_{H_{i}}| + \sum_{j=1}^{l} |\overline{Z}_{G_{j}}| + \sum_{j=1}^{l} |\overline{Z}_{H_{j}}| + \sum_{i=1}^{k} |Z_{G_{i}}| + \sum_{j=1}^{l} |\overline{Z}_{H_{j}}|,$$

$$\leq 3|D|.$$

To conclude the proof, we note that

$$2(d_H + d_G) = 2\sum_{i=1}^k |S_i| + 2\sum_{j=1}^l |\overline{S}_j| \le 3|D|,$$

$$2(kl) = \gamma_{pr}(G)\frac{\gamma_{pr}(H)}{2} \le 3|D|,$$

$$\gamma_{pr}(G)\gamma_{pr}(H) \le 6\gamma_{pr}(G\square H).$$

2.5 Proof of Theorem 5

Proof. For $i=1,\ldots,n$, let $k_i=\gamma_{pr}(A^i)/2$, and let $\{x_1^i,y_1^i,\ldots,x_{k_i}^i,y_{k_i}^i\}$ be a γ_{pr} -set of A^i , and $D_1^i,\ldots,D_{k_i}^i$ be the corresponding partitions (as defined in Theorem 4). Let $Q=\{D_1^1,\ldots,D_{k_1}^1\}\times\cdots\times\{D_1^n,\ldots,D_{k_n}^n\}$. Then Q forms a partition of $V(A^1\square\cdots\square A^n)$

with $|Q| = \prod_{i=1}^{n} \gamma_{pr}(A^{i})/2 = \frac{1}{2^{n}} \prod_{i=1}^{n} \gamma_{pr}(A^{i}).$

Let D be a γ_{pr} -set of $A^1 \square \cdots \square A^n$. Then, for each $u \in V(A^1 \square \cdots \square A^n)$, there exists an i such that $N_{\Box A^i}(u) \cap D$ is non-empty. We now proceed slightly differently than previously. Based on this observation (as in the 2-dimensional case), we define n different matrices F^i with $i=1,\ldots,n$, where each of the *n* matrices is an *n*-ary $|V(A^1)| \times \cdots \times |V(A^n)|$

matrix F^i such that:

$$F^{i}(u_{1},\ldots,u_{n}) = \begin{cases} i & \text{if } u_{1}\cdots u_{n} \in D, \\ j_{\min} & \text{where } j_{\min} = \min\{ j \mid N_{\square A^{j}}(u_{1}\cdots u_{n}) \cap D \neq \emptyset \}. \end{cases}$$

Thus, each of the n matrices F^i with $i=1,\ldots,n$ differs only in the entries that correspond to vertices in the paired dominating set D.

For $j=1,\ldots,n$ and $i=1,\ldots,n$, let $d_j^i\subseteq Q$ be the set of the elements in Q which are j-matrices in the matrix F^i . By Prop. 2, each element of Q belongs to at least one d_i^i -set for each $i=1,\ldots,n$. Now, if an element $q\in Q$ belongs to the d_i^i -set, then q also belongs to the d_i^j -set. To see this, if M_i and M_j are the submatrices determined by q with respect to the matrices F^i and F^j , respectively, then all the entries that do not match in M_i and M_j have value j in M_j . Thus, each $q \in Q$ belongs to at least one d_i^i -set for some

$$i \in \{1, \dots, n\}$$
. Then, $\frac{1}{2^n} \prod_{i=1}^n \gamma_{pr}(A^i) \le \sum_{i=1}^n |d_i^i|$.

 $i \in \{1, \dots, n\}$. Then, $\frac{1}{2^n} \prod_{i=1}^n \gamma_{pr}(A^i) \leq \sum_{i=1}^n |d_i^i|$. Similar to Q, let $B = \{D_1^1, \dots, D_{k_1}^1\} \times \dots \times \{D_1^{n-1}, \dots, D_{k_{n-1}}^{n-1}\}$. For convenience, we

denote B as
$$\{B_1, \dots, B_{|B|}\}$$
, where $|B| = \prod_{i=1}^{n-1} \gamma_{pr}(A^i)/2 = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \gamma_{pr}(A^i)$.

Since D is a γ_{pr} -set, the subgraph of $A^1 \stackrel{\iota^{-1}}{\square} \cdots \square A^n$ induced by D has a perfect matching. Let

$$D_i = \{u \in D \mid \text{the matching edge incident to } u \text{ is in } E_i\}$$
.

Then, D can be written as the disjoint union of the subsets D_i . For $p = 1, \ldots, |B|$ and $i=1,\ldots,n$, let $Z_p^i=D_i\cap(B_p\times A^n)$, and

$$S_p = \{D_x^n \mid \text{the submatrix of } F^n \text{ determined by } B_p \times D_x^n \text{ is an } n\text{-matrix},$$
 with $x \in \{1, \dots, k_n\}\}$.

Claim 6. For
$$p = 1, ..., |B|, 2|S_p| \le 2|Z_p^1| + \cdots + 2|Z_p^{n-1}| + |Z_p^n|$$
.

Proof. Let $S_p = \{D_{j_1}^n, D_{j_2}^n, \dots, D_{j_t}^n\}$, and for $j = 1, \dots, n$, let $V_j = \Phi_{A^n}(Z_p^j)$. Note that $|V_j| \leq |Z_p^j|$. Similar to the proof of Claim 5, let $C = \{x_j^n \mid j \notin \{j_1, j_2, \dots, j_t\}\} \cup \{y_j^n \mid j \notin \{j_1, j_2, \dots, j_t\}\}$ $\{j_1, j_2, \ldots, j_t\}\}.$

Let M be the matching on $V_n \cup C$ formed by taking all of the $\{x_i^n, y_i^n\}$ edges induced by the vertices in C, and then adding the edges from a maximal matching on the remaining unmatched vertices in V_n . Then, $E = V_1 \cup \cdots \cup V_n \cup C$ is a dominating set of A^n with Mas a matching.

Let $M_1 = V(M)$ and $M_2 = (V_n \cup C) \setminus M_1$. We note that M_1 consists of all the vertices in C plus the matched vertices from V_n , and M_2 contains only the unmatched vertices from V_n .

In order to obtain a perfect matching, we recursively modify E by choosing an unmatched vertex a in E, and then either matching it with an appropriate vertex, or removing it from E. Specifically, if $N_{A^n}(a) \setminus V(M)$ is non-empty, there exists a vertex $a' \in N_{A^n}(a) \setminus V(M)$ such that we can add a' to E and (a, a') to the matching M. Otherwise, a is incident on only matched vertices, and we can safely remove it from E without altering the fact that E is a dominating set.

Our recursively modified E (denoted by E_{rec}) is now a paired dominating set of A_n . Furthermore, in the worst case, we have doubled the unmatched vertices from V_n , and also doubled the vertices in V_1, \ldots, V_{n-1} . Thus,

$$2k_n \le |E_{rec}| \le 2|V_1| + \dots + 2|V_{n-1}| + |M_1| + 2|M_2|$$
.

This implies that
$$2k_n - |C| \le 2|V_1| + \dots + 2|V_{n-1}| + |Z_p^n|$$
. Since $2k_n - |C| = 2|S_p|$, therefore, $2|S_p| \le 2|V_1| + \dots + 2|V_{n-1}| + |Z_p^n| \le 2|Z_p^1| + \dots + 2|Z_p^{n-1}| + |Z_p^n|$.

To conclude the proof, we follow a similar method as in the proof of Theorem 4. We begin by noting that,

$$|d_n^n| = \sum_{p=1}^{|B|} |S_p|$$
.

Using Claim 6, we now see

$$2\sum_{p=1}^{|B|}|S_p| \le \sum_{p=1}^{|B|} \left(2\sum_{j=1}^n |Z_p^j| - |Z_p^n|\right) = 2|D| - \sum_{p=1}^{|B|} |Z_p^n| = 2|D| - |D_n|.$$

Therefore, $2|d_n^n| \leq 2|D| - |D_n|$. Similarly, we can show that $2|d_i^i| \leq 2|D| - |D_i|$ for i = 1, ..., n. To conclude the proof, we see

$$\frac{1}{2^{n-1}} \prod_{i=1}^{n} \gamma_{pr}(A_i) = 2(k_1 \cdots k_n) \le 2 \sum_{i=1}^{n} |d_i^i| \le 2n|D| - \sum_{i=1}^{n} |D_i| = (2n-1)|D|,$$

$$\prod_{i=1}^{n} \gamma_{pr}(A_i) \le 2^{n-1}(2n-1)\gamma_{pr}(A_1 \square \cdots \square A_n).$$

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