Secure quantum weak oblivious transfer against individual measurements

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In quantum weak oblivious transfer, Alice sends Bob two bits and Bob can learn one of the bits at his choice. It was found that the security of such a protocol is bounded by $2P^*_{Alice} + P^*_{Bob} \geq 2$, where P^*_{Alice} is the probability with which Alice can guess Bob's choice, and P^*_{Bob} is the probability with which Bob can guess both of Alice's bits given that he learns one of the bits with certainty. Here we propose a protocol and show that as long as Alice is restricted to individual measurements, then both P^*_{Alice} and P^*_{Bob} can be made arbitrarily close to $1/2$, so that maximal violation of the security bound can be reached. Even with some limited collective attacks, the security bound can still be violated. Therefore, although our protocol still cannot break the bound in principle when Alice has unlimited cheating power, it is sufficient for achieving secure quantum weak oblivious transfer in practice.

I. INTRODUCTION

Oblivious transfer (OT) [1, 2] is known to be an essential building block for two-party and multi-party protocols [3]. However, unconditionally secure OT was shown to be impossible even in quantum cryptography, because the adversary can always cheat with the so-called honest-but-curious attack [4–8]. To evade the problem, the concept "weak OT" was proposed recently [9], in which the security goals of OT are slightly modified, so that the honest-but-curious attack is no longer considered a successful cheating. Even so, it was found that a security bound exists for weak OT [9], thus it cannot be unconditionally secure either.

Nevertheless, we will point out below that the cheating strategy to weak OT has its own limitation too. By making use of this limitation, we can build quantum weak OT protocols which will violate the existing security bound when the cheater is restricted to individual measurements as well as some limited collective attacks. Therefore, while in principle the security bound still applies to our protocols, in practice the attack will be very difficult to be implemented.

Note that previously there was already a quantum OT protocol [10] which was considered secure against individual measurements [11]. However, the protocol calls for quantum bit commitment as a building block. Thus its security is unreliable, as it is widely believed [12, 13] that unconditionally secure quantum bit commitment does not exist.

In the next section, we will review the definitions of OT and weak OT, and the existing security bound of the latter. Our protocol will be proposed in Sect. III. Section IV is dedicated to the security analysis. We will show why the security bound still applies in principle. On the other hand, we will elaborate how to reach the maximal violation of the bound when only individual measurements are allowed. It will also be shown that the bound can still be violated for limited collective attacks. Some ideas on further improvement of our protocol will be discussed in Sect. V. Finally, in Sect. VI we summarize the result, and explain why it is important in practice to find a secure protocol against individual measurements.

II. DEFINITIONS AND THE SECURITY BOUND

There are many variations of OT. The most well-known ones are all-or-nothing OT [1] and one-out-of-two OT [2]. Here we are interested in the latter, which is defined as follows [14].

One-out-of-two Oblivious Transfer
(i) Alice knows two bits x_0 and x_1.
(ii) Bob gets bit x_b and not $x_{\bar{b}}$ with $Pr(b = 0) = Pr(b = 1) = 1/2$. (Here $\bar{b}$ denotes the bit-compliment of b.)
(iii) Bob knows which of x_0 or x_1 he got.
(iv) Alice does not know which x_b Bob got.

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More rigorously, this definition indicates that a secure protocol should guarantee that at the end of the process, Bob should get x_b with reliability 100%, i.e., the value he decoded matches Alice's actual input with certainty. Meanwhile, the amount of information he gains on $x_{\bar{b}}$ should be arbitrarily close to zero, so that he has to guess $x_{\bar{b}}$ by himself, which results in a reliability 50% for $x_{\bar{b}}$ since his guess stands a probability $1/2$ to be correct. However, as pointed out in [15], in the literature there is the lack of a self-consistent definition of OT specifically made for the quantum case. This is because with quantum methods, it is possible that Bob may accept a lower reliability of learning x_b, so that the reliability for $x_{\bar{b}}$ can be significantly raised. This is exactly what the honest-but-curious attack [4–7] achieves. In the above definition it is vague whether such a result is considered as successful cheating, making it hard to discuss the security of OT protocols in a precise way.

To mend the problem, weak OT is proposed [9], with an improved definition on Bob's cheating. Define the symbols

P^*_{Alice}: The maximum probability with which cheating-Alice can get honest-Bob's choice bit b and honest-Bob does not abort.

P^*_{Bob}: The maximum over $b \in \{0,1\}$ of the probability with which cheating-Bob can get $x_{\bar{b}}$ given that he gets x_b with certainty and honest-Alice does not abort.

For every protocol there will always be $P^*_{Alice}, P^*_{Bob} \geq 1/2$, as a cheating party can do no worse than a random guess. Then weak OT is defined as a kind of one-out-of-two OT which requires security only against cheating-Bob who gets one of honest-Alice's bits with certainty.

Note that the names "Alice" and "Bob" were used reversely in [9], comparing with the literature on QOT [1, 3–8, 10, 11, 14–17]. Here we follow the literature and use the names in the above way.

It was proven [9] that the optimal security bound for any quantum weak OT protocol is

$$2P^*_{Alice} + P^*_{Bob} \geq 2, \tag{1}$$

from which it follows that one of the two parties must be able to cheat with probability at least $2/3$.

In brief, Ref. [9] obtained this bound with the following method. First, consider Bob's cheating. Let ρ_{b,x_0,x_1} denote the reduced state of his portion of the system. Since a weak OT protocol should allow honest-Bob to learn x_b with certainty, there must be a non-destructive measurement that enables him to do so without disturbing the system. After Bob learned x_b with this measurement, his system will still remain in the state ρ_{b,x_0,x_1}. To gain some information on the other bit $x_{\bar{b}}$, he performs the Helstrom measurement to optimally distinguish the two states corresponding to $x_{\bar{b}} = 0$ and $x_{\bar{b}} = 1$, respectively. Thus his cheating can be successful with probability

$$P^*_{Bob} \geq \frac{1}{2} + \frac{1}{8}\Delta, \tag{2}$$

where

$$\Delta \equiv \frac{1}{2}\Big(\sum_{x_0 \in \{0,1\}} \|\rho_{0,x_0,0} - \rho_{0,x_0,1}\|_{Tr} + \sum_{x_1 \in \{0,1\}} \|\rho_{1,0,x_1} - \rho_{1,1,x_1}\|_{Tr} \Big). \tag{3}$$

Secondly, consider Alice's cheating. She implements a uniform superposition over x_0, x_1 of honest strategies by introducing two additional private qubits for storing the values of x_0 and x_1. Then she applies the controlled-unitary operations

$$controlled - U_0 : |\psi_{1,1,x_1}\rangle |1\rangle |x_1\rangle \rightarrow (I_A \otimes U_{0,x_1}) |\psi_{1,1,x_1}\rangle |1\rangle |x_1\rangle \tag{4}$$

and

$$controlled - U_1 : |\psi_{0,x_0,1}\rangle |x_0\rangle |1\rangle \rightarrow (I_A \otimes U_{1,x_0}) |\psi_{0,x_0,1}\rangle |x_0\rangle |1\rangle \tag{5}$$

for $x_1 \in \{0,1\}$ and $x_0 \in \{0,1\}$, respectively, where U_{0,x_1} and U_{1,x_0} satisfy

$$\begin{aligned} F(\rho_{1,0,x_1}, \rho_{1,1,x_1}) &= \langle\psi_{1,0,x_1}| (I_A \otimes U_{0,x_1}) |\psi_{1,1,x_1}\rangle, \\ F(\rho_{0,x_0,0}, \rho_{0,x_0,1}) &= \langle\psi_{0,x_0,0}| (I_A \otimes U_{1,x_0}) |\psi_{0,x_0,1}\rangle. \end{aligned} \tag{6}$$

Here $F(\rho, \xi) \equiv \left\|\sqrt{\rho}\sqrt{\xi}\right\|_{Tr}$ is the fidelity between ρ and ξ. Then the successful probability for her cheating was shown [9] to be

$$P^*_{Alice} \geq \frac{1}{2} + \frac{1}{16}F, \tag{7}$$

where

$$F \equiv \sum_{x_0 \in \{0,1\}} F(\rho_{0,x_0,0}, \rho_{0,x_0,1}) + \sum_{x_1 \in \{0,1\}} F(\rho_{1,0,x_1}, \rho_{1,1,x_1}). \tag{8}$$

Combining the Fuchs-van de Graaf inequalities

$$1 - \frac{1}{2} \|\rho - \xi\|_{Tr} \leq F(\rho, \xi) \leq \sqrt{1 - \frac{1}{4} \|\rho - \xi\|_{Tr}^2} \tag{9}$$

with equations (2) and (7), the security bound $2P^*_{Alice} + P^*_{Bob} \geq 2$ is finally obtained.

This bound was also shown to be optimal, as Ref. [9] exhibited a family of protocols whose cheating probabilities can be made arbitrarily close to any point on the P^*_{Alice} versus P^*_{Bob} tradeoff curve.

III.   THE PROTOCOL

A.   Limitation of the cheating strategy in existing protocols

Intriguingly, while the security bound $2P^*_{Alice} + P^*_{Bob} \geq 2$ indicates that in a protocol where Bob cannot cheat (i.e., $P^*_{Bob} = 1/2$), Alice can guess Bob's choice b at least with the probability $P^*_{Alice} = 3/4$, we must note that her cheating strategy has a serious drawback. That is, once Alice applies the cheating, she will not be able to determine the values of x_0, x_1. For example, consider the Chailloux-Kerenidis-Sikora (CKS) protocol proposed in Sect. 4 of [8] (also presented in Sect. 3.2 of [9] with reverse usage of the names "Alice" and "Bob"), as described below.

The CKS protocol
1. Bob randomly chooses $b \in \{0,1\}$ and prepares the two-qutrit state $|\phi_b\rangle = (|bb\rangle + |22\rangle)/\sqrt{2}$. He sends one of the qutrits to Alice.
2. Alice chooses $x_0, x_1 \in \{0,1\}$ and applies the unitary transformation $|0\rangle \to (-1)^{x_0}|0\rangle, |1\rangle \to (-1)^{x_1}|1\rangle, |2\rangle \to |2\rangle$ on Bob's qutrit.
3. Alice returns the qutrit to Bob who now has the state $|\psi_b\rangle = [(-1)^{x_b}|bb\rangle + |22\rangle]/\sqrt{2}$.
4. Bob performs the measurement $\{\Pi_0 = |\phi_b\rangle\langle\phi_b|, \Pi_1 = |\phi_b'\rangle\langle\phi_b'|, I - \Pi_0 - \Pi_1\}$ on the state $|\psi_b\rangle$, where $|\phi_b'\rangle = (|bb\rangle - |22\rangle)/\sqrt{2}$.
5. If the outcome is Π_0 then Bob learns with certainty that $x_b = 0$, if it is Π_1 then $x_b = 1$, otherwise he aborts.

This protocol can reach $P^*_{Bob} = 1/2$, but it is insecure against Alice's individual attacks. As shown in Sect. 4 of [8], Alice's optimal cheating strategy is simply to measure the qutrit she received in step 2 using the computational basis. If she gets outcome $|0\rangle$ ($|1\rangle$) then she knows with certainty that $b = 0$ ($b = 1$). If she gets outcome $|2\rangle$ then she guesses the value of b. Therefore on average, Alice can learn Bob's b correctly with the probability $P^*_{Alice} = 3/4$. After the measurement she returns the measured qutrit to Bob. Then Bob's state will be either $|bb\rangle$ or $|22\rangle$. With any of these two states, the outcome of Bob's measurement in step 4 will always be either Π_0 or Π_1. Hence he will never abort, so that Alice's cheating cannot be detected at all.

However, we can see that Alice cannot control, nor she can learn what will be the actual outcome of Bob's measurement in step 4, because $|bb\rangle$ and $|22\rangle$ can both be projected as Π_0 or Π_1. As a result, at the end of the protocol Bob gets a bit x_b, but its value is unknown to Alice. That is, once dishonest-Alice gains the information on b, she loses the information on x_b. Now we prove that this result is general for any Alice's cheating strategy.

Theorem 1: In the CKS protocol, Alice cannot learn x_b with reliability 1 and gain a non-trivial amount of information on b simultaneously, while escaping Bob's detection with probability 1.

Proof: Let α denote the ancillary system that dishonest-Alice introduced for her cheating, system β denote the qutrit that she received and then returns to Bob, and system β' denote the qutrit that Bob always keeps at his side. To ensure that Bob will never abort in step 5 so that Alice can pass Bob's detection with probability 1, the state of $\beta \otimes \beta'$ at this stage must be completely contained in the Hilbert space supported by $|\phi_b\rangle_{\beta\beta'}$ and $|\phi_b'\rangle_{\beta\beta'}$. Therefore, for any Alice's cheating strategy, at the end of step 3 the general form of the quantum system shared by Alice and Bob can always be written as

$$T(|e_{ini}\rangle_\alpha |\phi_b\rangle_{\beta\beta'}) = \lambda_b^{(0)} \left|e_b^{(0)}\right\rangle_\alpha |\phi_b\rangle_{\beta\beta'} + \lambda_b^{(1)} \left|e_b^{(1)}\right\rangle_\alpha |\phi_b'\rangle_{\beta\beta'}, \tag{10}$$

for $b = 0, 1$, with $|e_{ini}\rangle_\alpha$ ($\left|e_b^{(0)}\right\rangle_\alpha$ and $\left|e_b^{(1)}\right\rangle_\alpha$) being the normalized initial (final) state(s) of α, $\left|\lambda_b^{(0)}\right|^2 + \left|\lambda_b^{(1)}\right|^2 = 1$, and T is the operator that Alice applies for her cheating. Also, in step (4) of the protocol, Bob learns that $x_b = 0$ ($x_b = 1$) if he gets $|\phi_b\rangle_{\beta\beta'}$ ($|\phi_b'\rangle_{\beta\beta'}$). Therefore, if Alice wants to be able to learn x_b with reliability 1, she has to choose an operation T which can ensure that $\left|e_b^{(0)}\right\rangle_\alpha$ and $\left|e_b^{(1)}\right\rangle_\alpha$ are orthogonal.

By substituting $|\phi_b\rangle_{\beta\beta'} = (|bb\rangle_{\beta\beta'} + |22\rangle_{\beta\beta'})/\sqrt{2}$ and $|\phi_b'\rangle_{\beta\beta'} = (|bb\rangle_{\beta\beta'} - |22\rangle_{\beta\beta'})/\sqrt{2}$ into equation (10), we can rewrite it as

$$T(|e_{ini}\rangle_\alpha |\phi_b\rangle_{\beta\beta'}) = (|f_b\rangle_\alpha |bb\rangle_{\beta\beta'} + |f_b'\rangle_\alpha |22\rangle_{\beta\beta'})/\sqrt{2}, \tag{11}$$

where

$$\begin{aligned}
|f_b\rangle_\alpha &\equiv \lambda_b^{(0)} \left|e_b^{(0)}\right\rangle_\alpha + \lambda_b^{(1)} \left|e_b^{(1)}\right\rangle_\alpha, \\
|f_b'\rangle_\alpha &\equiv \lambda_b^{(0)} \left|e_b^{(0)}\right\rangle_\alpha - \lambda_b^{(1)} \left|e_b^{(1)}\right\rangle_\alpha,
\end{aligned} \tag{12}$$

To gain a non-trivial amount of information on b, equation (11) indicates that Alice needs to distinguish the reduced density matrices

$$\rho_{b=0} \equiv (|f_0\rangle_\alpha \langle f_0| + |f_0'\rangle_\alpha \langle f_0'|)/2 \tag{13}$$

and

$$\rho_{b=1} \equiv (|f_1\rangle_\alpha \langle f_1| + |f_1'\rangle_\alpha \langle f_1'|)/2. \tag{14}$$

While there could exist an operation T, which can ensure $\rho_{b=0} \neq \rho_{b=1}$ before Alice obtains x_b, we will show below that after Alice performed any operation M that can make her learn x_b with reliability 1, $\rho_{b=0}$ and $\rho_{b=1}$ will become equal to each other, so that they cannot be distinguished any more.

An important fact is that qutrit β' is always kept at Bob's side, so that it remains unchanged under Alice operation T. Thus we can write $T = U_{\alpha\beta} \otimes I_{\beta'}$, where $U_{\alpha\beta}$ applies on systems α and β only, while $I_{\beta'}$ is the identity operator on β'. Denoting

$$\begin{aligned}
U_{\alpha\beta}(|e_{ini}\rangle_\alpha |0\rangle_\beta) &= |\Psi_0\rangle_{\alpha\beta}, \\
U_{\alpha\beta}(|e_{ini}\rangle_\alpha |1\rangle_\beta) &= |\Psi_1\rangle_{\alpha\beta}, \\
U_{\alpha\beta}(|e_{ini}\rangle_\alpha |2\rangle_\beta) &= |\Psi_2\rangle_{\alpha\beta},
\end{aligned} \tag{15}$$

then we have

$$\begin{aligned}
T(|e_{ini}\rangle_\alpha |\phi_b\rangle_{\beta\beta'}) &= (U_{\alpha\beta} \otimes I_{\beta'})(|e_{ini}\rangle_\alpha (|bb\rangle_{\beta\beta'} + |22\rangle_{\beta\beta'})/\sqrt{2}) \\
&= (|\Psi_b\rangle_{\alpha\beta} |b\rangle_{\beta'} + |\Psi_2\rangle_{\alpha\beta} |2\rangle_{\beta'})/\sqrt{2}).
\end{aligned} \tag{16}$$

Comparing with equation (11), we yield

$$\begin{aligned}
|\Psi_b\rangle_{\alpha\beta} &= |f_b\rangle_\alpha |b\rangle_\beta, \\
|\Psi_2\rangle_{\alpha\beta} &= |f_b'\rangle_\alpha |2\rangle_\beta.
\end{aligned} \tag{17}$$

The latter indicates that $|f_b'\rangle_\alpha$ does not depend on b, i.e.,

$$|f_0'\rangle_\alpha = |f_1'\rangle_\alpha. \tag{18}$$

Then

$$\langle f_0' |f_1'\rangle_\alpha = 1. \tag{19}$$

With equation (12) we know

$$\begin{aligned}
\langle f_0' |f_1'\rangle_\alpha &= (\lambda_0^{(0)*} \left\langle e_0^{(0)}\right|_\alpha - \lambda_0^{(1)*} \left\langle e_0^{(1)}\right|_\alpha)(\lambda_1^{(0)} \left|e_1^{(0)}\right\rangle_\alpha - \lambda_1^{(1)} \left|e_1^{(1)}\right\rangle_\alpha) \\
&= \lambda_0^{(0)*} \lambda_1^{(0)} \left\langle e_0^{(0)} \middle| e_1^{(0)}\right\rangle_\alpha - \lambda_0^{(0)*} \lambda_1^{(1)} \left\langle e_0^{(0)} \middle| e_1^{(1)}\right\rangle_\alpha \\
&\quad - \lambda_0^{(1)*} \lambda_1^{(0)} \left\langle e_0^{(1)} \middle| e_1^{(0)}\right\rangle_\alpha + \lambda_0^{(1)*} \lambda_1^{(1)} \left\langle e_0^{(1)} \middle| e_1^{(1)}\right\rangle_\alpha
\end{aligned} \tag{20}$$

and

$$
\begin{aligned}
\langle f_0 \,|f_1\rangle_\alpha &= (\lambda_0^{(0)*}\left\langle e_0^{(0)}\right|_\alpha + \lambda_0^{(1)*}\left\langle e_0^{(1)}\right|_\alpha)(\lambda_1^{(0)}\left|e_1^{(0)}\right\rangle_\alpha + \lambda_1^{(1)}\left|e_1^{(1)}\right\rangle_\alpha) \\
&= \lambda_0^{(0)*}\lambda_1^{(0)}\left\langle e_0^{(0)}\,\middle|e_1^{(0)}\right\rangle_\alpha + \lambda_0^{(0)*}\lambda_1^{(1)}\left\langle e_0^{(0)}\,\middle|e_1^{(1)}\right\rangle_\alpha \\
&\quad + \lambda_0^{(1)*}\lambda_1^{(0)}\left\langle e_0^{(1)}\,\middle|e_1^{(0)}\right\rangle_\alpha + \lambda_0^{(1)*}\lambda_1^{(1)}\left\langle e_0^{(1)}\,\middle|e_1^{(1)}\right\rangle_\alpha .
\end{aligned}
\tag{21}
$$

Now if Alice performs any operation M on the state $T(|e_{ini}\rangle_\alpha \,|\phi_b\rangle_{\beta\beta'})$ so that x_b $(b = 0, 1)$ is obtained with reliability 1, the final state can still be written as equation (10), except that the coefficients $\lambda_0^{(0)}$, $\lambda_0^{(1)}$, $\lambda_1^{(0)}$ and $\lambda_1^{(1)}$ cannot stay non-vanishing simultaneously. Instead, one of $\lambda_0^{(0)}$ and $\lambda_0^{(1)}$ must become zero, and one of $\lambda_1^{(0)}$ and $\lambda_1^{(1)}$ must be zero too. In this case, we can see that in the right hand side of either equation (20) or (21), only one of the coefficients before the four terms $\left\langle e_0^{(0)}\,\middle|e_1^{(0)}\right\rangle_\alpha$, $\left\langle e_0^{(0)}\,\middle|e_1^{(1)}\right\rangle_\alpha$, $\left\langle e_0^{(1)}\,\middle|e_1^{(0)}\right\rangle_\alpha$ and $\left\langle e_0^{(1)}\,\middle|e_1^{(1)}\right\rangle_\alpha$ can remain non-vanishing. No matter which single term remains, there will always be either

$$
\langle f_0 \,|f_1\rangle_\alpha = \langle f_0' \,|f_1'\rangle_\alpha .
\tag{22}
$$

or

$$
\langle f_0 \,|f_1\rangle_\alpha = -\,\langle f_0' \,|f_1'\rangle_\alpha .
\tag{23}
$$

Combining with equation (19), they both give

$$
|f_0\rangle_\alpha \,\langle f_0| = |f_1\rangle_\alpha \,\langle f_1| .
\tag{24}
$$

Substituting it and equation (18) into equations (13) and (14), we finally obtain

$$
\rho_{b=0} = \rho_{b=1} .
\tag{25}
$$

Thus they provide absolutely zero knowledge on b. Therefore, once Alice performs the operation to learn x_b with reliability 1, she can no longer gain any information on b. This ends the proof of Theorem 1.

The above proof does not exclude the existence of other cheating strategies, in which Alice can learn x_b with a reliability less than 1, and/or Bob may abort in step 5 with a non-vanishing probability. But this will do no harm to our purpose, as it will be shown later in Sects. IV.B and IV.C.

In fact, besides the CKS protocol, some other QOT protocols [16, 17] also display the same feature described in Theorem 1. Let $(3/4, 1/2)$-*protocol* denote any QOT protocol of this kind, i.e., both $P_{Alice}^* = 3/4$ and $P_{Bob}^* = 1/2$ are satisfied exactly, and Alice cannot determine x_b with reliability 1 once she gain a non-trivial amount of information on b. It will be shown below that though a $(3/4, 1/2)$-protocol merely saturates the security bound $2P_{Alice}^* + P_{Bob}^* \geq 2$, it can be utilized to construct a compound protocol which can eventually violate this bound when Alice is limited to individual attacks.

B. Our protocol

Our method is to use such a $(3/4, 1/2)$-protocol as a building block, with which Alice transfers a series of bits $x_0^{(i)}$, $x_1^{(i)}$ (not the final x_0, x_1 that she wants to transfer) to Bob. The values of $x_0^{(i)}$, $x_1^{(i)}$ are *not* completely random. Instead, they must be chosen according to a certain rule. Then Bob uses many of $x_0^{(i)}$, $x_1^{(i)}$ to check whether Alice can determine their values correctly. Finally he uses one of the remaining pairs of $x_0^{(i)}$, $x_1^{(i)}$ and asks Alice to encode her x_0, x_1. The general form of the protocol is as follows.

Protocol A: weak OT for transferring (x_0, x_1)
A1. Alice and Bob discuss and agree on a set S of classical n-bit strings.
A2. Alice randomly chooses two strings
$X_0 = x_0^{(1)}x_0^{(2)}...x_0^{(i)}...x_0^{(n)}$ and $X_1 = x_1^{(1)}x_1^{(2)}...x_1^{(i)}...x_1^{(n)}$ from S. Note that at this stage, none of these $x_0^{(i)}$, $x_1^{(i)}$ have any specific relationship with the two final bits x_0, x_1 (we call them as *target bits* thereafter) that Alice wants to transfer to Bob as the goal of the weak OT.

A3. For each i $(i = 1, ..., n)$, Alice transfers $x_0^{(i)}$, $x_1^{(i)}$ to Bob using a $(3/4, 1/2)$-protocol. Bob randomly chooses $b_i \in \{0, 1\}$ and decodes x_{b_i}.

A4. Security check: among all these n runs of the $(3/4, 1/2)$-protocol, Bob picks m $(m < n)$ runs randomly. For each of these runs, he asks Alice to announce $x_0^{(i)}$ and $x_1^{(i)}$, and checks whether they are consistent with the value of x_{b_i} that he obtained in the $(3/4, 1/2)$-protocol. He also checks that there is at least two strings X_0' and X_1' in set S, such that all the m bits $x_0^{(i)}$ $(x_1^{(i)})$ that Alice announced are contained in X_0' (X_1').

A5. If Alice's announced values pass the above check, Bob picks one of the remaining $n - m$ unchecked runs (which is denoted as the $\hat{i}$-th run) of the $(3/4, 1/2)$-protocol. This run should satisfy the requirement that $x_0^{(\hat{i})} = 0$, $x_0^{(\hat{i})} = 1$, $x_1^{(\hat{i})} = 0$, and $x_1^{(\hat{i})} = 1$ are all allowed by set S. That is, in set S there is at least one string which contains all the m bits $x_0^{(i)}$ $(x_1^{(i)})$ that Alice announced in step A4 and $x_0^{(\hat{i})} = 0$ $(x_1^{(\hat{i})} = 0)$, and an equal number of strings, each of which also contains all these m bits $x_0^{(i)}$ $(x_1^{(i)})$ but with $x_0^{(\hat{i})} = 1$ $(x_1^{(\hat{i})} = 1)$ instead. This guarantees that the m bits $x_0^{(i)}$ $(x_1^{(i)})$ announced in step A4 are insufficient for Bob to deduce the value of $x_0^{(\hat{i})}$ $(x_1^{(\hat{i})})$ from set S. Alice checks that this requirement is met after Bob told her the value of $\hat{i}$.

A6. Alice completes the weak OT by using the $\hat{i}$-th run of the $(3/4, 1/2)$-protocol to encode the target bits x_0, x_1. That is, she announces $x_0 \oplus x_0^{(\hat{i})}$ and $x_1 \oplus x_1^{(\hat{i})}$ to Bob. Thus Bob can obtain either x_0 or x_1 depending on whether he has obtained $x_0^{(\hat{i})}$ or $x_1^{(\hat{i})}$ in the $\hat{i}$-th run of the $(3/4, 1/2)$-protocol.

C. A concrete example

To make our protocol easier for understanding and analyzing, here we provide a concrete example of our above protocol where the CKS protocol is used as the $(3/4, 1/2)$-protocol and the explicit form of set S is given.

Protocol B: a concrete example
B1. Alice and Bob run the CKS protocol for $n = 3k$ times. Every 3 runs of the CKS protocol are grouped together and we call it as a *triple run*. Let $x_0^{(i)}$, $x_1^{(i)}$ $(i = 1, 2, 3)$ denote the bits that Alice transfers to Bob in a triple run. The values of the strings $X_0 = x_0^{(1)} x_0^{(2)} x_0^{(3)}$ and $X_1 = x_1^{(1)} x_1^{(2)} x_1^{(3)}$ cannot be completely random. Instead, they are required to be chosen within the set $S = \{000, 001, 010, 100\}$.

B2. Security check: for every triple run, Bob randomly picks two runs of the CKS protocol, denotes them as the i_1-th and i_2-th runs. The remaining run that is not picked is denoted as the i_3-th run. Bob asks Alice to reveal $x_0^{(i_1)}$, $x_1^{(i_1)}$ and $x_0^{(i_2)}$, $x_1^{(i_2)}$. If $x_0^{(i_1)} = x_1^{(i_1)} = x_0^{(i_2)} = x_1^{(i_2)} = 0$ then Bob marks the corresponding triple run as a *useful run*, as both $x_0^{(i_3)}$ and $x_1^{(i_3)}$ can either be 0 or 1 according to the definition of set S, so that they may potentially be used for encoding the target bits x_0, x_1 later. Else if any of $x_0^{(i_1)}$, $x_1^{(i_1)}$, $x_0^{(i_2)}$, $x_1^{(i_2)}$ is 1, Bob asks Alice to reveal $x_0^{(i_3)}$, $x_1^{(i_3)}$ too, and checks whether both $X_0 = x_0^{(1)} x_0^{(2)} x_0^{(3)}$ and $X_1 = x_1^{(1)} x_1^{(2)} x_1^{(3)}$ belong to set S. He also checks that none of Alice's announced values conflicts with what he decoded from the CKS protocol.

B3. If Alice's data passes the above check, Bob picks one of the useful runs and asks Alice to complete the weak OT using this run. Then Alice announces $x_0 \oplus x_0^{(i_3)}$ and $x_1 \oplus x_1^{(i_3)}$ to Bob, so that he can obtain either the target bit x_0 or x_1 depending on whether he has obtained $x_0^{(i_3)}$ or $x_1^{(i_3)}$ in the corresponding run of the CKS protocol.

B4. For better security, Bob can further ask Alice to reveal $x_0^{(i_3)}$ and $x_1^{(i_3)}$ of all the rest useful runs which are not picked in step B3. Then he checks whether they conflict with what he decoded from the CKS protocol.

IV. SECURITY

A. The collective attack

The above protocols A and B are, unfortunately, still restricted by the security bound $2P_{Alice}^* + P_{Bob}^* \geq 2$ if Alice has unlimited computational power to apply collective attacks. Taking Protocol B as an example, her cheating strategy is as follows.

In step B1, for each triple run Alice introduces a 6-qubit system $C = c_0^{(1)} c_0^{(2)} c_0^{(3)} c_1^{(1)} c_1^{(2)} c_1^{(3)}$ to keep her choice of the strings $X_0 = x_0^{(1)} x_0^{(2)} x_0^{(3)}$ and $X_1 = x_1^{(1)} x_1^{(2)} x_1^{(3)}$ at the quantum level. The state of system C is initialized as

$$
\left| c_0^{(1)} c_0^{(2)} c_0^{(3)} c_1^{(1)} c_1^{(2)} c_1^{(3)} \right\rangle
$$
$$
= \frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |100\rangle)
$$
$$
\otimes \frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |100\rangle), \tag{26}
$$

where the first (last) three qubits are corresponding to the string X_0 (X_1). That is, it is a superposition of all the states allowed by set S.

In the i-th run ($i = 1, 2, 3$) of the CKS protocol during a triple run, let $\beta^{(i)}$ denote the qutrit that Bob sent to Alice, taken from his two-qutrit state $\left| \phi_b^{(i)} \right\rangle = (|bb\rangle + |22\rangle)/\sqrt{2}$. Alice uses $c_0^{(i)}$, $c_1^{(i)}$ as control qubits to determine her transformation on $\beta^{(i)}$. That is, on $c_0^{(i)} \otimes c_1^{(i)} \otimes \beta^{(i)}$ she applies the unitary transformation

$$
T_{c_0^{(i)} c_1^{(i)} \beta^{(i)}} = \sum_{x_0^{(i)}, x_1^{(i)} = 0}^{1} \left(\left| x_0^{(i)} \right\rangle_{c_0^{(i)}} \left\langle x_0^{(i)} \right| \otimes \left| x_1^{(i)} \right\rangle_{c_1^{(i)}} \left\langle x_1^{(i)} \right| \right.
$$
$$
\left. \otimes \begin{bmatrix} (-1)^{x_0^{(i)}} & 0 & 0 \\ 0 & (-1)^{x_1^{(i)}} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\beta^{(i)}} \right). \tag{27}
$$

By doing so, Alice manages to finish the transformation $|0\rangle \to (-1)^{x_0} |0\rangle$, $|1\rangle \to (-1)^{x_1} |1\rangle$, $|2\rangle \to |2\rangle$ on Bob's qutrit $\beta^{(i)}$, just as it is required in the CKS protocol when Alice is honest. The only difference is that in the current case, $x_0^{(i)}$ and $x_1^{(i)}$ do not have deterministic classical values. Instead, they are kept at the quantum level.

In step B2 whenever Bob picks one run of the CKS protocol and asks Alice to reveal the corresponding $x_0^{(i)}$ and $x_1^{(i)}$, Alice measures the qubits $c_0^{(i)}$ and $c_1^{(i)}$ in the computational basis $\{|0\rangle, |1\rangle\}$. If the result is $|0\rangle$ ($|1\rangle$) then she announces the corresponding $x_0^{(i)}$ or $x_1^{(i)}$ as 0 (1). From equation (27) it can be seen that Alice's announcement will never conflict with the values Bob decodes from the CKS protocol. Now recall that a useful run is defined as the triple run where $x_0^{(i_1)} = x_1^{(i_1)} = x_0^{(i_2)} = x_1^{(i_2)} = 0$. Therefore by combining equations (26) and (27), we know that the state of $c_0^{(i_3)} \otimes c_1^{(i_3)} \otimes \phi_b^{(i_3)}$ of any useful run at the end of step B2 becomes

$$
\left| c_0^{(i_3)} \otimes c_1^{(i_3)} \otimes \phi_b^{(i_3)} \right\rangle
$$
$$
= \frac{1}{2} \sum_{x_0^{(i_3)}, x_1^{(i_3)} = 0}^{1} \left\{ \left| x_0^{(i_3)} \right\rangle_{c_0^{(i_3)}} \otimes \left| x_1^{(i_3)} \right\rangle_{c_1^{(i_3)}} \right.
$$
$$
\left. \otimes \frac{1}{\sqrt{2}} [(-1)^{x_b^{(i_3)}} |bb\rangle + |22\rangle] \right\}. \tag{28}
$$

If a useful run is picked for the security check in step B4, Alice can simply measure the qubits $c_0^{(i_3)}$ and $c_1^{(i_3)}$ in the basis $\{|0\rangle, |1\rangle\}$ and reveal $x_0^{(i_3)}$ and $x_1^{(i_3)}$ correctly. On the other hand, if a useful run is picked in step B3 to encode Alice's target bits x_0, x_1, then she will have the freedom to choose whether to measure $c_0^{(i_3)}$ and $c_1^{(i_3)}$ in the basis $\{|0\rangle, |1\rangle\}$ and learn the values of x_0, x_1 as an honest Alice does, or to learn the value of Bob's b with a certain probability instead. In the latter case, she measures $c_0^{(i_3)}$ and $c_1^{(i_3)}$ in the basis $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

This is because equation (28) can be rewritten as

$$
\left| c_{\bar{b}}^{(i_3)} \otimes c_b^{(i_3)} \otimes \phi_b^{(i_3)} \right\rangle
$$

$$
= \frac{1}{2} \sum_{x_b^{(i_3)}=0}^{1} \{ (\sum_{x_{\bar{b}}^{(i_3)}=0}^{1} \left| x_{\bar{b}}^{(i_3)} \right\rangle_{c_{\bar{b}}^{(i_3)}}) \otimes \left| x_b^{(i_3)} \right\rangle_{c_b^{(i_3)}}
$$

$$
\otimes \frac{1}{\sqrt{2}} [(-1)^{x_b^{(i_3)}} |bb\rangle + |22\rangle] \}
$$

$$
= \frac{1}{\sqrt{2}} |+\rangle_{c_{\bar{b}}^{(i_3)}} \otimes \sum_{x_b^{(i_3)}=0}^{1} \{ \left| x_b^{(i_3)} \right\rangle_{c_b^{(i_3)}}
$$

$$
\otimes \frac{1}{\sqrt{2}} [(-1)^{x_b^{(i_3)}} |bb\rangle + |22\rangle] \}
$$

$$
= \frac{1}{\sqrt{2}} |+\rangle_{c_{\bar{b}}^{(i_3)}} \otimes [|-\rangle_{c_b^{(i_3)}} \otimes |bb\rangle + |+\rangle_{c_b^{(i_3)}} \otimes |22\rangle]. \tag{29}
$$

We can see that if Alice finds the outcome of her measurement on $c_0^{(i_3)}$ ($c_1^{(i_3)}$) is $|-\rangle$, then she knows with certainty that Bob's choice is $b=0$ ($b=1$). This will occur with the probability $1/2$. On the other $1/2$ case, the outcomes of Alice's measurements on both $c_0^{(i_3)}$ and $c_1^{(i_3)}$ are $|+\rangle$, thus she has to guess the value of b by herself. Therefore, the average probability that Alice can learn Bob's b correctly is still $P_{Alice}^* = 3/4$, which is the same as that of the original CKS protocol. As there is always $P_{Bob}^* \geq 1/2$ for any protocol, we can see that the security bound $2P_{Alice}^* + P_{Bob}^* \geq 2$ still holds in the current case.

B. Security against individual attacks

However, the above cheating requires the computational power to perform collective operations on many qubits/qutrits. More rigorously, equations (26) and (27) indicate that at the end of step B1, in every triple run Alice needs to make 6 qubits and 3 qutrits entangled together, even if Bob's half of his two-qutrit states is not counted. Here we will show that if Alice is limited to individual measurements, then the protocol can be secure.

In this scenario, during each run of the CKS protocol, Alice is not allowed to perform collective operations to entangle the qutrit β she received from Bob with her quantum ancillary system anymore. What she can do is to handle β individually. In general, such operations can be modeled as a channel $C_{\alpha\beta}$ which takes β as an input, then outputs a single qutrit state and a classical register α containing her measurement outcome. The effect of the channel $C_{\alpha\beta}$ can be written as

$$
C_{\alpha\beta}(|e_{ini}\rangle_\alpha |j\rangle_\beta) = \sum_{j'=0}^{2} \lambda_{jj'} |e_{jj'}\rangle_\alpha |j'\rangle_\beta , \tag{30}
$$

where $j = 0, 1, 2$, and $|e_{ini}\rangle_\alpha$ ($|e_{jj'}\rangle_\alpha$) is the initial (final) state of α. Comparing with equations (10) and (15), we can see that $C_{\alpha\beta}$ is actually a special case of the general cheating operation T studied in the proof of Theorem 1. The specialty in the current case is that α is classical, while in equation (10) it can be either classical or quantum. Therefore, by formulating the resultant state of $C_{\alpha\beta}$ as equation (10) and repeating the same proof in Sect. III.A, we find that the result of Theorem 1 still applies here. That is, if Alice can guess b with nonzero bias by applying channel $C_{\alpha\beta}$ to Bob's qutrit β and Bob never aborts, then she cannot learn x_b with reliability 1.

Moreover, as α is a classical register, there will be no alternative bases for measuring it. That is, once Alice decides on what kind of channel to apply, then the measurement basis for α is also fixed. No matter when Alice will measure α and extract the information stored in it, this information is already a deterministic classical object after the channel is applied, and there is only one choice of the basis for extracting it. This is different from an unlimited quantum attack, where Alice can apply the cheating operation and delay the measurement, then at a later time, if she wants to learn x_b, she measures α in a certain basis, while if she wants to learn b, she measures α in another basis. In the current case, even if the measurement could be delayed, there is only one basis for Alice (otherwise it will become a collective attack). Therefore, Alice needs to determine beforehand which basis to use, and picks the corresponding channel to apply. Dishonest-Alice will surely choose a basis which enables her to learn Bob's b with a nonzero bias, because this is the goal of her cheating. But then Theorem 1 guarantees that she cannot know with certainty the

value of x_b that Bob actually obtained. Consequently, if this run of the CKS protocol is picked for the security check in Protocol B, Alice will stand a non-vanishing probability ε to either announce a wrong value of x_b or cause Bob to abort (in case his measurement outcome is neither Π_0 nor Π_1 in step 5 of the CKS protocol).

Now suppose that Alice chooses to cheat in pn ($1/n \leq p \leq 1$) runs of the CKS protocol. While Alice can apply different strategies in these runs so that the value of ε can vary, we can define ε_m as the minimum of ε in any run that Alice cheats. Thus $1 - \varepsilon_m$ is the maximal probability for Alice to pass Bob's check in a single run. Since at the end of Protocol B, $n - 1$ runs of the CKS protocol will be checked, there can be two possibilities.

(I) The only one run that is not checked is picked among the pn runs that Alice cheats. Since this run is used for encoding the target bits in step B3, Alice can gain a non-trivial amount of information on b. As the CKS protocol ensures that Alice can learn b correctly with the probability $3/4$ at the most, and the other $pn - 1$ runs that Alice cheats are all checked, the maximal probability for Alice to learn b correctly and pass the checks successfully in this case is

$$P_{Alice}^{I} \leq \frac{3}{4}(1 - \varepsilon_m)^{pn-1}. \tag{31}$$

(II) The only one run that is not checked is not picked among the pn runs that Alice cheats. As Alice acts honestly in this run, she can only get b by guess, which can be correct with the probability $1/2$. Meanwhile, all the pn runs that Alice cheats are checked. Thus the maximal probability for Alice to learn b correctly and pass the checks successfully in this case is

$$P_{Alice}^{II} \leq \frac{1}{2}(1 - \varepsilon_m)^{pn}. \tag{32}$$

Note that cases (I) and (II) occur with the probabilities p and $1 - p$, respectively. Thus the total probability for Alice to pass the check while learning b correctly is

$$\begin{aligned} P_{Alice}^{*} &= p P_{Alice}^{I} + (1 - p) P_{Alice}^{II} \\ &\leq \frac{3}{4}p(1 - \varepsilon_m)^{pn-1} + \frac{1}{2}(1 - p)(1 - \varepsilon_m)^{pn}. \end{aligned} \tag{33}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial p}\left(\frac{3}{4}p(1 - \varepsilon_m)^{pn-1} + \frac{1}{2}(1 - p)(1 - \varepsilon_m)^{pn}\right) \\ &= ((\frac{3}{4}p + \frac{1}{2}(1 - p)(1 - \varepsilon_m))n \ln(1 - \varepsilon_m) \\ &\quad + \frac{3}{4} - \frac{1}{2}(1 - \varepsilon_m))(1 - \varepsilon_m)^{pn-1} \\ &< 0, \end{aligned} \tag{34}$$

higher P_{Alice}^{*} can be obtained by lowering p. The lowest nonzero p is $p = 1/n$, i.e., Alice cheats in $pn = 1$ run only and hopes that she is so lucky that this run is finally picked for encoding the target bits. In this case

$$\begin{aligned} P_{Alice}^{*} &\leq \frac{1}{2} + \frac{1}{4n} - \frac{\varepsilon_m}{2}(1 - \frac{1}{n}) \\ &\leq \frac{1}{2} + \frac{1}{4n}. \end{aligned} \tag{35}$$

As a result, for any arbitrarily small positive constant ζ, Bob can choose $n > 1/(4\zeta)$ and ask Alice to perform the corresponding Protocol B, which can achieve $P_{Alice}^{*} < 1/2 + \zeta$.

On the other hand, Bob's cheating probability remains the same as that of the CKS protocol. This is because in any useful run, the values of $x_0^{(i_1)}$, $x_1^{(i_1)}$, $x_0^{(i_2)}$, $x_1^{(i_2)}$ that Alice revealed are always 0. As set S is defined as $S = \{000, 001, 010, 100\}$, any value of $x_0^{(i_3)}$ and $x_1^{(i_3)}$ remains possible to Bob unless Alice reveals them. Thus the values of $x_0^{(i_1)}$, $x_1^{(i_1)}$, $x_0^{(i_2)}$, $x_1^{(i_2)}$ in a useful run do not provide any information for Bob to deduce $x_0^{(i_3)}$ and $x_1^{(i_3)}$. Also, the values of $x_0^{(i)}$, $x_1^{(i)}$ in different triple runs are chosen independently, so that the specific $x_0^{(i_3)}$, $x_1^{(i_3)}$ finally chosen for encoding the target bits x_0, x_1 are not affected by any $x_0^{(i)}$, $x_1^{(i)}$ from all the other runs. Consequently, Bob still has to decode the target bits via the corresponding run of the CKS protocol, without any help from other runs. Therefore, his cheating probability in our Protocol B is still $P_{Bob}^{*} = 1/2$, as what can be obtained in a single run of the original CKS protocol [8].

Putting things together, we can see that when Alice is limited to individual measurements, in our Protocol B $2P_{Alice}^{*} + P_{Bob}^{*}$ can be made arbitrarily close to $3/2$, which is the maximal violation of the security bound $2P_{Alice}^{*} + P_{Bob}^{*} \geq 2$ since the minimums for P_{Alice}^{*} and P_{Bob}^{*} are both $1/2$.

C. Security against limited collective attacks

If Alice is allowed to perform collective operations but it is restricted to a limited number of quantum systems only, then the security bound $2P^*_{Alice} + P^*_{Bob} \geq 2$ can also be violated to a certain degree.

Here we consider the case where Alice's collective operations are limited to the quantum systems in the same run of the CKS protocol only, i.e., the qutrit $\beta^{(i)}$ that Bob sends to her and the ancillary system she introduces (e.g., it can contain the two control qubits $c_0^{(i)}$, $c_1^{(i)}$ for keeping $x_0^{(i)}$, $x_1^{(i)}$ at the quantum level). She can still apply the transformation defined in equation (27) or other operations on these systems for cheating. Our discussion below will remain valid as long as this ancillary system cannot be entangled with the ancillary system for any other run (e.g., equation (26) is not allowed).

In this scenario, since the potential cheating strategies could be innumerous and much more complicated than the individual attacks, it is hard to prove the exact security bound of our protocol. But at least here we can obtain the loose upper and lower bounds of the security, which is $5/3 \leq 2P^*_{Alice} + P^*_{Bob} < 2$. It means that the probability for successful cheatings is higher than that of the individual attacks, but it still violates the security bound for the unlimited collective attack.

Let us prove the upper bound $2P^*_{Alice} + P^*_{Bob} < 2$ first. After the end of step B1, from Alice's point of view, the general form of the state of Alice's and Bob's combined system for each single run of the CKS protocol is

$$
\begin{aligned}
|\alpha \otimes \beta \otimes \beta'\rangle \;=\; & \lambda_b^{(0)} \left|e_b^{(0)}\right\rangle_\alpha |\phi_b\rangle_{\beta\beta'} + \lambda_b^{(1)} \left|e_b^{(1)}\right\rangle_\alpha |\phi_b'\rangle_{\beta\beta'} \\
& + \lambda_b^{(2)} \left|e_b^{(2)} \otimes \phi_b''\right\rangle_{\alpha\beta\beta'},
\end{aligned}
\tag{36}
$$

where the notations are the same as those in the proof of Theorem 1, with the additional $\left|e_b^{(2)} \otimes \phi_b''\right\rangle_{\alpha\beta\beta'}$, which represents the state orthogonal to both $\left|e_b^{(0)}\right\rangle_\alpha |\phi_b\rangle_{\beta\beta'}$ and $\left|e_b^{(1)}\right\rangle_\alpha |\phi_b'\rangle_{\beta\beta'}$. Note that the actual system may already collapse to one of the terms at the right hand side of the equation due to Bob's measurement on $\beta \otimes \beta'$. But Alice can still treat the whole state as the entangled form in this equation if she has not measured α. This is because Alice's and Bob's local operations are commutable for the entangled system $\alpha \otimes \beta \otimes \beta'$, so that it does not matter mathematically who performs the measurement first.

Since Bob learns that $x_b^{(i)} = 0$ ($x_b^{(i)} = 1$) if he gets $|\phi_b\rangle_{\beta\beta'}$ ($|\phi_b'\rangle_{\beta\beta'}$), otherwise he aborts, the above equation can be understood as

$$
\begin{aligned}
|\alpha \otimes \beta \otimes \beta'\rangle \;=\; & \lambda_b^{(0)} \left|e_b^{(0)}\right\rangle_\alpha \left|x_b^{(i)} = 0\right\rangle_{\beta\beta'} + \lambda_b^{(1)} \left|e_b^{(1)}\right\rangle_\alpha \left|x_b^{(i)} = 1\right\rangle_{\beta\beta'} \\
& + \lambda_b^{(2)} \left|e_b^{(2)} \otimes abort\right\rangle_{\alpha\beta\beta'},
\end{aligned}
\tag{37}
$$

Comparing with equation (10), it is even more general since it also includes the case where Bob may abort. Now if none of the coefficients $\lambda_b^{(0)}$ and $\lambda_b^{(1)}$ equals exactly to 1, then the value of $x_b^{(i)}$ is kept at the quantum level. That is, it will be determined by the uncertainty in quantum measurement, so that Alice cannot control with certainty which value can be obtained by Bob. Else if one of $\lambda_b^{(0)}$ and $\lambda_b^{(1)}$ equals exactly to 1, then the other one and $\lambda_b^{(2)}$ obviously have to be zero, and the value of $x_b^{(i)}$ becomes classically deterministic.

After the end of step B1, suppose that the values of $x_0^{(i)}$, $x_1^{(i)}$ in pn ($n = 3k$, $0 \leq p \leq 1$) runs of the CKS protocol in Protocol B are kept at the quantum level (the values of $\lambda_b^{(0)}$ and $\lambda_b^{(1)}$ depend on Alice's specific strategy, which can be different in each run). In the rest $(1-p)n$ runs, $x_0^{(i)}$, $x_1^{(i)}$ are no longer kept at the quantum level after step B1, but take deterministic classical values instead, so that Alice can ensure that the values of both X_0 and X_1 are presented in set S. Then these $(1-p)n$ runs are in fact executed honestly, as Theorem 1 ensures that Alice cannot use them to decode Bob's b. She can get b only if one of the other pn runs of the CKS protocol dishonestly executed is picked in step B3 for encoding the target bits x_0 and x_1 to complete the weak OT. This will occur with the probability p. Even if Alice uses the optimal cheating strategy so that she can still learn Bob's b with the probability $3/4$ (which is the maximum that can be obtained in the CKS protocol) for such a single dishonest run, the probability for (this run to be chosen) and (b is learned correctly) will drop down to $(3/4)p$. If any other non-optimal cheating strategy was used in this run, the probability is limited by this value too. In the rest $(1-p)$ occasions where one of the $(1-p)n$ honestly executed runs is chosen for encoding the target bits, Alice can only get Bob's b by guessing which has probability $1/2$

to be correct. Therefore, the total probability for Alice to cheat successfully in our Protocol B is bounded by

$$P^*_{Alice} = [\frac{3}{4}p + \frac{1}{2}(1-p)]p_c$$
$$= (\frac{1}{2} + \frac{1}{4}p)p_c, \tag{38}$$

where p_c is the probability that Alice can pass the security checks. Finding the tight bound for P^*_{Alice} requires a rigorous evaluation of p_c, which will depend on the specific cheating strategy Alice applies on the pn runs. But we can show that there will always be $P^*_{Alice} < 3/4$. This is because if $p < 1$, for a loose evaluation we can simply take the maximum $p_c = 1$ which surely covers any strategy. Then $P^*_{Alice} = 1/2 + p/4 < 3/4$. On the other hand, consider the case $p = 1$. As the collective attacks are limited to the quantum systems in each single run of the CKS protocol, the values of $x_0^{(i)}$, $x_1^{(i)}$ in different runs will not correlate with each other. Then in the current case, since $x_0^{(i)}$, $x_1^{(i)}$ in all the n runs are kept at the quantum level, any one of them can turn out to be either 0 or 1 during the measurement in the security check. The outcome is determined independently in each single run by quantum uncertainty, thus Bob cannot ensure with probability 100% that X_0 and X_1 will always take the legitimate values in set S in every single run in the security check. Therefore, we have $p_c < 1$ when $p = 1$. Then equation (38) gives $P^*_{Alice} = (1/2 + 1/4)p_c < 3/4$. Namely, no matter $p < 1$ or $p = 1$, P^*_{Alice} cannot equal exactly to $3/4$, i.e., it is always lower than that of the original CKS protocol.

Meanwhile, Bob's cheating probability still equals to that of the CKS protocol, i.e., $P^*_{Bob} = 1/2$, since the specific $x_0^{(i_3)}$, $x_1^{(i_3)}$ finally chosen for encoding the target bits x_0, x_1 are not affected by any $x_0^{(i)}$, $x_1^{(i)}$ from all the other runs, as it is elaborated in the previous subsection. Combining this P^*_{Bob} with $P^*_{Alice} < 3/4$, we can see that under the limited collective attack, our Protocol B can obtain

$$2P^*_{Alice} + P^*_{Bob} < 2 \times \frac{3}{4} + \frac{1}{2}. \tag{39}$$

Thus the upper bound $2P^*_{Alice} + P^*_{Bob} < 2$ is proven.

Now we prove the lower bound $2P^*_{Alice} + P^*_{Bob} \geq 5/3$. This is because there exists the following cheating strategy for Alice. In every triple run, she only keeps one pair of $x_0^{(i)}$, $x_1^{(i)}$ ($i = 1, 2, 3$) at the quantum level by using the collective operation described by equation (27). The other two pairs of $x_0^{(i)}$, $x_1^{(i)}$ are all taken as 0 beforehand, and the corresponding two runs of the CKS protocol are executed honestly. Then all triple runs can pass the security check with certainty. Meanwhile, when a useful run is finally picked for encoding the target bits, the pair $x_0^{(i)}$, $x_1^{(i)}$ kept at the quantum level stands a probability 1/3 to be chosen. Thus we have $p = 1/3$. Substitute it into equation (38) and we yield

$$2P^*_{Alice} + P^*_{Bob} = \frac{5}{3}, \tag{40}$$

so that this lower bound can be reached even when Alice is restricted to the limited collective attack. But we do not know whether this bound is tight at the present moment, as it is unclear whether there may exist an even better cheating strategy.

V. POTENTIAL IMPROVEMENTS

By observing the above cheating strategy that led to the lower bound equation (40), we can see that the probability $p = 1/3$ comes from the specific set S used in Protocol B, which is made of 3-bit strings only. In the more general form, i.e., our Protocol A, we can expect that choosing a more complicated set S may further reduced the value of p. For example, set S can be chosen as a classical error-correcting code, e.g., the binary linear (n, k, d)-code [18]. That is, S is taken as a set of classical n-bit strings. Each string is called a codeword. This set of strings has two features. (a) Among all the 2^n possible choices of n-bit strings, only a particular set of the size $\sim 2^k$ ($k < n$) is selected to construct this set. (b) The distance (i.e., the number of different bits) between any two codewords in this set is not less than d ($d < n$). With these features, it can be expected that by increasing n while fixing k/n and d/n, a dishonest Alice will have to introduce a much larger number of entangled control qubits for keeping more pairs of $x_0^{(i)}$, $x_1^{(i)}$ at the quantum level, so that $X_0 = x_0^{(1)}x_0^{(2)}...x_0^{(i)}...x_0^{(n)}$ and $X_1 = x_1^{(1)}x_1^{(2)}...x_1^{(i)}...x_1^{(n)}$ will appear as legitimate strings in set S no matter which bits are picked for the security check. Therefore with a properly chosen S, Protocol A may further lower the successful probability of limited collective attacks, and also raises the difficulty of implementing these attacks. However, the rigorous security bound will depend heavily on the structure of the specific set S used in the protocol. This analysis is left for future research.

VI. SUMMARY AND REMARKS

Thus we show that the security bound $2P^*_{Alice} + P^*_{Bob} \geq 2$ for weak OT can be violated for an Alice with limited computational power. As a rigorously checkable example, we proposed Protocol B which reaches the maximal violation $2P^*_{Alice} + P^*_{Bob} \to 3/2$ when only individual measurements are allowed. For attacks using collective operations on a limited number of quantum systems, there can still be $P^*_{Alice} < 3/4$ while $P^*_{Bob} = 1/2$. An even lower value of P^*_{Alice} could be expected from Protocol A.

Note that Ref. [9] obtained the security bound without limiting Alice to individual measurements. Thus our protocols does not really break the bound in principle. But it still has great practical significance. This is because in practice, any quantum storage devices can keep the quantum states faithfully for a limited time only. Suppose that τ is the maximal storage time available with state-of-the-art technology. Then during step B1 of Protocol B, Bob can require that every run of the CKS protocol is separated from each other by a time interval larger than τ, so that any ancillary system that a dishonest Alice may introduce to entangle with Bob's qutrit will suffer from errors, making Alice unable to pass the security check. In this case, Alice has to finish the measurement on Bob's qutrit (if she does not want to perform the honest unitary transformation) in each single run of the CKS protocol before the next run begins. Thus her cheating is actually reduced to individual measurements. So we can see that as long as our protocol is proven secure against individual measurements, then it naturally implies that we can use it as a secure protocol in practice.

[1] Rabin, M. O.: How to exchange secrets by oblivious transfer. Technical Report TR-81, Aiken Computation Laboratory, Harvard University (1981) (available online at http://eprint.iacr.org/2005/187.pdf)

[2] Even, S., Goldreich, O., Lempel, A.: A randomized protocol for signing contracts. In: Advances in Cryptology: Proc. Crypto '82, ed. Chaum, D., Rivest, R. L., Sherman, A. T., Plenum (1982), p. 205

[3] Kilian, J.: Founding crytpography on oblivious transfer. In: Proc. 1988 ACM Annual Symposium on Theory of Computing, ACM, New York (1988), p. 20

[4] Colbeck, R.: Impossibility of secure two-party classical computation. Phys. Rev. A **76**, 062308 (2007)

[5] Salvail, L., Schaffner, C., Sotakova, M.: On the power of two-party quantum cryptography. e-print. arXiv:0902.4036 (2009)

[6] Salvail, L., Sotakova, M.: Two-party quantum protocols do not compose securely against honest-but-curious adversaries. e-print. arXiv:0906.1671 (2009)

[7] Colbeck, R.: Quantum and relativistic protocols for secure multi-party computation. e-print. arXiv:0911.3814 (2009)

[8] Chailloux, A., Kerenidis, I., Sikora, J.: Lower bounds for quantum oblivious transfer. Quantum Inf. Comput. **13**, 158 (2013)

[9] Chailloux, A., Gutoski, G., Sikora, J.: Optimal bounds for quantum weak oblivious transfer. e-print. arXiv:1310.3262 (2013)

[10] Bennett, C. H., Brassard, G., Crépeau, C., Skubiszewska, M.-H.: Practical quantum oblivious transfer. In: Advances in Cryptology: CRYPTO '91, Lecture Notes in Computer Science, vol. **576**, ed. Feigenbaum, J., Springer-Verlag (1992) p. 351

[11] Mayers, D., Salvail, L.: Quantum oblivious transfer is secure against all individual measurements. In: Proc. Third Workshop on Physics and Computation - PhysComp '94, IEEE Computer Society Press, Dallas (1994), p. 69

[12] Mayers, D.: Unconditionally secure quantum bit commitment is impossible. Phys. Rev. Lett. **78**, 3414 (1997)

[13] Lo, H.-K., Chau, H. F.: Is quantum bit commitment really possible? Phys. Rev. Lett. **78**, 3410 (1997)

[14] Crépeau. C.: Equivalence between two flavours of oblivious transfers (abstract). In: Advances in Cryptology: CRYPTO '87, Lecture Notes in Computer Science, vol. **293**, ed. Pomerance, C., Springer-Verlag (1988), p. 350

[15] He, G. P.: Quantum key distribution based on orthogonal states allows secure quantum bit commitment. J. Phys. A: Math. Theor. **44**, 445305 (2011)

[16] He, G. P., Wang, Z. D.: Oblivious transfer using quantum entanglement. Phys. Rev. A **73**, 012331 (2006)

[17] Shimizu, K., Imoto, N.: Communication channels analogous to one out of two oblivious transfers based on quantum uncertainty. Phys. Rev. A **66**, 052316 (2002)

[18] Brassard, G., Crépeau, C., Jozsa, R., Langlois, D.: A quantum bit commitment scheme provably unbreakable by both parties. In: Proc. 34th Annual IEEE Symposium on Foundations of Computer Science, IEEE, Los Alamitos (1993), p. 362