Ground states of nonlinear Schrödinger equations with sum of periodic and inverse-square potentials

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Abstract

We study the existence of solutions of the following nonlinear Schrödinger equation

$$-\Delta u + \left(V(x) - \frac{\mu}{|x|^2}\right)u = f(x, u) \text{ for } x \in \mathbb{R}^N \setminus \{0\},$$

where $V: \mathbb{R}^N \to \mathbb{R}$ and $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are periodic in $x \in \mathbb{R}^N$. We assume that 0 does not lie in the spectrum of $-\Delta + V$ and $\mu < \frac{(N-2)^2}{4}$, $N \geq 3$. The superlinear and subcritical term f satisfies a weak monotonicity condition. For sufficiently small $\mu \geq 0$ we find a ground state solution as a minimizer of the energy functional on a natural constraint. If $\mu < 0$ and 0 lies below the spectrum of $-\Delta + V$, then ground state solutions do not exist.

MSC 2010: Primary: 35Q55; Secondary: 35J10, 35J20, 58E05

Key words: Schrödinger equation, ground state, variational methods, strongly indefinite functional, Cerami sequence, Nehari-Pankov manifold, inverse-square potential.

Introduction

The paper deals with the Schrödinger equation

$$(1.1) -\Delta u + \left(V(x) - \frac{\mu}{|x|^2}\right)u = f(x, u) \text{ for } x \in \mathbb{R}^N \setminus \{0\}.$$

where $V: \mathbb{R}^N \to \mathbb{R}$ is a periodic potential, $\mu < \overline{\mu} := \frac{(N-2)^2}{4}$, $N \geq 3$, and $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ has superlinear and subcritical growth. The Schrödinger equation appears in many physical models, for instance in nonlinear optics, where propagation of light through periodic optical structures with localized singular defect potentials has been intensively studied (see [4,13] and references therein). We focus on the localized inverse-square potential $-\frac{\mu}{|x|^2}$ which arises in

many other areas such as quantum mechanics, nuclear physics, molecular physics and quantum cosmology [8, 10, 11].

In this paper we study the nonlinearity f having superlinear and subcritical growth (see (F2)-(F4) below), for instance in dimension N=3 in the study of self-focusing Kerr-like optical media one has $f(x,u) = \Gamma(x)|u|^2u$ with $\Gamma \in L^{\infty}(\mathbb{R}^3)$ periodic, positive and bounded away from 0 (see [15, 22]). Nonlinear Schrödinger equations in \mathbb{R}^N with the inverse-square potentials have recently attracted a considerable attention in the mathematical literature, for example [5, 7, 9–12, 25, 26, 29] and all of these works concentrate on critically growing nonlinearites having a component of the form $f(x,u) = \Gamma(x)|u|^{2^*-2}u$, where $2^* := \frac{2N}{N-2}$ is the Sobolev exponent. This is well-justified since for $V=0, 0 \le \mu < \overline{\mu}$ and $f(x,u)=|u|^{p-2}u$ with $p \neq 2^*$, Terracini [29] Theorem 0.1 has shown that (1.1) admits only trivial solution u = 0in $\mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and nontrivial solutions appear in the critical case $p=2^*$. Our aim is to deal with the existence of nontrivial soultions in the subcritical case. Therefore we must ensure that $V \neq 0$ and our principal assumption is that 0 does not belong to the spectrum $\sigma(-\Delta+V)$ of the Schrödinger operator $-\Delta+V$. Since $\sigma(-\Delta+V)$ is bounded from below and consists of sum of disjoint closed intervals, then either 0 lies below the spectum or 0 lies in a finite spectral gap of the Schrödinger operator. In the both cases solutions to (1.1) represent the so-called standing gap solitons [15, 22].

Recall that if $\mu=0$ there is a broad literature treating the Schrödinger equations with periodic potentials V in the subcritical case, see for example [1,6,16,19–22,28,31] and references therein. If $0 \le \mu \le \overline{\mu}$ we may consider the following bilinear form

(1.2)
$$B_{\mu}(u,v) := \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle + \left(V(x) - \frac{\mu}{|x|^2} \right) uv \, dx$$

and to the best of our knowledge, all existing approaches to the problem (1.1) with $\mu > 0$ require the positive definiteness of B_{μ} in a suitable function space, where solutions are looked for. However, if $\mu \neq 0$, V is periodic and nonconstant, possibly sign-changing, then B_{μ} may be strongly indefinite and requires more delicate approach. Moreover the singular potential $-\frac{\mu}{|x|^2}$ does not belong to the Kato's class [24] and cannot be treated as a lower order perturbation term of $-\Delta + V$.

In what follows, throughout the paper we assume the following conditions:

- (V) $V \in L^{\infty}(\mathbb{R}^N)$, V is \mathbb{Z}^N -periodic in $x \in \mathbb{R}^N$ and $0 \notin \sigma(-\Delta + V)$. Here V is \mathbb{Z}^N -periodic in $x \in \mathbb{R}^N$ means V(x+y) = V(x) for any $y \in \mathbb{Z}^N$.
- (F1) $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^N$. Moreover f is \mathbb{Z}^N -periodic in x.

(F2) There are a > 0 and 2 such that

$$|f(x,u)| \le a(1+|u|^{p-1})$$
 for all $u \in \mathbb{R}$, $x \in \mathbb{R}^N$.

- (F3) f(x, u) = o(u) uniformly with respect to x as $|u| \to 0$.
- (F4) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$, where F is the primitive of f with respect to u, that is $F(x,u) = \int_0^u f(x,t)dt$.
- (F5) $u \mapsto f(x,u)/|u|$ is non-decreasing on $(-\infty,0)$ and $(0,+\infty)$.

The energy functional $\mathcal{J}: H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

(1.3)
$$\mathcal{J}_{\mu}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \left(V(x) - \frac{\mu}{|x|^2} \right) |u|^2 dx - \int_{\mathbb{R}^N} F(x, u) dx$$

is of C^1 -class and its critical points correspond to solutions of (1.1). In view of (V) spectral theory asserts that $\sigma(-\Delta+V)$ is purely continuous, bounded from below and consists of closed disjoint intervals [24]. Moreover there is an orthogonal decomposition of $X := H^1(\mathbb{R}^N) = X^+ \oplus X^-$, such that $B_0(\cdot, \cdot)$ given by (1.2) with $\mu = 0$ is positive definite on X^+ and negative definite on X^- . If 0 lies in a finite spectral gap, then both X^+ and X^- are infinite dimensional and the problem is strongly indefinite. For any $u \in X$ we denote by $u^+ \in X^+$ and $u^- \in X^-$ the corresponding summands so that $u = u^+ + u^-$.

We introduce the following constants

$$\mu(V)^{+} := \sup \Big\{ M > 0 : \int_{\mathbb{R}^{N}} |\nabla u|^{2} + V(x)|u|^{2} dx \ge M \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx \text{ for all } u \in X^{+} \Big\},$$

$$\mu(V)^{-} := \sup \Big\{ M > 0 : -\int_{\mathbb{R}^{N}} |\nabla u|^{2} + V(x)|u|^{2} dx \ge M \int_{\mathbb{R}^{N}} |\nabla u|^{2} dx \text{ for all } u \in X^{-} \Big\},$$

which play a crucial role in case $\mu \neq 0$ and (1.2) is indefinite. We show that $\mu(V)^+ \in (0,1]$, $\mu(V)^- > 0$ and clearly if $V(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$, then $\mu(V)^+ = 1$, $X^- = \{0\}$ and we may take $\mu(V)^- = \infty$ (see Lemma 2.1).

Our first main result reads as follows.

Theorem 1.1. If $0 \le \mu < \frac{(N-2)^2}{4}\mu(V)^+$, then (1.1) has a ground state solution.

By a ground state solution, we mean a nontrivial critical point u of \mathcal{J}_{μ} such that

$$\mathcal{J}_{\mu}(u) = \inf_{\mathcal{N}_{\mu}} \mathcal{J}_{\mu} > 0,$$

where

$$\mathcal{N}_{\mu} := \{ u \in X \setminus X^- : \ \mathcal{J}'_{\mu}(u)(u) = 0 \text{ and } \mathcal{J}'_{\mu}(u)(v) = 0 \text{ for any } v \in X^- \}$$

stands for the Nehari-Pankov manifold introduced in [22] and generalized approaches can be found in [3, 21, 28]. The set \mathcal{N}_{μ} is a natural constraint and since it contains all nontrivial critical points, then any ground state solution is a nontrivial critical point with the least possible energy \mathcal{J}_{μ} .

The existence of ground states of (1.1) without the Hardy term ($\mu = 0$) has been recently obtained by Szulkin and Weth [28] under more restrictive assumptions, in particular they assumed that $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty,0)$ and $(0,+\infty)$. Moreover Liu in [19] has assumed the weak monotonicity condition (F5) and a least energy solution to (1.1) has been obtained. The existence of ground states in case $\mu = 0$ and under assumptions (V), (F1)-(F5) follows from [21][Theorem 1.1]. Below we provide comparison of ground states levels of \mathcal{J}_{μ} and \mathcal{J}_{0} . Moreover we obtain a nonexistence result in case $X^{-} = \{0\}$, $\mu < 0$. The behaviour of ground state solutions in the limit $\mu \to 0^+$ requires the following additional technical condition.

(F6) $f(x,\cdot)$ is of \mathcal{C}^1 class for a.e. $x \in \mathbb{R}^N$ and there are b>0 and $2 < q \le p$ such that

$$f(x,u)u - 2F(x,u) \ge b|u|^q$$
 for all $u \in \mathbb{R}, x \in \mathbb{R}^N$,

where p is the same one appearing in (F2).

Theorem 1.2. Suppose that $-\frac{4}{(N-2)^2}\mu(V)^- < \mu < \frac{(N-2)^2}{4}\mu(V)^+$, u_μ is a ground state of \mathcal{J}_μ and u_0 is a ground state of \mathcal{J}_0 .

(a) Then there are t > 0 and $v \in X^-$ such that $tu_{\mu} + v \in \mathcal{N}_0$ and

(1.4)
$$\inf_{\mathcal{N}_0} \mathcal{J}_0 \le \inf_{\mathcal{N}_\mu} \mathcal{J}_\mu + \frac{1}{2} \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} |tu_\mu + v|^2 dx.$$

If $\mu \geq 0$, then there are t > 0 and $v \in X^-$ such that $tu_0 + v \in \mathcal{N}_{\mu}$ and

(1.5)
$$\inf_{\mathcal{N}_{\mu}} \mathcal{J}_{\mu} \leq \inf_{\mathcal{N}_{0}} \mathcal{J}_{0} - \frac{1}{2} \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}} |tu_{0} + v|^{2} dx.$$

Moreover

(1.6)
$$\lim_{\mu \to 0^+} \inf_{\mathcal{N}_{\mu}} \mathcal{J}_{\mu} = \inf_{\mathcal{N}_0} \mathcal{J}_0.$$

- (b) If in addition (F6) holds and $\mu_n \to 0^+$, then there is a sequence $(x_n) \subset \mathbb{Z}^N$ such that $u_{\mu_n}(\cdot + x_n)$ tends to a ground state u of \mathcal{J}_0 in the strong topology of X as $n \to \infty$.
- (c) If $\mu < 0$ and $0 < \inf \sigma(-\Delta + V)$, then \mathcal{J}_{μ} has no ground states.

The problem of the existence of ground states in case $\mu < 0$ and $\inf \sigma(-\Delta + V) < 0$ remains open.

The paper is organized as follows. In the next section we formulate our problem in a variational setting and we investigate the properties of the Nehari-Pankov manifold on which we minimize \mathcal{J}_{μ} to find a ground state. Since \mathcal{N}_{μ} need not to be of class \mathcal{C}^1 , we are not able to apply the standard minimizing method of \mathcal{J}_{μ} on \mathcal{N}_{μ} . The method of Szulkin and Weth [28] fails as well due to the weak monotonicity condition (F5). Our approach uses a linking argument and we find a Cerami sequence by means of the critical point theory developed in [21][Section 2]. Next, in Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2. The lack of compactness of Cerami sequences requires decompositions of these sequences which is provided in Lemma 3.3 and proved in the last Section 5.

2 Variational setting

In view of the spectral theory [24] we may introduce a new inner product in $X = H^1(\mathbb{R}^N)$ by the following formula

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \langle \nabla u^+, \nabla v^+ \rangle + V(x) \langle u^+, v^+ \rangle \ dx - \int_{\mathbb{R}^N} \langle \nabla u^-, \nabla v^- \rangle + V(x) \langle u^-, v^- \rangle \ dx$$

and a norm given by

$$||u||^2 := \langle u, u \rangle,$$

which is equivalent with the usual Sobolev norm in $H^1(\mathbb{R}^N)$, that is

$$||u||_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx.$$

Then X^+ and X^- are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ as well. If $\inf \sigma(-\Delta + V) > 0$ we have $X^- = \{0\}$ and

$$||u||^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx.$$

Lemma 2.1. $\mu(V)^+ \in (0,1]$ and $\mu(V)^- > 0$.

Proof. Since $B_0(\cdot,\cdot)$ is positive definite on X^+ , then

$$B_0(u, u) \ge C \|u\|_{H^1}^2 \ge C \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

for some constant C>0 and all $u\in X^+$. Thus $\mu(V)^+\geq C>0$. Similarly, $B_0(\cdot,\cdot)$ is negative definite on X^- and $\mu(V)^->0$. Now suppose that $\mu(V)^+>1$, take any open and bounded $\Omega\subset\mathbb{R}^N$ such that $\sup_{x\in\Omega}V(x)=V_0>0$. Let us take any $u\in X$ such that $\sup_{x\in\Omega}V(x)=0$ and

(2.1)
$$\int_{\Omega} |\nabla u|^2 dx < V_0(\mu(V)^+ - 1)^{-1} \int_{\Omega} |u|^2 dx.$$

Observe that

$$V_0 \int_{\Omega} |u|^2 dx \ge \int_{\mathbb{R}^N} V(x)|u|^2 dx \ge (\mu(V)^+ - 1) \int_{\mathbb{R}^N} |\nabla u|^2 dx = (\mu(V)^+ - 1) \int_{\Omega} |\nabla u|^2 dx$$
 and we obtain a contradiction with (2.1). \square

In addition to the norm topology $\|\cdot\|$ we need the topology \mathcal{T} on X which is induced by the norm

$$||u||_{\mathcal{T}} := \max \left\{ ||u^+||, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} |\langle u^-, e_k \rangle| \right\},$$

where (e_k) stands for a total orthonormal sequence in X^- . Recall that [2, 16, 21, 32]

$$||u^-|| \le ||u||_{\mathcal{T}} \le ||u|| \text{ for } u \in X$$

and on bounded subsets of X the topology \mathcal{T} coincides with the product of the norm topology in X^+ and the weak topology in X^- . The convergence of a sequence in \mathcal{T} topology will be denoted by $u_n \xrightarrow{\mathcal{T}} u$.

Assumptions (V), (F1) and (F2) allow to consider the energy functional \mathcal{J}_{μ} associated to (1.1) which is a well-defined \mathcal{C}^1 -map. Moreover critical points of \mathcal{J}_{μ} correspond to solutions to (1.1). Observe that (F3) and (F5) imply that

$$(2.2) f(x,u)u \ge 2F(x,u) \ge 0 \text{ for } x \in \mathbb{R}^N, \ u \in \mathbb{R}.$$

Hence, if $t \in \mathbb{R}$, $u_n \xrightarrow{\mathcal{T}} u_0$ and $\mathcal{J}_{\mu}(u_n) \geq t$ for any $n \geq 1$, then $t \leq \frac{1}{2} (\|u_n^+\|^2 - \|u_n^-\|^2)$ and $\|u_n\|$ is bounded. Therefore for $\mu \geq 0$ one easily verifies the following conditions:

- (A1) \mathcal{J}_{μ} is \mathcal{T} -upper semicontinuous, that is $\mathcal{J}_{\mu}^{-1}([t,\infty))$ is \mathcal{T} -closed for any $t \in \mathbb{R}$.
- (A2) \mathcal{J}'_{μ} is \mathcal{T} -to-weak* continuous in $\mathcal{J}^{-1}_{\mu}([0,\infty))$, that is $\mathcal{J}'_{\mu}(u_n) \rightharpoonup \mathcal{J}'_{\mu}(u_0)$ provided that $u_n \xrightarrow{\mathcal{T}} u_0$ and $\mathcal{J}_{\mu}(u_n) \geq 0$ for $n \geq 0$.

In Lemma 2.5 below we will check the geometrical conditions (A3) and (A4) provided that $0 \le \mu < \frac{(N-2)^2}{4} \mu(V)^+$.

- (A3) There exists r > 0 such that $m := \inf_{u \in X^+: ||u|| = r} \mathcal{J}_{\mu}(u) > 0$.
- (A4) For every $u \in X \setminus X^-$ there exists R(u) > r such that

$$\sup_{\partial M(u)} \mathcal{J}_{\mu} \le \mathcal{J}_{\mu}(0) = 0,$$

where

$$(2.3) M(u) := \{ tu + v \in X | v \in X^-, ||tu + v|| \le R(u), t \ge 0 \}.$$

We also require the following condition which is implied by Lemma 2.6 below for $\mu > -\frac{4}{(N-2)^2}\mu(V)^-$.

(A5) If $u \in \mathcal{N}_{\mu}$ then $\mathcal{J}_{\mu}(u) \geq \mathcal{J}_{\mu}(tu+v)$ for $t \geq 0$ and $v \in X^{-}$.

Finally we intend to apply the following linking theorem obtained by the second author in [21] (cf. [2, 18, 33]).

Theorem 2.2. If $\mathcal{J}_{\mu} \in \mathcal{C}^1(X, \mathbb{R})$ satisfies (A1)-(A4), then there exists a Cerami sequence (u_n) at level c_{μ} , that is $\mathcal{J}_{\mu}(u_n) \to c_{\mu}$ and $(1 + ||u_n||)\mathcal{J}'_{\mu}(u_n) \to 0$, such that

$$0 < m \le c_{\mu} \le \inf_{\mathcal{N}_{\mu}} \mathcal{J}_{\mu}$$

and

$$c_{\mu} := \inf_{u \in X \backslash X^{-}} \inf_{h \in \Gamma(u)} \sup_{u' \in M(u)} \mathcal{J}_{\mu}(h(u', 1)),$$

where $\Gamma(u)$ consists of $h \in \mathcal{C}(M(u) \times [0,1])$ satisfying the following conditions

- (h1) h is \mathcal{T} -continuous (with respect to norm $\|\cdot\|_{\mathcal{T}}$);
- (h2) h(u,0) = u for all $u \in M(u)$;
- (h3) $\mathcal{J}_{\mu}(u) \geq \mathcal{J}_{\mu}(h(u,t))$ for all $(u,t) \in M(u) \times [0,1]$;
- (h4) each $(u,t) \in M(u) \times [0,1]$ has an open neighborhood W in the product topology of (X,\mathcal{T}) and [0,1] such that the set $\{v-h(v,s):(v,s)\in W\cap (M(u)\times [0,1])\}$ is contained in a finite-dimensional subspace of X.

Moreover $c_{\mu} = \inf_{\mathcal{N}} \mathcal{J}_{\mu}$ provided that $c_{\mu} \geq \mathcal{J}_{\mu}(u)$ for some critical point $u \in X \setminus X^{-}$ and (A5) additionally holds.

In the last part of this section we check assumptions (A3)-(A5). Observe that in view of the Hardy inequality

(2.4)
$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \ge \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx$$

for any $u \in X$. Hence

(2.5)
$$\int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \, dx \ge \left(1 - \frac{4\mu}{(N-2)^2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,$$

where $1 - \frac{4\mu}{(N-2)^2} > 0$.

The following lemma is standard and follows from (F1) - (F3).

Lemma 2.3. For any $\varepsilon > 0$ there is $C_{\varepsilon} > 0$ such that for any $u \in H^1(\mathbb{R}^N)$

(2.6)
$$\int_{\mathbb{R}^N} F(x, u) \, dx \le \varepsilon |u|_2^2 + C_{\varepsilon} |u|_p^p,$$

where $|\cdot|_k$ stands for the norm in $L^k(\mathbb{R}^N)$ for any $k \geq 1$.

Lemma 2.4. Let $0 \le \mu < \frac{(N-2)^2}{4}\mu(V)^+$. Then for any $u \in X^+$

(2.7)
$$||u||_{\mu} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + \left(V(x) - \frac{\mu}{|x|^2} \right) |u|^2 dx \right)^{1/2}$$

satisfies inequalities

(2.8)
$$||u||^2 \ge ||u||_{\mu}^2 \ge \frac{1}{2} \Big(\mu(V)^+ - \frac{4\mu}{(N-2)^2} \Big) ||u||^2.$$

Hence $\|\cdot\|_{\mu}$ is a norm in X^+ equivalent with $\|\cdot\|$.

Proof. Observe that by (2.4)

$$\begin{split} \int_{\mathbb{R}^N} |\nabla u|^2 + \Big(V(x) - \frac{\mu}{|x|^2} \Big) |u|^2 \; dx & \geq & \int_{\mathbb{R}^N} \mu(V)^+ |\nabla u|^2 - \frac{\mu}{|x|^2} |u|^2 \; dx \\ & \geq & \left(\mu(V)^+ - \frac{4\mu}{(N-2)^2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \; dx \end{split}$$

for any $u \in X^+$. On the other hand by (2.5)

$$\int_{\mathbb{R}^N} |\nabla u|^2 + \left(V(x) - \frac{\mu}{|x|^2} \right) |u|^2 \, dx \ge \int_{\mathbb{R}^N} V(x) |u|^2 \, dx$$

and by Lemma 2.1 we get (2.8).

Now we show that \mathcal{J}_{μ} satisfies (A3) and (A4).

Lemma 2.5. Let $0 \le \mu < \frac{(N-2)^2}{4}\mu(V)^+$. For any $u_0 \in X \setminus X^-$ there are R > r > 0 such that

(2.9)
$$\inf_{u \in X^+: ||u|| = r} \mathcal{J}_{\mu}(u) > \mathcal{J}_{\mu}(0) = 0 \ge \sup_{\partial M(u_0)} \mathcal{J}_{\mu}.$$

Proof. Observe that by (2.8) and Lemma 2.3 we get

$$\mathcal{J}_{\mu}(u^{+}) \geq \frac{1}{4} \Big(\mu(V)^{+} - \frac{4\mu}{(N-2)^{2}} \Big) \|u^{+}\|^{2} - \int_{\mathbb{R}^{N}} F(x, u^{+}) dx \\
\geq \frac{1}{4} \Big(\mu(V)^{+} - \frac{4\mu}{(N-2)^{2}} \Big) \|u^{+}\|^{2} - \varepsilon |u^{+}|_{2}^{2} - C_{\varepsilon} |u^{+}|_{p}^{p}$$

for any $\varepsilon > 0$. Taking $\varepsilon > 0$ small enough, it is easy to see that there exists r > 0 small enough such that

$$\inf_{u \in X^+: ||u|| = r} \mathcal{J}_{\mu}(u) > \mathcal{J}_{\mu}(0) = 0.$$

Let $u_0 \in X \setminus X^-$ and $u = tu_0 + u^-, u^- \in X^-, t \ge 0$. Since

$$\mathcal{J}_{\mu}(u) \leq \mathcal{J}_{0}(u) = \frac{1}{2} \int_{\mathbb{R}^{N}} |\nabla u|^{2} + V(x)|u|^{2} dx - \int_{\mathbb{R}^{N}} F(x, u) dx,$$

then similarly as in [19,28] we obtain that $\mathcal{J}_{\mu}(u) \to -\infty$ if $||u|| \to \infty$. Moreover $\mathcal{J}_{\mu}(u^{-}) \leq 0$ and we find sufficiently large R > r such that

$$\sup_{\partial M(u_0)} \mathcal{J}_{\mu} \le 0.$$

Lemma 2.6. If $\mu > -\frac{4}{(N-2)^2}\mu(V)^-$, $u \in X \setminus X^-$, $v \in X^-$ and $t \ge 0$, then

(2.10)
$$\mathcal{J}_{\mu}(u) \geq \mathcal{J}_{\mu}(tu+v) - \mathcal{J}'_{\mu}(u) \left(\frac{t^2-1}{2}u+tv\right).$$

In particular (A5) holds.

Proof. Observe that

$$\mathcal{J}_{\mu}(tu+v) - \mathcal{J}_{\mu}(u) - \mathcal{J}'_{\mu}(u) \left(\frac{t^{2}-1}{2}u+tv\right) = -\frac{1}{2}\|v\|^{2} - \frac{1}{2}\int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}}|v|^{2} dx + \int_{\mathbb{R}^{N}} \varphi(t,x) dx
\leq -\frac{1}{2}\|v\|^{2} + \frac{1}{2}\max\left\{0, -\mu\frac{(N-2)^{2}}{4}\right\}\int_{\mathbb{R}^{N}} |\nabla v|^{2} dx + \int_{\mathbb{R}^{N}} \varphi(t,x) dx,$$

where

$$\varphi(t,x) := f(x,u) \left(\frac{t^2 - 1}{2} u + tv \right) + F(u) - F(tu + v).$$

Suppose that $u(x) \neq 0$. Similarly as in [21,28] we show that $\varphi(t,x) \leq 0$. Indeed, in view of (2.2) we get $\varphi(0,x) \leq 0$. By (F4), we obtain $\varphi(t,x) \to -\infty$ as $t \to \infty$. Let $t_0 \geq 0$ be such that

$$\varphi(t_0, x) = \max_{t>0} \varphi(t, x).$$

We may assume that $t_0 > 0$ and thus $\partial_t \varphi(t_0, x) = 0$. Thus

$$f(x,u)(t_0u + v) = f(x,t_0u + v)u.$$

If $t_0u + v = 0$ or $t_0u + v \neq 0$, then by (2.2) we obtain $\varphi(t_0, x) \leq 0$. Suppose that $u < t_0u + v$. Then, by (F5) a function

$$(0, +\infty) \ni s \mapsto \frac{f(x, s)}{s} \in \mathbb{R}$$

is constant on $(u, t_0u + v)$ and

$$F(x,u) - F(x,t_0u + v) = f(x,u)\frac{u^2 - (t_0u + v)^2}{2u}.$$

Thus

$$\varphi(t_0, x) = f(x, u) \left(\frac{t_0^2 - 1}{2} u + t_0 v \right) + f(x, u) \frac{u^2 - (t_0 u + v)^2}{2u}$$
$$= -\frac{f(x, u) u v^2}{2u^2} \le 0.$$

Similarly we check the case $u > t_0 u + v$. Therefore $\varphi(t,x) \leq 0$ for any $t \geq 0$ and (2.10) holds.

3 Proof of Theorem 1.1

We need the following lemma.

Lemma 3.1. If $|x_n| \to \infty$, then for any $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \frac{1}{|x|^2} |u(\cdot - x_n)|^2 dx \to 0 \text{ as } n \to \infty.$$

Proof. Let $\varphi_m \in C_0^{\infty}(\mathbb{R}^N)$ and $\varphi_m \to u$ in $H^1(\mathbb{R}^N)$ as $m \to \infty$. Take $R_m > 0$ such that $\sup (\varphi_m) \subset B(0, R_m)$. For any m we find n = n(m) such that $|x_n| - R_m \ge m$ and (n(m)) is an increasing sequence. Then we get

$$\int_{\mathbb{R}^N} \frac{1}{|x|^2} |\varphi_m(\cdot - x_n)|^2 dx = \int_{\mathbb{R}^N} \frac{1}{|x + x_n|^2} |\varphi_m|^2 dx = \int_{B(0, R_m)} \frac{1}{|x + x_n|^2} |\varphi_m|^2 dx
\leq \frac{1}{(|x_n| - R_m)^2} \int_{B(0, R_m)} |\varphi_m|^2 dx \leq \frac{1}{m^2} |\varphi_m|_2^2
\to 0.$$

In view of the Hardy inequality (2.4) we conclude.

Lemma 3.2. If $(u_n) \subset X$ and $(\mu_n) \subset [0, +\infty)$ are such that $\mu_n \leq \mu < \frac{(N-2)^2}{4}\mu(V)^+$, $(1 + \|u_n\|)\mathcal{J}'_{\mu_n}(u_n) \to 0$ and $\mathcal{J}_{\mu_n}(u_n)$ is bounded from above, then (u_n) is bounded. In particular any Cerami sequence of \mathcal{J}_{μ} is bounded for $0 \leq \mu < \frac{(N-2)^2}{4}\mu(V)^+$.

Proof. Suppose that $(1 + ||u_n||)\mathcal{J}'_{\mu_n}(u_n) \to 0$, $\mathcal{J}_{\mu_n}(u_n) \leq M$ and $||u_n|| \to \infty$ as $n \to \infty$. Let $v_n := \frac{u_n}{||u_n||}$. We may assume that $v_n \rightharpoonup v$ in X and $v_n(x) \to v(x)$ a.e. in \mathbb{R}^N . Moreover there is a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that

(3.1)
$$\liminf_{n \to \infty} \int_{B(y_n, 1)} |v_n^+|^2 dx > 0.$$

Otherwise, in view of Lions lemma (see [32][Lemma 1.21]) we get that $v_n^+ \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. By Lemma 2.3 we obtain $\int_{\mathbb{R}^N} F(x, sv_n^+) dx \to 0$ for any s > 0. Let us fix any s > 0. Observe that Lemma 2.6 implies that

$$\mathcal{J}_{\mu_n}(u_n) \ge \mathcal{J}_{\mu_n}(sv_n^+) + o(1)$$

and by (2.8)

(3.2)
$$M \geq \limsup_{n \to \infty} \mathcal{J}_{\mu_n}(u_n) \geq \limsup_{n \to \infty} \mathcal{J}_{\mu_n}(sv_n^+) \geq \frac{s^2}{2} \limsup_{n \to \infty} \|v_n^+\|_{\mu_n}^2$$
$$\geq \frac{s^2}{4} \left(\mu(V)^+ - \frac{4\mu}{(N-2)^2}\right) \limsup_{n \to \infty} \|v_n^+\|^2.$$

Note that by (2.2)

$$||u_n^+||^2 - ||u_n^-||^2 \ge \mathcal{J}'_{\mu_n}(u_n)(u_n).$$

Hence

$$2\|u_n^+\|^2 \geq \|u_n^+\|^2 + \|u_n^-\|^2 + \mathcal{J}'_{\mu_n}(u_n)(u_n) = \|u_n\|^2 + \mathcal{J}'_{\mu_n}(u_n)(u_n)$$

and, passing to a subsequence if necessary, $C := \limsup_{n \to \infty} \|v_n^+\|^2 > 0$. Then by (3.2)

$$M \ge \frac{s^2}{4}C\Big(\mu(V)^+ - \frac{4\mu}{(N-2)^2}\Big)$$

for any $s \geq 0$ and the obtained contradiction shows that (3.1) holds. We may assume that $(y_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^N$ and

$$\liminf_{n \to \infty} \int_{B(y_n, \rho)} |v_n^+|^2 dx > 0$$

for some $\rho > 1$. Therefore passing to a subsequence $v_n^+(\cdot + y_n) \to v^+$ in $L^2_{loc}(\mathbb{R}^N)$ and $v^+ \neq 0$. Note that if $v(x) \neq 0$ then $|u_n(x+y_n)| = |v_n(x+y_n)| ||u_n|| \to \infty$ and by (F4)

$$\frac{F(x, u_n(x+y_n))}{\|u_n\|^2} = \frac{F(x, u_n(x+y_n))}{|u_n(x+y_n)|^2} |v_n(x+y_n)|^2 \to \infty$$

as $n \to \infty$. Since $\mathcal{J}'_{\mu_n}(u_n)(u_n) \to 0$, then

$$||u_n^+||^2 - ||u_n^-||^2 - \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} |u_n|^2 dx + o(1) = \int_{\mathbb{R}^N} f(x, u_n) u_n dx \ge 0$$

and $\frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} |u_n|^2 dx$ is bounded. Therefore by the \mathbb{Z}^N -periodicity of F in $x \in \mathbb{R}^N$ and by Fatou's lemma

$$0 = \limsup_{n \to \infty} \frac{\mathcal{J}_{\mu_n}(u_n)}{\|u_n\|^2} = \limsup_{n \to \infty} \left(\frac{1}{2} (\|v_n^+\|^2 - \|v_n^-\|^2 - \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} |u_n|^2 dx \right)$$
$$- \int_{\mathbb{R}^N} \frac{F(x, u_n(x + y_n))}{\|u_n\|^2} dx$$
$$= -\infty.$$

Thus we get a contradiction.

Below we provide a decomposition of bounded Palais-Smale sequences of \mathcal{J}_{μ} .

Lemma 3.3. Assume that $0 \le \mu < \overline{\mu}$ and let (u_n) be a bounded Palais-Smale sequence of \mathcal{J}_{μ} at level $c \ge 0$, that is $\mathcal{J}'_{\mu}(u_n) \to 0$ and $\mathcal{J}_{\mu}(u_n) \to c$. Then there is $k \ge 0$ and there are sequences $(\bar{u}_i)_{i=0}^k \subset X$ and $(x_n^i)_{0 \le i \le k} \subset \mathbb{Z}^N$ such that $x_n^0 = 0$, $|x_n^i| \to +\infty$, $|x_n^i - x_n^j| \to +\infty$, $i \ne j, i, j = 1, 2, \dots, k$, and passing to a subsequence, the following conditions hold:

$$\mathcal{J}'_{\mu}(\bar{u}_{0}) = 0,
\mathcal{J}'_{0}(\bar{u}_{i}) = 0 \text{ and } \bar{u}_{i} \neq 0 \text{ for } i = 1, ..., k,
u_{n} - \sum_{i=0}^{k} \bar{u}_{i}(\cdot - x_{n}^{i}) \to 0 \text{ in } X \text{ and } ||u_{n}||^{2} \to \sum_{i=0}^{k} ||\bar{u}_{i}||^{2} \text{ as } n \to \infty,
(3.3)$$

$$c = \mathcal{J}_{\mu}(\bar{u}_{0}) + \sum_{i=1}^{k} \mathcal{J}_{0}(\bar{u}_{i}).$$

Since proof of Lemma 3.3 is technical we postpone it to Section 5.

Proof of Theorem 1.1. Applying Theorem 2.2 we find a Cerami sequence (u_n) of \mathcal{J}_{μ} at level $c_{\mu} > 0$ and in view Lemma 3.2, (u_n) is bounded in X. If $\mu = 0$, then by Theorem 2.2 we obtain

$$\inf_{\mathcal{N}_0} \mathcal{J}_0 \ge c_0 > 0$$

and Lemma 3.3 implies that there is a nontrivial critical point $u_0 \in \mathcal{N}_0$ of \mathcal{J}_0 such that $\mathcal{J}_0(u_0) = c_0$. Hence u_0 is a ground state of \mathcal{J}_0 , that is $\mathcal{J}_0(u_0) = \inf_{\mathcal{N}_0} \mathcal{J}_0$. Now let us assume that $0 < \mu < \frac{(N-2)^2}{4} \mu(V)^+$ and consider

$$M(u_0) := \{ u = tu_0 + v \in X | v \in X^-, ||u|| \le R(u_0), t \ge 0 \},\$$

where $R(u_0)$ is given by Lemma 2.5. Observe that, if $t_n u_0 + v_n \rightharpoonup t_0 u_0 + v_0$ in X and $t_n u_0 + v_n \in M(u_0)$ for $n \geq 1$, then passing to a subsequence we may assume that $t_n \to t_0$, $v_n \rightharpoonup v_0$ in $L^2(\mathbb{R}^N, \frac{1}{|x|^2}), v_n(x) \to v_0(x)$ a.e. on \mathbb{R}^N . Hence $t_0 u_0 + v_0 \in M(u_0)$ and by Fatou's lemma

$$\limsup_{n\to\infty} \mathcal{J}_{\mu}(t_n u_0 + v_n) \le \mathcal{J}_{\mu}(t_0 u_0 + v_0).$$

Therefore $M(u_0)$ is weakly sequentially closed and \mathcal{J}_{μ} is weakly sequentially upper semicontinuous. Then \mathcal{J}_{μ} attains its maximum in $M(u_0)$, that is, there is $t_0u_0 + v_0 \in M(u_0)$ such that

$$\mathcal{J}_{\mu}(t_0 u_0 + v_0) \ge \mathcal{J}_{\mu}(u)$$

for any $u \in M(u_0)$. Note that by Lemma 2.5, $\mathcal{J}_{\mu}(t_0u_0 + v_0) > 0$ and hence $t_0u_0 + v_0 \neq 0$. Define h(s, u) = u for $s \in [0, 1]$ and $u \in M(u_0)$ and note that (h1)-(h4) are satisfied, that is $h \in \Gamma(u_0)$. Then in view of Lemma 2.6 we have

(3.4)
$$c_0 = \mathcal{J}_0(u_0) \ge \mathcal{J}_0(t_0 u_0 + v_0) > \mathcal{J}_\mu(t_0 u_0 + v_0) = \max_{u \in M(u_0)} \mathcal{J}_\mu(h(1, u)) \ge c_\mu.$$

Thus by (3.3) we get k = 0 and $\mathcal{J}_{\mu}(\bar{u}_0) = c_{\mu} > 0$. Therefore \bar{u}_0 is a nontrivial critical point of \mathcal{J}_{μ} and by Theorem 2.2

$$c_{\mu} = \inf_{\mathcal{N}_{\mu}} \mathcal{J}_{\mu}.$$

4 Proof of Theorem 1.2

Lemma 4.1. If $0 \le \mu < \frac{(N-2)^2}{4}\mu(V)^+$ then for every $u \in X \setminus X^-$ there is t > 0 and $v \in X^-$ such that $tu + v \in \mathcal{N}_{\mu}$. If $X^- = \{0\}$ and $\mu < \bar{\mu}$ then for every $u \in X \setminus \{0\}$ there is t > 0 such that $tu \in \mathcal{N}_{\mu}$.

Proof. Let $u \in X \setminus X^-$ and consider a map $\xi : \mathbb{R}^+ \times X^- \to \mathbb{R}$ such that

$$\xi(t, v) = -\mathcal{J}_{\mu}(tu^{+} + v).$$

Similarly as in proof of Theorem 1.1 we show that ξ is weakly lower semicontinuous for $\mu \geq 0$. Since ξ is bounded from below and coercive (see proof of Lemma 2.5), then we find $t \geq 0$ and $v \in X^-$ such that

$$\mathcal{J}_{\mu}(tu+v) = \sup_{\mathbb{R}^+ u^+ \oplus X^-} \mathcal{J}_{\mu},$$

where $\mathbb{R}^+u^+ := \{tu^+ | t \geq 0\}$. In view of Lemma 2.5 and Lemma 2.6 (condition (A5)) we get t > 0, hence $tu + v \in \mathcal{N}_{\mu}$. Observe that if $X^- = \{0\}$, then ξ is continuous, coercive and bounded from below for any $\mu \in \mathbb{R}$. Since the first inequality in (2.9) actually holds for any $\mu < \bar{\mu}$, therefore there is t > 0 such that $tu \in \mathcal{N}_{\mu}$.

Proof of Theorem 1.2.

(a) Let $u_{\mu} \in \mathcal{N}_{\mu}$ be a ground state of \mathcal{J}_{μ} and $-\frac{4}{(N-2)^2}\mu(V)^- < \mu < \frac{(N-2)^2}{4}\mu(V)^+$. In view of Lemma 4.1 there is t > 0 and $v \in X^-$ such that $tu_{\mu} + v \in \mathcal{N}_0$. Then by Lemma 2.6

$$(4.1) c_{\mu} = \mathcal{J}_{\mu}(u_{\mu}) \ge \mathcal{J}_{\mu}(tu_{\mu} + v) = \mathcal{J}_{0}(tu_{\mu} + v) - \frac{1}{2} \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}} |tu_{\mu} + v|^{2} dx$$

$$\ge c_{0} - \frac{1}{2} \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}} |tu_{\mu} + v|^{2} dx$$

and we obtain (1.4). Now, let us assume that $u_0 \in \mathcal{N}_0$ is a ground of \mathcal{J}_0 . Similarly as above we show (1.5), that is,

$$c_{0} = \mathcal{J}_{0}(u_{0}) \geq \mathcal{J}_{0}(t'u_{0} + v') = \mathcal{J}_{\mu}(t'u_{0} + v') + \frac{1}{2} \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}} |t'u_{0} + v'|^{2} dx$$

$$\geq c_{\mu} + \frac{1}{2} \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}} |t'u_{0} + v'|^{2} dx$$

for any $0 \le \mu \le \frac{(N-2)^2}{4}\mu(V)^+$ and some t' > 0 and $v' \in X^-$ such that $t'u + v' \in \mathcal{N}_{\mu}$. Observe that we get

$$c_0 \ge c_\mu = \mathcal{J}_\mu(u_\mu)$$

and by Lemma 3.2 we have that (u_{μ}) is bounded if $\mu \to 0^+$. Take any sequence $\mu_n \to 0^+$ such that $\mu_n \leq \mu < \frac{4}{(N-2)^2}\mu(V)^+$ and denote $u_n := u_{\mu_n}$. Then there is a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that

(4.2)
$$\liminf_{n \to \infty} \int_{B(y_n, 1)} |u_n^+|^2 dx > 0.$$

Otherwise, in view of Lions lemma we get that $u_n^+ \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$ and since $u_n \in \mathcal{N}_{\mu_n}$, then

$$||u_n^+||^2 = \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} u_n u_n^+ dx + \int_{\mathbb{R}^N} f(x, u_n) u_n^+ dx \to 0$$

as $n \to \infty$. Hence $\limsup_{n \to \infty} J_{\mu_n}(u_n) \le 0$, but this contradicts the following inequalities

$$\mathcal{J}_{\mu_n}(u_n) \ge \mathcal{J}_{\mu_n}\left(\frac{r}{\|u_n^+\|}u_n^+\right) \ge \inf_{n \in \mathbb{N}} \inf_{u \in X^+, \|u\|=r} \mathcal{J}_{\mu_n}(u) > 0,$$

for sufficiently small r > 0, where the last inequality follows from similar arguments provided in Lemma 2.5. Therefore (4.2) holds and we may assume that $(y_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^N$ and

$$\liminf_{n \to \infty} \int_{B(y_n, \rho)} |u_n^+|^2 dx > 0$$

for some $\rho > 1$. Therefore passing to a subsequence we find $u \in X$ such that $u_n^+(\cdot + y_n) \to u^+$ in $L^2_{loc}(\mathbb{R}^N)$ and $u^+ \neq 0$. Moreover we may assume that

$$u_n(\cdot + y_n) \rightharpoonup u$$
 in X and $u_n(x + y_n) \rightarrow u(x), \ u_n^+(x + y_n) \rightarrow u^+(x)$ a.e. on \mathbb{R}^N .

Let $t_n u_n + v_n \in \mathcal{N}_0$ and $t_n > 0$, $v_n \in X^-$. Then by (2.2)

$$(4.3) ||u_n^+||^2 = ||u_n^- + v_n/t_n||^2 + \frac{1}{t_n^2} \int_{\mathbb{R}^N} f(x, t_n u_n + v_n) (t_n u_n + v_n) dx$$

$$\geq ||u_n^- + v_n/t_n||^2 + 2 \int_{\mathbb{R}^N} \frac{F(x, t_n (u_n + v_n/t_n))}{t_n^2} dx.$$

Therefore $||u_n^- + v_n/t_n||$ is bounded and we may assume that $u_n^-(x) + v_n(x)/t_n \to v(x)$ a.e. on \mathbb{R}^N for some $v \in X^-$. Hence, if $t_n \to \infty$, then $|t_n u_n(x) + v_n(x)| = t_n |u_n(x) + v_n(x)/t_n| \to \infty$ provided that $u^+(x) + v(x) \neq 0$. In view of Fatou's lemma and by (F4)

$$\int_{\mathbb{R}^N} \frac{F(x, t_n(u_n + v_n/t_n))}{t_n^2} dx \to \infty,$$

which contradicts (4.3). Therefore t_n is bounded, thus $||t_n u_n^+||$ and $||t_n u_n^-| + v_n||$ are bounded and by the Hardy inequality

$$\frac{1}{2} \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} |t_n u_n + v_n|^2 \, dx \to 0$$

as $n \to \infty$. Therefore by (4.1) we get (1.6).

(b) Let (u_n) be a sequence of ground states of \mathcal{J}_{μ_n} as in (a) and take $x_n := y_n$. For $(x, u) \in \mathbb{R}^N \times \mathbb{R}$

$$G(x, u) := \frac{1}{2}f(x, u)u - F(x, u).$$

Observe that for any $\phi \in X$

$$\mathcal{J}_0'(u_n(\cdot + x_n))(\phi) = \mathcal{J}_{\mu_n}'(u_n)(\phi(\cdot - x_n)) + \int_{\mathbb{R}^N} \frac{\mu_n}{|x|^2} u_n \phi(\cdot - x_n) dx$$

$$\to 0$$

as $n \to \infty$. In view of the Vitali convergence theorem $\mathcal{J}'_0(u_n(\cdot + x_n))(\phi) \to \mathcal{J}'_0(u)(\phi)$. Thus u is a nontrivial critical point of \mathcal{J}_0 . Observe that by (1.6) and Fatou's lemma

$$(4.4) c_0 = \liminf_{n \to \infty} \mathcal{J}_{\mu_n}(u_n) = \liminf_{n \to \infty} \left(\mathcal{J}_{\mu_n}(u_n) - \frac{1}{2} \mathcal{J}'_{\mu_n}(u_n)(u_n) \right)$$

$$= \liminf_{n \to \infty} \int_{\mathbb{R}^N} G(x, u_n) dx = \liminf_{n \to \infty} \int_{\mathbb{R}^N} G(x, u_n(x + x_n)) dx \ge \int_{\mathbb{R}^N} G(x, u) dx$$

$$= \mathcal{J}_0(u) \ge c_0.$$

Thus we obtain that u is a ground state of \mathcal{J}_0 . Let us denote $w_n := u_n(\cdot + x_n)$ and observe that

(4.5)
$$\int_{\mathbb{R}^{N}} G(x, w_{n}) - G(x, w_{n} - u) dx = \int_{\mathbb{R}^{N}} \int_{0}^{1} \frac{d}{dt} G(x, w_{n} - u + tu) dt dx$$
$$= \int_{0}^{1} \int_{\mathbb{R}^{N}} g(x, w_{n} - u + tu) u dx dt,$$

where $g(x,u) := \partial_u G(x,u)$ for $u \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$. Since $(w_n - u + tu)$ is bounded in X, then for any $\varepsilon > 0$ there is $\delta > 0$ such that for any Ω with the Lebesgue measure $|\Omega| < \delta$, we have

$$\int_{\Omega} |g(x, w_n - u + tu)u| \, dx < \varepsilon$$

for any $n \in \mathbb{N}$. Thus $(g(x, w_n - u + tu)u)$ is uniformly integrable. Moreover for any $\varepsilon > 0$ there is $\Omega \subset \mathbb{R}^N$, $|\Omega| < +\infty$, such that for any $n \in \mathbb{N}$

$$\int_{\mathbb{R}^N \setminus \Omega} |g(x, w_n - u + tu)u| \, dx < \varepsilon.$$

Hence a family $(g(x, w_n - u + tu)u)$ is tight over \mathbb{R}^N . Since $g(w_n - u + tu)u \to g(tu)u$ a.e. in \mathbb{R}^N , then in view of the Vitali convergence theorem g(x, tu)u is integrable and

$$\int_{\mathbb{R}^N} g(x, w_n - u + tu) u \, dx \to \int_{\mathbb{R}^N} g(x, tu) u \, dx$$

as $n \to \infty$. By (4.5) we obtain

$$\int_{\mathbb{R}^N} G(x, w_n) - G(x, w_n - u) dx \to \int_0^1 \int_{\mathbb{R}^N} g(x, tu) u dx dt = \int_{\mathbb{R}^N} G(x, u) dx$$

as $n \to \infty$. Taking into account (4.4) we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} G(x, w_n - u_0) \, dx = 0,$$

and by (F6) we have $w_n \to u$ in $L^q(\mathbb{R}^N)$. Since (w_n) is bounded in $L^2(\mathbb{R}^N)$ and in $L^{2^*}(\mathbb{R}^N)$, then by the interpolation inequalities $w_n \to u$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. Thus

$$||w_{n}^{+} - u^{+}||^{2} = \mathcal{J}'_{\mu_{n}}(u_{n})((w_{n}^{+} - u^{+})(\cdot - x_{n})) - \langle u^{+}, w_{n}^{+} - u^{+} \rangle$$

$$+ \int_{\mathbb{R}^{N}} \frac{\mu_{n}}{|x|^{2}} u_{n}(w_{n}^{+} - u^{+})(x - x_{n}) dx + \int_{\mathbb{R}^{N}} f(x, w_{n})(w_{n}^{+} - u^{+}) dx$$

$$\to 0$$

and

$$||w_{n}^{-} - u^{-}||^{2} = -\mathcal{J}'_{\mu_{n}}(u_{n})((w_{n}^{-} - u^{-})(\cdot - x_{n})) - \langle u^{-}, w_{n}^{-} - u^{-} \rangle$$

$$- \int_{\mathbb{R}^{N}} \frac{\mu_{n}}{|x|^{2}} u_{n}(w_{n}^{-} - u^{-})(x - x_{n}) dx - \int_{\mathbb{R}^{N}} f(x, w_{n})(w_{n}^{-} - u^{-}) dx$$

$$\to 0$$

as $n \to \infty$. Therefore $w_n \to u$ in X.

(c) Suppose that $\mu < 0 < \inf \sigma(-\Delta + V)$ and u_{μ} is a ground state of \mathcal{J}_{μ} and u_0 is a ground state of \mathcal{J}_0 . Then $X^- = \{0\}$, $\mu(V)^- = \infty$ and by (1.4) we have

$$(4.6) c_0 < c_{\mu},$$

since $tu_{\mu} + v \in \mathcal{N}_0$ and $tu_{\mu} + v \neq 0$. In view of Lemma 4.1, for any $y \in \mathbb{Z}^N$, we find $t_y > 0$ such that $t_y u_0(\cdot + y) \in \mathcal{N}_{\mu}$. Then by (2.2) and (F4)

$$(4.7) ||u_0||^2 - \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} |u_0(\cdot + y)|^2 dx = \frac{1}{t_y^2} \int_{\mathbb{R}^N} f(x, t_y u_0) t_y u_0 dx$$
$$\geq 2 \int_{\mathbb{R}^N} \frac{F(x, t_y u_0)}{t_y^2} dx,$$

where the last integral tends to ∞ as $t_y \to \infty$. Therefore (t_{y_n}) is bounded if $|y_n| \to \infty$ and we may assume that $t_{y_n} \to t_0 \ge 0$. Observe that by Lemma 3.1

$$\liminf_{n\to\infty} \left(\frac{1}{2}t_{y_n}^2 ||u_0||^2\right) \ge \liminf_{n\to\infty} \mathcal{J}_{\mu}(t_{y_n}u_0(\cdot + y_n)) \ge c_{\mu} > 0,$$

hence $t_0 > 0$. Again by Lemma 3.1 we get

$$\mathcal{J}'_{0}(t_{y_{n}}u_{0})(u_{0}) = \mathcal{J}'_{0}(t_{y_{n}}u_{0}(\cdot + y_{n}))(u_{0}(\cdot + y_{n}))
= \mathcal{J}'_{\mu}(t_{y_{n}}u_{0}(\cdot + y_{n}))(u_{0}(\cdot + y_{n})) + \int_{\mathbb{R}^{N}} \frac{\mu}{|x|^{2}}|u_{0}(x + y_{n})|^{2}t_{y_{n}} dx
\rightarrow 0.$$

Thus $\mathcal{J}'_0(t_0u_0)(u_0) = 0$ and $t_0u_0 \in \mathcal{N}_0$. Note that by Lemma 3.1 and Lemma 2.6 (condition (A5)) we obtain

$$c_{\mu} \leq \lim_{n \to \infty} \mathcal{J}_{\mu}(t_{y_n} u_0(\cdot + y_n))$$
$$= \mathcal{J}_0(t_0 u_0) = \mathcal{J}_0(u_0) = c_0,$$

which contradicts (4.6).

5 Decomposition of bounded Palais-Smale sequences

In this section we obtain a variant of [14] [Theorem 5.1] for bounded Palais-Smale sequences, hence also for Cerami sequences, of strongly indefinite functionals involving sum of periodic and inverse square potentials.

Proof of Lemma 3.3. Let (u_n) be a bounded Palais-Smale sequence of \mathcal{J}_{μ} at $c \geq 0$. Then (u_n) is bounded in $L^2(\mathbb{R}^N, \frac{1}{|x|^2})$ by (2.4) and we may assume that

$$u_n \rightarrow \overline{u}_0 \text{ in } X,$$
 $u_n \rightarrow \overline{u}_0 \text{ in } L^2(\mathbb{R}^N, \frac{1}{|x|^2}),$
 $u_n \rightarrow \overline{u}_0 \text{ in } L^2_{loc}(\mathbb{R}^N),$
 $u_n \rightarrow \overline{u}_0 \text{ a.e. on } \mathbb{R}^N.$

Then $\mathcal{J}'_{\mu}(\overline{u}_0) = 0$. Denote $v_n = u_n - \overline{u}_0$. Then $v_n^+ \rightharpoonup 0$ in X^+ , $v_n^- \rightharpoonup 0$ in X^- , and $v_n \rightharpoonup 0$ in $L^2(\mathbb{R}^N, \frac{1}{|x|^2})$, hence

$$||v_n^+||^2 = ||u_n^+||^2 - ||\overline{u}_0^+||^2 + o(1), ||v_n^-||^2 = ||u_n^-||^2 - ||\overline{u}_0^-||^2 + o(1),$$

(5.2)
$$||v_n||^2 = ||u_n||^2 - ||\overline{u}_0||^2 + o(1),$$

(5.3)
$$\int_{\mathbb{D}^N} \frac{v_n^2}{|x|^2} = \int_{\mathbb{D}^N} \frac{u_n^2}{|x|^2} - \int_{\mathbb{D}^N} \frac{\overline{u}_0^2}{|x|^2} + o(1).$$

Now we prove that

(5.4)
$$\int_{\mathbb{R}^N} F(x, v_n) = \int_{\mathbb{R}^N} F(x, u_n) - \int_{\mathbb{R}^N} F(x, \overline{u}_0) + o(1).$$

In fact, by (F2), (F3), Vitali's theorem implies

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$$\int_{\mathbb{R}^{N}} F(x, u_{n}) - F(x, u_{n} - \overline{u}_{0})$$

$$= -\int_{\mathbb{R}^{N}} \int_{0}^{1} \frac{d}{d\theta} F(x, u_{n} - \theta \overline{u}_{0}) = \int_{\mathbb{R}^{N}} \int_{0}^{1} f(x, u_{n} - \theta \overline{u}_{0}) \overline{u}_{0}$$

$$\rightarrow \int_{\mathbb{R}^{N}} \int_{0}^{1} f(x, \overline{u}_{0} - \theta \overline{u}_{0}) \overline{u}_{0} = \int_{\mathbb{R}^{N}} \int_{0}^{1} \frac{d}{d\theta} F(x, \overline{u}_{0} - \theta \overline{u}_{0}) = \int_{\mathbb{R}^{N}} F(x, \overline{u}_{0}).$$

Therefore (5.1), (5.3) and (5.4) give that

(5.5)
$$\mathcal{J}_{\mu}(v_n) = \mathcal{J}_{\mu}(u_n) - \mathcal{J}_{\mu}(\overline{u}_0) + o(1).$$

Now we distinguish two cases.

Case 1.
$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^2 dx = 0.$$

In view of Lion's lemma, we have $v_n \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. Since the orthogonal projection of X on X^+ is continuous in the L^t -norm [30], then $v_n^+ \to 0$ and $v_n^- \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. Moreover using $\mathcal{J}'_{\mu}(u_n) = o(1)$ and $\mathcal{J}'_{\mu}(\overline{u_0}) = 0$ we obtain

$$o(1) = \mathcal{J}'_{\mu}(u_{n})v_{n}^{+}$$

$$= \int_{\mathbb{R}^{N}} \nabla u_{n} \nabla v_{n}^{+} + V(x)u_{n}v_{n}^{+} dx - \mu \int_{\mathbb{R}^{N}} \frac{u_{n}v_{n}^{+}}{|x|^{2}} dx - \int_{\mathbb{R}^{N}} f(x, u_{n})v_{n}^{+} dx$$

$$= ||v_{n}^{+}||^{2} - \mu \int_{\mathbb{R}^{N}} \frac{v_{n}v_{n}^{+}}{|x|^{2}} + \mathcal{J}'_{\mu}(\overline{u}_{0})v_{n}^{+} + \int_{\mathbb{R}^{N}} f(x, \overline{u}_{0})v_{n}^{+} - \int_{\mathbb{R}^{N}} f(x, u_{n})v_{n}^{+}$$

$$= ||v_{n}^{+}||^{2} - \mu \int_{\mathbb{R}^{N}} \frac{v_{n}v_{n}^{+}}{|x|^{2}} + \int_{\mathbb{R}^{N}} f(x, \overline{u}_{0})v_{n}^{+} - \int_{\mathbb{R}^{N}} f(x, u_{n})v_{n}^{+},$$

which combining with (2.4) implies that

$$(5.6) \qquad \left(1 - \frac{4\mu}{(N-2)^2}\right) \|v_n^+\|^2 \le \mu \int_{\mathbb{R}^N} \frac{v_n^+ v_n^-}{|x|^2} + \int_{\mathbb{R}^N} f(x, u_n) v_n^+ - \int_{\mathbb{R}^N} f(x, \overline{u}_0) v_n^+ + o(1).$$

Similarly, we have

$$o(1) = \mathcal{J}'_{\mu}(u_n)v_n^-$$

$$= -\|v_n^-\|^2 - \mu \int_{\mathbb{D}^N} \frac{v_n v_n^-}{|x|^2} dx + \int_{\mathbb{D}^N} f(x, \overline{u}_0) v_n^- dx - \int_{\mathbb{D}^N} f(x, u_n) v_n^- dx$$

and

$$(5.7) ||v_n^-||^2 \le -\mu \int_{\mathbb{R}^N} \frac{v_n^+ v_n^-}{|x|^2} dx + \int_{\mathbb{R}^N} f(x, \overline{u}_0) v_n^- dx - \int_{\mathbb{R}^N} f(x, u_n) v_n^- dx + o(1).$$

Hence

(5.8)
$$\left(1 - \frac{4\mu}{(N-2)^2}\right) \|v_n\|^2 \leq \int_{\mathbb{R}^N} f(x, \overline{u}_0) v_n^- - \int_{\mathbb{R}^N} f(x, u_n) v_n^- dx + \int_{\mathbb{R}^N} f(x, u_n) v_n^+ dx - \int_{\mathbb{R}^N} f(x, \overline{u}_0) v_n^+ dx + o(1).$$

Since $v_n^+ \to 0$ and $v_n^- \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$, then $v_n \to 0$ in X and we end the proof with k = 0.

Case 2. There is a sequence $(y_n) \subset \mathbb{Z}^N$ such that

(5.9)
$$\liminf_{n \to \infty} \int_{B(y_n, \rho)} |v_n|^2 dx > 0$$

for some $\rho > 1$. Passing to a subsequence we may assume that $|y_n| \to +\infty$. Let $\widehat{u}_n = u_n(x+y_n)$ and note that by (5.9) we find $\overline{u}_1 \neq 0$ such that up to a subsequence

$$\widehat{u}_n \rightarrow \overline{u}_1 \text{ in } X,$$
 $\widehat{u}_n \rightarrow \overline{u}_1 \text{ in } L^2_{loc}(\mathbb{R}^N),$
 $\widehat{u}_n \rightarrow \overline{u}_1 \text{ a.e. on } \mathbb{R}^N.$

In view of the Hölder inequality and Lemma 3.1 we get for any $\phi \in X$

$$\int_{\mathbb{R}^N} \frac{1}{|x|^2} u_n(x) \phi(x - y_n) \, dx \to 0 \text{ as } n \to \infty.$$

Then

$$o(1) = \mathcal{J}'_{\mu}(u_n)(\phi(x - y_n))$$

$$= \int_{\mathbb{R}^N} \nabla u_n \nabla \phi(x - y_n) + V(x)u_n \phi(x - y_n) dx - \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u_n \phi(x - y_n) dx$$

$$- \int_{\mathbb{R}^N} f(x, u_n) \phi(x - y_n) dx$$

$$= \int_{\mathbb{R}^N} \nabla \widehat{u}_n \nabla \phi + V(x) \widehat{u}_n \phi dx - \int_{\mathbb{R}^N} f(x, \widehat{u}_n) \phi dx + o(1)$$

$$= \mathcal{J}'_{0}(\widehat{u}_n)(\phi) + o(1)$$

which implies that $\mathcal{J}'_0(\overline{u}_1)(\phi) = 0$ and \overline{u}_1 is a critical point of \mathcal{J}_0 . Now denote

$$z_n := u_n - \overline{u}_0 - \overline{u}_1(\cdot - y_n)$$

and since $\overline{u}_1(\cdot - y_n) \rightharpoonup 0$ in X then

$$(5.10) \quad \|z_n^+\|^2 = \|u_n^+\|^2 - \|\overline{u}_0^+\|^2 - \|\overline{u}_1^+\|^2 + o(1), \ \|z_n^-\|^2 = \|u_n^-\|^2 - \|\overline{u}_0^-\|^2 - \|\overline{u}_1^-\|^2 + o(1),$$

$$(5.11) \quad ||z_n||^2 = ||u_n||^2 - ||\overline{u}_0||^2 - ||\overline{u}_1||^2 + o(1).$$

In view of Lemma 3.1

(5.12)
$$\int_{\mathbb{R}^N} \frac{z_n^2}{|x|^2} dx = \int_{\mathbb{R}^N} \frac{(u_n - \overline{u}_0)^2}{|x|^2} dx - \int_{\mathbb{R}^N} \frac{\overline{u}_1^2(x - y_n)}{|x|^2} dx + o(1)$$
$$= \int_{\mathbb{R}^N} \frac{u_n}{|x|^2} dx - \int_{\mathbb{R}^N} \frac{\overline{u}_0^2}{|x|^2} dx + o(1).$$

Observe that $z_n(x+y_n) \to \overline{u}_1(x)$ a.e. on \mathbb{R}^N and by Vitali's theorem and (5.4) we get

$$(5.13) \int_{\mathbb{R}^{N}} F(x, z_{n}) dx = \int_{\mathbb{R}^{N}} F(x, z_{n}(x + y_{n})) dx$$

$$= \int_{\mathbb{R}^{N}} F(x, u_{n} - \overline{u}_{0}) dx - \int_{\mathbb{R}^{N}} F(x, \overline{u}_{1}) dx + o(1)$$

$$= \int_{\mathbb{R}^{N}} F(x, u_{n}) dx - \int_{\mathbb{R}^{N}} F(x, \overline{u}_{0}) dx - \int_{\mathbb{R}^{N}} F(x, \overline{u}_{1}) dx + o(1).$$

Now by (5.10)-(5.13), we have

(5.14)
$$\mathcal{J}_{\mu}(z_n) = \mathcal{J}_{\mu}(u_n) - \mathcal{J}_{\mu}(\overline{u}_0) - \mathcal{J}_0(\overline{u}_1) + o(1)$$

and we take $x_n^1 := y_n$. Now we replace v_n by z_n and repeat the above arguments in Case 1 and Case 2, that is if

(5.15)
$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |z_n|^2 dx = 0,$$

then $z_n \to 0$ in X and in view of (5.10) and (5.14) we take k = 1. Otherwise as in Case 2 we find $(y_n) \subset \mathbb{Z}^N$ such that (5.9) holds for (z_n) . Then passing to a subsequence $|y_n| \to \infty$ and $|y_n - x_n^1| \to \infty$ as $n \to \infty$. Similarly as above $\widehat{u}_n = u_n(x + y_n)$ and we find $\overline{u}_2 \neq 0$ such that up to a subsequence

$$\widehat{u}_n \rightarrow \overline{u}_2 \text{ in } X,$$
 $\widehat{u}_n \rightarrow \overline{u}_2 \text{ in } L^2_{loc}(\mathbb{R}^N),$
 $\widehat{u}_n \rightarrow \overline{u}_2 \text{ a.e. on } \mathbb{R}^N,$

 \overline{u}_2 is a critical point of \mathcal{J}_0 . Now denote new $z_n := u_n - \overline{u}_0 - \overline{u}_1(\cdot - x_n^1) - \overline{u}_2(\cdot - y_n)$ and similarly to (5.10), (5.11) and (5.14) we obtain

$$\begin{aligned} \|z_{n}^{+}\|^{2} &= \|u_{n}^{+}\|^{2} - \|\overline{u}_{0}^{+}\|^{2} - \|\overline{u}_{1}^{+}\|^{2} - \|\overline{u}_{2}^{+}\|^{2} + o(1), \\ \|z_{n}^{-}\|^{2} &= \|u_{n}^{-}\|^{2} - \|\overline{u}_{0}^{-}\|^{2} - \|\overline{u}_{1}^{-}\|^{2} - \|\overline{u}_{2}^{-}\|^{2} + o(1), \\ \|z_{n}\|^{2} &= \|u_{n}\|^{2} - \|\overline{u}_{0}\|^{2} - \|\overline{u}_{1}\|^{2} - \|\overline{u}_{2}\|^{2} + o(1), \\ \mathcal{J}_{\mu}(\overline{z}_{n}) &= \mathcal{J}_{\mu}(u_{n}) - \mathcal{J}_{\mu}(\overline{u}_{0}) - \mathcal{J}_{0}(\overline{u}_{1}) - \mathcal{J}_{0}(\overline{u}_{2}) + o(1), \end{aligned}$$

and $x_n^2 := y_n$. Again we repeat the above arguments in Case 1 and Case 2 and the iterations must stop after finite steps, since there is a constant $\rho_0 > 0$ such that

(5.16)
$$||u|| \ge \rho_0 \text{ for any } u \ne 0 \text{ such that } \mathcal{J}_0'(u) = 0.$$

Indeed $\mathcal{J}'_0(u)(u^+) = 0$, (F2) and (F3) imply that for any $\varepsilon > 0$ there is a constant $C_1 > 0$ such that

$$||u^+||^2 \le \int_{\mathbb{R}^N} |f(x,u)u^+| dx \le \varepsilon ||u^+|| ||u|| + C_1 ||u^+|| ||u||^{p-1}.$$

Similarly by $\mathcal{J}_0'(u)(u^-) = 0$ we get a constant $C_2 > 0$ such that

$$||u^-||^2 \le \int_{\mathbb{R}^N} |f(x,u)u^-| dx \le \varepsilon ||u^-|| ||u|| + C_2 ||u^-|| ||u||^{p-1}.$$

Hence

$$||u||^2 \le 2\varepsilon ||u||^2 + 2\max\{C_1, C_2\} ||u||^p$$

and (5.16) is satisfied, which completes the proof.

Acknowledgements. The first author was supported by the National Natural Science Foundation of China (Grant Nos.11271299, 11001221) and the Fundamental Research Funds for the Central Universities (Grant No. 3102015ZY069). The second author was partially supported by the NCN Grant no. 2014/15/D/ST1/03638 and he would like to thank the members of the Department of Applied Mathematics of the Northwestern Polytechnical University in Xi'an, where part of this work has been done, for their invitation and hospitality.

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