AN INTEGRAL FORMULA FOR AFFINE CONNECTIONS

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ABSTRACT. In this article, we introduce a 2-parameter family of affine connections and derive the Ricci curvature. We first establish an integral Bochner technique. On one hand, this technique yields a new proof to our recent work in [9] for substatic manifolds. On the other hand, this technique leads to various geometric inequalities and eigenvalue estimates under a much more general Ricci curvature conditions. The new Ricci curvature condition interpolates between static Ricci tensor and 1-Bakry-Émery Ricci, and also includes the conformal Ricci as an intermediate case.

1. INTRODUCTION

The classical Reilly formula is actually an integral Bochner formula for gradient vector fields on manifolds with boundary in references. It has been proven to be a quite useful tool in differential geometry.

Motivated by a work of Qiu and the second author [12], we have established a generalized Reilly type formula in previous work [9]. Such a generalization enabled us to prove a Heintze-Karcher-Ros-Brendle type inequality under a sub-static condition. Such kind of inequality, which could lead to an Alexandrov type rigidity theorem, has been proved first by Brendle [2]. See also recent work by Wang-Wang [17]. Moreover, the general formula has been used to prove several geometric inequalities in [9] and also applied by Chen-Wang-Wang-Yau [4] to prove the stability of quasi-local energy with respect to a static spacetime.

The formula was proved via very basic integration-by-parts with respect to the Levi-Civita connection, although the computation is complicated and tedious. The key point is that we introduced a "weight" function V, which was motivated by Brendle and Brende-Hung-Wang [2, 3].

In this article, we adapt a new point of view to recover the formula in [9]. We find that the formula in [9] is indeed an integral Bochner formula for some special vector fields with respect to a special *torsion-free affine* connection instead of the Levi-Civita connection. Moreover, this turns out to be a general phenomenon that a wide class of *torsion-free affine* connections give rise to a class of Reilly type formulas.

Let $(M^n, \bar{g})$ be an n-dimensional smooth Riemannian manifold and $\bar{\nabla}$ be the Levi-Civita connection of $\bar{g}$. Let $V = e^u$ be a positive smooth function on M, where u is a smooth function on M. We call $(M, \bar{g}, V)$ a Riemannian triple.

For $\alpha, \gamma \in \mathbb{R}$, we define a 2-parameter family of affine connections: given two vector fields X, Y on M, let

$$(1.1) \qquad D_X^{\alpha,\gamma} Y = \bar{\nabla}_X Y + \alpha du(X) Y + \alpha du(Y) X + \gamma \bar{g}(X, Y) \bar{\nabla} u.$$

For simplicity, we often omit the superscript α, γ when it is clear in the context. The Ricci curvature under $D^{\alpha,\gamma}$ is (see Proposition 2.3 below)

$$Ric^D := Ric - [(n-1)\alpha + \gamma] \bar{\nabla}^2 u + [(n-1)\alpha^2 - \gamma^2] du \otimes du$$
$$(1.2) \qquad + [\gamma \bar{\Delta} u + (\gamma^2 + (n-1)\alpha\gamma) |\bar{\nabla} u|^2] \bar{g}.$$

There are two trivial cases among all $D^{\alpha,\gamma}$. One is the Levi-Civita connection for $\bar{g}$ when $\alpha = \gamma = 0$, while the other is the Levi-Civita connection for the conformal metric $e^{2\alpha u} \bar{g}$ when $\alpha = -\gamma$. For other choices of α and γ, $D^{\alpha,\gamma}$ may not be a Levi-Civita connection for any Riemannian metric.

For the case $\alpha = 0, \gamma = 1$, one sees from (1.2) that

$$(1.3) \qquad Ric^{D^{0,1}} = Ric - \frac{\bar{\nabla}^2 V}{V} + \frac{\bar{\Delta} V}{V} \bar{g},$$

where Ric is the usual Ricci curvature for $\bar{g}$. We call $Ric^{D^{0,1}}$ *static Ricci tensor*. A Riemannian triple $(M, \bar{g}, V)$ satisfying $V Ric^{D^{0,1}} = 0$ is referred to a static manifold in the literature, see e.g. [5].

For the case $\alpha = \frac{1}{n-1}, \gamma = 0$, one sees from (1.2) that

$$(1.4) \qquad Ric^{D^{\frac{1}{n-1},0}} = Ric - \bar{\nabla}^2 u + \frac{1}{n-1} du \otimes du.$$

This is in fact the 1-*Bakry-Émery Ricci tensor* in the literature which was introduced by Bakry and Émery [1]. The fact that this affine connection gives rise to the 1-Bakry-Émery Ricci tensor has also been observed by Wylie-Yeroshkin [19] in their recent studies of manifolds with density.

The main result of this paper is the following Reilly type integral formula.

Theorem 1.1. *Let $(M^n, \bar{g}, V = e^u)$ be an n-dimensional Riemannian triple and $\alpha, \gamma \in \mathbb{R}$. Let $D = D^{\alpha,\gamma}$ be the affine connection defined as in (1.1) and $\tau = (n+1)\alpha + \gamma$. Let ϕ be a smooth function on a bounded domain $\Omega \subset M$ with smooth boundary Σ. Then the following integral formula holds:*

$$(1.5)$$
$$\int_\Omega V^\tau \left[|\bar{\Delta}^D \phi|^2 - |\bar{\nabla}^{2,D} \phi|_{\bar{g}}^2 \right] - V^\tau Ric^D(\bar{\nabla}^D \phi, \bar{\nabla}^D \phi) d\Omega$$
$$= \int_\Sigma V^\tau \left[H^D \langle \bar{\nabla}^D \phi, \nu \rangle^2 + (h - \gamma u_\nu g)(\nabla^D \phi, \nabla^D \phi) - 2 V^{-\gamma} \langle \nabla^D \phi, \nabla^D (V^\gamma \phi_\nu) \rangle \right] dA.$$

The notations $\bar{\nabla}^D$, $\bar{\nabla}^{2,D}$ and $\bar{\Delta}^D$ play the role of gradient, Hessian and Laplacian with respect to D, the exact definition will be given in Definition 2.4. $H^D := H + (n-1)\alpha u_\nu$ is the affine mean curvature, where H is the usual mean curvature.

Formula (1.5) reduces to Reilly's original formula for $\bar{g}$ in the case $\alpha = \gamma = 0$ and for $e^{2\alpha u} \bar{g}$ in the case $\alpha = -\gamma$. Moreover, when $\alpha = 0, \gamma = 1$, it reduces to the following

Theorem 1.2. *Let $(M^n, \bar{g}, V = e^u)$ be an n-dimensional Riemannian triple. Let ϕ be a smooth function on a bounded domain $\Omega \subset M$ with smooth boundary Σ. We have*

$$(1.6) \qquad \int_\Omega V^3 \Big[(\bar{\Delta}\phi + 2\bar{\nabla}u\bar{\nabla}\phi)^2 - |\bar{\nabla}_i\bar{\nabla}_j\phi + \bar{\nabla}_i u\bar{\nabla}\phi_j + \bar{\nabla}_j u\bar{\nabla}_i\phi|^2 \Big] d\Omega$$

$$= \int_\Sigma V^3 \Big(H\phi_\nu^2 + (h - u_\nu g)(\nabla\phi, \nabla\phi) - 2V^{-1}\langle \nabla\phi, \nabla(V\phi_\nu) \rangle \Big) dA$$

$$+ \int_\Omega V^3 (Ric - \frac{\bar{\nabla}^2 V}{V} + \frac{\bar{\Delta}V}{V}\bar{g})(\bar{\nabla}\phi, \bar{\nabla}\phi) d\Omega.$$

If we let $\phi = \frac{f}{V}$ in (1.6), then we recover Theorem 1.1 in [9] by a different method.

Let us illustrate the basic idea of the proof of Theorem 1.1. It is well known that a local Bochner formula holds for general vector fields under an affine connection. Since the connection is in general not metric compatible, we have only a divergent form instead of the Laplacian of some function in the Bochner identity, see Proposition 3.2. Nevertheless, we are able integrate this local Bochner formula to get an integral formula. To achieve an effective Reilly type formula, there are two innovative ingredients with this method. First, we choose a right volume form, which is a "weight", to get the divergent-free property. Second, we choose a suitable vector field X which satisfies DX is symmetric. It turns out that we choose $X = \bar{\nabla}^D\phi$ and the volume form $V^\tau d\Omega$, see Lemmata 3.3 and 3.4.

With the integral formula in Theorem 1.1 at hand, we are able to prove Heintze-Karcher type, Minkowski type, and Lichnerowicz type inequalities.

Theorem 1.3. *Let $(M^n, \bar{g}, V = e^u)$ be an n-dimensional Riemannian triple and $\alpha, \gamma \in \mathbb{R}$. Let $D = D^{\alpha,\gamma}$ be the affine connection defined as in (1.1) and $\tau = (n+1)\alpha + \gamma$. Then we have the following results.*

    (i) **Heintze-Karcher type inequality:** *if $Ric^D \geq 0$ and $H^D > 0$, then*

$$(1.7) \qquad n\int_\Omega V^\tau d\Omega \leq (n-1)\int_\Sigma \frac{V^\tau}{H^D} dA.$$

    Equality in (1.7) holds only if Σ is umbilical.
    (ii) **Minkowski inequality:** *If $Ric^D \geq 0$ and $h - \gamma u_\nu g > 0$, then*

$$(1.8) \qquad \left(\int_\Sigma V^{\tau-\alpha} dA \right)^2 \geq \frac{n}{n-1} \int_\Omega V^\tau d\Omega \int_\Sigma H^D V^{\tau-2\alpha} dA,$$

    Equality in (1.8) holds only if Σ is umbilical.
    (iii) **Lichnerowicz inequality:** *If $Ric^D \geq (n-1)V^{\alpha-\gamma}\bar{g}$ and*
        a) $\Sigma = \emptyset$, *then* $\lambda_1(\bar{\Delta}^D) \geq n$;
        b) $\Sigma \neq \emptyset$ *and* Σ *satisfies* $H^D \geq 0$, *then* $\lambda_1^{Dir}(\bar{\Delta}^D) \geq n$;
        c) $\Sigma \neq \emptyset$ *and* $h - \gamma u_\nu g \geq 0$, *then* $\lambda_1^{Neu}(\bar{\Delta}^D) \geq n$.

Here $\lambda_1, \lambda_1^{Dir}$ and λ_1^{Neu} indicate the closed, the Dirichlet and the Neumann first (nonzero) eigenvalue of the affine Laplacian $\bar{\Delta}^D$, i.e., there exists some non-trivial ϕ such that $\bar{\Delta}^D\phi = -\lambda_1\phi$ with Dirichlet boundary condition $\phi = 0$ or Neumann boundary condition $\phi_\nu = 0$.

Remark 1.4.

(i) *Theorem 1.3 reduces to Heintze-Karcher, Minkowski, and Lichnerowicz in-equalities for $\bar{g}$ in the case $\alpha = \gamma = 0$ or $e^{2\alpha u}\bar{g}$ in the case $\alpha = -\gamma$. See Section 2.1 for an overview.*

(ii) *In the case $\alpha = 0, \gamma = 1$, the Heintze-Karcher type inequalities were first proved by Brendle [2], then by Wang-Wang [17] for non-homologous static manifolds without warped product structure by using the same method and by the authors for general cases using Reilly type formulas in [9]. The Minkowski type inequalities have been proved in our previous work [9].*

(iii) *Theorem 1.3 gives new geometric inequalities under the condition of non-negative 1-Bakry-Émery Ricci, which is the case $\alpha = \frac{1}{n-1}, \gamma = 0$. To illustrate the idea, we only list the example of the Heintze-Karcher type inequality and the others hold true similarly.*

Corollary. Let $(M^n, \bar{g}, e^u d\Omega)$ be a smooth weighted Riemannian manifold and Ω be a bounded domain in M. If the 1-Bakry-Émery Ricci is nonnegative, namely,

$$Ric - \bar{\nabla}^2 u + \frac{1}{n-1} du \otimes du \geq 0,$$

and the weighted mean curvature $H + \langle \bar{\nabla}u, \nu \rangle > 0$, then the following inequality holds:

$$(1.9) \qquad n \int_\Omega e^{\tau u} d\Omega \leq (n-1) \int_\Sigma \frac{e^{\tau u}}{H + \langle \bar{\nabla}u, \nu \rangle} dA,$$

where $\tau = \frac{n+1}{n-1}$. Moreover, if equality in (1.7) holds, then Σ is umbilical.

We remark that the weight in (1.9) is $e^{\tau u}$ instead of e^u. The new weight volume form $e^{\tau u} d\Omega$ has a property that it is parallel under the affine connection $D^{\alpha,\gamma}$. In the special case of 1-Bakry-Émery Ricci curvature, this parallel property has been observed by Wylie-Yeroshkin [19] recently.

In particular, for a Riemannian triple (M, g, V) whose static Ricci tensor has a positive lower bound, we get the first eigenvalue estimate for the operator $V\bar{\Delta} - \bar{\Delta}V\cdot$.

Corollary 1.5. *Let $(M^n, \bar{g}, V)$ be an n-dimensional Riemannian triple. Let Ω be a bounded domain in M with smooth boundary Σ. Assume the static Ricci tensor satisfies*

$$V Ric - \bar{\nabla}^2 V + \bar{\Delta}V \bar{g} \geq (n-1)\bar{g}.$$

Then we have
a) *if $\Sigma = \emptyset$, then $\lambda_1(V\bar{\Delta} - \bar{\Delta}V\cdot) \geq n$;*
b) *if $\Sigma \neq \emptyset$ and Σ satisfies $H^D \geq 0$, then $\lambda_1^{Dir}(V\bar{\Delta} - \bar{\Delta}V\cdot) \geq n$;*
c) *if $\Sigma \neq \emptyset$ and $h - \gamma u_\nu g \geq 0$, then $\lambda_1^{Neu}(V\bar{\Delta} - \bar{\Delta}V\cdot) \geq n$.*

The rest of the paper is organized as follows. In section 2, we recall classical results, introduce our affine connections, fix the notations, and give the Ricci curvature under affine connections. In section 3, we establish the Bochner formula and prove the main theorem, Theorem 1.1. In section 4, we prove the Heintze-Karcher type and the Minkowski type inequalities of Theorem 1.3. In section 5, we prove the Poincare type and the Lichnerowicz type inequalities. In the **Appendix**, we prove Proposition 2.3.

2. Preliminaries and notations

2.1. **Classical results.** Let us first recall the classical Reilly's formula [13]. For a bounded domain Ω with boundary Σ in an n-dimensional smooth Riemannian manifold $(M, \bar{g})$ and $\phi \in C^\infty(\bar{\Omega})$, the classical Reilly's formula reads as

$$(2.1) \quad \int_\Omega (\bar{\Delta}\phi)^2 - |\bar{\nabla}^2\phi|^2 - Ric(\bar{\nabla}\phi, \bar{\nabla}\phi)d\Omega = \int_\Sigma H\phi_\nu^2 + h(\nabla\phi, \nabla\phi) + 2\phi_\nu\Delta\phi dA.$$

Here and throughout this paper, $\bar{\nabla}$ and $\bar{\Delta}$ denote the gradient and the Laplacian on Ω respectively, ∇ and Δ denote the gradient and the Laplacian on Σ respectively with respect to the induced metric from $\bar{g}$. $d\Omega$ and dA are the Riemannian volume form of $\bar{g}$ and the induced area element from $\bar{g}$ respectively. ν is the normal vector field of Σ and $\phi_\nu = \bar{\nabla}_\nu\phi$ is the normal derivative of ϕ. $h(X, Y) = \langle \bar{\nabla}_X\nu, Y\rangle$ is the classical second fundamental form of Σ and $H = \text{tr}_{\bar{g}}h$ is the usual mean curvature of Σ.

With the Reilly formula, some classical geometric inequalities can be readily proved.

Heintze-Karcher inequality: If $Ric \geq 0$ and Σ is mean convex, i.e., $H > 0$, then

$$(2.2) \qquad \int_\Sigma \frac{1}{H}dA \geq \frac{n}{n-1}Vol(\Omega).$$

Minkowski inequality: If $Ric \geq 0$ and Σ is convex, i.e., $h \geq 0$, then

$$(2.3) \qquad Area(\Sigma)^2 \geq \frac{n}{n-1}Vol(\Omega)\int_\Sigma HdA.$$

Lichnerowicz inequality: If $Ric \geq (n-1)\bar{g}$ and Σ is empty, then

$$(2.4) \qquad \lambda_1(\bar{\Delta}) \geq n.$$

Similar inequalities like (2.2) were first derived by Heintze-Karcher [6] using the classical approach of Jacobian fields from Riemanian geometry. Ros [16] proved the current form of this inequality using Reilly's formula. Inequality (2.3) was first derived by Minkowski [10] in the Euclidean case as a consequence of the famous Brunn-Minkowski theorem in convex geometry. Reilly [15] proved this inequality under the condition of nonnegative Ricci by using his formula. Recently, Wang-Zhang [20] gave an alternative proof of Minkowski inequality (2.3) using ABP method. Inequality (2.4) was proved by Lichnerowicz [7] using the classical Bochner technique.

2.2. **Notations under affine connections.** As in the introduction, a two parameter family of *torsion free* affine connections $D^{\alpha,\gamma}$ is defined on M for $\alpha, \gamma \in \mathbb{R}$:

$$D_X^{\alpha,\gamma}Y = \bar{\nabla}_X Y + \alpha du(X)Y + \alpha du(Y)X + \gamma\bar{g}(X, Y)\bar{\nabla}u.$$

One checks directly that $D^{\alpha,\gamma}$ is torsion-free. For a general affine connection, we adapt the following convention of Ricci curvature.

Definition 2.1. *Given an affine connection D, for any vector fields X, Y, we define the Ricci curvature as*

$$Ric^D(X, Y) = \omega^i\left(R^D(e_i, X)Y\right),$$

where $\{e_i\}$ is a local frame of the tangent bundle, $\{\omega^i\}$ is the dual 1-form of $\{e_i\}$ and the Riemann curvature operator R^D is defined as

$$R^D(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

Remark 2.2. *In the case we have a Riemannian metric $\bar{g}$, the Ricci curvature Ric^D of D can also be interpreted as*

$$Ric^D(X,Y) = \langle R^D(e_i, X)Y, e_i \rangle,$$

where $\{e_i\}$ is an orthonormal frame of the tangent bundle.

By direct computation, we derive the following representation of Ric^D for $D = D^{\alpha,\gamma}$ in terms of Levi-Civita connection $\bar{\nabla}$.

Proposition 2.3. *The Ricci curvature Ric^D of $D = D^{\alpha,\gamma}$ satisfies the following identity:*

$$\begin{aligned}
Ric^D =& Ric - [(n-1)\alpha + \gamma]\bar{\nabla}^2 u + [(n-1)\alpha^2 - \gamma^2]du \otimes du \\
&+ \left\{ \gamma\bar{\Delta}u + \gamma[(n-1)\alpha + \gamma]|\bar{\nabla}u|^2 \right\} \bar{g}.
\end{aligned}$$

We will prove Proposition 2.3 in **Appendix**. As already mentioned in the introduction, the Ricci curvature Ric^D of the new affine connection $D^{\alpha,\gamma}$ not only yields new Ricci curvature tensors, but also recovers some of known examples in the literature, which includes conformal Ricci tensor from conformal geometry, and static Ricci tensor raised from General Relativity and the 1-Bakry-Éemery Ricci tensor from manifolds with density.

Next we explain the notations in (1.5).

Definition 2.4.

(i) *The D-gradient on Ω and Σ are defined respectively by*

$$\bar{\nabla}^D \phi := V^{\gamma-\alpha}\bar{\nabla}\phi, \quad \nabla^D \phi := V^{\gamma-\alpha}\nabla\phi$$

(ii) *The D-Hessian $\bar{\nabla}^{2,D}\phi$ and D-Laplacian $\bar{\Delta}^D\phi$ on Ω are defined respectively by*

$$\begin{aligned}
(2.5) \quad \bar{\nabla}^{2,D}\phi \quad &:= \quad D(V^{\gamma-\alpha}\bar{\nabla}\phi) \\
&= \quad V^{\gamma-\alpha}\left[\bar{\nabla}^2\phi + \gamma du \otimes d\phi + \gamma d\phi \otimes du + \alpha\langle\bar{\nabla}u, \bar{\nabla}\phi\rangle\bar{g}\right],
\end{aligned}$$

and

$$\bar{\Delta}^D\phi := \text{tr}_{\bar{g}}(\bar{\nabla}^{2,D}\phi) = V^{\gamma-\alpha}\left[\bar{\Delta}\phi + (2\gamma + n\alpha)\langle\bar{\nabla}u, \bar{\nabla}\phi\rangle\right].$$

We note that in the case $\alpha = 1, \gamma = -1$, the D-gradient, the D-Hessian and the D-Laplacian are in consistence with the classical ones for conformal metric $e^{2u}\bar{g}$. By virtue of this, we believe Definition 2.4 is natural for $D^{\alpha,\gamma}$.

3. Bochner technique for general affine connections

In this section, we review a Bochner identity for general affine connection and prove Theorem 1.1.

It is well known that under Levi-Civita connection $\bar{\nabla}$, the following Bochner formula holds: for a smooth vector field X on M with the property that $\bar{\nabla}X$ is symmetric,

$$\bar{\Delta}\frac{1}{2}|X|^2 = |\bar{\nabla}X|^2 + \bar{\nabla}_X(\text{div}_{\bar{g}}X) + Ric(X,X),$$

see e.g. Petersen [11] Proposition 33, page 207.

Under an affine connection, the following Ricci identity holds.

Lemma 3.1. (Ricci identity) *Under local coordinates* $\{\partial_i\}$, *for any smooth vector field* X, *we have*

$$(3.1) \qquad D_i D_j X^i = D_j D_i X^i + R^D_{jk} X^k.$$

Proof. A vector filed X can be viewed as a $(1,0)$-tensor field and we have

$$D^2 X(V,W) = D_W(D_V X) - D_{D_W V} X,$$
$$(3.2) \qquad D^2 X(W,V) = D_V(D_W X) - D_{D_V W} X.$$

It follows that

$$(3.3) \qquad D^2 X(W,V) - D^2 X(V,W) = R^D(V,W)X.$$

Let $\{e_i\}$ be an local frame of the tangent bundle, $\{\omega^i\}$ is the dual 1-form of $\{e_i\}$. Then we have from (3.3)

$$(3.4) \qquad \omega^i \left(D^2 X(e_j, e_i) \right) - \omega^i \left(D^2 X(e_i, e_j) \right) = Ric^D(e_j, X).$$

We finish the proof. $\qquad\qquad\square$

Proposition 3.2. (Bochner formula) *Let* D *be an affine connection on* M *and* Ric^D *be the Ricci curvature of* D. *Let* X *be a smooth vector field on* M. *Then we have*

$$(3.5) \qquad \mathrm{div}^D(D_X X) = (DX)^t \cdot DX + d(\mathrm{div}^D X)(X) + Ric^D(X,X).$$

where div^D *is the divergence operator w.r.t.* D, $\mathrm{div}^D Y = D_i Y^i$ *for a vector field* Y, $(DX)^t$ *is the transpose of* DX.

Under local coordinates $\{\partial_i\}$, (3.5) reads as

$$(3.6) \qquad D_i(X^j D_j X^i) = D_i X^j D_j X^i + X^j D_j D_i X^i + R^D_{ji} X^i X^j.$$

Moreover,

$$(3.7) \qquad D_i(X^j D_j X^i - X^i D_j X^j) = D_i X^j D_j X^i - (D_j X^j)^2 + R^D_{ji} X^i X^j.$$

Proof. Using tensor calculus, we have

$$(3.8) \qquad D_i(X^j D_j X^i) = D_i X^j D_j X^i + X^j D_i D_j X^i.$$

Combining (3.8) with the Ricci identity (3.1) we get (3.6). We also have by using tensor calculus that

$$X^j D_j D_i X^i = D_j(X^j D_i X^i) - (D_j X^j)(D_i X^i)$$
$$(3.9) \qquad\qquad = D_i(X^i D_j X^j) - (D_i X^i)^2.$$

Inserting (3.9) into (3.6), we get (3.7). $\qquad\qquad\square$

Our aim is to derive an integral formula from the local Bochner formula (3.7). From now on, let $(M^n, \bar{g}, V)$ be an n-dimensional Riemannian triple and $\alpha, \gamma \in \mathbb{R}$. Let $D = D^{\alpha,\gamma}$ be the affine connection defined as in (1.1). Note that Ric^D is symmetric. In order to obtain a useful integral formula, we need the following two important ingredients:

(i) DX is symmetric, i.e.,

$$(3.10) \qquad D_i X^j = D_j X^i;$$

(ii) The left hand side of (3.7) needs to be a "divergent form" with respect to some choice of volume form.

In the following two lemmata, we will find an appropriate vector field X and also a compatible volume form.

Lemma 3.3. *Let ϕ be a smooth function on M^n. Let*

$$(3.11) \qquad\qquad X = \bar{\nabla}^D \phi = V^{\gamma - \alpha} \bar{\nabla} \phi.$$

Then

$$DX = V^{\gamma - \alpha} \left[\bar{\nabla}^2 \phi + \gamma du \otimes d\phi + \gamma d\phi \otimes du + \alpha \bar{g} \left(\bar{\nabla} u, \bar{\nabla} \phi \right) \bar{g} \right] = \bar{\nabla}^{2,D} \phi$$

is symmetric. Moreover,

$$\mathrm{div}^D X := D_i X^i = V^{\gamma - \alpha} \left(\bar{\Delta} \phi + (2\gamma + n\alpha) \langle \bar{\nabla} u, \bar{\nabla} \phi \rangle \right) = \bar{\Delta}^D \phi.$$

Proof. Recall under local coordinates,

$$(3.12) \qquad\qquad D_i X^j = \langle D_i X, \bar{g}^{jk} \partial_k \rangle.$$

Under normal coordinates, we have

$$
\begin{aligned}
D_i X_j &= \langle D_i (V^{\gamma - \alpha} \bar{\nabla} \phi), \partial_j \rangle \\
&= \bar{\nabla}_i (V^{\gamma - \alpha} \bar{\nabla}_j \phi) + \alpha V^{\gamma - \alpha} u_i \phi_j + \alpha V^{\gamma - \alpha} \bar{\nabla} u \bar{\nabla} \phi \delta_{ij} + \gamma V^{\gamma - \alpha} \phi_i u_j \\
(3.13) \qquad &= V^{\gamma - \alpha} \left(\bar{\nabla}_i \bar{\nabla}_j \phi + \gamma u_i \phi_j + \gamma \phi_i u_j + \alpha \langle \bar{\nabla} u, \bar{\nabla} \phi \rangle \delta_{ij} \right).
\end{aligned}
$$

Clearly, DX is symmetric. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.4. *Let W be any smooth vector field on M. Then*

$$(3.14) \qquad\qquad V^\tau D_i W^i = \bar{\nabla}_i (V^\tau W^i)$$

where $\tau = (n+1)\alpha + \gamma$, is a divergent form with respect to the Riemannian volume form $d\Omega$. ($d\Omega$ denotes the volume form induced by Riemannian metric $\bar{g}$ throughout this paper)

Proof. By definition of D, we have

$$
\begin{aligned}
D_i W^i &= \bar{\nabla}_i W^i + \alpha u_i W^i + \alpha u_k W^k \delta_i^i + \gamma W^i u_i \\
(3.15) \qquad &= \bar{\nabla}_i W^i + [(n+1)\alpha + \gamma] u_i W^i.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V^\tau D_i W^i &= V^\tau \bar{\nabla}_i W^i + [(n+1)\alpha + \gamma] V^{\tau - 1} V_i W^i \\
&= V^\tau \bar{\nabla}_i W^i + \tau V^{\tau - 1} V_i W^i \\
(3.16) \qquad &= \bar{\nabla}_i (V^\tau W^i).
\end{aligned}
$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$$

As an immediate corollary, we can show that the volume form $V^\tau d\Omega$ is parallel under the new affine connection $D^{\alpha,\gamma}$. We thank the referee for pointing out this fact to us. The special case of $\alpha = \frac{1}{n-1}$ and $\gamma = 0$ was proved in [19].

Corollary 3.5. *We have $D_X(V^\tau d\Omega) = 0$ for any smooth vector field X.*

Proof. Given an arbitrary vector field W with compact support in Ω. From Lemma 3.4,

$$\int_\Omega W^i D_i(V^\tau d\Omega) = -\int_\Omega V^\tau D_i W^i d\Omega = -\int_\Omega \bar{\nabla}_i(V^\tau W^i)d\Omega = 0.$$

Since W is arbitrary, we conclude that $D_i(V^\tau d\Omega) = 0$ for any i. $\qquad\square$

Choosing $X = \bar{\nabla}^D\phi$ in (3.7), we obtain

$$D_i\Big((\bar{\nabla}^D\phi)^j D_j((\bar{\nabla}^D\phi)^i) - (\bar{\nabla}^D\phi)^i D_j((\bar{\nabla}^D\phi)^j)\Big)$$

(3.17)
$$= |\bar{\nabla}^{2,D}\phi|_{\bar{g}}^2 - |\bar{\Delta}^D\phi|^2 + Ric^D(\bar{\nabla}^D\phi, \bar{\nabla}^D\phi).$$

Multiplying (3.17) with V^τ and integrating over a bounded domain $\Omega \subset M$, we have

$$\int_\Omega V^\tau D_i\Big((\bar{\nabla}^D\phi)^j D_j((\bar{\nabla}^D\phi)^i) - (\bar{\nabla}^D\phi)^i D_j((\bar{\nabla}^D\phi)^j)\Big)d\Omega$$

(3.18)
$$= \int_\Omega V^\tau \left[|\bar{\nabla}^{2,D}\phi|_{\bar{g}}^2 - |\bar{\Delta}^D\phi|^2\right] + V^\tau Ric^D(\bar{\nabla}^D\phi, \bar{\nabla}^D\phi)d\Omega.$$

Applying (3.14) and the Stokes' theorem on (3.18), we obtain the Reilly type integral formula.

Theorem 3.6 (Theorem 1.1). *Let $(M^n, \bar{g}, V = e^u)$ be an n-dimensional Riemannian triple and $\alpha, \gamma \in \mathbb{R}$. Let $D = D^{\alpha,\gamma}$ be the affine connection defined as in (1.1) and $\tau = (n+1)\alpha + \gamma$. Let ϕ be a smooth function on a bounded domain $\Omega \subset M$ with smooth boundary Σ. Then the following integral formula holds:*

$$\int_\Omega V^\tau \left[\left|\bar{\Delta}^D\phi\right|^2 - \left|\bar{\nabla}^{2,D}\phi\right|_{\bar{g}}^2\right] - V^\tau Ric^D(\bar{\nabla}^D\phi, \bar{\nabla}^D\phi)d\Omega$$
$$= \int_\Sigma V^\tau \left[H^D\langle\bar{\nabla}^D\phi, \nu\rangle^2 + (h - \gamma u_\nu g)(\nabla^D\phi, \nabla^D\phi) - 2V^{-\gamma}\langle\nabla^D\phi, \nabla^D(V^\gamma\phi_\nu)\rangle\right] dA.$$

Proof. Using (3.14), we have

$$\int_\Omega V^\tau D_i\Big((\bar{\nabla}^D\phi)^i D_j((\bar{\nabla}^D\phi)^j) - (\bar{\nabla}^D\phi)^j D_j((\bar{\nabla}^D\phi)^i)\Big)d\Omega$$
$$= \int_\Omega \bar{\nabla}_i\left[V^\tau\Big((\bar{\nabla}^D\phi)^i D_j((\bar{\nabla}^D\phi)^j) - (\bar{\nabla}^D\phi)^j D_j((\bar{\nabla}^D\phi)^i)\Big)\right]d\Omega$$
$$= \int_\Sigma V^\tau\Big(\langle\bar{\nabla}^D\phi, \nu\rangle D_j((\bar{\nabla}^D\phi)^j) - V^{\gamma-\alpha}\phi^j\langle D_j(\bar{\nabla}^D\phi), \nu\rangle\Big)dA.$$

Thus we only need to simplify the boundary term. At any fixed point $P \in \Sigma$, we choose normal coordinates with respect to $\bar{g}$ such that the indices $a = 1, \cdots, n-1$ represents coordinates on Σ and $j = 1, \cdots, n-1, \nu$ for coordinates on Ω. For simplicity, we will not distinguish upper and lower indexes and we denote ϕ_{ij} as the Hessian ϕ with respect to the Levi-Civita connection.

Using (3.13), we obtain

(3.19) $$\langle\bar{\nabla}^D\phi, \nu\rangle D_j((\bar{\nabla}^D\phi)^j) = V^{2(\gamma-\alpha)}\big(\bar{\Delta}\phi\phi_\nu + (2\gamma + n\alpha)\langle\bar{\nabla}u, \nabla\phi\rangle\phi_\nu\big),$$

(3.20) $$V^{\gamma-\alpha}\phi^j\langle D_j((\bar{\nabla}^D\phi)), \nu\rangle = V^{2(\gamma-\alpha)}\big(\phi_{j\nu}\phi_j + \gamma u_\nu\phi_j^2 + (\alpha+\gamma)\langle\bar{\nabla}u, \nabla\phi\rangle\phi_\nu\big).$$

Combining (3.19) and (3.20), we have

(3.21)
$$\langle \bar{\nabla}^D \phi, \nu \rangle D_j((\bar{\nabla}^D \phi)^j) - V^{\gamma-\alpha} \phi^j \langle D_j(\bar{\nabla}^D \phi), \nu \rangle$$
$$= V^{2(\gamma-\alpha)} \left(\phi_\nu \phi_{aa} - \phi_a \phi_{a\nu} + (n-1)\alpha u_\nu \phi_\nu^2 + (\gamma + (n-1)\alpha) u_a \phi_a \phi_\nu - \gamma \phi_a^2 u_\nu \right)$$
$$= V^{2(\gamma-\alpha)} \left(H\phi_\nu^2 + \phi_a h_{ab} \phi_b + \phi_\nu \Delta\phi - \phi_a \nabla_a \phi_\nu \right.$$
$$\left. + (n-1)\alpha u_\nu \phi_\nu^2 + (\gamma + (n-1)\alpha) u_a \phi_a \phi_\nu - \gamma \phi_a^2 u_\nu \right),$$

where we used in the last equality the Gauss-Weigarten formula

$$\phi_{aa} = \Delta\phi + H\phi_\nu, \qquad \phi_{a\nu} = \nabla_a \phi_\nu - h_{ab}\phi_b.$$

Using $H^D = H + (n-1)\alpha u_\nu$ in (3.21), we get

(3.22)
$$\int_\Sigma V^\tau \left(\langle \bar{\nabla}^D \phi, \nu \rangle D_j((\bar{\nabla}^D \phi)^j) - \phi^j \langle D_j(\bar{\nabla}^D \phi), \nu \rangle \right) dA$$
$$= \int_\Sigma V^{\tau+2(\gamma-\alpha)} \left(H^D \phi_\nu^2 + (h - \gamma u_\nu g)(\nabla\phi, \nabla\phi) \right)$$
$$+ \int_\Sigma V^{\tau+2(\gamma-\alpha)} \left(\phi_\nu \Delta\phi - \langle \nabla\phi, \nabla\phi_\nu \rangle + (\gamma + (n-1)\alpha) \langle \nabla u, \nabla\phi \rangle \phi_\nu \right) dA.$$

Integrating by parts for the last line of (3.22) and noting $-[\tau + 2(\gamma - \alpha)] + (\gamma + (n-1)\alpha) = -2\gamma$, we get the assertion. $\qquad\square$

4. Heintz-Karcher type and Minkowski type inequalities

In this section, we will give proofs to the Heintz-Karcher type and Minkowski type inequalities stated in Theorem 1.3.

Proof of Theorem 1.3 (i). Recall that $\bar{\Delta}^D \phi = V^{\gamma-\alpha} \left[\bar{\Delta}\phi + (2\gamma + n\alpha) \bar{g} \left(\bar{\nabla} u, \bar{\nabla}\phi \right) \right]$. We know from the standard elliptic PDE theory that the following Dirichlet boundary value problem

(4.1)
$$\begin{cases} \bar{\Delta}^D \phi &= 1 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \Sigma, \end{cases}$$

admits a unique smooth solution $\phi \in C^\infty(\overline{\Omega})$. We will use the solution ϕ of the Dirichlet problem (4.1) in (1.5). For $\tau = (n+1)\alpha + \gamma$, we have

$$\frac{n-1}{n} \int_\Omega V^\tau = \frac{n-1}{n} \int_\Omega V^\tau (\bar{\Delta}^D \phi)^2$$
$$\geq \int_\Omega V^\tau \left[|\bar{\Delta}^D \phi|^2 - |\bar{\nabla}^{2,D} \phi|^2 \right]$$

(4.2)
$$\geq \int_\Sigma V^\tau H^D \langle \bar{\nabla}^D \phi, \nu \rangle^2,$$

where in the first inequality, we have used the Cauchy-Schwarcz inequality, in the second inequality we have used integral formula (1.5), the nonnegativity of Ric^D, and the Dirichlet boundary condition.

On the other hand, using equation (4.1), divergent structure (3.14), and integration by parts, we have

$$(4.3) \qquad \int_\Omega V^\tau \;=\; \int_\Omega V^\tau \bar{\Delta}^D \phi$$

$$\;=\; \int_\Sigma V^\tau \langle \bar{\nabla}^D \phi, \nu \rangle.$$

Combining (4.2), (4.3) and using Hölder's inequality, we obtain

$$(4.4) \qquad \left(\int_\Omega V^\tau d\Omega \right)^2 \;=\; \left(\int_\Sigma V^\tau \langle \bar{\nabla}^D \phi, \nu \rangle dA \right)^2$$

$$\leq\; \int_\Sigma V^\tau H^D \langle \bar{\nabla}^D \phi, \nu \rangle^2 dA \int_\Sigma \frac{V^\tau}{H^D} dA$$

$$\leq\; \frac{n-1}{n} \int_\Omega V^\tau d\Omega \int_\Sigma \frac{V^\tau}{H^D} dA.$$

The assertion for the inequality follows. If the equality holds, we have

$$(4.5) \qquad \bar{\nabla}^{2,D} \phi = \frac{1}{n} \bar{g}.$$

Restricting (4.5) on Σ, using $\phi = 0$ on Σ and Gauss formula, we conclude that $h_{\alpha\beta} = \lambda g_{\alpha\beta}$ for some smooth function λ, i.e., Σ is umbilic. $\qquad \square$

Proof of Theorem 1.3 (ii). Consider the Neumann boundary value problem

$$(4.6) \qquad \begin{cases} \bar{\Delta}^D \phi \;=\; 1 & \text{in } \Omega, \\ V^\gamma \phi_\nu \;=\; c & \text{on } \Sigma, \end{cases}$$

where $c = \frac{\int_\Omega V^\tau}{\int_\Sigma V^{\tau-\alpha}}$. The existence and uniqueness follows from the Fredholm alternative as in standard elliptic PDE theory. We will apply the solution ϕ of (4.6) to the integral formula (1.5). By using the Cauchy-Schwarz inequality, the equation and boundary condition in (4.6) and the curvature assumptions, we get

$$\frac{n-1}{n} \int_\Omega V^\tau d\Omega \;\geq\; \int_\Sigma V^\tau H^D \langle \bar{\nabla}^D \phi, \nu \rangle^2 dA$$

$$\;=\; \int_\Sigma V^\tau H^D V^{2\gamma-2\alpha} \phi_\nu^2 dA$$

$$\;=\; c^2 \int_\Sigma V^{\tau-2\alpha} H^D dA.$$

Inserting the value of c we get the assertion.

If the equality holds, we have

$$(4.7) \qquad \bar{\nabla}^{2,D} \phi = \frac{1}{n} \bar{g},$$

and

$$(4.8) \qquad (h - \gamma u_\nu g)(\nabla \phi, \nabla \phi) = 0.$$

Since by assumption $(h_{\alpha\beta} - \gamma \frac{V_\nu}{V} g_{\alpha\beta}) > 0$ on Σ, it follows from (4.8) that $\phi = 0$ on Σ. Restricting (4.7) on Σ, we see that Σ is umbilical.

5. Poincaré type and Lichnerowicz type inequalities

Along the same line of the above results, we now prove a Poincare type inequality.

Theorem 5.1. *Let $(M^n, \bar{g}, V = e^u)$ be an n-dimensional Riemannian triple and $\alpha, \gamma \in \mathbb{R}$. Let Ω be a bounded domain in M with smooth boundary Σ. Assume Ric^D of $D = D^{\alpha,\gamma}$ is positive definite. For any $f \in C^\infty(M)$ and $\tau = (n+1)\alpha + \gamma$, if one of the following alternatives holds,*

(i) $\Sigma = \emptyset$ and $\int_\Omega fV^\tau d\Omega = 0$;

(ii) $\Sigma \neq \emptyset$, $f \equiv 0$ on Σ and $H^D \geq 0$;

(iii) $\Sigma \neq \emptyset$, $\int_\Omega fV^\tau d\Omega = 0$ and Σ satisfies $h - \gamma u_\nu g \geq 0$.

Then we have

$$(5.1) \qquad \frac{n}{n-1} \int_\Omega f^2 V^\tau d\Omega \leq \int_\Omega \left\langle (\mathrm{Ric}^D)^{-1} \bar{\nabla} f, \bar{\nabla} f \right\rangle V^\tau d\Omega.$$

Proof. The proof is similar as in [8] while we use Ric^D in this paper. In case (i), we solve PDE

$$(5.2) \qquad \bar{\Delta}^D \phi = f \text{ in } \Omega.$$

In case (ii), we solve the Dirichlet boundary value problem below,

$$(5.3) \qquad \begin{cases} \bar{\Delta}^D \phi &= f & \text{in } \Omega, \\ \phi &= 0 & \text{on } \Sigma. \end{cases}$$

In case (iii), we solve the Neumann boundary value problem

$$(5.4) \qquad \begin{cases} \bar{\Delta}^D \phi &= f & \text{in } \Omega, \\ \phi_\nu &= 0 & \text{on } \Sigma. \end{cases}$$

Problems (5.2) and (5.4) are solvable since $\int_\Omega fV^\tau d\Omega = 0$.

In all these three cases, we apply the integral formula (1.5) to the solutions of the PDEs, i.e. ϕ satisfying $\bar{\Delta}^D \phi = f$. By Cauchy-Schwarz inequality, we have
(5.5)
$$\begin{aligned} &\frac{n-1}{n} \int_\Omega f^2 V^\tau - \int_\Omega V^\tau Ric^D(\bar{\nabla}^D \phi, \bar{\nabla}^D \phi), \\ \geq &\int_\Sigma V^\tau \left[H^D \langle \bar{\nabla}^D \phi, \nu \rangle^2 + (h - \gamma u_\nu g)(\nabla^D \phi, \nabla^D \phi) - 2V^{-\gamma} \langle \nabla^D \phi, \nabla^D(V^\gamma \phi_\nu) \rangle \right] dA. \end{aligned}$$

where the right hand side contains only boundary integrations. Next, we will show that in all these cases, the right hand side boundary integrations are nonnegative.

In case (i), the boundary $\Sigma = \emptyset$ and the result follows immediately.

In case (ii), the first boundary integral in (5.5) is nonnegative since $H^D \geq 0$. Recall from Definition 2.4 (i), $\nabla^D \phi = V^{\gamma-\alpha} \nabla \phi$ where ∇ is the covariant derivative with respect to the induced metric of the boundary Σ. Thus we have $\nabla^D \phi \equiv 0$, since $\phi \equiv 0$ on Σ.

In case (iii), $\phi_\nu \equiv 0$ on Σ. We observe that $\langle \bar{\nabla}^D \phi, \nu \rangle = V^{\gamma-\alpha} \phi_\nu \equiv 0$ and $\nabla^D(V^\gamma \phi_\nu) = V^{\gamma-\alpha} \nabla(V^\gamma \phi_\nu) \equiv 0$. Thus the first and last boundary terms in (5.5) are all zero. Under the condition that $h - \gamma u_\nu g \geq 0$, we conclude that the right hand side boundary terms are all nonnegative.

Equivalently, we have shown, in all the three cases, the following inequality holds,

$$(5.6) \qquad \frac{n-1}{n} \int_{\Omega} f^2 V^{\tau} \geq \int_{\Omega} V^{\tau} Ric^D(\bar{\nabla}^D \phi, \bar{\nabla}^D \phi).$$

On the other hand, by integration by parts and recall Definition 2.4, also noting the divergence property (3.14), we have

$$\int_{\Omega} f^2 V^{\tau} = \int_{\Omega} f V^{\tau} \bar{\Delta}^D \phi = \int_{\Omega} f \bar{\nabla}(V^{\tau} \bar{\nabla}^D \phi)$$

$$(5.7) \qquad\qquad = \int_{\Sigma} f V^{\tau} \bar{\nabla}_{\nu}^D \phi - \int_{\Omega} V^{\tau} \langle \bar{\nabla}^D \phi, \bar{\nabla} f \rangle$$

$$= -\int_{\Omega} V^{\tau} \langle \bar{\nabla}^D \phi, \bar{\nabla} f \rangle,$$

where the last identity holds due to the following observations: case(i), $\Sigma = \emptyset$; case (ii), $f \equiv 0$ on Σ; case (iii), $\phi_{\nu} \equiv 0$ on Σ.

Applying Hölder's inequality to (5.7), we obtain

$$(5.8) \quad \left(\int_{\Omega} f^2 V^{\tau} \right)^2 \leq \left(\int_{\Omega} V^{\tau} Ric^D(\bar{\nabla}^D \phi, \bar{\nabla}^D \phi) \right) \left(\int_{\Omega} V^{\tau} \left\langle (Ric^D)^{-1} \bar{\nabla} f, \bar{\nabla} f \right\rangle \right).$$

Combining (5.6) and (5.8), we proved (5.1). $\qquad\square$

As a consequence, we get the Lichnerowicz type inequality for the first eigenvalue of D-Laplacian $\bar{\Delta}^D$.

Proof of Theorem 1.3 (iii). By using the divergence property (3.14), One sees directly that the first eigenvalues have the following variational representation

$$(5.9) \qquad \lambda_1^{Dir} = \inf_{\{f \in C^1(\overline{\Omega}); f \not\equiv 0, f|_{\Sigma}=0\}} \frac{\int_{\Omega} |\bar{\nabla} f|^2 V^{\tau+\gamma-\alpha} d\Omega}{\int_M f^2 V^{\tau} d\Omega},$$

$$(5.10) \qquad \lambda_1^{Neu} = \inf_{\{f \in C^1(\overline{\Omega}); f \not\equiv 0, \int_{\Omega} f V^{\tau}=0\}} \frac{\int_M |\bar{\nabla} f|^2 V^{\tau+\gamma-\alpha} d\Omega}{\int_M f^2 V^{\tau} d\Omega}.$$

The assertion follows immediately from Theorem 5.1 above by using the curvature condition $Ric^D \geq (n-1)V^{\alpha-\gamma} \bar{g}$. $\qquad\square$

Proof of Corollary 1.5. Let $\alpha = 0, \gamma = 1$. It is sufficient to observe the following fact: the Dirichlet boundary problem for PDEs

$$(5.11) \qquad \begin{cases} \bar{\Delta}^D \phi = -\lambda_1^{Dir} \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Sigma, \end{cases}$$

and

$$(5.12) \qquad \begin{cases} V \bar{\Delta} f - \bar{\Delta} V f = -\lambda_1^{Dir} f & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \end{cases}$$

are equivalent under the correspondence $\phi = \frac{f}{V}$. The same phenomenon holds for Neumann boundary problems. Then Corollary 1.5 follows from Theorem 1.3 (iii). $\qquad\square$

6. Appendix: Proof of Proposition 2.3

In this appendix we prove Proposition 2.3. We only need to prove it at any fixed point P under normal coordinates system $\{\partial_i\}_{i=1}^n$ such that $\bar{g}_{ij}(P) = \delta_{ij}$, $\partial_k \bar{g}_{ij}(P) = 0$, and $\bar{\nabla}_{\partial_i} \partial_j(P) = 0$. Throughout the proof we will use these properties implicitly. We use $u_i := du(\partial_i)$ and let u_{ij} be the Hessian of u with respect to $\bar{\nabla}$.

By definition, we first observe that

$$
\text{(6.1)} \qquad D_i \partial_j = \bar{\nabla}_i \partial_j + \alpha u_i \partial_j + \alpha u_j \partial_i + \gamma \delta_{ij} \bar{\nabla} u,
$$

and

$$
\text{(6.2)} \qquad D_k(D_i \partial_j) = \bar{\nabla}_k(D_i \partial_j) + \alpha u_k(D_i \partial_j) + \alpha du(D_i \partial_j)\partial_k + \gamma \bar{g}(\partial_k, D_i \partial_j)\bar{\nabla} u.
$$

Next, we compute each term on the right of (6.2). It turns out

$$
\bar{\nabla}_k(D_i \partial_j) = \bar{\nabla}_k(\bar{\nabla}_i \partial_j) + \alpha u_{ik}\partial_j + \alpha u_{jk}\partial_i + \gamma \delta_{ij}\bar{\nabla}_k\bar{\nabla} u,
$$

$$
\alpha u_k(D_i \partial_j) = \alpha^2 u_k u_i \partial_j + \alpha^2 u_k u_j \partial_i + \alpha \gamma \delta_{ij} u_k \bar{\nabla} u,
$$

$$
\alpha du(D_i \partial_j)\partial_k = 2\alpha^2 u_i u_j \partial_k + \alpha \gamma \delta_{ij} |\bar{\nabla} u|^2 \partial_k,
$$

$$
\text{(6.3)} \qquad \gamma \bar{g}(\partial_k, D_i \partial_j)\bar{\nabla} u = (\alpha \gamma u_i \delta_{jk} + \alpha \gamma u_j \delta_{ik})\bar{\nabla} u + \gamma^2 \delta_{ij} u_k \bar{\nabla} u.
$$

Adding up these four terms into (6.2), we have

$$
\begin{aligned}
D_k(D_i \partial_j) = &\bar{\nabla}_k(\bar{\nabla}_i \partial_j) + \alpha u_{ik}\partial_j + \alpha u_{jk}\partial_i + \gamma \delta_{ij}\bar{\nabla}_k\bar{\nabla} u \\
&+ \alpha^2 u_k u_i \partial_j + \alpha^2 u_k u_j \partial_i + \alpha \gamma \delta_{ij} u_k \bar{\nabla} u \\
&+ 2\alpha^2 u_i u_j \partial_k + \alpha \gamma \delta_{ij} |\bar{\nabla} u|^2 \partial_k \\
\text{(6.4)} \qquad &+ (\alpha \gamma u_i \delta_{jk} + \alpha \gamma u_j \delta_{ik})\bar{\nabla} u + \gamma^2 \delta_{ij} u_k \bar{\nabla} u.
\end{aligned}
$$

Using the metric tensor $\bar{g}$, we have

$$
\begin{aligned}
\langle D_k(D_i \partial_j), \partial_k \rangle_{\bar{g}} = &\left\langle \bar{\nabla}_k(\bar{\nabla}_i \partial_j), \partial_k \right\rangle_{\bar{g}} + 2\alpha u_{ij} + \gamma \bar{\Delta} u \delta_{ij} \\
&+ \left[\alpha^2 + (2n+1)\alpha^2 + 2\alpha\gamma \right] u_i u_j \\
\text{(6.5)} \qquad &+ \left[(n+1)\alpha\gamma + \gamma^2 \right] |\bar{\nabla} u|^2 \delta_{ij}.
\end{aligned}
$$

By swapping k and i in (6.4), we obtain

$$
\begin{aligned}
D_i(D_k \partial_j) = &\bar{\nabla}_i(\bar{\nabla}_k \partial_j) + \alpha u_{ik}\partial_j + \alpha u_{ji}\partial_k + \gamma \delta_{kj}\bar{\nabla}_i\bar{\nabla} u \\
&+ \alpha^2 u_k u_i \partial_j + \alpha^2 u_i u_j \partial_k + \alpha \gamma \delta_{kj} u_i \bar{\nabla} u \\
&+ 2\alpha^2 u_k u_j \partial_i + \alpha \gamma \delta_{kj} |\bar{\nabla} u|^2 \partial_i \\
\text{(6.6)} \qquad &+ (\alpha \gamma u_k \delta_{ij} + \alpha \gamma u_j \delta_{ik})\bar{\nabla} u + \gamma^2 \delta_{kj} u_i \bar{\nabla} u.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\langle D_i(D_k \partial_j), \partial_k \rangle_{\bar{g}} = &\left\langle \bar{\nabla}_i(\bar{\nabla}_k \partial_j), \partial_k \right\rangle_{\bar{g}} + \left[(n+1)\alpha + \gamma \right] u_{ij} + 2\alpha\gamma |\bar{\nabla} u|^2 \delta_{ij} \\
\text{(6.7)} \qquad &+ \left[(n+3)\alpha^2 + 2\alpha\gamma + \gamma^2 \right] u_i u_j.
\end{aligned}
$$

Combining (6.5) and (6.7), using Definition 2.1 we have

$$
\begin{aligned}
R_{ij}^D &= \left\langle R^D(\partial_k, \partial_i)\partial_j, \partial_k \right\rangle_{\bar{g}} \\
&= \langle D_k(D_i \partial_j), \partial_k \rangle_{\bar{g}} - \langle D_i(D_k \partial_j), \partial_k \rangle_{\bar{g}} \\
&= R_{ij} - \left[\alpha(n-1) + \gamma \right] u_{ij} + \left[(n-1)\alpha^2 - \gamma^2 \right] u_i u_j \\
\text{(6.8)} \qquad &+ \gamma \bar{\Delta} u \delta_{ij} + \left[(n-1)\alpha\gamma + \gamma^2 \right] |\bar{\nabla} u|^2 \delta_{ij}.
\end{aligned}
$$

This finishes the proof for Ric^D.

$\square$

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