Hamilton-Souplet-Zhang type gradient estimates for porous medium type equations on Riemannian manifolds

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ABSTRACT. In this paper, by employ the cutoff function and the maximum principle, some Hamilton-Souplet-Zhang type gradient estimates for porous medium type equation are deduced. As a special case, an Hamilton-Souplet-Zhang type gradient estimates of the heat equation is derived which is different from the result of Souplet-Zhang. Furthermore, our results generalize those of Zhu. As application, some Livillous theorems for ancient solution are derived.

1. Introduction and Main results

In the paper, let (M^n, g) be an *n*-dimensional complete Riemannian manifold. We consider the porous medium type equations

$$u_t = \Delta u^m + \lambda(x, t)u^l, m > 1 \tag{1.1}$$

on (M^n, g) , where l and m are two real numbers, and $\lambda(x, t) \geq 0$ is defined on $M^n \times [0, \infty)$ which is C^2 in the first variable and C^1 in the second variable.

The famous porous medium equations (PME for short)

$$u_t = \Delta u^m, m > 1 \tag{1.2}$$

are of great interest because of important in mathematics, physics, and applications in many other fields. For m=1 it is the famous heat equation. As m>1, it is called the porous medium equation, and it has arisen in different applications to model diffusive phenomena, such as, groundwater infiltration (Boussinesq's model, 1903, with m=2), flow of gas in porous media (Leibenzon-Muskat model, $m\geq 2$), heat radiation in plasmas (m>4), liquid thin films moving under gravity (m=4), and others. We can read a work by Cázquez [19] for basic theory and various applications of the porous medium equation in the Euclidean space. In the case m<1, it is said to be the fast diffusion equation.

In 1979, Aronson and Bénilan [1] obtained a famous second order differential inequality

$$\sum_{i} \frac{\partial}{\partial x_{i}} \left(m u^{m-2} \frac{\partial u}{\partial x_{i}} \right) \ge -\frac{\kappa}{t}, \quad \kappa = \frac{n}{n(m-1)+2}, \tag{1.3}$$

for all positive solutions of (1.2) on the Euclidean space \mathbb{R}^n with $m > 1 - \frac{2}{n}$.

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Generalized research on PME (1.2) also attracted many researchers' interest. In 1993, Hui [7] considered the asymptotic behaviour for solutions to equation

$$u_t = \Delta u^m - u^p \tag{1.4}$$

as $l \to \infty$. In 1994, Zhao and Yuan [23] proved the uniqueness of the solutions to equation (1.4) with initial datum a measure. In 1997, Kawanago [11] demonstrated existence and behaviour for solutions to equation

$$u_t = \Delta u^m + u^l. (1.5)$$

In 2001, E. Chasseigne [4] investigated the initial trace for the equation (1.4) in a cylinder $\Omega \times [0,T]$, where Ω is a regular, bounded open subset of \mathbb{R}^n and T>0, m>1, and q are constants. Recently, Xie, Zheng and Zhou [21] studied global existence for equation

$$u_t = \Delta u^m - u^{p(x)} \tag{1.6}$$

in $\Omega \times (0,T)$, where p(x) > 0 is continuous function satisfying $0 < p_- = \inf p(x) \le p(x) \le p_+ = \sup p(x) < \infty$.

Recently, regularity estimates of PME (1.2) on manifolds are investigated. In 2009, Lu, Ni, Vázquez and Villani [14] studied the PME on an n-dimensional complete manifold (M^n,g) , they obtained a local Aronson-Bénilan estimate. Huang, Huang and Li in [8] improved the part results of Lu, Ni, Vázquez and Villani. In this article, we will study Hamilton-Souplet-Zhang type gradient estimates to equation (1.1).

Let First recall some known results.

Theorem A (Hamilton [6]). Let (\mathbf{M}^n, g) be a closed Riemannian manifold with $Ricci(\mathbf{M}) \geq -k$ for some $k \geq 0$. Suppose that u is arbitrary positive solution to the heat equation

$$u_t = \Delta u \tag{1.7}$$

and $u \leq M$. Then

$$\frac{|\nabla u^2(x,t)|}{u^2(x,t)} \le C\left(\frac{1}{t} + 2k\right) \log \frac{M}{u(x,t)}.$$
(1.8)

In 2006, Souplet and Zhang [18] generalized Hamilton's result, and obtained the corresponding gradient estimate and Liouville theorem.

Theorem B (Souplet-Zhang [18]). Let (\mathbf{M}^n, g) be a Riemannian manifolds with $n \geq 2$ and $Ricci(\mathbf{M}) \geq -k$ for some $k \geq 0$. Suppose that u is arbitrary positive solution to the heat equation (1.7) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset \mathbf{M}^n \times (-\infty, \infty)$ and $u \leq M$ in $Q_{R,T}$. Then

$$\frac{|\nabla u(x,t)|}{u(x,t)} \le C\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right) \left(1 + \log\frac{M}{u(x,t)}\right) \tag{1.9}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$, where C is a dimensional constant.

In 2013, Zhu [26] deduced a Hamilton's gradient estimate and Liouville theorem for PME (1.2) on noncompact Riemannian manifolds. Huang, Xu and Zeng in [9] improve the result of Zhu.

Theorem C (Zhu [26]). Let (\mathbf{M}^n, g) be a Riemannian manifolds with $n \geq 2$ and $Ricci(\mathbf{M}) \geq -k$ for some $k \geq 0$. Suppose that u is arbitrary positive solution to the

PME (1.2) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset \mathbf{M}^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Then for $1 < m < 1 + \frac{1}{\sqrt{2n+1}}$

$$\frac{|\nabla v|}{v^{\frac{m-2}{4(m-1)}}} \le CM^{1+\frac{2-m}{4(m-1)}} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right). \tag{1.10}$$

Recently, Cao and Zhu [3] obtained some Aronson and Bénilan estimates for PME (1.2) under Ricci flow.

Our results of this paper are encouraged by the work in Ref. [10, 12, 14, 15, 16, 17, 18, ?, 21, 26]. We consider the porous medium type equation (1.1), and deduce some Hamilton-Souplet-Zhang type gradient estimates.

Our main results state as follows.

Theorem 1.1. Let (M^n,g) be a Riemannian manifold with dimensional n. Suppose that $Ric(M^n) \geq -k$ with $k \geq 0$. If u(x,t) is a positive solution of the equation (1.1) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Also suppose that there exist two positive numbers δ and ϵ such that $\lambda(x,t) \leq \delta$ and $|\nabla \lambda|^2 \leq \epsilon |\lambda|$. Then for $1 < m < 1 + \sqrt{\frac{1}{n-1}}$ and $l \geq 1 - m$,

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^{2}(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) + C_{3}\left(\delta^{\frac{1}{2}}M^{\frac{m+l-1}{2(m-1)}} + \epsilon^{\frac{1}{4}}M^{\frac{3m+l-2}{4(m-1)}}\right)$$
(1.11)

in $Q_{\frac{R}{2},\frac{T}{2}}$, where $\beta = -\frac{1}{m-1}$, $\gamma = \frac{8}{1-(m-1)^2(n-1)}$, $C_3 = C_3(m,n,l)$ and C is a constant.

When $\lambda(x,t)=0$, we get the following:

Corollary 1.1. Let (M^n, g) be a Riemannian manifold with dimensional n. Suppose that $Ric(M^n) \geq -k$ with $k \geq 0$. If u(x,t) is a positive solution of the PME (1.2) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1}u^{m-1}$ and $v \leq M$. Then for $1 < m < 1 + \sqrt{\frac{1}{n-1}}$

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^{2}(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right)$$
(1.12)

in $Q_{\frac{R}{2},\frac{T}{2}}$, where $\beta = -\frac{1}{m-1}$, $\gamma = \frac{8}{1-(m-1)^2(n-1)}$ and C is a constant.

Take $\lambda(x,t) = 0$ and $m \searrow 1$ in Corollary 1.1, the following estimate is derived.

Corollary 1.2. Let (M^n, g) be a Riemannian manifold of dimensional n. Suppose that $Ric(M^n) \ge -k$ with $k \ge 0$. If u(x, t) is a positive solution of the heat equation

$$u_t = \Delta u$$
,

in $Q_{R.T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Then we have for $u \leq M$

$$\frac{|\nabla u|}{\sqrt{u}}(x,t) \le C\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) \tag{1.13}$$

in $Q_{\frac{R}{2},\frac{T}{2}}$, where C is a constant.

4

Theorem 1.2. Let (M^n,g) be a Riemannian manifold with dimensional n. Suppose that $Ric(M^n) \ge -k$ with $k \ge 0$. If u(x,t) is a positive solution of the equation (1.1) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \le M$. Also suppose that there exist a positive number ϵ such that $|\nabla \lambda|^2 \le \epsilon |\lambda|$. Then for $1 < m < 1 + \sqrt{\frac{1}{n-1}}$ and $2 - 3m \le l \le 2 - \frac{3}{2}m$,

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^2(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) + C_3\epsilon^{\frac{1}{4}}M^{\frac{3m+l-2}{4(m-1)}}$$
(1.14)

in $Q_{\frac{R}{2},\frac{T}{2}}$, where $\beta=-\frac{1}{m-1}$, $\gamma=\frac{8}{1-(m-1)^2(n-1)}$, $C_3=C_3(m,n,l)$ and C is a constant.

Remark: (a) Since $1 + \sqrt{\frac{1}{n-1}} > 1 + \sqrt{\frac{1}{2n+1}}$, so the result of Corollary 1.1 in the paper generalize those of Zhu in [26].

- (b) When $\lambda(x,t) = 0$, the result of Corollary 1.1 in the paper is the result of Huang, Xie and Zeng in [9].
- (c) (1.13) is different from Souplet-Zhang's result in [18]. Moreover, our results in form seem to be simpler than Souplet-Zhang's result.

2. Preliminary

In this section, we derive a lemma.

Lemma 2.1. [21] Let $A = (a_{ij})$ be a nonzero $n \times n$ symmetric matrix with eigenvalues λ_k , for any $a, b \in \mathbb{R}$, then

$$\max_{A \in S(n); |v|=1} \left[\frac{aA + b \operatorname{tr} A I_n}{|A|} (v, v) \right]^2 = (a+b)^2 + (n-1)b^2.$$

Lemma 2.2. Let $1 < m < 1 + \sqrt{\frac{1}{n-1}}$ and $\theta = \frac{1 - (m-1)^2 (n-1)}{4(m-1)} > 0$. Then we have

$$(m-1)v\Delta w - w_t \ge \theta w^2 v^{\beta-1} - 2(m-1)kwv - m\nabla w \cdot \nabla v$$

$$+\lambda \left[\beta(m-1) - 2(m+l-2)\right] \left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} w$$

$$-(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} \left(|\lambda|w + \frac{|\nabla\lambda|^2}{|\lambda|} \cdot \frac{1}{v^{\beta-2}}\right). \tag{2.1}$$

Proof. Let $v = \frac{m}{m-1}u^{m-1}$, then

$$v_t = (m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1)\left(\frac{m-1}{m}\right)^{\frac{l-1}{m-1}}v^{1+\frac{l-1}{m-1}}.$$
 (2.2)

Let $w = \frac{|\nabla v|^2}{v^{\beta}}$, then

$$\begin{array}{lcl} w_t & = & \frac{2v_iv_{it}}{v^{\beta}} - \beta\frac{v_i^2v_t}{v^{\beta+1}} \\ & = & \frac{2v_i\left[(m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1)\left(\frac{m-1}{m}\right)^{\frac{l-1}{m-1}}v^{1+\frac{l-1}{m-1}}\right]_i}{v^{\beta}} \\ & & \frac{v_i^2\left[(m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1)\left(\frac{m-1}{m}\right)^{\frac{l-1}{m-1}}v^{1+\frac{l-1}{m-1}}\right]}{v^{\beta+1}} \end{array}$$

$$= 2(m-1)\frac{v_{i}^{2}v_{jj}}{v^{\beta}} + 2(m-1)\frac{v_{i}v_{jji}}{v^{\beta-1}} + 4\frac{v_{i}v_{ij}v_{j}}{v^{\beta}} + 2\lambda\frac{(m+l-2)(\frac{m-1}{m}v)^{\frac{l-1}{m-1}}v_{i}^{2}}{v^{\beta}} + 2(m-1)(\frac{m-1}{m}v)^{\frac{l-1}{m-1}}\frac{\nabla v \cdot \nabla \lambda}{v^{\beta-1}} - \beta(m-1)\frac{v_{i}^{2}v_{jj}}{v^{\beta}} - \beta\frac{v_{i}^{2}v_{j}^{2}}{v^{\beta+1}} - \lambda\beta(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}}, \quad (2.3)$$

$$w_j = \frac{2v_i v_{ij}}{v^\beta} - \beta \frac{v_i^2 v_j}{v^\beta},\tag{2.4}$$

$$w_{jj} = \frac{2v_{ij}^2}{v^{\beta}} + \frac{2v_i v_{ijj}}{v^{\beta}} - 4\beta \frac{v_i v_{ij} v_j}{v^{\beta+1}} - \beta \frac{v_i^2 v_{jj}}{v^{\beta+1}} + \beta(\beta+1) \frac{v_i^2 v_j^2}{v^{\beta+2}}.$$
 (2.5)

By (2.4) and (2.5)

$$(m-1)v\Delta w - w_t$$

$$=2(m-1)\frac{v_{ij}^{2}}{v^{\beta-1}}+2(m-1)\frac{v_{i}v_{ijj}}{v^{\beta-1}}-2(m-1)\frac{v_{i}v_{jji}}{v^{\beta-1}}-4\beta(m-1)\frac{v_{i}v_{ij}v_{j}}{v^{\beta}} +\beta (\beta+1)(m-1)\frac{v_{i}^{2}v_{j}^{2}}{v^{\beta+1}}-2(m-1)\frac{v_{i}^{2}v_{jj}}{v^{\beta}}-4\frac{v_{i}v_{ij}v_{j}}{v^{\beta}}+\beta \frac{v_{i}^{2}v_{j}^{2}}{v^{\beta+1}} -2\lambda(m+l-2)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}}-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{\nabla v \cdot \nabla \lambda}{v^{\beta-1}} +\lambda\beta(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}} =2(m-1)\frac{v_{i}v_{ij}v_{j}}{v^{\beta}}-2(m-1)\frac{v_{i}v_{ij}v_{j}}{v^{\beta}}-2(m-1)\frac{v_{i}^{2}v_{ijj}}{v^{\beta}} +\beta\left[(\beta+1)(m-1)+1\right]\frac{v_{i}v_{j}^{2}}{v^{\beta+1}}-2\lambda(m+l-2)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}} -2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}} -2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_{i}^{2}}{v^{\beta}}, \quad (2.6)$$

Since

$$\nabla w \cdot \nabla v = \frac{2v_i v_{ij} v_j}{v^{\beta}} - \beta \frac{v_i^2 v_j^2}{v^{\beta+1}}.$$
 (2.7)

Adding $\varepsilon \times (2.7)$ to (2.6),

$$(m-1)v\Delta w - w_t$$

$$\begin{split} &= 2(m-1)\frac{v_{ij}^2}{v^{\beta-1}} + 2(m-1)\frac{R_{ij}v_iv_j}{v^{\beta-1}} + \left[2\varepsilon - 4(1+\beta(m-1))\right]\frac{v_iv_{ij}v_j}{v^{\beta}} \\ &- 2(m-1)\frac{v_i^2v_{jj}}{v^{\beta}} + \beta\Big[(\beta+1)(m-1) + 1 - \varepsilon\Big]\frac{v^iv_j^2}{v^{\beta+1}} - \varepsilon\nabla w \cdot \nabla v \\ &- 2\lambda(m+l-2)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_i^2}{v^{\beta}} - 2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{\nabla v \cdot \nabla \lambda}{v^{\beta-1}} \\ &+ \lambda\beta(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{v_i^2}{v^{\beta}} \end{split}$$

$$=2(m-1)\frac{|A|^{2}}{v^{\beta-1}}+2(m-1)R_{ij}wv+\left[2\varepsilon-4(1+\beta(m-1)]\frac{A(e,e)}{|A|}w|A\right]\\ -2(m-1)\frac{\operatorname{tr}A}{|A|}w|A|+\beta\left[(\beta+1)(m-1)+1-\varepsilon\right]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ -2\lambda(m+l-2)\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}\frac{1}{w}+\lambda\beta(m-1)\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}w\\ -2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}\frac{1}{v^{2}}\frac{1}{v^{2}}\lambda v\\ -2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}\frac{1}{v^{2}}\frac{1}{v^{2}}\lambda v\\ =2(m-1)\frac{|A|^{2}}{v^{\beta-1}}+\left\{\left[2\varepsilon-4[1+\beta(m-1)]\right]\frac{A(e,e)}{|A|}-2(m-1)\frac{\operatorname{tr}A}{|A|}\right\}w|A|\\ +2(m-1)R_{ij}wv+\beta\left[(\beta+1)(m-1)+1-\varepsilon]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}\frac{1}{w^{2}}v^{2}-1-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)\left[\left(2\varepsilon-4[1+\beta(n-1)]\right)\frac{A(e,e)}{|A|}-2(m-1)\frac{\operatorname{tr}A}{|A|}\right]^{2}w^{2}v^{\beta-1}\\ +2(m-1)R_{ij}wv+\beta\left[(\beta+1)(m-1)+1-\varepsilon]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}\frac{1}{w^{2}}v^{2}-1\\ +2(m-1)R_{ij}wv+\beta\left[(\beta+1)(m-1)+1-\varepsilon]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{1}{m-1}}w-2(m-1)\frac{\operatorname{tr}A}{|A|}^{2}w^{2}v^{\beta-1}\\ +2(m-1)R_{ij}wv+\beta\left[(\beta+1)(m-1)+1-\varepsilon]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{m-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}\frac{\nabla v\cdot\nabla\lambda}{v^{\beta-1}},\\ \text{where }A_{ij}=(v_{ij})\text{ and }e=\nabla v/|\nabla v|.\text{ By applying Lemma 2.1 with }a=2\varepsilon-4[1+\beta(n-1)]v\Delta w\cdot wv\\ \geq -\frac{1}{8(m-1)}\left\{\left[2\varepsilon-4[1+\beta(m-1)]-2(m-1)\right]^{2}+4(m-1)^{2}(n-1)\right\}w^{2}v^{\beta-1}\\ +2(m-1)R_{ij}wv+\beta\left[(\beta+1)(m-1)+1-\varepsilon]w^{2}v^{\beta-1}-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}\frac{\nabla v\cdot\nabla\lambda}{v^{\beta-1}}\right\}\\ =-\frac{1}{8(m-1)}f(\beta,\varepsilon)w^{2}v^{\beta-1}+2(m-1)R_{ij}wv-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}\frac{\nabla v\cdot\nabla\lambda}{v^{\beta-1}}\\ =-\frac{1}{8(m-1)}f(\beta,\varepsilon)w^{2}v^{\beta-1}+2(m-1)R_{ij}wv-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}\frac{\nabla v\cdot\nabla\lambda}{v^{\beta-1}}\\ =-\frac{1}{8(m-1)}f(\beta,\varepsilon)w^{2}v^{\beta-1}+2(m-1)R_{ij}wv-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}\frac{\nabla v\cdot\nabla\lambda}{v^{\beta-1}}\\ =-\frac{1}{8(m-1)}f(\beta,\varepsilon)w^{2}v^{\beta-1}+2(m-1)R_{ij}wv-\varepsilon\nabla w\cdot\nabla v\\ +\lambda\left[\beta(m-1)-2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{i-1}{m-1}}w-2(m-1)\left(\frac{m-1}{m}$$

where

$$f(\beta,\varepsilon) = \left[2\varepsilon - 4\left[1 + \beta(m-1)\right] - 2(m-1)\right]^2 + 4(m-1)^2(n-1) - 8(m-1)\beta\left[(\beta+1)(m-1) + 1 - \varepsilon\right].$$
 (2.9)

For the purpose of showing that the coefficient of $w^2v^{\beta-1}$ is positive, we minimize the function $f(\beta,\varepsilon)$ by letting $\varepsilon=m$ and $\beta=-\frac{1}{m-1}$, such that

$$f(\beta, \varepsilon) = 4(m-1)^2(n-1) - 4.$$

Then (2.8) becomes

$$(m-1)v\Delta w - w_{t} \geq \frac{1 - (m-1)^{2}(n-1)}{4(m-1)}w^{2}v^{\beta-1} - 2(m-1)kwv - m\nabla w \cdot \nabla v$$

$$+ \lambda \left[\beta(m-1) - 2(m+l-2)\right] \left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} w$$

$$- 2(m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} \frac{|\nabla v| \cdot |\nabla \lambda|}{v^{\beta-1}}$$

$$= \theta w^{2}v^{\beta-1} - 2(m-1)kwv - m\nabla w \cdot \nabla v$$

$$+ \lambda \left[\beta(m-1) - 2(m+l-2)\right] \left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} w$$

$$- (m-1)\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} \left(|\lambda|w + \frac{|\nabla \lambda|^{2}}{|\lambda|} \cdot \frac{1}{v^{\beta-2}}\right), \quad (2.10)$$

where $\theta = \frac{1-(m-1)^2(n-1)}{4(m-1)} > 0$ as $1 < m < 1 + \sqrt{\frac{1}{n-1}}$, and in the last inequality we utilize the fact that

$$\begin{split} &-2(m-1)\Big(\frac{m-1}{m}v\Big)^{\frac{l-1}{m-1}}\frac{|\nabla v|\cdot|\nabla\lambda|}{v^{\beta-1}}\\ &\geq -(m-1)\Big(\frac{m-1}{m}v\Big)^{\frac{l-1}{m-1}}\left(|\lambda|w+\frac{|\nabla\lambda|^2}{|\lambda|}\cdot\frac{1}{v^{\beta-2}}\right). \end{split}$$

3. Proof of main results

From here, we will utilize the well-known cut-off function of Li and Yau to derive the desire bounds.

Proof of Theorem 1.1. Assume that a function $\Psi = \Psi(x,t)$ is a smooth cutoff function supported in $Q_{R,T}$, satisfying the following properties,

- (1) $\Psi = \Psi(d(x, x_0), t) \equiv \psi(r, t); \ \Psi(r, t) = 1 \text{ in } Q_{R/2, T/2}, \ 0 \le \Psi \le 1.$
- (2) Ψ is decreasing as a radial function in the spatial variables.
- (3) $|\partial_r \Psi|/\Psi^a \le C_a/R$, $|\partial_r^2 \Psi|/\Psi^a \le C_a/R^2$ when 0 < a < 1.
- (4) $|\partial_t \Psi| / \Psi^{1/2} < C/T$.

Assume that the maximum of Ψw is arrived at point (x_1, t_1) . By [13], we can suppose, without loss of generality, that x_1 is not on the cut-locus of \mathbf{M}^n . Therefore, at (x_1, t_1) , it yields $\Delta(\Psi w) \leq 0$, $(\Psi w)_t \geq 0$ and $\nabla(\Psi w) = 0$. Hence, by (2.1) and a straightforward calculation, it yields that

$$0 \ge \left[(m-1)v\Delta - \partial_t \right] (\Psi w)$$

8

$$=\Psi\left[(m-1)v\Delta - \partial_{t}\right]w + (m-1)vw\Delta\Psi - w\Psi_{t} + 2(m-1)\frac{v}{\Psi}\nabla\Psi\cdot\nabla(\Psi w) - 2(m-1)vw\frac{|\nabla\Psi|^{2}}{\Psi}$$

$$=\Psi\theta w^{2}v^{\beta-1} - 2(m-1)\Psi kwv - p\nabla(\Psi w)\cdot\nabla v + mw\nabla v\cdot\nabla\Psi + \Psi\lambda\left[\beta(m-1) - 2(m+l-2)\right]\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}w$$

$$-(m-1)\Psi\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\left(\lambda w + \frac{|\nabla\lambda|^{2}}{\lambda}\cdot\frac{1}{v^{\beta-2}}\right) + (m-1)vw\Delta\Psi - w\Psi_{t} + 2(m-1)\frac{v}{\Psi}\nabla\Psi\cdot\nabla(\Psi w) - 2(m-1)vw\frac{|\nabla\Psi|^{2}}{\Psi}.$$
(3.1)

Then (3.1) becomes at the point (x_1, t_1)

$$\Psi\theta w^{2}v^{\beta-1} \leq 2(m-1)\Psi kwv - mw\nabla v \cdot \nabla\Psi - (m-1)vw\Delta\Psi + 2(m-1)vw\frac{|\nabla\Psi|^{2}}{\Psi} + w\Psi_{t} - \Psi\lambda \left[\beta(m-1) - 2(m+l-2)\right] \left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}w + (m-1)\Psi\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} \left(\lambda w + \frac{|\nabla\lambda|^{2}}{\lambda} \cdot \frac{1}{v^{\beta-2}}\right).$$
(3.2)

Now setting $\theta = \frac{2}{\gamma} \cdot \frac{1}{m-1}$ and $\gamma = \frac{8}{1-(m-1)^2(n-1)}$, then (3.2) gives

$$\begin{split} 2\Psi w^{2} \leq & 2\gamma(m-1)^{2}\Psi kwv^{2-\beta} - \gamma(m-1)v^{1-\beta}mw\nabla v \cdot \nabla\Psi - (m-1)^{2}\gamma v^{2-\beta}w\Delta\Psi \\ & + 2(m-1)^{2}\gamma v^{2-\beta}w\frac{|\nabla\Psi|^{2}}{\Psi} + \gamma(m-1)w\Psi_{t}v^{1-\beta} \\ & - (m-1)\gamma\Psi\lambda\Big[(\beta-1)(m-1) - 2(m+l-2)\Big]\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}wv^{1-\beta} \\ & + (m-1)^{2}\gamma\Psi\Big(\frac{m-1}{m}v\Big)^{\frac{l-1}{m-1}}\frac{|\nabla\lambda|^{2}}{\lambda} \cdot \frac{v^{1-\beta}}{v^{\beta-2}}. \end{split} \tag{3.3}$$

Now, we need to search for an upper bound for each term on the right-hand side of (3.3). After a siample calculation, it is not diffucult to find the following estimates.

$$2\gamma(m-1)^{2}\Psi kwv^{2-\beta} \le \frac{1}{4}\Psi w^{2} + C\gamma^{2}(m-1)^{4}M^{4-2\beta}k^{2}, \tag{3.4}$$

$$-(m-1)\gamma v^{1-\beta} m w \nabla v \cdot \nabla \Psi \le \frac{1}{4} \Psi w^2 + C \gamma^2 (m-1)^4 M^{4-2\beta} \frac{1}{R^4}, \tag{3.5}$$

$$-(m-1)^{2}\gamma v^{2-\beta}w\Delta\Psi \le \frac{1}{4}\Psi w^{2} + C\gamma^{2}(m-1)^{4}M^{4-2\beta}\left(\frac{1}{R^{4}} + \frac{k}{R^{2}}\right), \quad (3.6)$$

$$2(m-1)^{2}\gamma v^{2-\beta}w\frac{|\nabla\Psi|^{2}}{\Psi} \le \frac{1}{4}\Psi w^{2} + C\gamma^{2}(m-1)^{4}M^{4-2\beta}\frac{1}{R^{4}},$$
(3.7)

$$(m-1)\gamma w\Psi_t v^{1-\beta} \le \frac{1}{4}\Psi w^2 + C\gamma^2 (m-1)^4 M^{4-2\beta} \frac{1}{T^2},\tag{3.8}$$

where C is a constant and we used the fact that $0 < v \le M$, $\beta = -\frac{1}{m-1} < 0$.

Applying $0 < v \le M$, $\beta = -\frac{1}{m-1} < 0$ and m > 1 we now give estimates for the last two items of (3.3).

$$-(m-1)\gamma\Psi\lambda\Big[(\beta+1)(m-1)-2(m+l-2)\Big]\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}wv^{1-\beta}$$

$$\leq \frac{1}{4}\Psi w^2 + C_1\delta^2 M^{\frac{2m+2l-2}{m-1}},$$
(3.9)

where $C_1 = C_1(m, n, l)$, and inequality holds for $l \leq 1 - m$.

$$(m-1)^{2} \gamma \Psi\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}} \frac{|\nabla \lambda|^{2}}{|\lambda|} \cdot \frac{v^{1-\beta}}{v^{\beta-2}} \le C_{2} \epsilon M^{\frac{3m+l-2}{m-1}}.$$
 (3.10)

where $C_2 = C_2(m, n, l)$, and inequality is valid for $l \ge 2 - 3m$. Hence, both (3.9) and (3.10) hold for l > 1 - m.

Substituting (3.4)– (3.10) into (3.3), we have for $l \ge 1 - m$ and $C_3 = C_3(m, n, l)$

$$2\Psi w^{2} \leq \frac{3}{2}\Psi w^{2} + C\gamma^{2}(m-1)^{4}M^{4-2\beta} \left(\frac{1}{R^{4}} + k^{2} + \frac{1}{T^{2}}\right) + C_{3}\left(\delta^{2}M^{\frac{2m+2l-2}{m-1}} + \epsilon M^{\frac{3m+l-2}{m-1}}\right), \tag{3.11}$$

which gives at the point (x_1, t_1)

$$\Psi w^{2} \leq C\gamma^{2}(m-1)^{4}M^{4-2\beta} \left(\frac{1}{R^{4}} + k^{2} + \frac{1}{T^{2}}\right) + C_{3}\left(\delta^{2}M^{\frac{2m+2l-2}{m-1}} + \epsilon M^{\frac{3m+l-2}{m-1}}\right).$$
(3.12)

Hnece, for all the point $(x,t) \in Q_{R,T}$,

$$(\Psi^{2}w^{2})(x,t) \leq (\Psi^{2}w^{2})(x_{1},t_{1}) \leq (\Psi w^{2})(x_{1},t_{1})$$

$$\leq C\gamma^{2}(m-1)^{4}M^{4-2\beta}\left(\frac{1}{R^{4}}+k^{2}+\frac{1}{T^{2}}\right)$$

$$+C_{3}(\delta^{2}M^{\frac{2m+2l-2}{m-1}}+\epsilon M^{\frac{3m+l-2}{m-1}}). \tag{3.13}$$

Notice that $\Psi = 1$ in $Q_{R/2,T/2}$ and $w = \frac{|\nabla v|^2}{v^{\beta}}$. Therefore, we have for $l \geq 1 - m$,

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^{2}(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) + C_{3}\left(\delta^{\frac{1}{2}}M^{\frac{m+l-1}{2(m-1)}} + \epsilon^{\frac{1}{4}}M^{\frac{3m+l-2}{4(m-1)}}\right).$$

Proof of Corollary 1.2. By taking $\lambda(x,t) = 0$ in (1.11), we deduce that

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^2(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) \tag{3.14}$$

Applying $v = \frac{m}{m-1}u^{m-1}$ to (3.14), we obtain

$$m \cdot m^{\frac{1}{2(m-1)}} \frac{|\nabla u|}{u^{\frac{3}{2}-m}}(x,t) \le C\gamma^2 \left[(m-1)M \right]^{1+\frac{1}{2(m-1)}} \left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right)$$
 (3.15)

Since $(m-1)v = mu^{m-1}$, we have $(m-1)v \to 1$ as $m \searrow 1$. Therefore, we follow $(m-1)M \to 1$ as $m \searrow 1$. A sample computation yields

$$\begin{split} \lim_{m \to 1^+} \left[(m-1)M \right]^{\frac{1}{4(m-1)}} &= \lim_{m \to 1^+} \left[1 + (m-1)M - 1 \right]^{\frac{1}{(m-1)M-1} \cdot \frac{(m-1)M-1}{2(m-1)}} = e^{\frac{1}{2}}, \\ \lim_{m \to 1^+} \left[m \right]^{\frac{1}{2(m-1)}} &= \lim_{m \to 1^+} \left[1 + m - 1 \right]^{\frac{1}{m-1} \cdot \frac{1}{2}} = e^{\frac{1}{2}}, \\ \lim_{m \to 1^+} \gamma &= \lim_{m \to 1^+} \frac{8}{1 - (m-1)^2 (n-1)} = 8. \end{split}$$

Hence as $m \setminus 1$, (3.15) becomes

$$\frac{|\nabla u|}{u^{\frac{1}{2}}}(x,t) \le C\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right),\,$$

where C = C(n).

Proof of Theorem 1.2. Since $\beta=-\frac{1}{m-1}<0$ and m>1, then $(\beta-1)(m-1)-2(m+l-2)\geq 0$ for $l\leq 2-\frac{3}{2}m$. Hence, (3.3) becomes

$$\begin{split} 2\Psi w^{2} \leq & 2\gamma(m-1)^{2}\Psi kwv^{2-\beta} - \gamma(m-1)v^{1-\beta}pw\nabla v \cdot \nabla\Psi - (m-1)^{2}\gamma v^{2-\beta}w\Delta\Psi \\ & + 2(m-1)^{2}\gamma v^{2-\beta}w\frac{|\nabla\Psi|^{2}}{\Psi} + \gamma(m-1)w\Psi_{t}v^{1-\beta} \\ & + (m-1)^{2}\gamma\Psi\left(\frac{m-1}{m}v\right)^{\frac{l-1}{m-1}}\frac{|\nabla\lambda|^{2}}{\lambda} \cdot \frac{v^{1-\beta}}{v^{\beta-2}}. \end{split} \tag{3.16}$$

A discussion of similar Theorem 1.1 from (3.4)-(3.8), (3.10) and (3.16), we have for $2 - 3m \le l \le 2 - \frac{3}{2}m$ and $C_3 = C_3(m, n, l)$

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \le C\gamma^2(m-1)M^{1-\frac{\beta}{2}}\left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) + C_3\epsilon^{\frac{1}{4}}M^{\frac{3m+l-2}{4(m-1)}}.$$

4. Applications

In this section, we will deduce some related Liouville type theorems. Applying Corollary 1.1, it follows the following Liouville type theorem.

Theorem 4.1 (Liouville type theorem). Let (M^n, g) be a complete, non-compact Riemannian manifold with nonnegative Ricci curvature. Suppose that u is a positive ancient solution of the equation (1.2) such that $v(x,t) = o(d(x) + \sqrt{T})^{\frac{2(m-1)}{2m-1}}$, where $v = \frac{m}{m-1}u^{m-1}$. Then u is a constant.

By utilize Corollary 1.2, the related Liouville type theorem is derived, as follows.

Theorem 4.2 (Liouville type theorem). Let (M^n, g) be a complete, non-compact Riemannian manifold with nonnegative Ricci curvature. Suppose that u is a positive ancient solution of the heat equation (1.7) such that $u(x,t) = o(d(x) + \sqrt{T})^2$. Then u is a constant.

The proof of Theorem 4.1 and Theorem 4.2 are the same. So we only prove Theorem 4.1.

Proof of Theorem 4.1. Fix (x_0, t_0) in space time. Assume that u(x, t) is a positive ancient solution to PME (1.2) such that $v(x, t) = o(d(x) + \sqrt{T})^{\frac{2(m-1)}{2m-1}}$ near infity. Applying (1.12) to u on the cube $B(x_0, R) \times [t_0 - R^2, t_0]$, then we have

$$v(x_0, t_0)^{\frac{1}{2(m-1)}} |\nabla v(x_0, t_0)| \le \frac{C}{R} \cdot o(R).$$

Let $R \to \infty$, we get $|\nabla v(x_0, t_0)| = 0$. Therefore, the result are derived.

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