The Curvature of the Hitchin Connection

Jørgen Ellegaard Andersen and Niccolo Skovgård Poulsen September 24, 2018

Abstract

In this paper we calculate the curvature of the Hitchin connection. We further show that a slight (possibly trivial) modification of the Hitchin connection has curvature equal to an explict given multiple of the Weil-Petersen symplectic form on Teichmüller space.

Dedicated to Nigel Hitchin at the conference *Hitchin70*, celebrating his 70'th Birthday.

1 Indroduction

In [10] Hitchin introduce a projectively flat connection in the bundle of quantizations of the moduli spaces M of flat SU(n)-connections over a surface of genus g>1 with central holonomy around a marked point on the surface. This connection was also constructed in [7] by Axelrod, Della Pietra and Witten from a more physical perspective, where it was also establish how it is related to quantum Chern-Simons theory. See also [3], where it was shown how these two constructions agree and can be slightly generalised. Let us here briefly recall the setup.

The moduli space M is compact and smooth in the co-prime case, i.e. in case when the central holonomy around the special marked point generates the centre of SU(n). In general it has a smooth part M', which consist of the irreducible connections (if n=2, then g>2 for this to be the case, since $M=\mathbb{P}^3$ in the case (g,n)=(2,2)). The smooth part M' has a natural symplectic form called the Seshadri-Atiyah-Bott-Goldmann symplectic form. The Chern-Simons line bundle \mathcal{L} over M is a prequantum line bundle for ω [8]. By the Narasimhan-Seshadri Theorem [12,13], the moduli space further has a natural Kähler structure once a complex structure on Σ has been choosen. This gives a family of complex structures on the moduli space M parametrized by the Teichmüller space of Σ , which we denote \mathcal{T} . Consider now the trivial $C^{\infty}(M, \mathcal{L}^k)$ -bundle $\mathcal{H}^{(k)}$ over Teichmüller space \mathcal{T} . Then a

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Hitchin connection is a connection in $\mathcal{H}^{(k)}$, which preserve the sub-bundle of holomorphic sections $H^{(k)}(M_{\sigma}, \mathcal{L}^k)$, $\sigma \in \mathcal{T}$. Further, we require it is given by adding a differential operator valued one form to the trivial connection in $\mathcal{H}^{(k)}$

$$\nabla_V^H = \nabla_V^t + u(V),$$

for all vector fields V on \mathcal{T} . Hitchin found an explicit formula for u, which in [3] is proven to be given by the following global differential operator

$$u(V) = \frac{-1}{2n + 4k} \left(\Delta_{G(V)} + 2\nabla_{G(V)dF_{\sigma}} + 4kV'[F]_{\sigma} \right).$$

Here F_{σ} is a Ricci potential for M_{σ} the moduli space with the Kähler structure given by the point $\sigma \in \mathcal{T}$. The notation V' indicate that we project V onto the holomorphic directions on \mathcal{T} . Finally the symetric two tensor, G(V) is given by $G(V) = V'[g_{M_{\sigma}}^{-1}]$ and the operator $\Delta_{G(V)}$ is given by

$$\Delta_{G(V)}: C^{\infty}(M, \mathcal{L}^{k}) \xrightarrow{\nabla_{\sigma}^{1,0}} C^{\infty}(M, T_{\sigma}^{*} \otimes \mathcal{L}^{k}) \xrightarrow{G \otimes Id} C^{\infty}(M, T_{\sigma} \otimes \mathcal{L}^{k})$$

$$\xrightarrow{\nabla_{\sigma}^{1,0} \otimes Id + Id \otimes \nabla_{\sigma}^{1,0}} C^{\infty}(M, T_{\sigma}^{*} \otimes T_{\sigma} \otimes \mathcal{L}^{k}) \xrightarrow{\operatorname{tr}} C^{\infty}(M, \mathcal{L}^{k})$$

For this Hitchin connection it was shown in [2], that the curvature is given by

Theorem 1.1 ([2, Theorem 4.8])

The curvature of the Hitchin connection acts by

$$F_{\nabla}^{2,0} = \frac{k}{(2k+2n)^2} P_k(\partial_{\mathcal{T}} c) \quad F_{\nabla}^{1,1} = \frac{ik}{2k+2n} (\theta - 2i\partial_{\mathcal{T}} \bar{\partial}_{\mathcal{T}} F) \quad F_{\nabla}^{0,2} = 0,$$

on sections of the bundle $H^{(k)}$.

Here θ is as defined below in (1). The one form c on \mathcal{T} with values in $C^{\infty}(M)$ is given by

$$c(V) = -\Delta_{G(V)}F - dFG(V)dF - 2nV'[F].$$

Finally, $P_k(\partial_{\mathcal{T}}c(V,W))$ is the prequantum operator associated with the function $\partial_{\mathcal{T}}c(V,W) \in C^{\infty}(M)$

$$P_k(\partial_{\mathcal{T}}c(V,W)) = \frac{i}{k} \nabla_{X_{\partial_{\mathcal{T}}c(V,W)}} + \partial_{\mathcal{T}}c(V,W),$$

where $X_{\partial_{\mathcal{T}}c(V,W)}$ is the Hamiltonian vector field of the function $\partial_{\mathcal{T}}c(V,W)$. In fact it was observe in [2], that since the curvature must preserve the holomorphic sections $X_{\partial_{\mathcal{T}}c(V,W)} = 0$ and so $d_M(\partial_{\mathcal{T}}c(V,W)) = 0$.

The form θ is given as follows

$$\theta(\mu_1, \bar{\mu_2}) = \frac{1}{4} g_{\mathcal{M}_{VP}^{n,k}}(G(\mu_1) \omega_{\mathcal{M}_{VP}^{n,k}} \bar{G}(\bar{\mu_2})), \tag{1}$$

In this paper we show that

Lemma 1.2

$$F_{\nabla^H}^{1,1} = \frac{ik}{2k+2n}(\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F) = -\frac{ik(n^2 - 1)}{12(k+n)\pi}\omega_{\mathcal{T}}$$

And using this we can find a 1-form \tilde{c} on \mathcal{T} and we consider

$$\tilde{\nabla}^H = \nabla^H + \tilde{c} \otimes \mathrm{Id}_{H^{(k)}}.$$

We remark that it might be that \tilde{c} is zero. In any case after this (possible trivial) modification, we can prove that

Theorem 1.3

The connection $\tilde{\nabla}^H$ is still a Hitchin connection and has pure (1,1) curvature given by

$$F_{\tilde{\nabla}^H} = \frac{ik(n^2 - 1)}{12(k+n)\pi} \omega_{\mathcal{T}}.$$

In section 2 we briefly recall our Kähler coordinate construction on the universal moduli space of vector bundles from [6]. In the following section 3, we compute the (1,1) part of the curvature of the Hitchin connection using the results of [6]. In final section 4 we modify the Hitchin connection by adding to it a scalar valued one-form on Teichmüller space tensor the identity of $H^{(k)}$, such that the resulting connection has only curvature of type (1,1).

2 The Moduli Space of Vector Bundles

In order to compute the curvature of the Hitchin connection, we will use the local coordinates of [6], which we will now briefly recall. Let Σ be a surface of genus two or greater. Pick a point in $\mathcal{T} \times M$, that is a Riemann surface X and a holomorphic vector bundle E over it. For an element

$$\mu \oplus \nu \in H^1(X, TX) \oplus H^1(X, \operatorname{End} E)$$

define a map

$$\chi^{\mu\oplus\nu}:\mathbb{H}\times\mathbf{SL}(n,\mathbb{C})\to\mathbb{H}\times\mathbf{SL}(n,\mathbb{C})$$

which is annihilated by the following differential operator

$$\bar{\partial}_{\mathbb{H}} \chi^{\mu \oplus \nu} = (\mu - \frac{1}{2} \tilde{g}_X^{-1} \operatorname{tr} \nu \otimes \nu) \cdot \partial_{\mathbb{H}} \chi^{\mu \oplus \nu} + \partial_{\mathbf{SL}(n,\mathbb{C})} \chi^{\mu \oplus \nu} \cdot \nu.$$

We will denote the projection to \mathbb{H} by $\chi_1^{\mu\oplus\nu}$ and the projection to $\mathbf{SL}(n,\mathbb{C})$ by $\chi_2^{\mu\oplus\nu}$.

The near by points contained in the coordinate neighbourhood in $\mathcal{T} \times M$ are represented by a pair of equivalence classes of representations into

 $\mathbf{PSL}(2,\mathbb{R})$ and $\mathbf{SU}(n)$ respectively. Let's say our base point corresponds to $\rho_{\mathbb{H}}: \pi_1(\Sigma) \to \mathbf{PSL}(2,\mathbb{R})$ and $\rho_E: \pi_1(\Sigma - p) \to \mathbf{SU}(n)$. Then the point corresponding to $\mu \oplus \nu$ is

$$(\rho_{\mathbb{H}}^{\mu\oplus\nu},\rho_E^{\mu\oplus\nu})(\gamma)=(\chi_1^{\mu\oplus\nu}(\rho_{\mathbb{H}}(\gamma)(\chi_1^{\mu\oplus\nu})^{-1}(z)),\chi_2^{\mu\oplus\nu}(\gamma z,e)\rho_E(\gamma)(\chi_2^{\mu\oplus\nu}(z,e))^{-1}).$$

We proved in [6] that this construction gives coordinates and moreover, we provided a Ricci potential for the total space and in particular, we showed in Theorem 4.2 in [6], that for the Ricci potential on M_{σ} , which is found in [15] fulfils

Lemma 2.1

For a pair of vector fields on \mathcal{T} represented by μ_1 and $\bar{\mu}_2$ we have that

$$2\bar{\partial}_{\mathcal{T}}\partial_{\mathcal{T}}F(\mu_{1},\bar{\mu}_{2}) = tr(\mu_{1}P_{\mathrm{End}E}^{1,0}\bar{\mu}_{2}P_{\mathrm{End}E}^{0,1}) - i\frac{n^{2}-1}{6\pi}\omega_{\mathcal{T}}(\mu_{1},\bar{\mu}_{2}).$$

Where $P_{\text{End}E}^{0,1}$ (resp. $P_{\text{End}E}^{1,0}$) is the projection on harmonic (0,1)-forms (resp. (1,0)-forms) with values in EndE.

3 The (1,1)-curvature of the Hitchin Connection

First we calculate $G(V_{\mu})$ in coordinates, here μ denotes the betrami differential corresponding to V by the Kodaira-Spencer map. We recall from Hitchin [10] that $G(V_{\mu})(\alpha,\beta) = \int_{\sigma} V'_{\mu}[-\star_{\sigma}] \operatorname{tr} \alpha \otimes \beta$. To calculate the variation of $-\star_{\sigma}$, we need to fix a harmonic 1-form, ν on Σ . We split it into $\nu = \nu_1 + \bar{\nu}_2^T$ at a point $X \in \mathcal{T}$ where ν_1, ν_2 are harmonic (0,1)-forms on X with values in EndE. Then we have that at a point $(X_{\mu \oplus 0}, E)$, we can use the quasiconformal maps $\chi_1^{\mu \oplus 0}$ to change the complex structure on X, so that the complex structure on $X_{\mu \oplus 0}$ is decribed by a quotient construction of \mathbb{H} with the standard structure. Then ν is given by

$$\begin{split} (\chi_1^{\mu\oplus 0})_*^{-1}\nu = & (\nu_1(\overline{\partial\chi_1^{\mu\oplus 0}})(d\bar{z} - \mu \frac{\partial\chi_1^{\mu\oplus 0}}{\partial\chi_1^{\mu\oplus 0}}dz) \\ & + \nu_2(\partial\chi_1^{\mu\oplus 0})(-\bar{\mu}\frac{\overline{\partial\chi_1^{\mu\oplus 0}}}{\partial\chi_1^{\mu\oplus 0}}d\bar{z} + dz)) \circ (\chi_1^{\mu\oplus 0})^{-1}. \end{split}$$

So we can find the harmonic representativ of ν at μ , which we denote ν^{μ} , using the projections on harmonic (1,0)-forms and (0,1)-forms on $X_{\mu\oplus 0}$ with values in EndE to obtain that

$$\nu^{\mu} = P_{\text{End}E}^{0,1}((\overline{\partial \chi_{1}^{\mu \oplus 0}})(\nu_{1}d\bar{z} - \bar{\mu}\nu_{2}d\bar{z})) \circ (\chi_{1}^{\mu \oplus 0})^{-1}) + P_{\text{End}E}^{1,0}((\partial \chi_{1}^{\mu \oplus 0}(\nu_{2}dz - \mu\nu_{1}dz)) \circ (\chi_{1}^{\mu \oplus 0})^{-1})$$

Now $I[\nu] = [-\star \nu^{\mu}]$ and as is seen in [10, Lemma 2.15] we have that $V_{\mu}(I)[\nu] = [V_{\mu}(-\star \nu^{\mu})]$, since $[-\star V_{\mu}\nu^{\mu}]$ is exact. To calculate $V_{\mu}[-\star \nu^{\mu}]$, we pull it back to X_0 with $\chi_1^{\mu \oplus 0}$ and find that

$$(\chi_{1}^{\mu \oplus 0})_{*}(-\star)\nu^{\mu} = iP_{\mathrm{End}E}^{0,1}((\overline{\partial \chi_{1}^{\mu \oplus 0}})(\nu_{1}d\bar{z} - \bar{\mu}\nu_{2}d\bar{z}))(-\mu(\partial \chi_{1}^{\mu \oplus 0})^{-1} + (\overline{\partial \chi_{1}^{\mu \oplus 0}})^{-1}) - iP_{\mathrm{End}E}^{1,0}((\partial \chi_{1}^{\mu \oplus 0})(\nu_{2}dz + \mu\nu_{1}dz))(\bar{\mu}(\overline{\partial \chi_{1}^{\mu \oplus 0}})^{-1} + (\partial \chi_{1}^{\mu \oplus 0})^{-1}).$$

When we evaluate this at $\varepsilon\mu$ and differentiate with respect to ε , then most of the terms have explicit factors of ε and are quickly seen to contribute $-iP_{\mathrm{End}E}^{1,0}\mu\nu_1 - i\mu P_{\mathrm{End}E}^{0,1}\nu_1$, at $\varepsilon = 0$. Now the only terms remaining are

$$P_{\mathrm{End}E}^{0,1}((\overline{\partial\chi_1^{\mu\oplus 0}})(\nu_1d\bar{z})(\overline{\partial\chi_1^{\mu\oplus 0}})^{-1})$$

and

$$P_{\text{End}\,E}^{1,0}((\partial \chi_1^{\mu \oplus 0})(\nu_2 dz))((\partial \chi_1^{\mu \oplus 0})^{-1}).$$

The harmonic projections are given as $P_{\mathrm{End}E}^{0,1}=I-\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$ and $P_{\mathrm{End}E}^{1,0}=I-\partial\Delta_0^{-1}\partial^*$. When we differentiated these with respect to ε the I's will disappear and either the first or last $\bar{\partial}$ or $\bar{\partial}^*$ (resp. ∂ or ∂^*) in $\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$ (resp. $\partial\Delta_0^{-1}\partial^*$) will not be differentiated. In the first case, we have an exact contribution, which does not change the cohomology class. In the second case the term will be zero, since $\nu \in \ker \bar{\partial}^*$ ($\bar{\nu}^T \in \ker \partial^*$). We now conclude that

$$V_{\mu}(I)[\nu] = [-iP_{\text{End}E}^{1,0}\mu\nu_1 - i\mu P_{\text{End}E}^{0,1}\nu_1].$$

And so we must have that

$$G(V_{\mu})(\nu_1, \nu_2) = -2i \int_{\Sigma} \mu \operatorname{tr} \nu_1 \nu_2,$$

and thus

$$G(V_{\bar{\mu}})(\bar{\nu}_1^T, \bar{\nu}_2^T) = 2i \int_{\Sigma} \bar{\mu} \mathrm{tr} \bar{\nu}_1^T \bar{\nu}_2^T.$$

Now that we have an expression in our coordinates for $G(V_{\mu})$ at the center point, we can calculate (1) in local coordinates

$$G(V_{\mu_1})\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(V_{\bar{\mu}_2})_{i\bar{j}}$$

$$= \left(\sum_{j,l} -2i \int_X \mu_1 \mathrm{tr}\bar{\nu}_i^T \bar{\nu}_j^T (-I \int \mathrm{tr}\nu_j \wedge \bar{\nu}_l^T) 2i \int_X \bar{\mu}_2 \mathrm{tr}\nu_l \nu_k\right).$$

Also recall that at the center point we have chosen our basis of ν_i 's to be orthonormal and so $P^{0,1}\alpha = -i\sum_i \nu_i \int_{\Sigma} \operatorname{tr} \alpha \wedge \nu_i$ and so we obtain that

$$G(V_{\mu_1})\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(V_{\bar{\mu}_2})_{i\bar{j}} = 4i\left(\int_X \mu_1 \mathrm{tr}\bar{\nu}_i^T P^{1,0}(\bar{\mu}_2 \nu_j)\right).$$

Contract with the metric and using that $\operatorname{tr} P^{0.1} F = \sum_i \int_{\Sigma} (F \nu_i) \wedge \bar{\nu}_i^T$, we get that

$$\theta(\mu_1, \bar{\mu}_2) = i \operatorname{tr}(\mu_1 P^{0,1} \bar{\mu}_2 P^{1,0})$$

Thus by Lemma 2.1 and Theorem 1.1, we have proved Lemma 1.2.

4 Modification of (2,0)-part of the Curvature

In this section we prove Theorem 1.3. First we observe that by the result of the previous section we can use the Bianchi idendity for the curvature to conclude that the (2,0)-part of the curvature of the Hitchin connection is $\bar{\partial}_{\mathcal{T}}$ closed, and hence $d_{\mathcal{T}}$ closed by the following argument. We let V',W' be holomorphic vector fields on \mathcal{T} and U'' anti-holomorphic. Then the Bianchi identity gives

$$0 = U''(F^{2,0}(V', W') - V'(F^{1,1}(W', U'')) + W'(F^{1,1}(U'', V')).$$

But since $F^{1,1}$ is proportional to the symplectic form on \mathcal{T} , we get that

$$-V'(F^{1,1}(W',U'')) + W'(F^{1,1}(U'',V')) = \partial_{\mathcal{T}}F^{1,1}(V',W',U'') = 0.$$

We conclude that $\bar{\partial}_{\mathcal{T}}F^{2,0}=0$. Finally we recall form [2] that $d_MF^{2,0}=0$ as well. Now use that $F^{2,0}$ is mapping class group invariant, so it pushes down to a closed (2,0)-form on the moduli space \mathcal{M}_q of genus g curves.

To proceed further we need to assume that Σ has genus three or greater, since this assumption will imply that the following two statement are true.

- The moduli space of genus $g \geq 3$ curves, \mathcal{M}_g , contains complete curves. This means that there exist a complex surface S and a holomorphic embedding $S \to \mathcal{M}_g$. For explicit construction see [19] for genus 3 and for higher genus references there in.
- The second thing we need is Harer's result [9], that for $g \geq 3$ the second cellular homology is

$$H_2(M_q,\mathbb{C}) \cong \mathbb{C}.$$

Harer's result implies that $H^2_{dR}(\mathcal{M}_g,\mathbb{C})\cong\mathbb{C}$, since it is dual to $H_2(M_g,\mathbb{C})$. We know that the generator must be $\omega_{\mathcal{T}}$, thus in order to prove that $F^{(2,0)}$ is exact, we need to show that it's class is 0. We can use the Surface S, which is a complex embedding submanifold and we can integrate $F^{(2,0)}$ over it and as it is a (2,0)-form the result is 0, at the same time we know that the integral of $\omega_{\mathcal{T}}$ is non-zero over S and so the cohomology class of $F^{(2,0)}$ is 0. This means that there exists a 1-from \tilde{c} on \mathcal{M}_g such that $F^{(2,0)} = -d_{\mathcal{M}_g}\tilde{c}$.

Now we can pull back \tilde{c} to \mathcal{T} and then define a sligtly modified, but still mapping class group invariant Hitchin connection, as discussed in the

introduction. We just need to check that it is still a Hitchin connection. By [3, Lemma 2.2] it is enough to prove that

$$\frac{i}{2}V[I](\nabla_{V}^{t})^{1,0}s + \nabla_{M_{\sigma}}^{0,1}(u(V) + \tilde{c}(V))s = 0$$

But since $\nabla^t + u(V)$ is a Hitchin connection, this reduce to showing $\bar{\partial}_{M_{\sigma}}\tilde{c}(W,V) = 0$. But that follows from the defining identity, since $d_{\mathcal{T}}\tilde{c} = \partial_{\mathcal{T}}c$ which is a (2,0) from. To calculate the curvature we se that

$$F_{\tilde{\nabla}}(V,W) = [\nabla_V + \tilde{c}(V), \nabla_W + \tilde{c}(W)] = [\nabla_V, \nabla_W] + [\tilde{c}(V), \nabla_W] + [\tilde{c}(V), \tilde{c}(W)] + [\tilde{c}(V), \tilde{c}(W)].$$

The first term is just the curvature calculated in Theorem 1.1. The two next terms only contribute $-W[\tilde{c}(V)] + V[\tilde{c}(W)] = d_{\mathcal{T}}\tilde{c}(W,V)$, since \tilde{c} does not depend on where we are in the moduli space of vector bundles and so commute with the differential operator u. The last term is also zero, since multiplication by functions commute, hence we conclude that

$$F_{\tilde{\nabla}}(V,W) = \frac{(n^2 - 1)k}{6\pi(k + n)}\omega_{\mathcal{T}}(V,W) + F_{\nabla}^{(2,0)}(V,W) + d_{\mathcal{T}}\tilde{c}(V,W) = \frac{(n^2 - 1)k}{6\pi(k + n)}\omega_{\mathcal{T}}(V,W)$$

where the last equality follows by the construction of \tilde{c} , since $F_{\nabla}^{(2,0)}(V,W) = -d_{\mathcal{T}}\tilde{c}(V,W)$. This concludes the proof of Theorem 1.3.

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