Leapfrogging vortex rings for the three dimensional Gross-Pitaevskii equation

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Abstract

Leapfrogging motion of vortex rings sharing the same axis of symmetry was first predicted by Helmholtz in his famous work on the Euler equation for incompressible fluids. Its justification in that framework remains an open question to date. In this paper, we rigorously derive the corresponding leapfrogging motion for the axially symmetric three-dimensional Gross-Pitaevskii equation.

1 Introduction

The goal of this paper is to describe a class of cylindrically symmetric solutions to the threedimensional Gross-Pitaevskii equation

$$i\partial_t u - \Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2)$$

for a complex-valued function $u: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$. In the regime which we shall describe, it turns out that the Gross-Pitaevskii equation bears some resemblance with the Euler equation for flows of incompressible fluids

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = -\nabla p \\ \operatorname{div} v = 0, \end{cases}$$

where $v: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the velocity field and $p: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the pressure field. In this analogy, the role of the velocity v is played by the current[†]

$$j(u) := u \times \nabla u = (iu, \nabla u) = \operatorname{Re}(u\nabla \bar{u})$$

and the vorticity field $\omega := \operatorname{curl} v$ therefore corresponds, up to a factor of two, to the Jacobian

$$J(u) := \frac{1}{2}\operatorname{curl} j(u) = \left(\partial_2 u \times \partial_3 u, \partial_3 u \times \partial_2 u, \partial_1 u \times \partial_2 u\right).$$

In his celebrated work [7, 8] on the Euler equation, Helmholtz considered with great attention the situation where the vorticity field ω is concentrated in a "circular vortex-filament of very small section", a thin vortex ring. A central question in Helmholtz's work, as far as dynamics is concerned, is related to the possible forms of stability of the family of such vortex rings, allowing a change in time of cross-section, radius, position or even possibly of inner profile, and a description of these evolutions. When only one vortex-filament is present, Helmholtz's conclusions are:

For $y \in \mathbb{C}$ and $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ we write $(y, z) := (\operatorname{Re}(y\bar{z}_1), \dots, \operatorname{Re}(y\bar{z}_k)) \in \mathbb{R}^k$ and $y \times z := (iy, z)$.

Hence in a circular vortex-filament of very small section in an indefinitely extended fluid, the center of gravity of the section has, from the commencement, an approximately constant and very great velocity parallel to the axis of the vortex-ring, and this is directed towards the side to which the fluid flows through the ring.

Instead, when two vortex-filaments interact, Helmholtz predicts the following:

We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

The motion described by Helmholtz, and illustrated in Figure 1 below, is often termed leapfrogging in the fluid mechanics community. Even though it has been widely studied since Helmholtz, as far as we know it has not been mathematically justified in the context of the Euler equation, even in the axi-symmetric case without swirl[†]. As a matter of fact, the interaction leading to the leapfrogging motion is somehow borderline in strength compared to the stability of isolated vortex rings.

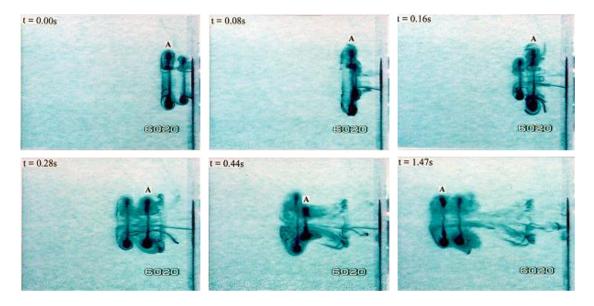


Figure 1: ©T.T. Lim, Phys. of Fluids, Vol. 9

Our main results in this paper, Theorem 1 and 2 below, provide a mathematical justification to the leapfrogging motion of two or more vortex rings in the context of the axi-symmetric three-dimensional Gross-Pitaevskii equation.

[†]We refer to [5, 17] for some attempts in that direction, and an account of the difficulties.

1.1 Reference vortex rings

A well-known particularity of the Gross-Pitaevskii equation is that vortex ring intensities are necessarily quantized. For stability reasons, we only consider simply quantized rings.

Let \mathcal{C} be a smooth oriented closed curve in \mathbb{R}^3 and let $\vec{\mathcal{J}}$ be the vector distribution corresponding to 2π times the circulation along \mathcal{C} , namely

$$\langle \vec{\mathcal{J}}, \vec{X} \rangle = 2\pi \int_{\mathcal{C}} \vec{X} \cdot \vec{\tau} \qquad \forall \vec{X} \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),$$

where $\vec{\tau}$ is the tangent vector to \mathcal{C} . To the "current density" $\vec{\mathcal{J}}$ is associated the "induction" \vec{B} , which satisfies the equations

$$\operatorname{div}(\vec{B}) = 0, \qquad \operatorname{curl}(\vec{B}) = \vec{\mathcal{J}} \qquad \text{in } \mathbb{R}^3,$$

and is obtained from $\vec{\mathcal{J}}$ by the Biot-Savart law. To \vec{B} is then associated a vector potential \vec{A} , which satisfies

$$\operatorname{div}(\vec{A}) = 0, \quad \operatorname{curl}(\vec{A}) = \vec{B} \quad \text{in } \mathbb{R}^3,$$

so that

$$-\Delta \vec{A} = \operatorname{curl} \operatorname{curl}(\vec{A}) = \vec{\mathcal{J}} \quad \text{in } \mathbb{R}^3.$$

Since we only consider axi-symmetric configurations in this paper, we let \mathbb{H} to be the half-space $\{(r,z) \mid r > 0, z \in \mathbb{R}\}$ and we denote by $r(\cdot)$ and $z(\cdot)$ the coordinate functions in \mathbb{H} . For $a \in \mathbb{H}$, let \mathcal{C}_a be the circle of radius r(a) parallel to the xy-plane in \mathbb{R}^3 , centered at the point (0,0,z(a)), and oriented so that its binormal vector points towards the positive z-axis. By cylindrical symmetry, we may write the corresponding vector potential as

$$\vec{A}_a \equiv A_a(r,z)\vec{e}_{\theta}.$$

The expression of the vector Laplacian in cylindrical coordinates yields the equation for the scalar function A_a :

$$\begin{cases} -\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_z^2\right)A_a = 2\pi\delta_a & \text{in } \mathbb{H} \\ A_a = 0 & \text{on } \partial\mathbb{H}, \end{cases}$$

or equivalently

$$\begin{cases} -\operatorname{div}\left(\frac{1}{r}\nabla\left(rA_{a}\right)\right) = 2\pi\delta_{a} & \text{in } \mathbb{H} \\ A_{a} = 0 & \text{on } \partial\mathbb{H}, \end{cases}$$

which can be integrated explicitly in terms of complete elliptic integrals † .

Up to a constant phase factor, there exists a unique unimodular map $u_a^* \in \mathcal{C}^{\infty}(\mathbb{H} \setminus \{a\}, S^1) \cap W^{1,1}_{loc}(\mathbb{H}, S^1)$ such that

$$r(iu_a^*, \nabla u_a^*) = rj(u_a^*) = -\nabla^{\perp}(rA_a).$$

[†]The integration is actually simpler in the original cartesian coordinates. A classical reference is the book of Jackson [10], an extended analysis can be found in the 1893 paper of Dyson [6]. See Appendix A for some details.

In the sense of distributions in \mathbb{H} , we have

$$\begin{cases} \operatorname{div}(rj(u_a^*)) &= 0\\ \operatorname{curl}(j(u_a^*)) &= 2\pi\delta_a, \end{cases}$$

and the function u_a^* corresponds therefore to a *singular* vortex ring. In order to describe a reference vortex ring for the Gross-Pitaevskii equation, we shall make the notion of core more precise. In \mathbb{R}^2 , the Gross-Pitaevskii equation possesses a distinguished stationary solution called vortex: in polar coordinates, it has the special form

$$u_{\varepsilon}(r,\theta) = f_{\varepsilon}(r) \exp(i\theta)$$

where the profile $f_{\varepsilon}: \mathbb{R}^+ \to [0,1]$ satisfies $f_{\varepsilon}(0) = 0$, $f_{\varepsilon}(+\infty) = 1$, and

$$\partial_{rr} f_{\varepsilon} + \frac{1}{r} \partial_{r} f_{\varepsilon} - \frac{1}{r^{2}} f_{\varepsilon} + \frac{1}{\varepsilon^{2}} f_{\varepsilon} (1 - f_{\varepsilon}^{2}) = 0.$$

Notice that ε has the dimension of a length, and since by scaling $f_{\varepsilon}(r) = f_1(\frac{r}{\varepsilon})$ it is the characteristic length of the core.

The reference vortex ring associated to the point $a \in \mathbb{H}$ is defined to be

$$u_{\varepsilon,a}^*(r,z) = f_{\varepsilon} (\|(r,z) - a\|) u_a^*(r,z).$$

More generally, when $a = \{a_1, \dots, a_n\}$ is a family of n distinct points in \mathbb{H} , we set

$$u_a^*(r,z):=\prod_{k=1}^n u_{a_k}^*(r,z),\quad \text{and}\quad u_{\varepsilon,a}^*(r,z):=\prod_{k=1}^n u_{\varepsilon,a_k}^*(r,z),$$

where the products are meant in \mathbb{C} . The field $u_{\varepsilon,a}^*$ hence corresponds to a collection of n reference vortex rings (sharing the same axis and oriented in the same direction), and is the typical kind of object which we shall study the evolution of. It can be shown that

$$\|Ju_{\varepsilon,a}^* - \pi \sum_{i=1}^n \delta_{a_i}\|_{\dot{W}^{-1,1}(\mathbb{H})} = O(\varepsilon) \text{ as } \varepsilon \to 0,$$

where here and in the sequel, for a complex function u on \mathbb{H} we denote by Ju its jacobian function $Ju = \partial_r u \times \partial_z u$.

1.2 The system of leapfrogging

Being an exact collection of (or even a single) reference vortex rings is not a property which is preserved by the flow of the Gross-Pitaevskii equation[†]. To carry out our analysis, we rely mainly on the energy density and the current density. For cylindrically symmetric solutions $u \equiv u(r, z, t)$, the Gross-Pitaevskii equation writes

$$(GP)_{\varepsilon}^{c} \qquad \begin{cases} ir\partial_{t}u - \operatorname{div}(r\nabla u) = \frac{1}{\varepsilon^{2}}ru(1 - |u|^{2}) & \text{in} \quad \mathbb{H} \times \mathbb{R}, \\ \partial_{r}u = 0 & \text{on} \quad \partial \mathbb{H} \times \mathbb{R}. \end{cases}$$

[†]Exact traveling waves having the form of vortex rings have been constructed in [4], these are very similar in shape but not exactly equal to reference vortex rings.

Equation $(GP)_{\varepsilon}^{c}$ is an hamiltonian flow for the (weighted) Ginzburg-Landau energy

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u) := \int_{\mathbb{H}} \left(\frac{|\nabla u|^2}{2} + \frac{(1 - |u|^2)}{4\varepsilon^2} \right) r \, dr dz,$$

and the Cauchy problem is known to be well-posed for initial data with finite energy. Classical computations leads to the estimate:

Lemma 1. It holds

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}\left(u_{\varepsilon,a}^{*}\right) = \sum_{i=1}^{n} r(a_{i}) \left[\pi \log\left(\frac{r(a_{i})}{\varepsilon}\right) + \gamma + \pi\left(3\log(2) - 2\right) + \pi \sum_{j \neq i} A_{a_{j}}(a_{i}) + O\left(\left(\frac{\varepsilon}{\rho_{a}}\right)^{\frac{2}{3}} \log^{2}\left(\frac{\varepsilon}{\rho_{a}}\right)\right)\right],$$

where

$$\rho_a := \frac{1}{4} \min \left(\min_{i \neq j} |a_i - a_j|, \min_i r(a_i) \right). \tag{1}$$

In Lemma 1, the constant γ is defined by (see [2])

$$\gamma := \liminf_{\varepsilon \to 0} \left[\mathcal{E}_{\varepsilon}(v_{\varepsilon}, B_1) - \pi |\log \varepsilon| \right] \text{ with } v_{\varepsilon} \in H^1(B_1, \mathbb{C}) \text{ and } v_{\varepsilon}(z) = z \text{ on } \partial B_1,$$

where B_1 is the unit disk in \mathbb{R}^2 and where for an open subset $\Omega \subset \mathbb{R}^2$ and $u \in H^1_{loc}(\Omega, \mathbb{C})$ we denote the unweighted two-dimensional Ginzburg-Landau energy of u in Ω by

$$\mathcal{E}_{\varepsilon}(u,\Omega) = \int_{\Omega} e_{\varepsilon}(u) d\mathcal{L}^2 := \int_{\Omega} \left(\frac{|\nabla u|^2}{2} + \frac{(1 - |u|^2)}{4\varepsilon^2} \right) d\mathcal{L}^2.$$

In light of Lemma 1, we define the quantity

$$H_{\varepsilon}(a_1, \cdots, a_n) := \sum_{i=1}^n r(a_i) \left[\pi \log \left(\frac{r(a_i)}{\varepsilon} \right) + \gamma + \pi \left(3 \log(2) - 2 \right) + \pi \sum_{i \neq i} A_{a_i}(a_i) \right],$$

and we consider the associated hamiltonian system

$$\dot{a}_i(s) = \frac{1}{\pi |\log \varepsilon|} \mathbb{J} \nabla_{a_i} H_{\varepsilon} (a_1(s), \cdots, a_n(s)), \qquad i = 1, \cdots, n,$$

where, with a slight abuse of notation,

$$\mathbb{J} := \begin{pmatrix} 0 & -\frac{1}{r(a_i)} \\ \frac{1}{r(a_i)} & 0 \end{pmatrix}.$$

In addition to the hamiltonian H_{ε} , the system $(LF)_{\varepsilon}$ also conserves the momentum

$$P(a_1, \dots, a_n) := \pi \sum_{k=1}^{n} r^2(a_k),$$

which may be interpreted as the total area of the disks determined by the vortex rings. As a matter of fact, note also that

$$\mathcal{P}(u_{\varepsilon,a}^*) := \int_{\mathbb{H}} Ju_{\varepsilon,a}^* \, r \, dr dz = \pi \sum_{k=1}^n r^2(a_k) + o(1),$$

as $\varepsilon \to 0$, and that, at least formally, the momentum \mathcal{P} is a conserved quantity for $(GP)^c_{\varepsilon}$.

When n=2, the system $(LF)_{\varepsilon}$ may be analyzed in great details. Since P is conserved and since H_{ε} is invariant by a joint translation of both rings in the z direction, it is classical to introduce the variables (η, ξ) by

$$\begin{cases} r^2(a_1) = \frac{P}{2} - \eta \\ r^2(a_2) = \frac{P}{2} + \eta \end{cases}, \qquad \xi = z(a_1) - z(a_2),$$

and to draw the level curves of the function H_{ε} in those two real variables, the momentum P being considered as a parameter.

The next figure illustrates the global behavior of the phase portrait, with three distinct regions which we have called "pass through", "attract then repel" and "leapfrogging". The leapfrogging region corresponds to the central part, where all solutions are periodic in time; its interpretation was discussed earlier in this introduction. In the pass through region, the first vortex ring always remains the smallest, hence quickest, of the two vortex rings: being initially located below the second vortex ring on the z-axis it first catches up, then passes inside the second and finally gets away in front of it. Instead, in the attract then repel region the first vortex ring initially starts to catch up, but doing so its circular radius increases whereas the one of the second vortex ring decreases, up to a point where both vortex rings have the same radius and the first still lag behind the second. From that point on, the first one has a larger radius than the second, and therefore the second increases its lead indefinitely. The behavior in those last two regions is actually very much reminiscent of two-solitons interactions in the Korteweg - de Vries equation, in particular the speeds at plus and minus infinity in time are equal or exchanged. Notice also that the two points at the common boundary of the three regions correspond, up to labeling, to the same situation : two vortex rings travel with the same constant speed at a special mutual distance[‡].

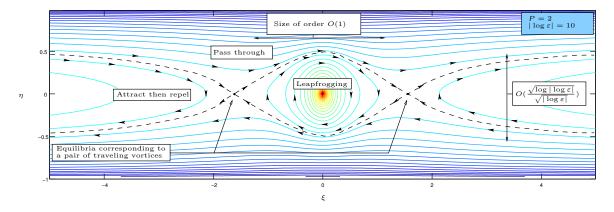


Figure 2: Phase portrait of the system $(LF)_{\varepsilon}$ for two vortex rings

The typical size of the leapfrogging region is also described in the figure. In particular, it shrinks and becomes more flat as ε decreases towards zero.

[†]A similar situation is described by Hicks [9] for a simplified vortex model introduced by Love [16] in 1894.

[‡]We stress that this holds at the level of the system $(LF)_{\varepsilon}$, we do not know whether such special solutions exist at the level of equation $(GP)_{\varepsilon}^{c}$.

1.3 Statement of the main results

We present two results in this section. The first one follows rather easily from the second, but its statement has the advantage of being somewhat simpler. On the other hand, it involves a limiting procedure $\varepsilon \to 0$, whereas the second one is valid for small but fixed values of ε .

In order to state those results, and in view of the size of the leapfrogging region mentioned at the end of the previous subsection, we fix some $(r_0, z_0) \in \mathbb{H}$, an integer $n \geq 1$, and n distinct points b_1^0, \dots, b_n^0 in \mathbb{R}^2 . The initial positions of the cores of the vortex rings are then set to be

$$a_{i,\varepsilon}^0 := \left(r_0 + \frac{r(b_i^0)}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{z(b_i^0)}{\sqrt{|\log \varepsilon|}}\right), \quad i = 1, \dots, n.$$

As a matter of fact, this is the appropriate scaling for which **relative** self-motion and interactions between vortex-rings are of the same magnitude. In any scaling in which $a_{i,\varepsilon}^0 - (r_0, z_0) = o(1)$ as $\varepsilon \to 0$ for all i, the "leading-order" vortex motion is expected to be a translation with constant velocity $1/r_0$ in the vertical direction and in the rescaled time. The above scaling is the appropriate one for which, in the next-order correction, the difference in the self-motion speeds (due to different values of the radii at the next order) and interaction between vortices are of the same magnitude. In the case of two vortices, for example, this will give rise to small-scale periodic corrections to a leading-order translation, which is the signature of "leapfrogging".

Note that $a_{i,\varepsilon}^0 \in \mathbb{H}$ provided ε is sufficiently small, which we assume throughout. Concerning their evolution, we consider the solution to the Cauchy problem for the system of ordinary differential equations

(LF)
$$\begin{cases} \dot{b}_i(s) = \sum_{j \neq i} \frac{(b_i(s) - b_j(s))^{\perp}}{\|b_i(s) - b_j(s)\|^2} - \frac{r(b_i(s))}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases} i = 1, \dots, n,$$

and we finally set

$$a_{i,\varepsilon}(s) := \left(r_0 + \frac{r(b_i(s))}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{s}{r_0} + \frac{z(b_i(s))}{\sqrt{|\log \varepsilon|}}\right). \tag{2}$$

System (LF) and (2) describe the main order asymptotic of (LF)_{ε} in the leapfrogging region, after a proper rescaling in time.

We will prove

Theorem 1. Let $(u_{\varepsilon}^0)_{\varepsilon>0}$ be a family of initial data for $(GP)_{\varepsilon}^c$ such that

$$\left\| J u_{\varepsilon}^{0} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}^{0}} \right\|_{\dot{W}^{-1,1}(\Omega)} = o\left(\frac{1}{|\log \varepsilon|}\right), \tag{3}$$

as $\varepsilon \to 0$, for any open subset Ω strongly included in \mathbb{H} . Assume also that

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u_{\varepsilon}^{0}) \leq H_{\varepsilon}(a_{1,\varepsilon}^{0}, \cdots, a_{n,\varepsilon}^{0}) + o(1), \ as \ \varepsilon \to 0.$$
 (4)

Then, for every $s \in \mathbb{R}$ and every open subset Ω strongly included in \mathbb{H} we have

$$\left\| J u_{\varepsilon}^{s} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(s)} \right\|_{\dot{W}^{-1,1}(\Omega)} = o\left(\frac{1}{\sqrt{|\log \varepsilon|}}\right)$$
 (5)

where we denote by u_{ε}^s the solution of $(GP)_{\varepsilon}^c$ with initial datum u_{ε}^0 and evaluated at time $t = s/|\log \varepsilon|$, and where the points $a_{i,\varepsilon}(s)$ are defined in (2) through the solution of the system (LF).

In the statement of Theorem 1, the $\dot{W}^{-1,1}$ norm is defined by

$$\|\mu\|_{\dot{W}^{-1,1}(\Omega)} = \sup \left\{ \int \varphi \, d\mu, \ \varphi \in W_0^{1,\infty}(\Omega), \ \|\nabla \varphi\|_{\infty} \le 1 \right\}.$$

Remark 1. Asymptotic formulas for the potential vectors A_{a_i} (see Appendix A) lead to the equivalence

$$H_{\varepsilon}(a_{1,\varepsilon},\cdots,a_{n,\varepsilon}) = \Gamma_{\varepsilon}(r_0,n) + W_{\varepsilon,r_0}(b_1,\cdots,b_n) + o(1) \text{ as } \varepsilon \to 0,$$
 (6)

where $\Gamma_{\varepsilon}(r_0, n) = nr_0(\pi |\log \varepsilon| + \gamma + \pi n \log r_0 + \pi n(3 \log 2 - 2) + \pi \frac{n-1}{2} \log |\log \varepsilon|)$ and

$$W_{\varepsilon,r_0}(b_1,\cdots,b_n) = \pi \sum_{i=1}^n r(b_i) \sqrt{|\log \varepsilon|} - \pi r_0 \sum_{i \neq j} \log|b_i - b_j|.$$
 (7)

Also, expansion of the squares leads directly to

$$P(a_{1,\varepsilon},\cdots,a_{n,\varepsilon}) = \pi n r_0^2 + 2\pi r_0 \sum_{i=1}^n \frac{r(b_i)}{\sqrt{|\log \varepsilon|}} + \pi \sum_{i=1}^n \frac{r(b_i)^2}{|\log \varepsilon|},$$

and therefore

$$\left(H_{\varepsilon} - \frac{|\log \varepsilon|}{2r_0}P\right)(a_{1,\varepsilon}, \cdots, a_{n,\varepsilon}) = -\frac{\pi}{2}nr_0|\log \varepsilon| + \Gamma_{\varepsilon}(r_0, n) + \pi r_0W(b_1, \cdots, b_n) + o(1), \quad as \ \varepsilon \to 0,$$

where

$$W(b_1, \dots, b_n) := -\sum_{i \neq j} \log|b_i - b_j| - \frac{1}{2r_0^2} \sum_{i=1}^n r(b_i)^2.$$

The function W, which does not depend upon ε , is precisely the hamiltonian for the system (LF). A second quantity preserved by (LF) is given by $Q(b_1, \dots, b_n) := \sum_{i=1}^n r(b_i)$. When n=2, all the solutions are (LF) are periodic in time.

We will now state a quantitative version of Theorem 1 which holds for small but fixed values of ε , not just asymptotically as $\varepsilon \to 0$. We fix positive constants K_0 and r_0 and we consider an arbitrary solution $a_{\varepsilon}(s) \equiv \{a_{i,\varepsilon}(s)\}_{1 \le i \le n}$ of the system (LF) $_{\varepsilon}$ on some time interval $[0, S_0], S_0 \ge 0$, which we assume to satisfy

$$\frac{K_0^{-1}}{\sqrt{|\log \varepsilon|}} \le \min_{\substack{s \in [0, S_0] \\ 1 \le j}} \min_{\substack{i \neq j}} |a_{i,\varepsilon}(s) - a_{j,\varepsilon}(s)| \le \max_{\substack{s \in [0, S_0] \\ s \in [0, S_0]}} \max_{\substack{i \neq j}} |a_{i,\varepsilon}(s) - a_{j,\varepsilon}(s)| \le \frac{K_0}{\sqrt{|\log \varepsilon|}}$$

$$\frac{r_0}{2} \le \min_{\substack{s \in [0, S_0] \\ i \in [0, S_0]}} \min_{\substack{i \in [0, S_0] \\ i \in [0, S_0]}} \max_{\substack{i \in [0, S_0] \\ i \in [0, S_0]}} \max_{\substack{i \in [0, S_0] \\ i \in [0, S_0]}} |a_{i,\varepsilon}(s) - a_{j,\varepsilon}(s)| \le \frac{K_0}{\sqrt{|\log \varepsilon|}}$$
(8)

We define the localization scale

$$r_a^0 := \left\| J u_{\varepsilon}^0 - \pi \sum_{i=1}^n \delta_{a_{i,\varepsilon}^0} \right\|_{\dot{W}^{-1,1}(\Omega_0)}, \tag{9}$$

where $\Omega_0 := \{r \geq \frac{r_0}{4}\}$, and the excess energy

$$\Sigma^{0} := \left[\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u_{\varepsilon}^{0}) - H_{\varepsilon}(a_{1,\varepsilon}^{0}, \cdots, a_{n,\varepsilon}^{0}) \right]^{+}$$

$$(10)$$

at the initial time.

Theorem 2. Let $a_{\varepsilon}(s) \equiv \{a_{i,\varepsilon}(s)\}_{1 \leq i \leq n}$ be a solution of the system $(LF)_{\varepsilon}$ on some time interval $[0, S_0]$, $S_0 \geq 0$, which satisfies (8). There exist positive numbers ε_0 , σ_0 and C_0 , depending only on r_0 , n, K_0 and S_0 with the following properties. Assume that $0 < \varepsilon \leq \varepsilon_0$ and that

$$r_a^0|\log\varepsilon| + \Sigma^0 \le \sigma_0,\tag{11}$$

then

$$\left\| Ju_{\varepsilon}^{s} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(s)} \right\|_{\dot{W}^{-1,1}(\Omega_{0})} \leq C_{0} \left(r_{a}0 + \frac{\Sigma^{0}}{\sqrt{|\log \varepsilon|}} + \frac{C_{\delta}}{|\log \varepsilon|^{1-\delta}} \right) e^{C_{0}s},$$

for every $s \in [0, S_0]$, where $\delta > 0$ can be chosen arbitrarily small.

To finish this introduction, let us mention that we have not analyzed the convergence of $(GP)_{\varepsilon}^{c}$ towards $(LF)_{\varepsilon}$ in the "pass through" and "attract then repel" regions. It is conceivable, yet probably difficult, to obtain closeness estimates valid for all times in those cases, reminiscent of what is sometimes called orbital stability of multi-solitons, e.g. in the Korteweg - de Vries equation [18] or the 1D Gross-Pitaevskii equation [3]. One would have to deal with algebraic rather than exponential interaction estimates.

Also, having in mind the initial question related to the Euler equation, let us mention that one crucial advantage in the analysis of the Gross-Pitaevskii equation is that it has an inherent core localization scale ε . On the other hand, Euler velocity fields are divergence free, whereas Gross-Pitaevskii ones only have small divergence when averaged in time. Analysis of leapfrogging for the Euler equation would therefore probably require a different strategy.

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2 Strategy for the proofs

The overall strategy follows many of the lines which we adopted in our prior work [11] on the inhomogeneous Gross-Pitaevskii equation[†] The effort is actually focused on Theorem 2 first, Theorem 1 can be deduced from it rather directly. The essential new ingredients with respect to [11] are refined approximation estimates (mainly Proposition 1) and the key observation in Proposition 5.

[†]Another work on the 2D inhomogeneous GP equation is a recent preprint of Kurzke et al [15], which studies a situation where the inhomogeneity and its derivatives are of order $|\log \varepsilon|^{-1}$. This is critical in the sense that interaction of vortices with the background potential and with each other are of the same order of magnitude. In the present work, by contrast, critical coupling occurs in hard-to-resolve corrections to the leading-order dynamics.

2.1 Localisation, excess energy and approximation by a reference field

In this section we present arguments which are not directly related to the time evolution but only to some assumptions on the energy density and on the Jacobian of a function u. In rough terms, we assume that u is known a priori to satisfy some localisation estimates and some energy upper bounds, and we will show, by combining them together, that under a certain approximation threshold this can be improved by a large amount, without any further assumption.

In order to state quantitative results, we assume here that $\{a_i\}_{1\leq i\leq n}$ is a collection of points in \mathbb{H} such that

$$\frac{8}{|\log \varepsilon|} \le \min_{i \ne j} |a_i - a_j| \le \max_{i \ne j} |a_i - a_j| \le K_1$$

$$\frac{r_0}{2} \le r(a_i) \le \max_i r(a_i) \le 2r_0.$$
(H₁)

We assume next that $u \in H^1_{loc}(\mathbb{H}, \mathbb{C})$ is such that its Jacobian Ju satisfies the rough localisation estimate

$$r_a := \|Ju - \pi \sum_{i=1}^n \delta_{a_i}\|_{\dot{W}^{-1,1}(\Omega_0)} < \frac{\rho_a}{4}, \tag{12}$$

where ρ_a is defined in (1). We finally define the excess energy relative to those points,

$$\Sigma_a := \left[\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u) - H_{\varepsilon}(a_1, \cdots, a_n) \right]^+. \tag{13}$$

We will show that if r_a and Σ_a are not too large then actually a much better form of localisation holds.

Proposition 1. Under the assumption (H_1) and (12), there exist constants $\varepsilon_1, \sigma_1, C_1 > 0$, depending only on n, r_0 and K_1 , with the following properties. If $\varepsilon \leq \varepsilon_1$ and

$$\Sigma_a^r := \Sigma_a + r_a |\log \varepsilon| \le \sigma_1 |\log \varepsilon|, \tag{14}$$

then there exist ξ_1, \dots, ξ_n in \mathbb{H} such that

$$||Ju - \pi \sum_{i=1}^{n} \delta_{\xi_i}||_{\dot{W}^{-1,1}\left(\left\{r \ge C_1(\Sigma_{\xi} + 1)/|\log \varepsilon|\right\}\right)} \le C_1 \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r}, \tag{15}$$

and

$$\int_{\mathbb{H}\setminus\cup_{i}B(\xi_{i},\varepsilon^{\frac{2}{3}})} r \left[e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j(u_{\xi}^{*}) \right|^{2} \right] \leq C_{1} \left(\Sigma_{\xi} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^{C_{1}} e^{C_{1} \Sigma_{a}^{r}} \right), \tag{16}$$

where we have written

$$\Sigma_{\xi} := \left[\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u) - H_{\varepsilon}(\xi_1, \cdots, \xi_n) \right]^+.$$

Moreover,

$$\Sigma_{\xi} \le \Sigma_a + C_1 r_a |\log \varepsilon| + C_1 \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r}, \tag{17}$$

and the values of ε_1 and σ_1 are chosen sufficiently small so that

$$C_1|\log \varepsilon|^{C_1} e^{C_1 \sigma_1|\log \varepsilon|} \le \varepsilon^{-\frac{1}{6}}, \quad and \quad C_1(\Sigma_{\xi} + 1)/|\log \varepsilon| \le \frac{r_0}{4}$$
 (18)

whenever $\varepsilon \leq \varepsilon_1$.

Remark 2. It is tempting to simplify somewhat the statement of Proposition 1 by replacing the term Σ_{ξ} in the right-hand side of (16) by Σ_{a}^{r} (in view of (17) this would be correct up to a possible change of C_{1}), and hence obtain error bounds that only depend on the input data. Yet, it turns out that (17) is not optimal in all cases and the key step of our subsequent analysis will make use of that difference.

We will now focus on estimates that are valid up to and including the cores. By definition (see Appendix A.1), we have

$$rj(u_{\varepsilon}^*) = -\nabla^{\perp}(r\Psi_{\varepsilon}^*).$$

Since the latter is singular at the points a_i and not in L^2_{loc} , there is no hope that estimate (16) in Proposition 1 could be extended to the whole of \mathbb{H} . For that purpose, we have to replace $j(u_{\xi}^*)$ by some mollified version. The function $j(u_{\xi,\varepsilon}^*)$ would be a natural candidate, but that would require that the vortex locations ξ_i are known to a precision at least as good as ε , which is not the case in view of (15). For that reason, instead we modify the function Ψ_{ξ}^* to a function Ψ_{ξ}^* in the following way (truncate $r\Psi_{\xi}^*$):

We write

$$r_{\xi} := C_1 \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r} \tag{19}$$

and for each $i=1,\dots,n$ we consider the connected component \mathcal{C}_i of the superlevel set $\{r\Psi_{\xi}^* \geq r\Psi_{\xi}^*(\xi_i + (r_{\xi}, 0))\}$ (by convention we include ξ_i , where Ψ_{ξ}^* is in principle not defined, in this set) which contains the point $\xi_i + (r_{\xi}, 0)$, and we set $r\Psi_{\xi}^{\natural} = r\Psi_{\xi}^*(\xi_i + (r_{\xi}, 0))$ inside \mathcal{C}_i . Next, we set $\Psi_{\xi}^{\natural} = \Psi_{\xi}^*$ on $\mathbb{H} \setminus \bigcup_{i=1}^n \mathcal{C}_i$ and finally we define

$$rj^{\natural}(u_{\xi}^{*}) = -\nabla^{\perp}(r\Psi_{\xi}^{\natural}). \tag{20}$$

Remark 3. Note that by construction $j^{\natural}(u_{\xi}^*)$ and $j(u_{\xi}^*)$ coincide everywhere outside $\cup_i \mathcal{C}_i$, that is everywhere except on a neighborhood of order r_{ξ} of the points ξ_i , and that $j^{\natural}(u_{\xi}^*) \equiv 0$ inside each \mathcal{C}_i . In the sense of distributions,

$$\operatorname{div}(rj^{\dagger}(u_{\varepsilon}^{*})) = 0 \tag{21}$$

and

$$\operatorname{curl}(j^{\natural}(u_{\xi}^{*})) = \sum_{i=1}^{n} |j(u_{\xi}^{*})| d\mathcal{H}^{1} \bigsqcup_{\partial \mathcal{C}_{i}}.$$
 (22)

Proposition 2. In addition to the statements of Proposition 1, there exists $\varepsilon_2 \leq \varepsilon_1$ such that if $\varepsilon \leq \varepsilon_2$ then we have

$$\int_{\mathbb{H}} r \left[e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j^{\natural}(u_{\xi}^*) \right|^2 \right] \le C_2 \left(\Sigma_a^r + \log|\log \varepsilon| \right), \tag{23}$$

where C_2 depends only on n, K_1 and r_0 .

The term $\log |\log \varepsilon|$ is not small and even diverging as $\varepsilon \to 0$, but since the main order for the energy in the core region is of size $|\log \varepsilon|$ that estimate will be sufficient for our needs. Away from the cores we will of course stick to estimate (16).

2.2 Time evolution of the Jacobian and conservation of momentum

For sufficiently regular solutions of $(GP)^c_{\varepsilon}$ we have

$$\partial_{t}(iv, \nabla v) = (i\partial_{t}v, \nabla v) - (v, \nabla i\partial_{t}v)
= (\frac{1}{r}\operatorname{div}(r\nabla v) + \frac{1}{\varepsilon^{2}}v(1 - |v|^{2}), \nabla v) - (v, \nabla(\frac{1}{r}\operatorname{div}(r\nabla v) + \frac{1}{\varepsilon^{2}}v(1 - |v|^{2})))
= \frac{2}{r}(\operatorname{div}(r\nabla v), \nabla v) - \nabla\left(\frac{1}{r}(v, \operatorname{div}(r\nabla v)) + \frac{1 - |v|^{4}}{2\varepsilon^{2}}\right).$$
(24)

Taking the *curl* of the previous identy and integrating against a test function φ with bounded support and which vanishes at r = 0 we obtain

$$\frac{d}{dt} \int_{\mathbb{H}} Jv \, \varphi \, dr dz = -\int_{\mathbb{H}} \varepsilon_{ij} \frac{1}{r} (\partial_{k}(r\partial_{k}v), \partial_{j}v) \partial_{i}\varphi
= -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_{k}r}{r} (\partial_{j}v, \partial_{k}v) \partial_{i}\varphi + \int_{\mathbb{H}} \varepsilon_{ij} (\partial_{j}v, \partial_{k}v) \partial_{ik}\varphi
+ \int_{\mathbb{H}} \varepsilon_{ij} \partial_{j} (\frac{\sum_{k} |\partial_{k}v|^{2}}{2}) \partial_{i}\varphi
= -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_{k}r}{r} (\partial_{j}v, \partial_{k}v) \partial_{i}\varphi + \int_{\mathbb{H}} \varepsilon_{ij} (\partial_{j}v, \partial_{k}v) \partial_{ik}\varphi$$
(25)

where we sum over repeated indices and since

$$\int_{\mathbb{H}} \varepsilon_{ij} \partial_j \left(\sum_k \frac{|\partial_k v|^2}{2} \right) \partial_i \varphi = \int_{\mathbb{H}} \varepsilon_{ij} \left(\sum_k \frac{|\partial_k v|^2}{2} \right) \partial_{ij} \varphi = 0$$

by anti-symmetry. In the sequel we will write

$$\mathcal{F}(\nabla v, \varphi) := -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} (\partial_j v, \partial_k v) \partial_i \varphi + \int_{\mathbb{H}} \varepsilon_{ij} (\partial_j v, \partial_k v) \partial_{ik} \varphi, \tag{26}$$

so that (25) is also rewritten as

$$\frac{d}{dt} \int_{\mathbb{H}} Jv \,\varphi \, dr dz = \mathcal{F}(\nabla v, \varphi), \tag{27}$$

and is the equation from which the dynamical law for the vortex cores will be deduced. For a real Lipschitz vector field $X = (X_r, X_z)$, we expand

$$\mathcal{F}(X,\varphi) = \int_{\mathbb{H}} -\frac{1}{r} X_r X_z \partial_r \varphi + \frac{1}{r} X_r^2 \partial_z \varphi + X_r X_z \partial_{rr} \varphi + X_z^2 \partial_{rz} \varphi - X_r^2 \partial_{rz} \varphi - X_r X_z \partial_{zz} \varphi. \tag{28}$$

Integrating by parts, we have

$$\begin{split} \int_{\mathbb{H}} X_r X_z \partial_{rr} \varphi &= \int_{\mathbb{H}} -\partial_r X_z X_r \partial_r \varphi - X_z \partial_r X_r \partial_r \varphi \\ &= \int_{\mathbb{H}} (-\partial_z X_r - \text{curl} X) X_r \partial_r \varphi + (\frac{1}{r} X_r + \partial_z X_z - \frac{1}{r} \text{div}(rX)) X_z \partial_r \varphi, \\ \int_{\mathbb{H}} X_z^2 \partial_{rz} \varphi &= \frac{1}{2} \int_{\mathbb{H}} X_z^2 \partial_{rz} \varphi + \frac{1}{2} \int_{\mathbb{H}} X_z^2 \partial_{rz} \varphi \\ &= \int_{\mathbb{H}} -\frac{1}{2} \partial_z (X_z^2) \partial_r \varphi - \frac{1}{2} \partial_r (X_z^2) \partial_z \varphi, \\ \int_{\mathbb{H}} -X_r^2 \partial_{rz} \varphi &= \int_{\mathbb{H}} \frac{1}{2} \partial_r (X_r^2) \partial_z \varphi + \frac{1}{2} \partial_z (X_r^2) \partial_r \varphi, \end{split}$$

and

$$\int_{\mathbb{H}} -X_r X_z \partial_{zz} \varphi = \int_{\mathbb{H}} (\partial_r X_z - \operatorname{curl} X) X_z \partial_z \varphi + X_r (\frac{1}{r} \operatorname{div}(rX) - \frac{1}{r} X_r - \partial_r X_r) \partial_z \varphi,$$

so that after summation and simplification

$$\mathcal{F}(X,\varphi) = \int_{\mathbb{H}} -(\operatorname{curl}X)X \cdot \nabla \varphi + \frac{1}{r}\operatorname{div}(rX)X \times \nabla \varphi. \tag{29}$$

Formally, the choice $\varphi = r^2$ in (27) leads to the conservation of the momentum along the z-axis

 $\frac{d}{dt} \int_{\Omega} Jv \, r^2 dr dz = 0,$

but its justification would require additional arguments at infinity. In the next section we shall consider a version of the momentum localized on some large but finite part of \mathbb{H} .

2.3 Expansion of the main terms in the dynamics

In this section we strengthen assumption (H_1) into

$$\frac{K_0^{-1}}{\sqrt{|\log \varepsilon|}} \le \min_{i \ne j} |a_i - a_j| \le \max_{i \ne j} |a_i - a_j| \le \frac{K_0}{\sqrt{|\log \varepsilon|}}$$

$$\frac{r_0}{2} \le \min_{i} r(a_i) \le \max_{i} r(a_i) \le 2r_0$$
(H₀)

which is nothing but the time independent version of (8), and we define r_a and Σ_a as in (12) and (13). We shall also always implicitly assume that

$$\max_{i} |z(a_i)| \le K_0.$$

Since the problem is invariant under translation along the z-axis, and since we have already assumed that all the points are close to each other (as expressed by the first line in (H_0)), it is clear that this is not really an assumption but just a convenient way to avoid the necessity for various translations along the z-axis in some our subsequent claims.

Note that for sufficiently small ε , and adapting the constant K_0 if necessary, the situation described by (H_0) indeed implies (H_1) , and therefore in the sequel we shall refer freely to the improved approximation points ξ_i whose existence was established in Proposition 1.

Our analysis in the next sections will make rigorous the fact that the main contribution in the dynamical law for the vortex cores is obtained from (27), with a suitable choice of test function φ , by replacing in the expression $\mathcal{F}(\nabla u, \varphi)$ the term ∇u by $j^{\natural}(u_{\xi}^*)$. Regarding φ , we assume that it satisfies

- φ is affine on each ball $B(\xi_i, \frac{1}{|\log \varepsilon|})$,
- φ is compactly supported in the union of disjoint balls $\bigcup_i B(\xi_i, 1/(2K_0\sqrt{|\log \varepsilon|})),$
- $|\nabla \varphi| \le C$ and $|D^2 \varphi| \le CK_0 \sqrt{|\log \varepsilon|}$,

where C is a universal constant for such a test function to exist. We will refer to the above requirement as condition (H_{φ}) .

Proposition 3. Under the assumptions (H_0) , (12), (14) and (H_{φ}) , there exist $\varepsilon_3 \leq \varepsilon_2$ and C_3 depending only on n and K_0 and r_0 such that if $\varepsilon \leq \varepsilon_3$ we have

$$\left| \mathcal{F}(j^{\natural}(u_{\xi}^*), \varphi) - \sum_{i=1}^n \mathbb{J} \nabla_{a_i} H_{\varepsilon}(\xi_1, \cdots, \xi_n) \cdot \nabla \varphi(\xi_i) \right| \leq C_3 \left(\sum_{i=1}^r \log |\log \varepsilon| \right).$$

The main task in the remaining sections will be to control the discrepandcy between $\mathcal{F}(\nabla u, \varphi)$ and $\mathcal{F}(j^{\natural}(u_{\xi}^*), \varphi)$; for that purpose we will have to use the evolution equation to a larger extent (up to now our analysis was constrained on fixed time slices).

2.4 Approximation of the momentum

As remarked earlier, the choice $\varphi = r^2$ in (27) formally leads to the conservation of the momentum $\int Jur^2 dr dz$. Yet, giving a clear meaning to the previous integral and proving its conservation in time is presumably not an easy task. Instead, we will localise the function r^2 by cutting-it off sufficiently far away from the origin and derive an approximate conservation law. More precisely, we set

$$R_{\varepsilon} := |\log \varepsilon|^2$$

and we let $0 \le \chi_{\varepsilon} \le 1$ be a smooth cut-off function with compact support in $[0, 2R_{\varepsilon}] \times [-2R_{\varepsilon}, 2R_{\varepsilon}]$ and such that $\chi_{\varepsilon} \equiv 1$ on $[0, R_{\varepsilon}] \times [-R_{\varepsilon}, R_{\varepsilon}]$ and $|\nabla \chi_{\varepsilon}| \le C/R_{\varepsilon}$. In the sequel we write

$$P_{\varepsilon}(u) := \int_{\mathbb{H}} Jur^2 \chi_{\varepsilon} \, dr dz. \tag{30}$$

Proposition 4. Under the assumption (H_0) and (14), there exist $\varepsilon_4 \leq \varepsilon_3$ such that if $\varepsilon \leq \varepsilon_4$ then we have :

$$|P_{\varepsilon}(u) - P(\xi_1, \cdots, \xi_n)| \le C_4 \frac{(1 + \Sigma_{\xi})^2}{|\log \varepsilon|^2}, \tag{31}$$

and

$$|\partial_t P_{\varepsilon}(u)| \le C_4 \frac{1 + \Sigma_{\xi}}{R_{\varepsilon}} = C_4 \frac{1 + \Sigma_{\xi}}{|\log \varepsilon|^2},\tag{32}$$

where C_4 depends only on n, K_0 and r_0 .

2.5 A key argument

Coming back to Remark 1 and Remark 2 we now state

Proposition 5. Under the assumptions (H_0) and (14), there exists $\varepsilon_5 \leq \varepsilon_4$ and $\sigma_5 > 0$, depending only on K_0 and n, such that if $\varepsilon \leq \varepsilon_5$ and if

$$\Sigma_a + |H_{\varepsilon}(a_1, \dots, a_n) - H_{\varepsilon}(\xi, \dots, \xi_n)| \le \sigma_5 |\log \varepsilon|, \tag{33}$$

then

$$\Sigma_{\xi} \le 2\Sigma_a + C_5 \left[r_a \sqrt{|\log \varepsilon|} + \frac{1}{|\log \varepsilon|} + |\log \varepsilon| |P_{\varepsilon}(u) - P(a_1, \cdots, a_n)| \right]$$
(34)

where C_5 depends only on n, K_0 and r_0 .

Proof. For a quantity f we temporarily write $\Delta f := |f(a_1, \dots, a_n) - f(\xi_1, \dots, \xi_n)|$ when the latter has a well defined meaning. By the triangle inequality we have

$$\Delta H_{\varepsilon} \le \Delta \left(H_{\varepsilon} - \frac{|\log \varepsilon|}{2r_0} P \right) + \frac{|\log \varepsilon|}{2r_0} \Delta P,$$

and also

$$\Delta P \leq |P_{\varepsilon}(u) - P(\xi_1, \dots, \xi_n)| + |P_{\varepsilon}(u) - P(a_1, \dots, a_n)|.$$

In view of the expansion in Remark 1 (the o(1) holds in particular in C^1 norm under assumption (H_0)), we have

$$\Delta\left(H_{\varepsilon} - \frac{|\log \varepsilon|}{2r_0}P\right) \leq C|(\xi_1 - a_1, \cdots, \xi_n - a_n)|\sqrt{|\log \varepsilon|}$$

and by (12) and (15)

$$|(\xi_1 - a_1, \cdots, \xi_n - a_n)| \le C(r_a + \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r}).$$

By (31) we also have

$$\frac{|\log \varepsilon|}{2r_0} |P_{\varepsilon}(u) - P(\xi_1, \cdots, \xi_n)| \le \frac{C_4}{2r_0} \frac{(1 + \Sigma_{\xi})^2}{|\log \varepsilon|} \le \frac{C_4}{2r_0} \frac{(1 + \Sigma_a + \Delta H_{\varepsilon})^2}{|\log \varepsilon|}$$

and

$$\frac{C_4}{2r_0} \frac{(1 + \Sigma_a + \Delta H_{\varepsilon})^2}{|\log \varepsilon|} \leq \frac{3C_4}{2r_0} \frac{1 + \Sigma_a^2 + (\Delta H_{\varepsilon})^2}{|\log \varepsilon|} \leq \frac{3C_4}{2r_0} \left(\frac{1}{|\log \varepsilon|} + \sigma_5(\Sigma_a + \Delta H_{\varepsilon}) \right).$$

By summation of all the inequalities gathered so far we obtain

$$\Delta H_{\varepsilon} \leq C \left((r_a + \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r}) \sqrt{|\log \varepsilon|} + \frac{1}{|\log \varepsilon|} + \Sigma_a + |\log \varepsilon| |P_{\varepsilon}(u) - P(a_1, \cdots, a_n)| \right) + \frac{3C_4}{2r_0} \sigma_5 \Delta H_{\varepsilon}.$$
(35)

We therefore choose σ_5 in such a way that $\frac{3C_4}{2r_0}\sigma_5 \leq \frac{1}{2}$, and we may then absorb the last term of the previous inequality in its left-hand side. Combined with the fact that $\Sigma_{\xi} \leq \Sigma_a + \Delta H_{\varepsilon}$ the conclusion (34) follows.

Remark 4. The main gain in (34) is related to the fact that in the right-hand side we have a term of the form $r_a\sqrt{|\log \varepsilon|}$ rather than (the easier) $r_a|\log \varepsilon|$ which would have followed from a crude gradient bound on H_{ε} . Note however that we have exploited here the assumption (H_0) , that is the fact that all the cores are of order $\sqrt{|\log \varepsilon|}$ apart from each other, whereas Proposition 1 holds under the weaker assumption (H_1) .

The right-hand side of (34) also contains a term involving $P_{\varepsilon}(u)$ and $P(a_1, \dots, a_n)$. When introducing time dependence in the next sections, we will take advantage of the fact that P is preserved by the ODE flow $(LF)_{\varepsilon}$ and that P_{ε} is almost preserved by the PDE flow $(GP)_{\varepsilon}^{c}$, as already expressed in (32).

2.6 Time dependence and Stopping time

In this section we introduce time dependence and go back to the setting of Theorem 2, that is we assume (8) and (11). For $s \in [0, S_0]$, we define the localization scales

$$r_a^s := \left\| J u_\varepsilon^s - \pi \sum_{i=1}^n \delta_{a_{i,\varepsilon}(s)} \right\|_{\dot{W}^{-1,1}(\Omega_0)},\tag{36}$$

and the excess energy

$$\Sigma^{s} := \left[\mathcal{E}_{\varepsilon}^{w}(u_{\varepsilon}^{s}) - H_{\varepsilon}(a_{1,\varepsilon}(s), \cdots, a_{n,\varepsilon}(s)) \right]^{+}$$
(37)

where we recall that $\Omega_0 = \{r \geq \frac{r_0}{4}\}$ and u_{ε}^s is the solution of $(GP)_{\varepsilon}^c$ evaluated at time $t = s/|\log \varepsilon|$.

Since $\mathcal{E}_{\varepsilon}^{\mathrm{w}}$ is preserved by the flow of $(\mathrm{GP})_{\varepsilon}^{c}$ and since H_{ε} is preserved by $(\mathrm{LF})_{\varepsilon}$, we have

$$\Sigma^s = \Sigma^0 \qquad \forall s \in [0, S_0]. \tag{38}$$

We introduce the stopping time

$$S_{\text{stop}} := \inf \left\{ S \in [0, S_0], \ r_a^s \le \frac{\rho_{\min}}{8}, \quad \forall s \in [0, S] \right\},$$
 (39)

where we have set, in view of (8),

$$\rho_{\min} := \frac{K_0^{-1}}{\sqrt{|\log \varepsilon|}}.$$
(40)

By (11) and continuity it is clear that $S_{\text{stop}} > 0$, at least provided ε_0 and σ_0 are chosen small enough. By construction, we also have

$$r_a^s < \frac{\rho_{a_{\varepsilon}(s)}}{4} \qquad \forall s \in [0, S_{\text{stop}}],$$

and likewise by (38)

$$\sum^{s} + r_{a}^{s} |\log \varepsilon| \le \sigma_{1} |\log \varepsilon|,$$

where σ_1 is given in Proposition 1. Applying Proposition 1 for each $s \in [0, S_{\text{stop}}]$, we get functions $s \mapsto \xi_i(s) \equiv \xi_i^s$, $i = 1, \dots, n$. By continuity of the flow map for $(GP)_{\varepsilon}^c$, and doubling C_1 if necessary, we may further assume that these maps are piecewise constant and hence measurable on $[0, S_{\text{stop}}]$. In the sequel, in view of (15), we set

$$r_{\xi}^{s} = C_{1} \varepsilon |\log \varepsilon|^{C_{1}} e^{C_{1}(\Sigma^{0} + r_{a}^{s}|\log \varepsilon|)} \stackrel{(18)}{\leq} \varepsilon^{\frac{5}{6}}, \tag{41}$$

for each $s \in [0, S_{\text{stop}}]$.

The following Proposition yields a first estimate on the time evolution of the vortex cores. At this stage it does not contain any information about the actual motion law, but only a rough (but essential) Lipschitz bound.

Proposition 6. For sufficiently small values of σ_0 and ε_0 , whose threshold may be chosen depending only on n, K_0 and r_0 , the following holds: There exist $C_6 > 0$, also depending only on n, K_0 and r_0 , such that for all $s_1, s_2 \in [0, S_{\text{stop}}]$ such that $s_1 \leq s_2 \leq s_1 + |\log \varepsilon|^{-1}$ we have

$$||Ju_{\varepsilon}^{s_1} - Ju_{\varepsilon}^{s_2}||_{\dot{W}^{-1,1}(\Omega_0)} \le C_6(|s_1 - s_2| + r_{\xi}^{s_1}), \tag{42}$$

$$r_{\xi}^{s_2} \le r_{\xi}^{s_1} \left(1 + C_6 \left(|s_2 - s_1| + \varepsilon^{\frac{5}{6}} \right) |\log \varepsilon| \right), \tag{43}$$

$$\{a_{i,\varepsilon}(s_2), \xi_i(s_2)\} \subset B(a_{i,\varepsilon}(s_1), \frac{\rho_{\min}}{4}).$$
 (44)

Moreover, if $r_a^{s_1} \le \rho_{\min}/16$, then $S_{\text{stop}} \ge s_1 + (C_6\sqrt{|\log \varepsilon|})^{-1}$.

2.7 Control of the discrepancy

The following proposition is the final ingredient leading to the proof of Theorem 2, it can be regarded as a discrete version of the Gronwall inequality for the quantity r_a^s .

Proposition 7. Assume that $s < S_{\text{stop}}$ and that $r_a^s \le \rho_{\min}/16$ and set

$$S := s + \frac{(r_{\xi}^s)^2}{\varepsilon}.$$

Then $S < S_{\text{stop}}$ and

$$\frac{r_a^S - r_a^s}{S - s} \le C_0 \left(r_a^s + \frac{\Sigma^0 + r_a^0 |\log \varepsilon|}{\sqrt{|\log \varepsilon|}} + \frac{C_\delta}{|\log \varepsilon|^{1 - \delta}} \right),\tag{45}$$

where C_0 depends only on n, K_0 and r_0 , $\delta > 0$ is arbitrary and C_{δ} depends only on δ .

Remark 5. The time step S-s on which the differential inequality (45) holds is not arbitrary, in view of (41) it satisfies

$$S - s = C_1^2 \varepsilon |\log \varepsilon|^{2C_1} e^{2C_1(\Sigma^0 + r_a^s |\log \varepsilon|)}$$

which, for ε sufficiently small, is both large with respect to ε and small with respect to lower powers of ε . The fact that it is large with respect to ε , as the proof of Proposition 7 will show, is essential in order to allow the averaging effects of the continuity equation (see (118)) to act. On the other hand, the fact that it is small with respect to lower powers of ε will allow us, when using it iteratively, to rely on the softer estimates of Proposition 6 to bridge the gaps between the discrete set of times so obtained and the full time interval $[0, S_0]$ which appears in the statement of Theorem 2.

3 Proofs

Proof of Lemma 1 It suffices to combine the expansion of Lemma (A.1) with those (see e.g. [2]) for the optimal Ginzburg-Landau profile f_{ε} .

Proof of Proposition 1 We divide the proof in several steps. We first set

$$\ell_{\epsilon,a} = 4 \max(\frac{1}{|\log \varepsilon|}, r_a).$$

Step 1: rough lower energy bounds on $B(a_i, \ell_{\epsilon,a})$. In view of our assumptions and the fact that the $\dot{W}^{-1,1}$ is decreasing with respect to the domain, we are in position, provided ε_1 and σ_1 are sufficiently small (depending only on r_0 and K_1), to apply Theorem B.1 after translation to the balls $B(a_i, \ell_{\epsilon,a})$. This yields the lower bounds

$$\mathcal{E}_{\varepsilon}(u_{\varepsilon}^{s}, B(a_{i}, \ell_{\epsilon, a})) \geq \pi \log \frac{\ell_{\epsilon, a}}{\varepsilon} + \gamma - \frac{C}{\ell_{\epsilon, a}} \left(\varepsilon \sqrt{\log(\ell_{\epsilon, a}/\varepsilon)} + r_{a} \right)$$

$$\geq \pi |\log \varepsilon| - C \left(r_{a} |\log \varepsilon| + \log |\log \varepsilon| \right),$$

$$(46)$$

for any $1 \le i \le n$, where C is universal provided we require that ε_1 is also sufficiently small so that $\log |\log \varepsilon| \ge 1$ for $\varepsilon \le \varepsilon_1$. From (46) and the global energy bound given by the assumption of Σ_a it follows, comparing the weight function r with its value $r(a_i)$, that

$$\mathcal{E}_{\varepsilon}^{W}(u, B(a_{i}, \ell_{\epsilon, a})) \ge \pi r(a_{i})|\log \varepsilon| - C\left(r_{a}|\log \varepsilon| + \log|\log \varepsilon|\right) \tag{47}$$

for any $1 \leq i \leq n$, and for a possibly larger constant C depending only on K_1 , r_0 and n. Step 2: rough upper energy bounds on $\mathbb{H} \setminus \bigcup_{i=1}^n B(a_i, \ell_{\epsilon,a}/2)$ and $B(a_i, 2\ell_{\epsilon,a})$. The equivalent of (47) with $\ell_{\epsilon,a}$ replaced by $\ell_{\epsilon,a}/2$, combined with the global upper energy bound given by the definition of Σ_a , yields the upper bound

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \mathbb{H} \setminus \bigcup_{i=1}^{n} B(a_{i}, \frac{\ell_{\epsilon, a}}{2}) \leq C\left(\Sigma_{a}^{r} + \log|\log \varepsilon|\right), \tag{48}$$

where C depends only K_0 and n. Also, combining (47) (for all but one i) with the definition of Σ_a , we obtain the upper bound

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, B(a_i, 2\ell_{\epsilon,a})) \leq \pi r(a_i) |\log \varepsilon| + C \left(\Sigma_a^r + \log |\log \varepsilon|\right),$$

for any $1 \le i \le n$, and therefore

$$\mathcal{E}_{\varepsilon}(u, B(a_i, 2\ell_{\epsilon, a})) \le \pi \log \frac{2\ell_{\epsilon, a}}{\varepsilon} + C\left(\Sigma_a^r + \log|\log \varepsilon|\right),\tag{49}$$

for any $1 \le i \le n$, where C depends only on K_0 and n.

Step 3: first localisation estimates. We apply Theorem B.4, after translation, to each of the balls $B(a_i, 2\ell_{\epsilon,a})$, and we denote by ξ_i the corresponding points. In view of (49), this yields

$$\sum_{i=1}^{n} \|Ju - \pi \delta_{\xi_i}\|_{\dot{W}^{-1,1}(B(a_i, 2\ell_{\epsilon, a}))} \le \varepsilon e^{C(\Sigma_a^r + \log|\log \varepsilon|)} \le \varepsilon |\log \varepsilon|^C e^{C\Sigma_a^r}.$$
 (50)

Note that from (50) and the definition of r_a in (12) we have the bound

$$\max_{i=1,\dots,n} |a_i - \xi_i| \le \frac{1}{\pi} \left(r_a + \varepsilon |\log \varepsilon|^C e^{C\Sigma_a^r} \right). \tag{51}$$

Provided ε_1 and σ_1 are sufficiently small, this also implies that

$$B(\xi_i, \ell_{\epsilon}) \subset B(a_i, \ell_{\epsilon,a}) \qquad \forall i = 1 \cdots, n,$$

where we have set

$$\ell_{\epsilon} := \frac{1}{|\log \varepsilon|}.$$

From now on we will rely entirely on the points ξ_i rather than on the a_i for our constructions. **Step 4: improved lower energy bounds close to the cores.** We apply Theorem B.1, after translation, to each of the balls $B(\xi_i, \rho)$, where $\ell_{\epsilon}/2 \ge \rho \ge \varepsilon^{\frac{4}{5}}$ is some free parameter which we will fix later. Since $\dot{W}^{-1,1}$ norms are monotone functions of the domain and since $B(\xi_i, \rho) \subset B(a_i, \ell_{\epsilon,a})$ by (51), in view of (50) we obtain

$$\mathcal{E}_{\varepsilon}(u, B(\xi_i, \rho)) \ge \pi \log \frac{\rho}{\varepsilon} + \gamma - C\varepsilon |\log \varepsilon|^C e^{C\Sigma_a^r} \rho^{-1}, \tag{52}$$

and therefore

$$\mathcal{E}_{\varepsilon}^{W}(u, B(\xi_{i}, \rho)) \ge r(\xi_{i}) \left(\pi \log \frac{\rho}{\varepsilon} + \gamma \right) - C(\varepsilon |\log \varepsilon|^{C} e^{C\Sigma_{a}^{r}} \rho^{-1} + \rho |\log \varepsilon|), \tag{53}$$

for $i = 1, \dots, n$.

Note that taking $\rho = \ell_{\epsilon}/2$ and then arguing exactly as in Step 2 yields the slight variant of (48):

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \mathbb{H} \setminus \bigcup_{i=1}^{n} B(\xi_{i}, \frac{\ell_{\epsilon}}{2}) \leq C\left(\Sigma_{a}^{r} + \log|\log \varepsilon|\right). \tag{54}$$

Yet at this point we wish to keep ρ as a free parameter.

Step 5: towards lower energy bounds away from the cores. In this step we compare u, away from the cores, with the singular vortex ring u_{ξ}^* . For convenience, we simply denote $j(u_{\xi}^*)$ by j^* , we let $0 \le \chi \le 1$ be a lipschitz function on \mathbb{H} , and we set

$$\mathbb{H}_{\xi,\rho} := \mathbb{H} \setminus \bigcup_{i=1}^{n} B(\xi_i,\rho). \tag{55}$$

The starting point is the pointwise equality

$$e_{\varepsilon}(u) = \frac{1}{2}|j_*|^2 + j_*\left(\frac{j(u)}{|u|} - j_*\right) + e_{\varepsilon}(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j_*\right|^2,\tag{56}$$

which holds almost everywhere in \mathbb{H} . Notice that all the terms in the right-hand side of (56) are pointwise non-negative except possibly the second one. We integrate (56) multiplied by χ^2 on $\mathbb{H}_{\xi,\rho}$ and estimate the corresponding terms.

We first write

$$\int_{\mathbb{H}_{\xi,\rho}} r j_* \left(\frac{j(u)}{|u|} - j_* \right) \chi^2 = \int_{\mathbb{H}_{\xi,\rho}} r j_* \left(j(u) - j_* \right) \chi^2 + \int_{\mathbb{H}_{\xi,\rho}} r j_* \left(\frac{j(u)}{|u|} - j(u) \right) \chi^2$$

and we readily estimate

$$\Big| \int_{\mathbb{H}_{\xi,\rho}} r j_* \Big(\frac{j(u)}{|u|} - j(u) \Big) \chi^2 \Big| \leq \frac{C}{\rho} \Big(\int_{\mathbb{H}_{\xi,\rho}} r \frac{j^2(u)}{|u|^2} \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{H}_{\xi,\rho}} r (1 - |u|)^2 \Big)^{\frac{1}{2}} \leq C \frac{\varepsilon}{\rho} |\log \varepsilon|,$$

where we have used the facts that $|j^*| \leq C/\rho$ on $\mathbb{H}_{\xi,\rho}$ and that the last two integral factors are dominated by (a constant multiple of) the weighted energy. By definition (see Appendix A.1), we have

$$rj_* = -\nabla^{\perp} (r\Psi_{\xi}^*).$$

We modify (truncate) the function Ψ_{ξ}^* to a function $\tilde{\Psi}_{\xi}^*$ in the following way: for each $i = 1, \dots, n$ we consider the connected component C_i of the superlevel set $\{\Psi_{\xi}^* \geq \Psi_{\xi}^*(\xi_i + (\rho, 0))\}$

(by convention we include ξ_i , where Ψ_{ξ}^* is in principle not defined, in this set) which contains the point $\xi_i + (\rho, 0)$, and we set $\tilde{\Psi}_{\xi}^* = \Psi_{\xi}^*(\xi_i + (\rho, 0))$ on C_i . Next, we set $\tilde{\Psi}_{\xi}^* = \Psi_{\xi}^*$ on $\mathbb{H} \setminus \bigcup_{i=1}^n C_i$. By construction,

$$-\nabla^{\perp}(r\tilde{\Psi}_{\xi}^*) = rj_* \mathbb{1}_{\mathbb{H}\setminus\bigcup_{i=1}^n C_i},$$

so that

$$-\nabla^{\perp} \left(r \tilde{\Psi}_{\xi}^* \chi^2\right) = r j_* \mathbb{1}_{\mathbb{H}_{\xi,\rho}} \chi^2 + r j_* \left[\sum_{i=1}^n (\mathbb{1}_{B(\xi_i,\rho)} - \mathbb{1}_{\mathcal{C}_i})\right] \chi^2 - 2r \tilde{\Psi}_{\xi}^* \chi \nabla^{\perp} \chi.$$

The latter and integration by parts yields

$$\left| \int_{\mathbb{H}_{\xi,\rho}} r j_{*} (j(u) - j_{*}) \chi^{2} \right| \leq \sum_{i=1}^{n} \int_{B(\xi_{i},\rho) \triangle C_{i}} r |j^{*}| |j(u) - j_{*}| + \int_{\mathbb{H}} 2r |\tilde{\Psi}_{\xi}^{*}| |\nabla^{\perp} \chi | \chi |j(u) - j_{*}| + \left| \int_{\mathbb{H}} 2r \tilde{\Psi}_{\xi}^{*} (J(u) - J(u_{\xi}^{*})) \chi^{2} \right|.$$
(57)

In order to bound the right-hand side of (57) we first remark that, from (122) and (123) in the Appendix, for each $i = 1, \dots, n$, we have

$$d_{\mathcal{H}}(\mathcal{C}_i, B(\xi_i, \rho)) \le \frac{\rho^2}{\rho_a} \log(\frac{\rho}{\rho_a}), \quad \text{and hence} \quad \mathcal{L}^2(\mathcal{C}_i \triangle B(\xi_i, \rho)) \le C \frac{\rho^3}{\rho_a} \log(\frac{\rho}{\rho_a}). \tag{58}$$

We write

$$\int_{B(\xi_i,\rho)\triangle C_i} r|j^*| \left|j(u)-j_*\right| \leq \int_{B(\xi_i,\rho)\triangle C_i} r|j^*| \left|\frac{j(u)}{|u|}-j_*\right| + \int_{B(\xi_i,\rho)\triangle C_i} r\varepsilon |j^*| \left|\frac{j(u)}{|u|}\right| \frac{||u|-1|}{\varepsilon},$$

and since $|j^*| \leq C/\rho$ on $C_i \triangle B(\xi_i, \rho)$, we deduce from (58), the Cauchy-Schwarz inequality, and global energy upper bounds, that

$$\sum_{i=1}^{n} \int_{B(\xi_{i},\rho) \triangle C_{i}} r|j^{*}| |j(u) - j_{*}| \le C \left(\frac{\rho}{\rho_{a}} \log(\frac{\rho}{\rho_{a}}) |\log \varepsilon| \right)^{\frac{1}{2}} + C \frac{\varepsilon}{\rho} |\log \varepsilon|.$$
 (59)

Concerning the second error term in (57), we first decompose it as

$$\int_{\mathbb{H}} r|\tilde{\Psi}_{\xi}^{*}||\nabla^{\perp}\chi||j(u) - j_{*}|\chi = \int_{\mathbb{H}} r|\tilde{\Psi}_{\xi}^{*}||\nabla^{\perp}\chi||\frac{j(u)}{|u|} - j_{*}|\chi + \int_{\mathbb{H}} r|\tilde{\Psi}_{\xi}^{*}||\nabla^{\perp}\chi||\frac{j(u)}{|u|}||1 - |u||\chi$$
(60)

and we write by Cauchy-Schwarz inequality on one hand

$$\int_{\mathbb{H}} r|\tilde{\Psi}_{\xi}^{*}||\nabla^{\perp}\chi||\frac{j(u)}{|u|} - j_{*}|\chi \leq C||\nabla\chi||_{\infty} \left(\int_{\operatorname{spt}(\nabla\chi)} r|\tilde{\Psi}_{\xi}^{*}|^{2}\right)^{\frac{1}{2}} \left(\int_{\operatorname{spt}(\nabla\chi)} r(e_{\varepsilon}(u) + e_{\varepsilon}(u_{\xi}^{*}))\chi^{2}\right)^{\frac{1}{2}},\tag{61}$$

and by direct comparison with the energy density on the other hand

$$\int_{\mathbb{H}} r|\tilde{\Psi}_{\xi}^{*}||\nabla^{\perp}\chi||\frac{j(u)}{|u|}||1-|u||\chi \leq C\varepsilon|\log\varepsilon|^{2}||\nabla\chi||_{\infty}.$$
 (62)

Coming back to (56), and taking into account (59)-(62), we conclude that

$$\int_{\mathbb{H}_{\xi,\rho}} re_{\varepsilon}(u)\chi^{2} \ge \int_{\mathbb{H}_{\xi,\rho}} \left[\frac{|j_{*}|^{2}}{2} + e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j_{*} \right|^{2} \right] \chi^{2} - \left| \int_{\mathbb{H}} 2r\tilde{\Psi}_{\xi}^{*} \left(J(u) - J(u_{\xi}^{*}) \right) \chi^{2} \right| - \operatorname{Err}(\chi^{2}), \tag{63}$$

where

$$\operatorname{Err}(\chi^{2}) \leq C \left[\left(\frac{\rho}{\rho_{a}} \log(\frac{\rho}{\rho_{a}}) |\log \varepsilon| \right)^{\frac{1}{2}} + \frac{\varepsilon}{\rho} |\log \varepsilon| + \|\nabla \chi\|_{\infty} \left(\int_{\operatorname{spt}(\nabla \chi)} r |\tilde{\Psi}_{\xi}^{*}|^{2} \right)^{\frac{1}{2}} \left(\int_{\operatorname{spt}(\nabla \chi)} r (e_{\varepsilon}(u) + e_{\varepsilon}(u_{\xi}^{*})) \chi^{2} \right)^{\frac{1}{2}} + \|\nabla \chi\|_{\infty} \varepsilon |\log \varepsilon|^{2} \right].$$

$$(64)$$

Step 6: improved lower energy bounds away from the cores. The right-hand side of estimate (63) contains quantities which we do not yet control: we need good localisation estimates for the jacobian Ju, also outside the cores, and we also have to get rid of the energy term due to the cut-off in (64). To deal with the localisation, we shall rely on Theorem B.2, but in view of the difference between $\mathcal{E}_{\varepsilon}$ and $\mathcal{E}_{\varepsilon}^{w}$ (the factor r), we only expect good localisation estimates when r is not too small. To quantify this, we define the set

$$\mathcal{S} = \left\{ s = 2^{-k}, k \in \mathbb{Z}, \text{ s.t. } \mathcal{E}_{\varepsilon}^{w} \left(u, \left\{ s \le r \le 2s \right\} \setminus \bigcup_{i=1}^{n} B(\xi_{i}, \frac{\ell_{\epsilon}}{2}) \right) \le \frac{\pi}{12} s |\log \varepsilon| \right\}$$

and the value

$$r_{\mathcal{S}} := \min \left\{ s = 2^{-k}, k \in \mathbb{Z}, \text{ s.t. } 2^{-\ell} \in \mathcal{S} \ \forall \ell \le k \right\}.$$

Note that by (54) we have

$$r_{\mathcal{S}} \le C \left(\frac{\sum_{a}^{r}}{|\log \varepsilon|} + \frac{\log|\log \varepsilon|}{|\log \varepsilon|} \right),$$
 (65)

which we will improve later on in (74). Also, whenever Ω is an open bounded subset contained in $\{s \leq r \leq 2s\} \setminus \bigcup_{i=1}^n B(\xi_i, \frac{\ell_{\epsilon}}{2})$ for some $s \geq r_{\mathcal{S}}$, covering it with two of the above slices we obtain

$$\mathcal{E}_{\varepsilon}(u,\Omega) \leq \frac{1}{s} \mathcal{E}_{\varepsilon}^{w}(u,\Omega) \leq \frac{\pi}{4} |\log \varepsilon|$$

and therefore by Theorem B.2

$$||Ju||_{\dot{W}^{-1,1}(\Omega)} \le C\mathcal{E}_{\varepsilon}(u,\Omega)\varepsilon^{\frac{3}{4}}.$$
 (66)

We now take $\rho := \rho_{\epsilon} = \varepsilon^{\frac{2}{3}}$, and in view of (65) we let $r_{\mathcal{S}} \leq \tilde{r} \leq C\left(\Sigma_a^r + \log|\log \varepsilon|\right)/|\log \varepsilon|$. We choose $0 \leq \chi \leq 1$ a lipschitz function supported in $\{r \geq \tilde{r}\}$ and such that $\chi \equiv 1$ on $\{r \geq 2\tilde{r}\}$ and $|\nabla \chi| \leq C/\tilde{r}$. We then invoke estimate (63) of Step 5, which we add-up with estimate (53) (note that $\chi \equiv 1$ on each $B(\xi_i, \rho)$ by definition of \tilde{r} , at least provided ε_1 and σ_1 are chosen small enough) to write

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r \ge \tilde{r}\}) \ge \int_{\mathbb{H}} re_{\varepsilon}(u)\chi^{2} \ge T_{1} - T_{2} - T_{3} + T_{4} \tag{67}$$

where

$$T_1 = \int_{\mathbb{H}_{\xi,\rho}} \frac{|j_*|^2}{2} + \sum_{i=1}^n r(\xi_i) \left(\pi \log \frac{\rho_{\epsilon}}{\varepsilon} + \gamma \right) - \int_{\{r \leq \tilde{r}\}} r \frac{|j_*|^2}{2},$$

$$T_2 = \operatorname{Err}(\chi^2)$$
 , $T_3 := \Big| \int_{\mathbb{H}} 2r \tilde{\Psi}_{\xi}^* \big(J(u) - J(u_{\xi}^*) \big) \chi^2 \Big|,$

and

$$T_4 = \int_{\mathbb{H}_{\varepsilon,\rho}} \left[e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j_* \right|^2 \right] \chi^2 \ge 0.$$

We invoke Lemma A.1 (with $a = \xi$) and the definition of H_{ε} to obtain

$$T_1 \ge H_{\varepsilon}(\xi) - C(\tilde{r}^2 + \varepsilon^{\frac{2}{3}} |\log \varepsilon|^3),$$
 (68)

where we have also used (H_1) in order to get rid of ρ_a wherever it appeared. Invoking (124) to compute some of the terms in (64), we also obtain

$$T_2 \le C \left(\varepsilon^{\frac{1}{3}} |\log \varepsilon|^{\frac{3}{2}} + \tilde{r} \left(\mathcal{E}_{\varepsilon}^{w} (u, \{ \tilde{r} \le r \le 2\tilde{r} \}) + \tilde{r} \right)^{\frac{1}{2}} + \frac{\varepsilon}{\tilde{r}} |\log \varepsilon|^{2} \right),$$

and since $\tilde{r} \geq r_{\mathcal{S}}$ the definition of the latter yields

$$T_2 \le C\left(\varepsilon^{\frac{1}{3}}|\log \varepsilon|^{\frac{3}{2}} + \tilde{r}^{\frac{3}{2}}|\log \varepsilon|^{\frac{1}{2}} + \frac{\varepsilon}{\tilde{r}}|\log \varepsilon|^2\right). \tag{69}$$

It remains to estimate T_3 for which we will rely on (50) and (66). To that purpose, we write

$$\chi^2 = \chi^2 \left(\sum_{i=1}^n \psi_i^{\text{in}} + \sum_{j \in \mathbb{N}} \psi_j^{\text{out}} \right)$$
 on \mathbb{H}

for an appropriate partition of unity on \mathbb{H} verifying the following:

- 1. Each function of the partition is \mathcal{C}^{∞} smooth and compactly supported, its support has a smooth boundary.
- 2. Each point of H is contained in the support of at most four functions of the partition.
- 3. We have

$$\operatorname{spt}(\psi_i^{\text{in}}) \subset B(\xi_i, \ell_{\epsilon}), \qquad |\nabla \psi_i^{\text{in}}| \leq C/\ell_{\epsilon}, \qquad \forall i = 1, \dots, n,$$
$$\operatorname{spt}(\psi_j^{\text{out}}) \subset \mathbb{H} \setminus \bigcup_{i=1}^n B(\xi_i, \ell_{\epsilon}/2), \qquad \forall j \in \mathbb{N}.$$

For each $j \in \mathbb{N}$, there exists $r_j > \tilde{r}/2$ such that

$$\operatorname{spt}(\psi_j^{\text{out}}) \subset \{r_j \le r \le 2r_j\} \quad \text{and} \quad |\nabla \psi_j^{\text{out}}| \le C\left(\frac{1}{r_j} + \frac{1}{\ell_{\epsilon}}\right).$$

The existence of such a partition can be obtained by covering \mathbb{H} with rectangular tiles with a step size close to being dyadic in the r direction and constant in the z direction and then arranging the round holes corresponding to the ξ_i 's. It may be necessary to shift a little the rectangular tiles so that the balls around the ξ_i 's do not meet their boundaries (this is the only reason of r_j not being exactly dyadic).

We use (50) for the terms involving ψ_i^{in} and (66) for those with ψ_j^{out} . Since χ vanishes at r=0 we have $|\chi(r,\cdot)| \leq r \|\nabla\chi\|_{\infty}$ and in the dual norm we may crudely estimate

$$\|r\tilde{\Psi}_{\xi}^*\chi^2\psi_j^{\mathrm{out}}\|_{W^{1,\infty}} \leq \frac{C}{\ell_{\varepsilon}} \leq C|\log \varepsilon|, \qquad \|r\tilde{\Psi}_{\xi}^*\chi^2\psi_i^{\mathrm{in}}\|_{W^{1,\infty}} \leq \frac{C}{\rho_{\varepsilon}} \leq C\varepsilon^{-\frac{2}{3}},$$

so that we finally obtain

$$T_3 \le C\left(\varepsilon^{\frac{1}{3}}|\log \varepsilon|^C e^{C\Sigma_a^r} + \varepsilon^{\frac{3}{4}}|\log \varepsilon|^2 \mathcal{E}_{\varepsilon}^{w}\left(u, \mathbb{H} \setminus \bigcup_{i=1}^n B(\xi_i, \ell_{\epsilon}/2)\right) \le C\varepsilon^{\frac{1}{3}}|\log \varepsilon|^C e^{C\Sigma_a^r}.$$
 (70)

Combining (68),(69) and (70) in (67) we derive

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r \geq \tilde{r}\}) \geq H_{\varepsilon}(\xi) + T_4 - C\left(\varepsilon^{\frac{1}{3}}|\log \varepsilon|^C e^{C\Sigma_a^r} + \tilde{r}^{\frac{3}{2}}|\log \varepsilon|^{\frac{1}{2}} + \frac{\varepsilon}{\tilde{r}}|\log \varepsilon|^2\right), \tag{71}$$

and combining the latter with the definition of Σ_{ξ} yields the upper bound

$$\mathcal{E}_{\varepsilon}^{w}(u, \{r \leq \tilde{r}\}) + T_{4} \leq \Sigma_{\xi} + C\left(\varepsilon^{\frac{1}{3}}|\log \varepsilon|^{C}e^{C\Sigma_{a}^{r}} + \tilde{r}^{\frac{3}{2}}|\log \varepsilon|^{\frac{1}{2}} + \frac{\varepsilon}{\tilde{r}}|\log \varepsilon|^{2}\right). \tag{72}$$

On the other hand, by definition of $r_{\mathcal{S}}$ we also have the lower bound

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r \le r_{\mathcal{S}}\}) \ge \mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r_{\mathcal{S}}/2 \le r \le r_{\mathcal{S}}\}) \ge \frac{\pi}{24} r_{\mathcal{S}} |\log \varepsilon|.$$
 (73)

The comparison of (72) specified for $\tilde{r} = r_{\mathcal{S}}$ and (73) leads to the conclusion that

$$r_{\mathcal{S}} \le C\left(\frac{\Sigma_{\xi}}{|\log \varepsilon|} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^C e^{C\Sigma_a^r}\right).$$
 (74)

Step 7: improved closeness and upper energy bounds. We now choose $\tilde{r} = C(\Sigma_{\xi}/|\log \varepsilon| + \varepsilon^{\frac{1}{3}}|\log \varepsilon|^C e^{C\Sigma_a^r})$ in (72) to obtain

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r \leq \tilde{r}\}) + \int_{\mathbb{H}_{\xi, \rho}} \left[e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j_{*} \right|^{2} \right] \chi^{2} \leq C \left(\sum_{\xi} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^{C} e^{C \sum_{a}^{r}} \right). \tag{75}$$

The same estimate with \tilde{r} replaced by half its value, combined with the fact that in the integral of (75) the integrand is pointwise dominated by the one of $\mathcal{E}_{\varepsilon}^{w}$, allows, in view of the first term of (75), to get rid of χ^{2} in the integrand and conclude that

$$\mathcal{E}_{\varepsilon}^{\mathbf{w}}(u, \{r \leq \tilde{r}\}) + \int_{\mathbb{H}_{\xi, \rho}} \left[e_{\varepsilon}(|u|) + \left| \frac{j(u)}{|u|} - j_* \right|^2 \right] \leq C\left(\Sigma_{\xi} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^C e^{C\Sigma_a^r}\right), \tag{76}$$

which yields (16), for a suitable value of C_1 , by taking $\rho = \rho_{\epsilon} = \varepsilon^{\frac{2}{3}}$. Note that combining the lower bound (63) (with the error terms now controlled) with the lower bounds (53) (used

for $\rho = \rho_{\epsilon} = \varepsilon^{\frac{2}{3}}$ and for all except one i) and Lemma A.1, we also obtain, in view of the definition of Σ_a ,

$$\mathcal{E}_{\varepsilon}^{W}(u, B(\xi_{i}, \rho_{\epsilon})) \leq r(\xi_{i}) \left(\pi \log \frac{\rho_{\epsilon}}{\varepsilon} + \gamma \right) + C\left(\Sigma_{\xi} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^{C} e^{C\Sigma_{a}^{r}} \right), \tag{77}$$

so that

$$\mathcal{E}_{\varepsilon}(u, B(\xi_i, \rho_{\epsilon})) \le \pi \log \frac{\rho_{\epsilon}}{\varepsilon} + \gamma + C(\Sigma_{\xi} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^C e^{C\Sigma_a^r}), \tag{78}$$

for any $1 \leq i \leq n$. Inequality (17) is a direct consequence of (51) and the explicit form of H_{ε} . Finally, it remains to improve the local estimate (50) to the more global one (15). For that purpose, it suffices to use a (possibly countable) partition of unity, exactly as we did in Step 6, and to rely either on (15) or on Theorem B.2. By the chain rule, the $W^{-1,1}$ norms after the test function is multiplied by the functions of the partition are increased at most by a factor being the sup norm of the gradients of the partition, which in our case is bounded by $C|\log \varepsilon|$. Estimate (15) then follows by summation as in (70), and adapting C_1 if necessary.

Proof of Proposition 2. First notice that in view of Remark 3 and estimate 16, it suffices to establish an inequality like (23) only on each of the balls $B(\xi_i, \varepsilon^{\frac{2}{3}})$. The proof is very reminiscent of Step 5 in the proof of Proposition 1. We decompose the energy as in (56), but with j_* replaced by j^{\natural} (here and in the sequel for simplicity we write j^{\natural} in place of $j^{\natural}(u_{\varepsilon}^*)$):

$$e_{\varepsilon}(u) = \frac{1}{2}|j^{\sharp}|^2 + j^{\sharp}\left(\frac{j(u)}{|u|} - j^{\sharp}\right) + e_{\varepsilon}(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j^{\sharp}\right|^2.$$
 (79)

Recall that $\rho_{\epsilon} = \varepsilon^{\frac{2}{3}}$ and let χ_i be a cut-off function with compact support in $B(\xi_i, 2\rho_{\epsilon})$ and such that $\chi_i \equiv 1$ on $B(\xi_i, \rho_{\epsilon})$ and $|\nabla \chi_i| \leq C/\rho_{\epsilon}$. On one hand, similar to (77) we have the upper bound

$$\int re_{\varepsilon}(u)\chi_{i} \leq r(\xi_{i}) \left(\pi \log \frac{2\rho_{\epsilon}}{\varepsilon} + \gamma\right) + C\left(\Sigma_{a}^{r} + \varepsilon^{\frac{1}{3}} |\log \varepsilon|^{C} e^{C\Sigma_{a}^{r}}\right). \tag{80}$$

On the other hand, by direct computation and the definition (19) of r_{ξ} we have the lower bound

$$\int r \frac{|j^{\natural}|^2}{2} \chi_i \ge \pi r(\xi_i) \log \frac{2\rho_{\epsilon}}{r_{\xi}} - C \ge \pi r(\xi_i) \log \frac{\rho_{\epsilon}}{\varepsilon} - C(\Sigma_a^r + \log|\log \varepsilon|). \tag{81}$$

To conclude, it suffices then to control the cross-term in (79). We write

$$\int rj^{\natural} \left(\frac{j(u)}{|u|} - j^{\natural}\right) \chi_i = \int rj^{\natural} \left(j(u) - j^{\natural}\right) \chi_i + \int rj^{\natural} \frac{j(u)}{|u|} (|u| - 1) \chi_i$$

and then for arbitrary $\kappa \in \mathbb{R}$,

$$\int rj^{\natural} (j(u) - j^{\natural}) \chi_{i} = \int \nabla^{\perp} (r\Psi_{\xi}^{\natural} - \kappa) (j(u) - j^{\natural}) \chi_{i}$$

$$= -\int (r\Psi_{\xi}^{\natural} - \kappa) \operatorname{curl}(j(u) - j^{\natural}) \chi_{i} - \int (r\Psi_{\xi}^{\natural} - \kappa) (j(u) - j^{\natural}) \nabla^{\perp} \chi_{i}.$$

Finally, we split

$$\int (r\Psi_{\xi}^{\natural} - \kappa)(j(u) - j^{\natural}) \nabla^{\perp} \chi_{i} = \int (r\Psi_{\xi}^{\natural} - \kappa)(\frac{j(u)}{|u|} - j^{\natural}) \nabla^{\perp} \chi_{i} + \int (r\Psi_{\xi}^{\natural} - \kappa)\frac{j(u)}{|u|}(|u| - 1) \nabla^{\perp} \chi_{i}.$$

We choose κ to be the mean value of $r\Psi_{\xi}^{\sharp}$ over the support of $\nabla^{\perp}\chi_{i}$, and therefore in view of the logarithmic nature of Ψ_{ξ}^{\sharp} we have the upper bound $|r\Psi_{\xi}^{\sharp} - \kappa| \leq C$ on the support of $\nabla^{\perp}\chi_{i}$. As in Proposition 1, by Cauchy-Schwarz and the L^{∞} bound on j^{\sharp} , we estimate

$$\left| \int r j^{\natural} \frac{j(u)}{|u|} (|u| - 1) \chi_i \right| \le \frac{C}{r_{\xi}} \varepsilon \mathcal{E}_{\varepsilon}^{\mathbf{w}} (u, B(\xi_i, 2\rho_{\epsilon})) \le C$$

and

$$|\int (r\Psi_{\xi}^{\natural} - \kappa) \frac{j(u)}{|u|} (|u| - 1) \nabla^{\perp} \chi_{i}| \leq C \frac{\varepsilon}{\rho_{\epsilon}} \mathcal{E}_{\varepsilon}^{w} (u, B(\xi_{i}, 2\rho_{\epsilon})) \leq C.$$

Next, we have

$$\begin{split} |\int (r\Psi_{\xi}^{\natural} - \kappa)(\frac{j(u)}{|u|} - j^{\natural})\nabla^{\perp}\chi_{i}| &\leq C \|\frac{j(u)}{|u|} - j^{\natural}\|_{L^{2}(\operatorname{supp}(\nabla\chi_{i}))}\|\nabla\chi_{i}\|_{L^{2}} \\ &\leq C \left(\Sigma_{a}^{r} + \varepsilon^{\frac{1}{3}}|\log\varepsilon|^{C}e^{C\Sigma_{a}^{r}}\right)^{\frac{1}{2}} \\ &\leq C(\Sigma_{a}^{r} + 1). \end{split}$$

For the last term, we write

$$\begin{split} |\int (r\Psi_{\xi}^{\natural} - \kappa) \mathrm{curl}(j(u) - j^{\natural}) \chi_{i}| &\leq C \|2J(u) - \mathrm{curl}j^{\natural}\|_{W^{-1,1}(B(\xi_{i}, 2\rho_{\epsilon}))} \|(r\Psi_{\xi}^{\natural} - \kappa) \chi_{i}\|_{W^{1,\infty}} \\ &\leq C r_{\xi} \frac{1}{r_{\xi}} \leq C, \end{split}$$

where we have used (15) and the fact that by construction

$$\|\operatorname{curl} j^{\natural} - 2\pi \sum_{i=1}^{n} \delta_{\xi_{i}} \|_{\dot{W}^{-1,1}(B(\xi_{i},2\rho_{\epsilon}))} \le Cr_{\xi}.$$

The conclusion follows.

Proof of Proposition 3. Since $j^{\natural}(u_{\xi}^*)$ is not sufficiently regular across the boundaries of the sets C_i , defined after (19), the computation which follows (28) does not hold as is with X replaced by $j^{\natural}(u_{\xi}^*)$ and we need instead to divide the integration domain \mathbb{H} into the union of the pieces C_i and of the complement of this union. Performing the integration by parts then imply (only) some boundary terms, which actually end up in justifying (29) provided curlX is understood in a weak sense according to (22) and div(rX) according to (21), namely

$$\mathcal{F}(j^{\natural}(u_{\xi}^*),\varphi) = -\sum_{i=1}^n \int_{\partial \mathcal{C}_i} |j(u_{\xi}^*)| j(u_{\xi}^*) \cdot \nabla \varphi. \tag{82}$$

For each fixed i, to compute the boundary term on ∂C_i we use a reference polar frame (ρ, θ) centered at ξ_i . First by construction of C_i and (123)-(124) we have

$$\rho = r_{\mathcal{E}} + O(r_{\mathcal{E}}^2 \log(r_{\mathcal{E}})), \tag{83}$$

so that C_i is close to being a circle, and then by (123), (124) and (83),

$$j^{\natural}(u_{\xi}^*) = \frac{e_{\theta}}{r_{\xi}} + \sum_{i} \mathbb{J} \nabla_{a_{j}} H_{\varepsilon}(\xi_{1}, \cdots, \xi_{n}) + O(\Sigma_{a}^{r} + \log|\log \varepsilon|) \quad \text{on } \partial C_{i},$$
 (84)

where the main error term, of order $\Sigma_a^r + \log|\log \varepsilon|$, comes from the difference between $|\log \varepsilon|$ (as appearing in the definition of H_{ε}) and $\log r_{\xi}$ (from the value of Ψ_{ξ}^* on C_i). The computation of the right-hand-side of (82) is then a direct consequence of (83) and (84), with a cancellation at main order since $\frac{e_{\theta}}{\rho}$ integrates to zero on a circle. The actual details are left to the reader.

Proof of Proposition 4. Since estimate (15) is only valid for r not too close to zero, we shall split χ_{ε} into two pieces. More precisely, we write $1 = \Psi_1 + \Psi_2$ where Ψ_1 is supported in $r \leq 2\tau \equiv 2C_1(\Sigma_a^r + 1)/|\log \varepsilon|$, Ψ_2 is supported in $r \geq \tau$, and $|\nabla \Psi_1| + |\nabla \Psi_2| \leq 10/\tau$. Using (15) we immediately obtain

$$\left| \int Jur^2 \chi_{\varepsilon} \Psi_2 - \pi \sum_{i=1}^n r(\xi_i)^2 \right| \le C_1 \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r} ||r^2 \chi_R \Psi_2||_{W^{1,\infty}} \le C \varepsilon |\log \varepsilon|^{C_1} e^{C_1 \Sigma_a^r} R_{\varepsilon}^2.$$

To estimate the part involving Ψ_1 , and in particular the singularity at r=0, we use Theorem B.3 (more precisely its higher dimensional extension - see e.g. [12]) in the 3D cylinder in cartesian coordinates corresponding to $r \leq 2\tau$ and $|z| \leq 2R_{\varepsilon}$. Writing back its statement in cylindrical coordinates yields

$$\left| \int Jur^{2}\chi_{\varepsilon}\Psi_{1} \right| \leq C \frac{\mathcal{E}_{\varepsilon}^{W}(u, \{r \leq 2\tau\})}{|\log \varepsilon|} ||r\chi_{\varepsilon}\Psi_{1}||_{\infty}$$

$$+ C\varepsilon^{\frac{1}{24}} (1 + \frac{\mathcal{E}_{\varepsilon}^{W}(u, \{r \leq 2\tau\})}{|\log \varepsilon|}) (1 + C\tau^{2}R_{\varepsilon}) ||r\chi_{\varepsilon}\Psi_{1}||_{\mathcal{C}^{0,1}}$$

$$\leq C \frac{1 + \Sigma_{a}^{r}}{|\log \varepsilon|} \tau$$

$$\leq C \frac{(1 + \Sigma_{a}^{r})^{2}}{|\log \varepsilon|^{2}},$$

$$(85)$$

provided ε is required to be sufficiently small. By summation we obtain (31). To obtain (32), we notice that in the expansion (28) the terms for which the derivatives of φ fall onto r^2 exactly cancel (that would correspond without cut-off to the conservation of the momentum) and the remaining ones (where the derivatives fall onto χ_{ε}) are pointwise bounded by $Ce_{\varepsilon}(u)r|\nabla\chi_{\varepsilon}|$, so that the conclusion follows by integration and (16).

Proof of Proposition 6. In this proof $\|\cdot\|$ is understood to mean $\dot{W}^{-1,1}(\Omega_0)$ and $|\cdot|$ refers to the Euclidean norm on \mathbb{H} .

We write

$$||Ju_{\varepsilon}^{s_{1}} - Ju_{\varepsilon}^{s_{2}}|| \leq ||Ju_{\varepsilon}^{s_{1}} - \pi \sum_{i=1}^{n} \delta_{\xi_{i}(s_{1})}|| + ||Ju_{\varepsilon}^{s_{2}} - \pi \sum_{i=1}^{n} \delta_{\xi_{i}(s_{2})}|| + ||\pi \sum_{i=1}^{n} (\delta_{\xi_{i}(s_{1})} - \delta_{\xi_{i}(s_{2})})||$$

$$\leq r_{\xi}^{s_{1}} + r_{\xi}^{s_{2}} + \pi \sum_{i=1}^{n} |\xi_{i}(s_{1}) - \xi_{i}(s_{2})|.$$
(86)

If C_6 is chosen sufficiently large, it follows from the separation assumption (8), the finite speed of propagation of $(LF)_{\varepsilon}$, and the definition of S_{stop} , that

$$\xi_i(s) \in B(a_{i,\varepsilon}(s_1), \frac{\rho_{\min}}{4}) \quad \forall s \in [s_1, s_2].$$

Let

$$\varphi(x) = \sum_{i=1}^{n} \frac{(x - a_{i,\varepsilon}(s_1)) \cdot (\xi_i(s_2) - \xi_i(s_1))}{|\xi_i(s_2) - \xi_i(s_1)|} \chi\Big(|x - a_{i,\varepsilon}(s_1)|\Big), \tag{87}$$

where $\chi \in \mathcal{C}^{\infty}(\mathbb{R}^+, [0, 1])$ is such that $\chi \equiv 1$ on $[0, \rho_{\min}/4]$, $\chi \equiv 0$ on $[\frac{\rho_{\min}}{2}, +\infty)$. By construction and the definition of ρ_{\min} , we have $\varphi \in \mathcal{D}(\Omega_0)$ and it follows that

$$\pi \sum_{i=1}^{n} |\xi_{i}(s_{1}) - \xi_{i}(s_{2})| = \langle \pi \sum_{i=1}^{n} (\delta_{\xi_{i}(s_{2})} - \delta_{\xi_{i}(s_{1})}), \varphi \rangle$$

$$\leq (r_{\xi}^{s_{1}} + r_{\xi}^{s_{2}}) \|\varphi\|_{W^{1,\infty}} + \langle Ju_{\varepsilon}^{s_{2}} - Ju_{\varepsilon}^{s_{1}}, \varphi \rangle.$$
(88)

Combining this with (86), we conclude that

$$||Ju_{\varepsilon}^{s_2} - Ju_{\varepsilon}^{s_1}|| \le C(r_{\varepsilon}^{s_1} + r_{\varepsilon}^{s_2}) + \langle Ju_{\varepsilon}^{s_2} - Ju_{\varepsilon}^{s_1}, \varphi \rangle.$$
(89)

By (27),

$$|\langle Ju_{\varepsilon}^{s_2} - Ju_{\varepsilon}^{s_1}, \varphi \rangle| \le \frac{1}{|\log \varepsilon|} |\int_{s_1}^{s_2} \mathcal{F}(\nabla u_{\varepsilon}^s, \varphi) \, ds|. \tag{90}$$

Recall that

$$\mathcal{F}(\nabla u_{\varepsilon}^{s},\varphi) := -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_{k} r}{r} (\partial_{j} u_{\varepsilon}^{s}, \partial_{k} u_{\varepsilon}^{s}) \partial_{i} \varphi + \int_{\mathbb{H}} \varepsilon_{ij} (\partial_{j} u_{\varepsilon}^{s}, \partial_{k} u_{\varepsilon}^{s}) \partial_{ik} \varphi, \tag{91}$$

and that by (87) we have

$$\partial_{ik}\varphi \equiv 0$$
 on $\bigcup_i B(a_{i,\varepsilon}(s_1), \frac{\rho_{\min}}{4}).$

Since $|\nabla \varphi| \leq C$ and $|D^2 \varphi| \leq C/\rho_{\min}$, we have

$$\left| \int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} (\partial_j u_{\varepsilon}^s, \partial_k u_{\varepsilon}^s) \partial_i \varphi \right| \leq C \mathcal{E}_{\varepsilon}^{\mathbf{w}} (u_{\varepsilon}^s) \leq C |\log \varepsilon|$$

and by (16), (34) and (39)

$$\left| \int_{\mathbb{H}} \varepsilon_{ij} (\partial_j u_{\varepsilon}^s, \partial_k u_{\varepsilon}^s) \partial_{ik} \varphi \right| \leq \frac{C}{\rho_{\min}} \left(\Sigma^0 + r_a^s |\log \varepsilon| + \log |\log \varepsilon| \right) \leq C |\log \varepsilon|.$$

Going back to (90) we thus obtain

$$|\langle Ju_{\varepsilon}^{s_2} - Ju_{\varepsilon}^{s_1}, \varphi \rangle| \le C|s_1 - s_2| \tag{92}$$

and therefore

$$||Ju_{\varepsilon}^{s_1} - Ju_{\varepsilon}^{s_2}|| \le C(r_{\varepsilon}^{s_1} + r_{\varepsilon}^{s_2} + |s_1 - s_2|).$$
 (93)

It remains to estimate $r_{\xi}^{s_2}$. For that purpose, we write

$$r_{a}^{s_{2}} = \|Ju_{\varepsilon}^{s_{2}} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(s_{2})}\|$$

$$\leq \|Ju_{\varepsilon}^{s_{2}} - Ju_{\varepsilon}^{s_{1}}\| + \|Ju_{\varepsilon}^{s_{1}} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(s_{1})}\| + \|\pi \sum_{i=1}^{n} (\delta_{a_{i,\varepsilon}(s_{1})} - \delta_{a_{i,\varepsilon}(s_{2})})\|$$

$$\leq r_{a}^{s_{1}} + C(|s_{1} - s_{2}| + r_{\xi}^{s_{1}} + r_{\xi}^{s_{2}}),$$

$$(94)$$

where we have used (93) and (88). By the definition (41) of $r_{\xi}^{s_2}$ and (94) we obtain

$$r_{\xi}^{s_{2}} \leq C_{1}\varepsilon |\log \varepsilon|^{C_{1}} e^{C_{1}(\Sigma^{0} + r_{a}^{s_{2}}|\log \varepsilon|)}$$

$$\leq r_{\xi}^{s_{1}} e^{C(|s_{2} - s_{1}| + 2\varepsilon^{\frac{5}{6}})|\log \varepsilon|}$$

$$\leq r_{\xi}^{s_{1}} (1 + C\left(|s_{2} - s_{1}| + \varepsilon^{\frac{5}{6}}\right)|\log \varepsilon|),$$

$$(95)$$

where we have used the fact that $|s_2 - s_1| \le |\log \varepsilon|^{-1}$ by assumption. It remains to prove the last assertion of the statement, namely that if $r_a^{s_1} < \rho_{\min}/16$ then $S_{\text{stop}} \ge s_1 + (C_6\sqrt{|\log \varepsilon|})^{-1}$. By definition of S_{stop} , the latter follows easily from (94) and (95), increasing the value of C_6 if necessary.

Proof of Proposition 7. The proof follows very closely the strategy used in [11] Proposition 7.1. By (43) and the definition of S we first remark that

$$r_{\xi}^{\tau} \le 2r_{\xi}^{s} \qquad \forall \tau \in [s, S].$$
 (96)

Next, note that

$$r_{a}^{S} - r_{a}^{s} = \|Ju_{\varepsilon}^{S} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(S)}\| - \|Ju_{\varepsilon}^{s} - \pi \sum_{i=1}^{n} \delta_{a_{i,\varepsilon}(s)}\|$$

$$\leq \pi \sum_{i=1}^{n} (|\xi_{i}(S) - a_{i,\varepsilon}(S)| - |\xi_{i}(s) - a_{i,\varepsilon}(s)|) + r_{\xi}^{S} + r_{\xi}^{s}$$

$$\leq \pi \sum_{i=1}^{n} \nu_{i} \cdot (\xi_{i}(S) - \xi_{i}(s) + a_{i,\varepsilon}(s) - a_{i,\varepsilon}(S)) + r_{\xi}^{S} + r_{\xi}^{s}$$
(97)

for $\nu_i = \frac{\xi_i(S) - a_{i,\varepsilon}(S)}{|\xi_i(S) - a_{i,\varepsilon}(S)|}$ (unless $\xi_i(S) - a_{i,\varepsilon}(S) = 0$, in which case ν_i can be any unit vector). We let

$$\varphi(x) = \sum_{i} \nu_i \cdot (x - a_{i,\varepsilon}(s)) \chi(|x - a_{i,\varepsilon}(s)|)$$

for $\chi \in C^{\infty}(\mathbb{R}^+, [0, 1])$ such that $\chi \equiv 1$ on $[0, \frac{1}{4}\rho_{\min}]$ and $\chi \equiv 0$ on $(\frac{1}{2}\rho_{\min}, \infty)$. It follows from (44) that

$$\pi \sum_{i=1}^{n} \nu_i \cdot \left(\xi_i(S) - \xi_i(s) + a_{i,\varepsilon}(s) - a_{i,\varepsilon}(S) \right) = \pi \sum_{i=1}^{n} \left[\varphi(\xi_i(S)) - \varphi(\xi_i(s)) - \varphi(a_{i,\varepsilon}(S)) + \varphi(a_{i,\varepsilon}(s)) \right],$$

so that (97) and the definition of r_{ξ}^{S} imply that

$$r_a^S - r_a^s \le \langle \varphi, Ju_\varepsilon^S - Ju_\varepsilon^S \rangle - \pi \sum_{i=1}^n \left[\varphi(a_{i,\varepsilon}(S)) - \varphi(a_{i,\varepsilon}(S)) \right] + C(r_\xi^S + r_\xi^S). \tag{98}$$

Our main task in the sequel is therefore to provide an estimate for the quantity $\langle \varphi, Ju_{\varepsilon}^S - Ju_{\varepsilon}^s \rangle$. By (27) (and taking into account the $|\log \varepsilon|$ change of scale in time) we have

$$\langle \varphi, J u_{\varepsilon}^{S} - J u_{\varepsilon}^{s} \rangle = \int_{s}^{S} \frac{1}{|\log \varepsilon|} \mathcal{F}(\nabla u_{\varepsilon}^{\tau}, \varphi) \, d\tau. \tag{99}$$

In the sequel for the ease of notation we write u in place of u_{ε}^{τ} , for $\tau \in [s, S]$. Similar to what we did in (56), and in view of the definition (26) of \mathcal{F} , we decompose here

$$(\partial_{j}u, \partial_{k}u) = \partial_{j}|u|\partial_{k}|u| + \frac{j(u)_{j}}{|u|}\frac{j(u)_{k}}{|u|}$$

$$= \partial_{j}|u|\partial_{k}|u| + \left(\frac{j(u)}{|u|} - j^{\sharp}\right)_{j}\left(\frac{j(u)}{|u|} - j^{\sharp}\right)_{k}$$

$$+ \left(\frac{j(u)}{|u|} - j^{\sharp}\right)_{j}\left(j^{\sharp}\right)_{k} + \left(\frac{j(u)}{|u|} - j^{\sharp}\right)_{k}\left(j^{\sharp}\right)_{j}$$

$$+ \left(j^{\sharp}\right)_{j}\left(j^{\sharp}\right)_{k},$$

$$(100)$$

where

$$j^{\natural} \equiv j^{\natural}(u_{\xi(\tau)}^*).$$

Hence,

$$\mathcal{F}(\nabla u, \varphi) = \mathcal{F}(j^{\natural}, \varphi) + \sum_{p=1}^{4} T_p$$
 (101)

where

$$\begin{split} T_1 &:= -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} \left[\partial_j |u| \partial_k |u| + \left(\frac{j(u)}{|u|} - j^{\natural} \right)_j \left(\frac{j(u)}{|u|} - j^{\natural} \right)_k \right] \partial_i \varphi, \\ T_2 &:= \int_{\mathbb{H}} \varepsilon_{ij} \left[\partial_j |u| \partial_k |u| + \left(\frac{j(u)}{|u|} - j^{\natural} \right)_j \left(\frac{j(u)}{|u|} - j^{\natural} \right)_k \right] \partial_{ik} \varphi, \\ T_3 &:= -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} \left[\left(\frac{j(u)}{|u|} - j^{\natural} \right)_j \left(j^{\natural} \right)_k + \left(\frac{j(u)}{|u|} - j^{\natural} \right)_k \left(j^{\natural} \right)_j \right] \partial_i \varphi, \end{split}$$

and

$$T_4 := \int_{\mathbb{H}} \varepsilon_{ij} \left[\left(\frac{j(u)}{|u|} - j^{\natural} \right)_j \left(j^{\natural} \right)_k + \left(\frac{j(u)}{|u|} - j^{\natural} \right)_k \left(j^{\natural} \right)_j \right] \partial_{ik} \varphi.$$

By Proposition 3 we already know that

$$\left| \mathcal{F}(j^{\natural}, \varphi) - \sum_{i=1}^{n} \mathbb{J} \nabla_{a_i} H_{\varepsilon}(\xi_1(\tau), \cdots, \xi_n(\tau)) \cdot \nabla \varphi(\xi_i(\tau)) \right| \leq C_3 \left(\Sigma^0 + r_a^{\tau} |\log \varepsilon| + \log |\log \varepsilon| \right),$$

moreover since $S \leq S_{\text{stop}}$ we have for any $i = 1, \dots, n$,

$$|\nabla_{a_i} H_{\varepsilon}(\xi_1(\tau), \cdots, \xi_n(\tau)) - \nabla_{a_i} H_{\varepsilon}(a_{1,\varepsilon}(\tau), \cdots, a_{n,\varepsilon}(\tau))| \le C|\log \varepsilon| r_a^{\tau}$$

and since φ is affine there,

$$\nabla \varphi(\xi_i(\tau)) = \nabla \varphi(a_{i,\varepsilon}(\tau)),$$

so that after integration and using the fact that the points $a_{i,\varepsilon}$ evolve according to the ODE $(LF)_{\varepsilon}$ we obtain

$$\left| \int_{s}^{S} \frac{1}{|\log \varepsilon|} \mathcal{F}(j^{\natural}, \varphi) d\tau - \pi \sum_{i=1}^{n} \left[\varphi(a_{i,\varepsilon}(S)) - \varphi(a_{i,\varepsilon}(s)) \right] \right| \leq C(S-s) \left(r_{a}^{s} + \frac{\Sigma^{0}}{|\log \varepsilon|} + \frac{\log|\log \varepsilon|}{|\log \varepsilon|} \right).$$

Now that we have accounted for the main order, we need to control all the terms T_p (at least integrated in time between s and S). We begin with the terms T_1 and T_2 for which we already have good estimates (pointwise in time) thanks to Proposition 1 and Proposition 2. Indeed, by Proposition 2 and since $|\nabla \varphi| \leq C$, we have

$$\frac{|T_1|}{|\log \varepsilon|} \le C \left(r_a^s + \frac{\Sigma^0}{|\log \varepsilon|} + \frac{\log|\log \varepsilon|}{|\log \varepsilon|} \right). \tag{102}$$

By Proposition 1 and (34), and since $|D^2\varphi| \leq C\sqrt{|\log \varepsilon|}$, we have

$$\frac{|T_2|}{|\log \varepsilon|} \le C \left(r_a^s + \frac{\Sigma^0}{\sqrt{|\log \varepsilon|}} + |\log \varepsilon|^{-\frac{3}{2}} + \sqrt{|\log \varepsilon|} \Big| P_{\varepsilon}(u) - P(a_{1,\varepsilon}(\tau) \cdots, a_{n,\varepsilon}(\tau)) \Big| \right). \tag{103}$$

In order to deal with the last term involving P and P_{ε} , recall first that P is preserved by the flow $(LF)_{\varepsilon}$, so that

$$P(a_{1,\varepsilon}(\tau),\cdots,a_{n,\varepsilon}(\tau)) = P(a_{1,\varepsilon}(0),\cdots,a_{n,\varepsilon}(0)), \tag{104}$$

and that P_{ε} is almost preserved by $(GP)_{\varepsilon}^{c}$, as expressed by (32), so that

$$\left| P_{\varepsilon}(u) - P_{\varepsilon}(u_{\varepsilon}^{0}) \right| \leq C \frac{S_{\text{stop}}}{\left| \log \varepsilon \right|^{3}} \left(\Sigma^{0} + \sqrt{\left| \log \varepsilon \right|} \right) \leq C S_{\text{stop}} \left| \log \varepsilon \right|^{-\frac{5}{2}}$$
 (105)

where we have used the rough bound (17) for Σ_{ξ} , the rough estimate $r_a^{\tau} \leq C/\sqrt{|\log \varepsilon|}$ which follows from the definition of S_{stop} , and where we have taken into account the factor $|\log \varepsilon|^{-1}$ which arises from the change of time scale which we have here with respect to the one of (27). On the other hand, at the initial time by (31) and the bound (14) on r_a^0 and Σ^0 we have

$$\left| P_{\varepsilon}(u_{\varepsilon}^{0}) - P(\{a_{i,\varepsilon}(0)\}) \right| \le \left| P_{\varepsilon}(u_{\varepsilon}^{0}) - P(\{\xi_{i}(0)\}) \right| + Cr_{a}^{0} \le C \left(r_{a}^{0} + \frac{1}{|\log \varepsilon|^{2}} \right). \tag{106}$$

In total, similar to (102) we obtain

$$\frac{|T_2|}{|\log \varepsilon|} \le C \left(r_a^s + \frac{\Sigma^0 + r_a^0 |\log \varepsilon|}{\sqrt{|\log \varepsilon|}} + \frac{\log |\log \varepsilon|}{|\log \varepsilon|} \right), \tag{107}$$

where we have absorbed some of the above error terms by the term $\log |\log \varepsilon|/|\log \varepsilon|$. We decompose

$$T_3 = T_{3,1} + T_{3,2} + T_{3,3}$$

where

$$T_{3,1} := -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} \left[\left(\frac{j(u)}{|u|} - j^{\natural} \right)_j \left(j^{\natural} - j^{\natural} (u_{\xi(s)}^*) \right)_k + \left(\frac{j(u)}{|u|} - j^{\natural} \right)_k \left(j^{\natural} - j^{\natural} (u_{\xi(s)}^*) \right)_j \right] \partial_i \varphi,$$

$$T_{3,2} := -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} \left[\left(j(u) - j^{\natural} \right)_j \left(j^{\natural} (u_{\xi(s)}^*) \right)_k + \left(j(u) - j^{\natural} \right)_k \left(j^{\natural} (u_{\xi(s)}^*) \right)_j \right] \partial_i \varphi,$$

and

$$T_{3,3} := -\int_{\mathbb{H}} \varepsilon_{ij} \frac{\partial_k r}{r} \left[\left(\frac{j(u)}{|u|} \right)_j \left(j^{\natural}(u_{\xi(s)}^*) \right)_k + \left(\frac{j(u)}{|u|} \right)_k \left(j^{\natural}(u_{\xi(s)}^*) \right)_j \right] (1 - |u|) \partial_i \varphi,$$

and accordingly we decompose $T_4 = T_{4,1} + T_{4,2} + T_{4,3}$. We first deal with $T_{3,3}$ and $T_{4,3}$, where invoking the inequality

$$\frac{|j(u)|}{|u|}(1-|u|) \le \frac{\varepsilon}{2} \left(\frac{|j(u)|^2}{|u|^2} + \frac{(1-|u|)^2}{\varepsilon^2} \right) \le C\varepsilon e_{\varepsilon}(u)$$
(108)

combined with the global $|\log \varepsilon|$ bound on the energy and the L^{∞} bound $|j^{\natural}| \leq Cr_{\xi}^{-1} \leq C\varepsilon^{-1}/|\log \varepsilon|^{C_1}$ we directly infer (increasing C_1 if necessary) that

$$\frac{T_{3,3} + T_{4,3}}{|\log \varepsilon|} \le \frac{C}{|\log \varepsilon|}.\tag{109}$$

We next turn to the terms $T_{3,1}$ and $T_{4,1}$, for which we rely on Proposition 6 and the definition of S to get the upper bound

$$\sum_{i=1}^{n} |\xi_i(s) - \xi_i(\tau)| \le C \left(r_{\xi}^s + \frac{(r_{\xi}^s)^2}{\varepsilon} \right) \le C \frac{(r_{\xi}^s)^2}{\varepsilon}. \tag{110}$$

Using the almost explicit form of j^{\natural} (more precisely (122), (123) and the definition of the cut-off at the scale r_{ε}^{s}), we compute that

$$\int_{\operatorname{supp}(\varphi)} \left| j^{\sharp} - j^{\sharp}(u_{\xi(s)}^*) \right|^2 \le C(1 + \log\left(\frac{\frac{(r_{\xi}^s)^2}{\varepsilon}}{r_{\xi}^s}\right)) \le C\left(\Sigma^0 + r_a^s |\log \varepsilon| + \log|\log \varepsilon|\right). \tag{111}$$

The previous inequality combined with Proposition 2 and the Cauchy-Schwarz inequality then yields

$$\frac{T_{3,1}}{|\log \varepsilon|} \le C \left(r_a^s + \frac{\Sigma^0}{|\log \varepsilon|} + \frac{\log|\log \varepsilon|}{|\log \varepsilon|} \right).$$

For $T_{4,1}$, since the integration domain does no longer contain the cores we obtain the stronger estimate

$$\int_{\text{supp}(D^2\varphi)} \left| j^{\natural} - j^{\natural} (u_{\xi(s)}^*) \right|^2 \le C \frac{\left(\sum_{i=1}^n |\xi_i(s) - \xi_i(\tau)| \right)^2}{\rho_{\min}^2} \le C \varepsilon^{\frac{2}{3}} |\log \varepsilon|, \tag{112}$$

where we have used (110) and (18) for the last inequality. Using once more the Cauchy-Schwarz inequality, combined here with Proposition 1 (or even simply the crude $|\log \varepsilon|$ global energy bound) and the L^{∞} bound $|D^2 \varphi| \leq C \sqrt{|\log \varepsilon|}$ we obtain

$$\frac{T_{4,1}}{|\log \varepsilon|} \le C\varepsilon^{\frac{1}{3}} \sqrt{|\log \varepsilon|}.$$

At this stage we are left to estimate $T_{3,2}$ and $T_{4,2}$, which we will only be able to do after integration in time. To underline better the time dependence, it is convenient here to write

 j_{τ}^{\natural} in place of j^{\natural} and j_{s}^{\natural} in place of $j^{\natural}(u_{\xi}^{s})$. The main ingredient in the argument is then to perform a Helmholtz type decomposition of $j(u) - j_{\tau}^{\natural}$. More precisely, we first fix a cut-off function χ with compact smooth support B in $\{r \geq \frac{r_0}{8}\}$, which is identically equal to 1 on the support of φ and which satisfies $|\nabla \chi| \leq C$ (its only aim is to get rid of boundary terms, of spatial infinity, and of the singularity at r = 0). For every $\tau \in [s, S]$, we then set

$$\chi(j(u) - j_{\tau}^{\natural}) = \nabla f^{\tau} + \frac{1}{r} \nabla^{\perp} g^{\tau} \quad \text{in } B,$$
 (113)

where f^{τ} and g^{τ} are the unique solutions of the Neumann

$$\operatorname{div}(r\nabla f^{\tau}) = \operatorname{div}\left(\chi r(j(u) - j_{\tau}^{\natural})\right) \quad \text{in } B,$$

$$\partial_{n} f^{\tau} = 0 \quad \text{on } \partial B,$$

$$\int_{B} f^{\tau} = 0,$$
(114)

and Dirichlet

$$-\operatorname{div}\left(\frac{1}{r}\nabla g^{\tau}\right) = \operatorname{curl}\left(\chi(j(u) - j_{\tau}^{\natural})\right) \quad \text{in } B,$$

$$q^{\tau} = 0 \qquad \qquad \text{on } \partial B$$
(115)

boundary value problems. By construction,

$$\int_{s}^{S} \frac{T_{3,2}}{|\log \varepsilon|} d\tau = -\frac{1}{|\log \varepsilon|} \int_{B} \varepsilon_{ij} \frac{\partial_{k} r}{r} \left[(\nabla F + \frac{\nabla^{\perp} G}{r})_{j} (j_{s}^{\natural})_{k} + (\nabla F + \frac{\nabla^{\perp} G}{r})_{k} (j_{s}^{\natural})_{j} \right] \partial_{i} \varphi \quad (116)$$

and

$$\int_{s}^{S} \frac{T_{4,2}}{|\log \varepsilon|} d\tau = -\frac{1}{|\log \varepsilon|} \int_{B} \varepsilon_{ij} \left[(\nabla F + \frac{\nabla^{\perp} G}{r})_{j} (j_{s}^{\natural})_{k} + (\nabla F + \frac{\nabla^{\perp} G}{r})_{k} (j_{s}^{\natural})_{j} \right] \partial_{ik} \varphi \qquad (117)$$

where $F = \int_s^S f^{\tau} d\tau$ and $G = \int_s^S g^{\tau} d\tau$. Integrating (114), we split $F = F_1 + F_2 + F_3$ where

$$\operatorname{div}\left(r\nabla F_{p}\right)=L_{p} \text{ in } B, \qquad \partial_{n}F_{p}=0 \text{ on } \partial B, \qquad \int_{B}F_{p}=0,$$

for p = 1, 2, 3, and where

$$L_{1} := \chi \int_{s}^{S} \operatorname{div}(rj(u)), \qquad L_{2} := r\nabla \chi \cdot \int_{s}^{S} \frac{j(u)}{|u|} (|u| - 1), \qquad L_{3} := r\nabla \chi \cdot \int_{s}^{S} (\frac{j(u)}{|u|} - j_{\tau}^{\natural}).$$

Similarly, integrating (115) we split $G = G_1 + G_2 + G_3$ where

$$\operatorname{div}\left(\frac{1}{r}\nabla G_p\right) = M_p \text{ in } B, \qquad M_p = 0 \text{ on } \partial B,$$

for p = 1, 2, 3, and where

$$M_1 := \chi \int_{\mathfrak{s}}^{S} \operatorname{curl}(j(u) - j_{\tau}^{\natural}), \qquad M_2 := \nabla^{\perp} \chi \cdot \int_{\mathfrak{s}}^{S} \frac{j(u)}{|u|} (|u| - 1), \qquad M_3 := \nabla^{\perp} \chi \cdot \int_{\mathfrak{s}}^{S} (\frac{j(u)}{|u|} - j_{\tau}^{\natural}).$$

Before we state precise bounds for each of them, we note that it should be clear at this stage that all the terms L_p and M_p are small in some sense, except perhaps for the term L_1 which

requires some more explanation. For that last term, we rely on the continuity equation (and this is the main reason for the integration in time)

$$\partial_t |u|^2 = -\frac{2}{r} \operatorname{div}(rj(u)) \tag{118}$$

which is a consequence of $(GP)_{\varepsilon}^{c}$, and from which we infer that

$$L_1 = \varepsilon \frac{\chi}{|\log \varepsilon|} \left[\frac{(|u|^2 - 1)}{\varepsilon} \right]_s^S, \tag{119}$$

so that

$$||L_1||_{L^2} \le C \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \le C \frac{(S-s)}{\sqrt{|\log \varepsilon|}} \frac{\varepsilon^2}{(r_\xi^s)^2} \le C(S-s)|\log \varepsilon|^{-2C_1},$$

where we have used the definitions of S and r_{ξ}^{s} . Regarding L_{2} and M_{2} , using (108) we easily obtain

$$||L_2||_{L^1} + ||M_2||_{L^1} \le C\varepsilon(S-s)|\log\varepsilon|.$$

For L_3 and M_3 , we use the fact that $\nabla \chi$ lives away from the cores so that Proposition 1 and Proposition 5 yield (we bound the terms involving P in (34) exactly as we did to simplify (103) into (107))

$$||L_3||_{L^2} + ||M_3||_{L^2} \le C(S-s) \left(r_a^s \sqrt{|\log \varepsilon|} + \Sigma^0 + r_a^0 |\log \varepsilon| + |\log \varepsilon|^{-1} \right)^{\frac{1}{2}} \le C(S-s).$$

Finally, regarding M_1 , we have on one side using (15) and (96)

$$||M_1||_{W^{-1,1}} \le C(S-s)r_{\varepsilon}^s,$$

and on the other side using the pointwise inequality $Ju \leq Ce_{\varepsilon}(u)$ for an arbitrary function u and the global energy bound

$$||M_1||_{L^1} < C(S-s)|\log \varepsilon|.$$

By interpolation, it follows that for any 1

$$||M_1||_{W^{-1,p}} \le C_p(S-s)(r_{\varepsilon}^s)^{\theta} |\log \varepsilon|^{1-\theta},$$

where $\frac{1}{p} = \theta + \frac{1-\theta}{2}$. These bounds on L_i and M_i turn into bounds on F_i and G_i , since by standard elliptic estimates we have

$$\|\nabla F_1\|_{L^p} \le C_p \|L_1\|_{L^2} \qquad \text{for all } 1 \le p < +\infty, \|\nabla F_2\|_{L^p} + \|\nabla G_2\|_{L^p} \le C_p (\|L_2\|_{L^1} + \|M_2\|_{L^1}) \qquad \text{for all } 1 \le p < 2, \|\nabla F_3\|_{L^p} + \|\nabla G_3\|_{L^p} \le C_p (\|L_3\|_{L^2} + \|M_3\|_{L^2}) \qquad \text{for all } 1 \le p < +\infty, \|\nabla G_1\|_{L^p} \le C_p \|M_1\|_{W^{-1,p}} \qquad \text{for all } 1
$$(120)$$$$

To estimate (116), we then simply input (120) into (116) where we use the Hölder inequality with j_s^{\natural} estimated in $L^{p'}$ (and all the other weights other than F_i or G_i in L^{∞}). The largest contribution arises from G_1 since $||j_s^{\natural}||_{L^{p'}} \simeq (r_{\xi}^s)^{-\theta}$ when p < 2 and the final $|\log \varepsilon|^{-1}$

bound is obtained by choosing p arbitrarilly close to 1 (and hence θ arbitrarilly close to 1). For the terms involving p > 2 we use the straightforward bound $\|j_s^{\natural}\|_{L^{p'}} \leq C$.

The estimate of (117) is at first sight slightly more difficult since $\partial_{ik}\varphi$ is diverging like $\sqrt{|\log \varepsilon|}$ whereas in (116) $\partial_i \varphi$ was bounded in absolute value. On the other hand, the integrand only lives on the support of $D^2 \varphi$, which is both away from the cores and of Lebesgues measure of order $|\log \varepsilon|^{-1}$. More precisely, for F_1 and G_1 we rely exactly on the same estimate as in (120), whereas, since the forcing terms L_2, L_3, M_2 and M_3 have a support disjoint from that of $D^2 \varphi$, it follows by elliptic regularity that

$$\|\nabla F_i\|_{L^{\infty}(\text{supp}(D^2\varphi))} \le C\|\nabla F_i\|_{L^1}, \quad \|\nabla G_i\|_{L^{\infty}(\text{supp}(D^2\varphi))} \le C\|\nabla G_i\|_{L^1}, \quad i = 2, 3.$$
 (121)

We then combine (121) with our previous estimate (120), and therefore in the Hölder estimate of (117) for these four terms we can take $p = \infty$ and hence p' = 1. Finally, regarding j_s^{\natural} we have for $p = \infty$

$$||j_s^{\sharp}||_{L^1(\operatorname{supp}(D^2\varphi))} \le ||j_s^{\sharp}||_{L^{\infty}(\operatorname{supp}(D^2\varphi))} \mathcal{L}^2(\operatorname{supp}(D^2\varphi))) \le C\rho_{\min},$$

and for p < 2

$$||j_s^{\natural}||_{L^{p'}(\operatorname{supp}(D^2\varphi))} \le C(\rho_{\min})^{-\theta}$$

where $\frac{1}{p} = \theta + \frac{1-\theta}{2}$. The conclusion then follows by summation.

Proofs of Theorem 2 and Theorem 1 Theorem 2 follows very directly from Proposition 7. Indeed, the iterative use of Proposition 7 leads to a discrete Gronwall inequality which is a forward Euler scheme for the corresponding classical (continuous) Gronwall inequality, and the latter has convex solutions which are therefore greater than their discrete equivalent. The actual details can be taken almost word for word from the ones used in [11] Proof of Theorem 1.3, and are therefore not repeated here.

Finally, Theorem 1 is also easily deduced from Theorem 2. The only point which deserves additional explanation is the fact that in the assumptions of Theorem 1 only local norms $\|\cdot\|_{\dot{W}^{-1,1}(\Omega)}$ with Ω being of compact closure in the interior of \mathbb{H} are used whereas the definition of r_a^0 for Theorem 2 involves the unbounded set Ω_0 . As the proof of Proposition 1 shows (more precisely its Step 6), the closeness estimates expressed in (15) (which hold in expanding domains whose union ends up covering the whole of \mathbb{H} as ε tends to zero) only require a first localisation estimate in a neigborhood of size $1/\sqrt{|\log \varepsilon|}$ of the points $a_{i,\varepsilon}$, which is of course implied by the assumptions of Theorem 1.

A Vector potential of loop currents

In the introduction we have considered the inhomogeneous Poisson equation

$$\begin{cases} -\operatorname{div}\left(\frac{1}{r}\nabla\left(rA_{a}\right)\right) = 2\pi\delta_{a} & \text{in } \mathbb{H}, \\ A_{a} = 0 & \text{on } \mathbb{H} \end{cases}$$

Its integration is classical (see e.g. [10]) and yields

$$A_a(r,z) = \frac{r(a)}{2} \int_0^{2\pi} \frac{\cos(t)}{\sqrt{r(a)^2 + r^2 + (z - z(a))^2 - 2r(a)r\cos(t)}} dt,$$

which in turn simplifies to

$$A_a(r,z) = \sqrt{\frac{r(a)}{r}} \frac{1}{k} \left[(2 - k^2)K(k^2) - 2E(k^2) \right]$$

where

$$k^{2} = \frac{4r(a)r}{r(a)^{2} + r^{2} + (z - z(a))^{2} + 2r(a)r}$$

and where E and K denote the complete elliptic integrals of first and second kind respectively (see e.g. [1]). Note that $A_{\lambda a}(\lambda r, \lambda z) = A_a(r, z)$ for any $\lambda > 0$ and that we have the asymptotic expansions [1] of the complete elliptic integrals as $s \to 1$:

$$K(s) = -\frac{1}{2}\log(1-s)\left(1 + \frac{1-s}{4}\right) + \log(4) + O(1-s),$$

$$E(s) = 1 - \log(1-s)\frac{1-s}{4} + O(1-s),$$

and similarly for their derivatives. For $(r, z) \in \mathbb{H} \setminus \{a\}$, direct computations therefore yield

$$A_a(r,z) = \left(\log(\frac{r(a)}{\rho}) + 3\log(2) - 2\right) + O\left(\frac{\rho}{r(a)}|\log(\frac{\rho}{r(a)})|\right) \quad \text{as } \frac{\rho}{r(a)} \to 0, \tag{122}$$

and

$$\partial_{\rho}A_{a} = -\frac{1}{\rho} + O\left(\frac{1}{r(a)}\right) \quad \text{as } \frac{\rho}{r(a)} \to 0,$$
 (123)

where $\rho := |a - (r, z)|$.

Concerning the asymptotic close to r = 0, we have

$$A_a(r,z) \simeq \frac{rr(a)^2}{r(a)^3 + |z|^3}$$
 as $\frac{r}{r(a)} \to 0$. (124)

A.1 Singular unimodular maps

When $a = \{a_1, \dots, a_n\}$ is a family of n distinct points in \mathbb{H} , we define the function Ψ_a^* on $\mathbb{H}_a := \mathbb{H} \setminus a$ by

$$\Psi_a^* = \sum_{i=1}^n A_{a_i},$$

so that

$$\begin{cases}
-\operatorname{div}\left(\frac{1}{r}\nabla(r\Psi_a^*)\right) = 2\pi \sum_{i=1}^n \delta_{a_i} & \text{on } \mathbb{H}, \\
\Psi_a^* = 0 & \text{on } \partial \mathbb{H}.
\end{cases}$$

Up to a constant phase shift, there exists a unique unimodular map $u_a^* \in \mathcal{C}^{\infty}(\mathbb{H}_a, S^1) \cap W^{1,1}_{loc}(\mathbb{H}, S^1)$ such that

$$r(iu_a^*,\nabla u_a^*)=rj(u_a^*)=-\nabla^\perp(r\Psi_a^*).$$

In the sense of distributions in \mathbb{H} , we have

$$\begin{cases} \operatorname{div}(rj(u_a^*)) &= 0\\ \operatorname{curl}(j(u_a^*)) &= 2\pi \sum_{i=1}^n \delta_{a_i}. \end{cases}$$

Let

$$\rho_a := \frac{1}{4} \min \left(\min_{i \neq j} |a_i - a_j|, \min_i r(a_i) \right),$$

for $\rho \leq \rho_a$ we set

$$\mathbb{H}_{a,\rho} := \mathbb{H} \setminus \bigcup_{i=1}^n B(a_i,\rho).$$

Lemma A.1. Under the above assumptions we have

$$\int_{\mathbb{H}_{a,\rho}} \frac{|j(u_a^*)|^2}{2} r \, dr dz = \pi \sum_{i=1}^n r(a_i) \left[\log\left(\frac{r(a_i)}{\rho}\right) + \sum_{j \neq i} A_{a_j}(a_i) + \left(3\log(2) - 2\right) + O\left(\frac{\rho}{\rho_a}\log^2(\frac{\rho}{\rho_a})\right) \right].$$

Proof. We have the pointwise equality

$$|j(u_a^*)|^2 r = \frac{1}{r} |\nabla^{\perp}(r\Psi_a^*)|^2 = \frac{1}{r} |\nabla(r\Psi_a^*)|^2,$$

so that after integration by parts

$$\int_{\mathbb{H}_{a,\rho}} \frac{|j(u_a^*)|^2}{2} r \, dr dz = -\frac{1}{2} \int_{\mathbb{H}_{a,\rho}} \operatorname{div}\left(\frac{1}{r} \nabla (r \Psi_a^*)\right) r \Psi_a^* \, dr dz + \frac{1}{2} \int_{\partial \mathbb{H}_{a,\rho}} \Psi_a^* \nabla (r \Psi_a^*) \cdot \vec{n},$$

and the first integral of the right-hand side in the previous identity vanishes by definition of Ψ_a^* and $\mathbb{H}_{a,\rho}$. We next decompose the boundary integral as

$$\frac{1}{2} \int_{\partial \mathbb{H}_{a,\rho}} \Psi_a^* \nabla(r \Psi_a^*) \cdot \vec{n} = \frac{1}{2} \sum_{i,j,k=1}^n \int_{\partial B(a_i,\rho)} A_{a_j} \nabla(r A_{a_k}) \cdot \vec{n},$$

and for fixed i, j, k we write

$$A_{a_i}\nabla(rA_{a_k})\cdot\vec{n} = \left(-A_{a_i}\partial_\rho A_{a_k}r + A_{a_i}A_{a_k}n_r\right).$$

Using (122), we have

$$\left| \int_{\partial B(a_i,\rho)} A_{a_j} A_{a_k} n_r \right| \le r(a_i) O\left(\frac{\rho}{\rho_a} \log^2(\frac{\rho}{\rho_a})\right).$$

When i = j = k, we have by (122) and (123)

$$-\frac{1}{2} \int_{\partial B(a_i,\rho)} A_{a_j} \partial_{\rho} A_{a_k} r = \pi r(a_i) \left(\log(\frac{r(a_i)}{\rho}) + 3\log(2) - 2 + O\left(\frac{\rho}{\rho_a} \log(\frac{\rho}{\rho_a})\right) \right)$$

while when $i = k \neq j$ we have

$$-\frac{1}{2} \int_{\partial B(a_i,\rho)} A_{a_j} \partial_{\rho} A_{a_k} r = \pi r(a_i) \left(A_{a_j}(a_i) + O\left(\frac{\rho}{\rho_a}\right) \right).$$

Finally, when $i \neq k$ we have

$$\left| \frac{1}{2} \int_{\partial B(a_i, \rho)} A_{a_j} \partial_{\rho} A_{a_k} r \right| \le r(a_i) O\left(\left(\frac{\rho}{\rho_a} \right)^2 \log\left(\frac{\rho}{\rho_a} \right) \right).$$

The conclusion follows by summation.

If we next fix some constant $K_0 > 0$ and we assume that the points a_i are of the form

$$a_i := \left(r_0 + \frac{r(b_i)}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{z(b_i)}{\sqrt{|\log \varepsilon|}}\right), \quad i = 1, \dots, n,$$

for some $r_0 > 0$, $z_0 \in \mathbb{R}$ and n points $\{b_1, \dots, b_n\} \in \mathbb{R}^2$ which satisfy

$$\max_{i} |b_i| \le K_0,$$
 and $\min_{i \ne j} \operatorname{dist}(b_i, b_j) \ge \frac{1}{K_0},$

we directly deduce from Lemma A.1, (122) and (123):

Lemma A.2. Under the above assumptions we have

$$\int_{\mathbb{H}_{a,\rho}} \frac{|j(u_a^*)|^2}{2} r \, dr dz = \pi n r_0 \Big(|\log \rho| + n \log r_0 + n \Big(3 \log(2) - 2 \Big) + \frac{n-1}{2} \log |\log \varepsilon| \Big)$$

$$+ \pi r_0 \Big(\sum_i \frac{r(b_i)}{r_0} \frac{|\log \rho|}{\sqrt{|\log \varepsilon|}} - \sum_{i \neq j} \log |b_i - b_j| \Big)$$

$$+ O_{K_0,r_0} \Big(\frac{1}{\sqrt{|\log \varepsilon|}} \Big).$$

B Jacobian and Excess for 2D Ginzburg-Landau functional

For the ease of reading, we recall in this appendix a few results from [12], [13] and [14] which we use in our work.

Theorem B.1 (Thm 1.3 in [13] - **Lower energy bound**). There exists an absolute constant C > 0 such that for any $u \in H^1(B_r, \mathbb{C})$ satisfying $||Ju - \pi \delta_0||_{\dot{W}^{-1,1}(B_r)} < r/4$ we have

$$\mathcal{E}_{\varepsilon}(u, B_r) \ge \pi \log \frac{r}{\varepsilon} + \gamma - \frac{C}{r} \left(\varepsilon \sqrt{\log \frac{r}{\varepsilon}} + \|Ju - \pi \delta_0\|_{\dot{W}^{-1,1}(B_r)} \right).$$

Theorem B.2 (from Thm 1.1 in [13] - **Jacobian estimate without vortices**). There exists an absolute constant C > 0 with the following property. If Ω is a bounded domain, $u \in H^1(\Omega, \mathbb{C}), \ \varepsilon \in (0,1]$ and $\mathcal{E}_{\varepsilon}(u,\Omega) < \pi |\log \varepsilon|$, then

$$||Ju||_{\dot{W}^{-1,1}(\Omega)} \le \varepsilon C \mathcal{E}_{\varepsilon}(u,\Omega) \exp\left(\frac{1}{\pi} \mathcal{E}_{\varepsilon}(u,\Omega)\right).$$

Theorem B.3 (Thm 2.1 in [12] - **Jacobian estimate with vortices**). There exists an absolute constant C > 0 with the following property. If Ω is a bounded domain, $u \in H^1(\Omega, \mathbb{C})$, and $\varphi \in \mathcal{C}^{0,1}_c(\Omega)$, then for any $\lambda \in (1,2]$ and any $\varepsilon \in (0,1)$,

$$\left| \int_{\Omega} \varphi Ju \, dx \right| \leq \pi d_{\lambda} \|\varphi\|_{\infty} + \|\varphi\|_{\mathcal{C}^{0,1}} h^{\varepsilon}(\varphi, u, \lambda)$$

where

$$d_{\lambda} = \left\lfloor \frac{\lambda}{\pi} \frac{\mathcal{E}_{\varepsilon}(u, \operatorname{spt}(\varphi))}{|\log \varepsilon|} \right\rfloor,$$

|x| denotes the greatest integer less than or equal to x, and

$$h^{\varepsilon}(\varphi, u, \lambda) \leq C \varepsilon^{\frac{\lambda - 1}{12\lambda}} \Big(1 + \frac{\mathcal{E}_{\varepsilon}(u, \operatorname{spt}(\varphi))}{|\log \varepsilon|} \Big) \Big(1 + \mathcal{L}^{2}(\operatorname{spt}(\varphi)).$$

Theorem B.4 (Thm 1.2' in [14] - **Jacobian localization for a vortex in a ball**). There exists an absolute constant C > 0, such that for any $u \in H^1(B_r, \mathbb{C})$ satisfying

$$||Ju - \pi \delta_0||_{\dot{W}^{-1,1}(B_r)} < r/4,$$

if we write

$$\Xi = \mathcal{E}_{\varepsilon}(u, B_r) - \pi \log \frac{r}{\varepsilon}$$

then there exists a point $\xi \in B_{r/2}$ such that

$$||Ju - \pi \delta_{\xi}||_{\dot{W}^{-1,1}(B_r)} \le \varepsilon C(C + \Xi) \left[(C + \Xi)e^{\Xi/\pi} + \sqrt{\log \frac{r}{\varepsilon}} \right].$$

Theorem B.5 (Thm 3 in [14] - **Jacobian localization for many vortices**). Let Ω be a bounded, open, simply connected subset of \mathbb{R}^2 with \mathcal{C}^1 boundary. There exists constants C and K, depending on $\operatorname{diam}(\Omega)$, with the following property: For any $u \in H^1(\Omega, \mathbb{C})$, if there exists $n \geq 0$ distinct points a_1, \dots, a_n in Ω and $d \in \{\pm 1\}^n$ such that

$$||Ju - \pi \sum_{i=1}^{n} d_i \delta_{a_i}||_{\dot{W}^{-1,1}(\Omega)} \le \frac{\rho_a}{Kn^5},$$

where

$$\rho_a := \frac{1}{4} \min_i \left\{ \min_{j \neq i} |a_i - a_j|, \operatorname{dist}(a_i, \partial \Omega) \right\},\,$$

and if in addition $\mathcal{E}_{\varepsilon}(u,\Omega) \geq 1$ and

$$\frac{n^5}{\rho_a} \mathcal{E}_{\varepsilon}(u, \Omega) + \frac{n^{10}}{\rho_a^2} \sqrt{\mathcal{E}_{\varepsilon}(u, \Omega)} \le \frac{1}{\varepsilon},$$

then there exist ξ_1, \dots, ξ_d in Ω such that

$$||Ju - \pi \sum_{i=1}^{n} d_i \delta_{\xi_i}||_{\dot{W}^{-1,1}(\Omega)} \le C\varepsilon \left[n(C + \Xi_{\Omega}^{\varepsilon})^2 e^{\Xi_{\Omega}^{\varepsilon}/\pi} + (C + \Xi_{\Omega}^{\varepsilon}) \frac{n^5}{\rho_a} + \mathcal{E}_{\varepsilon}(u,\Omega) \right],$$

where

$$\Xi_{\Omega}^{\varepsilon} := \mathcal{E}_{\varepsilon}(u,\Omega) - n(\pi \log \frac{1}{\varepsilon} + \gamma) + -\pi \Big(\sum_{i \neq j} d_i d_j \log |a_i - a_j| + \sum_{i,j} d_i d_j H_{\Omega}(a_i, a_j) \Big)$$

and H_{Ω} is the Robin function of Ω .

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