Counting the number of trigonal curves of genus five over finite fields

Tom Wennink

Abstract

The trigonal curves form a subvariety of M_5 , the moduli space of smooth curves of genus five. The cohomological data of these spaces can be found by counting their numbers of points over finite fields. The trigonal curves of genus five can be represented by projective plane quintics that have one singularity, which is of delta-invariant one. We use a partial sieve method for plane curves to count the number of trigonal curves over any finite field. The result agrees with the findings of a computer program we have written that counts the number of trigonal curves over the finite fields of two and three elements.

1 Introduction.

Inside the moduli space M_5 of smooth curves of genus five there is the subvariety \mathcal{T}_5 of trigonal curves. In this article we count $\#\mathcal{T}_5(k)$ for all finite fields k.

Theorem 7.2. The number of smooth trigonal curves of genus five over a finite field \mathbb{F}_q is given by

$$\#\mathcal{T}_5(\mathbb{F}_q) = q^{11} + q^{10} - q^8 + 1.$$

We can find cohomological data of a space from knowing its numbers of points over finite fields. Let \mathcal{X} be a Deligne-Mumford stack over \mathbb{Z} of dimension d. Bogaart and Edixhoven have shown in [1, Theorem 2.1] that under certain conditions we have $\#\mathcal{X}(\mathbb{F}_q) = P(q)$, where $P(t) = \sum P_i t^i$ is a polynomial over \mathbb{Q} of degree d such that $P_i = P_{d-i}$. Furthermore the cohomology of \mathcal{X} is determined by P.

From Theorem F in [2, p.15] and the fact that \overline{M}_5 is unirational it follows that \overline{M}_5 satisfies the required conditions. By purity, knowing the cohomology of \overline{M}_5 is equivalent to knowing the Hodge Euler characteristic of \overline{M}_5 . This can be computed from the Hodge Euler characteristic of M_5 and the \mathbb{S}_n -equivariant Hodge Euler characteristics of the $M_{g,n}$ occurring in the description of the boundary Δ_5 . Not all the $\#M_{4,n}(\mathbb{F}_q)$ are known

yet but again with the help of Theorem F from [2, p.15] we find that all $\#M_{g,n}(\mathbb{F}_q)$ are polynomial. (We say a function of \mathbb{F}_q is polynomial if it is an element of $\mathbb{Q}[q]$.) From this it follows that $\#M_5(\mathbb{F}_q)$ is polynomial. We have counted the trigonal curves in this article and for hyperelliptic curves we know $\#H_5(\mathbb{F}_q) = q^9$. The curves in $M_5(\mathbb{F}_q)$ that are neither hyperelliptic nor trigonal have a canonical embedding as a complete intersection of three quadrics in \mathbb{P}^4 . The number of these curves over finite fields \mathbb{F}_q is given by a polynomial $P(q) = \sum P_i q^i$. The dimension of M_5 is 12 so if we are able to compute P_i for $6 \leq i \leq 12$ then we know P(q) (using Poincaré Duality for \overline{M}_5).

There is a bijection between smooth trigonal curves of genus five and plane quintics that have one singularity of delta-invariant one and no other singularities. We fix a point P and then aim to count quintics that have a singularity at P of delta-invariant one and no other singularities, see Section 3. First we take all quintics that have a singularity at P of delta-invariant one and then apply a sieve method, see Section 4. This is done by adding and subtracting various loci of quintics with singular points besides P. The computation for these loci is found in Section 5. With each step of the sieve method more and more curves that have singular points besides P will have been removed exactly once. This process does not terminate so we stop at some point. We then have to correct the count so that all curves that have singularities besides P have been removed exactly once. The necessary computations for this are found in Section 6. Section 7 combines the results and contains a weblink to a computer program we have written that counts $\#\mathcal{T}_5(\mathbb{F}_2)$ and $\#\mathcal{T}_5(\mathbb{F}_3)$. The results of this program confirm Theorem 7.2.

During our counting we come across a lot of cases. We have computed all the cases we need and have listed the results. The methods we use are illustrated by a number of examples in which we describe how we compute a single case.

Acknowledgements

I would like to thank Carel Faber for his help and support. He was the advisor for my master thesis and afterwards helped me transform it into this article.

2 Preliminaries and tools.

Our notation for partitions and tuples of points is in the style of [3].

Definition 2.1. We write $\lambda = [1^{\lambda_1}, \dots, v^{\lambda_v}]$ for the partition where i appears λ_i times. This partition has weight $|\lambda| := \sum_{i=1}^v i \cdot \lambda_i$. We consider the empty partition [] to have weight 0. For the sake of notation we leave out the zero powers, e.g. $[1^2, 2^0, 3^0, 4^1]$ is the same as $[1^2, 4^1]$.

Notation 2.2. With k we denote a finite field with q elements. We define k_i to be the finite field extension of k that has q^i elements.

Notation 2.3. When we talk about the Frobenius map \mathcal{F} we mean the geometric Frobenius.

Let \mathbb{P}^2 be the projective plane over k. The geometric Frobenius on \mathbb{P}^2 is the endomorphism $\mathcal{F}: \mathbb{P}^2 \to \mathbb{P}^2$ defined by $(x:y:z) \mapsto (x^q:y^q:z^q)$.

Definition 2.4. Let X be a scheme defined over k. An n-tuple (x_1, \ldots, x_n) of distinct subschemes of $X_{\bar{k}}$ is called a conjugate n-tuple if $\mathcal{F}(x_i) = x_{i+1}$ for $1 \leq i < n$ and $\mathcal{F}(x_n) = x_1$, where \mathcal{F} is the Frobenius map.

A $|\lambda|$ -tuple $(x_1, \ldots, x_{|\lambda|})$ of distinct subschemes of $X_{\bar{k}}$ is called a λ -tuple if it consists of λ_1 conjugate 1-tuples, followed by λ_2 conjugate 2-tuples, etc.

For a scheme X defined over k we write $X(\lambda)$ for the set of λ -tuples of points of X.

Let X be a k-scheme, $N_r := |X(k_r)|$ and let $(\mu * N)(n) = \sum_{d|n} \mu(\frac{n}{d}) N_d$ denote the convolution of N_- with the Möbius function.

Lemma 2.5. For any scheme X defined over k, the number of λ -tuples of points of X is equal to

$$|X(\lambda)| = \prod_{i=1}^{v} {\binom{(\mu * N)(i)}{i} \choose \lambda_i}.$$

Definition 2.6. For any scheme X defined over k and $w \in \mathbb{Z}_{\geq 0}$ we define

$$\pi_w(X) := \sum_{|\lambda|=w} (-1)^{\sum_i \lambda_i} \cdot |X(\lambda)|.$$

Lemma 2.7. The inverse Hasse-Weil zeta function generates π , that is if X is a scheme of finite type over k we have

$$\frac{1}{Z(X;t)} = \sum_{w=0}^{\infty} \pi_w(X)t^w.$$

Proof. We have

$$\sum_{w=0}^{\infty} \pi_w(X) t^w = \sum_{w=0}^{\infty} \sum_{|\lambda|=w} \prod_{i=1}^{v} {\binom{(\mu*N)(i)}{i}}{\lambda_i} (-1)^{\lambda_i} t^{i \cdot \lambda_i}$$

$$= \prod_{i=1}^{\infty} \sum_{k=0}^{\infty} {\binom{(\mu*N)(i)}{i}}{(-t^i)^k}$$

$$= \prod_{i=1}^{\infty} \exp\left(\frac{(\mu*N)(i)}{i} \cdot \log(1 - t^i)\right)$$

$$= \exp\left(\sum_{i=1}^{\infty} \frac{(\mu*N)(i)}{i} \sum_{k=1}^{\infty} \frac{-t^{ik}}{k}\right)$$

$$= \exp\left(-\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{(\mu*N)(i) \cdot t^{ik}}{ik}\right)$$

$$= \exp\left(-\sum_{i=1}^{\infty} \sum_{d|i} \frac{(\mu*N)(d) \cdot t^i}{i}\right)$$

$$= \exp\left(-\sum_{i=1}^{\infty} N_i \frac{t^i}{i}\right)$$

$$= \frac{1}{Z(X;t)}$$

Using the fact that $Z(\mathbb{P}^n;t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^nt)}$ and $Z(\mathbb{P}^n - \{\mathbf{P}\};t) = \frac{1}{(1-qt)\cdots(1-q^nt)}$ we get the following corollaries.

Corollary 2.8. $\pi_w(\mathbb{P}^n) = 0$ for $w \geq n+2$ and $\pi_w(\mathbb{P}^n - \{\mathbf{P}\}) = 0$ for $w \geq n+1$.

Corollary 2.9.

$$\sum_{w=0}^{\infty} \pi_w(\mathbb{P}^n) = 0.$$

3 Plane curves.

Let a smooth trigonal curve of genus five be given. The dual system of the g_3^1 is a g_5^2 . This gives the following.

Proposition 3.1. There is a bijection between the smooth trigonal curves of genus five and projective plane quintics that have precisely one singularity which is of delta-invariant one.

We want to count

$$\#\mathcal{T}_5(k) = \sum_{C/k} \frac{1}{|\operatorname{Aut}_k(C)|}$$

where the sum is over representatives of k-isomorphism classes of smooth trigonal curves of genus five over k. By the above proposition it is equivalent to sum over representatives of k-isomorphism classes of plane quintics that have precisely one singularity which is of delta-invariant one.

Let C be a smooth trigonal curve of genus five. Because the g_3^1 and the g_5^2 are unique the automorphism group of C as an abstract curve is isomorphic to the automorphism group of C as a plane curve. Plane curve automorphisms are elements of PGL₃. So the automorphism group of C is the stabilizer of C for the action of PGL₃ on \mathbb{P}^2 . We get

$$\sum_{C/k} \frac{1}{|\mathrm{Aut}_k(C)|} = \sum_{C/k} \frac{1}{|\mathrm{Stab}_k(C)|} = \sum_{C/k} \frac{|\mathrm{Orb}_k(C)|}{|\mathrm{PGL}_3(k)|} = \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{C \in T(k)} 1.$$

Here T is the set of plane quintics with exactly one singularity which has delta-invariant one. This enables us to simply count plane curves rather than plane curves up to k-isomorphism.

Definition 3.2. Let C be a plane quintic with polynomial equation $\sum_{i=0}^{5} F_i(x,y)z^{5-i}$ and let L,L' be lines. We say C has tangents L,L' at (0:0:1) if C has a singularity of multiplicity 2 at (0:0:1) and $L \cdot L' = F_2(x,y)$. We say C has tangents L,L' at a point P if there exists a linear transformation ϕ such that $\phi(P) = (0:0:1)$ and $\phi(C)$ has tangents $\phi(L), \phi(L')$ at (0:0:1).

The curves in T(k) can be separated into those which have an ordinary cusp and those which have an ordinary node. The curves that have an ordinary node can then be further separated into those that have both tangents defined over k, in which case it is a split node and those for which the tangents are a conjugate 2-tuple, a non-split node. We will count these three types of curves separately.

Let C be a curve in T(k) and let P be its singular point. Since P is the only singular point of C it is defined over k. If P is a split node then we can apply a k-linear coordinate change such that P gets mapped to (0:0:1) and the tangents at P get mapped to the lines x and y. We can count the number of curves in T(k) that have a split node by first counting the curves that have a singularity at (0:0:1) with tangents x, y and then dividing by a suitable subset of $PGL_3(k)$. We develop some notation for this.

Notation 3.3. We write **P** for the fixed point (0:0:1).

Notation 3.4. We write **C** for the space of plane k-quintics, which is a $\mathbb{P}^{20}(k)$.

Definition 3.5. We define C_{split} to be the subset of C consisting of curves that have a singularity at P of multiplicity 2 with tangents x and y.

Definition 3.6. Let \mathbf{C}' be a subset of \mathbf{C} and let λ be a partition. Here we allow for λ to be an infinite partition $[1^{\lambda_1}, \ldots, i^{\lambda_i}, \ldots]$. We define $\mathbf{C}'(\lambda)$ to be the subset of \mathbf{C}' consisting of those curves whose singularities form precisely a λ -tuple of points. We also define $\mathbf{C}'(\mathbf{P}, \lambda)$ to be the subset of \mathbf{C}' consisting of those curves whose singularities besides \mathbf{P} form precisely a λ -tuple of points.

Remark 3.7. Note that $\mathbf{C}(\lambda)$ as defined above is different from the set of λ -tuples of points $\mathbb{P}^{20}(\lambda)$, we trust that this difference will be clear from the context.

With this notation $\mathbf{C}_{\mathrm{split}}(\mathbf{P},[])$ consists of the curves that have no singularities besides \mathbf{P} . Let $\mathrm{Stab}_k(\mathbf{P},\{x,y\})$ be the subgroup of $\mathrm{PGL}_3(k)$ that fixes \mathbf{P} and fixes the set $\{x,y\}$. It is the group that fixes $\mathbf{C}_{\mathrm{split}}(\mathbf{P},[])$ so we get

$$\frac{|\{C \in T(k) \mid C \text{ has a split node}\}|}{|\operatorname{PGL}_3(k)|} = \frac{|\mathbf{C}_{\operatorname{split}}(\mathbf{P},[])|}{|\operatorname{Stab}_k(\mathbf{P},\{x,y\})|}.$$

We do the corresponding thing for non-split nodes and cusps.

Definition 3.8. Let α be a fixed element of k_2 that is not in k. The set $\mathbf{C}_{\text{non-split}} \subset \mathbf{C}$ is given by those curves that have a singularity at \mathbf{P} of multiplicity 2 with tangents $x + \alpha y$ and $x + \mathcal{F}(\alpha)y$.

Definition 3.9. The set $\mathbf{C}_{\text{cusp}} \subset \mathbf{C}$ is given by those curves that have a singularity at \mathbf{P} of multiplicity 2 with double tangent y and nonzero coefficient for x^3z^2 .

We have

$$\frac{|T(k)|}{|\operatorname{PGL}_3(k)|} = \frac{|\mathbf{C}_{\operatorname{split}}(\mathbf{P},[])|}{|\operatorname{Stab}_k(\mathbf{P},\{x,y\})|} + \frac{|\mathbf{C}_{\operatorname{non-split}}(\mathbf{P},[])|}{|\operatorname{Stab}_k(\mathbf{P},\{x+\alpha y,x+\mathcal{F}(\alpha)y\})|} + \frac{|\mathbf{C}_{\operatorname{cusp}}(\mathbf{P},[])|}{|\operatorname{Stab}_k(\mathbf{P},\{y\})|}.$$

We compute the sizes of the three stabilizer groups. The matrices that fix P, x and y have the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{pmatrix}$$

where $ab \neq 0$. There are $q^2(q-1)^2$ such matrices. We can also permute x and y which adds a factor two so $|\operatorname{Stab}_k(\mathbf{P}, \{x,y\})| = 2q^2(q-1)^2$.

If $\operatorname{char}(k) \neq 2$ then the field k has a quadratic nonresidue r and we can take α such that $\alpha^2 = r$. The matrices that fix P, $x + \alpha y$ and $x + \mathcal{F}(\alpha)y$ have the form

$$\begin{pmatrix} a & rb & 0 \\ b & a & 0 \\ c & d & 1 \end{pmatrix} \text{ if } \operatorname{char}(k) \neq 2, \quad \begin{pmatrix} a & \frac{\alpha \cdot \mathcal{F}(\alpha) \cdot (a+b)}{\alpha + \mathcal{F}(\alpha)} & 0 \\ \frac{a+b}{\alpha + \mathcal{F}(\alpha)} & b & 0 \\ c & d & 1 \end{pmatrix} \text{ if } \operatorname{char}(k) = 2.$$

In either case we get $q^2 \cdot (q^2 - 1)$ matrices so with the tangent permutation we have $|\operatorname{Stab}_k(\mathbf{P}, \{x + \alpha y, x + \mathcal{F}(\alpha)y\})| = 2(q^4 - q^2)$.

The matrices that fix P and y have the form

$$\begin{pmatrix}
a & b & 0 \\
0 & c & 0 \\
d & e & 1
\end{pmatrix}$$

where $ac \neq 0$. Since we have no permutation of tangents here we get $|\operatorname{Stab}_k(\mathbf{P}, \{y\})| = q^3(q-1)^2$.

4 The partial sieve method.

We want to compute $|\mathbf{C}_{\mathrm{split}}(\mathbf{P},[])|$, $|\mathbf{C}_{\mathrm{non-split}}(\mathbf{P},[])|$ and $|\mathbf{C}_{\mathrm{cusp}}(\mathbf{P},[])|$ using the sieve principle. In this section we focus on $|\mathbf{C}_{\mathrm{split}}(\mathbf{P},[])|$, the other two cases are similar. We take $|\mathbf{C}_{\mathrm{split}}|$ and then sieve by adding and subtracting various loci of curves with singular points besides \mathbf{P} .

Definition 4.1. Let \mathbf{C}' be a subset of \mathbf{C} and let $S \subset \mathbb{P}^2$ be a set of points. We define $\mathbf{C}'(S)$ to be the subset of \mathbf{C}' consisting of those curves that have a singularity at each point in S.

The locus of singular curves in $\mathbf{C}_{\mathrm{split}}$ is the union of all $\mathbf{C}_{\mathrm{split}}(S)$ where $S \subset \mathbb{P}^2 - \{\mathbf{P}\}$ is a conjugate n-tuple of points for some $n \geq 1$. We can use the sieve principle to compute the elements of this union. That is, we sum $(-1)^{m+1}|\mathbf{C}_{\mathrm{split}}(S_1) \cap \ldots \cap \mathbf{C}_{\mathrm{split}}(S_m)|$ over all $m \geq 1$ and all unordered choices of distinct $S_1, \ldots, S_m \subset \mathbb{P}^2 - \{\mathbf{P}\}$ where every S_i is a conjugate n_i -tuple of points.

Since $\mathbf{C}_{\mathrm{split}}(S_1) \cap \mathbf{C}_{\mathrm{split}}(S_2) = \mathbf{C}_{\mathrm{split}}(S_1 \cup S_2)$ we find that when we subtract this sum from $|\mathbf{C}_{\mathrm{split}}|$ we get

$$\sum_{\lambda} \left((-1)^{\sum_{i} \lambda_{i}} \cdot \sum_{S \in (\mathbb{P}^{2} - \{\mathbf{P}\})(\lambda)} |\mathbf{C}_{\text{split}}(S)| \right).$$

This sum won't terminate because there are curves that have infinitely many singularities. Another problem with computing this sum is that $|\mathbf{C}_{\text{split}}(S)|$ becomes increasingly hard to compute as the number of points in S grows. So instead we apply a partial sieve principle where we pick a number N and only sieve for all λ with $|\lambda| \leq N$.

Definition 4.2. We define

$$\mathcal{S}_{\mathrm{split},N} := \sum_{|\lambda| \le N} \left((-1)^{\sum_i \lambda_i} \cdot \sum_{S \in (\mathbb{P}^2 - \{\mathbf{P}\})(\lambda)} |\mathbf{C}_{\mathrm{split}}(S)| \right).$$

We also have to account for the curves that have more than N singularities besides \mathbf{P} . This is done by explicitly counting the number of curves that have precisely a λ -tuple of singlarities besides \mathbf{P} for all λ such that $|\lambda| > N$. For each curve we have to subtract it as often as it has been counted in the computation of $\mathcal{S}_{\text{split}}$.

Definition 4.3. We define

$$\sigma_N(\lambda) := -\sum_{\substack{\mu \subset \lambda \\ |\mu| \le N}} (-1)^{\sum_i \mu_i} \prod_{i=1}^v \binom{\lambda_i}{\mu_i},$$

where $\mu \subset \lambda$ means that $\mu_i \leq \lambda_i$ for all i.

We have

$$|\mathbf{C}_{\mathrm{split}}(\mathbf{P},[])| = \mathcal{S}_{\mathrm{split},N} + \sum_{|\lambda| > N} |\mathbf{C}_{\mathrm{split}}(\mathbf{P},\lambda)| \cdot \sigma_N(\lambda).$$

Definition 4.4. Let C be a projective plane curve, we define $\#\delta_k^1(C)$ to be the number of (isolated) singularities over k on C of delta-invariant one.

Let $[\lambda, 1^1]$ be the partition μ where $\mu_1 = \lambda_1 + 1$ and $\mu_i = \lambda_i$ for i > 1. Since $\operatorname{Stab}_k(\mathbf{P}, \{x, y\})$ is the group that fixes $\mathbf{C}_{\text{split}}(\mathbf{P}, \lambda)$ we have

$$\frac{|\mathbf{C}_{\mathrm{split}}(\mathbf{P},\lambda)|}{|\mathrm{Stab}_k(\mathbf{P},\{x,y\})|} = \frac{1}{|\mathrm{PGL}_3(k)|} \cdot \sum_{C \in \mathbf{C}([\lambda,1^1])} \#\{\text{ordinary split nodes over } k \text{ on } C\}.$$

This means we get

$$\begin{split} \frac{|T(k)|}{|\operatorname{PGL}_3(k)|} = & \frac{\mathcal{S}_{\operatorname{split},N}}{|\operatorname{Stab}_k(\mathbf{P},\{x,y\})|} + \frac{\mathcal{S}_{\operatorname{non-split},N}}{|\operatorname{Stab}_k(\mathbf{P},\{x+\alpha y,x+\mathcal{F}(\alpha)y\})|} + \frac{\mathcal{S}_{\operatorname{cusp},N}}{|\operatorname{Stab}_k(\mathbf{P},\{y\})|} \\ & + \frac{1}{|\operatorname{PGL}_3(k)|} \sum_{|\lambda| > N} \sum_{C \in \mathbf{C}([\lambda,1^1])} \# \delta_k^1(C) \cdot \sigma_N(\lambda). \end{split}$$

The choice of N is made by considering the difficulty of computing $S_{\text{split},N}$ versus the above sum over $|\lambda| > N$. We choose N to be five.

5 The sieving part.

In this section we will compute $S_{\text{split},5}$, $S_{\text{non-split},5}$ and $S_{\text{cusp},5}$.

5.1 A split node.

To know $S_{\text{split},5}$ we have to compute $|\mathbf{C}_{\text{split}}(S)|$ for all S such that $|S| \leq 5$. The space $\mathbf{C}_{\text{split}}(S)$ is an \mathbb{A}^{15} with $3 \cdot |S|$ linear conditions on it. For different choices of S we will need to prove the independence of a subset of these conditions. To do this look at the matrix that has the monomial coefficients of degree five as columns and the linear conditions as rows. To make this explicit we need to decide what the three conditions are for each point. We have the condition that the point is on the curve and we pick two from the three conditions that we get by demanding the derivatives with respect to x, y, z to be zero at the point. When we have constructed this matrix we take a subset of the columns such that we get a square matrix. If the determinant is nonzero then the conditions are independent.

We also want the conditions to be independent from the condition that there must be a split node at **P** with tangents x, y. This means that in our choice of a subset of columns we must avoid the columns corresponding to the coefficients of $x^2z^3, xyz^3, y^2z^3, xz^4, yz^4, z^5$.

Because S is a λ -tuple of points for some partition λ , the Frobenius morphism maps S to itself and all linear conditions can be expressed using coefficients over k. The k-dimension of $\mathbf{C}_{\mathrm{split}}(S)$ as a vector space over k is therefore the same as the \bar{k} -dimension of $\mathbf{C}_{\mathrm{split}}(S)$ as a vector space over \bar{k} . So the dimension of $\mathbf{C}_{\mathrm{split}}(S)$ over k does not change when we apply a \bar{k} -change of coordinates, that fixes \mathbf{P} , to the points in S.

Note that after applying such a transformation the conditions that $x^2z^3 = y^2z^3 = 0$ and $xyz^3 \neq 0$ might change, but they will still be conditions on the coefficients of x^2z^3 , xyz^3 , y^2z^3 .

Example 5.1. Let |S|=5 so that together with **P** we have six points such that there are no four points on a line, no six on an irreducible conic and no three on a line through **P**. We apply a \bar{k} -linear transformation that fixes **P** and maps the points in S to $(1:0:0), (0:1:0), (1:1:1), (1:\alpha:\beta), (1:\gamma:\delta)$ for some $\alpha, \beta, \gamma, \delta$ where $\alpha, \gamma \neq 0, 1$ and $\alpha \neq \gamma$. We also have $\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta \neq 0$ for otherwise there would be six points on a conic.

We construct a matrix as described above where we take derivatives w.r.t. x and z. This gives us a 15×21 matrix. If we leave out the columns corresponding to x^2z^3 , xyz^3 , y^2z^3 , xz^4 , yz^4 , z^5 we get a square matrix with determinant

$$(\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta)^4(\alpha - \gamma)(\alpha - 1)\alpha^2(\gamma - 1)\gamma^2.$$

So the conditions are independent and $\mathbf{C}_{\mathrm{split}}(S) = \mathbb{A}^0$.

The above argument also ensures independence for the cases where we have $|S| \leq 4$ such that together with **P** there are no four points on a line and no three on a line through **P**.

Example 5.2. Let us have two lines and let |S| = 5 be such that the intersection point of the lines is in S, one of the lines has three more points from S on it, and the other line has \mathbf{P} and the last point in S on it.

By Bézout any quintic that has singularities at the points in S has the two lines as a component. If the line through \mathbf{P} is not either x or y then we get a contradiction with the tangent conditions and $\mathbf{C}_{\mathrm{split}}(S) = \emptyset$. Without loss of generality we take the line to be y. Denote the intersection point of the two lines by Q. Now $\mathbf{C}_{\mathrm{split}}(S)$ is isomorphic to the space of cubics that pass through the points in $S - \{Q\}$ and pass through \mathbf{P} with tangent x.

We use a k-linear transformation that keeps **P** fixed and maps the points in S to $(1:0:0), (0:1:0), (1:1:0), (1:\alpha:0), (1:0:1)$ where $\alpha \neq 0, 1$. Taking the columns for x^3, x^2y, xy^2, y^3 gives determinant $-\alpha(\alpha-1)$. So the conditions are independent and $\mathbf{C}_{\text{split}}(S) = \mathbb{A}^3$.

Example 5.3. Let S have at least four points on a line that passes through \mathbf{P} . By Bézout the line is a double component of any quintic in $\mathbf{C}_{\mathrm{split}}(S)$. This contradicts the fact that \mathbf{P} is an ordinary node so $\mathbf{C}_{\mathrm{split}}(S) = \emptyset$.

We list the possible cases for S where after every case we describe $\mathbf{C}_{\text{split}}(S)$. The cases where $\mathbf{C}_{\text{split}}(S) = \emptyset$ have been left out. Whenever we have three points (including \mathbf{P}) on a line through \mathbf{P} , we leave implicit that the line is either x or y (see Example 5.2).

- No four points on a line, no six on an irreducible conic and no three on a line through **P**: $\mathbb{A}^{15-3\cdot|S|}$
- **P** and two more points on a line and $0 \le n \le 3$ points outside the line, no four points on a line and no six on a conic: \mathbb{A}^{10-3n}
- P and two more points on a line and the three other points on another line, such that none of the points is on the intersection of the two lines:
- **P** and two points on one line and **P** and two points on another line, and $0 \le n \le 1$ points outside these lines: \mathbb{A}^{5-3n}
- **P** and three more points on a line and $0 \le n \le 2$ points outside the line such that there are no three on another line through **P**: \mathbb{A}^{9-3n}
- ${\bf P}$ and three points on one line and ${\bf P}$ and two points on another line: ${\mathbb A}^4$
- Four points on a line and **P** and $0 \le n \le 1$ points outside the line such that there are no three on a line through **P**: \mathbb{A}^{5-3n}

- Four points on a line and \mathbf{P} and another point outside the line such that there are three on a line through \mathbf{P} : \mathbb{A}^3
- Five points on a line and **P** outside the line: \mathbb{A}^4
- Six points on an irreducible conic: A³

Now we have to compute the number of possible λ -tuples of points belonging to each case.

Example 5.4. For each λ of weight five we look at how many choices of λ -tuples of points there are such that there are 5 points on a line and the point **P** outside the line.

Pick a line that does not pass through \mathbf{P} , there are q^2 such lines. We then take a λ -tuple of points on the chosen line. Because of Corollary 2.8 the contribution to $\mathcal{S}_{\text{split}}$ is given by

$$\sum_{|\lambda|=5} (-1)^{\sum_i \lambda_i} \cdot q^2 \cdot \sum_{S \in \mathbb{P}^1(\lambda)} |\mathbf{C}_{\mathrm{split}}(S)| = q^2 \cdot |\mathbb{A}^4(k)| \cdot \sum_{|\lambda|=5} (-1)^{\sum_i \lambda_i} \cdot |\mathbb{P}^1(\lambda)|$$
$$= q^2 \cdot |\mathbb{A}^4(k)| \cdot \pi_5(\mathbb{P}^1)$$
$$= 0$$

In the above list all cases besides the first one have four points on a line, six on an irreducible conic or three on a line through **P**. Using Corollary 2.8 it turns out that all these cases contribute zero. This means we only need to know for every partition λ with $|\lambda| \leq 5$, the number of λ -tuples such that there are no four points on a line, six on an irreducible conic or three on a line through **P**. This is given by $|\mathbb{P}^2(\lambda)|$ minus the number of choices of λ -tuples of points corresponding to all other cases. However these numbers all become zero when we sum them with a factor $(-1)^{\sum_i \lambda_i}$. So we can just pretend we don't have to subtract anything at all.

$$\begin{split} \frac{\mathcal{S}_{\text{split},5}}{|\text{Stab}_{k}(\mathbf{P},\{x,y\})|} &= \frac{1}{2q^{2}(q-1)^{2}} \sum_{|\lambda| \leq 5} (-1)^{\sum_{i} \lambda_{i}} \cdot \sum_{S \in (\mathbb{P}^{2} - \{\mathbf{P}\})(\lambda)} |\mathbb{A}^{15 - 3 \cdot |S|}| \\ &= \frac{1}{2q^{2}(q-1)^{2}} \sum_{i=1}^{5} \pi_{i} (\mathbb{P}^{2} - \{\mathbf{P}\}) \cdot q^{15 - 3i} \\ &= \frac{q^{15} - q^{14} - q^{13} + q^{12}}{2q^{2}(q-1)^{2}} \\ &= \frac{1}{2} (q^{11} + q^{10}) \end{split}$$

5.2 A non-split node.

If S is a set of points such that **P** and two points in S are on a line then by Bézout that line is part of any curve in $C_{\text{non-split}}(S)$, so it is one of

the tangent lines. Because the sets of points we consider are closed under Frobenius there have to be just as many points on the other tangent line. So for non-split nodes we only have the following non-empty cases:

- No four points on a line, no six on an irreducible conic and no three on a line through \mathbf{P} : $\mathbb{A}^{15-3\cdot|S|}$
- **P** and two points on one line and **P** and two points on another line, and $0 \le n \le 1$ points outside these lines: \mathbb{A}^{5-3n}
- Four points on a line and **P** and $0 \le n \le 1$ points outside the line such that there are no three on a line through **P**: \mathbb{A}^{5-3n}
- Four points on a line and \mathbf{P} and another point outside the line such that there are three on a line through \mathbf{P} : \mathbb{A}^3
- Five points on a line and **P** outside the line: \mathbb{A}^4
- Six points on an irreducible conic: A³

Because we used \bar{k} -linear transformations for the computations of $\mathbf{C}_{\text{split}}(S)$ their results also apply to $\mathbf{C}_{\text{non-split}}(S)$ for these cases.

Choosing four points (closed under Frobenius) such that we have \mathbf{P} and two points on one tangent line and \mathbf{P} and two points on the other line is the same as choosing two non- \mathbf{P} points on one of the tangent lines and then taking their conjugates. So if we work over k_2 then we can use the fact that $\pi_2(\mathbb{P}^1 - \{\mathbf{P}\}) = 0$. Choosing five non- \mathbf{P} points on a k_2 -conic can also be done over k_2 . These two cases become zero and the other cases are exactly the same as they are for split nodes. This means we get the result

$$\frac{\mathcal{S}_{\text{non-split},5}}{|\text{Stab}_k(\mathbf{P}, \{x + \alpha y, x + \mathcal{F}(\alpha)y\})|} = \frac{q^{15} - q^{14} - q^{13} + q^{12}}{2(q^4 - q^2)} = \frac{1}{2}(q^{11} - q^{10}).$$

5.3 A cusp.

For $\mathbf{C}_{\text{cusp}}(S)$ we can not as easily reuse the previous computations since we have the added condition that the coefficient for x^3z^2 is nonzero. This means that we need to keep the line y fixed when we do a linear transformation. If S is a set of points such that \mathbf{P} and two points in S are on a line then by Bézout that line is part of any curve in $\mathbf{C}_{\text{cusp}}(S)$. For any such curve the cusp at \mathbf{P} is not ordinary and so $\mathbf{C}_{\text{cusp}}(S)$ is empty. Similarly $\mathbf{C}_{\text{cusp}}(S)$ is empty for any set of points S such that there are five points on a conic that has tangent y at \mathbf{P} . We have the following non-empty cases:

• No four points on a line, no six on an irreducible conic, no five on a conic that has tangent y at \mathbf{P} , and no three on a line through \mathbf{P} : $\mathbb{A}^{15-3\cdot|S|} = \mathbb{A}^{14-3\cdot|S|}$

- Four points on a line and **P** and $0 \le n \le 1$ points outside the line such that there are no three on a line through **P**: $\mathbb{A}^{5-3n} \mathbb{A}^{4-3n}$
- Five points on a line and **P** outside the line: $\mathbb{A}^4 \mathbb{A}^3$

Example 5.5. Let S have no four points on a line, no six on an irreducible conic, no five on a conic that has tangent y at P, and no three on a line through P.

If |S|=4 then we can use a \bar{k} -linear transformation that fixes **P** and y and maps the points in S to $(0:1:0), (1:1:1), (1:\alpha:\beta), (1:\gamma:\delta)$ where $\alpha, \gamma \neq 1, \alpha \neq \gamma$ and $\alpha\beta\gamma - \alpha\gamma\delta - \alpha\beta + \gamma\delta + \alpha - \gamma \neq 0$. Taking derivatives w.r.t. y and z and taking the columns for $y^3z^2, xy^3z, x^2y^2z, x^3yz, x^4z, x^2y^3, x^3y^2, x^4y, x^5, xy^4, y^4z, y^5$ gives determinant

$$-(\alpha\beta\gamma - \alpha\gamma\delta - \alpha\beta + \gamma\delta + \alpha - \gamma)^3(\alpha - \gamma)^2(\alpha - 1)^2(\gamma - 1)^2.$$

If |S| = 5 then we want to use the same arguments as in Example 5.1. But there we don't consider the coefficient for x^3z^2 so we might be counting some non-ordinary cusps. We show that there are actually no non-ordinary cusps on any curves that have a double tangent y at \mathbf{P} and singularities at all the points in S. If \mathbf{P} is a non-ordinary cusp then it lies on an irreducible quartic or quintic component or on the intersection of two irreducible components. If it lies on an irreducible quartic then having six singularities in total implies that there will be four on a line. If there are six singularities on an irreducible quintic then they all need to have delta-invariant 1. Finally if it lies on the intersection of two irreducible components then one of those components is a line or a conic, and we will have \mathbf{P} and two more points on a line or five points on a conic that has tangent y at \mathbf{P} .

As with nodes all cases besides the first one listed become zero and we have

$$\frac{\mathcal{S}_{\text{cusp,5}}}{|\text{Stab}_k(\mathbf{P}, \{y\})|} = \frac{1}{q^3(q-1)^2} \sum_{i=1}^5 \pi_i(\mathbb{P}^2 - \{\mathbf{P}\}) \cdot (q^{15-3i} - q^{14-3i})$$

$$= \frac{q^{15} - 2q^{14} + 2q^{12} - q^{11}}{q^3(q-1)^2}$$

$$= q^{10} - q^8.$$

6 The explicit part.

In this section we will compute

$$-\frac{1}{|\operatorname{PGL}_{3}(k)|} \sum_{|\lambda| > 5} \sum_{C \in \mathbf{C}([\lambda, 1^{1}])} \# \delta_{k}^{1}(C) \cdot \sigma_{5}(\lambda). \tag{1}$$

Definition 6.1. We define the type of a projective plane quintic over k to be given by the following information:

- the degree and multiplicity of the irreducible components
- the fields over which the irreducible components are defined
- the number of points in which every pair of irreducible components intersects
- the number of singularities of each irreducible component
- the delta-invariants of singularities of each irreducible components
- the fields over which the singularities of each irreducible component are defined

Remark 6.2. This definition is naturally rather analogous to [3, Definition 7.2].

By knowing the type of a curve we know over what fields its singularities are defined and the delta-invariants of its singularities. If we can count how many curves there are of each type then that gives us enough information to compute (1). It is sufficient to compute the number of curves for those types where the singularities form a $[\lambda, 1^1]$ -tuple such that $|\lambda| > 5$, $\sigma_5(\lambda) \neq 0$ and there is at least one (isolated) singularity of delta-invariant one.

We first separate the types by the degree and multiplicity of their irreducible components. Section 6.1 considers types where there is at least one component of higher multiplicity. In Sections 6.2 to 6.7 all components will have multiplicity one.

The results will be listed in tables where rows correspond to the various types. The column marked $\#\delta_k^1$ contains the number of (isolated) singularities of delta-invariant one. The column marked $\#\{C\}$ contains the number of curves of each type divided by $|\operatorname{PGL}_3(k)|$. Note that our tables will not contain the types for which there are no singularities of delta-invariant one or for which $\sigma_5(\lambda) = 0$.

Remark 6.3. It is not necessary for our purpose but in general it might be interesting to separately count split nodes, non-split nodes and cusps. The main reason we do not do this is that for the types of curves we have to count it is hard to determine whether a singularity on an irreducible component is a split node, a non-split node or a cusp.

6.1 A component of higher multiplicity.

Let λ be an infinite partition such that there is a curve $C \in \mathbf{C}([\lambda, 1^1])$ with $\#\delta_k^1(C) \neq 0$. We need an isolated singularity of delta-invariant one so the

double component C can only be one double (or triple) line. Let η be the partition such that the singularities of C that are not on the double line form a η -tuple of points. Let ρ be the infinite partition such that ρ_i is the number of conjugate i-tuples in \mathbb{P}^1 .

$$\sigma_{5}(\lambda) = -\sum_{\substack{\mu \subset \lambda \\ |\mu| \leq 5}} (-1)^{\sum_{i} \mu_{i}} \prod_{i=1}^{v} {\lambda_{i} \choose \mu_{i}}$$

$$= -\sum_{\substack{\nu \subset \eta \\ |\nu| \leq 5}} \sum_{\substack{\mu \subset \rho \\ |\mu| \leq 5 - |\nu|}} (-1)^{\sum_{i} \mu_{i} + \nu_{i}} \prod_{i=1}^{v} {\rho_{i} \choose \mu_{i}} {\eta_{i} \choose \nu_{i}}$$

$$= -\sum_{\substack{\nu \subset \eta \\ |\nu| < 5}} (-1)^{\sum_{i} \nu_{i}} \prod_{i=1}^{v} {\eta_{i} \choose \nu_{i}} \sum_{|\mu| \leq 5 - |\nu|} (-1)^{\sum_{i} \mu_{i}} |\mathbb{P}^{1}(\mu)|$$

Since $|\nu| \leq |\eta| \leq 3$ we can use Corollaries 2.8 and 2.9 to get

$$\sum_{|\mu| \le 5 - |\nu|} (-1)^{\sum_i \mu_i} |\mathbb{P}^1(\mu)| = \sum_{i=0}^{5 - |\nu|} \pi_i(\mathbb{P}^1) = \sum_{i=0}^{\infty} \pi_i(\mathbb{P}^1) = 0.$$

So the types of curves that have a multiple component contribute zero.

6.2 Five lines.

The five lines can be in general position and intersect in ten points or there can be three lines through one point so that we get eight points. Any other type of curve consisting of five distinct lines has fever than seven intersection points.

6.2.1 General position.

Example 6.4. We count the number of curves of the type where the lines are all defined over k. First we take two k-points and then through each of these points a pair of k-lines such that none of the lines passes through both points. This can be done in $\binom{q^2+q+1}{2}\binom{q}{2}^2$ ways. The four lines have six intersection points. Take as the fifth line any line over k that does not go through any of these points. There are (q-2)(q-3) such lines. We can end up with the same five lines if we start with another choice of two of the intersection points such that none of the five lines passes through both points. There are 15 such choices so we get

$$\frac{1}{|PGL_3(k)|} \cdot \frac{1}{15} {q^2 + q + 1 \choose 2} {q \choose 2}^2 (q - 2)(q - 3) = \frac{1}{120} (q - 2)(q - 3).$$

Whenever λ is a finite partition we can use

$$\sigma_{N}(\lambda) = -\sum_{\substack{\mu \subset \lambda \\ |\mu| \leq N}} (-1)^{\sum_{i} \mu_{i}} \prod_{i=1}^{v} {\lambda_{i} \choose \mu_{i}}$$

$$= \sum_{\substack{\mu \subset \lambda \\ |\mu| > N}} (-1)^{\sum_{i} \mu_{i}} \prod_{i=1}^{v} {\lambda_{i} \choose \mu_{i}}$$

$$= \sum_{\substack{\mu \subset \lambda \\ |\mu| < |\lambda| - N}} (-1)^{\sum_{i} (\lambda_{i} - \mu_{i})} \prod_{i=1}^{v} {\lambda_{i} \choose \lambda_{i} - \mu_{i}}$$

$$= (-1)^{\sum_{i} \lambda_{i}} \sum_{\substack{\mu \subset \lambda \\ |\mu| < |\lambda| - N}} (-1)^{\sum_{i} \mu_{i}} \prod_{i=1}^{v} {\lambda_{i} \choose \mu_{i}}.$$

So here we get

$$\sigma_5([1^9]) = (-1) \cdot (1 - {9 \choose 1} + {9 \choose 2} - {9 \choose 3}) = 56.$$

lines	$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$
$[1^{5}]$	$[1^{10}]$	56	10	$\frac{1}{120}(q-2)(q-3)$
$[1^3, 2^1]$	$[1^4, 2^3]$	6	4	$\frac{1}{12}q(q-1)$
$[1^2, 3^1]$	$[1^1, 3^3]$	2	1	$\frac{1}{6}(q+1)q$

6.2.2 Three of the lines intersect in one point P.

lines	$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta^1_k$	$\#\{C\}$
$[1^5]$	[18]	6	7	$\frac{1}{12}(q-2)$
$[1^3, 2^1]$, both k_2 -lines through P	$[1^4, 2^2]$	2	3	$\frac{1}{4}q$

6.3 One conic and three lines.

One or two of the lines can be tangent to the conic, two lines can intersect on the conic, or three lines can go through one point. We only need to consider these as separate cases because combining them will result in fever than seven singularities. We say the components are in general position when none of the above cases applies.

6.3.1 Two of the lines are tangent to the conic.

Example 6.5. We count the number of curves of the type where all components and singularities are defined over k. The plane conics form a \mathbb{P}^5 and every reducible k-conic is either a pair of different k-lines, a conjugate pair of lines, or a double k-line. So the number of irreducible plane conics over k is given by

$$\frac{q^6 - 1}{q - 1} - \binom{q^2 + q + 1}{2} - \frac{1}{2}(q^4 - q) - (q^2 + q + 1) = (q^2 + q + 1)q^2(q - 1).$$

Pick two k-points A, B on the conic and take the tangent lines at these points, they meet in a point P. Choose two more k-points on the conic and take the line through them. We need to subtract the case where P is on all three lines. If $\operatorname{char}(k) \neq 2$ then the lines through A and B are the only lines through P that are tangent to the conic. There are q-1 other k-points on the conic that each determine a line through P. Each line gets determined twice this way so we get

$$\frac{1}{|\mathrm{PGL}_3(k)|} \cdot (q^2 + q + 1)q^2(q - 1)\binom{q + 1}{2} \left(\binom{q - 1}{2} - \frac{q - 1}{2}\right) = \frac{1}{4}(q - 3).$$

If char(k) = 2 then all lines through P are tangent to the conic so we get

$$\frac{1}{|\mathrm{PGL}_3(k)|} \cdot (q^2 + q + 1)q^2(q - 1) \binom{q + 1}{2} \binom{q - 1}{2} = \frac{1}{4}(q - 2).$$

lines	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $char(k) = 2$
$[1^3]$	$[1^7]$	1	5	$\frac{1}{4}(q-3)$	$\frac{1}{4}(q-2)$
	$[1^5, 2^1]$	-1	3	$\frac{1}{4}(q-1)$	$\frac{1}{4}q$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	3	$\frac{1}{4}(q-1)$	$\frac{1}{4}q$
	$[1^1, 2^3]$	-1	1	$\frac{1}{4}(q-3)$	$\frac{1}{4}(q-2)$

6.3.2 One of the lines is tangent to the conic.

lines	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta^1_k$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $char(k) = 2$
$[1^3]$	[18]	6	7	$\frac{1}{8}(q-3)^2$	$\frac{1}{8}(q-2)(q-4)$
	$[1^6, 2^1]$	-4	5	$\frac{1}{4}(q-1)^2$	$\frac{1}{4}q(q-2)$
	$[1^4, 2^2]$	2	3	$\frac{1}{8}(q-1)^2$	$\frac{1}{8}q(q-2)$

6.3.3 Two of the lines intersect on the conic.

lines	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$
$[1^3]$	$[1^7]$	1	6	$\frac{1}{4}(q-2)(q-3)$
	$[1^5, 2^1]$	-1	4	$\frac{1}{4}q(q-1)$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	2	$\frac{1}{4}q(q-1)$

6.3.4 Three lines through one point.

Example 6.6. We count the number of curves of the type where all components and singularities are defined over k. We first consider the case where $\operatorname{char}(k) \neq 2$. Choose an irreducible conic and two k-points on the conic. Take the tangent lines to the conic at the two points and denote their intersection by P. There are $\frac{q-1}{2}$ lines through P that intersect the conic in two k-points. We can also choose an irreducible conic and a conjugate pair of points on the conic. Let P be the intersection of the corresponding tangent lines. There are now $\frac{q+1}{2}$ lines through P that intersect the conic in two k-points. Putting both cases together we get

$$\frac{1}{|\operatorname{PGL}_3(k)|} \cdot (q^2 + q + 1)q^2(q - 1) \left(\binom{q+1}{2} \binom{\frac{q-1}{2}}{3} + \frac{q^2 - q}{2} \binom{\frac{q+1}{2}}{3} \right) = \frac{1}{48} (q - 3)^2.$$

If $\operatorname{char}(k) = 2$ then there is one k-point outside the conic that is the intersection of all tangent lines to the conic. Through any other k-point outside the conic there are $\frac{q}{2}$ lines that intersect the conic in two k-points. So we have

$$\frac{1}{|PGL_3(k)|} \cdot (q^2 + q + 1)q^2(q - 1)(q^2 - 1)\binom{\frac{q}{2}}{3} = \frac{1}{48}(q - 2)(q - 4).$$

lines	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $char(k) = 2$
$[1^3]$	$[1^7]$	1	6	$\frac{1}{48}(q-3)^2$	$\frac{1}{48}(q-2)(q-4)$
	$[1^5, 2^1]$	-1	4	$\frac{1}{16}(q-1)^2$	$\frac{1}{16}q(q-2)$
	$[1^3, 2^2]$	1	2	$\frac{1}{16}(q-1)^2$	$\frac{1}{16}q(q-2)$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	2	$\frac{1}{8}(q^2 - 2q - 1)$	$\frac{1}{8}q(q-2)$
	$[1^3, 4^1]$	-1	2	$\frac{1}{8}(q+1)(q-1)$	$\frac{1}{8}q^2$

6.3.5 General position.

Example 6.7. We count the number of curves of the type where all components and singularities are defined over k and $\operatorname{char}(k) \neq 2$. We choose an irreducible k-conic and six k-points on the conic. There are 15 ways to choose three lines connecting these six points. The lines should not intersect in one point so we subtract the results from Example 6.6 to get

$$\frac{1}{|PGL_3(k)|} \cdot (q^2 + q + 1)q^2(q - 1) \binom{q+1}{6} \cdot 15 - \frac{1}{48}(q - 3)^2 = \frac{1}{48}(q^2 - 7q + 11)(q - 3).$$

lines	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $\operatorname{char}(k) = 2$
$[1^3]$	$[1^{9}]$	21	9	$\frac{1}{48}(q^2 - 7q + 11)(q - 3)$	$\frac{1}{48}(q-2)(q-4)^2$
	$[1^7, 2^1]$	-9	7	$\frac{1}{16}(q^2 - 3q + 1)(q - 1)$	$\frac{1}{16}q(q-2)^2$
	$[1^5, 2^2]$	1	5	$\frac{1}{16}(q^3 - 2q^2 - 1)$	$\frac{1}{16}q^2(q-2)$
	$[1^3, 2^3]$	3	3	$\frac{1}{48}(q^2 - 3q + 1)(q - 1)$	$\frac{1}{48}q(q-2)^2$
$[1^1, 2^1]$	$[1^3, 2^3]$	3	3	$\frac{1}{8}(q^2 - q - 1)(q - 1)$	$\frac{1}{8}q^2(q-2)$
	$[1^1, 2^4]$	-3	1	$\frac{1}{8}(q^2 - q - 3)(q - 3)$	$\frac{1}{8}(q^2 - 2q - 4)(q - 2)$
	$[1^3, 2^1, 4^1]$	-1	3	$\frac{1}{8}(q^2 - q + 1)(q + 1)$	$\frac{1}{8}q^3$
	$[1^1, 2^2, 4^1]$	1	1	$\frac{1}{8}(q^2 - q - 1)(q - 1)$	$\frac{1}{8}q^2(q-2)$

6.4 Two conics and one line.

The conics can intersect in three or in four points. If they intersect in four points then the line can intersect the conics in three or in four points. Any other type will result in fever than seven singularities. We say the components are in general position if the conics intersect in four points and the line intersects the conics in four points.

In the tables the first column will denote the fields over which the intersection points of the conics are defined.

6.4.1 General position and the conics are defined over k.

Example 6.8. We count the number of curves of the type where all singularities are defined over k and $\operatorname{char}(k) \neq 2$. There are $\frac{1}{6}\binom{q^2+q+1}{2}(q^4-2q^3+q^2)$ ways to pick four k-points P_1, P_2, P_3, P_4 such that there are no three on a line. We write \mathcal{P} for the pencil of conics through P_1, P_2, P_3, P_4 . There are six lines through P_1, P_2, P_3, P_4 and these lines intersect in seven k-points. Let Q_1, Q_2, Q_3 denote the other three intersection points.

Let L be any line such that none of P_1, P_2, P_3, P_4 is on L. We have a degree two map $\mathbb{P}^1 \to \mathbb{P}^1$ that sends a point R on L to the conic in \mathcal{P} through R. By Hurwitz's theorem there are two branch points, we call these two conics the *tangent conics* for L.

There are 3 k-lines through two of Q_1, Q_2, Q_3 . Let L be such a line. Both tangent conics for L are reducible and each one intersects L in one of the Q_i . There are q-1 other k-points on L so there are $\frac{q-1}{2}$ conics in $\mathcal{P}(k)$ that intersect L in two k-points. One of these conics is reducible so we have $\frac{q-3}{2}$ irreducible conics.

There are 3(q-3) k-lines through precisely one of Q_1, Q_2, Q_3 and not through any of P_1, P_2, P_3, P_4 . Let L be such a line. One of the tangent lines for L intersects it in the Q_i point on L so the other tangent conic for L is also defined over k. There are $\frac{q-5}{2}$ irreducible conics in $\mathcal{P}(k)$ that intersect L in two k-points. (This is not well defined when q=3 but that is not a problem since this case occurs 3(q-3) times. The same idea applies to the next two cases.)

There are q-2 irreducible conics in $\mathcal{P}(k)$ that each have q-3 k-points besides P_1, P_2, P_3, P_4 . This means we have $\frac{1}{2}((q-2)(q-3)-3(q-3))=\frac{1}{2}(q-5)(q-3)$ k-lines that have two irreducible tangent conics over k. For every such line there are $\frac{q-7}{2}$ irreducible conics in $\mathcal{P}(k)$ that intersect it in two k-points.

There are $(q-3)^2$ k-lines not through any of $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3$. So there are $(q-3)^2 - \frac{1}{2}(q-5)(q-3) = \frac{1}{2}(q-3)(q-1)$ k-lines that have a conjugate pair of irreducible tangent conics. For every such line there are $\frac{q-5}{2}$ irreducible conics in $\mathcal{P}(k)$ that intersect it in two k-points.

Putting everything together gives us

$$\frac{1}{|\operatorname{PGL}_3(k)|} \frac{1}{6} {q^2 + q + 1 \choose 2} (q^4 - 2q^3 + q^2) \left(3 {\frac{q-3}{2} \choose 2} + 3(q-3) {\frac{q-5}{2} \choose 2} \right)
+ \frac{1}{2} (q-3)(q-5) {\frac{q-7}{2} \choose 2} + \frac{1}{2} (q-1)(q-3) {\frac{q-5}{2} \choose 2} \right) = \frac{1}{192} (q^2 - 9q + 17)(q-3)(q-5).$$

Example 6.9. We count the number of curves of the type where all singularities are defined over k for $\operatorname{char}(k) = 2$. Let us choose P_1, P_2, P_3, P_4 and define \mathcal{P} and Q_1, Q_2, Q_3 as in Example 6.8. There is one line through Q_1, Q_2, Q_3 and it intersects every conic in \mathcal{P} in one point. For any other line not through any of P_1, P_2, P_3, P_4 there is precisely one conic in \mathcal{P} that intersects it in one point.

There are 3(q-2) k-lines through one of Q_1, Q_2, Q_3 and not through any of P_1, P_2, P_3, P_4 . For any such line there are $\frac{q-4}{2}$ irreducible conics in $\mathcal{P}(k)$ that intersect it in two k-points.

There are (q-2)(q-4) k-lines not through any of $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3$. For any such line there are $\frac{q-6}{2}$ irreducible conics in $\mathcal{P}(k)$ that intersect it in two k-points.

Putting these two cases together we get

$$\frac{1}{|\operatorname{PGL}_3(k)|} \frac{1}{6} \binom{q^2 + q + 1}{2} (q^4 - 2q^3 + q^2) \left(3(q - 2) \binom{\frac{q - 4}{2}}{2} + (q - 2)(q - 4) \binom{\frac{q - 6}{2}}{2} \right)$$

$$= \frac{1}{192} (q - 2)(q - 4)(q - 5)(q - 6).$$

con pts	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $char(k) = 2$
$[1^4]$	[18]	6	8	$\frac{1}{192}(q^2 - 9q + 17)(q - 3)(q - 5)$	$\frac{1}{192}(q-2)(q-4)(q-5)(q-6)$
	$[1^6, 2^1]$	-4	6	$\frac{1}{96}(q^2 - 5q + 3)(q - 1)(q - 3)$	$\frac{1}{96}q(q-2)(q-3)(q-4)$
	$[1^4, 2^2]$	2	4	$\frac{1}{192}(q^2 - q + 1)(q - 1)(q - 3)$	$\frac{1}{192}q(q-1)(q-2)^2$
$[1^2, 2^1]$	$[1^6, 2^1]$	-4	6	$\frac{1}{32}(q^2 - 3q + 1)(q - 1)(q - 3)$	$\frac{1}{32}q(q-2)^2(q-3)$
	$[1^4, 2^2]$	2	4	$\frac{1}{16}(q^3 - 2q^2 - 1)(q - 1)$	$\frac{1}{16}q^2(q-1)(q-2)$
$[1^1, 3^1]$	$[1^5, 3^1]$	3	5	$\frac{1}{24}(q+1)q(q-1)(q-2)$	$\frac{1}{24}q(q+1)(q-1)(q-2)$
	$[1^3, 2^1, 3^1]$	-1	3	$\frac{1}{12}q^2(q+1)(q-1)$	$\frac{1}{12}q^2(q+1)(q-1)$
	$[1^1, 2^2, 3^1]$	-1	1	$\frac{1}{24}(q+1)q(q-1)(q-2)$	$\frac{1}{24}q(q+1)(q-1)(q-2)$
$[2^2]$	$[1^4, 2^2]$	2	4	$\frac{1}{64}(q^3 - 2q^2 - 2q - 1)(q - 3)$	$\frac{1}{64}q(q^2 - 3q - 2)(q - 2)$
$[4^{1}]$	$[1^4,4^1]$	-2	4	$\frac{1}{32}(q^2 - q - 1)(q + 1)(q - 1)$	$\frac{1}{32}q^2(q+1)(q-2)$

6.4.2 General position and the conics form a conjugate pair.

con pts	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $\operatorname{char}(k) = 2$
$[1^4]$	$[1^4, 2^2]$	2	4	$\frac{1}{96}(q^3 - 4q^2 + 4q + 3)(q - 1)$	$\frac{1}{96}q(q-1)(q-2)^2$
	$[1^4, 4^1]$	-2	4	$\frac{1}{96}(q^2 - 3q + 3)(q + 1)(q - 1)$	$\frac{1}{96}q^2(q-1)(q-2)$
$[1^1, 3^1]$	$[1^1, 2^2, 3^1]$	-1	1	$\frac{1}{12}(q+1)q(q-1)(q-2)$	$\frac{1}{12}(q+1)q(q-1)(q-2)$
	$[1^1, 3^1, 4^1]$	1	1	$\frac{1}{12}(q+1)q^2(q-1)$	$\frac{1}{12}(q+1)q^2(q-1)$

6.4.3 The line is tangent to one of the conics.

con pts	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $char(k) = 2$
$[1^4]$	$[1^7]$	1	6	$\frac{1}{48}(q-3)(q-4)(q-5)$	$\frac{1}{48}(q-2)(q-4)(q-6)$
	$[1^5, 2^1]$	-1	4	$\frac{1}{48}(q-1)(q-2)(q-3)$	$\frac{1}{48}q(q-2)(q-4)$
$[1^2, 2^1]$	$[1^5, 2^1]$	-1	4	$\frac{1}{8}(q-1)^2(q-2)$	$\frac{1}{8}q(q-2)^2$
	$[1^3, 2^2]$	1	2	$\frac{1}{8}q(q-1)^2$	$\frac{1}{8}q^2(q-2)$
$[1^1, 3^1]$	$[1^4, 3^1]$	1	3	$\frac{1}{6}(q+1)q(q-1)$	$\frac{1}{6}(q+1)q(q-1)$
	$[1^2, 2^1, 3^1]$	-1	1	$\frac{1}{6}(q+1)q(q-1)$	$\frac{1}{6}(q+1)q(q-1)$
$[2^2]$	$[1^3, 2^2]$	1	2	$\frac{1}{16}q(q-1)(q-3)$	$\frac{1}{16}q(q-2)^2$
$[4^{1}]$	$[1^3, 4^1]$	-1	2	$\frac{1}{8}(q+1)q(q-1)$	$\frac{1}{8}q^3$

6.4.4 Two k-conics intersect in three points.

con pts	$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $\operatorname{char}(k) = 2$
$[1^2]$	$[1^7]$	1	6	$\frac{1}{16}(q-3)^2(q-5)$	$\frac{1}{16}(q-2)(q-4)(q-5)$
	$[1^5, 2^1]$	-1	4	$\frac{1}{8}(q-1)^2(q-3)$	$\frac{1}{8}q(q-2)(q-3)$
	$[1^3, 2^2]$	1	2	$\frac{1}{16}(q-1)^3$	$\frac{1}{16}q(q-1)(q-2)$
$[2^{1}]$	$[1^5, 2^1]$	-1	4	$\frac{1}{16}(q-1)^2(q-3)$	$\frac{1}{16}q(q-2)(q-3)$
	$[1^3, 2^2]$	1	2	$\frac{1}{8}(q-1)^3$	$\frac{1}{8}q(q-1)(q-2)$

6.4.5 A conjugate pair of conics intersects in three point.

$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $\operatorname{char}(k) = 2$
$[1^3, 2^2]$	1	2	$\frac{1}{8}(q^2 - 2q - 1)(q - 1)$	$\frac{1}{8}q(q-1)(q-2)$
$[1^3, 4^1]$	-1	2	$\frac{1}{8}(q+1)(q-1)^2$	$\frac{1}{8}q^2(q-1)$

6.5 A cubic and two lines.

The cubic can be smooth or singular. If the cubic is singular then one of the lines can be tangent to the cubic. Any other type will have fever than seven singularities.

6.5.1 A singular cubic and no lines are tangent to it.

If the lines are defined over k_2 then $\sigma_5(\lambda)$ will always be zero so we need only consider the types where the lines are defined over k.

Example 6.10. We count the number of curves of the type where all singularities are defined over k. Take two k-lines and a k-point P outside the lines. Then choose three k-points on each line, such that no two of these six points are on a line through P and none of the points is the intersection of the two lines. There is precisely one irreducible cubic through the six points that has a singularity at P of delta-invariant one. We get

$$\frac{1}{|PGL_3(k)|} {q^2 + q + 1 \choose 2} (q^2 - q) {q \choose 3} {q - 3 \choose 3} = \frac{1}{72} (q - 2)(q - 3)(q - 4)(q - 5).$$

$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$
[18]	6	8	$\frac{1}{72}(q-2)(q-3)(q-4)(q-5)$
$[1^6, 2^1]$	-4	6	$\frac{1}{12}q(q-1)(q-2)(q-3)$
$[1^4, 2^2]$	2	4	$\frac{1}{8}(q+1)q(q-1)(q-2)$
$[1^5, 3^1]$	3	5	$\frac{1}{18}(q+1)q(q-1)(q-2)$
$[1^3, 2^1, 3^1]$	-1	3	$\frac{1}{6}(q+1)q^2(q-1)$

6.5.2 A smooth cubic.

Example 6.11. We count the number of curves of the type where the components and singularities are all defined over k. Take two k-lines L, L' and three k-points on each line that are not the intersection point. There is a \mathbb{P}^3 of cubics through the six points. From this we subtract the reducible cubics through the six points. We get a reducible cubic by taking any k-line together with the lines L, L'. Or we can take a line though two of the points and an irreducible conic through the other four points. Finally we can take three lines connecting the six points such that none of the lines is L or L'.

We also subtract the singular irreducible cubics we counted in Example 6.10. This gives us

$$\frac{1}{|PGL_3(k)|} {q^2 + q + 1 \choose 2} {q \choose 3}^2 (|\mathbb{P}^3| - |\mathbb{P}^2| - 9(q - 2) - 6)$$
$$-\frac{1}{72} (q - 2)(q - 3)(q - 4)(q - 5) = \frac{1}{72} (q^4 - 3q^3 + 3q^2 - 17q + 36)(q - 2).$$

lines	$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta^1_k$	$\#\{C\}$
$[1^2]$	$[1^7]$	1	7	$\frac{1}{72}(q^4 - 3q^3 + 3q^2 - 17q + 36)(q - 2)$
	$[1^5, 2^1]$	-1	5	$\frac{1}{12}(q^3 - q^2 + q - 3)q(q - 2)$
	$[1^3, 2^2]$	1	3	$\frac{1}{8}(q^2 - q + 2)(q + 1)q(q - 1)$
	$[1^4,3^1]$	1	4	$\frac{1}{18}(q^2 - q + 1)(q + 1)q(q - 2)$
	$[1^1, 3^2]$	1	1	$\frac{1}{18}(q^3 - 2)(q+1)q$
	$[1^2, 2^1, 3^1]$	-1	2	$\frac{1}{6}(q^2 - q + 1)(q + 1)q^2$
$[2^{1}]$	$[1^1, 2^3]$	-1	1	$\frac{1}{12}(q^3 - q^2 + 4)(q^2 - 3)$
	$[1^1, 2^1, 4^1]$	1	1	$\frac{1}{4}(q+1)q^2(q-1)^2$
	$[1^1,6^1]$	-1	1	$\frac{1}{6}(q^3 - q^2 - 2)q^2$

6.5.3 A singular cubic and one line is tangent to it.

$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$
$[1^7]$	1	6	$\frac{1}{6}(q-2)(q-3)(q-4)$
$[1^5, 2^1]$	-1	4	$\frac{1}{2}q(q-1)(q-2)$
$[1^4, 3^1]$	1	3	$\frac{1}{3}(q+1)q(q-1)$

6.6 A cubic and a conic.

To get seven singularities the cubic must be singular and it has to intersect the conic in six points.

Example 6.12. We count the number of curves of the type where the singularities are all defined over k for $\operatorname{char}(k) \neq 2$. Choose an irreducible conic and two k-points A, B on the conic. Take the tangents to the conic at A and B and denote their intersection by P. We want to pick six points on the conic such that there are no two on a line through P. There are three ways we can do this: We can take A, B and four of the other points, either A or B and five of the other points, or six of the other points. There is exactly one irreducible cubic through the six points with a singularity at P. We can also choose an irreducible conic and a conjugate pair of points on the conic. We then again take P to be the intersection point of the corresponding tangent lines and choose six k-points on the conic such that

there are no two on a line through P. Combining these cases gives us

$$\frac{1}{|PGL_3(k)|}(q^2+q+1)q^2(q-1)$$

$$\left(\binom{q+1}{2}(q-1)(q-3)(q-5)(q-7)\left(\frac{1}{4!}+2\frac{1}{5!}(q-9)+\frac{1}{6!}(q-9)(q-11)\right)$$

$$+\frac{q^2-q}{2}\frac{1}{6!}(q+1)(q-1)(q-3)(q-5)(q-7)(q-9)\right) = \frac{1}{720}(q^2-9q+15)(q-3)(q-5)(q-7).$$

$[\lambda, 1^1]$ pts	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$ for $\operatorname{char}(k) \neq 2$	$\#\{C\}$ for $\operatorname{char}(k) = 2$
$[1^7]$	1	7	$\frac{1}{720}(q^2 - 9q + 15)(q - 3)(q - 5)(q - 7)$	$\frac{1}{720}(q-2)(q-4)^2(q-6)(q-8)$
$[1^5, 2^1]$	-1	5	$\frac{1}{48}(q^3 - 6q^2 + 10q - 1)(q - 1)(q - 3)$	$\frac{1}{48}(q-2)^3q(q-4)$
$[1^3, 2^2]$	1	3	$\frac{1}{16}(q^2 - q - 1)(q + 1)(q - 1)(q - 3)$	$\frac{1}{16}(q^2 - 2q - 4)q^2(q - 2)$
$[1^1, 2^3]$	-1	1	$\frac{1}{48}(q^2 - q - 3)(q^2 - 2q - 7)(q - 3)$	$\frac{1}{48}(q^2 - 2q - 4)(q + 2)(q - 2)(q - 4)$
$[1^4,3^1]$	1	4	$\frac{1}{18}(q^2 - 3q + 3)(q+1)q(q-1)$	$\frac{1}{18}(q+1)q(q-1)^2(q-2)$
$[1^1, 3^2]$	1	1	$\frac{1}{18}(q^2 + 2q + 3)q^2(q - 2)$	$\frac{1}{18}(q^2+2q+3)(q+1)(q-1)(q-2)$
$[1^2, 2^1, 3^1]$	-1	2	$\frac{1}{6}(q^2 - q - 1)(q + 1)q(q - 1)$	$\frac{1}{6}(q+1)^2q(q-1)(q-2)$
$[1^3,4^1]$	-1	3	$\frac{1}{8}(q^2 - q + 1)(q + 1)^2(q - 1)$	$\frac{1}{8}(q^2-2)q^3$
$[1^1, 2^1, 4^1]$	1	1	$\frac{1}{8}(q^2 - q - 1)(q + 1)(q - 1)^2$	$\frac{1}{8}(q^2-2)q^2(q-2)$
$[1^2,5^1]$	1	2	$\frac{1}{5}(q^2+1)q^2(q+1)$	$\frac{1}{5}(q^2+1)(q+1)^2(q-1)$
$[1^1,6^1]$	-1	1	$\frac{1}{6}(q^2+q+2)q^2(q-1)$	$\frac{1}{6}(q^2+q+2)(q+1)(q-1)^2$

6.7 A quartic and a line.

To get seven singularities the quartic must have three singularities and it has to intersect the line in four points.

quart. sing.	$[\lambda, 1^1]$ points	$\sigma_5(\lambda)$	$\#\delta_k^1$	$\#\{C\}$
$[1^3]$	$[1^7]$	1	7	$\frac{1}{144}(q-2)(q-3)^2(q-4)(q-5)$
	$[1^5, 2^1]$	-1	5	$\frac{1}{24}q(q-1)^2(q-2)(q-3)$
	$[1^3, 2^2]$	1	3	$\frac{1}{48}(q+1)q(q-1)(q-2)(q-3)$
	$[1^4,3^1]$	1	4	$\frac{1}{18}(q+1)q^2(q-1)(q-2)$
	$[1^3,4^1]$	-1	3	$\frac{1}{24}(q+1)q^2(q-1)^2$
$[1^1, 2^1]$	$[1^5, 2^1]$	-1	5	$\frac{1}{48}q(q-1)(q-2)(q-3)^2$
	$[1^3, 2^2]$	1	3	$\frac{1}{8}(q+1)q(q-1)^2(q-2)$
	$[1^1, 2^3]$	-1	1	$\frac{1}{16}(q^2 - q - 4)(q + 1)(q - 2)(q - 3)$
	$[1^2, 2^1, 3^1]$	-1	2	$\frac{1}{6}(q+1)q^3(q-1)$
	$[1^1, 2^1, 4^1]$	1	1	$\frac{1}{8}(q+1)q^2(q-1)^2$
$[3^1]$	$[1^4, 3^1]$	1	4	$\frac{1}{72}(q+1)q(q-1)(q-2)(q-3)$
	$[1^2, 2^1, 3^1]$	-1	2	$\frac{1}{12}(q+1)q^2(q-1)^2$
	$[1^1, 3^2]$	1	1	$\frac{1}{9}(q^3 - q - 3)(q + 1)q$

7 Results.

When we add the results together then for all characteristics of the field k we get

$$\sum \sigma(\lambda) \cdot \# \delta_k^1 \cdot \# \{C\} = 1,$$

where the sum is over the rows of all the tables in Section 6.

Remark 7.1. If we compute $\sum \sigma(\lambda) \cdot \# \delta_k^1 \cdot \# \{C\}$ where the sum is only over the rows of one the tables then for most tables we get the same result for all characteristics of k. The only exceptions are the tables in Sections 6.3.1 and 6.3.2.

Combining our results from the sieving part and the explicit part we get

$$\frac{|T(k)|}{|\mathrm{PGL}_3(k)|} = \frac{1}{2}(q^{11} + q^{10}) + \frac{1}{2}(q^{11} - q^{10}) + q^{10} - q^8 + 1,$$

which proves our theorem.

Theorem 7.2. The number of smooth trigonal curves of genus five over a finite field \mathbb{F}_q is given by

$$\#\mathcal{T}_5(\mathbb{F}_q) = q^{11} + q^{10} - q^8 + 1.$$

We have written computer programs that loop over all plane quintics over \mathbb{F}_2 and \mathbb{F}_3 that have an ordinary split node/non-split node/cusp with fixed tangents at **P**. For each curve we test for all points besides **P** whether they are singular or not. This way we count

$$|\mathbf{C}_{\text{split}}(\mathbf{P}, \lambda)|, \quad |\mathbf{C}_{\text{non-split}}(\mathbf{P}, \lambda)| \quad \text{and} \quad |\mathbf{C}_{\text{cusp}}(\mathbf{P}, \lambda)|$$
 (2)

for all partitions λ . From the results for $\lambda = []$ we can now easily compute $\#\mathcal{T}_5(\mathbb{F}_2)$ and $\#\mathcal{T}_5(\mathbb{F}_3)$. As an extra check the programs use the information from (2) to count the subresults

$$\sum_{|\lambda|=w} \left((-1)^{\sum_i \lambda_i} \cdot \sum_{S \in (\mathbb{P}^2 - \{\mathbf{P}\})(\lambda)} |\mathbf{C}_{\mathrm{split}}(S)| \right)$$

for $0 \le w \le 5$ and

$$\sum_{|\lambda|=w} |\mathbf{C}_{\mathrm{split}}(\mathbf{P},\lambda)| \cdot \sigma_N(\lambda)$$

for $6 \le w \le 9$.

The results of these computer counts agree with the results in this article. The programs are written in the C programming language and the source code is available online together with lists of the results for (2) and the above subresults.

The address is: https://github.com/Wennink/countingtrigonalcurves

References

- [1] T. van den Bogaart and B. Edixhoven, Algebraic stacks whose number of points over finite fields is a polynomial, Number Fields and Function Fields Two Parallel Worlds, Progress in Mathematics, Vol. 239, Birkhäuser Boston, Boston, 2005.
- [2] Gaëtan Chenevier and Jean Lannes, Formes automorphes et voisins de Kneser des réseaux de Niemeier, http://arxiv.org/abs/1409.7616, 2014.
- [3] J. Bergström, Cohomology of moduli spaces of curves of genus three via point counts, J. Reine Angew. Math., 622:155187, 2008.