TROPICAL LIMIT OF LOG-INFLECTION POINTS FOR PLANAR CURVES

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ABSTRACT. The paper describes the behavior of log-inflection points (i.e. points of inflection with respect to the parallelization of $(\mathbb{C}^{\times})^2$ given by the multiplicative group law) of curves in $(\mathbb{C}^{\times})^2$ under passing to the tropical limit. Assuming that the limiting tropical curve is smooth, we show that log-inflection points accumulate by pairs at the midpoints of bounded edges.

1. Introduction

Denote by $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ the complex torus and let $f: (\mathbb{C}^{\times})^2 \to \mathbb{C}$ be a Laurent polynomial in two variables with the Newton polygon Δ . The zero locus

$$V_f = \{(z, w) \in (\mathbb{C}^{\times})^2 \mid f(z, w) = 0\} \subset (\mathbb{C}^{\times})^2$$

is a non-compact complex curve. For non-singular V_f the logarithmic Gauss map (see [Kap91])

$$(1) \gamma_f: V_f \to \mathbb{C}P^1$$

is defined by $\gamma_f(z,w) = (z\frac{\partial f}{\partial z}:w\frac{\partial f}{\partial w})$. This map depends only on V_f and not on f itself. It is the map that takes a point of V_f to the slope of its tangent plane with respect to the parallelization of $(\mathbb{C}^\times)^2$ given by the multiplicative translations $(z,w) \mapsto (\alpha z, \beta w), \alpha, \beta \in \mathbb{C}^\times$.

Definition 1. The critical points of γ_f are called the log-inflection points of V_f .

Denote the set of critical points by $\rho V_f \subset V_f$. If all coefficients of the Laurent polynomial f are real we define

$$\mathbb{R}V_f = \{(z, w) \in (\mathbb{R}^{\times})^2 \mid f(z, w) = 0\} \subset (\mathbb{R}^{\times})^2.$$

We have

$$\gamma_f|_{\mathbb{R}V_f}: \mathbb{R}V_f \to \mathbb{R}P^1,$$

and we denote by $\rho \mathbb{R}V_f \subset \mathbb{R}V_f$ the set of critical points of $\gamma_f|_{\mathbb{R}V_f}$ (called real log-inflection points). Recall the definition of tropical limits (see e.g. [IKMZ], [BIMS15] as well as [Mik06]). Let

$$\{f_t = \sum_{j,k \in \mathbb{Z}} a_{jk,t} z^j w^k\}_{t \in A}$$

be a family of polynomials parameterized by an (infinite) set A. Suppose that the degree of f_t is universally bounded in z, z^{-1} , w and w^{-1} . Let $\alpha : A \to \mathbb{R}$ be a map whose image is unbounded from above. The pair (f_t, α) is called a *scaled sequence* of polynomials f_t (α is the scaling).

The scaled sequence of polynomials is said to *converge tropically* if the limit

$$a_{jk}^{\mathrm{trop}} = \lim_{t \in A} \log_{\alpha(t)} |a_{jk,t}| \quad \in \mathbb{R} \cup \{-\infty\}$$

exists for all $j, k \in \mathbb{Z}$. Here the limit is taken with respect to the directed set A, where the order is given by α . Define the limiting Newton polygon

$$\Delta = \text{ConvexHull}\{(j,k) \in \mathbb{Z}^2 \mid a_{jk}^{\text{trop}} \neq -\infty\} \subset \mathbb{R}^2.$$

It is a bounded convex lattice polygon. The set $\mathbb{R} \cup \{-\infty\}$ is called the set of tropical numbers and denoted by \mathbb{T} . Denote by $\text{Log}: (\mathbb{C}^{\times})^2 \to \mathbb{R}^2$ the map defined by

$$Log(z, w) = (\log |z|, \log |w|).$$

If a scaled sequence of polynomials (f_t, α) converges tropically, then the limit

$$C = \lim_{t \in A} \operatorname{Log}_{\alpha(t)}(V_t)$$

exists in the Hausdorff sense, i.e. as the limit of closed subsets of \mathbb{R}^2 with the topology given by the Hausdorff distance. It is called the tropical limit of $\{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A}$.

More generally, given a scaled sequence (V_t, α) of curves $V_t \subset (\mathbb{C}^*)^n$ we say that it converges tropically to $C \subset \mathbb{R}^n$ if C is the limit of $\text{Log}_{\alpha(t)}(V_t)$ in the Hausdorff sense.

It can be proved (see [Mik05], and [IKMZ] for a more general statement) that the limit C is a rectilinear graph in \mathbb{R}^n . The edges E of C are straight intervals with rational slopes, and can be prescribed integer weights w(E) (coming from V_t) so that for every vertex $v \in C$ it holds the balancing condition

$$\sum_{E\ni v} w(E)u(E) = 0,$$

where $u(E) \in \mathbb{Z}^n$ is the primitive vector parallel to E in the direction away from v. A rectilinear graph $C \subset \mathbb{R}^n$ with these properties is called a tropical curve. If n=2 then C defines a lattice subdivision of the Newton polygon Δ . If each polygon of the subdivision is a triangle of area $\frac{1}{2}$ (the minimal possible area for a lattice polygon) then C is called smooth. For details we refer to [IMS09] and [BIMS15].

Definition 2. Let C be a smooth plane tropical curve. The tropical parabolic locus of C is the set $\rho C \subset C$ formed by the midpoints of all bounded edges of C.

The main result of this paper is the following.

Theorem 3. Let $C \subset \mathbb{R}^2$ be a smooth tropical curve, and $\{V_t \subset \mathbb{R}^2\}$ $(\mathbb{C}^{\times})^2\}_{t\in A}$ be a family of complex curves parameterised by the scaled sequence $\alpha: A \to \mathbb{R}$. Suppose that V_t tropically converges to C and that the limiting Newton polygon Δ coincides with the Newton polygons of V_t for large $\alpha(t)$. Then $\operatorname{Log}_{\alpha(t)}(\rho V_t)$ converges to ρC in the Hausdorff metric when $\alpha(t) \to +\infty$. Furthermore, a small neighborhood of a point from ρC contains exactly two points of $\operatorname{Log}_{\alpha(t)}(\rho V_t)$ for large $\alpha(t)$.

In other words, $2\rho C$ is the tropical limit of ρV_t .

Remark 4. Theorem 3 provides a 2-1 correspondence $\Phi_t: \rho V_t \to \rho C$ for large $\alpha(t)$. If the family $\{V_t\}_{t\in A}$ is defined over \mathbb{R} , then ρV_t is invariant under the involution conj of complex conjugation in $(\mathbb{C}^{\times})^2$. In particular, the pair $\Phi_t^{-1}(p)$ is conj-invariant for each $p \in \rho C$.

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2. The case of real curves

Let $\{V_t \subset (\mathbb{C}^{\times})^2\}_{t \in A}$ be a family of real curves (i.e. invariant with respect to the involution of complex conjugation) tropically convergent to a smooth tropical curve $C \subset \mathbb{R}^2$ with respect to a scaling $\alpha : A \to \mathbb{R}$. Assume that the limiting Newton polygon coincides with the Newton polygons of V_t for large $\alpha(t)$. Let E be a bounded edge of C. Denote by $p \in E$ the midpoint of E, and let $t \in A$ such that $\alpha(t)$ is so large that the 2-1 correspondence $\Phi_t: \rho V_t \to \rho C$ is defined.

Definition 5. The edge E is called V_t -twisted if conj preserves $\Phi_t^{-1}(p)$ pointwise. Otherwise conj induces a non-trivial permutation of $\Phi_t^{-1}(p)$, and we call E V_t -untwisted.

Clearly E is V_t -twisted if $\Phi_t^{-1}(p) \subset \mathbb{R}V_t$, and V_t -untwisted if $\Phi_t^{-1}(p)$ forms a complex-conjugated pair.

Definition 6. (cf. [BIMS15])

A subset T of the set of bounded edges of a smooth tropical curve $C \subset$ \mathbb{R}^2 is called twist-admissible if they satisfy the following condition: for any cycle γ of C (considered as a graph), if E_1, \dots, E_k denote edges in $\gamma \cap T$, and if $u(E_i)$ is a primitive integer in the direction of E_i , then

(2)
$$\sum_{i=1}^{k} u(E_i) = (0,0) \mod 2.$$

Recall that a patchworking polynomial may be defined as

$$f_t(z_1, z_2) = \sum_{j \in B} \beta_j(t) (\alpha(t))^{a_j} z^j.$$

Here $j=(j_1,j_2),\ z^j=(z_1^{j_1},z_2^{j_2}),\ j\mapsto a_j$ is a strictly convex function from a finite subset $B\subset\mathbb{Z}^2$ to \mathbb{R} , and $(\beta_j(t))_{t\in A}\subset\mathbb{R}$ is a family converging (with respect to the directed set A) to a non-zero real number. Denote

$$\sigma_j = \operatorname{sign}(\lim_{t \in A} \beta_j(t)) \in \{-1, +1\}.$$

The family $(V_t)_{t\in A}$ tropically converge a tropical curve C defined by the tropical polynomial with the coefficients a_j (see e.g. [IMS09]). According to Viro's patchworking theorem (cf. [GKZ94], [Vir01]), if the tropical curve C is smooth then the rigid isotopy class of the curve $V_t = \{(x,y) \in (\mathbb{R}^{\times})^2 \mid f_t(x,y) = 0\}$ for large value of $\alpha(t)$ is determined by C together with the signs σ_j (see e.g. [BIMS15]).

Proposition 7. Let V_t be a family of real curves given by a patchworking polynomial as above, and let C be its tropical limit. Assume that C is smooth. For each bounded edge E of C, denote by v_1^E and v_2^E the two vertices of the segment Δ_E dual to E in the subdivision of Δ defined by C. The segment Δ_E is adjacent to two other triangles of the subdivision of Δ . Denote by v_3^E and v_4^E the vertices of these two triangles different from v_1^E and v_2^E . Then E is V_t -twisted if and only if

- $\sigma_{v_1^E}\sigma_{v_2^E}\sigma_{v_3^E}\sigma_{v_4^E} > 0$ if the coordinates modulo 2 of v_3^E and v_4^E are distinct.
- $\sigma_{v_3^E}\sigma_{v_4^E} < 0$ if the coordinates modulo 2 of v_3^E and v_4^E are the same.

The following corollary follows easily from Proposition 7 and known facts about patchworking (see [BIMS15]).

Corollary 8. The set of V_t -twisted edges of C is twist-admissible. Conversely, for any twist-admissible subset T of bounded edges of a smooth tropical curve C there exist a family $\{V_t \subset (\mathbb{C}^\times)^2\}_{t \in A}$ of real curves such that T is the set of V_t -twisted edges.

3. Tropical limit of logarithmic Gauss map

Let $\{V_t\}_{t\in A} = \{f_t = 0\}_{t\in A} \subset (\mathbb{C}^{\times})^2$ be a family of complex curves parameterised by the scaled sequence $\alpha: A \to \mathbb{R}$. Let \widetilde{V}_t , $t \in A$, be the graph of the logarithmic Gauss map γ_{f_t} (which we denote by γ_t for brevity) given by

$$\widetilde{V}_t = \{(z_t, \gamma_t(z_t)) | z_t \in V_t\} \subset (\mathbb{C}^\times)^2 \times \mathbb{CP}^1.$$

Denote by $\pi_1: \widetilde{V}_t \to V_t \subset (\mathbb{C}^{\times})^2$ the projection on the first factor and by $\pi_2: \widetilde{V}_t \to \mathbb{CP}^1$ the projection on the second factor. The map π_1 is an isomorphism and the map π_2 is of degree $2\text{Area}(\Delta)$ by the

Bernstein-Kouchnirenko formula (see [Kou76] and also [Mik00], Lemma 2). Denote by \mathbb{TP}^n the tropical projective space of dimension n (see for example [BIMS15]). Denote by $\pi_1^{trop}: \mathbb{R}^2 \times \mathbb{TP}^1 \to \mathbb{R}^2$ the projection on the first factor and by $\pi_2^{trop}: \mathbb{R}^2 \times \mathbb{TP}^1 \to \mathbb{TP}^1$ the projection on the second factor.

Proposition 9. There exists a subfamily $A' \subset A$ with unbounded $\alpha(A') \subset \mathbb{R}$ such that the subsequence $(V_t)_{t \in A'}$ tropically converges to a tropical curve \widetilde{C} in $\mathbb{R}^2 \times \mathbb{TP}^1$. Furthermore, we have $\pi_1^{trop}(\widetilde{C}) = C$.

Proof. The proposition is a special case of Theorem 38 of [IKMZ]. We apply this theorem to the projective curves V_t , $t \in A$, obtained as the closures of $\widetilde{V}_t \cap (\mathbb{C}^{\times})^3$ in \mathbb{CP}^3 . We set \widetilde{C} to be the closure in $\mathbb{R}^2 \times \mathbb{TP}^1$ of $\bar{C} \cap \mathbb{R}^3$, where $\bar{C} \subset \mathbb{TP}^3$ is the tropical limit of a subfamily from Theorem 38 of [IKMZ]. By the uniqueness of tropical limit we have $C=\pi_1^{trop}(\widetilde{C}).$

Definition 10. We refer to the map $\pi_2^{trop}|_{\widetilde{C}}: \widetilde{C} \to \mathbb{TP}^1$ as a tropical limit of the family γ_t , and we call it a tropical Gauss map.

Remark 11. This construction extends in the case of hypersurfaces of higher dimensions. However in this text, we will focus on the case of curves.

Remark 12. A priori, the tropical curve C not only depends on C but also on an approximation $(V_t)_{t\in A}$ of C, and even on the choice of the subfamily $A' \subset A$. Nevertheless, the tropical limit of the ramification locus of γ_t is a well-defined subset of C and does not depend on these choices as shown in this paper.

We now define the degree of a projection $\pi:\Gamma\subset\mathbb{R}^n\mapsto\mathbb{R}^k$, where Γ is a tropical curve, and where we identify \mathbb{R}^n with $\mathbb{R}^k \times \mathbb{R}^{n-k}$. If the image $\pi(\Gamma)$ is a point, we say that π is of degree 0. Assume that $\pi(\Gamma)$ is not reduce to a point. Then if $x \in \pi(\Gamma)$ is a generic point, the fiber $\pi^{-1}(x)$ intersects Γ transversally (meaning that any point of intersection is in the relative interior of an edge of Γ) at finitely many points. Denote by e the edge of $\pi(\Gamma)$ containing x and denote by \mathbb{Z}_e the sublattice of $\mathbb{Z}^k \times \mathbb{Z}^{n-k}$ generated by $0 \times \mathbb{Z}^{n-k}$ and $(\overrightarrow{v}, 0)$, where \overrightarrow{v} is a primitive vector contained in e. Let $z \in \Gamma \cap \pi^{-1}(x)$, and denote by \tilde{e} the edge of Γ containing z. Define the local tropical intersection number $\iota_z(\Gamma, \pi^{-1}(x))$ as the index of the sublattice in \mathbb{Z}_e generated by \mathbb{Z}^{n-k} and a primitive vector contained in \widetilde{e} (multiplied by the weight of \widetilde{e}). The degree of π is defined as the sum

$$\deg(\pi) := \sum_{z \in \pi^{-1}(x) \cap \Gamma} \iota_z(\Gamma, \pi^{-1}(x)).$$

It follows from the balancing condition that the degree is independent of the point x (see also Lemma 40 in [BIMS15]).

The following proposition follows directly from the proof of Proposition 42 of [IKMZ] (the degree of π_1 is 1 and the degree of π_2 is $2\text{Area}(\Delta)$).

Proposition 13. One has $deg(\pi_1^{trop}) = 1$ and $deg(\pi_2^{trop}) = 2Area(\Delta)$.

One can define similarly the local degree of π . Let $s \in \Gamma_1$ and let U_s be a small neighborhood of s only containing points from edges adjacent to s. If π maps the neighborhood U_s to a point, we say that π is of local degree 0 at s. If not, let $x \in \pi(\Gamma) \cap \pi(U_s)$ be a generic point. Define the local degree of π at s as the sum of local tropical intersection number $\iota_z(\Gamma, \pi^{-1}(x))$ over all $z \in U_s$.

4. First properties of tropical Gauss map

Proposition 14. Let $\{V_t \subset (\mathbb{C}^\times)^2\}_{t\in A}$ be a family of complex curves parameterised by the scaled sequence $\alpha: A \to \mathbb{R}$. Assume that $\{V_t \subset (\mathbb{C}^\times)^2\}_{t\in A}$ tropically converges to a smooth plane tropical curve C. Let e be an edge of C of slope $p \in \mathbb{QP}^1$, and let x belongs to the relative interior of e. If $x_t \in V_t$ satisfies $\lim_{t\in A} \operatorname{Log}_{\alpha(t)}(x_t) = x$, then

$$\lim_{t \in A} \gamma_t(x_t) = p.$$

Proof. Denote by $\{f_t = \sum_{j,k \in \Delta \cap \mathbb{Z}^2} a_{jk,t} z^j w^k\}_{t \in A}$ a family of polynomials defining the family of complex curves $(V_t)_{t \in A}$. Consider the tropical curve C_x obtained by translating the tropical curve C along the vector $-x \in \mathbb{R}^2$. Denote by $\{f_{t,x}\}_{t \in A}$ the family of polynomials given by

$$f_{t,x}(z,w) = \alpha(t)^{-b} f_t(\alpha(t)^{x_1} z, \alpha(t)^{x_2} w) = \alpha(t)^{-b} \sum_{k} a_{jk,t} z^j w^k \alpha(t)^{b_{j,k}},$$

where $x = (x_1, x_2)$ and $b_{i,k} = jx_1 + kx_2$, and

$$b = b_{j_1,k_1} + a_{j_1,k_1}^{trop} = b_{j_2,k_2} + a_{j_2,k_2}^{trop},$$

where (j_1, k_1) , and (j_2, k_2) are the integer points on the edge (of the subdivision of Δ) dual to the edge e.

The family of complex curves $\{V_{t,x}\}_{t\in A} = \{f_{t,x} = 0\}_{t\in A}$ converges tropically to C_x . Furthermore, denoting by Φ_x the isomorphism

$$\begin{array}{ccc} V_{t,x} & \xrightarrow{\Phi_x} & V_t \\ (z,w) & \mapsto & (\alpha(t)^{x_1}z, \alpha(t)^{x_2}w) \end{array}$$

one has the following commutative diagram

$$V_{t,x} \xrightarrow{\Phi_x} V_t .$$

$$\uparrow_{t,x} \downarrow \qquad \qquad \downarrow \gamma_t$$

$$\mathbb{CP}^1 \xrightarrow{\mathrm{Id}} \mathbb{CP}^1$$

It is then enough to prove the proposition for the point $(0,0) \in C_x$ and the family $V_{t,x}$. Since $b > b_{j,k} + a_{j,k}^{trop}$ for all $(j,k) \in (\Delta \cap \mathbb{Z}^2) \setminus$

 $\{(j_1, k_1), (j_2, k_2)\}$, there exist real numbers $c_{j,k}$ such that $c_{j_1,k_1} = c_{j_2,k_2} = 0$ and $c_{j,k} > 0$ otherwise such that

$$f_{t,x}(z,w) = \sum_{(j,k)\in\Delta\cap\mathbb{Z}^2} a_{jk,t}\alpha(t)^{-a_{j,k}^{trop}-c_{j,k}} z^j w^k.$$

Since the tropical curve C is smooth, there exists a \mathbb{Z} -affine automorphism G of the plane such that $G(j_1, k_1) = (0, 0)$ and $G(j_2, k_2) = (1, 0)$. Write $G = M + (v_1, v_2)$, where $M \in GL_2(\mathbb{Z})$ and $(v_1, v_2) \in \mathbb{Z}^2$. The map M give rise to a multiplicative automorphism Φ_M of $(\mathbb{C}^{\times})^2$. Define

$$g_t(z, w) = z^{v_1} w^{v_2} f_{t,x}(\Phi_M(z, w)),$$

and

$$W_t = \{g_t = 0\}.$$

One has $\Phi_M(W_t) = V_{t,x}$ and the family of complex curves $(W_t)_{t \in A}$ tropically converges to the tropical curve D which is the image of C by ${}^tM^{-1}$. The edge of slope p containing (0,0) is then mapped to an edge of slope ∞ . Furthermore one has the following commutative diagram:

$$W_{t} \xrightarrow{\Phi_{M}} V_{t,x} .$$

$$\uparrow_{g_{t}} \downarrow \qquad \qquad \downarrow^{\gamma_{t,x}} \\ \mathbb{CP}^{1} \xrightarrow{M^{-1}} \mathbb{CP}^{1}$$

One deduce that it is enough to prove the statement for the point (0,0) and the edge of slope ∞ in the tropical limit of the family of complex curves $(W_t)_{t\in A}$ given by the family of polynomials $g_t(z, w)$. One has

$$g_t(z, w) = a_{j_1 k_1, t} \alpha(t)^{-a_{j_1 k_1}^{trop}} + a_{j_2 k_2, t} \alpha(t)^{-a_{j_2 k_2}^{trop}} z + R_t(z, w),$$

where $R_t(z, w)$ is a Laurent polynomial such that all coefficients $r_{jk,t}$ satisfies $r_{jk,t} = O(\alpha(t)^c)$, where c < 0 is some real number. Let's divide g_t by $a_{j_1k_1,t}\alpha(t)^{-a_{j_1k_1}^{trop}}$ and make the change of variables $z \mapsto a_{j_2k_2,t}^{-1}\alpha(t)^{a_{j_2k_2}^{trop}}z$. It does not affect the limiting tropical curve since the tropical limit of $a_{j_2k_2,t}\alpha(t)^{-a_{j_2k_2}^{trop}}$ is zero. We end with a polynomial h_t of the form

$$h_t(z, w) = 1 + z + S_t(z, w),$$

where as above $S_t(z, w)$ is a Laurent polynomial such that all coefficients $s_{jk,t}$ satisfies $s_{jk,t} = O(\alpha(t)^c)$. One has then

$$\gamma_{h_t}(z_t, w_t) = \left[z_t (1 + \frac{\partial S_t}{\partial z}) : w_t \frac{\partial S_t}{\partial w} \right].$$

Let $x_t = (z_t, w_t) \in \{h_t = 0\}$ such that $\lim_{t \to +\infty} \text{Log}_t(x_t) = (0, 0)$. Since $z_t = -1 - S_t(z_t, w_t)$, one obtains that $\lim_{t \to +\infty} z_t = -1$. One has also $\lim_{t \to +\infty} (z_t \frac{\partial S_t}{\partial z}) = \lim_{t \to +\infty} (w_t \frac{\partial S_t}{\partial w}) = 0$, which proves the proposition. Taking the logarithm with base $\alpha(t)$, we obtain the following corollary, where $1_{\mathbb{T}} = 0$ is the neutral element for the tropical multiplication.

Corollary 15. Let $\{V_t \subset (\mathbb{C}^{\times})^2\}_{t\in A}$ be a family of complex curves parametrised by the scaled sequence $\alpha: A \to \mathbb{R}$ which tropically converges to a smooth plane tropical curve C. Let e be an edge of C of slope $p \in \mathbb{QP}^1$, x be a point of the relative interior of e, and $x_t \in V_t$ be a sequence such that $\lim_{t\in A} \operatorname{Log}_{\alpha(t)}(x_t) = x$. Then up to passing to a subsequence one has

- if $p \neq 0$ or ∞ , then $\lim_{t \in A} \operatorname{Log}_{\alpha(t)}(\gamma_t(x_t)) = [1_{\mathbb{T}} : 1_{\mathbb{T}}] \in \mathbb{TP}^1$,
- if p = 0, then $\lim_{t \in A} \operatorname{Log}_{\alpha(t)}(\gamma_t(x_t)) = [a : 1_{\mathbb{T}}] \in \mathbb{TP}^1$, for some $a \in [-\infty, 0]$,
- if $p = \infty$, then $\lim_{t \in A} \operatorname{Log}_{\alpha(t)}(\gamma_t(x_t)) = [1_{\mathbb{T}} : a] \in \mathbb{TP}^1$, for some $a \in [-\infty, 0]$.

Since the tropical map π_1^{trop} is of degree 1, one has a continuous inclusion $i_C: C \hookrightarrow \widetilde{C}$ such that $\pi_1^{trop} \circ i_C = id$.

Proposition 16. We have $\pi_2^{trop}(s) = [1_{\mathbb{T}} : 1_{\mathbb{T}}] \in \mathbb{TP}^1$ for any $s \in i_C(\operatorname{Vert}(C))$.

Proof. At least one of the edges adjacent to s must have the slope different from 0 and ∞ . The proposition follows then immediately from Corollary 15.

Proposition 17. For any $s \in i_C(Vert(C))$, the tropical map π_2^{trop} is of local degree 1 at s.

Proof. Since the logarithmic Gauss map commutes with multiplicative translations, one can assume that $s = i_C(0,0)$. Performing a multiplicative automorphism Φ_M as in the proof of Proposition 14, denote by $(W_t)_{t\in A}$ the family of complex curves such that $\Phi_M(W_t) = V_t$. Denote by W_t the graph of the logarithmic Gauss map on W_t and by $\Phi := (\Phi_M, {}^tM^{-1})$ the map sending \widetilde{W}_t to \widetilde{V}_t . Denote by \widetilde{D} the tropical limit (up to passing to a subsequence) of \widetilde{W}_t , by $D = \pi_1^{trop}(\widetilde{D})$ and by $s'=i_D(0,0)$. By definition, the map Φ_M sends $\operatorname{Log}_{\alpha(t)}^{-1}(0,0)\subset W_t$ to $\operatorname{Log}_{\alpha(t)}^{-1}(0,0) \subset V_t$. Since \widetilde{W}_t (resp., \widetilde{V}_t) is a graph over W_t (resp., V_t), there exist a small neighborhood $U_{s'}$ of s' and a small neighborhood U_s of s such that $\Phi(\operatorname{Log}_{\alpha(t)}^{-1}(U_{s'})) \subset \operatorname{Log}_{\alpha(t)}^{-1}(U_s)$. However, let $y \in \pi_2^{trop}(U_s)$ be a generic point, and let $(y_t)_{t \in A} \in \mathbb{CP}^1$ such that $\lim_{t\in A} \operatorname{Log}_{\alpha(t)}(y_t) = y$. It follows from Proposition 43 of [BIMS15] that the local degree of π_2^{trop} at s is equal to the intersection of \widetilde{V}_t and $\pi_2^{-1}(y_t)$ in $\operatorname{Log}_{\alpha(t)}^{-1}(U_s)$, for sufficiently large $\alpha(t)$. Applying the map Φ , one deduce that it is enough to prove the statement for the family of complex curves $(W_t)_{t\in A}$ given by the family of polynomials

$$h_t(z, w) = 1 + z + w + S_t(z, w),$$

where $S_t(z, w)$ is a Laurent polynomial such that all coefficients $s_{jk,t}$ satisfies $s_{jk,t} = O(\alpha(t)^c)$. The graph of the logarithmic Gauss map over W_t is then given by

$$\widetilde{W}_t = \left\{ z, w, \left[z(1 + \frac{\partial S_t}{\partial z}) : w(1 + \frac{\partial S_t}{\partial w}) \right] \mid (z, w) \in W_t \right\}$$

and the graph of the logarithmic Gauss map on the line $\mathcal{L} = \{1 + z + w = 0\}$ is given by

 $\widetilde{\mathcal{L}} = \{z, w, [z:w] \mid (z, w) \in \mathcal{L}\}.$

Denote by \widetilde{L} the tropical limit of $\widetilde{\mathcal{L}}$. It follows from the description of \widetilde{W}_t and $\widetilde{\mathcal{L}}$ that there exists a small neighborhood U of the point (0,0,0)in \mathbb{R}^3 such that $\widetilde{D} \cap U = \widetilde{L} \cap U$. But the map $\pi_2^{trop} : \widetilde{L} \to \mathbb{TP}^1$ is of degree one, which proves the proposition.

Example 18. In the left hand side of Figure 1, we draw the tropical limit (when $t \to +\infty$) of V_t , for $V_t = \{1 + x + y + t^{-1}xy\}$. We draw also the image of this tropical limit under the first projection π_1^{trop} . In the right hand side of Figure 1, we did the same for $V_t = \{1+x+y+t^{-1}x^2\}$.

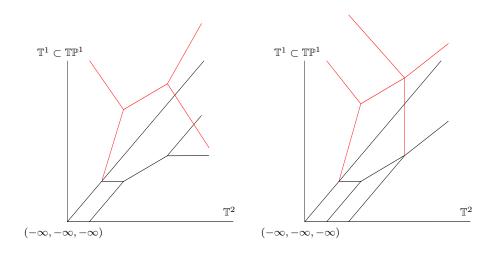


FIGURE 1. Tropical limit of \widetilde{V}_t for $V_t = \{1+x+y+t^{-1}xy\}$ and for $V_t = \{1 + x + y + t^{-1}x^2\}.$

5. Proof of Theorem 3, Proposition 7 and Corollary 8

The tropical Gauss map π_2^{trop} contracts a priori some edges of $i_C(C)$. It is then not immediately clear how to locate the tropical limit of the logarithmic inflection points on \widetilde{C} . To overcome this difficulty, we embed \widetilde{C} and \mathbb{TP}^1 in higher dimensional spaces such that the new tropical Gauss map does not contract any edges coming from C (see Proposition 19). Then we prove that the tropical Gauss map is of local degree 2 at each point coming from a midpoint of C (see Proposition 20).

Denote by m_1, \dots, m_l the midpoints of bounded edges of C, and by x_t^1, \dots, x_t^l some points in V_t such that $\lim_{t \in A} \operatorname{Log}_{\alpha(t)}(x_t^i) = m_i$, for $1 \leq i \leq l$. Put $\gamma_t(x_t^i) = [u_t^i : v_t^i]$, for $1 \leq i \leq l$ and denote by

$$\begin{array}{cccc} \Phi_t: & \mathbb{CP}^1 & \to & \mathbb{CP}^{l+1} \\ & [u:v] & \mapsto & \left[u:v:\varphi_t^1(u,v):\cdots:\varphi_t^l(u,v)\right] \end{array},$$

where $\varphi_t^i(u,v) = uv_t^i - vu_t^i$ are linear maps. Consider the line $\mathcal{L}_t = \Phi_t(\mathbb{CP}^1)$, and denote by L its tropical limit in \mathbb{TP}^{l+1} (after passing to a subsequence), which exists by compactness theorem (see [IKMZ], Section 3.4). Introduce also

$$\widehat{V}_t = \{ (z_t, \Phi_t \circ \gamma_t(z_t)) \mid z_t \in V_t \} \subset (\mathbb{C}^\times)^2 \times \mathbb{CP}^{l+1},$$

and denote by $\widehat{C} \subset \mathbb{R}^2 \times \mathbb{TP}^{l+1}$ its tropical limit (after passing to a subsequence). One has isomorphisms $\widehat{V}_t \simeq \widetilde{V}_t$ and $\mathcal{L}_t \simeq \mathbb{CP}^1$ given by projections. The projection on the second factor $\mathbb{R}^2 \times \mathbb{TP}^{l+1} \to$ \mathbb{TP}^{l+1} restricts to a projection $\pi^{trop}: \hat{C} \to L$ of degree $2\text{Area}(\Delta)$, the projection $\mathbb{R}^2 \times \mathbb{TP}^{l+1} \to \mathbb{R}^2 \times \mathbb{TP}^1$ restricts to a projection $p: \hat{C} \to \widetilde{C}$ of degree 1 and the projection $\mathbb{TP}^{l+1} \to \mathbb{TP}^1$ restricts to a projection $q: \widehat{C} \to \widehat{C}$ $L \to \mathbb{TP}^1$ of degree 1. Furthermore, the following diagram commutes.

$$\widehat{C} \xrightarrow{\pi^{trop}} L$$

$$\downarrow^{p} \qquad \downarrow^{q}$$

$$\widetilde{C} \xrightarrow{\pi_{2}^{trop}} \mathbb{TP}^{1}$$

$$\downarrow^{\pi_{1}^{trop}}$$

$$C$$

Denote by $i_p: C \to \widehat{C}$ the continuous inclusion that is right-inverse to the projection $\pi_1^{trop} \circ p$, and denote by s_L the image in L of $[1_{\mathbb{T}} : 1_{\mathbb{T}}] \in$ \mathbb{TP}^1 by the continuous inclusion $\mathbb{TP}^1 \hookrightarrow L$ that is right-inverse to q. It follows from Proposition 17 that for any $s \in Vert(C)$, one has $\pi^{trop}(i_p(s)) = s_L$. In fact, if $\pi^{trop}(i_p(s)) \neq s_L$, a small neighborhood of $i_C(s)$ would be contracted to $[1_T:1_T]$ by π_2^{trop} , contradicting Proposition 17. Moreover, since π_2^{trop} is locally of degree 1 at $i_C(s)$ for any $s \in Vert(C)$ and since the projections p and q are both of degree 1, we deduce that the tropical map π^{trop} is locally of degree 1 at $i_p(s)$ for any $s \in Vert(C)$.

Proposition 19. Let s be a vertex of C and let \hat{e} be a bounded edge of $i_p(C)$ adjacent to $i_p(s)$. Then \hat{e} is not contracted by π^{trop} .

Proof. Assume that \widehat{e} is contracted by π^{trop} , and denote by e the edge of C containing $\pi_1^{trop} \circ p(\widehat{e})$. Let us first prove that $i_p(e) = \widehat{e}$.

Assume that $\widehat{e} \subsetneq i_p(e)$. Then \widehat{e} is adjacent to a vertex $s' \in Vert(\widehat{C}) \setminus i_p(Vert(C))$. Since \widehat{e} is contracted by π^{trop} , one has $\pi^{trop}(s') = \pi^{trop}(s) = s_L$. Since by definition every vertex is at least 3-valent, there exists an edge adjacent to s' not belonging to $i_p(Edge(C))$. Such an edge is contracted by $\pi_1^{trop} \circ p$ and so is not contracted by π^{trop} . Then the local degree of π^{trop} at s' is at least 1. Since every vertex in $i_p(Vert(C))$ already contributes to 1 to the degree of π^{trop} and since $\#(Vert(C)) = 2\operatorname{Area}(\Delta)$, we conclude that the degree of π^{trop} is at least $2\operatorname{Area}(\Delta) + 1$, which is impossible. Then one has $i_p(e) = \widehat{e}$.

It follows from the construction of \widehat{C} that for any bounded edge e_C of C, the set $i_p(e_C)$ consists of at least two edges (if m_C is the middle point of e_C , there is a point of \widehat{C} in a boundary divisor of \mathbb{TP}^{l+1} projecting to m_C). Then $i_p(e_C) \neq \widehat{e}$ and \widehat{e} cannot be contracted by π^{trop} .

We call a tropical map f finite over a subset F if it does not contract any edge in F.

Proposition 20. Let e be a bounded edge of C, and let m be the midpoint of e. The map π^{trop} is finite over $i_p(e)$ and is of local degree 1 at any point of $i_p(e)$ except at $i_p(m)$, where it is of local degree 2.

Proof. Denote by T the image of $i_p(e)$ by the map π^{trop} . The set T is a tree since $T \subset L$. Let us first show that s_L is a vertex of valence 1 of T. Assume that the valence of s_L is at least 2 and denote by a_1, a_2 two edges of T adjacent to s_L . Since T is a tree, the sets $a_1 \setminus s_L$ and $a_2 \setminus s_L$ are in two different connected components of $T \setminus s_L$. Since $i_p(e)$ is connected, it means that there are at least 3 vertices of $i_p(e)$ mapped to s_L . As in the proof of Proposition 19 this give a contradiction with the degree of π^{trop} . Denote by e_L the edge of T adjacent to s_L .

Assume first that the edges e_1^1 and e_1^2 of $i_p(e)$ adjacent to the vertices over s_L join over the edge e_L , see Figure 2.

In this case, one has $i_p(e) = e_1^1 \cup e_1^2$. Since the local degree of π^{trop} at the two vertices in $i_p(Vert(C))$ adjacent to $i_p(e)$ is equal to one, the local degree of π^{trop} on the edges e_1^1 and e_1^2 is equal to one. Then the vertex adjacent to e_1^1 and e_1^2 is the vertex $i_p(m)$ (the lengths of e_1^1 and e_1^2 are equal). Moreover, the map π^{trop} is of local degree 2 at $i_p(m)$. In fact, if the local degree would be more than 2, then by the balancing condition s_L would have at least 3 preimages by π^{trop} over the set $(\pi_1^{trop} \circ p)^{-1}(e)$, which is impossible.

Assume now that the two edges e_1^1 and e_1^2 do not join over e_L , and denote by s_1 the other vertex of T adjacent to e_L , see Figure 3.

In this case, the preimage of s_1 consists of 2 vertices at which the map

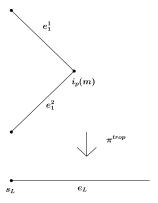


FIGURE 2. The two edges e_1^1 and e_1^2 join over the edge e_L .

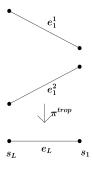


FIGURE 3. The two edges e_1^1 and e_1^2 do not join over the edge e_L .

 π^{trop} is of local degree one. As in the proof of Proposition 19, one see that every edge of $i_e(C)$ adjacent to $i_e(s_1)$ is not contracted by π^{trop} . One conclude then easily by induction.

Proof of Theorem 3. Let e be a bounded edge of C, and denote by v_1 and v_2 the vertices adjacent to e. Consider the fibration $\lambda_t: V_t \to C$ as defined in [Mik04]. It follows from Proposition 17 that there exists a small neighborhood \mathcal{N}_i of v_i , i=1,2 in C such that for t big enough, the map $\gamma_t|_{\lambda_t^{-1}(\mathcal{N}_i)}$ is of degree one on its image, see Figure 4. It follows from Proposition 14 that the image by γ_t of a boundary component of $\lambda_t^{-1}(\mathcal{N}_i)$ converges to the slope of the edge supporting this boundary component. Consider a small circle $\Gamma \subset \mathbb{CP}^1$ centered at the slope of e and of radius e. The image by e0 the boundary component of e1.

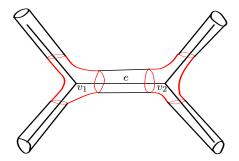


FIGURE 4. The sets $\lambda_t^{-1}(\mathcal{N}_i)$ in red inside V_t .

supported above e is contained in the same connected component of $\mathbb{CP}^1 \setminus \Gamma$ as the slope s_e of e. Then for t big enough, Γ is contained in $\gamma_t \left(\lambda_t^{-1}(\mathcal{N}_i) \right)$, and the set $\gamma_t^{-1}|_{\lambda_t^{-1}(\mathcal{N}_i)}(\Gamma)$ is a circle $\Gamma_i \subset \lambda_t^{-1}(\mathcal{N}_i)$ (see Figure 5). Note that one of the two connected components of $\lambda_t^{-1}(\mathcal{N}_i)$

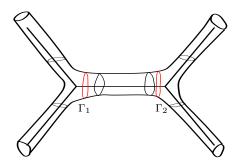


FIGURE 5. The circles Γ_i in red inside the sets $\lambda_t^{-1}(\mathcal{N}_i)$.

 Γ_i only contains the boundary component of $\lambda_t^{-1}(\mathcal{N}_i)$ supported on e. If not, then there would exist a path in $V_t \setminus \Gamma_i$ with ends a_t and b_t , such that the tropical limit of a_t lie in the interior of e and the tropical limit of b_t lie in the interior of another edge adjacent to v_i (see Figure 6). But it follows from Proposition 14 that for t big enough, such a path

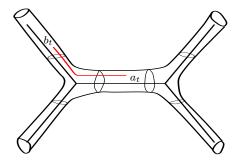


FIGURE 6. The path with ends a_t and b_t inside V_t . should intersect Γ_i .

The restriction of γ_t to the annulus \mathcal{A} supported on e with boundary $\Gamma_1 \cup \Gamma_2$ gives us a map of degree 2 to the disc containing s_e with boundary Γ . In fact, the preimage of Γ by γ_t in \mathcal{A} consists exactly of $\Gamma_1 \cup \Gamma_2$ since the map γ_t is of degree $2\text{Area}(\Delta)$ and since from Proposition 17 for any vertex v of C there exists a small neighborhood \mathcal{N} of v such that the map γ_t restricted to $\lambda_t^{-1}(\mathcal{N})$ is of degree 1. By the Riemann-Hurwitz formula, such a map has two critical points. The tropical limit of a scaled sequence of critical points must be a point with the local degree of π^{trop} greater than one. Proposition 20 now implies the theorem as ρV_t are the critical points of γ_t .

Proof of Proposition 7. Suppose that $\sigma_{v_1^E}\sigma_{v_2^E}=-1$. Then there is a branch of $\mathbb{R}V_t$ corresponding to E in the positive quadrant. The tropical limit of the branch of $\mathbb{R}V_t$ corresponding to E in the positive quadrant is non-convex (see Figure 7) if and only if $\sigma_{v_2^E}\sigma_{v_4^E}=-1$. Thus

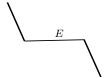


FIGURE 7. Non-convex tropical limit of a branch of $\mathbb{R}V_t$ corresponding to an edge E.

both log-inflection points corresponding to E must be real (and belong to two different quadrants).

If $\sigma_{v_3^E}\sigma_{v_4^E}=1$ then the tropical limit of both corresponding branches are convex (see Figure 8). Suppose that one of the branches \mathcal{B} of $\mathbb{R}V_t$

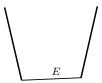


FIGURE 8. Convex tropical limit of a branch of $\mathbb{R}V_t$ corresponding to an edge E.

corresponding to E contains two real log-inflection points. Consider the second branch \mathcal{B}' of $\mathbb{R}V_t$ corresponding to E. If \mathcal{B}' is also logarithmically non-convex then we have more than two inflection points for E, which contradicts Theorem 3. If \mathcal{B}' of $\mathbb{R}V_t$ is logarithmically convex then we can find a tangent line to \mathcal{B}' parallel to an inflection point of \mathcal{B} (in the logarithmic coordinates). This gives us at least three points (counted with multiplicity) with the same image under the logarithmic Gauss map γ_t . There are $2\text{Area}(\Delta) - 2$ vertices of C not adjacent to E. It follows from Proposition 17 that each of them contributes one point

of V_t with the same image of γ_t . We arrive to a contradiction with $\deg \gamma_t = 2 \operatorname{Area}(\Delta)$.

This finishes the proof of the proposition in the case $\sigma_{v_1^E}\sigma_{v_2^E}=-1$. The case $\sigma_{v_1^E}\sigma_{v_2^E}=1$ is obtained by applying the coordinate change $z\mapsto -z$ or/and $w\mapsto -w$.

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