ALMOST DUALITY FOR SAITO STRUCTURE AND COMPLEX REFLECTION GROUPS

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ABSTRACT. We reformulate Dubrovin's almost duality of the Frobenius structure to the setting of the Saito structures without metric. Then we formulate and study the existence and uniqueness problem of the natural Saito structure on the orbit spaces of finite complex reflection groups from the viewpoint of the almost duality. We give a complete answer to the problem for the irreducible groups.

1. Introduction

- 1.1. Almost duality for Frobenius structures. Dubrovin introduced the Frobenius structure in [4]. Later he defined the almost Frobenius structure in [6]. These are both structures on the tangent bundles of manifolds consisting of flat metrics (i.e. nondegenerate bilinear forms whose Levi–Civita connections are flat), multiplications and nonzero vector fields. However, they must satisfy slightly different conditions. Dubrovin showed that given a Frobenius structure, one can construct an almost Frobenius structure and vice versa. He called it the almost duality in the title of the article [6]. Later, Arsie and Lorenzoni introduced the notion of bi-flat F-manifold [1]. They showed, in semisimple case, that it can be regarded as an extension of the almost duality to the Frobenius structure without metric [2, §4].
- 1.2. Complex reflection groups. Let V be a complex vector space of finite dimension n. A finite complex reflection group (or a unitary reflection group) $G \subset GL(V)$ is a finite group generated by pseudo-reflections on V. Reducible finite complex reflection groups are direct products of irreducible ones, and irreducible finite complex reflection groups were classified by Shephard and Todd in 1954 [16]. The number of minimal generators for the irreducible groups are either n or n+1 and those with n minimal generators are said to be well-generated. They are also called duality groups because there is a duality between the degrees and the codegrees of such groups. The duality groups include the finite Coxeter groups. See [10] and [12] for finite complex reflection groups.

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- 1.3. Frobenius structures on the orbit spaces of Coxeter groups. In 1979, Saito [15] showed the existence of flat coordinates on the orbit spaces of finite Coxeter groups. See also Saito-Yano-Sekiguchi [14]. In 1993, Dubrovin [4] constructed Frobenius structures on them. See also [5]. Later Dubrovin [6] generalized the construction to the Shephard groups which include the Coxeter groups and form a proper subclass of the duality groups. These constructions may be regarded as applications of the almost duality.
- 1.4. Saito structures on the orbit spaces of the duality groups. The Saito structure without metric introduced by Sabbah [13] is a structure weaker than the Frobenius structure, consisting of a torsion-free flat connection and a multiplication on the tangent bundle together with a nonzero vector field. We call it Saito structure for short throughout this paper. If a Saito structure admits a compatible metric, it makes a Frobenius structure. Recently, Kato, Mano and Sekiguchi showed that there exist certain polynomial Saito structures on the orbit spaces of the duality groups [9]. See also [8]. In [2, §5], Arsie and Lorenzoni studied the same polynomial Saito structures based on their theory of bi-flat F-manifold [1] and computed many examples.
- 1.5. Aim and results of the paper. The aim of this article is to formulate and study the existence and uniqueness problem of the "natural" Saito structure for finite complex reflection groups from the viewpoint of the almost duality. We give a complete answer to the problem for the irreducible groups. Especially, it contains another proof of Kato–Mano–Sekiguchi's result (i.e. the case of duality groups).

The paper consists of two parts. The first part is devoted to formulating the almost duality for the Saito structure. We introduce the almost Saito structure in $\S 2$ and show the duality between the Saito structure and the almost Saito structure in $\S 3$ (Theorem 3.11). In $\S 4.1$, we recall the definition of Frobenius manifold and Dubrovin's almost duality and explain the relationship with $\S 2$ and $\S 3$. In $\S 4.2$, we recall the definition of Arsie–Lorenzoni's bi-flat F-manifold [1] and show that the almost duality for the Saito structure is a notion equivalent to the bi-flat F-manifold. $\S 5$ is devoted to matrix representations of (almost) Saito structures.

In the second part, we apply the formulation to the orbit spaces of finite complex reflection groups. In $\S 6$ we characterize, using the almost duality, the Saito structures we are looking for. We call it the natural Saito structure because it comes from the trivial connection on TV. In $\S 7$, we show that unique natural Saito structures exist for a certain class of finite complex reflection groups including the duality groups (Theorem 7.5, Corollary 7.6). For the duality groups, they coincide with the Saito structures studied

¹ In the case of Shephard groups which are not Coxeter groups, the natural Saito structures do not coincide in general with the underlying Saito structures of Dubrovin's Frobenius structures.

by Kato-Mano-Sekiguchi [9] and by Arsie-Lorenzoni [2, §5]. We also show that four irreducible groups do not admit natural Saito structures (§9.4.1). For the remaining irreducible groups, it turns out that all natural Saito structures are those induced from branched covering maps from the orbit spaces of some duality groups (§8, §9.4.2, §9.4.3).

In appendix §C, the natural Saito structure for the rank two cases are listed. §A and §B contain proofs of some technical results.

- 1.6. Conventions. Throughout this paper, a manifold means a complex manifold. For a manifold M, TM denotes the holomorphic tangent bundle and \mathcal{T}_M its sheaf of local sections. We write $x \in \mathcal{T}_M$ to mean that x is a local section of TM. A multiplication on TM means an \mathcal{O}_M -bilinear map $\mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$.
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2. The Almost Saito structure

In this section, we first recall the definition of the Saito structure in §2.1 and give a definition of the almost Saito structure in §2.2. In §2.3, we introduce the regular almost Saito structure which is characterized by the property that the multiplication is completely determined by the connection and the vector field e. The regularity will play an important role in application to the finite complex reflection groups. In §2.4, we explain that an almost Saito structure is always accompanied with a two-parameter family of almost Saito structures. Roughly speaking, one parameter λ corresponds to the "twist" of the multiplication and another parameter ν corresponds to the shift of the parameter r.

2.1. Saito structure. The following notion was introduced by Sabbah in [13].

Definition 2.1. A Saito structure (without a metric) on a manifold M consists of

- a torsion-free flat connection ∇ on TM,
- an associative commutative multiplication * on TM with a unit $e \in \Gamma(M, \mathcal{T}_M)$,
- a vector field $E \in \Gamma(M, \mathcal{T}_M)$ called the Euler vector field,

satisfying the following conditions.

(SS1)
$$\nabla_x(y*z) - y*\nabla_x z - \nabla_y(x*z) + x*\nabla_y z = [x,y]*z \quad (x,y,z \in \mathcal{T}_M),$$

(SS2)
$$[E, x * y] - [E, x] * y - x * [E, y] = x * y \quad (x, y \in \mathcal{T}_M)$$
,

$$(\mathbf{SS3}) \qquad \nabla e = 0 \ ,$$

(SS4)
$$\nabla_x \nabla_y E - \nabla_{\nabla_x y} E = 0 \quad (x, y \in \mathcal{T}_M)$$
.

Remark 2.2. If $(\nabla, *, E)$ is a Saito structure with a unit e, then $(\nabla, *, E - \lambda e)$ ($\lambda \in \mathbb{C}$) is also a Saito structure. Therefore we may be able to think that a Saito structure is always accompanied with a one-parameter family.

We list a few formulas which will be useful later.

Lemma 2.3. Let $(\nabla, *, E)$ be a Saito structure on M with a unit e. Then the following holds. For $x, y, z \in \mathcal{T}_M$,

$$[e, y * z] = y * [e, z] + [e, y] * z ,$$

$$[E, e] = -e, \quad [e, E * x] = E * [e, x] + x,$$

(2.3)
$$\nabla_{y*z} E - y * \nabla_z E - \nabla_y (E * z) + E * \nabla_y z + y * z = 0,$$

$$(2.4) (E * \nabla_x (y * z) - y * \nabla_x (E * z) + y * \nabla_{x*z} E) - (x \leftrightarrow y) = E * [x, y] * z.$$

Proof. (2.1): substituting x = e into (SS1),

$$\nabla_e(y*z) - y*\nabla_e z = [e, y]*z$$

Using the torsion freeness of ∇ and (SS3), we obtain

$$[e, y * z] - y * [e, z] = [e, y] * z$$
.

(2.2): the first equation immediately follows from (SS2) if we substitute x = y = e. Substituting y = E into (2.1), we obtain the second equation:

$$[e, E*z] = E*[e, z] + [e, E]*z = E*[e, z] + z.$$

(2.3): substituting x = E into (SS1),

$$\nabla_E(y * z) - y * \nabla_E z - \nabla_y(E * z) + E * \nabla_y z - [E, y] * z = 0.$$

Changing (x, y) to (y, z) in (SS2),

$$[E, y * z] - [E, y] * z - y * [E, z] - y * z = 0$$
.

Subtracting, we obtain (2.3).

(2.4):

$$(2.5) 0 = x * (2.3) - (x \leftrightarrow y) = x * (\nabla_{y*z} E - \nabla_y (E * z) + E * \nabla_y z) - (x \leftrightarrow y) .$$

Then we have

$$0 = (2.3) - E * (SS1)$$

= $(x * \nabla_{y*z} E - x * \nabla_y (E * z) + E * \nabla_y (x * z)) - (x \leftrightarrow y) + E * [x, y] * z .$

Now assume that $(\nabla, *, E)$ is a Saito structure on a manifold M with a unit e. If one chooses a nonzero constant $c \in \mathbb{C}^*$, one can construct a new multiplication *' by $x *' y = cx * y \ (x, y \in \mathcal{T}_M)$. Then it is easy to see that $(\nabla, *', E)$ is also a Saito structure on M with a unit $c^{-1}e$. Therefore we introduce the following equivalence relation.

Definition 2.4. Two Saito structures $(\nabla, *, E)$ and $(\nabla', *', E')$ on M are said to be equivalent if $\nabla = \nabla'$, E = E' and if there exists a nonzero constant $c \in \mathbb{C}^*$ such that

$$x *' y = cx * y \quad (x, y \in \mathcal{T}_M)$$
.

2.2. The Almost Saito structure.

Definition 2.5. An almost Saito structure on a manifold N with parameter $r \in \mathbb{C}$ consists of

- a torsion-free flat connection ∇ on TN,
- an associative commutative multiplication \star on TN with a unit $E \in \Gamma(N, \mathcal{T}_N)$,
- a nonzero vector field $e \in \Gamma(N, \mathcal{T}_N)$

satisfying the following conditions.

(ASS1)
$$\nabla_x(y \star z) - y \star \nabla_x z - \nabla_y(x \star z) + x \star \nabla_y z = [x, y] \star z \quad (x, y, z \in \mathcal{T}_N)$$
.

(ASS2)
$$[e, x \star y] - [e, x] \star y - x \star [e, y] + e \star x \star y = 0 \quad (x, y \in \mathcal{T}_N) .$$

$$(\mathbf{ASS3}) \qquad \nabla_x E = rx \quad (x \in \mathcal{T}_N) \ .$$

(ASS4)
$$\nabla_x \nabla_y e - \nabla_{\nabla_x y} e + \nabla_{x \star y} e = 0 \quad (x, y \in \mathcal{T}_N)$$
.

Let us list some formulas.

Lemma 2.6. Let (∇, \star, e) be an almost Saito structure on N with parameter r and a unit E. Then the following holds. For $x, y, z \in \mathcal{T}_N$,

(2.6)
$$[E, y \star z] = y \star [E, z] + [E, y] \star z ,$$

$$(2.7) [e, E] = e , [E, e \star z] = e \star [E, z] - e \star z ,$$

(2.8)
$$\nabla_x(e \star y) - e \star \nabla_x y - \nabla_{x \star y} e + x \star \nabla_y e + e \star x \star y = 0.$$

(2.9)
$$(e \star \nabla_x (y \star z) - y \star \nabla_x (e \star z) + y \star \nabla_{x \star z} e) - (x \leftrightarrow y) = e \star [x, y] \star z .$$

Proof. (2.6): substituting x = E into (ASS1),

$$\nabla_E(y \star z) - y \star \nabla_E z = [E, y] \star z$$
.

Using the torsion freeness of ∇ and (ASS3), we obtain

$$[E, y \star z] - y \star [E, z] = [E, y] \star z.$$

(2.7): the first equation immediately follows from (ASS2) if we substitute x = y = E. Substituting y = e into (2.6), we obtain the second equation:

$$[E, e \star z] = e \star [E, z] + [E, e] \star z = e \star [E, z] - e \star z$$
.

(2.8): changing (x, y, z) to (e, x, y) in (ASS1), we have

$$\nabla_e(x \star y) - x \star \nabla_e y - \nabla_x(e \star y) + e \star \nabla_x y - [e, x] \star y = 0.$$

Subtracting this equation from (ASS2), we obtain (2.8).

$$(2.9)$$
 is obtained by

$$e \star (\mathbf{ASS1}) + x \star (\mathrm{change}\ (x,y)\ \mathrm{to}\ (y,z)\ \mathrm{in}\ (2.8)) - y \star (\mathrm{change}\ (x,y)\ \mathrm{to}\ (x,z)\ \mathrm{in}\ (2.8))$$
.

Let $c \in \mathbb{C}$ be a nonzero constant. If (∇, \star, e) is an almost Saito structure on a manifold N with parameter r, (∇, \star, ce) is also an almost Saito structure on N with parameter r. So we introduce the following equivalence relation.

Definition 2.7. Two almost Saito structures (∇, \star, e) and (∇', \star', e') on N are said to be equivalent if $\nabla = \nabla'$, $\star = \star'$ and if there exists a nonzero constant $c \in \mathbb{C}$ such that e' = ce.

2.3. The Regular almost Saito structure. Let N be a manifold. Given a pair (∇, e) consisting of a connection ∇ on TN and a vector field $e \in \Gamma(N, \mathcal{T}_N)$, define $Q \in \operatorname{Hom}_{\mathcal{O}_N}(\mathcal{T}_N, \mathcal{T}_N)$ by

$$Q(x) = \nabla_x e \quad (x \in \mathcal{T}_N)$$

We say that the pair (∇, e) is regular if \mathcal{Q} is an isomorphism.

Lemma 2.8. Let e be a vector field on a manifold N and let ∇ be a torsion free, flat connection on TN. If the pair (∇, e) is regular, then a multiplication \star on TN satisfying (ASS4) is unique and it is given by

(2.10)
$$x \star y = -\mathcal{Q}^{-1}(\nabla_x \nabla_y e) + \nabla_x y.$$

Proof. Immediate.

Proposition 2.9. Let ∇ be a torsion free, flat connection on TN and let $e \in \Gamma(N, \mathcal{T}_N)$ be a vector field on N. Assume that the pair (∇, e) is regular. Define the multiplication $\star \in \operatorname{Hom}_{\mathcal{O}_N}(\mathcal{T}_N \otimes \mathcal{T}_N, \mathcal{T}_N)$ by (2.10).

- $(1) \star is commutative.$
- (2) \star is associative if and only if (ASS1) holds.
- (3) (∇, \star, e) is an almost Saito structure if and only if it satisfies (ASS1), (ASS2) and (ASS3).

Proof. (1) The commutativity of \star follows from the flatness and the torsion freeness of ∇ :

$$-\mathcal{Q}(x \star y - y \star x) = \nabla_x \nabla_y e - \nabla_y \nabla_x e - \mathcal{Q}(\nabla_x y - \nabla_y x)$$
$$= \nabla_{[x,y]} e - \mathcal{Q}([x,y]) = 0.$$

(2) By the commutativity of \star , we have

$$\begin{aligned} & \mathcal{Q}(y \star (z \star x) - (y \star z) \star x) \\ &= \mathcal{Q}(y \star (x \star z) - x \star (y \star z)) \\ &= -\nabla_y \nabla_{x \star z} e + \mathcal{Q}(\nabla_y (x \star z)) - (x \leftrightarrow y) \\ &= -\nabla_y (-\nabla_x \nabla_z e + \mathcal{Q}(\nabla_x z)) + \mathcal{Q}(\nabla_y (x \star z)) - (x \leftrightarrow y) \\ &= -\nabla_{[x,y]} \nabla_z e - \nabla_y \nabla_{\nabla_x z} e + \nabla_x \nabla_{\nabla_y z} e + \mathcal{Q}(\nabla_y (x \star z) - \nabla_x (y \star z)) \ . \end{aligned}$$

In passing to the last line, we used the flatness of ∇ . Moreover, we have

$$Q(-y \star \nabla_x z + x \star \nabla_y z - [x, y] \star z)$$

$$= \nabla_y \nabla_{\nabla_x z} e - \nabla_x \nabla_{\nabla_y z} e + \nabla_{[x, y]} \nabla_z e + Q(\nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z)$$

$$= \nabla_y \nabla_{\nabla_x z} e - \nabla_x \nabla_{\nabla_y z} e + \nabla_{[x, y]} \nabla_z e .$$

Here we used the flatness of ∇ . Adding the above two equations, we obtain

$$\mathcal{Q}(y\star(z\star x)-(y\star z)\star x)=\mathcal{Q}(-\nabla_x(y\star z)+y\star\nabla_x\,z-(x\leftrightarrow y)+[x,y]\star z)\;.$$

Since Q is an isomorphism, it follows that the associativity of \star is equivalent to (ASS1). (3) This follows form (1) and (2).

Definition 2.10. We say that an almost Saito structure (∇, \star, e) on N is regular if (∇, e) is regular.

For a regular almost Saito structure (∇, \star, e) , we sometimes omit \star , and call (∇, e) a regular almost Saito structure.

Remark 2.11. Let (∇, \star, e) be an almost Saito structure on N with parameter r and a unit E. Then by $(\mathbf{ASS3})$ and (2.7),

$$Q(E) = \nabla_E e = \nabla_e E + [E, e] = (r - 1)e.$$

Thus, if an almost Saito structure (∇, \star, e) has parameter r = 1, then it is not regular.

2.4. **Two-parameter family.** The next proposition shows that if there exists one almost Saito structure, then there exists a two-parameter family of almost Saito structures.

Let (∇, \star, e) be an almost Saito structure on a manifold N with parameter $r \in \mathbb{C}$ and a unit E. Take $\lambda \in \mathbb{C}$ and define $\mathcal{I}_{\lambda} \in \text{Hom}_{\mathcal{O}_N}(\mathcal{T}_N, \mathcal{T}_N)$ by $\mathcal{I}_{\lambda}(x) = (E - \lambda e) \star x$. Assume that

$$N^{\lambda} = \{ p \in N \mid I_{\lambda} : T_p N \to T_p N \text{ has rank dim } N \}$$

is a nonempty subset of N. Here I_{λ} is the endomorphism of TN corresponding to \mathcal{I}_{λ} .

Proposition 2.12. Take $\nu \in \mathbb{C}$ and define a new multiplication \star_{λ} and a new connection $\nabla^{[\lambda,\nu]}$ on TN^{λ} by

$$x \star_{\lambda} y = \mathcal{I}_{\lambda}^{-1}(x \star y) ,$$

$$\nabla_{x}^{[\lambda,\nu]} y = \nabla_{x} y + \nu x \star_{\lambda} y + \lambda \nabla_{x \star_{\lambda} y} e \quad (x, y \in \mathcal{T}_{N^{\lambda}}).$$

Then $(\nabla^{[\lambda,\nu]}, \star_{\lambda}, e)$ is an almost Saito structure on N^{λ} with parameter $r + \nu$ and the unit $E - \lambda e$.

Proof. In this proof, $x, y, z \in \mathcal{T}_{N^{\lambda}}$. First we show a technical lemma. Set

$$\mathcal{C}(x, y, z; \lambda) = \nabla_{x}(y \star_{\lambda} z) - y \star_{\lambda} \nabla_{x} z - \lambda y \star_{\lambda} \nabla_{x \star_{\lambda} z} e ,$$

$$\mathcal{E}(x, y; \lambda) = \nabla_{x}(e \star_{\lambda} y) - e \star_{\lambda} \nabla_{x} y - \nabla_{x \star_{\lambda} y} e + x \star_{\lambda} \nabla_{y} e + e \star_{\lambda} x \star_{\lambda} y$$

$$+ \lambda x \star_{\lambda} \nabla_{e \star_{\lambda} y} e - \lambda e \star_{\lambda} \nabla_{x \star_{\lambda} y} e .$$

Lemma 2.13.

$$\mathcal{C}(x, y, z; \lambda) - \mathcal{C}(y, x, z; \lambda) = [x, y] \star_{\lambda} z,$$

$$\mathcal{E}(x, y; \lambda) = 0.$$

Proof. (2.11): Put $z' = \mathcal{I}_{\lambda}^{-1}(z)$.

$$C(x, y, z; \lambda) = \mathcal{I}_{\lambda}^{-1} \left((E - e\lambda) \star \nabla_{x} (y \star z') - y \star \nabla_{x} ((E - \lambda e) \star z') - \lambda y \star \nabla_{x \star z'} e \right)$$

$$= \mathcal{I}_{\lambda}^{-1} \left(\nabla_{x} (y \star z') - y \star \nabla_{x} z' - \lambda (e \star \nabla_{x} (y \star z') - y \star \nabla_{x} (e \star z') + y \star \nabla_{x \star z'} e) \right).$$

So

$$\mathcal{C}(x, y, z; \lambda) - \mathcal{C}(y, x, z; \lambda) = \mathcal{I}_{\lambda}^{-1} \left(\nabla_{x} (y \star z') - y \star \nabla_{x} z' - \nabla_{y} (x \star z') + x \star \nabla_{y} z' - \lambda (e \star \nabla_{x} (y \star z') - y \star \nabla_{x} (e \star z') + y \star \nabla_{x \star z'} e - (x \leftrightarrow y)) \right)$$

$$= \mathcal{I}_{\lambda}^{-1} ([x, y] \star z' - \lambda e \star [x, y] \star z') = \mathcal{I}_{\lambda}^{-1} ([x, y] \star z) = [x, y] \star_{\lambda} z.$$

In the last line, we used (ASS1) and (2.9).

(2.12): put $y' = \mathcal{I}_{\lambda}^{-1}(y)$. Then

$$\mathcal{E}(x, y; \lambda) = \mathcal{I}_{\lambda}^{-1} \big((E - \lambda e) \star \nabla_{x} (e \star y') - e \star \nabla_{x} (E - \lambda e) \star y' - (E - \lambda e) \star \nabla_{x \star y'} e$$

$$+ x \star \nabla_{(E - \lambda e) \star y'} e + e \star x \star y' + \lambda x \star \nabla_{e \star y'} e - \lambda e \star \nabla_{x \star y'} e \big)$$

$$= \mathcal{I}_{\lambda}^{-1} \big(\nabla_{x} (e \star y') - e \star \nabla_{x} y' - \nabla_{x \star y'} e + x \star \nabla_{y'} e + e \star x \star y' \big)$$

$$= 0 ,$$

because of (2.8).

The commutativity and the associativity of \star_{λ} , and the torsion freeness of $\nabla^{[\lambda,\nu]}$ follow from those of \star , ∇ . As for the flatness of $\nabla^{[\lambda,\nu]}$,

$$\nabla_{x}^{[\lambda,\nu]} \nabla_{y}^{[\lambda,\nu]} z - \nabla_{y}^{[\lambda,\nu]} \nabla_{x}^{[\lambda,\nu]} z$$

$$= \nabla_{x} \nabla_{y} z - \nabla_{y} \nabla_{x} z + (\nu \operatorname{Id} + \lambda \mathcal{Q}) (\mathcal{C}(x, y, z; \lambda) - \mathcal{C}(y, x, z; \lambda))$$

$$= \nabla_{[x,y]} z + (\nu \operatorname{Id} + \lambda \mathcal{Q}) ([x, y] \star_{\lambda} z) = \nabla_{[x,y]}^{[\lambda,\mu]} z.$$

Here we used (ASS4), the flatness of ∇ and (2.11).

Next let us check the conditions (ASS1)-(ASS4). (ASS1) follows from (2.11):

$$\nabla_x^{[\lambda,\nu]}(y\star_{\lambda}z) - y\star_{\lambda}\nabla_x^{[\lambda,\nu]}z - (x\leftrightarrow y) = \mathcal{C}(x,y,z;\lambda) - \mathcal{C}(y,x,z;\lambda) = [x,y]\star_{\lambda}z.$$

(ASS2): put $x' = \mathcal{I}_{\lambda}^{-1}(x)$ and $y' = \mathcal{I}_{\lambda}^{-1}(y)$. Then by (ASS2) and (2.7),

$$[e, x] = [e, (E - \lambda e) \star x'] = (E - \lambda e) \star [e, x'] + \lambda e \star e \star x'.$$

Therefore

$$[e, x \star_{\lambda} y] - [e, x] \star_{\lambda} y - x \star_{\lambda} [e, y] + e \star_{\lambda} x \star_{\lambda} y$$
$$= (E - \lambda e) \star ([e, x' \star y'] - [e, x'] \star y' - x' \star [e, y'] + e \star x' \star y') = 0.$$

(ASS3): Given that $E - \lambda e$ is a unit of \star_{λ} , we have

$$\nabla_x^{[\lambda,\nu]}(E-\lambda e) = \nabla_x(E-\lambda e) + \nu x + \lambda \nabla_x e = (r+\nu)x$$
.

(ASS4):

$$\nabla_{x}^{[\lambda,\nu]} \nabla_{y}^{[\lambda,\nu]} e - \nabla_{\nabla_{x}^{[\lambda,\nu]} y}^{[\lambda,\nu]} e + \nabla_{x\star_{\lambda} y}^{[\lambda,\nu]} e = \nabla_{x} \nabla_{y} e - \nabla_{\nabla_{x} y} e + \nabla_{x\star_{y}} e + (\nu \operatorname{Id} + \lambda \mathcal{Q}) (\mathcal{E}(x,y;\lambda)) + \mathcal{Q}(x\star_{\lambda} y - x\star_{y} - \lambda x\star_{y} \star_{\lambda} e) = 0.$$

This completes the proof of Proposition 2.12.

The next lemma shows how local flat coordinates of ∇ and the $\nabla^{[0,-1]}$ are related.

Lemma 2.14. Let (∇, \star, e) be an almost Saito structure with parameter $r \in \mathbb{C}$ on a manifold N. Let u^1, \ldots, u^n be local flat coordinates with respect to ∇ and let $e^i = e(u^i)$. If (∇, e) is regular, e^1, \ldots, e^n are flat local coordinates with respect to $\nabla^{[0,-1]}$.

Proof. First let us show that e^1, \ldots, e^n are local coordinates. By the definition of e^i , the vector field e is written as

$$e = \sum_{i=1}^{n} e^{i} \partial_{u^{i}} .$$

So the matrix representation of $Q = \nabla e$ with respect to $(\partial_{u^1}, \dots, \partial_{u^n})$ is given by

$$\left(\frac{\partial e^j}{\partial u^k}\right)_{i,k} .$$

Then by the regularity of (∇, e) , this matrix is invertible. This implies that e^1, \ldots, e^n are local coordinates.

The above matrix representation also implies that $\mathcal{Q}(\partial_{e^i}) = \partial_{u^i}$. Therefore by (ASS4),

$$\nabla_x \, \partial_{e^i} = \mathcal{Q}^{-1}(\nabla_x \mathcal{Q}(\partial_{e^i})) + x \star \partial_{e^i} = \mathcal{Q}^{-1}(\nabla_x \, \partial_{u^i}) + x \star \partial_{e^i} = x \star \partial_{e^i} \quad (x \in \mathcal{T}_N) .$$
Thus $\nabla_x^{[0,-1]} \partial_{e^i} = 0$.

3. Almost duality for the Saito structure

In §3.1, we show that one can construct a two-parameter family of almost Saito structures from a given Saito structure. In §3.2, we also show that one can construct a Saito structure if given an almost Saito structure. In §3.3, we explain that these constructions can be seen as inverse operations. We call this phenomenon the almost duality for the Saito structure.

In fact, a Saito structure is always accompanied with a one-parameter family (Remark 2.2) while an almost Saito structure is accompanied with a two-parameter family (§2.4). How they correspond via these constructions is mentioned in Remark 3.10.

3.1. From a Saito structure to an almost Saito structure. Let $(\nabla, *, E)$ be a Saito structure on a manifold M and let e be its unit. Take $\lambda \in \mathbb{C}$ and define $\mathcal{U}_{\lambda} \in \operatorname{Hom}_{\mathcal{O}_M}(\mathcal{T}_M, \mathcal{T}_M)$ by $\mathcal{U}_{\lambda}(x) = (E - \lambda e) * x$. Let

$$M_{\lambda} = \{ p \in M \mid U_{\lambda} : T_pM \to T_pM \text{ has rank dim } M \}.$$

Here U_{λ} is the endomorphism of TM corresponding to \mathcal{U}_{λ} . We put the assumption that $M_{\lambda} \subset M$ is not empty.

Proposition 3.1. Choose $r \in \mathbb{C}$ and define a multiplication \star_{λ} and a connection $\nabla^{(\lambda,r)}$ on TM_{λ} by

(3.1)
$$x \star_{\lambda} y = \mathcal{U}_{\lambda}^{-1}(x * y) ,$$
$$\nabla_{x}^{(\lambda,r)} y = \nabla_{x} y + r x \star_{\lambda} y - \nabla_{x \star_{\lambda} y} E .$$

Then

- (1) \star_{λ} is commutative and associative with the unit $E \lambda e$.
- (2) $\nabla^{(\lambda,r)}$ is torsion free and flat.
- (3) $(\nabla^{(\lambda,r)}, \star_{\lambda}, e)$ is an almost Saito structure on M_{λ} with parameter r.
- (4) Define $\mathcal{P}_{\lambda} \in \text{Hom}_{\mathcal{O}_{M_{\lambda}}}(\mathcal{T}_{M_{\lambda}}, \mathcal{T}_{M_{\lambda}})$ by $\mathcal{P}_{\lambda}(x) = e \star_{\lambda} x$. Then for $x, y \in \mathcal{T}_{M_{\lambda}}$,

$$\mathcal{P}_{\lambda}^{-1}(x \star_{\lambda} y) = x * y , \quad \boldsymbol{\nabla}_{x}^{(\lambda,r)} y - \boldsymbol{\nabla}_{x*y}^{(\lambda,r)} e = \nabla_{x} y .$$

Remark 3.2. The appearance of the two parameters in Proposition 3.1 is due to the existence of the two-parameter family explained in §2.4. If one considers the almost Saito structure obtained from $(\nabla^{(0,0)}, \star_0, e)$ by twisting by λ and shifting by ν as in Proposition 2.12, then it is nothing but $(\nabla^{(\lambda,\nu)}, \star_{\lambda}, e)$ in Proposition 3.1. (However, in this way, $(\nabla^{(\lambda,\nu)}, \star_{\lambda}, e)$ is given on $M_0 \cap (M_0)^{\lambda}$ which may be smaller than M_{λ} .)

Remark 3.3. The 2-parameter family of connections $\nabla^{(\lambda,r)}$ had been considered before in the case of the Frobenius manifolds under the name of the second structure connection [7, §9.2]. The flatness and torsion freeness are proved in Theorem 9.4 of *loc.cit*. and the condition (ASS3) can be derived easily from formula (9.12) in *loc.cit*..

To prove Proposition 3.1, we first prove a technical lemma. Set

$$\mathcal{C}_{2}(x, y, z; \lambda) = \nabla_{x}(y \star_{\lambda} z) - y \star_{\lambda} \nabla_{x} z + y \star_{\lambda} \nabla_{x \star_{\lambda} z} E ,$$

$$\mathcal{E}_{2}(x, y; \lambda) = \nabla_{x}(e \star_{\lambda} y) - e \star_{\lambda} \nabla_{x} y - x \star_{\lambda} \nabla_{e \star_{\lambda} y} E + e \star_{\lambda} \nabla_{x \star_{\lambda} y} E + e \star_{\lambda} x \star_{\lambda} y .$$

Lemma 3.4.

(3.2)
$$\mathcal{C}_2(x, y, z; \lambda) - \mathcal{C}_2(y, x, z; \lambda) = [x, y] \star_{\lambda} z,$$

(3.3)
$$\mathcal{E}_2(x,y;\lambda) = 0.$$

Proof. (3.2): put $z' = \mathcal{U}_{\lambda}^{-1}(z)$. Then

$$C_{2}(x, y, z; \lambda) - C_{2}(y, x, z; \lambda)$$

$$= \mathcal{U}_{\lambda}^{-1} \left((E - \lambda e) * \nabla_{x}(y * z') - y * \nabla_{x}((E - \lambda e) * z') + y * \nabla_{x*z'} E \right) - (x \leftrightarrow y)$$

$$= \mathcal{U}_{\lambda}^{-1} \left(E * \nabla_{x}(y * z') - y * \nabla_{x}(E * z') + y * \nabla_{x*z'} E - \lambda(\nabla_{x}(y * z') - y * \nabla_{x} z') \right) - (x \leftrightarrow y)$$

$$= \mathcal{U}_{\lambda}^{-1} (E * [x, y] * z' - \lambda[x, y] * z') = [x, y] \star_{\lambda} z.$$

In passing to the last line, we used (SS1) and (2.4).

(3.3): put $y' = \mathcal{U}_{\lambda}^{-1}(y)$. Then

$$\mathcal{E}_{2}(x,y;\lambda) = \mathcal{U}_{\lambda}^{-1} \big((E - \lambda e) * \nabla_{x} y' - x * \nabla_{y'} E - \nabla_{x} (E - \lambda e) * y' + \nabla_{x*y'} E + x * y' \big)$$

= $\mathcal{U}_{\lambda}^{-1} (E * \nabla_{x} y' - x * \nabla_{y'} E - \nabla_{x} (E * y') + \nabla_{x*y'} E + x * y') = 0$,

due to
$$(2.3)$$
.

Proof of Proposition 3.1. (1) The commutativity and associativity of * imply the same properties for \star_{λ} . By definition of \star_{λ} , it is clear that $E - \lambda e$ is its unit.

(2) The torsion freeness of ∇ and the commutativity of \star_{λ} imply the torsion freeness of $\nabla^{(\lambda,r)}$. As for the flatness, we have

$$\nabla_{x}^{(\lambda,r)} \nabla_{y}^{(\lambda,r)} z - \nabla_{y}^{(\lambda,r)} \nabla_{x}^{(\lambda,r)} z$$

$$= \nabla_{x} \nabla_{y} z - \nabla_{y} \nabla_{x} z + (r \operatorname{Id} - \mathcal{W}) (\mathcal{C}_{2}(x, y, z; \lambda) - \mathcal{C}_{2}(y, x, z; \lambda))$$

$$= \nabla_{[x,y]} z + (r \operatorname{Id} - \mathcal{W}) ([x, y] \star_{\lambda} z) = \nabla_{[x,y]}^{(\lambda,r)} z.$$

Here $W(x) = \nabla_x E$. We used (SS4), the flatness of ∇ and (3.2).

(3) Let us check the conditions (ASS1)–(ASS4).

Condition (ASS1):

$$\nabla_x^{(\lambda,r)}(y \star_{\lambda} z) - y \star_{\lambda} \nabla_x^{(\lambda,r)} z - \nabla_y^{(\lambda,r)}(x \star_{\lambda} z) + x \star_{\lambda} \nabla_y^{(\lambda,r)} z$$

$$= C_2(x, y, z; \lambda) - C_2(y, x, z; \lambda)$$

$$= [x, y] \star_{\lambda} z.$$

Condition (ASS2): put $x' = \mathcal{U}_{\lambda}^{-1}(x)$ and $y' = \mathcal{U}_{\lambda}^{-1}(y)$. By (2.1),

$$[e, x] = [e, (E - \lambda e) * x'] = (E - \lambda e) * [e, x'] + [e, E] * x'$$
.

Therefore

$$[e, x \star_{\lambda} y] - [e, x] \star_{\lambda} y - x \star_{\lambda} [e, y] + e \star_{\lambda} x \star_{\lambda} y$$

= $(E - \lambda e) * ([e, x' * y'] - [e, x'] * y' - x' * [e, y']) = 0$.

In the last line, we used (2.1) again.

Condition (ASS3): given that $E - \lambda e$ is a unit of \star_{λ} ,

$$\nabla_x^{(\lambda,r)} (E - \lambda e) = \nabla_x E + rx - \nabla_x E = rx$$
.

Condition (ASS4):

$$\nabla_x^{(\lambda,r)} \nabla_y^{(\lambda,r)} e - \nabla_{\nabla_x^{(\lambda,r)} y}^{(\lambda,r)} e + \nabla_{x \star_{\lambda} y}^{(\lambda,r)} e = (r \operatorname{Id} - \mathcal{W}) (\mathcal{E}_2(x,y;\lambda)) \stackrel{(3.3)}{=} 0.$$

(4) Multiplication: notice that $\mathcal{P}_{\lambda} = \mathcal{U}_{\lambda}^{-1}$. Therefore

$$x \star_{\lambda} \mathcal{P}_{\lambda}^{-1}(y) = x * (\mathcal{U}_{\lambda}^{-1} \circ \mathcal{P}_{\lambda}^{-1})(y) = x * y$$
.

Connection: since $x \star_{\lambda} y = x * y \star_{\lambda} e$, it is easy to see that

$$\nabla_x^{(\lambda,r)} y - \nabla_{x*y}^{(\lambda,r)} e = \nabla_x y + rx \star_{\lambda} y - \nabla_{x\star_{\lambda}y} E - \nabla_{x*y} e - rx * y \star_{\lambda} e + \nabla_{x*y\star_{\lambda}e} E$$
$$= \nabla_x y .$$

This completes the proof of Proposition 3.1.

Definition 3.5. We call $(\nabla^{(0,r)}, \star_0, e)$ in Proposition 3.1 the dual almost Saito structure of $(\nabla, *, E)$ with parameter $r \in \mathbb{C}$.

Remark 3.6. It would be simpler if we define $(\nabla^{(0,0)}, \star_0, e)$ as the dual almost Saito structure of $(\nabla, *, E)$. However, we adopt Definition 3.5 for the sake of application to the complex reflection groups.

3.2. From an almost Saito structure to a Saito structure. Let (∇, \star, e) be an almost Saito structure with parameter $r \in \mathbb{C}$ on a manifold N and let E be its unit. Define an endomorphism $\mathcal{P} \in \text{Hom}_{\mathcal{O}_N}(\mathcal{T}_N, \mathcal{T}_N)$ by $\mathcal{P}(x) = e \star x$. Let

$$N_0 = \{ p \in N \mid P : T_p N \to T_p N \text{ has rank dim } N \}$$
.

Here P is the endomorphism of TN corresponding to \mathcal{P} . We put the assumption that N_0 is not empty.

Proposition 3.7. Define a multiplication * and a connection ∇ on TN_0 by

(3.4)
$$x * y = x \star \mathcal{P}^{-1}(y) = \mathcal{P}^{-1}(x) \star y ,$$
$$\nabla_x y = \nabla_x y - \nabla_{x*y} e .$$

Then

- (1) * is commutative and associative with the unit e.
- (2) ∇ is torsion free and flat.
- (3) $(\nabla, *, E)$ is a Saito structure on N_0 .
- (4) Define $\mathcal{U} \in \operatorname{Hom}_{\mathcal{O}_{N_0}}(\mathcal{T}_{N_0}, \mathcal{T}_{N_0})$ by $\mathcal{U}(x) = E * x$. Then for $x, y \in \mathcal{T}_{N_0}$,

$$x * \mathcal{U}^{-1}(y) = \mathcal{U}^{-1}(x) * y = x \star y$$
,

$$\nabla_x y + rx \star y - \nabla_{x \star y} E = \nabla_x y .$$

Before proving Proposition 3.7, we show the next auxiliary lemma. Let $x, y, z \in \mathcal{T}_{N_0}$. We set

$$\mathcal{C}_3(x,y,z) = \nabla_x(y*z) - y*\nabla_x z + y*\nabla_{x*z} e ,$$

$$\mathcal{E}_3(x,y) = \nabla_x(E*y) - E*\nabla_x y + E*\nabla_{x*y} e - x*\nabla_{E*y} e - x*y .$$

Lemma 3.8.

(3.5)
$$C_3(x, y, z) - C_3(y, x, z) = [x, y] * z$$
,

$$\mathcal{E}_3(x,y) = 0.$$

Proof. (3.6): put $z' = \mathcal{P}^{-1}(z)$. Then

$$C_{3}(x, y, z) - C_{3}(y, x, z)$$

$$= \mathcal{P}^{-1} \left(e \star \nabla_{x} (y \star z') - y \star \nabla_{x} (e \star z') + y \star \nabla_{x \star z'} e \right) - (x \leftrightarrow y)$$

$$\stackrel{(2.9)}{=} \mathcal{P}^{-1} (e \star [x, y] \star z') = [x, y] * z$$

(3.6): put $y' = \mathcal{P}^{-1}(y)$. Then, given that E is a unit of \star ,

$$\mathcal{E}_3(x,y) = \mathcal{P}^{-1} \left(e \star \nabla_x y' - \nabla_x (e \star y') + \nabla_{x \star y'} e - x \star \nabla_{y'} e - e \star x \star y' \right)$$

$$\stackrel{(2.8)}{=} 0.$$

Proof of Proposition 3.7. (1) The commutativity and associativity of \star imply the same property for *. By the definition, it is clear that e is a unit of *.

(2) The torsion freeness of ∇ and the commutativity of * imply the torsion freeness of ∇ . The flatness is shown as follows.

$$\nabla_x \nabla_y z - \nabla_y \nabla_x z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \mathcal{Q} (\mathcal{C}_3(x, y, z) - \mathcal{C}_3(y, x, z))$$
$$= \nabla_{[x,y]} z - \mathcal{Q}([x, y] * z) = \nabla_{[x,y]} z.$$

Here $Q(x) = \nabla_x e$. We used (ASS4), the flatness of ∇ and (3.5).

(3) Let us check the conditions (SS1)-(SS4).

Condition (SS1):

$$\nabla_x (y * z) - y * \nabla_x z - \nabla_y (x * z) + x * \nabla_y z - [x, y] * z$$

$$= \mathcal{C}_3(x, y, z) - \mathcal{C}_3(y, x, z) - [x, y] * z \stackrel{\text{(3.5)}}{=} 0 .$$

Condition (SS2): putting $x' = \mathcal{P}^{-1}(x)$ and $y' = \mathcal{P}^{-1}(y)$, we have

$$\begin{split} [E,x*y] - [E,x]*y - x*[E,y] - x*y \\ &= [E,e\star x'\star y'] - [E,e\star x']\star y' - x'\star [E,e\star y'] - e\star x'\star y' \\ \stackrel{(2.7)}{=} e\star \left([E,x'\star y'] - [E,x']\star y' - x'\star [E,y']\right) \stackrel{(2.6)}{=} 0 \ . \end{split}$$

Condition (SS3): given that e is a unit of *,

$$\nabla_x e = \nabla_x e - \nabla_{x*e} e = 0$$
.

Condition (SS4):

$$\nabla_x \nabla_y E - \nabla_{\nabla_x y} \stackrel{\text{(ASS4)}}{=} - \mathcal{Q}(\mathcal{E}_3(x,y)) \stackrel{(3.6)}{=} 0$$
.

(4) Multiplication: since $\mathcal{U} = \mathcal{P}^{-1}$,

$$x * \mathcal{U}^{-1}(y) = x \star (\mathcal{P}^{-1} \circ \mathcal{U}^{-1})(y) = x \star y.$$

Connection: since $x * y = x \star \mathcal{P}^{-1}(y) = x \star (E * y) = x \star y * E$, it is easy to see that

$$\nabla_x y + rx \star y - \nabla_{x \star y} E = \nabla_x y - \nabla_{x \star y} e + rx \star y - \nabla_{x \star y} E + \nabla_{x \star y \star E} e$$
$$= \nabla_x y + rx \star y - \nabla_{x \star y} E \stackrel{(ASS3)}{=} \nabla_x y .$$

This completes the proof of Proposition 3.7.

Definition 3.9. We call $(\nabla, *, E)$ in Proposition 3.7 the dual Saito structure of (∇, \star, e) .

Remark 3.10. Let (∇, \star, e) be an almost Saito structure and denote $(\nabla, *, E)$ its dual Saito structure. If one considers a member of the two-parameter family of almost Saito structures (§2.4), $(\nabla^{[\lambda,\nu]}, \star_{\lambda}, e)$, then its dual Saito structure is given by $(\nabla, *, E - \lambda e)$.

Similarly, for the almost Saito structure $(\nabla^{(\lambda,r)}, \star_{\lambda}, e)$ constructed from a Saito structure $(\nabla, *, E)$ as in Proposition 3.1, its dual Saito structure is $(\nabla, *, E - \lambda e)$.

3.3. Almost duality for the Saito structure. Propositions 3.1-(4) and 3.7-(4) imply the duality between the Saito structure and the almost Saito structure. The results in §3.1 and §3.2 can be summarized as follows.

Theorem 3.11. (i) Let $(\nabla, *, E)$ be a Saito structure with a unit e and let $(\nabla^{(0,r)}, \star_0, e)$ be its dual almost Saito structure with parameter $r \in \mathbb{C}$. Then the dual Saito structure of $(\nabla^{(0,r)}, \star_0, e)$ is $(\nabla, *, E)$.

(ii) Let (∇, \star, e) be an almost Saito structure with parameter $r \in \mathbb{C}$ and a unit E, and denote $(\nabla, *, E)$ its dual Saito structure. Then the dual almost Saito structure of $(\nabla, *, E)$ with parameter r is (∇, \star, e) .

Remark 3.12. In §2.1, §2.2, we defined the equivalence relations for the Saito structure and the almost Saito structure. Notice that the constructions in §3.1, §3.2 preserve those equivalence relations.

- 4. On the almost duality of the Frobenius manifold and on the bi-flat ${\it F}$ -manifold
- 4.1. Relationship to the almost duality of Frobenius manifolds [6]. A Frobenius structure [4] on a manifold M of charge $D \in \mathbb{C}$ is a Saito structure $(\nabla, *, E)$ on M together with a nondegenerate symmetric bilinear form η on TM satisfying

$$(4.1) x(\eta(y,z)) = \eta(\nabla_x y, z) + \eta(y, \nabla_x z) (x, y, z \in \mathcal{T}_M),$$

(4.2)
$$\eta(x*y,z) = \eta(x,y*z) \quad (x,y,z \in \mathcal{T}_M) ,$$

(4.3)
$$E\eta(x,y) - \eta([E,x],y) - \eta(x,[E,y]) = (2-D)\eta(x,y) \quad (x,y \in \mathcal{T}_M).$$

An almost Frobenius manifold [6, §3] of charge $D \in \mathbb{C}$ is an almost Saito structure (∇, \star, e) with parameter $r = \frac{1-D}{2}$ on a manifold N together with a nondegenerate symmetric bilinear form g on TN satisfying

$$(4.4) x(g(y,z)) = g(\nabla_x y, z) + g(y, \nabla_x z) (x, y, z \in \mathcal{T}_N),$$

$$(4.5) g(x \star y, z) = g(x, y \star z) \quad (x, y, z \in \mathcal{T}_N) ,$$

$$(4.6) e g(x,y) - g([e,x],y) - g(x,[e,y]) + g(e \star x,y) = 0 (x,y \in \mathcal{T}_N).$$

Here we stated the definition in a slightly different way from Dubrovin's. His axiom **AFM3** is equivalent to (**ASS4**) (see Lemma 2.14) and axiom **AFM2** is equivalent to (**ASS1**)–(**ASS3**). (Although he excluded the case D = 1 in his definition, it is not necessary if stated in this way.)

Notice that in the definition of the Frobenius structure, ∇ is the Levi–Civita connection of η and that in the definition of the almost Frobenius manifold, ∇ is the Levi–Civita connection of g.

Dubrovin showed the following duality between the Frobenius structure and the almost Frobenius manifold [6]: if $(\eta, *, E)$ is a Frobenius structure on M of charge D, then

$$g(x,y) = \eta(x,\mathcal{U}_0^{-1}(y)) \ , \quad x \star y = x * \mathcal{U}_0^{-1}(y)$$

and the unit e of * make an almost Frobenius structure of charge D on M_0 . Here $\mathcal{U}_0 = E*$ and $M_0 \subset M$ are the same as §3.1. g is called the intersection form of $(\eta, *, E)$. Dubrovin gave a formula for the Levi–Civita connection of g in [6, Proposition 2.3]. The connection $\nabla^{(\lambda,r)}$ (with $\lambda = 0$) given in §3.1 is nothing but this Levi–Civita connection (see (5.8)). He also showed that if (g, \star, e) is an almost Frobenius manifold structure on N of charge D, then

$$\eta(x,y) = g(x,\mathcal{P}^{-1}(y)) \ , \quad x*y = x \star \mathcal{P}^{-1}(y)$$

and the unit E of \star make a Frobenius structure on $N_0 \subset N$ of charge D. Here $\mathcal{P} = e \star$ and $N_0 \subset N$ are the same as §3.2. It is not difficult to see that the almost duality of

the Frobenius manifold implies the almost duality between the underlying Saito structure and the almost Saito structure in the sense of Theorem 3.11.

Dubrovin also noted that given a Frobenius structure $(\eta, *, E)$ of charge D, then the metric η and the intersection form g form a flat pencil of metrics $g^* - \lambda \eta^*$ on the cotangent bundle T^*M_{λ} ($\lambda \in \mathbb{C}$) [6, Remark 1 in §2]. (Here * means taking the dual between the tangent bundle and the cotangent bundle.) In §3.1, we give a construction of a two-parameter family of almost Saito structures from a given Saito structure. This is nothing but a generalization of Dubrovin's flat pencil of metrics: in our notation, the Levi–Civita connection of the metric $(g^* - \lambda \eta^*)^*$ on TM_{λ} is given by $\nabla^{(\lambda,r)}$ in Proposition 3.1, with $r = \frac{1-D}{2}$.

4.2. Relationship to the bi-flat F-manifold. In [1], Arsie and Lorenzoni introduced the notion of bi-flat F-manifold, generalizing Dubrovin's flat pencils of metrics.

Definition 4.1. A bi-flat F-manifold $(M, \nabla, \nabla, *, *, *, e, E)$ consists of a manifold M and

- torsion-free flat connections ∇ , ∇ on TM,
- associative commutative multiplications * and * on TM with the units $e, E \in \Gamma(M, \mathcal{T}_M)$,

satisfying (SS1),(SS2),(SS3),(ASS1) and (ASS3) with r=0, together with the following two conditions.

- $\mathcal{U} \in \operatorname{Hom}_{\mathcal{O}_M}(\mathcal{T}_M, \mathcal{T}_M)$ given by $\mathcal{U}(x) = E * x$ is an isomorphism and E * x * y = x * y.
- " ∇ and ∇ are almost hydrodynamically equivalent", i.e.

$$\nabla_x(y*z) - \nabla_y(x*z) = \nabla_x(y*z) - \nabla_y(x*z) \quad (x,y,z \in \mathcal{T}_M)$$
.

The bi-flat F-manifold and the almost duality for Saito structure are equivalent notions. Precise statements are given in the next two lemmas.

Lemma 4.2. Let $(M, \nabla, \nabla, *, *, *, e, E)$ be a bi-flat F-manifold. Then $(\nabla, *, E)$ is a Saito structure on M and

(4.7)
$$\nabla_x y = \nabla_x y - \nabla_{x \star y} E .$$

In other words, $(\nabla, *, E)$ is a Saito structure on M and (∇, \star, e) is the dual almost Saito structure on M with parameter r = 0.

Proof. We write $E * x = \mathcal{U}(x)$ and $\nabla_x E = \mathcal{W}(x)$ $(x \in \mathcal{T}_M)$. Notice that the equations (2.1),(2.2),(2.3) (2.4) and (2.5) hold since they are derived from (SS1),(SS2),(SS3).

Let $x, y, z \in \mathcal{T}_M$. First we show (4.7). Let us put $z' = \mathcal{U}^{-1}(z)$. Substituting z by z' in (SS1) and subtracting it from (ASS1), we have

$$0 = (\nabla_x - \nabla_x)(y * z') - (\nabla_y - \nabla_y)(x * z')$$
$$+ \mathcal{U}^{-1}(x * \nabla_y z - y * \nabla_y z - x * E * \nabla_y z' + y * E * \nabla_x z')$$
$$= \mathcal{U}^{-1}(x * (\nabla_y z - \nabla_y z + \nabla_{y \star z} E) - (x \leftrightarrow y)).$$

In passing to the last line, we applied the almost hydrodynamically equivalent condition and (2.5). Since \mathcal{U} is assumed to be invertible on M, we obtain

(4.8)
$$x * (\nabla_y z - \nabla_y z + \nabla_{y \star z} E) = y * (\nabla_x z - \nabla_x z + \nabla_{x \star z} E).$$

Now, if we set y to E, the LHS of (4.8) becomes

$$x * (\nabla_E z - \nabla_E z + \nabla_z E) = x * ([E, z] - [E, z]) = 0.$$

Here we used the torsion freeness of ∇ , ∇ and $\nabla E = 0$. Therefore the RHS of (4.8) is zero when y = E. This implies (4.7) since the multiplication $E* = \mathcal{U}$ is invertible. (SS4): To show that $(\nabla, *, E)$ is a Saito structure on M, it is enough to show that (SS4)

holds. Expanding the flatness condition for ∇ by using (4.7), we have

$$0 = (\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]})(E * z)$$

$$= (\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]})(E * z)$$

$$- \nabla_x \nabla_{y*z} E + \nabla_y \nabla_{x*z} E$$

$$- \mathcal{W} \circ \mathcal{U}^{-1} (x * \nabla_y (E * z) - x * \nabla_{y*z} E - (x \leftrightarrow y)) - \mathcal{W}([x,y] * z)$$

$$\stackrel{(2.5)}{=} - \nabla_x \nabla_{y*z} E + \nabla_y \nabla_{x*z} E - \mathcal{W}(x * \nabla_y z - y * \nabla_x z - [x,y] * z) .$$

Here we used the flatness of ∇ . Now if we set y to e, this equation becomes

(4.9)
$$0 = -\nabla_x \nabla_z E + \nabla_e \nabla_{x*z} E - \mathcal{W}(x * [e, z] - \nabla_x z + [e, x] * z) .$$

Here we used the condition $\nabla e = 0$ and the torsion freeness of ∇ . Using the flatness of ∇ , we can write the second term in the RHS as follows.

$$\nabla_e \nabla_{x*z} E = \nabla_{x*z} \underbrace{\nabla_e E}_{=[e,E]=e} + \mathcal{W}([e,x*z]) = \mathcal{W}([e,x*z]) .$$

Here we used the torsion freeness of ∇ , (2.2) and $\nabla e = 0$. Substituting this into (4.9), we obtain

$$0 = -\nabla_x \nabla_z E + \mathcal{W}([e, x * z] - x * [e, z] + \nabla_x z - [e, x] * z)$$

$$\stackrel{(2.1)}{=} -\nabla_x \nabla_z E + \mathcal{W}(\nabla_x z) .$$

This equation is equivalent to (SS4).

The converse of Lemma 4.2 also holds.

Lemma 4.3. Let $(\nabla, *, E)$ be a Saito structure on M with a unit e. Then

$$(M_{\lambda}, \nabla, \boldsymbol{\nabla}^{(\lambda,0)}, *, \star_{\lambda}, e, E - \lambda e) \quad (\lambda \in \mathbb{C})$$

is a bi-flat F-manifold, where M_{λ} , $\nabla^{(\lambda,0)}$, \star_{λ} are the same as in §3.1.

Proof. It is enough to check that the connections ∇ and $\nabla^{(\lambda,0)}$ are almost hydrodynamically equivalent. This is immediate from (3.1):

$$\begin{split} \boldsymbol{\nabla}_{x}^{(\lambda,0)}(y*z) - \boldsymbol{\nabla}_{y}^{(\lambda,0)}(x*z) &= \nabla_{x}(y*z) - \nabla_{x\star_{\lambda}(y*z)} - (x \leftrightarrow y) \\ &= \nabla_{x}(y*z) - \nabla_{y}(x*z) - \nabla_{\mathcal{U}_{\lambda}^{-1}(x*y*z)}E + \nabla_{\mathcal{U}_{\lambda}^{-1}(y*x*z)}E \\ &= \nabla_{x}(y*z) - \nabla_{y}(x*z). \end{split}$$

5. Matrix representations

In this section, we describe the conditions for a Saito structure and an almost Saito structure by matrix representations.

Let x^{α} $(1 \leq \alpha \leq n)$ be local coordinates of M and denote $\partial_{x^{\alpha}} = \partial_{\alpha} \in \mathcal{T}_{M}$ $(1 \leq \alpha \leq n)$ the corresponding basis. The identity matrix and the zero matrix of order n are denoted I_{n} and O_{n} . For matrices A, B of order n, [A, B] = AB - BA.

5.1. Saito structure. Let $(\nabla, *, E = \sum_{\mu=1}^{n} E^{\mu} \partial_{\mu})$ be a Saito structure on M with a unit e. Let us write

$$E = \sum_{\mu=1}^{n} E^{\mu} \partial_{\mu} , \quad e = \sum_{\mu=1}^{n} e^{\mu} \partial_{\mu} , \quad \partial_{\alpha} * \partial_{\beta} = \sum_{\gamma=1}^{n} C_{\alpha\beta}^{\gamma} \partial_{\gamma} , \quad \nabla_{\alpha} (\partial_{\beta}) = \sum_{\gamma=1}^{n} \Gamma_{\alpha\beta}^{\gamma} \partial_{\gamma} .$$

Let $W, Q, C_{\alpha}, \Gamma_{\alpha}$ be matrices whose entries are given by

$$W^{\mu}{}_{\alpha} = \partial_{\alpha} E^{\mu} \;, \quad Q^{\mu}{}_{\alpha} = \partial_{\alpha} e^{\mu} \;, \quad (C_{\alpha})^{\gamma}{}_{\beta} = C^{\gamma}{}_{\alpha\beta} \;, \quad (\Gamma_{\alpha})^{\gamma}{}_{\beta} = \Gamma^{\gamma}{}_{\alpha\beta} \;.$$

Then the commutativity and associativity of * are expressed as follows.

$$C_{\alpha\beta}^{\gamma} = C_{\beta\alpha}^{\gamma} , \quad [C_{\alpha}, C_{\beta}] = O_n .$$

That e is a unit of * is expressed as

$$\sum_{\alpha=1}^{n} e^{\alpha} C_{\alpha} = I_{n} .$$

The torsion freeness and the flatness of ∇ are expressed as follows.

$$\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}$$
, $\partial_{\alpha}\Gamma_{\beta} - \partial_{\beta}\Gamma_{\alpha} + [\Gamma_{\alpha}, \Gamma_{\beta}] = O_n$.

The conditions (SS1)–(SS4) are expressed as follows.

(5.1)
$$\partial_{\alpha} C_{\beta} + [\Gamma_{\alpha}, C_{\beta}] = \partial_{\beta} C_{\alpha} + [\Gamma_{\beta}, C_{\alpha}].$$

(5.2)
$$\partial_{\alpha}(E \cdot C) + [\Gamma_{\alpha}, E \cdot C] = [W + E \cdot \Gamma, C_{\alpha}] + C_{\alpha}.$$

$$(5.3) Q + e \cdot \Gamma = O_n .$$

(5.4)
$$\partial_{\alpha}(W + E \cdot \Gamma) + [\Gamma_{\alpha}, W + E \cdot \Gamma] = O_n.$$

Here

$$E \cdot C = \sum_{\mu=1}^{n} E^{\mu} C_{\mu} , \quad E \cdot \Gamma = \sum_{\mu=1}^{n} E^{\mu} \Gamma_{\mu} , \quad e \cdot \Gamma = \sum_{\mu=1}^{n} e^{\mu} \Gamma_{\mu} .$$

Together with (5.1), (5.2) can be simplified to

(5.5)
$$E(C_{\alpha\beta}^{\gamma}) = -\sum_{\mu=1}^{n} W^{\mu}{}_{\alpha} C_{\mu\beta}^{\gamma} + [W, C_{\alpha}]^{\gamma}{}_{\beta} + C_{\alpha\beta}^{\gamma}.$$

Together with the flatness of ∇ , (5.4) can be simplified to

(5.6)
$$E(\Gamma_{\alpha\beta}^{\gamma}) = -\sum_{\mu=1}^{n} W^{\mu}{}_{\beta} \Gamma_{\mu\beta}^{\gamma} + [W, \Gamma_{\alpha}]^{\gamma}{}_{\beta} - \partial_{\alpha} W^{\gamma}{}_{\beta}.$$

If one constructs an almost Saito structure $(\nabla^{(\lambda,r)}, \star_{\lambda}, e)$ by (3.1), then the new multiplication \star_{λ} is expressed as

(5.7)
$$\partial_{\alpha} \star_{\lambda} \partial_{\beta} = \sum_{\gamma=1}^{n} B_{\alpha\beta}^{\gamma}(\lambda) \partial_{\gamma}$$
, where $B_{\alpha}(\lambda) = (B_{\alpha\beta}^{\gamma}(\lambda)) = C_{\alpha}(E \cdot C - \lambda I_{n})^{-1}$.

The new connection $\nabla^{(\lambda,r)}$ is given by

(5.8)
$$\boldsymbol{\nabla}_{\alpha}^{(\lambda,r)}(\partial_{\beta}) = \sum_{\gamma=1}^{n} \left(\Gamma_{\alpha} + (rI_{n} - W - E \cdot \Gamma) B_{\alpha}(\lambda) \right)_{\beta}^{\gamma} \partial_{\gamma} .$$

5.2. Almost Saito structure. Next consider an almost Saito structure (∇, \star, e) with a unit E. Let us write

$$e = \sum_{\mu=1}^{n} e^{\mu} \partial_{\mu} , \quad E = \sum_{\mu=1}^{n} E^{\mu} \partial_{\mu} \quad \partial_{\alpha} \star \partial_{\beta} = \sum_{\gamma=1}^{n} B_{\alpha\beta}^{\gamma} \partial_{\gamma} , \quad \nabla_{\alpha}(\partial_{\beta}) = \sum_{\gamma=1}^{n} \Omega_{\alpha\beta}^{\gamma} \partial_{\gamma} .$$

Let $Q, W, B_{\alpha}, \Omega_{\alpha}$ be matrices whose entries are given by

$$Q^{\mu}_{\alpha} = \partial_{\alpha} e^{\mu}, \quad W^{\mu}_{\alpha} = \partial_{\alpha} E^{\mu}, \quad (B_{\alpha})^{\gamma}_{\beta} = B^{\gamma}_{\alpha\beta}, \quad (\Omega_{\alpha})^{\gamma}_{\beta} = \Omega^{\gamma}_{\alpha\beta}.$$

Then the commutativity and associativity of \star are expressed as follows.

$$B_{\alpha\beta}^{\gamma} = B_{\beta\alpha}^{\gamma} , \quad [B_{\alpha}, B_{\beta}] = O_n .$$

That E is a unit of \star is expressed as

$$\sum_{\alpha=1}^{n} E^{\alpha} B_{\alpha} = I_{n} .$$

The torsion freeness and the flatness of ∇ are expressed as follows.

$$\Omega_{\alpha\beta}^{\gamma} = \Omega_{\beta\alpha}^{\gamma} , \quad \partial_{\alpha}\Omega_{\beta} - \partial_{\beta}\Omega_{\alpha} + [\Omega_{\alpha}, \Omega_{\beta}] = O_n .$$

The conditions (ASS1)–(ASS4) are expressed as follows.

(5.9)
$$\partial_{\alpha} B_{\beta} + [\Omega_{\alpha}, B_{\beta}] = \partial_{\beta} B_{\alpha} + [\Omega_{\beta}, B_{\alpha}].$$

$$(5.10) \partial_{\alpha}(e \cdot B) + [\Omega_{\alpha}, e \cdot B] = [Q + e \cdot \Omega, B_{\alpha}] - B_{\alpha}(e \cdot B) .$$

$$(5.11) W + E \cdot \Omega = rI_n .$$

$$(5.12) \partial_{\alpha}(Q + e \cdot \Omega) + [\Omega_{\alpha}, Q + e \cdot \Omega] + (Q + e \cdot \Omega)B_{\alpha} = O_n.$$

Here

$$e \cdot B = \sum_{\mu=1}^{n} e^{\mu} B_{\mu} , \quad e \cdot \Omega = \sum_{\mu=1}^{n} e^{\mu} \Omega_{\mu} , \quad E \cdot \Omega = \sum_{\mu=1}^{n} E^{\mu} \Omega_{\mu} .$$

Together with (5.9), (5.10) can be simplified as follows.

(5.13)
$$e(B_{\alpha\beta}^{\gamma}) = -\sum_{\mu=1}^{n} Q^{\mu}{}_{\alpha} B_{\beta\mu}^{\gamma} + [Q, B_{\alpha}]^{\gamma}{}_{\beta} - (B_{\alpha}(e \cdot B))^{\gamma}{}_{\beta}.$$

Together with the flatness of ∇ , (5.12) can also be simplified as follows.

$$(5.14) e(\Omega_{\alpha\beta}^{\gamma}) = -\sum_{\mu=1}^{n} Q^{\mu}{}_{\alpha} \Omega_{\mu\beta}^{\gamma} + [Q, \Omega_{\alpha}]^{\gamma}{}_{\beta} - \partial_{\alpha} Q^{\gamma}{}_{\beta} - ((Q + e \cdot \Omega)B_{\alpha})^{\gamma}{}_{\beta}.$$

If one constructs a Saito structure $(\nabla, *, E)$ by (3.4), then the new multiplication * is expressed as

$$\partial_{\alpha} * \partial_{\beta} = \sum_{\gamma=1}^{n} C_{\alpha\beta}^{\gamma} \partial_{\gamma}$$
, where $C_{\alpha} = (C_{\alpha\beta}^{\gamma}) = B_{\alpha} (e \cdot B)^{-1}$.

The new connection ∇ is given by

$$\nabla_{\alpha}(\partial_{\beta}) = \sum_{\alpha=1}^{n} \left(\Omega_{\alpha} - (Q + e \cdot \Omega) C_{\alpha} \right)^{\gamma}{}_{\beta} \partial_{\gamma} .$$

6. The orbit spaces of complex reflection groups

6.1. Orbit spaces of finite complex reflection groups. Let n be a positive integer and let $V = \mathbb{C}^n$ be equipped with the standard Hermitian metric. The standard coordinates of V are denoted u^1, \ldots, u^n and the ring of polynomials on V is denoted $\mathbb{C}[V] = \mathbb{C}[u^1, \ldots, u^n] = \mathbb{C}[u]$.

Let $G \subset GL(V)$ be a (not necessarily irreducible) finite complex reflection group. Denote $\mathbb{C}[V]^G$ the ring of G-invariant polynomials and let $M = \operatorname{Spec} \mathbb{C}[V]^G$ be the space of G-orbits. It is well-known that $\mathbb{C}[V]^G$ is a polynomial ring generated by $n = \dim V$ homogeneous polynomials. Such generators are called a set of basic invariants. Fix a set of basic invariants $x^1, \ldots, x^n \in \mathbb{C}[V]^G$. So $\mathbb{C}[V]^G = \mathbb{C}[x^1, \ldots, x^n] = \mathbb{C}[x]$. The degrees d_1, \ldots, d_n of x^1, \ldots, x^n are uniquely determined by G and called the degrees of G. Except for §9.3.1, we always assume that the degrees are in descending order:

$$d_1 \geq \ldots \geq d_n \geq 1$$
,

for any set of basic invariants x^1, \ldots, x^n .

We regard $\mathbb{C}[x]$ as a graded ring with the grading deg $x^{\alpha} = d_{\alpha}$ $(1 \leq \alpha \leq n)$. Note that for a homogeneous polynomial $f \in \mathbb{C}[u]$ or for a weighted homogeneous polynomial $f \in \mathbb{C}[x]$, it holds that

(6.1)
$$\sum_{\alpha=1}^{n} d_{\alpha} x^{\alpha} \frac{\partial f}{\partial x^{\alpha}} = \sum_{k=1}^{n} u^{k} \frac{\partial f}{\partial u_{k}} = (\deg f) f.$$

We set

(6.2)
$$E_{\text{deg}} = \sum_{\alpha=1}^{n} d_{\alpha} x^{\alpha} \frac{\partial}{\partial x^{\alpha}}$$

Let \mathcal{A} be the arrangement of reflection hyperplanes of G. For each hyperplane $H \in \mathcal{A}$, fix a linear form $L_H \in V^*$ such that $H = \ker L_H$. Let e_H be the order of the cyclic subgroup of G fixing H pointwise. Let us define polynomials $\Pi, \delta \in \mathbb{C}[V]$ by

(6.3)
$$\Pi = \prod_{H \in \mathcal{A}} L_H^{e_H - 1}, \qquad \delta = \prod_{H \in \mathcal{A}} L_H^{e_H}.$$

Then it is known that Π is skew invariant and δ is invariant (i.e. $g(\Pi) = \det(g)\Pi$ and $g(\delta) = \delta$ for any $g \in G$). Since

(6.4)
$$\frac{\partial(x^1,\ldots,x^n)}{\partial(u^1,\ldots,u^n)} = \text{const.} \cdot \Pi ,$$

the discriminant locus of the orbit map $\omega: V \to M$, $\omega(u^1, \dots, u^n) = (x^1(u), \dots, x^n(u))$, is given by $\{\Delta = 0\} \subset M$. Here we write Δ for δ when we regard it as a polynomial in x, i.e. $\Delta(x(u)) = \delta(u)$. We set

$$M_0 = M \setminus \{\Delta = 0\} .$$

6.2. Natural connection on TM_0 . The trivial holomorphic connection on TV induces a connection ∇^V on TM_0 . Explicitly, it is given as follows.

$$\nabla_{\partial_{\alpha}}^{V}(\partial_{\beta}) = \sum_{\gamma=1}^{n} \Omega_{\alpha\beta}^{\gamma} \, \partial_{\gamma} \, ,$$

$$\Omega_{\alpha\beta}^{\gamma} = -\sum_{i,j=1}^{n} \frac{\partial u^{i}}{\partial x^{\alpha}} \frac{\partial u^{j}}{\partial x^{\beta}} \frac{\partial^{2} x^{\gamma}}{\partial u^{i} \partial u^{j}} \in \mathbb{C}(x) \, , \quad \Omega_{\alpha} = (\Omega_{\alpha\beta}^{\gamma})_{\gamma,\beta} \, .$$

The induced connection ∇^V is torsion free and flat.

Let us list a few properties of the connection matrix Ω_{α} . Below, $M(n, \mathbb{C}[x])$ denotes the ring of $n \times n$ matrices with coefficients in $\mathbb{C}[x]$.

Proposition 6.1.

$$(6.5) \Delta\Omega_{\alpha\beta}^{\gamma} \in \mathbb{C}[x] .$$

(6.6)
$$\Delta \cdot \det \Omega_{\alpha} \in \mathbb{C}[x]. Moreover \det \Omega_{\alpha} = \frac{const.}{\Delta} if \deg \delta = nd_{\alpha}.$$

(6.7)
$$\Omega_{\mu}^{-1} \in M(n, \mathbb{C}[x]) \text{ if deg } \delta = nd_{\mu} \text{ and if det } \Omega_{\mu} \neq 0.$$

(6.8)
$$(\Omega_{\mu})^{-1}\Omega_{\alpha} \in M(n, (\mathbb{C}[x]) \text{ if deg } \delta = nd_{\mu} \text{ and if det } \Omega_{\mu} \neq 0.$$

Lemma 6.2. Let

$$J^{\gamma}{}_{i} = \frac{\partial x^{\gamma}}{\partial u^{i}} \ .$$

If $\det \Omega_{\mu} \neq 0$, then

$$J^{-1}\Omega_u^{-1} \in M(n, \mathbb{C}[u]) .$$

Proposition 6.1 and Lemma 6.2 can be proven by studying pole orders along each reflection hyperplane $H \in \mathcal{A}$. See §A.

Lemma 6.3.

(6.9)
$$\sum_{\alpha=1}^{n} d_{\alpha} x^{\alpha} \Omega_{\alpha\beta}^{\gamma} = \begin{cases} (1 - d_{\gamma}) & (\beta = \gamma) \\ 0 & (\beta \neq \gamma) \end{cases}.$$

Proof. By (6.1),

$$-\sum_{\alpha=1}^{n} d_{\alpha} x^{\alpha} \Omega_{\alpha\beta}^{\gamma} = \sum_{i,j=1}^{n} \left(\sum_{\alpha=1}^{n} d_{\alpha} x^{\alpha} \frac{\partial u^{i}}{\partial x^{\alpha}} \right) \frac{\partial u^{j}}{\partial x^{\beta}} \frac{\partial^{2} x^{\gamma}}{\partial u^{i} \partial u^{j}}$$

$$= \sum_{j=1}^{n} \frac{\partial u^{j}}{\partial x^{\beta}} \cdot \sum_{i=1}^{n} u^{i} \frac{\partial}{\partial u^{i}} \left(\frac{\partial x^{\gamma}}{\partial u^{j}} \right)$$

$$= (d_{\gamma} - 1) \sum_{j=1}^{n} \frac{\partial u^{j}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial u^{j}}$$

$$= (6.9) .$$

Notice that (6.9) implies that

(6.10)
$$\nabla_x^V E_{\text{deg}} = x \quad (x \in \mathcal{T}_{M_0}) .$$

6.3. The Natural (Almost) Saito Structure. Let \mathcal{X}_{-d_1} be the \mathbb{C} -vector space of polynomial vector fields on M of degree $-d_1$. Here d_1 is the maximal degree of G. Let x_1, \ldots, x_n be a set of basic invariants of G. Then the space \mathcal{X}_{-d_1} is spanned by $\partial_{x^{\alpha}}$ with $d_{\alpha} = d_1$.

Definition 6.4. (1) An almost Saito structure (∇, \star, e) on M_0 is a natural almost Saito structure for G if it satisfies the following conditions.

- $\bullet \ \boldsymbol{\nabla} = \boldsymbol{\nabla}^{V}.$
- A unit of \star is $\frac{1}{d_1}E_{\text{deg}}$ and hence the parameter is $r = \frac{1}{d_1}$ (by (6.10)).
- $e \neq 0$ and $e \in \mathcal{X}_{-d_1}$, i.e. e is a nonzero vector field on M of degree $-d_1$.
- (2) A Saito structure $(\nabla, *, E)$ on an open subset of M is a natural Saito structure for G if its dual almost Saito structure with parameter $r = \frac{1}{d_1}$ is a natural almost Saito structure for G. (Especially, the Euler vector field E must be $\frac{1}{d_1}E_{\text{deg}}$.)
- (3) A Saito structure $(\nabla, *, E)$ on M is called a polynomial Saito structure if
 - there exist ∇ -flat coordinates t^1, \ldots, t^n which form a set of basic invariants of G.
 - all entries of the matrix representations of $\partial_{t^{\alpha}} * (1 \leq \alpha \leq n)$ with respect to the basis $\partial_{t^1}, \ldots, \partial_{t^n}$ are polynomials in t.

Remark 6.5. In the case when G is a Coxeter group, the Saito structure considered in [15], [14] and [4] can be characterized as follows.

- The intersection form g (i.e. the metric of the dual almost Frobenius structure) is the complexification of the standard Euclidean metric.
- The Euler vector field is $\frac{1}{d_1}E_{\text{deg}}$.
- The unit vector field e is the vector field ∂_{x^1} corresponding to a basic invariant x^1 of the maximal degree.

The Levi–Civita connection of the above intersection form g is the natural connection ∇^V . Therefore the three conditions in Definition 6.4-(1) appear as a result of a straightforward generalization of the case of the Coxeter groups.

Let us mention that the Frobenius manifold structures for the Shephard groups [6, §5.3] is a result of a generalization in a different direction, in which one takes $\frac{\partial x^n}{\partial u^i \partial u^j}$ $(1 \le i, j \le n)$ as $g(\partial_{u^i}, \partial_{u^j})$.

Lemma 6.6. Let $e \in \mathcal{X}_{-d_1}$ be a nonzero vector field such that the pair (∇^V, e) is regular on M_0 . Then the pair (∇^V, e) makes a natural almost Saito structure for G if and only if it satisfies the conditions (ASS1) and (ASS2).

Proof. By (6.10), it is clear that ∇^V and $\frac{1}{d_1}E_{\text{deg}}$ satisfy the condition (**ASS3**). If the pair (∇^V, e) is regular, the multiplication \star obtained from (**ASS4**) (or (2.10)) has the unit $\frac{1}{d_1}E_{\text{deg}}$ since

$$x \star E_{\text{deg}} = -\mathcal{Q}^{-1}(\boldsymbol{\nabla}_{x}^{V} \boldsymbol{\nabla}_{E_{\text{deg}}}^{V} e) + \boldsymbol{\nabla}_{x}^{V} E_{\text{deg}}$$
$$= -\mathcal{Q}^{-1}(\boldsymbol{\nabla}_{x}^{V} (1 - d_{1}) e) + x = d_{1}x.$$

Therefore the statement follows from Proposition 2.9.

Lemma 6.6 can be used to show the nonexistence of the natural Saito structure for certain irreducible groups. See §9.4.1.

- 7. When the discriminant Δ is a monic of degree n in x^1
- 7.1. **Assumptions.** Let $G \subset GL(V)$ be a finite complex reflection group. In this section, we assume that G satisfies the conditions (i) and (ii) below.
 - (i) $d_{\alpha} > 1$ $(1 \le \alpha \le n = \dim V)$.
 - (ii) there exists a set of basic invariants x^1, \ldots, x^n such that the discriminant $\Delta \in \mathbb{C}[x]$ is a monic polynomial of degree n as a polynomial in x^1 .

The assumption (i) implies that V does not contain a trivial representation of G.

Remark 7.1. If G is irreducible, then (i) (ii) are equivalent to the condition that G is a duality group. See §9.2.

We will see that the pair $(\nabla^V, \partial_{x^1})$ is regular on M_0 (Corollary 7.3). Moreover we show that it makes a regular natural almost Saito structure for G and that its dual Saito structure is a polynomial Saito structure defined on the whole orbit space M (Corollary 7.6). We do this by first constructing a polynomial Saito structure on the whole orbit space M, and show that its dual almost Saito structure is $(\nabla^V, \partial_{x^1})$ (Theorem 7.5).

For the sake of simplicity, we write $\mathbb{C}[x'] = \mathbb{C}[x^2, \dots, x^n]$.

7.2. Constructing a polynomial Saito structure on M. By (6.5), $\Delta\Omega_{\alpha\beta}^{\gamma}$ is a polynomial in x. By assumption (i), deg $\Delta = nd_1$. So

$$\deg \Delta \Omega_{\alpha\beta}^{\gamma} = d_{\gamma} - d_{\alpha} - d_{\beta} + nd_1 < (n+1)d_1.$$

Therefore it is at most of degree n as a polynomial in x^1 . So we can write it in the following form:

(7.1)
$$\Delta\Omega_{\alpha\beta}^{\gamma} = (x^1)^n \Gamma_{\alpha\beta}^{\gamma} + (x^1)^{n-1} D_{\alpha\beta}^{\gamma} + \cdots, \quad (\Gamma_{\alpha\beta}^{\gamma}, D_{\alpha\beta}^{\gamma} \in \mathbb{C}[x']) .$$

Here \cdots means terms of degree less than n-1 in x^1 . We also set

(7.2)
$$\Gamma_{\alpha} = (\Gamma_{\alpha\beta}^{\gamma})_{\gamma,\beta} , \quad D_{\alpha} = (D_{\alpha\beta}^{\gamma})_{\gamma,\beta} .$$

Lemma 7.2. Under the assumptions (i) and (ii), the following holds.

(7.3)
$$\Gamma_{\alpha\beta}^{\gamma} = 0 \text{ if } d_{\gamma} \leq d_{\beta} , \quad \Gamma_{1} = O_{n} .$$

(7.4)
$$D_1 = \frac{1}{d_1} \operatorname{diag}[1 - d_1, \dots, 1 - d_n] - \sum_{\alpha=2}^n \frac{d_\alpha}{d_1} x^{\alpha} \Gamma_{\alpha}.$$

(7.5)
$$\Delta \det \Omega_1 = \det D_1 = \prod_{\gamma=1}^n \frac{1 - d_{\gamma}}{d_1} \neq 0.$$

(7.6)
$$\Omega_1^{-1} \in M(n, \mathbb{C}[x]) \text{ and } \Omega_1^{-1} - D_1^{-1}x^1 \in M(n, \mathbb{C}[x']).$$

(7.7)
$$\Omega_1^{-1}\Omega_\alpha \in M(n,\mathbb{C}[x]) \text{ and } \Omega_1^{-1}(\Omega_\alpha - \Gamma_\alpha) \in M(n,\mathbb{C}[x']) .$$

Proof. (For (7.3) and (7.4), the assumption (i) is not necessary.)

(7.3): let us compute the degree of $\Gamma_{\alpha\beta}^{\gamma} \in \mathbb{C}[x']$.

$$\deg \Gamma_{\alpha\beta}^{\gamma} = \deg(\Delta \Omega_{\alpha\beta}^{\gamma}) - nd_1 = d_{\gamma} - d_{\alpha} - d_{\beta}.$$

Therefore $\Gamma_{\alpha\beta}^{\gamma} = 0$ if $d_{\gamma} - d_{\alpha} - d_{\beta} < 0$. A sufficient condition for this is $d_{\gamma} - d_{\beta} \leq 0$. When $\alpha = 1$, this holds for any (β, γ) since $d_1 \geq d_{\gamma}$ and $d_{\beta} \geq 1$.

(7.4) By the expansion (7.1) and (7.3),

$$\Delta \sum_{\alpha=1}^{n} \frac{d_{\alpha}}{d_{1}} x^{\alpha} \Omega_{\alpha} = \left(D_{1} + \sum_{\alpha=2}^{n} \frac{d_{\alpha}}{d_{1}} x^{\alpha} \Gamma_{\alpha} \right) (x^{1})^{n} + \cdots$$

Here \cdots means terms of smaller degree in x^1 . On the other hand, by (6.9),

$$\Delta \sum_{i=1}^{n} \frac{d_{\alpha}}{d_{1}} x^{\alpha} \Omega_{\alpha} = \frac{\Delta}{d_{1}} \operatorname{diag}[1 - d_{1}, \dots, 1 - d_{n}] = \frac{(x^{1})^{n}}{d_{1}} \operatorname{diag}[1 - d_{1}, \dots, 1 - d_{n}] + \cdots$$

The last equality follows from the assumption (ii). Comparing the coefficients of $(x^1)^n$, we obtain (7.4).

(7.5): by the expansion (7.1),

$$\Delta \det \Omega_1 = \frac{1}{\Lambda^{n-1}} (\det D_1 \cdot (x^1)^{n(n-1)} + \cdots) .$$

On the other hand, since $\deg \Delta = nd_1$, $\Delta \det \Omega_1$ is a constant by (6.6). Therefore in the RHS, the denominator must divide the numerator. Given their degrees, $\Delta \det \Omega_1 = \det D_1$. By (7.3) (7.4), D_1 is triangular with diagonal entries $\frac{1-d_{\gamma}}{d_1}$ ($1 \leq \gamma \leq n$). Therefore

$$\Delta \det \Omega_1 = \det D_1 = \prod_{\gamma=1}^n \frac{1 - d_{\gamma}}{d_1} \neq 0.$$

The last inequality holds since we assumed $d_{\alpha} > 1$.

(7.6): since $\deg \Delta = nd_1$ and $\det \Omega_1 \neq 0$ by (7.5), $\Omega_1^{-1} \in M(n, \mathbb{C}[x])$ follows from (6.7). Given that

$$\deg (\Omega_1^{-1})^{\gamma}{}_{\beta} = d_1 + d_{\gamma} - d_{\beta} < 2d_1 ,$$

 $(\Omega_1^{-1})^{\gamma}{}_{\beta}$ is at most degree one as a polynomial in x^1 . So let us write $\Omega_1^{-1} = Ax^1 + B$ $(A, B \in M(n, \mathbb{C}[x']))$. Then by the expansion (7.1),

$$\Delta \cdot I_n = (\Delta \Omega_1) \Omega_1^{-1} = (D_1(x^1)^{n-1} + \cdots) (Ax^1 + B) = (D_1 A(x^1)^n + \cdots).$$

Comparing the coefficients of $(x^1)^n$, $AD_1 = I_n$.

(7.7): by (7.6) and the expansion (7.1),

$$\Omega_1^{-1}(\Omega_\alpha - \Gamma_\alpha) = \frac{1}{\Delta} (x^1 D_1^{-1} + B)((x^1)^{n-1} (D_\alpha - a\Gamma_\alpha) + \cdots)$$
$$= \frac{1}{\Delta} ((x^1)^n D_1^{-1} (D_\alpha - a\Gamma_\alpha) + \cdots) .$$

Here $a \in \mathbb{C}[x']$ is the coefficient of $(x^1)^{n-1}$ in Δ , i.e.,

$$\Delta = (x^1)^n + a(x^1)^{n-1} + \cdots$$

On the other hand, $\Omega_1^{-1}, \Omega_1^{-1}\Omega_{\alpha} \in M(n, \mathbb{C}[x])$ by (7.6) and (6.8), and $\Gamma_{\alpha} \in M(n, \mathbb{C}[x'])$. So the LHS is a matrix with polynomial entries. Therefore in the RHS, the denominator Δ must divide the numerator and its quotient is $D_1^{-1}(D_{\alpha} - a\Gamma_{\alpha})$. Since $D_1, D_{\alpha}, a, \Gamma_{\alpha}$ do not depend on x^1 ,

$$\Omega_1^{-1}(\Omega_\alpha - \Gamma_\alpha) = D_1^{-1}(D_\alpha - a\Gamma_\alpha) \in M(n, \mathbb{C}[x'])$$
.

The proof of Lemma 7.2 is finished.

Corollary 7.3. The pair $(\nabla^V, \partial_{x^1})$ is regular.

Proof. The endomorphism $\mathcal{Q} = \nabla^V(\partial_{x^1}) \in \text{Hom}(\mathcal{T}_{M_0}, \mathcal{T}_{M_0})$ is represented by the matrix Ω_1 . By (7.5), it is invertible on M_0 . Thus \mathcal{Q} is an isomorphism.

Lemma 7.4. Assume (i) and (ii). Let

$$C_{\alpha} = \Omega_1^{-1}(\Omega_{\alpha} - \Gamma_{\alpha}) \in M(n, \mathbb{C}[x']) , \quad C_{\alpha\beta}^{\gamma} = (C_{\alpha})_{\beta}^{\gamma} , \quad U = \sum_{\alpha=1}^{n} \frac{d_{\alpha}}{d_1} x^{\alpha} C_{\alpha} .$$

Then the following holds.

$$(7.8) C_1 = I_n .$$

(7.9)
$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha} , \quad C^{\gamma}_{\alpha\beta} = C^{\gamma}_{\beta\alpha} .$$

(7.10)
$$\partial_{\alpha}\Gamma_{\beta} - \partial_{\beta}\Gamma_{\alpha} + [\Gamma_{\alpha}, \Gamma_{\beta}] = O_{n} .$$

(7.11)
$$\partial_{\alpha} C_{\beta} + [\Gamma_{\alpha}, C_{\beta}] = \partial_{\beta} C_{\alpha} + [\Gamma_{\beta}, C_{\alpha}].$$

$$(7.12) C_{\alpha}C_{\beta} = C_{\beta}C_{\alpha} .$$

(7.13)
$$U = \Omega_1^{-1} D_1 , \quad \det U = \Delta .$$

Proof. (7.8) holds since $\Gamma_1 = O_n$.

(7.9) follows from the torsion freeness of ∇^V , i.e. $\Omega_{\alpha\beta}^{\gamma} = \Omega_{\beta\alpha}^{\gamma}$.

(7.10) (7.11) (7.12) follow from the flatness of ∇^V as follows. Substituting $\Omega_{\alpha} = \Omega_1 C_{\alpha} + \Gamma_{\alpha}$ into

$$\partial_{\alpha}\Omega_{\beta} - \partial_{\beta}\Omega_{\alpha} + [\Omega_{\alpha}, \Omega_{\beta}] = O_n \quad (1 \le \alpha, \beta \le n) ,$$

we obtain

$$O_{n} = \underbrace{\partial_{\alpha} \Gamma_{\beta} - \partial_{\beta} \Gamma_{\alpha} + [\Gamma_{\alpha}, \Gamma_{\beta}]}_{S_{1}} + \underbrace{\Omega_{1} \left(\partial_{\alpha} C_{\beta} + [\Gamma_{\alpha}, C_{\beta}] - \partial_{\beta} C_{\alpha} - [\Gamma_{\beta}, C_{\alpha}] \right)}_{S_{2}} + \underbrace{\left(\partial_{\alpha} \Omega_{1} + [\Gamma_{\alpha}, \Omega_{1}] \right) C_{\beta} - \left(\partial_{\beta} \Omega_{1} + [\Gamma_{\beta}, \Omega_{1}] \right) C_{\alpha} + [\Omega_{1} C_{\alpha}, \Omega_{1} C_{\beta}]}_{S_{3}}.$$

Let us simplify S_3 . Using $\partial_{\alpha}\Omega_1 = \partial_1\Omega_{\alpha} - [\Omega_{\alpha}, \Omega_1]$,

$$S_3 = (\partial_1 \Omega_\alpha - [\Omega_1 C_\alpha, \Omega_1]) C_\beta - (\partial_1 \Omega_\beta - [\Omega_1 C_\beta, \Omega_1]) C_\alpha + [\Omega_1 C_\alpha, \Omega_1 C_\beta]$$
$$= \partial_1 \Omega_\alpha C_\beta - \partial_1 \Omega_\beta C_\alpha + \Omega_1^2 [C_\alpha, C_\beta] .$$

Noticing that

$$O_n = \partial_1 C_\alpha = D_1^{-1} (\Omega_\alpha - \Gamma_\alpha) + \Omega_1^{-1} \partial_1 \Omega_\alpha \ \Rightarrow \ \partial_1 \Omega_\alpha = -\Omega_1 D_1^{-1} \Omega_1 C_\alpha \ ,$$

we have

$$S_3 = \Omega_1(I_n - D_1^{-1})\Omega_1[C_{\alpha}, C_{\beta}]$$
.

Now regard S_1, S_2, S_3 as functions in x^1 . Then S_1 is a constant matrix. S_2 and S_3 depend on x^1 only through the factor Ω_1 and $\Omega_1(I_n - D_1^{-1})\Omega_1$. Since entries of Ω_1 are rational functions whose numerator and denominator have degrees n-1 and n, entries of $S_2 + S_3$ are rational functions whose numerator and denominator have degrees 2n-1 and 2n. Therefore $O_n = S_1 + S_2 + S_3$ implies that $S_1 = O_n$ and $S_2 + S_3 = O_n$. Applying a similar argument to $S_2 + S_3 = O_n$, we obtain $S_2 = O_n$ and $S_3 = O_n$. (7.13): by (6.9) and (7.4),

$$U = \sum_{\alpha=1}^{n} \frac{d_{\alpha}}{d_{1}} x^{\alpha} C_{\alpha} = \Omega_{1}^{-1} \sum_{\alpha=1}^{n} \frac{d_{\alpha}}{d_{1}} x^{\alpha} (\Omega_{\alpha} - \Gamma_{\alpha}) = \Omega_{1}^{-1} D_{1}.$$

Therefore by (7.5),

$$\det U = \frac{\det D_1}{\det \Omega_1} = \Delta .$$

Theorem 7.5. Assume that a finite complex reflection group G satisfies the conditions (i) and (ii) in §7.1. Let ∇ and * be a connection and a multiplication on the tangent bundle TM of the orbit space M of G given by

$$\nabla_{\alpha}(\partial_{\beta}) = \sum_{\gamma=1}^{n} \Gamma_{\alpha\beta}^{\gamma} \cdot \partial_{\gamma} , \quad \partial_{\alpha} * \partial_{\beta} = \sum_{\gamma=1}^{n} C_{\alpha\beta}^{\gamma} \cdot \partial_{\gamma} .$$

Then

- (1) ∇ is torsion-free and flat.
- (2) * is commutative and associative and its unit is $e = \partial_{x^1}$.
- (3) $(\nabla, *, \frac{1}{d_1}E_{\text{deg}})$ is a polynomial Saito structure defined on the whole orbit space M.
- (4) Its dual almost Saito structure with the parameter $r = \frac{1}{d_1}$ is the regular natural almost Saito structure $(\nabla^V, \partial_{x^1})$ for G.

Proof. We check the conditions by using the matrix representations in §5.

- (1) follows from (7.9) (7.10).
- (2) follows from (7.8) (7.9) and (7.12).
- (3) Let us check the conditions (5.1), (5.3),(5.5),(5.6). The condition (5.1) is equivalent to (7.11). The condition (5.3) holds because $Q = \Gamma_1 = O_n$.

The condition (5.5): computing the LHS by the degree formula (6.1),

LHS =
$$\frac{1}{d_1} E_{\text{deg}}(C_{\alpha\beta}^{\gamma}) = \frac{1}{d_1} (\text{deg } C_{\alpha\beta}^{\gamma}) C_{\alpha\beta}^{\gamma} = \frac{d_1 + d_{\gamma} - d_{\alpha} - d_{\beta}}{d_1} C_{\alpha\beta}^{\gamma}$$
.

On the other hand, since $W^{\gamma}_{\beta} = \frac{d_{\beta}}{d_{1}} \delta_{\gamma_{\beta}}$,

$$RHS = -\frac{d_{\alpha}}{d_{1}}C_{\alpha\beta}^{\gamma} + \frac{d_{\gamma} - d_{\beta}}{d_{1}}C_{\alpha\beta}^{\gamma} + C_{\alpha\beta}^{\gamma} = LHS.$$

The calculation of the condition (5.6) is similar:

LHS =
$$\frac{1}{d_1} E_{\text{deg}}(\Gamma_{\alpha\beta}^{\gamma}) = \frac{1}{d_1} \text{deg } \Gamma_{\alpha\beta}^{\gamma} = \frac{d_{\gamma} - d_{\alpha} - d_{\beta}}{d_1} \Gamma_{\alpha\beta}^{\gamma}$$
.

On the other hand,

$$RHS = -\frac{d_{\alpha}}{d_{1}}\Gamma^{\gamma}_{\alpha\beta} + \frac{d_{\gamma} - d_{\beta}}{d_{1}}\Gamma^{\gamma}_{\alpha\beta} = LHS.$$

Next we show that there exist ∇ -flat coordinates t^1, \ldots, t^n which are basic invariants. Recall that the matrices Γ_{μ} $(1 \leq \mu \leq n)$ are strictly upper triangular (see (7.3)). Therefore there exists a unique upper triangular matrix $X \in M(n, \mathbb{C}[x])$ satisfying

$$X^{\gamma}{}_{\gamma} = 1$$
, $\deg X^{\gamma}{}_{\beta} = d_{\gamma} - d_{\beta}$, $\frac{\partial}{\partial x^{\mu}} X + \Gamma_{\mu} X = O_n$ $(1 \le \mu \le n)$.

(One can solve the system of differential equations starting from $(\gamma, \gamma + 1)$ entries and then moving to $(\gamma, \gamma + 2)$ entries, and so on. The integrability of the equations follows

from the flatness of ∇ .) It is clear that X is invertible and that $X^{-1} \in M(n, \mathbb{C}[x])$. Then unique homogeneous solutions t^1, \ldots, t^n of the equations

$$dt^{\alpha} = \sum_{\beta=1}^{n} (X^{-1})^{\alpha}{}_{\beta} dx^{\beta} \quad (1 \le \alpha \le n)$$

are ∇ -flat coordinates. They are of the form $t^{\alpha} = x^{\alpha} + (a \text{ polynomial in } x^{\beta} \ (\beta > \alpha))$ and have degrees $\deg t^{\alpha} = d_{\alpha}$. Thus they are basic invariants. The matrix representations of $\partial_{t^{\alpha}}*$ with respect to the basis $\partial_{t^{1}}, \ldots, \partial_{t^{n}}$ are given by $X^{-1}(\sum_{\mu=1}^{n} X^{\mu}{}_{\alpha}C_{\mu})X$. It is clear that their entries are polynomials in x.

(4) The matrix representation of $\frac{1}{d_1}E_{\text{deg}}*$ is given by U in Lemma 7.4. Therefore the dual almost Saito structure $(\nabla^{(0,\frac{1}{d_1})},\star,\partial_1)$ with the parameter $\frac{1}{d_1}$ is defined on the subset where U has rank n, that is, on $M_0 = M \setminus \{\Delta = 0\}$ by (7.13). By (7.13), the matrix representation B_{α} of the multiplication $\partial_{\alpha}\star$ is given by

$$B_{\alpha} = U^{-1}C_{\alpha} = U^{-1}\Omega_1^{-1}(\Omega_{\alpha} - \Gamma_{\alpha}) = D_1^{-1}(\Omega_{\alpha} - \Gamma_{\alpha}) .$$

Therefore by (5.8), the matrix representation of the dual connection $\nabla_{\alpha}^{(0,\frac{1}{d_1})}$ $(1 \le \alpha \le n)$ is given by

$$\Gamma_{\alpha} + \frac{1}{d_1} (I_n - \operatorname{diag}[d_1, \dots, d_n] - E_{\operatorname{deg}} \cdot \Gamma) B_{\alpha} \stackrel{(7.4)}{=} \Gamma_{\alpha} + D_1 B_{\alpha} = \Omega_{\alpha}$$
.

Thus $\nabla^{(0,\frac{1}{d_1})} = \nabla^V$. Regularity of the pair $(\nabla^V, \partial_{x^1})$ is already shown in Lemma 7.3. \square

Theorem 7.5 can be restated in the following way.

Corollary 7.6. If a finite complex reflection group G satisfies the assumptions (i) and (ii), then $(\nabla^V, \partial_{x^1})$ makes a regular natural almost Saito structure for G. Its dual Saito structure can be extended to a polynomial Saito structure on the whole orbit space M.

Moreover if $d_1 > d_2$, then $(\nabla^V, \partial_{x^1})$ is a unique natural almost Saito structure for G up to equivalence.

Proof. The uniqueness follows since $\dim_{\mathbb{C}} \mathcal{X}_{-d_1} = 1$ if $d_1 > d_2$.

7.3. Basic derivations and relationship to Kato–Mano–Sekiguchi's work [9]. First let us recall the definitions of the basic derivations, the codegrees and the discriminant matrix (see [12, Chapter 6]). For a finite complex reflection group G, the set of G-invariant polynomial vector fields on V is a free $\mathbb{C}[V]^G$ -module of rank $n = \dim V$ (see Lemma 6.48 of loc.cit). A homogeneous basis of this module is called a set of basic derivations for G. The degrees of basic derivations are called the codegrees of G and denoted

$$0 = d_1^* \le d_2^* \le \dots \le d_n^* .$$

Definition 7.7. The discriminant matrix of G with respect to a set of basic invariants x^1, \ldots, x^n and a set of basic derivations $\tilde{X}_1, \ldots, \tilde{X}_n$ is the matrix $M = (M^{\alpha}{}_{\beta})$ $(M^{\alpha}{}_{\beta} \in \mathbb{C}[x])$ defined by

$$M^{\alpha}{}_{\beta} = dx^{\alpha}(X_{\beta}) = X_{\beta}(x^{\alpha})$$
.

Here $X_{\beta} = \omega_* \tilde{X}_{\beta}$ is a polynomial vector field on M corresponding to \tilde{X}_{β} and $\omega : V \to M$ is the orbit map.

Now assume that G is a finite complex reflection group satisfying the assumptions (i) and (ii) in §7.1. Take a set of basic invariants x^1, \ldots, x^n satisfying the assumption (i). Let $(\nabla, *, \frac{1}{d_1}E_{\text{deg}})$ be the natural Saito structure on M with the unit ∂_{x^1} , given in Theorem 7.5. Define vector fields X_1, \ldots, X_n on M by

(7.14)
$$X_{\alpha} = \frac{1}{d_1} E_{\text{deg}} * \partial_{x^{\alpha}} .$$

Notice that $\deg X_{\alpha} = d_1 - d_{\alpha}$ and notice also that $X_1 = \frac{1}{d_1} E_{\text{deg}}$. In this setting, the following proposition holds.

Proposition 7.8. (1) For each $1 \le \alpha \le n$, there exists unique homogeneous G-invariant polynomial vector field \tilde{X}_{α} of degree $d_1 - d_{\alpha}$ on V such that $\omega_* \tilde{X}_{\alpha} = X_{\alpha}$. Here $\omega : V \to M$ is the orbit map.

(2) Moreover if $d_{\alpha} + d_{\alpha}^* = d_1$ ($1 \leq \alpha \leq n$) holds, $\tilde{X}_1, \ldots, \tilde{X}_n$ are basic derivations for G and the matrix U defined in Lemma 7.4 is the discriminant matrix with respect to x^1, \ldots, x^n and $\tilde{X}_1, \ldots, \tilde{X}_n$.

Proof. (1) Since the matrix representation of $\frac{1}{d_1}E_{\text{deg}}*$ is given by U,

(7.15)
$$X_{\alpha} = \sum_{\beta=1}^{n} U^{\beta}{}_{\alpha} \, \partial_{x^{\beta}} \stackrel{(7.13)}{=} \sum_{\beta=1}^{n} (\Omega_{1}^{-1} D_{1})^{\beta}{}_{\alpha} \partial_{x^{\beta}} .$$

Therefore its (local) lift to V is

$$\sum_{\beta,i=1}^{n} (\Omega_1^{-1} D_1)^{\beta}{}_{\alpha} \frac{\partial u^i}{\partial x^{\beta}} \partial_{u^i} = \sum_{i=1}^{n} (J^{-1} \Omega_1^{-1} D_1)^i{}_{\alpha} \partial_{u^i}.$$

By Lemma 6.2, entries of $J^{-1}\Omega_1^{-1}$ are polynomials in u. Entries of D_1 are polynomials in x by definition, and hence they are polynomials in u. Therefore this is a polynomial vector field globally defined on V. This is the \tilde{X}_{α} in the proposition. Notice that \tilde{X}_{α} are independent on the complement of reflection hyperplanes of V since the matrix U is invertible on M_0 .

(2) If $d_{\alpha} + d_{\alpha}^* = d_1$ holds, $\deg \tilde{X}_{\alpha} = d_{\alpha}^*$. Therefore $\tilde{X}_1, \ldots, \tilde{X}_n$ are a set of basic derivations

for G. By (7.15), it is immediate to see that entries of the discriminant matrix with respect to x^1, \ldots, x^n and X_1, \ldots, X_n are given by

$$dx^{\alpha}(X_{\beta}) = U^{\alpha}{}_{\beta}$$
.

Therefore the discriminant matrix is U.

Now we will explain the relationship to Kato-Mano-Sekiguchi's result [9, §6].

In the rest of this subsection, we assume that G is a duality group (see §9.2 and [10, §12.6] for the definition). Notice that Proposition 7.8 holds for G since any duality group satisfies the assumptions (i) and (ii) and satisfies also the requirement in Proposition 7.8 (2). Let x^1, \ldots, x^n be a set of basic invariants satisfying the assumption (i) and let $(\nabla, *, \frac{1}{d_1} E_{\text{deg}})$ be the natural Saito structure on M. (Recall that a set of ∇ -flat basic invariants t^1, \ldots, t^n exists and that t^{α} is of the form $t^{\alpha} = x^{\alpha} + (\text{a polynomial in } x^{\beta}, \beta > \alpha)$. See the proof of Theorem 7.5-(3).) We can assume that the basic invariants x^1, \ldots, x^n are ∇ -flat coordinates.

Denote by $(\nabla^V)^*$ the connection on the cotangent bundle T^*M_0 which is dual to the natural connection ∇^V . Given that ∇^V is induced from the trivial connection on TV, it is clear that $(\nabla^V)^*$ has local flat sections du^1, \ldots, du^n where u^1, \ldots, u^n are the standard coordinates of $V = \mathbb{C}^n$.

Let $X_{\alpha} = \frac{1}{d_1} E_{\text{deg}} * \partial_{\alpha}$ for $1 \leq \alpha \leq n$ as in (7.14). They form a basis of \mathcal{T}_{M_0} since $\frac{1}{d_1} E_{\text{deg}} *$ is invertible on $M_0 \subset M$. So denote the dual basis of $\mathcal{T}_{M_0}^*$ by X^1, \ldots, X^n . Let us write down the flatness equation for du^i $(1 \leq i \leq n)$ using this basis.

Lemma 7.9. For $1 \le i \le n$,

$$(\nabla^{V})^{*}(du^{i}) = 0$$

$$\Leftrightarrow \partial_{x^{\alpha}}y^{i} = y^{i}U^{-1}C_{\alpha}\left(\left(\frac{1}{d_{1}} + 1\right)I_{n} - \frac{1}{d_{1}}\operatorname{diag}(d_{1}, \dots, d_{n})\right) \quad (1 \leq \alpha \leq n) .$$

Here y^i is the row vector $(X_1(u^i), \dots, X_n(u^i))$, C_{α} $(1 \le \alpha \le n)$ and $U = \frac{1}{d_1} \sum_{\alpha=1}^n d_{\alpha} x_{\alpha} C_{\alpha}$ are the matrix representations of the multiplications $\partial_{\alpha} *$ and $\frac{1}{d_1} E_{\text{deg}} *$ with respect to the basis $\partial_{x^1}, \dots, \partial_{x^n}$.

Proof. It is enough to show that the connection matrix of $\nabla_{\partial_{\alpha}}^{V}$ with respect to the basis X_1, \ldots, X_n is the matrix appearing in the RHS of (7.16).

Since ∇^V is the connection of the dual almost Saito structure with parameter $r = \frac{1}{d_1}$, the connection matrix Ω_{α} of $\nabla^V_{\partial_{\alpha}}$ with respect to $\partial_{x^1}, \ldots, \partial_{x^n}$ is given by (5.8):

$$\Omega_{\alpha} = \left(\frac{1}{d_1}I_n - \frac{1}{d_1}\operatorname{diag}(d_1, \dots, d_n)\right)C_{\alpha}U^{-1}.$$

Then the connection matrix Ω_{α}^{X} with respect to the basis X_{1}, \ldots, X_{n} is given by

$$\Omega_{\alpha}^{X} = U^{-1}\Omega_{\alpha}U + U^{-1}\partial_{\alpha}U$$

$$\stackrel{(5.2)}{=} U^{-1}C_{\alpha}\left(\left(\frac{1}{d_{1}} + 1\right)I_{n} - \frac{1}{d_{1}}\operatorname{diag}(d_{1}, \dots, d_{n})\right).$$

It is not difficult to see that (7.16) is nothing but the Okubo type equation (102) in [9, Theorem 6.1] and that ∇ -flat basic invariants x^{α} ($1 \leq \alpha \leq n$) here correspond to "uniquely defined special G-invariant homogeneous polynomials $F'_{n-\alpha}(u)$ " there. Thus the polynomial Saito structure for the duality group G in loc.cit. agrees with the natural Saito structure obtained in this section.

7.4. Relationship to Arsie-Lorenzoni's standard bi-flat structure. In this subsection, we explain that Arsie-Lorenzoni's standard bi-flat structure obtained in [2, §5] is the same as the natural Saito structure.

As in the previous section, we assume that G is a finite complex reflection group satisfying the assumptions (i) and (ii) in §7.1. Take a set of basic invariants x^1, \ldots, x^n satisfying the assumption (i). Let $(\nabla, *, \frac{1}{d_1}E_{\text{deg}})$ be the natural Saito structure on M with the unit ∂_{x^1} , given in Theorem 7.5. Modifying if necessary, we assume that the basic invariants x^1, \ldots, x^n are ∇ -flat coordinates.

By (5.8), the matrix representation B_{α} of $\partial_{x^{\alpha}} \star$ and the connection matrix Ω_{α} of the dual connection ∇^{V} with respect to the basis $\partial_{x^{1}}, \ldots, \partial_{x^{n}}$ satisfies the relation

$$\Omega_{\alpha\beta}^{\gamma} = \left(\frac{1}{d_1}(I_n - \operatorname{diag}(d_1, \dots, d_n))B_{\alpha}\right)_{\beta}^{\gamma}.$$

Therefore

$$B_{\alpha\beta}^{\gamma} = \left(d_1 \left(I_n - \operatorname{diag}(d_1, \dots, d_n)\right)^{-1} \Omega_{\alpha}\right)_{\beta}^{\gamma}$$
$$= \frac{d_1}{d_{\gamma} - 1} \sum_{i,j=1}^{n} \frac{\partial u^i}{\partial x^{\alpha}} \frac{\partial u^j}{\partial x^{\beta}} \frac{\partial^2 x^{\gamma}}{\partial u^i \partial u^j} ,$$

and

$$\partial_{u^{i}} \star \partial_{u^{j}} = \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{k=1}^{n} \frac{\partial u^{i}}{\partial x^{\alpha}} \frac{\partial u^{j}}{\partial x^{\beta}} B_{\alpha\beta}^{\gamma} \frac{\partial u^{k}}{\partial x^{\gamma}} \partial_{u^{k}}$$
$$= \frac{d_{1}}{d_{\gamma} - 1} \sum_{k=1}^{n} \frac{\partial^{2} x^{\gamma}}{\partial u^{i} \partial u^{j}} \frac{\partial u^{k}}{\partial x^{\gamma}} \partial_{u^{k}} .$$

This agrees with the formula of the structure constants in [2, Theorem 5.3].

8. The Case of branched covering

8.1. Setting. In this section, G is a finite complex reflection group acting on an n-dimensional complex vector spaces $V = \mathbb{C}^n$ and K is a normal reflection subgroup of G. Denote the degrees of K and G by $d_1^K \geq d_2^K \geq \ldots \geq d_n^K$ and $d_1^G \geq d_2^G \geq \ldots \geq d_n^G$ respectively. Since K is a normal subgroup, the action of G on $\mathbb{C}[V]$ preserves the subring $\mathbb{C}[V]^K$ of K-invariant polynomials and $\mathbb{C}[V]^G = (\mathbb{C}[V]^K)^{G/K}$. Hence G/K acts on the orbit space $M_K = \operatorname{Spec} \mathbb{C}[V]^K$ as automorphisms of the branched covering map $\pi: M_K \to M_G = \operatorname{Spec} \mathbb{C}[V]^G$. We write Δ_K (resp. Δ_G) for the discriminant of K (resp. G). Let $\Delta_{G/K} := \Delta_G/\Delta_K \in \mathbb{C}[V]^K$.

Lemma 8.1. The ramification locus of π is $\{\Delta_{G/K} = 0\}$.

Proof. Let $p \in V$ and let $K_p = \{g \in K \mid g(p) = p\}$, $G_p = \{g \in G \mid g(p) = p\}$ be the stabilizer subgroups. Then $\omega_K(p) \in M_K$ is contained in the ramification locus of π if and only if $K_p \subsetneq G_p$. Here $\omega_K : V \to M_K$ is the orbit map. By the theorem of Steinberg (see [10, §9.7]), K_p (resp. G_p) is generated by reflections of K (resp. G_p) which fixes G_p . Therefore $G_p : G_p \Leftrightarrow (\delta_G/\delta_K)(p) = 0$ (see (6.3)).

Let us set $M_K' = M_K \setminus \{\Delta_{G/K} = 0\}$ and $M_G' = \operatorname{Im} \pi|_{M_K'}$. Therefore $\pi|_{M_K'} : M_K' \to M_G'$ is an unramified Galois covering with the Galois group G/K. Moreover, let us set $M_{K,0}' = M_K \setminus \{\Delta_G = 0\}$ and $M_{G,0} = M_G \setminus \{\Delta_G = 0\}$. Then it follows that $\pi|_{M_{K,0}'} : M_{K,0}' \to M_{G,0}$ is an unramified Galois covering with the Galois group G/K. In the next Definition 8.2, either $(U_K, U_G) = (M_K', M_G')$ or $(M_{K,0}', M_{G,0})$.

Definition 8.2. (1) A connection ∇ on TU_K is G/K-invariant if

$$\nabla_{g_*x}(g_*y) = g_*(\nabla_x y) \quad (g \in G/K \ , \ x, y \in \mathcal{T}_{U_K}) \ .$$

(2) A multiplication \ast on TU_K is G/K-invariant if

$$(g_*x)*(g_*y) = g_*(x*y) \quad (g \in G/K, x, y \in \mathcal{T}_{U_K}).$$

(3) We say that a Saito structure $(\nabla, *, E)$ on U_K is G/K-invariant if $\nabla, *, E$ are G/K-invariant. We also say that an almost Saito structure (∇, \star, e) on U_K is G/K-invariant if ∇, \star, e are G/K-invariant.

If ∇ is a G/K-invariant connection on TU_K , then the covering map π induces a connection ∇^{π} on TU_G as follows. Let $p \in U_G$ and take an open neighborhood U of p such that $\pi^{-1}(U)$ consists of |G/K| disjoint open sets, say $U_1, \ldots, U_{|G/K|}$. Then $\pi: U_i \to U$ is an isomorphism $(1 \le i \le |G/K|)$. For $x, y \in \Gamma(U, \mathcal{T}_{U_G})$, take their lifts $x_1, y_1 \in \Gamma(U_1, \mathcal{T}_{U_K})$ to U_1 and define ∇^{π} by

$$\nabla_x^{\pi} y = \pi_*(\nabla_{x_1} y_1) .$$

This definition does not depend on the choice of the lifts. For, if $x_2, y_2 \in \Gamma(U_2, T_{U_K})$ are the lifts of x, y to U_2 , then there exists $g \in G/K$ such that $g_*x_1 = x_2, g_*y_1 = y_2$. Then $\nabla_{x_2} y_2 = g_*(\nabla_{x_1} y_1)$ follows from the G/K-invariance. Similarly, if * is a G/K-invariant multiplication on TU_K , π induces a multiplication $*_{\pi}$ on TU_G by

$$x *_{\pi} y = \pi_*(x_1 * y_1)$$
.

Therefore, given that the conditions for the Saito structures and almost Saito structures are local conditions, it is easy to see the following statements.

- (1) If $(\nabla, *, E)$ is a G/K-invariant Saito structure on U_K , then π induces a Saito structure $(\nabla^{\pi}, *_{\pi}, \pi_* E)$ on U_G .
- (2) If (∇, \star, e) is a G/K-invariant almost Saito structure on U_K , then π induces an almost Saito structure $(\nabla^{\pi}, \star_{\pi}, \pi_* e)$ on U_G .
- (3) If the above $(\nabla, *, E)$ and (∇, \star, e) are dual to each other, then $(\nabla^{\pi}, *_{\pi}, \pi_{*}E)$ and $(\nabla^{\pi}, \star_{\pi}, \pi_{*}e)$ are dual to each other.

8.2. Natural (almost) Saito structure via covering. We first give criterions for a natural almost Saito structure for K to be G/K invariant.

Denote $\nabla^{V,K}$ (resp. $\nabla^{V,G}$) the natural connection on $TM_{K,0}$ (resp. $TM_{G,0}$) induced from the trivial connection d on TV by the orbit maps ω_K, ω_G . It is clear that $\nabla^{V,K}$ is G/K-invariant and that the π -induced connection $(\nabla^{V,K})^{\pi}$ is nothing but $\nabla^{V,G}$. Notice also that $E_{\text{deg}}^K = \sum_{\alpha=1}^n d_{\alpha}^K y^{\alpha} \frac{\partial}{\partial y^{\alpha}}$ is G/K-invariant and

$$\pi_* E_{\text{deg}}^K = \sum_{\alpha=1}^n d_{\alpha}^G x^{\alpha} \frac{\partial}{\partial x^{\alpha}} = E_{\text{deg}}^G .$$

Lemma 8.3. Assume that a vector field e on M_K is G/K-invariant. Assume moreover that the pair $(\nabla^{V,K}, e)$ is regular. If \star is the multiplication determined by (ASS4) (or (2.10)), then \star is G/K-invariant.

Proof. The G/K-invariance of \star easily follows from those of $\nabla^{V,K}$ and e.

Let k_K be the number of the degrees of K which are equal to d_1^K :

$$d_1^K = \dots = d_{k_K}^K > d_{k_K+1}^K \ge \dots \ge d_n^K$$
.

We put the following assumptions on the pair (G, K).

- (iii) either $k_K = 1$ (i.e. $d_1^K > d_2^K$) or G/K is abelian.
- (iv) $d_1^K = d_1^G$

Lemma 8.4. If the assumption (iii) holds, there exists a set of basic invariants y^1, \ldots, y^n of K such that y^1, \ldots, y^{k_K} are semi-invariants of G. Moreover if the assumption (iv) holds, such y^1, \ldots, y^{k_K} are G-invariant.

The proof of Lemma 8.4 is given in §B. Notice that under the assumptions (iii) (iv), any vector field e on M_K of degree $-d_1^K$ is G/K-invariant.

We arrive at the following

Proposition 8.5. Assume that K satisfies the assumptions (i) and (ii) in §7.1 and let y^1, \ldots, y^n be a set of basic invariants satisfying (ii). Denote $(\nabla^{V,K}, \partial_{y^1})$ the associated regular natural Saito structure. Assume also that (G, K) satisfies the assumptions (iii)(iv).

- (1) The projection $\pi: M_K \to M_G$ induces a regular natural almost Saito structure $(\nabla^{V,G}, \pi_*(\partial_{y^1}))$ for G. Its parameter is $r = \frac{1}{d_1^G}$ and a unit is $E = \frac{1}{d_1^G} E_{\text{deg}}^G$.
- (2) Denote by $(\nabla, *, \frac{1}{d_1} E_{\text{deg}}^K)$ the natural Saito structure on M_K which is dual to $(\nabla^{V,K}, \partial_{y^1})$. Then π induces a natural Saito structure $(\nabla^{\pi}, *_{\pi}, \frac{1}{d_1} E_{\text{deg}}^G)$ on M'_G . Moreover it is dual to $(\nabla^{V,G}, \pi_*(\partial_{y^1}))$.

9. Irreducible finite complex reflection groups

In this section, notations are the same as in §6 and §8.

9.1. Classification. Let $m, n \in \mathbb{Z}_{>0}$ and let p > 0 be a divisor of m. Recall that the monomial group G(m, p, n) is defined as the semidirect product of

$$A(m, p, n) = \{ (\theta_1, \dots, \theta_n) \in \underbrace{\mu_m \times \dots \times \mu_m}_{n\text{-times}} \mid (\theta_1 \theta_2 \dots \theta_n)^{m/p} = 1 \},$$

with the symmetric group \mathfrak{S}_n acting by permutations of the factors. Here μ_m is the cyclic group of m-th roots of unity. The irreducible finite complex reflection groups are classified by Shephard–Todd [16]:

- $G(1,1,n) = \mathfrak{S}_n$ for $n \geq 2$ regarded as acting on the (n-1)-dimensional invariant subspace i.e. the Weyl group of type A_{n-1} ,
- G(m, p, n) for m > 1, n > 1 and $(m, p, n) \neq (2, 2, 2)$,
- $G(m, p, 1) = G(m/p, 1, 1) = \mu_{m/p}$
- 34 exceptional cases named G_4, \ldots, G_{37} .

G(2,2,2) does not appear in the list since $G(2,2,2) = A_1 \times A_1$.

- 9.2. **Duality groups.** For an irreducible finite complex reflection group G, the following conditions are equivalent [3] [11].
 - \bullet G satisfies the assumptions (i) and (ii) in §7.1.
 - $d_{\alpha} + d_{\alpha}^* = d_1 \ (1 \le \alpha \le n).$
 - \bullet G is generated by n reflections.

Such G is said to be well-generated, or G is called a duality group. The duality groups are the monomial groups $G(m, 1, 1) \cong \mu_m$ $(m \geq 1)$, $G(1, 1, n) = \mathfrak{S}_n \cong A_{n-1}$ $(n \geq 2)$,

G(m,1,n) $(m,n \ge 2)$, G(m,m,n) $(m \ge 3$ and $n \ge 2$ or m = 2 and $n \ge 3$), and exceptional groups G_4 to G_{37} except for $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$.

If G is one of these groups, $d_1 > d_2$. So dim $\mathcal{X}_{-d_1} = 1$ and a choice of the nonzero vector field $e \in \mathcal{X}_{-d_1}$ is unique up to scalar multiplication. Moreover the pair (∇^V, e) is regular. Therefore a natural almost Saito structure (and hence a natural Saito structure) exists uniquely up to equivalence by Corollary 7.6. This was already proved by Kato, Mano and Sekiguchi [9]. See §7.3 for how their result and our formulation are related.

In [2, §5], Arsie and Lorenzoni computed the bi-flat F-manifold structures equivalent to our natural (almost) Saito structures when G is a duality groups of rank ≤ 3 . They also found a universal formula [2, Theorem 6.1] for the multiplication \star , which is a generalization of the formula for the Coxeter case (see [6, Eq. (5.21)]).

For the duality groups of rank two, we list the natural Saito structure in Tables 5, 6, 7, 8. These results agree with the computations in [2]. From these tables, we can see that some admit a Frobenius structure, but some do not.

9.3. G(m, p, n) which are not duality groups. The degrees of G(m, p, n) are

$$m, 2m, \ldots, (n-1)m, \frac{nm}{p}.$$

In fact, $\sigma_i := e_i\left((u^1)^m, \ldots, (u^n)^m\right)$ $(1 \le i \le n-1)$ together with $\sigma_n^{m,p} := e_n(u_1, \ldots, u_n)^{m/p}$, where e_k denotes the k-th elementary symmetric polynomial, form a set of basic invariants. If p > 1 and n > 1, the maximal degree is (n-1)m which is of multiplicity one if $(p,n) \ne (2,2)$. In the following, we consider the cases of irreducible G(m,p,n)'s which are not duality groups.

9.3.1. The case when the maximal degree has multiplicity one. First, we assume that either $1 and <math>n \ge 3$ or 2 and <math>n = 2. In these cases, a natural (almost) Saito structure is unique if exists since the maximal degree has multiplicity one. The group K = G(m, m, n) is a normal reflection subgroup of G = G(m, p, n) and $G/K = \mu_{m/p}$. If p > 1, they satisfy the assumptions (i)–(iv). We apply the construction of Proposition 8.5 to (G, K) = (G(m, p, n), G(m, m, n)). Let us set

$$x^{i} := \begin{cases} \sigma_{n-i} & (1 \le i \le n-1) \\ \sigma_{n}^{m,p} & (i=n) \end{cases}, \qquad y^{i} := \begin{cases} \sigma_{n-i} & (1 \le i \le n-1) \\ \sigma_{n}^{m,m} & (i=n) \end{cases}.$$

We put $d_i^G := \deg x^i$ and $d_i^K := \deg y^i$. The map

$$\pi: M_K = \operatorname{Spec} \mathbb{C}[y_1, \dots, y_n] \to M_G = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$$

defined by $x^i = y^i$ $(1 \le i \le n-1)$ and $x^n = (y^n)^{m/p}$ is a $\mu_{m/p}$ -covering branched at $\{x^n = 0\}$. Let $(\nabla, *, \frac{1}{d_1} E_{\text{deg}}^K)$ be the natural polynomial Saito structure on M_K and

 $(\nabla^{\pi}, *_{\pi}, \frac{1}{d_1} E_{\text{deg}}^G)$ be the natural Saito structure on M'_G induced by π . Then we have the following.

Proposition 9.1. (i) The multiplication $*_{\pi}$ is defined on the whole orbit space M_G . (ii) The connection ∇^{π} has a logarithmic pole along $\{x^n = 0\}$.

Proof. (i) Define C_{ij}^k (resp. $(C_\pi)_{\alpha\beta}^{\gamma}$) by

$$\partial_{y^i} * \partial_{y^j} = \sum_k C_{ij}^k \partial_{y^k} \quad \left(\text{resp. } \partial_{x^\alpha} * \partial_{x^\beta} = \sum_{\gamma} (C_\pi)_{\alpha\beta}^{\gamma} \partial_{x^\gamma} \right) .$$

We show that $(C_{\pi})_{\alpha\beta}^{\gamma}$ is a polynomial in x. It is easy to see that

$$(C_{\pi})_{\alpha\beta}^{\gamma} = \sum_{i,j,k} C_{ij}^{k} \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial y^{j}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial y^{k}} = \begin{cases} C_{\alpha\beta}^{\gamma} & \text{if } \alpha \neq n, \ \beta \neq n, \ \gamma \neq n \\ C_{\alpha\beta}^{n} \frac{m}{p} (y^{n})^{m/p-1} & \text{if } \alpha \neq n, \ \beta \neq n, \ \gamma = n \\ C_{n\beta}^{\gamma} \frac{1}{m/p} (y^{n})^{1-m/p} & \text{if } \alpha = n, \ \beta \neq n, \ \gamma \neq n \\ C_{n\beta}^{n} & \text{if } \alpha = n, \ \beta \neq n, \ \gamma = n \\ C_{nn}^{\gamma} \frac{1}{(m/p)^{2}} (y^{n})^{2(1-m/p)} & \text{if } \alpha = n, \ \beta = n, \ \gamma \neq n \\ C_{nn}^{n} \frac{1}{m/p} (y^{n})^{1-m/p} & \text{if } \alpha = n, \ \beta = n, \ \gamma = n \end{cases}$$

The omitted cases are equivalent to one of the above cases under the symmetry of α and β . On the other hand, $\mu_{m/p}$ -invariance of the multiplication * implies that $(y^n)^k$ appears in the polynomial $C_{\alpha\beta}^{\gamma}$ only if

$$k \equiv \left\{ \begin{array}{ll} 0 & \text{if } \alpha \neq n, \ \beta \neq n, \ \gamma \neq n \\ 1 & \text{if } \alpha \neq n, \ \beta \neq n, \ \gamma = n \\ \frac{m}{p} - 1 & \text{if } \alpha = n, \ \beta \neq n, \ \gamma \neq n \\ 0 & \text{if } \alpha = n, \ \beta \neq n, \ \gamma = n \\ \frac{m}{p} - 2 & \text{if } \alpha = n, \ \beta = n, \ \gamma \neq n \\ \frac{m}{p} - 1 & \text{if } \alpha = n, \ \beta = n, \ \gamma = n \end{array} \right. \pmod{\frac{m}{p}} \; .$$

Comparing the above two equations, it is clear that $C_{\alpha\beta}^{\gamma}$ is a polynomial in x^n except for the case $\alpha = n, \ \beta = n, \ \gamma \neq n$. To see that $C_{nn}^{\gamma} \ (\gamma \neq n)$ is a polynomial, it is enough to show that deg $C_{nn}^{\gamma} > (\frac{m}{p} - 2)d_n^K$. This holds since

$$\deg\,C_{nn}^{\gamma} - (\frac{m}{p}-2)d_n^K = \left(d_1^K + d_{\gamma}^K - 2d_n^K\right) - (\frac{m}{p}-2)d_n^K = d_1^G + d_{\gamma}^G - d_n^G > 0\ .$$

Note that the last inequality holds for any $\gamma \neq n$ if and only if p>1 .

(ii) Let Γ_{ij}^k (resp. $(\Gamma^{\pi})_{\alpha\beta}^{\gamma}$) be the Christoffel symbol of ∇ (resp. ∇^{π}) defined by

$$\nabla_{\partial_{y^i}} \partial_{y^j} = \sum_k \Gamma^k_{ij} \partial_{y^k} \quad \left(\text{resp. } \nabla^\pi_{\partial_{x^\alpha}} \partial_{x^\beta} = \sum_{\gamma} (\Gamma^\pi)^{\gamma}_{\alpha\beta} \partial_{x^\gamma} \right) .$$

We have

$$(\Gamma^{\pi})_{\alpha\beta}^{\gamma} = \sum_{i,j,k} \Gamma_{ij}^{k} \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial y^{j}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial y^{k}} + \sum_{l} \frac{\partial^{2} y^{l}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial y^{l}} = \Gamma_{\alpha\beta}^{\gamma} \frac{\partial y^{\alpha}}{\partial x^{\alpha}} \frac{\partial y^{\beta}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial y^{\gamma}} + \frac{\partial^{2} y^{\gamma}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial y^{\gamma}}.$$

By the same argument as above, the $\mu_{m/p}$ -invariance of ∇ implies that the first term of the above equation is a polynomial except for the case $\alpha = n, \ \beta = n, \ \gamma \neq n$. But in fact one can show that $\Gamma_{nn}^{\gamma} = 0$ for any γ , see Lemma 9.2 below. So the first term is regular for any α , β , γ . For the second term, it is clear that the term is zero unless $\alpha = \beta = \gamma = n$ and

$$\frac{\partial^2 y^n}{\partial x^n \partial x^n} \frac{\partial x^n}{\partial y^n} = \frac{1}{x^n} .$$

Hence $(\Gamma^{\pi})_{\alpha\beta}^{\gamma}$ is regular unless $\alpha = \beta = \gamma = n$ and $(\Gamma^{\pi})_{nn}^{n}$ has a logarithmic pole along $\{x^{n} = 0\}.$

Lemma 9.2. Under the same notation as in the above proof, we have $\Gamma_{nn}^{\gamma} = 0$ for any γ .

Proof. Let K = G(m, m, n) and G = G(m, 1, n). Note that their discriminants are related as $\Delta_G = x^n \Delta_K$ under $x^i = y^i$ $(1 \le i \le n-1)$ and $x^n = (y^n)^m$. Let $(\Omega^K)_{\alpha\beta}^{\gamma}$ (resp. $(\Omega^G)_{\alpha\beta}^{\gamma}$) be the Christoffel symbol of the natural connection $\nabla^{V,K}$ (resp. $\nabla^{V,G}$) on $M_{K,0}$ (resp. $M_{G,0}$). For $\gamma \ne n$, we have

$$(\Omega^G)_{nn}^{\gamma} = \frac{1}{m^2} (y^n)^{2(1-m)} (\Omega^K)_{nn}^{\gamma}.$$

So we have

$$\Delta_G(\Omega^G)_{nn}^{\gamma} = \frac{1}{m^2} (y^n)^{2-m} \Delta_K(\Omega^K)_{nn}^{\gamma}.$$

By Proposition 6.1, we have $\Delta_G(\Omega^G)_{nn}^{\gamma} \in \mathbb{C}[x]$. It follows that $\Delta_K(\Omega^K)_{nn}^{\gamma}$ is divisible by $(y^n)^{m-2}$. It then follows that Γ_{nn}^{γ} is also divisible by $(y^n)^{m-2}$, see (7.1). In particular, if $\deg \Gamma_{nn}^{\gamma} = d_{\gamma}^K - 2d_n^K < (m-2)d_n^K$ then $\Gamma_{nn}^{\gamma} = 0$. The above inequality certainly holds for $\gamma \neq n$ since it is equivalent to $d_{\gamma}^K < md_n^K = mn$. Thus we have $\Gamma_{nn}^{\gamma} = 0$ for $\gamma \neq n$. By the degree reason, we also have $\Gamma_{nn}^n = 0$.

Remark 9.3. One can also consider the case p = 1, i.e. the case when G = G(m, 1, n) and K = G(m, m, n). In this case too, the μ_m -covering $\pi : M_K = \operatorname{Spec} \mathbb{C}[y_1, \ldots, y_n] \to M_G = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ branched along $\{x^n = 0\}$ induces a Saito structure on $M'_G = M_G \setminus \{x^n = 0\}$. However this is not a natural Saito structure for G since the maximal degrees of G and K are different. Moreover both the multiplication and the connection have a logarithmic pole along $\{x^n = 0\}$.

9.3.2. The case when the maximal degree has multiplicity two: G(2k,2,2) with k > 1. The degrees of G(2k,2,2) satisfy $d_1 = d_2$, so the space of the vector fields \mathcal{X}_{-d_1} of degree $-d_1$ is two-dimensional. We can see that (∇^V, e) is regular for any nonzero $e \in \mathcal{X}_{-d_1}$. We can also see that there exist only three lines $l_1, l_2, l_3 \subset \mathcal{X}_{-d_1}$ such that $e \in l_i$ ($e \neq 0$) and ∇^V make a natural almost Saito structure. They are the ones induced by branched covering maps $\pi: M_K \to M_{G(2k,2,2)}$ from the orbit spaces of certain duality groups K. See Table 10 for the covering maps. The connections of the natural Saito structures are logarithmic along the ramification loci of the branched covering maps. However, one can check that all the multiplications are defined over the whole space $M_{G(2k,2,2)}$.

9.4. Exceptional groups which are not duality groups.

9.4.1. Nonexistence of natural Saito structures for G_{12} , G_{13} , G_{22} and G_{31} . Let G be one of G_{12} , G_{13} , G_{22} (of rank n = 2) and G_{31} (of rank n = 4). The degrees satisfy $d_1 > d_2$. So a choice of a nonzero vector field $e = \partial_{x^1}$ of degree $-d_1$ is unique up to scalar multiplication. The discriminant Δ of G is monic of degree n + 1 in x^1 (and hence G is not a duality group).

The pair (∇^V, e) is regular since the matrix representation Ω_1 of $\nabla^V e$ is invertible on M_0 . Then the multiplication \star on TM_0 defined by (ASS4) or (2.10) is represented by matrices (see (5.14))

$$B_{\alpha} = -\Omega_1^{-1} \partial_{x^1} \Omega_{\alpha} \quad (1 \le \alpha \le n) .$$

We can check that these B_{α} 's do not satisfy (ASS2) (or (5.13)). Thus by Lemma 6.6, G does not admit any natural almost Saito structure, hence it does not admit any natural Saito structure.

- 9.4.2. The case of G_{15} . The group G_{15} satisfies the condition $d_1 > d_2$. Therefore a natural almost Saito structure is unique if exists. It contains G_{14} as a normal subgroup and the pair $(G, K) = (G_{15}, G_{14})$ satisfy the assumptions (i)–(iv). See Table 13. Therefore by Proposition 8.5, the branched covering map $\pi: M_{G_{14}} \to M_{G_{15}}$ induces a unique natural almost Saito structure and a unique natural Saito structure for G_{15} from that for G_{14} . So the situation is similar to the cases in §9.3.1. One can check that the connection of the natural Saito structure for G_{15} is logarithmic along the branch locus of π but the multiplication is defined all over the orbit space $M_{G_{15}}$.
- 9.4.3. The cases of G_7 , G_{11} and G_{19} . These rank 2 groups are not duality groups. Let G be one of the three groups. The degrees of G satisfy $d_1 = d_2$. We can see that (∇^V, e) is regular for any nonzero $e \in \mathcal{X}_{-d_1}$. So the situation is completely analogous to that of G(2k, 2, 2) with k > 1 (cf. §9.3.2). One can show that there exist only three lines l_1, l_2, l_3 in the 2-dimensional space \mathcal{X}_{-d_1} such that $e \in l_i$ ($e \neq 0$) and ∇^V make a natural almost Saito

structure. They are the ones induced by branched covering maps $\pi: M_K \to M_G$ from the orbit spaces of certain duality groups K. See Tables 11, 12, 14. The connections of the natural Saito structures are logarithmic along the branch loci but the multiplications are defined over the whole space M_G .

Appendix A. Proofs of Proposition 6.1 and Lemma 6.2

In this section, notations are the same as in §6.1.

Let us fix a reflection hyperplane of G, say, $H_1 \in \mathcal{A}$ and set $e_1 = e_{H_1}$, the order of the cyclic subgroup G_{H_1} preserving H_1 pointwise. Let us fix a basis e_1, \ldots, e_n of V as follows: e_1 is an eigenvector of $g \in G_{H_1}$ ($g \neq \operatorname{Id}$) whose eigenvalue is not one; e_2, \ldots, e_n form a basis of H_1 which is the eigenspace of g with eigenvalue one. Denote by (v_1, \ldots, v_n) the associated complex coordinates of V. It is clear that $L_{H_1} = \operatorname{const.} v_1$.

A.1. Lemmas. Let us set

$$\Pi' = \prod_{H \in \mathcal{A} \setminus \{H_1\}} L_H^{e_H - 1} = \frac{\Pi}{L_{H_1}^{e_1 - 1}}.$$

Lemma A.1. Π' is G_{H_1} -invariant.

Proof. The action of $g \in G_{H_1}$ can be written, with some e_1 -th root of unity μ , as $g\mathbf{e}_1 = \mu\mathbf{e}_1$ and $g\mathbf{e}_i = \mathbf{e}_i$ $(i \neq 1)$. Therefore $g(v_1) = \mu^{-1}v_1$, $g(v_i) = v_i$ $(i \neq 1)$. Since Π is skew-invariant, $g(\Pi) = \mu\Pi$. Therefore $g(\Pi') = g(\Pi)/g(L_{H_1})^{e_1-1} = g(\Pi')$.

For the sake of convenience, we introduce the notation

$$f \succ g \stackrel{\text{def.}}{\Longrightarrow} \frac{f}{g} \in \mathbb{C}(v_2, \dots, v_n)[v_1^{e_1}] \quad (f, g \in \mathbb{C}(v_1, \dots, v_n)).$$

Note the following facts.

- If $f_1 \succ g_1$ and $f_2 \succ g_2$ then $f_1 f_2 \succ g_1 g_2$.
- If $f \in \mathbb{C}[v]$ is a G_{H_1} -invariant polynomial, then $f \succ 1$.
- If $f \succ 1$ then

$$\frac{\partial f}{\partial v_1} \succ v_1^{e_1 - 1} , \quad \frac{\partial f}{\partial v_i} \succ 1 \quad (i \neq 1) .$$

• If $f \succ v^p \ (p \neq 0)$, then

$$\frac{\partial f}{\partial v_1} \succ v_1^{p-1} , \quad \frac{\partial f}{\partial v_i} \succ v^p \quad (i \neq 1) .$$

For $1 \le \alpha \le n$ and $1 \le i \le n$, we put

$$w_{\alpha}^{i} = \frac{\partial v^{i}}{\partial x^{\alpha}}, \quad z_{\alpha,j}^{i} = \frac{\partial w_{\alpha}^{i}}{\partial v^{j}}, \quad z_{\alpha} = (z_{\alpha,j}^{i})_{i,j}.$$

Lemma A.2.

(A.1)
$$\frac{\partial x^{\alpha}}{\partial v_i} \succ \begin{cases} v_1^{e_1 - 1} & (i = 1) \\ 1 & (i \neq 1) \end{cases}.$$

(A.2)
$$w_{\alpha}^{i} \succ \begin{cases} v_{1}^{1-e_{1}} & (i=1) \\ 1 & (i \neq 1) \end{cases}$$

(A.3)
$$z_{\alpha,j}^{i} \succ \begin{cases} v_{1}^{-e_{1}} & (i=j=1) \\ v_{1}^{1-e_{1}} & (i=1, j \neq 1) \\ v_{1}^{e_{1}-1} & (i \neq 1, j = 1) \\ 1 & (i \neq 1, j \neq 1) \end{cases} .$$

(A.4)
$$\det z_{\alpha} \succ v_1^{-e_1}.$$

(A.5)
$$(z_{\alpha}^{-1})_{i}^{j} \succ \frac{1}{\det z_{\alpha}} \times \begin{cases} 1 & (i = j = 1) \\ v_{1}^{e_{1}-1} & (i = 1, j \neq 1) \\ v_{1}^{1-e_{1}} & (i \neq 1, j = 1) \\ v_{1}^{-e_{1}} & (i, j \neq 1) \end{cases}$$

In (A.5), we assume $\det z_{\alpha} \neq 0$.

Proof. (A.1) holds since x^{α} is a G-invariant polynomial.

(A.2): since the determinant of the Jacobian matrix is proportional to Π ,

$$\begin{split} \frac{\partial v^i}{\partial x^\alpha} &= \det \left(\frac{\partial x^\alpha}{\partial v^i}\right)^{-1} \times \text{the } (\alpha,i) \text{ cofactor of } \left(\frac{\partial x^\alpha}{\partial v^i}\right) \\ & \succ \frac{1}{\Pi} \times \begin{cases} 1 & (i=1) \\ v_1^{e_1-1} & (i\neq 1) \end{cases} \\ & \succ \begin{cases} v_1^{1-e_1} & (i=1) \\ 1 & (i\neq 1) \end{cases} \end{split}$$

(A.3) follows from (A.2). (A.4) and (A.5) follow from (A.3).

Corollary A.3. For $1 \le \alpha, \beta, \mu, \gamma \le n$ and $1 \le i, j \le n$,

(A.6)
$$\frac{\partial x^{\gamma}}{\partial v^{i}} z_{\alpha,j}^{i} w_{\beta}^{j} \succ \begin{cases} v_{1}^{-e_{1}} & (i=j=1) \\ 1 & ((i,j) \neq (1,1)) \end{cases}.$$

(A.7)
$$\frac{\partial x^{\gamma}}{\partial v^{j}} (z_{\mu}^{-1})^{j}{}_{i} w_{\beta}^{i} \succ \frac{1}{\det z_{\mu}} \times \begin{cases} 1 & (i = 1 \text{ or } j = 1) \\ v_{1}^{-e_{1}} & (i \neq 1 \text{ and } j \neq 1) \end{cases}$$

$$(A.8) \quad \frac{\partial x^{\gamma}}{\partial v^{j}} (z_{\mu}^{-1})^{j}{}_{i} z_{\alpha,k}^{i} w_{\beta}^{k} \quad \succ \quad \frac{1}{\det z_{\mu}} \times \begin{cases} v_{1}^{-e_{1}} & (i = k = 1 \text{ or } i \neq 1 \text{ and } j \neq 1) \\ 1 & (else) \end{cases}$$

(A.9)
$$(z_{\mu}^{-1})^{j}{}_{i}w^{i}{}_{\gamma} \succ \begin{cases} v_{1} & (j=1) \\ 1 & (j \neq 1) \end{cases}$$

In (A.7) (A.8) (A.9), we assume $\det z_{\mu} \not\equiv 0$.

A.2. **Proof of Proposition 6.1.** Notice that $\Omega_{\alpha\beta}^{\gamma}$ can have poles only along reflection hyperplanes and that

$$\Omega_{\alpha\beta}^{\gamma} = \sum_{i,i=1}^{n} \frac{\partial x^{\gamma}}{\partial v^{i}} z_{\alpha,j}^{i} w_{\beta}^{j}, \quad \det \Omega_{\alpha} = \det z_{\alpha}, \quad \delta \succ v_{1}^{e_{1}}.$$

(6.5): by (A.6), $\delta \cdot \Omega_{\alpha\beta}^{\gamma}$ does not have a pole along the hyperplane H_1 . Given that $H_1 \in \mathcal{A}$ can be taken arbitrarily, $\delta \cdot \Omega_{\alpha\beta}^{\gamma}$ does not have a pole along any reflection hyperplanes. This implies that $\delta \cdot \Omega_{\alpha\beta}^{\gamma}$ is a polynomial in v. Since both δ and $\Omega_{\alpha\beta}^{\gamma}$ are G-invariant, so is $\delta \cdot \Omega_{\alpha\beta}^{\gamma}$.

(6.6): by (A.4), $\delta \det \Omega_{\alpha} = \delta \det z_{\alpha}$ does not have a pole along H_1 . Given that H_1 is taken arbitrary, $\delta \det \Omega_{\alpha}$ is a polynomial in v. Since both δ and $\det \Omega_{\alpha}$ are G-invariant, so is $\delta \det \Omega_{\alpha}$. Since $\det (\delta \cdot \det \Omega_{\alpha}) = \det \delta - nd_{\alpha}$, $\delta \cdot \det \Omega_{\alpha}$ is a constant if $\det \delta = nd_{\alpha}$.

(6.7) (6.8): notice that

$$\sum_{i,j=1}^{n} \frac{\partial x^{\gamma}}{\partial v^{j}} (z_{\mu}^{-1})^{j}_{i} w_{\beta}^{i} = (\Omega_{\mu}^{-1})^{\gamma}_{\beta} , \quad \sum_{i,j,k=1}^{n} \frac{\partial x^{\gamma}}{\partial v^{j}} (z_{\mu}^{-1})^{j}_{i} z_{\alpha,k}^{i} w_{\beta}^{k} = (\Omega_{\mu}^{-1} \Omega_{\alpha})^{\gamma}_{\beta} .$$

If deg $\delta = nd_{\mu}$ and det $\Omega_{\mu} \neq 0$, then by (6.6),

$$\det z_{\mu} = \det \Omega_{\mu} = \frac{\text{nonzero const.}}{\delta} .$$

Then (A.7) (A.8) imply that $(\Omega_{\mu}^{-1})_{\beta}^{\gamma}$ and $(\Omega_{\mu}^{-1}\Omega_{\alpha})_{\beta}^{\gamma}$ do not have poles along H_1 . So they do not have poles along any reflection hyperplanes. Therefore they are G-invariant polynomials. This completes the proof of Proposition 6.1.

A.3. **Proof of Lemma 6.2.** Notice that $\Omega_{\mu}J = -Jz_{\mu}$. Therefore by Corollary A.9, each entry of $J^{-1}\Omega_{\mu}^{-1} = -z_{\mu}^{-1}J^{-1}$ does not have pole along any hyperplane. Thus it is a polynomial on V.

APPENDIX B. PROOF OF LEMMA 8.4

We use the same notation as §8. For the case when G/K is abelian, let us show the next sub-lemma.

Lemma B.1. If G/K is abelian, then there exists a set of basic invariants $y^1, \ldots, y^n \in \mathbb{C}[V]^K$ of K such that each y^{α} is a semi-invariant of G/K.

Proof. First, we claim that there exists a set of basic invariants z^1, \ldots, z^n of K of deg $z^{\alpha} = d_{\alpha}^K$ such that

(B.1)
$$g(z^{\alpha}) = \chi_{\alpha}(g)z^{\alpha} + \text{ a polynomial in } z^{\alpha+1}, \dots, z^n$$

holds for some characters χ_{α} of G/K.

Let d be one of the degrees of K and denote by $I_d \subset \{1, \ldots, n\}$ the set of integers α such that deg $z^{\alpha} = d$. Set $R_d := \bigoplus_{\alpha \in I_d} \mathbb{C}z^{\alpha}$. Since the action of G preserves the grading on $\mathbb{C}[V]^K = \mathbb{C}[z^1, \ldots, z^n]$, the action of $g \in G/K$ on z^{α} ($\alpha \in I_d$) can be written as

(B.2)
$$g(z^{\alpha}) = \sum_{\beta \in I_d} A_{\alpha\beta}^g z^{\beta} + \text{a polynomial in } z^{\gamma} \text{'s with deg } z^{\gamma} < d, \quad (A_{\alpha\beta}^g \in \mathbb{C}).$$

By the algebraic independence of z^{α} , it follows that R_d is a representation of G/K via $A^g = (A^g_{\alpha\beta})$. Since the group G/K is finite and abelian, we may assume that z^{α} ($\alpha \in I_d$) are eigenvectors of A^g for any $g \in G_K$. The claim follows from this.

Next, we put

(B.3)
$$y^{\alpha} = \frac{1}{|G/K|} \sum_{g \in G/K} \chi_{\alpha}(g)^{-1} g(z^{\alpha}) .$$

Then it is immediate to show that $g(y^{\alpha}) = \chi_{\alpha}(g)y^{\alpha}$. Moreover, we have

$$y^{\alpha} = z^{\alpha} + \text{ a polynomial in } z^{\alpha+1}, \dots, z^n$$
.

So y^1, \ldots, y^n are algebraically independent and form a set of basic invariants of K.

Lemma B.2. If $d_1^K > d_2^K$, then there exists a set of basic invariants y^1, \ldots, y^n of K of $\deg y^{\alpha} = d_{\alpha}^K$ such that y^1 is a semi-invariant of G.

Proof. If z^1, \ldots, z^n are a set of basic invariants of K of deg $z^{\alpha} = d_{\alpha}^K$, then

(B.4)
$$g(z^1) = A^g z^1 + \text{ a polynomial in } z^2, \dots, z^n, \quad (A^g \in \mathbb{C})$$

holds for any $g \in G/K$ because the action of G/K preserves the grading. Then by the same argument as in Lemma B.1, we can show that $\chi_1 : G/K \to \mathbb{C}^*$ given by $\chi_1(g) = A^g$

is a character of G/K. Therefore if we take y^1 as in (B.3), then $g(y^1) = \chi_1(g)y^1$ and y^1, z^2, \ldots, z^n form a set of basic invariants of K.

Now we prove Lemma 8.4. By the above lemmas, the first part was already proved. Therefore assume $d_1^K = d_1^G$. Let k_G be the number of degrees of G which are equal to d_1^G . Note that $k_G \geq k_K$ since $\mathbb{C}[V]^G \subset \mathbb{C}[V]^K = \mathbb{C}[y^1, \ldots, y^n]$, where y^1, \ldots, y^{k_K} are G-semi-invariants with characters $\chi_1, \ldots, \chi_{k_K}$. Take a set of basic invariants x^1, \ldots, x^n of G with deg $x^\alpha = d_\alpha^G$. Express x^1, \ldots, x^{k_G} as polynomials in y^1, \ldots, y^n :

$$x^{\alpha} = \sum_{\beta=1}^{k_K} X_{\alpha\beta} y^{\beta} + \text{a polynomial in } y^{k_K+1}, \dots, y^n, \quad (X_{\alpha\beta} \in \mathbb{C}) .$$

If the rank of the $k_G \times k_K$ matrix $X = (X_{\alpha\beta})$ is smaller than k_K , then this contradicts the algebraic independence of x and y. Therefore rank $X = k_K$ and we may assume that x^1, \ldots, x^{k_G} are chosen so that

$$x^{\alpha} = \begin{cases} y^{\alpha} + \text{a polynomial in } y^{\alpha+1}, \dots, y^{n} & (\text{if } 1 \leq \alpha \leq k_{K}) \\ \text{a polynomial in } y^{k_{K}+1}, \dots, y^{n} & (\text{if } \alpha > k_{K}) \end{cases}.$$

Now look at the action of $g \in G/K$ on x^{α} $(1 \le \alpha \le k_K)$:

(B.5)
$$x^{\alpha} = g(x^{\alpha}) = \chi_{\alpha}(g)y^{\alpha} + \text{a polynomial in } y^{\alpha+1}, \dots, y^{n}.$$

Then $\chi_{\alpha}(g) = 1$ follows from the algebraic independence of y. Thus y^1, \ldots, y^{k_K} are G/K-invariant.

APPENDIX C. RANK TWO EXAMPLES

This section contains tables for the irreducible finite complex reflection groups of rank two. In Tables 1, 2, 3, 4, the degrees d_1, d_2 , the discriminant Δ , a set of basic invariants x, y are shown for each G.

For the duality groups, we list a unique (up to equivalence) natural Saito structure in Tables 5, 6, 7, 8. Flat coordinates are denoted t^1, t^2 and ∂_{t^1} is assumed to be a unit of the multiplication *. For the remaining groups G = G(m, p, 2), G(2k, 2, 2) and $G_7, G_{11}, G_{15}, G_{19}$, we list covering maps $\pi : M_K \to M_G$ which induce natural (almost) Saito structures. In the tables, (x, y) denotes the set of basic invariants for G listed in Tables 1, 2, 3, 4 and (x', y') denotes the set of basic invariants for the normal subgroup K.

Let
$$i = \sqrt{-1}$$
, $\zeta_m = e^{\frac{2\pi i}{m}}$, $\tau = \zeta_5 + \zeta_5^{-1} + 1$ and
$$\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_m = \begin{pmatrix} \zeta_m & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_m = \tau_m^{-1} \rho \tau_m , \quad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$r_1 = \frac{\zeta_3}{2} \begin{pmatrix} -1 - i & 1 - i \\ -1 - i & -1 + i \end{pmatrix}, \quad r_2 = \frac{\zeta_3}{2} \begin{pmatrix} -1 + i & -1 + i \\ 1 + i & -1 - i \end{pmatrix}, \quad s = \begin{pmatrix} \zeta_8^{-1} & 0 \\ 0 & \zeta_8 \end{pmatrix}$$
$$r_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad r_4 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad r_5 = \frac{\zeta_5^2}{\sqrt{2}} \begin{pmatrix} -\tau + i & -\tau + 1 \\ \tau - 1 & -\tau - i \end{pmatrix}.$$

G		degrees	Δ	x	y	duality group?
G(m, 1, 2)	$(m \ge 2)$	2m, m	$u^m v^m (u^m - v^m)^2$	$u^m v^m$	$u^m + v^m$	yes
G(m, m, 2)	$(m \ge 3)$	m, 2	$(u^m - v^m)^2$	$u^m + v^m$	uv	yes
G(kp, p, 2)	(p > 2, k > 1)	kp, 2k	$u^k v^k (u^{kp} - v^{kp})^2$	$u^{kp} + v^{kp}$	$u^k v^k$	no
G(2k, 2, 2)	(k > 1)	2k, 2k	$u^k v^k (u^{2k} - v^{2k})^2$	$u^{2k} + v^{2k}$	$u^k v^k$	no

Table 1. The monomial groups of rank two.

G	degrees	x	y	Δ	duality group?
G_4	6, 4	$t_{\mathcal{T}}$	$f_{\mathcal{T}}$	$h_{\mathcal{T}}^3$	yes
G_5	12,6	$f_{\mathcal{T}}^3$	$t_{\mathcal{T}}$	$f_{\mathcal{T}}^3 h_{\mathcal{T}}^3$	yes
G_6	12, 4	$t_{\mathcal{T}}^2$	$f_{\mathcal{T}}$	$h_{\mathcal{T}}^3 t_{\mathcal{T}}^2$	yes
G_7	12 , 12	$f_{\mathcal{T}}^3$	$t_{\mathcal{T}}^2$	$f_{\mathcal{T}}^3 h_{\mathcal{T}}^3 t_{\mathcal{T}}^2$	no

$$f_{\mathcal{T}}(u,v) = u^4 + 2i\sqrt{3}u^2v^2 + v^4,$$

$$h_{\mathcal{T}}(u,v) = u^4 - 2i\sqrt{3}u^2v^2 + v^4,$$

$$t_{\mathcal{T}}(u,v) = u^5v - uv^5.$$

Table 2. The exceptional groups of type \mathcal{T} . Note $f_{\mathcal{T}}^3 - h_{\mathcal{T}}^3 = 12i\sqrt{3}\,t_{\mathcal{T}}^2$.

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G	degrees	x	y	Δ	duality group?
G_8	12,8	$t_{\mathcal{O}}$	$h_{\mathcal{O}}$	$f_{\mathcal{O}}^4$	yes
G_9	24,8	$t_{\mathcal{O}}^2$	$h_{\mathcal{O}}$	$f_{\mathcal{O}}^4 t_{\mathcal{O}}^2$	yes
G_{10}	24 , 12	$h_{\mathcal{O}}^3$	$t_{\mathcal{O}}$	$f_{\mathcal{O}}^4 h_{\mathcal{O}}^3$	yes
G_{11}	24, 24	$h_{\mathcal{O}}^3$	$t_{\mathcal{O}}^2$	$f_{\mathcal{O}}^4 h_{\mathcal{O}}^3 t_{\mathcal{O}}^2$	no
G_{12}	8,6	$h_{\mathcal{O}}$	$f_{\mathcal{O}}$	$t_{\mathcal{O}}^2$	no
G_{13}	12,8	$f_{\mathcal{O}}^2$	$h_{\mathcal{O}}$	$f_{\mathcal{O}}^2 t_{\mathcal{O}}^2$	no
G_{14}	24,6	$t_{\mathcal{O}}^2$	$f_{\mathcal{O}}$	$h_{\mathcal{O}}^3 t_{\mathcal{O}}^2$	yes
G_{15}	24 , 12	$t_{\mathcal{O}}^2$	$f_{\mathcal{O}}^2$	$f_{\mathcal{O}}^2 h_{\mathcal{O}}^3 t_{\mathcal{O}}^2$	no

$$f_{\mathcal{O}} = u^5 v - u v^5 ,$$

$$h_{\mathcal{O}} = u^8 + 14u^4 v^4 + v^8 ,$$

$$t_{\mathcal{O}} = u^{12} - 33u^8 v^4 - 33u^4 v^8 + v^{12} .$$

Table 3. The exceptional groups of type \mathcal{O} . Note $h_{\mathcal{O}}^3 - t_{\mathcal{O}}^2 = 108 f_{\mathcal{O}}^4$.

G	degrees	x	y	Δ	duality group?
G_{16}	30, 20	$t_{\mathcal{I}}$	$h_{\mathcal{I}}$	$f_{\mathcal{I}}^{5}$	yes
G_{17}	60, 20	$t_{\mathcal{I}}^2$	$h_{\mathcal{I}}$	$f_{\mathcal{I}}^5 t_{\mathcal{I}}^2$	yes
G_{18}	60, 30	$h_{\mathcal{I}}^3$	$t_{\mathcal{I}}$	$f_{\mathcal{I}}^5 h_{\mathcal{I}}^3$	yes
G_{19}	60,60	$h_{\mathcal{I}}^3$	$t_{\mathcal{I}}^2$	$f_{\mathcal{I}}^5 h_{\mathcal{I}}^3 t_{\mathcal{I}}^2$	no
G_{20}	30 , 12	$t_{\mathcal{I}}$	$f_{\mathcal{I}}$	$h_{\mathcal{I}}^3$	yes
G_{21}	60 , 12	$t_{\mathcal{I}}^2$	$f_{\mathcal{I}}$	$h_{\mathcal{I}}^3 t_{\mathcal{I}}^2$	yes
G_{22}	20 , 12	$h_{\mathcal{I}}$	$f_{\mathcal{I}}$	$t_{\mathcal{I}}^2$	no

$$f_{\mathcal{I}} = u^{12} + \frac{22u^{10}v^2}{\sqrt{5}} - 33u^8v^4 - \frac{44u^6v^6}{\sqrt{5}} - 33u^4v^8 + \frac{22u^2v^{10}}{\sqrt{5}} + v^{12} ,$$

$$h_{\mathcal{I}} = \frac{\sqrt{5}}{5808} \text{hess} (f_{\mathcal{I}}) = u^{20} + \cdots ,$$

$$t_{\mathcal{I}} = -\frac{1}{480\sqrt{5}} \frac{\partial (f_{\mathcal{I}}, h_{\mathcal{I}})}{\partial (u, v)} = u^{29}v + \cdots ,$$

Table 4. The exceptional groups of type \mathcal{I} . Note that $f_{\mathcal{I}}^5 = h_{\mathcal{I}}^3 + 60\sqrt{5}\,t_{\mathcal{I}}^2$.

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G	t^1	t^2	$\partial_{t^2} * \partial_{t^2}$	Frobenius manifold?
G(m, 1, 2)	$u^m v^m - \frac{(u^m + v^m)^2}{4m}$	$u^m + v^m$	$\frac{m-1}{4m^2}(t^2)^2\partial_{t^1} - \frac{m-2}{2m}t^2\partial_{t^2}$	no if $m \neq 2$
G(m, m, 2)	$u^m + v^m$	uv	$m^2(t^2)^{m-2}\partial_{t^1}$	yes

TABLE 5. Natural Saito structures for G(m, 1, 2) and G(m, m, 2).

G	t^1	t^2	$\partial_{t^2} * \partial_{t^2}$	Frobenius manifold?
G_4	$t_{\mathcal{T}}$	$f_{\mathcal{T}}$	$-rac{i\sqrt{3}}{16}t^2\partial_{t^1}$	yes
G_5	$f_{\mathcal{T}}^3 - 6i\sqrt{3}t_{\mathcal{T}}^2$	$t_{\mathcal{T}}$	$-432(t^2)^2\partial_{t^1}$	yes
G_6	$t_{\mathcal{T}}^2 + \frac{5i}{96\sqrt{3}} f_{\mathcal{T}}^3$	$f_{\mathcal{T}}$	$-\frac{5}{2^{10}}(t^2)^4\partial_{t^1}-\frac{i}{16\sqrt{3}}(t^2)^2\partial_{t^2}$	no

Table 6. Natural Saito structure for the duality groups of type \mathcal{T} .

G	t^1	t^2	$\partial_{t^2} * \partial_{t^2}$	Frobenius manifold?
G_8	$t_{\mathcal{O}}$	$h_{\mathcal{O}}$	$\frac{9}{4}t^2\partial_{t^1}$	yes
G_9	$t_{\mathcal{O}}^2 - \frac{11}{16}h_{\mathcal{O}}^3$	$h_{\mathcal{O}}$	$\frac{495}{2^8}(t^2)^4\partial_{t^1} + \frac{9}{8}(t^2)^2\partial_{t^2}$	no
G_{10}	$h_{\mathcal{O}}^3 - \frac{7}{12}t_{\mathcal{O}}^2$	$t_{\mathcal{O}}$	$\frac{35}{36}(t^2)^2\partial_{t^1} + \frac{1}{3}t^2\partial_{t^2}$	no
G_{14}	$t_{\mathcal{O}}^2 + 66f_{\mathcal{O}}^4$	$f_{\mathcal{O}}$	$44352(t^2)^6\partial_{t^1} - 96(t^2)^3\partial_{t^2}$	no

Table 7. Natural Saito structures for duality groups of type \mathcal{O} .

G	t^1	t^2	$\partial_{t^2} * \partial_{t^2}$	Frobenius manifold?
G_{16}	$t_{\mathcal{I}}$	$h_{\mathcal{I}}$	$-rac{3}{80\sqrt{5}}t^2\partial_{t^1}$	yes
G_{17}	$t_{\mathcal{I}}^2 + \frac{29}{2400\sqrt{5}}h_{\mathcal{I}}^3$	$h_{\mathcal{I}}$	$\frac{319}{32 \cdot 10^5} (t^2)^4 \partial_{t^1} - \frac{9}{400\sqrt{5}} (t^2)^2 \partial_{t^2}$	no
G_{18}	$h_{\mathcal{I}}^3 + 38\sqrt{5}t_{\mathcal{I}}^2$	$t_{\mathcal{I}}$	$16720(t^2)^2\partial_{t^1} - 32\sqrt{5}t^2\partial_{t^2}$	no
G_{20}	$t_{\mathcal{I}}$	$f_{\mathcal{I}}$	$\frac{\sqrt{5}}{48}(t^2)^3\partial_{t^1}$	yes
G_{21}	$t_{\mathcal{I}}^2 - \frac{29}{2880\sqrt{5}} f_{\mathcal{I}}^5$	$f_{\mathcal{I}}$	$\frac{551}{2^{12} \cdot 3^4 \cdot 5} (t^2)^8 \partial_{t^1} + \frac{\sqrt{5}}{288} (t^2)^4 \partial_{t^2}$	no

Table 8. Natural Saito structures for the duality groups of type \mathcal{I} .

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e	K	$K \to G(kp, p, 2)$	$M_K \to M_{G(kp,p,2)}$
∂_x	$G(kp, kp, 2) = \langle \sigma_{kp}, \rho \rangle$	$g \mapsto g$	$(x',y')\mapsto (x',(y')^k)$

Table 9. A covering map for $G(kp, p, 2) = \langle \sigma_{kp}, \tau_{kp}^p, \rho \rangle \ (p > 2, k > 1)$

e	K	$K \to G(2k, 2, 2)$	$M_K \to M_{G(2k,2,2)}$
∂_x	$G(2k, 2k, 2) = \langle \sigma_{2k}, \rho \rangle$	$g \mapsto g$	$(x',y') \mapsto (x',(y')^k)$
$-2\partial_x + \partial_y$	$G(k,1,2) = \langle \tau_{2k}^2, \rho \rangle$	$g \mapsto g$	$(x', y') \mapsto ((y')^2 - 2x', x')$
$-2\partial_x - \partial_y$	$G(k,1,2) = \langle \tau_{2k}^2, \rho \rangle$	$g \mapsto \tau^{-1}g\tau$	$(x', y') \mapsto ((y')^2 - 2x', -x')$

Table 10. Covering maps for $G(2k, 2, 2) = \langle \sigma_{2k}, \tau_{2k}^2, \rho \rangle$ (k > 1).

e	K	$K \to G_7$	$M_K o M_{G_7}$
∂_x	$G_5 = \langle r_1, rr_2r \rangle$	$g \mapsto g$	$(x',y')\mapsto (x',(y')^2)$
∂_y	$G_6 = \langle r, r_1 \rangle$	$g \mapsto g$	$(x', y') \mapsto ((y')^3, x')$
$12i\sqrt{3}\partial_x + \partial_y$	$G_6 = \langle r, r_1 \rangle$	$g \mapsto s^{-1}gs$	$(x', y') \mapsto (-(y')^3 + 12i\sqrt{3}x', x')$

Table 11. Covering maps for $G_7 = \langle r, r_1, r_2 \rangle$.

e	K	$K \to G_{11}$	$M_K \to M_{G_{11}}$
∂_x	$G_{10} = \langle r_1, r_3^{-1} r_4 r_3 \rangle$	$g \mapsto g$	$(x', y') \mapsto (x', (y')^2)$
∂_y	$G_9 = \langle r_3, r_4 \rangle$	$g \mapsto g$	$(x',y') \mapsto ((y')^3,x')$
$\partial_x + \partial_y$	$G_{14} = \langle r_1, r_4^2 r_3 r_4^2 \rangle$	$g \mapsto g$	$(x', y') \mapsto (x' + 108(y')^4, x')$

Table 12. Covering maps for $G_{11} = \langle r_1, r_3, r_4 \rangle$.

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e	K	$K \to G_{15}$	$M_K \to M_{G_{15}}$
∂_x	$G_{14} = \langle r_1, rr_3 r \rangle$	$g \mapsto g$	$(x', y') \mapsto (x', (y')^2)$

Table 13. A covering map for $G_{15} = \langle r, r_1, r_3 \rangle$.

e	K	$K \to G_{19}$	$M_K o M_{G_{19}}$
∂_x	$G_{18} = \langle r_1^2, r_5 \rangle$	$g \mapsto g$	$(x',y')\mapsto (x',(y')^2)$
∂_y	$G_{17} = \langle r, r_5 \rangle$	$g \mapsto g$	$(x', y') \mapsto ((y')^3, x')$
$-60\sqrt{5}\partial_x + \partial_y$	$G_{21} = \langle r, r_5 r_1 r_5^{-1} \rangle$	$g \mapsto g$	$(x', y') \mapsto (-60\sqrt{5}x' + (y')^5, x')$

Table 14. Covering maps for $G_{19} = \langle r, r_1, r_5 \rangle$.

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