HOMOLOMORPHIC JACOBI MANIFOLDS

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Abstract. Holomorphic Jacobi manifolds are morally equivalent to homogeneous holomorphic Poisson manifolds. On another hand, they encompass holomorphic Poisson manifolds as a special case. In this paper, we first develop holomorphic Jacobi structures and show that they yield a much richer framework than that of holomorphic Poisson structures. We then discuss their relationship with generalized contact bundles and Jacobi Nijenhuis structures.

1. Introduction

In this paper, we study special cases of generalized contact bundles (see [27]). Namely, these are called holomorphic Jacobi manifolds. By a holomorphic Jacobi manifold, we mean a complex manifold X equipped with a holomorphic line bundle $L \to X$ together with a holomorphic bi-derivation J of L so that (L, J) is a Jacobi structure (Definition 37, see also [14, 19, 21, 6] for more details about real Jacobi manifolds). Equivalently, J is completely determined by the data of a Jacobi $\mathcal{O}_X$-module on the sheaf $\mathcal{L}$ of holomorphic sections of L, where $\mathcal{O}_X$ is the standard sheaf of holomorphic functions on X. In other words, for each open set $U \subset X$, $\mathcal{L}(U)$ is an $\mathcal{O}_X(U)$-module endowed with a Lie bracket $\{-,-\} : \mathcal{L}(U) \times \mathcal{L}(U) \to \mathcal{L}(U)$ and a Lie algebra homomorphism $R_U : \mathcal{L}(U) \to \operatorname{Der} \mathcal{O}_X(U)$ satisfying:

$$\{\lambda_1, a\lambda_2\} = \big(R_U(\lambda_1)a\big)\lambda_2 + a\{\lambda_1, \lambda_2\},$$

for all $\lambda_1, \lambda_2 \in \mathcal{L}(U)$ and for all $a \in \mathcal{O}_X(U)$. In addition, these Lie brackets and homomorphisms R_U are all required to be compatible with restrictions. Non-degenerate holomorphic Jacobi structures are complex contact structures which naturally appear in the twistor theory of quaternionic manifolds [18, 24, 22].

Recently, holomorphic Poisson structures were intensively studied due to their close relationship with generalized complex geometry. In fact, it was proved in [1] that, locally, any generalized complex manifold is the product of a symplectic manifold by a holomorphic Poisson manifold. Moreover, deformations of holomorphic Poisson manifolds were investigated by several mathematicians [13, 9, 8] while their integration problem was considered in [16]. A quite natural question is whether all these results on holomorphic Poisson structures can be extended to the setting of Jacobi geometry. But this question still remained open despite the fact that Poisson manifolds and Jacobi manifold are quite interrelated. In fact, the category of Poisson manifolds can be viewed as a subcategory of the category of abstract Jacobi manifolds. Meanwhile, Jacobi manifolds can be regarded as *homogeneous Poisson manifolds*. Here, by a *homogeneous Poisson manifold* [7], we mean a manifold M equipped with a Poisson bi-vector π together with a vector field η such that $[\eta, \pi]^{SN} = \mathcal{L}_\eta \pi = -\pi$, where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket. Nonetheless, almost nothing is known about generic holomorphic Jacobi structures. The present paper is the first in a series aiming at filling this gap.

The paper is divided into three parts. In Section 2, we explore basic definitions and results needed for a better understanding of holomorphic Jacobi structures. Section 3 defines holomorphic Jacobi structures and unravels their properties as well as their relationship with Jacobi Nijenhuis [23], generalized contact structures [27] and analogous *homogeneous* structures in the realm of Poisson geometry. In sum, we explain the following diagram:

$$\begin{tikzcd}
\text{Jacobi Nijenhuis} \arrow[d, Leftrightarrow] & \text{holomorphic Jacobi} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{generalized contact} \arrow[d, Leftrightarrow] \\
\begin{array}{c}\text{homogeneous}\\ \text{Poisson Nijenhuis}\end{array} & \begin{array}{c}\text{homogeneous}\\ \text{holomorphic Poisson}\end{array} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \begin{array}{c}\text{homogeneous}\\ \text{generalized complex}\end{array}
\end{tikzcd}$$

Finally, Section 4 studies the Lie algebroid of a holomorphic Jacobi manifold.

In this paper, undecorated tensor products and homomorphisms are over $\mathbb{C}$ or complex valued smooth functions, unless otherwise stated. Moreover, if E is a (real) vector bundle, we denote by $E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ its *complexification*.

2. Basics definitions and results

2.1. **Holomorphic Poisson, Poisson Nijenhuis and generalized complex manifolds.** In this section we recall the basic definitions and results from [16]. Let M be a (real) manifold M equipped with a complex structure j.

Definition 1. *A* holomorphic Poisson structure *on the complex manifold $X = (M, j)$ is a* holomorphic Poisson bi-vector Π*, that is a bi-vector $\Pi \in \Gamma(\wedge^2 T^{1,0}X)$ satisfying:*

$$\overline{\partial}\Pi = 0 \qquad \textit{and} \qquad [\Pi, \Pi]^{SN} = 0$$

(where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket *of complex multivectors). A* holomorphic Poisson manifold *is a complex manifold equipped with a holomorphic Poisson structure.*

Remark 2. *A holomorphic Poisson manifold is nothing but a complex manifold X equipped with the structure of a sheaf of Poisson algebra on its sheaf $\mathcal{O}_X$ of holomorphic sections. Given a holomorphic Poisson manifold (X, Π), the Poisson bracket of two holomorphic functions f, g on X, is given by $\{f, g\} = \Pi(df, dg) = \Pi(\partial f, \partial g)$.*

Characterizations of holomorphic Poisson structures in terms of Poisson Nijenhuis structures and generalized complex structures of special type were given in [16] (see Theorem 6 below). Before reviewing these characterizations, we will recall the basic definitions.

First of all, every bi-vector π on a manifold M determines a skew-symmetric bracket $[-,-]_\pi$ on 1-forms given by

$$[\rho,\sigma]_\pi := \mathcal{L}_{\pi^\sharp\rho}\sigma - \mathcal{L}_{\pi^\sharp\sigma}\rho - d\pi(\rho,\sigma), \tag{1}$$

for all $\rho,\sigma \in \Omega^1(M)$, where $\pi^\sharp : T^*M \to TM$ consists in "raising an index via π", i.e. $\pi^\sharp\rho := \pi(\rho,-)$. A direct computation shows that π is a Poisson bi-vector if and only if $[-,-]_\pi$ is a Lie bracket. In this case, $[-,-]_\pi$ is the Lie bracket on sections of the *cotangent algebroid* $(T^*M)_\pi$ of the Poisson manifold (M,π). Now, let $\phi : TM \to TM$ be a $(1,1)$-tensor, and let $\phi^* : T^*M \to T^*M$ be its transpose. If $\pi^\sharp \circ \phi^* = \phi \circ \pi^\sharp$, then

$$\pi_\phi := \pi(\phi-,-) \tag{2}$$

is a well-defined bi-vector such that $\pi_\phi^\sharp = \pi^\sharp \circ \phi^*$.

Let π bi a Poisson bi-vector, and let ϕ be a $(1,1)$ tensor on M. We say that ϕ is *compatible* with π if

$$\pi^\sharp \circ \phi^* = \phi \circ \pi^\sharp, \tag{3}$$

hence π_ϕ is well-defined by (2), and

$$\phi^*[\rho,\sigma]_\pi = [\phi^*\rho,\sigma]_\pi + [\rho,\phi^*\sigma]_\pi - [\rho,\sigma]_{\pi_\phi}, \tag{4}$$

for all $\rho,\sigma \in \Omega^1(M)$.

Definition 3. *A* Poisson Nijenhuis manifold *is a manifold M equipped with a* Poisson Nijenhuis structure*, i.e. a pair (π,ϕ), where π is a Poisson bi-vector, and ϕ is a compatible $(1,1)$ tensor whose* Nijenhuis torsion *$\mathcal{N}_\phi : \wedge^2 TM \to TM$:*

$$\mathcal{N}_\phi(\xi,\zeta) := [\phi(\xi),\phi(\zeta)] + \phi^2[\xi,\zeta] - \phi[\phi(\xi),\zeta] - \phi[\xi,\phi(\zeta)],$$

$\xi,\zeta \in \mathfrak{X}(M)$, vanishes identically.

Proposition 4. *Let (π,ϕ) be a Poisson Nijenhuis structure. Then (π,π_ϕ) is a* bi-Hamiltonian *structure, i.e. π, π_ϕ and $\pi+\pi_\phi$ are all Poisson bi-vectors.*

We now recall the definition of a generalized complex manifold [12, 11]. Let M be a manifold. Denote by $\mathbb{R}_M := M \times \mathbb{R} \to M$ the trivial line bundle. The *generalized tangent bundle* $\mathbb{T}M := TM \oplus T^*M$ is canonically equipped with the following structures:

- the projection $\mathrm{pr}_T : \mathbb{T}M \to TM$,
- the symmetric bilinear form $\langle\langle -,- \rangle\rangle : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{R}_M$:
$$\langle\langle (\xi,\rho),(\zeta,\sigma) \rangle\rangle := \sigma(\xi) + \rho(\zeta),$$
- the *Dorfman bracket* $[\![-,-]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$:
$$[\![(\xi,\rho),(\zeta,\sigma)]\!] := ([\xi,\zeta], \mathcal{L}_\xi\sigma - \mathcal{L}_\zeta\rho + d\rho(\zeta)),$$

$\xi, \zeta \in \mathfrak{X}(M)$, $\rho, \sigma \in \Omega^1(M)$. With the above three structures $\mathbb{T}M$ is a *Courant algebroid.*

Definition 5. *A* generalized complex manifold *is a manifold M equipped with a* generalized complex structure*, i.e. a vector bundle endomorphism $\mathcal{J} : \mathbb{T}M \to \mathbb{T}M$ such that*

- $\mathcal{J}$ *is* almost complex*, i.e.* $\mathcal{J}^2 = -\mathbb{1}$*,*
- $\mathcal{J}$ *is* skew-symmetric*, i.e.*

$$\langle\!\langle \mathcal{J}\alpha, \beta \rangle\!\rangle + \langle\!\langle \alpha, \mathcal{J}\beta \rangle\!\rangle = 0, \quad \alpha, \beta \in \Gamma(\mathbb{T}M),$$

- $\mathcal{J}$ *is* integrable*, i.e.*

$$[\![\mathcal{J}\alpha, \mathcal{J}\beta]\!] - [\![\alpha, \beta]\!] - \mathcal{J}[\![\mathcal{J}\alpha, \beta]\!] + \mathcal{J}[\![\alpha, \mathcal{J}\beta]\!] = 0, \quad \alpha, \beta \in \Gamma(\mathbb{T}M).$$

Let $(M, \mathcal{J})$ be a generalized complex manifold. Using the direct sum decomposition $\mathbb{T}M = TM \oplus T^*M$, and the definition, one can see that

$$\mathcal{J} = \begin{pmatrix} \phi & \pi^\sharp \\ \omega_\flat & -\phi^* \end{pmatrix}$$

where π is a Poisson bi-vector, $\phi : TM \to TM$ is an endomorphism compatible with π, and ω is a 2-form, with associated vector bundle morphism $\omega_\flat : TM \to T^*M$, satisfying additional compatibility conditions [5, 25]. In particular, when $\omega = 0$, then ϕ is a complex structure, and (π, ϕ) is a Poisson Nijenhuis structure. More precisely we have the following

Theorem 6 (Characterization of holomorphic Poisson structures [16])**.** *Let $X = (M, j)$ be a complex manifold, and let $\Pi \in \Gamma(\wedge^2(TM)^{\mathbb{C}})$ be a complex bi-vector on M. Denote by π', π the real and the imaginary part of Π respectively: $\Pi = \pi' + i\pi$, $\pi', \pi \in \Gamma(\wedge^2 TM)$. The following conditions are equivalent*

(1) Π *is a holomorphic Poisson structure on X,*
(2) (π, j) *is a Poisson Nijenhuis structure on M, and $\pi' = \pi_j$ (see Equation 2),*
(3) $\mathcal{J} := \left(\begin{smallmatrix} j & \pi^\sharp \\ 0 & -j^* \end{smallmatrix}\right)$ *is a generalized complex structure on M, and $\pi' = \pi_j$.*

2.2. **homogeneous holomorphic Poisson, homogeneous Poisson Nijenhuis and homogeneous generalized complex manifolds.** In this section we consider holomorphic Poisson, Poisson Nijenhuis, and generalized complex manifolds equipped with an additional compatible structure: what we call a *homogeneity vector field.* We will also discuss the relationship between these three notions in presence of a homogeneity vector field. In particular, we will analyze homogeneous generalized complex manifolds. Later on in the paper we will remark that homogeneous holomorphic Poisson, homogeneous Poisson Nijenhuis and homogeneous generalized complex manifolds on one side are closely related to holomorphic Jacobi, Jacobi Nijenhuis and generalized contact bundles on the other side.

Definition 7. *A* homogeneous holomorphic Poisson manifold *is a complex manifold $X = (M, j)$ equipped with a holomorphic Poisson structure Π together with a holomorphic vector field H such that $[H, \Pi]^{SN} = \mathcal{L}_H \Pi = -\Pi$. The pair (Π, H) is called a* homogeneous holomorphic Poisson structure on X*, and we say that* Π *is* homogeneous *with respect to H.*

Example 8. *Let $\mathfrak{g}$ be a complex Lie algebra. Its complex dual $\mathfrak{g}^*$ is equipped with the holomorphic Lie-Poisson structure Π. By linearity, the holomorphic Euler vector field H on $\mathfrak{g}^*$ is a homogeneity vector field for Π. Hence, $(\mathfrak{g}^*, \Pi, H)$ is a homogeneous holomorphic Poisson manifold.*

Example 9. *Let $X = (M, j)$ be a complex manifold, with local holomorphic coordinates (z^i). The cotangent bundle T^*X is coordinatized by the z^i's and their conjugated momenta p_i. There is a canonical complex symplectic structure Ω on T^*X locally given by $\Omega = dp_i \wedge dz^i$. The associated Poisson structure Π is locally given by $\Pi = \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial p_i}$. The holomorphic Euler vector field H on T^*X is locally given by $H = p_i \frac{\partial}{\partial p_i}$ and it is a homogeneity vector field for Π. Hence, (T^*X, Π, H) is a homogeneous holomorphic Poisson manifold. More generally, let (X, Ω) be a complex symplectic manifold, with associated Poisson structure $\Pi = \Omega^{-1}$. Additionally, let H be a* homogeneity vector field *for Ω, i.e. H is a holomorphic vector field on X such that $\mathcal{L}_H \Omega = \Omega$. Then H is clearly a homogeneity vector field for Π, hence (X, Π, H) is a homogeneous holomorphic Poisson manifold.*

Example 10. *The present example encompasses Examples 8 and 9 as special cases. Let $A \to X$ be a holomorphic Lie algebroid (see Definition 15 below, and reference* [16] *for more details about holomorphic Lie algebroids). Its complex dual $A^* \to X$ is equipped with the holomorphic Lie-Poisson structure Π. By linearity, the holomorphic Euler vector field H on A^* is a homogeneity vector field for Π. Hence, (A^*, Π, H) is a homogeneous holomorphic Poisson manifold.*

Definition 11. *A* homogeneous Poisson Nijenhuis manifold *is a Poisson Nijenhuis manifold (M, π, ϕ) equipped with a* homogeneity vector field *for (π, ϕ), i.e. a vector field η such that $\mathcal{L}_\eta \pi = -\pi$, and $\mathcal{L}_\eta \phi = 0$. The triple (π, ϕ, η) is called a* homogeneous Poisson Nijenhuis structure.

Definition 12. *A* homogeneous generalized complex manifold *is a generalized complex manifold $(M, \mathcal{J})$ equipped with a homogeneity vector field for*

$$\mathcal{J} = \begin{pmatrix} \phi & \pi^\sharp \\ \omega_\flat & -\phi^* \end{pmatrix}$$

i.e. a vector field η such that, 1) $\mathcal{L}_\eta \pi = -\pi$, 2) $\mathcal{L}_\eta \phi = 0$, and 3) $\mathcal{L}_\eta \omega = \omega$. The pair $(\mathcal{I}, \eta)$ is called a homogeneous generalized complex structure.

Theorem 13. *Let $X = (M, j)$ be a complex manifold, let $\Pi \in \Gamma(\wedge^2 (TM)^{\mathbb{C}})$ be a complex bi-vector on M, and let $H \in \Gamma((TM)^{\mathbb{C}})$ be a complex vector field. Denote by π', π the* *real and the imaginary part of* Π *respectively:* $\Pi = \pi' + i\pi$, $\pi', \pi \in \Gamma(\wedge^2 TM)$. *Finally, let* η *and* η' *be twice the real and the imaginary part of* H *respectively:* $H = \frac{1}{2}(\eta + i\eta')$, $\eta, \eta' \in \mathfrak{X}(M)$. *Then, the following three conditions are equivalent*

1. (X, Π, H) *is a homogeneous holomorphic Poisson manifold,*
2. (M, π, j, η) *is a homogeneous Poisson Nijenhuis manifold,* $\pi' = \pi_j$, *and* $\eta' = -j\eta$.
3. $(M, \mathcal{J}, \eta)$, *where* $\mathcal{J} := \begin{pmatrix} j & \pi^\sharp \\ 0 & -j^* \end{pmatrix}$, *is a homogeneous generalized complex manifold,* $\pi' = \pi_j$, *and* $\eta' = -j\eta$.

Proof.

(1) $\Leftrightarrow$ (2). From Theorem 6, (X, Π) is a holomorphic Poisson manifold if and only if (M, π, j) is a Poisson Nijenhuis manifold and $\pi' = \pi_j$. Moreover, H is a holomorphic vector field if and only if it is a section of $T^{1,0}X$, whence $\eta' = -j\eta$, $\overline{\partial} H = 0$, and $\mathcal{L}_\eta j = 0$. Now, let $\Pi = \pi_j + i\pi$ be a holomorphic Poisson bi-vector and $H = \frac{1}{2}(\eta - ij\eta)$ a holomorphic vector field on X. It remains to check that $\mathcal{L}_H \Pi = -\Pi$ if and only if $\mathcal{L}_\eta \pi = -\pi$. To see this, for any complex multivector field $Z \in \Gamma(\wedge^\bullet (TM)^{\mathbb{C}})$, let $Z^{k,l}$ be its projection onto $\Gamma(\wedge^k T^{1,0}X \oplus \wedge^l T^{0,1}X)$. Let $\overline{H} = \frac{1}{2}(\eta + ij\eta) \in \Gamma(T^{0,1}X)$ be the complex conjugate of H and notice that,

$$\mathcal{L}_{\overline{H}} \Pi = \overline{\partial}_{\overline{H}} \Pi + (\mathcal{L}_{\overline{H}} \Pi)^{1,1} + (\mathcal{L}_{\overline{H}} \Pi)^{0,2}.$$

The latter expression vanishes identically. Indeed the first summand vanishes because Π is holomorphic, the second summand vanishes because H is holomorphic, hence $\overline{H}$ is anti-holomorphic. The third summand vanishes because $\Pi \in \Gamma(\wedge^2 T^{1,0}X)$ (use, e.g., local coordinates). It follows that

$$\mathcal{L}_\eta \pi + \mathcal{L}_{j\eta} \pi_j = 2\,\mathrm{Im}(\mathcal{L}_{\overline{H}} \Pi) = 0. \tag{5}$$

Now, since Π and H are holomorphic, then $\mathcal{L}_H \Pi$ is holomorphic as well. In particular $\mathcal{L}_H \Pi$ and $-\Pi$ agree if and only if their imaginary parts agree. Finally

$$\mathrm{Im}(\mathcal{L}_H \Pi) = \frac{1}{2}(\mathcal{L}_\eta \pi - \mathcal{L}_{j\eta} \pi_j) = \mathcal{L}_\eta \pi,$$

where we used (5). Hence $\mathcal{L}_H \Pi = -\Pi$ if and only if $\mathcal{L}_\eta \pi = -\pi$. This concludes the proof.

(2) $\Leftrightarrow$ (3). It immediately follows from [5, Proposition 2.2]. □

Remark 14. *Let* (X, Π, H) *be a homogeneous holomorphic Poisson manifold, with* $X = (M, j)$, $\Pi = \pi_j + i\pi$ *and* $H = \frac{1}{2}(\eta - ij\eta)$. *Then* (π, π_j, η) *is a* homogeneous bi-Hamiltonian structure, *i.e.* (π, π_j) *is a bi-Hamiltonian structure and* η *is a homogeneity vector field for both* π *and* π_j. *Additionally, looking at the real part of identities* $\mathcal{L}_H \Pi = -\Pi$ *and* $\mathcal{L}_{\overline{H}} \Pi = 0$ *one easily sees that*

$$\mathcal{L}_{j\eta} \pi_j = \pi, \quad \textit{and} \quad \mathcal{L}_{j\eta} \pi = -\pi_j. \tag{6}$$

2.3. **Holomorphic vector bundles, linear complex structures and the real gauge algebroid.** In next subsection, we discuss the *holomorphic gauge algebroid* of a holomorphic vector bundle (Definition 26) from the real geometric point of view. It turns out that this description is simpler if one first revisits holomorphic vector bundles in terms of *linear complex structures* on vector bundles (over a complex manifold). This is done in Lemma 20. In its turn, the holomorphic gauge algebroid plays a key role for holomorphic Jacobi and related structures (much as the real gauge algebroid plays a key role for standard Jacobi and related structures [17, 26, 27]).

Let $X = (M, j)$ be a complex manifold. Both $T^{1,0}X$ and $T^{0,1}X$ are complex Lie algebroids, and a *holomorphic vector bundle* $E \to X$ over X can be seen as a complex vector bundle $E \to M$ equipped with a flat $T^{0,1}X$-connection. In particular there is an operator $\overline{\partial} : \Gamma(E) \to \Omega^{0,1}(X, E)$ whose kernel consists of *holomorphic sections of* E (see, e.g., [16]).

Definition 15. *A* holomorphic Lie algebroid *over X is a holomorphic vector bundle $A \to X$ equipped with a* holomorphic anchor *i.e. a holomorphic vector bundle morphism $\rho : A \to TX$, and a $\mathbb{C}$-linear Lie bracket $[-,-] : \Gamma_A \times \Gamma_A \to \Gamma_A$ on its sheaf Γ_A of holomorphic sections such that*

$$[\alpha_1, f\alpha_2] = \rho(\alpha_1)(f)\alpha_2 + f[\alpha_1, \alpha_2]$$

for all $\alpha_1, \alpha_2 \in \Gamma_A$ and $f \in \mathcal{O}_X$.

In [16] the authors prove that a holomorphic Lie algebroid $A \to X$ is equivalent to a holomorphic vector bundle $A \to X$ equipped with a holomorphic vector bundle map $\rho : A \to TX$ and a real Lie algebroid structure, with anchor ρ itself, and Lie bracket such that it restricts to a $\mathbb{C}$-bilinear Lie bracket on holomorphic sections. In what follows, given a holomorphic Lie algebroid $A \to X$, we denote by $A_{\mathrm{Re}} \to M$ the underlying real Lie algebroid. Now, denote by $j_A : A_{\mathrm{Re}} \to A_{\mathrm{Re}}$ the complex structure on A_{Re}. The (Lie algebroid) Nijenhuis torsion of j_A vanishes, i.e.

$$[j_A\alpha, j_A\beta] - [\alpha, \beta] - j_A[j_A\alpha, \beta] - j_A[\alpha, j_A\beta] = 0,$$

for all $\alpha, \beta \in \Gamma(A_{\mathrm{Re}})$. From a torsionless endomorphism $\phi : A_{\mathrm{Re}} \to A_{\mathrm{Re}}$ of a real Lie algebroid A_{Re} one can always define a new Lie algebroid structure $(A_{\mathrm{Re}})_\phi$ on A_{Re} with anchor $\rho_\phi := \rho \circ \phi$ and bracket $[-,-]_\phi$ defined by

$$[\alpha, \beta]_\phi := [\phi\alpha, \beta] + [\alpha, \phi\beta] - \phi[\alpha, \beta].$$

In particular, a holomorphic Lie algebroid $A \to X$ defines another real Lie algebroid $(A_{\mathrm{Re}})_{j_A}$. We call A_{Re} and $(A_{\mathrm{Re}})_{j_A}$ the *real* and the *imaginary Lie algebroids* of $A \to X$ respectively.

Finally, complexifying A_{Re} and decomposing into the eigenbundles of the complex structure, one also defines from $A \to X$ two complex Lie algebroids $A^{1,0} \to M$ and $A^{0,1} \to M$ with anchors $\rho^{1,0} : A^{1,0} \to T^{1,0}X$ and $\rho^{0,1} : A^{0,1} \to T^{0,1}X$, and Lie brackets $[-,-]^{1,0}$ and $[-,-]^{0,1}$. We refer to [16] for the details.

Example 16. *The tangent bundle $TX \to X$ is a holomorphic Lie algebroid, with underlying real Lie algebroid $TM \to M$, imaginary Lie algebroid $(TM)_j \to M$, and associated complex Lie algebroids $T^{1,0}X$ and $T^{0,1}X$.*

Given a holomorphic vector bundle $E \to X$ one can define its *holomorphic gauge algebroid* $DE \to E$ (Definition 26), encoding infinitesimal automorphisms of E. As already mentioned, we do this in next subsection. Now recall what the real gauge algebroid of a real vector bundle is. Given a real vector bundle $E \to M$, and a point $x \in M$, the fiber over x of the real gauge algebroid of E consists of $\mathbb{R}$-linear maps $\Delta : \Gamma(E) \to E_x$ satisfying the following Leibniz rule: $\Delta(fe) = \xi(f)e_x + f(x)\Delta(e)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$. We denote by $D_{\mathbb{R}}E$ the real gauge algebroid of $E \to M$ to distinguish it from the holomorphic gauge algebroid of $E \to X$ (that will be simply denoted by DE). Sections of $D_{\mathbb{R}}E$ are *derivations of* E (also called *covariant differential operators* in [20], see also [15]), i.e. $\mathbb{R}$-linear operators $\Delta : \Gamma(E) \to \Gamma(E)$ satisfying the following *Leibniz rule*: $\Delta(fe) = \xi(f)e + f\Delta(e)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$, for a (necessarily unique) vector field $\xi \in \mathfrak{X}(M)$, also called the *symbol* of Δ, and denoted $\sigma(\Delta)$. The gauge algebroid is a Lie algebroid with anchor given by the symbol $\sigma : DE \to TM$, and Lie bracket given by the commutator of derivations. The kernel of the symbol, consists of $C^\infty(M)$-linear derivations, i.e. endomorphisms of E covering the identity of M. Hence there is a short exact sequence of vector bundles:

$$0 \longrightarrow \operatorname{End}_{\mathbb{R}} E \longrightarrow D_{\mathbb{R}}E \xrightarrow{\sigma} TM \longrightarrow 0.$$

For more details about the gauge algebroid (including its functorial properties) we refer to [17, 26]. We only recall here that sections of the gauge algebroid $DE \to M$ are in one-to-one correspondence with infinitesimal automorphisms of E, or, equivalently, *linear vector fields* on (the total space of) E, i.e. vector fields $\xi \in \mathfrak{X}(E)$ preserving fiber-wise linear functions on E. We denote by $\mathfrak{X}_{\mathrm{lin}}(E)$ the Lie algebra of linear vector fields on E. Every linear vector field $\xi \in \mathfrak{X}_{\mathrm{lin}}(E)$ projects to a vector field $\underline{\xi} \in \mathfrak{X}(M)$, and the linear vector field $\xi \in \mathfrak{X}(E)$ corresponds to the derivation $\Delta_\xi : \Gamma(E) \to \Gamma(E)$ implicitly defined by

$$\langle \varphi, \Delta_\xi e\rangle := \underline{\xi}\langle \varphi, e\rangle - \langle \xi(\varphi), e\rangle,$$

for all $\varphi \in \Gamma(E^*_{\mathbb{R}})$ and $e \in \Gamma(E)$ (here we identified sections of the real dual vector bundle $E^*_{\mathbb{R}}$ of E with fiber-wise linear functions on E). The correspondence $\xi \mapsto \Delta_\xi$ is a Lie algebra isomorphism and, additionally, $\sigma(\Delta_\xi) = \underline{\xi}$. Finally, $C^\infty(M)$-linear derivations of E correspond to vertical linear vector fields on E.

Before giving a precise definition of the holomorphic gauge algebroid it is convenient to discuss *linear* $(1,1)$ *tensors on a vector bundle*. Thus, let $E \to M$ be a vector bundle. Recall that TE is a double vector bundle, with side vector bundles E and TM (see, e.g., [20, Chapter 9]).

Definition 17. *A* $(1,1)$ *tensor* $\phi : TE \to TE$ *is* linear *if it is a double vector bundle morphism, or, equivalently, if it preserves linear vector fields on* E.

Lemma 18. *There is a $C^\infty(M)$-linear one-to-one correspondence $\phi \mapsto \phi_{DE}$, between linear $(1,1)$ tensors ϕ on E and endomorphisms $\psi : DE \to DE$ with the following two properties:*

(1) *there is a $(1,1)$ tensor $\underline{\psi} : TM \to TM$ such that $\underline{\psi} \circ \sigma = \sigma \circ \psi$, and*
(2) *there is an endomorphism $\psi_E \in \Gamma(\operatorname{End}_{\mathbb{R}} E)$ such that $\psi(h) = \psi_E \circ h$ for every endomorphism $h \in \Gamma(\operatorname{End}_{\mathbb{R}} E)$.*

Correspondence $\phi \mapsto \phi_{DE}$ preserves the compositions, i.e. $(\phi \circ \phi')_{DE} = \phi_{DE} \circ \phi'_{DE}$ for every two linear $(1,1)$ tensors on E.

Proof. It immediately follows from the definition that a linear $(1,1)$ tensor $\phi : TE \to TE$ defines an endomorphism $\phi_{DE} : D_{\mathbb{R}}E \to D_{\mathbb{R}}E$ just by restriction to linear vector fields. Since ϕ is a morphism of double vector bundles, it descends to a $(1,1)$ tensor $\underline{\phi} : TM \to TM$ on M, and preserves vertical tangent vectors. Hence ϕ_{DE} has property (1) in the statement, and preserves $\operatorname{End}_{\mathbb{R}} E$. Put $\phi_E := \phi_{DE}(\mathrm{id}_E) \in \Gamma(\operatorname{End}_{\mathbb{R}} E)$. It is easy to see, e.g. in local coordinates, that $\phi_{DE}(h) = \phi_E \circ h$ for all $h \in \Gamma(\operatorname{End}_{\mathbb{R}} E)$. Hence $\phi_D E$ has also property (2). Since linear vector fields generate the whole $\mathfrak{X}(E)$, then the correspondence $\phi \mapsto \phi_{DE}$ is injective. Surjectivity is easily checked in local coordinates. The last part of the proposition is obvious. □

Remark 19. *Since the isomorphism $\mathfrak{X}_{\mathrm{lin}}(E) \to \Gamma(D_{\mathbb{R}}E)$ intertwines the Lie brackets, and linear vector fields generate all vector fields on E, then a linear $(1,1)$ tensor $\phi : TE \to TE$ is torsionless if and only if $\phi_{DE} : D_{\mathbb{R}}E \to D_{\mathbb{R}}E$ is torsionless.*

Now we need to further describe features of holomorphic vector bundles. This is done in Lemma 20 below:

Lemma 20. *Let M be a smooth manifold. The following data are equivalent:*

(1) *a complex structure j on M and a holomorphic vector bundle $E \to X = (M, j)$;*
(2) *a complex structure j on M and a complex vector bundle $E \to X = (M, j)$ equipped with a flat $T^{0,1}X$ connection;*
(3) *a real vector bundle $E \to M$ equipped with a linear complex structure $j_E^{\mathrm{tot}} : TE \to TE$ in its total space;*
(4) *a complex structure j on M and a complex vector bundle $E \to M$, with complex structure $j_E : E \to E$, equipped with a torsionless complex structure j_{DE} in its real gauge algebroid $D_{\mathbb{R}}E$ such that*
    (4b) *the symbol $\sigma : D_{\mathbb{R}}E \to TM$ intertwines j_{DE} and j,*
    (4c) *the restriction of j_{DE} to endomorphisms $\operatorname{End}_{\mathbb{R}} E$ agrees with the map $\operatorname{End}_{\mathbb{R}} E \to \operatorname{End}_{\mathbb{R}} E$, $\phi \mapsto j_E \circ \phi$.*

Proof. The equivalence (1) ⇔ (2) is standard and has already been mentioned at the beginning of this section.

Let us prove (2) ⇔ (3). Start with a complex vector bundle $E \to M$ over a complex manifold $X = (M, j)$, and a flat $T^{0,1}X$-connection $\overline{\partial}$ in E. We denote by $j_E : E \to E$ the complex structure on E. Let $E^* \to M$ be the complex dual to E. Connection $\overline{\partial}$

induces a flat connection in E^*, also denoted by $\overline{\partial}$. In order to define j_E^{tot}, denote by $\pi : E \to M$ the projection, let $e \in E$ and $x = \pi(e)$. Every tangent vector $\xi \in T_eE$ is a derivation $\xi : C^\infty(E) \to \mathbb{R}$, and can be extended to a self-conjugate complex derivation, also denoted by $\xi : C^\infty(E, \mathbb{C}) \to \mathbb{C}$, by $\mathbb{C}$-linearity. On the other hand, every self-conjugate complex derivation $\xi : C^\infty(E, \mathbb{C}) \to \mathbb{C}$ determines a tangent vector $\xi \in T_eE$ by restriction to real functions. A self-conjugate derivation $\xi : C^\infty(E, \mathbb{C}) \to \mathbb{C}$ is completely determined by its action on fiber-wise constant functions, i.e. functions in $C^\infty(M, \mathbb{C})$, and on fiber-wise linear functions, i.e. sections of E^* (notice, however, that this is not so for non self-conjugate derivations). Conversely, a triple $(e, \underline{\eta}, \eta)$ consisting of

(1) a point $e \in E$,
(2) a tangent vector $\underline{\eta} \in T_xM$, $x = \pi(e)$, and
(3) a $\mathbb{C}$-linear operator $\eta : \Gamma(E^*) \to \mathbb{C}$,

such that

$$\eta(f\varphi) = \underline{\eta}(f)\langle \varphi_x, e\rangle + f(x)\eta(\varphi), \tag{7}$$

for all $f \in C^\infty(M, \mathbb{C})$ and $\varphi \in \Gamma(E^*)$, comes from a unique $\xi \in T_eE$ such that $\underline{\eta} = (d\pi)(\xi)$, and η is the restriction of $\xi : C^\infty(E, \mathbb{C}) \to \mathbb{C}$ to fiber-wise linear complex functions. So let $e \in E$, and $\xi \in T_eE$, and define

$$\underline{\eta} \in T_xM,$$
$$\eta : \Gamma(E^*) \to \mathbb{C},$$

as follows. Firstly, put $\underline{\eta} := j(\pi_*\xi)$. Put also $(\pi_*\xi)^{0,1} := p^{0,1}(\pi_*\xi) = (\pi_*\xi - i\underline{\eta})/2$ where $p^{0,1} : T_xM \otimes_{\mathbb{R}} \mathbb{C} \to T_x^{0,1}X$ is the projection. Secondly, put

$$\eta(\varphi) := i\left(\xi(\varphi) - 2\langle \overline{\partial}_{(\pi_*\xi)^{0,1}}\varphi, e\rangle\right) \tag{8}$$

for all $\varphi \in \Gamma(E^*)$. An easy computation shows that η satisfies Leibniz rule (7). Hence $(e, \underline{\eta}, \eta)$ defines a tangent vector $\xi' \in T_eE$. We put $j_E^{\mathrm{tot}}\xi := \xi'$. Checking that $(j_E^{\mathrm{tot}})^2 = -\mathbb{1}$ is straigthforward. Thus we have an almost complex structure $j_E^{\mathrm{tot}} : TE \to TE$ on E. Now, integrability of j_E^{tot} follows from flatness of $\underline{\partial}$. The linearity of j_E^{tot} immediately follows from (8) and the fact that $d\pi$ maps ξ' to $\underline{\eta}$. In particular, j_E^{tot} descends to j under $d\pi : TE \to TM$ and agrees with j_E on fibers.

Conversely, let $j_E^{\mathrm{tot}} : TE \to TE$ be a linear complex structure. Then it descends to a complex structure j on M under $d\pi : TE \to TM$ and restricts to a complex structure $j_E : E \to E$ on fibers. Define a $T^{0,1}X$-connection in E^* as follows. For $\varphi \in \Gamma(E^*)$, and $\underline{\eta}^{0,1} \in T^{0,1}M$ put

$$\langle \overline{\partial}_{\underline{\eta}^{0,1}}\varphi, e\rangle = \tilde{\xi}(\varphi), \tag{9}$$

where $\tilde{\xi} \in T^{0,1}E$ is any tangent vector such that $\pi_*\tilde{\xi} = \underline{\eta}^{0,1}$. The $\mathbb{C}$-linearity of φ guarantees that $\overline{\partial}_{\underline{\eta}^{0,1}}\varphi$ is independent of the choice of $\tilde{\xi}$. Now, linearity in the argument e follows from the linearity of j_E^{tot}. Linearity in the argument $\underline{\eta}^{0,1}$ is obvious, and the Leibniz rule with respect to the argument φ follows from the fact that $d\pi$ intertwines

j_E^{tot} and j (by definition of j). Flatness of $\overline{\partial}$ follows from the integrability of j_E^{tot}. So $\overline{\partial}$ is a flat $T^{0,1}X$-connection in E^*. By duality, it induces a flat $T^{0,1}X$-connection in E. Comparing (8) and (9) we see that this construction inverts the above contruction of j_e^{tot} from $\overline{\partial}$.

Finally (3) $\Leftrightarrow$ (4) immediately follows from Proposition 18 and Remark 19.

$\square$

Remark 21. *A direct computation exploiting Equation (8) shows that*

$$j_{DE}\Delta = j_E \circ (\Delta - 2\overline{\partial}_{\sigma(\Delta)^{0,1}}), \tag{10}$$

for all $\Delta \in D_{\mathbb{R}}E$. Formula (10) can be used to prove directly the equivalence between (2) and (4) in the above proposition. Notice that, despite $j_{DE}\Delta \in D_{\mathbb{R}}E$ for all $\Delta \in D_{\mathbb{R}}E$, none of the two summands in the right hand side of (10) is in $D_{\mathbb{R}}E$. Finally, it immediately follows from Formula (10) that j_{DE} preserves $\mathbb{C}$-linear sections of $D_{\mathbb{R}}E$, i.e. those sections commuting with j_E.

2.4. **The holomorphic gauge algebroid.** Let $E \to X$ be a holomorphic vector bundle over a complex manifold $X = (M, j)$. Proposition (20) shows that the gauge algebroid $D_{\mathbb{R}}E$ is equipped with a torsionless complex structure j_{DE}. Additionally j_{DE} restricts to the subbundle DE consisting of $\mathbb{C}$-linear derivations (see Remark 21). We will show below that DE is a holomorphic Lie algebroid over X, whose underlying real Lie algebroid structure $(DE)_{\mathrm{Re}}$ is obtained from $D_{\mathbb{R}}E$ by restriction. To see this, we first describe DE in an alternative way.

First of all, sections of the complexified tangent bundle $(TM)^{\mathbb{C}}$ can be seen as derivations of the complex algebra $C^\infty(M, \mathbb{C})$. They are also real derivations ξ of the real vector bundle $\mathbb{C}_M := M \times \mathbb{C} \to M$ such that $X(1) = X(i) = 0$, and we denote them by $\mathfrak{X}_{\mathbb{C}}(M)$. Clearly, $(TM)^{\mathbb{C}}$ is a complex Lie algebroid whose anchor is the identity and whose Lie bracket is the commutator. Now, let $E \to M$ be a complex vector bundle over a smooth manifold (beware not yet a holomorphic vector bundle over a complex manifold). Denote by $D_{\mathbb{C}}E \to M$ the bundle whose sections are $\mathbb{C}$-linear operators $\Delta : \Gamma(E) \to \Gamma(E)$ such that there exists (a necessarily unique) $\xi \in \mathfrak{X}_{\mathbb{C}}(M)$, also denoted $\sigma(\Delta)$, such that

$$\Delta(fe) = \xi(f)e + f\Delta(e),$$

for all $f \in C^\infty(M, \mathbb{C})$, and all $e \in \Gamma(E)$. Sections of $D_{\mathbb{C}}E$ are called *complex derivations* of E. Clearly, $D_{\mathbb{C}}E$ is a complex Lie algebroid with anchor σ and Lie bracket the commutator of complex derivations. Notice that the complex structure on the vector bundle $D_{\mathbb{C}}E \to M$ is the composition with $j_E : E \to E$. The kernel of the symbol map $\sigma : D_{\mathbb{C}}E \to (TM)^{\mathbb{C}}$ is the bundle of $\mathbb{C}$-linear endomorphisms of E. Summarizing, there is a short exact sequence of complex vector bundles:

$$0 \longrightarrow \operatorname{End} E \longrightarrow D_{\mathbb{C}}E \xrightarrow{\sigma} (TM)^{\mathbb{C}} \longrightarrow 0,$$

where the endomorphisms are over $\mathbb{C}$.

Remark 22. *In general, complex derivations of E are not derivations in the real sense, because their symbol needs not to be real.*

Now, we assume that M is equipped with a complex structure j and put $X = (M, j)$. In this case, $(TM)^{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X$. Denote $D^{1,0}E := \sigma^{-1}(T^{1,0}M)$ and $D^{0,1}E := \sigma^{-1}(T^{0,1}M)$. We have

$$D_{\mathbb{C}}E = D^{1,0}E + D^{0,1}E, \tag{11}$$

but $D^{1,0}E \cap D^{0,1}E = \operatorname{End}_{\mathbb{C}} E$, so (11) is not a direct sum decomposition. However, we can "correct it" to a direct sum decomposition if E is a holomorphic vector bundle over X, i.e. if there is a flat $T^{0,1}X$-connection $\overline{\partial}$ in E. Indeed, connection $\overline{\partial}$ splits the short exact sequence $0 \to \operatorname{End}_{\mathbb{C}} E \to D^{0,1}E \to T^{0,1}X \to 0$, hence there is a direct sum decomposition

$$D_{\mathbb{C}}E = D^{1,0}E \oplus T^{0,1}X, \tag{12}$$

and formula

$$\overline{\partial}{}^{DE}_{\xi}\Delta := [\overline{\partial}_{\xi}, \Delta]^{1,0},$$

$\xi \in \Gamma(T^{0,1}X)$, $\Delta \in \Gamma(D^{1,0}E)$, defines a flat $T^{0,1}X$ connection in $D^{1,0}E$, which makes it a holomorphic vector bundle over X. In the following we denote by $\Delta \mapsto \Delta^{1,0}$ the projection $D_{\mathbb{C}}E \to D^{1,0}E$ with kernel $\operatorname{im}\overline{\partial} \simeq T^{0,1}M$.

Lemma 23. *There is a natural complex vector bundle isomorphism $\iota : D^{1,0}E \to DE$ such that*

(1) *diagram*

$$\begin{tikzcd} D^{1,0}E \arrow[r, "\iota"] \arrow[d, "\sigma"'] & DE \arrow[d, "\sigma"] \\ T^{1,0}X \arrow[r, "2\operatorname{Re}"] & TM \end{tikzcd},$$

commutes

(2)

$$\iota[\Delta_1, \Delta_2] = [\iota\Delta_1, \iota\Delta_2]$$

for every two holomorphic *sections of $D^{1,0}E \to X$.*

Corollary 24. *There is a holomorphic Lie algebroid structure $DE \to X$ such that*

(1) *the underlying real Lie algebroid structure is induced from that of $D_{\mathbb{R}}E$ by restriction (of the anchor and the bracket),*
(2) *the associated complex Lie algebroid $(DE)^{1,0}$ coincides with $D^{1,0}E$.*

Proof of Lemma 23. Define $\iota : D^{1,0}E \to DE$ by

$$\iota\Delta := \Delta + \overline{\partial}_{\overline{\sigma(\Delta)}},$$

where by $\overline{\xi}$ we mean the complex conjugate of the complex vector field $\xi \in \mathfrak{X}_{\mathbb{C}}(M)$. Clearly, ι is a well-defined homomorphism of complex vector bundles. It is easy to see that it is injective. Indeed it follows from $\Delta + \overline{\partial}_{\overline{\sigma(\Delta)}} = 0$, that

$$0 = \sigma(\Delta + \overline{\partial}_{\overline{\sigma(\Delta)}}) = 2\,\mathrm{Re}(\sigma(\Delta)).$$

Since $\sigma(\Delta) \in \Gamma(T^{1,0}X)$, this is enough to have $\sigma(\Delta) = 0$. Hence $\Delta = \Delta + \overline{\partial}_{\overline{\sigma(\Delta)}} = \iota\Delta = 0$. Since DE and $D^{1,0}E$ have the same complex rank $\dim_{\mathbb{C}} M + (\mathrm{rank}_{\mathbb{C}}\, E)^2$, then ι is an isomorphism.

Now, (1) follows from

$$\sigma(\iota\Delta) = \sigma(\Delta + \overline{\partial}_{\overline{\sigma(\Delta)}}) = 2\,\mathrm{Re}(\sigma(\Delta)).$$

Condition (2) can be easily checked by a similar direct computation. □

Remark 25. *The inverse $\iota^{-1} : DE \to D^{1,0}E$ of the isomorphism ι is given by*

$$\iota^{-1}\Delta = \Delta - \overline{\partial}_{\sigma(\Delta)^{0,1}} = \frac{1}{2}(\Delta - j_E \circ j_{DE}\Delta),$$

$\Delta \in \Gamma(DE)$.

Definition 26. *The holomorphic Lie algebroid $DE \to X$ is the* holomorphic gauge algebroid *of the holomorphic vector bundle E.*

Remark 27. *Holomorphic sections of $DE \to X$ are precisely derivations of the sheaf of holomorphic sections of $E \to X$.*

Remark 28. *The holomorphic gauge algebroid DE, its associated complex algebroid $D^{1,0}E$, and the* complex gauge algebroid *$D_{\mathbb{C}}E$ fit in the following triangle of short exact sequences:*

$$\begin{tikzcd}
& 0 \arrow[rr] & & \mathrm{End}\, E \arrow[rr] \arrow[dd, equal] & & D^{1,0}E \arrow[rr] \arrow[dd, "\iota"] & & T^{1,0}X \arrow[rr] \arrow[dd, "2\,\mathrm{Re}"] & & 0 \\
0 \arrow[rr] & & \mathrm{End}\, E \arrow[ur, equal] \arrow[dr, equal] \arrow[rr] & & D_{\mathbb{C}}E \arrow[ur] \arrow[rr] & & (TM)^{\mathbb{C}} \arrow[ur] \arrow[rr] & & 0 & \\
& 0 \arrow[rr] & & \mathrm{End}\, E \arrow[rr] & & DE \arrow[rr] \arrow[ul] & & TX \arrow[rr] \arrow[ul] & & 0
\end{tikzcd}.$$

Here endomorphisms are taken over $\mathbb{C}$.

2.5. **Holomorphic jets.** In this section we briefly discuss first holomorphic jets of a holomorphic vector bundle from the real differential geometric point of view. In the theory of Jacobi structures the first jet bundle plays a *dual role* to the gauge algebroid. First of all, let $E \to M$ be a real vector bundle. We denote by $\mathfrak{J}^1_{\mathbb{R}}E$ the first (real) jet bundle of $E \to M$, and by $\mathsf{j}^1_{\mathbb{R}} : \Gamma(E) \to \Gamma(\mathfrak{J}^1_{\mathbb{R}}E)$, $e \mapsto \mathsf{j}^1_{\mathbb{R}}e$ the first (real) jet prolongation of sections of $E \to M$. Recall that $\mathfrak{J}^1_{\mathbb{R}}E$ fits in the short exact sequence of vector bundles over M:

$$0 \longrightarrow TM \otimes_{\mathbb{R}} E \xrightarrow{\gamma} \mathfrak{J}^1_{\mathbb{R}}E \longrightarrow E \longrightarrow 0,$$

where the second arrow is the embedding $\gamma : TM \otimes_{\mathbb{R}} E \to \mathfrak{J}^1_{\mathbb{R}} E$, $df \otimes e \mapsto \mathfrak{j}^1_{\mathbb{R}}(fe) - f\mathfrak{j}^1_{\mathbb{R}} e$, for all $f \in C^\infty(M)$, and $e \in \Gamma(E)$. Additionally, every section $\theta \in \Gamma(\mathfrak{J}^1_{\mathbb{R}} E)$ can be uniquely written in the form

$$\theta = \mathfrak{j}^1_{\mathbb{R}} e + \gamma(\omega),$$

where $e \in \Gamma(E)$, and $\omega \in \Gamma(T^*M \otimes_{\mathbb{R}} E)$.

Now let $X = (M, j)$ be a complex manifold and let $E \to X$ be a holomorphic vector bundle over it. In particular, $E \to M$ is a complex vector bundle, and $\mathfrak{J}^1_{\mathbb{R}} E \to M$ inherits a (fiber-wise) complex structure from it: $i \cdot \mathfrak{j}^1_{\mathbb{R}} e := \mathfrak{j}^1_{\mathbb{R}} ie$, for all $e \in \Gamma(E)$.

Remark 29. *Vector bundle $\mathfrak{J}^1_{\mathbb{R}} E \to M$ is canonically equipped with another complex vector bundle structure $j_{\mathfrak{J}^1 E} : \mathfrak{J}^1_{\mathbb{R}} E \to \mathfrak{J}^1_{\mathbb{R}} E$. To see this, first notice that there is a well-defined embedding $(T^{0,1}X)^* \otimes E \to T^*M \otimes_{\mathbb{R}} E$ given by the composition:*

$$(T^{0,1}X)^* \otimes E \longrightarrow (T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes E \xrightarrow{\simeq} T^*M \otimes_{\mathbb{R}} E.$$

Embedding $(T^{0,1}X)^* \otimes E \to T^*M \otimes_{\mathbb{R}} E$ is right inverse to projection $T^*M \otimes_{\mathbb{R}} E \to (T^{0,1}X)^* \otimes E$, $\omega \otimes_{\mathbb{R}} e \mapsto \frac{1}{2}(\omega + ij^*\omega) \otimes e$. In the following we will understand this embedding and interpret $(T^{0,1}X)^* \otimes E$ as a subbundle of $T^*M \otimes_{\mathbb{R}} E$. Then $j_{\mathfrak{J}^1 E}$ is defined on sections of $\mathfrak{J}^1_{\mathbb{R}} E$ by

$$j_{\mathfrak{J}^1 E}(\mathfrak{j}^1_{\mathbb{R}} e + \gamma(\omega)) := \mathfrak{j}^1_{\mathbb{R}}(ie) + \gamma(j^*\omega - 2i\overline{\partial} e).$$

Connection $\overline{\partial} : \Gamma(E) \to \Gamma((T^{0,1}M)^* \otimes E)$ determines a morphism $\Phi_{\overline{\partial}} : \mathfrak{J}^1_{\mathbb{R}} E \to (T^{0,1}M)^* \otimes E$, $\mathfrak{j}^1_{\mathbb{R}} e \mapsto \overline{\partial} e$, of real vector bundles.

Definition 30. *Vector bundle $\mathfrak{J}^1 E := \ker \Phi_{\overline{\partial}}$ is the bundle of* first holomorphic jets *of sections of $E \to X$.*

The above definition reflects the idea that *holomorphic jets* are *jets of holomorphic sections*.

Proposition 31. *The first holomorphic jet bundle $\mathfrak{J}^1 E$ is a holomorphic vector bundle over X.*

Proof. First of all $\mathfrak{J}^1 E$ is a complex subbundle of $\mathfrak{J}^1_{\mathbb{R}} E$. Secondly, the inclusion $(T^{0,1}X)^* \otimes E \to \mathfrak{J}^1_{\mathbb{R}} E$ given by the composition

$$(T^{0,1}X)^* \otimes E \longrightarrow T^*M \otimes_{\mathbb{R}} E \xrightarrow{\gamma} \mathfrak{J}^1_{\mathbb{R}} E$$

splits the exact sequence

$$0 \longrightarrow \mathfrak{J}^1 E \longrightarrow \mathfrak{J}^1_{\mathbb{R}} E \xrightarrow{\Phi_{\overline{\partial}}} T^{0,1}X \otimes E \longrightarrow 0.$$

Hence there is a direct sum decomposition

$$\mathfrak{J}^1_{\mathbb{R}} E = \mathfrak{J}^1 E \oplus (T^{0,1}X)^* \otimes E. \tag{13}$$

In particular, there are a projection $\mathfrak{J}^1_{\mathbb{R}} E \to \mathfrak{J}^1 E$, denoted $\theta \mapsto \theta^{1,0}$, and a short exact sequence

$$0 \longrightarrow (T^{1,0}X)^* \otimes E \xrightarrow{\gamma^{1,0}} \mathfrak{J}^1 E \longrightarrow E \longrightarrow 0,$$

where $\gamma^{1,0}$ is given by the composition

$$(T^{1,0}X)^* \otimes E \longrightarrow (T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes E \xrightarrow{\simeq} T^*M \otimes_{\mathbb{R}} E \xrightarrow{\gamma} \mathfrak{J}^1_{\mathbb{R}} E \longrightarrow \mathfrak{J}^1 E.$$

Denote by $\mathrm{j}^{1,0} : \Gamma(E) \to \Gamma(\mathfrak{J}^1 E)$ the composition of the first jet prolongation $\mathrm{j}^1_{\mathbb{R}} : \Gamma(E) \to \Gamma(\mathfrak{J}^1_{\mathbb{R}} E)$ followed by the projection $\Gamma(\mathfrak{J}^1_{\mathbb{R}} E) \to \Gamma(\mathfrak{J}^1 E)$, i.e. $\mathrm{j}^{1,0} e = (\mathrm{j}^1_{\mathbb{R}} e)^{1,0} = \mathrm{j}^1_{\mathbb{R}} e - \gamma(\overline{\partial} e)$, for all $e \in \Gamma(E)$. Let $e \in \Gamma(E)$, and $f \in C^\infty(M, \mathbb{C})$. Then

$$\gamma^{1,0}(df \otimes e) = \mathrm{j}^{1,0}(fe) - f \mathrm{j}^{1,0} e. \tag{14}$$

Moreover, it is clear that every section $\theta \in \Gamma(\mathfrak{J}^1 E)$ can be uniquely written in the form

$$\theta = \mathrm{j}^{1,0} e + \gamma^{1,0}(\omega),$$

where $e \in \Gamma(E)$, and $\omega \in \Gamma((T^{1,0}X)^* \otimes E)$.

We are now in the position to define a flat $T^{0,1}X$-connection $\overline{\partial}^{\mathfrak{J}^1 E}$ in $\mathfrak{J}^1 E$. Namely, put

$$\overline{\partial}^{\mathfrak{J}^1 E}_\xi (\mathrm{j}^{1,0} e + \gamma^{1,0}(\omega)) := \mathrm{j}^{1,0} \left(\overline{\partial}_\xi e \right) + \gamma^{1,0} \left(\overline{\partial}_\xi \omega - i_{\partial \xi} \overline{\partial} e \right),$$

for all $e \in \Gamma(E)$, $\omega \in \Gamma((T^{0,1}X)^* \otimes E)$, and $\xi \in \Gamma(T^{1,0}X)$. A direct computation exploiting (14) shows that $\overline{\partial}^{\mathfrak{J}^1 E}$ is a well-defined flat $T^{0,1}X$-connection. This concludes the proof. □

Remark 32. *Recall the direct sum decomposition $\mathfrak{J}^1_{\mathbb{R}} E = \mathfrak{J}^1 E \oplus (T^{0,1}X \otimes E)$. It is easy to see that the two complex structures on $\mathfrak{J}^1_{\mathbb{R}} E$ agree on the first summand and they "anti-agree" on the second summand.*

2.6. **Multidifferential calculus on a holomorphic line bundle.** In this section we discuss the Schouten-Jacobi algebra of a holomorphic line bundle. It is analogue to the Schouten-Nijenhuis algebra of multivector fields on a manifold. The role of vector fields is now played by derivations, i.e. sections of the gauge algebroid. As the "integrability condition" of a Poisson bi-vector can be expressed in terms of the Schouten-Nijenhuis bracket, the "intergability condition" of a Jacobi structure can be expressed in terms of the Schouten-Jacobi bracket. For details on the Schouten-Jacobi algebra of a real line bundle and its role in Jacobi geometry we refer to [17, Appendix B]. Let $L \to M$ be a complex line bundle.

Definition 33. *A* complex k-multiderivation *of L is a $\mathbb{C}$-multilinear, skewsymmetric operator $\Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L)$ which is a complex derivation in each entry. Complex k-multiderivations of L are sections of a complex vector bundle, denoted by $D^k_{\mathbb{C}} L \to M$. We also put $D^\bullet_{\mathbb{C}} L = \bigoplus_k D^k_{\mathbb{C}} L$. An element in $\Gamma(D^\bullet_{\mathbb{C}} L)$ is, simply, a* multiderivation*.*

Remark 34. *Clearly $D^1_{\mathbb{C}} L = D_{\mathbb{C}} L \simeq \operatorname{Hom}(\mathfrak{J}^1_{\mathbb{R}} L, L)$, where the homomorphisms are taken over $\mathbb{C}$ and $\mathfrak{J}^1_{\mathbb{R}} L$ is equipped with the fiber-wise complex structure given by $i \cdot \mathrm{j}^1_{\mathbb{R}} \lambda := \mathrm{j}^1_{\mathbb{R}}(i\lambda)$. More generally*

$$D^k_{\mathbb{C}} L \simeq \operatorname{Hom}(\wedge^k \mathfrak{J}^1_{\mathbb{R}} L, L), \tag{15}$$

where k-multiderivation Δ corresponds to homomorphism $\Phi_\Delta : \wedge^k \mathfrak{J}^1_{\mathbb{R}} L \to L$ given by

$$\Phi_\Delta(\mathrm{j}^1_{\mathbb{R}} \lambda_1, \ldots, \mathrm{j}^1_{\mathbb{R}} \lambda_k) := \Delta(\lambda_1, \ldots, \lambda_k).$$

Remark 35. *Since the rank of the complex vector bundle $L \to M$ plays no role in the definition of multiderivations, the latter is valid for any complex vector bundle.*

There is a degree zero graded Lie bracket $[-,-]^{SJ}$ on the graded vector space $\Gamma(D^\bullet_{\mathbb{C}}L)[1]$ ($\Gamma(D^\bullet_{\mathbb{C}}L)$ shifted by 1) given by

$$[\Delta_1, \Delta_2]^{SJ} := (-)^{k_1 k_2}\Delta_1 \circ \Delta_2 - \Delta_2 \circ \Delta_1,$$

for all $\Delta_i \in \Gamma(D^{k_i+1}_{\mathbb{C}}L)$, $i = 1, 2$, where $\Delta_1 \circ \Delta_2$ is given by the following "*Gerstenhaber formula*":

$$\begin{aligned}
&(\Delta_1 \circ \Delta_2)(\lambda_1, \ldots, \lambda_{k_1+k_2+1}) \\
&= \sum_{\tau \in S_{k_2+1,k_1}} (-)^\tau \Delta_1(\Delta_2(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(k_2+1)}), \lambda_{\tau(k_2+2)}, \ldots, \lambda_{\tau(k_1+k_2+1)}),
\end{aligned}$$

for all $\lambda_1, \ldots, \lambda_{k_1+k_2+1} \in \Gamma(L)$. Here $S_{k,l}$ denotes *unshuffles*. The bracket $[-,-]^{SJ}$ is called the *Schouten-Jacobi bracket* (see [17, Appendix B] for more details).

In what follows we denote by $\mathbb{C}_M := M \times \mathbb{C} \to M$ the trivial complex line bundle over M. When $L \to X$ is a holomorphic line bundle over a complex manifold $X = (M, j)$, then the exact diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathbb{C}_M \arrow[u] \arrow[r] & D^{1,0}L \arrow[u] \arrow[r, "\sigma"] \arrow[d, bend left] & T^{1,0}X \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C}_M \arrow[u, equal] \arrow[r] & D_{\mathbb{C}}L \arrow[u] \arrow[r, "\sigma"] & (TM)^{\mathbb{C}} \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] \arrow[r] & T^{0,1}X \arrow[u, "\overline{\partial}_{(-)}"] \arrow[r, equal] & T^{0,1}X \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

is obtained from exact diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & L \arrow[l] \arrow[d, equal] & \mathfrak{J}^1 L \arrow[l] \arrow[d, shift right] & (T^{1,0}X)^* \otimes L \arrow[l, "\gamma^{1,0}"'] \arrow[d] & 0 \arrow[l] \\
0 & L \arrow[l] \arrow[d] & \mathfrak{J}^1_{\mathbb{R}} L \arrow[l] \arrow[d, "\Phi_{\overline{\partial}}"'] \arrow[u, bend right, shift right] & TM \otimes_{\mathbb{R}} L \arrow[l, "\gamma"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & (T^{0,1}X)^* \otimes L \arrow[l] \arrow[d] & (T^{0,1}X)^* \otimes L \arrow[l, equal] \arrow[d] & 0 \arrow[l] \\
 & & 0 & 0 &
\end{tikzcd}
$$

applying the functor $\operatorname{Hom}(-, L)$ (here we used $\operatorname{End} L \simeq \mathbb{C}_M$). Splitting $\mathfrak{J}^1_{\mathbb{R}} L = \mathfrak{J}^1 L \oplus (T^{0,1}X)^* \otimes L$ (see (13)), and isomorphism (15), now determine a factorization

$$D^{\bullet}_{\mathbb{C}} L = D^{\bullet,0} L \otimes T^{0,\bullet} X, \tag{16}$$

where $D^{k,0}L = \wedge^k (\mathfrak{J}^1 L)^* \otimes L$, and $D^{\bullet,0}L := \bigoplus_k D^{k,0}L$. Factorization (16) extends splitting $D_{\mathbb{C}}L = D^{1,0}L \oplus T^{0,1}X$ (see (12)). Finally, denote by $\Delta \mapsto \Delta^{k,0}$ the projection $D_{\mathbb{C}}L \to D^{k,0}L$. Vector bundle $D^{k,0}L$ is a holomorphic vector bundle over X with flat $T^{0,1}X$-connection also denoted by $\overline{\partial}^{DE}$ and given by

$$\overline{\partial}^{DE}_{\xi} \Delta := \left([\overline{\partial}_{\xi}, \Delta]^{SJ} \right)^{k,0},$$

for all $\xi \in \Gamma(T^{0,1}X)$, and $\Delta \in \Gamma(D^{k,0}L)$.

3. Holomorphic Jacobi structures

3.1. **Holomorphic Jacobi manifolds.** Holomorphic Jacobi manifolds are the main objects in this paper. Before giving their full definition, we recall that a (real) *Jacobi manifold* is a triple $(M, L, \{-,-\})$, where M is a manifold, $L \to M$ is a line bundle, and $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ is a (skew-symmetric) bi-derivation satisfying the Jacobi identity. The bracket $\{-,-\}$ is also called a *Jacobi bracket* on $L \to M$, and the pair $(L, \{-,-\})$ is called a *Jacobi bundle*.

Example 36. *Let M be an odd dimensional manifold and let $\mathcal{C} \subset TM$ be a contact structure on it. Then the normal line bundle $L := TM/\mathcal{C} \to M$ is canonically equipped with a Jacobi bracket. See, e.g.,* [6] *for details (see also* [26, Section 3]*).*

Definition 37. *A* holomorphic Jacobi manifold *is a complex manifold $X = (M, j)$ equipped with a* holomorphic Jacobi structure, *i.e. a pair (L, J), where $L \to X$ is a holomorphic line bundle over X and J is a* holomorphic Jacobi bi-derivation *of L, i.e. a bi-derivation $J \in \Gamma(D^{2,0}L)$ such that 1) $\overline{\partial}^{DL} J = 0$, and 2) $[J, J]^{SJ} = 0$ (where*

$[-,-]^{SJ}$ *is the* Schouten-Jacobi bracket *of complex multiderivations of L). The pair (L,J) is also called a* holomorphic Jacobi bundle *over X.*

There is a more algebraic definition of a Jacobi manifold (see Lemma 39 below). To see this, first recall that a *Jacobi algebra* is a commutative algebra with unit equipped with a Lie bracket which is a first order diffferential operator in each entry. We here propose a slightly more general

Definition 38. *A* Jacobi module *over a commutative algebra with unit $\mathcal{A}$ (or a* Jacobi $\mathcal{A}$-module*), is an $\mathcal{A}$-module $\mathcal{L}$, equipped with*

(1) *a Lie bracket $\mathcal{L}\times\mathcal{L}\to\mathcal{L}$, written $(\lambda_1,\lambda_2)\mapsto\{\lambda_1,\lambda_2\}$, and*
(2) *a Lie algebra homomorphism $\mathcal{L}\to\operatorname{Der}\mathcal{A}$, written $\lambda\mapsto R_\lambda$, such that*

$$\{\lambda_1, a\lambda_2\} = R_{\lambda_1}(a)\lambda_2 + a\{\lambda_1,\lambda_2\},$$

for all $\lambda_1,\lambda_2\in\mathcal{L}$ and $a\in\mathcal{A}$.

Lemma 39. *A holomorphic Jacobi manifold is the same as a holomorphic line bundle $L\to X$ equipped with the structure of a Jacobi $\mathcal{O}_X$-module on its sheaf Γ_L of holomorphic sections, i.e.*

(1) *for all open subsets $U\subset X$, a structure of Jacobi $\mathcal{O}_X(U)$-module on $\Gamma_L(U)$, such that*
(2) *both the Lie bracket $\{-,-\}:\Gamma_L(U)\times\Gamma_L(U)\to\Gamma_L(U)$ and the Lie algebra homomorphisms $R:\Gamma_L(U)\to\operatorname{Der}\mathcal{O}_X(U)$ are compatible with restrictions.*

Proof. Suppose $X=(M,j)$ is a complex manifold and (L,J) is a holomorphic Jacobi structure on it. Restricting $J:\Gamma(L)\times\Gamma(L)\to\Gamma(L)$ to holomorphic sections we get a Jacobi bundle structure. Conversely, suppose $L\to X$ is a holomorphic line bundle equipped with the structure of a Jacobi $\mathcal{O}_X$-module on its sheaf Γ_L of holomorphic sections. First of all R can be uniquely extended to a $\mathbb{C}$-linear, first order differential operator, also denoted R, from $\Gamma(L)$ to $\Gamma(T^{1,0}X)$, as follows. For $\lambda\in\Gamma(L)$ first define R_λ (locally) on holomorphic functions. Thus, let $U\subset X$ be an open ball (i.e. an open subset bi-holomorphic to an open ball in $\mathbb{C}^n$), let μ be a holomorphic generator of $\Gamma(L|_U)$, and let $f\in\mathcal{O}_X(U)$. A section $\lambda\in\Gamma(L|_U)$ can be always written as $\lambda=g\mu$ with $g\in C^\infty(M,\mathbb{C})$. Put

$$R_\lambda(f) := fR_\mu(g) + gR_\mu(f) - R_{f\mu}(g).$$

By construction, R_λ is a well defined derivation of $\mathcal{O}_X(U)$ (with values in $C^\infty(U,\mathbb{C})$), hence it extends uniquely to a (non-necessarily holomorphic) vector field, also denoted by R_λ, in $\Gamma(T^{1,0}U)$. Finally, define $\{-,-\}:\Gamma(L|_U)\times\Gamma(L|_U)\to\Gamma(L|_U)$ by

$$\{f\mu, g\mu\} = R_{f\mu}(g)\mu - gR_\mu(f)\mu,$$

for all $f,g\in C^\infty(U,\mathbb{C})$. It is clear that the local data defined in this way, define a global J such that (X,L,J) is a holomorphic Jacobi manifold. □

Example 40 (Complex contact manifolds)**.** *Recall that a* complex contact structure *on a complex manifold* $X = (M, j)$ *of (complex dimension)* $2n+1$ *is given by a holomorphic vector subbundle* $\mathcal{C} \subset T^{1,0}X$ *of rank* $2n$ *which is completely non-integrable in the sense that the* Frobenius map*:*

$$\wedge^2 \mathcal{C} \longrightarrow T^{1,0}X/\mathcal{C}, \quad (\xi, \eta) \longmapsto [\xi, \eta] \operatorname{mod} \mathcal{C},$$

is everywhere nondegenerate. Now, let $(X, \mathcal{C})$ *be a complex contact manifold, i.e. a complex manifold equipped with a complex contact structure, and consider the holomorphic line bundle* $L = T^{1,0}X/\mathcal{C}$*, called the* contact line bundle *in the sequel, along with the exact sequence:*

$$0 \longrightarrow \mathcal{C} \longrightarrow T^{1,0}X \xrightarrow{\theta_{\mathcal{C}}} L \longrightarrow 0,$$

where $\theta_{\mathcal{C}}$ *is the canonical projection. In fact,* $\theta_{\mathcal{C}}$ *could be viewed as a holomorphic* 1*-form on* X *with values in* L*. When* $L \simeq \mathbb{C}_X$*, then* $\theta_{\mathcal{C}}$ *can be viewed as a standard holomorphic* 1*-form on* X*, called a (*global*)* contact 1*-form. Contact* 1*-forms do always exist locally.*

The contact line bundle $L \to X$ *of a complex contact manifold* $(X, \mathcal{C})$ *is naturally equipped with a holomorphic Jacobi bracket that can be constructed in the same way as in the real case of Example 36 (see, e.g.,* [6]*, see also* [26, Section 3]*). Even more, complex contact structures with fixed contact line bundle* $L \to X$*, are in one-to-one correspondence with* non-degenerate *holomorphic Jacobi bi-derivations of* L [26, Section 3]*.*

Example 41. *Let* $A \to X$ *be a holomorphic Lie algebroid over a complex manifold* $X = (M, j)$*, and let* $L \to X$ *be a holomorphic line bundle equipped with a flat holomorphic* A*-connection. Then the complex manifold* $A^* \otimes L$ *is canonically equipped with a holomorphic Jacobi bundle* $\tau^* L \to A^* \otimes L$*, where* $\tau : A^* \otimes L \to X$ *is the projection. The Jacobi bracket between holomorphic sections of* $\tau^* L \to A^* \otimes L$ *can be defined in the same way as in the real case* [17, Subsection 2.3]*. Actually, given a holomorphic complex bundle* $A \to X$ *and a holomorphic line bundle* $L \to X$*, the following sets of data are equivalent:*

- *a holomorphic Lie algebroid structure on* $A \to X$ *and a flat holomorphic* A*-connection in* L*;*
- *a* fiber-wise linear *holomorphic Jacobi bundle structure on* $\tau^* A \to A^* \otimes L$

(see [17, Subsection 2.3] *for the notion of* fiber-wise linear Jacobi brackets*).*

3.2. **Jacobi Nijenhuis manifolds and generalized contact bundles.** In Section 3.5 we show that holomorphic Jacobi manifolds are equivalent to

(1) (real) Jacobi Nijenhuis manifolds,
(2) generalized contact bundles,

of a certain kind. Here we present the notions in (1) and (2). Regarding Jacobi Nijenhuis manifold, they were first defined in [23]. Here we take a slightly more general point of view, where the Jacobi structure lives on a non-necessarily trivial line bundle.

First of all, we fix our notation. Let $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ be a bi-derivation of a real line bundle $L \to M$. We can regard $\{-,-\}$ as a vector bundle morphism $\wedge^2 \mathfrak{J}^1_{\mathbb{R}} L \to L$ (15). When doing this we will use the symbol J. Summarizing, from now on, unless otherwise stated, $\{-,-\}$ will denote a skew-symmetric bracket on $\Gamma(L)$ which is a derivation in each entry, while J will denote the corresponding L-valued skew-symmetric bilinear form on $\mathfrak{J}^1_{\mathbb{R}} L$, so that $\{-,-\}$ and J contain the same information. To express this fact we also write $J \equiv \{-,-\}$ (or $\{-,-\} \equiv J$). The 2-form J determines an obvious vector bundle morphism $J^\sharp : \mathfrak{J}^1_{\mathbb{R}} L \to D_{\mathbb{R}} L = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{J}^1_{\mathbb{R}} L, L)$. Notice that, for any $\Delta \in \Gamma(D_{\mathbb{R}} L)$ there is a unique derivation $\mathcal{L}_\Delta : \Gamma(\mathfrak{J}^1_{\mathbb{R}} L) \to \Gamma(\mathfrak{J}^1_{\mathbb{R}} L)$, such that 1) Δ and $\mathcal{L}_\Delta$ share the same symbol, and 2) $\mathcal{L}_\Delta \mathrm{j}^1_{\mathbb{R}} \lambda = \mathrm{j}^1_{\mathbb{R}} \Delta \lambda$ for all $\lambda \in \Gamma(L)$. Derivation $\mathcal{L}_\Delta$ is the *Lie derivative along* Δ. We are now ready to see that a bi-derivation $\{-,-\} \equiv J$ of L determines a skew-symmetric bracket $[-,-]_J$ on $\Gamma(\mathfrak{J}^1_{\mathbb{R}} L)$ given by

$$[\rho, \sigma]_J := \mathcal{L}_{J^\sharp \rho} \sigma - \mathcal{L}_{J^\sharp \sigma} \rho - \mathrm{j}^1_{\mathbb{R}} J(\rho, \sigma), \tag{17}$$

for all $\rho, \sigma \in \Gamma(\mathfrak{J}^1_{\mathbb{R}} L)$. A direct computation shows that J is a Jacobi bi-derivation if and only if $[-,-]_J$ is a Lie bracket. In this case $[-,-]_J$ is the Lie bracket on sections of the *jet algebroid* $(\mathfrak{J}^1_{\mathbb{R}} L)_J$ of the Jacobi manifold (M, L, J) [17]. Now, let $\phi : D_{\mathbb{R}} L \to D_{\mathbb{R}} L$ be an endomorphism, and let $\phi^\dagger : \mathfrak{J}^1_{\mathbb{R}} L \to \mathfrak{J}^1_{\mathbb{R}} L$ be its adjoint, i.e. $\langle \phi^\dagger(\rho), \Delta \rangle := \langle \rho, \phi(\Delta) \rangle$, for all $\rho \in \mathfrak{J}^1_{\mathbb{R}} L$, and $\Delta \in D_{\mathbb{R}} L$, where $\langle -,- \rangle : \mathfrak{J}^1_{\mathbb{R}} L \times D_{\mathbb{R}} L \to L$ is the "duality pairing". If $J^\sharp \circ \phi^\dagger = \phi \circ J^\sharp$, then $J_\phi := J(\phi -, -)$ is a well-defined L-valued skew-symmetric form on $\mathfrak{J}^1_{\mathbb{R}} L$ such that $J^\sharp_\phi = J^\sharp \circ \phi^\dagger$. Denote by $\{-,-\}_\phi$ the bi-derivation corresponding to J_ϕ: $J_\phi \equiv \{-,-\}_\phi$.

Now, let $\{-,-\}$ bi a bi-derivation of L, let $J : \wedge^2 \mathfrak{J}^1_{\mathbb{R}} L \to L$ be the associated 2-form, and let $\phi : D_{\mathbb{R}} L \to D_{\mathbb{R}} L$ be an endomorphism. We say that ϕ is *compatible* with J if

$$J^\sharp \circ \phi^\dagger = \phi \circ J^\sharp, \tag{18}$$

(hence J_ϕ is well-defined) and

$$\phi^\dagger [\rho, \sigma]_J = [\phi^\dagger \rho, \sigma]_J + [\rho, \phi^\dagger \sigma]_J - [\rho, \sigma]_{J_\phi}, \tag{19}$$

for all $\rho, \sigma \in \Gamma(J^1_{\mathbb{R}} L)$.

Definition 42. *A* Jacobi Nijenhuis manifold *is a manifold M equipped with a* Jacobi Nijenhuis structure*, i.e. a triple $(L, \{-,-\}, \phi)$, where $L \to M$ is a line bundle, $\{-,-\}$ is a Jacobi bracket on L, and $\phi : D_{\mathbb{R}} L \to D_{\mathbb{R}} L$ is a compatible endomorphism whose* Nijenhuis torsion $\mathcal{N}_\phi : \wedge^2 D_{\mathbb{R}} L \to D_{\mathbb{R}} L$*:*

$$\mathcal{N}_\phi(\Delta_1, \Delta_2) := [\phi(\Delta_1), \phi(\Delta_2)] + \phi^2[\Delta_1, \Delta_2] - \phi[\phi(\Delta_1), \Delta_2] - \phi[\Delta_1, \phi(\Delta_2)].$$

$\Delta_1, \Delta_2 \in \Gamma(D_{\mathbb{R}} L)$, vanishes identically.

Proposition 43. *Let $(L, \{-,-\} \equiv J, \phi)$ be a Jacobi Nijenhuis structure. Then $(L, \{-,-\}, \{-,-\}_\phi)$ is a* Jacobi bi-Hamiltonian *structure, i.e. $\{-,-\}$, $\{-,-\}_\phi$ and $\{-,-\} + \{-,-\}_\phi$ are all Jacobi brackets.*

We now recall the definition of a generalized contact bundle from [27]. Let $L \to M$ be a line bundle. The *omni-Lie algebroid* [2, 4] $\mathbb{D}L := D_{\mathbb{R}}L \oplus \mathfrak{J}^1_{\mathbb{R}}L$ is canonically equipped with the following structures:

- the projection $\mathrm{pr}_D : \mathbb{D}L \to D_{\mathbb{R}}L$,
- the symmetric bilinear form $\langle\!\langle -, - \rangle\!\rangle : \mathbb{D}L \otimes \mathbb{D}L \to \mathbb{L}$:
$$\langle\!\langle (\Delta, \rho), (\nabla, \sigma) \rangle\!\rangle := \langle \sigma, \Delta \rangle + \langle \rho, \nabla \rangle,$$
- the *Dorfman-Jacobi bracket* $[\![-, -]\!] : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \to \Gamma(\mathbb{D}L)$:
$$[\![(\Delta, \rho), (\nabla, \sigma)]\!] := ([\Delta, \nabla], \mathcal{L}_\Delta \sigma - \mathcal{L}_\nabla \rho + \mathfrak{j}^1_{\mathbb{R}} \langle \rho, \nabla \rangle),$$

$\Delta, \nabla \in \Gamma(D_{\mathbb{R}}L)$, $\rho, \sigma \in \Omega^1(M)$. With the above three structures $\mathbb{D}L$ is an *L-Courant algebroid* [3] and a *contact Courant algebroid* [10].

Definition 44 ([27])**.** *A* generalized contact bundle *is a line bundle $L \to M$ equipped with a* generalized contact structure*, i.e. a vector bundle endomorphism $\mathcal{I} : \mathbb{D}L \to \mathbb{D}L$ such that*

- $\mathcal{I}$ *is* almost complex*, i.e.* $\mathcal{I}^2 = -1$*,*
- $\mathcal{I}$ *is* skew-symmetric*, i.e.*
$$\langle\!\langle \mathcal{I}\alpha, \beta \rangle\!\rangle + \langle\!\langle \alpha, \mathcal{I}\beta \rangle\!\rangle = 0, \quad \alpha, \beta \in \Gamma(\mathbb{D}L),$$
- $\mathcal{I}$ *is* integrable*, i.e.*
$$[\![\mathcal{I}\alpha, \mathcal{I}\beta]\!] - [\![\alpha, \beta]\!] - \mathcal{I}[\![\mathcal{I}\alpha, \beta]\!] + \mathcal{I}[\![\alpha, \mathcal{I}\beta]\!] = 0, \quad \alpha, \beta \in \Gamma(\mathbb{D}L).$$

Let $(L \to M, \mathcal{I})$ be a generalized contact bundle. Using the direct sum decomposition $\mathbb{D}L = D_{\mathbb{R}}L \oplus \mathfrak{J}^1_{\mathbb{R}}L$, and the definition, one can see that
$$\mathcal{I} = \begin{pmatrix} \phi & J^\sharp \\ \omega_\flat & -\phi^\dagger \end{pmatrix}$$
where J is a Jacobi bi-derivation, $\phi : D_{\mathbb{R}}L \to D_{\mathbb{R}}L$ is an endomorphism compatible with J, and $\omega : \wedge^2 D_{\mathbb{R}}L \to L$ is a 2-form, with associated vector bundle morphism $\omega_\flat : D_{\mathbb{R}}L \to \mathfrak{J}^1_{\mathbb{R}}L$, satisfying additional compatibility conditions [27]. In particular, when $\omega = 0$, then ϕ is a complex structure in the vector bundle $D_{\mathbb{R}}L \to M$, and (L, J, ϕ) is a Jacobi Nijenhuis structure. Generalized contact bundles are supported by odd dimensional manifolds and their geometry is an odd dimensional analogue of generalized complex geometry.

3.3. **Jacobi Nijenhuis and homogeneous Poisson Nijenhuis, generalized contact and homogeneous generalized complex.** Jacobi manifolds are *morally* equivalent to homogeneous Poisson manifolds. The equivalence is provided by the *Poissonization construction* in one direction, and by Proposition 45 in the other direction. There are similar *moral equivalences* between Jacobi Nijenhuis manifolds, resp. generalized contact bundles, on one side and homogeneous Poisson Nijenhuis manifolds, resp. homogeneous generalized complex manifolds, on the other side (Theorem 47, resp. Theorem 48). To see this, let $L \to M$ be a (real) line bundle. The manifold $\widetilde{M} := L^* \smallsetminus 0$ is a principal bundle over M with structure group the multiplicative group $\mathbb{R}^\times := \mathbb{R} \smallsetminus \{0\}$. Denote by $p : \widetilde{M} \to M$ the projection. Let η be the restriction to $\widetilde{M}$ of the *Euler vector field* on L^*. Then η is the fundamental vector field corresponding to the canonical generator 1 in the Lie algebra $\mathbb{R}$ of the structure group $\mathbb{R}^\times$. Clearly, sections of L are in one-to-one correspondence with *homogeneous functions* on $\widetilde{M}$, i.e. functions $f \in C^\infty(\widetilde{M})$ such that $\mathcal{L}_\eta f = f$. For any section $\lambda \in \Gamma(L)$, we denote by $\widetilde{\lambda}$ the corresponding homogeneous function on $\widetilde{M}$. Now, let $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ be a bi-derivation, i.e. a skew-symmetric bracket which is a first order differential operator, hence a derivation, in each argument. It is easy to see that there is a unique bi-vector π on $\widetilde{M}$ such that

$$\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\}_{\widetilde{M}} = \widetilde{\{\lambda_1, \lambda_2\}},$$

for all $\lambda_1, \lambda_2 \in \Gamma(L)$, where $\{-,-\}_{\widetilde{M}} : C^\infty(\widetilde{M}) \times C^\infty(\widetilde{M}) \to C^\infty(\widetilde{M})$ is the skew bracket determined by π, i.e. $\{f_1, f_2\}_{\widetilde{M}} = \pi(df_1, df_2)$. Additionally, $\mathcal{L}_\eta \pi = -\pi$. Finally, π is a *Poisson bi-vector*, hence (π, η) is a homogeneous Poisson structure on $\widetilde{M}$, if and only if $\{-,-\}$ is a Jacobi bracket, i.e. it satisfies the Jacobi identity. In this case, the homogeneous Poisson manifold $(\widetilde{M}, \pi, \eta)$ is sometimes called the *Poissonization* of the Jacobi manifold $(M, L, \{-,-\})$. Notice that π comes from a symplectic structure if and only if J comes from a contact structure.

Proposition 45. *Every homogeneous Poisson manifold is the Poissonization of a canonical Jacobi manifold around every non-singular point of the homogeneity vector field.*

Proof. Let (N, π_N, η_N) be an $(n+1)$-dimensional homogeneous Poisson manifold, and let $x \in N$ be a point such that $(\eta_N)_x \neq 0$. Then η_N is everywhere non-zero in a whole neighborhood U of x. Even more, U can be chosen so that there is a diffeomorphism $F : U \to (a,b) \times M$, where (a,b) is an interval with $0 < a < b$, M in an n-dimensional manifold, and F intertwines η_N and $t\frac{d}{dt}$, t being the canonical coordinate on (a,b). In the following, we use F to identify U with $(a,b) \times M$, and η_N with $t\frac{d}{dt}$. Then, the space of orbits of η_N in U is simply M, and the projection $p : U = (a,b) \times M \to M$ is the projection onto the second factor. Smooth functions f on U that are homogeneous with respect to η_N, i.e. such that $\mathcal{L}_{\eta_N} f = f$, are linear functions in the coordinate t, and they form a $C^\infty(M)$-module $\mathcal{L}$. The value of any such function on a fiber $\mathcal{F}$ of p is completely determined by its value on a point of $\mathcal{F}$. Hence $\mathcal{L}$ is the module of sections of a line bundle $L \to M$. Since π_N is homogeneous with respect to η_N, then the corresponding Poisson bracket restricts to a Jacobi bracket $\{-,-\}$ on L. Our final aim is to show that U can be embedded in $\widetilde{M}$ as an an *open homogeneous, Poisson submanifold*. Clearly, there is a unique smooth Poisson map $i : U \to \widetilde{M}$ intertwining η_N and η, such that

$$\widetilde{\lambda}(i(x)) = \lambda(x),$$

for all $x \in U$ and $\lambda \in \Gamma(L) = \mathcal{L} \subset C^\infty(U)$. Since $\mathcal{L}$ generates $C^\infty(U)$ as a C^∞-algebra, then i is the inclusion of an open (homogeneous, Poisson) submanifold. □

Remark 46. *Proposition 45 should be compared with a similar one: Proposition 2.3 in* [7]*, where M is viewed as a submanifold of N rather then a quotient. Notice that we can work with any pair (π_N, η_N) such that π_N is a (not necessarily Poisson) bi-vector on N and η_N is a vector field on N such that $\mathcal{L}_{\eta_N}\pi_N = -\pi_N$. In this case, $\{-,-\}$ is still a well-defined bi-derivation of $L \to M$, and it satisfies the Jacobi identity if and only if π_N is a Poisson bi-vector.*

Theorem 47. *The Poissonization of a Jacobi Nijenhuis manifold is a homogeneous Poisson Nijenhuis manifold in a natural way. Conversely, every homogeneous Poisson Nijenhuis manifold is the Poissonization of a canonical Jacobi Nijenhuis manifold around every non-singular point of the homogeneity vector field.*

Proof. Let $L \to M$ be a line bundle and let $\{-,-\}$ be a bi-derivation of L. Consider the slit dual $\widetilde{M} = L^* \smallsetminus 0$, the Euler vector field $\eta \in \mathfrak{X}(\widetilde{M})$ and the bi-vector π determined by $\{-,-\}$ on $\widetilde{M}$ as in the beginning of the present section (if, in particular, $(M, L, \{-,-\})$ is a Jacobi manifold, then (π, η) is its Poissonization). Finally let $\phi : D_\mathbb{R}L \to D_\mathbb{R}L$ be an endomorphism. From [26, Proposition A.1] there is an embedding $\iota : \Gamma(D_\mathbb{R}L) \to \mathfrak{X}(\widetilde{M})$, $\Delta \mapsto \widetilde{\Delta}$ of $C^\infty(M)$-modules, and Lie algebras, uniquely determined by $\widetilde{\Delta}(\widetilde{\lambda}) = \widetilde{\Delta(\lambda)}$, for all $\lambda \in \Gamma(L)$, and there is a canonical isomorphism $p^*D_\mathbb{R}L \simeq T\widetilde{M}$ of vector bundles over $\widetilde{M}$ such that ι agrees with the pull-back along p of sections of DL. In particular, there is a unique (1,1) tensor $\widetilde{\phi}$ on $T\widetilde{M}$ such that $\widetilde{\phi}(\widetilde{\Delta}) = \widetilde{\phi(\Delta)}$ for all $\Delta \in \Gamma(D_\mathbb{R}L)$. It is not hard to see that $(\widetilde{M}, \pi, \widetilde{\phi}, \eta)$ is a homogeneous Poisson Nijenhuis manifold if and only if $(M, L, \{-,-\}, \phi)$ is a Jacobi Nijenhuis manifold. We leave the details to the reader.

Conversely, let $(N, \pi_N, \phi_N, \eta_N)$ be a homogeneous Poisson Nijenhuis manifold and let $p : U \to M$, $(M, L, \{-,-\})$, and $i = U \to \widetilde{M}$ be as in the proof of Proposition 45. From $\mathcal{L}_{\eta_N}\phi_N = 0$, it follows that ϕ_N restricts to an endomorphism $\phi : D_\mathbb{R}L \to D_\mathbb{R}L$. Moreover, (M, L, J, ϕ) is a Jacobi Nijenhuis manifold. This concludes the proof. □

Now, let $(L \to M, \mathcal{I})$ be a generalized contact bundle, $\mathcal{I} = \left(\begin{smallmatrix} \phi & J^\sharp \\ \omega_\flat & -\phi^\dagger \end{smallmatrix}\right)$. Consider $\widetilde{M} = L^* \smallsetminus 0$ and the Euler vector field η on it. Since L is equipped with a Jacobi structure J, then $\widetilde{M}$ is canonically equipped with a homogeneous Poisson structure π. We call $(\widetilde{M}, \pi, \eta)$ the *Poissonization* of $(L \to M, \mathcal{I})$.

Theorem 48. *The Poissonization of a generalized contact bundle is a homogeneous generalized complex manifold in a natural way. Conversely, every homogeneous generalized complex manifold is the Poissonization of a canonical generalized contact bundle around every non-singular point of the homogeneity vector field.*

Proof. We discussed in [27, Remark 3.6, arXiv version] that π can be canonically completed to a homogeneous generalized complex structure $(\mathcal{J}, \eta)$ (Definition 12): $\mathcal{J} = \begin{pmatrix} \widetilde{\phi} & \pi^\sharp \\ \widetilde{\omega}_\flat & -\widetilde{\phi}^* \end{pmatrix}$. The second part of the statement can be proved along similar lines as in the proof of Theorem 47. We leave the details to the reader. □

Remark 49. *Let $L \to M$ be a real line bundle and let $\widetilde{M} = L^* \smallsetminus 0$ be its* slit dual. *Denote by η the Euler vector field on $\widetilde{M}$. We proved that, given a bi-derivation J of L and an endomorphism $\phi : D_{\mathbb{R}}L \to D_{\mathbb{R}}L$ we can canonically construct a bi-vector π and a (1,1) tensor $\widetilde{\phi}$ on $\widetilde{M}$. This construction establishes a one-to-one correspondence between pairs (J, ϕ) consisting of a bi-derivation J of L and an endomorphism ϕ of DL, and pairs $(\pi, \widetilde{\phi})$ consisting of a bi-vector and a (1,1) tensor on $\widetilde{M}$ such that $\mathcal{L}_\eta \pi = -\pi$, and $\mathcal{L}_\eta \widetilde{\phi} = 0$. Finally, this correspondence identifies Jacobi Nijenhuis structure (L, J, ϕ) (resp. generalized contact structures $(L, \mathcal{I})$) on M, and homogeneous Poisson Nijenhuis structures $(\pi, \widetilde{\phi}, \eta)$ (resp. homogeneous generalized complex structures $(\mathcal{J}, \eta)$) on $\widetilde{M}$.*

3.4. **Holomorphic Jacobi and homogeneous holomorphic Poisson manifolds.** In this section we show that, similarly as for real Jacobi manifolds, holomorphic Jacobi manifolds are morally equivalent to homogeneous holomorphic Poisson manifolds. In their turn homogeneous holomorphic Poisson manifolds are related to homogeneous Poisson Nijenhuis manifolds and to homogeneous generalized complex manifolds (Theorem 13). Finally homogeneous Poisson Nijenhuis manifolds are morally equivalent to Jacobi Nijenhuis manifolds (Theorem 47), and homogeneous generalized complex manifolds are morally equivalent to generalized contact bundles (Theorem 48). In this way, we depict the following picture

$$\begin{array}{ccccc} \text{Jacobi Nijenhuis} & \Longleftarrow & \text{holomorphic Jacobi} & \Longrightarrow & \text{generalized contact} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \begin{array}{c}\text{homogeneous}\\ \text{Poisson Nijenhuis}\end{array} & \Longleftarrow & \begin{array}{c}\text{homogeneous}\\ \text{holomorphic Poisson}\end{array} & \Longrightarrow & \begin{array}{c}\text{homogeneous}\\ \text{generalized complex}\end{array} \end{array}$$

However, there is a difference between the holomorphic Jacobi and the holomorphic Poisson case. Namely, every holomorphic Poisson manifold is a (real) Poisson Nijenhuis manifold. On the other hand, a holomorphic Jacobi structure on a complex manifold $X = (M, j)$ gives rise to a Jacobi Nijenhuis structure, but not on M, rather on a larger manifold, specifically, a circle bundle over M.

3.4.1. *From holomorphic Jacobi to homogeneous holomorphic Poisson manifolds.* Let $L \to X$ be a holomorphic line bundle over a complex manifold $X = (M, j)$ and let L^* be its complex dual. In the following we denote by $\widetilde{M}$ the *slit complex dual* of L, i.e. $\widetilde{M} := L^* \smallsetminus 0$ (beware that in the previous section, we denoted by the same symbol $\widetilde{M}$ a different object, namely the *slit real dual* of a *real* line bundle). The real manifold

$\widetilde{M}$ is equipped with a complex structure $\widetilde{j}$ induced by the complex structure on L, so that $\widetilde{X} = (\widetilde{M}, \widetilde{j})$ is a holomorphic principal bundle over X with structure group the multiplicative $\mathbb{C}^\times := \mathbb{C} \smallsetminus 0$. Denote by $p : \widetilde{X} \to X$ the projection. Let $H \in \Gamma(T^{1,0}\widetilde{X})$ be the restriction to $\widetilde{X}$ of the holomorphic Euler vector field on L^*. Then H is the fundamental vector field corresponding to the canonical generator 1 in the complex Lie algebra $\mathbb{C}$ of the structure group $\mathbb{C}^\times$. It is easy to see that $H = \frac{1}{2}(\eta - i\widetilde{j}\eta)$, where $\eta \in \mathfrak{X}(\widetilde{M})$ is the (restriction of) the real Euler vector field of the real (rank 2) vector bundle $L^* \to M$.

Proposition 50. *A holomorphic Jacobi bracket $J \equiv \{-,-\}$ on $L \to X$ determines canonically a homogeneous holomorphic Poisson structure (Π, H) on $\widetilde{X}$. Additionally, J comes from a complex contact structure (see Example 40) if and only if Π comes from a complex symplectic structure.*

Proof. The proof is similar to that of the analogous statement in the smooth category. We report a sketch here for completeness. First of all, holomorphic sections of $L \to X$ are in one-to-one correspondence with homogeneous holomorphic functions on $\widetilde{X}$, i.e. functions $f \in \mathcal{O}_{\widetilde{X}}$ such that $\mathcal{L}_H f = f$. For any holomorphic section λ of $L \to X$, we denote by $\widetilde{\lambda}$ the corresponding homogeneous holomorphic function on $\widetilde{X}$. Now, let $J \equiv \{-,-\} \in \Gamma(D^{2,0}_{\mathbb{C}}L)$ be a holomorphic bi-derivation of $L \to X$, i.e. $\overline{\partial}^{DL}J = 0$. It is easy to see that there exists a unique holomorphic bivector Π on $\widetilde{X}$ such that

$$\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\}_{\widetilde{X}} = \widetilde{\{\lambda_1, \lambda_2\}},$$

for all holomorphic sections λ_1, λ_2 of L, where $\{-,-\}_{\widetilde{X}} : \mathcal{O}_{\widetilde{X}} \times \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$ is the skewsymmetric bracket corresponding to π, i.e. $\{f_1, f_2\}_{\widetilde{X}} = \Pi(\partial f_1, \partial f_2)$. Additionally, $\mathcal{L}_H \Pi = -\Pi$. Finally, Π is a *holomorphic Poisson bi-vector*, hence (Π, H) is a homogeneous holomorphic Poisson structure on $\widetilde{X}$, if and only if J is a Jacobi bracket, i.e. it satisfies the Jacobi identity.

The second part of the statement can be proved, e.g., in local coordinates. We leave the details to the reader. We only remark that, if μ is a holomorphic section which generates $\Gamma(L)$ locally in its domain, and $\mathcal{C} \subset T^{1,0}X$ is a complex contact structure with $L = T^{1,0}X/\mathcal{C}$, then, in view of the *complex contact Darboux lemma* there are complex coordinates (t, z^i, P_i) on X such that

$$\theta_{\mathcal{C}} = \left(dt - P_k dz^k\right) \otimes \mu.$$

where $\theta_{\mathcal{C}} : T^{1,0}X \to L$ is the projection. It is now easy to see that the corresponding holomorphic (homogeneous) symplectic structure Ω on $\widetilde{X}$ is locally given by

$$\Omega = d\widetilde{\mu} \wedge (dt - P_k dz^k) - \widetilde{\mu}\, dP_k \wedge dz^k.$$

□

Remark 51. *Let $L \to X$ be a holomorphic line bundle and let $\widetilde{X} = L^* \smallsetminus 0$ be its* slit complex dual*. Denote by H the Euler vector field on $\widetilde{X}$. The proof of Proposition 50 shows that, given a holomorphic bi-derivation J of L, we can canonically construct a holomorphic bi-vector Π on $\widetilde{X}$. This construction establishes a one-to-one correspondence between holomorphic bi-derivations of L and holomorphic bi-vectors on $\widetilde{X}$. Additionally, this correspondence identifies holomorphic Jacobi structures (L, J) on X, and homogeneous holomorphic Poisson structures (Π, H) on $\widetilde{X}$.*

Example 52. *Let $\mathfrak{g}$ be a complex Lie algebra and let (Π, H) be the homogeneous holomorphic Poisson structure on its complex dual $\mathfrak{g}^*$ (see Example 8). We also consider the complex projective space $\mathbb{CP}(\mathfrak{g}^*)$. Call it X, and denote by $L \to X$ the complex dual of the tautological (line) bundle over $\mathbb{CP}(\mathfrak{g}^*)$. Clearly, $\widetilde{X} = \mathfrak{g}^* \smallsetminus 0$. Additionally H identifies with the Euler vector field on $\widetilde{X}$. Together with Remark 51, this shows that $L \to X$ is equipped with a canonical Jacobi bundle structure.*

Example 53. *Let X_0 be a complex manifold and recall that the cotangent bundle T^*X_0 is equipped with a canonical homogeneous symplectic structure (Ω, H) where Ω is the canonical complex symplectic form and H is the holomorphic Euler vector field (see Example 9). We also consider the complex projective bundle $\mathbb{CP}(T^*X_0)$. Call it X, and denote by $L \to X$ the complex dual of the tautological (line) bundle over $\mathbb{CP}(T^*X_0)$. Clearly, $\widetilde{X} = T^*X_0 \smallsetminus 0$, and H identifies with the Euler vector field on $\widetilde{X}$. Hence there is a canonical contact structure on $\mathbb{CP}(T^*X_0)$. The latter agrees with the standard contact structure on the complex manifold of complex contact elements in X_0.*

Example 54. *More generally, let $A \to X_0$ be a holomorphic Lie algebroid and let (Π, H) be the homogeneous holomorphic Poisson structure on its complex dual A^* (see Example 10). We also consider the complex projective bundle $\mathbb{CP}(A^*)$. Call it X, and denote by $L \to X$ the complex dual of the tautological (line) bundle over $\mathbb{CP}(A^*)$. Clearly, $\widetilde{X} = A^* \smallsetminus 0$. Additionally H identifies with the Euler vector field on $\widetilde{X}$. This shows that $L \to X$ is equipped with a canonical Jacobi bundle structure.*

Definition 55. *The holomorphic Poisson manifold $(\widetilde{X}, \Pi, H)$ of Proposition 50 is called the* Poissonization *of the holomorphic Jacobi manifold (X, L, J).*

3.4.2. *From homogeneous holomorphic Poisson to holomorphic Jacobi manifolds.*

Proposition 56. *Every homogeneous holomorphic Poisson manifold is the Poissonization of a canonical holomorphic Jacobi manifold around a non-singular point of the homogeneity vector field.*

Proof. The proof is similar to that of Proposition 45. We report it here for completeness. So, let (N, j_N) be a complex manifold, and let (Π_N, H_N) be a homogeneous holomorphic Poisson structure on it. Denote by η_N twice the real part of H so that $H = \frac{1}{2}(\eta_N - ij_N\eta_N)$ and let $x \in N$ be a point such that $(H_N)_x \neq 0$. Then η_N and $j_N\eta_N$ span a two dimensional distribution D on a whole neighborhood U of x. Since H is holomorphic, then distribution D is integrable. Let M be the space of leaves of D. We assume, which is always possible, upon shrinking U if necessary, that M is a smooth manifold and the projection $p : U \to M$ is a surjective submersion. Clearly, j_N descends to a complex structure j_M on M. Let $X = (M, j_M)$. Upon shrinking U again if necessary, we can always achieve the following situation: 1) there is a biholomorphism $F : U \to B \times X$, where $B \subset \mathbb{C}$ is a ball not containing 0, and 2) F intertwines H_N and $w\frac{d}{dw}$, w being the canonical complex coordinate on B. In the following, we use F to identify U with $B \times X$, and H_N with $w\frac{d}{dw}$. Then, the projection $p : U = B \times X \to X$ is the projection onto the second factor. Holomorphic functions f on U that are homogeneous with respect to H_N, i.e. such that $\mathcal{L}_{H_N} f = f$, are holomorphic functions that are linear in the coordinate w, and they form an $\mathcal{O}_X$-module $\mathcal{L}$. The value of any such function on a fiber $\mathcal{F}$ of p is completely determined by its value on a point of $\mathcal{F}$. Hence $\mathcal{L}$ is the module of holomorphic sections of a holomorphic line bundle $L \to X$. Since Π_N is homogeneous with respect to H_N, then the corresponding Poisson bracket restricts to a Jacobi bracket $\{-,-\}$ on L. Finally, similarly as in the real case, U can be embedded into the Poissonization of $(X, L, \{-,-\})$ as an open homogeneous holomorphic Poisson submanifold. □

3.5. **Holomorphic Jacobi, Jacobi Nijenhuis and generalized contact manifolds.** Let $X = (M, j)$ be a complex manifold, let $L \to X$ be a holomorphic line bundle, and let $J \in \Gamma(D^{2,0}_{\mathbb{C}} L)$ be a holomorphic bi-derivation of L. Construct $(\widetilde{X}, \Pi, H)$ as in the proof of Proposition 50, $\widetilde{X} = (\widetilde{M}, \widetilde{j})$ (if, in particular, (X, L, J) is a holomorphic Jacobi manifold, then $(\widetilde{X}, \Pi, H)$ is its Poissonization). We have $\Pi = \pi' + i\pi$ for some real Poisson bi-vectors π, π' on $\widetilde{M}$, and $H = \frac{1}{2}(\eta - i\widetilde{j}\eta)$, where η is the real Euler vector field on $\widetilde{M}$. Additionally, $\mathcal{L}_\eta \pi = -\pi$, $\mathcal{L}_\eta \pi' = -\pi'$, and $\mathcal{L}_\eta \widetilde{j} = 0$. Now, notice that $\widetilde{M}$ can be given the structure of a principal $\mathbb{R}^\times$-bundle so that η is the fundamental vector field corresponding to $1 \in \mathbb{R}$. Indeed, consider the real projective bundle $\mathbb{RP}(L^*)$ (of the rank two real vector bundle $L \to M$). Denote it by $\widehat{M}$. Let $\ell \to \widehat{M}$ be the real dual of the real tautological line bundle over $\widehat{M} = \mathbb{RP}(L^*)$. Clearly, $\widetilde{M}$ identifies canonically with the slit real dual of $\ell \to \widehat{M}$, i.e. $\widetilde{M} = \ell^* \smallsetminus 0$. Additionally, η identifies with the (restriction of the) Euler vector field on (the total space of the real line bundle) $\ell \to \widehat{M}$. It follows that (see Remark 49)

(1) π, π' correspond to bi-derivations $\widehat{J}, \widehat{J}'$ of $\ell \to \widehat{M}$,
(2) $\widetilde{j}$ corresponds to an endomorphism $\widehat{j} : D_{\mathbb{R}}\ell \to D_{\mathbb{R}}\ell$.

Theorem 57. *Let $X = (M, j)$ be a complex manifold, let $L \to X$ be a holomorphic line bundle, and let $J \in \Gamma(D^2_{\mathbb{C}} L)$ be a complex bi-derivation of the complex line bundle $L \to M$. Consider the bi-derivations $\widehat{J}, \widehat{J}'$ of the real line bundle $\ell \to \widehat{M}$, and the endomorphism $\widehat{j} : D_{\mathbb{R}}\ell \to D_{\mathbb{R}}\ell$, as constructed above. Then, the following conditions are equivalent*

(1) (L, J) *is a holomorphic Jacobi structure on* X*,*
(2) $(\widehat{J}, \widehat{j})$ *is a Jacobi Nijenhuis structure on* $\widehat{M}$*, and* $\widehat{J}' = \widehat{J}_{\widehat{j}}$*,*
(3) $\begin{pmatrix} \widehat{j} & \widehat{J}^\sharp \\ 0 & -\widehat{j}^\dagger \end{pmatrix}$ *is a generalized contact structure on* $\ell \to \widehat{M}$*, and* $\widehat{J}' = \widehat{J}_{\widehat{j}}$*.*

Additionally, $\widehat{J}$ *(and hence* $\widehat{J}'$*) comes from a contact structure if and only if* J *comes from a complex contact structure.*

Proof. The first part of the statement immediately follows from Proposition 56, and Theorems 6, 47, 48 (see also Remarks 49 and 51). The last part can be easily proved in local coordinates (see also Example 60 below). □

The situation is summarized in the following diagram

$$\begin{tikzcd}
\text{homogeneous holomorphic Poisson} & & \widetilde{X} \arrow[dl, "\mathbb{R}^\times"'] \arrow[dd, "\mathbb{C}^\times"] \\
\text{Jacobi Nijenhuis / generalized contact} & \widehat{M} \arrow[dr, "S^1"'] & \\
\text{holomorphic Jacobi} & & X
\end{tikzcd} .$$

Remark 58. *Let* (X, L, J) *be a holomorphic Jacobi manifold, and let* $\widehat{J}, \widehat{j}$ *be as in Theorem 57. From Proposition 43,* $(\ell, \widehat{J}, \widehat{J}_{\widehat{j}})$ *is a Jacobi bi-Hamiltonian structure on* $\widehat{M}$*, i.e.* $\widehat{J}, \widehat{J}_{\widehat{j}}$ *and* $\widehat{J} + \widehat{J}_{\widehat{j}}$ *are all Jacobi structures.*

Remark 59. *Let* (X, L, J) *be a holomorphic Jacobi manifold, and let* $\eta, \pi, \widetilde{j}, \widehat{J}, \widehat{j}$ *be as above. From* $[\widetilde{j}\eta, \eta] = 0$*, it follows that* $\widetilde{j}\eta$ *corresponds to a derivation* Δ *of* $\ell \to \widehat{M}$*. It is easy to see that* $\Delta = \widehat{j}\mathbb{1}$*, where* $\mathbb{1} : \Gamma(L) \to \Gamma(L)$ *is the identity derivation. From*

$$\mathcal{L}_{\widetilde{j}\eta}\pi_{\widetilde{j}} = \pi, \quad \textit{and} \quad \mathcal{L}_{\widetilde{j}\eta}\pi = -\pi_{\widetilde{j}}$$

(see (6)) now follows that

$$[\widehat{j}\mathbb{1}, \widehat{J}_{\widehat{j}}]^{SJ} = \widehat{J} \quad \textit{and} \quad [\widehat{j}\mathbb{1}, \widehat{J}]^{SJ} = -\widehat{J}_{\widehat{j}},$$

where $[-,-]^{SJ}$ *is the* Schouten-Jacobi bracket*.*

Example 60. *Let* $X = \mathbb{C}^{2n+1}$*, with complex coordinates*

$$(t = r + is, z^k = x^k + iy^k, P_k = m_k + iq_k)$$

and let

$$\theta = dt - P_k dz^k$$

be the canonical complex contact 1-form on it. Then $\widetilde{X} = \mathbb{C}^{2n+1} \times \mathbb{C}^\times$*, and it has an additional (nowhere vanishing) complex coordinate* $w = u + iv$*. The holomorphic Euler vector field is then* $H = w\frac{\partial}{\partial w}$ *and the homogeneous symplectic structure on* $\widetilde{X}$ *is*

$$\Omega = dw \wedge (dt - P_k dz^k) - w dP_i \wedge dz^k.$$

The real Euler vector field is $\eta = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$*. It is actually convenient to use a polar representation for the coordinate* w *and write* $w = \rho e^{i\varphi/2}$*. Then* $\eta = \rho\frac{\partial}{\partial \rho}$ *and* φ *can be seen as the fiber coordinate in the* S^1*-bundle*

$$\mathbb{RP}(\mathbb{C}^{2n+1} \times \mathbb{C} \to \mathbb{C}^{2n+1}) \simeq \mathbb{C}^{2n+1} \times S^1 \to \mathbb{C}^{2n+1}.$$

Theorem 57 now implies that there are two contact structures ϑ *and* ϑ_j *on* $\mathbb{C}^{2n+1} \times S^1$*. They are locally given by*

$$\vartheta = \cos(\varphi/2)\vartheta_r - \sin(\varphi/2)\vartheta_s$$

and

$$\vartheta_j = -\sin(\varphi/2)\vartheta_r - \cos(\varphi/2)\vartheta_s,$$

where

$$\vartheta_r := dr - m_k dx^k + q_k dy^k, \quad \text{and} \quad \vartheta_s = ds - m_k dy^k - q_k dx^k.$$

4. Lie algebroids of a holomorphic Jacobi manifold

Recall that a Poisson bi-vector π on a real manifold M determines a Lie algebroid structure on T^*M, denoted by $(T^*M)_\pi$ and called the *cotangent Lie algebroid of the Poisson manifold* (M, π). The anchor of $(T^*M)_\pi$ is given by $\pi^\sharp : T^*M \to TM$ and the Lie bracket is $[-,-]_\pi$ (see (1)).

Similarly, a holomorphic Poisson structure Π on a complex manifold $X = (M, j)$ determines a holomorphic Lie algebroid structure on $T^*X \to X$, denoted by $(T^*X)_\Pi$ and called the *cotangent Lie algebroid of* (X, Π) [16]. Specifically, the anchor $\rho : T^*X \to TX$ is given by

$$\rho(\omega) := 2\,\mathrm{Re}\left(\Pi^\sharp(\omega - ij^*\omega)\right)$$

for all $\omega \in \Omega^1(X)$, and the bracket $[-,-]_\Pi$ acts on two holomorphic sections ω_1, ω_2 of $T^*X \to X$ as

$$[\omega_1, \omega_2]_\Pi = \mathcal{L}_{\rho(\omega_1)}\omega_2 - \mathcal{L}_{\rho(\omega_2)}\omega_1 - d\langle\rho(\omega_1), \omega_2\rangle.$$

Let $\Pi = \pi_j + i\pi$, with π, π_j real Poisson structures on M.

Proposition 61 (Laurent-Gengoux, Stiénon, and Xu [16])**.** *The real (resp. imaginary) Lie algebroid of the holomorphic Lie algebroid* $(T^*X)_\Pi$ *is* $(T^*M)_{4\pi_j}$ *(resp.* $(T^*M)_{4\pi}$*).*

Now recall that a Jacobi stucture J on a real line bundle $L \to M$ does also determine a Lie algebroid structure on the first jet bundle $\mathfrak{J}^1_{\mathbb{R}}L$, denoted $(\mathfrak{J}^1_{\mathbb{R}}L)_J$ and called the *jet algebroid of the Jacobi manifold* (M, L, J). The ancor of $(\mathfrak{J}^1_{\mathbb{R}}L)_J$ is given by the composition of $J^\sharp : \mathfrak{J}^1_{\mathbb{R}}L \to D_{\mathbb{R}}L$ followed by the symbol $\sigma : D_{\mathbb{R}}L \to TM$, and the Lie bracket is $[-,-]_J$ (see (17)).

Similarly, a holomorphic Jacobi structure (L, J) on a complex manifold $X = (M, j)$ determines a holomorphic Lie algebroid structure on $\mathfrak{J}^1 L \to X$, denoted by $(\mathfrak{J}^1 X)_J$ and called the *jet Lie algebroid of* (X, L, J). The anchor $\rho : \mathfrak{J}^1 L \to TX$ is given by

$$\rho(\theta) := 2\,\mathrm{Re}\left((\sigma \circ J^\sharp)(\theta)\right)$$

for all $\theta \in \Gamma(\mathfrak{J}^1 L)$, and the bracket $[-,-]_J$ acts on two holomorphic sections θ_1, θ_2 of $\mathfrak{J}^1 L \to X$ as

$$[\theta_1, \theta_2]_J = \mathcal{L}_{J^\sharp\theta_1}\theta_2 - \mathcal{L}_{J^\sharp\theta_2}\theta_1 - \mathrm{j}^{1,0} J(\theta_1, \theta_2).$$

Our next aim is to find the relationship between the real (resp. the imaginary) Lie algebroid of the holomorphic Lie algebroid $(\mathfrak{J}^1 L)_J \to X$ and the Lie algebroid $(\mathfrak{J}^1_{\mathbb{R}} L)_{\widehat{J}_j} \to \widehat{M}$ (resp. $(\mathfrak{J}^1_{\mathbb{R}} L)_{\widehat{J}} \to \widehat{M}$, see previous section for a definition of $\widehat{J}$). Notice that, unlike the holomorphic Poisson case, the two cannot be simply equal, because their base manifolds are different. So the situation is more complicated than in the holomorphic Poisson case. We start discussing the relationship between vector bundles $\mathfrak{J}^1 L \to X$ and $\mathfrak{J}^1_{\mathbb{R}} L \to \widehat{M}$.

Denote by

$$\begin{aligned} \widetilde{p} &: \widetilde{X} \longrightarrow X, \\ \widehat{p} &: \widehat{M} \longrightarrow X, \\ q &: \widetilde{X} \longrightarrow \widehat{M} \end{aligned}$$

the projections. Additionally, for a complex manifold $X = (M, j)$, denote by $(T^*X)^{1,0}$ the $+i$-eigenbundle of $j^* : (T^*M)^{\mathbb{C}} \to (T^*M)^{\mathbb{C}}$, and by $\Omega^{1,0}(X)$ its sections.

Proposition 62.

(1) *There are canonical isomorphisms of complex vector bundles*

$$(T^*\widetilde{X})^{1,0} \simeq \widetilde{p}^* \mathfrak{J}^1 L \simeq q^* \mathfrak{J}^1_{\mathbb{R}} \ell, \quad \textit{and} \quad \mathfrak{J}^1_{\mathbb{R}} \ell \simeq p^* \mathfrak{J}^1 L.$$

(2) *Isomorphism* $(T^*\widetilde{X})^{1,0} \simeq \widetilde{p}^* \mathfrak{J}^1 L$ *identifies pull-back sections with* 1*-forms* $\omega \in \Omega^{1,0}(\widetilde{X})$ *such that* $\mathcal{L}_H \omega = \omega$ *and* $\mathcal{L}_{\overline{H}} \omega = 0$.

(3) *Isomorphism* $(T^*\widetilde{X})^{1,0} \simeq q^* \mathfrak{J}^1_{\mathbb{R}} \ell$ *identifies pull-back sections with* 1*-forms* $\omega^{1,0}$ *of the kind* $\omega^{1,0} = \omega - i\widetilde{j}^* \omega$, *with* $\omega \in \Omega^1(\widetilde{X})$ *a* 1*-form homogeneous with respect to* η, *i.e.* $\mathcal{L}_\eta \omega = \omega$.

(4) *Isomorphism* $\mathfrak{J}^1_{\mathbb{R}} \ell \simeq \widehat{p}^* \mathfrak{J}^1 L$ *identifies pull-back sections with section* θ *of* $\mathfrak{J}^1_{\mathbb{R}} \ell \to \widehat{M}$ *such that* $\mathcal{L}_{\widehat{j}\mathbb{1}} \theta = \widehat{j}^\dagger \theta$.

Proof. Isomorphism $(T^*\widetilde{X})^{1,0} \simeq \widetilde{p}^* \mathfrak{J}^1 L$ identifies $\mathrm{j}^{1,0}\lambda$ with $\partial\widetilde{\lambda}$, where $\lambda \in \Gamma(L)$, and claim (2) can be easily checked, e.g., in local coordinates.

Isomorphism $(T^*\widetilde{X})^{1,0} \simeq q^* \mathfrak{J}^1_{\mathbb{R}} \ell$ is obtained composing isomorphism $(T^*\widetilde{X})^{1,0} \to T^*\widetilde{X}$, $\omega \mapsto 2\,\mathrm{Re}\,\omega$ with the isomorphism $T^*\widetilde{M} \simeq p^* \mathfrak{J}^1_{\mathbb{R}} \ell$ in [26, Proposition A.3.(3)]. Claim (3) is then obvious from the explicit form of isomorphism $T^*\widetilde{M} \simeq p^* \mathfrak{J}^1_{\mathbb{R}} \ell$.

For isomorphism $\mathfrak{J}^1_{\mathbb{R}}\ell \simeq \widehat{p}^*\mathfrak{J}^1L$, notice, first of all, that $\mathfrak{J}^1_{\mathbb{R}}\ell$ and $\mathfrak{J}^1L$ have the same complex rank. Now, from claim (2), sections of $\mathfrak{J}^1L$ are in one-to-one correspondence with 1-forms $\omega^{1,0} \in \Omega^{1,0}(\widetilde{X})$ such that

$$\mathcal{L}_H\omega^{1,0} = \omega^{1,0}, \quad \text{and} \quad \mathcal{L}_{\overline{H}}\omega^{1,0} = 0. \tag{20}$$

Let $\omega^{1,0} = \omega - i\widetilde{j}^*\omega$ for some $\omega \in \Omega^1(\widetilde{M})$. Then conditions (20) are equivalent to $\mathcal{L}_\eta\omega = \omega$ and $\mathcal{L}_{\widetilde{j}\eta}\omega = \widetilde{j}^*\omega$. Hence, using claim (3), we find that sections of $\mathfrak{J}^1L$ are in one-to-one correspondence with sections θ of $\mathfrak{J}^1_{\mathbb{R}}\ell$ such that $\mathcal{L}_{\widehat{j}\mathbb{1}}\theta = \widehat{j}^\dagger\theta$. This correspondence is $C^\infty(M,\mathbb{C})$-linear, so it comes from a morphism of vector bundles $\widehat{p}^*\mathfrak{J}^1L \to \mathfrak{J}^1_{\mathbb{R}}\ell$. It is easy to see that this morphism is injective, hence it is an isomorphism. This completes the proof of claim (1) and proves claim (4). □

Theorem 63. *The real (resp. imaginary) Lie algebroid of the holomorphic Lie algebroid $\mathfrak{J}^1L$ acts naturally on the fibration $\widehat{M} \to M$, and $(\mathfrak{J}^1_{\mathbb{R}}\ell)_{4\widehat{J}_j}$ (resp. $(\mathfrak{J}^1_{\mathbb{R}}\ell)_{4\widehat{J}}$) is the associated action Lie algebroid.*

We refer to [15, Section 3] for the notions of an (*infinitesimal*) *action of a Lie algebroid on a fibration* and the associated *action Lie algebroid*.

Proof of Theorem 63. First of all, the holomorphic Lie algebroid $(\mathfrak{J}^1L)_J$ acts on the holomorphic line bundle $L \to X$ via flat (holomorphic) connection ∇ defined by

$$\nabla_\theta := (\iota \circ J^\sharp)(\theta), \tag{21}$$

for all $\theta \in \Gamma(\mathfrak{J}^1L)$, where $\iota : D^{1,0}L \to DL$ is the canonical isomorphism. Since the symbol of ∇_θ is precisely $\rho(\theta)$, then ∇ is well-defined. To check that it is flat it is enough to check that $[\nabla_{\theta_1}, \nabla_{\theta_2}]$ and $\nabla_{[\theta_1,\theta_2]_J}$ agree on holomorphic sections of L, whenever θ_1, θ_2 are of the form $\mathrm{j}^1\lambda_1, \mathrm{j}^1\lambda_2$ for some holomorphic section λ_1, λ_2 of L. This easily follows from the Jacobi identity for the Jacobi bracket $\{-,-\}$ and formula

$$[\mathrm{j}^{1,0}\lambda_1, \mathrm{j}^{1,0}\lambda_2] = \mathrm{j}^{1,0}\{\lambda_1, \lambda_2\}.$$

Now, $(\mathfrak{J}^1L)_J$ acts on the complex dual line bundle $L^* \to X$ as well via the dual connection. When a holomorphic Lie algebroid acts on a holomorphic vector bundle E, then its real Lie algebroid acts on E as well and the action is uniquely defined by the same formula on holomorphic sections [16]. Hence the real Lie algebroid of $(\mathfrak{J}^1L)_J$ acts on $L^* \to X$. Denote by ∇ again the flat $(\mathfrak{J}^1L)_J$-connection in L^*.

For every $\theta \in \Gamma(\mathfrak{J}^1L)$, ∇_θ is a section of the holomorphic gauge algebroid of L^*. In particular, it is a (real) derivation of the (real) vector bundle $L^* \to X$. Accordingly, it corresponds to a linear vector field on the total space L^* of $L^* \to X$. It is easy to see that linear vector fields on L^* descend to $\widehat{p}$-projectable vector fields on $\widehat{M} = \mathbb{RP}(L^*)$. Hence the real Lie algebroid of $(\mathfrak{J}^1L)_J$ acts on the manifold $\widehat{M}$, and the pull-back $\widehat{p}^*\mathfrak{J}^1L = \mathfrak{J}^1_{\mathbb{R}}\ell \to \widehat{M}$ is equipped with the action Lie algebroid structure. We claim that the latter coincides with $(\mathfrak{J}^1_{\mathbb{R}}\ell)_{4\widehat{J}_j}$. To see this it is enough to check that the structure maps of the two Lie algebroids agree on pull-back sections of $\widehat{p}^*\mathfrak{J}^1 L \to \widehat{M}$. Even more, it is enough to check that the anchors and the brackets agree on sections of the form $\widehat{p}^*\mathsf{j}^{1,0}\lambda$, with λ a holomorphic sections of $L \to X$. To do this we pass through $\widetilde{X}$. Namely, $\widetilde{p}^*\mathsf{j}^{1,0}\lambda$ identifies with $\partial\widetilde{\lambda}$ through the isomorphism $\widetilde{p}^*\mathfrak{J}^1 L \simeq (T^{1,0}X)^*$ in Proposition 62, and we have $\mathcal{L}_H\widetilde{\lambda} = \widetilde{\lambda}$. Hence $\mathcal{L}_\eta f_\lambda = f_\lambda$, where $f_\lambda = \operatorname{Re}\widetilde{\lambda}$. In particular, f_λ corresponds to a section $\widehat{\lambda}$ of $\ell \to \widetilde{M}$, and $\widehat{p}^*\mathsf{j}^{1,0}\lambda$ corresponds to $\mathsf{j}^1_{\mathbb{R}}\widehat{\lambda}$ via isomorphism $\widehat{p}^*\mathfrak{J}^1 L \simeq \mathfrak{J}^1_{\mathbb{R}}\ell$ (see Proposition 62 again). Now, let λ_1, λ_2 be holomorphic sections of $L \to X$ and compute

$$[\mathsf{j}^1_{\mathbb{R}}\widehat{\lambda}_2, \mathsf{j}^1_{\mathbb{R}}\widehat{\lambda}_2]_{4\widehat{J}_j} = \mathsf{j}^1_{\mathbb{R}}\{\widehat{\lambda}_1, \widehat{\lambda}_2\}_{4\widehat{J}_j}$$

where $\{-,-\}_{4\widehat{J}_j}$ is the Jacobi bracket corresponding to the Jacobi structure $4\widehat{J}_j$. To complete with the brackets, it is enough to check that

$$\{\widehat{\lambda}_1, \widehat{\lambda}_2\}_{4\widehat{J}_j} = \widehat{\{\lambda_1, \lambda_2\}}.$$

This is easy, indeed $\{\lambda_1, \lambda_2\}$ corresponds to the homogeneous function $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\}_\Pi$ on $\widetilde{X}$ with respect to H. Now from [16, Corollary 2.4]

$$\operatorname{Re}\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\}_\Pi = \{\operatorname{Re}\widetilde{\lambda}_1, \operatorname{Re}\widetilde{\lambda}_2\}_{4\pi_j}$$

which is the homogeneous function with respect to η corresponding to $\{\widehat{\lambda}_1, \widehat{\lambda}_2\}_{4\widehat{J}_j}$, as claimed.

It remains to check that the anchor ρ of $(\mathfrak{J}^1_{\mathbb{R}}\ell)_{4\widehat{J}_j}$ and the anchor ρ' of the action Lie algebroid $\widehat{p}^*\mathfrak{J}^1 L \to \widehat{M}$ agree on sections of the form $\widehat{p}^*\mathsf{j}^{1,0}\lambda$, where λ is a holomorphic section of $L \to X$. Using the same notations as above, compute

$$\rho(\mathsf{j}^1_{\mathbb{R}}\widehat{\lambda})(f) = \sigma(\{\widehat{\lambda}, -\}_{4\widehat{J}_j})(f). \tag{22}$$

for all (local) functions f on $\widehat{M}$. The right hand side of (22) is the push forward of the function

$$\{\operatorname{Re}\widetilde{\lambda}, f\}_{4\pi_j}$$

on $\widetilde{X}$, the latter being constant along fibers of $q : \widetilde{X} \to \widehat{M}$. In fact, it is enough to choose f to be the (pushforward of the) quotient g_1/g_2 of two fiber-wise $\mathbb{R}$-linear functions g_1, g_2 on L^*. We can even choose g_1, g_2 to be the real parts of two fiber-wise $\mathbb{C}$-linear, holomorphic functions γ_1, γ_2 on L^*. So γ_1, γ_2 are actually holomorphic sections of $L \to X$. In this case, from [16, Corollary 2.4] again,

$$\{\operatorname{Re}\widetilde{\lambda}, f\}_{4\pi_j} = \frac{\operatorname{Re}\{\widetilde{\lambda}, \widetilde{\gamma}_1\}_\Pi g_2 - \operatorname{Re}\{\widetilde{\lambda}, \widetilde{\gamma}_2\}_\Pi g_1}{g_2^2}.$$

On the other hand

$$\rho'(\mathsf{j}^1_{\mathbb{R}}\widehat{\lambda})(f) = \frac{\operatorname{Re}\left(\iota\{\lambda, -\}(\gamma_1)\right) g_2 - \operatorname{Re}\left(\iota\{\lambda, -\}(\gamma_2)\right) g_1}{g_2^2},$$

and, to conclude, it is enough to notice that, from λ, γ being holomorphic, it follows that $\iota\{\lambda, -\}(\gamma) = \{\lambda, \gamma\}$ corresponds to the homogeneous function $\{\widetilde{\lambda}, \widetilde{\gamma}\}_\Pi$ on $\widetilde{X}$, $\gamma = \gamma_1, \gamma_2$.

As for the Lie algebroid $(\mathfrak{J}^1_{\mathbb{R}}\ell)_{\widehat{J}}$, it is enough to notice that, if $J \in \Gamma(D^{2,0}L)$ is a holomorphic Jacobi structure on L, then iJ is also a holomorphic Jacobi structure and $\widehat{iJ} = \widehat{J}_j$. We leave details to the reader. We only remark that the action of the imaginary Lie algebroid of $(\mathfrak{J}^1L)_J$ on $\widehat{M} \to X$ is induced by a linear action on L, given by a flat connection ∇' defined by

$$\nabla'_\theta := (\iota \circ J^\sharp)(i\theta) = (\iota \circ J^\sharp)(j_{\mathfrak{J}^1L}\theta),$$

for all $\theta \in \Gamma(\mathfrak{J}^1L)$. □

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