Nontautological Bielliptic Cycles

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Let $[\overline{\mathcal{B}}_{2,0,20}]$ and $[\mathcal{B}_{2,0,20}]$ be the classes of the loci of stable resp. smooth bielliptic curves with 20 marked points where the bielliptic involution acts on the marked points as the permutation (1 2)...(19 20). Graber and Pandharipande proved in [GP03] that these classes are nontatoulogical. In this note we show that their result can be extended to prove that $[\overline{\mathcal{B}}_g]$ is nontautological for $g \geq 12$ and that $[\mathcal{B}_{12}]$ is nontautological.

1 Introduction

The system of tautological rings $\{R^{\bullet}(\overline{\mathcal{M}}_{g,n})\}$ is defined to be the minimal system of \mathbb{Q} -subalgebras of the Chow rings $A^{\bullet}(\overline{\mathcal{M}}_{g,n})$ closed under pushforward (and hence pullback) along the natural gluing and forgetful morphisms

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

$$\overline{\mathcal{M}}_{g,n+2} \longrightarrow \overline{\mathcal{M}}_{g+1,n},$$

$$\overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

The tautological ring $R^{\bullet}(\mathcal{M}_{g,n})$ of the moduli space of smooth curves is the image of $R^{\bullet}(\overline{\mathcal{M}}_{g,n})$ under the localization morphism $A^{\bullet}(\overline{\mathcal{M}}_{g,n}) \to A^{\bullet}(\mathcal{M}_{g,n})$. We will denote by $RH^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ the image of $R^{\bullet}(\overline{\mathcal{M}}_{g,n})$ under the cycle map $A^{\bullet}(\overline{\mathcal{M}}_{g,n}) \to H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and define $RH^{2\bullet}(\mathcal{M}_{g,n})$ accordingly. We say a cohomology class is *tautological* if it lies in the tautological subring of its cohomology ring, otherwise we say it is *nontautological*. In this note we will work over \mathbb{C} and all Chow and cohomology rings are assumed to be taken with rational coefficients.

These tautological rings are relatively well understood. An additive set of generators for the groups $R^{\bullet}(\overline{\mathcal{M}}_{g,n})$ is given by decorated boundary strata and there exists an algorithm for computing the intersection product (see [GP03]). The class of many "geometrically defined" loci can be shown to be tautological, for example this is the case for the class of the locus $\overline{\mathcal{H}}_g$ of hyperelliptic curves in $\overline{\mathcal{M}}_g$ (see [FP05, Theorem 1]).

Any odd cohomology class of $\overline{\mathcal{M}}_{g,n}$ is nontautological by definition. Deligne proved that $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$, thus providing a first example of the existence of nontautological classes. In fact it is known that $H^{\bullet}(\overline{\mathcal{M}}_{0,n}) = RH^{\bullet}(\overline{\mathcal{M}}_{0,n})$ (see [Kee92]) and that $H^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = RH^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ for all n (see [Pet14, Corollary 1.2]).

Examples of geometrically defined loci which can be proven to be nontautological are still relatively scarce. In [GP03] Graber and Pandharipande hunt for algebraic classes in $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and $H^{2\bullet}(\mathcal{M}_{g,n})$ which are nontautological. In particular they show that the classes of the loci $\overline{\mathcal{B}}_{2,0,20}$ and $\mathcal{B}_{2,0,20}$ of stable resp. smooth bielliptic curves of genus 2 with 20 marked points where the bielliptic involution acts on the set of marked points as the permutation (1 2)...(19 20)

are nontautological. They also show that for sufficiently high odd genus h the class of the locus of stable curves of genus 2h admitting a map to a curve of genus h is nontautological in $H^{2\bullet}(\overline{\mathcal{M}}_{2h})$. Their result relies on the existence of odd cohomology in $H^{\bullet}(\overline{\mathcal{M}}_{h,1})$ which has been proven to exist in [Pik95] for all $h \geq 8069$. A recent survey of different methods of obtaining nontautological classes can be found in [FP13].

In this note we prove the following two new results.

Theorem 1. The cohomology class $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \geq 12, 0 \leq n \leq 2g-2$ and $g \geq 2$.

Theorem 2. The cohomology class $[\mathcal{B}_{g,0,2m}]$ is nontautological when g+m=12 and $g\geq 2$.

With Theorem 1 we improve the genus for which algebraic nontautological classes on $\overline{\mathcal{M}}_g$ are known to exists from 16138 to 12. As far as the author is aware, Theorem 2 provides the first example of a nontautological algebraic class on \mathcal{M}_g .

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2 Preliminaries

Admissible double covers were introduced to compactify moduli spaces of double covers of smooth curves, let us recall the definition:

Definition 3. Let $(S, x_1, ..., x_k, y_1, ..., y_{2m})$ be a stable pointed curve of arithmetic genus g. An admissible double cover is the data of a stable pointed curve $(T, x'_1, ..., x'_k, y'_1, ..., y'_m)$ of arithmetic genus g' and a 2-to-1 map $f: S \to T$ satisfying the following conditions:

- the restriction to the smooth locus $f^{\text{sm}} : S^{\text{sm}} \to T^{\text{sm}}$ is branched exactly at the points $x'_1, ..., x'_k$ and the inverse image of x'_i is x_i for all i = 1, ..., k,
- the inverse image of y'_i under f is $\{y_{2i}, y_{2i+1}\},$
- \bullet the image under f of each node is a node.

We call S the source curve and T the target curve of the admissible cover. An admissible hyperelliptic structure on S is an admissible cover where g' = 0 and an admissible bielliptic structure on S is an admissible cover with g' = 1. Note that the admissible double cover $S \to T$ induces an involution on S fixing the points $x_1, ..., x_k$ and permuting the points $y_1, ..., y_{2m}$ pairwise.

One can define families of admissible double covers and isomorphisms between them (see [ACV03, Section 4]). By using the Riemann-Hurwitz formula and by induction on the number of nodes we can deduce that the number k in the above definition equals 2g + 2 - 4g'. We denote the moduli stack of admissible bielliptic covers with 2m marked points switched by the involution by $\overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}}$. When m=0 we simply write $\overline{\mathcal{B}}_g^{\mathrm{Adm}}$.

A natural target map and source map from each moduli space of admissible double covers can be defined as follows. The target map is a finite surjective map which sends each admissible cover to the target stable pointed curve $(T, x'_1, ..., x'_k, y'_1, ..., y'_m) \in \overline{\mathcal{M}}_{g',k+m}$. From the properness of

 $\overline{\mathcal{M}}_{g',k+m}$ we deduce that the space of such admissible covers is proper. The dimension of the space of such admissible double covers equals 2g - g' + 2m - 1. In the bielliptic case we get

$$\dim \overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}} = 2g - 2 + m.$$

The source map forgets all the structure of an admissible double cover except for

$$(S, x_1, ..., x_k, y_1, ..., y_{2m}) \in \overline{\mathcal{M}}_{q,k+2m}.$$

In the bielliptic case this gives a map $\overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}} \to \overline{\mathcal{M}}_{g,2g-2+2m}$. We can compose this map with a composition of forgetful maps $\overline{\mathcal{M}}_{g,2g-2+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ which forgets the first 2g-2-n points (which therefore correspond to the first 2g-2-n ramification points of the admissible bielliptic covers) and stabilizes. We denote by $\overline{\mathcal{B}}_{g,n,2m}$ the image substack of $\overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}}$ in $\overline{\mathcal{M}}_{g,n+2m}$. The above discussion can be summarized in the following diagram:

$$\overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}} \longrightarrow \overline{\mathcal{B}}_{g,n,2m} \hookrightarrow \overline{\mathcal{M}}_{g,n+2m}$$

$$\downarrow \qquad \qquad \downarrow$$
 $\overline{\mathcal{M}}_{1,2g-2+m}$

The moduli stack $\mathcal{B}_{g,2m}^{\mathrm{Adm}}$ is the open dense substack of $\overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}}$ of admissible bielliptic covers of smooth curves and we denote its image stack in $\mathcal{M}_{g,n+2m}$ by $\mathcal{B}_{g,n,2m}$. We have well defined Chow classes

$$[\overline{\mathcal{B}}_{g,n,2m}] \in A^{g-1+n+m}(\overline{\mathcal{M}}_{g,n+2m})$$

 $[\mathcal{B}_{g,n,2m}] \in A^{g-1+n+m}(\mathcal{M}_{g,n+2m}).$

We will abuse notation and also denote the image of these classes in the respective cohomology rings by $[\overline{\mathcal{B}}_{g,n,2m}]$ and $[\mathcal{B}_{g,n,2m}]$. In a completely analogous way, we can define spaces of admissible hyperelliptic covers $\overline{\mathcal{H}}_{g,2m}^{\text{Adm}}$ and the loci $\overline{\mathcal{H}}_{g,n,2m}$ and $\mathcal{H}_{g,n,2m}$ in $\overline{\mathcal{M}}_{g,n+2m}$ and $\mathcal{M}_{g,n+2m}$ for all $0 \le n \le 2g + 2$.

Notation 4. We will denote by $\overline{\mathcal{M}}_{g,n}^D$ (resp. $\mathcal{M}_{g,n}^D$) the moduli stack parameterizing trivial étale double covers

$$f: (C_1; y_{1,1}, ..., y_{n,1}) \cup (C_2; y_{1,2}, ..., y_{n,2}) \rightarrow (C; y_1, ..., y_n)$$

mapping two isomorphic stable (resp. smooth) curves $(C_1; y_{1,1}, ..., y_{n,1}) \simeq (C_2; y_{1,2}, ..., y_{n,2})$ to a curve $(C; y_1, ..., y_n) \simeq (C_1; y_{1,1}, ..., y_{n,1})$ such that $f^{-1}(y_i) = (y_{i,1}, y_{i,2})$.

Our proof of Theorem 1 relies on the following result for pullbacks along gluing morphisms.

Proposition 5 ([GP03, Proposition 1]). Let $\xi \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ be the gluing morphism and let $\gamma \in RH^{\bullet}(\overline{\mathcal{M}}_{q_1+q_2,n_1+n_2})$, then

$$\xi^*(\gamma) \in RH^{\bullet}(\overline{\mathcal{M}}_{g_1,n_1+1}) \otimes RH^{\bullet}(\overline{\mathcal{M}}_{g_2,n_2+1}).$$

We say that a cycle $\lambda \in H^{\bullet}(\overline{\mathcal{M}}_{g_1,n_1}) \otimes H^{\bullet}(\overline{\mathcal{M}}_{g_2,n_2})$ admits a tautological Künneth decomposition if $\lambda \in RH^{\bullet}(\overline{\mathcal{M}}_{g_1,n_1}) \otimes RH^{\bullet}(\overline{\mathcal{M}}_{g_2,n_2})$.

3 Proof of Theorem 1 and 2

We are now ready to prove Theorem 1. We start by proving the following weaker result.

Proposition 6. We have

$$[\overline{\mathcal{B}}_{g,0,2m}] \notin RH^{\bullet}(\overline{\mathcal{M}}_{g,2m})$$

for g + m = 12 and $g \ge 2$.

Proof. Let $\iota_1 \colon \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \to \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ be the inclusion and $\iota_2 \colon \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m}$ the gluing morphism which glues the corresponding first g-1 points of the two factors and orders the remaining points by sending the k'th marked point of the first curve to 2k-1 and the k'th marked point of the second curve to 2k. Let ι be the composition $\iota_2 \circ \iota_1$ and let Δ resp. Δ_o be the diagonal of $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ resp. $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ so that $\iota_1^*([\Delta]) = [\Delta_o]$. In Lemma 7 we will prove that $\iota^*([\overline{\mathcal{B}}_{g,0,2m}]) = \alpha[\Delta_o]$ for some $\alpha \in \mathbb{Q}_{>0}$. Let $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) := ((\partial \overline{\mathcal{M}}_{1,11}) \times \overline{\mathcal{M}}_{1,11}) \cup (\overline{\mathcal{M}}_{1,11} \times (\partial \overline{\mathcal{M}}_{1,11}))$. Since the sequence

$$A^{10}(\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})) \longrightarrow A^{11}(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \xrightarrow{\iota_1^*} A^{11}(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \longrightarrow 0$$

is exact there exists a class $B \in A^{10}(\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}))$ such that $\iota_2^*([\overline{\mathcal{B}}_{g,0,2m}]) = \alpha[\Delta] + B$.

The class B admits a tautological Künneth decomposition by Lemma 8.i. Given a basis $\{e_i\}_{i\in I}$ for $H^{\bullet}(\overline{\mathcal{M}}_{1,11})$ with dual basis $\{\hat{e}_i\}_{i\in I}$ the cohomology class of the diagonal can be written as

$$[\Delta] = \sum_{i \in I} (-1)^{\deg e_i} e_i \otimes \hat{e_i}.$$

In particular since $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$ the diagonal $[\Delta]$ does not admit a tautological Künneth decomposition. Since the pullback of a tautological class along a (composition of) gluing morphisms admits a tautological Künneth decomposition by Proposition 5, this shows that $[\overline{\mathcal{B}}_{g,0,2m}]$ is nontautological.

Lemma 7. Consider the composition of gluing morphisms $\iota \colon \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \to \overline{\mathcal{M}}_{g,2m}$ defined above. We have $\iota^*(\overline{\mathcal{B}}_{g,2m}) = \alpha[\Delta_o]$ for some $\alpha \in \mathbb{Q}_{>0}$.

Proof. Consider the fiber diagram

$$F \xrightarrow{\qquad} \overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}} \downarrow^{\phi} \\ \downarrow^{\mathcal{M}_{1,11}} \times \mathcal{M}_{1,11} \xrightarrow{\iota} \overline{\mathcal{M}}_{g,2m}$$

We will describe the fiber product F, or rather the push forward of its class to $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$. Consider the moduli stack $\mathcal{M}_{1,11}^D$, there is a closed embedding $\mathcal{M}_{1,11}^D \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$, $(C_1 \cup C_2 \to C) \mapsto (C_1, C_2)$ with image the diagonal Δ_o . We define a map $\eta \colon \mathcal{M}_{1,11}^D \to \overline{\mathcal{B}}_{g,2m}^{\mathrm{Adm}}$ as follows: on the source curves η attaches rational bridges R_i between the corresponding marked points $y_{i,1}$ of C_1 and $y_{i,2}$ of C_2 for all $1 \le i \le g-1$ and on the target curve it attaches a rational curve R_i' with two marked points to the corresponding marked point y_i of C. The trivial double cover $C_1 \cup C_2 \to C$ then induces an admissible double cover

$$\left(C_1 \cup C_2 \cup \bigcup_{i=1}^{g-1} R_i \; ; \; y_{g,1}, y_{g,2}, ..., y_{11,1}, y_{11,2}\right) \longrightarrow \left(C \cup \bigcup_{i=1}^{g-1} R_i' \; ; \; y_g, ..., y_{11}\right),$$

branched at the marked points of each R'_i , which maps each pair of marked points $y_{i,1}$, $y_{i,2}$ of $C_1 \cup C_2 \cup \bigcup_{i=1}^{g-1} R_i$ to the corresponding marked point y_i of $C \cup \bigcup_{i=1}^{g-1} R'_i$. By the universal property of fiber products we get a map $\mathcal{M}^{D}_{1,11} \to F$. We claim that the

By the universal property of fiber products we get a map $\mathcal{M}_{1,11}^D \to F$. We claim that the composition $\mathcal{M}_{1,11}^D \to F \to F^{\text{red}}$ is a finite¹ surjective morphism. The map $F \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ is proper since properness is stable under base extension, the map $\mathcal{M}_{1,11}^D \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ is proper because $\overline{\mathcal{M}}_{1,11}^D \to \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ is proper. It follows that $\mathcal{M}_{1,11}^D \to F$ is proper. Since the map $\mathcal{M}_{1,11}^D \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ is quasifinite so is $\mathcal{M}_{1,11}^D \to F$. Since $\mathcal{M}_{1,11}^D \to F^{\text{red}}$ is proper and quasifinite and F^{red} is of finite type (and reduced) it remains to check that this map induces a surjection on closed points.

By definition an object of F over $\operatorname{Spec} \mathbb{C}$ consists of a curve $\tilde{C} := (\tilde{C}_1, \tilde{C}_2) \in \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}(\mathbb{C})$, an object $(S \to T) \in \overline{\mathcal{B}}_{g,2m}^{\operatorname{Adm}}(\mathbb{C})$ and an isomorphism $\gamma \colon \iota(\tilde{C}) \xrightarrow{\sim} \phi(S \to T)$. To prove the claim we will show that $(\tilde{C}, (S \to T), \gamma)$ is isomorphic to an object in the image of $\mathcal{M}_{1,11}^D(\mathbb{C})$. Let $f \colon \tilde{C}_1 \cup \tilde{C}_2 \to \iota(\tilde{C})$ be the map of curves induced by ι , set $C := \iota(\tilde{C})$, $C_1 := f(\tilde{C}_1)$ and $C_2 := f(\tilde{C}_2)$, let τ be the involution on C induced by the bielliptic involution of $S \to T$ and let Q_i be the node of C corresponding to the i'th marking of \tilde{C}_1 and \tilde{C}_2 .

Since C_1 and C_2 are smooth there are two possibilities for the action of τ on C: Either it fixes C_1 and C_2 or it switches the whole of C_1 with the whole of C_2 . Suppose τ fixes C_1 and C_2 . By construction the involution τ maps marked points lying on C_1 to marked points lying on C_2 so this is only possible if C has no marked points at all. In this case τ must fix the different strands of C at each Q_i . If the inverse image of Q_i in S were to be a rational bridge R_i then this rational bridge would have 2 marked ramification points which are not nodes, but this would imply that τ switches the nodes on the rational bridge and therefore switches the strands of C at Q_i . It follows that the inverse image of each Q_i in S is a single node \hat{Q}_i . Since C_1 and C_2 are smooth, τ induces an involution on the set of nodes $\{\hat{Q}_1, ..., \hat{Q}_{11}\}$. We can thus find distinct \hat{Q}_i , $\hat{Q}_j \neq \tau(\hat{Q}_i)$ such that $S - \{\hat{Q}_i, \tau(\hat{Q}_i), \hat{Q}_j, \tau(\hat{Q}_j)\}$ is connected. But this means that there are at least two nodes P_i and P_j of T such that $T - \{P_i, P_j\}$ is connected. This would imply that the arithmetic genus of T is at least 2, which is a contradiction.

We can therefore assume τ maps C_1 to C_2 . Let us first suppose that τ does not fix all nodes, so there exist some distinct i, j such that $\tau(Q_i) = Q_j$. Let P be the image of $\{Q_i, Q_j\}$ under the bielliptic map. Like before we see that $T \setminus \{P\}$ is connected and it therefore has arithmetic genus 0 (since by assumption the arithmetic genus of T is 1). However the arithmetic genus of $C_1 \setminus \{Q_i, Q_j\}$ is 1 and the bielliptic map restricts to an isomorphism $C_1 \setminus \{Q_i, Q_j\} \to T \setminus \{P\}$, which is a contradiction.

We have thus proven that τ switches the components C_1 and C_2 and fixes the nodes Q_i , which implies that $((\tilde{C}_1, \tilde{C}_2), (S \to T), \gamma)$ is isomorphic to an object in the image of $\overline{\mathcal{M}}_{1,11}^D(\mathbb{C})$. This proves that the map $\mathcal{M}_{1,11}^D \to F^{\text{red}}$ is surjective.

It follows that the pushforward of Δ_o to $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ equals the pushforward of F to $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ up to a scalar. Since

$$\operatorname{codim}_{\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}} \Delta_o = 11 = \operatorname{codim}_{\overline{\mathcal{M}}_{g,2m}} \overline{\mathcal{B}}_{g,2m}^{\operatorname{Adm}}$$

we see that $\iota^*[\overline{\mathcal{B}}_{g,2m}] = \alpha[\Delta_o]$ for some $\alpha \in \mathbb{Q}_{>0}$ and g + m = 12.

Lemma 8. i Every algebraic class of codimension 11 in $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ supported on $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ admits a tautological Künneth decomposition.

¹As in [Vis89, Definition 1.8].

ii Every algebraic class on $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ of complex codimension less than 11 admits a tautological Künneth decomposition.

Proof. This is a slightly weaker version of [GP03, Lemma 3], the proof given there required that $RH^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = H^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ and $H^k(\overline{\mathcal{M}}_{1,n}) = 0$ for n < 11, for which there was no reference at the time of [GP03]. The first equation is [Pet14, Corollary 1.2]. The second condition follows from Getzlers' computations for n < 11 in [Get98].

9. We have now concluded the proof of Proposition 6. To prove Theorem 1 it remains to show that $[\overline{\mathcal{B}}_{q,n,2m}]$ is nontautological for all n, g, m with g+m>12.

Proof of Theorem 1. We will show in Lemma 10 and 11 that if $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological then so are $[\overline{\mathcal{B}}_{g,n+1,2m}]$ for $n \leq 2g-3$, and $[\overline{\mathcal{B}}_{g,n,2m+2}]$. In Lemma 12 we will show that if $[\overline{\mathcal{B}}_{g,1,0}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g+1}]$. Using these statements inductively, with base case the statement of Proposition 6, we conclude that $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \geq 12$. \square

Lemma 10. If $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological and $n \leq 2g-3$ then so is $[\overline{\mathcal{B}}_{g,n+1,2m}]$.

Proof. Let $\pi \colon \overline{\mathcal{M}}_{g,n+1+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the morphism which forgets the first point and stabilizes. Since $\pi(\overline{\mathcal{B}}_{g,n+1,2m}) = \overline{\mathcal{B}}_{g,n,2m}$ and $\dim \overline{\mathcal{B}}_{g,n+1,2m} = \dim \overline{\mathcal{B}}_{g,n,2m}$ we have $\pi_*[\overline{\mathcal{B}}_{g,n+1,2m}] = \alpha[\overline{\mathcal{B}}_{g,n,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Because the push forward of a tautological class by the forgetful morphism is tautological, the result follows.

Lemma 11. If $[\overline{\mathcal{B}}_{q,n,2m}]$ is nontautological then so is $[\overline{\mathcal{B}}_{q,n,2m+2}]$.

Proof. Suppose n < 2g - 2 then by the previous result $[\overline{\mathcal{B}}_{g,n+1,2m}]$ is nontautological. Consider the gluing morphism

$$\sigma \colon \overline{\mathcal{M}}_{g,n+2m+1} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+2}$$

which glues the first points of both curves together, then $\sigma^{-1}(\overline{\mathcal{B}}_{g,n,2m+2}) = \overline{\mathcal{B}}_{g,n+1,2m}$. Since $\operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+2}} \overline{\mathcal{B}}_{g,n,2m+2} = \operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+1}} \overline{\mathcal{B}}_{g,n+1,2m}$ it follows that $\sigma^*[\overline{\mathcal{B}}_{g,n,2m+2}] = \alpha[\overline{\mathcal{B}}_{g,n+1,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Since σ is a gluing morphism and the pullback of a tautological class along σ admits tautological Künneth decomposition $[\overline{\mathcal{B}}_{g,n,2m+2}]$ is nontautological.

If n=2g-2 we can first prove that $[\overline{\mathcal{B}}_{g,n-1,2m+2}]$ is nontautological in the same way by pulling back through the map $\overline{\mathcal{M}}_{g,n+2m} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+1}$ and then use Lemma 10. \square

Lemma 12. If $[\overline{\mathcal{B}}_{g,1,0}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g+1}]$.

Proof. Let $\epsilon \colon \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{g+1}$ be the gluing morphism. From the description of the boundary divisors of $\overline{\mathcal{B}}_{g+1}^{\mathrm{Adm}}$ (see [Pag16, Page 1275-1276]) it follows that there exists $\alpha, \beta \in \mathbb{Q}_{>0}$ such that

$$\epsilon^*[\overline{\mathcal{B}}_{g+1}] = \alpha[\overline{\mathcal{B}}_{g,1,0} \times \overline{\mathcal{M}}_{1,1}] + \beta[\overline{\mathcal{H}}_{g-1,0,2} \times \overline{\mathcal{M}}_{1,1}^D] \in H^{\bullet}(\overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1}).$$

The class $[\overline{\mathcal{H}}_{g-1,0,2} \times \overline{\mathcal{M}}_{1,1}^D]$ admits a tautological Künneth decomposition (since the class of the hyperelliptic locus is tautological by [FP05, Theorem 1] and therefore so is its pushforward under a gluing morphism with a tautological class). The class $[\overline{\mathcal{B}}_{g,1} \times \overline{\mathcal{M}}_{1,1}]$ does not admit a tautological Künneth decomposition by assumption. It follows by Proposition 5 that $[\overline{\mathcal{B}}_{g+1}]$ is nontautological.

13. We will now prove a similar result for the open locus of $\overline{\mathcal{M}}_{g,2m}$ where g+m=12.

Proof of Theorem 2. The case where g=2 is treated in [GP03, Section 3]. We use a similar argument to prove the remaining cases. The proof runs by contradiction. Suppose $[\mathcal{B}_{g,0,2m}] \in RH^{\bullet}(\mathcal{M}_{g,2m})$ then there is some collection of cycles Z_i in $\overline{\mathcal{M}}_{g,2m}$, of complex codimension 11 and supported on $\partial \overline{\mathcal{M}}_{g,2m}$ such that $\sum [Z_i] + [\overline{\mathcal{B}}_{g,0,2m}]$ is a tautological class. Consider again the gluing morphism $\iota_2 \colon \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m}$ as above. By assumption the pullback of $\sum [Z_i] + [\overline{\mathcal{B}}_{g,2m}]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ admits a tautological Künneth decomposition whereby the pullback of $\sum [Z_i]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ must be nontautological.

We shall use the usual notation that Δ_j is the locus of curves in $\overline{\mathcal{M}}_{g,2m}$ consisting of two curves, one of which has genus j, glued together in a single node, and Δ_{irr} is the locus that generically parametrizes irreducible singular curves. Since $\iota_2(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ does not have a separating node we see that $\iota_2(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \not\subset \Delta_j$. The intersection

$$\Delta_i \cap (\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$$

therefore lies in $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$. It follows by Lemma 8.i that $\iota_2^*[Z_i]$ admits a tautological Künneth decomposition if Supp $Z_i \subset \Delta_j$.

Consider now the Z_i with support inside Δ_{irr} . We can decompose the map ι_2 as

$$\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \xrightarrow{\iota_2''} \overline{\mathcal{M}}_{g-1,2m+2} \xrightarrow{\iota_2'} \overline{\mathcal{M}}_{g,2m}$$

Then there exist cycles Y_i in $\overline{\mathcal{M}}_{g-1,2m+2}$ such that $\iota'_{2*}[Y_i] = [Z_i]$. Now

$$\iota_2^*[Z_i] = \iota_2''^* \iota_1'^*[Z_i]$$

= $\iota_2''^*(c_1(N_{\overline{\mathcal{M}}_{g-1}, 2m+2}, \overline{\mathcal{M}}_{g,2m}) \cap [Y_i]).$

We see that $\iota_2^*[Z_i]$ decomposes as a product of algebraic classes of codimension less than 11, which admit tautological Künneth decomposition by Lemma 8.ii.

We conclude that all the $[Z_i]$ have tautological Künneth decomposition when pulled back to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$. Therefore $\iota_2^*(\sum [Z_i] + [\overline{\mathcal{B}}_{g,0,2m}])$ does not admit a tautological Künneth decomposition. It follows by Proposition 5 that $[\mathcal{B}_{g,2m}]$ is nontautological.

References

- [ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. *Communications in Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman.
- [FP05] Carel Faber and Rahul Pandharipande. Relative maps and tautological classes. *JEMS*, 7:13–49, 2005.
- [FP13] Carel Faber and Rahul Pandharipande. Tautological and non-tautological cohomology of the moduli space of curves. *Handbook of Moduli*, Volume I:293–330, 2013.
- [Get98] Ezra Getzler. The semi-classical approximation for modular operads. *Communications* in Mathematical Physics, 194(2):481–492, 1998.
- [GP03] Tom Graber and Rahul Pandharipande. Constructions of nontautological classes of moduli spaces of curves. *Michigan Math. J.*, 51(1):93–110, 04 2003.
- [Kee92] Sean Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. Transactions of the American Mathematical Society, 330(2):545–574, 1992.

- [Pag16] Nicola Pagani. Moduli of abelian covers of elliptic curves. *Journal of Pure and Applied Algebra*, 220(3):1258 1279, 2016.
- [Pet14] Dan Petersen. The structure of the tautological ring in genus one. Duke Math. J., 163(4):777-793, $03\ 2014$.
- $[{\rm Pik95}] \quad {\rm Martin~Pikaart.~An~orbifold~partition~of} ~\overline{M}_g^n.~ \it{The~Moduli~Space~of~Curves}, 1995.$
- [Vis89] Angelo Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. Inventiones mathematicae, 97(3):613–670, 1989.
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