DIRICHLET-TO-NEUMANN SEMIGROUP WITH RESPECT TO A GENERAL SECOND ORDER EIGENVALUE PROBLEM

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ABSTRACT. In this paper we study a Dirichlet-to-Neumann operator with respect to a second order elliptic operator with measurable coefficients, including first order terms, namely, the operator on $L^2(\partial\Omega)$ given by $\varphi \mapsto \partial_{\nu} u$ where u is a weak solution of

$$\begin{cases} -\mathrm{div}\left(a\nabla u\right) + b\cdot\nabla u - \mathrm{div}\left(cu\right) + du = \lambda u \text{ on } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

Under suitable assumptions on the matrix-valued function a, on the vector fields b and c, and on the function d, we investigate positivity, sub-Markovianity, irreducibility and domination properties of the associated Dirichlet-to-Neumann semigroups.

1. Introduction

Form methods for evolution equations date back to the pioneering works by D. Hilbert on integral equations in the beginning of the twentieth century. However, it was not until the late 1950s that they have been systematically developed towards its applications to evolution equations. At this early period two schools have emerged, one centered around J.-L. Lions (elliptic forms) and other around T. Kato (sectorial forms). Both notions (elliptic and sectorial forms) turn out to be essentially equivalent, being different descriptions of the same ideas. Standard references for both theories include R. Dautray and J.-L. Lions's book [13] and T. Kato's book [16]; more recent developments have been documented in E.-M. Ouhabaz's book [20].

In a recent paper, W. Arendt and A.F.M. ter Elst [7] have extended the classical form method in many respects. In the case these authors call the 'complete case', which corresponds to Lions's elliptic forms, the form domain V is allowed to be a Hilbert space (over $\mathbb{K} = \mathbb{R}$ or \mathbb{C}) not necessarily embedded in the reference space, say H, provided there is a bounded linear operator $j:V\to H$ with dense range; if $\mathfrak{a}:V\times V\to \mathbb{K}$ is a continuous sesquilinear form which is j-elliptic in the sense that

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega \|j(u)\|_{H}^{2} \geqslant \alpha \|u\|_{V}^{2} \quad (u \in V)$$

for some constants $\omega \in \mathbb{R}$ and $\alpha > 0$, then an operator A on H can be associated to \mathfrak{a} in such a way that

(1.1)
$$x \in \mathcal{D}(A)$$
 and $Ax = f$ if, and only if

$$x = j(u)$$
 for some $u \in V$ and $\mathfrak{a}(u, v) = (f|j(v))_H$ for all $v \in V$.

A further consequence for the so called 'incomplete case', which corresponds to Kato's sectorial forms, is that an *m*-sectorial operator (and therefore, a holomorphic semigroup generator) can be associated to a densely defined sectorial form, regardless it is closable or not.

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This new form method allows an elegant treatment of the so-called Dirichlet-to-Neumann operator. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. By definition, the Dirichlet-to-Neumann operator is the operator D_0 acting on $L^2(\partial\Omega)$ with the property that $\varphi \in \mathcal{D}(D_0)$ and $D_0\varphi = h$ if, and only if there is a weak solution $u \in H^1(\Omega)$ of

$$\begin{cases} \Delta u = 0 \text{ on } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

such that $\partial_{\nu}u = h$ in a weak sense; an element $u \in H^1(\Omega)$ with distributional Laplacian $\Delta u \in L^2(\Omega)$ is said to have a weak normal derivative if there exists $h \in L^2(\partial\Omega)$ such that Green's formula holds, meaning that the identity

$$\int_{\Omega} (\Delta u) \overline{v} \, dx + \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx = \int_{\partial \Omega} h \overline{v} \, d\sigma$$

holds for every $v \in H^1(\Omega)$. In this case we set $\partial_{\nu}u := h$. By showing that D_0 is associated with a *j*-elliptic form, namely, the classical Dirichlet form

$$\mathfrak{a}(u,v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx \quad (u,v \in H^1(\Omega))$$

with $j: H^1(\Omega) \to L^2(\partial\Omega)$ being the trace, Arendt & ter Elst have provided an interesting application of their theory where a non-injective j appears in a natural way.

In this paper we study the Dirichlet-to-Neumann operator, to be denoted by $D_{\lambda}^{\mathscr{A}}$, defined by $\varphi \mapsto \partial_{\nu} u$ where $u \in H^{1}(\Omega)$ is a weak solution of the eigenvalue problem

(1.2)
$$\begin{cases} -\operatorname{div}(a\nabla u) + b \cdot \nabla u - \operatorname{div}(cu) + du = \lambda u & \text{on } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

and $\partial_{\nu}u$ is the 'weak conormal derivative' which, in the smooth case, coincides with the classical conormal derivative $(a\nabla u + cu) \cdot \nu$. In Problem (1.2), a is a matrix-valued function, b and c are vector fields, d is measurable function and λ is a number; for the precise hypotheses on these data, see Theorem 1.1 below. The difficulty here lies, of course, in the presence of the first order terms ' $b \cdot \nabla u$ ' and 'div (cu)'. To the best of our knowledge, Dirichlet-to-Neumann operators whose internal dynamics includes first order terms have been first considered in [9] in connection with Calderón's inverse problem which asks, roughly speaking, whether $\mathscr A$ can be determined from $D_{\lambda}^{\mathscr A}$. Following Arendt & ter Elst approach to the Dirichlet-to-Neumann operator D_0 through form methods it is clear that $D_{\lambda}^{\mathscr A}$ should be, at best, the associated operator, in the sense of (1.1), to the sesquilinear form $\mathfrak{a}_{\lambda}: H^1(\Omega) \times H^1(\Omega) \to \mathbb{K}$ defined by

$$(1.3) \quad \mathfrak{a}_{\lambda}(u,v) = \int_{\Omega} a \nabla u \cdot \overline{\nabla v} \, dx + \int_{\Omega} (b \cdot \nabla u) \overline{v} \, dx + \int_{\Omega} u(c \cdot \overline{\nabla v}) \, dx + \int_{\Omega} du \overline{v} \, dx - \lambda \int_{\Omega} u \overline{v} \, dx.$$

However, as shown in [7] (cf. also [4]), this form is not j-elliptic in general, even when a=I, b=c=0 and d=0. This lack of ellipticity can be circumvented by a general procedure. Roughly speaking, to any sesquilinear form $\mathfrak{a}: V \times V \to \mathbb{K}$ an m-sectorial operator A can still be associated to \mathfrak{a} in the sense of (1.1) provided \mathfrak{a} is j-elliptic on a suitable closed subspace of V which complements $\mathcal{N}(j)$; the precise statement will be recalled below in Proposition 3.1. Moreover, the recent theory of 'compactly elliptic forms' introduced in Arendt et al [8] makes this task even easier and we briefly describe how this theory can also be used in the construction of our Dirichlet-to-Neumann operator.

The present work is motivated by some results in [5] (cf. also [4]) where it has been shown, among other things, that the semigroup generated by the Dirichlet-to-Neumann operator with

respect to the eigenvalue problem

(1.4)
$$\begin{cases} -\Delta u = \lambda u \text{ on } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

which corresponds to Problem (1.2) with $a=I,\,b=c=0$ and d=0, is positive and irreducible whenever $\lambda<\lambda_1^{\rm D},\,\lambda_1^{\rm D}$ being the first eigenvalue of the Dirichlet Laplacian given in variational terms by

(1.5)
$$\lambda_1^{\mathcal{D}} = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u^2| dx}.$$

The question whether this semigroup remains positive or not for $\lambda > \lambda_1^{\rm D}$ is a major research topic; recent contributions to this issue include e.g. the paper by D. Daners [12]. Here we do not address this question but focus on the problem whether some properties, having positivity at their center, of the Dirichlet-to-Neumann semigroup is preserved under first order perturbations of Problem (1.4). With little additional effort we can also describe when these semigroups are in fact sub-Markovian. Moreover, we also consider irreducibility and some domination properties. Some of these questions have been also studied, in connection with Calderón's problem, in [19].

Let $e^{-tD_{\lambda}^{\mathscr{A}}}$ be the semigroup on $L^2(\partial\Omega)$ generated by $-D_{\lambda}^{\mathscr{A}}$. In the following, A_{D} denotes the realization of \mathscr{A} with Dirichlet boundary conditions (see next section). For simplicity, we also consider real scalars.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with Lipschitz boundary. Suppose the matrix-valued function $a \in L^{\infty}(\Omega; \mathbb{R}^{N \times N})$ is symmetric and uniformly positive-definite in the sense that, for some $\kappa > 0$,

$$a(x)\xi \cdot \xi \geqslant \kappa |\xi|^2 \quad (\xi \in \mathbb{R}^N, a.e. \ x \in \Omega).$$

Suppose the vector fields $b, c \in C^1(\overline{\Omega})^N$ are real and satisfy $\operatorname{div} b = \operatorname{div} c = 0$ and $b \cdot \nu = c \cdot \nu = 0$. Let $d \in L^{\infty}(\Omega)$ be real-valued. Suppose $\lambda \in \mathbb{R} \setminus \sigma(A_D)$.

- (a) If $4\kappa^{-1}||b-c||_{L^{\infty}(\Omega)^{N}}^{2} + ||d^{-}||_{L^{\infty}(\Omega)} + \lambda < \kappa \lambda_{1}^{D}$ then $e^{-tD_{\lambda}^{\mathscr{A}}}$ is positive.
- (b) If $4\kappa^{-1}\|b-c\|_{L^{\infty}(\Omega)^{N}}^{2} + \|d^{-}\|_{L^{\infty}(\Omega)} + \lambda < \kappa \lambda_{1}^{D} \text{ and } \lambda \leqslant d \text{ then } e^{-tD_{\lambda}^{\mathscr{A}}} \text{ is sub-Markovian.}$
- (c) If $||d^-||_{L^{\infty}(\Omega)} + \lambda < \kappa \lambda_1^D$ then $e^{-tD_{\lambda}^{\mathscr{A}}}$ is irreducible.
- (d) If b = c and $\lambda_2 \leqslant \lambda_1 < \kappa \lambda_1^D \|d^-\|_{L^{\infty}(\Omega)}$ then $0 \leqslant e^{-tD_{\lambda_2}^{\mathscr{A}}} \leqslant e^{-tD_{\lambda_1}^{\mathscr{A}}}$ in the sense of positive operators, i.e.

$$0\leqslant e^{-tD^{\mathscr{A}}_{\lambda_2}}\varphi\leqslant e^{-tD^{\mathscr{A}}_{\lambda_1}}\varphi\quad (t>0,\ 0\leqslant\varphi\in L^2(\partial\Omega)).$$

The regularity required on the boundary $\partial\Omega$ has two purposes: first, it guarantees that elements in $H^1(\Omega)$ have a well-defined trace on the boundary and the trace operator $j:H^1(\Omega)\to L^2(\partial\Omega)$ is compact; second, the divergence theorem holds, that is, there is an outward unit normal ν defined a.e. on $\partial\Omega$ and

$$\int_{\Omega} \partial_j u \, dx = \int_{\partial \Omega} u \nu_j \, d\sigma \quad (u \in H^1(\Omega)).$$

Let us finish this introduction by briefly describing how the paper is organized. In Section 2 we define realizations of a second order operator $\mathscr{A} = -\mathrm{div}\,(a\nabla u) + b\cdot\nabla u - \mathrm{div}\,(cu) + du$ under various boundary conditions, which will play in Problem (1.2) the same role as the Laplacian does in Problem (1.4). We also recall the basic definitions and relevant properties of positive, irreducible and sub-Markovian semigroups which are needed in the sequel. In Section 3 we define the main object of study here, namely, the Dirichlet-to-Neumann operator with respect to Problem (1.2), denoted by $D_{\lambda}^{\mathscr{A}}$, and prove the analogous version of the folklore result which

relates its spectrum to the spectrum of the realization of $\mathscr A$ with Robin boundary conditions. The proof of Theorem 1.1 is carried out in Section 4.

2. Preliminaries

Let us start by formulating the following hypothesis which we assume throughout this section. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} .

Hypothesis 2.1. The matrix-valued function $a \in L^{\infty}(\Omega; \mathbb{C}^{N \times N})$ is Hermitian and uniformly positive-definite in the sense that

$$\operatorname{Re} a(x)\xi \cdot \overline{\xi} \geqslant \kappa |\xi|^2 \quad (\xi \in \mathbb{C}^N, \text{a.e. } x \in \Omega).$$

The vector fields $b,c\in L^\infty(\Omega)^N$ as well the measurable function $d\in L^\infty(\Omega)$ are, possibly, complex-valued.

We will consider suitable realizations of the second order elliptic operator in divergence form

$$\mathcal{A}u = \sum_{j,k=1}^{N} -\partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{N} b_j\partial_j u - \sum_{j=1}^{N} \partial_j(c_j u) + du$$
$$= -\operatorname{div}(a\nabla u) + b \cdot \nabla u - \operatorname{div}(cu) + du$$

on $L^2(\Omega)$ under Dirichlet and Robin boundary conditions; such realizations play the role of the Dirichlet and Robin Laplacian as we pass from the Dirichlet-to-Neumann operator D_{λ} with respect to Problem (1.4) to the one relative to Problem (1.2). Clearly the associated form is

(2.1)
$$\mathfrak{a}(u,v) = \int_{\Omega} \left(\sum_{j,k=1}^{N} a_{jk} \partial_{k} u \overline{\partial_{j} v} + \sum_{j=1}^{N} b_{j} \partial_{j} u \overline{v} + \sum_{j=1}^{N} c_{j} u \overline{\partial_{j} v} + du \overline{v} \right) dx$$
$$= \int_{\Omega} (a \nabla u) \cdot \overline{\nabla v} \, dx + \int_{\Omega} (b \cdot \nabla u) \overline{v} \, dx + \int_{\Omega} u (c \cdot \overline{\nabla v}) \, dx + \int_{\Omega} du \overline{v} \, dx$$

with domain $H^1(\Omega)$. For an element $u \in H^1(\Omega)$ we say that $\mathscr{A}u \in L^2(\Omega)$ if there exists an element $f \in L^2(\Omega)$, in which case we write $\mathscr{A}u = f$, such that

$$\int_{\Omega} f \overline{v} \, dx = \mathfrak{a}(u, v) \quad (v \in C_{c}^{\infty}(\Omega)).$$

Proposition 2.2. Let $\Omega \subset \mathbb{R}^N$ be an open set and assume Hypothesis 2.1. Let A_D be the operator on $L^2(\Omega)$ defined by

$$\mathscr{D}(A_D) := \{ u \in H_0^1(\Omega) : \mathscr{A}u \in L^2(\Omega) \},$$

$$A_D u := \mathscr{A}u \quad (u \in \mathscr{D}(A_D)).$$

Then $-A_D$ is the generator of a quasi-contrative C_0 -semigroup. If $\mathbb{K} = \mathbb{C}$ then $-A_D$ generates a cosine operator function and hence a holomorphic semigroup of angle $\pi/2$.

Before we go into the proof we quote the following result which has been noted in [17]. We write $\mathfrak{a}(u)$ for $\mathfrak{a}(u,u)$ throughout this paper.

Proposition 2.3. Let V and H be Hilbert spaces and let $j:V\to H$ be a bounded linear operator with dense range. Let $\mathfrak{a}:V\times V\to \mathbb{C}$ be a j-elliptic form with associated operator A on H. If there exists an $M\geqslant 0$ such that

$$(2.2) |\text{Im } \mathfrak{a}(u)| \leq M ||u||_V ||j(u)||_H \quad (u \in V)$$

then -A generates a cosine operator function and hence a holomorphic semigroup of angle $\pi/2$.

In fact, estimate (2.2) implies that the numerical range of A lies in a parabola with vertex on the real axis and opened in the direction of the positive real axis. Thus the assertion that -A generates a cosine operator function follows from a theorem due to M. Crouzeix. Moreover, it is known that every generator of a cosine operator function also generates a holomorphic semigroup of angle $\pi/2$; for more details and references, see [17, Proposition 2.4].

Proof of Proposition 2.2. Let $\mathfrak{a}: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{K}$ be the sesquilinear form defined by the same formula as in Eq. (2.1), but restricted to $H_0^1(\Omega)$. Recall that, under Hypothesis 2.1, $b, c \in L^{\infty}(\Omega)^N$. We first estimate

(2.3)
$$\operatorname{Re} \int_{\Omega} ((b+\overline{c}) \cdot \nabla u) \overline{u} \, dx \geqslant -\|b+\overline{c}\|_{L^{\infty}(\Omega)^{N}} \|\nabla u\|_{L^{2}(\Omega)^{N}} \|u\|_{L^{2}(\Omega)} \\ \geqslant -\|b+\overline{c}\|_{L^{\infty}(\Omega)^{N}} (\varepsilon \|\nabla u\|_{L^{2}(\Omega)^{N}}^{2} + c_{\varepsilon} \|u\|_{L^{2}(\Omega)}^{2}).$$

If we choose $\varepsilon > 0$ such that $||b + \overline{c}||_{L^{\infty}(\Omega)^{N}} \varepsilon = \frac{\kappa}{2}$ then we get the estimate

$$\operatorname{Re}\mathfrak{a}(u) + (\|b + \overline{c}\|_{L^{\infty}(\Omega)^{N}} c_{\varepsilon} + \|(\operatorname{Re} d)^{-}\|_{L^{\infty}(\Omega)}) \|u\|_{L^{2}(\Omega)}^{2} \geqslant \frac{\kappa}{2} \int_{\Omega} |\nabla u|^{2} dx.$$

From this, with $\omega_1 = \|b + \overline{c}\|_{L^{\infty}(\Omega)^N} c_{\varepsilon} + \|(\operatorname{Re} d)^-\|_{L^{\infty}(\Omega)} + \frac{\kappa}{2}$, we obtain

(2.4)
$$\operatorname{Re} \mathfrak{a}(u) + \omega_1 \|u\|_{L^2(\Omega)}^2 \geqslant \frac{\kappa}{2} \|u\|_{H^1(\Omega)}^2 \quad (u \in H_0^1(\Omega)).$$

Thus, \mathfrak{a} is $L^2(\Omega)$ -elliptic. It is elementary to check that $A_{\rm D}$ is the operator associated with \mathfrak{a} ; thus the assertion that it generates a quasi-contrative C_0 -semigroup follows from the general theory, cf. e.g [3, Theorem 5.7].

Moreover,

$$|\operatorname{Im}\mathfrak{a}(u)| = \left| \operatorname{Im} \left(\int_{\Omega} (b \cdot \nabla u) \overline{u} \, dx + \int_{\Omega} u(c \cdot \overline{\nabla u}) \, dx + \int_{\Omega} d|u|^2 \, dx \right) \right|$$

$$\leq (\|b - \overline{c}\|_{L^{\infty}(\Omega)^{N}} + \|\operatorname{Im} d\|_{L^{\infty}(\Omega)}) \|u\|_{H^{1}(\Omega)} \|u\|_{L^{2}(\Omega)},$$

thus the last assertion follows from Proposition 2.3.

Remark 2.4. Note that in the derivation of estimate (2.4) no special property of $H_0^1(\Omega)$ is used, so that it is still valid for elements $u \in H^1(\Omega)$ whenever the form \mathfrak{a} defined by Eq. (2.1) is considered on $H^1(\Omega)$. We use this in the following without further ado.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. In this case elements in $H^1(\Omega)$ have a well-defined trace on the boundary. We say an element $u \in H^1(\Omega)$ with $\mathscr{A}u \in L^2(\Omega)$ has a weak conormal derivative if there exists $h \in L^2(\partial\Omega)$ such that

(2.5)
$$-\int_{\Omega} (\mathscr{A}u)\overline{v} \, dx + \mathfrak{a}(u,v) = \int_{\partial\Omega} h\overline{v} \, d\sigma$$

holds for every $v \in H^1(\Omega)$. In this case we put $\partial_{\nu}u := h$; this definition is natural in the sense that it reduces to the classical notion of conormal derivative (smooth case) and also to the definition of weak normal derivative introduced in [7] (see also [6] and [8]). By repeating the same proof above we can define a realization of \mathscr{A} with Neumann boundary conditions, namely, an operator A_N on $L^2(\Omega)$ given by

$$\mathscr{D}(A_{\mathrm{N}}) := \{ u \in H^{1}(\Omega) : \mathscr{A}u \in L^{2}(\Omega), \ `\partial_{\nu}u = 0` \},$$

$$A_{\mathrm{N}}u := \mathscr{A}u \quad (u \in \mathscr{D}(A_{\mathrm{N}})),$$

where ' $\partial_{\nu}u = 0$ ' means ' $\mathfrak{a}(u,v) = \int_{\Omega} (\mathscr{A}u)\overline{v} \,dx$ for all $v \in H^1(\Omega)$ ', or, equivalently, that the weak conormal derivative exists and equals zero. The operator A_N , however, will not be relevant in this paper.

Next we consider a realization of \mathscr{A} with the Robin boundary condition $\partial_{\nu}u + \beta u|_{\partial\Omega} = 0$, where $\beta \in L^{\infty}(\partial\Omega)$. Let $\mathfrak{a}^{\beta}: H^{1}(\Omega) \times H^{1}(\Omega) \to \mathbb{K}$ be the sesquilinear form defined by

(2.6)
$$\mathfrak{a}^{\beta}(u,v) = \mathfrak{a}(u,v) + \int_{\partial\Omega} \beta u \overline{v} \, d\sigma.$$

Since, under our present assumptions on Ω , the trace is compact from $H^1(\Omega)$ to $L^2(\partial\Omega)$, it follows from Lions's lemma [18, Ch. 2, Lemma 6.1] that there exists $c_1 \ge 0$ such that

$$\|(\operatorname{Re}\beta)^{-}\|_{L^{\infty}(\partial\Omega)} \int_{\partial\Omega} |u|^{2} d\sigma \leqslant \frac{\kappa}{4} \|u\|_{H^{1}(\Omega)}^{2} + c_{1} \int_{\Omega} |u|^{2} dx,$$

thus from estimate (2.4) (cf. Remark 2.4) we get

Re
$$\mathfrak{a}^{\beta}(u) + (\omega_1 + c_1) \|u\|_{L^2(\Omega)}^2 \geqslant \frac{\kappa}{4} \|u\|_{H^1(\Omega)}^2 \quad (u \in H^1(\Omega)).$$

Therefore, \mathfrak{a}^{β} is $L^2(\Omega)$ -elliptic. Moreover, by a standard trace inequality the form \mathfrak{a}^{β} clearly satisfies an estimate of the form (2.2). We have thus proved the following.

Proposition 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, let $\beta \in L^{\infty}(\partial\Omega)$ and assume Hypothesis 2.1. Let A_{β} be the operator on $L^2(\Omega)$ defined by

$$\mathscr{D}(A_{\beta}) := \{ u \in H^{1}(\Omega) : \mathscr{A}u \in L^{2}(\Omega), \ \partial_{\nu}u + \beta u|_{\partial\Omega} = 0 \},$$
$$A_{\beta}u := \mathscr{A}u \quad (u \in \mathscr{D}(A_{\beta})).$$

Then $-A_{\beta}$ is the generator of a quasi-contrative C_0 -semigroup. If $\mathbb{K} = \mathbb{C}$ then $-A_{\beta}$ generates a cosine operator function and hence a holomorphic semigroup of angle $\pi/2$.

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $1 \leq p < \infty$. A semigroup T on $L^p(\Omega, \mu)$ is positive if $T(t)f \geqslant 0$ whenever $f \geqslant 0$ and t > 0. A semigroup T on $L^p(\Omega, \mu)$ is sub-Markovian if, besides being positive, it is also L^∞ -contractive, meaning that $||T(t)f||_\infty \leqslant ||f||_\infty$ for all $f \in L^p \cap L^\infty(\Omega)$. This property is crucial in connection with the problem of extrapolating the semigroup to the L^p scale which is the starting point for further investigations of their spectral properties. In order to establish these properties, which are equivalent to the invariance of certain convex and closed sets, we employ the following j-elliptic version of Ouhabaz's invariance theorem, proved in [7, Proposition 2.9]. Actually, the version stated and proved in [7, Proposition 2.9] assumes in addition that the form is accretive and the consequence of this is that item (c) below is possible with $\omega = 0$. This may be interesting if one is concerned with item (c) as a necessary condition but here we are interested in the implication '(c) \Rightarrow (a)' so that the version stated below, which can be proven by adapting the proof in [3, Theorem 9.20], is more convenient. Moreover, $P: H \to C$ is the minimizing projection, see e.g. [11, Theorem 5.2].

Proposition 2.6. Let V and H be Hilbert spaces and let $j:V \to H$ be a bounded linear operator with dense range. Let $\mathfrak{a}:V\times V\to \mathbb{K}$ be a j-elliptic and continuous sesquilinear form with associated operator A. Let T be the semigroup generated by -A. Let $C\subset H$ be a non-empty closed convex set with minimizing projection $P:H\to C$. Then the following assertions are equivalent.

- (a) C is invariant under T.
- (b) For all $u \in V$ there exists $w \in V$ such that

$$Pj(u) = j(w)$$
 and $Re \mathfrak{a}(w, u - w) \geqslant 0$.

(c) For all $u \in V$ there exists $w \in V$ such that

$$Pj(u) = j(w)$$
 and $\operatorname{Re} \mathfrak{a}(u, u - w) \geqslant -\omega ||j(u) - j(w)||_H^2$

for some $\omega \in \mathbb{R}$ depending only on the form \mathfrak{a} .

The celebrated Krein-Rutman theorem asserts that if the generator A of a positive semigroup has compact resolvent and $s(A) > -\infty$ (s(A) is the spectral bound of A), then -s(A) is the first eigenvalue $\lambda_1(-A)$ of -A and admits a positive eigenfunction. We refer the interested reader to [1, Lecture 10] for more information.

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $1 \leq p < \infty$. By definition, a semigroup T on $L^p(\Omega, \mu)$ is called irreducible if the only closed ideals $\mathscr{I} \subset L^p(\Omega, \mu)$ (they are necessarily of the form $L^p(\omega)$, for some $\omega \in \mathscr{A}$) which are invariant by T (in the sense that $T\mathscr{I} \subset \mathscr{I}$) are $\mathscr{I} = \{0\}$ and $\mathscr{I} = L^p(\Omega, \mu)$ itself. The following well known result is often used to deduce irreducibility for a large class of elliptic operators. See Remark 2.8.

Proposition 2.7. Let $\Omega \subset \mathbb{R}^N$ be an open connected set. Let T be a strongly continuous semigroup on $L^2(\Omega)$ associated to an elliptic form $\mathfrak{a}: V \times V \to \mathbb{K}$, where $V \subset H^1(\Omega)$ is a subspace containing $C_c^{\infty}(\Omega)$. Then T is irreducible.

Proof. Suppose T leaves $L^2(\omega)$ invariant for some measurable $\omega \subset \Omega$. By Ouhabaz invariance theorem, $\chi_{\omega}u \in V$ whenever $u \in V$. As in the proof of [20, Theorem 4.5] (cf. also [1, Proposition 11.1.2]), it follows from connectedness that either ω has measure zero or $\Omega \setminus \omega$ has measure zero; this is the hard part of the proof but it is well known (see the references just given). Thus, either $L^2(\omega) = \{0\}$ or $L^2(\omega) = L^2(\Omega)$.

Remark 2.8. The statement and proof of Proposition 2.7 may be surprising to some readers, due to its simplicity. It is convenient to say some words about this. Experts know very well that related results on irreducibility as stated in Ouhabaz's book include hypotheses on positivity. To understand why, it is important to observe that in [20, Definition 2.8] irreducibility is defined as follows. A holomorphic semigroup T (in particular, a semigroup associated with an elliptic form) on $L^2(\Omega, \mu)$ is irreducible if and only if

(2.7)
$$T(t)f > 0$$
 a.e. on Ω $(t > 0)$ whenever $0 \neq f \in L^2(\Omega, \mu)_+$.

This defining property is easily seen to imply the invariance property we have used to define irreducibility and already implies that, in particular, an irreducible semigroup is positive. On the other hand, for positive semigroups, both concepts of irreducibility are equivalent; this is the content of [20, Theorem 2.9]. To summarize, our definition of irreducibility here dispenses with hypotheses on positivity because these hypotheses are usually required only to go from the invariance property we have used to define irreducibility to the property expressed in Eq. (2.7). Moreover, locality of the form is also not needed to arrive at the conclusion in the hard part of the proof above (although the forms to which we apply the result are local); this can in part be explained since the form domain is very special, namely, a subspace of $H^1(\Omega)$ and Ω is connected.

Finally, we will also need some monotonicity properties of the semigroups $e^{-tA_{\beta}}$ when different β 's are considered. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Suppose that the form \mathfrak{a} in Eq. (2.1) has real coefficients. By [20, Theorem 4.2] the semigroup generated by $-A_{\beta}$ is positive. Moreover, it follows from well-known comparison results, see e.g. [20, Theorem 2.24], that if $\beta_0, \beta_1 \in L^{\infty}(\partial\Omega)$ and $0 < \beta_0 \leqslant \beta_1$ then $e^{-tA_{\beta_1}} \leqslant e^{-tA_{\beta_0}}$ for all $t \geqslant 0$. If Ω is also connected then the semigroup generated by $-A_{\beta}$ is irreducible, by Proposition 2.7. In this case, the Krein-Rutman theorem (more precisely, the monotonicity result [1, Theorem 10.2.10]) allows us to summarize this discussion in the following.

Proposition 2.9. Let $\Omega \subset \mathbb{R}^N$ be a bounded open connected set with Lipschitz boundary. Suppose the form \mathfrak{a} in Eq. (2.1) has real coefficients and let $\beta_0, \beta_1 \in L^{\infty}(\partial\Omega)$ with $0 < \beta_0 \leqslant \beta_1$. Then

$$\lambda_1(A_{\beta_0}) = \lambda_1(A_{\beta_1})$$
 if, and only if $A_{\beta_0} = A_{\beta_1}$.

3. The Dirichlet-to-Neumann operator $D_{\lambda}^{\mathscr{A}}$

Now we turn to the definition of the Dirichlet-to-Neumann operator with respect to Problem (1.2). Under suitable assumptions on Ω and \mathscr{A} (see Proposition 3.2 below) we define the Dirichlet-to-Neumann operator $D_{\lambda}^{\mathscr{A}}$ as the operator on $L^{2}(\partial\Omega)$ such that $\varphi\in\mathscr{D}(D_{\lambda}^{\mathscr{A}})$ and $D_{\lambda}^{\mathscr{A}}\varphi=h$ if, and only if there is a weak solution $u\in H^1(\Omega)$ of Problem (1.2) with $\partial_{\nu}u=h$ (weak conormal derivative). As we have anounced in the introduction (see paragraph before Eq. (1.3)), let us see that $D_{\lambda}^{\mathscr{A}}$ is the operator associated to the form \mathfrak{a}_{λ} defined in Eq. (1.3) when $j: H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator. The form \mathfrak{a}_{λ} is not j-elliptic in general and the fact that it admits a well-defined associated operator with good properties can be established in a reasonably easy way either by appealing to the theory of compactly elliptic forms (which we recall at the end of this section) or by using the following result.

For a bounded sequilinear form $\mathfrak{a}: V \times V \to \mathbb{K}$, let $V_i(\mathfrak{a})$ be the closed subspace

$$V_j(\mathfrak{a}) := \{ u \in V : \mathfrak{a}(u, v) = 0 \text{ for all } v \in \mathcal{N}(j) \}.$$

Proposition 3.1 (cf. [7], Corollary 2.2). Let V, H be Hilbert spaces and let $j \in \mathcal{L}(V, H)$ have dense range. Let $\mathfrak{a}: V \times V \to \mathbb{K}$ be a continuous sesquilinear form and suppose that

- (i) $V = V_j(\mathfrak{a}) + \mathcal{N}(j)$; (ii) there exists $\omega \in \mathbb{R}$ and $\alpha > 0$ such that $\operatorname{Re} \mathfrak{a}(u) + \omega ||j(u)||_H^2 \geqslant \alpha ||u||_V^2$ for all $u \in V_j(\mathfrak{a})$.

Then \mathfrak{a} admits an associated operator A in the sense of (1.1) which is m-sectorial.

The following result describes the Dirichlet-to-Neumann operator $D_{\lambda}^{\mathscr{A}}$ along with its basic properties.

Proposition 3.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and assume the conditions stated in Hypothesis 2.1. Suppose $\hat{\lambda} \notin \sigma(A_D)$. Then the operator $D_{\lambda}^{\mathscr{A}}$ on $L^2(\partial\Omega)$ given by

$$\mathscr{D}(D_{\lambda}^{\mathscr{A}}) = \{ \varphi \in L^{2}(\partial\Omega) : \text{there exists } u \in H^{1}(\Omega) \text{ such that}$$

$$\mathscr{A}u = \lambda u, \ u|_{\partial\Omega} = \varphi \text{ and } \partial_{\nu}u \in L^{2}(\partial\Omega) \}$$

$$D_{\lambda}^{\mathscr{A}}\varphi = \partial_{\nu}u,$$

is quasi-m-accretive and has compact resolvent. If $b = \overline{c}$ and d is real then $D_{\lambda}^{\mathscr{A}}$ is self-adjoint. If $\mathbb{K} = \mathbb{C}$ then $D_{\lambda}^{\mathscr{A}}$ is quasi-m-sectorial. Moreover, $D_{\lambda}^{\mathscr{A}}$ is the operator associated with the restrictions of \mathfrak{a}_{λ} and j to

$$V_j(\mathfrak{a}_{\lambda}) = \{ u \in H^1(\Omega) : \mathscr{A}u = \lambda u \text{ in the distributional sense} \}.$$

Proof. Let us first show that (i) in Proposition 3.1 is satisfied. To this end, we follow the same reasoning used in [5, Lemma 2.2] which concerns the case a = I, b = c = 0 and d = 0. With our notation, we have to prove that $V = H_0^1(\Omega) \oplus V_j(\mathfrak{a}_{\lambda})$ if $\lambda \notin \sigma(A_D)$. Define $\mathscr{L}: H_0^1(\Omega) \to H^{-1}(\Omega)$ by $\langle \mathcal{L}u,v\rangle=\mathfrak{a}(u,v)$. Under the usual identifications in the Gel'fand triple $H_0^1(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow$ $H^{-1}(\Omega)$ we see that $A_{\rm D}$ is the part of \mathscr{L} on $L^2(\Omega)$ so that $\sigma(\mathscr{L}) = \sigma(A_{\rm D})$ by [2, Proposition 3.10.3]. Thus $\lambda - \mathcal{L}$ is invertible. Fix $u \in H^1(\Omega)$ and define $\Phi \in H^{-1}(\Omega)$ by $\Phi(v) = \mathfrak{a}_{\lambda}(u,v)$. There exists $u_0 \in H_0^1(\Omega)$ such that $\lambda u_0 - \mathcal{L}u_0 = -\Phi$, that is,

$$\lambda \int_{\Omega} u_0 \overline{v} \, dx - \mathfrak{a}(u_0, v) = -\mathfrak{a}_{\lambda}(u, v) \quad (v \in H_0^1(\Omega)),$$

thus $\mathfrak{a}_{\lambda}(u-u_0,v)=0$ for all $v\in H_0^1(\Omega)$. Therefore $u-u_0\in V_j(\mathfrak{a}_{\lambda})$ and it follows that $H^1(\Omega) = H^1_0(\Omega) + V_j(\mathfrak{a}_{\lambda})$. Since $\lambda \notin \sigma(A_D)$ we also have $H^1_0(\Omega) \cap V_j(\mathfrak{a}_{\lambda}) = \{0\}$.

Now let us prove the ellipticity of $\mathfrak{a}_{\lambda}|_{V_{i}(\mathfrak{a}_{\lambda})}$, that is, condition (ii) in Proposition 3.1. Lions's lemma applied to the (compact) immersion $V_i(\mathfrak{a}_{\lambda}) \hookrightarrow L^2(\Omega)$ and to the (injective) restriction of the trace $j: V_j(\mathfrak{a}_{\lambda}) \to L^2(\partial\Omega)$ gives, for each $\delta > 0$, a constant $c_{\delta} \geq 0$ such that

(3.1)
$$\int_{\Omega} |u|^2 dx \leqslant \delta ||u||_{H^1(\Omega)}^2 + c_{\delta} \int_{\partial \Omega} |u|^2 d\sigma,$$

for all $u \in V_i(\mathfrak{a}_{\lambda})$, so that

$$\int_{\Omega} |u|^2 dx \leqslant \frac{\delta}{1-\delta} \int_{\Omega} |\nabla u|^2 dx + \frac{c_{\delta}}{1-\delta} \int_{\partial \Omega} |u|^2 d\sigma \quad (u \in V_j(\mathfrak{a}_{\lambda})).$$

From estimate (2.4) (see Remark 2.4) we have

Re
$$\mathfrak{a}_{\lambda}(u) + (|\lambda| + \omega_1) ||u||_{L^2(\Omega)}^2 \ge \frac{\kappa}{2} ||u||_{H^1(\Omega)}^2 \quad (u \in H^1(\Omega)).$$

Combining the above two estimates we obtain

$$\operatorname{Re} \mathfrak{a}_{\lambda}(u) + \frac{(|\lambda| + \omega_{1})\delta}{1 - \delta} \int_{\Omega} |\nabla u|^{2} dx + \frac{(|\lambda| + \omega_{1})c_{\delta}}{1 - \delta} \int_{\partial \Omega} |u|^{2} d\sigma \geqslant \frac{\kappa}{2} ||u||_{H^{1}(\Omega)}^{2} \quad (u \in V_{j}(\mathfrak{a}_{\lambda})).$$

By choosing $\delta > 0$ satisfying $\frac{(|\lambda| + \omega_1)\delta}{1 - \delta} = \frac{\kappa}{4}$ and then ω to be the corresponding number $\frac{(|\lambda| + \omega_1)c_\delta}{1 - \delta}$ we arrive at the estimate

(3.2)
$$\operatorname{Re} \mathfrak{a}_{\lambda}(u) + \omega \|u\|_{\partial\Omega}\|_{L^{2}(\partial\Omega)}^{2} \geqslant \frac{\kappa}{4} \|u\|_{H^{1}(\Omega)}^{2} \quad (u \in V_{j}(\mathfrak{a}_{\lambda})).$$

This is precisely the ellipticity of $\mathfrak{a}_{\lambda}|_{V_{j}(\mathfrak{a}_{\lambda})}$ with respect to the trace.

The description of $D_{\lambda}^{\mathscr{A}}$ follows from standard arguments, based on the definition of weak conormal derivative in Eq. (2.5); see e.g. [3, Lecture 8].

Proposition 3.3. Under the same assumptions as in Proposition 3.2, let $\lambda \notin \sigma(A_D)$ and $\beta \in \mathbb{K}$.

- (a) $\mathcal{N}(\beta + D_{\lambda}^{\mathscr{A}})$ and $\mathcal{N}(\lambda A_{\beta})$ are isomorphic. (b) $-\beta \in \sigma(D_{\lambda}^{\mathscr{A}})$ if, and only if $\lambda \in \sigma(A_{\beta})$.

Proof. (a). We prove that $\mathcal{N}(\lambda - A_{\beta}) \ni u \mapsto u|_{\partial\Omega} \in \mathcal{N}(\beta + D_{\lambda}^{\mathscr{A}})$ is an isomorphism.

First, we check this operator is well-defined. Let \mathfrak{a}^{β} be the form defined by Eq. (2.6). Thus, $u \in \mathcal{N}(\lambda - A_{\beta})$ if, and only if $\mathfrak{a}^{\beta}(u, v) = \lambda(u|v)_{L^{2}(\Omega)}$ for all $v \in H^{1}(\Omega)$, or

(3.3)
$$\mathfrak{a}(u,v) + \beta \int_{\partial\Omega} u\overline{v} \, d\sigma = \lambda \int_{\Omega} u\overline{v} \, dx \quad (v \in H^1(\Omega)).$$

Equivalently, $\mathfrak{a}_{\lambda}(u,v) = (-\beta u|v)_{L^{2}(\partial\Omega)}$ for all $v \in H^{1}(\Omega)$, which means that $u|_{\partial\Omega} \in \mathscr{D}(D_{\lambda}^{\mathscr{A}})$ and $D_{\lambda}^{\mathscr{A}}u|_{\partial\Omega} = -\beta u|_{\partial\Omega}$, that is, $u|_{\partial\Omega} \in \mathscr{N}(\beta + D_{\lambda}^{\mathscr{A}})$.

Now, we prove the above operator is surjective. If $\varphi \in \mathcal{N}(\beta + D_{\lambda}^{\mathscr{A}})$ then $D_{\lambda}^{\mathscr{A}}\varphi = -\beta\varphi$ and this means that, for some $u \in H^1(\Omega)$, $u|_{\partial\Omega} = \varphi$ and $\mathfrak{a}_{\lambda}(u,v) = (-\beta \varphi|v)_{L^2(\partial\Omega)}$ for all $v \in H^1(\Omega)$. This is clearly equivalent to Eq. (3.3); thus, $\varphi = u|_{\partial\Omega}$ for some $u \in \mathcal{N}(\lambda - A_{\beta})$.

Finally, we show injectivity. If $u \in \mathcal{N}(\lambda - A_{\beta})$ and $u|_{\partial\Omega} = 0$ then, by (3.3), $\mathfrak{a}_{\lambda}(u, v) = 0$ for all $v \in H^1(\Omega)$; in particular, $\mathscr{A}u = \lambda u$ (in the distributional sense). Thus, $A_D u = \lambda u$ and, since $\lambda \notin \sigma(A_{\rm D})$, we conclude that u=0.

(b). Follows from (a) and the fact that $\sigma(D_{\lambda}^{\mathscr{A}})$ (resp. $\sigma(A_{\beta})$) is a pure point spectrum, since $D_{\lambda}^{\mathscr{A}}$ (resp. A_{β}) has compact resolvent.

A sequilinear form $\mathfrak{a}: V \times V \to \mathbb{K}$ is said to be *compactly elliptic* if there exists a Hilbert space H and a compact operator $j:V\to H$ such that $\mathfrak a$ is 'j-elliptic' in the sense that, for some constant $\tilde{\alpha} > 0$,

$$\operatorname{Re} \mathfrak{a}(u) + \|\widetilde{j}(u)\|_{\widetilde{H}}^2 \geqslant \widetilde{\alpha} \|u\|_V^2 \quad (u \in V).$$

This notion has been introduced in [8]. Now, consider the condition

(3.4) if
$$u \in V$$
, $j(u) = 0$ and $\mathfrak{a}(u, v) = 0$ for all $v \in \mathcal{N}(j)$, then $u = 0$.

If \mathfrak{a} is compactly elliptic and satisfies (3.4) then it follows from Lions's lemma that condition (ii) in Proposition 3.1 is satisfied. In fact, (3.4) implies that the restriction $j:V_j(\mathfrak{a})\to H$ is injective and the operador $\widetilde{j}:V_j(\mathfrak{a})\to \widetilde{H}$ is compact by hypothesis. From Lions's lemma,

$$\|\widetilde{j}(u)\|_{\widetilde{H}}^2 \leqslant \frac{\widetilde{\alpha}}{2} \|u\|_V^2 + c\|j(u)\|_H^2 \quad (u \in V_j(\mathfrak{a}))$$

for some constant $c \ge 0$. The aforementioned condition (ii) follows with $\alpha = \frac{\tilde{\alpha}}{2}$ by combining the above with the compactly ellipticity estimate. Note that this argument is nothing more that an abstract counterpart of the reasoning leading to estimate (3.1). It is easy to see that, under condition (3.4), a well-defined operator can be associated to \mathfrak{a} (compact ellipticity is not required for this) through (1.1). The following result tell us that much more can be derived from this construction.

Proposition 3.4 (cf. [3], Proposition 8.10 and Theorem 8.11). Let V, H be Hilbert spaces and let $j \in \mathcal{L}(V, H)$ have dense range. Let $\mathfrak{a}: V \times V \to \mathbb{K}$ be a continuous sesquilinear form. If \mathfrak{a} is compactly elliptic and satisfies (3.4) then conditions (i) and (ii) in Proposition 3.1 hold. Moreover, the associated operator on H concides with the operator associated to the elliptic form obtained from \mathfrak{a} and j by restriction to $V_j(\mathfrak{a})$.

Let us see that \mathfrak{a}_{λ} is compactly elliptic and satisfies (3.4) with $V = H^1(\Omega)$ and j being the trace from $H^1(\Omega)$ to $L^2(\partial\Omega)$. On the one hand, from the estimate in the proof of Proposition 2.2 (see also Remark 2.4) we can infer that

$$\operatorname{Re} \mathfrak{a}_{\lambda}(u) + (|\lambda| + \omega_1) ||u||_{L^2(\Omega)}^2 \geqslant \frac{\kappa}{2} ||u||_{H^1(\Omega)}^2 \quad (u \in H^1(\Omega)),$$

which implies that \mathfrak{a}_{λ} is compactly elliptic with $H = L^2(\Omega)$ and \tilde{j} being the multiplication of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ by $\sqrt{|\lambda| + \omega_1}$. On the other hand, condition (3.4) means here that if $u \in H^1(\Omega)$, $u|_{\partial\Omega} = 0$, i.e. $u \in H^1_0(\Omega)$, and $\mathfrak{a}_{\lambda}(u,v) = 0$ for all $v \in H^1_0(\Omega)$, then u = 0. Accordingly, let $u \in H^1_0(\Omega)$ and suppose $\mathfrak{a}_{\lambda}(u,v) = 0$ for all $v \in H^1_0(\Omega)$. Thus $u \in H^1_0(\Omega)$ and $\mathfrak{a}(u,v) = \lambda \int_{\Omega} u\overline{v} dx$ for all $v \in C_c^{\infty}(\Omega)$, which means that $A_D u = \lambda u$. Therefore, if $\lambda \notin \sigma(A_D)$ then u = 0, that is, \mathfrak{a}_{λ} satisfies (3.4). In view of Proposition 3.4, the above arguments give another proof of Proposition 3.2.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and assume Hypothesis 2.1. Let us define the number

$$(4.1) \lambda_{1}^{\mathcal{D}}(\mathfrak{a}) := \inf_{u \in H_{0}^{1}(\Omega), u \neq 0} \frac{\operatorname{Re} \mathfrak{a}(u)}{\|u\|_{L^{2}(\Omega)}^{2}} \\ = \inf_{u \in H_{0}^{1}(\Omega), u \neq 0} \frac{\int_{\Omega} a \nabla u \cdot \overline{\nabla u} \, dx + \operatorname{Re} \int_{\Omega} ((b + \overline{c}) \cdot \nabla u) \overline{u} \, dx + \int_{\Omega} (\operatorname{Re} d) |u|^{2} \, dx}{\int_{\Omega} |u|^{2} \, dx}.$$

The number $\lambda_1^{\rm D}(\mathfrak{a})$ coincides with the first eigenvalue $\lambda_1^{\rm D}$ of the Dirichlet Laplacian when $\mathscr{A} = -\Delta$ (that is, a = I, b = c = 0 and d = 0). From the variational characterization of $\lambda_1^{\rm D}$ in Eq. (1.5) it follows from the estimate (2.3) in the proof of Proposition 2.2 that

$$\operatorname{Re}\mathfrak{a}(u) \geqslant \kappa \int_{\Omega} |\nabla u|^2 \, dx - \|b + \overline{c}\|_{L^{\infty}(\Omega)^N} \|\nabla u\|_{L^2(\Omega)^N} \|u\|_{L^2(\Omega)} - \|(\operatorname{Re} d)^-\|_{L^{\infty}} \int_{\Omega} |u|^2 \, dx$$

$$\geqslant \kappa \int_{\Omega} |\nabla u|^2 dx - \frac{\|b + \overline{c}\|_{L^{\infty}(\Omega)^N}}{\sqrt{\lambda_1^D}} \|\nabla u\|_{L^2(\Omega)^N}^2 - \|(\operatorname{Re} d)^-\|_{L^{\infty}} \int_{\Omega} |u|^2 dx$$

$$\geqslant \left(\kappa - \frac{\|b + \overline{c}\|_{L^{\infty}(\Omega)^N}}{\sqrt{\lambda_1^D}}\right) \lambda_1^D \int_{\Omega} |u|^2 dx - \|(\operatorname{Re} d)^-\|_{L^{\infty}(\Omega)} \int_{\Omega} |u|^2 dx,$$

provided $||b + \overline{c}||_{L^{\infty}(\Omega)^{N}} < \kappa \sqrt{\lambda_{1}^{D}}$; in this case

$$\lambda_1^{\mathrm{D}}(\mathfrak{a}) \geqslant \kappa \lambda_1^{\mathrm{D}} - \|b + \overline{c}\|_{L^{\infty}(\Omega)^N} \sqrt{\lambda_1^{\mathrm{D}}} - \|(\operatorname{Re} d)^-\|_{L^{\infty}(\Omega)}.$$

In the following remarks we show how a more explicit estimate can be obtained under additional assumptions on b, c and d.

Remark 4.1. The following is well known to the experts and is included here for the convenience of the reader. Suppose, in addition to the standing assumptions on Ω , a and d, that $b, c \in C^1(\overline{\Omega})^N$ are real vector fields.

(a) If $u, v \in H^1(\Omega)$ then

$$\begin{split} \int_{\Omega} (b \cdot \nabla u) \overline{v} \, dx &= \sum_{j=1}^{N} \int_{\Omega} b_{j} \partial_{j} u \overline{v} \, dx \\ &= \sum_{j=1}^{N} \int_{\Omega} \partial_{j} (b_{j} u \overline{v}) \, dx - \sum_{j=1}^{N} \int_{\Omega} u \partial_{j} (b_{j} \overline{v}) \, dx \\ &= \int_{\partial \Omega} u \overline{v} b \cdot \nu \, d\sigma - \int_{\Omega} (\operatorname{div} b) u \overline{v} \, dx - \int_{\Omega} u (b \cdot \overline{\nabla v}). \end{split}$$

Therefore, if div b=0 and $b\cdot \nu=0$ then $b\cdot \nabla$ is skew-Hermitian in the sense that

$$\int_{\Omega} (b \cdot \nabla u) \overline{v} \, dx = -\int_{\Omega} (b \cdot \overline{\nabla v}) u \, dx \quad (u, v \in H^{1}(\Omega)).$$

In particular, Re $\int_{\Omega} (b \cdot \nabla u) \overline{u} \, dx = 0$ for all $u \in H^1(\Omega)$.

(b) It follows from the computation in (a) above (with b replaced by b+c) that

$$\lambda_{1}^{\mathcal{D}}(\mathfrak{a}) \geqslant \inf_{u \in H_{0}^{1}(\Omega), u \neq 0} \frac{\int_{\Omega} a \nabla u \cdot \overline{\nabla u} \, dx + \int_{\Omega} (\operatorname{Re} d) |u|^{2} \, dx}{\|u\|_{L^{2}(\Omega)}^{2}} + \frac{1}{2} \inf_{u \in H_{0}^{1}(\Omega), u \neq 0} \frac{\int_{\partial \Omega} |u|^{2} (b+c) \cdot \nu \, d\sigma - \int_{\Omega} \operatorname{div}(b+c) |u|^{2} \, dx}{\|u\|_{L^{2}(\Omega)}^{2}}.$$

Thus, under the hypotheses in Theorem 1.1, $\lambda_1^{\mathrm{D}}(\mathfrak{a}) \geqslant \kappa \lambda_1^{\mathrm{D}} - \|(\operatorname{Re} d)^-\|_{L^{\infty}(\Omega)}$. In particular, if $\operatorname{Re} d \geqslant 0$ then $\lambda_1^{\mathrm{D}}(\mathfrak{a}) \geqslant \kappa \lambda_1^{\mathrm{D}}$.

The rest of this paper is devoted to the proof of Theorem 1.1. For simplicity, we restrict to real scalars, so that, in particular,

$$\int_{\Omega} (b \cdot \nabla u) v \, dx = - \int_{\Omega} (b \cdot \nabla v) u \, dx \quad (u, v \in H^{1}(\Omega)).$$

It may be interesting to carry out all the details concerning the case of complex scalars; we leave this task to the interested reader. Note that, in view of Remark 4.1(b), the condition $4\kappa^{-1}\|b-c\|_{L^{\infty}(\Omega)^{N}}^{2}+\|d^{-}\|_{L^{\infty}(\Omega)}+\lambda<\kappa\lambda_{1}^{D}$ stated in Theorem 1.1(a)(b) implies the more 'abstract' condition $4\kappa^{-1}\|b-c\|_{L^{\infty}(\Omega)^{N}}^{2}+\lambda<\lambda_{1}^{D}(\mathfrak{a})$, which is actually what we use in the proofs below. The same applies to the conditions in Theorem 1.1(c)(d).

Proof of Theorem 1.1(a). We apply Ouhabaz invariance criterion to the closed convex set $C := \{u \in L^2(\Omega; \mathbb{R}) : u \geq 0\}$, whose minimizing projection is $Pu = u^+$.

From the lattice properties of $H^1(\Omega)$ and properties of the trace we know that if $u \in H^1(\Omega)$ then $u^+ \in H^1(\Omega)$ and $(u|_{\partial\Omega})^+ = u^+|_{\partial\Omega}$. A similar statement holds for u^- . On the other hand, there exists $u_0, \widetilde{u_0} \in H^1_0(\Omega)$ and $u_1, u_2 \in V_j(\mathfrak{a}_{\lambda})$ such that

$$u^+ = u_0 + u_1$$
 and $u^- = \widetilde{u_0} + u_2$.

But $u=u^+-u^-=(u_0-\widetilde{u_0})+(u_1-u_2)$, thus $u_0=\widetilde{u_0}$ if $u\in V_j(\mathfrak{a}_\lambda)$; we can assume that $u\in V_j(\mathfrak{a}_\lambda)$ from the start, since our Dirichlet-to-Neumann operator is also associated to the restrictions of the trace and the form \mathfrak{a}_λ to $V_j(\mathfrak{a}_\lambda)$ (see Proposition 3.4). Therefore $(u|_{\partial\Omega})^+=u_1|_{\partial\Omega}$ and $(u|_{\partial\Omega})^-=u_2|_{\partial\Omega}$; this means that $P_j(u)=j(u_1)$ and it suffices to show that $\mathfrak{a}_\lambda(u,u-u_1)\geqslant -\omega\|j(u-u_1)\|_{L^2(\partial\Omega)}^2$ for some $\omega\in\mathbb{R}$ (depending only on \mathfrak{a}_λ); since $u-u_1=-u_2$, this amounts to show that $\mathfrak{a}_\lambda(u,u_2)\leqslant \omega\|j(u_2)\|_{L^2(\partial\Omega)}^2$ or, equivalently,

$$\mathfrak{a}_{\lambda}(u_1, u_2) \leqslant \mathfrak{a}_{\lambda}(u_2) + \omega \int_{\partial \Omega} |u_2|^2 d\sigma.$$

From estimate (3.2), it is enough to show that $\mathfrak{a}_{\lambda}(u_1, u_2) \leqslant \frac{\kappa}{4} ||u_2||_{H^1(\Omega)}^2$. Let us show that this can always be achieved under the hypotheses in Theorem 1.1(a).

First, note that

$$\mathfrak{a}_{\lambda}(u_1, u_2) = \mathfrak{a}_{\lambda}(u^+, u^-) - \mathfrak{a}_{\lambda}(u_0, u_2) - \mathfrak{a}_{\lambda}(u_1, u_0) - \mathfrak{a}_{\lambda}(u_0, u_0).$$

Clearly,

$$\mathfrak{a}_{\lambda}(u^{+}, u^{-})$$

$$= \int_{\Omega} a \nabla u^{+} \cdot \nabla u^{-} dx + \int_{\Omega} (b \cdot \nabla u^{+}) u^{-} dx + \int_{\Omega} u^{+} (c \cdot \nabla u^{-}) dx + \int_{\Omega} du^{+} u^{-} dx - \lambda \int_{\Omega} u^{+} u^{-} dx$$

$$= 0.$$

Moreover, since $u_1 \in V_j(\mathfrak{a}_{\lambda})$ and $u_0 \in H_0^1(\Omega)$, we have $\mathfrak{a}_{\lambda}(u_1, u_0) = 0$. Besides,

$$\begin{split} &-\mathfrak{a}_{\lambda}(u_{0},u_{2})\\ &=-\int_{\Omega}a\nabla u_{0}\cdot\nabla u_{2}\,dx-\int_{\Omega}(b\cdot\nabla u_{0})u_{2}\,dx-\int_{\Omega}u_{0}(c\cdot\nabla u_{2})\,dx-\int_{\Omega}du_{0}u_{2}\,dx+\lambda\int_{\Omega}u_{0}u_{2}\,dx\\ &\stackrel{(1)}{=}-\int_{\Omega}a\nabla u_{2}\cdot\nabla u_{0}\,dx+\int_{\Omega}(b\cdot\nabla u_{2})u_{0}\,dx+\int_{\Omega}u_{2}(c\cdot\nabla u_{0})\,dx-\int_{\Omega}du_{2}u_{0}\,dx+\lambda\int_{\Omega}u_{2}u_{0}\,dx\\ &\stackrel{(2)}{=}2\Big(\int_{\Omega}(b\cdot\nabla u_{2})u_{0}\,dx+\int_{\Omega}u_{2}(c\cdot\nabla u_{0})\,dx\Big)\\ &=2\int_{\Omega}((b-c)\cdot\nabla u_{2})u_{0}\,dx\\ &\leqslant2\|b-c\|_{L^{\infty}(\Omega)}\|\nabla u_{2}\|_{L^{2}(\Omega)^{N}}\|u_{0}\|_{L^{2}(\Omega)}\\ &\leqslant\frac{\kappa}{4}\|\nabla u_{2}\|_{L^{2}(\Omega)^{N}}^{2}+\frac{4\|b-c\|_{L^{\infty}(\Omega)}^{2}}{\kappa}\|u_{0}\|_{L^{2}(\Omega)}^{2}. \end{split}$$

Above, identity (1) follows from Remark 4.1(a), which asserts that

$$\int_{\Omega} (b \cdot \nabla u_0) u_2 \, dx = -\int_{\Omega} (b \cdot \nabla u_2) u_0 \, dx$$

with an analogous identity for c. Identity (2) follows from the fact that $\mathfrak{a}_{\lambda}(u_2, u_0) = 0$ (since $u_2 \in V_j(\mathfrak{a}_{\lambda})$ and $u_0 \in H_0^1(\Omega)$) and the estimates are the usual Hölder and Young's inequalities,

respectively. Thus,

$$\begin{split} \mathfrak{a}_{\lambda}(u_{1},u_{2}) &\leqslant \frac{\kappa}{4} \|\nabla u_{2}\|_{L^{2}(\Omega)^{N}}^{2} \\ &- \int_{\Omega} a \nabla u_{0} \cdot \nabla u_{0} \, dx - \int_{\Omega} (b \cdot \nabla u_{0}) u_{0} \, dx - \int_{\Omega} u_{0}(c \cdot \nabla u_{0}) \, dx - \int_{\Omega} du_{0}^{2} \, dx \\ &+ \left(\lambda + 4\kappa^{-1} \|b - c\|_{L^{\infty}(\Omega)}^{2}\right) \int_{\Omega} |u_{0}|^{2} \, dx \\ &\leqslant \frac{\kappa}{4} \|u_{2}\|_{H^{1}(\Omega)}^{2} \\ &- \int_{\Omega} a \nabla u_{0} \cdot \nabla u_{0} \, dx - \int_{\Omega} (b \cdot \nabla u_{0}) u_{0} \, dx - \int_{\Omega} u_{0}(c \cdot \nabla u_{0}) \, dx - \int_{\Omega} du_{0}^{2} \, dx \\ &+ \lambda_{1}^{D}(\mathfrak{a}) \int_{\Omega} |u_{0}|^{2} \, dx \\ &\leqslant \frac{\kappa}{4} \|u_{2}\|_{H^{1}(\Omega)}^{2}, \end{split}$$

where the last estimate above follows from the definition of $\lambda_1^{\rm D}(\mathfrak{a})$ in (4.1).

Now, we turn to the proof of Theorem 1.1(b). As it is well known, the sub-Markovian property is equivalent to the invariance of the closed convex set $C = \{u \in L^2(\Omega) : u \leq 1\}$. We then apply Ouhabaz's invariance criterion with $Pu = u \wedge 1$.

Proof of Theorem 1.1(b). From the lattice properties of $H^1(\Omega)$ and properties of the trace we know that if $u \in H^1(\Omega)$ then $u \wedge 1 \in H^1(\Omega)$ and $(u \wedge 1)|_{\partial\Omega} = u|_{\partial\Omega} \wedge 1$. On the other hand, there exists $u_0, \widetilde{u_0} \in H^1_0(\Omega)$ and $u_1, u_2 \in V_i(\mathfrak{a}_{\lambda})$ such that

$$u \wedge 1 = u_0 + u_1$$
 and $-(u-1)^+ = \widetilde{u_0} + u_2$.

But $u = u \wedge 1 + (u - 1)^+ = (u_0 - \widetilde{u_0}) + (u_1 - u_2)$, thus $u_0 = \widetilde{u_0}$ if $u \in V_j(\mathfrak{a}_{\lambda})$. Therefore $(u|_{\partial\Omega}) \wedge 1 = u_1|_{\partial\Omega}$ (and $(u|_{\partial\Omega} - 1)^+ = -u_2|_{\partial\Omega}$), so that $P_j(u) = j(u_1)$, $u - u_1 = -u_2$ and, as before, we must show that $\mathfrak{a}_{\lambda}(u_1, u_2) \leqslant \frac{\kappa}{4} ||u_2||_{H^1(\Omega)}^2$. Using the estimates in the proof of (a) above we find that

$$\begin{split} \mathfrak{a}_{\lambda}(u_1,u_2) \\ &= \mathfrak{a}_{\lambda}(u \wedge 1, -(u-1)^+) - \mathfrak{a}_{\lambda}(u_0,u_2) - \mathfrak{a}_{\lambda}(u_1,u_0) - \mathfrak{a}_{\lambda}(u_0,u_0) \\ \leqslant &- \int_{\Omega} a \nabla (u \wedge 1) \cdot \nabla (u-1)^+ \, dx - \int_{\Omega} [b \cdot \nabla (u \wedge 1)](u-1)^+ \, dx \\ &- \int_{\Omega} (u \wedge 1)[c \cdot \nabla (u-1)^+] \, dx - \int_{\Omega} d(u \wedge 1)(u-1)^+ \, dx + \lambda \int_{\Omega} (u \wedge 1)(u-1)^+ \, dx \\ &+ \frac{\kappa}{4} \int_{\Omega} |\nabla u_2|^2 \, dx \\ &- \int_{\Omega} a \nabla u_0 \cdot \nabla u_0 \, dx - \int_{\Omega} (b \cdot \nabla u_0)u_0 \, dx - \int_{\Omega} u_0(c \cdot \nabla u_0) \, dx - \int_{\Omega} du_0^2 \, dx \\ &+ \left(\lambda + \frac{4\|b-c\|_{L^{\infty}(\Omega)}^2}{\kappa}\right) \int_{\Omega} |u_0|^2 \, dx \\ \leqslant &- \int_{\Omega} c \cdot \nabla (u-1)^+ \, dx - \int_{\Omega} d(u-1)^+ \, dx + \lambda \int_{\Omega} (u-1)^+ \, dx \\ &+ \frac{\kappa}{4} \int_{\Omega} |\nabla u_2|^2 \, dx \end{split}$$

$$-\int_{\Omega} a\nabla u_0 \cdot \nabla u_0 \, dx - \int_{\Omega} ((b+c) \cdot \nabla u_0) u_0 \, dx - \int_{\Omega} d|u_0|^2 \, dx + \lambda_1^{\mathrm{D}}(\mathfrak{a}) \int_{\Omega} |u_0|^2 \, dx.$$

Since div c = 0, $\lambda \leq d$ and $c \cdot \nu \geq 0$ we have

$$\int_{\Omega} (-c \cdot \nabla \varphi - d\varphi + \lambda \varphi) \, dx = \int_{\Omega} (\operatorname{div} c - d + \lambda) \varphi \, dx - \int_{\partial \Omega} c \cdot \nu \varphi \, d\sigma \leqslant 0$$

for all $\varphi \in C^1(\overline{\Omega})_+$, which is dense in $H^1(\Omega)_+$. The conclusion follows.

Next, we prove Theorem 1.1(c). We follow the arguments in [5, Theorem 4.2] in order to transfer the irreducibility from A_{β} to $D_{\lambda}^{\mathscr{A}}$.

Proof of Theorem 1.1(c). Let $\Gamma_1 \subset \partial \Omega$ be a Borel set with non-zero measure, put $\Gamma_0 := \partial \Omega \backslash \Gamma_1$, and suppose that $L^2(\Gamma_1) = \{ \varphi \in L^2(\partial \Omega) : \varphi|_{\Gamma_0} = 0 \text{ a.e.} \}$ is invariant under $e^{-tD_\lambda^\mathscr{A}}$. The restriction $e^{-tD_\lambda^\mathscr{A}}|_{L^2(\Gamma_1)}$ is positive and its generator has compact resolvent (since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact). By the Krein-Rutman theorem the first eigenvalue, say $-\beta_1$, of its generator admits an eigenfunction $0 < \varphi \in L^2(\Gamma_1)$, which turns to be also a positive eigenfunction of $D_\lambda^\mathscr{A}$, that is, $D_\lambda^\mathscr{A}\varphi = -\beta_1\varphi$. By definition, there exists $u \in H^1(\Omega)$ such that $\mathscr{A}u = \lambda u$, $u|_{\partial\Omega} = \varphi$ and $\partial_\nu u = -\beta_1\varphi$; thus,

$$-\lambda \int uv \, dx + \mathfrak{a}(u,v) = -\beta_1 \int_{\partial \Omega} uv \, d\sigma \quad (v \in H^1(\Omega)).$$

Since $u|_{\partial\Omega}=\varphi\geqslant 0$ we have $u^-\in H^1_0(\Omega)$. Inserting $v=u^-$ into the above identity, we obtain

$$\lambda \int_{\Omega} |u^{-}|^{2} dx = \lambda \int_{\Omega} u u^{-} dx = \mathfrak{a}(u, u^{-}) = \mathfrak{a}(u^{-}) \geqslant \lambda_{1}^{\mathcal{D}}(\mathfrak{a}) \int_{\Omega} |u^{-}|^{2} dx$$

where the last inequality follows from the definition in (4.1). Since $\lambda < \lambda_1^{\rm D}(\mathfrak{a})$, we must have $u^- = 0$, that is, $u \geq 0$. By Proposition 3.3, $A_{\beta_1}u = \lambda u$ and then, from the Krein-Rutman theorem, we infer that $\lambda = \lambda_1(A_{\beta_1})$, the first eigenvalue of A_{β_1} .

Now, choose a number $\beta_0 < \beta_1$ and consider the function $\beta \in L^{\infty}(\partial\Omega)$ given by $\beta = \beta_0 \mathbf{1}_{\Gamma_0} + \beta_1 \mathbf{1}_{\Gamma_1}$. Since $u|_{\Gamma_0} = 0$ we have

$$\mathfrak{a}^{\beta}(u,v) = \mathfrak{a}(u,v) + \int_{\partial\Omega} \beta uv \, d\sigma = \mathfrak{a}(u,v) + \beta_1 \int_{\partial\Omega} uv \, d\sigma = \lambda \int_{\Omega} uv \, dx$$

for all $v \in H^1(\Omega)$, which implies that $u \in \mathcal{D}(A_\beta)$ and $A_\beta u = \lambda u$. Again, the Krein-Rutman theorem allows us to infer that $\lambda = \lambda_1(A_\beta)$, the first eigenvalue of A_β . We have thus shown that $\lambda_1(A_{\beta_1}) = \lambda_1(A_\beta)$; since $\beta \leqslant \beta_1$, it follows that $e^{-tA_{\beta_1}} \leqslant e^{-tA_\beta}$ and then, by Proposition 2.9, that $A_\beta = A_{\beta_1}$. Therefore, $\mathfrak{a}^\beta = \mathfrak{a}^{\beta_1}$; in particular, $\mathfrak{a}^\beta(v) = \mathfrak{a}^{\beta_1}(v)$ for all $v \in H^1(\Omega)$, that is,

$$\int_{\Gamma_0} \beta_0 v^2 d\sigma + \int_{\Gamma_1} \beta_1 v^2 d\sigma = \int_{\partial \Omega} \beta_1 v^2 d\sigma$$

for all $v \in H^1(\Omega)$. Hence $\int_{\Gamma_0} v^2 d\sigma = 0$ for all $v \in H^1(\Omega)$. Then, by the Stone-Weierstrass theorem, $\int_{\partial\Omega} \varphi^2 \mathbf{1}_{\Gamma_0} d\sigma = 0$ for all $\varphi \in C(\partial\Omega)$, which implies that the Borel measure $\mathbf{1}_{\Gamma_0} d\sigma$ is zero; this is the same as saying that $\sigma(\Gamma_0) = 0$.

Finally, we prove Theorem 1.1(d). To this end we observe that, alternatively, the operator $D_{\lambda}^{\mathscr{A}}$ is also associated with an embedded form, namely, the form \mathfrak{b}_{λ} with domain $\mathscr{D}(\mathfrak{b}_{\lambda}) = j(H^{1}(\Omega)) = j(V_{j}(\mathfrak{a}_{\lambda}))$ given by

$$\mathfrak{b}_{\lambda}(j(u),j(v)) := \mathfrak{a}_{\lambda}(u,v) \quad (u,v \in V_{i}(\mathfrak{a}_{\lambda})).$$

We actually prove below a slightly stronger statement, namely, that the domination property $0 \leqslant e^{-tD_{\lambda_2}^{\mathscr{A}_2}} \leqslant e^{-tD_{\lambda_1}^{\mathscr{A}_1}}$ holds whenever $\lambda_i < \kappa \lambda_1^{\mathrm{D}} - \|d_i^-\|_{L^{\infty}(\Omega)}$ $(i=1,2), \ \lambda_2 \leqslant \lambda_1$ and $d_2 \geqslant d_1$; this generalizes [14, Theorem 2.4]. To make the dependence on d explicit, we write \mathfrak{a}_{λ}^d instead of

 \mathfrak{a}_{λ} (and similarly for \mathfrak{b}_{λ}). By $e^{-tD_{\lambda_{i}}^{\mathscr{A}_{i}}}$ we mean the Dirichlet-to-Neumann semigroup with respect to $\mathfrak{a}_{\lambda_{i}}^{d_{i}}$ for i=1,2.

Proof of Theorem 1.1(d). By [20, Theorem 2.24] it is enough to show that $\mathfrak{b}_{\lambda_2}^{d_2}(\varphi, \psi) \geqslant \mathfrak{b}_{\lambda_1}^{d_1}(\varphi, \psi)$ for all $0 \leqslant \varphi, \psi \in j(H^1(\Omega)) = j(V_j(\mathfrak{a}_{\lambda_1}^{d_1})) = j(V_j(\mathfrak{a}_{\lambda_2}^{d_2}))$.

Let φ, ψ be as above. There exists $u_1, v_1 \in V_j(\mathfrak{a}_{\lambda_1}^{d_1})$ and $u_2, v_2 \in V_j(\mathfrak{a}_{\lambda_2}^{d_2})$ such that $\varphi = j(u_1) = j(u_2)$ and $\psi = j(v_1) = j(v_2)$. Since $u_1 - u_2 \in H_0^1(\Omega)$, it follows as in the proof of (a), and by taking into account that we are assuming b = c, that

$$\mathfrak{a}_{\lambda_2}^{d_2}(u_2 - u_1, v_2) = -2\Big(\int_{\Omega} ((b - c) \cdot \nabla v_2)(u_2 - u_1) \, dx = 0,$$

that is, $\mathfrak{a}_{\lambda_2}^{d_2}(u_2, v_2) = \mathfrak{a}_{\lambda_2}^{d_2}(u_1, v_2)$. Clearly, we have $\mathfrak{a}_{\lambda_1}^{d_1}(u_1, v_2) = \mathfrak{a}_{\lambda_1}^{d_1}(u_1, v_1)$. Moreover, as in the proof of (c), we can show that $u_1, v_2 \geqslant 0$. Thus,

$$\begin{split} \mathfrak{b}_{\lambda_2}^{d_2}(\varphi,\psi) &= \mathfrak{a}_{\lambda_2}^{d_2}(u_2,v_2) \\ &= \mathfrak{a}_{\lambda_1}^{d_1}(u_1,v_1) + \int_{\Omega} (d_2 - d_1) u_1 v_2 \, dx + \int_{\Omega} (\lambda_1 - \lambda_2) u_1 v_2 \, dx \\ &\geqslant \mathfrak{b}_{\lambda_1}^{d_1}(\varphi,\psi). \end{split}$$

Domination properties of the semigroups $e^{-tD_{\lambda}^{\mathscr{A}}}$ in the nonself-adjoint case $b \neq c$ seem to be more difficult, at least under the present hypotheses and by using the methods employed in this paper. Two deeper problems would be (i) to investigate the graph $D_{\lambda}^{\mathscr{A}}$ when $\lambda \in \sigma(A_{\mathrm{D}})$ along the lines of [8] and [10] and (ii) to investigate the operator (or graph) $D_{\lambda}^{\mathscr{A}}$ on rough domains in the spirit of e.g. [6] or [15]. These questions will be investigated elsewhere.

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