Microlocal regularity of Besov type for solutions to quasi-elliptic non linear partial differential equations

Gianluca Garello and Alessandro Morando

Abstract. Using a standard linearization technique and previously obtained microlocal properties for pseudodifferential operators with smooth coefficients, the authors state results of microlocal regularity in generalized Besov spaces for solutions to non linear PDE.

1. Introduction

In previous papers, [4], [5], [6], [7], the authors studied the problem of L^p and Besov continuity and local regularity for pseudodifferential operators with smooth and non smooth symbols, whose derivatives decay at infinity in non homogeneous way. Particularly in [6], [7] emphasis is given on symbols with quasi-homogeneous decay; in [8] also microlocal properties were studied.

Pseudodifferential operators whose smooth symbols have a quasi-homogeneous decay at infinity were first introduced in 1977 in Lascar [9], where their microlocal properties in the L^2 -framework were studied.

Symbol classes of quasi-homogeneous type and several related problems have been developed in the meantime, see e.g. Segàla [10] for the local solvability, Garello [2] for symbols with decay of type (1,1), Yamazaki [13] where non smooth symbols in the L^p -framework are introduced and studied under suitable restrictive conditions on the Fourier transform of the symbols themselves.

The aim of the present paper is to apply the previous results to the study of microlocal properties of fully non linear equations, by means of the linearization techniques introduced by M. Beals and M.C. Reeds in [1] and well described in [11], [3]. Namely, consider the non linear equation

$$F(x, \partial^{\alpha} u)_{\alpha \in \mathcal{I}} = 0, \tag{1.1}$$

where $F(x,\zeta) \in C^{\infty}(\mathbb{R}^n \times \mathbb{C}^N)$ for suitable positive integer N, and \mathcal{I} is a bounded subset of multi-indices in \mathbb{Z}^n_+ . After the linearization obtained by differentiating with respect to the x_j variable:

$$\sum_{\alpha \in \mathcal{I}} \frac{\partial F}{\partial \zeta^{\alpha}} (x, \partial^{\beta} u)_{\beta \in \mathcal{I}} \partial^{\alpha} \partial_{x_{j}} u = -\frac{\partial F}{\partial_{x_{j}}} (x, \partial^{\beta} u)_{\beta \in \mathcal{I}}, \tag{1.2}$$

we reduce the study of (1.1) to the following linear equation

$$\sum_{\alpha \in \mathcal{I}} a_{\alpha}(x) \partial^{\alpha} u_j = f_j(x), \quad u_j = \partial_{x_j} u, \tag{1.3}$$

where the coefficients $a_{\alpha}(x)$ and the forcing term $f_j(x)$ are clearly non smooth, but their regularity depends on u itself. Precisely here we are considering the regularity of solutions to (1.1) in the framework of quasi-homogeneous Besov spaces $B_{\infty,\infty}^{s,M}$, which are introduced in Section 2, by means of a suitable decomposition of \mathbb{R}^n in anisotropic dyadic crowns.

In §3 pseudodifferential operators and symbol classes are defined and in §4 we introduce the microlocal properties of Besov type $B_{\infty,\infty}^{s,M}$ for pseudodifferential operators with smooth symbols, obtained in [8]. Such results apply in §5 to the study of the microlocal regularity for solution to equations of type (1.3), with coefficients of Besov type and, in the last section, to quasi-linear and fully non linear equations.

2. Quasi-homogeneous Besov spaces

In the following $M=(m_1,\ldots,m_n)$ is a weight vector with positive integer components, such that $\min_{1\leq j\leq n}m_j=1$ and

$$|\xi|_M := \left(\sum_{j=1}^n \xi_j^{2m_j}\right)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^n$$
 (2.1)

is called quasi-homogeneous weight function on \mathbb{R}^n .

We set $m^* := \max_{1 \le j \le n} m_j$, $\frac{1}{M} := \left(\frac{1}{m_1}, \dots, \frac{1}{m_n}\right)$, $\alpha \cdot \frac{1}{M} = \sum_{j=1}^n \frac{\alpha_j}{m_j}$ and $\langle \xi \rangle_M^2 := (1 + |\xi|_M^2)$. Clearly the usual euclidean norm $|\xi|$ corresponds to the quasi-homogeneous weight in the case $M = (1, \dots, 1)$. By easy computations, see e.g. [6] we obtain the following

Proposition 2.1. For any weight vector M there exists a suitable positive constant C such that

i)
$$\frac{1}{C}\langle \xi \rangle \le \langle \xi \rangle_M \le C \langle \xi \rangle^{m^*}, \quad \xi \in \mathbb{R}^n,$$

ii)
$$|\xi + \eta|_M \le C(|\xi|_M + |\eta|_M), \quad \xi, \eta \in \mathbb{R}^n;$$

- iii) (quasi-homogeneity) for any t > 0, $|t^{1/M}\xi|_M = t|\xi|_M$, where $t^{1/M}\xi = (t^{1/m_1}\xi_1, \dots, t^{1/m_n}\xi_n)$;
- iv) $\xi^{\gamma} \partial^{\alpha+\gamma} |\xi|_M \leq C_{\alpha,\gamma} \langle \xi \rangle_M^{1-\alpha \cdot \frac{1}{M}}$, for any $\alpha, \gamma \in \mathbb{Z}_+^n$ and $\xi \neq 0$.

For t > 0, $h \ge -1$ integer, we introduce the notations: $t^{\frac{h}{|M|}} = t^{\frac{h}{m_1}} \dots t^{\frac{h}{m_n}}$ and $t^{\frac{h}{M}} \xi = \left(t^{\frac{h}{m_1}} \xi_1, \dots, t^{\frac{h}{m_n}} \xi_n\right)$.

In the following $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int e^{-ix\cdot\xi}u(x)\,dx$ stands for both the Fourier transform of $u\in\mathcal{S}(\mathbb{R}^n)$ and its extension to $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.2. Consider $u \in L^{\infty}(\mathbb{R}^n)$, R > 0, such that $\operatorname{supp} \hat{u} \subset B_R^M := \{\xi \in \mathbb{R}^n \; ; \; |\xi|_M \leq R\}$. Then for any $\alpha \in \mathbb{Z}_+^n$ there exists $c_{\alpha} > 0$ independent of R such that

$$\|\partial^{\alpha} u\|_{L^{\infty}} \le c_{\alpha} R^{\alpha \cdot \frac{1}{M}} \|u\|_{L^{\infty}}. \tag{2.2}$$

Proof. Consider $\phi \in C^{\infty}(\mathbb{R}^n)$ such that $\operatorname{supp} \phi \subset B_2^M$, $\phi(x) = 1$ in B_1^M and set $\phi_R(\xi) = \phi\left(R^{-\frac{1}{M}}\xi\right)$. Since $\phi_R(\xi) = 1$ in B_R^M , we obtain $\hat{u}(\xi) = \phi_R(\xi)\hat{u}(\xi)$. Thus

$$u = \mathcal{F}^{-1}(\phi_R \hat{u}) = \mathcal{F}^{-1}\phi_R * u =$$

$$= (2\pi)^{-n} R^{\frac{1}{|M|}} \left(\int e^{i\left(R^{\frac{1}{M}}\cdot\right)\cdot\eta} \phi(\eta) \, d\eta * u \right)$$

$$= (2\pi)^{-n} R^{\frac{1}{|M|}} \left(\hat{\phi}\left(-R^{\frac{1}{M}}\cdot\right) * u \right) \in C^{\infty}(\mathbb{R}^n),$$

$$(2.3)$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform. Moreover

$$\partial^{\alpha} u = (2\pi)^{-n} (-1)^{|\alpha|} R^{\frac{1}{|M|}} R^{\alpha \cdot \frac{1}{M}} (\partial^{\alpha} \hat{\phi}) \left(-R^{\frac{1}{M}} \cdot \right) * u. \tag{2.4}$$

Then

$$\|\partial^{\alpha}u\|_{L^{\infty}} \leq (2\pi)^{-n} R^{\frac{1}{|M|}} \|(\partial^{\alpha}\hat{\phi})\left(-R^{\frac{1}{M}}\cdot\right)\|_{L^{1}} R^{\alpha\cdot\frac{1}{M}} \|u\|_{L^{\infty}}$$

$$= (2\pi)^{-n} R^{\frac{1}{|M|}} \int \left|\left(\partial^{\alpha}\hat{\phi}\right)\left(-R^{\frac{1}{M}}\xi\right)\right| d\xi R^{\alpha\cdot\frac{1}{M}} \|u\|_{L^{\infty}}$$

$$= (2\pi)^{-n} \int \left|\partial^{\alpha}\hat{\phi}(\eta)\right| d\eta R^{\alpha\cdot\frac{1}{M}} \|u\|_{L^{\infty}} = c_{\alpha} R^{\alpha\cdot\frac{1}{M}} \|u\|_{L^{\infty}}.$$

$$(2.5)$$

Proposition 2.3 (Quasi-homogeneous dyadic decomposition). For some K>1 let us consider the cut-off function $\phi(t)\in C_0^\infty\left([0,+\infty]\right)$ such that $0\leq \phi(t)\leq 1$, $\phi(t)=1$ for $0\leq t\leq \frac{1}{2K}$, $\phi(t)=0$, when t>K. Set now $\varphi_0(\xi)=\phi\left(\left|2^{-1/M}\xi\right|_M\right)-\phi(\left|\xi\right|_M)$ and

$$\varphi_{-1}(\xi) = \phi(|\xi|_M), \quad \varphi_h(\xi) = \varphi_0\left(2^{-h/M}\xi\right) \text{ for } h = 0, 1, \dots$$
 (2.6)

Then for any $\alpha, \gamma \in \mathbb{Z}_+^n$ a positive constant $C_{\alpha,\gamma,K}$ exists such that:

$$supp \, \varphi_{-1} \subset C_{-1}^{K,M} := B_K^M$$

$$supp \, \varphi_h \subset C_h^{K,M} := \left\{ \xi \in \mathbb{R}^n \; ; \; \frac{1}{K} 2^{h-1} \le |\xi|_M \le K 2^{h+1} \right\}, \; h \ge 0;$$

$$(2.7)$$

$$\sum_{h=-1}^{\infty} \varphi_h(\xi) = 1, \text{ for all } \xi \in \mathbb{R}^n;$$
(2.8)

$$\left| \xi^{\gamma} \partial^{\alpha + \gamma} \varphi_h(\xi) \right| \le C_{\alpha, \gamma, K} 2^{-\left(\alpha \cdot \frac{1}{M}\right)h}, \quad \xi \in \mathbb{R}^n, \ h = -1, 0, \dots$$
 (2.9)

Moreover for any fixed $\xi \in \mathbb{R}^n$ the sum in (2.8) reduces to a finite number of terms, independent of the choice of ξ itself. Setting now for every $u \in \mathcal{S}'(\mathbb{R}^n)$

$$u_h = \varphi_h(D)u := \mathcal{F}^{-1}(\varphi_h \hat{u}), \qquad (2.10)$$

we obtain:

$$\sum_{h=-1}^{\infty} u_h = u, \quad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n); \tag{2.11}$$

and, for every integer $k \geq 0$ there exists $C_k > 0$ such that

$$\frac{1}{C_k} 2^{hk} \|u_h\|_{L^{\infty}} \le \sum_{\alpha \cdot \frac{1}{M} = k} \|\partial^{\alpha} u_h\|_{L^{\infty}} \le C_k 2^{hk} \|u_h\|_{L^{\infty}}, \quad h = 0, 1 \dots$$
 (2.12)

Proof. It is trivial to prove (2.7). For every fixed $\xi \in \mathbb{R}^n$ we have $\phi\left(\left|2^{-h/M}\xi\right|_M\right) = 1$ for any suitably large integer h; then (2.8), (2.11) follow. For every integer $h \geq 0$ we obtain

$$\left|\xi^{\gamma}\partial^{\alpha+\gamma}\varphi_{h}(\xi)\right| = \left|\left(2^{-h/M}\xi\right)^{\gamma}\left(\partial^{\alpha+\gamma}\varphi_{0}\right)\left(2^{-h/M}\xi\right)\right|2^{-h\left(\alpha\cdot\frac{1}{M}\right)} \leq C_{\alpha,\gamma,K}2^{-h\left(\alpha\cdot\frac{1}{M}\right)},$$

where $C_{\alpha,\gamma,K} = \max_{n} |\eta^{\gamma} \partial^{\alpha+\gamma} \varphi_0(\eta)|$ is independent of h; thus (2.9) is proved.

In order to prove at the end (2.12), let us consider $\chi(\xi) \in C_0^{\infty}(\mathbb{R}^n)$ identically equal to one in a suitable neighborhood of supp φ_0 . We can then write

$$\varphi_0(\xi) = \left(\sum_{\alpha \cdot \frac{1}{M} = k} \xi^{\alpha} \chi_{\alpha}(\xi)\right) \varphi_0(\xi), \tag{2.13}$$

with

$$\chi_{\alpha}(\xi) = \frac{\xi^{\alpha} \chi(\xi)}{\sum_{\alpha \cdot \frac{1}{M} = k} (\xi^{\alpha})^2} \in C_0^{\infty}(\mathbb{R}^n).$$
 (2.14)

Thus we have:

$$\hat{u}_{h}(\xi) = \varphi_{0} \left(2^{-\frac{h}{M}} \xi \right) \hat{u}(\xi) = \sum_{\alpha \cdot \frac{1}{M} = k} \left(2^{-\frac{h}{M}} \xi \right)^{\alpha} \chi_{\alpha} \left(2^{-\frac{h}{M}} \xi \right) \hat{u}(\xi)$$

$$= \sum_{\alpha \cdot \frac{1}{M} = k} 2^{-h\alpha \cdot \frac{1}{M}} \xi^{\alpha} \chi_{\alpha} \left(2^{-\frac{h}{M}} \xi \right) \hat{u}_{h}(\xi)$$

$$= 2^{-hk} \sum_{\alpha \cdot \frac{1}{M} = k} \chi_{\alpha} \left(2^{-\frac{h}{M}} \xi \right) \widehat{D^{\alpha} u_{h}}(\xi).$$

$$(2.15)$$

We have then verified.

$$2^{hk}u_h = \sum_{\alpha \cdot \frac{1}{M} = k} 2^{\frac{h}{|M|}} \left(\left(\mathcal{F}^{-1} \chi_{\alpha} \right) \left(2^{\frac{h}{M}} \cdot \right) \right) * D^{\alpha} u_h, \tag{2.16}$$

which in view of the Young inequality and Proposition 2.2 shows (2.12).

We call the sequences $\varphi := \{\varphi_h\}_{h=-1}^{\infty}$, defined in (2.6), and $\{u_h\}_{h=-1}^{\infty}$, defined in (2.10), respectively quasi-homogeneous partition of unity and quasi-homogeneous dyadic decomposition of u.

Following the arguments in [12, §10.1] we can introduce now the classes of quasi-homogeneous Besov functions and state their properties in suitable way.

Definition 2.4. For any $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ we say that u belongs to the quasi-homogeneous Besov space $B^{s,M}_{\infty,\infty}$ if

$$||u||_{B_{\infty,\infty}^{s,M}}^{\varphi} := \sup_{h=-1,\dots} 2^{sh} ||u_h||_{L^{\infty}} < \infty$$
 (2.17)

is satisfied for some quasi-homogeneous partition of unity φ .

Different choices of the partition of unity φ in (2.17) give raise to equivalent norms, noted by $\|\cdot\|_{B^{s,M}_{\infty,\infty}}$. The space $B^{s,M}_{\infty,\infty}$ has Banach structure and when $M=(1,\ldots,1)$ and s>0, it is the usual Hölder-Zygmund space.

Proposition 2.5. Let us consider a sequence of Schwartz distributions $\{u_h\}_{h=-1}^{\infty} \subset \mathcal{S}'(\mathbb{R}^n)$ and a constant K > 1 such that supp $\hat{u}_h \subset C_h^{K,M}$ for any $h \geq -1$. Set now $u := \sum_{h=-1}^{\infty} u_h$.

The following properties are satisfied:

$$\sup_{h \geq -1} \left(2^{rh} \|u_h\|_{L^{\infty}} \right) < \infty \Rightarrow u \in B^{r,M}_{\infty,\infty}, \quad r \in \mathbb{R},$$
and
$$\|u\|_{B^{r,M}_{\infty,\infty}} \leq C \sup_{h \geq -1} \left(2^{rh} \|u_h\|_{L^{\infty}} \right),$$
(2.18)

where the constant C is independent of the sequence $\{u_h\}_{h=-1}^{\infty}$. When r > 0, (2.18) is true for all the sequences of Schwartz distributions $\{u_h\}_{h=-1}^{\infty}$ with supp $\hat{u}_h \subseteq B_h^{K,M} := B_{K2^{h+1}}^M = \{\xi \in \mathbb{R}^n : |\xi|_M \le K2^{h+1}\}, h = -1, 0, \ldots$ **Proposition 2.6 (Quasi-homogeneous Meyer multipliers).** Consider a family of smooth functions $\{m_h\}_{h=-1}^{\infty}$ such that for any $\alpha \in \mathbb{Z}_+^n$:

$$\|\partial^{\alpha} m_h\|_{L^{\infty}} \le C_{\alpha} 2^{h \alpha \cdot \frac{1}{M}}. \tag{2.19}$$

Then the linear operator $L = \sum_{h=-1}^{\infty} m_h(x) \varphi_h(D)$ maps continuously $B_{\infty,\infty}^{s,M}$ into itself, for any s > 0.

Proof. Consider the quasi-homogeneous partition of unity in Proposition 2.3, with K=1, and for any $h=-1,0,\ldots$ and T>2 write:

$$\hat{m}_h = \sum_{k=-1}^{\infty} \varphi_k \left(\left(2^h T \right)^{-\frac{1}{M}} \cdot \right) \hat{m}_h = \sum_{k=-1}^{\infty} \hat{m}_{h,k} \,. \tag{2.20}$$

Notice that $\hat{m}_{h,-1}(\xi) = \phi\left(\left|\left(2^hT\right)^{-\frac{1}{M}}\xi\right|_M\right)\hat{m}_h(\xi)$, and when $h \geq 0$, $\hat{m}_{h,k}(\xi) = \varphi_0\left(\left(2^{h+k}T\right)^{-\frac{1}{M}}\xi\right)\hat{m}_h(\xi)$.

Thus for any $u \in \mathcal{S}'(\mathbb{R}^n)$, by setting $M_k u = \sum_{h=-1}^{\infty} m_{h,k} u_h$ for $k \geq -1$, we have:

$$Lu = \sum_{h=-1}^{\infty} m_h \varphi_h(D) u = \sum_{h=-1}^{\infty} m_h u_h = \sum_{h=-1}^{\infty} m_{h,k} u_h = \sum_{k=-1}^{\infty} M_k u.$$
 (2.21)

Notice now that for any $h, k \ge -1$:

 $||m_{h,k}u_h||_{L^{\infty}} \leq ||m_{h,k}||_{L^{\infty}} ||u_k||_{L^{\infty}};$

$$\operatorname{supp} \widehat{m_{h,-1}u_h} \subset B_h^{T,M} + C_h^{1,M} \subset B_h^{K,M}; \tag{2.22}$$

 $\operatorname{supp} \widehat{m_{h,k}u_h} \subset C_{h+k}^{T,M} + C_h^{1,M} \subset C_{h+k}^{K,M}, \quad \text{ for suitable constants } T, K.$

Using now (2.12) and (2.19), for any integer l > 0 there exist positive constants $C_l > 0$ such that

$$||m_{h,k}||_{L^{\infty}} \le C_l \sum_{\alpha \cdot \frac{1}{M} = l} ||\partial^{\alpha} m_{h,k}||_{L^{\infty}} 2^{-(h+k)l} \le C_l 2^{-kl}.$$
 (2.23)

Thus for any s > 0:

$$2^{s(h+k)} \| m_{h,k} u_h \|_{L^{\infty}} \leq 2^{s(h+k)} \| m_{h,k} \|_{L^{\infty}} \| u_h \|_{L^{\infty}}$$

$$\leq C_l 2^{sh} 2^{(s-l)k} \| u_h \|_{L^{\infty}} \leq C_l 2^{(s-l)k} \| u \|_{B^{s,M}_{\infty,\infty}}.$$

$$(2.24)$$

Thus for any s > 0, $l \ge 1$ and $k \ge -1$, in view of Proposition 2.5, we get

$$||M_k u||_{B^{s,M}_{\infty,\infty}} \le C_l 2^{(s-l)k} ||u||_{B^{s,M}_{\infty,\infty}}.$$
 (2.25)

Then, by choosing l > s, in view of (2.21) and (2.25) we conclude that $||Lu||_{B^{s,M}_{\infty,\infty}} \le C_s ||u||_{B^{s,M}_{\infty,\infty}}$.

Theorem 2.7. Consider $F \in C^{\infty}(\mathbb{C})$ such that F(0) = 0, s > 0. Then, for any $u \in B^{s,M}_{\infty,\infty}$ and suitable $C = C(F, ||u||_{L^{\infty}})$, we have:

$$F(u) \in B^{s,M}_{\infty,\infty} \quad and \quad \|F(u)\|_{B^{s,M}_{\infty,\infty}} \leq C \|u\|_{B^{s,M}_{\infty,\infty}}. \tag{2.26}$$

Proof. Using the notations in Proposition 2.3 let us define for any integer $p \ge 0$, $\Psi_p u = \Psi_p(D) u$, where $\Psi_p(\xi) = \sum_{-1 \le h \le p-1} \varphi_h(\xi)$. Since F(0) = 0 setting moreover

 $\Psi_{-1}(\xi) = 0$ we can consider the telescopic expansion:

$$F(\Psi_0 u) + \sum_{p=0}^{\infty} \left(F(\Psi_{p+1} u) - F(\Psi_p u) \right) = \sum_{p=-1}^{\infty} \left(F(\Psi_{p+1} u) - F(\Psi_p u) \right) . \tag{2.27}$$

By means of standard computations we have for any $p \geq 0$

$$F(\Psi_{p+1}u) - F(\Psi_p u) = u_p \int_0^1 F'(\Psi_p u + t u_p) dt.$$
 (2.28)

Thus by setting

$$m_p(x) = \int_0^1 F'(\Psi_p u + t u_p) dt,$$
 (2.29)

we obtain $F(u) = \sum_{p=-1}^{\infty} m_p u_p = Lu$. It is now sufficient to verify that m_p defined in (2.29) is a Meyer multiplier.

Without any loss of generality, it is enough to consider $\tilde{m}_p = G(\Psi_p u)$, with $G = F' \in C^{\infty}$. Then

$$\partial^{\alpha} G\left(\Psi_{p} u\right) = \sum G^{(q)}\left(\Psi_{p} u\right) \left(\partial^{\gamma_{1}} \Psi_{p} u\right) \dots \left(\partial^{\gamma_{q}} \Psi_{p} u\right),$$

where $1 \le q \le |\alpha|$, $\gamma_1 + \dots + \gamma_q = \alpha$ and $|\gamma_j| \ge 1$, $j = 1, \dots, q$. It follows from the Proposition 2.2 that for any multi-index γ_j :

$$\|\partial^{\gamma}\Psi_{p}u\|_{L^{\infty}} \leq C2^{p(\gamma_{j}\cdot\frac{1}{M})}\|\Psi_{p}u\|_{L^{\infty}}.$$

Then for a suitable positive constant C depending on α , G and $||u||_{L^{\infty}}$:

$$\|\partial^{\alpha} G(\psi_{p} u)\|_{L^{\infty}} \leq C 2^{p\left(\gamma_{1} \cdot \frac{1}{M} + \dots \gamma_{q} \cdot \frac{1}{M}\right)} \leq C 2^{p\left(\alpha \cdot \frac{1}{M}\right)},$$

which ends the proof.

Remark 2.8. Set $\tilde{F}(t) = F(t) - F(0)$, with $F \in C^{\infty}(\mathbb{C})$. Since the constant functions belong to $B^{s,M}_{\infty,\infty}$, we obtain that for any $u \in B^{s,m}_{\infty,\infty}$, F(u) fulfills (2.26), for any s > 0.

3. Quasi-homogeneous symbols

In this section, we recall the definition of some symbol classes which are well behaved on the quasi-homogeneous structure of the spaces $B_{\infty,\infty}^{s,M}$. Here we just collect some basic definitions and a few related results, referring the reader to [6, 8] for a more detailed analysis. Let $M = (m_1, \ldots, m_n)$ be a vector with positive integer components obeying the assumptions of the previous §2.

Definition 3.1. For $m \in \mathbb{R}$ and $\delta \in [0,1]$, $S_{M,\delta}^m$ will be the class of functions $a(x,\xi) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all $\alpha,\beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha,\beta} > 0$ such that:

$$|\partial_x^{\beta} \partial_{\xi}^{\alpha} a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle_M^{m-\alpha \cdot 1/M + \delta\beta \cdot 1/M}, \quad \forall x, \, \xi \in \mathbb{R}^n \,. \tag{3.1}$$

We also set $S_M^m := S_{M,0}^m$.

For each symbol $a \in S_{M,\delta}^m$, the pseudodifferential operator $a(x,D) = \operatorname{Op}(a)$ is defined on $\mathcal{S}(\mathbb{R}^n)$ by the usual quantization

$$a(x,D)u = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$
 (3.2)

It is well-known that (3.2) defines a linear bounded operator from $\mathcal{S}(\mathbb{R}^n)$ to itself. In the following, we will denote by $\operatorname{Op} S^m_{M,\delta}$ the set of pseudodifferential operators with symbol in $S_{M,\delta}^m$ (and set $\operatorname{Op} S_M^m := \operatorname{Op} S_{M,0}^m$ according to Definition 3.1). From Proposition 2.1, iv), it is clear that $\langle \xi \rangle_M^m \in S_M^m$, for every $m \in \mathbb{R}$.

For pseudodifferential operators in $\operatorname{Op} S^m_{M,\delta}$, a suitable symbolic calculus is developed in [8, Propositions 2.3-2.5] under the restriction $\delta < \frac{1}{m^*}$; in particular the composition a(x,D)b(x,D) of two operators $a(x,D) \in \operatorname{Op} S^m_{M,\delta}, b(x,D) \in \operatorname{Op} S^{m'}_{M,\delta}$ belongs to Op $S_{M,\delta}^{m+m'}$, for all $m,m' \in \mathbb{R}$ as long as $\delta < \frac{1}{m^*}$.

The analysis of linear partial differential equations with rough coefficients needs the introduction of non smooth symbols studied in [8]. We recall the definitions and the main properties.

Definition 3.2. For r > 0, $m \in \mathbb{R}$ and $\delta \in [0,1]$, $B_{\infty,\infty}^{r,M} S_{M,\delta}^m$ is the set of measurable functions $a(x,\xi)$ such that for every $\alpha \in \mathbb{Z}_+^n$

$$|\partial_{\xi}^{\alpha} a(x,\xi)| \le C_{\alpha} \langle \xi \rangle_{M}^{m-\alpha \cdot 1/M}, \quad \forall x, \, \xi \in \mathbb{R}^{n};$$
 (3.3)

$$|\partial_{\xi}^{\alpha} a(x,\xi)| \leq C_{\alpha} \langle \xi \rangle_{M}^{m-\alpha \cdot 1/M}, \quad \forall x, \, \xi \in \mathbb{R}^{n};$$

$$\|\partial_{\xi}^{\alpha} a(\cdot,\xi)\|_{B_{\infty,\infty}^{r,M}} \leq C_{\alpha} \langle \xi \rangle_{M}^{m-\alpha \cdot 1/M+\delta r}, \quad \forall \xi \in \mathbb{R}^{n}.$$

$$(3.3)$$

As in the case of smooth symbols, we set for brevity $B_{\infty,\infty}^{r,M} S_M^m := B_{\infty,\infty}^{r,M} S_{M,0}^m$.

Theorem 3.3. If r > 0, $m \in \mathbb{R}$, $\delta \in [0,1]$ and $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_{M,\delta}$, then for all $s \in](\delta - 1)r, r[$

$$a(x,D): B^{s+m,M}_{\infty,\infty} \to B^{s,M}_{\infty,\infty}$$
 (3.5)

is a linear continuous operator.

If in addition $\delta < 1$, then the mapping property (3.5) is still true for s = r.

Since the inclusion $S_{M,\delta}^m \subset B_{M,\infty}^{r,M} S_{M,\delta}^m$ is true for all r > 0, a straightforward consequence of Theorem 3.3 is the following

Corollary 3.4. If $a \in S_{M,\delta}^m$, for $m \in \mathbb{R}$ and $\delta \in [0,1[$, then (3.5) is true for all $s \in \mathbb{R}$. If $\delta = 1$, (3.5) is true for all s > 0.

4. Microlocal properties

In this section we review some known microlocal tools and properties concerning the pseudodifferential operators introduced above. For the proofs of the results collected below, the reader is addressed to [8]. In the sequel, we will set $T^{\circ}\mathbb{R}^{n} := \mathbb{R}^{n} \times (\mathbb{R}^{n} \setminus \{0\})$, and $M = (m_{1}, \ldots, m_{n})$ will be a vector under the assumptions of $\S 2$.

Definition 4.1. We say that a set $\Gamma_M \subseteq \mathbb{R}^n \setminus \{0\}$ is M-conic, if

$$\xi \in \Gamma_M \implies t^{1/M} \xi \in \Gamma_M , \forall t > 0.$$

Definition 4.2. A symbol $a \in S_{M,\delta}^m$ is microlocally M-elliptic at $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$ if there exist an open neighborhood U of x_0 and an M-conic open neighborhood Γ_M of ξ_0 such that for $c_0 > 0$, $\rho_0 > 0$:

$$|a(x,\xi)| \ge c_0 \langle \xi \rangle_M^m, \quad \forall (x,\xi) \in U \times \Gamma_M, \quad |\xi|_M > \rho_0.$$
 (4.1)

Moreover the characteristic set of $a \in S_{M,\delta}^m$ is $\operatorname{Char}(a) \subset T^{\circ}\mathbb{R}^n$ defined by

$$(x_0, \xi_0) \in T^{\circ} \mathbb{R}^n \setminus \text{Char}(a) \iff a \text{ is microlocally } M\text{-elliptic at } (x_0, \xi_0).$$
 (4.2)

Definition 4.3. We say that $a \in \mathcal{S}'(\mathbb{R}^n)$ is microlocally regularizing on $U \times \Gamma_M$ if $a_{|U \times \Gamma_M} \in C^{\infty}(U \times \Gamma_M)$ and for every m > 0 and all $\alpha, \beta \in \mathbb{Z}_+^n$ a positive constant $C_{m,\alpha,\beta} > 0$ exists in such a way that:

$$|\partial_{\xi}^{\alpha} \partial_{x}^{\beta} a(x,\xi)| \le C_{m,\alpha,\beta} (1+|\xi|)^{-m} \,, \quad \forall (x,\xi) \in U \times \Gamma_{M} \,. \tag{4.3}$$

Proposition 4.4. (Microlocal parametrix). Assume that $0 \le \delta < 1/m^*$. Then $a \in S^m_{M,\delta}$ is microlocally M-elliptic at $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$ if and only if there exist symbols $b,c \in S^{-m}_{M,\delta}$ such that

$$a(x, D)b(x, D) = I + r(x, D)$$
 and $c(x, D)a(x, D) = I + l(x, D)$, (4.4)

being I the identity operator and the symbols $r(x, \xi)$, $l(x, \xi)$ microlocally regularizing at (x_0, ξ_0) .

Definition 4.5. For $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$, $s \in \mathbb{R}$, we define $mclB^{s,M}_{\infty,\infty}(x_0, \xi_0)$ as the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that:

$$\psi(D)(\phi u) \in B^{s,M}_{\infty,\infty}, \tag{4.5}$$

where $\phi \in C_0^{\infty}(\mathbb{R}^n)$ is identically one in a neighborhood of $x_0, \psi(\xi) \in S_M^0$ is a symbol identically one on $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$, for $0 < \varepsilon_0 < |\xi_0|_M$, and finally $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ is an M-conic neighborhood of ξ_0 .

Under the same assumptions, we also write

$$(x_0,\xi_0) \notin WF_{B^{s,M}_{\infty,\infty}}(u)$$
.

The set $WF_{B^{s,M}_{\infty,\infty}}(u) \subset T^{\circ}\mathbb{R}^n$ is called the $B^{s,M}_{\infty,\infty}$ -wave front set of u.

We say that a distribution satisfying the previous definition is microlocally in $B_{\infty,\infty}^{s,M}$ at (x_0,ξ_0) . Moreover the closed set $WF_{B_{\infty,\infty}^{s,M}}(u)$ is M-conic in the ξ variable.

Finally we say that $x_0 \notin B^{s,M}_{\infty,\infty}$ – singsupp (u) if and only if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi \equiv 1$ in some open neighborhood of x_0 , such that $\phi u \in B^{s,M}_{\infty,\infty}$.

Proposition 4.6. If $u \in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$, with $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$, then for any $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, such that $\varphi(x_0) \neq 0$, $\varphi u \in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$.

Proposition 4.7. Let π_1 be the canonical projection of $T^{\circ}\mathbb{R}^n$ onto \mathbb{R}^n , $\pi_1(x,\xi) = x$. For every $u \in \mathcal{S}'(\mathbb{R}^n)$ and $s \in \mathbb{R}$ we have:

$$B_{\infty,\infty}^{s,M} - \operatorname{singsupp}(u) = \pi_1(WF_{B_{\infty,\infty}^{s,M}}(u)).$$

Theorem 4.8. Let $a \in S_{M,\delta}^m$ for $\delta \in [0, 1/m^*[, m \in \mathbb{R} \text{ and } (x_0, \xi_0) \in T^{\circ}\mathbb{R}^n]$. Then for all $s \in \mathbb{R}$

$$u \in mcl B^{s+m,M}_{\infty,\infty}(x_0, \xi_0) \quad \Rightarrow \quad a(x, D)u \in mcl B^{s,M}_{\infty,\infty}(x_0, \xi_0).$$
 (4.6)

Theorem 4.9. Let $a \in S_{M,\delta}^m$, for $m \in \mathbb{R}$, $\delta \in [0,1/m^*[$, be microlocally M-elliptic at $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$. For $s \in \mathbb{R}$ assume that $u \in \mathcal{S}'(\mathbb{R}^n)$ fulfills $a(x,D)u \in mclB_{\infty,\infty}^{s,M}(x_0,\xi_0)$. Then $u \in mclB_{\infty,\infty}^{s+m,M}(x_0,\xi_0)$.

As a consequence of Theorems 4.8, 4.9, the following holds.

Corollary 4.10. For $a \in S_{M,\delta}^m$, $m \in \mathbb{R}$, $\delta \in [0, 1/m^*[$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, the inclusions

$$WF_{B^{s,M}_{\infty,\infty}}(a(x,D)u)\subset WF_{B^{s+m,M}_{\infty,\infty}}(u)\subset WF_{B^{s,M}_{\infty,\infty}}(a(x,D)u)\cup \operatorname{Char}(a) \qquad (4.7)$$
 hold true for every $s\in\mathbb{R}$.

5. Non regular symbols

In this section, the microlocal regularity results discussed in §4 are applied to obtain microlocal regularity results for a linear partial differential equation of quasi-homogeneous order $m \in \mathbb{N}$ of the form

$$A(x,D)u := \sum_{\alpha \cdot 1/M \le m} a_{\alpha}(x)D^{\alpha}u = f(x), \qquad (5.1)$$

where $D^{\alpha} := (-i)^{|\alpha|} \partial^{\alpha}$ and the coefficients a_{α} belong to the Besov space $B_{\infty,\infty}^{r,M}$ of positive order r. It is clear that $A(x,\xi) = \sum_{\alpha \cdot 1/M \leq m} a_{\alpha}(x) \xi^{\alpha} \in B_{\infty,\infty}^{r,M} S_{M}^{m}$. We assume that A(x,D) is microlocally M-elliptic at a given point $(x_{0},\xi_{0}) \in T^{\circ}\mathbb{R}^{n}$; according to Definition 4.2 and the quasi-homogeneity of the norm $|\xi|_{M}$, this means that there exist an open neighborhood U of x_{0} and an open M-conic neighborhood Γ_{M} of ξ_{0} such that the M-principal symbol of A(x,D) satisfies

$$A_m(x,\xi) = \sum_{\alpha \cdot 1/M = m} a_{\alpha}(x)\xi^{\alpha} \neq 0, \quad \text{for } (x,\xi) \in U \times \Gamma_M.$$
 (5.2)

The forcing term f is assumed to be in some $B^{s,M}_{\infty,\infty}$, with a suitable order of smoothness s, microlocally at (x_0, ξ_0) (cf. Definition 4.5).

Theorem 5.1. Let A(x,D)u = f be a linear partial differential equation, as in (5.1), with coefficients in the space $B^{r,M}_{\infty,\infty}$ of positive order r. Assume that A(x,D) is microlocally M-elliptic at $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$. If $f \in mclB^{s-m,M}_{\infty,\infty}(x_0,\xi_0)$ and $u \in B^{s-\delta r,M}_{\infty,\infty}$, for $0 < \delta < 1/m^*$ and $(\delta - 1)r + m < s \le r + m$, then $u \in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$.

Remark 5.2. Assuming in (5.1) A(x,D) with coefficients in $B^{r,M}_{\infty,\infty}$, r>0, u a priori in $B^{s-\delta r,M}_{\infty,\infty}$ for $(\delta-1)r+m< s\leq r+m$, $\delta\in]0,1/m^*[$, we obtain

$$WF_{B^{s,M}_{\infty,\infty}}(u) \subset WF_{B^{s-m,M}_{\infty,\infty}}(A(x,D)u) \cup \operatorname{Char}(A)$$
.

Following [11], [7], non smooth symbols in $B^{r,M}_{\infty,\infty}S^m_M$ can be decomposed, for a given $\delta \in]0,1]$, into the sum of a smooth symbol in $S^m_{M,\delta}$ and a non smooth symbol of lower order. Namely, let ϕ be a fixed C^∞ function such that $\phi(\xi)=1$ for $\langle \xi \rangle_M \leq 1$ and $\phi(\xi)=0$ for $\langle \xi \rangle_M > 2$. For given $\varepsilon > 0$ we set $\phi(\varepsilon^{1/M}\xi):=\phi(\varepsilon^{1/m_1}\xi_1,\ldots,\varepsilon^{1/m_n}\xi_n)$.

Any symbol $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_M$ can be split in

$$a(x,\xi) = a^{\#}(x,\xi) + a^{\natural}(x,\xi),$$
 (5.3)

where for some $\delta \in]0,1]$

$$a^{\#}(x,\xi) := \sum_{h=-1}^{\infty} \phi(2^{-h\delta/M}D_x)a(x,\xi)\varphi_h(\xi).$$

One can prove the following proposition (see [7, Proposition 3.9] and [11] for the proof):

Proposition 5.3. If $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_M$, with r > 0, $m \in \mathbb{R}$, and $\delta \in]0,1]$, then $a^\#(x,\xi) \in S^m_{M,\delta}$ and $a^{\natural}(x,\xi) \in B^{r,M}_{\infty,\infty}S^{m-r\delta}_{M,\delta}$.

Proposition 5.4. Assume that $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_M$, $m \in \mathbb{R}$, is microlocally M-elliptic at $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$, then for any $\delta \in]0,1]$, $a^{\#}(x,\xi) \in S^m_{M,\delta}$ is still microlocally M-elliptic at (x_0,ξ_0) .

Proof. The microlocal M-ellipticity of $a(x,\xi)$ yields the existence of positive constants c_1, ρ_1 such that

$$|a(x,\xi)| \ge c_1 \langle \xi \rangle_M^m$$
, when $(x,\xi) \in U \times \Gamma_M$ and $|\xi|_M > \rho_1$, (5.4)

where U is a suitable open neighborhood of x_0 and Γ_M an open M-conic neighborhood of ξ_0 . On the other hand, for any $\rho_0 > 0$ we can find a positive integer h_0 , which increases together with ρ_0 , such that $\varphi_h(\xi) = 0$ as long as $|\xi|_M > \rho_0$ and $h = -1, \ldots, h_0 - 1$. We can then write:

$$a^{\#}(x,\xi) = \sum_{h=h_0}^{\infty} \phi\left(2^{-h\delta/M} D_x\right) a(x,\xi) \varphi_h(\xi), \quad |\xi|_M > \rho_0.$$
 (5.5)

Set for brevity $\phi\left(2^{-h\delta/M}\cdot\right) = \phi_h(\cdot)$.

By means of (5.5), the Cauchy-Schwarz inequality and [7, Lemma 3.8], when $|\xi|_M > \rho_0$ we can estimate

$$\begin{split} &|a^{\#}(x,\xi) - a(x,\xi)|^2 \\ &= \left|\sum_{h=h_0}^{\infty} (\phi_h(D_x) - I) \, a(x,\xi) \varphi_h(\xi)\right|^2 \\ &= \sum_{h=h_0}^{\infty} \sum_{k=h-N_0}^{h+N_0} \left\langle (\phi_h(D_x)) - I \right\rangle \, a(x,\xi) \varphi_h(\xi), \\ &= \sum_{k=-N_0}^{N_0} \sum_{h=h_0}^{\infty} \left\langle (\phi_h(D_x) - I) \, a(x,\xi) \varphi_h(\xi), (\phi_{h+t}(D_x) - I) \, a(x,\xi) \varphi_{h+t}(\xi) \right\rangle \\ &\leq \sum_{t=-N_0}^{N_0} \sum_{h=h_0}^{\infty} \left\| (\phi_h(D_x)) - I \right\rangle \, a(\cdot,\xi) \|_{L^{\infty}} |\varphi_h(\xi)| \\ &\times \left\| (\phi_{h+t}(D_x) - I) \, a(\cdot,\xi) \right\|_{L^{\infty}} |\varphi_{h+t}(\xi)| \\ &\leq C^2 \sum_{t=-N_0}^{N_0} \sum_{h=h_0}^{\infty} 2^{-h\delta r} 2^{-(h+t)\delta r} \|a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}}^2 \\ &\leq C^2 \sum_{h=h_0}^{\infty} 2^{-2h\delta r} \|a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}}^2 \leq C^2 2^{-2h_0\delta r} \|a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}}^2, \end{split}$$

where C denotes different positive constants depending only on δ , N_0 and r. Since $\|a(\cdot,\xi)\|_{B^{r,M}_{\infty,\infty}} \leq c^* \langle \xi \rangle_M^m$, let us fix ρ_0 large enough to have $C2^{-h_0\delta r} < \frac{c_1}{2c^*}$ (with c_1 from (5.4)). Then for $(x,\xi) \in U \times \Gamma_M$ and $|\xi|_M > \max\{\rho_0,\rho_1\}$

$$|a^{\#}(x,\xi)| \ge |a(x,\xi)| - |a^{\#}(x,\xi) - a(x,\xi)| \ge \frac{c_1}{2} \langle \xi \rangle_M^m$$
 (5.6)

follows and the proof is concluded.

Proof of Theorem 5.1

Consider now the linear partial differential equation (5.1), with A(x, D) microlocally M-elliptic at (x_0, ξ_0) . For an arbitrarily fixed $\delta \in]0, 1/m^*[$, we split the symbol $A(x, \xi)$ as $A(x, \xi) = A^{\#}(x, \xi) + A^{\natural}(x, \xi)$, according to Proposition 5.3. In view of Propositions 5.4, 4.4 there exists a smooth symbol $B(x, \xi) \in S_{M,\delta}^{-m}$ such that

$$B(x, D)A^{\#}(x, D) = I + R(x, D)$$
,

where R(x, D) is microlocally regularizing at (x_0, ξ_0) .

Applying now B(x, D) to both sides of (5.1), on the left, we obtain:

$$u = B(x, D)f - R(x, D)u - B(x, D)A^{\dagger}(x, D)u.$$

$$(5.7)$$

Assume that $f \in mclB^{s-m,M}_{\infty,\infty}(x_0,\xi_0)$ and $u \in B^{s-\delta r,M}_{\infty,\infty}$ for $(\delta-1)r+m < s \le r+m$. Since $A^{\natural}(x,\xi) \in B^{r,M}_{\infty,\infty}S^{m-r\delta}_{M,\delta}$, one can apply Theorem 3.3 and Corollary 3.4 to find that $B(x,D)A^{\natural}(x,D)u \in B^{s,M}_{\infty,\infty}$; moreover Theorem 4.8 and Corollary 3.4 give $B(x,D)f \in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$ and $R(x,D)u \in B^{s,M}_{\infty,\infty}$. This shows the result of Theorem 5.1. By means of the argument stated above, we obtain the following general result for non regular pseudodifferential operators.

Corollary 5.5. For $a(x,\xi) \in B^{r,M}_{\infty,\infty}S^m_M$, r > 0, u belonging a priori to $B^{s-\delta r,M}_{\infty,\infty}$, for $(\delta - 1)r + m < s \le r + m$, $\delta \in]0,1/m^*[$, we have

$$WF_{B^{s,M}_{\infty,\infty}}(u) \subset WF_{B^{s-m,M}_{\infty,\infty}}(a(x,D)u) \cup \operatorname{Char}(a)$$
.

6. Some applications to non linear equations

In this section, we apply the previous results to the study of microlocal properties for a class of quasi-linear and fully non linear partial differential equation of weighted elliptic type.

For $M = (m_1, \ldots, m_n)$, satisfying the assumptions in §2, and a given positive integer m, let us first consider the quasi-linear equation of quasi-homogeneous type

$$\sum_{\alpha \cdot 1/M \le m} a_{\alpha}(x, D^{\beta}u)_{\beta \cdot 1/M \le m-1} D^{\alpha}u = f(x), \qquad (6.1)$$

where $a_{\alpha}(x,\zeta) \in C^{\infty}(\mathbb{R}^{n} \times \mathbb{C}^{N})$ are given functions of the vectors $x \in \mathbb{R}^{n}$, $\zeta = (\zeta_{\beta})_{\beta \cdot 1/M \leq m-1} \in \mathbb{C}^{N}$ and f(x) is a given forcing term. We assume that the equation (6.1) is microlocally M-elliptic at a given point $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$, $\sum_{\alpha \cdot 1/M \le m} a_{\alpha}(x,\zeta) \xi^{\alpha} \text{ of the }$ meaning that the M-principal symbol $A_m(x,\xi,\zeta) :=$

differential operator in the left-hand side of the equation satisfies

$$A_m(x,\xi,\zeta) \neq 0 \text{ for } (x,\xi) \in U \times \Gamma_M,$$
 (6.2)

where U is a suitable neighborhood of x_0 and Γ_M a suitable M-conic neighborhood of ξ_0 .

Under the previous assumptions, we may prove the following

Theorem 6.1. Consider r > 0, $0 < \delta < \frac{1}{m^*}$, $\sigma < s \le r + m$, where $\sigma = \sigma_{r,\delta,m} := \max\{(\delta-1)r + m, r + m - 1\}$. Let $u \in B^{r+m-1,M}_{\infty,\infty} \cap B^{s-\delta r,M}_{\infty,\infty}$ be a solution to the equation (6.1), microlocally $M-\text{elliptic at }(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$, with $f \in mclB^{s-m,M}_{\infty,\infty}(x_0,\xi_0)$. Then $u \in mclB^{s,M}_{\infty,\infty}(x_0,\xi_0)$.

Proof. In view of Theorems 3.3 and 2.7, from $u \in B_{\infty,\infty}^{r+m-1,M}$ it follows that

Then, since $u \in B^{s-\delta r,M}_{\infty,\infty}$, as long as $\beta \cdot 1/M \le m-1$, hence $a_{\alpha}(\cdot,D^{\beta}u)_{\beta \cdot 1/M} \in B^{r,M}_{\infty,\infty}$. Then, since $u \in B^{s-\delta r,M}_{\infty,\infty}$, $0 < \delta < \frac{1}{m^*}$ and $(\delta - 1)r + m < s \le r + m$, we can apply Theorem 5.1 to $A(x,\xi) := \sum_{\alpha \cdot 1/M \le m} a_{\alpha}(x,D^{\beta}u)_{\beta \cdot 1/M \le m-1} \xi^{\alpha} \in B^{r,M}_{\infty,\infty} S^m_M$, which is

microlocally M-elliptic at (x_0, ξ_0) because of (6.2). This shows the result.

We observe that if $r\delta \geq 1$, then $B^{r+m-1,M}_{\infty,\infty} \cap B^{s-\delta r,M}_{\infty,\infty} = B^{r+m-1,M}_{\infty,\infty}$, since $s-\delta r \leq 1$ $r+m-\delta r \leq r+m-1$. If $r>m^*$, we may always find $\delta^* \in]0,1/m^*[$ such that $r\delta^* \geq 1$ 1, the minimum admissible value being $\delta^* = \frac{1}{r}$. Then the microregularity result of Theorem 6.1 applies to an arbitrary solution $u \in B^{r+m-1,M}_{\infty,\infty}$ of the equation (6.1) with $\delta^* = \frac{1}{r}$ (note that $\sigma = r + m - 1$ when $r > m^*$). We can then state the following

Corollary 6.2. For $r > m^*$, $r + m - 1 < s \le r + m$, let $u \in B^{r+m-1,M}_{\infty,\infty}$ be a solution to the equation (6.1), microlocally M-elliptic at $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$, with $f \in mclB^{s-m,M}_{\infty,\infty}(x_0, \xi_0)$.

Let us consider now the fully non linear equation

$$F(x, D^{\alpha}u)_{\alpha \cdot 1/M \le m} = f(x), \qquad (6.3)$$

where m is a given positive integer, $F(x,\zeta) \in C^{\infty}(\mathbb{R}^n \times \mathbb{C}^N)$ is a known function of $x \in \mathbb{R}^n$, $\zeta = (\zeta_{\beta})_{\beta,1/M \le m-1} \in \mathbb{C}^N$.

Let the equation (6.3) be microlocally M-elliptic at $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$, meaning that the linearized M-principal symbol $A_m(x, \xi, \zeta) := \sum_{\alpha \cdot 1/M = m} \frac{\partial F}{\partial \zeta_{\alpha}}(x, \zeta) \xi^{\alpha}$ satisfies

$$\sum_{\alpha:1/M=m} \frac{\partial F}{\partial \zeta_{\alpha}}(x,\zeta)\xi^{\alpha} \neq 0 \text{ for } (x,\xi) \in U \times \Gamma_{M}, \tag{6.4}$$

for U a suitable neighborhood of x_0 and Γ_M a suitable M-conic neighborhood of ξ_0 . Under the assumptions above, we may prove the following

Theorem 6.3. For r > 0, $0 < \delta < \frac{1}{m^*}$, assume that $u \in B^{r+m,M}_{\infty,\infty}$, satisfying in addition

$$\partial_{x_j} u \in B^{r+m-\delta r,M}_{\infty,\infty}, \quad j = 1,\dots,n,$$
 (6.5)

is a solution to the equation (6.3), microlocally M-elliptic at $(x_0, \xi_0) \in T^{\circ}\mathbb{R}^n$. If moreover the forcing term satisfies

$$\partial_{x_j} f \in mcl B_{\infty,\infty}^{r,M}(x_0, \xi_0), \quad j = 1, \dots, n,$$

$$(6.6)$$

 $we\ obtain$

$$\partial_{x_j} u \in mcl B_{\infty,\infty}^{r+m,M}(x_0,\xi_0), \quad j = 1,\dots, n.$$

$$(6.7)$$

Proof. For each j = 1, ..., n, we differentiate (6.3) with respect to x_j finding that $\partial_{x_j} u$ must solve the linearized equation

$$\sum_{\alpha \cdot 1/M \le m} \frac{\partial F}{\partial \zeta_{\alpha}} (x, D^{\beta} u)_{\beta \cdot 1/M \le m} D^{\alpha} \partial_{x_{j}} u = \partial_{x_{j}} f - \frac{\partial F}{\partial x_{j}} (x, D^{\beta} u)_{\beta \cdot 1/M \le m} .$$
 (6.8)

From Theorems 3.3 and 2.7, $u \in B_{\infty,\infty}^{r+m,M}$ yields that $\frac{\partial F}{\partial \zeta_{\alpha}}(\cdot, D^{\beta}u)_{\beta \cdot 1/M \leq m} \in B_{\infty,\infty}^{r,M}$. Because of the hypotheses (6.5), (6.6), for each $j = 1, \ldots, n$, Theorem 5.1 applies to $\partial_{x_j} u$, as a solution of the equation (6.8) (which is microlocally M-elliptic at (x_0, ξ_0) in view of (6.4)), taking s = r + m. This proves the result.

Lemma 6.4. For every $s \in \mathbb{R}$, assume that $u, \partial_{x_j} u \in B^{s,M}_{\infty,\infty}$ for all $j = 1, \ldots, n$. Then $u \in B^{s+1/m^*,M}_{\infty,\infty}$. The same is still true if the Besov spaces $B^{s,M}_{\infty,\infty}, B^{s+1/m^*,M}_{\infty,\infty}$ are replaced by $mclB^{s,M}_{\infty,\infty}(x_0,\xi_0), mclB^{s+1/m^*,M}_{\infty,\infty}(x_0,\xi_0)$ at a given point $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$.

Proof. Let us argue for simplicity in the case of the spaces $B_{\infty,\infty}^{s,M}$, the microlocal case being completely analogous.

In view of Theorem 3.3, that u belongs to $B_{\infty,\infty}^{s+1/m^*,M}$ is completely equivalent to show that $\langle D \rangle_M^{1/m^*} u \in B_{\infty,\infty}^{s,M}$. By the use of the known properties of the Fourier transform, we may rewrite $\langle D \rangle_M^{1/m^*} u$ in the form

$$\langle D \rangle_M^{1/m^*} u = \langle D \rangle_M^{1/m^*-2} u + \sum_{j=1}^n \Lambda_{j,M}(D)(D_{x_j}u),$$

where $\Lambda_{j,M}(D)$ is the pseudodifferential operator with symbol $\langle \xi \rangle_M^{1/m^*-2} \xi_j^{2m_j-1}$, that is

$$\Lambda_{j,M}(D)v := \mathcal{F}^{-1}\left(\langle \xi \rangle_M^{1/m^*-2} \xi_j^{2m_j-1} \widehat{v}\right), \quad j = 1, \dots, n.$$

Since $\langle \xi \rangle_M^{1/m^*-2} \xi_j^{2m_j-1} \in S_M^{1/m^*-1/m_j}$, the result follows at once from Corollary 3.4.

As a straightforward application of the previous lemma, the following consequence of Theorem 6.3 can be proved.

Corollary 6.5. Under the same assumptions of Theorem 6.3 we have that $u \in mclB^{r+m+\frac{1}{m^*},M}_{\infty,\infty}(x_0,\xi_0)$.

Remark 6.6. We notice that if $r\delta \geq 1$ then every function $u \in B^{r+m,M}_{\infty,\infty}$ automatically satisfies the condition (6.5); indeed one can compute $\partial_{x_j} u \in B^{r+m-1/m_j,M}_{\infty,\infty} \subset B^{r+m-r\delta,M}_{\infty,\infty}$ being $1/m_j \leq 1 \leq r\delta$ for each $j=1,\ldots,n$. As already observed before, for $r>m^*$ we can always find $\delta^* \in]0,1/m^*[$ such that $r\delta^* \geq 1$ (it suffices to choose an arbitrary $\delta^* \in [1/r,1/m^*[)$; hence, applying Theorem 6.3 with such a δ^* we conclude that if $r>m^*$ and the right-hand side f of the equation (6.3) obeys the condition (6.6) at a point $(x_0,\xi_0) \in T^{\circ}\mathbb{R}^n$ then every solution $u \in B^{r+m,M}_{\infty,\infty}$ to such an equation satisfies the condition (6.7); in particular $u \in mcl B^{r+m+1/m^*,M}_{\infty,\infty}(x_0,\xi_0)$.

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Gianluca Garello
Department of Mathematics
University of Torino
Via Carlo Alberto 10
I- 10123 Torino, Italy
e-mail: gianluca.garello@unito.it

Alessandro Morando
DICATAM
University of Brescia
Via Valotti 9
I-25133 Brescia, Italy
e-mail: alessandro.morando@unibs.it