SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN POROUS MEDIUM

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ABSTRACT. We investigate systems of degenerate parabolic equations idealizing reactive solute transport in porous media. Taking advantage of the inherent structure of the system that allows to deduce a scalar Generalized Porous Medium Equation for the sum of the solute concentrations, we show existence of a unique weak solution to the coupled system and derive regularity estimates. We also prove that the system supports solutions propagating with finite speed thus giving rise to free boundaries and interaction of compactly supported initial concentrations of different species.

1. INTRODUCTION

The transport of pollutants in subsurface environments is a complex process modeled by advection-diffusion-reaction equations that describe the evolution of contaminant concentrations in porous medium through advection, dispersion, diffusion and adsorption. More than often the adsorption, accumulation of a pollutant on the solid matrix at the fluid-solid interface, is in fact the main mechanism responsible for the contaminant transport in soil.

In this work, we address the case of multicomponent contaminant transport by considering a competitive adsorption process of Freundlich type between different species $z_1, \ldots z_N$. For expediency, we write the Freundlich multicomponent equilibrium isotherm as

$$\mathbf{b}_a(\mathbf{z}) = |\mathbf{z}|_1^{p-1}\mathbf{z} \in \mathbb{R}^N, \quad p \in (0,1), \quad \mathbf{z} = (z_1, \ldots, z_N),$$

and consider the model problem

$$(1.1) \qquad\qquad \partial_t \mathbf{b}(\mathbf{z}) = \Delta \mathbf{z} + \mathbf{f},$$

where $|\mathbf{z}|_1 = \sum\limits_{i=1}^{N} |z_i|$ is the usual l^1 norm in $\mathbb{R}^N$, $\mathbf{b}(\mathbf{z}) = \phi\mathbf{z} + (1-\phi)\mathbf{b}_a(\mathbf{z})$, and $\phi \in [0,1)$ is the medium constant porosity. For a more thorough discussion on the physical background and derivation of this model, see Section 1.4 below.

While keeping in mind that the $\mathbf{b}$-term in problem (1.1) arises from the multicomponent adsorption with the Freundlich isotherm, we will allow for more general nonlinearities and denote hereafter the Freundlich nonlinearity as $\mathbf{b}_f(\mathbf{z}) = (\phi + (1-\phi)|\mathbf{z}|_1^{p-1})\mathbf{z}$.

1.1. General isotherms. Let us assume that

$$\mathbf{b}(\mathbf{z}) = B(|\mathbf{z}|_1)\,\mathbf{z},$$

where $B : \mathbb{R}^+ \to \mathbb{R}^+$ is such that

$$\beta(r) := B(|r|)r \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R} \setminus \{0\})$$

and

$$(H_1) \qquad\qquad \beta(0) = 0, \quad \beta(\pm\infty) = \pm\infty, \quad \beta'(r) > 0 \text{ for } r \neq 0.$$

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Note that this nonlinearity includes the Freundlich one $\mathbf{b}_f(\mathbf{z}) = (\phi + (1-\phi)|\mathbf{z}|_1^{p-1})\mathbf{z}$, allows for blow up $D_{\mathbf{z}}\mathbf{b}(0) \sim \infty$, and that by definition $|\mathbf{b}(\mathbf{z})|_1 = \beta(|\mathbf{z}|_1)$. Most importantly, this type of nonlinearity possesses a structure which will allow us to derive a particular scalar equation from the system. Since β is monotone increasing, the inverse

$$\Phi = \beta^{-1}$$

is well-defined and continuous with $\Phi(0) = 0$. We further assume that $\Phi \in \mathcal{C}^1(\mathbb{R})$ satisfies the structural conditions

$$(H_2) \qquad\qquad s \in \mathbb{R}: \qquad 1 \le \frac{s\Phi'(s)}{\Phi(s)} \le \frac{1}{a}$$

and

$$(H_3) \qquad\qquad s \in \mathbb{R}: \qquad \frac{s\Phi''(s)}{\Phi'(s)} \ge -\frac{1}{a}$$

for some structural constant $a \in (0,1)$. In the scalar case $\partial_t b(z) = \Delta z + (\ldots)$ the change of variables $u = b(z)$ produces the well-known Generalized Porous Media Equation (GPME) $\partial_t u = \Delta\Phi(u) + (\ldots)$ for which (H_2) is a standard assumption, see [8, 30] and the references therein. It is well-known that the scalar GPME is degenerate at $u = 0$ if $\Phi'(0) = 0$ and strictly parabolic if $\Phi'(0) > 0$, which can be seen from the divergence form $\Delta\Phi(u) = \mathrm{div}(\Phi'(u)\nabla u)$. Note in particular that the structural lower bound in (H_2) includes both the degenerate *slow diffusion* $\Phi'(0) = 0$ and the nondegenerate case $\Phi'(0) > 0$, but the assumption $\Phi \in \mathcal{C}^1(\mathbb{R})$ excludes *fast diffusion* $\Phi'(0) = +\infty$. We will shortly perform a similar change of variables $\mathbf{u} = \mathbf{b}(\mathbf{z})$ for the multicomponent problem in order to move the nonlinearity from the time derivative to the spatial ones.

We shall deal with the degenerate and the nondegenerate diffusions simultaneously in a unified framework, except in Section 4 where we discuss the existence of free boundaries and consequently restrict ourselves to *slow diffusions* $\Phi'(0) = 0$. Note also that in view of (H_2) the function $\Phi(s)/s$ is continuous monotone and nondecreasing in $\mathbb{R}$ with at most algebraic growth

$$(1.2) \qquad\qquad 0 < s_1 < s_2: \qquad \frac{\Phi(s_2)/s_2}{\Phi(s_1)/s_1} \le \left(\frac{s_2}{s_1}\right)^{\frac{1}{a}-1},$$

all properties that will be crucial in the subsequent analysis just as in the standard theory for GPME. The structural assumptions (H_1)-(H_2)-(H_3) are easily verified for the physical Freundlich isotherm when $p \in (0,1)$. With these structural assumptions the blowup $D_{\mathbf{z}}\mathbf{b}(0) \sim \infty$ corresponds now to slow diffusion $\Phi'(0) = 0$, but linear diffusion $\Phi(s) = s$ is also allowed. In fact, the Freundlich isotherm $\mathbf{b}_f(\mathbf{z}) = (\phi + (1-\phi)|\mathbf{z}|_1^{p-1})\mathbf{z}$ behaves like $\phi\mathbf{z}$ for large $|\mathbf{z}|_1$, hence $\beta(r) \sim \phi r$ and $\phi(s) = \beta^{-1}(s) \sim s/\phi$ for large r, s. In other words, the system $\partial_t\mathbf{b}_f(\mathbf{z}) = \Delta\mathbf{z} + \mathbf{f}$ behaves as N uncoupled linear heat equations $\phi\partial_t\mathbf{z} \approx \Delta\mathbf{z} + \mathbf{f}$ for large $|\mathbf{z}|_1$. From the physical point of view, roughly speaking, this means that for very large concentrations the porous rock matrix saturates and the adsorption phenomena become negligible compared to inertial effects.

The monotonicity assumption (H_1) allows us to invert

$$\mathbf{b}(\mathbf{z}) = \mathbf{u} \Leftrightarrow \mathbf{z} = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}\mathbf{u},$$

so that problem (1.1) can be recast as a degenerate parabolic system of (generalized) porous medium type

$$(1.3) \qquad\qquad \partial_t\mathbf{u} = \Delta\left(\frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}\mathbf{u}\right) + \mathbf{f}.$$

We shall refer to $\mathbf{u}$ as density whereas we shall speak of concentration when dealing with the original $\mathbf{z}$ variable.

1.2. Systems of decoupled Cauchy-Dirichlet problems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial \Omega$ and define $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial \Omega \times (0,T)$ for fixed $T > 0$. For given boundary data $\mathbf{z}^D(x,t) = (z_1^D, \ldots, z_N^D)(x,t)$, initial condition $\mathbf{z}^0(x) = (z_1^0, \ldots, z_N^0)(x)$, and the resultant of forcing terms $\mathbf{f}(x,t) = (f_1, \ldots, f_N)(x,t)$, we consider the following two equivalent formulations, the first written for the original concentrations $\mathbf{z} = (z_1, \ldots, z_N)$

$$(1.4) \qquad \begin{cases} \partial_t \mathbf{b}(\mathbf{z}) = \Delta \mathbf{z} + \mathbf{f} & \text{in } Q_T \\ \mathbf{z}(x,0) = \mathbf{z}^0(x) & \text{in } \Omega \\ \mathbf{z} = \mathbf{z}^D & \text{in } \Sigma_T \end{cases}$$

and the second one for the densities $\mathbf{u} = (u_1, \ldots, u_N)$

$$(1.5) \qquad \begin{cases} \partial_t \mathbf{u} = \Delta \left(\frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} \mathbf{u} \right) + \mathbf{f} & \text{in } Q_T \\ \mathbf{u}(x,0) = \mathbf{u}^0(x) & \text{in } \Omega \\ \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} \mathbf{u} = \mathbf{z}^D & \text{in } \Sigma_T \end{cases}$$

where $\mathbf{b}(\mathbf{z}) = \mathbf{u} \Leftrightarrow \mathbf{z} = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} \mathbf{u}$. As usual for the scalar GPME, the boundary conditions in the density formulation (1.5) are enforced in terms of the physical concentration $\frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} \mathbf{u} = \mathbf{b}^{-1}(\mathbf{u}) = \mathbf{z}^D$ rather than $\mathbf{u} = \mathbf{u}^D = \mathbf{b}(\mathbf{z}^D)$. It is easy to see formally that non-negative data $f_i, u_i^0, z_i^D \geq 0$ should lead to non-negative solutions $u_i \geq 0 \iff z_i \geq 0$ and, therefore, we shall only deal with such non-negative data and solutions. This is of course consistent with the fact that z_i represent physical concentrations and should stay non-negative when time evolves. Summing the equations in (1.5) we recognize that $w = |\mathbf{u}|_1 = u_1 + \ldots + u_N$ is a non-negative solution to

$$(\text{GPME}) \qquad w = |\mathbf{u}|_1 : \quad \begin{cases} \partial_t w = \Delta(\Phi(w)) + F & \text{in } Q_T, \\ w(x,0) = w^0(x) & \text{in } \Omega, \\ \Phi(w) = g^D & \text{in } \Sigma_T \end{cases}$$

with $F = f_1 + \ldots + f_N \geq 0$, $g^D = |\mathbf{z}^D|_1 \geq 0$, and $w^0 = |\mathbf{u}^0|_1 \geq 0$. Note that the boundary data is written for $\Phi(w)$ rather than for w as is common for the scalar GPME.

The initial condition and inhomogeneity should satisfy

$$(1.6) \qquad \forall i = 1 \ldots N : \quad 0 \leq u_i^0 \leq M \text{ and } 0 \leq f_i \leq M \quad \text{a.e. } (x,t) \in Q_T$$

for some finite $M > 0$. The boundary data will always assumed to be non-negative and bounded as well, but we shall sometimes assume the following. If $\gamma : \Omega \to \partial \Omega$ is the usual trace operator then there exists $\mathbf{Z}^D(x,t)$ such that

$$(1.7) \quad \mathbf{z}^D = \gamma(\mathbf{Z}^D) : \quad 0 \leq Z_i^D \in L^\infty(Q_T) \cap L^2(0,T; H^1(\Omega)) \text{ and } \partial_t Z_i^D \in L^\infty(Q_T)$$

(we shall indistinctly write $\mathbf{z}^D$ or $\mathbf{Z}^D$ both for the trace boundary values or their extension to Ω).

1.3. Main results.

Let us now first introduce our main theorem, which addresses existence, uniqueness, and regularity.

Theorem 1.1. *Assume that* (H_2) *holds. For any data* $0 \leq u_i^0 \leq M$, $0 \leq z_i^D \leq M$, *and* $0 \leq f_i \leq M$ *there exists a unique non-negative bounded very weak solution* $\mathbf{u}$ *to* (1.5). *Moreover,* $w = |\mathbf{u}|_1$ *is the unique non-negative bounded very weak*

solution to (GPME) *and there exist positive constants* $\alpha = \alpha(a,n) \in (0,1)$ *and* $C = C(a,T,n,N,M)$ *such that*

$$(1.8) \qquad \|\mathbf{u}\|_{C^{\alpha,\alpha/2}(Q')} \le C(1 + 1/d' + 1/\sqrt{\tau})$$

holds in all parabolic subdomains $Q' = \Omega' \times (\tau,T)$ *with* $0 < \tau < T$, $\Omega' \subset\subset \Omega$, *and* $d' = \mathrm{dist}(\overline{\Omega'}, \partial\Omega)$.

Assume in addition that (H_3) *holds and that the data satisfy* (1.6)-(1.7). *Then* w *is a global weak energy solution to* (4.2) *and* $\mathbf{u}$ *is a local weak energy solution to* (4.1) *in the sense that*

$$(1.9) \qquad \|\nabla(\varrho u_i)\|_{L^2(Q_T')} \le C(1 + 1/d') \qquad \forall\, i = 1 \dots N\,,$$

where $\varrho = \frac{\Phi(w)}{w} = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}$, *holds in any* $Q_T' = \Omega' \times (0,T)$ *with some constant* $C = C(a,T,n,N,M) > 0$.

The proof of the theorem can be found from Section 3, where one also find definitions of very weak and energy solutions. It is also worth stressing that if the initial and boundary data are compatible, then the local regularity (1.8) can be improved to global regularity up to the bottom and lateral boundaries, see Proposition 2.4 later on.

Let us immediately comment the content of the theorem. First of all, in the theory of (possibly degenerate) *scalar* diffusion equations such as (GPME) the so-called *pressure* variable $p = p(w) = \Phi'(w)$ plays an important role, as can be seen from the divergence form $\Delta\Phi(w) = \mathrm{div}(\Phi'(w)\nabla w)$. In view of (1.5) another pressure variable of interest is clearly

$$(1.10) \qquad \varrho = \varrho(w) = \frac{\Phi(w)}{w}, \qquad w = |\mathbf{u}|_1.$$

Note that our structural assumption (H_2) bounds the ratio p/ϱ away from zero and from above, roughly meaning that the degeneracy of p in (GPME) should be comparable to the degeneracy of ϱ in (1.5). The natural energy space for (GPME) is $\nabla\Phi(w) = p\nabla w = \nabla(\varrho w) \in L^2(Q_T)$, whereas that for (1.5) is rather here $\nabla(\varrho u_i) \in L^2(Q_T)$. Our structural assumption (H_3) will later allow us to derive such an estimate for each u_i from that estimate obtained for w. In other words, bounds for the scalar quantity w suffice to control the system in terms of energy considerations. This idea of controlling the vector-valued $\mathbf{u}$ by means of the scalar w will be the cornerstone of our analysis, and will also appear in the study of the Hölder regularity and of the free-boundaries.

Second, concerning the existence, in their celebrated work [2], Alt and Luckhaus studied systems of elliptic-parabolic PDEs which included the **b**-term as in (1.1) and also nonlinear p-Laplacian type diffusion, Stefan problems, and reaction terms $\mathbf{f} = \mathbf{f}(x,t,\mathbf{z})$. Their analysis requires however a particular monotone structure which restricts the **b**-term to be of the form $\mathbf{b} = D_{\mathbf{z}}\varphi$ for some convex potential φ satisfying certain structural assumptions. In our case, the dependence of the Freundlich nonlinearity $\mathbf{b}_f$ on $\mathbf{z}$ through the $l^1(\mathbb{R}^N)$-norm precludes any such monotonicity and, therefore, the results of [2] seem to be of no use here. Though system (4.1) is formally parabolic for non-negative solutions, it is readily checked that the ellipticity fails for signed solutions due to the dependence on the $l^1(\mathbb{R}^N)$ norm. A direct approach by Galerkin approximation as in [2] produces here approximative solutions whose componentwise sign cannot be controlled uniformly. Since ellipticity fails for signed solutions the sequence of projected solutions does not enjoy enough compactness, hence the method from [2] cannot be adapted. In order to tackle this issue we use instead the specific structure of the system allowing to control each component u_i in terms of the scalar quantity $w = |\mathbf{u}|_1$.

The method of proof for Theorem 1.1 is classical for scalar problems, but requires technical work for system (4.1): we first establish existence of positive classical solutions $\mathbf{u}^k$ for approximated positive data $u^{0,k}, \mathbf{z}^{D,k}, \mathbf{f}^k$ and derive some a priori regularity and energy estimates. Taking $k \to \infty$ finally gives the desired solution $\mathbf{u} = \lim \mathbf{u}^k$, which inherits regularity and energy estimates from the previous ones.

1.4. **Physical background.** Based on a continuum approach at a macroscopic level by homogenization, the mass conservation written for the concentration of one contaminant component $z = z(t, x)$ can be written as, cf. [6],

$$(1.11) \qquad \phi \frac{\partial z}{\partial t} + \rho(1 - \phi)\frac{\partial b_a}{\partial t} + \phi \nabla \cdot (z\mathbf{V} - \mathbf{D}\nabla z) = f \,.$$

Here, we have made the assumptions of saturated flow and constant porosity $\phi \in (0, 1)$, denoted the advective water flux by $z\mathbf{V}$ ($\mathbf{V}$ is the Darcy velocity), the bulk density of the solid matrix by $\rho > 0$, the hydrodynamic dispersion matrix describing both the molecular diffusion and the mechanical dispersion by $\mathbf{D}$, and modeled the source or sink terms by f. Moreover, $b_a = b_a(z)$ describes the concentration of contaminant adsorbed on the solid matrix through a reactive adsorption process that can be assumed to be either fast (equilibrium) or slow (non-equilibrium).

An adsorption isotherm $b_a(z)$ relates the concentration of the adsorbed component to its concentration in the fluid phase at constant temperature. One of the most commonly used nonlinear equilibrium isotherms for a single species is the Freundlich isotherm expressed as, cf. [6, 31]

$$b_a(z) = K\,z^p \quad K > 0\,, \quad p \in (0, 1)\,, \quad z \geq 0\,.$$

The Freundlich exponent $p \in (0, 1)$ makes equation (1.11) singular at $z = 0$ because, at least formally, $\partial_t b_a(z) = b_a'(z)\partial_t z$ and $b_a'(0) = \infty$. The equation may thus exhibit finite speed of propagation of compactly supported initial solutions giving rise to free boundaries that separate the region where the solute concentration vanishes from that with positive concentration. This is in marked contrast with the behavior of solutions when the Freundlich exponent p equals or exceeds one since the equation becomes nonsingular for $p \geq 1$ and the information propagates with infinite speed, as usual for uniformly parabolic equations.

Equation (1.11), complemented with suitable initial and boundary conditions, and all its variants arising from different equilibrium and non-equilibrium, linear or non-linear, isotherms, have attracted considerable attention over the last 20 years, both from an analysis and numerical simulation point of view, see, *e.g.*, [29, 11, 10, 4, 5, 24, 1]. It is, however, the above equilibrium Freundlich isotherm which makes the problem most challenging due to the degeneracy and similarity to the porous medium equation.

A competitive adsorption process between different species $z_1, \ldots z_N$, can be modeled by a multicomponent isotherm of Freundlich type, cf. [27, 22, 32], and [28] for a review of competitive equilibrium adsorption modeling. As an idealization of the physical model proposed, e.g., in [27], the multicomponent Freundlich equilibrium isotherm can be expressed as

$$\mathbf{b}_a(\mathbf{z}) = |\mathbf{z}|_1^{p-1}\mathbf{z} \in \mathbb{R}^N\,, \quad p \in (0, 1)\,, \quad \mathbf{z} = (z_1, \ldots, z_N)\,,$$

where $|\mathbf{z}|_1 = \sum\limits_{i=1}^{N} |z_i|$. Scaling the density, neglecting the hydrodynamical effects ($\mathbf{V} = 0$), assuming that $\mathbf{D} = \mathrm{Id}$ and writing $\mathbf{b}(\mathbf{z}) = \phi\mathbf{z} + (1 - \phi)\mathbf{b}_a(\mathbf{z})$, we obtain from (the multicomponent version of) (1.11) the model problem (1.1), i.e.,

$$\partial_t \mathbf{b}(\mathbf{z}) = \Delta\mathbf{z} + \mathbf{f}\,.$$

1.5. **The content.** The paper is organized as follows. In Section 2 we consider smooth positive data, construct corresponding smooth positive solutions to (1.5), and establish a priori energy as well as Hölder estimates. The Hölder estimates are based on the celebrated method of intrinsic scaling [15], a standard technique at least for scalar problems. In Section 3 we consider more general data, introduce different notions of weak solutions, and prove Theorem 1.1. Approximating the data suitably we show existence of a unique weak solution to problem (1.5) which inherits Hölder regularity and energy estimates from the smooth positive solutions constructed in Section 2. Finally in Section 4 we impose the degeneracy condition $\Phi'(0) = 0$ and consider the problem in the whole space without the forcing term and with compactly supported initial data. We show that the corresponding Cauchy problem is well posed and admits free boundary solutions and, moreover, investigate the finite speed of propagation of the free boundaries and the evolution and interaction of distinct compactly supported initial concentrations.

2. SMOOTH POSITIVE SOLUTIONS AND A PRIORI ESTIMATES

We will assume throughout this section that the data is smooth and (componentwise) positive. Solutions of (1.5) and (GPME) corresponding to such data are shown to be classical and positive and satisfy certain a priori energy and locally uniform Hölder estimates.

Proposition 2.1 (Existence of positive classical solutions). *Assume that $\mathbf{z}^D$ and $\mathbf{u}^0$ are smooth and positive and $\mathbf{f}$ is smooth and non-negative. Moreover, let $F := |\mathbf{f}|_1$ and assume that*

$$0 < m = \min\left\{\operatorname*{ess\,inf}_{\overline{Q_T}} |\mathbf{u}^0|_1, \quad \operatorname*{ess\,inf}_{\overline{\Sigma_T}} |\mathbf{z}^D|_1\right\},$$

$$0 < M = \max\left\{\operatorname*{ess\,sup}_{\overline{Q_T}} |\mathbf{u}^0|_1, \quad \operatorname*{ess\,sup}_{\overline{\Sigma_T}} |\mathbf{z}^D|_1, \quad \operatorname*{ess\,sup}_{\overline{Q_T}} F\right\}.$$

Then there exists a classical solution $\mathbf{u} \in \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T)) \cap \mathcal{C}^{2,1}(\Omega \times [0,T]) \cap \mathcal{C}^{\infty}(Q_T)$ to (1.5) with $u_i > 0$ on $\overline{Q_T}$. Moreover, defining $w = |\mathbf{u}|_1 = u_1 + \ldots + u_N$, $w \in \mathcal{C}^{2,1}(\overline{Q_T}) \cap \mathcal{C}^{\infty}(Q_T)$ is a classical solution to (GPME) and

$$0 < m \le w(x,t) \le M(1+T) \quad in \ \overline{Q_T}.$$

Remark 2.1. We do not impose any compatibility conditions on the initial and boundary data at $\partial\Omega \times \{t = 0\}$. Although this limits the boundary regularity it has no importance in the sequel. Note also that we could prove uniqueness of positive classical solutions at this stage. However since we will later establish a stronger uniqueness result (within the class of non-negative very weak solutions) we postpone the uniqueness issue until then.

Proof. We will exploit the diagonal structure of the system by first showing existence of a classical solution w to (GPME), then reconstructing $\mathbf{u}$ by solving N independent linear parabolic equations for the u_i, and finally checking that $w = |\mathbf{u}|_1$ as desired.

For smooth positive data let $w^0 := |\mathbf{u}^0|_1 > 0$ in $\overline{\Omega}$ and $g^D := |\mathbf{z}^D|_1 > 0$ in $\overline{\Sigma_T}$. Write $\Delta\Phi(w) = \operatorname{div}(\Phi'(w)\nabla w)$ and observe that hypothesis (H_2) implies that $\Phi'(w) > 0$ is bounded away from zero and from above as long as $0 < m \le w \le C$ so that equation (GPME) is uniformly parabolic for such values of w. Therefore, after approximating Φ by a globally Lipschitz function Φ_ε such that $\Phi_\varepsilon(0) = 0$, $\Phi(s) = \Phi_\varepsilon(s)$ for $|s| \in (\varepsilon, 1/\varepsilon)$ and $\Phi'_\varepsilon > 0$, well known results for quasilinear

parabolic equations (cf. [25]) guarantee the existence of a positive classical solution $w_\varepsilon(x,t)$ to the ε-problem. A standard comparison principle with $0 \leq F \leq M$ and $0 < m \leq w^0, g^D \leq M$ shows moreover that

$$0 < m \leq w_\varepsilon \leq \max\left\{\|w^0\|_{L^\infty(\Omega)}, \|g^D\|_{L^\infty(\Sigma_T)}\right\} + T\|F\|_{L^\infty(Q^T)} \leq M(1+T)$$

as in our statement. In particular, for $\varepsilon > 0$ small enough there holds $\Phi_\varepsilon(w_\varepsilon) = \Phi(w_\varepsilon)$ so that w_ε is in fact a classical solution to the original problem. The argument is standard for scalar equations and we refer, e.g., to [30, 8] for more details.

Once there exists a smooth positive solution w to (GPME), the pressure $\varrho = \frac{\Phi(w)}{w}$ becomes smooth in the interior, belongs to $\mathcal{C}^{2,1}(\overline{\Omega} \times (0,T)) \cap \mathcal{C}^{2,1}(\Omega \times [0,T])$, and the bounds

$$(2.1) \qquad\qquad 0 < C_1 \leq \varrho \leq C_2 \quad \text{in } \overline{Q^T}$$

hold for some $C_1, C_2 > 0$ depending on a, m, M, T only. Then standard results [25] on *linear* parabolic equations allow us to solve

$$(2.2) \qquad \begin{cases} \partial_t u_i = \Delta(\varrho u_i) + f_i = \text{div}(\varrho \nabla u_i) + \text{div}(u_i \nabla \varrho) + f_i & \text{in } Q_T, \\ \varrho u_i = z_i^D & \text{in } \Sigma_T, \\ u_i(x,0) = u_i^0(x) & \text{in } \Omega, \end{cases}$$

for fixed $i = 1, \ldots N$ and show that $u_i \in \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T)) \cap \mathcal{C}^{2,1}(\Omega \times [0,T]) \cap \mathcal{C}^\infty(Q_T)$ (up to the corners if the data are compatible). Indeed, from (2.2) it follows that the equation is uniformly parabolic and the boundary condition reads simply as $u_i = \frac{z_i^D}{\varrho}$ on Σ_T. The assumptions on the data and the strong maximum principle ensure moreover that $u_i > 0$ in $\overline{Q_T}$.

Let now $\tilde{w} = |\mathbf{u}|_1$ and observe that $\partial_t w = \Delta\Phi(w) + F = \Delta(\varrho w) + F$. Because $u_i > 0$ we can write $\tilde{w} = u_1 + \ldots + u_N$. Summing (2.1) over $i = 1 \ldots N$, we obtain $\partial_t \tilde{w} = \Delta(\varrho \tilde{w}) + F$. In other words, $\tilde{w}$ is a positive classical solution to the same equation as w with the same initial and boundary data. By standard uniqueness argument for smooth positive solutions we conclude that $w = \tilde{w}$. In particular $\varrho = \frac{\Phi(w)}{w} = \frac{\Phi(\tilde{w})}{\tilde{w}} = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}$ in (2.1) and the proof is complete. $\qquad\square$

Proposition 2.2 (A priori energy estimates). *Assume that hypotheses* (H_1), (H_2) *and* (H_3) *hold, let* $\mathbf{u} \in \mathcal{C}^{2,1}(\overline{Q_T})$ *be a classical positive solution corresponding to smooth positive data and assume that*

$$\|\mathbf{u}^0\|_{L^\infty(\Omega)} + \|\mathbf{z}^D\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t \mathbf{z}^D\|_{L^\infty(Q_T)} + \|\mathbf{f}\|_{L^\infty(Q_T)} \leq M,$$

for some $M > 0$. *Then we have the bounds*

$$(2.3) \qquad\qquad \|\nabla(\varrho w)\|_{L^2(Q_T)} \leq C,$$

where $w = |\mathbf{u}|_1$ *and* $\varrho = \frac{\Phi(w)}{w}$, *and*

$$(2.4) \qquad \|\nabla(\varrho u_i)\|_{L^2(Q_T')} \leq C(1 + 1/d'), \qquad \forall\, i = 1 \ldots N.$$

Here, the constant $C > 0$ *depends on* a, T, n, N, M *only, and* $Q_T' = \Omega' \times (0,T)$ *with* $\Omega' \subset\subset \Omega$ *and* $d' = \text{dist}(\overline{\Omega'}, \partial\Omega) > 0$.

Remark 2.2. We were not able to establish energy estimates for $\nabla(\varrho u_i)$ up to the boundary as the dependence on $1/d'$ in (2.4) shows. This will not be an issue later on since in Proposition 3.1 we shall prove uniqueness within the class of very weak solutions and estimate (2.4) as well as assumption (H_3), which is only used in proving (2.4), can be dispensed with while considering very weak solutions. On the other hand, estimate (2.4) is sufficient for our purposes in Section 4 where the problem is considered in $\mathbb{R}^n$ with compactly supported initial data.

Observe also that the validity of estimate (2.4) up to the boundary would directly yield (2.3) since $w = |\mathbf{u}|_1 = \sum_i u_i$ for non-negative solutions.

Proof. We will first establish (2.3) for the scalar variable w and then show how the particular structure of system (1.5) allows us to derive (2.4) for each component u_i. We shall denote by C any positive constant depending, as in the statement, only on a, T, n, N, M whereas the primed constants C' are also allowed to depend on $d' = \text{dist}(\overline{\Omega'}, \partial\Omega)$.

Step 1. The assumptions on $\mathbf{u}^0, \mathbf{z}^D, \mathbf{f}$ translate into similar properties for the data $w^0 = |\mathbf{u}^0|_1, g^D = |\mathbf{z}^D|_1, F = |\mathbf{f}|_1$ so by the comparison principle for solutions to (GPME) we have

$$0 \le u_i \le |\mathbf{u}|_1 = w \le C(M, N, T).$$

Recalling that $\varrho w = \Phi(w)$, inequality (2.3) is nothing but the usual (global) energy estimate for the GPME leading to the usual concept of weak *energy* solutions. For smooth positive solutions, bound (2.3) is easily derived for $\|\nabla\Phi(w)\|_{L^2(Q_T)} = \|\nabla(\varrho w)\|_{L^2(Q_T)}$ by taking $\varphi = (\Phi(w) - g^D) \in L^2(0, T; H_0^1(\Omega))$ as a test function in (GPME), cf. [8, 30] for further details.

Step 2. Since $0 \le w \le C$, the structural assumptions imply that

$$0 \le \varrho = \frac{\Phi(w)}{w} \le \frac{1}{a}\Phi'(w) \le C(a, M, T).$$

The $L^\infty(Q_T)$-norm of any term involving u_i, w, ϱ can thus be bounded by a constant $C = C(a, T, n, N, M) > 0$ only. Now fix $i \in \{1, \ldots, N\}$ and choose a cutoff function $\chi = \chi(x) \in \mathcal{C}_c^\infty(\Omega)$ such that $0 \le \chi \le 1$ in Ω, $\chi \equiv 1$ in Ω' and $|\nabla\chi| \le 2/d'$ where $\Omega' \subset\subset \Omega$ and $d' = \text{dist}(\overline{\Omega'}, \partial\Omega)$. Multiplying the i-th equation in (1.5) by a test function $\varphi = \chi^2 \varrho u_i$, integrating over Q_T and by parts in the Laplacian term, we obtain

$$\int_{Q_T} \chi^2 |\nabla(\varrho u_i)|^2 \, \mathrm{d}x\, \mathrm{d}t = -2 \int_{Q_T} \varrho u_i \chi \nabla\chi \cdot \nabla(\varrho u_i) \, \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_{Q_T} \chi^2 \varrho u_i f_i \, \mathrm{d}x\, \mathrm{d}t - \int_{Q_T} \chi^2 \varrho u_i \partial_t u_i \, \mathrm{d}x\, \mathrm{d}t.$$

Integrating the last term by parts in t and using $0 \le u_i, \varrho \le C$ to bound the limit terms at $t = 0, T$, gives

$$\int_{Q_T} \chi^2 |\nabla(\varrho u_i)|^2 \, \mathrm{d}x\, \mathrm{d}t$$
$$\le 2\|\varrho u_i \nabla\chi\|_{L^2(Q_T)} \|\chi\nabla(\varrho u_i)\|_{L^2(Q_T)} + C + \left(C + \frac{1}{2}\int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, \mathrm{d}x\, \mathrm{d}t\right)$$
$$\le \frac{1}{2}\|\chi\nabla(\varrho u_i)\|_{L^2(Q_T)}^2 + 8\|\varrho u_i \nabla\chi\|_{L^2(Q_T)}^2 + C + \frac{1}{2}\int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, \mathrm{d}x\, \mathrm{d}t,$$

where we have also taken into account that $0 \le \chi^2 \varrho u_i f_i \le C$ and used Young's inequality. Estimating $\|\varrho u_i \nabla\chi\|_{L^2(Q_T)}^2 \le C/(d')^2 \le C'$ then yields the bound

$$(2.5) \qquad \|\chi\nabla(\varrho u_i)\|_{L^2(Q_T)}^2 \le C' + \underbrace{\int_{Q_T} \chi^2 u_i^2 \partial_t \varrho \, \mathrm{d}x\, \mathrm{d}t}_{:=A}.$$

We exploit now the structure of the system to control A. Indeed, since $\varrho = \frac{\Phi(w)}{w}$ one easily computes for smooth positive solutions

$$\partial_t \varrho = \frac{d}{dw}\left(\frac{\Phi(w)}{w}\right)\partial_t w = \frac{w\Phi'(w) - \Phi(w)}{w^2}\big(\Delta(\varrho w) + F\big).$$

Thus integrating by parts gives

$$A = \int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)\big(\Delta(\varrho w) + F\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\int_{Q_T} \nabla(\chi^2)(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{Q_T} \chi^2\nabla(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{Q_T} \chi^2(\varrho u_i)^2 \nabla\left(\frac{1}{(\varrho w)^2}\right)\big(w\Phi'(w) - \Phi(w)\big)\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\nabla\big(w\Phi'(w) - \Phi(w)\big)\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)F\,\mathrm{d}x\,\mathrm{d}t$$

$$= A_1 + A_2 + A_3 + A_4 + B.$$

Observing that $0 \le \varrho u_i \le \varrho w = \Phi(w)$ and that the hypothesis (H_2) implies that $0 \le w\Phi'(w) - \Phi(w) \le C(a)\Phi(w)$, we can control the first term as

$$A_1 = -2\int_{Q_T} \chi\nabla\chi\cdot(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\|\nabla\chi\|_{L^2(Q_T)}\|\nabla(\varrho w)\|_{L^2(Q_T)} \le C'.$$

The second term is bounded similarly as follows

$$A_2 = -2\int_{Q_T} \chi^2(\varrho u_i)\nabla(\varrho u_i)\cdot\frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$\le 2\|\chi\nabla(\varrho u_i)\|_{L^2(Q_T)}\left\|\chi\frac{\varrho u_i}{\varrho w}\frac{w\Phi'(w) - \Phi(w)}{\varrho w}\nabla(\varrho w)\right\|_{L^2(Q_T)}$$

$$\le C\|\chi\nabla(\varrho u_i)\|_{L^2(Q_T)}\|\nabla(\varrho w)\|_{L^2(Q_T)}$$

$$\le \frac{1}{2}\|\chi\nabla(\varrho u_i)\|^2_{L^2(Q_T)} + C\|\nabla(\varrho w)\|^2_{L^2(Q_T)}$$

$$\le \frac{1}{2}\|\chi\nabla(\varrho u_i)\|^2_{L^2(Q_T)} + C,$$

where we have also used the Young's inequality (the first term on the right-hand side will be reabsorbed into (2.5)). The third quantity is controlled as

$$A_3 = 2\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{\nabla(\varrho w)}{(\varrho w)^3}\cdot\big(w\Phi'(w) - \Phi(w)g\big)\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t$$

$$= 2\int_{Q_T} \chi^2\frac{(\varrho u_i)^2}{(\varrho w)^2}\frac{w\Phi'(w) - \Phi(w)}{\Phi(w)}|\nabla(\varrho w)|^2\,\mathrm{d}x\,\mathrm{d}t \le C\|\nabla(\varrho w)\|^2_{L^2(Q_T)} \le C.$$

In the fourth term we write $\nabla(\varrho w) = \nabla\Phi(w) = \Phi'(w)\nabla w$ and use (H_3) to get

$$
\begin{aligned}
A_4 &= -\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\nabla\big(w\Phi'(w) - \Phi(w)\big)\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}w\Phi''(w)\nabla w\cdot\nabla(\varrho w)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\frac{w\Phi''(w)}{\Phi'(w)}|\nabla(\varrho w)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{1}{a}\|\nabla(\varrho w)\|_{L^2(Q_T)} \leq C.
\end{aligned}
$$

Remark 2.3. Note that an upper bound for A_4 is obtained here by using the lower bound (H_3) for $\frac{s\Phi''(s)}{\Phi'(s)}$. If in particular $\Phi''(s) \geq 0$ for all $s \geq 0$, which is the typical case for the PME nonlinearity $\Phi(s) = |s|^{m-1}s$ in the range $m \geq 1$, then $A_4 \leq 0$. This convexity condition is also valid for the Freundlich isotherm since $\beta_f(r) = \phi r + (1-\phi)r^p$ is concave and thus $\Phi_f = \beta_f^{-1}$ is convex.

For the last term we obtain

$$
B = \int_{Q_T} \chi^2(\varrho u_i)^2 \frac{1}{(\varrho w)^2}\big(w\Phi'(w) - \Phi(w)\big)F\,\mathrm{d}x\,\mathrm{d}t \leq C(a)\int_{Q_T}\Phi(w)F\,\mathrm{d}x\,\mathrm{d}t \leq C.
$$

Plugging the above estimates back into (2.5) finally yields

$$
\|\nabla(\varrho u_i)\|^2_{L^2(Q'_T)} \leq \|\chi\nabla(\varrho u_i)\|^2_{L^2(Q_T)} \leq C'.
$$

Keeping track of the dependence of the estimates on $d' = \mathrm{dist}\,\big(\overline{\Omega'}, \partial\Omega\big)$ and optimizing all inequalities, one easily sees that $C' = C(1 + 1/d')$ with $C = C(a, T, n, N, M)$ only and the proof is complete. $\qquad\square$

We will next address the regularity issue.

Proposition 2.3. *Let* $\mathbf{u}$ *and* M *be as in Proposition 2.1. There exist* $\alpha = \alpha(a, n) \in (0, 1)$ *and* $C = C(a, T, n, N, M) > 0$ *such that the estimate*

$$
\|\mathbf{u}\|_{C^{\alpha,\alpha/2}(Q')} \leq C(1 + 1/d' + 1/\sqrt{\tau}),
$$

holds for any parabolic subdomain $Q' = \Omega' \times (\tau, T)$, *where* $0 < \tau < T$, $\Omega' \subset\subset \Omega$, *and* $d' = \mathrm{dist}(\overline{\Omega'}, \partial\Omega)$.

Remark 2.4. We would like to stress that our proof handles both the nondegenerate $\Phi'(0) > 0$ and degenerate $\Phi'(0) = 0$ cases in a unified framework.

Proof. The proof goes in several steps and is based on the intrinsic scaling method, cf. [15]. As usual, we can assume after translation that the intrinsic cylinders are centered at the origin $(x_0, t_0) = (0, 0)$. We write $w = |\mathbf{u}|_1$ and recall that w is positive and bounded with the L^∞ bounds depending on M, T only. We begin by considering the scalar problem.

Step 1: Alternatives. Suppose that $\sup_{Q_r^\mu} w \leq \mu$ in an intrinsic cylinder

$$
Q_r^\mu := B_r \times (-\tau_r^\mu, 0)\,, \qquad \tau_r^\mu := \frac{\mu}{\Phi(\mu)}r^2\,,
$$

and define

$$
\widetilde{Q}_r^\mu := B_{3r/4} \times \big(-\frac{3}{4}\tau_r^\mu, -\frac{1}{2}\tau_r^\mu\big) \subset Q_r^\mu.
$$

Now consider the following two alternatives

$$(2.6) \qquad \left| \widetilde{Q}_r^\mu \cap \{w \le \mu/2\} \right| \le \delta |\widetilde{Q}_r^\mu| \,,$$

$$(2.7) \qquad \left| \widetilde{Q}_r^\mu \cap \{w \le \mu/2\} \right| > \delta |\widetilde{Q}_r^\mu| \,,$$

where $\delta \in (0,1)$ is a small parameter to be fixed shortly. The first one is the nondegenerate alternative and the second is the degenerate alternative. We will analyze them separately.

Step 2: Nondegenerate alternative 1. Set

$$r_j = \left(\frac{1}{4} + \frac{1}{4^{j+1}} \right) r \,, \qquad k_j := \left(\frac{1}{4} + \frac{1}{4^{j+1}} \right) \mu \,, \qquad w_j := (k_j - w)_+ \,.$$

Set also

$$\widetilde{Q}^j := B_{r/4+r_j} \times \left(-\frac{1}{2}\tau_r^\mu - \tau_r^\mu \left(\frac{r_j}{r} \right)^2 , -\frac{1}{2}\tau_r^\mu \right) \,,$$

and let ϕ_j be a cut-off function such that ϕ_j is smooth, $0 \le \phi_j \le 1$, ϕ_j vanishes on the parabolic boundary $\partial_p \widetilde{Q}^j$ (bottom and lateral), is one on $\widetilde{Q}^{j+1}$ and $|\nabla \phi_j|^2 + (\partial_t \phi_j^2)_+ \le r^{-2} 16^{j+1}$. Since w solves the equation $\partial_t w - \Delta \Phi(w) = F$ and $F \ge 0$, it is easy to check that w_j is a weak subsolution to the equation $\partial_t w_j - \mathrm{div}(\Phi'(w)\nabla w_j) = 0$, and testing the latter with $w_j \phi_j^2$ leads to the Caccioppoli inequality

$$\sup_{-\tau_r^\mu < t < 0} \fint_{B_r} \frac{w_j^2 \phi_j^2}{\tau_r^\mu} \, dx + \fint_{Q_r^\mu} \Phi'(w) |\nabla(w_j \phi_j)|^2 \, dx \, dt$$

$$\le c \fint_{Q_r^\mu} \left[\Phi'(w) w_j^2 |\nabla \phi_j|^2 + w_j^2 (\partial_t \phi_j^2)_+ \right] \, dx \, dt \,.$$

Setting

$$\bar{w}_j := \begin{cases} k_j - k_{j+1} & \text{if } w \le k_{j+1} \\ w_j & \text{if } w > k_{j+1} \end{cases}$$

and recalling from (H_2) that $\Phi(s)/s$ is monotone non-decreasing with at most algebraic growth, we see that

$$\frac{r^2}{\tau_r^\mu} |\nabla \bar{w}_j|^2 = \frac{\Phi(\mu)}{\mu} |\nabla \bar{w}_j|^2 \le c(a) \frac{\Phi(k_{j+1})}{k_{j+1}} |\nabla \bar{w}_j|^2 \le c(a) \Phi'(w) |\nabla w_j|^2$$

since in the support of $\nabla \bar{w}_j$ we have $w \ge k_{j+1} \ge \mu/4$. Similarly,

$$\Phi'(w) w_j^2 |\nabla \phi_j|^2 \le c(a) 16^j \frac{1}{r^2} \frac{\Phi(\mu)}{\mu} w_j^2 \le \frac{c(a) 16^j}{\tau_r^\mu} \mu^2 \chi_{\{w < k_j\}} \,.$$

Collecting estimates we arrive at

$$\sup_{-\tau_r^\mu < t < 0} \fint_{B_r} \bar{w}_j^2 \phi_j^2 \, dx + r^2 \fint_{Q_r^\mu} |\nabla(\bar{w}_j \phi_j)|^2 \, dx \, dt \le c 16^j \mu^2 \left(\frac{|\widetilde{Q}^j \cap \{w < k_j\}|}{|\widetilde{Q}^j|} \right)$$

Next, the parabolic Sobolev embedding (see [15, Proposition 3.1, p.7]) gives us

$$\fint_{Q_r^\mu} (\bar{w}_j \phi_j)^{2(1+2/n)} \, dx \, dt$$

$$\le c(n) \left(\sup_{-\tau_r^\mu < t < 0} \fint_{B_r} \bar{w}_j^2 \phi_j^2 \, dx + r^2 \fint_{Q_r^\mu} |\nabla(\bar{w}_j \phi_j)|^2 \, dx \, dt \right)^{1+2/n} \,.$$

Since

$$(\bar{w}_j \phi_j)^{2(1+2/n)} \ge 4^{-6j} \mu^{2(1+2/n)} \chi_{\widetilde{Q}^{j+1} \cap \{w < k_{j+1}\}} \,,$$

we get

$$E_{j+1} \leq \bar{c}(a,n)4^{8j}E_j^{1+2/n} \qquad \text{with} \quad E_j := \frac{|\widetilde{Q}^j \cap \{w < k_j\}|}{|\widetilde{Q}^j|}.$$

A standard iteration lemma on fast geometric convergence of series ([15, p.12]]) shows that if $E_0 \leq \bar{c}^{-n/2}4^{-2n^2}$ then E_j tends to zero as $j \to \infty$. Indeed, choosing $\delta := \bar{c}^{-n/2}4^{-2n^2}$, it follows from (2.6) that $w \geq \mu/4$ in $B_{r/2}\times(-9\tau_r^\mu/16, -\tau_r^\mu/2)$.

Step 3: Nondegenerate alternative 2. We test $\partial_t w = \text{div}(\Phi'(w)\nabla w) + F$ with $(1/w - 4/\mu)_+\xi^2$, where $\xi \in C_0^\infty(B_{r/2})$, $0 \leq \xi \leq 1$, $\xi \equiv 1$ in $B_{r/4}$ and $|\nabla\xi| \leq 8/r$. Note that the chosen test function vanishes on $B_{r/2} \times \{-\tau_r^\mu/2\}$ by Step 2. Taking advantage of $-F\left(\frac{1}{w} - \frac{4}{\mu}\right)_+\xi^2 \leq 0$, straightforward manipulations then lead to

$$\sup_{-\tau_r^\mu/2 < t < 0} \fint_{B_{r/2}} \log\left(\frac{\mu/4}{w}\right)_+ \xi^2\, dx$$
$$+ \frac{\tau_r^\mu}{4} \fint_{B_{r/2}\times(-\tau_r^\mu/2,0)} \Phi'(w)\frac{|\nabla(w-\mu/4)_+|^2}{w^2}\xi^2\, dx\, dt$$
$$\leq \tau_r^\mu \fint_{B_{r/2}\times(-\tau_r^\mu/2,0)} \Phi'(w)|\nabla\xi|^2\, dx\, dt + \fint_{B_{r/2}} \xi^2\, dx$$
$$\leq \tau_r^\mu \fint_{B_{r/2}\times(-\tau_r^\mu/2,0)} \frac{1}{a}\frac{\Phi(w)}{w}|\nabla\xi|^2\, dx\, dt + \fint_{B_{r/2}} \xi^2\, dx$$
$$\leq \tau_r^\mu \frac{1}{a}\frac{\Phi(\mu)}{\mu} \fint_{B_{r/2}\times(-\tau_r^\mu/2,0)} |\nabla\xi|^2\, dx\, dt + \fint_{B_{r/2}} \xi^2\, dx \leq \frac{65}{a},$$

where we used successively (H_2), the monotonicity of $\Phi(s)/s$ with $w \leq \mu$, the definition $\tau_r^\mu \frac{\Phi(\mu)}{\mu} = r^2$, and the cutoff function properties $|\nabla\xi| \leq 8/r$, $\xi^2 \leq 1$. As a consequence, we readily obtain

$$\sup_{-\tau_r^\mu/2 < t < 0} |B_{r/4} \cap \{w(\cdot,t) < 4^{-1-m}\mu\}| \leq \frac{1}{m}\frac{2^n 65}{a\log 4}|B_{r/4}|,$$

and in particular

$$\frac{|B_{r/4} \times (-\tau_r^\mu/2,0) \cap \{w < 4^{-1-m}\mu\}|}{|B_{r/4} \times (-\tau_r^\mu/2,0)|} \leq \frac{1}{m}\frac{2^n 65}{a\log 4},$$

for any $m \in \mathbb{N}$.

Next, redefine $k_j := 4^{-m-1}(2^{-1} + 2^{-1-j})\mu$ and $r_j = (2^{-3} + 2^{-3-j})r$, and set

$$\hat{Q}^j := \hat{B}^j \times (-\tau_r^\mu/2,0), \qquad \hat{B}^j := B_{r_j}(0).$$

Choose $\xi_j \in C_0^\infty(\hat{B}^j)$ in such a way that $0 \leq \xi_j \leq 1$, $\xi_j = 1$ in $\hat{B}^{j+1}$ and $|\nabla\xi_j| \leq 2^{4+j}/r$. The Caccioppoli estimate then takes the form

$$\sup_{-\tau_r^\mu/2 < t < 0} \fint_{\hat{B}^j} \frac{w_j^2\xi_j^2}{\tau_r^\mu}\, dx + \fint_{\hat{Q}^j} \Phi'(w)|\nabla(w_j\xi_j)|^2\, dx\, dt \leq c \fint_{\hat{Q}^j} \Phi'(w)w_j^2|\nabla\xi_j|^2\, dx\, dt,$$

because ξ_j is independent of time and the newly defined w_j vanishes on the initial boundary of $\hat{Q}^j$ by Step 2. Since $s \mapsto \Phi(s)/s$ is a nondecreasing function and $1/\tau_r^\mu = \Phi(\mu)/\mu$, it follows that

$$\frac{\Phi(k_j)}{k_j} \sup_{-\tau_r^\mu/2 < t < 0} \fint_{\hat{B}^j} w_j^2\xi_j^2\, dx + \fint_{\hat{Q}^j} \Phi'(w)|\nabla(w_j\xi_j)|^2\, dx\, dt \leq \hat{c}(a)\frac{4^j}{r^2}\Phi(k_j)k_j\hat{E}_j,$$

where this time $\hat{E}_j := |\hat{Q}^j \cap \{w_j > 0\}|/|\hat{Q}^j|$. Analogously to Step 2, we then arrive at $\hat{E}_{j+1} \leq c(n,a)4^{8j}E_j^{1+2/n}$, and by choosing $m \equiv m(n,a)$ large enough, i.e.

$$\hat{E}_0 \leq \frac{1}{m}\frac{2^n 65}{a \log 4} \leq \hat{c}^{-n/2}4^{-2n^2},$$

we conclude that $w \geq 4^{-m-2}\mu$ in $Q_{r/8}^{\mu}$.

Step 4: Degenerate alternative. Let us then analyze the occurrence of (2.7). For this, set $v = \mu/2 - (w - \mu/2)_+ + \|F\|_{L^\infty(Q_r^\mu)}(t + \tau_r^\mu)$, which is a nonnegative weak supersolution to $\partial_t v - [\Phi(\mu)/\mu]\mathrm{div}(b(x,t)\nabla v) \geq 0$ in Q_r^μ with $b(x,t) := \mu\Phi'(w(x,t))/\Phi(\mu)$. By definition of v and $\mu = \sup w$ we have in the support of ∇v

$$\nabla v(x,t) \neq 0 \quad \Rightarrow \quad \mu/2 \leq w \leq \mu \quad \Rightarrow \quad c(a)^{-1} \leq b(x,t) \leq c(a).$$

Redefining b to be one on $\{\nabla v(x,t) = 0\}$ and scaling b and v as $\bar{b}(x,t) = b(rx, \tau_r^\mu t)$ and $\bar{v}(x,t) = v(rx, \tau_r^\mu t)$, we see that $\bar{v}$ is a weak supersolution to the equation $\partial \bar{v} - \mathrm{div}(\bar{b}\nabla \bar{v}) = 0$ in $B_1 \times (-1,0)$ with measurable coefficient $\bar{b}$ bounded uniformly from below and from above by positive constants depending only on a. By the weak Harnack principle we then obtain

$$\fint_{B_{3/4}\times(-3/4,-1/2)} \bar{v}\,dx \leq c(a) \inf_{B_{1/2}\times(-1/4,0)} \bar{v}.$$

Scaling back to v and recalling its definition in terms of w, we finally get by (2.7)

$$\sup_{Q_{r/2}^\mu} w \leq \mu\left(1 - \frac{\delta}{2c}\right) + \tau_r^\mu\|F\|_{L^\infty(Q_r^\mu)}\left(1 - \frac{1}{8c}\right).$$

Step 5: Conclusion from alternatives. Taking

$$\sigma = \sigma(a,n) := \min\left\{4^{-m-2}, \frac{\delta}{4c}, \frac{1-16^{-a}}{2}\right\} \in (0,1),$$

we deduce from the previous alternatives that

$$\text{either} \quad \inf_{Q_{r/8}^\mu} w \geq \sigma\mu \quad \text{or} \quad \sup_{Q_{r/2}^\mu} w \leq (1-2\sigma)\mu + \tau_r^\mu|F|_{L^\infty(Q_r^\mu)}$$

holds provided that $\sup_{Q_r^\mu} w \leq \mu$. Choose now any $R > 0$ such that $B_R \times (-R^2, 0) \subset Q_T$ (after translation), let $R_j = 8^{-j}R$,

$$\mu_0 := 1 + \Phi^{-1}\left(\sigma^{-1}R^2\|F\|_{L^\infty(B_R\times(-R^2,0))}\right) + \sup_{B_R\times(-R^2,0)} w,$$

and then inductively

$$\mu_{j+1} := (1-2\sigma)\mu_j + \frac{\mu_j}{\Phi(\mu_j)}R_j^2\|F\|_{L^\infty(B_R\times(-R^2,0))}$$

for $j \geq 0$. Clearly $\mu_j \geq (1-2\sigma)^j\mu_0$. Using the algebraic growth (1.2) leads to

$$\frac{R_j^2\|F\|_{L^\infty(B_R\times(-R^2,0))}}{\Phi(\mu_j)} = \frac{\Phi(\mu_0)}{\Phi(\mu_j)}R_j^2\frac{\|F\|_{L^\infty(B_R\times(-R^2,0))}}{\Phi(\mu_0)}$$

(2.8)

$$\leq \left(\frac{(1-2\sigma)^{-1/a}}{65}\right)^j \frac{R^2\|F\|_{L^\infty(B_R\times(-R^2,0))}}{\Phi(\mu_0)} \leq \sigma,$$

where the last inequality follows from the definition of σ and from the bound

$$\mu_0 \geq \Phi^{-1}\left(\sigma^{-1}R^2\|F\|_{L^\infty}\right) \implies \Phi(\mu_0) \geq \sigma^{-1}R^2\|F\|_{L^\infty}.$$

Similarly, we get

$$\frac{\mu_{j+1}}{\Phi(\mu_{j+1})}\left(\frac{R_j}{8}\right)^2 = \frac{1}{16}\frac{\mu_{j+1}}{\mu_j}\frac{\Phi(\mu_j)}{\Phi(\mu_{j+1})}\frac{\mu_j}{\Phi(\mu_j)}\left(\frac{R_j}{2}\right)^2 \le \frac{\mu_j}{\Phi(\mu_j)}\left(\frac{R_j}{2}\right)^2.$$

Therefore, we conclude that

$$(1-2\sigma)\mu_j \le \mu_{j+1} \le (1-\sigma)\mu_j \qquad \text{and} \qquad Q_{R_{j+1}}^{\mu_{j+1}} \subset Q_{R_j/2}^{\mu_j},$$

and alternatives reduce to

$$\text{either} \qquad \inf_{Q_{R_{j+1}}^{\mu_j}} w \ge \sigma\mu_j \qquad \text{or} \qquad \sup_{Q_{R_{j+1}}^{\mu_{j+1}}} w \le \mu_{j+1},$$

provided that $\sup_{Q_{R_j}^{\mu_j}} w \le \mu_j$. Observe that $\sup_{Q_{R_0}^{\mu_0}} w \le \mu_0$ by the definition of μ_0. Since we are considering positive solutions $w > 0$ and $\mu_j \to 0$ as $j \to \infty$ the degenerate alternative can clearly occur at most a finite number of times, thus

$$\sigma\mu_J \le \inf_{Q_{R_J/8}^{\mu_J}} w \le \sup_{Q_{R_J/8}^{\mu_J}} w \le \mu_J \quad \text{for some finite } J.$$

Since $\sigma = \sigma(a,n)$ only, the algebraic growth (1.2) then readily implies that

$$\frac{1}{c(a,n)}\frac{\Phi(\mu_J)}{\mu_J} \le \frac{\Phi(w)}{w} \le c(a,n)\frac{\Phi(\mu_J)}{\mu_J} \qquad \text{in} \quad Q_{R_J/8}^{\mu_J}.$$

Scaling as in step 4 and writing $\partial_t u_i = \Delta_\varrho u_i + f_i = \text{div}(\varrho\nabla u_i) + (\ldots)$, where $\varrho = \Phi(w)/w$, we see that each scaled component $\overline{u}_i$ solves in $B_{1/8} \times (-1,0)$ a uniformly parabolic linear equation $\partial_t \overline{u}_i = \text{div}(\overline{\varrho}\nabla\overline{u}_i) + (\ldots)$ in divergence form with a measurable coefficient $\overline{\varrho}$ satisfying $c(a,n) \le \overline{\varrho} \le c(a,n)^{-1}$. In particular, $\overline{u}_i$ are Hölder continuous and satisfy a DeGiorgi-Nash-Moser oscillation estimate which, scaling back to u_i, takes the explicit form

$$\underset{Q_{\theta R_J}^{\mu_J}}{\text{osc}}\ u_i \le c\theta^\beta\mu_J + c\frac{\mu_J}{\Phi(\mu_J)}(\theta R_J)^2\|f_i\|_{L^\infty(B_R\times(-R^2,0))},$$

for some $\beta = \beta(a,n)$, $c = c(a,n)$ only and for all $\theta \in (0,1)$. Observing that $0 \le f_i \le |\mathbf{f}|_1 = F$ and recalling estimate (2.8) which holds for all j and with $\sigma = \sigma(a,n)$, we obtain

$$\forall\,\theta \in (0,1): \qquad \underset{Q_{\theta R_J}^{\mu_J}}{\text{osc}}\ u_i \le c\,(\theta^\beta + \theta^2)\mu_J.$$

Setting

$$\alpha = \alpha(a,n) := \min\{\beta, -\log(1-\sigma)/\log 8, 2\}$$

and increasing the constant c by a factor depending only on n,a, we conclude that

$$\underset{B_r\times(-r^2,0)}{\text{osc}}\ u_i \le c\left(\frac{r}{R}\right)^\alpha\left(\frac{\Phi(\mu_0)}{\mu_0}\right)^{\alpha/2}\mu_0$$

for all $r \in (0,R]$. This yields the desired interior Hölder continuity estimate after standard manipulations. $\qquad\square$

Assuming regularity and compatibility from the data, the solution can be shown to be Hölder continuous up to the boundary.

Proposition 2.4. *Let $\mathbf{u}$, M be as in Proposition 2.1 and $\alpha = \alpha(a,n)$ as in Proposition 2.3. Assume further that the initial and boundary data are compatible and β-Hölder continuous with some $\beta \in (0,1)$, i.e. there is $\mathbf{U} \in \mathcal{C}^{\beta,\beta/2}(\overline{Q^T})$ such that $\mathbf{u}^0 = \mathbf{U}(\,.\,,0)$ and $\mathbf{z}^D = \frac{\Phi(|\mathbf{U}|_1)}{|\mathbf{U}|_1}\mathbf{U}$ in Σ_T. Then $\mathbf{u} \in \mathcal{C}^{\gamma,\gamma/2}(\overline{Q^T})$, with $\gamma = \min(\alpha,\beta)$. Moreover, the Hölder norm of $\mathbf{u}$ depends only on a,n,N,M,T and on the β-Hölder norm of the data.*

Proof. Our assumptions on the data turn into similar compatibility and regularity conditions for the scalar problem (GPME). A straightforward modification of our interior argument (see, e.g., [15, 30]) shows that w is $\mathcal{C}^{\gamma,\gamma/2}$ up to the boundary, which in turn yields the same regularity for u_i through the linear parabolic equation $\partial_t u_i = \Delta(\varrho u_i) + f_i = \mathrm{div}(\varrho \nabla u_i) + \mathrm{div}(u_i \nabla \varrho) + f_i$. $\qquad\square$

3. Weak solutions

Let us first introduce different notions of solutions.

Definition 1 (weak solutions)**.**

   (i) A non-negative function $w \in L^\infty(Q_T)$ is called bounded very weak solution of (GPME) if the equality

(3.1)
$$\int_{Q_T} \{w\partial_t\varphi + \Phi(w)\Delta\varphi + F\varphi\}\,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega w^0(x)\varphi(x,0)\,\mathrm{d}x + \int_{\Sigma_T} g^D\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}x\,\mathrm{d}t$$

      holds for all $\varphi \in \mathcal{C}^{2,1}(\overline{Q_T})$ vanishing on Σ_T and in $\Omega \times \{t = T\}$.

  (ii) A non-negative function $w \in L^\infty(Q_T)$ is called a bounded weak energy solution of (GPME) if $\Phi(w) \in L^2(0,T;H^1(\Omega))$, the trace $\gamma(\Phi(w)) = g^D$ in $L^2(0,T;H^{1/2}(\partial\Omega))$ and the equality

(3.2)
$$\int_{Q_T} \{w\partial_t\varphi - \nabla\Phi(w)\cdot\nabla\varphi + F\varphi\}\,\mathrm{d}x\mathrm{d}t = -\int_\Omega w^0(x)\varphi(x,0)\,\mathrm{d}x$$

      holds for all $\varphi \in \mathcal{C}^{2,1}(\overline{Q_T})$ vanishing on Σ_T and in $\Omega \times \{t = T\}$.

 (iii) A function $\mathbf{u} = (u_1,\ldots,u_N) \in L^\infty(Q_T)$ is called a (non-negative) bounded very weak solution of (1.5) if $u_i \geq 0$ a.e. in Q_T and the equality

(3.3)
$$\int_{Q_T} \{u_i\partial_t\varphi + \varrho u_i\Delta\varphi + f_i\varphi\}\,\mathrm{d}x\mathrm{d}t + \int_\Omega u_i^0(x)\varphi(x,0)\,\mathrm{d}x = \int_{\Sigma_T} z_i^D\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}x\mathrm{d}t,$$

      where $\varrho = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}$, holds for any $i = 1,\ldots,N$ and for all $\varphi \in \mathcal{C}^{2,1}(\overline{Q_T})$ vanishing on Σ_T and in $\Omega \times \{t = T\}$.

  (iv) A function $\mathbf{u} \in L^\infty(Q_T)$ is called a (non-negative) bounded weak energy solution of (1.5) if $u_i \geq 0$ a.e. in Q_T, $(\varrho u_i) \in L^2(0,T;H^1(\Omega))$, the trace $\gamma(\varrho u_i) = z_i^D$ in $L^2(0,T;H^{1/2}(\partial\Omega))$, and the equality

(3.4)
$$\int_{Q_T} \{w\partial_t\varphi - \nabla(\varrho u_i)\cdot\nabla\varphi + f_i\varphi\}\,\mathrm{d}x\mathrm{d}t = -\int_\Omega z_i^0(x)\varphi(x,0)\,\mathrm{d}x,$$

      where $\varrho = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}$, holds for any $i = 1,\ldots,N$ and for all $\varphi \in \mathcal{C}^{2,1}(\overline{Q_T})$ vanishing on Σ_T and in $\Omega \times \{t = T\}$.

The notion of very weak solutions preserves the diagonal structure of the system as shown in the following lemma.

Lemma 3.1. *If $\mathbf{u}$ is a non-negative bounded very weak (resp. energy) solution of system (1.5) in the sense of Definition 1 then $w = |\mathbf{u}|_1$ is a non-negative bounded very weak (resp. energy) solution to problem (GPME) in the sense of Definition 1.*

Proof. Sum equalities (3.3) over i from 1 to N and observe that by definition $\sum_i \varrho u_i = \varrho \sum_i u_i = \varrho|\mathbf{u}|_1 = \Phi(w)$, $\sum_i f_i = F$, $\sum_i u_i^0 = |\mathbf{u}^0|_1 = w^0$, and $\sum_i z_i^D = |\mathbf{z}^D|_1 = g^D$. $\qquad\square$

We will now address uniqueness. Note that the following proposition guarantees uniqueness also within the class of energy solutions since weak energy solutions are in particular very weak solutions.

Proposition 3.1 (Uniqueness). *Given the non-negative and bounded data* $\mathbf{f}, \mathbf{u}^0, \mathbf{z}^D$ *there exists at most one non-negative bounded very weak solution to problem* (1.5) *in the sense of Definition 1.*

Proof. Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two solutions to problem (1.5), corresponding to the same initial and boundary data. It follows from the previous lemma that $w^1 = |\mathbf{u}^1|_1$ and $w^2 = |\mathbf{u}^2|_1$ are both bounded very weak solutions to the same Cauchy-Dirichlet problem (GPME). A standard comparison result for such solutions [30, Theorem 6.5] provides uniqueness within this class. Thus $w^1 = w^2 = w$ and, in particular, the pressures coincide, $\varrho^1 = \varrho^2 = \varrho = \frac{\Phi(w)}{w}$. Next, we use a duality proof, as in proving the comparison results for GMPE, to show that $\mathbf{u}^1 = \mathbf{u}^2$. In fact, the situation here is simpler because we already know that $\varrho^1 = \varrho^2$. For the sake of completeness, we nonetheless give the details.

Fixing any $i \in 1 \ldots N$, denoting $\tilde{u} = u_i^1 - u_i^2$, and subtracting the weak formulation (3.3) satisfied by u^2 from that satisfied by u^1, we see that

$$
(3.5) \qquad \int_{Q_T} \{\tilde{u}\partial_t\varphi + \varrho\tilde{u}\Delta\varphi\}\ \mathrm{d}x\mathrm{d}t = 0
$$

for all $\varphi \in \mathcal{C}^{2,1}(\overline{Q_T})$ vanishing on $\Sigma_T \cup \{t = T\}$. Fix some arbitrary $\theta \in \mathcal{C}_c^\infty(Q_T)$, choose $\varepsilon > 0$, and let $\varrho_\varepsilon = \max\{\varrho, \varepsilon\}$. Since $\mathbf{u}^1$ and $\mathbf{u}^2$ are bounded so is $\varrho = \varrho^1 = \varrho^2$, and we can construct a smooth approximation $\{\varrho_{\varepsilon,k}\}_{k\in\mathbb{N}}$ to ϱ_ε such that $\varepsilon \le \varrho_{\varepsilon,k} \le C$. For fixed ε, k we can then solve the approximate dual backward equation

$$
(3.6) \qquad \begin{cases} \partial_t\varphi + \varrho_{\varepsilon,k}\Delta\varphi = \theta & \text{in } Q_T \\ \varphi = 0 & \text{in } \Sigma_T \\ \varphi(.,T) = 0 & \text{in } \Omega. \end{cases}
$$

for a unique $\varphi = \varphi_{\varepsilon,k} \in \mathcal{C}^{2,1}(\overline{Q_T}) \cap \mathcal{C}^\infty(Q_T)$. Since φ vanishes by construction on Σ_T and in $\Omega \times \{t = T\}$ it is admissible as a test function in (3.5). This gives

$$
\left|\int_{Q_T} \tilde{u}\theta\ \mathrm{d}x\mathrm{d}t\right| = \left|\int_{Q_T} \tilde{u}(\varrho - \varrho_{\varepsilon,k})\Delta\varphi\ \mathrm{d}x\mathrm{d}t\right|
$$

$$
\le \left(\int_{Q_T} \tilde{u}^2 \frac{|\varrho - \varrho_{\varepsilon,k}|^2}{\varrho_{\varepsilon,k}}\ \mathrm{d}x\mathrm{d}t\right)^{1/2} \left(\int_{Q_T} \varrho_{\varepsilon,k}|\Delta\varphi|^2\ \mathrm{d}x\mathrm{d}t\right)^{1/2}
$$

$$
\le \frac{C}{\varepsilon^{1/2}}\|\varrho - \varrho_{\varepsilon,k}\|_{L^2(Q_T)} \left(\int_{Q_T} \varrho_{\varepsilon,k}|\Delta\varphi|^2\ \mathrm{d}x\mathrm{d}t\right)^{1/2},
$$

because $\tilde{u} \in L^\infty$ and $\varrho_{\varepsilon,k} \ge \varepsilon$. Since φ is smooth, a straightforward computation shows that (cf. [30, Theorem 6.5])

$$
\left(\int_{Q_T} \varrho_{\varepsilon,k}|\Delta\varphi|^2\ \mathrm{d}x\mathrm{d}t\right)^{1/2} \le C\|\nabla\theta\|_{L^2(Q_T)}
$$

for some $C > 0$ independent of ε, k, θ. For fixed $\varepsilon > 0$ we can then choose k large enough such that $|\varrho_\varepsilon - \varrho_{\varepsilon,k}|_{L^2(Q_T)} \le \varepsilon$. By definition of the cutoff function ϱ_ε, we

have $0 \leq \varrho_\varepsilon - \varrho \leq \varepsilon$. Hence $|\varrho - \varrho_\varepsilon|_{L^2(Q_T)} \leq \varepsilon |Q_T|^{1/2}$ so that $|\varrho - \varrho_{\varepsilon,k}|_{L^2(Q_T)} \leq |\varrho - \varrho_\varepsilon|_{L^2(Q_T)} + |\varrho_\varepsilon - \varrho_{\varepsilon,k}|_{L^2(Q_T)} \leq C\varepsilon$, and we obtain

$$\left| \int_{Q_T} \tilde{u}\theta \, \mathrm{d}x\mathrm{d}t \right| \leq C\varepsilon^{1/2} \|\nabla\theta\|_{L^2(Q_T)}.$$

Because $\theta \in \mathcal{C}_c^\infty(Q_T)$ was arbitrary and ε was independent of θ we conclude letting $\varepsilon \to 0$ that $\tilde{u} = u_i^1 - u_i^2 = 0$ a.e. in Q_T and the proof is complete. □

Remark 3.1. The above uniqueness proof does not really require $L^\infty(Q_T)$ bounds but merely that $\mathbf{u}, \varrho\mathbf{u} \in L_{loc}^2(Q_T)$. In fact, scalar parabolic equations such as (GPME) benefit usually from smoothing properties that should allow one to extend the theory to L^1 data. Due to the coupled vectorial nature of the problem and the lack of space we shall not pursue this direction here.

Theorem 1.1 allows for the initial data $\mathbf{u}^0$ to vanish identically in some ball $B_r(x_0) \subset \Omega$. As we will see in Section 4, this leads to free boundaries in the degenerate case $\Phi'(0) = 0$. It will also become clear in the proof that the structural condition (H_3) needs to be enforced only to get the estimate (1.9). In fact, this energy estimate plays no role whatsoever in the analysis so one may actually dispense with it and limit oneself to very weak solutions of (1.5).

Proof of Theorem 1.1. Uniqueness follows from Proposition 3.1. The existence argument is based on a "lifting" technique, classical for scalar GPME and working here thanks to the diagonal structure of the system.

We first lift and approximate the bounded non-negative data $\mathbf{u}^0, \mathbf{f}, \mathbf{z}^D$ componentwise by smooth functions $u_i^{0,k}, f_i^k$ and $z_i^{D,k}$ such that $\frac{1}{k} \leq u_i^{0,k}, f_i^k, z_i^{D,k} \leq C + \frac{1}{k}$ for some constant $C > 0$ depending only on the data, and

$$\|\mathbf{u}^{0,k} - \mathbf{u}^0\|_{L^1(\Omega)} + \|\mathbf{f}^k - \mathbf{f}\|_{L^1(Q_T)} + \|\mathbf{z}^{D,k} - \mathbf{z}^D\|_{L^1(\Sigma_T)} \to 0$$

as $k \to \infty$. By Proposition 2.1, given the smooth data $\mathbf{u}^{0,k}, \mathbf{f}^k, \mathbf{z}^{D,k}$ there exists a positive classical solution $\mathbf{u}^k$ to (1.5) which is bounded in Q_T uniformly in k. By virtue of Proposition 2.3 $\{\mathbf{u}^k\}_k$ is also bounded in $\mathcal{C}^{\alpha,\alpha/2}(Q')$ for any subdomain Q' and for some $\alpha = \alpha(a,n) \in (0,1)$. By diagonal extraction we may then assume that $\mathbf{u}^k \to \mathbf{u}$ in $\mathcal{C}_{\mathrm{loc}}(Q_T)$ with the limit function $\mathbf{u}$ satisfying the local $\mathcal{C}^{\alpha,\alpha/2}$-estimate (1.8). In particular $\mathbf{u}^k(x,t) \to \mathbf{u}(x,t) \geq 0$ pointwise in Q_T. From the continuity of Φ with $\lim_{s \to 0} \frac{\Phi(s)}{s} = \Phi'(0)$ it then follows that $\varrho^k = \frac{\Phi(|\mathbf{u}^k|_1)}{|\mathbf{u}^k|_1} \to \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} = \varrho$ a.e. in Q_T. Since $\mathbf{u}^k, \varrho^k$ are bounded uniformly in $L^\infty(Q_T)$ we conclude by dominated convergence that $u_i^k \to u_i$ and $\varrho^k u_i^k \to \varrho u_i$ in $L^p(Q_T)$ for all $p \in [1, \infty)$. Given that $\mathbf{u}^k$ is a smooth positive solution and that for all $i = 1, \ldots, N$ it holds

$$\int_{Q_T} \{u_i^k \partial_t \varphi + \varrho^k u_i^k \Delta\varphi + f_i^k \varphi\} \, \mathrm{d}x\mathrm{d}t + \int_\Omega u_i^{0,k}(x)\varphi(x,0)\mathrm{d}x = \int_{\Sigma_T} z_i^{D,k} \frac{\partial\varphi}{\partial\nu} \mathrm{d}x\mathrm{d}t.$$

The previous strong $L^p(Q_T)$ convergence and convergence of the data allow one to send $k \to \infty$ to obtain (3.3). Similarly by Lemma 3.1 we see that $w = \lim w^k = \lim |\mathbf{u}^k|_1 = |\mathbf{u}|_1$ is a very weak solution to (GPME).

Regarding the energy estimates, if the data satisfy (1.7) and (1.6), then they can be approximated as before by smooth positive data satisfying in addition

$$\|\mathbf{u}^{0,k}\|_{L^\infty(Q_T)} + \|\mathbf{f}^k\|_{L^\infty(Q_T)}$$
$$+ \|\mathbf{z}^{D,k}\|_{L^\infty(Q_T)} + \|\mathbf{z}^{D,k}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t \mathbf{z}^{D,k}\|_{L^\infty(Q_T)} \leq C.$$

By Proposition 2.2 we get

$$\|\nabla(\varrho^k w^k)\|_{L^2(Q_T)} \leq C$$

and
$$\|\nabla(\varrho^k u_i^k)\|_{L^2(Q_T')} \le C(1+1/d') \qquad \forall i = 1 \ldots N$$

uniformly in k for some $C = C(a, n, N, T, M)$ only. Since $\varrho^k u_i^k, \varrho^k w^k \to \varrho u_i, \varrho w$ we conclude that $\nabla(\varrho u_i), \nabla(\varrho w)$ satisfy the same L^2 bounds and the proof is complete.
$\square$

4. Free boundaries

In this section we set $Q = \mathbb{R}^n \times (0, \infty)$, $Q^{\tau, T} = \mathbb{R}^n \times (\tau, T)$, $Q^T = \mathbb{R}^n \times (0, T)$, and consider the Cauchy Problem

$$(4.1) \qquad \begin{cases} \partial_t \mathbf{u} = \Delta \left(\frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1} \mathbf{u} \right) & \text{in } \mathbb{R}^n \times (0, \infty) \\ \mathbf{u}(x, 0) = \mathbf{u}^0(x) & \text{in } \mathbb{R}^n \end{cases}$$

with a non-negative, bounded and compactly supported initial data $\mathbf{u}^0$. If the modulus of ellipticity $\varrho = \frac{\Phi(|\mathbf{u}|_1)}{|\mathbf{u}|_1}$ in (4.1) vanishes when $|\mathbf{u}|_1 = 0$ (the degenerate case), compactly supported solutions should evolve from compactly supported initial data. By analogy with scalar equations, the free boundary $\Gamma(t) := \partial \operatorname{supp} \mathbf{u}(\,.\,, t)$ should then propagate with finite speed in the sense that, at any $x_0 \notin \operatorname{supp} \mathbf{u}(\,.\,, t_0)$ we should have $x_0 \notin \operatorname{supp} \mathbf{u}(\,.\,, t_0 + h)$ for small enough $h > 0$. Although this behaviour is well understood for scalar equations, the coupled nature of system (4.1) prevents us from just recalling known results. Instead, we will again resort to our central idea, based on the particular structure of the system, that controlling $w = |\mathbf{u}|_1$ controls each individual species u_i. Indeed, $0 \le u_i \le |\mathbf{u}|_1 = w$, thus $\operatorname{supp} u_i \subset \operatorname{supp} w$ and $\mathbf{u}$ will propagate with finite speed as long as w does. This will in turn be ensured by looking at the scalar problem

$$(4.2) \qquad \begin{cases} \partial_t w = \Delta \Phi(w) & \text{in } \mathbb{R}^n \times (0, \infty) \\ w(x, 0) = w^0(x) & \text{in } \mathbb{R}^n \end{cases}$$

with a non-negative, bounded and compactly supported initial data $w^0 = |\mathbf{u}^0|_1$.

Given that assumption (H_2) does not rule out nondegenerate diffusion (we may have $\Phi'(0) > 0$) for which the finite speed of propagation obviously fails, we need to impose an extra degeneracy condition. For the scalar Cauchy problem (4.2) this is normally done through replacing the structural condition (H_2) by the *slow diffusion* hypothesis

$$(S_a) \qquad\qquad s > 0: \qquad 1 + a \le \frac{s\Phi'(s)}{\Phi(s)} \le \frac{1}{a}.$$

One readily sees that condition (S_a) implies $\Phi'(0) = 0$ and the algebraic behaviour $0 \le \Phi(1)s^{\frac{1}{a}} \le \Phi(s) \le \Phi(1)s^{1+a}$ for small s, which usually provides information on the speed of propagation in terms of a. However, since our analysis should include the Freundlich isotherm $\mathbf{b}_f(\mathbf{z}) = (\phi + (1 - \phi)|\mathbf{z}|_1^{p-1})\mathbf{z}$ for which the corresponding $\Phi_f(s)$ behaves linearly at infinity, we cannot assume (S_a) globally in $s > 0$ as the lower bound $1 < cst \le \frac{s\Phi'(s)}{\Phi(s)}$ is not admissible for large s. In fact, given that the degeneracy is essentially a local feature at the level sets $\{s \approx 0\}$ we could require condition (S_a) to hold only for $0 < s \le s_0$ which is certainly true for our Freundlich isotherm with $1 + a = 1/p > 1$. However, we prefer to avoid this technical path and, instead, impose the less restrictive degeneracy condition

$$(S_m) \qquad \forall s > 0: \qquad \int_0^s \Phi(s')ds' \ge c\,\Phi(s)^{\frac{m+1}{m}} \qquad \text{for some } m > 1.$$

This implies that $\Phi'(0) = 0$ and is valid for the pure PME nonlinearity $\Phi(s) = s^m$ with $c = m + 1$ if $m > 1$.

For technical reasons it will be convenient to reformulate (S_m) in terms of the original concentration $\mathbf{z} = \mathbf{b}^{-1}(\mathbf{u})$. It is easy to see that the change of variables $r = \Phi(s) = \beta^{-1}(s)$ turns (S_m) into the equivalent condition

$$(S'_m) \quad \forall r > 0 : \qquad f(r) := r\beta(r) - \int_0^r \beta(r')dr' \geq cr^{\frac{m+1}{m}} \qquad \text{for some } m > 1,$$

from which it follows that $\beta'(0) = \infty$, as expected since $\beta = \Phi^{-1}$. An explicit computation shows that (S'_m) holds true globally in $r > 0$ for the Freundlich isotherm $\beta_f(r) = \phi r + (1 - \phi)r^p$ with $c = (1 - \phi)^{-1/p}$ and $m = \frac{1}{p} > 1$, showing that (S_m), equivalently (S'_m), is indeed weaker than (S_a).

In the case of pure PME nonlinearity $\Phi(s) = s^m$ the Cauchy problem (4.2) has been widely studied and the qualitative and quantitative theory of free boundaries is now well understood, see e.g. [19, 20] and references therein. Partial results [12, 9] also hold for general nonlinearities $\Phi(s)$ but to the best of our knowledge one always assumes the degeneracy condition in the form (S_a), which fails for the Freundlich isotherm.

We will start our analysis with a standard statement and prove it assuming only the weaker condition (S_m).

Proposition 4.1. *Assume that conditions (H_1)-(H_2) and (S_m) hold for some $a \in (0,1)$ and $m > 1$, and let the initial datum $0 \leq w^0(x) \leq M$ be compactly supported in B_{R_0} for some $R_0 > 0$. Then the Cauchy problem (4.2) admits a unique weak energy solution $0 \leq w(x,t) \leq M$ and $\|\nabla\Phi(w)\|_{L^2(Q)} \leq C(a, R_0, M, n)$. Moreover, $w(\,.\,,t)$ is compactly supported for all $t > 0$, the free boundary $\Gamma(t) = \partial \operatorname{supp} w(\,.\,,t)$ propagates with finite speed, and*

$$(4.3) \qquad \forall t \geq 0 : \qquad \operatorname{supp} w(\,.\,,t) \subseteq B_{R(t)} \quad \text{with } R(t) := R_0 + C_1 t^\lambda,$$

with some constants $C_1(m, a, M, R_0) > 0$ and $\lambda = \lambda(m, n) > 0$.

Proof. Existence and uniqueness are proven in [30]. Since $0 \leq w^0 \leq M$, the comparison principle gives $0 \leq w \leq M$ in Q. The $L^2(Q)$-bound for $\nabla\Phi(u)$ easily follows from letting $t \to \infty$ in the classical energy identity

$$(4.4) \qquad \int_{B_R} \Psi(w(t,x))\,\mathrm{d}x + \int_0^t \int_{B_R} |\nabla\Phi(w(x,\tau))|^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_{B_R} \Psi(w^0(x))\,\mathrm{d}x$$

where $\Psi(s) := \int_0^s \Phi(s')\,\mathrm{d}s'$ (see [30] for details). Indeed with our assumptions w^0 is bounded and compactly supported hence $\|\Psi(w^0)\|_{L^1(\mathbb{R}^n)} \leq \Psi(M)\operatorname{meas}(B_{R_0}) = C(a, M, R_0, n)$.

As for the speed of propagation of the support, we go back to the original concentration formulation and recall from [13] two results based on the energy methods introduced in [3]. To verify that the assumptions in [13] are satisfied, we set $z := \Phi(w) = \beta^{-1}(w)$ and observe that $\partial_t\beta(z) = \Delta z$ in Q. Making the change of variables $r = \Phi(s)$ in (S'_m) shows that $f(z(x,t)) = \Psi(w(x,t))$, and the energy identity (4.4) readily gives

$$E(z;t) := \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^n} f(z(x,\tau))\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^n} |\nabla z(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C(a, M, R_0, n).$$

Defining $j(r) := \int_0^r \beta(r')\,\mathrm{d}r'$ and making the change $r = \Phi(s)$, we obtain for all $0 \leq w_1, w_2 \leq M$ the bound

$$|j(\Phi(w_1)) - j(\Phi(w_2))| = \left| \int_{\Phi(w_2)}^{\Phi(w_1)} \beta(r)\,\mathrm{d}r \right| = \left| \int_{w_1}^{w_2} s\Phi'(s)\,\mathrm{d}s \right|$$

$$\leq \left| \int_{w_1}^{w_2} \frac{1}{a} \Phi(s) \, \mathrm{d}s \right| \leq \frac{\Phi(M)}{a} |w_1 - w_2|.$$

Since $0 \leq w(x, t) \leq M$ and $z = \Phi(w) \leq \Phi(M) \leq C(a, M)$ we have in particular

$$\forall t_1, t_2 \in [0, T]: \qquad \|j(z(t_1)) - j(z(t_2))\|_{L^1(\mathbb{R}^n)} \leq C \|w(t_1) - w(t_2)\|_{L^1(\mathbb{R}^n)}.$$

Given our assumptions on w^0, we can infer from the classical theory [30] for the scalar Cauchy problem (4.2) that $w \in \mathcal{C}([0, T]; L^1(\mathbb{R}^n))$. This in turn yields $j(z) \in \mathcal{C}(0, T; L^1(\mathbb{R}^n))$.

The energy estimate $E(z; t) \leq C$ and the continuity of $j(z)$ allow us to apply [13, Corollary 3.1] and we conclude that $z(\,.\,, t)$ is compactly supported, satisfies (4.3). Similarly, applying [13, Theorem 3.1] shows that the free boundary $\Gamma(t) = \partial \operatorname{supp} w(\,.\,, t)$ propagates with finite speed and the proof is complete. $\qquad \square$

We can now establish the corresponding result on the multicomponent Cauchy problem (4.1).

Theorem 4.1 (Free Boundary solutions). *Let conditions* (H_1)-(H_2)-(H_3) *and* (S'_m) *hold for some* $a \in (0, 1)$ *and* $m > 1$. *Assume that* $\mathbf{u}^0 \in L^\infty(\mathbb{R}^n)$ *is componentwise non-negative with* $w^0 = |\mathbf{u}^0|_1 \leq M$, *and such that* $\operatorname{supp} w^0 \subseteq B_{R_0}$ *for some* $R_0 > 0$. *Then there exists a unique non-negative very weak solution* $\mathbf{u} \in L^\infty(Q)$ *to* (4.1). *Moreover,*

(i) $w = |\mathbf{u}|_1$ *is the unique weak energy solution to* (4.2), $0 \leq w \leq M$, *and*

$$\|\nabla \Phi(w)\|_{L^2(Q)} \leq C(a, M, R_0)$$

(ii) $\mathbf{u}$ *is a local energy solution to* (4.1) *in the sense that for all* $T > 0$ *we have*

$$\|\nabla(\varrho u_i)\|_{L^2(Q^T)} \leq C(a, M, R_0, T) \qquad \forall i = 1, \ldots N,$$

where $\varrho = \frac{\Phi(w)}{w}$.

(iii) $\operatorname{supp} w(\,.\,, t)$ *propagates with finite speed, and*

$$\forall t \geq 0, \, i = 1 \ldots N: \qquad \operatorname{supp} u_i(\,.\,, t) \subseteq \operatorname{supp} w(\,.\,, t) \subseteq B_{R(t)}$$

with $R(t) = R_0 + C_1 t^\lambda$ *for some* $C_1(m, a, M, R_0) > 0$ *and* $\lambda = \lambda(m, n) > 0$ *only.*

(iv) *There is* $\alpha = \alpha(a, n) \in (0, 1)$ *such that* $\mathbf{u}$ *is* $(\alpha, \alpha/2)$-*Hölder continuous in any strip* $Q^{\tau, T} = \mathbb{R}^n \times (\tau, T)$, $0 < \tau < T$, *and*

$$\|\mathbf{u}\|_{C^{\alpha, \alpha/2}(Q^{\tau, T})} \leq C(1 + 1/\sqrt{\tau}).$$

for some $C(a, T, n, N, M) > 0$ *only.*

(v) *If* Φ *is smooth in* $\mathbb{R}^+$ *then* u_i *is smooth in* $\{u_i > 0\} \cap \{t > 0\}$.

Remark 4.1. As in Theorem 1.1, the structural condition (H_3) is only needed to get (ii) and can be relaxed by restricting (4.1) to very weak solutions instead of energy solutions. Moreover, as in Proposition 2.4, the Hölder regularity estimate (iv) can be extended up to $t = 0^+$ if we assume further $\mathcal{C}^\beta(\mathbb{R}^n)$ regularity from $\mathbf{u}^0$. Note in particular that in (iii) we only claim that w has finite speed of propagation but not that the individual species propagate with finite speed. In fact the support of each species should in general be discontinuous in time, see the discussion at the end of this section.

Proof. Arguing exactly as in the proof of Proposition 3.1 but choosing now test function $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^n \times (0, \infty))$ it is easy to show uniqueness within the class of very weak solutions. It is therefore enough to prove existence in finite time intervals $[0, T]$ for any fixed $T > 0$.

Given that the free boundary should a priori propagate with finite speed, solutions to the Cauchy problem in the whole space should agree with solutions of the

Cauchy-Dirichlet problem in $B_R \times (0,T)$ with zero boundary conditions, as long as $R > 0$ is large enough so that the free boundary stays at a positive distance from ∂B_R for all $t \leq T$. We should therefore be able to construct solutions to the Cauchy problem in $\mathbb{R}^n \times (0,T)$ by considering auxiliary Dirichlet problems in large balls and using Theorem 1.1.

From Proposition 4.1 we can define the unique solution $\overline{w}$ of the Cauchy problem (4.2) in $\mathbb{R}^n \times (0,\infty)$ with initial data $w^0 := |\mathbf{u}^0|_1$. For any fixed $T > 0$ choose $R > 0$ large enough so that $R(T) \leq R/2$ and

$$\forall t \leq T : \qquad \operatorname{supp} \overline{w}(\,.\,,t) \subseteq B_{R(T)} \subseteq B_{R/2},$$

where $R(T) = R_0 + C_1 T^\lambda$ and the constants C_1 and λ are as in Proposition 4.1. Next, let $\mathbf{u} = \mathbf{u}_R$ be the unique solution to problem (1.5) in $B_R \times (0,T)$ corresponding to the initial data $\mathbf{u}^0$ and zero boundary values on ∂B_R and given by Theorem 1.1. It also follows from Theorem 1.1 that $w := |\mathbf{u}|_1$ is a weak solution to the corresponding Cauchy-Dirichlet problem in $B_R \times (0,T)$ with zero boundary values. Given the definition of R we thus see that $\overline{w}$ remains at a distance $R/2$ away from ∂B_R. It is easy to see that the restriction $\overline{w}_{|B_R \times (0,T)}$ is also a weak solution to the same Cauchy-Dirichlet problem as w. By standard uniqueness theorem for weak solutions of (GPME) we conclude that $w = \overline{w}$ in $B_R \times (0,T)$. In particular

$$\forall\, t \leq T,\, i = 1 \ldots N : \qquad \operatorname{supp} u_i(\,.\,,t) \subseteq \operatorname{supp} w(\,.\,,t) = \operatorname{supp} \overline{w}(\,.\,,t) \subseteq B_{R(t)}$$

where $R(t) = R_0 + C_1 t^\lambda$ and the distance between the support of $\mathbf{u}$ and ∂B_R is at least $R/2 > 0$ for all $t \leq T$. Extending $\mathbf{u}$ and w outside B_{R_0} by zero for all $t \in (0,T)$ it is then a simple exercise to verify that these extensions satisfy the weak formulations of (4.1) and (4.2) in the whole space, whence existence of free-boundary solutions in $\mathbb{R}^n \times (0,T)$ for arbitrary $T > 0$. The energy estimate (i) and propagation properties (iii) immediately follow from the definition of $\overline{w}$ and Proposition 4.1.

For fixed $T > 0$ take now $R > 0$ large enough so that $\mathbf{u}$ stays supported in B_R for all $t \leq T$. Viewing $\mathbf{u}$ as the unique solution to the Cauchy-Dirichlet problem in B_{R+1} and taking $\Omega = B_{R+1}$, $\Omega' = B_R$ in Theorem 1.1 we have $d' = 1$ in (1.9), thus

$$\|\nabla(\varrho u_i)\|_{L^2(\mathbb{R}^n \times (0,T))} = \|\nabla(\varrho u_i)\|_{L^2(B_R \times (0,T))} \leq C(a,n,N,M,T)$$

as claimed in (ii).

Assertion (iv) is proven similarly by considering the Cauchy-Dirichlet problem in B_{R+1}, choosing $\Omega' = B_R \subset B_{R+1} = \Omega$ in Theorem 1.1 and taking $d' = 1$ in estimate (1.8).

To prove (v) we use a local bootstrap argument. If $w > 0$ in some $B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$, $t_0 > 0$, then in particular the pressure $p = \Phi'(w) > 0$ there. Since w is Hölder continuous and Φ is smooth also p is Hölder continuous. Moreover, w solves a uniformly parabolic equation in divergence form: $\partial_t w = \Delta \Phi(w) = \operatorname{div}(p\nabla w)$. Hence $w \in C^{1+\beta}$ for some β. By bootstrapping we immediately see that w is locally smooth. Consider now any species u_i and observe that if $u_i > 0$ in $B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$ then $w = |\mathbf{u}|_1 \geq u_i > 0$ and also $\varrho = \Phi(w)/w > 0$. Since u_i solves $\partial_t u_i = \Delta(\varrho u_i) = \operatorname{div}(\varrho \nabla u_i) + \operatorname{div}(u_i \nabla \varrho)$ with now smooth coefficients we conclude that u_i is smooth and the proof is complete. $\qquad \square$

Although degenerate, problem (4.2) is nonetheless diffusive in nature and we expect that the information cannot propagate backwards as confirmed by the following result.

Proposition 4.2 (Persistence property)**.** *Under the hypotheses of Theorem 4.1, let us further assume that for any $M > 0$ there exists $\overline{a} = \overline{a}(M) > 0$ such that*

$$\forall s \in (0, M]: \qquad 1 + \overline{a} \leq \frac{s\Phi'(s)}{\Phi(s)}.$$

Then the support $\Omega(t) := \{x : w(x,t) > 0\}$ of $w = |\mathbf{u}|_1$ is non-contracting in time.

Remark 4.2. This (weaker) degeneracy condition is easily checked for the Freundlich isotherm. In fact, $s\Phi'(s)/\Phi(s) = \beta(r)/(r\beta'(r))$ is strictly greater than one for $\beta_f(r) = \phi r + (1-\phi)r^p$ in any finite interval (but not in the limit $s \to \infty$ because $\Phi_f(s)$ becomes linear).

Proof. Dahlberg and Kenig [7] proved that nonnegative solutions to (4.2) satisfy certain Harnack inequality provided the strong condition (S_a) holds. In this case positivity of w at (x_0, t_0) implies positivity at (x_0, t) for all later $t \geq t_0$ and the support is non-contracting. Unfortunately, as already discussed, (S_a) does not hold globally in $s > 0$.
In order to tackle this technical detail we recall from Theorem 4.1 that $0 \leq w \leq M$, with $M = \|w^0\|_{L^\infty(\mathbb{R}^n)}$. Now the assumption on $\Phi(s)$ allows one to construct a $\mathcal{C}^1$-function $\overline{\Phi}$ which satisfies (S_a) globally in $s > 0$ for some $a = a(M) \in (0, 1)$ and such that $\overline{\Phi}(s) = \Phi(s)$ for all $s \in [0, M]$. By construction w is a solution to $\partial_t w = \Delta\overline{\Phi}(w)$, and the assertion follows from [7] (for a precise statement see e.g. [12, Corollary 1.5]). □

We end this section with a "divide and rule" result.

Proposition 4.3. *Assume that (H_1), (H_2), (H_3) and (S'_m) hold true for some $a \in (0,1)$ and $m > 1$. Let $k \in \{1, \dots, N\}$, $\hat{\mathbf{u}}^0 = (u^0_1, \dots, u^0_k) \in L^\infty(\mathbb{R}^n; \mathbb{R}^k)$ and $\check{\mathbf{u}}^0 = (u^0_{k+1}, \dots, u^0_N) \in L^\infty(\mathbb{R}^n; \mathbb{R}^{N-k})$ be non-negative and compactly supported, with $d = \mathrm{dist}(\mathrm{supp}(\hat{\mathbf{u}}^0), \mathrm{supp}(\check{\mathbf{u}}^0)) > 0$. Set $\mathbf{u}^0 = (\hat{\mathbf{u}}^0, \check{\mathbf{u}}^0) \in L^\infty(\mathbb{R}^n; \mathbb{R}^{k+(N-k)})$. Moreover, let $\hat{\mathbf{u}}(x,t)$ be the unique solution of the k-dimensional Cauchy problem (4.1) with the initial data $\hat{\mathbf{u}}^0$, $\check{\mathbf{u}}(x,t)$ the unique solution of the $(N-k)$-dimensional Cauchy problem (4.1) with the initial data $\check{\mathbf{u}}^0$ and assume finally that $\mathbf{u}(x,t)$ is the unique solution of the N-dimensional Cauchy problem (4.1) with the initial data $\mathbf{u}^0 = (\hat{\mathbf{u}}^0, \check{\mathbf{u}}^0)$. Then there exists $T > 0$ such that $\mathbf{u} \equiv (\hat{\mathbf{u}}, \check{\mathbf{u}})$ in $Q^T = \mathbb{R}^n \times (0, T)$. More precisely, if $\hat{w}$ and $\check{w}$ are the two solutions of the scalar Cauchy problem (4.2) with the respective initial data $\hat{w}^0 = |\hat{\mathbf{u}}^0|_{l^1(\mathbb{R}^k)}$ and $\check{w}^0 = |\check{\mathbf{u}}^0|_{l^1(\mathbb{R}^{N-k})}$, then*

$$T = \inf\{t \geq 0 : \ \mathrm{supp}\,\hat{w}(\,.\,,t) \cap \mathrm{supp}\,\check{w}(\,.\,,t) \neq \emptyset\} \in (0, \infty].$$

Remark 4.3. As is clear from the above definition, T is the first time when the supports of $\hat{w}, \check{w}$ touch. Our statement can be reformulated simply as: if the initial data can be separated into two distinct patches of k and $N-k$ species then it suffices to solve two independent lower dimensional systems of order k and $N - k$ as long as their respective supports do not touch. Note that we do not claim anything for what happens after $t = T$ because when the supports touch the two patches start interacting and the situation becomes more involved.

Proof. From Theorem 4.1 it is clear that the supports of $\hat{\mathbf{u}}(\,.\,,t), \check{\mathbf{u}}(\,.\,,t)$ propagate with finite speed, which implies that $T > 0$. Letting now $\tilde{\mathbf{u}} = (\hat{\mathbf{u}}, \check{\mathbf{u}}) \in L^\infty(\mathbb{R}^n \times (0, T); \mathbb{R}^{k+(N-k)})$, we observe that by definition of T

$$(x, t) \in \mathrm{supp}(\hat{\mathbf{u}}) \cap \{t \leq T\} \quad \Rightarrow \quad \begin{cases} \tilde{\mathbf{u}} = (\hat{\mathbf{u}}, 0) \\ |\tilde{\mathbf{u}}|_{l^1(\mathbb{R}^N)} = |(\hat{\mathbf{u}}, 0)|_{l^1(\mathbb{R}^{k+(N-k)})} = |\hat{\mathbf{u}}|_{l^1(\mathbb{R}^k)} \end{cases}$$

and

$$(x,t) \in \mathrm{supp}(\check{\mathbf{u}}) \cap \{t \le T\} \quad \Rightarrow \quad \begin{cases} \tilde{\mathbf{u}} = (0, \check{\mathbf{u}}) \\ |\tilde{\mathbf{u}}|_{l^1(\mathbb{R}^N)} = |(0, \check{\mathbf{u}})|_{l^1(\mathbb{R}^{k+(N-k)})} = |\check{\mathbf{u}}|_{l^1(\mathbb{R}^{N-k})} \end{cases} .$$

It is then easy to check that $\tilde{\mathbf{u}} = (\hat{\mathbf{u}}, \check{\mathbf{u}})$ solves the global N-dimensional system in Q^T (but a priori *not* for later times). From uniqueness in Theorem 1.1 (restricted to finite time intervals) we conclude that $\tilde{\mathbf{u}} = \mathbf{u}$ in Q^T. $\qquad\square$

One can refine Proposition 4.3 by considering an arbitrary number j, say, of initial patches as follows: 1) as long as the supports do not intersect, solve j independent systems, 2) when two or more patches touch, glue them together into a single higher-dimensional patch and resume with $j' < j$ patches, 3) keep iterating as long as there is more than one patch left.

This "divide and rule" behaviour should lead to infinite speed of propagation (discontinuity in time) of the supports of each individual species, even though the support of $w = |\mathbf{u}|_1$ does propagate with finite speed as stated in Theorem 4.1. Consider for example the case of only two species $\mathbf{u} = (u_1, u_2)$ with the initial (compact) supports at a positive distance from each other. We know from Proposition 4.3 that we only have to solve two scalar problems as long as the supports do not intersect. Assume that this happens at $t = T$ and that at this particular moment the two supports look like two tangent balls (thus only one species is present in each ball). If at least one of the balls were expanding at time $t = T^-$, then for $t = T^+$ the balls should intersect and the support of w would thus look like an 8-shaped domain with a thin but non-empty interior neck connecting the balls. Moreover, the diffusion coefficient $\varrho = \frac{\Phi(w)}{w}$ becomes positive in the entire 8-shaped domain, in particular in the neck. Since $\partial_t u_i = \mathrm{div}(\varrho \nabla u_i) + (\ldots)$ with the support of u_i in either of the two balls at time $t = T$, the diffusion occurring in the neck will ensure infinite speed of propagation between the two balls. Thus the support of u_i should jump from only one ball at $t = T$ to the whole 8-shaped domain at $t = T^+$. This instantaneous invasion phenomenon is beyond the scope of this paper and will be investigated elsewhere.

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