A GENERALIZATION OF THE GAUSS-BONNET AND HOPF-POINCARÉ THEOREMS. PART II

F. A. ARIAS AND M. MALAKHALTSEV

Abstract. This paper is a continuation of [1]. Let $\pi : E \to M$ be a locally trivial fiber bundle over a two-dimensional manifold M, and $\Sigma \subset M$ be a discrete subset. A subset $Q \subset E$ is called a *n-sheeted branched section of the bundle* π if $Q' = \pi^{-1}(M \setminus \Sigma) \cap Q$ is a n-sheeted covering of $M \setminus \Sigma$. The set Σ is called the *singularity set* of the branched section Q. We define the index of a singularity point of a branched section, and give examples of its calculation, in particular for branched sections of the projective tangent bundle of M determined by binary differential equations. Also we define a resolution of singularities of a branched section, and prove an analog of Hopf-Poincaré-Gauss-Bonnet theorem for the branched sections admitting a resolution.

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1. Introduction

Let us recall that a branched covering is a smooth map $f : X \to Y$, where X and Y are compact n-dimensional manifolds, such that $df_x : T_xX \to T_{f(x)}Y$ is an isomorphism for all points $x \in X \setminus A$ for some subset $A \subset X$ of dimension less or equal to $n-2$. In this case, if $X' = X \setminus f^{-1}(f(A))$ and $Y' = Y \setminus f(A)$, then the induced map $f' : X' \to Y'$ is a finite-sheeted covering map. The points of the set $f(A)$ are called the *branch points* of the branch covering f ([2], Section 18.3).

Now let $\xi = \{\pi_E : E \to M\}$ be a fiber bundle. Let Σ be a closed subset of M, $M' = M \setminus \Sigma$, and $E' = \pi^{-1}(M')$.

Definition 1. An *n-sheeted branched section* of the bundle ξ is a subset $Q \subset E$ such that $Q' = Q \cap E'$ is an embedded submanifold of E and $\pi_E|_{Q'} : Q' \to M'$ is a n-sheeted covering. The set Σ is called the *singularity set* of the branched section Q.

Example 1. Let V be a section of the tangent bundle $\pi_{TN} : TN \to N$, and $f : N \to M$ be a k-sheeted covering, then we can construct a branched section $df(V)$ of the tangent bundle $\pi_{TM} : TM \to M$ in the following way. Let us consider the subset $Q = \{df_y(V(y)) \mid y \in N\} \subset TM$. For each $x \in M$, let us set $\mathcal{V}(x) = \{df_y(V(y)) \mid y \in f^{-1}(x)\} \subset T_xM$. Take the subset $\Sigma \subset M$ consisting of points $x \in M$ such that the number of elements of the set $\mathcal{V}(x)$ is less than k. Then $M' = M \setminus \Sigma$ is open, $Q' = Q \cap \pi^{-1}_{TM}(M')$ is a submanifold of TM and f induces a k-sheeted covering $f' : Q' \to M'$. Indeed, for each $x \in M'$ there exists a neighborhood $U \subset M'$ of x such that $f^{-1}(U) = \bigsqcup_{j=1}^{k} \widetilde{U}_j \subset N$ and, for each $j = \overline{1,k}$, the application $f_j = f|_{\widetilde{U}_j} : \widetilde{U}_j \to U$ is a diffeomorphism. Therefore $df_j : T\widetilde{U}_j \subset TN \to TU \subset TM$ is also a diffeomorphism. As $V : \widetilde{U}_j \to V(\widetilde{U}_j) \subset TN$ is a diffeomorphism onto its image,

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the map $\theta_j = df_j \circ V \circ f_j^{-1} : U \to df_j(V(\widetilde{U}_j)) \subset Q \subset TM$, $j = \overline{1,k}$, is a diffeomorphism onto its image, as well. Note that, for each $y \in U \subset M'$, we have that the set $f^{-1}(y) = \left\{p_j \in \widetilde{U}_j\right\}_{j=\overline{1,k}}$ consists of k distinct points, and the set $\left\{df_{p_j}(V(p_j))\right\}_{j=\overline{1,k}}$ consists of k distinct vectors, by the definition of M'. Therefore, $\theta_i(U) \cap \theta_j(U) = \emptyset$, for $i \neq j$. Thus $\pi_{TM}^{-1}(U) \cap Q' = \bigsqcup_{j=1}^{k} \theta_j(U)$, this means that U is simply covered in Q', and Q' is a k-sheeted covering of M'.

The branched sections naturally appear in the theory of differential equations over manifolds. Our main example in this paper is the following one.

Example 2. Let M be a connected compact oriented manifold and let ω be a symmetric tensor of order n over M. Recall that such a tensor can be written locally as follows

$$\omega_{(x,y)} = a_0(x,y)dx^n + a_1(x,y)dx^{n-1}dy + \cdots + a_n(x,y)dy^n, \tag{1}$$

where (x, y) are coordinate functions on an open set $U \subset M$, and $a_i : U \to \mathbb{R}$ are smooth functions defined in U. In what follows, we suppose that ω has the following properties:

(1) The function $\omega_{(x,y)}$ is identically zero if and only if $a_i(x, y) = 0$ for $0 \leq i \leq n$. We set $\Sigma = \{p \in M : \omega_p = 0\}$.
(2) On $M \setminus \Sigma$, the tensor ω has the form $\omega = \lambda_1\lambda_2 \cdots \lambda_n$, where $\lambda_i \in \Omega(M \setminus \Sigma)$ pairwise linearly independent.

Statement 1. *The n-form ω determines a branched section of the bundle $\pi : PTM \to M$*

Proof. Let Q be the solution on PTM of the equation (1). We will prove that Q is a branched section of π. Let $E' = \pi^{-1}(M \setminus \Sigma)$ and $Q' = Q \cap E'$. It follows from the property (2) that the set $F_p = Q \cap \pi^{-1}(p)$, $p \in M \setminus \Sigma$ has exactly n elements, therefore each fiber of the surjective map $\pi' := \pi|_{Q'} : Q' \to M \setminus \Sigma$ is finite with n elements. On the other hand, if $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}P^1$ is a trivialization of PTM on U, then the restriction $\varphi' := \varphi|_{\pi^{-1}(U)\cap Q'} : \pi^{-1}(U) \cap Q' \to U \times \mathbb{R}P^1$ is a homeomorphism on its image. Since $\pi|_{\pi^{-1}(U)\cap Q'} : \pi^{-1}(U) \cap Q' \to U \cap (M \setminus \Sigma)$ has finite fiber with n elements over each point $p \in U \cap (M \setminus \Sigma)$, from the following commutative diagram

$$\begin{tikzcd} & \varphi'(\pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q') \arrow[dl, "pr_1"'] \\ U \cap (M \setminus \Sigma) & \pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q'. \arrow[l, "\pi'"'] \arrow[u, "\varphi'"'] \end{tikzcd} \tag{2}$$

It follows that $\pi|_{Q'} : Q' \to M \setminus \Sigma$ is a local diffeomorphism. Therefore, $\pi|_{Q'} : Q' \to M \setminus \Sigma$ is a n-sheeted branched covering, and so Q is a branched section of PTM. □

Geometrically Q determines an n-web at the points of $M \setminus \Sigma$.

Example 3. Let $\xi = \left\{\pi : \overline{P} \to M\right\}$ be a $\overline{G}$-principal bundle which reduces to a finite subgroup $G \subset \overline{G}$ over $M \setminus \Sigma$, where $\Sigma \subset M$ is a closed subset. Then the corresponding G-principal bundle $P \subset \overline{P}$ is a branched section of the bundle ξ with singularity set Σ.

For example, let M be a two-dimensional oriented Riemannian manifold, and $\overline{P} = SO(M)$, the $SO(2)$-principal bundle of orthonormal positively oriented frames of M. Any finite subgroup $G \subset SO(2)$ is a cyclic group $G \cong \mathbb{Z}_m$ generated by the rotation $R_{2\pi/m}$.

If $P \subset SO(M') \subset SO(M)$ is a reduction of $SO(M)$ to G over $M' = M \setminus \Sigma$, then at each point $x \in M'$ we have the set $\mathcal{N}(x) = \{e \in T_xM \mid (e, R_{\pi/2}e) \in P\}$, which consists of m unit vectors such that the angle between any two of them is $2\pi l/m$. The set $\mathcal{N}(x)$ defines a regular m-polygon $P_m \subset T_xM'$ inscribed in a unit circle centered at $0 \in T_xM$.

It is clear that, vice versa, if at any point of $M' = M \setminus \Sigma$, we are given a unitary m-polygon $P_m \subset T_pM'$ and the field of these polygons is smooth (these means that locally we can choose m unitary vector fields whose values are the vertices of the polygons P_m), then the bundle $SO(M)$ reduces to the subgroup $G \cong \mathbb{Z}_m$ of the Lie group $SO(2)$.

This situation occurs, for example, when M is a surface in $\mathbb{R}^3$, and Σ is the set of umbilic points of M. Then at each point of M' we have two orthogonal eigenspaces of the shape operator of the surface, which determine a square in T_pM with vertices at points where these eigenspaces meet the unit circle centered at $0 \in T_pM$. Therefore, over $M' = M \setminus \Sigma$ the bundle $SO(M)$ reduces to the subgroup $G \cong \mathbb{Z}_4$ generated by the rotation $R_{\pi/2}$. The corresponding principal subbundle P, the branched section of the bundle $SO(M) \to M$, consists of oriented orthogonal frames such that the frame vectors span the eigenspaces.

Moreover, as the difference of the principal curvatures never vanish on M', we can order the principal curvatures in such a way that $k_1(p) > k_2(p)$ at any $p \in M'$. Let $L_a(p)$, $a = 1, 2$ be the eigenspace corresponding to the principal curvature $k_a(p)$, $a = 1, 2$. Then we can choose the subbundle $P \subset SO(M)$ in such a way that, for $\{e_1, e_2\} \in P$, the vector e_a spans L_a, $a = 1, 2$, therefore in this case the bundle $SO(M) \to M$ reduces to the group $G \cong \mathbb{Z}_2$.

Also, note that this example is related to Example 2. Indeed if we have the reduction of $P \subset SO(M)$ to the subgroup $G \cong \mathbb{Z}_m$ over M', then at each point $p \in M'$ we have m (or $m/2$) subspaces spanned by the vector e_1 from the frame $\{e_1, e_2\} \in P$. Then we can take the binary differential equation (1) such that these subspaces are the roots of the corresponding algebraic equation.

The paper is organized as follows. In Section 2 we define the index of an isolated singular point of a branched section of locally trivial bundle $\xi = \{\pi_E : E \to M\}$ over a two-dimensional oriented manifold M (see Definition 2), this definition generalizes the definition of the index of a singular point of a section from ([1], Section 2.2, Definition 1). In Section 3 we define a resolution of a branched section (see Definition 3), and give various examples of resolutions (see Examples 6 – 9). And, finally, in Section 3 we prove an analogue of the Gauss-Bonnet theorem for a branched section which admits resolution (see Theorem 1).

2. The index of a singular isolated point

2.1. **Local monodromy group.** Let M be a two-dimensional closed oriented manifold. Let $\xi = \{\pi_E : E \to M\}$ be a fiber bundle with oriented typical fiber F.

Let us consider a k-sheeted branched section Q of ξ (see Definition 1) with singularity set Σ, and let $\pi_Q = \pi_E|_Q : Q \to M$. Recall that $M' = M \setminus \Sigma$, $E' = \pi^{-1}(M')$, and $Q' = Q \cap E'$.

Assume that $x \in \Sigma$ is an isolated point of Σ. Let us take a neighborhood $U(x)$ such that $U'(x) = U(x) \setminus \{x\}$ is an open subset of M' and there exists a diffeomorphism $\varphi : (D, 0) \to (U(x), x)$, where $D \subset \mathbb{R}^2$ is the standard open 2-disk centered at the origin $0 \in \mathbb{R}^2$. We will call $U(x)$ a *disk neighborhood* of x and assume that φ sends the standard orientation of D to the orientation of $U(x)$ induced by the orientation of M.

By Definition 1, the map $\pi_Q|_{\pi_Q^{-1}(U'(x))} : \pi_Q^{-1}(U'(x)) \to U'(x)$ is a k-sheeted covering.

If $U(x)$ is a disk neighborhood of an isolated point $x \in \Sigma$, then for each point $y \in U'(x)$, the fundamental group $\Pi_1(y) = \pi_1(U'(x), y)$ is isomorphic to $\mathbb{Z}$. There are two generators of $\Pi_1(y)$: $[\gamma_+]$ and $[\gamma_-]$, where $\gamma_\pm = \phi(C_\pm)$ and $C_\pm$ is a circle in D passing through the point $\varphi^{-1}(y)$ and enclosing the origin, and having positive (negative, respectively) orientation. We will call the element $[\gamma_\pm] \in \Pi_1(y)$ the positive (the negative, respectively) generator of $\Pi_1(y)$.

The group $\Pi_1(y)$ acts on the fiber $Q_y = \pi_Q^{-1}(y)$ in the following way: for an element $[\gamma] \in \Pi_1(y)$ and $q \in Q_y$ we set $[\gamma] \cdot q = \bar{q}$ if the lift $\widetilde{\gamma}$ of γ starting at q terminates in $\bar{q}$. This action is well defined, this means that if γ_1 and γ_2 represent the same element in $\Pi_1(y)$, then the lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ starting at a same point q terminate at a same point $\bar{q}$.

This action is a homomorphism of the group $\Pi_1(y)$ to the group of permutations of the fiber Q_y and its image is called the *local monodromy group* of the branched section Q at the point $y \in M'$.

Statement 2. *The local monodromy group does not depend on a choice of the disk neighborhood $U(x)$.*

Proof. Let $U(x)$ and $V(x)$ be two disk neighborhoods of x, and y lies in $U(x) \cap V(x)$. Then $\Pi_1^U(y) = \pi_1(U'(x), y) = \Pi_1^V(y) = \pi_1(V'(x), y)$ because for each class $[\gamma] \in \pi_1(V'(x), y)$ or $[\gamma] \in \pi_1(U'(x), y)$ one can find a representative $\gamma_1 \in [\gamma]$ which takes values in $U'(x) \cap V'(x)$. □

Statement 3. *Let γ be a loop in $U'(x)$ based at a point $y \in U'(x)$ such that its homotopic class represents the positive generator of $\Pi_1(y)$. Then for each orbit O of the local monodromy group action on Q_y and each point $q \in O$, there exists a loop $\widetilde{\gamma}$ in $\pi_Q^{-1}(U'(x))$ based at q which passes through each point of the orbit once and only once and such that $\pi_1(\pi_E)([\widetilde{\gamma}]) = [\gamma]^k$, where k is the number $\#O$ of elements of the orbit O. Here $\pi_1(\pi_E) : \pi_1(\pi_Q^{-1}(U'(x)), q) \to \pi_1(U(x), y)$ is the homomorphism of the fundamental groups induced by the map π_E.*

Proof. First of all note that if we have an action of the group $\mathbb{Z}$ on a finite set, then we can enumerate elements of each orbit in such a way that the action of the group generator 1 on this orbit is represented by the cycle $\sigma = (2, 3, \cdots, 1)$. Indeed, let O be an orbit of the action, and $q \in O$. The map $F : \mathbb{Z}/H_q \to O$, $[g] \to g \cdot q$, where H_q is the isotropy subgroup of the action, is an equivariant bijection. The group H_q is a cyclic group, this means that there exists $k \in \mathbb{Z}$, $k \geq 0$ such that $H_q = \{km \mid m \in \mathbb{Z}\}$, therefore $\mathbb{Z}/H_q = \{[0], [1], \cdots, [k-1]\}$, and the action of the generator $1 \in \mathbb{Z}$ on $\mathbb{Z}/H_p$ is given exactly by the cycle σ.

Now, for a point $y \in U'(x)$, let $[\gamma]$, $\gamma : [0, 1] \to U'(x)$, be the positive generator of $\Pi_1(y)$. Let us take an orbit O of the local monodromy group action on Q_y and a point $q \in O$. Let k be the number of elements of O. As we have seen, the action of $[\gamma]$ on O is represented by the cycle σ, this means we can enumerate the points of the orbit O in such a way that $q_1 = q$, $[\gamma]q_1 = q_2$, …, $[\gamma]q_{k-1} = q_k$, and $[\gamma]q_k = q_1 = q$. Therefore, by the construction of the action of $\Pi_1(y)$ on Q_y, for the lift $\widetilde{\gamma}_1$ of γ to Q' such that $\widetilde{\gamma}_1(0) = q_1$ we have that $\widetilde{\gamma}_1(1) = q_2$, for the lift $\widetilde{\gamma}_2$ of γ to Q' such that $\widetilde{\gamma}_2(0) = q_2$ we have that $\widetilde{\gamma}_1(1) = q_3$, …, and finally for the lift $\widetilde{\gamma}_k$ of γ to Q' such that $\widetilde{\gamma}_k(0) = q_k$ we have that $\widetilde{\gamma}_k(1) = q_k = q$.

What do we do in fact is that we take a point $q_1 = q \in Q_y$, then construct the points $q_2 = [\gamma]q_1$, $q_3 = [\gamma]q_2$, …, up to $[\gamma]q_k = q_1$. Then the set $\{q_1, q_2, \cdots, q_k\}$ is the orbit O of the point q.

It is clear that $\widetilde{\gamma} = \widetilde{\gamma}_k \cdot \widetilde{\gamma}_{k-1} \cdot \cdots \cdot \widetilde{\gamma}_1$, where $\cdot$ is the path composition, is a loop in $\pi_Q^{-1}(U'(x))$ at the point $q_1 = q$, $\widetilde{\gamma}$ passes once and only once through each point of O, and $\pi_1(\pi_E)([\widetilde{\gamma}]) = [\gamma]^k$. Thus $\widetilde{\gamma}$ is the required loop. □

2.2. **The index of isolated singular point.** Let M be an oriented two-dimensional manifold. Let $\xi = \{\pi_E : E \to M\}$ be a locally trivial fiber bundle with standard fiber F and a connected structure Lie group G.

Let Q be a branched section of ξ with singularity set Σ, and x be an isolated point of Σ. Take a disk neighborhood $U(x)$, and for a point $y \in U'(x)$, let $\mathcal{O}_y$ be the set of orbits of local monodromy group action on Q_y. Take an orbit $O \in \mathcal{O}(y)$ and a point $q \in O$. Let $[\gamma]$ be a positive generator of the group $\Pi_1(y)$, and $\widetilde{\gamma}$ the loop at q constructed in Statement 3.

Let $\psi : \pi_E^{-1}(U(x)) \to U(x) \times F$ be a trivialization of the bundle ξ, and $p : \pi_E^{-1}(U(x)) \to F$ be the corresponding projection. Then the element $[p \circ \widetilde{\gamma}] \in \pi_1(F)$ is called the *index of the branched section Q at the singular point x corresponding to the orbit $O \in \mathcal{O}_y$*, call it $ind_x(Q; y, O)$.

Statement 4.

a) *The index $ind_x(Q; y, O)$ does not depend on a choice of the loop $\gamma : [0,1] \to U(x')$ representing the positive generator of the group $\Pi_1(y)$.*
b) *The index $ind_x(Q; y, O)$ does not depend on a trivialization.*
c) *The index $ind_x(Q; y, O)$ does not depend on a choice of the disk neighborhood $U(x)$, this means that, if $U(x)$ and $V(x)$ are two disk neighborhoods of x, and $y \in U(x) \cap V(x)$, then the constructions of $ind_x(Q; y, O)$ performed for $U(x)$ and for $V(x)$ result in the same element in $\pi_1(F)$.*

Proof. a) If γ and μ are two representatives of the positive generator of $\Pi_1(y)$, then γ is homotopic to μ, therefore γ^k is homotopic to μ^k, therefore the lift $\widetilde{\mu}$ of μ^k is homotopic to the lift $\widetilde{\gamma}$ of γ^k, hence $p\widetilde{\gamma}$ is homotopic to $p\widetilde{\mu}$.

b) This is because the gluing functions are homotopic to the identity as the structure group is connected.

c) This follows directly from the fact that $\Pi_1^U(y) = \Pi_1^V(y)$ (see the proof of Statement 3), and from a). □

Example 4. Let us consider the trivial bundle $\xi = \{\pi_E : E = \mathbb{C} \times \mathbb{C}^* \to M = \mathbb{C}\}$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\pi_E(z, w) = z$. Let us take the subset $Q = \{(z, w) \mid z^2 = w^3\} \subset E$.

As $\pi_E|_Q : Q \to M \setminus \{z = 0\}$ is a 3-sheeted covering, we see that Q is a 3-sheeted branched section of the bundle ξ.

It is clear that the singularity set of Q is $\Sigma = \{0\}$, so Q has only one singular point $z = 0$ and this point is isolated. For the disk neighborhood of the isolated singular point $z = 0$ we take the entire $M = \mathbb{C}$.

Let us take $y = 1$, then $Q_y = \{a = (1,1), b = (1, \varepsilon), c = (1, \varepsilon^2)\}$, where $\varepsilon = \exp(2\pi i/3)$. The loop $\gamma(t) = \exp(2\pi it)$, $t \in [0,1]$, represents the positive generator of the group $\Pi(y = 1)$, and the lift $\widetilde{\gamma}_a$ of γ which starts at the point $a = (1,1)$ is given by $\widetilde{\gamma}_a(t) = (\exp(2\pi it), \exp(\frac{4}{3}\pi it)$, $t \in [0,1]$. Therefore $[\gamma]a = c$. In the same manner one can prove that $[\gamma]b = a$, $[\gamma]c = b$.

Thus, the orbit O of the point a is $Q_{y=1} = \{a, b, c\}$, and for a representative of the class $[\widetilde{\gamma}]$ constructed in Statement 3 we can take the loop $\widetilde{\gamma} = (\exp(6\pi it), \exp(4\pi it))$ for $t \in [0,1]$.

Therefore, the loop $p\widetilde{\gamma} : [0,1] \to \mathbb{C}^*$ is given by $p\widetilde{\gamma} = \exp(4\pi it))$ for $t \in [0,1]$. Hence

$$ind_0(Q; y = 1, Q_{y=1}) = 2 \in \mathbb{Z} = \pi_1(\mathbb{C}^*). \tag{3}$$

Let us consider the finite set of elements of $\pi_1(F)$:

$$ind_x(Q; y) = \{ind_x(Q; y, O) \mid O \in \mathcal{O}(y)\}. \tag{4}$$

Statement 5. *The set $ind_x(Q; y)$ does not depend on $y \in U'(x)$.*

Proof. Let $y, \bar{y}$ be two points in $U'(x)$. Take a curve $\delta : [0, 1] \to U'(x)$ such that $\delta(0) = y$, $\delta(1) = \bar{y}$.

The curve δ defines the group isomorphism $\psi_\delta : \Pi_1(y) \to \Pi_1(\bar{y})$, $[\gamma] \mapsto [\delta^{-1} \cdot \gamma \cdot \delta]$, where $\delta^{-1}(t) = \delta(1-t)$. Also, δ defines the bijection $\widetilde{\psi}_\delta : Q_y \to Q_{\bar{y}}$, $q \in Q_y \mapsto \bar{q} \in Q_{\bar{y}}$, such that for the lift $\widetilde{\delta}$ of δ to Q with $\widetilde{\delta}(0) = q$ we have that $\widetilde{\delta}(1) = \bar{q}$. In addition, the bijection $\widetilde{\psi}_\delta$ is equivariant is sense that $\widetilde{\psi}_\delta([\gamma]q) = \psi_\delta([\gamma])\widetilde{\psi}_\delta(q)$.

Therefore $\widetilde{\delta}$ induces a bijection $\alpha_\delta : \mathcal{O}(y) \to \mathcal{O}(\bar{y})$, $O_q \mapsto O_{\widetilde{\psi}_\delta(q)}$, where O_q is the $\Pi_1(y)$-orbit of the point $q \in Q_y$ and $O_{\widetilde{\psi}_\delta(q)}$ is the $\Pi_1(\bar{y})$-orbit of the point $\widetilde{\psi}_\delta(q) \in Q_{\bar{y}}$.

Let us prove that the loop $\widetilde{\gamma}$ which passes through the points of the orbit $O_q \in \mathcal{O}(y)$ constructed in Statement 3 is homotopic in $\pi^{-1}(U'(x))$ to the corresponding loop of the orbit $O_{\widetilde{\psi}(q)} \in \mathcal{O}(\bar{y})$.

Let γ be a loop at $y \in U'(x)$ which represents the positive generator of $\Pi_1(y)$. The loop $\widetilde{\gamma}$ constructed in Statement 3 is homotopic to the lift of the loop γ^k starting at a point $q \in Q_y$. As the loop γ^k is freely homotopic to the loop $\bar{\gamma}^k$, where $\bar{\gamma} = \delta^{-1}\gamma\delta$, the lift $\widetilde{\gamma}$ is freely homotopic to the lift of $\bar{\gamma}^k$ starting at the point $\widetilde{\psi}_\delta(q)$, but this lift is in turn homotopic to the loop $\widetilde{\bar{\gamma}}$.

Therefore the loops $p\widetilde{\gamma}$ and $p\widetilde{\bar{\gamma}}$ are freely homotopic in F, therefore define the same element in $\pi_1(F)$. Thus we have that $ind_x(Q; y, O) = ind_x(Q; \bar{y}, \alpha_\delta(O))$ for all $O \in \mathcal{O}(y)$, hence $ind_x(Q; y) = ind_x(Q; \bar{y})$. $\square$

Corollary 1. *The set $ind_x(Q)$ does not depend on the disk neighborhood $U(x)$, this means if $U_1(x)$ and $U_2(x)$ are disk neighborhoods of an isolated singular point $x \in \Sigma$, and $y_1 \in U_1'(x)$ and $y_2 \in U_2'(x)$, then the set $ind_x(Q; y_1)$ constructed via $U_1(x)$ and the set $ind_x(Q; y_2)$ constructed via $U_2(x)$ coincide.*

Proof. Follows from Statement 6 $\square$

From Statement 5 it follows that we can give the following definition.

Definition 2. Let Q be a branched section of the bundle ξ. The *index of Q at $x \in M$* is

$$ind_x(Q) = ind_x(Q; y), \tag{5}$$

where y is a point of $U'(x)$, where $U(x)$ is a disk neighborhood of x.

Let us fix an element $a \in H^1(F)$. The index of Q at a point x with respect to a is

$$ind_x(Q; a) = \sum_{O \in O(y)} \frac{1}{\#O} \langle a, ind_x(Q; y, O) \rangle = \sum_{O \in O(y)} \frac{1}{\#O} \int_{\gamma(Q;y,O)} \alpha, \tag{6}$$

where $\alpha \in \Omega^1(F)$ represents $a \in H^1(F)$ and $\gamma(Q; y, O)$ represents the class $ind_x(Q; y, O) \in \pi_1(F)$.

Example 5. Let M be a connected compact oriented manifold and let ω be a symmetric tensor of order n over M. In Example 2 we have constructed a branched section $Q \subset PTM$ determined by the binary differential equation (1).

If we consider the covering $q : \mathbb{S}^1TM \to PTM$ given by $q((p, \vec{v})) = [\vec{v}]$, we see that $q \circ \pi : \mathbb{S}^1TM \to M$ is a fiber bundle and $q^{-1}(Q)$ is a $2n$-sheeted branched covering of the bundle $\mathbb{S}^1TM \to M$. Let

$p \in \Sigma$ be a singular point, $U'(p)$ be a neighborhood disk of p, and $\mathcal{O}_p = \{O_1, \cdots, O_r\}$ the set of the orbits of the action of $\pi_1(U'(p))$ on $\pi^{-1}(p)$. From the equation (6) it follows that the index of $q^{-1}(Q)$ at the singular point $p \in \Sigma$ with respect the cohomology class $a = [\frac{1}{2\pi} d\theta] \in H^1(\mathbb{S}^1)$, where $d\theta$ is the angular form on $\mathbb{S}^1$ is given by

$$ind_p(Q; a) = \sum_{i=1}^{r} \frac{1}{2\pi k_i} \int_{\gamma_i} d\theta, \tag{7}$$

where k_i is the number of elements of the orbit O_i, and γ_i is the index of the point p corresponding to the orbit O_i. Let us choose a frame (e_1, e_2) along the curve γ, and we consider a unit vector field $X(t), 0 \leq t \leq 1$ such that $\omega_{\gamma(t)}(X(t))$ around the curve $\gamma : I \to U'(p)$. If $\tilde{\theta}$ is the angle between e_1 and $X(0)$, we obtain that the index of Q at the point p with respect to the form a can be also calculated in terms of this rotation angle by the formula

$$ind_p(Q, O_i, a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{2\pi k_i}. \tag{8}$$

Note that if the action of $\pi_1(U'(p))$ on $\pi^{-1}(p)$ is transitive, then the equation (7) reduces to the following

$$ind_p(Q; a) = \frac{1}{4\pi n} \int_{\gamma} d\theta, \tag{9}$$

where γ is the index of p in $\pi^{-1}(p)$, and it is also true that

$$ind_p(Q, \pi^{-1}(p), a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{4n\pi}. \tag{10}$$

The equation (10) coincides with the index of a binary differential n-form given in [3].

Now, we note that the index of Q at a singular point x seen as a singularity of the bundle $\pi : PTM \to M$ is twice the index of the same point as a singular point of the bundle $\pi \circ q : \mathbb{S}^1 TM \to M$.

Remark 1. This construction can be used to calculate an index of singular points of singular distributions over a two dimensional manifold M. In [[5], pages 218-223], the author gives another constructions of indexes of singular points of 1-dimensional singular distributions and branched covering of two sheets defined by such a distributions.

3. Resolution of a branched section

Let M be a two-dimensional oriented manifold, and $\xi = \{\pi_E : E \to M\}$ be a fiber bundle. Let Σ be a discrete subset of the manifold M.

Definition 3. Let Q be an n-sheeted branched section of the bundle ξ with singularity set Σ, $M' = M \setminus \Sigma$, $E' = \pi^{-1}(M')$, and $Q' = Q \cap E'$. A *resolution of* Q is a map $\iota : S \to E$, where S is an oriented two-dimensional manifold with boundary, such that

(1) $\iota(S) = Q$;
(2) $\pi = \pi_E \circ \iota : S \to M$ is surjective;
(3) the map ι is an embedding of $S' = S \setminus \partial S$ onto Q'.

In case M is compact, we assume S to be compact, too.

Remark 2. From Definition 3 it follows that $\pi_E(Q) = M$ and $\pi_E(\partial S) = \Sigma$.

Example 6. Let $M = \mathbb{R}^2$, $E = \mathbb{P}T\mathbb{R}^2$ and a branched section is the solution of the differential equation $xydx^2 - (x^2 - y^2)dxdy - xydy^2 = 0$. As the discriminant of this equation is $(x^2 - y^2)^2 - 4(xy)^2 = (x^2 + y^2)^2$, this differential equation is a binary differential equation (see Example 2). This differential equation is represented in the form $(xdx + ydy)(ydx - xdy) = 0$, therefore its solution Q consists of two 1-dimensional distributions L_1 and L_2 on $\mathbb{R}^2$ given respectively by the equations $xdx + ydy = 0$ and $ydx - xdy = 0$. One can easily see that these equations determine sections with singularities s_1 and s_2 of the bundle E, which admit resolutions (see [1]), call them S_1 and S_2, so the manifold $S_1 \sqcup S_2$ is a resolution of the branched section Q.

Example 7. Let $M = \mathbb{R}^2$, $E = PT\mathbb{R}^2$ and the branched section Q is the solution of the binary differential equation

$$ydx^2 - 2xdxdy - ydy^2 = 0. \tag{11}$$

The discriminant of equation (11) is $4(x^2 + y^2)$, therefore this equation has two real roots for all (x, y) different from the origin, and at the origin all the coefficients vanish. That is why, equation (11) is a binary differential equation (see Example 2).

The standard coordinates (x, y) on $\mathbb{R}^2$ induce a trivialization of the bundle $\pi_E = E = PT\mathbb{R}^2 \to M = \mathbb{R}^2$, namely for the one-dimensional subspace $l \in PT_{(x,y)}\mathbb{R}^2$ spanned by a vector $p\partial_x + q\partial_y$, we assign the point $(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1$. Thus, $PT\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}P^1$, and

$$Q = \left\{(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1 \mid yp^2 - 2xpq - yq^2 = 0\right\} \tag{12}$$

In this case

$$\Sigma = (0, 0), \quad Q' = \left\{(x, y, [p : q]) \in Q \mid x^2 + y^2 > 0\right\}, \quad M' = \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{13}$$

The projection $\pi_{PT\mathbb{R}^2} : PT\mathbb{R}^2 \to \mathbb{R}^2$ restricted to Q' is a trivial (as a fiber bundle) double covering of M'. Indeed, take the following open sets U_1 and U_2:

$$U_1 = M' \setminus (-\infty, 0) \times \{0\} \text{ and } U_2 = M' \setminus (0, \infty) \times \{0\} \tag{14}$$

It is clear that $M' = U_1 \cup U_2$. Also, at the points of U_1 we have $x + \sqrt{x^2 + y^2} > 0$, and at the points of U_2 we have $x - \sqrt{x^2 + y^2} > 0$.

Now let us take two sections of the bundle $\pi_{PT\mathbb{R}^2} : PT\mathbb{R}^2 \to \mathbb{R}^2$ defined on M':

$$s_1 : (x, y) \mapsto \begin{cases} (x, y, [x + \sqrt{x^2 + y^2} : y]), & (x, y) \in U_1, \\ (x, y, [-y : x - \sqrt{x^2 + y^2}]), & (x, y) \in U_2, \end{cases} \tag{15}$$

and

$$s_2 : (x, y) \mapsto \begin{cases} (x, y, [-y : x + \sqrt{x^2 + y^2}]), & (x, y) \in U_1, \\ (x, y, [x - \sqrt{x^2 + y^2} : y]), & (x, y) \in U_2, \end{cases} \tag{16}$$

Note that over $U_1 \cap U_2$ there holds

$$[x + \sqrt{x^2 + y^2} : y] = [-y : x - \sqrt{x^2 + y^2}] \text{ and } [-y : x + \sqrt{x^2 + y^2}] = [x - \sqrt{x^2 + y^2} : y], \tag{17}$$

therefore the sections s_1 and s_2 are well defined. One can easily prove that $s_i(M') \subset Q'$, $i = 1, 2$, and $s_1(M') \cap s_2(M') = \emptyset$. Therefore Q' is a trivial double covering of M'.

Now let us construct a resolution of the branched section Q. Recall that $\mathbb{S}^1 = \{(u,v) \mid u^2+v^2=1\}$, then let take the diffeomorphism

$$f : \mathbb{S}^1 \to \mathbb{R}P^1, (u,v) \mapsto \begin{cases} [u+1 : v], & u > -1, \\ [-v : u-1], & u < 1, \end{cases} \tag{18}$$

and then the diffeomorphism f "rotated" at the angle $\pi/2$ gives the diffeomorphism,

$$g : \mathbb{S}^1 \to \mathbb{R}P^1, (u,v) \mapsto \begin{cases} [-v : u+1], & u > -1, \\ [u-1 : v], & u < 1. \end{cases} \tag{19}$$

We take $S_1 = S_2 = \mathbb{R}_+ \times \mathbb{S}^1 = [0,\infty) \times \mathbb{S}^1$, and $S'_1 = S'_2 = (0,\infty)$. We set $S = S_1 \sqcup S_2$, then $S' = S'_1 \sqcup S'_2$. Then $\iota : S \to \mathbb{R}^2 \times \mathbb{R}P^1$ is given by

$$\iota|_{S_1}(r,(u,v)) = (ru, rv, f(u,v)), \quad \iota|_{S_2}(r,(x,y)) = (ru, rv, g(u,v)). \tag{20}$$

One can easy see that $\iota|_{S'_i} : S'_i \to Q'_i$, $i = 1,2$ is a diffeomorphism. For example, any point $(x,y,[p:q]) \in V_{11}$, is the image of the point $(r,(u,v))$ under the map $\iota|_{S_1}$, where

$$u = \frac{x}{\sqrt{x^2+y^2}}, \quad v = \frac{y}{\sqrt{x^2+y^2}}, \quad r = \sqrt{x^2+y^2}. \tag{21}$$

Example 8. As a generalization of Examples 6 and 7 one can take n sections with singularities [1] of a bundle $\xi = \pi_E : E \to M$ which have the same set of singularities Σ, call them s_i, $i = \overline{1,n}$. These sections define a branched section Q of the bundle ξ: $Q = \{s_i(x) \mid x \in M \setminus \Sigma\}$. If S_i is a resolution of s_i, then $S = \sqcup S_i$ is a resolution of Q.

Example 9. Let us present an example of branched section, where the covering $\pi_Q|_{Q'} : Q' \to M'$ is not trivial. Take $M = \mathbb{R}^2 = \mathbb{C}$, $E = \mathbb{S}^1(\mathbb{C}) = \mathbb{C} \times \mathbb{S}^1$, the bundle of unit vectors over M, and let

$$Q = \left\{(z,w) \in \mathbb{C} \times \mathbb{S}^1 \mid |z|w^2 = z\right\}. \tag{22}$$

Then $M' = \mathbb{C} \setminus \{0\}$, $Q' = \{(z,w) \mid w^2 = z/|z|\}$, and it is well known that $\pi_Q|_{Q'} : Q' \to M'$ is a non trivial double covering. Now let us take

$$S = [0,\infty) \times \mathbb{S}^1, \text{ and } \iota : S \to E, \quad (r, e^{i\varphi}) \mapsto (re^{2i\varphi}, e^{i\varphi}) \tag{23}$$

Then $S' = (0,\infty)$, and it is clear that the properties (1)–(3) of Definition 3 hold true for ι.

Example 10. Let us present another example of branched section, where the covering $\pi_Q|_{Q'} : Q' \to M'$ is not trivial. Take $M = \mathbb{R}^2 = \mathbb{C}$, $E = PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}P^1 = \mathbb{C} \times \mathbb{R}P^1$, and let

$$Q = \left\{(z,[w] \mid |w| = 1 \text{ and } |z|^2 w^4 = z^2\right\} \tag{24}$$

Then $M' = \mathbb{C} \setminus \{0\}$, $Q' = \{(z,w) \mid w^4 = z^2/|z|^2\}$, and it is clear that $\pi_Q|_{Q'} : Q' \to M'$ is a non trivial double covering. Now let us take

$$S = [0,\infty) \times \mathbb{R}P^1, \text{ and } \iota : S \to E, \quad (r,[w]) \mapsto (rw^2, [w]), \tag{25}$$

where $|w| = 1$. Then $S' = (0,\infty)$, and it is clear that the properties (1)–(3) of Definition 3 hold true for ι.

Remark 3. In Examples 9–10, for each $x \in M$, the set S_x is a discrete set if $x \in M \setminus \Sigma$, or is diffeomorphic to a circle $\mathbb{S}^1$ if $x \in \Sigma$.

Now let us consider a point $x \in \Sigma$. Then, according to Definition 3, $S_x = \pi^{-1}(x)$ consists of the connected components of the boundary ∂S. Let us denote by $C(S_x)$ the set of connected components of S_x. As S_x is compact, the set $C(S_x)$ is finite, and each element of this set is diffeomorphic to a circle $\mathbb{S}^1$.

Statement 6. *Let C be a connected component of a boundary. Then there exists a neighborhood $N(C)$ of C and a diffeomorphism $f_C : N(C) \to \mathbb{S}^1 \times [0,1]$ such that $f_C(C) = \mathbb{S}^1 \times \{0\}$ and $U(x) = \pi(N(C)$ is a disk neighborhood of x. For each $y \in U'(x)$, the set of orbits $\mathcal{O}_y$ consists of only one element. In this cases the curve $\widehat{\gamma}$ corresponding to the orbit by Statement 3 is a generator of the group $\pi_1(N(C)) \cong \mathbb{Z}$.*

Proof. Indeed, $N(C) \setminus C$ is homeomorphic to a ring and $U'(x)$ is homeomorphic to a ring as well. The map $N(C) \setminus C \to U'(x)$ induced by π is a n-fold covering therefore $\pi_* : \pi_1(N(C)) \cong \mathbb{Z} \to \pi_1(U'(x))$ has the form $m \to km$. At the same time $\pi_*([\widetilde{\gamma}]) = \gamma^k$, thus $[\widetilde{\gamma}]$ is a generator of the group $\pi_1(N(C))$. □

Corollary 2. *The curve $\widetilde{\gamma}$ is homotopic in $N(C) \subset S$ to the curve $C \subset E_x$. Therefore the curve C represents $ind_x(Q, O)$.*

4. Connection and the Gauss-Bonnet theorem

Let $\xi = (\pi_E : E \to M)$ be a locally trivial fiber bundle with standard fiber F and structure group G. Assume that G is a connected Lie group.

Let $(U, \psi : \pi^{-1}(U) \to U \times F)$ be a chart of the atlas of ξ. Let

$$\eta = p_F \circ \psi : \pi^{-1}(U) \to F, \tag{26}$$

where $p_F : U \times F \to F$ is the canonical projection onto F. For each $x \in U$ the map η restricted to $F_x = \pi^{-1}(x)$ induces a diffeomorphism $\eta_x : F_x \to F$, and let $i_x : F \to F_x$ be the inverse of η_x. Note that if we take another chart $(U', \psi' : \pi^{-1}(U') \to U' \times F)$, and $\eta' : \pi^{-1}(U') \to F$ is the corresponding map, then on $\pi^{-1}(U \bigcap U')$ we have that

$$\psi' \circ \psi^{-1} : (U \cap U') \times F \to (U \cap U') \times F, \quad (x, y) \mapsto (x, g(x)y), \tag{27}$$

where $g : U \cap U' \to G$ is the gluing map of the charts. Now, for any $x \in U \cap U'$, we have $\eta'_x \circ \eta_x^{-1}(y) = g(x)y$, and, as G is connected, $\eta'_x \circ \eta_x^{-1} : F \to F$ is homotopic to the identity map. This means that for any $x \in m$ we have well defined isomorphisms of the homotopy and (co)homology groups:

$$\begin{aligned} &\pi_*(\eta_x) : \pi_*(F_x) \to \pi_*(F), \\ &H_*(\eta_x) : H_*(F_x) \to H_*(F), \quad H^*(\eta_x) : H^*(F) \to H^*(F_x), \end{aligned} \tag{28}$$

which do not depend on the chart.

In [1], for a locally trivial bundle with standard fiber F and structure Lie group G, we have proved the following statement ([1], Statement 1):

Statement 7. *Let $a \in H^1(F)$ and H be a connection in E. There exists a 1-form $\alpha \in \Omega^1(E)$ such that*

1. $\alpha|_H = 0$;
2. *for each $x \in M$, $di_x^*\alpha = 0$ and $[i_x^*\alpha] = H^1(\eta_x)a$.*

The decomposition $TE = H \oplus V$ gives a bicomplex representation of the complex $\Omega(E)$, then the form α lies in $\Omega^{(0,1)}(E)$ and $d\alpha = \theta_{(1,1)} + \theta_{(2,0)}$, where $\theta_{(1,1)} \in \Omega^{(1,1)}$ and$\theta_{(2,0)} \in \Omega^{(2,0)}$, and

$$\theta_{(1,1)}(X,Y) = (L_X\alpha)(Y), \quad \theta_{(2,0)} = \widetilde{\alpha}(\Omega). \tag{29}$$

where L_X is the Lie derivative with respect to the vector field X, and Ω is the curvature form of the connection H (for details see [1], Section 3).

Now let Q be a branched section of the bundle ξ which admits a resolution $\iota : S \to E$ (see Definition 3). Let us fix an element $a \in H^1(F)$, and let $\alpha \in \Omega^1(E)$ be the corresponding 1-form (see Statement 7). Then, by the Stokes theorem we have

$$\int_{\partial S} \iota^*\alpha = \int_S \iota^* d\alpha. \tag{30}$$

By Remark 2 we have that $\pi_E(\partial S) = \Sigma$. For $x \in \Sigma$, let $C(S_x)$ be the set of connected components of $\pi_E^{-1}(x)$.

From Corollary 2, it follows that, for $C \in C(S_x)$, we have

$$\int_C \alpha = \int_{\gamma(Q;y,O(C)} i_x^*\alpha, \tag{31}$$

where $\gamma(Q;y,O(C))$ represents the class $ind_x(Q;y,O(C)) \in \pi_1(F)$, and $O(C)$ is the orbit of the local monodromy group corresponding to C. Therefore, from (6) we have that

$$ind_x(Q;a) = \sum_{C\in C(S_x)} \frac{1}{\#O(C)} \int_C \alpha. \tag{32}$$

If all the orbits of the local monodromy group corresponding to the components $C \in C(S_x)$ have the same number of elements $N(x)$, then

$$\int_{\partial S} \iota^*\alpha = \sum_{x\in\Sigma} \sum_{C\in C(S_x)} \int_C \alpha = \sum_{x\in\Sigma} N(x)\ ind_x(Q;a) \tag{33}$$

Thus we get the following theorem

Theorem 1 (Gauss-Bonnet-Hopf-Poincaré for branched sections)**.** *If, for any $x \in \Sigma$, all the orbits of the local monodromy group corresponding to the components $C \in C(S_x)$ have the same number of elements $N(x)$, then*

$$\int_S \iota^*\theta_{(1,1)} + \iota^*\theta_{(2,0)} = \sum_{x\in\pi(\partial S)} N(x)\ ind_x(Q).$$

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Universidad de los Andes, Bogotá, Colombia
E-mail address: fa.arias44@uniandes.edu.co; mikarm@uniandes.edu.co