# Projective Curvature Tensors of Second Type Almost Geodesic Mappings

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#### Abstract

We consider equitorsion second type almost geodesic mappings of a non-symmetric affine connection space in this article. Using different computational methods, we obtained some invariants of these mappings. Last generalized Thomas projective parameter and Weyl projective tensor as invariants of a second type almost geodesic mapping of a non-symmetric affine connection space are further generalized here.

### 1 Introduction

A lot of research papers and monographs are dedicated to developments of the theory of differential geometry [1,2,6,8–21] and its applications [3–5,7]. Einstein (see [3–5]) concluded the symmetric affine connection theory covers researches about a gravitation. The theory electromagnetism is covered by anti-symmetric parts of affine connections. The research about non-symmetric affine connected spaces is started by L. P. Eisenhart [6].

An N-dimensional manifold  $\mathcal{M}_N$  endowed with a non-symmetric affine connection  $\nabla$  (affine connection coefficients  $L^i_{jk}$  and  $L^i_{kj}$  are different) is said to be the non-symmetric affine connection space  $\mathbb{G}\mathbb{A}_N$ . Because of the previous mentioned non-symmetry of affine connection coefficients it exists symmetric and anti-symmetric part of these coefficients respectively defined as:

$$\widetilde{S}_{jk}^{i} = \frac{1}{2}(L_{jk}^{i} + L_{kj}^{i}) \text{ and } \widetilde{T}_{jk}^{i} = \frac{1}{2}(L_{jk}^{i} - L_{kj}^{i}).$$
 (1.1)

<sup>2010</sup> Math. Subj. Classification: Primary: 53A55; Secondary: 53B05, 53C15, 53C22 Key words: almost geodesic mapping, curvature tensor, generalization, Weyl projective tensor, Thomas projective parameter

This research is supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174012

A symmetrization and an anti-symmetrization without division by indices i and j will be denoted as  $(i \dots j)$  and  $[i \dots j]$  respectively.

The magnitude  $\widetilde{T}^i_{jk}$  is a torsion tensor of the space  $\mathbb{G}\mathbb{A}_N$ . An affine connection space  $\mathbb{A}_N$  endowed with an affine connection S which coefficients coincide with the symmetric part  $\widetilde{S}^i_{jk}$  of the affine connection coefficients  $L^i_{jk}$  of the space  $\mathbb{G}\mathbb{A}_N$  is said to be the associated space of the space  $\mathbb{G}\mathbb{A}_N$ .

There are a lot of researchers interested for a development of the non-symmetric affine connection space theory. Some significant results in this subject are obtained into the papers [10,11,14–19,21].

Four kinds of covariant differentiation (see [10]) with regard to an affine connection of a non-symmetric affine connection space  $\mathbb{G}\mathbb{A}_N$  are defined as:

$$a_{j|k}^{i} = a_{j,k}^{i} + L_{\alpha k}^{i} a_{j}^{\alpha} - L_{jk}^{\alpha} a_{\alpha}^{i}, \qquad a_{j|k}^{i} = a_{j,k}^{i} + L_{k\alpha}^{i} a_{j}^{\alpha} - L_{kj}^{\alpha} a_{\alpha}^{i}, \quad (1.2)$$

$$a_{j|k}^{i} = a_{j,k}^{i} + L_{\alpha k}^{i} a_{j}^{\alpha} - L_{kj}^{\alpha} a_{\alpha}^{h}, \qquad a_{j|k}^{i} = a_{j,k}^{i} + L_{k\alpha}^{i} a_{j}^{\alpha} - L_{jk}^{\alpha} a_{\alpha}^{i}, \quad (1.3)$$

for a partial derivative denoted by comma and an indexed magnitude  $a_i^i$ .

All of these covariant derivatives become restricted to a covariant derivative

$$a_{j;k}^{i} = a_{j,k}^{i} + \widetilde{S}_{\alpha k}^{i} a_{j}^{\alpha} - \widetilde{S}_{jk}^{\alpha} a_{\alpha}^{i}, \tag{1.4}$$

of the magnitude  $a_j^i$  with regard to an affine connection of the associated space  $\mathbb{A}_N$  of the space  $\mathbb{G}\mathbb{A}_N$ .

For this reason, it exists only one curvature tensor

$$R_{jmn}^{i} = \widetilde{S}_{jm;n}^{i} - \widetilde{S}_{jn;m}^{i}, \tag{1.5}$$

of the associated space  $\mathbb{A}_N$ .

# 1.1 Almost geodesic mappings of a space $\mathbb{G}\mathbb{A}_N$

In an attempt to generalize the term of geodesics N. S. Sinyukov (see [12]) defined an almost geodesic line of a symmetric affine connection space  $\mathbb{A}_N$ . Consequently, he defined a term of an almost geodesic mapping f between symmetric affine connection spaces  $\mathbb{A}_N$  and  $\overline{\mathbb{A}}_N$ . Sinyukov noticed three types  $\pi_1, \pi_2, \pi_3$  of almost geodesic mappings between symmetric affine connection spaces. His research has been directly developed by many authors in a lot of papers [1,2,8,9,13,20].

The Sinyukov's generalization of geodesics is primary developed for the case of a generalized affine connection space  $\mathbb{GA}_N$  in [14–16]. In this space it exists four kinds of covariant differentiation but these covariant derivatives are reduced onto first two ones (1.2) for the case of any contra-variant tensor.

For this reason, there are two kinds of almost geodesic lines of the space  $\mathbb{GA}_N$  [14–19, 21] defined as a curve  $\ell = \ell(t)$  which tangential vector  $\lambda^i = d\ell^i/dt \neq 0$  satisfies the following equations:

$$\overline{\lambda}_{\theta}^{i}(2) = \overline{a}(t)\lambda^{i} + \overline{b}(t)\overline{\lambda}_{\theta}^{i}(1), \quad \overline{\lambda}_{\theta}^{i}(1) = \lambda_{\parallel\alpha}^{i}\lambda^{\alpha}, \quad \overline{\lambda}_{\theta}^{i}(2) = \overline{\lambda}_{\theta}^{i}(1)_{\parallel\alpha}^{i}\lambda^{\alpha}, \quad (1.6)$$

 $\theta = 1, 2$ , for covariant differentiation of the  $\theta$ -th kind with regard to affine connection of the space  $\mathbb{G}\overline{\mathbb{A}}_N$  denoted by  $\parallel$ .

Because of two kinds of almost geodesic lines of this space a mapping  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  is the almost geodesic mapping of a  $\theta$ -th kind,  $\theta = 1, 2$ , if any geodesic line of the space  $\mathbb{G}\mathbb{A}_N$  it turns into an almost geodesic line of the  $\theta$ -th kind of the space  $\mathbb{G}\overline{\mathbb{A}}_N$ . For this reason, there are three types of almost geodesic mappings of the space  $\mathbb{G}\mathbb{A}_N$  and any of these three types have two kinds. A class of almost geodesic mappings of a  $\tau$ -th type,  $\tau = 1, 2, 3$ , and of a  $\theta$ -th kind  $\theta = 1, 2$  of the space  $\mathbb{G}\mathbb{A}_N$  is denoted as  $\mathbb{F}_{\theta}$ .

Basic equations of a second type almost geodesic mapping  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  of a  $\theta$ -th kind,  $\theta = 1, 2$ , are [15]:

$$\overline{L}_{jk}^{i} = L_{jk}^{i} + \psi_{j} \delta_{k}^{i} + \psi_{k} \delta_{j}^{i} + \sigma_{j} F_{k}^{i} + \sigma_{k} F_{j}^{i} + \xi_{jk}^{i}, \tag{1.7}$$

$$F_{j|k}^{i} + F_{k|j}^{i} + F_{\alpha}^{i} F_{j}^{\alpha} \sigma_{k} + F_{\alpha}^{i} F_{k}^{\alpha} \sigma_{j} + (-1)^{\theta} \left( \xi_{j\alpha}^{i} F_{k}^{\alpha} + \xi_{k\alpha}^{i} F_{j}^{\alpha} \right)$$

$$= \mu_{j} F_{k}^{i} + \mu_{k} F_{j}^{i} + \nu_{j} \delta_{k}^{i} + \nu_{k} \delta_{j}^{i},$$

$$(1.8)$$

for covariant vectors  $\mu_j, \nu_j$ , an affinor  $F_j^i$  and an anti-symmetric tensor  $\xi_{jk}^i$ .

A second type almost geodesic mapping  $f: \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  of a  $\theta$ -th kind,  $\theta = 1, 2$ , satisfies the property of reciprocity (it is an element of the class  $\pi_2(e)$ ) if it saves the affinor  $F_j^i$  and its inverse mapping is a second type almost geodesic mapping of the  $\theta$ -th kind. An almost geodesic mapping f of the space  $\mathbb{GA}_N$  satisfies the property of reciprocity (see [15]) if and only if the affinor  $F_j^i$  satisfies a relation

$$F_{\alpha}^{i}F_{i}^{\alpha} = e\delta_{i}^{i}, \qquad e = 0, \pm 1. \tag{1.9}$$

# 2 Invariants of second type almost geodesic mappings

The aim of this paper is to find some new invariants of almost geodesic mappings of a second type which satisfy the property of reciprocity. The results in this subject obtained until now are about the theories of special subclasses of the classes  $\pi_2$ ,  $\theta = 1, 2$ .

Motivated by Sinyukov's results, it is obtained (see [15]) magnitudes

$$T_{1jk}^{i} = \widetilde{S}_{jk}^{i} - \frac{1}{e - F^{2}} \left( \left( F \widetilde{S}_{k\alpha}^{\alpha} - F_{k}^{\alpha} \widetilde{S}_{\alpha\beta}^{\beta} \right) F_{j}^{i} + \left( F \widetilde{S}_{j\alpha}^{\alpha} - F_{j}^{\alpha} \widetilde{S}_{\alpha\beta}^{\beta} \right) F_{k}^{i} \right), \quad (2.10)$$

$$\hat{T}_{2jk}^{i} = T_{jk}^{i} + e F_{\alpha}^{i} \left( F_{(j|k)}^{\alpha} - \widetilde{T}_{\beta(k}^{\alpha} F_{j)}^{\beta} \right)$$

$$- \frac{e}{1 + N} F_{\alpha}^{\beta} \left( \left( F_{\beta|j}^{\alpha} - \widetilde{T}_{\gamma(\beta}^{\alpha} F_{j)}^{\gamma} \right) \delta_{k}^{i} + \left( F_{\beta|k}^{\alpha} - \widetilde{T}_{\gamma(\beta}^{\alpha} F_{k)}^{\gamma} \right) \delta_{j}^{i} \right), \quad (2.11)$$

for  $F = F_{\alpha}^{\alpha}$ ,  $e - F^2 \neq 0$  and Thomas projective parameter  $T_{jk}^i$  of the associated space  $\mathbb{A}_N$  in the expression of the invariant  $T_{jk}^i$ , are invariants of a canonical second type almost geodesic mapping of the first kind of the space  $\mathbb{G}\mathbb{A}_N$ .

Moreover, Weyl projective tensor of the space  $\mathbb{G}\hat{\mathbb{A}}_N$  which affine connection coefficients are  $\hat{L}^i_{jk} = L^i_{jk} + eF^i_{\alpha}F^{\alpha}_{(j|k)} - eT^{\alpha}_{\beta(j}F^{\beta}_k)F^i_{\alpha}$  is an invariant of the canonical second type almost geodesic mapping f of the first kind. The aim of our following research is to find some other more general invariants of special second type almost geodesic mappings of the space  $\mathbb{G}\mathbb{A}_N$ .

Let a mapping  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of a  $\theta$ -th kind,  $\theta = 1, 2$ , which satisfies the property of reciprocity. The composition (1.9) involved into the basic equation (1.8) together with using of the fact the mapping f is an equitorsion one  $(\xi_{jk}^i = 0)$  involved into the both of basic equations (1.7, 1.8) proves it is satisfied relations

$$\overline{L}_{ik}^{i} = L_{ik}^{i} + \psi_{i} \delta_{k}^{i} + \psi_{k} \delta_{i}^{i} + \sigma_{i} F_{k}^{i} + \sigma_{k} F_{i}^{i}, \tag{2.12}$$

$$F_{j|k}^{i} + F_{k|j}^{i} = \mu_{j} F_{k}^{i} + \mu_{k} F_{j}^{i} + (\nu_{j} - e\sigma_{j}) \delta_{k}^{i} + (\nu_{k} - e\sigma_{k}) \delta_{j}^{i}.$$
 (2.13)

It is proved a following proposition is satisfied in this way.

**Proposition 2.1** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of a  $\theta$ -th kind,  $\theta = 1, 2$ , which satisfies the property of reciprocity. The equations (2.12, 2.13, 1.9) are basic equations of this mapping.  $\square$ 

Based on the fact the second type almost geodesic mapping f satisfies the property of reciprocity the corresponding magnitudes  $\overline{\psi}_i, \overline{\sigma}_i, \overline{F}^i_j$  which determine an inverse mapping  $f^{-1}$  of the mapping f are [15]

$$\overline{\psi}_i = -\psi_i, \quad \overline{\sigma}_i = -\sigma_i, \quad \overline{F}_j^i = F_j^i.$$

After contracting the basic equation (2.12) by indices i and k and using the fact it is satisfied a relation  $\sigma_j = \frac{1}{2}(\sigma_j - \overline{\sigma}_j)$  we obtain it is satisfied an equation

$$\psi_{j} = \frac{1}{N+1} \left( \overline{L}_{j\alpha}^{\alpha} - L_{j\alpha}^{\alpha} \right) + \frac{1}{2(N+1)} \left[ \left( \overline{\sigma}_{j} \overline{F} + \overline{\sigma}_{\alpha} \overline{F}_{j}^{\alpha} \right) - \left( \sigma_{j} F + \sigma_{\alpha} F_{j}^{\alpha} \right) \right], \quad (2.14)$$

for  $F = F_{\alpha}^{\alpha}$  as above.

Using the previous expression of the magnitude  $\psi_j$  we conclude the basic equation (2.12) has a form

$$\overline{L}_{jk}^{i} = L_{jk}^{i} + \overline{\omega}_{jk}^{i} - \omega_{jk}^{i}, \qquad (2.15)$$

for

$$\omega_{jk}^{i} = -\frac{1}{2}(\sigma_{j}F_{k}^{i} + \sigma_{k}F_{j}^{i}) + \frac{1}{N+1}(L_{j\alpha}^{\alpha}\delta_{k}^{i} + L_{k\alpha}^{\alpha}\delta_{j}^{i}) + \frac{1}{2(N+1)}((\sigma_{j}F + \sigma_{\alpha}F_{j}^{\alpha})\delta_{k}^{i} + (\sigma_{k}F + \sigma_{\alpha}F_{k}^{\alpha})\delta_{j}^{i}),$$

$$(2.16)$$

and the magnitude  $\overline{\psi}_j$  defined in the same manner as a function of the corresponding elements of the space  $\mathbb{G}\overline{\mathbb{A}}_N$ .

The equation (2.15) proves it is satisfied an equality

$$\overline{\mathcal{T}}_{jk}^i = \mathcal{T}_{jk}^i,$$

for

$$\underline{\mathcal{T}}_{jk}^i = L_{jk}^i - \omega_{jk}^i \quad \text{and} \quad \overline{\underline{\mathcal{T}}}_{jk}^i = \overline{L}_{jk}^i - \overline{\omega}_{jk}^i.$$
(2.17)

It is proved a following lemma in this way.

**Lemma 2.1** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion almost geodesic mapping of a second type which satisfies the property of reciprocity. A magnitude  $\mathcal{T}^i_{2jk}$  defined in the first of the expressions (2.17) is an invariant of the mapping f.  $\square$ 

The invariant  $\mathcal{T}^i_{2jk}$  of a second type almost geodesic mapping  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  which satisfies the property of reciprocity is said to be the  $\pi_2$ -generalized Thomas projective parameter.

Let us generalize Weyl projective tensor of an equitorsion almost geodesic mapping  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  of the first type which satisfies the property of reciprocity. This generalization will be realized just for first type almost geodesic mappings. A result for the case of second type almost geodesic mappings may be obtained in the same manner.

First of all, we can observe the magnitude  $\omega^i_{jk}$  defined into the equation (2.16) is symmetric by indices j and k. For this reason, symmetric parts  $\widetilde{S}^i_{jk}$  and  $\overline{\widetilde{S}}^i_{jk}$  of affine connection coefficients  $L^i_{jk}$  and  $\overline{L}^i_{jk}$  of the spaces  $\mathbb{G}\mathbb{A}_N$  and  $\mathbb{G}\overline{\mathbb{A}}_N$  satisfy a relation

$$\frac{\widetilde{S}_{jk}^{i}}{\widetilde{S}_{jk}^{i}} = \widetilde{S}_{jk}^{i} + \overline{\omega}_{jk}^{i} - \omega_{jk}^{i}, \qquad (2.18)$$

for the above defined magnitudes  $\omega^i_{jk}$  and  $\overline{\omega}^i_{jk}$ .

Using the covariant derivative of the first kind (1.2) we obtain a curvature tensor  $R_{imn}^i$  of the associated space  $\mathbb{A}_N$  has a form:

$$R_{jmn}^{i} = \widetilde{S}_{jm|n}^{i} - \widetilde{S}_{jn|m}^{i} - \widetilde{T}_{\alpha n}^{i} \widetilde{S}_{jm}^{\alpha} - \widetilde{T}_{jm}^{\alpha} \widetilde{S}_{\alpha n}^{i}$$

$$+ \widetilde{T}_{jn}^{\alpha} \widetilde{S}_{\alpha m}^{i} + \widetilde{T}_{\alpha m}^{i} \widetilde{S}_{jn}^{\alpha} + 2\widetilde{T}_{mn}^{\alpha} \widetilde{S}_{j\alpha}^{i}.$$

$$(2.19)$$

Motivated by this result we are going to find a rule of change of the curvature tensor  $R_{imn}^i$  bellow. Let us involve following substitutions:

$$\mathcal{U}_{2jk}^{i} = \frac{1}{2} \left( \mathcal{T}_{2jk}^{i} + \mathcal{T}_{2kj}^{i} \right) \quad \text{and} \quad \overline{\mathcal{U}}_{2jk}^{i} = \frac{1}{2} \left( \overline{\mathcal{T}}_{2jk}^{i} + \overline{\mathcal{T}}_{2kj}^{i} \right), \tag{2.20}$$

for the above obtained invariant  $\mathcal{T}^i_{jk} = \overline{\mathcal{T}}^i_{jk}$  of the mapping f.

From the equations (2.17) and (2.20) together with the above mentioned symmetry of the magnitude  $\omega^i_{jk}$  from the equation (2.16) by indices j and k we conclude it is satisfied equalities

$$\mathcal{Q}_{jjk}^{i} = \widetilde{S}_{jk}^{i} - \omega_{jk}^{i} \quad \text{and} \quad \overline{\mathcal{Q}}_{jk}^{i} = \widetilde{\overline{S}}_{jk}^{i} - \overline{\omega}_{jk}^{i}.$$
(2.21)

It is easy to be obtained covariant derivatives

$$\mathcal{U}_{2jm|n}^{i} = \frac{1}{2} \left( \mathcal{T}_{2jm|n}^{i} + \mathcal{T}_{2mj|n}^{i} \right) \quad \text{and} \quad \overline{\mathcal{U}}_{2jm|n}^{i} = \frac{1}{2} \left( \overline{\mathcal{T}}_{2jm|n}^{i} + \overline{\mathcal{T}}_{2mj|n}^{i} \right), \quad (2.22)$$

of the previous defined magnitudes  $\mathcal{U}^i_{jm}$  and  $\overline{\mathcal{U}}^i_{2jm}$  satisfy a relation

$$\overline{\mathcal{U}}_{2jm\parallel n}^{i} = \mathcal{U}_{2jm\parallel n}^{i} + \widetilde{\overline{S}}_{\alpha n}^{i} \overline{\mathcal{U}}_{jm}^{\alpha} - \widetilde{\overline{S}}_{jn}^{\alpha} \overline{\mathcal{U}}_{\alpha m}^{i} - \widetilde{\overline{S}}_{mn}^{\alpha} \overline{\mathcal{U}}_{2j\alpha}^{i} \\
- \widetilde{S}_{\alpha n}^{i} \mathcal{U}_{jm}^{\alpha} + \widetilde{S}_{jn}^{\alpha} \mathcal{U}_{\alpha m}^{i} + \widetilde{S}_{mn}^{\alpha} \mathcal{U}_{2j\alpha}^{i}.$$
(2.23)

The equations (2.21, 2.23) prove it is satisfied a following proposition.

**Proposition 2.2** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces  $\mathbb{G}\mathbb{A}_N$  and  $\mathbb{G}\overline{\mathbb{A}}_N$ . Covariant derivatives  $\widetilde{S}^i_{jm|n}$  and  $\widetilde{\overline{S}}^i_{jm|n}$  of symmetric

parts  $\widetilde{S}^i_{jm}$  and  $\widetilde{\overline{S}}^i_{jm}$  of the corresponding affine connection coefficients  $L^i_{jm}$  and  $\overline{L}^i_{jm}$  satisfy a relation

$$\begin{split} \widetilde{\overline{S}}_{jm\parallel n}^{i} &= \widetilde{S}_{jm\parallel n}^{i} + \overline{\omega}_{jm\parallel n}^{i} - \omega_{jm\parallel n}^{i} + \widetilde{\overline{S}}_{\alpha n}^{i} \overline{\mathcal{U}}_{jm}^{\alpha} - \widetilde{\overline{S}}_{jn}^{\alpha} \overline{\mathcal{U}}_{\alpha m}^{i} - \widetilde{\overline{S}}_{mn}^{\alpha} \overline{\mathcal{U}}_{j\alpha}^{i} \\ &- \widetilde{S}_{\alpha n}^{i} \mathcal{U}_{jm}^{\alpha} + \widetilde{S}_{jn}^{\alpha} \mathcal{U}_{\alpha m}^{i} + \widetilde{S}_{mn}^{\alpha} \mathcal{U}_{j\alpha}^{i}, \end{split} \tag{2.24}$$

for the magnitudes  $\mathcal{U}^i_{jk}$  and  $\overline{\mathcal{U}}^i_{jk}$  defined above.  $\square$ 

Using the invariance  $\widetilde{\overline{T}}_{jk}^i = \widetilde{T}_{jk}^i$  and the consequent invariances  $\overline{\overline{T}}_{2jm}^{\alpha}\widetilde{\overline{T}}_{\alpha n}^i = \underline{T}_{2jm}^{\alpha}\widetilde{T}_{\alpha n}^i$  such as  $\overline{\overline{T}}_{2j\alpha}^i\widetilde{\overline{T}}_{mn}^{\alpha} = \underline{T}_{2j\alpha}^i\widetilde{T}_{mn}^{\alpha}$  we obtain it is satisfied a following proposition.

**Proposition 2.3** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces  $\mathbb{G}\mathbb{A}_N$  and  $\mathbb{G}\overline{\mathbb{A}}_N$ . Magnitudes  $\widetilde{S}^{\alpha}_{jm}\widetilde{T}^i_{\alpha n}, \widetilde{S}^i_{\alpha j}\widetilde{T}^{\alpha}_{mn}$  and its deformations  $\widetilde{\overline{S}}^{\alpha}_{jm}\widetilde{T}^i_{\alpha n}, \widetilde{\overline{S}}^i_{\alpha j}\widetilde{T}^{\alpha}_{mn}$  under the mapping f satisfy equations

$$\widetilde{\overline{S}}_{jm}^{\alpha}\widetilde{\overline{T}}_{\alpha n}^{i} = \widetilde{S}_{jm}^{\alpha}\widetilde{T}_{\alpha n}^{i} + \overline{\omega}_{jm}^{\alpha}\widetilde{\overline{T}}_{\alpha n}^{i} - \omega_{jm}^{\alpha}\widetilde{T}_{\alpha n}^{i}, \qquad (2.25)$$

$$\widetilde{\overline{S}}_{j\alpha}^{i}\widetilde{\overline{T}}_{mn}^{\alpha} = \widetilde{S}_{j\alpha}^{i}\widetilde{T}_{mn}^{\alpha} + \overline{\omega}_{j\alpha}^{i}\widetilde{\overline{T}}_{mn}^{\alpha} - \omega_{j\alpha}^{i}\widetilde{T}_{mn}^{\alpha}, \qquad (2.26)$$

for the magnitude  $\omega^i_{jk}$  defined into the equation (2.16) and the corresponding one  $\overline{\omega}^i_{jk}$ .  $\square$ 

If we contract the basic equation (2.13) by the indices i and k we conclude it is satisfied a relation

$$F_{j} = \mu_j F + \mu_\alpha F_j^\alpha + (N+1)(\nu_j - e\sigma_j) - F_{j}^\alpha_{\alpha},$$
 (2.27)

 $\theta = 1, 2$ . This equation proves it is satisfied a following proposition.

**Proposition 2.4** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of the first kind between non-symmetric affine connection spaces  $\mathbb{G}\mathbb{A}_N$  and  $\mathbb{G}\overline{\mathbb{A}}_N$  which satisfies the property of reciprocity. A covariant derivative  $\omega^i_{jm|n}$  of the magnitude  $\omega^i_{jk}$  defined into the equation (2.16) satisfies an equality

$$\omega_{jm_{1}^{i}n}^{i} = \frac{1}{N+1} \left( L_{j\alpha_{1}^{i}n}^{\alpha} \delta_{m}^{i} + L_{m\alpha_{1}^{i}n}^{\alpha} \delta_{j}^{i} \right) + \frac{1}{2} (\nu_{n} - e\sigma_{n}) (\sigma_{j} \delta_{m}^{i} + \sigma_{m} \delta_{j}^{i}) 
- \frac{1}{2} \left( \sigma_{j_{1}^{i}n} F_{m}^{i} + \sigma_{m_{1}^{i}n} F_{j}^{i} + \sigma_{j} F_{m_{1}^{i}n}^{i} + \sigma_{m} F_{j_{1}^{i}n}^{i} \right) 
+ \frac{1}{2(N+1)} \left( \sigma_{j_{1}^{i}n} F + \sigma_{\alpha_{1}^{i}n} F_{j}^{\alpha} + \sigma_{\alpha} F_{j_{1}^{i}n}^{\alpha} \right) \delta_{m}^{i} 
+ \frac{1}{2(N+1)} \left( \sigma_{m_{1}^{i}n} F + \sigma_{\alpha_{1}^{i}n} F_{m}^{\alpha} + \sigma_{\alpha} F_{m_{1}^{i}n}^{\alpha} \right) \delta_{j}^{i} 
+ \frac{1}{2(N+1)} \left( \mu_{n} F + \mu_{\alpha} F_{n}^{\alpha} - F_{n_{1}^{i}\alpha}^{\alpha} \right) (\sigma_{j} \delta_{m}^{i} + \sigma_{m} \delta_{j}^{i}),$$
(2.28)

for magnitudes  $\mu_i, \nu_i$  used into the basic equation (2.13).  $\square$ 

A difference  $\widehat{\Delta}^i_{jmn} = \overline{\omega}^i_{jm|n} - \omega^i_{jm|n}$ , of the magnitudes  $\overline{\omega}^i_{jm|n}$  and  $\omega^i_{jm|n}$  satisfies a relation

$$\widehat{\Delta}_{jmn}^{i} = \frac{1}{N+1} \left( \left( \widetilde{\overline{S}}_{j\alpha|n}^{\alpha} - \widetilde{S}_{j\alpha|n}^{\alpha} \right) \delta_{m}^{i} + \left( \widetilde{\overline{S}}_{m\alpha|n}^{\alpha} - \widetilde{S}_{m\alpha|n}^{\alpha} \right) \delta_{j}^{i} \right) + \widehat{\rho}_{jmn}^{i} - \widehat{\rho}_{jmn}^{i},$$

$$(2.29)$$

for

$$2\hat{\rho}_{jmn}^{i} = -2L_{jn}^{\beta} \tilde{T}_{\beta\alpha}^{\alpha} \delta_{m}^{i} - 2L_{mn}^{\beta} \tilde{T}_{\beta\alpha}^{\alpha} \delta_{j}^{i} + (\nu_{n} - e\sigma_{n})(\sigma_{j}\delta_{m}^{i} + \sigma_{m}\delta_{j}^{i})$$

$$- (\sigma_{j|n}F_{m}^{i} + \sigma_{m|n}F_{j}^{i} + \sigma_{j}F_{m|n}^{i} + \sigma_{m}F_{j|n}^{i})$$

$$+ \frac{1}{N+1} (\sigma_{j|n}F + \sigma_{\alpha|n}F_{j}^{\alpha} + \sigma_{\alpha}F_{j|n}^{\alpha})\delta_{m}^{i}$$

$$+ \frac{1}{N+1} (\sigma_{m|n}F + \sigma_{\alpha|n}F_{m}^{\alpha} + \sigma_{\alpha}F_{m|n}^{\alpha})\delta_{j}^{i}$$

$$+ \frac{1}{N+1} (\mu_{n}F + \mu_{\alpha}F_{n}^{\alpha} - F_{n|\alpha}^{\alpha})(\sigma_{j}\delta_{m}^{i} + \sigma_{m}\delta_{j}^{i}),$$

$$(2.30)$$

a magnitude  $\hat{\overline{\rho}}_{jmn}^i$  from the space  $\mathbb{G}\overline{\mathbb{A}}_N$  analogue to the magnitude  $\hat{\rho}_{jmn}^i$ , the magnitudes  $\mu_i, \nu_i$  from the equation (2.13) and the corresponding ones  $\overline{\mu}_i, \overline{\nu}_i$ .

From the equations (2.29, 2.30) we conclude it exists a magnitude

$$\hat{v}_{1}ij = -\frac{1}{N+1} \widetilde{S}_{i\alpha_{\parallel}j}^{\alpha} + \overline{L}_{ij}^{\beta} \widetilde{T}_{\beta\alpha}^{\alpha} - (\overline{\nu}_{j} - e\overline{\sigma}_{j})\sigma_{i} 
+ \frac{1}{2(N+1)} (\overline{\sigma}_{i\parallel j} \overline{F} + \overline{\sigma}_{\alpha\parallel j} \overline{F}_{i}^{\alpha} + \overline{\sigma}_{\alpha} \overline{F}_{i\parallel j}^{\alpha}) 
- \frac{1}{2(N+1)} (\overline{\mu}_{j} \overline{F} + \overline{\mu}_{\alpha} \overline{F}_{j}^{\alpha} - \overline{F}_{j\parallel \alpha}^{\alpha}) \overline{\sigma}_{i} 
+ \frac{1}{N+1} \widetilde{S}_{i\alpha_{\parallel}j}^{\alpha} - L_{ij}^{\beta} \widetilde{T}_{\beta\alpha}^{\alpha} + (\nu_{j} - e\sigma_{j})\sigma_{i} 
- \frac{1}{2(N+1)} (\sigma_{i\parallel j} F + \sigma_{\alpha\parallel j} F_{i}^{\alpha} + \sigma_{\alpha} F_{i\parallel j}^{\alpha}) 
+ \frac{1}{2(N+1)} (\mu_{j} F + \mu_{\alpha} F_{j}^{\alpha} - F_{j\parallel \alpha}^{\alpha}) \sigma_{i},$$
(2.31)

such that the equation (2.24) has a form

$$\begin{split} &\widetilde{\overline{S}}_{jm\parallel n}^{i} = \widetilde{S}_{jm\parallel n}^{i} - \delta_{m}^{i} \hat{\psi}_{jn} - \delta_{j}^{i} \hat{\psi}_{mn} \\ &- \frac{1}{2} \left( \overline{\sigma}_{j\parallel n} \overline{F}_{m}^{i} + \overline{\sigma}_{m\parallel n} \overline{F}_{j}^{i} + \overline{\sigma}_{j} \overline{F}_{m\parallel n}^{i} + \overline{\sigma}_{m} \overline{F}_{j\parallel n}^{i} \right) \\ &+ \frac{1}{2} \left( \sigma_{j\parallel n} F_{m}^{i} + \sigma_{m\parallel n} F_{j}^{i} + \sigma_{j} F_{m\parallel n}^{i} + \sigma_{m} F_{j\parallel n}^{i} \right) \\ &+ \widetilde{\overline{S}}_{\alpha n}^{i} \overline{\mathcal{U}}_{jm}^{\alpha} - \widetilde{\overline{S}}_{jn}^{\alpha} \overline{\mathcal{U}}_{\alpha m}^{i} - \widetilde{\overline{S}}_{mn}^{\alpha} \overline{\mathcal{U}}_{j\alpha}^{i} - \widetilde{S}_{\alpha n}^{i} \overline{\mathcal{U}}_{jm}^{\alpha} + \widetilde{S}_{jn}^{\alpha} \mathcal{U}_{\alpha m}^{i} + \widetilde{S}_{mn}^{\alpha} \mathcal{U}_{j\alpha}^{i}. \end{split} \tag{2.32}$$

Using the equations (2.20, 2.25, 2.26, 2.32) and the expression (2.19) of the curvature tensors  $R^i_{jmn}$  and  $\overline{R}^i_{jmn}$  of the associated spaces  $\mathbb{A}_N$  and  $\overline{\mathbb{A}}_N$  we obtain it is satisfied the following equation

$$\begin{split} \overline{R}_{jmn}^{i} &= R_{jmn}^{i} - \delta_{m}^{i} \hat{v}_{jn} + \delta_{n}^{i} \hat{v}_{jm} - \delta_{j}^{i} \hat{v}_{lmn}] \\ &- \frac{1}{2} \left( \overline{\sigma}_{j\parallel n} \overline{F}_{m}^{i} + \overline{\sigma}_{m\parallel n} \overline{F}_{j}^{i} + \overline{\sigma}_{j} \overline{F}_{m\parallel n}^{i} + \overline{\sigma}_{m} \overline{F}_{j\parallel n}^{i} \right) \\ &+ \frac{1}{2} \left( \sigma_{j\parallel n} F_{m}^{i} + \sigma_{m\parallel n} F_{j}^{i} + \sigma_{j} F_{m\parallel n}^{i} + \sigma_{m} F_{j\parallel n}^{i} \right) \\ &+ \frac{1}{2} \left( \overline{\sigma}_{j\parallel m} \overline{F}_{n}^{i} + \overline{\sigma}_{n\parallel m} \overline{F}_{j}^{i} + \overline{\sigma}_{j} \overline{F}_{n\parallel m}^{i} + \overline{\sigma}_{n} \overline{F}_{j\parallel m}^{i} \right) \\ &+ \frac{1}{2} \left( \overline{\sigma}_{j\parallel m} \overline{F}_{n}^{i} + \overline{\sigma}_{n\parallel m} \overline{F}_{j}^{i} + \overline{\sigma}_{j} \overline{F}_{n\parallel m}^{i} + \overline{\sigma}_{n} \overline{F}_{j\parallel m}^{i} \right) \\ &+ \frac{1}{2} \left( \overline{\sigma}_{j\parallel m} F_{n}^{i} + \sigma_{n\parallel m} F_{j}^{i} + \sigma_{j} F_{n\parallel m}^{i} + \sigma_{n} F_{j\parallel m}^{i} \right) \\ &- \frac{1}{2} \left( \sigma_{j\parallel m} F_{n}^{i} + \sigma_{n\parallel m} F_{j}^{i} + \sigma_{j} F_{n\parallel m}^{i} + \sigma_{n} F_{j\parallel m}^{i} \right) \\ &+ 2 \widetilde{\overline{S}}_{\alpha n}^{i} \widetilde{\overline{S}}_{jm}^{\alpha} - 2 \widetilde{\overline{S}}_{jn}^{\alpha} \widetilde{\overline{S}}_{\alpha m}^{i} - 2 \widetilde{\overline{S}}_{n}^{\alpha} \widetilde{\overline{S}}_{jm}^{i} + 2 \widetilde{\overline{S}}_{n}^{\alpha} \widetilde{\overline{S}}_{nm}^{i} \\ &- \overline{\omega}_{jm}^{\alpha} \overline{L}_{\alpha n}^{i} + \omega_{jm}^{\alpha} L_{\alpha n}^{i} - \overline{\omega}_{in}^{i} \overline{L}_{\alpha m}^{i} - \omega_{jn}^{i} L_{\alpha m}^{i} + 2 \overline{\omega}_{jn}^{i} \widetilde{\overline{T}}_{mn}^{\alpha} - 2 \omega_{j\alpha}^{i} \widetilde{T}_{mn}^{\alpha} \\ &+ \overline{\omega}_{\alpha m}^{i} \overline{L}_{jn}^{\alpha} - \omega_{\alpha m}^{i} L_{jn}^{\alpha} + \overline{\omega}_{jn}^{\alpha} \overline{L}_{\alpha m}^{i} - \omega_{jn}^{\alpha} L_{\alpha m}^{i} + 2 \overline{\omega}_{jn}^{i} \widetilde{T}_{mn}^{\alpha} - 2 \omega_{j\alpha}^{i} \widetilde{T}_{mn}^{\alpha} \\ &= R_{jmn}^{i} - \delta_{m}^{i} \hat{v}_{jn}^{i} + \delta_{n}^{i} \hat{v}_{jm}^{i} - \delta_{j}^{i} \hat{v}_{[mn]}^{i} + F_{jmn}^{i} - \overline{F}_{jmn}^{i}, \end{split}$$

for

$$\mathcal{F}_{1jmn}^{i} = \frac{1}{2} \left( \sigma_{j|n} F_{m}^{i} + \sigma_{m|n} F_{j}^{i} + \sigma_{j} F_{m|n}^{i} + \sigma_{m} F_{j|n}^{i} \right) 
- \frac{1}{2} \left( \sigma_{j|m} F_{n}^{i} + \sigma_{n|m} F_{j}^{i} + \sigma_{j} F_{n|m}^{i} + \sigma_{n} F_{j|m}^{i} \right) 
- 2\widetilde{S}_{\alpha n}^{i} \widetilde{S}_{jm}^{\alpha} + 2\widetilde{S}_{jn}^{\alpha} \widetilde{S}_{\alpha m}^{i} 
+ \omega_{jm}^{\alpha} L_{\alpha n}^{i} + \omega_{\alpha n}^{i} L_{jm}^{\alpha} - \omega_{\alpha m}^{i} L_{jn}^{\alpha} - \omega_{jn}^{\alpha} L_{\alpha m}^{i} - 2\omega_{j\alpha}^{i} \widetilde{T}_{mn}^{\alpha},$$
(2.34)

the magnitude  $\omega^i_{jk}$  defined into the equation (2.16) and the corresponding magnitude  $\overline{\mathcal{F}}^i_{1\,jmn}$ .

After contracting the equation (2.33) by indices i and n we conclude Ricci tensors  $R_{jm}$  and  $\overline{R}_{jm}$  of the associated spaces  $A_N$  and  $\overline{A}_N$  satisfy a relation

$$\overline{R}_{jm} = R_{jm} + (N-1)\hat{v}_{jm} + \hat{v}_{1[jm]} + \mathcal{F}_{jm} - \overline{\mathcal{F}}_{jm},$$
(2.35)
$$for \ \mathcal{F}_{jm} = \mathcal{F}_{jm\alpha}^{\alpha} \text{ and } \overline{\mathcal{F}}_{jm} = \overline{\mathcal{F}}_{jm\alpha}^{\alpha}.$$

After alternating the equation (2.35) by indices j and m we conclude it is satisfied a relation

$$(N+1)\hat{v}_{[jm]} = \overline{R}_{[jm]} - R_{[jm]} - \mathcal{F}_{[jm]} + \overline{\mathcal{F}}_{[jm]}. \tag{2.36}$$

From the equations (2.35) and (2.36) we conclude it is satisfied an expression

$$(N^{2}-1)\hat{v}_{jm} = (N\overline{R}_{jm} + \overline{R}_{mj}) - (NR_{jm} + R_{mj}) + (N\overline{\mathcal{F}}_{jm} + \overline{\mathcal{F}}_{mj}) - (N\mathcal{F}_{jm} + \mathcal{F}_{mj}).$$

$$(2.37)$$

After involving the results (2.36, 2.37) in the equation (2.33) we obtain it is satisfied an equality

$$\overline{\mathcal{W}}_{jmn}^i = \mathcal{W}_{jmn}^i,$$

where we denoted by

$$\mathcal{W}_{jmn}^{i} = R_{jmn}^{i} + \frac{1}{N+1} \delta_{j}^{i} R_{[mn]} + \frac{N}{N^{2}-1} \delta_{[m}^{i} R_{jn]} + \frac{1}{N^{2}-1} \delta_{[m}^{i} R_{n]j} + \frac{1}{N^{2}-1} \delta_{[m}^{i} F_{n]j} + \frac{1}{N^{2}-1} \delta_{[m}^{i} F_{n]j} + \frac{1}{N^{2}-1} \delta_{[m}^{i} F_{n]j},$$
(2.38)

a geometric object of the space  $\mathbb{G}\mathbb{A}_N$ , where the magnitude  $\mathcal{F}^i_{1jmn}$  is defined into the equation (2.34) and for the corresponding magnitude  $\mathcal{F}_{1ij\alpha} = \mathcal{F}^{\alpha}_{1ij\alpha}$ . The corresponding magnitude  $\overline{\mathcal{W}}^i_{jmn}$  of the space  $\mathbb{G}\overline{\mathbb{A}}_N$  is defined in the same manner.

It is proved a following theorem is satisfied in this way.

**Theorem 2.1** Let  $f: \mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$  be an equitorsion second type almost geodesic mapping of the first kind which satisfies the property of reciprocity. The magnitude  $\mathcal{W}^i_{2\ jmn}$  defined into the equation (2.38) is an invariant of this mapping.  $\square$ 

# 3 Acknowledgements

This paper is financially supported by the Serbian Ministry of Education, Science and Technological Developments, Grant. No. 174012.

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