Formalism of operators for Laguerre-Gauss modes

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Abstract

In this work apply the algebra of operators of quantum mechanics in the Helmholtz wave equation in cylindrical coordinates in paraxial approximation to describe the raising and lowering operators for the Laguerre-Gauss modes.

I. INTRODUCTION

It is well known that the momentum of electromagnetic wave contains a linear and angular contribution [1]. In 1909, Poynting [2] showed that circularly polarised beams carries angular momentum (spin). In 1936, Beth [3] made the first experimental observation of the change in the spin angular momentum of polarized light. The angular momentum can be separated into two components [4]: the orbital (associated with the azimuthal phase dependence of the beam) and the spin (associated with the polarisation). Recently much interest theoretical and experimental [5] have been given to study the orbital angular momentum (OAM) of the radiation. In 1992, Allen et al. [6] showed that any beam with helical wavefronts carried OAM along the direction of propagation. An example of light physically realisable with this distribution are the Laguerre-Gauss beams (LG). The amplitude in the modes appears the term $exp(-il\phi)$, where l is a integer number. Experimentally these modes can be produced through spiral phase plates [7], cylindrical lens mode converters [8] and through computer generated holograms [9]. In LG beams each photon carry $l\hbar$ of OAM [6]. For polarized light beams, the spin per photon is $\sigma_z \hbar$, where the value of σ_z (-1 $\leq \sigma_z \leq$ 1) depends on the polarization. The LG beams are solutions of the paraxial wave equation [4] in cylindrical coordinate, this equation is identical the Schrödinger equation for a free particle where the time variable was change for propagation variable z of beam. The paraxial approximation assumes that the structure transversal of beam varies slowly with the propagation. In this paper we write the raising and lowering operators in cylindrical coordinates for LG modes using the analogy with the quantum harmonic oscillator.

II. THE PARAXIAL EQUATION

The electric field of a light beam propagating in free space is described by Maxwell's wave equation

$$\nabla^2 \vec{E}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial^2 \vec{E}(\vec{r}, t)}{\partial t^2}$$
 (1)

with c is the light velocity. The electric field of a monochromatic wave propagating in direction z can be written as

$$\vec{E}(\vec{r},t) = \hat{\epsilon}u(\vec{r},t)e^{i(kz-\omega t)}$$
(2)

where $\hat{\epsilon}$ is the vector of polarization, $u(\vec{r},t)$ is the field amplitude, $k=nk_0$ is the wave number, where n is the index of refraction of the medium and $k_0 = \omega/c$, ω is the angular frequency. The function $u(\vec{r},t)$ ($\vec{r}=(x,y,z)$ or $\vec{r}=(r,\phi,z)$, depends on the coordinate system adopted, rectangular or cylindrical) specifies the transverse structure of the wave. Substituting Eq. (2) in the wave equation (1) results

$$\frac{i}{2k}\nabla^2 u(\vec{r},t) = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)u(\vec{r},t) \tag{3}$$

where ∇^2 is the Laplacian operator.

The paraxial approximation describe nicely the propagation of some types beams that appear in lasers. The approximation is made disregarding the derivative of the second order in z in relation derived from the first order in (3). Then we will have to cylindrical coordinates

$$\left| \frac{\partial^2 u}{\partial z^2} \right| \ll \left| \frac{\partial u}{\partial z} \right|, \left| \frac{\partial^2 u}{\partial r^2} \right|, \left| \frac{\partial^2 u}{\partial \phi^2} \right| \tag{4}$$

The dependence of the amplitude is essentially due to diffraction effects and is generally retarded compared with the variation transversal the width of beam.

The paraxial approximation above describes the propagation of light beams whose transverse dimensions are much smaller than the typical longitudinal distance over which the field changes in magnitude, ie, the beam transversal profile varies slowly with the direction propagation z. Otherwise, when a wave is paraxial light beams form small angles with the direction of propagation. The scalar Helmholtz's Eq. (3) in paraxial approximation is then

$$\nabla_t^2 u(\vec{r}, z) + 2ik\partial_z u(\vec{r}, z) = 0$$
 (5)

where ∇_t contains only derivative in the transversal variables (x, y or r, ϕ). In polares coordinates (r, ϕ, z) and considering azimuthal symmetry $\nabla_t^2 = \frac{1}{r^2} \frac{\partial^2}{\partial^2 \phi} + \frac{1}{r} \frac{\partial}{\partial r} r(\frac{\partial}{\partial r})$.

III. LAGUERRE-GAUSS MODES

The Laguerre-Gauss modes are solutions of (5) in cylindrical coordinates the same way that are Hermite-Gauss modes in rectangular coordinates. The Laguerre-Gauss modes (LG) form a complete orthonormal basis set of solutions for paraxial light beams. These modes

normalized are describes for complex scalar function

$$u_{lp}^{LG}(r,\phi,z) = c_{lp} \left[\frac{r\sqrt{2}}{w(z)} \right]^{|l|} exp \left[-\frac{r^2}{w^2(z)} \right] L_p^{|l|} \left(\frac{2r^2}{w^2(z)} \right) exp \left[-i \left(\frac{kr^2}{2R(z)} + l\phi - \varphi(z) \right) \right]$$
(6)

where c_{lp} is the normalization constant given by

$$c_{lp} = \sqrt{\frac{2p!}{\pi w^2(z)(p+|l|)!}}$$
 (7)

and $L_p^{|l|}$ is the generalised Laguerre polynomial defined by expression

$$L_p^{|l|}(x) = \frac{e^x x^{-|l|}}{p!} \frac{d^p}{dx^p} \left(e^{-x} x^{p+|l|} \right) = \sum_{m=0}^p (-1)^m \binom{p+|l|}{p-m} \frac{x^p}{m!}$$
 (8)

the terms that appear in (6) are given by

$$w^2(z) = \frac{2(z^2 + b^2)}{kb} \tag{9}$$

$$R(z) = \frac{z^2 + b^2}{z} \tag{10}$$

$$\varphi(z) = (2p + |l| + 1) \tan^{-1} \left(\frac{z}{b}\right) \tag{11}$$

where w(z) is the radius of the beam, R(z) is the wavefront radius of curvature, $\psi(z)$ is the Gouy phase, b is the Rayleigh range, and the beam waist is at z=0. The index p is the number of radial modes (number of additional concentric rings around the central zone) and l is the number of intertwinded helices (relates to the azimuthal phase), when l=0 and p=0, the LG beam is identical to the fundamental Gaussian beam. The fundamental mode u_{00}^{LG} is written as

$$u_{00}^{LG}(r,\phi,z) = \sqrt{\frac{2}{\pi w^2(z)}} exp\left[-\frac{r^2}{w^2(z)}\right] exp\left[-i\left(\frac{kr^2}{2R(z)} - tan^{-1}\left(\frac{z}{b}\right)\right)\right]$$
(12)

IV. THE RAISING AND LOWERING OPERATORS

As can be observed the equation (5) is identical the Schrödinger's equation for a free particle in two dimensions, where the propagation variable z make the function the time. The formalism of quantum mechanics can be perfectly applied to the classical beams in paraxial approximation as was done by Stoler [10]. So we can to get higher-order modes from of fundamental mode utilizing the operators algebra. Making a analogy with the

two-dimensional harmonic oscillator. Defining the operators [12] for z=0.

$$A_x(0) = \frac{1}{\sqrt{2bk}} (kr\cos\phi + b\cos\phi\partial r - br\sin\phi\partial\phi)$$
 (13)

$$A_y(0) = \frac{1}{\sqrt{2bk}} (krsin\phi + bsin\phi\partial r + brcos\phi\partial\phi)$$
 (14)

The evolution these operators in z is given by the propagator operator U(z) for $\partial_{\phi}\partial_{r} = \partial_{r}\partial_{\phi}$ [11]

$$U(z) = exp\left[-\frac{i}{2k}P^2z\right] = exp\left[\frac{-i}{2k}\left(\partial_r^2 + r^2\partial_\phi^2\right)z\right]$$
 (15)

then

$$A_x(z) = U(z)A_x(0)U^{\dagger}(z) \tag{16}$$

$$A_y(z) = U(z)A_y(0)U^{\dagger}(z) \tag{17}$$

The ladder operators are defined by [11]

$$A_{\pm}(z) = \frac{1}{\sqrt{2}} \left[A_x(z) \mp i A_y(z) \right]$$
 (18)

$$A_{\pm}(z) = \frac{1}{2\sqrt{bk}} \left[ke^{\mp i\phi}r + (be^{\mp i\phi} + 2ze^{\pm i\phi})(\partial_r + r\partial_\phi)\right]$$
(19)

These operators obey the commutation rules.

$$[A_{\pm}(z), A_{\pm}^{\dagger}(z)] = 1 \quad and \quad [A_{\pm}(z), A_{\pm}^{\dagger}(z)] = 0$$
 (20)

As for harmonic oscillator the higher-order modes can be obtained from the fundamental mode (12) using the Eq. (23).

$$u_{lp}^{LG}(z) = \sqrt{\frac{1}{m!n!}} [A_{-}^{\dagger}(z)]^{n} [A_{+}^{\dagger}(z)]^{m} u_{00}^{LG}(z)$$
(21)

where m = (l+p)/2 and n = (l-p)/2.

V. CONCLUSION

The analogy between the wave equation in paraxial approximation and the equation for a free particle Schrondiger allowed us to use the operator formalism of quantum mechanics to calculate the raising and lowering operators that describe the modes of Laguerre-Gaussian beams. As in the case of the quantum harmonic oscillator can meet with the operator of a survey oscillation mode from any fundamental way.

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