DIFFEOMORPHISM CLASSES OF CALABI-YAU VARIETIES

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ABSTRACT. In this article we investigate diffeomorphism classes of Calabi-Yau three-folds. In particular, we focus on those embedded in toric Fano manifolds. Along the way, we give various examples and conclude with a curious remark regarding mirror symmetry.

1. INTRODUCTION

A longstanding problem in geometry is the classification of geometric objects up to isomorphism. For example, from a topological point of view, we are interested in classifying objects up to homeomorphism. In Differential Geometry, the classification is up to diffeomorphism and in complex geometry, we look for a classification up to (analytic) isomorphism.

This is the starting point for the construction of the moduli space. The main goal is the classification of families of these geometric objects (up to equivalence) so that the classifying space, the so called *moduli space*, is a reasonable geometric space. Roughly speaking, the moduli space is a parameter space that classifies these objects, in the sense that its points parametrise the geometric objects that we are considering. One of the easiest examples is the collection of all the lines (through the origin) in three dimensional space. The space that classifies this collection is well known and has a nice geometric structure: it is the projective plane (a smooth and compact manifold). As another example, we can consider the space that classifies, up to isomorphism, smooth rational curves of genus zero with 3 distinct marked points. It turns out that this space is just a point, since any triple of distinct points on a projective line can be sent in a distinct triple by an automorphism.

Unfortunately, the general situation turns out to be very complicated. In complex dimension one, we would like to classify all smooth curves, i.e., Riemann surfaces up to isomorphism. The classification can be carried out by using the genus g of the curve. For $g \geq 1$ the moduli space $\mathcal{M}_g$ is well understood and has a rich geometric structure. We also observe that in this case all the objects are projective, i.e., all smooth curves embedded in projective space.

In dimension two, the classification of compact complex surfaces is more involved than that in dimension one. It turns out that it is convenient to classify birational classes of surfaces. Then, for every birational class there is a unique minimal model, that has to be classified.

In dimension higher or equal than three, the classification is quite far from being complete. The idea is to generalize the technique used for dimension two and this has developed the so called Minimal Model Program. This classification is not concluded yet and already in dimension three there are many technical issues that have to be understood such as the uniqueness of the minimal model.

Motivated by a better understanding of this classification, we are interested in the role played by Calabi-Yau manifolds. First of all, the classification of 3-dimensional algebraic varieties has still some gaps due to the lack of understanding of Calabi-Yau threefolds. Moreover, the moduli space of Calabi-Yau varieties has received attention by theoretical

physicists, since these geometric objects are important for mirror symmetry, cohomological quantum field theory and string theory, branches of physics dealing with general relativity and quantum mechanics.

In dimension one, Calabi-Yau curves are genus 1 curves and they are all homeomorphic each other. These are not isomorphic and they are classified by the so called j-invariant. In dimension two, Calabi-Yau surfaces are called K3 surfaces and they are all homeomorphic each other. Also in this case they are not all isomorphic; moreover, there exist K3 surfaces that are not projective. We also remark that K3 surfaces are extensively studied and they play a central role in algebraic and complex geometry.

In higher dimension, the classification of Calabi-Yau manifolds is quite hard and many questions are still open also in the topological setting. For example, the topology of Calabi-Yau manifolds is not uniquely determined for dimension greater or equal than three. It is also not known if there are only finitely many topological types of Calabi-Yau threefolds.

From the differential point of view, C.T.C. Wall described the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology [18]. In particular, the Hodge data, the triple intersection in cohomology and the second Chern class completely determine the diffeomorphism type of a simply connected Calabi-Yau threefold. Recently, A. Kanazawa, P. M. H. Wilson [12], refined the theorem by Wall, providing some inequalities on the invariants, which hold in the case of Calabi-Yau threefolds.

The setting is very complicated. An interesting task is to find new examples of Calabi-Yau threefolds. The best known example of Calabi-Yau threefolds is the smooth quintic hypersurface in projective space $\mathbb{P}^4$, which is defined by a homogeneous polynomial of degree five. Actually, this example can be generalized to construct the majority of all known Calabi-Yau varieties. Indeed, projective space $\mathbb{P}^4$ is a particular example of smooth toric Fano varieties and these manifolds play a fundamental role in the construction of examples of Calabi-Yau. Once we have a toric Fano manifold, there always exists a submanifold of codimension one that is a Calabi-Yau manifold (see Section 2.2).

The toric set-up is an algebraic property that can be analyzed in terms of combinatorial algebra. Indeed, the classification of smooth toric Fano varieties of dimension n up to isomorphism turns out to be equivalent to the classification of combinatorial objects, namely some special polytopes in $\mathbb{R}^n$ up to linear unimodular transformation.

In [1], V. Batyrev describes a combinatorial criterion in terms of reflexive polyhedra for a hypersurface in a toric variety to be Calabi-Yau. He also investigates mirror symmetry in terms of an exchange of a dual pair of reflexive lattice polytopes. Moreover, he also provides the complete biregular classification of all 4-dimensional smooth toric Fano varieties: there are exactly 123 different types [2].

Using a computer program, M. Kreuzer and H. Sharke are able to describe all the reflexive polyhedra that exist in dimension four. They are about $500,000,000$ [13]. In particular, they find more than $30,000$ topological distinct Calabi-Yau threefolds with distinct pairs of the Hodge numbers (a, b), where $h^{1,1}(X) = a$ and $h^{1,2}(X) = b$ (see Section 2.1). Furthermore, in [3] the authors find 210 reflexive 4-polytopes defining 68 topologically different Calabi-Yau varieties of dimension 3 with the Hodge number $a = 1$.

Therefore, it is interesting to investigate the set-up of toric Fano manifolds and try to answer some questions that naturally arise. For example, if the Hodge numbers (a, b) of two Calabi-Yau manifolds X_1 and X_2 are different, then they are not homeomorphic. It is interesting to understand the converse. If X_1 and X_2 have the same Hodge numbers, we wonder if they are homeomorphic or even diffeomorphic or isomorphic.

First of all, we deal with Calabi-Yau manifolds X_1 and X_2 contained in the same toric Fano manifold. In this specific context, we are able to prove that if X_1 and X_2 are

deformation equivalent as abstract manifolds, then they are deformation equivalent as embedded manifolds.

Then, we review the Theorems by C.T.C. Wall (Theorem 4.1) and by A. Kanazawa, P. M. H. Wilson (Theorem 4.2), and we investigate some examples of simply connected Calabi-Yau manifolds with Hodge number $a = 1$.

From the point of view of moduli spaces, it is an interesting problem to understand the behaviour of the moduli space of Calabi-Yau manifolds. In dimension two, the moduli space of K3 surfaces is an irreducible 20 dimensional space and many properties are known. For Calabi-Yau threefolds, it is not known whether the moduli space is irreducible or not: M. Reid's conjecture predicts that this space should not behave too bad [15].

Then, instead of studying the moduli space of all Calabi-Yau threefolds, M.-C. Chang and H.I. Kim propose to investigate the space $M_{m,c}$ [6] that classifies Calabi-Yau threefolds with fixed invariants m and c, which are related to the invariant used by Wall (see Section 5). In this context, we describe an example of Calabi-Yau threefold and its mirror lying in the same $M_{5,50}$. In particular, we provide an example of two Calabi-Yau threefolds lying in $M_{5,50}$ that are neither diffeomorphic nor deformation equivalent.

With the aim of providing an introduction to the subject, Section 2 is devoted to recalling some preliminaries on Calabi-Yau manifolds and toric Fano manifolds. In Section 3, we compare the embedded deformations of a Calabi-Yau manifold in a toric Fano manifold with the abstract ones. Section 4 recalls Wall's Theorem on the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology. We also describe some examples. In Section 5, we make some remarks on the relation between Calabi-Yau and mirror symmetry.

Notation. Throughout the paper, we will work over the field of complex numbers.

2. Preliminaries

In this section, we recall the main definitions of Calabi-Yau and toric Fano manifolds.

2.1. **Calabi-Yau Manifolds.** Let X be a complex manifold and denote by T_X its holomorphic tangent bundle. X is a Calabi-Yau manifold of dimension n if it is a projective manifold with trivial canonical bundle and without holomophic p-forms, i.e., $K_X := \Omega_X^n \cong \mathcal{O}_X$ and $H^0(X, \Omega_X^p) = 0$ for p in between 0 and n.

If X has dimension 3, we have $\Omega_X^3 \cong \mathcal{O}_X$. Since Ω_X^1 is isomorphic to the dual of T_X, this implies that $\Omega_X^2 \cong T_X$, and, by duality, that $H^0(X, \Omega_X^2) = H^2(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = H^0(X, \Omega_X^1) = 0$.

Denoting by $h^{i,j}(X) = \dim_{\mathbb{C}} H^j(X, \Omega_X^i)$ and fixing $h^{1,1}(X) = a$ and $h^{1,2}(X) = b$, we can collect the information above in the so-called *Hodge diamond*:

$$
\begin{array}{ccccccc}
& & & 1 & & & \\
& & 0 & & 0 & & \\
& 0 & & a & & 0 & \\
1 & & b & & b & & 1 \\
& 0 & & a & & 0 & \\
& & 0 & & 0 & & \\
& & & 1. & & & \\
\end{array}
$$

This shows that the topological type of Calabi-Yau manifold is not uniquely determined for dimension 3. If X_1 and X_2 are two Calabi-Yau threefolds with different a and b then they cannot be homeomorphic.

Next, consider the case where the Hodge numbers (a, b) are the same. Let X_1 and X_2 be two Calabi-Yau threefolds, with the same Hodge numbers a and b, i.e., $h^{1,1}(X_i) = a$ and $h^{1,2}(X_i) = b$, for $i = 1, 2$. Then, we wonder if X_1 and X_2 are diffeomorphic. Indeed, if the Calabi-Yau threefolds are diffeomorphic, then they have the same numbers Hodge numbers (a, b) but nothing is known about the other implication.

To understand the problem, we focus our attention on the class of Calabi-Yau manifolds embedded in toric Fano manifolds.

2.2. **Toric Fano manifolds.** Let F be a smooth toric Fano variety of dimension n. A Fano manifold is a projective manifold F, whose anticanonical line bundle $-K_F := \wedge^n T_F$ is ample.

If F is also a toric variety, then $-K_F$ is very ample (so base point free) [14, Lemma 2.20]. Therefore, by Bertini's Theorem [9, Corollary III.10.9], the generic section of $\mathcal{O}_F(-K_F)$ gives a smooth connected hypersurface $X \subseteq F$, such that $X \in |-K_F|$. Thus, X is a smooth Calabi-Yau variety [7, Proposition 11.2.10]. This shows that once we have a toric Fano manifold, then there always exists a submanifold of codimension 1 that is a Calabi-Yau manifold.

In particular, if F has dimension 4, X is a smooth complex Calabi-Yau threefold. This is actually one of the most fruitful way to construct examples of Calabi-Yau threefolds [3].

Example 2.1. The projective space $\mathbb{P}^4$ is a smooth toric Fano manifold of dimension 4. The general quintic hypersurface is a smooth Calabi-Yau threefold. This is the most extensively studied example of Calabi-Yau threefold. In this case, it can be proved that $a = 1$ and $b = 101$.

In Proposition 3.4, we investigate the infinitesimal deformations of smooth complex Calabi-Yau threefolds, which are obtained as anticanonical hypersurfaces in a Fano manifold.

3. Abstract vs Embedded Deformations

In this section, we review the notion of deformations of a submanifold X in a manifold F. In particular, we are interested in the infinitesimal deformations of X as an abstract manifold and in the embedded deformations of X in F. For more details, we refer the reader to [17, Sections 2.4 and 3.2].

We denote by Def_X the functor of infinitesimal deformations of X as an abstract variety, i.e.,

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set},$$

where $\mathrm{Def}_X(A)$ is the set of isomorphism classes of commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ \downarrow & & \downarrow{\scriptstyle p_A} \\ \mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(A), \end{array}$$

where i is a closed embedding and p_A is a flat morphism.

Remark 3.1. In our setting, X is smooth, then all the fibers of p_A are diffeomorphic by Ereshman's Theorem. Thus, an infinitesimal deformation of X is nothing else than a deformation of the complex structure over the same differentiable structure of X. In

particular, if X_1 and X_2 are deformation equivalent then they are diffeomorphic, i.e., $X_1 \sim_{def} X_2 \implies X_1 \cong_{dif} X_2$. The converse is not true: $X_1 \cong_{dif} X_2 \not\Longrightarrow X_1 \sim_{def} X_2$. There are examples of diffeomorphic Calabi-Yau threefolds that are not deformation equivalent [8, 16].

Remark 3.2. If X is a Calabi-Yau manifold, then Bogomolov-Tian-Todorov Theorem implies that the functor Def_X is smooth. This property implies that the moduli space is smooth at the point corresponding to X.

We denote by H_X^F the functor of infinitesimal embedded deformations of X in F, i.e.,

$$H_X^F : \mathbf{Art} \to \mathbf{Set},$$

where $H_X^F(A)$ is the set of commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;i\;\;} & X_A \subset F \times \mathrm{Spec}(A) \\
\downarrow & & \downarrow{\scriptstyle p_A} \\
\mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(A),
\end{array}
$$

where i is a closed embedding, $X_A \subset F \times \mathrm{Spec}(A)$ and p_A is a flat morphism induced by the projection $F \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$.

In particular, the following forgetful morphism of functors is well defined:

$$\phi : H_X^F \to \mathrm{Def}_X;$$

moreover, the image of an infinitesimal deformation of X in F is the isomorphism class of the deformation of X, viewed as an abstract deformation [17, Section 3.2.3].

Example 3.3. Let $n \geq 4$ and X be the general anticanonical hypersurface in $\mathbb{P}^n$. Note that $\mathbb{P}^n$ is a smooth toric Fano variety and X a smooth Calabi-Yau manifold.

For every Calabi-Yau manifold X in a projective space $\mathbb{P}^n$, the embedded deformations of X in $\mathbb{P}^n$ are unobstructed [10, Corollary A.2].

Therefore, the functor Def_X and the morphism ϕ are both smooth and this implies that $H_X^{\mathbb{P}^n}$ is also smooth [17, Corollary 2.3.7].

In particular, this implies that all the infinitesimal deformations of the general anti-canonical hypersurface X as an abstract variety are obtained as embedded deformations of X inside $\mathbb{P}^n$. The following proposition shows that a similar property is true for any smooth toric Fano variety and not only for $\mathbb{P}^n$.

Proposition 3.4. *Let F be a smooth toric Fano variety with $\dim F > 3$ and denote by X a smooth connected hypersurface in F such that $X \in |-K_F|$. Then, the forgetful morphism*

$$\phi : H_X^F \to \mathrm{Def}_X$$

is smooth.

Proof. The varieties F and X are both smooth, so we have the exact sequence

$$0 \to T_X \to T_{F|X} \to N_{X/F} \to 0$$

that induces the following exact sequence in cohomology, namely:

$$\cdots \to H^0(X, N_{X/F}) \xrightarrow{\alpha} H^1(X, T_X) \to H^1(X, T_{F|X}) \to H^1(X, N_{X/F}) \xrightarrow{\beta} H^2(X, T_X) \to \cdots .$$

The morphism α is the map induced by ϕ on the tangent spaces and β is an obstruction map for ϕ [17, Proposition 3.2.9]. Applying the standard smoothness criterion [17, Proposition 2.3.6], it is enough to prove that α is surjective and β is injective; in particular, it suffices to prove that $H^1(X, T_{F|X}) = 0$.

For this purpose, consider the exact sequence

$$0 \to \mathcal{O}_F(-X) \to \mathcal{O}_F \to \mathcal{O}_X \to 0$$

and tensor it with T_F, thus yielding

$$0 \to T_F \otimes \mathcal{O}_F(-X) \to T_F \to T_{F|X} \to 0$$

and the induced exact sequence in cohomology, namely:

$$\cdots \to H^1(F, T_F) \to H^1(F, T_{F|X}) \to H^2(F, T_F \otimes \mathcal{O}_F(-X)) \to H^2(F, T_F) \to \cdots .$$

If F is a smooth toric Fano variety, then $H^i(F, T_F) = 0$, for all $i > 0$ [5, Proposition 4.2]. Since $\mathcal{O}_F(-X) \cong \mathcal{O}_F(K_F)$, we are reduced to prove that $H^2(F, T_F \otimes \mathcal{O}_F(K_F)) = 0$. This follows from Lemma 3.5.

$\square$

Lemma 3.5. *Let F be a smooth toric Fano variety with* $\dim F > 3$. *Then the following holds:*

$$H^2(F, T_F \otimes \mathcal{O}_F(K_F)) = 0.$$

Proof. As for projective space, there exists a generalized Euler exact sequence for the tangent bundle of toric varieties [7, Theorem 8.1.6]:

$$0 \to Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F \to \oplus_i \mathcal{O}_F(D_i) \to T_F \to 0,$$

where $K_F = -\sum_i D_i$ [7, Theorem 8.2.3]. We note also that $Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F \cong \mathcal{O}_F^{rank}$, where *rank* denotes the *rank* of $Pic(F)$. By tensoring with $\mathcal{O}_F(K_F)$, we obtain

$$0 \to Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F(K_F) \to \oplus_i \mathcal{O}_F(D_i + K_F) \to T_F \otimes \mathcal{O}_F(K_F) \to 0$$

and so

$$\cdots \to H^2(F, Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F(K_F)) \to$$

$$\cdots \to \oplus_i H^2(F, \mathcal{O}_F(D_i + K_F)) \to H^2(F, T_F \otimes \mathcal{O}_F(K_F)) \to H^3(F, Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F(K_F)) \to \cdots$$

Since $-K_F$ is ample, by Kodaira vanishing Theorem, $H^j(F, \mathcal{O}_F) = 0, j > 0$. Moreover, by Serre duality $H^j(F, \mathcal{O}_F(K_F)) = 0, j \neq \dim F$. Therefore, if $\dim F > 3$, then $H^2(F, Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F(K_F)) = H^3(F, Pic(F) \otimes_{\mathbb{Z}} \mathcal{O}_F(K_F)) = 0$ and

$$\oplus_i H^2(F, \mathcal{O}_F(D_i + K_F)) \cong H^2(F, T_F \otimes \mathcal{O}_F(K_F)).$$

By Serre duality, $H^2(F, \mathcal{O}_F(D_i + K_F)) \cong H^{\dim F - 2}(F, \mathcal{O}_F(-D_i))^\nu$, for all i.

Using the following exact sequence

$$0 \to \mathcal{O}_F(-D_i) \to \mathcal{O}_F \to \mathcal{O}_{D_i} \to 0$$

and the fact that $H^j(F, \mathcal{O}_F) = 0$, for $j > 0$, we conclude that $H^{\dim F - 2}(F, \mathcal{O}_F(-D_i)) \cong H^{\dim F - 3}(D_i, \mathcal{O}_{D_i})$, for all i.

Therefore, we are left to prove that $\oplus_i H^{\dim F - 3}(D_i, \mathcal{O}_{D_i}) = 0$.

Consider the following exact sequence on a toric variety [7, Theorem 8.1.4]

$$0 \to \Omega_F^1 \to M \otimes_Z \mathcal{O}_F \to \oplus_i \mathcal{O}_{D_i} \to 0,$$

where M is a lattice related to the toric structure; here we only need that $M \otimes_Z \mathcal{O}_F \cong \mathcal{O}_F^r$, for some $r \in \mathbb{N}$.

This induces

$$\cdots \to H^{\dim F - 3}(F, M \otimes_Z \mathcal{O}_F) \to \oplus_i H^{\dim F - 3}(D_i, \mathcal{O}_{D_i}) \to H^{\dim F - 2}(F, \Omega_F^1) \to \cdots .$$

Since $H^j(F, \mathcal{O}_F) = 0$, for $j > 0$ and $\dim F > 3$, we have $H^{\dim F - 3}(F, M \otimes_Z \mathcal{O}_F) = H^{\dim F - 2}(F, \Omega_F^1) = 0$ [7, Theorem 9.3.2]. This implies $\oplus_i H^{\dim F - 3}(D_i, \mathcal{O}_{D_i}) = 0$.

$\square$

Remark 3.6. Proposition 3.4 shows that all the infinitesimal deformations of X as an abstract variety are obtained as infinitesimal deformations of X inside the smooth toric Fano manifold F. Moreover, since every deformation of a Calabi-Yau manifold is smooth (Bogomolov-Tian-Todorov Theorem), we conclude that the deformations of X inside F are also smooth.

4. Diffeomorphic Three-dimensional Calabi-Yau varieties

In this section, we focus on the diffeomorphism class of three dimensional Calabi-Yau manifolds.

In 1966, C.T.C. Wall described the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology.

Theorem 4.1. [18] *Diffeomorphism classes of simply-connected, spin, oriented, closed 6-manifolds X with torsion-free cohomology correspond bijectively to isomorphism classes of systems of invariants consisting of*

(1) *free Abelian groups $H^2(X, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$,*
(2) *a symmetric trilinear form $\mu \colon H^2(X, \mathbb{Z})^{\otimes 3} \to H^6(X, \mathbb{Z}) \cong \mathbb{Z}$ defined by $\mu(x, y, z) := x \cup y \cup z$,*
(3) *a linear map $p_1 \colon H^2(X, \mathbb{Z}) \to H^6(X, \mathbb{Z}) \cong \mathbb{Z}$, defined by $p_1(x) := p_1(X) \cup x$, where $p_1(X) \in H^4(X, \mathbb{Z})$ is the first Pontrjagin class of X, satisfying,*

for any $x, y \in H^2(X, \mathbb{Z})$, the following conditions

$$\mu(x, x, y) + \mu(x, y, y) \equiv 0 \pmod{2} \qquad 4\mu(x, x, x) - p_1(x) \equiv 0 \pmod{24}.$$

The symbol $\cup$ denotes the cup product of differential forms and the isomorphism $H^6(X, \mathbb{Z}) \cong \mathbb{Z}$ above is given by pairing a cohomology class with the fundamental class of X with natural orientation.

Let X be a Calabi-Yau threefold. In [12], the authors investigate the interplay between the trilinear form μ and the Chern classes $c_2(X)$ and $c_3(X)$ of X, providing the following numerical relation.

Theorem 4.2. [12] *Let (X, H) be a very ample polarized Calabi-Yau threefold, i.e., $x = H$ is a very ample divisor on X. Then the following inequalities holds:*

$$(4.1) \qquad -36\mu(x, x, x) - 80 \leq \frac{c_3(X)}{2} = h^{1,1}(X) - h^{2,2} \leq 6\mu(x, x, x) + 40.$$

Note, that if X is a Calabi-Yau threefold, then $p_1(X) = -2c_2(X) \in H^4(X, \mathbb{Z})$ and

$$\int_X c_3(X) = \chi(X) = \sum_{i=0}^{6} \dim H^i(X, \mathbb{R}) = 2h^{1,1}(X) - 2h^{1,2}(X).$$

Remark 4.3. By Wall's Theorem, if X is simply-connected, spin, oriented, closed 6-manifolds with torsion-free cohomology, then the diffeomorphism class is determined by the free Abelian groups $H^2(X, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$, and the form μ and p_1. For any data we have a diffeormphism class. If X is Calabi Yau, then μ and p_1 have to satisfy the numerical conditions of Equation (4.1). Note that, having μ and p_1 on X that satisfy all the numerical conditions, it does not imply that X is a Calabi Yau.

In particular, let X_1 and X_2 be two simply connected Calabi-Yau threefolds with torsion-free cohomology and the same Hodge numbers $h^{1,1}(X) = a$ and $h^{1,2}(X) = b$. To be diffeomorphic, they should have the same μ and p_1, that satisfy the numerical conditions.

Corollary 4.4. *Let X be a Calabi-Yau threefold, with torsion-free cohomology and $h^{1,1}(X) = 1$ and $h^{1,2}(X) = h^{2,1}(X) = b$, for some $b \in \mathbb{N}$; hence, we have $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ and $H^3(X, \mathbb{Z}) \cong \mathbb{Z}^{2+b}$. Fix a generator $H \in H^2(X, \mathbb{Z})$ and set $\mu(H, H, H) = m \in \mathbb{Z}$. Then the following holds:*

$$m \geq \frac{b - 81}{36}.$$

Proof. Set $p_1(X) = -2c_2(X) \in H^4(X, \mathbb{Z})$; so there exists $c \in \mathbb{Z}$ such that $c_2(X) = cH^*$. Therefore, the linear form p_1 reduces to

$$p_1 : H^2(X, \mathbb{Z}) \to H^6(X, \mathbb{Z}) \cong \mathbb{Z} \qquad p_1(xH) := -2c_2(X) \cup xH = -2cxH^* \cup H = -2cx.$$

The numerical constraints of Theorem 4.1 reduce to

$$mx^2 y + mxy^2 \equiv 0 \pmod 2 \qquad 2mx^3 + cx \equiv 0 \pmod{12}$$

for any $x, y \in \mathbb{Z}$. The former condition is always verified while the latter congruence is equivalent to $2m + c \equiv 0 \pmod{12}$.

As for the numerical restriction of Theorem 4.2, Equation (4.1) reduces to

$$-36\mu(x, x, x) - 80 \leq 1 - b \leq 6\mu(x, x, x) + 40.$$

and so

$$-36m - 80 \leq 1 - b \leq 6m + 40.$$

In particular,

$$b \leq 81 + 36m \qquad -39 - 6m \leq b;$$
$$m \geq \frac{b - 81}{36} \qquad m \geq \frac{-39 - b}{6}.$$

Since b is positive, they reduce to

$$m \geq \frac{b - 81}{36}.$$

$\square$

Example 4.5. If $m = 5$, then $b \leq 261$. In [11], Appendix 1, there are three examples that satisfy this condition, namely $b = 51, 101, 156$. For $b = 101$ we obtain the general quintic threefold in $\mathbb{P}^4$. Projective models for the remaining two are still mysterious, as indicated by the question mark in the table in [11].

5. Some remarks on the moduli space of Calabi-Yau manifolds and mirror symmetry

Let X be a Calabi-Yau variety. Denote by H a primitive ample divisor. As in [6], let $M_{m,c}$ be the space of polarized varieties (X, H) such that $H^3 = m$ and $c_2(X)H = c$ for integers m and c. Little is known on the geometric structure of $M_{m,c}$. Some information can be found in [6].

Here we make the following remarks. Let X be a general quintic in $\mathbb{P}^4$. A hyperplane section on X is a (very) ample divisor H such that $H^3 = 5$. On the Calabi-Yau manifold X the Grothendieck-Riemann-Roch Theorem reads as follows:

$$\chi(H) = \frac{H^3}{6} + \frac{1}{12}c_2(X)H.$$

Since H is a divisor on X, we have

$$\chi(\mathcal{O}_X) + \chi(\mathcal{O}_H(H)) = \chi(H).$$

The first term on the left-hand side is zero because X is a Calabi-Yau; the second term can be computed via Noether's formula, namely:

$$\chi(\mathcal{O}_H(H)) = \frac{K_H^2 + c_2(H)}{12}.$$

A linear section of a quintic is a quintic surface in three-dimensional projective space. As well known, the second Betti number is 53, so the Euler characteristic is 55. Therefore, we get

$$\chi(\mathcal{O}_H(H)) = 5.$$

Hence, we get

$$5 = \frac{H^3}{6} + \frac{1}{12}c_2(X)H,$$

which yields $c_2(X)H = 50$.

This means that the pair (X, H) belongs to the space $M_{5,50}$, where X is a quintic in $\mathbb{P}^4$ and H is a hyperplane section.

The Hodge numbers of X are given by $(a, b) = (1, 101)$. The Hodge numbers of a mirror manifold X' are given by $(101, 1)$.

Proposition 5.1. *There exists a primitive ample divisor D on X' such that (X', D) belongs to $M_{5,50}$.*

Proof. In fact, as recalled in [4], a mirror of X can be found as a crepant resolution of a singular quintic in $\mathbb{P}^4$. Denote by D the pull-back of the hyperplane divisor on projective space. Clearly, $D^3 = 5$. Now, we need to compute

$$\chi(D) = \frac{D^3}{6} + \frac{1}{12}c_2(X')D.$$

Like before, we have

$$\chi(\mathcal{O}_D(D)) = \frac{K_D^2 + c_2(D)}{12}.$$

Notice that

$$K_D^2 = D^2 D = 5.$$

As mentioned before, the divisor D is the pull-back of the hyperplane divisor. We can take a member of it that does not intersect the blown up locus. Thus, $c_2(D)$ is again the Euler characteristic of a quintic surface in three-dimensional projective space. Therefore, the claim follows.

$\square$

Example 5.2. For the general quintic threefold X in $\mathbb{P}^4$, we have $a = 1$, $b = 101$, $m = 5$ and $c = 50$, that satisfy the previous conditions. Therefore, X lies in the space $M_{5,50} = \{(X, H) \mid H^3 = 5, c_2(X) \cdot H = 50\}$ introduced in [6]. The mirror $\tilde{X}$ of X is a smooth Calabi-Yau threefolds with $a = 101$, $b = 1$, $m = 5$ and $c = 50$. So it lies in the same space $M_{5,50}$ but the Hodge numbers are exchanged: this implies that X and $\tilde{X}$ are neither diffeomorphic nor deformation equivalent!

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