# LOSS OF CONTINUITY OF THE SOLUTION MAP FOR THE EULER EQUATIONS IN $\alpha$ -MODULATION AND HÖLDER SPACES

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ABSTRACT. We study the incompressible Euler equations in the  $\alpha$ -modulation  $M_{p,q}^{s,\alpha}$  and Hölder  $C^{1+\sigma}$  spaces on the plane. We show that for these spaces the associated data-to-solution map is not continuous on bounded sets.

#### 1. Introduction

In this paper we study the Cauchy problem for the non-periodic Euler equations of incompressible hydrodynamics

$$u_t + u \cdot \nabla u + \nabla p = 0, t \ge 0, \ x \in \mathbb{R}^n$$
 (E) 
$$\operatorname{div} u = 0$$
 
$$u(0) = u_0$$

with initial data in the  $\alpha$ -modulation spaces. In particular, our results apply to the Besov spaces including the classical Hölder-Zygmund spaces. According to the standard notion of well-posedness due to Hadamard a Cauchy problem is said to be locally (in time) well-posed in a Banach space X if given any initial data  $u_0$  in X there is a time T>0 and a unique solution u in a Banach space  $Y\subset C([0,T),X)$  which depends continuously on the initial data. Otherwise the Cauchy problem is said to be locally ill-posed.

Ill-posedness results establishing loss of continuity of the solution map  $u_0 \to u$  for the Euler equations in the  $C^1$  space and the borderline Besov space  $B^1_{\infty,1}$  have been proved recently in [18]. Here we refine the techniques of that paper to obtain ill-posedness results of this type in  $\alpha$ -modulation spaces  $M^{s,\alpha}_{p,q}$  for 1 < s < 2 and in  $C^{1+\sigma}$  for  $0 < \sigma < 1$ . More precisely, following the approach of Bourgain and Li [3] we construct a Lagrangian flow with a large gradient and then choose a suitable high-frequency perturbation of the initial vorticity to show that the assumption of continuity of the solution map  $u_0 \to u$  in the above spaces necessarily leads to a contradiction with the results of Kato and Ponce [13, 14]. We will work with the vorticity equations which in two dimensions assume the form

(1.1) 
$$\omega_t + u \cdot \nabla \omega = 0, \qquad t \ge 0, \ x \in \mathbb{R}^2$$
$$u = K * \omega = \nabla^{\perp} \Delta^{-1} \omega$$
$$\omega(0) = \omega_0$$

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where

$$K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2} \quad \text{and} \quad \nabla^{\perp} = \left( -\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right)$$

denote the Biot-Savart kernel and the symplectic gradient, respectively.

The first rigorous results on the Cauchy problem for the incompressible Euler equations go back to Gyunter [10], Lichtenstein [15] and Wolibner [23]. A survey of those and numerous further results can be found for example in Majda and Bertozzi [16], Constantin [5] or Bahouri, Chemin and Danchin [1]. For the most recent progress on local ill-posedness in borderline spaces such as  $C^1$ ,  $W^{n/p+1,p}$ ,  $B_{p,q}^{n/p+1}$  as well as  $C^k$ ,  $C^{k-1,1}$  for integer  $k \geq 1$  we refer to the papers of Bourgain and Li [3, 4], Elgindi and Masmoudi [8] and the authors [18]. For earlier results in  $C^{\sigma}$  with  $0 < \sigma < 1$ ,  $B_{p,\infty}^s$  for s > 0, p > 2 and s > n(2/p-1),  $1 \leq p \leq 2$  or the logarithmic Lipschitz spaces  $\log \operatorname{Lip}^{\alpha}$  for  $0 < \alpha < 1$  we refer to Bardos and Titi [2], Cheskidov and Shvydkoy [6] and the authors [17].

Our main goal is to prove the following result.

**Main Theorem.** Let  $0 < \sigma < 1$ ,  $0 < \alpha \le 1$ ,  $2 \le p \le \infty$ ,  $1 \le q \le \infty$  and suppose that  $M_{p,q}^{1+\sigma,\alpha}$  is continuously embedded in  $C^1$ . Then the solution map of the incompressible Euler equations (E) is not continuous on bounded subsets of  $M_{p,q}^{1+\sigma,\alpha}$ .

Thus, the Euler equations are in general locally ill-posed in  $\alpha$ -modulation spaces in the sense of Hadamard given above.<sup>1</sup> For the definition of the  $\alpha$ -modulation spaces see Section 2 below.

Remark 1. Observe that  $M_{p,q}^{s,1}$  coincides with the usual Besov space  $B_{p,q}^s$ . Therefore, somewhat surprisingly, Theorem 1 also yields ill-posedness (in the sense that the data-to-solution map loses its continuity properties) even in the classical Hölder spaces  $B_{\infty,\infty}^{1+\sigma} = C^{1+\sigma}$  for  $0 < \sigma < 1$ .

Continuous dependence results for the Euler equations (in the strong topology) have been obtained for initial data in Sobolev spaces  $H^s$  and more generally  $W^{s,p}$  with s>n/p+1 for example in Ebin and Marsden [8], Kato and Lai [12] and Kato and Ponce [14]. However, this is a rather difficult part of the local well-posedness theory which has not yet been satisfactorily resolved.

Remark 2. A different mechanism involving a gradual loss of regularity of the solution map is described by Morgulis, Shnirelman and Yudovich [19].

Remark 3. In this context it is also worth pointing out that neither for the critical Besov space  $B^1_{\infty,1}(\mathbb{R}^n)$  nor for the space  $B^{1+p/n}_{p,1}(\mathbb{R}^n)$  are the Euler equations strongly ill-posed in the sense of Bourgain and Li [3]. This can be seen by examining the arguments given in Pak and Park [20] and Vishik [22].

Our general strategy will be similar to that employed in [18] which we will use as the main reference. The remainder of the paper is organized as follows. In Section 2 we describe the general set up and prove several technical lemmas. The whole of Section 3 is then devoted to the proof of Theorem 1. Although the constructions in Sections 2 and 3 will be carried out in 2D they can be readily adapted to the 3D case. Rather than doing that in Section 4 we give a direct proof of ill-posedness in  $C^{1+s}$  by using a 3D shear flow argument.

<sup>&</sup>lt;sup>1</sup>For example, we have  $M_{\infty,q}^{1+\sigma,\alpha}\subset C^1$  whenever  $\sigma>2(1-\alpha)(1-1/q)$ .

#### 2. Basic Setup: Vorticity and Lagrangian Flow

We first recall the definition of  $\alpha$ -modulation spaces. For a more detailed account the reader is referred for example to [9]. A countable set  $\mathcal{Q}$  of subsets  $Q \subset \mathbb{R}^n$  is called an admissible covering if  $\mathbb{R}^n = \bigcup_{Q \in \mathcal{Q}} Q$  and if there is  $n_0 < \infty$  such that  $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$  for all  $Q \in \mathcal{Q}$ . Let

$$r_Q = \sup\{r \in \mathbb{R} : B(c_r, r) \subset Q, c_r \in \mathbb{R}^n\}$$
  
 $R_Q = \inf\{R \in \mathbb{R} : Q \subset B(c_R, R), c_R \in \mathbb{R}^n\}.$ 

Given  $0 \le \alpha \le 1$ , an admissible covering is an  $\alpha$ -covering of  $\mathbb{R}^n$  if  $|Q| \sim (1+|x|^2)^{\alpha n/2}$  (uniformly) for all  $Q \in \mathcal{Q}$  and all  $x \in Q$  and where  $\sup_{Q \in \mathcal{Q}} R_Q/r_Q \le K$  for some  $K < \infty$ . Let  $\mathcal{Q}$  be an  $\alpha$ -covering of  $\mathbb{R}^n$ . A bounded admissible partition of unity of order p (abbreviated p-BAPU) corresponding to  $\mathcal{Q}$  is a family of smooth functions  $\{\psi_Q\}_{Q \in \mathcal{Q}}$  satisfying

$$\psi_Q : \mathbb{R}^n \to [0, 1], \quad \text{supp } \psi_Q \subset Q,$$

$$\sum_{Q \in \mathcal{Q}} \psi_Q(\xi) \equiv 1, \quad \xi \in \mathbb{R}^n,$$

$$\sup_{Q \in \mathcal{Q}} |Q|^{1/p-1} ||\mathcal{F}^{-1}\psi_Q||_{L^p} < \infty$$

where  $\mathcal{F}$  denotes the Fourier transform.

For any  $1 \leq p, q \leq \infty$ ,  $s \in \mathbb{R}$  and  $0 \leq \alpha \leq 1$  the  $\alpha$ -modulation space  $M_{p,q}^{s,\alpha}(\mathbb{R}^n)$  is the space of all tempered distributions f for which the following norm

$$||f||_{M_{p,q}^{s,\alpha}} = \begin{cases} \left( \sum_{Q \in \mathcal{Q}} (1 + |\xi_Q|^2)^{qs/2} ||\mathcal{F}^{-1}\psi_Q \mathcal{F} f||_{L^p(\mathbb{R}^n)}^q \right)^{1/q} & \text{if } 1 \le q < \infty \\ \sup_{Q \in \mathcal{Q}} (1 + |\xi_Q|^2)^{s/2} ||\mathcal{F}^{-1}\psi_Q \mathcal{F} f||_{L^p} & \text{if } q = \infty \end{cases}$$

is finite, where  $\{\xi_Q \in Q : Q \in \mathcal{Q}\}$  is an arbitrary sequence. One shows that this definition is independent of an  $\alpha$ -covering  $\mathcal{Q}$  and of p-BAPU.

The following embedding results for  $\alpha$ -modulation spaces are known. Suppose that  $\alpha_1 < \alpha_2 < 1$  and  $1 \le p \le \infty$ . Then

$$M_{p,1}^{s,\alpha_1}\subset M_{p,1}^{s,\alpha_2}$$

and, in particular, for any  $\alpha < 1$  and  $1 \le p \le \infty$  we have

$$M_{p,1}^{s,\alpha} \subset B_{p,1}^s$$

The proofs of these results can be found e.g. in [11]; see Thm. 4.1 and Thm. 4.2. We next proceed to choose the initial vorticity  $\omega_0$  in (1.1). Given any radial

bump function  $0 \le \varphi \le 1$  with support in B(0, 1/4) define

(2.1) 
$$\varphi_0(x_1, x_2) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \varphi(x_1 - \varepsilon_1, x_2 - \varepsilon_2).$$

Fix a positive integer  $N_0 \in \mathbb{Z}_+$  and for any  $M \gg 1$  let

(2.2) 
$$\omega_0(x) = M^{-2} N^{-\frac{1}{r}} \sum_{N_0 \le k \le N_0 + N} \varphi_k(x),$$

where 1 , <math>N = 1, 2, 3... and where

$$\varphi_k(x) = 2^{(-1 + \frac{2}{r})k} \varphi_0(2^k x).$$

Clearly, the function  $\varphi_0$  is odd in both variables  $x_1, x_2$  and for any  $k \geq 1$  the supports of  $\varphi_k$  are disjoint and contained in a bounded set

(2.3) 
$$\operatorname{supp} \varphi_k \subset \bigcup_{\varepsilon_1, \varepsilon_2 = \pm 1} B((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}).$$

It is easy to check that  $\omega_0 \in W^{1,r}(\mathbb{R}^2) \cap C_c^{\infty}(\mathbb{R}^2)$ . In fact, we have

**Lemma 4.** For any  $2 < r < \infty$  and any positive integer N, we have

$$\|\omega_0\|_{W^{1,r}} \lesssim M^{-2}$$

*Proof.* A straightforward calculation is omitted.

Let  $u = \nabla^{\perp} \Delta^{-1} \omega \in W^{2,r} \cap C^{\infty}$  be the associated velocity field and consider its Lagrangian flow  $\eta(t)$ , i.e., the solution of the initial value problem

(2.5) 
$$\frac{d\eta}{dt}(t,x) = u(t,\eta(t,x))$$
$$\eta(0,x) = x.$$

It can be checked that  $\eta(t)$  is smooth and preserves the coordinate axes  $x_1$ ,  $x_2$  as well as the symmetries of the initial vorticity  $\omega_0$  in (2.2). In fact, the flow is hyperbolic near the origin (a stagnation point) and we have the following

**Proposition 5.** Given  $M \gg 1$  we have

$$\sup_{0 < t < M^{-3}} \|D\eta(t)\|_{\infty} > M$$

for any sufficiently large integer N>0 in (2.2) and any  $2 < r < \infty$  sufficiently close to 2.

Proof. See [18]; Prop. 6. 
$$\Box$$

In order to proceed we need the following simple comparison result for the derivatives of solutions of the Lagrangian flow equations.

**Lemma 6.** Let u and v be smooth divergence-free vector fields on  $\mathbb{R}^2$ . If  $\eta$  and  $\tilde{\eta}$  are the solutions of (2.5) corresponding to u and u + v respectively, then

$$\sup_{0\leq t\leq 1}\sum_{i=0,1}\left\|D^i\eta(t)-D^i\tilde{\eta}(t)\right\|_{\infty}\leq C\sup_{0\leq t\leq 1}\sum_{i=0,1}\left\|D^iv(t)\right\|_{\infty}$$

for some C > 0 depending only on the  $L^{\infty}$  norms of u and its derivatives.

*Proof.* Follows at once by applying Gronwall's inequality to the equation satisfied by the difference  $\eta(t) - \tilde{\eta}(t)$ .

### 3. The Proof of the main Theorem

As in the previous section let  $\omega(t) \in W^{1,r} \cap C^{\infty}$  be the solution of the vorticity equations (1.1) with the initial condition (2.2) and let  $\eta(t)$  be the Lagrangian flow of the velocity field  $u = \nabla^{\perp} \Delta^{-1} \omega$  as above. Our main goal in this section will be to prove

**Theorem 7.** Let r > 2. Assume that the incompressible Euler equations are well-posed in the  $\alpha$ -modulation space  $M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2)$  for any  $p \geq 2$ ,  $q \geq 1$ ,  $0 < \alpha \leq 1$  and  $0 < \sigma < 1$  in the sense of Hadamard. Moreover, assume that  $M_{p,q}^{1+\sigma,\alpha}$  is topologically embedded in  $C^1(\mathbb{R}^2)$ . Then there exist a T > 0 and a sequence  $\omega_{0,n}$  in  $C_c^{\infty}$  with the following properties.

- 1. There is a constant C > 0 such that  $\|\omega_{0,n}\|_{W^{1,r}} \leq C$  for sufficiently large positive integers n.
- 2. For any  $M \gg 1$  there is  $0 < t_0 \le T$  such that  $\|\omega_n(t_0)\|_{W^{1,r}} \ge M^{1/3}$  for sufficiently large n and for all r sufficiently close to 2.

Since Hadamard's notion entails continuity of the data-to-solution map we deduce from Theorem 7 that continuity cannot hold in  $M_{p,q}^{1+s,\alpha}(\mathbb{R}^2)$  or else we get a contradiction with the following result.

**Theorem** (Kato-Ponce [14]). Let  $1 < r < \infty$  and s > 1 + 2/r. For any  $\omega_0 \in W^{s-1,r}(\mathbb{R}^2)$  and any T > 0 there exists a constant  $K = K(T, \|\omega_0\|_{W^{s-1,r}}) > 0$  such that

$$\sup_{0 \le t \le T} \|\omega(t)\|_{W^{s-1,r}} \le K.$$

Our Main Theorem will be a direct consequence of Theorem 7.

Proof of Theorem 7. Given any large number  $M \gg 1$  pick  $T \leq M^{-3}$ . Observe that if  $\|\omega_0(t_0)\|_{W^{1,r}} > M^{1/3}$  for some  $0 < t_0 \leq M^{-3}$  then there is nothing to prove and therefore we may assume that

(3.1) 
$$\|\omega(t)\|_{W^{1,r}} \le M^{1/3}, \quad 0 \le t \le M^{-3}.$$

Next, by Proposition 5 we can find  $0 \le t_0 \le M^{-3}$  and a point  $x^* = (x_1^*, x_2^*)$  in  $\mathbb{R}^2$  for which the absolute value of one of the entries in the Jacobi matrix  $D\eta(t_0, x^*)$  is at least as large as M. Because the velocity field u is in  $W^{2,r}$  so is the associated Lagrangian flow<sup>2</sup> and hence by continuity in some sufficiently small  $\delta$ -neighbourhood of  $x^*$  we have e.g.,

(3.2) 
$$\left| \frac{\partial \eta_2}{\partial x_2}(t_0, x) \right| > M \quad \text{for all} \quad |x - x^*| < \delta.$$

We proceed to construct a sequence of high-frequency perturbations of the initial vorticity in  $W^{1,r}$ . To this end we choose a smooth bump function  $0 \le \hat{\chi} \le 1$  with compact support in the unit ball B(0,1) in the Fourier space and normalized by  $\int_{\mathbb{R}^2} \hat{\chi}(\xi) d\xi = 1$ . Using this function we set

(3.3) 
$$\hat{\rho}(\xi) = \hat{\chi}(\xi - \xi_0) + \hat{\chi}(\xi + \xi_0), \qquad \xi \in \mathbb{R}^2, \quad \xi_0 = (2, 0)$$

so that supp  $\hat{\rho} \subset B(-\xi_0, 1) \cup B(\xi_0, 1)$  with

(3.4) 
$$\rho(0) = \int_{\mathbb{R}^2} \hat{\rho}(\xi) \, d\xi = 2$$

and observe that for any a > 4 we have

(3.5) 
$$\operatorname{supp} \hat{\rho}(\cdot \pm a, \cdot) \cap B(0, 1) = \emptyset.$$

Define

$$(3.6) \qquad \beta_{k,\lambda}^{\tilde{\alpha},r}(x) = \frac{\lambda^{-1+\frac{2}{r}}}{k^{1-\tilde{\alpha}}} \sum_{\varepsilon_1,\varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x_{\epsilon}^*)) \sin kx_1, \qquad k \in \mathbb{Z}^+, \ \lambda > 0$$

where  $x_{\epsilon}^* = (\varepsilon_1 x_1^*, \varepsilon_2 x_2^*)$  and  $\lambda > 0$  and  $0 < \tilde{\alpha} < 1$  will be further specified below.

<sup>&</sup>lt;sup>2</sup>E.g., by the wellposedness theory of [14].

Remark 8. Note that the parameter  $\lambda$  in (3.6) relates to the speed with which the support of the function  $\rho$  is spreading in the Fourier space while k expresses the speed of its translation. In the standard modulation space  $M_{\infty,1}^{1+\sigma,0}(\mathbb{R}^2)$  one would need to set  $\lambda=1$  but in that case the spreading speed of the support of  $\rho$  (its shrinking speed in physical space) is zero and hence the arguments we apply in the present paper break down. Therefore, the case of the standard modulation space  $M_{\infty,1}^{1+\sigma,0}(\mathbb{R}^2)$  remains an open problem.

Before defining a suitable perturbation of  $\omega_0$  we need to derive several estimates for  $\beta_{k,\lambda}^{\tilde{\alpha},r}$  which we collect in the following lemma.

**Lemma 9.** Let  $2 \le p \le \infty$ ,  $2 < r < \infty$  and  $0 < \sigma < 1$ . For any  $k \in \mathbb{Z}^+$  and  $\lambda > 0$  sufficiently large, we have

1. 
$$\|\beta_{k_{\lambda}}^{\tilde{\alpha},r}\|_{W^{1,r}} \lesssim k^{-1+\tilde{\alpha}} + k^{\tilde{\alpha}}\lambda^{-1}$$

2. 
$$\|\Delta^{1/2+\sigma}\partial_j\Delta^{-1}\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1+2(1/r-1/p)}(\lambda^{\sigma}+k^{\sigma})$$

3. 
$$\|\partial_j \Delta^{-1} \beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^p} \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+2(1/r-1/p)}$$

where j = 1, 2.

*Proof.* We need to compute the  $L^r$  norms of  $\beta_{k,\lambda}^{\tilde{\alpha},r}$  and its first derivative  $\partial_i \beta_{k,\lambda}^{\tilde{\alpha},r}$ . By the triangle inequality and the fact that  $\hat{\rho}$  has compact support we have

$$\|\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{L^r} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1} \sum_{\varepsilon_1,\varepsilon_2} \left( \int_{\mathbb{R}^2} \left| \rho(\lambda(x-x_{\varepsilon}^*)) \right|^r \lambda^2 dx \right)^{1/r} \lesssim k^{-1+\tilde{\alpha}}\lambda^{-1}.$$

For the first derivatives, we have

$$\begin{split} \left\| \frac{\partial \beta_{k,\lambda}^{\tilde{\alpha},r}}{\partial x_1} \right\|_{L^r} &\lesssim k^{-1+\tilde{\alpha}} \lambda^{2/r} \sum_{\varepsilon_1,\varepsilon_2} \left\| \frac{\partial \rho}{\partial x_1} \left( \lambda (\cdot - x_{\varepsilon}^*) \right) \right\|_{L^r} + k^{\tilde{\alpha}} \lambda^{-1+2/r} \sum_{\varepsilon_1,\varepsilon_2} \left\| \rho \left( \lambda (\cdot - x_{\varepsilon}^*) \right) \right\|_{L^r} \\ &\simeq k^{-1+\tilde{\alpha}} \left\| \frac{\partial \rho}{\partial x_1} \right\|_{L^r} + k^{\tilde{\alpha}} \lambda^{-1} \|\rho\|_{L^r} \lesssim k^{-1+\tilde{\alpha}} + k^{\tilde{\alpha}} \lambda^{-1} \end{split}$$

and similarly

$$\left\| \frac{\partial \beta_{k,\lambda}^{\tilde{\alpha},r}}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\tilde{\alpha}} \left\| \frac{\partial \rho}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\tilde{\alpha}}.$$

Combining these estimates gives the bound for  $\|\beta_{k,\lambda}^{\tilde{\alpha},r}\|_{W^{1,r}}$ .

In order to derive the estimates in the remaining cases it will be convenient to use the Fourier transform

(3.7) 
$$\hat{\beta}_{k,\lambda}^{\tilde{\alpha},r}(\xi) = \frac{\lambda^{-1+2/r}}{k^{1-\tilde{\alpha}}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \sum_{j=1,2} \varepsilon_1 \varepsilon_2 \frac{(-1)^{j+1}}{2i\lambda^2} \hat{\rho}(\lambda^{-1}\xi_j^k) e^{-2\pi i \langle x_{\varepsilon}^*, \xi_j^k \rangle}$$

where  $\xi_j^k = (\xi_1 + \frac{(-1)^j}{2\pi}k, \xi_2)$ . Let p' be the conjugate exponent to p. Applying the Hausdorff-Young inequality we obtain

$$\begin{split} \left\| \Delta^{\frac{1+\sigma}{2}} \partial_j \Delta^{-1} \beta_{k,\lambda}^{\tilde{\alpha},r} \right\|_{L^p} &\lesssim \left\| \left| \cdot \right|^{\sigma} \hat{\beta}_{k,\lambda}^{\tilde{\alpha},r} \right\|_{L^{p'}} \\ &\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+2/r} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \lambda^{-2p'} |\xi|^{\sigma p'} \left| \hat{\rho}(\lambda^{-1} \xi_j^k) \right|^{p'} d\xi \right)^{1/p'} \end{split}$$

and changing the variables we further estimate by

$$\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-2(1-\frac{1}{p'})} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \eta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + \eta_2^2 \right)^{\frac{\sigma p'}{2}} |\hat{\rho}(\lambda^{-1}\eta)|^{p'} \frac{d\eta}{\lambda^2} \right)^{1/p'} \\
\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-\frac{2}{p}} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \lambda \zeta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + (\lambda \zeta_2)^2 \right)^{\frac{\sigma p'}{2}} |\hat{\rho}(\zeta)|^{p'} d\zeta \right)^{1/p'} \\
\lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+2\left(\frac{1}{r}-\frac{1}{p}\right)} (\lambda^{\sigma} + k^{\sigma})$$

for any  $\sigma \geq 0$ .

By the same calculation as above, we also have

$$\begin{split} & \| \partial_j \Delta^{-1} \beta_{k,\lambda}^{\tilde{\alpha},r} \|_{L^p} \lesssim \| |\cdot|^{-1} \hat{\beta}_{k,\lambda} \|_{L^{p'}} \\ & \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{2}{r}-\frac{2}{p}} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \lambda \zeta_1 - \frac{(-1)^j}{2\pi} k \right)^2 + (\lambda \zeta_2)^2 \right)^{-\frac{p'}{2}} \left| \hat{\rho}(\zeta) \right|^{p'} d\zeta \right)^{1/p'} \\ & \leq k^{-1+\tilde{\alpha}} \lambda^{-2+2\left(\frac{1}{r}-\frac{1}{p}\right)} \end{split}$$

for sufficiently large k and  $\lambda$ .

From now on we will restrict to the case

(3.8) 
$$\lambda = k^{\tilde{\alpha}}, \quad k = n \quad \text{and} \quad 0 < \tilde{\alpha} \le \alpha \le 1$$

and observe that it is possible to choose the integers n are sufficiently large so that, in particular, the assumptions of the previous lemma hold. Let  $\beta_n = \beta_{k,\lambda}^{\tilde{\alpha},r}$  and define a sequence of initial vorticities by

(3.9) 
$$\omega_{0,n}(x) = \omega_0(x) + \beta_n(x), \qquad n \gg 10.$$

Combining the first part of Lemma 9 with equation (2.4) of Lemma 4 we find that  $\omega_{0,n}$  belong to  $W^{1,r}$ , namely

for any sufficiently large n. This proves the first assertion of Theorem 7.

Denote by  $\omega_n(t)$  the sequence of vorticity solutions of (1.1) with initial data  $\omega_{0,n}$  and as before let  $\eta_n(t)$  be the Lagrangian flows of the corresponding velocity fields  $u_n = \nabla^{\perp} \Delta^{-1} \omega_n$ . The following lemma will be crucial in what follows.

**Lemma 10.** Let  $0 < \sigma < 1$ ,  $0 < \alpha \le 1$  and  $2 \le p \le \infty$ . For any  $1 \le q \le \infty$  we have

$$\|\nabla^{\perp}\Delta^{-1}\beta_{n}\|_{M_{p,q}^{1+\sigma,\alpha}} \simeq \|\nabla^{\perp}\Delta^{-1}\beta_{n}\|_{L^{p}} + \|\Delta^{\frac{1+\sigma}{2}}\nabla^{\perp}\Delta^{-1}\beta_{n}\|_{L^{p}}$$

for any sufficiently large  $n \in \mathbb{Z}^+$ .

*Proof.* From (3.5) and (3.7) we see that for any sufficiently large integer  $n \in \mathbb{Z}^+$  the subsets supp  $\hat{\beta}_n$  and B(0,1) are disjoint. Thus, it suffices to consider the case  $\sigma = 0$ , that is

$$\|\beta_n\|_{L^p} \simeq \|\beta_n\|_{M^{0,\alpha}_{p,q}}.$$

Let  $\mathcal{Q}$  be an admissible  $\alpha$ -covering by sets of size  $|Q| \sim (1+|x|^2)^{\alpha}$ . Using (3.7), (3.8) and the fact that supp  $\hat{\rho} \subset B(0,3)$  we find

$$\operatorname{supp} \hat{\beta}_{n=2^j} \subset B((2^j, 0), 2^{\tilde{\alpha}j}) \subset B((2^j, 0), 2^{\alpha j})$$

so that for any  $j \in \mathbb{Z}_+$  there is a  $Q \in \mathcal{Q}$  with supp  $\hat{\beta}_{2^j} \subset Q$  and we have

$$\|\beta_{2^j}\|_{M_{p,q}^{0,\alpha}} = \left(\sum_{Q\in\mathcal{Q}} \|\mathcal{F}^{-1}\psi_Q\mathcal{F}\beta_{2^j}\|_p^q\right)^{1/q} \simeq \|\beta_{2^j}\|_{L^p}$$

for  $q < \infty$ . Note that the case  $q = \infty$  is analogous.

Suppose now that the data-to-solution map for the Euler equations (E) is continuous from bounded subsets in  $M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2)$  into  $C([0,1],M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2))$ . Choose  $0<\tilde{\alpha}\leq\alpha$  so that

$$-1 + \sigma + 2\tilde{\alpha}(1/r - 1/p) < 0.$$

Then, from estimates 2 and 3 of Lemma 9 we have

$$\|\nabla^{\perp}\Delta^{-1}\beta_n\|_{L^p} + \|\Delta^{\frac{1+\sigma}{2}}\nabla^{\perp}\Delta^{-1}\beta_n\|_{L^p} \to 0 \quad \text{as } n \to \infty$$

where  $\beta_n$  is the perturbation sequence defined in (3.6) and combining (3.9) with Lemma 10 we obtain

(3.11) 
$$\|\nabla^{\perp}\Delta^{-1}(\omega_{0,n}-\omega_0)\|_{M_{\alpha,\sigma}^{1+\sigma,\alpha}}\longrightarrow 0 \quad \text{as } n\to\infty.$$

The continuity assumption on the solution map and (3.11) now imply

(3.12) 
$$\sup_{0 < t < T} \|\nabla^{\perp} \Delta^{-1}(\omega_n(t) - \omega(t))\|_{M_{p,q}^{1+\sigma,\alpha}} \longrightarrow 0 \quad \text{as } n \to \infty$$

from which, using the embedding assumption  $M_{p,q}^{1+\sigma} \subset C^1$  and Lemma 6, we obtain

(3.13) 
$$\sup_{0 \le t \le T} \sum_{i=0,1} \|D^i \eta_n(t) - D^i \eta(t)\|_{\infty} \longrightarrow 0 \quad \text{as } n \to \infty$$

where as before  $\eta(t)$  is the Lagrangian flow of the (smooth) divergence-free vector field  $u = \nabla^{\perp} \Delta^{-1} \omega$  of Proposition 5 and  $\eta_n(t)$  is the flow of  $u_n = \nabla^{\perp} \Delta^{-1} \omega_n$  whose initial vorticities are given by (2.2) and (3.9) respectively. The rest of the argument is completely analogous to the proof of Theorem 3 in [18]. Thus we omit the details.

## 4. A direct proof for $C^{1+\sigma}(\mathbb{R}^3)$ based on shear flow

In this section we present a short and direct argument showing the loss of continuity of the data-to-solution map of (E) in the classical Hölder space  $C^{1+\sigma}$  with  $0 < \sigma < 1$ . It is inspired by conversations with A. Shnirelman and C. Bardos from whom we learned about the DiPerna-Majda shear flow techniques.

Consider two 3D shear flows

$$u(t,x) = (f(x_2), 0, h(x_1 - tf(x_2)))$$
 and  $v(t,x) = (g(x_2), 0, h(x_1 - tg(x_2))).$ 

Let f, g and h be bounded functions in  $C^{1+\sigma}(\mathbb{R}^3)$  with h chosen so that in addition its derivative satisfies

$$h'(x_1) = |x_1|^{\sigma}$$
 for  $-2a \le x_1 \le 2a$ 

where  $a = \max\{\sup_{x_1} |f(x_1)|, \sup_{x_1} |g(x_1)|\}$ . It is easy to verify that both u(t) and v(t) solve the 3D Euler equations with initial conditions

$$u_0(x) = (f(x_2), 0, h(x_1))$$
 and  $v_0(x) = (g(x_2), 0, h(x_1)).$ 

Now, given any  $\epsilon > 0$  adjust f and q so that

$$||u_0 - v_0||_{C^{1+\sigma}} = ||f - g||_{C^{1+\sigma}} < \epsilon$$

and consider at any time  $0 \le t \le 1$  the norm of the difference of the corresponding solutions

$$||u(t) - v(t)||_{C^{1+\sigma}} = ||f - g||_{C^{1+\sigma}} + ||h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot))||_{C^{1+\sigma}}$$

$$\geq ||\nabla (h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot)))||_{C^{\sigma}}$$

$$= ||h'(\cdot - tf(\cdot)) - h'(\cdot - tg(\cdot))||_{C^{\sigma}}$$

which can be further bounded from below by

$$\geq \sup_{\substack{x \neq y \\ x, y \in [-b,b]^2}} \frac{\left| \left( |x_1 - tf(x_2)|^{\sigma} - |x_1 - tg(x_2)|^{\sigma} \right) - \left( |y_1 - tf(y_2)|^{\sigma} - |y_1 - tg(y_2)|^{\sigma} \right) \right|}{|x - y|^{\sigma}}$$

where  $b = \min\{\sup_{x_1} |f(x_1)|, \sup_{x_1} |g(x_1)|\}.$ 

Finally, pick  $x_2 = y_2 = c$  for some arbitrary constant c so that the expression above becomes

ove becomes
$$\geq \sup_{\substack{x_1 \neq y_1 \\ x, y \in [-b,b]^2}} \frac{|(|x_1 - tf(c)|^{\sigma} - |x_1 - tg(c)|^{\sigma}) - (|y_1 - tf(c)|^{\sigma} - |y_1 - tg(c)|^{\sigma})|}{|x_1 - y_1|^{\sigma}}$$

and evaluate it once again from below by choosing  $x_1 = tg(c)$  and  $y_1 = tf(c)$  to get the bound

$$\geq \frac{t^{\sigma}|g(c)-f(c)|^{\sigma}+t^{\sigma}|f(c)-g(c)|^{\sigma}}{t^{\sigma}|f(c)-g(c)|^{\sigma}}=2.$$

This shows local ill-posedness of the 3D Euler equations in  $C^{1+\sigma}$  in the Hadamard sense considered here.  $\Box$ 

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