ON TOTAL MEAN CURVATURES OF FOLIATED HALF-LIGHTLIKE SUBMANIFOLDS IN SEMI-RIEMANNIAN MANIFOLDS

FORTUNÉ MASSAMBA*, SAMUEL SSEKAJJA**

ABSTRACT. We derive total mean curvature integration formulae of a three co-dimensional foliation \mathcal{F}^n on a screen integrable half-lightlike submanifold, M^{n+1} in a semi-Riemannian manifold \overline{M}^{n+3} . We give generalized differential equations relating to mean curvatures of a totally umbilical half-lightlike submanifold admitting a totally umbilical screen distribution, and show that they are generalizations of those given by [4].

1. Introduction

The rapidly growing importance of lightlike submanifolds in semi-Riemannian geometry, particularly Lorentzian geometry, and their applications to mathematical physics—like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [3] and [4] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in (4+m)-dimensional spacetime manifold, where m is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [3] and [4], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [3], Duggal-Sahin [4] and Kupeli [13]. It is upon those books that many other researchers, including but not limited to [2], [5], [6], [7], [8], [9], [10], [11], have extended their theories.

Lightlike geometry rests on a number of operators, like shape and algebraic invariants derived from them, such as trace, determinants, and in general the r-th mean curvature S_r . There is a great deal of work so far on the case r=1 (see some in [3], [4] and many more) and as far as we know, very little has been done for the case r>1. This is partly due to the non-linearity of S_r for r>1, and hence very complicated to study. A great deal of research on higher order mean curvatures S_r in Riemannian geometry has been done with numerous applications, for instance see [1] and [12]. This gap has motivated our introduction of lightlike geometry of S_r for r>1. In this paper we have considered a half-lightlike submanifold admitting an integrable screen distribution, of a semi-Riemannian manifold. On it we

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have focused on a codimension 3 foliation of its screen distribution and thus derived integral formulae of its total mean curvatures (see Theorems 4.9 and 4.10). Furthermore, we have considered totally umbilical half-lightlike submanifolds, with a totally umbilical screen distribution and generalized Theorem 4.3.7 of [4] (see Theorem 5.2 and its Corollaries). The paper is organized as follows; In section 2 we summarize the basic notions on lightlike geometry necessary for other sections. In section 3 we give some basic information on Newton transformations of a foliation $\mathcal F$ of the screen distribution. Section 4 focuses on integration formulae of $\mathcal F$ and their consequences. In section 5 we discus screen umbilical half-lightlike submanifolds and generalizations of some well-known results of [4].

2. Preliminaries

Let (M^{n+1},g) be a two-co-dimensional submanifold of a semi-Riemannian manifold $(\overline{M}^{n+3},\overline{g})$, where $g=\overline{g}|_{TM}$. The submanifold (M^{n+1},g) is called a half-lightlike if the radical distribution $\operatorname{Rad}TM=TM\cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle TM^{\perp} of M, with rank one. Let S(TM) be a screen distribution which is a semi-Riemannian complementary distribution of $\operatorname{Rad}TM$ in TM, and also choose a screen transversal bundle $S(TM^{\perp})$, which is semi-Riemannian and complementary to $\operatorname{Rad}TM$ in TM^{\perp} . Then,

$$TM = \operatorname{Rad} TM \perp S(TM), \quad TM^{\perp} = \operatorname{Rad} TM \perp S(TM^{\perp}).$$
 (2.1)

We will denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ . It is well-known from [3] and [4] that for any null section E of $\operatorname{Rad} TM$, there exists a unique null section N of the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that g(E,N)=1, it follows that there exists a lightlike *transversal vector bundle* $\operatorname{ltr}(TM)$ locally spanned by N. Let $W\in \Gamma(S(TM^{\perp}))$ be a unit vector field, then $\overline{g}(N,N)=\overline{g}(N,Z)=\overline{g}(N,W)=0$, for any $Z\in \Gamma(S(TM))$.

Let tr(TM) be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$. Then we have the following decompositions of tr(TM) and $T\overline{M}$

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M} = S(TM) \perp S(TM^{\perp}) \perp \{ \operatorname{Rad} TM \oplus ltr(TM) \}.$$
 (2.3)

It is important to note that the distribution S(TM) is not unique, and is canonically isomorphic to the factor vector bundle $TM/\mathrm{Rad}\,TM$ [3]. Let P be the projection of TM on to S(TM). Then the local Gauss-Weingarten equations of M are the following;

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)W, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X) N + \rho(X) W, \tag{2.5}$$

$$\overline{\nabla}_X W = -A_W X + \phi(X) N, \tag{2.6}$$

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E, \tag{2.7}$$

$$\nabla_X E = -A_E^* X - \tau(X) E, \tag{2.8}$$

for all $E \in \Gamma(\operatorname{Rad}TM)$, $N \in \Gamma(\operatorname{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where ∇ and ∇^* are induced linear connections on TM and S(TM), respectively, B and D are called the local second fundamental forms of M, C is the local second fundamental form on S(TM). Furthermore, $\{A_N, A_W\}$ and A_E^* are the shape operators on TM and S(TM) respectively, and τ , ρ , ϕ and δ are differential 1-forms on TM. Notice that ∇^* is a metric connection on S(TM) while ∇ is generally not a metric connection. In fact, ∇ satisfies the following relation

$$(\nabla_X g)(Y, Z) = B(X, Y)\lambda(Z) + B(X, Z)\lambda(Y), \tag{2.9}$$

for all $X,Y,Z\in\Gamma(TM)$, where λ is a 1-form on TM given $\lambda(\cdot)=\overline{g}(\cdot,N)$. It is well-known from [3] and [4] that B and D are independent of the choice of S(TM) and they satisfy

$$B(X, E) = 0, \quad D(X, E) = \phi(X), \quad \forall X \in \Gamma(TM). \tag{2.10}$$

The local second fundamental forms B, D and C are related to their shape operators by the following equations

$$g(A_E^*X, Y) = B(X, Y), \quad \overline{g}(A_E^*X, N) = 0,$$
 (2.11)

$$g(A_W X, Y) = \varepsilon D(X, Y) + \phi(X)\lambda(Y), \tag{2.12}$$

$$g(A_N X, PY) = C(X, PY), \ \overline{g}(A_N X, N) = 0,$$
 (2.13)

$$\overline{g}(A_W X, N) = \varepsilon \rho(X), \text{ where } \varepsilon = \overline{g}(W, W),$$
 (2.14)

for all $X,Y\in \Gamma(TM)$. From equations (2.11) we deduce that A_E^* is S(TM)-valued, self-adjoint and satisfies $A_E^*E=0$. Let \overline{R} denote the curvature tensor of \overline{M} , then

$$\overline{g}(\overline{R}(X,Y)PZ,N) = g((\nabla_X A_N)Y, PZ) - g((\nabla_Y A_N)X, PZ) + \tau(Y)C(X, PZ) - \varepsilon\tau(X)C(Y, PZ)\{\rho(Y)D(X, PZ) - \rho(X)D(Y, PZ)\}, \quad \forall X, Y, Z \in \Gamma(TM).$$
 (2.15)

A half-lightlike submanifold (M,g) of a semi-Riemannian manifold \overline{M} is said to be totally umbilical [4] if on each coordinate neighborhood $\mathcal U$ there exist smooth functions $\mathcal H_1$ and $\mathcal H_2$ on $l\mathrm{tr}(TM)$ and $S(TM^\perp)$ respect such that

$$B(X,Y) = \mathcal{H}_1 g(X,Y), \quad D(X,Y) = \mathcal{H}_2 g(X,Y), \quad \forall X,Y \in \Gamma(TM).$$
 (2.16)

Furthermore, when ${\cal M}$ is totally umbilical then the following relations follows by straightforward calculations

$$A_E^*X = \mathcal{H}_1 PX, \ P(A_W X) = \varepsilon \mathcal{H}_2 PX, \ D(X, E) = 0, \ \rho(E) = 0,$$
 (2.17) for all $X, Y \in \Gamma(TM)$.

Next, we suppose that M is a half-lightlike submanifold of \overline{M} , with an integrable screen distribution S(TM). Let M' be a leaf of S(TM). Notice that for any screen integrable half-lightlike M, the leaf M' of S(TM) is a co-dimension 3 submanifold of \overline{M} whose normal bundle is $\{\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)\} \perp S(TM^{\perp})$. Now, using (2.4) and (2.7) we have

$$\overline{\nabla}_X Y = \nabla_X^* Y + C(X, PY)E + B(X, Y)N + D(X, Y)W, \tag{2.18}$$

for all $X,Y\in \Gamma(TM')$. Since S(TM) is integrable, then its leave is semi-Riemannian and hence we have

$$\overline{\nabla}_X Y = \nabla_X^{*'} Y + h'(X, Y), \quad \forall X, Y \in \Gamma(TM'), \tag{2.19}$$

where $\underline{h'}$ and $\nabla^{*'}$ are second fundamental form and the Levi-Civita connection of M' in \overline{M} . From (2.18) and (2.19) we can see that

$$h'(X,Y) = C(X,PY)E + B(X,Y)N + D(X,Y)W,$$
 (2.20)

for all $X,Y \in \Gamma(TM')$. Since S(TM) is integrable, then it is well-known from [4] that C is symmetric on S(TM) and also A_N is self-adjoint on S(TM) (see Theorem 4.1.2 for details). Thus, h' given by (2.20) is symmetric on TM'.

Let $L \in \Gamma(\{\operatorname{Rad} TM \oplus l \operatorname{tr}(TM)\} \perp S(TM^{\perp}))$, then we can decompose L as

$$L = aE + bN + cW, (2.21)$$

for non-vanishing smooth functions on \overline{M} given by $a=\overline{g}(L,N), b=\overline{g}(L,E)$ and $c=\varepsilon\overline{g}(L,W)$. Suppose that $\overline{g}(L,L)>0$, then using (2.21) we obtain a unit normal vector \widehat{W} to M' given by

$$\widehat{W} = \frac{1}{\overline{g}(L,L)}(aE + bN + cW) = \frac{1}{\overline{g}(L,L)}L. \tag{2.22}$$

Next we define a (1,1) tensor $\mathcal{A}_{\widehat{W}}$ in terms of the operators A_E^* , A_N and A_W by

$$\mathcal{A}_{\widehat{W}}X = \frac{1}{\overline{g}(L,L)}(aA_E^*X + bA_NX + cA_WX), \tag{2.23}$$

for all $X \in \Gamma(TM)$. Notice that $\mathcal{A}_{\widehat{W}}$ is self-adjoint on S(TM). Applying $\overline{\nabla}_X$ to \widehat{W} and using equations (2.23) (2.4) and (2.11)-(2.13), we have

$$g(\mathcal{A}_{\widehat{W}}X, PY) = -\overline{g}(\overline{\nabla}_X \widehat{W}, PY), \ \forall X, Y \in \Gamma(TM).$$
 (2.24)

Let $\nabla^{*\perp}$ be the connection on the normal bundle $\{\operatorname{Rad} TM \oplus l\operatorname{tr}(TM)\} \perp S(TM^{\perp})$. Then from (2.24) we have

$$\overline{\nabla}_X \widehat{W} = -\mathcal{A}_{\widehat{W}} X + \nabla_X^{*\perp} \widehat{W}, \quad \forall X \in \Gamma(TM), \tag{2.25}$$

where

$$\begin{split} \nabla_X^{*\perp} \widehat{W} &= -\frac{1}{\overline{g}(L,L)} X(\overline{g}(L,L)) \widehat{W} + \frac{1}{\overline{g}(L,L)} \left[\{ X(a) - a\tau(X) \} E \right. \\ &\left. + \{ X(b) + b\tau(X) + c\phi(X) \} N + \{ X(c) + aD(X,E) + b\rho(X) \} W \right]. \end{split}$$

Example 2.1. Let $\overline{M}=(\mathbb{R}^5_1,\overline{g})$ be a semi-Riemannian manifold, where \overline{g} is of signature (-,+,+,+,+) with respect to canonical basis $(\partial x_1,\partial x_2,\partial x_3,\partial x_4,\,\partial x_5)$, where (x_1,\cdots,x_5) are the usual coordinates on \overline{M} . Let M be a submanifold of \overline{M} and given parametrically by the following equations

$$x_1 = \varphi_1$$
, $x_2 = \sin \varphi_2 \sin \varphi_3$, $x_3 = \varphi_1$, $x_4 = \cos \varphi_2 \sin \varphi_3$, $x_5 = \cos \varphi_3$, where $\varphi_2 \in [0, 2\pi]$ and $\varphi_3 \in (0, \pi/2)$.

Then we have $TM = \text{span}\{E, Z_1, Z_2\}$ and $ltr(TM) = \text{span}\{N\}$, where

$$E = \partial x_1 + \partial x_3$$
, $Z_1 = \cos \varphi_3 \partial x_2 - \sin \varphi_2 \sin \varphi_3 \partial x_5$,

$$Z_2 = \cos \varphi_3 \partial x_4 - \cos \varphi_2 \sin \varphi_3 \partial x_5$$
 and $N = \frac{1}{2}(-\partial x_1 + \partial x_3)$.

Also, by straightforward calculations, we have

$$W = \sin \varphi_2 \sin \varphi_3 \partial x_2 + \cos \varphi_2 \sin \varphi_3 \partial x_4 + \cos \varphi_3 \partial x_5.$$

Thus, $S(TM^{\perp}) = \operatorname{span}\{W\}$ and hence M is a half-lightlike submanifold of \overline{M} . Furthermore we have $[Z_1,Z_2] = \cos \varphi_2 \sin \varphi_3 \partial x_2 - \sin \varphi_2 \sin \varphi_3 \partial x_4$, which leads to $[Z_1,Z_2] = \cos \varphi_2 \tan \varphi_3 Z_1 - \sin \varphi_2 \tan \varphi_3 Z_2 \in \Gamma(S(TM))$. Thus, M is a screen integrable half-lightlike submanifold of \overline{M} . Finally, it is easy to see that A_N is self-adjoint operator on S(TM).

In the next sections we shall consider screen integrable half-lightlike submanifolds of semi-Riemannian manifold \overline{M} and derive special integral formulae for a foliation of S(TM), whose normal vector is \widehat{W} and with shape operator $\mathcal{A}_{\widehat{W}}$.

3. Newton transformations of $\mathcal{A}_{\widehat{W}}$

Let $(\overline{M}^{m+3}, \overline{g})$ be a semi-Riemannian manifold and let (M^{n+1}, g) be a screen integrable half-lightlike submanifold of \overline{M} . Then S(TM) admits a foliation and let $\mathcal F$ be a such foliation. Then, the leaves of $\mathcal F$ are co-dimension three submanifolds of \overline{M} , whose normal bundle is $S(TM)^{\perp}$. Let \widehat{W} be unit normal vector to $\mathcal F$ such that the orientation of \overline{M} coincides with that given by $\mathcal F$ and \widehat{W} . The Levi-Civita connection $\overline{\nabla}$ on the tangent bundle of \overline{M} induces a metric connection ∇' on $\mathcal F$. Furthermore, h' and $\mathcal A_{\widehat{W}}$ are the second fundamental form and shape operator of $\mathcal F$. Notice that $\mathcal A_{\widehat{W}}$ is self-adjoint on $T\mathcal F$ and at each point $p\in \mathcal F$ has n real eigenvalues (or principal curvatures) $\kappa_1(p),\cdots,\kappa_n(p)$. Attached to the shape operator $\mathcal A_{\widehat{W}}$ are n algebraic invariants

$$S_r = \sigma_r(\kappa_1, \cdots, \kappa_n), \ 1 \le r \le n,$$

where $\sigma_r: M^{'n} \to \mathbb{R}$ are symmetric functions given by

$$\sigma_r(\kappa_1, \cdots, \kappa_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} \kappa_{i_1} \cdots \kappa_{i_r}. \tag{3.1}$$

Then, the characteristic polynomial of $\mathcal{A}_{\widehat{W}}$ is given by

$$\det(\mathcal{A}_{\widehat{W}} - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_r t^{n-\alpha},$$

where \mathbb{I} is the identity in $\Gamma(T\mathcal{F})$. The normalized r-th mean curvature H_r of M' is defined by

$$H_r = \binom{n}{r}^{-1} S_r$$
 and $H_0 = 1$. (a constant function 1).

In particular, when r=1 then $H_1=\frac{1}{n}\mathrm{tr}(\mathcal{A}_{\widehat{W}})$ which is the *mean curvature* of a \mathcal{F} . On the other hand, H_2 relates directly with the (intrinsic) scalar curvature of \mathcal{F} . Moreover, the functions S_r (H_r respectively) are smooth on the whole M and, for any point $p\in\mathcal{F}$, S_r coincides with the r-th mean curvature at p. In this paper, we shall use S_r instead of H_r .

Next, we introduce the Newton transformations with respect to the operator $\mathcal{A}_{\widehat{W}}$. The Newton transformations $T_r:\Gamma(T\mathcal{F})\to\Gamma(T\mathcal{F})$ of a foliation \mathcal{F} of a screen integrable half-lightlike submanifold M of an (n+3)-dimensional semi-Riemannian manifold \overline{M} with respect to $A_{\widehat{W}}$ are given by the inductive formula

$$T_0 = \mathbb{I}, \quad T_r = (-1)^r S_r \mathbb{I} + \mathcal{A}_{\widehat{W}} \circ T_{r-1}, \quad 1 \le r \le n.$$
 (3.2)

By Cayley-Hamiliton theorem, we have $T_n = 0$. Moreover, T_r are also self-adjoint and commutes with $\mathcal{A}_{\widehat{W}}$. Furthermore, the following algebraic properties of T_r are well-known (see [1], [12] and references therein for details).

$$tr(T_r) = (-1)^r (n-r)S_r,$$
 (3.3)

$$\operatorname{tr}(\mathcal{A}_{\widehat{W}} \circ T_r) = (-1)^r (r+1) S_{r+1}, \tag{3.4}$$

$$\operatorname{tr}(\mathcal{A}_{\widehat{W}}^2 \circ T_r) = (-1)^{r+1} (-S_1 S_{r+1} + (r+2) S_{r+2}), \tag{3.5}$$

$$\operatorname{tr}(T_r \circ \nabla_X' \mathcal{A}_{\widehat{W}}) = (-1)^r X(S_{r+1}) = (-1)^r \overline{g}(\nabla' S_{r+1}, X),.$$
 (3.6)

for all $X \in \Gamma(T\overline{M})$. We will also need the following divergence formula for the operators T_r

$$\operatorname{div}^{\nabla'}(T_r) = \operatorname{tr}(\nabla' T_r) = \sum_{\beta=1}^n (\nabla'_{Z_\beta} T_r) Z_\beta, \tag{3.7}$$

where $\{Z_1, \dots, Z_n\}$ is a local orthonormal frame field of $T\mathcal{F}$.

4. Integration formulae for \mathcal{F}

This section is devoted to derivation of integral formulas of foliation \mathcal{F} of S(TM) with a unit normal vector \widehat{W} given by (2.22). By the fact that $\overline{\nabla}$ is a metric connection then $\overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\widehat{W})=0$. This implies that the vector field $\overline{\nabla}_{\widehat{W}}\widehat{W}$ is always tangent to \mathcal{F} . Our main goal will be to compute the divergence of the vectors $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}$ and $T_r\overline{\nabla}_{\widehat{W}}\widehat{W}+(-1)^rS_{r+1}\widehat{W}$. The following technical lemmas are fundamentally important to this paper. Let $\{E,Z_i,N,W\}$, for $i=1,\cdots,n$ be a quasi-orthonormal field of frame of $T\overline{M}$, such that $S(TM)=\mathrm{span}\{Z_i\}$ and $\epsilon_i=\overline{g}(Z_i,Z_i)$.

Lemma 4.1. Let M be a screen integrable half-lightlike submanifold of \overline{M}^{n+3} and let M' be a foliation of S(TM). Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where \widehat{W} is a unit normal vector to \mathcal{F} . Then

$$\overline{g}((\nabla'_X \mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(Y, (\nabla'_X \mathcal{A}_{\widehat{W}})Z), \quad \overline{g}((\nabla'_X T_r)Y, Z) = \overline{g}(Y, (\nabla'_X T_r)Z),$$
 for all $X, Y, Z \in \Gamma(T\mathcal{F})$.

Proof. By simple calculations we have

$$\overline{g}((\nabla_X' \mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(\nabla_X' (\mathcal{A}_{\widehat{W}}Y), Z) - \overline{g}(\nabla_X' Y, \mathcal{A}_{\widehat{W}}Z). \tag{4.1}$$

Using the fact that ∇' is a metric connection and the symmetry of $\mathcal{A}_{\widehat{W}}$, (4.1) gives

$$\overline{g}((\nabla_X' \mathcal{A}_{\widehat{W}})Y, Z) = \overline{g}(Y, \nabla_X' (\mathcal{A}_{\widehat{W}}Z)) - \overline{g}(Y, \mathcal{A}_{\widehat{W}}(\nabla_X' Z)). \tag{4.2}$$

Then, from (4.2) we deduce the first relation of the lemma. A proof of the second relation follows in the same way, which completes the proof.

Lemma 4.2. Let M be a screen integrable half-lightlike submanifold of \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM). Let $\mathcal{A}_{\widehat{W}}$ be its shape operator, where \widehat{W} is a unit normal vector to \mathcal{F} . Denote by \overline{R} the curvature tensor of \overline{M} . Then

$$\operatorname{div}^{\nabla'}(T_0) = 0,$$

$$\operatorname{div}^{\nabla'}(T_r) = \mathcal{A}_{\widehat{W}}\operatorname{div}^{\nabla'}(T_{r-1}) + \sum_{i=1}^n \epsilon_i(\overline{R}(\widehat{W}, T_{r-1}Z_i)Z_i)',$$

where $(\overline{R}(\widehat{W},X)Z)'$ denotes the tangential component of $\overline{R}(\widehat{W},X)Z$ for $X,Z\in\Gamma(T\mathcal{F})$. Equivalently, for any $Y\in\Gamma(T\mathcal{F})$ then

$$\overline{g}(\operatorname{div}^{\nabla'}(T_r), Y) = \sum_{j=1}^r \sum_{i=1}^n \epsilon_i \overline{g}(\overline{R}(T_{r-1}Z_i, \widehat{W})(-\mathcal{A}_{\widehat{W}})^{j-1}Y, Z_i). \tag{4.3}$$

Proof. The first equation of the lemma is obvious since $T_0 = \mathbb{I}$. We turn to the second relation. By direct calculations using the recurrence relation (3.2) we derive

$$\operatorname{div}^{\nabla'}(T_r) = (-1)^r \operatorname{div}^{\nabla'}(S_r \mathbb{I}) + \operatorname{div}^{\nabla'}(\mathcal{A}_{\widehat{W}} \circ T_{r-1})$$
$$= (-1)^r \nabla' S_r + \mathcal{A}_{\widehat{W}} \operatorname{div}^{\nabla'}(T_{r-1}) + \sum_{i=1}^n \epsilon_i (\nabla'_{Z_i} \mathcal{A}_{\widehat{W}}) T_{r-1} Z_i. \quad (4.4)$$

Using Codazzi equation

$$\overline{g}(\overline{R}(X,Y)Z,\widehat{W}) = \overline{g}((\nabla'_Y \mathcal{A}_{\widehat{W}})X,Z) - \overline{g}((\nabla'_X \mathcal{A}_{\widehat{W}})Y,Z),$$

for any $X, Y, Z \in \Gamma(T\mathcal{F})$ and Lemma 4.1, we have

$$\overline{g}((\nabla'_{Z_{i}}\mathcal{A}_{\widehat{W}})Y,T_{r-1}Z_{i}) = \overline{g}((\nabla'_{Y}\mathcal{A}_{\widehat{W}})Z_{i},T_{r-1}Z_{i}) + \overline{g}(\overline{R}(Y,Z_{i})T_{r-1}Z_{i},\widehat{W})$$

$$= \overline{g}(T_{r-1}(\nabla'_{Y}\mathcal{A}_{\widehat{W}})Z_{i},Z_{i}) + \overline{g}(\overline{R}(\widehat{W},T_{r-1}Z_{i})Z_{i},Y), \quad (4.5)$$

for any $Y \in \Gamma(T\mathcal{F})$. Then applying (4.4) and (4.5) we get

$$\overline{g}(\operatorname{div}^{\nabla'}(T_r), Y) = (-1)^r \overline{g}(\nabla' S_r, Y) + \operatorname{tr}(T_{r-1}(\nabla'_Y \mathcal{A}_{\widehat{W}}))
+ \overline{g}(\operatorname{div}^{\nabla'}(T_{r-1}), Y) + \overline{g}(Y, \sum_{i=1}^n \epsilon_i \overline{R}(\widehat{W}, T_{r-1} Z_i) Z_i).$$
(4.6)

Then, applying (4.6) and (3.6) we get the second equation of the lemma. Finally, (4.3) follows immediately by an induction argument.

Notice that when the ambient manifold is a space form of constant sectional curvature, then $(\overline{R}(\widehat{W},X)Y)'=0$ for each $X,Y\in\Gamma(T\mathcal{F})$. Hence, from Lemma (4.2) we have $\operatorname{div}^{\nabla'}(T_r)=0$.

Lemma 4.3. Let M be a screen integrable half-lightlike submanifold of \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM). Let $A_{\widehat{W}}$ be its shape operator, where \widehat{W} is a unit normal vector to \mathcal{F} . Let $\{Z_i\}$ be a local field such $(\nabla'_X Z_i)p = 0$, for $i = 1, \dots, n$ and any vector field $X \in \Gamma(T\overline{M})$. Then at $p \in \mathcal{F}$ we have

$$g(\nabla'_{Z_i} \overline{\nabla}_{\widehat{W}} \widehat{W}, Z_j) = g(\mathcal{A}_{\widehat{W}}^2 Z_i, Z_j) - \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j) + g(\overline{\nabla}_{\widehat{W}} \widehat{W}, Z_i) g(Z_j, \overline{\nabla}_{\widehat{W}} \widehat{W}).$$

Proof. Applying $\overline{\nabla}_{Z_i}$ to $g(\overline{\nabla}_{\widehat{W}}\widehat{W},Z_j)$ and $\overline{g}(\widehat{W},\overline{\nabla}_{\widehat{W}}Z_j)$ in turn and then using the two resulting equations, we have

$$-\overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W}, \overline{\nabla}_{Z_{i}}Z_{j}) = g(\overline{\nabla}_{Z_{i}}\overline{\nabla}_{\widehat{W}}\widehat{W}, Z_{j}) + \overline{g}(\overline{\nabla}_{Z_{i}}\widehat{W}, \overline{\nabla}_{\widehat{W}}Z_{j}) + \overline{g}(\widehat{W}, \overline{\nabla}_{Z_{i}}\overline{\nabla}_{\widehat{W}}Z_{j}). \tag{4.7}$$

Furthermore, by direct calculations using $(\nabla'_X Z_i)p = 0$ we have

$$\overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_i,Z_j) = \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W},\overline{Z_i}Z_j) + \overline{g}(\widehat{W},\overline{\nabla}_{\widehat{W}}\overline{Z_i}Z_j),$$

and hence

$$g(\mathcal{A}_{\widehat{W}}^{2}Z_{i}, Z_{j}) - \overline{g}(\overline{R}(Z_{i}, \widehat{W})Z_{j}, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}}\mathcal{A}_{\widehat{W}})Z_{i}, Z_{j})$$

$$= g(\mathcal{A}_{\widehat{W}}^{2}Z_{i}, Z_{j}) - \overline{g}(\overline{R}(Z_{i}, \widehat{W})Z_{j}, \widehat{W})$$

$$- \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W}, \overline{Z_{i}}Z_{j}) - \overline{g}(\widehat{W}, \overline{\nabla}_{\widehat{W}}\overline{Z_{i}}Z_{j})$$

$$= g(\mathcal{A}_{\widehat{W}}^{2}Z_{i}, Z_{j}) - \overline{g}(\overline{\nabla}_{Z_{i}}Z_{j}, \overline{\nabla}_{\widehat{W}}\widehat{W})$$

$$- \overline{g}(\overline{\nabla}_{Z_{i}}\overline{\nabla}_{\widehat{W}}Z_{j}, \widehat{W}) + \overline{g}(\overline{\nabla}_{[Z_{i}:\widehat{W}]}Z_{j}, \widehat{W}). \tag{4.8}$$

Now, applying (4.7), the condition at p and the following relations

$$\overline{\nabla}_{Z_i}\widehat{W} = \sum_{k=1}^n \epsilon_k \overline{g}(\overline{\nabla}_{Z_i}\widehat{W}, Z_k) Z_k, \quad \overline{\nabla}_{\widehat{W}} Z_j = \overline{g}(\overline{\nabla}_{\widehat{W}} Z_j, \widehat{W}) \widehat{W},$$

and $g(\mathcal{A}_{\widehat{W}}^2 Z_i, Z_j) = -\sum_{k=1}^n \epsilon_k \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, Z_k) \overline{g}(\overline{\nabla}_{Z_k} Z_j, \widehat{W})$ to the last line of (4.8) and the fact that S(TM) is integrable we get

$$g(\mathcal{A}_{\widehat{W}}^2 Z_i, Z_j) - \overline{g}(\overline{R}(Z_i, \widehat{W}) Z_j, \widehat{W}) - \overline{g}((\nabla'_{\widehat{W}} \mathcal{A}_{\widehat{W}}) Z_i, Z_j)$$

= $g(\nabla'_{Z_i} \overline{\nabla}_{\widehat{W}} \widehat{W}, Z_j) - g(\overline{\nabla}_{\widehat{W}} \widehat{W}, Z_i) g(Z_j, \overline{\nabla}_{\widehat{W}} \widehat{W}),$

from which the lemma follows by rearrangement.

Notice that, using parallel transport, we can always construct a frame field from the above lemma.

Proposition 4.4. Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold \overline{M} and let \mathcal{F} be a foliation of S(TM). Then

$$\operatorname{div}^{\nabla'}(T_{r}\overline{\nabla}_{\widehat{W}}\widehat{W}) = \overline{g}(\operatorname{div}^{\nabla'}(T_{r}), \overline{\nabla}_{\widehat{W}}\widehat{W}) + (-1)^{r+1}\widehat{W}(S_{r+1})$$

$$+ (-1)^{r+1}(-S_{1}S_{r+1} + (r+2)S_{r+2}) - \sum_{i=1}^{n} \epsilon_{i}\overline{g}(\overline{R}(Z_{i},\widehat{W})T_{r}Z_{i},\widehat{W})$$

$$+ \overline{g}(\overline{\nabla}_{\widehat{W}}\widehat{W}, T_{r}\overline{\nabla}_{\widehat{W}}\widehat{W}),$$

where $\{Z_i\}$ is a field of frame tangent to the leaves of \mathcal{F} .

Proof. From (3.7), we deduce that

$$\operatorname{div}^{\nabla'}(T_r Z) = \overline{g}(\operatorname{div}^{\nabla'}(T_r), Z) + \sum_{i=1}^n \epsilon_i \overline{g}(\nabla'_{Z_i} Z, T_r Z_i), \tag{4.9}$$

for all $Z \in \Gamma(T\mathcal{F})$. Then using (4.9), Lemmas 4.2 and 4.3, we obtain the desired result. Hence the proof.

From Proposition 4.4 we have

Theorem 4.5. Let M be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM). Then

$$\operatorname{div}^{\overline{\nabla}}(T_r \overline{\nabla}_{\widehat{W}} \widehat{W}) = \overline{g}(\operatorname{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}} \widehat{W}) + (-1)^{r+1} \widehat{W}(S_{r+1}) + (-1)^{r+1} (-S_1 S_{r+1} + (r+2) S_{r+2}) - \sum_{i=1}^n \epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W}) T_r Z_i, \widehat{W}).$$

Proof. A proof follows easily form Proposition 4.4 by recognizing the fact that

$$\operatorname{div}^{\overline{\nabla}}(T_r \overline{\nabla}_{\widehat{W}} \widehat{W}) = \operatorname{div}^{\nabla'}(T_r \overline{\nabla}_{\widehat{W}} \widehat{W}) - \overline{g}(\overline{\nabla}_{\widehat{W}} \widehat{W}, T_r \overline{\nabla}_{\widehat{W}} \widehat{W}),$$

which completes the proof.

Theorem 4.6. Let M be a screen integrable half-lightlike submanifold of \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM). Then,

$$\operatorname{div}^{\overline{\nabla}}(T_r \overline{\nabla}_{\widehat{W}} \widehat{W} + (-1)^r S_{r+1} \widehat{W}) = \overline{g}(\operatorname{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}} \widehat{W}) + (-1)^{r+1} (r+2) S_{r+2} - \sum_{i=1}^n \epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W}) T_r Z_i, \widehat{W}).$$

Proof. By straightforward calculations we have

$$S_1 = \operatorname{tr}(\mathcal{A}_{\widehat{W}}) = -\sum_{i=1}^n \epsilon_i \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, Z_i) = -\sum_{i=1}^{n+1} \epsilon_i \overline{g}(\overline{\nabla}_{Z_i} \widehat{W}, Z_i) = -\operatorname{div}^{\overline{\nabla}}(\widehat{W}),$$

where $Z_{n+1} = \widehat{W}$. From this equation we deduce

$$\operatorname{div}^{\overline{\nabla}}(S_{r+1}\widehat{W}) = -S_1 S_{r+1} + \widehat{W}(S_{r+1}). \tag{4.10}$$

Then from (4.10) and Theorem 4.5 we get our assertion, hence the proof.

Next, we let dV denote the volume form \overline{M} . Then from Theorem 4.6 we

Corollary 4.7. Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM). Then

$$\int_{\overline{M}} \overline{g}(\operatorname{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}}\widehat{W})dV$$

$$= \int_{\overline{M}} ((-1)^r (r+2)S_{r+2} + \sum_{i=1}^n \epsilon_i \overline{g}(\overline{R}(Z_i, \widehat{W})T_r Z_i, \widehat{W})dV$$

Setting r = 0 in the above corollary we get

Corollary 4.8. Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold \overline{M} and let \mathcal{F} be a co-dimension three foliation of S(TM) with mean curvatures S_r . Then for r=0 we have

$$\int_{\overline{M}} 2S_2 dV = \int_{\overline{M}} \overline{Ric}(\widehat{W}, \widehat{W}) dV,$$

where
$$\overline{Ric}(\widehat{W},\widehat{W}) = \sum_{i=1}^{n} \epsilon_i \overline{g}(\overline{R}(Z_i,\widehat{W})\widehat{W},Z_i).$$

Notice that the equation in Corollary 4.8 is the lightlike analogue of (3.5) in [1] for co-dimension one foliations on Riemannian manifolds.

Next, we will discuss some consequences of the integral formulas developed so far.

A semi-Riemannian manifold \overline{M} of constant sectional curvature c is called a *semi-Riemannian space form* [3, 4] and is denoted by $\overline{M}(c)$. Then, the curvature tensor \overline{R} of $\overline{M}(c)$ is given by

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = c\{\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y}\}, \quad \forall \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M}). \tag{4.11}$$

Theorem 4.9. Let M be a screen integrable half-lightlike submanifold of a compact semi-Riemannian space form $\overline{M}(c)$ of constant sectional curvature c. Let $\mathcal F$ be a co-dimension three foliation of its screen distribution S(TM). If V is the total volume of \overline{M} , then

$$\int_{\overline{M}} S_r dV = \begin{cases} 0, & r = 2k+1, \\ c^{\frac{r}{2}} {\frac{n}{2} \choose \frac{r}{2}} V, & r = 2k, \end{cases}$$
(4.12)

for positive integers k.

Proof. By setting $\overline{X} = Z_i$, $\overline{Y} = \widehat{W}$ and $Z = T_r Z_i$ in (4.11) we deduce that $\overline{R}(Z_i, \widehat{W}) T_r Z_i = -cg(Z_i, T_r Z_i) \widehat{W}$. Then substituting this equation in Corollary 4.7 we obtain

$$\int_{\overline{M}} \overline{g}(\operatorname{div}^{\nabla'}(T_r), \overline{\nabla}_{\widehat{W}} \widehat{W}) dV = \int_{\overline{M}} ((-1)^r (r+2) S_{r+2} - c \operatorname{tr}(T_r)) dV.$$

Since \overline{M} is of constant sectional curvature c, then Lemma 4.2 implies that $T_r = 0$ for any r and hence the above equation simplifies to

$$(r+2)\int_{\overline{M}} S_{r+2}dV = c(n-r)\int_{\overline{M}} S_r dV.$$
 (4.13)

Since $S_1=-{\rm div}^{\overline{\nabla}}(\widehat{W})$ and that \overline{M} is compact, then $\int_{\overline{M}} S_1 dV=0$. Using this fact together with (4.13), mathematical induction gives $\int_{\overline{M}} S_r dV=0$ for all r=2k+1 (i.e., r odd). For r even we will consider r=2m and n=2l (i.e., both M and \overline{M} are odd dimensional). With these conditions, (4.13) reduces to

$$\int_{\overline{M}} S_{2m+2} dV = c \frac{l-m}{1+m} \int_{\overline{M}} S_{2m} dV.$$
 (4.14)

Now setting $m = 0, 1, \cdots$ and $S_0 = 1$ in (4.14) we obtain

$$\int_{\overline{M}} S_2 dV = clV, \quad \int_{\overline{M}} S_4 dV = c^2 \frac{(l-1)l}{2} V,$$

and more generally

$$\int_{\overline{M}} S_{2k} dV = c^k \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V.$$
 (4.15)

Hence, our assertion follows from 4.15, which completes the proof.

Next, when \overline{M} is Einstein i.e., $\overline{Ric} = \mu \overline{g}$ we have the following.

Theorem 4.10. Let M be a screen integrable half-lightlike submanifold of an Einstein compact semi-Riemannian manifold \overline{M} . Let $\mathcal F$ be a co-dimension three foliation of its screen distribution S(TM) with totally umbilical leaves. If V is the total volume of \overline{M} , then

$$\int_{\overline{M}} S_r dV = \begin{cases} 0, & r = 2k+1, \\ \left(\frac{\mu}{n}\right)^{\frac{n}{2}} \left(\frac{\frac{n}{2}}{\frac{r}{2}}\right) V, & r = 2k, \end{cases}$$
(4.16)

for positive integers k.

Proof. Suppose that $\mathcal{A}_{\widehat{W}} = \frac{1}{n} S_r \mathbb{I}$. Then by direct calculations using the formula for T_r we derive $T_r = (-1)^{r+1} \frac{(n-r)}{n} S_r \mathbb{I}$. Then, under the assumptions of the theorem we obtain $\overline{Ric}(\widehat{W}, \overline{\nabla}_{\widehat{W}}\widehat{W}) = 0$ and $\overline{Ric}(\widehat{W}, \widehat{W}) = \mu$ and hence, Corollary 4.7 reduces to

$$n(r+2)\int_{\overline{M}} S_{r+2}dV = \lambda(n-r)\int_{\overline{M}} S_rdV.$$
 (4.17)

Notice that (4.17) is similar to (4.13) and hence following similar steps as in the previous theorem we get $\int_{\overline{M}} S_r dV = 0$ for r odd and for r even we get

$$\int_{\overline{M}} S_{2k} dV = \left(\frac{\mu}{n}\right)^k \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V,$$

which complete the proof.

5. SCREEN UMBILICAL HALF-LIGHTLIKE SUBMANIFOLDS

In this section we consider totally umbilical half-lightlike submanifolds of semi-Riemannian manifold, with a totally umbilical screen distribution and thus, give a generalized version of Theorem 4.3.7 of [4] and its Corollaries, via Newton transformations of the operator A_N .

A screen distribution S(TM) of a half-lightlike submanifold M of a semi-Riemannian manifold \overline{M} is said to be totally umbilical [4] if on any coordinate neighborhood $\mathcal U$ there exist a function K such that

$$C(X, PY) = Kg(X, PY), \quad \forall X, Y \in \Gamma(TM).$$
 (5.1)

In case K=0, we say that S(TM) is totally geodesic. Furthermore, if S(TM) is totally umbilical then by straightforward calculations we have

$$A_N X = P X, \quad C(E, P X) = 0, \quad \forall X \in \Gamma(T M).$$
 (5.2)

Let $\{E, Z_i\}$, for $i = 1, \dots, n$, be a quasi-orthonormal frame field of TM which diagonalizes A_N . Let l_0, l_1, \dots, l_n be the respective eigenvalues (or principal curvatures). Then as before, the r-th mean curvature S_r is given by

$$S_r = \sigma_r(l_0, \cdots, l_n)$$
 and $S_0 = 1$.

The characteristic polynomial of A_N is given by

$$\det(A_N - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_r t^{n-\alpha},$$

where \mathbb{I} is the identity in $\Gamma(TM)$. The normalized r-th mean curvature H_r of M is defined by $\binom{n}{r}H_r=S_r$ and $H_0=1$. The Newton transformations $T_r:\Gamma(TM)\to\Gamma(TM)$ of A_N are given by the inductive formula

$$T_0 = \mathbb{I}, \quad T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 1 \le r \le n.$$
 (5.3)

By Cayley-Hamiliton theorem, we have $T_{n+1} = 0$. Also, T_r satisfies the following properties.

$$tr(T_r) = (-1)^r (n+1-r)S_r, (5.4)$$

$$tr(A_N \circ T_r) = (-1)^r (r+1) S_{r+1}, \tag{5.5}$$

$$\operatorname{tr}(A_N^2 \circ T_r) = (-1)^{r+1} (-S_1 S_{r+1} + (r+2) S_{r+2}), \tag{5.6}$$

$$\operatorname{tr}(T_r \circ \nabla_X A_N) = (-1)^r X(S_{r+1}).$$
 (5.7)

for all $X \in \Gamma(TM)$.

Proposition 5.1. Let (M,g) be a totally umbilical half-lightlike submanifold of a semi-Riemannian manifold \overline{M} of constant sectional curvature c. Then

$$\begin{split} g(\mathrm{div}^{\nabla}(T_{r}),X) &= (-1)^{r-1}\lambda(X)E(S_{r}) - \tau(X)\mathrm{tr}(A_{N}\circ T_{r-1}) \\ &- c\lambda(X)\mathrm{tr}(T_{r-1}) + g(\mathrm{div}^{\nabla}(T_{r-1}),A_{N}X) + g((\nabla_{E}A_{N})T_{r-1}E,X) \\ &+ \sum_{i=1}^{n} \epsilon_{i}\{-\lambda(X)B(Z_{i},A_{N}(T_{r-1}Z_{i})) \\ &+ \varepsilon\tau(Z_{i})C(X,T_{r-1}Z_{i})\{\rho(X)D(Z_{i},T_{r-1}Z_{i}) - \rho(Z_{i})D(X,T_{r-1}Z_{i})\}\}, \end{split}$$
 for any $X \in \Gamma(TM)$.

Proof. From the recurrence relation (5.3), we derive

$$g(\operatorname{div}^{\nabla}(T_r), X) = (-1)^r PX(S_r) + g((\nabla_E A_N) T_{r-1} E, X)$$

$$+ g(\operatorname{div}^{\nabla}(T_{r-1}), A_N X) + \sum_{i=1}^n \epsilon_i g((\nabla_{Z_i} A_N) T_{r-1} Z_i, X).$$
(5.8)

for any $X \in \Gamma(TM)$. But

$$g((\nabla_{Z_{i}}A_{N})T_{r-1}Z_{i}, X) = g(T_{r-1}Z_{i}, (\nabla_{Z_{i}}A_{N})X) + g(\nabla_{Z_{i}}A_{N}(T_{r-1}Z_{i}), X)$$

$$-g(\nabla_{Z_{i}}(A_{N}X), T_{r-1}Z_{i}) + g(A_{N}(\nabla_{Z_{i}}X), T_{r-1}Z_{i})$$

$$-g(A_{N}(\nabla_{Z_{i}}T_{r-1}Z_{i}), X),$$
(5.9)

for all
$$X \in \Gamma(TM)$$
.

Then applying (2.9) to (5.9) while considering the fact that A_N is screen-valued, we get

$$g((\nabla_{Z_i} A_N) T_{r-1} Z_i, X) = g(T_{r-1} Z_i, (\nabla_{Z_i} A_N) X) - \lambda(X) B(Z_i, A_N(T_{r-1} Z_i))$$
(5.10)

Furthermore, using (2.15) and (4.11), the first term on the right hand side of (5.10) reduces to

$$g(T_{r-1}Z_i,(\nabla_{Z_i}A_N)X) = -c\lambda(X)g(Z_i,T_{r-1}Z_i) + g((\nabla_X A_N)Z_i,T_{r-1}Z_i) - \tau(X)C(Z_i,T_{r-1}Z_i) + \varepsilon\tau(Z_i)C(X,T_{r-1}Z_i)\{\rho(X)D(Z_i,T_{r-1}Z_i) - \rho(X)D(X,T_{r-1}Z_i)\},$$
(5.11)

for any $X \in \Gamma(TM)$. Finally, replacing (5.11) in (5.10) and then put the resulting equation in (5.8) we get the desired result.

Next, from Proposition 5.1 we have the following.

Theorem 5.2. Let (M,g) be a half-lightlike submanifold of a semi-Riemannian manifold $\overline{M}(c)$ of constant curvature c, with a proper totally umbilical screen distribution S(TM). If M is also totally umbilical, then the r-th mean curvature S_r , for $r=0,1,\cdots,n$, with respect to A_N are solution of the following differential equation

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_r = \mathcal{H}_1(r+1)S_{r+1}.$$

Proof. Replacing X with E in the Proposition 5.1 and then using (2.16) and (5.2) we obtain, for all $r = 0, 1, \dots, n$,

$$E(S_{r+1}) - (-1)^r \tau(E) \operatorname{tr}(A_N \circ T_r) - c(-1)^r \operatorname{tr}(T_r) = (-1)^r \mathcal{H}_1 \operatorname{tr}(A_N \circ T_r),$$

from which the result follows by applying (5.4) and (5.5).

Corollary 5.3. Under the hypothesis of Theorem 5.2, the induced connection ∇ on M is a metric connection, if and only if, the r-th mean curvature S_r with respect to A_N are solution of the following equation

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_r = 0.$$

Also the following holds.

Corollary 5.4. Under the hypothesis of Theorem 5.2, $\overline{M}(c)$ is a semi-Euclidean space, if and only if, the r-th mean curvature S_r with respect to A_N are solution of the following equation

$$E(S_{r+1}) - \tau(E)(r+1)S_{r+1} = \mathcal{H}_1(r+1)S_{r+1}.$$

Notice that Theorem 5.2 and Corollary 5.3 are generalizations of Theorem 4.3.7 and Corollary 4.3.8, respectively, given in [4].

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* SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE UNIVERSITY OF KWAZULU-NATAL PRIVATE BAG X01, SCOTTSVILLE 3209 SOUTH AFRICA

E-mail address: massfort@yahoo.fr, Massamba@ukzn.ac.za

** SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE UNIVERSITY OF KWAZULU-NATAL PRIVATE BAG X01, SCOTTSVILLE 3209 SOUTH AFRICA E-mail address: ssekajja.samuel.buwaga@aims-senegal.org