Design of coherent quantum observers for linear quantum systems

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Abstract. Quantum versions of control problems are often more difficult than their classical counterparts because of the additional constraints imposed by quantum dynamics. For example, the quantum LQG and quantum H^∞ optimal control problems remain open. To make further progress, new, systematic and tractable methods need to be developed. This paper gives three algorithms for designing coherent quantum observers, i.e., quantum systems that are connected to a quantum plant and their outputs provide information about the internal state of the plant. Importantly, coherent quantum observers avoid measurements of the plant outputs. We compare our coherent quantum observers with a classical (measurement-based) observer by way of an example involving an optical cavity with thermal and vacuum noises as inputs.

Contents

1. Introduction

Feedback control of quantum systems can be broadly categorized into two schemes: 'classical' (or measurement-based) and 'coherent' control. Classical control involves making measurements on the plant (for example homodyne or heterodyne detection in the case of optical systems) and then generating feedback control signals based on these measurements. For a treatment of this topic see for example [1]. In this paper, we are concerned with coherent control which uses controllers that are themselves quantum systems, coupled directly to the plant. One advantage of coherent control schemes is that they avoid the loss of quantum information that occurs during measurements. Coherent quantum control is an active research area [2–10] and recent results [10] indicate regimes in which coherent controllers perform better than the optimal classical controllers. Coherent quantum observers represent an important building block in developing systematic and tractable approaches to coherent control problems.

Despite recent progress, quantum versions of the H^∞ [2,4] and LQG [3,5] optimal control problems remain open. These problems are difficult because of the constraints on the class of allowable controllers: quantum systems must evolve unitarily and preserve commutation relations [11,12]. These constraints lead to the notion of *physical*

realizability [2,13–15]. Current approaches [2–5] are only tractable for relatively simple examples.

Feedback control schemes require that the controller has access to information about the internal state of the plant. For example, in the classical LQG problem, the Kalman filter [16,17] is used to obtain an optimal estimate of the plant's internal state from a series of noisy measurements. In coherent control schemes, the controller does not have access to measurements, rather it must make use of information from its direct coupling with the plant. A coherent quantum observer is a quantum system that is designed
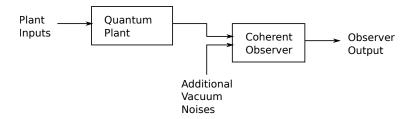

Figure 1. Quantum plant and coherent quantum observer.

such that when directly coupled with a plant, its outputs provide information about the internal state of the plant. The coherent quantum observer can in turn be directly coupled to control inputs of the plant, possibly via intermediate quantum systems, to achieve desired control outcomes whilst avoiding measurement.

Different approaches to design coherent quantum observers are discussed in [2,18,19]. However, as with the quantum control problems mentioned above, the main difficulties come from the fact that a coherent quantum observer should satisfy *physical realizability* constraints.

In this paper, we extend previous *physical realizability* results [13–15] to obtain algorithms for designing coherent quantum observers. The previous results we use demonstrate how strictly proper, linear time invariant (LTI) systems can be made physically realizable by allowing additional quantum noises. We apply these results to construct coherent quantum observers using a Kalman filter which is modified by adding quantum noises as prescribed in [13–15]. This is different to the approach taken in [2] because we make use of the stronger results from [13–15] so that only necessary quantum noises are added. We also incorporate novel refinements not considered in [2,18,19]. Our approach is tractable and can lead to better performance, however the coherent observers obtained are generally suboptimal.

We now outline three algorithms that we propose. The first algorithm is based on the Kalman filter which is modified by allowing additional vacuum noise inputs such that the resulting system is physically realizable. The second algorithm attempts to improve on the first by incorporating a free parameter over which we optimize. The purpose of this parameter is to compensate for the effect of the additional quantum vacuum noises. The third algorithm attempts to find a state transformation of the Kalman filter such that it can be made physically realizable with the minimal number

of additional quantum noises. Despite being suboptimal estimations, these algorithms provide a systematic and tractable approach to coherent quantum observer design. Like the celebrated Kalman filter, it is envisaged that the coherent quantum observer will play an important role in the solution to coherent control problems.

The main contribution of this paper is to give two additional algorithms for coherent quantum observer design which incorporate novel refinements not considered in [2, 18, 19]. We compare the performance of our three different algorithms using a metric corresponding to the steady-state expected value of the symmetrized error covariance matrix and show that the novel refinements which we propose can lead to better perfomance than implementing a Kalman filter as a quantum system as in our first algorithm.

The remainder of this paper is structured as follows. In Section 2, we introduce models to describe the quantum systems given by linear quantum stochastic differential equations. In Section 3, we formally define *Physical Realizability* of such linear quantum systems. We are then able to give our problem formulation in Section 4 where we also present relevant existing results which we will utilize. Section 5 contains the main contribution of the paper, we present three algorithms for designing coherent quantum observers for linear quantum systems. In Section 6, we present a measurement-based (classical) observer. The three algorithms are then compared with each other and the measurement-based alternative by way of an example in Section 7. Finally, we give our conclusion in Section 8.

2. Linear quantum system models

Consider a quantum system defined on a Hilbert space $\mathcal{H}$, and its environment modeled by the bosonic or symmetric Fock space over the Hilbert space $L^2(\mathbb{R}_+)$ of square integrable wave functions on the real positive line, corresponding to a single boson field mode. The evolution of the composite system, which is a closed system, can be described by a unitary operator U acting on the tensor product $\mathcal{H} \otimes \mathcal{F}$ that obeys the following quantum stochastic differential equations (QSDEs) as described by Hudson and Parthasarathy [20]

$$dU(t) = \left(db^\dagger L - L^\dagger db - \frac{1}{2} L^\dagger L \, dt - iH \, dt \right) U(t), \quad U(0) = I.$$

Here, H corresponds to the Hamiltonian of the system, L describes the coupling between the system and the environment, and $X^\dagger$ denotes the adjoint of an operator X. The operators b and $b^\dagger$ are the annihilation and creation processes defined on $\mathcal{F}$.

In the Heisenberg picture, the evolution of a self-adjoint operator x is described by

$$x(t) = U(t)^\dagger (x(0) \otimes I) U(t). \tag{1}$$

Using the input-output formalism of [21], we also have

$$y(t) = U(t)^\dagger (I \otimes w(t)) U(t) \tag{2}$$

where $y(t)$ is the output of the system and $w(t)$ is its input. Here, the self-adjoint entries of the vector $w(t)$ which act on the Boson Fock space $\mathcal{F}$ correspond to the quantum noises driving the system [20]. The noise increments $dw(t)$ in quadrature form are given by

$$dw = \begin{bmatrix} db(t) + db(t)^\dagger \\ i(db(t)^\dagger - db(t)) \end{bmatrix}. \tag{3}$$

Generally speaking, the QSDEs for a given quantum system can be obtained by applying quantum Itō rules to x and y which satisfy dynamics (1) and (2) respectively, and using the following quantum Itō multiplication table [20, 22]:

$$db\, db = 0, \quad db\, db^\dagger = (1 + k_n)dt, \quad db^\dagger\, db = k_n dt, \quad \text{and} \quad db^\dagger\, db^\dagger = 0. \tag{4}$$

Also, by using $(dt)^2 = 0$, and $dtdb = 0 = dtdb^\dagger$. Here, k_n is a parameter describing the intensity of the thermal noise input. The special case where $k_n = 0$ corresponds to an input being a vacuum noise.

In the case of open quantum harmonic oscillators, which we consider in this paper, the Hamiltonian H is quadratic and the coupling operator L is linear. This leads to linear QSDEs of the form

$$\begin{aligned} dx(t) &= Ax(t)\, dt + B\, dw(t), \\ dy(t) &= Cx(t)\, dt + D\, dw(t), \end{aligned} \tag{5}$$

where A, B, C, and D are real matrices which are supposed to be in $\mathbb{R}^{n_x \times n_x}$, $\mathbb{R}^{n_x \times n_w}$, $\mathbb{R}^{n_y \times n_x}$ and $\mathbb{R}^{n_y \times n_w}$ respectively, and n_x, n_w, and n_y are positive integers. However, not all QSDEs of the form (5) correspond to open quantum harmonic oscillators. When they do, they are said to be physically realizable. This is explained further in the following section where we also give explicit expressions for A, B, C, and D.

The state variables $x(t)$ of a physical realizable system of the form (5) should satisfy the following equal-time commutation relations described by the real antisymmetric matrix Θ

$$[x_i(t), x_j(t)] = x_i(t)x_j(t) - x_j(t)x_i(t) = 2i\Theta_{ij}, \quad \forall t \geq 0 \tag{6}$$

where Θ can be of the two following forms:

(i) Canonical, if $\Theta = \mathrm{diag}(J, \cdots, J)$, which is a block diagonal matrix with each diagonal block equal to $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, or

(ii) Degenerate canonical, if $\Theta = \mathrm{diag}(0_{n' \times n'}, \cdots, J)$, with $0 < n' \leq n_x$.

Also, we have the following Itō table for dw:

$$dw(t)\, dw(t)^T = F_w\, dt.$$

Here F_w is a non-negative hermitian matrix and $F_w = S_w + iT_w$, where S_w and T_w are the real and imaginary parts of F_w. The commutation relations for $dw(t)$ are determined

by T_w and the intensity of the noise processes is described by S_w. We will consider the case where the inputs are thermal noises with S_w, a block diagonal matrix with each diagonal block equal to

$$\begin{bmatrix} 1 + 2k_n & 0 \\ 0 & 1 + 2k_n \end{bmatrix}.$$

The matrix S_w was derived by using the quantum Itō multiplication table given in (4).

We set the following conventions:

(i) the dimensions n_x, n_y and n_w are even; and

(ii) $n_y \leq n_w$.

Also, without loss of generality, we restrict our attention to quantum plants (5) with canonical commutation relations. This just fixes the choice of basis for $x(t)$. For systems with degenerate canonical Θ, there exists an equivalent description (5) with canonical Θ which can be obtained by applying the appropriate state transformation (see [2] for more details).

3. Physical realizability

Not all QSDEs of the form (5) represent the dynamics of physically meaningful open quantum systems. As in [2,15], QSDEs that describe open quantum harmonic oscillators are said to be *physically realizable*. Physical Realizability is equivalent to the condition that Equation (5) can be derived from a unitary adapted quantum stochastic evolution as described in (1) and (2). We restrict our attention to quantum plants which are physically realizable and as such the QSDEs (5) are assumed to be physically realizable.

In some of the literature (e.g. [2]), the term *physical realizability* is used to describe physically meaningful systems with both classical and quantum degrees of freedom. In the following, we give the definition of *physical realizability* for QSDEs representing fully quantum systems. For a more general definition of *physical realizability* see [2, Definition 3.1].

Definition 1 ([2]) The system described by (5) is *physically realizable* if it has canonical commutation relations and it represents an open quantum harmonic oscillator. The system (5) describes an open quantum harmonic oscillator if there exists a quadratic Hamiltonian $H = \frac{1}{2}x(0)^T Rx(0)$, with a real, symmetric, $n_x \times n_x$ matrix R, and a coupling operator $L = \Lambda x(0)$, with a complex-valued $\frac{1}{2}n_w \times n_x$ coupling matrix Λ such that

$$\begin{aligned} x_k(t) &= U(t)^\dagger (x_k(0) \otimes 1)U(t), \quad k = 1, \cdots, n_x \\ y_l(t) &= U(t)^\dagger (1 \otimes w_l(t))U(t), \quad l = 1, \cdots, n_y \end{aligned} \tag{7}$$

where $\{U(t), \quad t \geq 0\}$ is an adapted process of unitary operators satisfying the following QSDE [20]

$$dU(t) = \left(-iH\,dt - \frac{1}{2}L^\dagger L\,dt + [-L^\dagger \ L^T]\Gamma dw(t) \right) U(t), \quad U(0) = I.$$

In this case, the matrices A, B, C and D are given by

$$A = 2\Theta\left(R + \mathfrak{Im}\left(\Lambda^\dagger\Lambda\right)\right),$$

$$B = 2i\Theta\left[\begin{array}{cc} -\Lambda^\dagger & \Lambda^T \end{array}\right]\Gamma,$$

$$C = P^T\left[\begin{array}{cc} \Sigma & 0 \\ 0 & \Sigma \end{array}\right]\left[\begin{array}{c} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{array}\right],$$

$$D = \left[\begin{array}{cc} I_{n_y \times n_y} & 0_{n_y \times (n_w - n_y)} \end{array}\right].$$

Here, Γ is a $n_w \times n_w$ matrix and

$$\Gamma = P\text{diag}(M),$$

$$M = \frac{1}{2}\left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array}\right],$$

$$\Sigma = \left[\begin{array}{cc} I_{\frac{1}{2}n_y \times \frac{1}{2}n_y} & 0_{\frac{1}{2}n_y \times \frac{1}{2}(n_w - n_y)} \end{array}\right].$$

P is the appropriately dimensioned square permutation matrix such that

$$P\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{2m} \end{array}\right] = \left[\begin{array}{cccccccc} a_1 & a_3 & \cdots & a_{2m-1} & a_2 & a_4 & \cdots & a_{2m} \end{array}\right]$$

and $\text{diag}(M)$ is the appropriately dimensioned square block diagonal matrix with the matrix M occurring along the diagonal. (Note: dimensions of P and $\text{diag}(M)$ can always be determined from the context in which they appear.) Also, $\mathfrak{Im}(.)$ denotes the imaginary part of a matrix, $X^\#$ denotes the complex conjugate of a matrix X, and $X^\dagger$ denotes the complex conjugate transpose of a matrix X.

Note that the equal-time canonical commutation relations (6) can be derived from the above definition.

For clarity's sake, from this point onward, we will often omit time dependence in our notation. We will use x in place of $x(t)$, etc. However, the reader should bear in mind that in general all operators (x, y, dw, etc.) are time dependent. Matrices (A, B, etc.) are time invariant.

4. Problem formulation

In this section, we first define a coherent quantum observer and introduce a class of coherent quantum observers that we consider in this paper (see also [18, 19]). We then present the design approaches that we will make use of them in the following section.

4.1. Coherent quantum observers

Denote the initial state of the coherent quantum observer by $\xi(0)$ which satisfies the canonical commutation relation, i.e.,

$$\xi(0)\xi(0)^T - (\xi(0)\xi(0)^T))^T = 2i\Theta.$$

Also, take the notation $\langle X \rangle_\rho = \mathrm{tr}(\rho X)$ corresponding to the quantum expectation of an observable X over the density matrix ρ for the initial joint plant and observer states (see e.g., [2, 23]). Then, a coherent quantum observer is a quantum system with an internal state ξ that is designed to estimate the internal variable of the plant's dynamics (5) with the following properties:

(i) it is designed such that the tracking error estimation $\langle x - \xi \rangle_\rho$ of the plant dynamics (5) exponentially converges to zero in the sense of expected values;

(ii) the following limit exists,

$$\bar{J} = \lim_{t \to \infty} \frac{1}{2} \left[\left\langle (x - \xi)(x - \xi)^T \right\rangle_\rho + \left\langle \left((x - \xi)(x - \xi)^T \right)^T \right\rangle_\rho \right]. \qquad (8)$$

Here $\bar{J}$ is a performance metric which corresponds to the steady-state quantum expectation of the symmetrized error covariance matrix;

(iii) it is physically realizable.

In this paper, we consider the class of coherent quantum observers described by QSDEs of the following form,

$$
\begin{aligned}
d\xi &= \hat{A}\,\xi\,dt + \hat{B}\,dy + B_{v_1}\,dv_1 + B_{v_2}\,dv_2, \\
d\eta &= \hat{C}\xi\,dt + \ dv_1,
\end{aligned}
\qquad (9)
$$

which are a special case of the QSDEs (5). Here $\xi, d\eta, dv_1$ and dv_2 are column vectors with dimensions n_ξ, n_η, n_{v_1} and n_{v_2} respectively, where n_ξ, n_η, n_{v_1} and n_{v_2} are even. Also, we assume $n_\xi = n_x$ and $n_{v_1} = n_\eta$. The matrices $\hat{A}$, $\hat{B}$, B_{v_1}, B_{v_2} and $\hat{C}$ are real. The vectors $\xi, d\eta, dv_1,$ and dv_2 each consist of entries which are self-adjoint operators acting on the tensor product Hilbert Space $\mathcal{H} \otimes \mathcal{F} \otimes \mathcal{H}_2 \otimes \mathcal{F}_2$. The complex Hilbert space $\mathcal{H}_2$ is the initial space of the coherent quantum observer $\xi(0)$ and $\mathcal{F}_2$ is the Boson Fock space which corresponds to the fields other than the output of the plant, which also interact with coherent quantum observer. The vectors dy, dv_1 and dv_2 represent the input fields which interact with the coherent quantum observer. The vector dy corresponds to the output of the plant (5). The entries of dv_1 and dv_2 correspond to the quadratures of the annihilation and creation processes which act on the boson Fock Space $\mathcal{F}_2$ which are supposed initially in the vacuum states. As such, dv_1 and dv_2 correspond to quantum vacuum noises. For notational convenience, we separate the vacuum noises into the two vectors dv_1 and dv_2 such that $n_{v_1} = n_\eta$ and $n_{v_2} \geq 0$. These quantum vacuum noises satisfy the Itō relations

$$dv_k\,dv_k{}^T = F_{v_k}\,dt, \quad \text{for} \quad k = 1, 2,$$

where F_{v_k} is a block diagonal matrix with each block equal to

$$
\begin{bmatrix}
1 & i \\
-i & 1
\end{bmatrix}.
$$

As was the case for the plant, without loss of generality, we restrict our attention to coherent quantum observers with canonical commutation relations.

The coherent quantum observer may incorporate additional inputs (other than those connected to the plant outputs) driven by quantum vacuum noises. These may be required to ensure *physical realizability*. Note that the convergence of the coherent quantum observer is independent of any additional quantum noises in the observer. However, these quantum noises can have an important effect in the value of the performance defined in (8).

4.2. Approaches to design coherent quantum observers

Finding an optimal estimation of the plant's state is difficult because of the requirement for *physical realizability* and the constraints that this imposes. We restrict our attention to design of coherent quantum observers of the form (9), which provide suboptimal solutions to such an estimation.

In the following, we will make use of the following results to make the coherent quantum observers proposed in Equation (9) physically realizable.

Theorem 1 *(See [15, Theorem 3])* Consider an LTI system of the form (9) where $\hat{A}, \hat{B}, \hat{C}$ are given. Then, there exists B_{v_1} and B_{v_2} such that the system is physically realizable with canonical commutation matrix Θ, and with $n_{v_2} = r$, where r is the rank of the matrix $\left(\Theta\hat{B}\Theta\hat{B}^T\Theta - \Theta\hat{A} - \hat{A}^T\Theta - \hat{C}^T\Theta\hat{C} \right)$. Conversely, suppose that there exists B_{v_1} and B_{v_2} such that the system (9) is physically realizable with canonical commutation matrix Θ. Then $n_{v_2} \geq r$, where r is the rank of the matrix $\left(\Theta\hat{B}\Theta\hat{B}^T\Theta - \Theta\hat{A} - \hat{A}^T\Theta - \hat{C}^T\Theta\hat{C} \right)$. This means that it is not possible to choose B_{v_1} and B_{v_2} such that the system is physically realizable and the dimension of dv_2 is less than r.

In [15], during the proof of this theorem, we give a method for constructing B_{v_1} and B_{v_2}. We described such a method in Appendix A. Again, this method results in the smallest possible dimension for dv_2 (for a given $\hat{A}, \hat{B}, \hat{C}$) such that (9) is physically realizable.

Below, we give another theorem that we will need in the following.

Theorem 2 *(See [13, Theorem 2])* Consider an LTI system of the form (9), where $\hat{A}, \hat{B}, \hat{C}$ are given and the system commutation matrix Θ is canonical. Suppose that the Riccati equation

$$X\hat{B}\Theta\hat{B}^T X - \hat{A}^T X - X\hat{A} - \hat{C}^T\Theta\hat{C} = 0 \tag{10}$$

has a solution X which is skew-symmetric and suppose that there exists a real non-singular matrix T such that $X = T^T\Theta T$. Then, there exists a system described by $\left\{ \tilde{A}, \tilde{B}, \tilde{C} \right\}$ with the same transfer function as the system $\left\{ \hat{A}, \hat{B}, \hat{C} \right\}$ which can be physically realized without the $B_{v_2} \, dv_2$ term (i.e. with $n_{v_2} = 0$) and where

$$X \quad = T^T\Theta T,$$

$$\tilde{A} = T\hat{A}T^{-1}, \quad \tilde{B} = T\hat{B}, \quad \tilde{C} = \hat{C}T^{-1}, \quad \text{and}$$
$$\tilde{B}_{v_1} = \Theta\tilde{C}^T \text{diag}(J).$$

In [14], sufficient conditions are given for the existence of a suitable solution to (10). The accompanying proof leads to a numerical process for obtaining the solution X that we include in Appendix B.

In the following section, we apply the classical Kalman filtering results. This means that we chose $\hat{A} = A - KC$ and $\hat{B} = K$ (K corresponds to Kalman gain) such that $A - KC$ be a Hurwitz matrix which ensures that the coherent quantum observer (9) tracks exponentially the plant dynamics (5) in the sense of expected values. Moreover, the fact that $A - KC$ is Hurwitz, guaranties that the limit defined in (8) exists. Also, we choose $\hat{C} = I$.

5. Algorithms to design coherent quantum observers

In this section, we give three algorithms to design coherent quantum observers. The motivation behind these algorithms is to treat the quantum plants classically to obtain Kalman filters and then obtain physically realizable quantum systems by allowing minimal additional quantum vacuum noises. Also, thanks to the performance metric defined in (8), we are able to compare the error of convergence for these different algorithms.

Algorithm 1

This is the simplest algorithm that we present. The quantum noises dw driving the plant (5) are treated as classical Wiener processes with intensity $S_w = \mathfrak{Re}\left[F_w\right]$ where $\mathfrak{Re}[.]$ denotes the real part of a matrix. We first obtain a standard Kalman filter as follows (for details of this, see for example [16])

$$d\hat{x} = (A - KC)\hat{x}\,dt + K\,dy,$$
$$d\hat{y} = \hat{x}\,dt, \tag{11}$$

where $K = (QC^T + V_{12})V_2^{-1}$. Here Q is the solution to the following algebraic Riccati equation,

$$(A - V_{12}V_2^{-1}C)Q + Q(A - V_{12}V_2^{-1}C)^T - QC^TV_2^{-1}CQ + V_1 - V_{12}V_2^{-1}V_{12}^T = 0, \tag{12}$$

where Q is the steady state of the error covariance matrix given as follows

$$Q = \lim_{t\to\infty}\left\langle (x - \hat{x})(x - \hat{x})^T \right\rangle_\rho$$

and

$$V = \begin{bmatrix} V_1 & V_{12} \\ V_{12}{}^T & V_2 \end{bmatrix}$$

describes the intensity of the joint process $\begin{bmatrix} B\,dw \\ D\,dw \end{bmatrix}$. This means that,

$$\mathbb{E}\left[\begin{bmatrix} B \\ D \end{bmatrix} dw\,dw^T \begin{bmatrix} B \\ D \end{bmatrix}^T\right] = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix} dt.$$

In particular,

$$\begin{aligned}
V_1 &= BS_wB^T, \\
V_2 &= DS_wD^T, \quad \text{and} \\
V_{12} &= BS_wD^T.
\end{aligned}$$

Note that the Kalman filter (11) is not physically realizable.

Now consider the observer (9) and replace $\hat{A}$, $\hat{B}$, and $\hat{C}$ by the following values

$$\begin{aligned}
\hat{A} &= A - KC, \\
\hat{B} &= K, \text{and} \\
\hat{C} &= I.
\end{aligned}$$

We find

$$\begin{aligned}
d\xi &= (A - KC)\,\xi\,dt + K\,dy + B_{v_1}\,dv_1 + B_{v_2}\,dv_2, \\
d\eta &= \xi\,dt + dv_1.
\end{aligned} \tag{13}$$

By Theorem 1, there exists B_{v_1} and B_{v_2} such that the system (13) is physically realizable. This system is a coherent quantum observer. Furthermore, within the class of quantum systems described by the QSDEs (9), this coherent quantum observer has the minimum number of additional quantum noises $(n_{v_1} + n_{v_2})$ for our choice of $\left\{\hat{A}, \hat{B}, \hat{C}\right\}$. Details for constructing B_{v_1} and B_{v_2} are included in Appendix A.

To see that (13) is a coherent quantum observer for the plant (5), it only remains to show that $\langle x - \xi \rangle_\rho$ converges to zero exponentially.

The combined plant and observer satisfy the following dynamics

$$\begin{aligned}
\begin{bmatrix} dx \\ d\xi \end{bmatrix} &= \mathcal{A}\begin{bmatrix} x \\ \xi \end{bmatrix} dt + \mathcal{B}\begin{bmatrix} dw \\ dv_1 \\ dv_2 \end{bmatrix}; \\
d\eta &= \xi\,dt + dv_1; \\
\mathcal{A} &= \begin{bmatrix} A & 0 \\ KC & A - KC \end{bmatrix}; \\
\mathcal{B} &= \begin{bmatrix} B & 0 & 0 \\ KD & B_{v_1} & B_{v_2} \end{bmatrix}.
\end{aligned} \tag{14}$$

From (14), making the necessary substitutions, $x - \xi$ satisfies

$$\begin{aligned}
d(x - \xi) &= (A - KC)\,(x - \xi)\,dt \\
&\quad + (B - KD)\,dw - B_{v_1}\,dv_1 - B_{v_2}\,dv_2.
\end{aligned}$$

Hence,

$$d\langle x - \xi \rangle_\rho = (A - KC)\langle x - \xi \rangle_\rho \, dt.$$

As a result, $\langle x - \xi \rangle_\rho$ converges exponentially to zero if $(A - KC)$ is Hurwitz. The fact that $(A - KC)$ is Hurwitz, follows from the properties of the classical Kalman filter which was used to choose K.

As $A - KC$ is Hurwitz, the limit in (8) converges, and $\bar{J}$ is the unique symmetric positive definite solution of the following Lyapunov equation

$$\begin{aligned}
0 &= \mathcal{A}_e \bar{J} + \bar{J}\mathcal{A}_e^T + \mathcal{B}_e S_{w,v}\mathcal{B}_e^T, \\
\mathcal{A}_e &= (A - KC), \\
\mathcal{B}_e &= \left[\begin{array}{ccc} (B - KD) & -B_{v_1} & -B_{v_2} \end{array} \right],
\end{aligned} \tag{15}$$

where

$$\left[\begin{array}{c} dw \\ dv_1 \\ dv_2 \end{array} \right] \left[\begin{array}{ccc} dw^T & dv_1^T & dv_2^T \end{array} \right] = F_{w,v}\, dt \qquad \text{and} \qquad S_{w,v} = \mathfrak{Re}\left[F_{w,v} \right].$$

Finally, the system (13) so obtained is a coherent quantum observer.

Algorithm 2

This algorithm is a refinement of the first, introducing a free parameter ρ, over which we optimize. The purpose of this parameter is to take into account the impact of the noise terms $B_{v_1}\, dv_1(t)$ and $B_{v_2}\, dv_2(t)$ when designing the Kalman filter. These noise terms are equivalent to additional measurement noise in the plant (5), however they cannot be calculated until after the Kalman filter is designed and hence are not available to the design process.

Compared to Algorithm 1, before calculating the Kalman filter, we first introduce an additional term into the plant model (5) to obtain the *modified plant*

$$\begin{aligned}
dx &= Ax\, dt + B\, dw, \\
dy &= Cx\, dt + D\, dw + \rho\, d\tilde{w}.
\end{aligned} \tag{16}$$

Here, $d\tilde{w}$ is a vacuum noise source with Itō product

$$d\tilde{w}\, d\tilde{w}^T = F_{\tilde{w}}\, dt,$$

where $F_{\tilde{w}}$ is a block diagonal matrix with each block equal to

$$\left[\begin{array}{cc} 1 & i \\ -i & 1 \end{array} \right].$$

Take $S_{\tilde{w}}$ as the real part of $F_{\tilde{w}}$. The noise sources dw and $d\tilde{w}$ are independent.

In effect, we inflate the value of the plant measurement noise when designing the Kalman filter to compensate for the unknown noise terms $B_{v_1}\, dv_1(t)$ and $B_{v_2}\, dv_2(t)$.

We now state Algorithm 2. The following procedure is repeated for different values of $\rho > 0$.

- Obtain the Kalman filter (11) for the modified plant (16) with K given by

$$K = (QC^T + V_{12})V_2^{-1},$$

  where Q is the solution to the Riccati equation (12) with

$$\begin{aligned} V_1 &= BS_wB^T, \\ V_2 &= DS_wD^T + \rho^2 I_{2\times 2}, \quad \text{and} \\ V_{12} &= BS_wD^T. \end{aligned}$$

- Obtain B_{v_1} and B_{v_2} as in Algorithm 1, such that the system

$$\begin{aligned} \mathrm{d}\xi &= (A - KC)\,\xi\,\mathrm{d}t + K\,\mathrm{d}y + B_{v_1}\,\mathrm{d}v_1 + B_{v_2}\,\mathrm{d}v_2, \\ \mathrm{d}\eta &= \xi\,\mathrm{d}t + \mathrm{d}v_1 \end{aligned} \tag{17}$$

  is physically realizable. This system is a coherent quantum observer.

- Calculate the performance metric $\bar{J}$ as in Algorithm 1 by solving the Lyapunov Equation (15). ($\bar{J}$ is calculated for the actual plant (5) and not for the modified plant (16)).

Finally, we choose the coherent quantum observer (17) which gives the least value of $\bar{J}$.

To see that each iteration results in a coherent quantum observer, consider the following: from the properties of the classical Kalman filter, $(A - KC)$ remains Hurwitz for $\rho \geq 0$.

Algorithm 3

Our final algorithm attempts to improve performance by reducing the number of additional quantum noises incorporated in the coherent quantum observer. Under certain sufficient conditions, it is possible to obtain a coherent quantum observer from a state transformation of the Kalman filter obtained in Algorithm 1. This coherent observer incorporates the minimum number of additional noises possible for a system of the form (9): $n_{v_2} = 0$.

Algorithm 3 proceeds as follows.

- Obtain the Kalman filter (11) as in Algorithm 1.
- Attempt to find a transformation T :

$$\tilde{\xi} = T\hat{x}, \quad \tilde{A} = T\hat{A}T^{-1}, \quad \tilde{B} = T\hat{B}, \quad \tilde{C} = \hat{C}T^{-1}$$

  such that the system

$$\begin{aligned} \mathrm{d}\tilde{\xi} &= \tilde{A}\tilde{\xi}\,\mathrm{d}t + \tilde{B}\,\mathrm{d}y + \tilde{B}_{v_1}\,\mathrm{d}v_1, \\ \mathrm{d}\eta &= \tilde{C}\tilde{\xi}\,\mathrm{d}t + \mathrm{d}v_1, \end{aligned} \tag{18}$$

  is physically realizable for some $\tilde{B}_{v_1}$. From Theorem 2, if the Riccati equation

$$X\hat{B}\Theta\hat{B}^T X - \hat{A}^T X - X\hat{A} - \hat{C}^T\Theta\hat{C} = 0$$

has a non-singular, real, skew-symmetric solution X, then such a T exists. Sufficient conditions and a construction for T are included in Appendix B. If the sufficient conditions for T are not satisfied, we revert to Algorithm 1.

Now take $\xi = \tilde{C}\tilde{\xi}$. Then, we have

$$d\xi = \hat{A}\xi\,dt + \hat{B}\,dy + \tilde{C}\tilde{B}_{v_1}\,dv_1$$
$$d\eta = \xi\,dt + dv_1,$$

which is equivalent to the following

$$d\xi = (A - KC)\xi\,dt + K\,dy + T^{-1}\tilde{B}_{v_1}\,dv_1$$
$$d\eta = \xi\,dt + dv_1. \tag{19}$$

The combined plant, observer dynamics can be described as follows

$$\begin{bmatrix} dx \\ d\xi \end{bmatrix} = \mathcal{A}_2 \begin{bmatrix} x \\ \xi \end{bmatrix} dt + \mathcal{B}_2 \begin{bmatrix} dw \\ dv_1 \end{bmatrix},$$
$$d\eta = \xi\,dt + dv_1,$$
$$\mathcal{A}_2 = \begin{bmatrix} A & 0 \\ KC & A - KC \end{bmatrix},$$
$$\mathcal{B}_2 = \begin{bmatrix} B & 0 \\ KD & T^{-1}\tilde{B}_{v_1} \end{bmatrix}.$$

Now we show that $\langle x - \xi \rangle_\rho$ converges exponentially to zero. Making the appropriate substitutions, we obtain

$$d(x - \xi) = (A - KC)(x - \xi)\,dt + (B - KD)\,dw - T^{-1}\tilde{B}_{v_1}\,dv.$$

Then, we find

$$d\langle x - \xi \rangle_\rho = (A - KC)\langle x - \xi \rangle_\rho\,dt.$$

From the properties of the Kalman filter (11), $(A - KC)$ is Hurwitz, therefore $\langle x - \xi \rangle_\rho$ converges exponentially to zero for arbitrary initial states and (19) is a coherent quantum observer.

We use the same performance metric (8) as previously. Once again, the limit in (8) converges because $A - KC$ is Hurwitz. Finally, $\bar{J}$ is the unique symmetric positive definite solution to the Lyapunov equation:
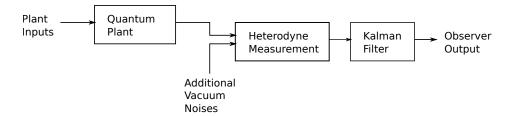
$$0 = \mathcal{A}_e\bar{J} + \bar{J}\mathcal{A}_e^T + \mathcal{B}_e S_{w,v_1}\mathcal{B}_e^T,$$
$$\mathcal{A}_e = (A - KC),$$
$$\mathcal{B}_e = \begin{bmatrix} (B - KD) & -T^{-1}\tilde{B}_{v_1} \end{bmatrix},$$

where

$$\begin{bmatrix} dw \\ dv_1 \end{bmatrix} \begin{bmatrix} dw^T & dv_1^T \end{bmatrix} = F_{w,v_1}\,dt \qquad \text{and} \qquad S_{w,v_1} = \mathfrak{Re}\left[F_{w,v_1}\right].$$

Figure 2. Quantum plant and classical observer consisting of heterodyne measurement and a Kalman filter.

6. Measurement-based (classical) observer

In the example which follows, we compare the coherent quantum observers designed in the previous section with the following classical observer which consists of heterodyne measurement and a Kalman filter as depicted in figure 2.

The output of the heterodyne measurement is described by the equation

$$\mathrm{d}y_H = \mathrm{d}y + \mathrm{d}w_H. \tag{20}$$

where, $\mathrm{d}w_H$ is a vacuum noise source of dimension $n_{w_H} = n_y$ and with Itō product

$$\mathrm{d}w_H \, \mathrm{d}w_H^T = F_{w_H} \, \mathrm{d}t,$$

where F_{w_H} is a block diagonal matrix with each block equal to

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix},$$

and S_{w_H} is the real part of F_{w_H}.

The following Kalman filter is applied to the output y_H of the heterodyne measurement

$$\begin{aligned} \mathrm{d}\hat{x} &= (A - KC)\hat{x} \, \mathrm{d}t + K \, \mathrm{d}y_H, \\ \mathrm{d}\hat{y} &= \hat{x} \, \mathrm{d}t. \end{aligned} \tag{21}$$

Here, $K = (QC^T + V_{12})V_2^{-1}$, and Q is the solution to the Riccati equation (12) with

$$\begin{aligned} V_1 &= BS_wB^T, \\ V_2 &= DS_wD^T + S_{w_H}, \quad \text{and} \\ V_{12} &= BS_wD^T. \end{aligned}$$

By combining equations (5), (20) and (21), we obtain the following dynamics for the combined plant and classical observer

$$\begin{aligned} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}\hat{x} \end{bmatrix} &= \mathcal{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \mathrm{d}t + \mathcal{B} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}w_H \end{bmatrix}, \\ \mathcal{A} &= \begin{bmatrix} A & 0 \\ KC & A - KC \end{bmatrix}, \\ \mathcal{B} &= \begin{bmatrix} B & 0 \\ KD & K \end{bmatrix}. \end{aligned}$$

The performance metric

$$J = \lim_{t \to \infty} \left\langle (x - \hat{x})(x - \hat{x})^T \right\rangle_\rho,$$

for the classical observer is the solution to the following Lyapunov equation:

$$
\begin{aligned}
0 &= \mathcal{A}_e J + J \mathcal{A}_e^T + \mathcal{B}_e S_{w,w_H} \mathcal{B}_e^T, \\
\mathcal{A}_e &= (A - KC), \\
\mathcal{B}_e &= \left[\begin{array}{cc} B - KD & -K \end{array}\right],
\end{aligned}
$$

where

$$
\left[\begin{array}{c} \mathrm{d}w \\ \mathrm{d}w_H \end{array}\right] \left[\begin{array}{cc} \mathrm{d}w^T & \mathrm{d}w_H^T \end{array}\right] = F_{w,w_H}\,\mathrm{d}t \qquad \text{and} \qquad S_{w,w_H} = \mathfrak{Re}\left[F_{w,w_H}\right].
$$

7. Example
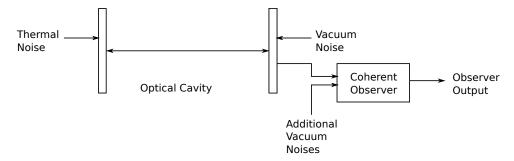
Figure 3. Plant and coherent quantum observer configuration.

Consider the quantum plant depicted in Figure 3. This plant consists of an optical cavity with thermal and vacuum noise inputs. Its dynamics are described by the following QSDEs of the form (5)
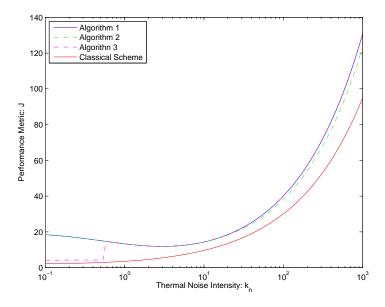
$$
\begin{aligned}
\mathrm{d}x &= -\frac{1}{2}\left(\kappa_1 + \kappa_2\right) x\,\mathrm{d}t - \sqrt{\kappa_1}\,\mathrm{d}w_1 - \sqrt{\kappa_2}\,\mathrm{d}w_2, \\
\mathrm{d}y &= \sqrt{\kappa_1}x\,\mathrm{d}t + \mathrm{d}w_1.
\end{aligned}
\tag{22}
$$

Here, κ_1, κ_2 are related to the mirror reflectances, $\mathrm{d}w_1$ is vacuum noise and $\mathrm{d}w_2$ is thermal noise of intensity k_n,

$$
S_{w_1} = I_{2\times2} \qquad \text{and} \qquad S_{w_2} = (1 + 2k_n)I_{2\times2}.
$$

We consider three scenarios, each with different values for κ_1, κ_2. For each scenario, we apply our algorithms to obtain coherent quantum observers across a range of thermal noise intensities k_n.
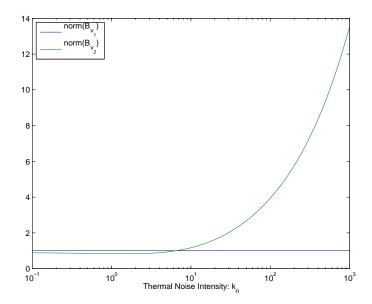
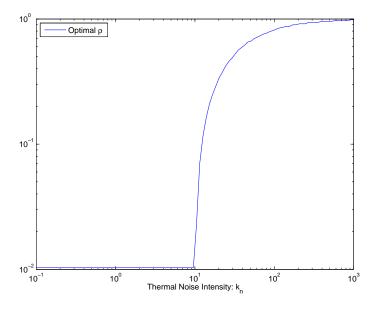**Figure 4.** Comparison of observers for $\kappa_1 = \kappa_2 = 0.1$.

7.1. Scenario 1: $\kappa_1 = \kappa_2 = 0.1$

Figure 4 compares the performance metric J for each of our coherent quantum observers with that for a classical observer consisting of heterodyne measurement and a Kalman filter. The classical observer performs best. This is not surprising as each of our coherent quantum observers introduces at least as much additional quantum noise as does the heterodyne measurement in the classical observer. Furthermore, the classical observer is optimal solution with respect to the output of the heterodyne measurement whereas the coherent quantum observers we consider are suboptimal. Notwithstanding this result, it is still of interest to develop tractable methods for designing coherent quantum observers as other considerations may favour the use of coherent quantum observers over measurement-based observers, in particular when the controllers are added.

The performance of Algorithm 1 never exceeds that of Algorithm 2. This is because, in Algorithm 2, $\rho = 0$ results in the same coherent quantum observer as Algorithm 1. Recall that Algorithm 1 does not take into account the B_{v_1} and B_{v_2} terms when designing the Kalman filter. Figure 5 shows the matrix norms for B_{v_1} and B_{v_2} for different values of k_n. It seems reasonable that as B_{v_2} becomes more significant, there is greater scope for Algorithm 2 to outperform Algorithm 1. This explains why Algorithm 2's relative performance increases with k_n. Figure 6 shows how the optimal ρ varies with k_n in Algorithm 2. We now turn our attention to Algorithm 3. For small values of k_n, a suitable transformation matrix T was found and a coherent quantum observer obtained with $n_{v_2} = 0$. In this regime, Algorithm 3 outperforms the other algorithms and produces a coherent quantum observer which approaches the performance of the classical observer. The discontinuity in Algorithm 3's performance corresponds to the point above which, no suitable T was found. In this regime the algorithm produces the

Figure 5. Significance of B_{v_1} and B_{v_2} for different values of k_n.

Figure 6. Choice of ρ for different values of k_n.

same coherent quantum observer as Algorithm 1. The range of k_n for which a suitable T exists is dependent on κ_1 and κ_2 as demonstrated in the scenarios which follow.

Note that Algorithm 3 produces a coherent quantum observer with a different value for B_{v_1} than that from Algorithm 1. In the following scenarios we shall see that in some regimes, despite introducing a smaller number of vacuum noises, Algorithm 3 does not perform better than Algorithm 1.

Finally, we briefly comment on the performance of the observers in the limit as

k_n approaches zero (that is, as the noise input dw_2 approaches a vacuum noise). For $k_n = 0$, the Kalman filter gain K, obtained in our algorithms, is zero. When dw_2 is a vacuum noise, the output of the plant gives no useful information about the internal state of the plant. In this special case, the optimal coherent quantum observer is the trivial one: a vacuum noise source. See [24] for a discussion of a class of plants driven solely by vacuum noises for which the authors show that the optimal controllers (and by implication the optimal observers) are trivial ones.

7.2. Scenario 2: $\kappa_1 = 0.5$; $\kappa_2 = 0.01$

Figure 7 shows the performance of the observers obtained for $\kappa_1 = 0.5$ and $\kappa_2 = 0.01$. (Compared to Scenario 1, mirror 1 is more lossy, while mirror 2 is less lossy.) For these mirrors, Algorithm 3 performs better than Algorithm 1 for greater noise intensities k_n. The discontinuity where no suitable state transformation T was found in Algorithm 3
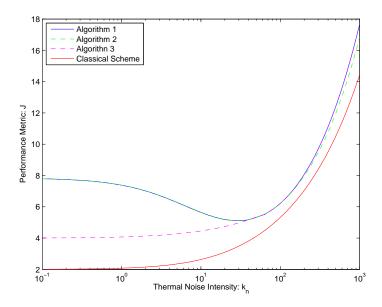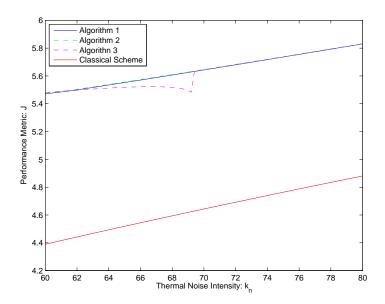

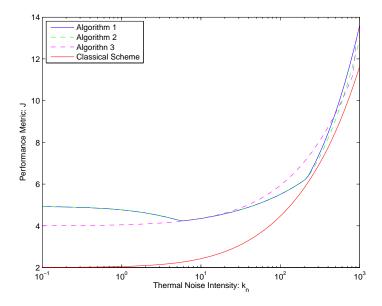**Figure 7.** Comparison of observers for $\kappa_1 = 0.5, \kappa_2 = 0.01$.

occurs at $k_n = 69$ and is shown in more detail in Figure 8.

7.3. Scenario 3: $\kappa_1 = 0.8$; $\kappa_2 = 0.01$

Figure 9 shows the performance of the observers obtained for $\kappa_1 = 0.8$ and $\kappa_2 = 0.01$. Compared to the previous scenarios, mirror 1 is even more lossy. As a result, the discontinuity in Algorithm 3's performance, above which no suitable state transformation T was found, occurs at the increased noise intensity $k_n = 910$. Below this point, Algorithm 3 gives a coherent quantum observer with $n_{v_2} = 0$ while above this point it gives a coherent quantum observer with $n_{v_2} = 2$. Algorithms 1 and 2 give coherent quantum observers with $n_{v_2} = 2$ for all considered values of k_n.

Figure 8. Comparison of observers for $\kappa_1 = 0.5, \kappa_2 = 0.01$.

Figure 9. Comparison of observers for $\kappa_1 = 0.8, \kappa_2 = 0.01$.

This scenario demonstrates a region where Algorithm 2 performs better than Algorithm 3 despite the latter giving a coherent quantum observer with less quantum noise sources. This is because, in this region, the impact of the B_{v_1} term obtained in Algorithm 3 is more significant than the combined impact of both the B_{v_1} and B_{v_2} terms in Algorithm 2.

Finally, this scenario suggests that the performance metric J obtained for Algorithms 1 and 2 is not necessarily smooth with respect to k_n. Obtaining an

explanation for this observation remains the subject of future research.

8. Conclusions

Like the celebrated Kalman filter in the context of classical feedback control problems, it is envisaged that coherent quantum observers will play a pivotal role in solving coherent quantum feedback control problems. Here, we have proposed three algorithms for the design of coherent quantum observers. The key idea behind each of our algorithms was to first treat the quantum plants classically to obtain a Kalman filter. We then made use of previous results to obtain a physically realizable system by taking the Kalman filter obtained and allowing additional vacuum noise sources in its quantum implementation. Algorithms 2 and 3 incorporate refinements to Algorithm 1 in an attempt to improve performance.

We compare the performance of the coherent quantum observers obtained with a measurement-based (classical) observer by way of an example involving an optical cavity with thermal and vacuum noise inputs. For each of the scenarios considered, the classical observer performs best. Algorithm 2 always performs at least as well as Algorithm 1. Algorithm 3 can potentially give a coherent quantum observer with a smaller number of quantum vacuum noise inputs than the other algorithms, however this does not guarantee better performance.

Acknowledgments

Appendix A.

Suppose we have a system of the form (9) with canonical commutation matrix Θ and where $\hat{A}, \hat{B}$, and $\hat{C}$ are given. The following construction for B_{v_1} and B_{v_2} results in a physically realizable system. It is not possible to construct B_{v_1} and B_{v_2} with smaller n_{v_2} such that (9) is physically realizable. For further details see [15].

- Construct the matrix

$$\tilde{S} = \Theta\hat{B}\Theta\hat{B}^T\Theta - \Theta\hat{A} - \hat{A}^T\Theta - \hat{C}^T\Theta\hat{C}.$$

  (Here Θ is the canonical commutation matrix of dimension $n_x \times n_x$)

- Find the rank of the matrix $\tilde{S}$: $n_{v_2} = \text{rank}\left[\tilde{S}\right]$.

- Calculate $S = \frac{i}{4}\tilde{S}$.

- Diagonalize S: $S = U^\dagger D U$. Here D is diagonal and U is unitary.
- Construct $\hat{D}$ by replacing each element of D with its absolute value.
- Construct $W = \left(\hat{D} + D\right)^{\frac{1}{2}} U$.
- Construct B_{v_1} and B_{v_2} as follows:

$$B_{v_1} = \Theta \hat{C}^T \text{diag}(J);$$
$$B_{v_2} = 2i\theta \begin{bmatrix} -W^\dagger & W^T \end{bmatrix} P\text{diag}(M).$$

Appendix B.

Here we give a numerical process for obtaining a suitable solution X to the Riccati equation (10) in Theorem 2. The three assumptions in the following, guarantee the existence of the solution X. For further details see [14].

- Construct

$$Z = \begin{bmatrix} \hat{A} & -\hat{B}\Theta\hat{B}^T \\ -\hat{C}^T\Theta\hat{C} & -\hat{A}^T \end{bmatrix}.$$

- Find the eigenvalues and eigenvectors of Z.
- Assumption 1: That Z has no purely imaginary eigenvalues. In practice, this means checking that the real part of each eigenvalue has magnitude greater than some small numerical tolerance.
- Construct the matrix

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

such that its columns are the eigenvectors of Z that correspond to eigenvalues with negative real part.

- Assumption 2: That X_1 is non-singular.
- Calculate $X = X_2 X_1^{-1}$.
- Assumption 3: That X is non-singular.
- Find the eigenvalues and eigenvectors of X. Hence, construct diagonal Λ with diagonal entries the eigenvalues of X and V with columns the corresponding eigenvectors normalized to length 1.
- Construct the $n_\xi \times n_\xi$ diagonal matrix $\tilde{\Lambda}$ with alternating diagonal entries i and $-i$.
- Construct the $n_\xi \times n_\xi$ block diagonal matrix $\tilde{V}$ with each diagonal block corresponding to $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.
- Calculate $D = \left(\tilde{\Lambda}^{-1}\Lambda\right)^{\frac{1}{2}}$.
- Calculate $T = \tilde{V} D V^\dagger$.

- Construct

$$\tilde{A} \quad = T\hat{A}T^{-1}, \quad \tilde{B} = T\hat{B}, \quad \tilde{C} = \hat{C}T^{-1}, \quad \text{and}$$
$$\tilde{B}_{v_1} = \Theta\tilde{C}^T\text{diag}(J).$$

The system $\left\{\tilde{A}, \tilde{B}, \tilde{C}\right\}$ has the same transfer function as $\left\{\hat{A}, \hat{B}, \hat{C}\right\}$ and is *physically realizable* with $n_{v_2} = 0$ and with $\tilde{B}_{v_1}$ as constructed above.

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