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Transmutations for Darboux transformed operators with applications

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Abstract

We solve the following problem. Given a continuous complex-valued potential q_1 defined on a segment $[-a, a]$ and let q_2 be the potential of a Darboux transformed Schrödinger operator, that is $q_2 = -q_1 + 2\left(\frac{f'}{f}\right)^2$ where f is a nonvanishing solution of the equation $A_1 f = \left(\frac{d^2}{dx^2} - q_1(x)\right)f = 0$. Suppose a transmutation operator $\mathbf{T}_1$ is known such that $A_1 \mathbf{T}_1 u = \mathbf{T}_1 \frac{d^2}{dx^2} u$ for any $u \in C^2[-a, a]$. Find an analogous transmutation operator for $A_2 = \frac{d^2}{dx^2} - q_2(x)$.

It is well known that the transmutation operators can be realized in the form of Volterra integral operators with continuously differentiable kernels. Given a kernel K_1 of the transmutation operator $\mathbf{T}_1$ we find the kernel K_2 of $\mathbf{T}_2$ in a closed form in terms of K_1. As a corollary interesting commutation relations between $\mathbf{T}_1$ and $\mathbf{T}_2$ are obtained which then are used in order to construct the transmutation operator for the one-dimensional Dirac system with a scalar potential.

1 Introduction

Transmutation operators are a widely used tool in the theory of linear differential equations (see, e.g., [2], [4], [23], [25], [35] and the recent review [33]). It is well known that under certain quite general conditions the transmutation operator transmuting the operator $A = -\frac{d^2}{dx^2} + q(x)$ into $B = -\frac{d^2}{dx^2}$ is a Volterra integral operator with good properties. Its kernel can be obtained as a solution of a certain Goursat problem for the Klein-Gordon equation with a variable coefficient. In the book [8] another approach to the transmutation was developed. It was shown that to every (regular) linear second-order ordinary differential operator L one can associate a linear space spanned on a so-called L-basis – an infinite family of functions $\{\varphi_k\}_{k=0}^\infty$ such that $L\varphi_k = 0$ for $k = 0, 1$, $L\varphi_k = k(k-1)\varphi_{k-2}$, for $k = 2, 3, \dots$ and all φ_k satisfy certain prescribed initial conditions. Then the operator of transmutation was introduced as an operation transforming functions from one such linear space corresponding to a certain operator L to functions from another linear space corresponding to another operator M, and the transformation consists in substituting the L-basis with the M-basis preserving the same coefficients in the expansion.

In [3] the relation between the transmutation operator in the form of a Volterra integral operator and as a transmutation of the bases was clarified. In particular, it was shown that the corresponding Volterra integral transmutation operator transmutes the powers of the independent variable x^k into the elements φ_k of an appropriate L-basis. In order to assure such an important mapping property

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of the transmutation operator a parametrized family of transmutation operators for the Schrödinger operator A was introduced. This parametrized family resulted to be a necessary and natural tool for solving the problem of construction of the transmutation operator for a Darboux transformed Schrödinger operator. The solution to this problem is the main aim of the present paper.

We give an explicit representation for the kernel of the transmutation operator corresponding to the Darboux transformed potential in terms of the transmutation kernel for its superpartner (Theorem 19). Moreover, this result leads to interesting commutation relations between the two transmutation operators (Corollary 25) which in their turn allow us to obtain a transmutation operator for the one-dimensional Dirac system with a scalar potential as well as to prove (Theorem 26) the main property of the transmutation operator under less restrictive conditions than it has been proved until now. Namely, we show that the considered transmutation operators transmute A into B for any continuous (and not necessarily differentiable) potential q admitting a nonvanishing, in general, complex-valued solution (and this condition is always fulfilled, e.g., when q is real valued). We give several examples of explicitly constructed kernels of transmutation operators. It is worth mentioning that in the literature there are very few explicit examples and even in the case when q is a constant such kernel was presented recently in [3]. The results of the present paper allow us to enlarge considerably the list of available examples and give a relatively simple tool for constructing Darboux related sequences of the transmutation kernels.

In Section 2 we introduce two main objects, the transmutation operators and the systems of recursive integrals or the L-bases. Besides recalling some recent results from [3] on the parametrized family of the transmutation operators we prove (Theorem 6) a criterion for a function to be a kernel of a transmutation operator from the parametrized family as well as a result on the explicit form of the inverse transmutation operator (Theorem 10). In Section 3 we present the construction of the transmutations for the Darboux transformed operators and obtain the commutation relations for them. We use the relations to weaken the conditions on the potential under which the transmutation operators preserve their main property (Theorem 26). The section ends with several examples of explicitly constructed transmutation kernels. In Section 4 the results of the preceding section are used in order to obtain the transmutation operator for the one-dimensional Dirac system with a scalar potential. Finally, Section 5 contains some concluding remarks.

2 Transmutation operators for Sturm-Liouville equations and systems of recursive integrals

2.1 Transmutation operators

According to the definition given by Levitan [23], let E be a linear topological space, A and B be linear operators: $E \to E$. Let E_1 and E_2 be closed subspaces of E.

Definition 1 *A linear invertible operator T defined on the whole E and acting from E_1 to E_2 is called a transmutation operator for the pair of operators A and B if it fulfills the following two conditions.*

1. *Both the operator T and its inverse T^{-1} are continuous in E;*
2. *The following operator equality is valid*

$$AT = TB \tag{1}$$

or which is the same

$$A = TBT^{-1}.$$

Our main interest concerns the situation when $A = -\frac{d^2}{dx^2} + q(x)$, $B = -\frac{d^2}{dx^2}$, and q is a continuous complex-valued function. Hence for our purposes it will be sufficient to consider the functional space $E = C^2[a, b]$ with the topology of uniform convergence. For simplicity we will assume that the interval is symmetric with respect to the origin, thus $E = C^2[-a, a]$.

An operator of transmutation for such A and B can be realized in the form (see, e.g., [23] and [25]) of a Volterra integral operator

$$Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)dt \tag{2}$$

where $K(x,t) = H\left(\frac{x+t}{2}, \frac{x-t}{2}\right)$ and H is the unique solution of the Goursat problem

$$\frac{\partial^2 H(u,v)}{\partial u\, \partial v} = q(u+v)H(u,v), \tag{3}$$

$$H(u,0) = \frac{1}{2}\int_0^u q(s)\, ds, \qquad H(0,v) = 0. \tag{4}$$

If the potential q is continuously differentiable, the kernel K itself is the solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right) K(x,t) = \frac{\partial^2}{\partial t^2} K(x,t), \tag{5}$$

$$K(x,x) = \frac{1}{2}\int_0^x q(s)\, ds, \qquad K(x,-x) = 0. \tag{6}$$

If the potential q is n times continuously differentiable, the kernel $K(x,t)$ is $n+1$ times continuously differentiable with respect to both independent variables (see [25]).

An important property of this transmutation operator consists in the way how it maps solutions of the equation

$$v'' + \omega^2 v = 0 \tag{7}$$

into solutions of the equation

$$u'' - q(x)u + \omega^2 u = 0 \tag{8}$$

where ω is a complex number. Denote by $e_0(i\omega, x)$ the solution of (8) satisfying the initial conditions

$$e_0(i\omega, 0) = 1 \qquad \text{and} \qquad e_0'(i\omega, 0) = i\omega. \tag{9}$$

The subindex "0" indicates that the initial conditions correspond to the point $x = 0$ and the letter "e" reminds us that the initial values coincide with the initial values of the function $e^{i\omega x}$.

The transmutation operator (2) maps $e^{i\omega x}$ into $e_0(i\omega, x)$,

$$e_0(i\omega, x) = T[e^{i\omega x}] \tag{10}$$

(see [25, Theorem 1.2.1]).

Following [25] we introduce the following notations

$$K_c(x,t;h) = h + K(x,t) + K(x,-t) + h \int_t^x \{K(x,\xi) - K(x,-\xi)\}d\xi$$

where h is a complex number, and

$$K_s(x,t;\infty) = K(x,t) - K(x,-t).$$

Theorem 2 ([25]) *Solutions $c(\omega, x; h)$ and $s(\omega, x; \infty)$ of equation (8) satisfying the initial conditions*

$$c(\omega, 0; h) = 1, \qquad\qquad c_x'(\omega, 0; h) = h \tag{11}$$

$$s(\omega, 0; \infty) = 0, \qquad\qquad s_x'(\omega, 0; \infty) = 1 \tag{12}$$

can be represented in the form

$$c(\omega, x; h) = \cos \omega x + \int_0^x K_c(x, t; h) \cos \omega t \, dt \tag{13}$$

and

$$s(\omega, x; \infty) = \frac{\sin \omega x}{\omega} + \int_0^x K_s(x, t; \infty) \frac{\sin \omega t}{\omega} \, dt. \tag{14}$$

Denote by

$$T_c u(x) = u(x) + \int_0^x K_c(x, t; h) u(t) dt$$

and

$$T_s u(x) = u(x) + \int_0^x K_s(x, t; \infty) u(t) dt$$

the corresponding integral operators. As was pointed out in [3] they are not, in general, transmutations on the whole space $C^2[-a, a]$.

2.2 A parametrized family of transmutation operators

In [3] we introduced the parametrized family of operators $\mathbf{T}_h$, $h \in \mathbb{C}$, given by the integral expression

$$\mathbf{T}_h u(x) = u(x) + \int_{-x}^x \mathbf{K}(x, t; h) u(t) dt \tag{15}$$

where

$$\mathbf{K}(x, t; h) = \frac{h}{2} + K(x, t) + \frac{h}{2} \int_t^x \big(K(x, s) - K(x, -s) \big) \, ds. \tag{16}$$

They are related to operators T_s and T_c (with the parameter h in the kernel of the latter operator) by

$$\mathbf{T}_h = T_c P_e + T_s P_o, \tag{17}$$

where $P_e f(x) = \big(f(x) + f(-x) \big)/2$ and $P_o f(x) = \big(f(x) - f(-x) \big)/2$ are projectors to even and odd functions, respectively. In this subsection we show that the operators $\mathbf{T}_h$ are transmutations, summarize their properties and later, in Theorem 15 we show how they act on powers of x.

Let us notice that $\mathbf{K}(x, t; 0) = K(x, t)$ and that the expression

$$\mathbf{K}(x, t; h) - \mathbf{K}(x, -t; h) = K(x, t) - K(x, -t) + \frac{h}{2} \int_{-t}^t \big(K(x, s) - K(x, -s) \big) \, ds = K(x, t) - K(x, -t)$$

does not depend on h. Thus, it is possible to compute $\mathbf{K}(x, t; h)$ for any h by a given $\mathbf{K}(x, t; h_1)$ for some particular value h_1. We formulate this result as the following statement.

Theorem 3 ([3]) *The integral kernels $\mathbf{K}(x, t; h)$ and $\mathbf{K}(x, t; h_1)$ are related by the expression*

$$\mathbf{K}(x, t; h) = \frac{h - h_1}{2} + \mathbf{K}(x, t; h_1) + \frac{h - h_1}{2} \int_t^x \big(\mathbf{K}(x, s; h_1) - \mathbf{K}(x, -s; h_1) \big) \, ds. \tag{18}$$

Using (10) let us check how the operators $\mathbf{T}_h$ act on solutions of (7).

Proposition 4 *The operator $\mathbf{T}_h$ maps a solution v of an equation $v'' + \omega^2 v = 0$, where ω is a complex number, into the solution u of the equation $u'' - q(x)u + \omega^2 u = 0$ with the following correspondence of initial values*

$$u(0) = v(0), \qquad u'(0) = v'(0) + hv(0).$$

Proof. It follows from (17) and Theorem 2 that the operator $\mathbf{T}_h$ indeed maps a solution of $v'' + \omega^2 v = 0$ into a solution of $u'' - q(x)u + \omega^2 u = 0$. It is clear from the definition (15) that

$$\mathbf{T}_h v(0) = v(0). \tag{19}$$

As for the derivative, we have

$$\left(\mathbf{T}_h v\right)'(x) = v'(x) + \int_{-x}^{x} \mathbf{K}'_x(x, t; h)v(t)\, dt + \mathbf{K}(x, x; h)v(x) + \mathbf{K}(x, -x; h)v(-x),$$

therefore

$$\left(\mathbf{T}_h v\right)'(0) = v'(0) + K(0, 0; h)u(0) + K(x, -x; h)v(0) = v'(0) + hv(0). \tag{20}$$

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Remark 5 *As can be seen from the proof, formulas (19) and (20) are valid for any function* $v \in C^1[-a, a]$.

We know that the kernel of the transmutation operator T is related to the solution of the Goursat problem (3)–(4). We show that a similar result holds for the operators $\mathbf{T}_h$.

Theorem 6 *In order for the function* $K(x, t; h)$ *to be the kernel of a transmutation operator acting as described in Proposition 4, it is necessary and sufficient that* $H(u, v; h) := K(u + v, u - v; h)$ *be a solution of the Goursat problem*

$$\frac{\partial^2 H(u, v; h)}{\partial u\, \partial v} = q(u + v)H(u, v; h), \tag{21}$$

$$H(u, 0; h) = \frac{h}{2} + \frac{1}{2}\int_0^u q(s)\, ds, \qquad H(0, v; h) = \frac{h}{2}. \tag{22}$$

If the potential q is continuously differentiable, the function $K(x, t; h)$ *itself should be the solution of the Goursat problem*

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right)K(x, t; h) = \frac{\partial^2}{\partial t^2}K(x, t; h), \tag{23}$$

$$K(x, x; h) = \frac{h}{2} + \frac{1}{2}\int_0^x q(s)\, ds, \qquad K(x, -x; h) = \frac{h}{2}. \tag{24}$$

Proof. If some transmutation operator T_h acts as described in Proposition 4, then on a common domain of definition it coincides with the constructed operator $\mathbf{T}_h$, because both operators are continuous and act identically on the dense subset of functions, namely, on linear combinations of solutions of equations (7). Hence it is sufficient to verify that the kernel $\mathbf{K}(x, t; h)$ satisfies conditions of the theorem.

Suppose $q \in C^1[-a, a]$. Then the kernel $K(x, t)$ is twice continuously differentiable and we can check (23) directly. In the following calculation $K'_x(x, \pm x)$ means $K'_x(x, t)\big|_{t=\pm x}$ and $K'_t(x, \pm x)$ means

$K_t'(x, t)\big|_{t=\pm x}$.

$$\left(\partial_x^2 - \partial_t^2\right) \mathbf{K}(x, t; h) = \left(\partial_x^2 - \partial_t^2\right) \left[\frac{h}{2} + K(x, t) + \frac{h}{2} \int_t^x \left(K(x, s) - K(x, -s)\right) ds\right] =$$

$$= q(x)K(x, t) + \frac{h}{2}\partial_x \left(\int_t^x \left(K_x'(x, s) - K_x'(x, -s)\right) ds + K(x, x) - K(x, -x)\right) +$$

$$+ \frac{h}{2}\partial_t\left(K(x, t) - K(x, -t)\right) = q(x)K(x, t) + \frac{h}{4}q(x) +$$

$$+ \frac{h}{2}\left(\int_t^x \left(K_{xx}''(x, s) - K_{xx}''(x, -s)\right) ds + K_x'(x, x) - K_x'(x, -x) + K_t'(x, t) + K_t'(x, -t)\right) =$$

$$= q(x)K(x, t) + \frac{h}{4}q(x) + \frac{h}{2}\left(\int_t^x \left(K_{ss}''(x, s) - K_{ss}''(x, -s) + q(x)(K(x, s) - K(x, -s))\right) ds +$$

$$+ K_x'(x, x) - K_x'(x, -x) + K_t'(x, t) + K_t'(x, -t)\right) = q(x)\left(K(x, t) + \frac{h}{4} + \int_t^x \left(K(x, s) - K(x, -s)\right) ds\right) +$$

$$+ \frac{h}{2}\left(K_t'(x, x) - K_t'(x, t) + K_t'(x, -x) - K_t'(x, -t) + K_x'(x, x) - K_x'(x, -x) + K_t'(x, t) + K_t'(x, -t)\right) =$$

$$= q(x)\left(K(x, t) + \frac{h}{2} + \int_t^x \left(K(x, s) - K(x, -s)\right) ds\right) = q(x)\mathbf{K}(x, t; h),$$

where we have used (5), (6) and the equalities $K_x'(x, x) + K_t'(x, x) = \partial_x(K(x, x)) = 1/2q(x)$, $K_x'(x, -x) - K_t'(x, -x) = \partial_x(K(x, -x)) = 0$. Equalities (24) are valid by the definition (16) of $\mathbf{K}(x, t; h)$. Clearly, the function $\mathbf{H}(u, v; h) = \mathbf{K}(u + v, u - v; h)$ is also the solution of the Goursat problem (21)–(22).

Suppose now that $q \in C[-a, a]$. Consider a sequence of functions $\{q_n\}_{n\in\mathbb{N}}$, where $q_n \in C^1[-a, a]$, $n \in \mathbb{N}$ and $q_n \to q$, $n \to \infty$ uniformly on $[-a, a]$. Denote by $H_n(u, v)$ the solutions of Goursat problems (3)–(4) with the potentials q_n, and by $H(u, v)$ the solution with the potential q. It is known (see, e.g., [25]), that $H_n(u, v) \to H(u, v)$, $n \to \infty$ uniformly. The same is valid for the functions $K_n(x, t) = H_n\left(\frac{x+t}{2}, \frac{x-t}{2}\right)$ and $K(x, t)$. Denote by $\mathbf{K}_n(x, t; h)$ the integral kernels constructed from $K_n(x, t)$ by means of (16) and let $\mathbf{H}_n(u, v; h) = \mathbf{K}_n(u + v, u - v; h)$. Then $\mathbf{H}_n(u, v; h) \to \mathbf{H}(u, v; h) = \mathbf{K}(u + v, u - v; h)$, $n \to \infty$ uniformly. Similarly to [25, 36], the function $\mathbf{H}_n(u, v; h)$ is the solution of the Goursat problem (21)–(22) with the potential q_n if and only if $\mathbf{H}_n(u, v; h)$ satisfies the integral equation

$$\mathbf{H}_n(u, v; h) = \frac{h}{2} + \frac{1}{2}\int_0^u q_n(y)\, dy + \int_0^u d\alpha \int_0^v q_n(\alpha + \beta)\mathbf{H}_n(\alpha, \beta; h)\, d\beta. \tag{25}$$

From the first part of the proof we know that the functions $\mathbf{H}_n(u, v; h)$ satisfy (25). Passing to the limit when $n \to \infty$ in (25), we obtain that $\mathbf{H}(u, v; h)$ satisfies (25) with the potential q and hence satisfies the Goursat problem (21)–(22).

For the second part of the theorem, suppose that a function $H(u, v; h)$ satisfies the Goursat problem (21)–(22). Consider the Goursat problem (3)–(4). It is well posed, and from its solution using (16) we can construct the kernel $\mathbf{K}(x, t; h)$. From the first part of the proof we have that the function $\mathbf{H}(u, v; h) = \mathbf{K}(u + v, u - v; h)$ satisfies the Goursat problem (21)–(22), and from the uniqueness of the solution it follows that $H = \mathbf{H}$ and H gives us the transmutation operator acting as described in Proposition 4. $\blacksquare$

Example 7 *Consider a function $k(x, t) = \frac{t-1}{2(x+1)}$ (later, in Subsection 3.3 it is explained how it can be obtained). We have*

$$(\partial_x^2 - \partial_t^2)k(x, t) = \frac{t-1}{(x+1)^3} = \frac{2}{(x+1)^2} \cdot \frac{t-1}{2(x+1)},$$

$$k(x, -x) = \frac{-x-1}{2(x+1)} = -\frac{1}{2} \qquad and \qquad k(x, x) = \frac{x-1}{2(x+1)} = -\frac{1}{2} + \frac{1}{2}\int_0^x \frac{2}{(s+1)^2}\, ds,$$

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thus the function $k(x,t)$ satisfies the Goursat problem (23)–(24) with $q(x) = 2/(x+1)^2$ and $h = -1$ and by Theorem 6 is the kernel of the transmutation operator $\mathbf{T}_{-1}$.

Consider two solutions of the equation $u'' = 0$, $u_1 = 1$ and $u_2 = x$. We have that their images

$$v_1 = \mathbf{T}_{-1} u_1 = 1 + \int_{-x}^{x} \frac{t-1}{2(x+1)} \, dt = \frac{1}{x+1},$$

$$v_2 = \mathbf{T}_{-1} u_2 = x + \int_{-x}^{x} \frac{(t-1)t}{2(x+1)} \, dt = \frac{x^3 + 3x^2 + 3x}{3(x+1)},$$

are solutions of the equation $v'' - 2/(x+1)^2 v = 0$ with the initial values in agreement with Proposition 4. For the solution $u_3 = \cos x$ of the equation $u'' = -u$ we get that

$$v_3 = \mathbf{T}_{-1} u_3 = \cos x + \int_{-x}^{x} \frac{(t-1)\cos t}{2(x+1)} \, dt = \cos x - \frac{\sin x}{x+1},$$

is a solution of the equation $v'' - \frac{2}{(x+1)^2} v = -v$.

For the case $q \in C^1[-a, a]$ the Volterra-type integral operator (2) is a transmutation in the sense of Definition 1 on the space $C^2[-a, a]$ if and only if the integral kernel $K(x,t)$ satisfies the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x) \right) K(x,t) = \frac{\partial^2}{\partial t^2} K(x,t),$$

$$K(x,x) = C + \frac{1}{2} \int_0^x q(s) \, ds, \qquad K(x,-x) = C,$$

where C is a constant (see [24, 35]). The proof of this fact consists in differentiating of the integral and integrating by parts, similar to the proof of Theorem 6. Due to Theorem 6 we see that $\mathbf{T}_h$ is indeed a transmutation.

Theorem 8 *Let $q \in C^1[-a, a]$. Then the operator $\mathbf{T}_h$ given by (15) satisfies the equality*

$$\left(-\frac{d^2}{dx^2} + q(x) \right) \mathbf{T}_h[u] = \mathbf{T}_h \left[-\frac{d^2}{dx^2}(u) \right] \tag{26}$$

for any $u \in C^2[-a, a]$.

Remark 9 *This theorem was proved in [3] under the additional assumption that there exists a non-vanishing solution f of $f'' - qf = 0$ on $[-a, a]$. Later, in Theorem 26 we show that such additional assumption allows one to weaken the requirement on the potential q to $q \in C[-a, a]$ keeping valid equality (26).*

As can be seen from (15), to define the transmutation operator we have to know its integral kernel in the region $-a < x < a$, $|t| < |x|$. The corresponding function H should be defined in two domains, upper-right and lower-left triangles $\Pi_1 : u > 0, v > 0, u + v < a$ and $\Pi_3 : u < 0, v < 0, |u| + |v| < a$ meanwhile the Goursat problem (21)–(22) may be considered in four domains, triangles Π_1 and Π_3 and triangles $\Pi_2 : u < 0, v > 0, |u| + |v| < a$, $\Pi_4 : u > 0, v < 0, |u| + |v| < a$. The potential q is continuous in each of these domains, boundary conditions (22) are continuously differentiable, therefore the corresponding solutions of the Goursat problems (see [36]) belong to $C^1(\bar{\Pi}_i)$, $i = 1 \ldots 4$. Hence the function $\mathbf{H}(u, v; h)$ is defined in the combined domain $|u| + |v| \le a$ and belongs to the class C^1 on it. Consequently, the function $\mathbf{K}(x, t; h) = \mathbf{H}\left(\frac{x+t}{2}, \frac{x-t}{2}; h \right)$ is defined on the domain $\bar{\Pi} : -a \le x \le a, -a \le t \le a$ and is continuously differentiable there. For the rest of this paper we assume that the kernel $\mathbf{K}$ is defined on this larger domain $\bar{\Pi}$.

Let us find the inverse operator $\mathbf{T}_h^{-1}$. Since $\mathbf{T}_h$ is the Volterra integral operator its inverse (see [24]) is again a Volterra integral operator

$$\mathbf{T}_h^{-1}u(x) = u(x) + \int_{-x}^{x} L(x,t;h)u(t)\, dt,$$

where $L(x,t;h)$ satisfies (for $q \in C^1[-a,a]$) the Goursat problem

$$\frac{\partial^2}{\partial x^2}L(x,t;h) - \left(\frac{\partial^2}{\partial t^2} - q(t)\right) L(x,t;h) = 0, \tag{27}$$

$$L(x,x;h) = C_1 - \frac{1}{2}\int_0^x q(s)\, ds, \qquad L(x,-x;h) = C_1. \tag{28}$$

To determine the value of the constant C_1 we use Proposition 4. Namely, we observe that the integral operator with the kernel L satisfying the Goursat problem (27)–(28) gives us the following correspondence of initial values

$$v(0) = u(0), \qquad v'(0) = u'(0) + 2C_1 u(0),$$

where $v = \mathbf{T}_h^{-1}u$. Therefore for the operator to be the inverse of the operator $\mathbf{T}_h$ we should take $C_1 = -h/2$. Comparing the Goursat problem (27)–(28) with $C_1 = -h/2$ with (23)–(24) we conclude that $L(x,t;h) := -\mathbf{K}(t,x;h)$ is the unique solution of (27)–(28) and hence it is the kernel of the inverse operator. The assumption $q \in C^1[-a,a]$ is not essential and can be easily overcome by considering an approximating sequence of continuously differentiable potentials q_n. Thus, the following statement is proved.

Theorem 10 *The inverse operator $\mathbf{T}_h^{-1}$ can be represented as the Volterra integral operator*

$$\mathbf{T}_h^{-1}u(x) = u(x) - \int_{-x}^{x} \mathbf{K}(t,x;h)u(t)\, dt. \tag{29}$$

Example 11 *For the operator $\mathbf{T}_{-1}$ with the kernel $K(x,t)$ and functions u_i and v_i, $i = 1,2,3$ from Example 7 it is easy to see that indeed*

$$u_i(x) = \mathbf{T}_{-1}^{-1}[v_i](x) = v_i(x) - \int_{-x}^{x} \frac{x-1}{2(t+1)}v_i(t)\, dt, \qquad i = 1,2,3.$$

2.3 A complete system of recursive integrals

Let $f \in C^2(a,b) \cap C^1[a,b]$ be a complex valued function and $f(x) \neq 0$ for any $x \in [a,b]$. The interval (a,b) is supposed to be finite. Let us consider the following auxiliary functions

$$\widetilde{X}^{(0)}(x) \equiv X^{(0)}(x) \equiv 1, \tag{30}$$

$$\widetilde{X}^{(n)}(x) = n\int_{x_0}^{x} \widetilde{X}^{(n-1)}(s) \left(f^2(s)\right)^{(-1)^{n-1}} ds, \tag{31}$$

$$X^{(n)}(x) = n\int_{x_0}^{x} X^{(n-1)}(s) \left(f^2(s)\right)^{(-1)^{n}} ds, \tag{32}$$

where x_0 is an arbitrary fixed point in $[a,b]$. We introduce the infinite system of functions $\{\varphi_k\}_{k=0}^{\infty}$ defined as follows

$$\varphi_k(x) = \begin{cases} f(x)X^{(k)}(x), & k \text{ odd}, \\ f(x)\widetilde{X}^{(k)}(x), & k \text{ even}, \end{cases} \tag{33}$$

where the definition of $X^{(k)}$ and $\widetilde{X}^{(k)}$ is given by (30)–(32) with x_0 being an arbitrary point of the interval $[a,b]$.

Example 12 *Let* $f \equiv 1$, $a = 0$, $b = 1$. *Then it is easy to see that choosing* $x_0 = 0$ *we have* $\varphi_k(x) = x^k$, $k \in \mathbb{N}_0$ *where by* $\mathbb{N}_0$ *we denote the set of non-negative integers.*

In [19] it was shown that the system $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $L_2(a, b)$ and in [20] its completeness in the space of piecewise differentiable functions with respect to the maximum norm was obtained and series expansions in terms of the functions φ_k were studied.

The system (33) is closely related to the notion of the L-basis introduced and studied in [8]. Here the letter L corresponds to a linear ordinary differential operator. This becomes more transparent from the following result obtained in [17] (for additional details and simpler proof see [18] and [21]) establishing the relation of the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ to Sturm-Liouville equations.

Theorem 13 ([17]) *Let q be a continuous complex valued function of an independent real variable* $x \in [a, b]$, λ *be an arbitrary complex number. Suppose there exists a solution f of the equation*

$$f'' - qf = 0 \tag{34}$$

on (a, b) such that $f \in C^2(a, b)$ together with $1/f$ are bounded on $[a, b]$. Then the general solution of the equation

$$u'' - qu = \lambda u \tag{35}$$

on (a, b) has the form

$$u = c_1 u_1 + c_2 u_2$$

where c_1 and c_2 are arbitrary complex constants,

$$u_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \varphi_{2k} \quad and \quad u_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \varphi_{2k+1} \tag{36}$$

and both series converge uniformly on $[a, b]$.

Remark 14 *It is easy to see that by definition the solutions u_1 and u_2 satisfy the following initial conditions*

$$u_1(x_0) = f(x_0), \qquad u_1'(x_0) = f'(x_0), \tag{37}$$

$$u_2(x_0) = 0, \qquad u_2'(x_0) = 1/f(x_0). \tag{38}$$

2.4 Transmutations and systems of recursive integrals

In this subsection we show the connection between the transmutation operators $\mathbf{T}_h$ and the functions φ_k. All the results of this subsection were proved in [3]. Here we recall them with explanations.

We suppose that f is a solution of (34) fulfilling the condition of Theorem 13 on a finite interval $(-a, a)$. We normalize f in such a way that $f(0) = 1$, and let $f'(0) = h$ where h is some complex constant. Let us obtain the expansion of the solution $c(\omega, x; h)$ from Subsection 2.1 in terms of the functions φ_k. According to Remark 14 the solutions (36) of equation (35) have the following initial values

$$u_1(0) = 1, \qquad u_1'(0) = h, \qquad u_2(0) = 0, \qquad u_2'(0) = 1.$$

Hence due to (11) we obtain $c(\omega, x; h) = u_1(x)$. From (13) and (36) we have the equality

$$\sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k)!} \varphi_{2k}(x) = \sum_{j=0}^{\infty} \frac{(i\omega)^{2j} x^{2j}}{(2j)!} + \int_0^x \left(K_c(x, t; h) \sum_{j=0}^{\infty} \frac{(i\omega)^{2j} t^{2j}}{(2j)!} \right) dt.$$

As the series under the sign of integral converges uniformly and the kernel $K_c(x, t; h)$ is at least continuously differentiable (for a continuous q [25]) we obtain the following relation

$$\sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k)!} \varphi_{2k}(x) = \sum_{j=0}^{\infty} \frac{(i\omega)^{2j}}{(2j)!} \left(x^{2j} + \int_0^x K_c(x, t; h) t^{2j} dt \right).$$

This equality holds for any ω hence we obtain the termwise relations

$$\varphi_{2k} = T_c[x^{2k}], \quad k \in \mathbb{N}_0. \tag{39}$$

Similarly, due to (12) we observe that $s(\omega, x; \infty) = u_2(x)$ and from (14) we find that

$$\varphi_{2k+1} = T_s[x^{2k+1}], \quad k \in \mathbb{N}_0. \tag{40}$$

Hence from the last two equalities and (17) we conclude that the following statement is true.

Theorem 15 ([3]) *Let q be a continuous complex valued function of an independent real variable $x \in [-a, a]$, and f be a particular solution of (34) such that $f \in C^2(-a, a)$ together with $1/f$ are bounded on $[-a, a]$ and normalized as $f(0) = 1$, $f'(0) = h$, where h is a complex number. Then the operator (15) with the kernel defined by (16) transforms x^k into $\varphi_k(x)$ for any $k \in \mathbb{N}_0$.*

Thus, we clarified what is the result of application of the transmutation $\mathbf{T}_h$ to the powers of the independent variable. This is very useful due to the fact that as a rule the construction of the kernel $\mathbf{K}(x, t; h)$ in a more or less explicit form up to now is impossible. Our result offers an algorithm for transmuting functions which can be represented or at least approximated by finite or infinite polynomials in the situation when $\mathbf{K}(x, t; h)$ is unknown.

Remark 16 *Let f be the solution of (34) satisfying the initial conditions*

$$f(0) = 1 \quad and \quad f'(0) = 0. \tag{41}$$

If it does not vanish on $[-a, a]$ then from Theorem 15 we obtain that the original transmutation operator T transmutes powers of the independent variable into φ_k: $\varphi_k = T[x^k]$ for any $k \in \mathbb{N}_0$. In general, of course f may have zeros on $[-a, a]$ and hence one can guarantee that the operator T transmutes the powers of x into φ_k whose construction is based on the solution f satisfying (41) only in some neighborhood of the origin. If we construct φ_k by a solution f such that $f(0) = 1$ and $f'(0) = h$, then the original transmutation operator T does not map powers of the independent variable exactly to the functions φ_k. As it was shown in [3] the following equalities are valid

$$\varphi_k = T[x^k] \quad when \ k \ is \ odd,$$

$$\varphi_k - \frac{h}{k+1}\varphi_{k+1} = T[x^k] \quad when \ k \in \mathbb{N}_0 \ is \ even.$$

This observation explains the necessity to consider the parametrized family of transmutation operators whose mapping properties are well adjusted to corresponding families of the functions φ_k.

Example 17 *For the operator $\mathbf{T}_{-1}$ from Example 7, consider the function $f = \mathbf{T}_{-1}[1] = \frac{1}{x+1}$ as a solution of (34) such that $f(0) = 1$ and $f'(0) = h = -1$, nonvanishing on any $[-a, a] \subset (-1, 1)$. The first 3 functions φ_k are given by*

$$\varphi_0 = f = \frac{1}{x+1}, \quad \varphi_1 = \frac{x^3 + 3x^2 + 3x}{3(x+1)}, \quad \varphi_2 = \frac{2x^3 + 3x^2}{3(x+1)}.$$

We have seen already that $\varphi_0 = \mathbf{T}_{-1}[1]$ and $\varphi_1 = \mathbf{T}_{-1}[x]$. We have also $\mathbf{T}_{-1}[x^2] = x^2 - \frac{x^3}{3(x+1)} = \varphi_2$.

3 Transmutation operators for Darboux transformed equations

3.1 The Darboux transformation

Consider a Sturm-Liouville operator $A_1 := \frac{d^2}{dx^2} - q_1(x)$, where q_1 is a continuous complex-valued function on some segment $[-a, a]$. Suppose a solution f of the equation $A_1 f = 0$ is given such that

$f(x) \neq 0$, $x \in [-a, a]$, it is normalized as $f(0) = 1$ and $h := f'(0)$ is some complex number. Then the operator $A_2 := \frac{d^2}{dx^2} - q_2(x)$, where $q_2(x) = 2\left(\frac{f'(x)}{f(x)}\right)^2 - q_1(x)$, is known as the Darboux transformation of the operator A_1.

Initially the Darboux transformation served for establishing a relation between the general solution u of the equation $A_1 u = \lambda u$ and the general solution v of the equation $A_2 v = \lambda v$ by a known particular solution f of the equation $A_1 u = 0$. This relation is given by the formula $v(x) = u'(x) - u(x)\frac{f'(x)}{f(x)}$. Later on it was found that the Darboux transformation was closely related to the factorization of the Schrödinger equation, and nowadays it is used in dozens of works, see e.g. [7, 10, 26, 30] in connection with solitons and integrable systems, e.g. [1, 11, 28, 29] and the review [31] of applications to quantum mechanics.

We remind some well known facts about the Darboux transformation. First, $1/f$ is the solution of $A_2 u = 0$. Second, it is closely related to the factorization of Sturm-Liouville and one-dimensional Schrödinger operators. Namely, we have

$$A_1 = \frac{d^2}{dx^2} - q_1(x) = \left(\partial_x + \frac{f'}{f}\right)\left(\partial_x - \frac{f'}{f}\right) = \frac{1}{f}\partial_x f^2 \partial_x \frac{1}{f} \cdot, \tag{42}$$

$$A_2 = \frac{d^2}{dx^2} - q_2(x) = \left(\partial_x - \frac{f'}{f}\right)\left(\partial_x + \frac{f'}{f}\right) = f\partial_x \frac{1}{f^2}\partial_x f \cdot. \tag{43}$$

Suppose that u is a solution of the equation $A_1 u = \omega u$ for some $\omega \in \mathbb{C}$. Then the function $v = \left(\partial_x - \frac{f'}{f}\right)u = \left(f\partial_x \frac{1}{f}\right)u$ is a solution of the equation $A_2 v = \omega v$, and vice versa, given a solution v of $A_2 v = \omega v$, the function $u = \left(\partial_x + \frac{f'}{f}\right)v = \left(\frac{1}{f}\partial_x f\right)v$ is a solution of $A_1 u = \omega u$.

3.2 Construction of transmutation operators for Darboux transformed equations

Consider the operators A_1, A_2 and a particular solution f of the equation $A_1 f = 0$ as in the previous subsection. Suppose that the operator $\mathbf{T}_{1;h}$ which transmutes the operator A_1 into the operator $B = d^2/dx^2$ is known in the sense that its kernel $\mathbf{K}_1(x, t; h)$ is given. As before $h = f'(0)$ and $\mathbf{T}_{1;h}$ transforms solutions according to Proposition 4. As the parameter h is fixed by the value $f'(0)$ we will simply write $\mathbf{T}_1$ instead of $\mathbf{T}_{1;h}$. We know that the function $1/f$ is the non-vanishing solution of the equation $A_2 u = 0$ satisfying $1/f(0) = 1$ and $(1/f)'(0) = -h$. Hence it is natural to look for the operator $\mathbf{T}_{2;-h}$ transmuting the operator A_2 into the operator B. We will simply write $\mathbf{T}_2$ further in this paper.

Let us explain the idea for obtaining the operator $\mathbf{T}_2$. We want to find an operator transforming solutions of the equation $Bu + \omega^2 u = 0$ into solutions of the equation $A_2 u + \omega^2 u = 0$, see the first diagram below. Starting with a solution σ of the equation $(\partial_x^2 + \omega^2)\sigma = 0$, by application of $\mathbf{T}_1$ we get a solution of $(A_1 + \omega^2)u = 0$, and the expression $\left(f\partial_x \frac{1}{f}\right)\mathbf{T}_1 \sigma$ is a solution of $(A_2 + \omega^2)v = 0$. But the operator $\left(f\partial_x \frac{1}{f}\right)\mathbf{T}_1$ is unbounded and hence cannot coincide with the operator $\mathbf{T}_2$. In order to find the required bounded operator we may consider the second copy of the equation $(\partial_x^2 + \omega^2)u = 0$, which is a result of the Darboux transformation applied to $(\partial_x^2 + \omega^2)\sigma = 0$ with respect to the particular solution $g \equiv 1$ and construct the operator $\mathbf{T}_2$ by making the second diagram commutative. In order to obtain a bounded operator $\mathbf{T}_2$, instead of using $f\partial_x \frac{1}{f}$ for the last step, we will use the inverse of $\frac{1}{f}\partial_x f$, i.e. $\frac{1}{f}\left(\int_0^x f(s) \cdot ds + C\right)$.

That explains how to obtain the following theorem.

Theorem 18 *The operator T_2, acting on solutions u of equations $(\partial_x^2 + \omega^2)u = 0$, $\omega \in \mathbb{C}$ by the rule*

$$T_2[u](x) = \frac{1}{f(x)} \left(\int_0^x f(\eta) \mathbf{T}_1[u'](\eta) \, d\eta + u(0) \right) \tag{44}$$

coincides with the transmutation operator $\mathbf{T}_{2;-h}$.

Proof. According to Proposition 4 the operator $\mathbf{T}_{2;-h}$ transforms a solution u of the equation $(\partial_x^2 + \omega^2)u = 0$ into a solution of the equation $(A_2 + \omega^2)v = (\partial_x^2 - q_2(x) + \omega^2)v = 0$ with the correspondence of initial conditions

$$\mathbf{T}_{2;-h}[u](0) = u(0), \quad (\mathbf{T}_{2;-h}[u])'(0) = u'(0) - hu(0). \tag{45}$$

Hence we have to verify the same properties for the operator T_2. Consider a solution u of the equation $(\partial_x^2 + \omega^2)u = 0$ for some particular $\omega \in \mathbb{C}$. The function u' is again a solution of this equation, and by Proposition 4, $\mathbf{T}_1 u$ is a solution of the equation $(\partial_x^2 - q_1 + \omega^2)v = 0$, so with the aid of the factorization (42) we obtain

$$\frac{1}{f} \partial_x f^2 \partial_x \frac{1}{f} \mathbf{T}_1[u'] = -\omega^2 \mathbf{T}_1[u']. \tag{46}$$

Let us apply the operator A_2 to the function $T_2 u$ and use the factorization (43).

$$A_2 T_2[u](x) = \left(f \partial_x \frac{1}{f^2} \partial_x f \right) \left(\frac{1}{f(x)} \left(\int_0^x f(s) \mathbf{T}_1[u'](s) \, ds + u(0) \right) \right) = \left(f \partial_x \frac{1}{f} \right) (\mathbf{T}_1[u']). \tag{47}$$

Applying $\frac{1}{f} \partial_x f$ to both sides of (47) due to (46) we obtain

$$\left(\frac{1}{f} \partial_x f \right) \left(A_2 T_2[u] \right) = \left(\frac{1}{f} \partial_x f^2 \partial_x \frac{1}{f} \right) (\mathbf{T}_1[u']) = -\omega^2 \mathbf{T}_1[u'] = -\omega^2 \left(\frac{1}{f} \partial_x f \right) (T_2[u]).$$

Hence the function $A_2 T_2[u]$ may differ from the function $-\omega^2 T_2[u]$ only by c/f, where c is a constant. To find the value of the constant we compute the values of both expressions for $x = 0$. We have

$$T_2[u](0) = u(0)$$

and

$$A_2 T_2[u](x) = \left(f \partial_x \frac{1}{f} \right) (\mathbf{T}_1[u'](x)) = -\frac{f'(x)}{f(x)} \mathbf{T}_1[u'](x) + (\mathbf{T}_1[u'](x))'.$$

As u' is a solution of $(\partial_x^2 + \omega^2)v = 0$, by Proposition 4 we have $\mathbf{T}_1[u'](0) = u'(0)$, $(\mathbf{T}_1[u'])'(0) = u''(0) + hu'(0) = -\omega^2 u(0) + hu'(0)$ (the last equality holds since the function u is also a solution of $(\partial_x^2 + \omega^2)v = 0$). Therefore

$$A_2 T_2[u](0) = -hu'(0) + \omega u(0) + hu'(0) = -\omega^2 u(0).$$

Hence $c = 0$ and the operator T_2 maps solutions of the equation $(\partial_x^2 + \omega^2)u = 0$ into solutions of the equation $(A_2 + \omega^2)v = 0$ for any $\omega \in \mathbb{C}$. To finish the proof, we have to check conditions (45) for the operator T_2. We have

$$T_2[u](0) = \frac{u(0)}{f(0)} = u(0)$$

and

$$(T_2[u])'(x) = -\frac{f'(x)}{f(x)} T_2[u](x) + \frac{1}{f(x)} \cdot f(x) \mathbf{T}_1[u'](x) = -\frac{f'(x)}{f(x)} T_2[u](x) + \mathbf{T}_1[u'](x),$$

hence for $x = 0$ we obtain

$$\big(T_2[u]\big)'(0) = -hT_2[u](0) + \mathbf{T}_1[u'](0) = -hu(0) + u'(0),$$

which finishes the proof. ∎

Now we show that the operator T_2 can be written as a Volterra integral operator and, as a consequence, extended by continuity to a wider class of functions.

Theorem 19 *The operator T_2 admits a representation as the Volterra integral operator*

$$T_2[u](x) = u(x) + \int_{-x}^{x} \mathbf{K}_2(x, t; -h) u(t)\, dt, \tag{48}$$

with the kernel
$$\mathbf{K}_2(x, t; -h) = -\frac{1}{f(x)} \bigg(\int_{-t}^{x} \partial_t \mathbf{K}_1(s, t; h) f(s)\, ds + \frac{h}{2} f(-t) \bigg). \tag{49}$$

Proof. Consider the expression

$$\int_0^x f(t) \mathbf{T}_1[u'](t)\, dt + u(0) = \int_0^x f(t) u'(t)\, dt + \int_0^x f(t) \int_{-t}^{t} \mathbf{K}_1(t, s; h) u'(s)\, ds\, dt + u(0).$$

Suppose that $x > 0$ (the opposite case is similar). We integrate by parts the first integral and change the order of integration in the second integral,

$$\int_0^x f(t) \mathbf{T}_1[u'](t)\, dt + u(0) =$$

$$f(x)u(x) - f(0)u(0) - \int_0^x f'(t)u(t)\, dt + \int_{-x}^{x} u'(s) \int_{|s|}^{x} \mathbf{K}_1(t, s; h) f(t)\, dt\, ds + u(0) =$$

$$f(x)u(x) - \int_0^x f'(t)u(t)\, dt + u(s) \int_{|s|}^{x} \mathbf{K}_1(t, s; h) f(t)\, dt \bigg|_{-x}^{x} - \int_{-x}^{x} u(s) \frac{d}{ds} \bigg(\int_{|s|}^{x} \mathbf{K}_1(t, s; h) f(t)\, dt \bigg) ds =$$

$$f(x)u(x) - \int_0^x f'(t)u(t)\, dt - \int_{-x}^{x} u(s) \frac{d}{ds} \bigg(\int_{|s|}^{x} \mathbf{K}_1(t, s; h) f(t)\, dt \bigg) ds. \tag{50}$$

To continue the proof we will use the extension of the kernel $\mathbf{K}_1$ onto the square $|x| \leq a, |t| \leq a$ as a continuously differentiable function. Consider the integral

$$\int_{-s}^{|s|} \mathbf{K}_1(t, s; h) f(t)\, dt.$$

For $s \leq 0$ it equals zero. For $s > 0$ note that by Theorem 15, $\mathbf{T}_1[1](x) = f(x)$, so we obtain from (29)

$$1 = \mathbf{T}_1^{-1}[f](x) = f(x) - \int_{-x}^{x} \mathbf{K}_1(t, x; h) f(t)\, dt.$$

Hence

$$\int_{-s}^{|s|} \mathbf{K}_1(t, s; h) f(t)\, dt = \begin{cases} 0, & \text{if } s \leq 0, \\ f(s) - 1, & \text{if } s > 0. \end{cases} \tag{51}$$

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Combining (50) and (51), we get

$$\int_0^x f(t)\mathbf{T}_1[u'](t)\,dt + u(0) = f(x)u(x) - \int_0^x f'(t)u(t)\,dt -$$

$$- \int_{-x}^x u(s)\frac{d}{ds}\left(\int_{-s}^x \mathbf{K}_1(t,s;h)f(t)\,dt - \int_{-s}^{|s|} \mathbf{K}_1(t,s;h)f(t)\,dt \right) ds =$$

$$= f(x)u(x) - \int_0^x f'(t)u(t)\,dt - \int_{-x}^x u(s)\frac{d}{ds}\left(\int_{-s}^x \mathbf{K}_1(t,s;h)f(t)\,dt \right) ds + \int_0^x u(s)f'(s)\,ds =$$

$$= f(x)u(x) - \int_{-x}^x u(s)\left(\int_{-s}^x \frac{d}{ds}\mathbf{K}_1(t,s;h)f(t)\,dt + \mathbf{K}_1(-s,s;h)f(-s) \right) ds =$$

$$= f(x)u(x) - \int_{-x}^x u(s)\left(\int_{-s}^x \frac{d}{ds}\mathbf{K}_1(t,s;h)f(t)\,dt + \frac{h}{2}f(-s) \right) ds.$$

Hence

$$T_2[u](x) = \frac{1}{f(x)}\left(\int_0^x f(\eta)\mathbf{T}_1[u'](\eta)\,d\eta + u(0) \right) = u(x) + \int_{-x}^x \mathbf{K}_2(x,t;-h)u(t)\,dt,$$

where the kernel $\mathbf{K}_2$ is given by (49). ∎

Remark 20 *Note that in the proof of Theorem 19 we made use only of the fact that $u \in C^1[-a,a]$ and never required that u be a solution of the equation $\partial_x^2 u + \omega^2 u = 0$. Therefore both representations for the operator T_2 obtained in Theorems 18 and 19 coincide on any function $u \in C^1[-a,a]$.*

Remark 21 *If the integral kernel $\mathbf{K}_1$ is known only in the domain $|x| \le a, |t| \le |x|$, from (50) it is also possible to obtain the expression of the integral kernel for the Volterra integral operator representing T_2. Since*

$$\frac{d}{dt}\left(\int_{|t|}^x \mathbf{K}_1(s,t;h)f(s)\,ds \right) = \int_{|t|}^x \frac{d}{dt}\mathbf{K}_1(s,t;h)f(s)\,ds - \mathbf{K}_1(|t|,t;h)f(|t|) \cdot (|t|)' =$$

$$= \begin{cases} \int_{|t|}^x \frac{d}{dt}\mathbf{K}_1(s,t;h)f(s)\,ds - \frac{h}{2}f(t) - \frac{f(t)}{2}\int_0^t q_1(s)\,ds & \text{for } t \ge 0, \\ \int_{|t|}^x \frac{d}{dt}\mathbf{K}_1(s,t;h)f(s)\,ds + \frac{h}{2}f(-t) & \text{for } t < 0, \end{cases}$$

the integral kernel for $x > 0$ is given by the expression

$$\mathbf{K}_2(x,t;-h) = \begin{cases} -\frac{1}{f(x)}\left(f'(t) + \int_{|t|}^x \frac{d}{dt}\big(\mathbf{K}_1(s,t;h)\big)f(s)\,ds - \frac{h}{2}f(t) - \frac{f(t)}{2}\int_0^t q_1(s)\,ds \right) & \text{if } t \ge 0, \\ -\frac{1}{f(x)}\left(\int_{|t|}^x \frac{d}{dt}\big(\mathbf{K}_1(s,t;h)\big)f(s)\,ds + \frac{h}{2}f(-t) \right) & \text{if } t < 0. \end{cases}$$

Corollary 22 *The operator T_2 given by (48) with the kernel (49) coincides with $\mathbf{T}_2$ on $C[-a,a]$.*

Proof. By Theorems 18 and 19 the Volterra operators T_2 and $\mathbf{T}_2$ coincide on the set of finite linear combinations of solutions of the equations $(\partial_x^2 + \omega^2)u = 0$, $\omega \in \mathbb{C}$. Since this set is dense in $C[-a,a]$, by continuity of T_2 and $\mathbf{T}_2$ we obtain that they coincide on the whole $C[-a,a]$. ∎

The next corollary follows immediately from Remark 20.

Corollary 23 *The operator T_2 given by (44) coincides with $\mathbf{T}_2$ on $C^1[-a,a]$.*

Operator A_1 is the Darboux transformation of the operator A_2 with respect to the solution $1/f$, hence we obtain another relation between the operators $\mathbf{T}_1$ and $\mathbf{T}_2$.

Corollary 24 *For any function $u \in C^1[-a, a]$ the equality*

$$\mathbf{T}_1[u](x) = f(x)\left(\int_0^x \frac{1}{f(\eta)}\mathbf{T}_2[u'](\eta)\, d\eta + u(0)\right) \tag{52}$$

is valid.

From the second commutative diagram at the beginning of this subsection we may deduce some commutation relations between the operators $\mathbf{T}_1$, $\mathbf{T}_2$ and d/dx. The proof immediately follows from (44) and (52).

Corollary 25 *The following operator equalities hold on $C^1[-a, a]$:*

$$\partial_x f \mathbf{T}_2 = f \mathbf{T}_1 \partial_x \tag{53}$$

$$\partial_x \frac{1}{f}\mathbf{T}_1 = \frac{1}{f}\mathbf{T}_2\partial_x. \tag{54}$$

Commutation relations (53) and (54) allow us to prove a more general version of [3, Theorem 11] which is also Theorem 8 with different conditions on the potential q.

Theorem 26 *Under the conditions of Theorem 15 the operator* (15) *with the kernel defined by* (16) *satisfies*

$$\left(-\frac{d^2}{dx^2} + q(x)\right)\mathbf{T}_h[u] = \mathbf{T}_h\left[-\frac{d^2u}{dx^2}\right]$$

for any $u \in C^2[-a, a]$.

Proof. Let $\mathbf{T}_2$ be the transmutation operator of the Darboux transformation of the operator $A = -\frac{d^2}{dx^2} + q(x)$ with respect to the solution f. Let $u \in C^2[-a, a]$, then $u' \in C^1[-a, a]$ and we obtain by (53), (54) and (42)

$$\mathbf{T}_h\left[\frac{d^2u}{dx^2}\right] = \frac{1}{f}\frac{d}{dx}\left[f\mathbf{T}_2 u'\right] = \frac{1}{f}\frac{d}{dx}f^2\frac{d}{dx}\frac{1}{f}\mathbf{T}_h u = \left(\frac{d^2}{dx^2} - q(x)\right)\mathbf{T}_h u.$$

■

In [20] the following notion of generalized derivatives was introduced. Consider a function g assuming that both f and g possess the derivatives of all orders up to the order n on the segment $[-a, a]$. Then in $[-a, a]$ the following generalized derivatives are defined

$$\gamma_0(g)(x) = g(x),$$

$$\gamma_k(g)(x) = \left(f^2(x)\right)^{(-1)^{k-1}}\left(\gamma_{k-1}(g)\right)'(x)$$

for $k = 1, 2, \ldots, n$.

Let a function u be defined by the equality

$$g = \frac{1}{f}\mathbf{T}_1 u,$$

and assume that $u \in C^n[-a, a]$. Note that below we do not necessarily require that the functions f and g be from $C^n[-a, a]$. With the use of (53) and (54) we have

$$\gamma_1(g) = f^2 \cdot \left(\frac{1}{f}\mathbf{T}_1 u\right)' = f^2 \cdot \frac{1}{f}\mathbf{T}_2 u' = f\mathbf{T}_2 u',$$

$$\gamma_2(g) = \frac{1}{f^2} \cdot \left(f\mathbf{T}_2 u'\right)' = \frac{1}{f^2} \cdot f\mathbf{T}_1 u'' = \frac{1}{f}\mathbf{T}_1 u''.$$

By induction we obtain the following corollary.

Corollary 27 *Let $u \in C^n[-a, a]$ and $g = \frac{1}{f}\mathbf{T}_1 u$. Then*

$$\gamma_k(g) = f\mathbf{T}_2 u^{(k)} \qquad \text{if } k \text{ is odd, } k \leq n,$$

and

$$\gamma_k(g) = \frac{1}{f}\mathbf{T}_1 u^{(k)} \qquad \text{if } k \text{ is even, } k \leq n.$$

To finish this subsection, let us consider how the operator $\mathbf{T}_{2;-h}$ acts on powers of the independent variable. By Theorem 15 we have to construct the system of functions $\{\varphi_k\}$ by formulas (30)–(32) and (33) starting with the function $1/f$. As it can be seen from (30)–(32), for any nonnegative integer k we have

$$\widetilde{X}^{(k)}_{1/f} = X^{(k)}_f \qquad \text{and} \qquad X^{(k)}_{1/f} = \widetilde{X}^{(k)}_f,$$

where the subindex f or $1/f$ corresponds to the starting function used in (30)–(32). Therefore constructing the functions $X^{(k)}$, $\widetilde{X}^{(k)}$ according to (30)–(32) and defining

$$\psi_k(x) = \begin{cases} \frac{1}{f(x)} X^{(k)}, & k \text{ even,} \\ \frac{1}{f(x)} \widetilde{X}^{(k)}, & k \text{ odd,} \end{cases} \tag{55}$$

(here the "second half" of the formal powers (30)–(32) is used, cf. (33)) we obtain the following statement.

Proposition 28 *Let a potential q_1 and a function f be as in Theorem 15. Let the operator $\mathbf{T}_{2;-h}$ be the transmutation operator for the Darboux transformed operator A_2. Then $\mathbf{T}_{2;-h}$ transforms x^k into $\psi_k(x)$ for any $k \in \mathbb{N}_0$.*

3.3 Examples

We start with the operator $A_0 = d^2/dx^2$. We have to pick up such a solution f of the equation $A_0 f = 0$ that $f'/f \neq 0$. This is in order to obtain an operator $A_1 \neq A_0$ as a result of the Darboux transformation of A_0. As such solution consider, e.g., $f_0(x) = x+1$. Both f_0 and $1/f_0$ are bounded on any segment $[-a, a] \subset (-1; 1)$ and the Darboux transformed operator has the form $A_1 = \frac{d^2}{dx^2} - \frac{2}{(x+1)^2}$.

The transmutation operator T for A_0 is obviously an identity operator and $K_0(x, t; 0) = 0$. Since $f'_0(0) = 1$, we look for the parametrized operator $\mathbf{T}_{0;1}$. Its kernel is given by (18): $\mathbf{K}_0(x, t; 1) = 1/2$. From Theorem 19 we obtain the transmutation kernel for the operator A_1

$$\mathbf{K}_1(x, t; -1) = -\frac{1}{x+1} \cdot \frac{1-t}{2} = \frac{t-1}{2(x+1)}, \tag{56}$$

the kernel from Example 7.

To obtain a less trivial example consider again the operator $A_1 = \frac{d^2}{dx^2} - \frac{2}{(x+1)^2}$ and the function $f_1(x) = (x+1)^2$ as a solution of $A_1 f = 0$. Since $h = f'_1(0) = 2$, we compute $\mathbf{K}_1(x, t; 2)$ from (56) using (18)

$$\mathbf{K}_1(x, t; 2) = \frac{3x^2 + 6x + 4 - 3t^2 + 2t}{4(x+1)}.$$

The Darboux transformation of the operator A_1 with respect to the solution f_1 is the operator $A_2 = \frac{d^2}{dx^2} - \frac{6}{(x+1)^2}$ and by Theorem 19 the transmutation operator $\mathbf{T}_{2;-2}$ for A_2 is given by the Volterra integral operator (15) with the kernel

$$\mathbf{K}_2(x, t; -2) = -\frac{1}{(x+1)^2}\left(\int_{-t}^x \frac{-3t+1}{2(s+1)}(s+1)^2\, ds + (1-t)^2\right) = \frac{(3t-1)(x+1)^2 - 3(t-1)^2(t+1)}{4(x+1)^2}.$$

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This procedure may be continued iteratively. Consider operators

$$A_n := \frac{d^2}{dx^2} - \frac{n(n+1)}{(x+1)^2}.$$

$f_n(x) = (x+1)^{n+1}$ is a solution of the equation $A_n f = 0$. The Darboux transformation of the operator A_n with respect to the solution f_n is the operator

$$\frac{d^2}{dx^2} - 2\Big(\frac{f_n'(x)}{f_n(x)}\Big)^2 + \frac{n(n+1)}{(x+1)^2} = \frac{d^2}{dx^2} - \frac{(n+1)(n+2)}{(x+1)^2},$$

i.e., exactly the operator A_{n+1}. If we know $\mathbf{K}_n(x,t;-n)$ for the operator A_n, by (18) we compute the kernel $\mathbf{K}_n(x,t;n+1)$ corresponding to the solution $f_n(x)$ and by Theorem 19 we may calculate the kernel $\mathbf{K}_{n+1}(x,t;-n-1)$. Careful analysis shows that we have to integrate only polynomials in all integrals involved, so the described procedure can be performed up to any fixed n.

Consider the Schrödinger equation

$$u'' + 2\operatorname{sech}^2(x)\, u = u. \tag{57}$$

This equation appears in soliton theory and as an example of a reflectionless potential in the one-dimensional quantum scattering theory (see, e.g. [22]). Equation (57) can be obtained as a result of the Darboux transformation of the equation $u'' = u$ with respect to the solution $f(x) = \cosh x$. The transmutation operator for the operator $A_1 = \partial_x^2 - 1$ was calculated in [3, Example 3]. Its kernel is given by the expression

$$\mathbf{K}_1(x,t;0) = -\frac{1}{2}\frac{\sqrt{x^2 - t^2} I_1(\sqrt{x^2 - t^2})}{x - t},$$

where I_1 is the modified Bessel function of the first kind. Hence from Theorem 19 we obtain the transmutation kernel for the operator $A_2 = \partial_x^2 + 2\operatorname{sech}^2 x - 1$

$$\mathbf{K}_2(x,t;0) = \frac{1}{2\cosh(x)} \int_{-t}^{x} \left(\frac{I_0(\sqrt{s^2 - t^2})t}{s - t} + \frac{\sqrt{s^2 - t^2} I_1(\sqrt{s^2 - t^2})}{(s - t)^2} \right) \cosh s\, ds.$$

4 Transmutation operator for the one-dimensional Dirac equation with a Lorentz scalar potential

One-dimensional Dirac equations with Lorentz scalar potentials are widely studied (see, for example, [5, 6, 12, 13, 14, 15, 16, 27, 32, 34] and [28] for intertwining techniques for it).

According to [27] the Dirac equation in one space dimension with a Lorentz scalar potential can be written as

$$(\partial_x + m + S(x))\psi_1 = E\psi_2, \tag{58}$$
$$(-\partial_x + m + S(x))\psi_2 = E\psi_1, \tag{59}$$

where m ($m > 0$) is the mass and $S(x)$ is a Lorentz scalar. Denote $\eta = m + S$ and write the system (58), (59) in a matrix form as

$$\begin{pmatrix} \partial_x + \eta & 0 \\ 0 & \partial_x - \eta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{60}$$

In order to apply the results on the transmutation operators and factorizations (42), (43) we consider a function f such that

$$\frac{f'(x)}{f(x)} = -\eta = -m - S(x).$$

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We can take $f(x) = \exp\left(-\int_0^x (m + S(s))\, ds\right)$, then $f(0) = 1$ and f does not vanish. Suppose the operators $\mathbf{T}_1$ and $\mathbf{T}_2$ are transmutations for the operators $A_1 = \left(\partial_x + \frac{f'}{f}\right)\left(\partial_x - \frac{f'}{f}\right)$ and $A_2 = \left(\partial_x - \frac{f'}{f}\right)\left(\partial_x + \frac{f'}{f}\right)$ respectively (corresponding to functions f and $1/f$ in the sense of Proposition 4). We will look for a solution of equation (60) in the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where u_1 and u_2 are some functions. From the commutation relations (53) and (54) we have

$$\begin{pmatrix} \partial_x - \frac{f'}{f} & 0 \\ 0 & \partial_x + \frac{f'}{f} \end{pmatrix} \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f\partial_x \frac{1}{f}\mathbf{T}_1 & 0 \\ 0 & \frac{1}{f}\partial_x f \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_2 & 0 \\ 0 & \mathbf{T}_1 \end{pmatrix} \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

hence

$$\begin{pmatrix} \mathbf{T}_2 & 0 \\ 0 & \mathbf{T}_1 \end{pmatrix} \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

By multiplying both sides by the inverse matrix $\begin{pmatrix} \mathbf{T}_2^{-1} & 0 \\ 0 & \mathbf{T}_1^{-1} \end{pmatrix}$ we obtain

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E \begin{pmatrix} \mathbf{T}_2^{-1} & 0 \\ 0 & \mathbf{T}_1^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =$$
$$= E \begin{pmatrix} 0 & \mathbf{T}_2^{-1} \\ -\mathbf{T}_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Therefore the operator $\begin{pmatrix} \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix}$ transmutes any solution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the system

$$u_1' = E u_2 \tag{61}$$
$$u_2' = -E u_1 \tag{62}$$

into the solution $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of the system (58), (59) with the initial conditions $\psi_1(0) = u_1(0)$, $\psi_2(0) = u_2(0)$. And vice versa if $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a solution of the system (58), (59), then the operator $\begin{pmatrix} \mathbf{T}_1^{-1} & 0 \\ 0 & \mathbf{T}_2^{-1} \end{pmatrix}$ transmutes it into the solution $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of (61), (62) such that $u_1(0) = \psi_1(0)$, $u_2(0) = \psi_2(0)$.

5 Conclusions

An explicit representation for the transmutation operator corresponding to a Darboux transformed Schrödinger operator is given in terms of the transmutation kernel for its superpartner and as a corollary the transmutation operator for the one-dimensional Dirac system with a scalar potential is obtained. Several examples of explicitly constructed transmutation operators are given. We expect that the techniques developed in the present paper will be used in practical applications of the transmutation operators.

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