Deformation Expression for Elements of Algebras (V) –Diagonal matrix calculus and ∗-special functions–

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In this note, we are interested in the $*$-version of various special functions. Noting that many special functions are defined by integrals involving the exponential functions, we define $*$-special functions by similar integral formula replacing exponential functions by $*$-exponential functions.

As it is seen in previous notes, a $*$-exponential function of a nondegenerate quadratic form has remarkable properties which are not seen in ordinary exponential functions.

To treat these it is convenient to use diagonal matrix expressions developed in this note.

1 Summary of fundamental properties of $*$-exponential functions

The Weyl algebra $(W_2; *)$ is the algebra generated by u, v with $u*v-v*u = -i\hbar$. For an arbitrary fixed 2×2-complex symmetric matrix K, we set $\Lambda = K+J$ where J is the standard skew-symmetric matrix $J=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Setting $(u,v)=(u_1,u_2)$, we define a product $*_K$ on the space of polynomials $\mathbb{C}[u_1, u_2]$ by the formula

$$(1.1) \qquad f *_K g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}}\Lambda^{ij}\overrightarrow{\partial_{u_j}})}g = \sum_k \frac{(i\hbar)^k}{k!2^k}\Lambda^{i_1 j_1}\cdots\Lambda^{i_k j_k}\partial_{u_{i_1}}\cdots\partial_{u_{i_k}}f\ \partial_{u_{j_1}}\cdots\partial_{u_{j_k}}g.$$

It is known and not hard to prove that $(\mathbb{C}[u_1, u_2], *_K)$ is isomorphic to $(W_2; *)$. If K is fixed, then every element $A\in(W_2; *)$ is expressed in the form of ordinary polynomial, which we denote by $:A:_K \in \mathbb{C}[u_1, u_2]$. Thus, for every K, $(\mathbb{C}[u_1, u_2], *_K)$ gives a concrete representation of the Weyl algebra, and the product formula (1.1) offers several ways to calculate transcendental elements.

All $(\mathbb{C}[u_1, u_2], *_K)$ are mutually isomorphic, that is, there is an isomorphism

$$I_K^{K'} : (\mathbb{C}[u_1, u_2], *_K) \to (\mathbb{C}[u_1, u_2], *_{K'})$$

called **intertwiner** for every K and K'. This is given by

$$I_K^{K'}(f(u_1, u_2)) = \Big(\exp \frac{i\hbar}{4} \sum (K'_{ij}-K_{ij})\partial_i\partial_j\Big)f(u_1, u_2).$$

The intertwiner $I_K^{K'}$ extends for several class of transcendental elements. This works pretty well for exponential functions of linear functions, and it is still amenable for exponential functions of

quadratic functions (cf.[16], [17]), but in general the extended intertwiner $I_K^{K'}$ is neither continuous nor 1-to-1.

For an element $H_* \in (W_2; *)$, the $*$-exponential function of H_* is defined by the evolution equation

$$(1.2) \qquad \frac{d}{dt} f_t = :H_*:_K * f_t, \quad f_0 = 1.$$

If the real analytic solution exists then the solution is denoted by $:e_*^{tH_*}:_K$. If the real analytic solution of the initial data g exists then the solution is denoted by $:e_*^{tH_*} * g:_K$.

In what follows we mainly concern the case $H_* = \frac{1}{i\hbar} u \circ v$, where $u \circ v = \frac{1}{2}(u*v+v*u)$.

For $K = \begin{bmatrix} \delta & c \\ c & \delta' \end{bmatrix}$, by setting $\Delta = e^t + e^{-t} - c(e^t - e^{-t})$, the solution of (1.2) is given by

$$(1.3) \qquad :e_*^{t\frac{1}{i\hbar}2u\circ v}:_K = \frac{2}{\sqrt{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'}} \, e^{\frac{1}{i\hbar} \frac{e^t - e^{-t}}{\Delta^2 - (e^t - e^{-t})^2 \delta \delta'}\left((e^t - e^{-t})(\delta' u^2 + \delta v^2) + 2\Delta uv\right)}.$$

As $:e_*^{z\frac{1}{i\hbar}2u\circ v}:_K$ has double branched singularities in general, we have to prepare two $\pm$ sheets with slits. Hence, we have two origins 0_+, 0_-. First, note that $:e_*^{0\frac{1}{i\hbar}2u\circ v}:_K = \sqrt{1}$. Thus, we set

$$(1.4) \qquad :e_*^{0_+\frac{1}{i\hbar}u\circ v}:_K = 1, \quad :e_*^{0_-\frac{1}{i\hbar}u\circ v}:_K = -1.$$

Note that $:e_*^{\pm\pi i\frac{1}{i\hbar}2u\circ v}:_K = \sqrt{1}$, but this is *not* an absolute scalar. The $\pm$ sign depends on K and the path form 0 to πi by setting $:e_*^{0\frac{1}{i\hbar}2u\circ v}:_K = 1$. As this is a scalar-like element belonging to a one parameter subgroup, we call it a *q-scalar*.

If $t = \pm\frac{\pi i}{2}$ in (1.3), then $:e_*^{\pm\frac{\pi i}{2}\frac{1}{i\hbar}2u\circ v}:_K$ is called the **polar element** and denoted by ε_{00} [15], [16]:

$$(1.5) \qquad :\varepsilon_{00}:_K = \frac{1}{\sqrt{c^2 - \delta\delta'}} e^{\frac{1}{i\hbar}\frac{1}{c^2 - \delta\delta'}(\delta' u^2 + \delta v^2 - 2cuv)}.$$

Let $Hol(\mathbb{C}^2)$ be the space of all holomorphic functions of $(u, v) \in \mathbb{C}^2$ with the uniform convergent topology on each compact subset. In [17], we summarize properties for generic expression parameters K as follows:

(**a**) : $e_*^{z\frac{2}{i\hbar}u\circ v}$ has no singular point on the real axis and the pure imaginary axis.

(**b**) $e_*^{z\frac{2}{i\hbar}u\circ v}$ is rapidly decreasing along any line parallel to the real line.

(**c**) $e_*^{z\frac{2}{i\hbar}u\circ v}$ is a $Hol(\mathbb{C}^2)$-valued $2\pi i$-periodic function, i.e. $:e_*^{(z+2\pi i)\frac{2}{i\hbar}u\circ v}:_K = :e_*^{z\frac{2}{i\hbar}u\circ v}:_K$. More precisely, it is πi-periodic or alternating πi-periodic depending on the real part of z

(**d**) As a result, $e_*^{z\frac{2}{i\hbar}u\circ v}$ must have periodic singular points. But the singular points are double branched. Hence $e_*^{z\frac{2}{i\hbar}u\circ v}$ is double valued with the sign ambiguity. Singular point set Σ_K is distributed πi-periodically along the two lines parallel to the imaginary axis.

(**e**) By requesting 1 at $z = 0$, i.e. $:e_*^{0\frac{2}{i\hbar}u\circ v}:_K = 1$, the value $:e_*^{[0\sim z]\frac{2}{i\hbar}u\circ v}:_K$ is determined uniquely, where $[0\sim z]$ is a path from 0 to z avoiding Σ_K and evaluating at z. Thus in spite of the double valued nature, the exponential law

$$e_*^{z\frac{2}{i\hbar}u\circ v} * e_*^{z'\frac{2}{i\hbar}u\circ v} = e_*^{(z+z')\frac{2}{i\hbar}u\circ v}$$

holds under the calculation such that $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

1.1 Exchanging interval and idempotent elements

As the pattern of periodicity depends on how the circle $\{re^{i\theta}; \theta \in \mathbb{R}\}$ round the singular points, it depends delicately on K. As it is mentioned in (**d**), there is two lines $\mathrm{Re}\,z=a$, $\mathrm{Re}\,z=b$ on which singular points of $e_*^{z\frac{2}{i\hbar}u\circ v}$ are sitting πi-periodically. The interval (a,b) depends on the expression parameter K. We denote this by $I_\circ(K)=(a,b)$, and we call it the (sheet) **exchanging interval** of $e_*^{z\frac{1}{i\hbar}2u\circ v}$.

As the exponential law shows that $e_*^{z\frac{1}{i\hbar}2u*v}$, $e_*^{z\frac{1}{i\hbar}2v*u}$ have singular points on the same two lines, $I_\circ(K)$ is called also the exchanging interval of these.

The periodicity of $:e_*^{(s+it)\frac{1}{i\hbar}2u\circ v}:_K$ w.r.t. t depends on s as follows:

If $s<a$, or $s>b$, then $e_*^{(s+it)\frac{1}{i\hbar}2u\circ v}$ is alternating π-periodic, and if $a<s<b$, then $e_*^{(s+it)\frac{1}{i\hbar}2u\circ v}$ is π-periodic.

There are three disjoint open subsets $\mathfrak{K}_\pm$ and $\mathfrak{K}_0$ of the space of 2×2 symmetric of all expression parameters such that

(i) $\mathfrak{K}_+ \cup \mathfrak{K}_- \cup \mathfrak{K}_0$ is dense.

(ii) If $K \in \mathfrak{K}_0$, then $a<0<b$. $:\varepsilon_{00}^2:_K = 1$, and $\varpi_*(0) = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{it\frac{1}{i\hbar}u\circ v}dt$ is called the **pseudo-vacuum**. Cauchy's integral theorem shows that $\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u\circ v}dt$ is independent of s whenever $a<s<b$.

(iii) If $K \in \mathfrak{K}_+$, then $0<a$ and $:\varepsilon_{00}^2:_K = -1$. $\varpi_{00} = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{it\frac{1}{i\hbar}(u\circ v-\frac{1}{2})}dt$ is called the **vacuum**. Cauchy's integral theorem shows that $\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{(s+it)(\frac{1}{i\hbar}u\circ v-\frac{1}{2})}dt$ is independent of s whenever $s<a$.

(iv) If $K \in \mathfrak{K}_-$, then $b<0$ and $:\varepsilon_{00}^2:_K = -1$. $\overline{\varpi}_{00} = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{it\frac{1}{i\hbar}(u\circ v+\frac{1}{2})}dt$ is called the **bar-vacuum**. Cauchy's integral theorem shows that $\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{(s+it)(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}dt$ is independent of s whenever $b<s$.

Idempotent elements

It is not hard to show the idempotent properties of ϖ_{00}, $\overline{\varpi}_{00}$, $\varpi_*(0)$. As the double integral $\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{(it+i\tau)\frac{1}{i\hbar}(u\circ v-\frac{1}{2})}dtd\tau$ is welldefined, the exponential law and the change of variables gives that if $0<a$, then

$$\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{it\frac{1}{i\hbar}(u\circ v-\frac{1}{2})}dt*\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{i\tau\frac{1}{i\hbar}(u\circ v-\frac{1}{2})}d\tau=\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{it\frac{1}{i\hbar}(u\circ v-\frac{1}{2})}dt.$$

Others are obtained similarly. These are given by periodic integrals along the imaginary axis.

Besides these, if $|\mathrm{Re}\,z| < 1/2$ then the integral in generic ordered expression

$$\int_{-\infty}^{\infty} :e_*^{t(z+\frac{1}{i\hbar}2u\circ v+i\xi)}:_K ds=\int_{-\infty}^{\infty} \frac{e^{-it(iz-\xi)}}{\sqrt{\Delta^2-(e^t-e^{-t})^2\delta\delta'}}\, e^{\frac{1}{i\hbar}\frac{e^t-e^{-t}}{\Delta^2-(e^t-e^{-t})^2\delta\delta'}\left((e^t-e^{-t})(\delta'u^2+\delta v^2)+2\Delta uv\right)}dt.$$

is rapidly decreasing w.r.t. $\xi \in \mathbb{R}$. We denote this by

$$(1.6) \qquad \int_{-\infty}^{\infty} e^{s(z+\frac{1}{i\hbar}u\circ v+i\xi)}ds = \int_{-\infty}^{\infty} e^{-is(iz+\frac{1}{\hbar}u\circ v-\xi)}ds = \delta_*(iz+\frac{1}{\hbar}u\circ v-\xi), \quad |\mathrm{Re}\,z| < 1/2.$$

Moreover, the property similar to idempotency

(1.7) $$\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi)*\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi') = \delta(\xi-\xi')\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi), \quad |\mathrm{Re}\,z| < 1/2,$$

is proved directly as follows: As the double integral is welldefined, we have

$$\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi)*\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi') = \iint e_*^{-it(iz+\frac{1}{\hbar}u\circ v-\xi)}*e_*^{-is(iz+\frac{1}{\hbar}u\circ v-\xi')}dtds$$

$$= \iint e^{it\xi+is\xi'}e_*^{-i(t+s)(iz+\frac{1}{\hbar}u\circ v)}dtds = \iint e^{is(\xi'-\xi)}e_*^{-i\sigma(iz-\xi+\frac{1}{\hbar}u\circ v)}dsd\sigma = \delta(\xi'-\xi)\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi).$$

Similar to the properties (ii), (iii) and (iv), $\int_{-\infty}^{\infty} e^{(s+it)(z+\frac{1}{i\hbar}u\circ v+i\xi)}ds$ is independent of t whenever $|t|$ is very small by Cauchy's integral theorem. Thus, by differentiating these we have

$$(\frac{1}{i\hbar}u\circ v)*\delta_*(\frac{1}{\hbar}u\circ v) = 0, \quad (\frac{1}{i\hbar}u\circ v-\frac{1}{2})*\varpi_{00} = 0, \quad (\frac{1}{i\hbar}u\circ v+\frac{1}{2})*\overline{\varpi}_{00} = 0, \quad (\frac{1}{i\hbar}u\circ v)*\varpi_*(0) = 0.$$

But note that differentiations are not by complex variable, but by the real/pure imaginary variables. Now, recall the equality given in [17]: In generic ordered expression K, let $I_\circ(K) = (a,b)$ be the exchanging interval. We have then

(1.8) $$\frac{1}{4\pi}\int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}:_K d\sigma = \begin{cases} :\varpi_{00}:_K, & s < a \\ 0, & a < s \end{cases}$$

$$\frac{1}{4\pi}\int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)\frac{1}{i\hbar}v*u}:_K d\sigma = \begin{cases} 0, & s < b \\ :\overline{\varpi}_{00}:_K, & b < s. \end{cases}$$

Hence we see

(1.9) $$:(\frac{1}{i\hbar}u\circ v-\frac{1}{2})*\frac{1}{4\pi}\int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)(\frac{1}{i\hbar}u\circ v-\frac{1}{2})}:_K d\sigma = -\delta(s-a):\varpi_{00}:_K$$

$$:(\frac{1}{i\hbar}u\circ v+\frac{1}{2})*\frac{1}{4\pi}\int_{-2\pi}^{2\pi} :e_*^{(s+i\sigma)(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}:_K d\sigma = \delta(s-b):\overline{\varpi}_{00}:_K$$

It is easy to see that first equality of (1.8) is the solution of

$$\frac{d}{ds}f(s) = -\delta(s-a)\varpi_{00}*f(s), \quad f(-\infty) = \varpi_{00}.$$
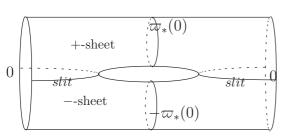
Similarly, we have
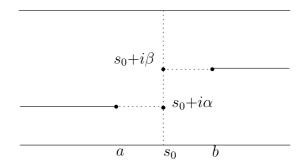
$$\frac{1}{4\pi}\int_0^{4\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}:_K dt = \begin{cases} 0, & s < a \\ :\tilde{D}_0:_K, & a < s < b, \\ 0, & b < s \end{cases}$$

where $\tilde{D}_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ which is independent of s, $a < s < b$ by Cauchy's integration theorem. Furthermore, $\tilde{D}_0 = \varpi_*(0)$ if $K \in \mathfrak{K}_0$. This gives

(1.10) $$:(\frac{1}{i\hbar}u\circ v)*\frac{1}{4\pi}\int_0^{4\pi} :e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}dt:_K = (\delta(s-a)-\delta(s-b)):\tilde{D}_0:_K$$

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To consider the discontinuity of $:\int_{\mathbb{R}} e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}ds:_K$ with respect to t, we have to fix several notations about singular points as in the r.h.s. figure:

It is not hard to see

$$(1.11) \qquad :(\frac{1}{i\hbar}u\circ v)*\int_{\mathbb{R}} e_*^{(s+it)(\frac{1}{i\hbar}u\circ v)}ds:_K = (\delta(t-\alpha) - \delta(t-\beta))\varpi_*(0)$$

1.2 Matrix elements

Three idempotent elements ϖ_{00}, $\overline{\varpi}_{00}$ and $\varpi_*(0)$ give matrix elements respectively.

Proposition 1.1 *In generic ordered expressions,* $E_{p,q} = \frac{1}{\sqrt{p!q!(i\hbar)^{p+q}}}u^p*\varpi_{00}*v^q$ *is the (p,q)-matrix element, that is* $E_{p,q}*E_{r,s} = \delta_{q,r}E_{p,s}$. *The K-expression* $:E_{p,q}:_K$ *of* $E_{p,q}$ *will be denoted by* $E_{p,q}(K)$. *Note that* $E_{0,0}(K)=:\varpi_{00}:_K$.

Proposition 1.2 $\overline{E}_{p,q} = \frac{\sqrt{-1}^{p+q}}{\sqrt{p!q!(i\hbar)^{p+q}}}v^p*\overline{\varpi}_{00}*u^q$ *is the (p,q)-matrix element. The K-expression of* $\overline{E}_{p,q}$ *will be denoted by* $\overline{E}_{p,q}(K)$. *Note that* $\overline{E}_{0,0}(K)=:\overline{\varpi}_{00}:_K$.

Different from the ordinary vacuum or bar-vacuum, we see $u*\varpi_*(0)\neq0$, $v*\varpi_*(0)\neq0$. But note that the bumping identity gives

$$(1.12) \qquad u^n*e_*^{it(\frac{1}{i\hbar}u\circ v)}=e_*^{it(\frac{1}{i\hbar}u\circ v-n)}*u^n, \quad v^n*e_*^{it(\frac{1}{i\hbar}u\circ v)}=e_*^{it(\frac{1}{i\hbar}u\circ v+n)}*v^n.$$

It is convenient to use the convention

$$(1.13) \qquad \zeta^k = \begin{cases} u^k, & k \geq 0 \\ v^{|k|}, & k < 0, \end{cases} \qquad \hat{\zeta}^\ell = \begin{cases} v^\ell, & \ell \geq 0 \\ u^{|\ell|}, & \ell < 0, \end{cases}$$

Proposition 1.3 *If* $K\in\mathfrak{K}_0$, *then* $:e_*^{it(\frac{1}{i\hbar}u\circ v)}:_K$ *is 2π-periodic and*

$$D_{k,\ell}(K) = \frac{1}{\sqrt{(\frac{1}{2})_k(\frac{1}{2})_\ell(i\hbar)^{|k|+|\ell|}}}\zeta^k*:\varpi_*(0):_K*\hat{\zeta}^\ell, \quad :\varpi_*(0):_K=\frac{1}{2\pi}\int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u\circ v)}dt:_K dt$$

are matrix elements for every $k,\ell\in\mathbb{Z}$, *where* $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$, *and extending convention* $(a)_{-n} = (a-1)(a-2)\cdots(a-n)$. *Note that* $D_{n,n}(K)=\frac{1}{2\pi}\int_0^{2\pi} :e_*^{it(\frac{1}{i\hbar}u\circ v-n)}:_K dt$.

Proof As the exponential law shows that the double integral $\iint (e_*^{it(\frac{1}{i\hbar}u\circ v)} * e_*^{i\tau(\frac{1}{i\hbar}u\circ v)})dtd\tau$ is wellde-fined, we see that if $k\neq\ell$, then the change of variables gives

$$\int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u\circ v+k)}ds * \int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u\circ v+\ell)}dt = \int_0^{2\pi} e^{it(k-\ell)}dt \int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u\circ v+\ell)}ds=0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u\circ v+k)}dt*\frac{1}{2\pi}\int_0^{2\pi} e_*^{is(\frac{1}{i\hbar}u\circ v+k)}ds=\frac{1}{2\pi}\int_0^{2\pi} e_*^{it(\frac{1}{i\hbar}u\circ v+k)}dt.$$

It is easy to see that the $*$-product $P(u,v)*\varpi_*(0)*Q(u,v)$ by any polynomials $P(u,v)$, $Q(u,v)$ is reduced to the shape $\phi*\varpi_*(0)*\psi$ where ϕ, ψ are polynomials of single variable u or v. Using (1.12), we have the desired result. $\qquad\square$

Every element of the Weyl algebra is represented by a matrix by the next theorems:

Theorem 1.1 *In the space* $Hol(\mathbb{C}^2)$,

$$\sum_{k=0}^{\infty} E_{k,k}(K)=1, \ \ if \ K\in\mathfrak{K}_+, \ \ \sum_{n=-\infty}^{\infty} D_{n,n}(K)=1, \ \ if \ K\in\mathfrak{K}_0, \ \ \sum_{k=0}^{\infty} \overline{E}_{k,k}(K)=1, \ \ if \ K\in\mathfrak{K}_-.$$

Precisely, they should be written as $\sum_{k=0}^{\infty} E_{k,k}(K)=:1:_K$ etc. for 1 here is not an absolute scalar.

Note We easily see that

(1.14) $$E_{p,q}*\overline{E}_{r,s} = 0 = \overline{E}_{r,s}*E_{p,q}$$

by using $\varpi_{00}*\overline{\varpi}_{00} = 0$ proved in [17], but here we repeat the calculation.

$$\int_{-\pi}^{\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v}d\sigma* \int_{-\pi}^{\pi} e_*^{(s'+i\sigma')\frac{1}{i\hbar}v*u}d\sigma' = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e_*^{(s+i\sigma)\frac{1}{i\hbar}u*v+(s'+i\sigma')\frac{1}{i\hbar}v*u}d\sigma d\sigma'$$

can be defined always to give 0, for by using $\frac{1}{i\hbar}u*v = \frac{1}{i\hbar}u\circ v-\frac{1}{2}$, and $\frac{1}{i\hbar}v*u = \frac{1}{i\hbar}u\circ v+\frac{1}{2}$, the change of variables gives

$$\int_{-\pi}^{\pi} e^{+i\sigma}d\sigma \int_{-\pi}^{\pi} e^{\frac{1}{2}(s'-s-i\tau)}e_*^{(s+s'+i\tau)\frac{1}{i\hbar}u\circ v}d\tau = 0.$$

Similarly one can prove $E_{k,k}(K)*D_{n,n}(K) = 0$ by direct calculations, but computing

$$D_{n,n}(K)*_K:\frac{1}{i\hbar}u\circ v:_K *_K E_{k,k}(K)$$

two ways $(a*b)*c = a*(b*c)$. We see $D_{n,n}(K)*E_{k,k}(K)=0$. That is, identities above are requested to protect associativity.

2 Fourier expansion of $:e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K$

Recall that the periodicity of $:e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K$ w.r.t. t depends on s, but it is 4π-periodic. Every $f(\theta)$ 4π-periodic function is written as the sum of a 2π-periodic and a alternating 2π-periodic functions:

$$f(\theta) = f_0(\theta)+f_-(\theta), \quad f_0(\theta)=\frac{1}{2}(f(\theta)+f(\theta+2\pi)), \quad f_-(\theta)=\frac{1}{2}(f(\theta)-f(\theta+2\pi)).$$

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The Fourier series of $f(\theta)$ is given as

$$f(\theta) = \sum_n \frac{1}{4\pi} \int_0^{4\pi} f(t)e^{-\frac{1}{2}int}dt\, e^{i\frac{n}{2}\theta}$$

$$= \sum_n \int_0^{2\pi} f_0(t)e^{-int}dt\, e^{in\theta} + \sum_n \frac{1}{2\pi}\int_0^{2\pi} f_-(t)e^{-i(n+\frac{1}{2})t}dt\, e^{i(n+\frac{1}{2})\theta}.$$

As the periodicity of $e_*^{(s+it)\frac{1}{i\hbar}(u\circ v)}$ w.r.t. t depends on s, we have to use Fourier basis depending on s. We denote the Fourier expansion

$$(2.1) \qquad :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K = \begin{cases} \sum_{k=0}^{\infty} \tilde{E}_k(K)e^{(s+it)(k+\frac{1}{2})}, & s < a \\ \sum_{n=-\infty}^{\infty} \tilde{D}_n(K)e^{(s+it)n}, & a < s < b \\ \sum_{k=0}^{\infty} \tilde{E}_{-k}(K)e^{-(s+it)(k+\frac{1}{2})}, & b < s \end{cases}$$

where

$$\tilde{E}_k(K) = \frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{-(s+it)(k+\frac{1}{2})}dt, \quad \tilde{D}_n(K) = \frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{-(s+it)n}dt$$

$$\tilde{E}_{-k}(K) = \frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{-(s+it)(-k-\frac{1}{2})}dt,$$

Cauchy's integral theorem shows that these are independent of s whenever $s < a$, $a < s < b$ and $b < s$ respectively.

Thus, taking $s \to -\infty$ (resp. $s \to \infty$) we see that

$$(2.2) \qquad \tilde{E}_k(K) = E_{k,k}(K), \quad \tilde{E}_{-k}(K) = \overline{E}_{k,k}(K), \quad \text{if } a < 0 < b, \text{ then } \tilde{D}_n(K) = D_{n,n}(K).$$

However, it should be noted that these identities do not directly imply the idempotency

$$\tilde{E}_k(K)*\tilde{E}_k(K) = \tilde{E}_k(K), \quad \tilde{E}_{-k}(K)*\tilde{E}_{-k}(K) = \tilde{E}_{-k}(K), \quad \tilde{D}_n(K)*\tilde{D}_n(K) = \tilde{D}_n(K).$$

Note $\tilde{D}_n(K)$ was denoted by $\tilde{D}_{n,n}(K)$ in the previous note [17]. As it is confusing, we changed the notation as above.

It is clear that $\tilde{D}_n(K) = D_{n,n}(K)$ if $a < 0 < b$. In general $\tilde{D}_n(K)$ may not be idempotent. However, $\tilde{D}_n(K)$ has an almost idempotent property as follows:

Lemma 2.1 *For $a < s, s' < b$, if there are s, s' such that $a < s+s' < b$ then*

$$\frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt *_K \frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s'+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt = \frac{1}{2\pi}\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt.$$

If such s, s' can not be selected, then $\int_0^{2\pi} :e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt * \int_0^{2\pi} :e_*^{(s'+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt$ is not welldefined.

Note Suppose $s, s' \in I_\circ(K)$ but $s+s' \notin I_\circ(K)$. By the 2π-periodicity, we have

$$\frac{1}{2\pi}\int_0^{2\pi} :e_*^{(\sigma+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt = \frac{1}{4\pi}\int_0^{4\pi} :e_*^{(\sigma+it)\frac{1}{i\hbar}u\circ v}:_K e^{(s+it)n}dt, \text{ for } \sigma = s, s.'$$

However, the product $e_*^{(s+s'+i(t+t'))\frac{1}{i\hbar}u\circ v}$ is alternating 2π-periodic hence

$$\left(\frac{1}{4\pi}\right)^2\int_0^{4\pi}\int_0^{4\pi} e_*^{(s+s'+i(t+t'))\frac{1}{i\hbar}u\circ v}dtdt' = 0.$$

But

$$\left(\frac{1}{2\pi}\right)^2\int_0^{2\pi}\int_0^{2\pi} e_*^{(s+s'+i(t+t'))\frac{1}{i\hbar}u\circ v}dtdt' = \frac{1}{2\pi}\int_0^{2\pi} e_*^{(s+s'+it)\frac{1}{i\hbar}u\circ v}dt \neq 0.$$

This sounds strange, for $a = b$ does not give $a^2=b^2$. Such strange phenomena caused by the discontinuity of $\int_0^{4\pi} e_*^{(s+s'+it)\frac{1}{i\hbar}u\circ v}dt$.

To avoid the possible confusion, it is better to use notations similar to (1.9), (1.10): Setting

$$V_n^{(+)}(s; K) = \frac{1}{4\pi}:\int_{-2\pi}^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u\circ v}e^{-(s+it)(n+\frac{1}{2})}dt:_K, \quad V_n^{(0)}(s; K) = \frac{1}{4\pi}:\int_{-2\pi}^{2\pi} e_*^{(s+it)\frac{1}{i\hbar}u\circ v}e^{-(s+it)n}dt:_K,$$

we denote

(2.3)
$$\left(\frac{1}{i\hbar}u\circ v\right)*V_n^{(+)}(s; *) = -\delta(s-a)\tilde{E}_n+\delta(s-b)\tilde{E}_{-n}$$
$$\left(\frac{1}{i\hbar}u\circ v\right)*V_n^{(0)}(s; *) = \delta(s-a)\tilde{D}_n-\delta(s-b)\tilde{D}_{-n}.$$

2.1 Element given by Fourier series

Suppose c_n is given as Fourier coefficients of a Schwartz distribution $h(z)$ defined on $|z| = 1$. Using its Fourier series we define three functions:

$$h_0(t)=\sum_{n=\infty}^{\infty} c_n e^{itn}, \quad h_+(t) = e^{-\frac{1}{2}it}h(t)=\sum_{n=\infty}^{\infty} c_n e^{it(n-\frac{1}{2})}, \quad h_-(t) = e^{\frac{1}{2}it}h(t)=\sum_{n=\infty}^{\infty} c_n e^{it(n+\frac{1}{2})}.$$

$h_0(t)$ is 2π-periodic in w.r.t. t, and others are 2π-alternating periodic. Hence the pairing with $:e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K$ defined below give respectively an element of $Hol(\mathbb{C}^2)$:

(1) If $a < s < b$, then

$$h_{0*}(s, \frac{1}{i\hbar}u\circ v) = \frac{1}{2\pi}\int_0^{2\pi} h(t):e_*^{(s+it)\frac{1}{i\hbar}u\circ v}:_K dt= \sum_{n=-\infty}^{\infty} c_{-n}e^{sn}\tilde{D}_n(K).$$

This is an element of $Hol(\mathbb{C}^2)$.

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(2) If $s < a$, then

$$h_{+*}(s, \frac{1}{i\hbar}u{\circ}v){=}\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{1}{2}it}h(t){:}e_*^{(s+it)\frac{1}{i\hbar}u{\circ}v}{:}_K dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sum_{n\in\mathbb{Z}} c_n e^{it(n-\frac{1}{2})}\sum_{k\geq 0} E_{k,k}(K)e^{(s+it)(k+\frac{1}{2})}dt{=}\sum_{n=0}^{\infty} c_{-n}e^{s(n+\frac{1}{2})}E_{n,n}(K)$$

as ${:}e_*^{(s+it)\frac{1}{i\hbar}u{\circ}v}{:}_K$ have only positive Fourier coefficients. For $s < a$, $\sum_{n=0}^{\infty} c_{-n}e^{s(n+\frac{1}{2})}E_{n,n}(K)$ is an element of $Hol(\mathbb{C}^2)$, but each $c_{-n}e^{s(n+\frac{1}{2})}E_{n,n}(K)$ is an element of $Hol(\mathbb{C}^2)$ for any s. Note also that $\frac{1}{i\hbar}u{\circ}v-\frac{1}{2} = \frac{1}{i\hbar}u{*}v$ to understand the meaning of $\frac{1}{2}$.

(3) If $b < s$, then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{2}it}h(t){:}e_*^{(s+it)\frac{1}{i\hbar}u{\circ}v}{:}_K dt{=}\frac{1}{2\pi}\int_0^{2\pi}\sum_{n\in\mathbb{Z}} c_n e^{it(n+\frac{1}{2})}\sum_{k\geq 0}\overline{E}_{k,k}(K)e^{-(s+it)(k+\frac{1}{2})}dt$$

$$=\sum_{n=0}^{\infty} c_n e^{-s(n+\frac{1}{2})}\overline{E}_{n,n}(K)$$

as $e^{\frac{1}{2}(s+it)}{:}e_*^{(s+it)\frac{1}{i\hbar}u{\circ}v}{:}_K$ is holomorphic on the on the outside of the disk $|e^{s+it}|{\geq}e^b$ having only negative Fourier coefficients. $\sum_{n=0}^{\infty} c_n e^{-s(n+\frac{1}{2})}\overline{E}_{n,n}(K)$ is an element of $Hol(\mathbb{C}^2)$, but each $c_n e^{-s(n+\frac{1}{2})}\overline{E}_{n,n}(K)$ is an element of $Hol(\mathbb{C}^2)$ for any s.

Then, these converge for $s < a$, $a < s < b$ and $b < s$ respectively in $C^\infty(S^1, Hol(\mathbb{C}^2))$ ($Hol(\mathbb{C}^2)$-valued smooth functions on S^1 with C^∞-topology).

Summarizing these we have

Theorem 2.1 *Let $\{c_n\}_{n\in\mathbb{Z}}$ be Fourier coefficients of a Schwartz distribution on S^1. Then in generic ordered expression, $\sum_{n=-\infty}^{\infty} c_{-n}e^{sn}\tilde{D}_n(K)$, $\sum_{n=0}^{\infty} c_{-n}e^{s(n+\frac{1}{2})}E_{n,n}(K)$, $\sum_{n=0}^{\infty} c_n e^{-s(n+\frac{1}{2})}\overline{E}_{n,n}(K)$ are elements of $Hol(\mathbb{C}^2)$ respectively for $a < s < b$, $s < a$ and $b < s$.*

It is remarkable that there is no singular point in the diagonal matrix expressions.

Applying Theorem2.1 to a function $h(z, t) = \sum_{n=-\infty}^{\infty} \frac{1}{(z+\frac{1}{2})_{-n}}e^{itn}$ depending on z, we see for instance that

$$\sum_{n=0}^{\infty}\frac{1}{(z+\frac{1}{2})_n}e^{s(n+\frac{1}{2})}E_{n,n}(K)$$

converges to an element of $Hol(\mathbb{C}^2)$, where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$ and $(a)_{-n} = (a-1)(a-2)\cdots(a-n)$.

2.2 Analytic continuation of $\delta_*(iz+\frac{1}{\hbar}u{\circ}v)$

In this section, we show that $\delta_*(iz+\frac{1}{\hbar}u{\circ}v)$ behaves like an idempotent element in the calculation based on the complex integral and residue calculus. It is known that $\delta_*(iz+\frac{1}{\hbar}u{\circ}v)$ is given by the difference of two different inverses:

$$\delta_*(iz+\frac{1}{\hbar}u{\circ}v) = (z+\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1} - (z+\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}$$

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which is holomorphic on the domain $|\mathrm{Re}\,z| < \frac{1}{2}$. By making analytic continuations of inverses, we have seen in [17] the next

Theorem 2.2 $\delta_*(iz+\frac{1}{\hbar}u\circ v)$ *is analytically continued on the space* $\mathbb{C}\backslash(\mathbb{Z}+\frac{1}{2})$ *as an* $Hol(\mathbb{C}^2)$-*valued holomorphic functions of* z *with simple poles.*

The residues of $:\delta_*(iz+\frac{1}{\hbar}u\circ v):_K$ *at* $-(n+\frac{1}{2})$, $(n+\frac{1}{2})$ *are* $E_{n,n}(K)$, $-\overline{E}_{n,n}(K)$ *respectively.*

Theorem 2.2 can be proved directly by taking integral of (2.1) as follows:

$$:\delta_*(iz+\frac{1}{\hbar}u\circ v):_K = \sum_{k=0}^{\infty}\frac{e^{a(z+k+\frac{1}{2})}}{z+(k+\frac{1}{2})}\tilde{E}_k(K)+ \sum_{n=-\infty}^{\infty}\frac{e^{b(z+n)}-e^{a(z+n)}}{z+n}\tilde{D}_n(K)- \sum_{k=0}^{\infty}\frac{e^{b(z-(k+\frac{1}{2})}}{z-(k+\frac{1}{2})}\tilde{E}_{-k}(K)$$

Note that $\frac{e^{b(z+n)}-e^{a(z+n)}}{z+n}\tilde{D}_n(K)$ is not singular at $z=-n$, and the singular points and the residues are calculated by the terms $\tilde{E}_k(K)$, $k \in \mathbb{Z}$. Thus, analytically continued $\delta_*(\zeta+\frac{1}{\hbar}u\circ v)$ may be viewed in generic ordered expression K that

$$:\delta_*(\zeta+\frac{1}{\hbar}u\circ v):_K = \sum_{n=0}^{N}\frac{i}{\zeta+i(n+\frac{1}{2})}E_{n,n}(K) - \sum_{n=0}^{N}\frac{i}{\zeta-i(n+\frac{1}{2})}\overline{E}_{n,n}(K) + \Phi_K^N(\zeta),$$

where $\Phi_K^N(\zeta)$ is holomorphic on $|\mathrm{Im}\,\zeta| < N+\frac{1}{2}$. Theorem 2.2 gives also the following:

Proposition 2.1 *For every holomorphic function* $f(z)$ *defined on a simply connected domain* D, *the following equality holds for generic ordered expression: For every smooth simple closed curve* C *in* D:

$$(2.4) \qquad \frac{1}{2\pi i}\int_C f(w)\delta_*(z-w+\frac{1}{\hbar}u\circ v)dw = \sum_n f(z+i(n+\frac{1}{2}))E_{n,n} - \sum_n f(z-i(n+\frac{1}{2}))\overline{E}_{n,n}$$

where the each summation runs through all integers $n \geq 0$ *such that* $-i(n+\frac{1}{2})$ *and* $i(n+\frac{1}{2})$ *are inside of* C *respectively.*

Thus, we define

$$(2.5) \qquad f_*^C(z+\frac{1}{\hbar}u\circ v) = \frac{1}{2\pi i}\int_C f(w)\delta_*(z-w+\frac{1}{\hbar}u\circ v)dw$$

for every holomorphic function f and

$$1_*^C = \frac{1}{2\pi i}\int_C \delta_*(z-w+\frac{1}{i\hbar}u\circ v)dw = \sum_k E_{k,k}(K) - \sum_k \overline{E}_{k,k}(K).$$

Now recalling that $E_{n,n}*\overline{E}_{k,k} = 0 = \overline{E}_{n,n}*E_{k,k}$, we have the idempotency

$$\frac{1}{2\pi i}\int_C \delta_*(z-w+\frac{1}{i\hbar}u\circ v)dw * \frac{1}{2\pi i}\int_C \delta_*(z-w+\frac{1}{i\hbar}u\circ v)dw = \frac{1}{2\pi i}\int_C \delta_*(z-w+\frac{1}{i\hbar}u\circ v)dw.$$

Hence we have

$$(2.6) \qquad f_*^C(z+\frac{1}{\hbar}u\circ v)*g_*^C(z+\frac{1}{\hbar}u\circ v) = \frac{1}{2\pi i}\int_C f(w)g(w)\delta_*(z-w+\frac{1}{\hbar}u\circ v)dw.$$

This is confirmed also by the resolvent calculus as follows:

Lemma 2.2 *If $w \neq w'$, then*

$$\delta_*(z-w+\frac{1}{\hbar}u\circ v)*\delta_*(z-w'+\frac{1}{\hbar}u\circ v) = 0.$$

Proof Let

$$\delta_*(z-w+\frac{1}{\hbar}u\circ v) = (z-w+\frac{1}{\hbar}u\circ v)^{-1}_{*+} - (z-w+\frac{1}{\hbar}u\circ v)^{-1}_{*-},$$

$$\delta_*(z-w'+\frac{1}{\hbar}u\circ v) = (z-w'+\frac{1}{\hbar}u\circ v)^{-1}_{*+} - (z-w'+\frac{1}{\hbar}u\circ v)^{-1}_{*-}.$$

In the resolvent calculus, the product of the inverse is given by

$$\frac{1}{w'-w}\left((z-w+\frac{1}{\hbar}u\circ v)^{-1}_{*\pm} - (z-w'+\frac{1}{\hbar}u\circ v)^{-1}_{*\pm}\right).$$

Hence the result follows by a direct computation. $\qquad\square$

Recalling (2.6), one may write this by

$$(2.7) \qquad \delta_*(z-w+\frac{1}{\hbar}u\circ v)*\delta_*(z-w'+\frac{1}{\hbar}u\circ v) = \delta(w'-w)\delta_*(z-w+\frac{1}{\hbar}u\circ v), \quad cf.(1.7).$$

Here, note that $\delta(w'-w)$ is the delta function of complex variables regarded as a *formal distribution*.
Note Formal distribution is the extended notion of distribution, using Laurent polynomials as test functions and residues as the integrations. This is convenient to algebraic calculations.

Since

$$(\frac{1}{i\hbar}u\circ v)*E_{n,n}(K) = (n+\frac{1}{2})E_{n,n}(K), \quad (\frac{1}{i\hbar}u\circ v)*\overline{E}_{n,n}(K) = -(n+\frac{1}{2})\overline{E}_{n,n}(K),$$

we have also that

$$:(\frac{1}{i\hbar}u\circ v)*\delta_*(z+\frac{1}{\hbar}u\circ v):_K = \sum_{n=0}^{\infty}\frac{i(n+\frac{1}{2})}{z+i(n+\frac{1}{2})}E_{n,n}(K) - \sum_{n=0}^{\infty}\frac{i(n+\frac{1}{2})}{z-i(n+\frac{1}{2})}\overline{E}_{n,n}(K) + (\frac{1}{i\hbar}u\circ v)*\Phi_K(z)$$

$$= -z:\delta_*(z+\frac{1}{\hbar}u\circ v):_K + 1^C + (\frac{1}{i\hbar}u\circ v)*\Phi_K(z).$$

Since the extended $\delta_*(z-w+\frac{1}{\hbar}u\circ v)$ is given by the difference of two different inverses, we have

$$(z+\frac{1}{\hbar}u\circ v)*\frac{1}{2\pi i}\int_C \delta_*(z-w+\frac{1}{\hbar}u\circ v)dw = \frac{1}{2\pi i}\int_C w\delta_*(z-w+\frac{1}{\hbar}u\circ v)dw = (z+\frac{1}{\hbar}u\circ v)^C_*.$$

Note The procedure of taking $\frac{1}{2\pi i}\int_C dw$ is useful when we concern only on residues.

2.2.1 Matrix elements

The proof of Lemma 2.2 extends to give

Lemma 2.3 *If $w-w'$ is not an integer, then for every integer $n \geq 0$*

$$\delta_*(z-w+\frac{1}{\hbar}u\circ v)*u^n*\delta_*(z-w'+\frac{1}{\hbar}u\circ v) = 0 = \delta_*(z-w+\frac{1}{\hbar}u\circ v)*v^n*\delta_*(z-w'+\frac{1}{\hbar}u\circ v).$$

Proof Recall the formula

$$(2.8) \qquad (\tfrac{1}{i\hbar})^n u^n * v^n = (\tfrac{1}{i\hbar}u\circ v - \tfrac{1}{2}) * (\tfrac{1}{i\hbar}u\circ v - \tfrac{3}{2}) * \cdots * (\tfrac{1}{i\hbar}u\circ v - \tfrac{2n-1}{2}).$$

Using $\int_0^\infty e_*^{t(\frac{1}{i\hbar}u\circ v - \alpha)} dt$, one can obtain an inverse $(u^n * v^n)_{*-}^{-1}$.

Under the independent $\pm$ sign, we consider $(z-w+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1} * (u^n * (z-w'+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1} * v^n)$. As $v^n * u^n$ commutes with $(z-w'+\tfrac{1}{\hbar}u\circ v)$, we see this is an inverse of

$$(u^n * v^n)_{*-}^{-2}(u^n*(z-w'+\tfrac{1}{i\hbar}u\circ v)*v^n)*(z-w+\tfrac{1}{i\hbar}u\circ v) = (u^n * v^n)_{*-}^{-1}(z-w'-n+\tfrac{1}{i\hbar}u\circ v)*(z-w+\tfrac{1}{i\hbar}u\circ v)$$

On the other hand, the resolvent calculus gives that this inverse is obtained by

$$\frac{1}{w'-w+n}\left(u^n*(z-w'+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1}*v^n - u^n*v^n*(z-w+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1}\right),$$

for

$$(u^n*v^n)_{*-}^{-2}(u^n*(z-w'+\tfrac{1}{i\hbar}u\circ v)*v^n)*(z-w+\tfrac{1}{i\hbar}u\circ v)$$
$$*\left(u^n*(z-w'+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1}*v^n - u^n*v^n*(z-w+\tfrac{1}{i\hbar}u\circ v)_{*\pm}^{-1}\right)$$
$$= (z-w+\tfrac{1}{i\hbar}u\circ v)-(z-w'-n+\tfrac{1}{i\hbar}u\circ v)=w'+n-w$$

By the similar calculations as in Lemma 2.2, we see

$$\delta_*(z-w+\tfrac{1}{\hbar}u\circ v)*(u^n*\delta_*(z-w'+\tfrac{1}{\hbar}u\circ v)*v^n)=0.$$

As $v_*^\circ = u*\int_{-\infty}^0 e_*^{s\frac{1}{i\hbar}v*u}ds$ is a left inverse of v i.e. $v*v_*^\circ = 1$, $v_*^\circ*v=1-\varpi_{00}$, we have

$$\delta_*(z-w+\tfrac{1}{\hbar}u\circ v)*u^n*\delta_*(z-w'+\tfrac{1}{\hbar}u\circ v) = \delta_*(z-w+\tfrac{1}{\hbar}u\circ v)*(u^n*\delta_*(z-w'+\tfrac{1}{\hbar}u\circ v)*v^n)*(v_*^\circ)^n=0.$$

$\square$

In what follows, we use the convention (1.13) and the notations

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1, \quad (a)_{-n} = (a-1)(a-2)\cdots(a-n).$$

Note also that

$$(2.9) \qquad \begin{aligned} (\tfrac{1}{i\hbar})^n v^n * u^n &= (\tfrac{1}{i\hbar}u\circ v + \tfrac{1}{2})*(\tfrac{1}{i\hbar}u\circ v + \tfrac{3}{2})*\cdots*(\tfrac{1}{i\hbar}u\circ v + \tfrac{2n-1}{2})=(\tfrac{1}{i\hbar}u\circ v + \tfrac{1}{2})_{n*}, \\ (\tfrac{1}{i\hbar})^n u^n * v^n &= (\tfrac{1}{i\hbar}u\circ v - \tfrac{1}{2})*(\tfrac{1}{i\hbar}u\circ v - \tfrac{3}{2})*\cdots*(\tfrac{1}{i\hbar}u\circ v - \tfrac{2n-1}{2})=(\tfrac{1}{i\hbar}u\circ v + \tfrac{1}{2})_{-n*}. \end{aligned}$$

As $\delta_*(z-w+\tfrac{1}{\hbar}u\circ v)$ is a difference of two different inverses, we see

$$(\tfrac{1}{i\hbar})^n \hat{\zeta}^n * \zeta^n * \delta_*(z-w+\tfrac{1}{\hbar}u\circ v) = (i(w-z)+\tfrac{1}{2})_n \delta_*(z-w+\tfrac{1}{\hbar}u\circ v), \quad n\in\mathbb{Z}.$$

Set

$$(2.10) \qquad \mathfrak{D}_{p,q}(z+\tfrac{1}{\hbar}u\circ v) = \frac{1}{2\pi i}\int_C \frac{1}{\sqrt{(i\hbar)^{p+q}(i(w-z)+\tfrac{1}{2})_p(i(w-z)+\tfrac{1}{2})_q}}\zeta^p*\delta_*(z-w+\tfrac{1}{\hbar}u\circ v)*\hat{\zeta}^q dw,$$

where C is a closed curve. It is not hard to see that $\mathfrak{D}_{p,q}(z+\tfrac{1}{\hbar}u\circ v)$ form matrix elements.

$$\mathfrak{D}_{p,q}(z+\tfrac{1}{\hbar}u\circ v)*\mathfrak{D}_{r,s}(z'+\tfrac{1}{\hbar}u\circ v)=\delta_{q,r}\delta(z-z')\mathfrak{D}_{p,s}(z+\tfrac{1}{\hbar}u\circ v).$$

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3 Integrals along real axis

So far we have mainly concerned with the Fourier expansion of $e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ w.r.t. the variable t, and we have seen there is a big difference between differentiations by real variables and by pure imaginary variables. Another remarkable feature of $e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ is that if $|\mathrm{Re}\, z| < 1/2$ then

$$(3.1) \qquad \int_{-\infty}^{\infty} e^{s(z+\frac{1}{i\hbar}u\circ v+i\xi)}ds = \int_{-\infty}^{\infty} e^{-is(iz+\frac{1}{\hbar}u\circ v-\xi)}ds = \delta_*(iz+\frac{1}{\hbar}u\circ v-\xi), \quad |\mathrm{Re}\, z| < 1/2.$$

is rapidly decreasing w.r.t. $\xi \in \mathbb{R}$. In this section we discuss what this property produces.

For every tempered distribution $f(\xi)$ on $\mathbb{R}$, we define

$$(3.2) \qquad f_*(z+\frac{1}{\hbar}u\circ v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\delta_*(iz-\xi+\frac{1}{\hbar}u\circ v)d\xi.$$

By (1.7), if the pointwise product $f(\xi)g(\xi)$ is defined as a tempered distribution, then we have a formula similar to (2.6)

$$(3.3) \qquad f_*(z+\frac{1}{\hbar}u\circ v)*g_*(z+\frac{1}{\hbar}u\circ v) = \int_{-\infty}^{\infty} f(\xi)g(\xi)\delta_*(iz-\xi+\frac{1}{\hbar}u\circ v)d\xi.$$

For the characteristic function $\chi_U(\xi)$ of an open subset $\chi_{U*}(z+\frac{1}{\hbar}u\circ v)$ is an idempotent element.

Let $Y_\pm(t)$ be the characteristic function of $(0,\infty)$, $(-\infty,0)$. Let $\hat{Y}_\pm(xi)$ be the Fourier transform of $Y_\pm(t)$. Then, we have

$$(3.4) \qquad (z+\frac{1}{\hbar}u\circ v)_{*\pm}^{-1} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{Y}_\pm(xi)\delta_*(iz-\xi+\frac{1}{\hbar}u\circ v)d\xi.$$

Similarly, we have for every $a \in \mathbb{R}$

$$e_*^{ai(iz+\frac{1}{\hbar}u\circ v)} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ai\xi}\delta_*(iz-\xi+\frac{1}{\hbar}u\circ v)d\xi.$$

Since $\lim_{n\to\infty}(1+\frac{aiw}{n})^n = e^{aiw}$ as tempered distribution, we have in generic ordered expression that

$$\lim_{n\to\infty}\left(1+\frac{ai(iz+\frac{1}{\hbar}u\circ v)}{n}\right)_*^n = e_*^{ai(iz+\frac{1}{\hbar}u\circ v)}.$$

Extended properties of $\delta_*(iz+\frac{1}{\hbar}u\circ v-\xi)$ and matrix elements

As in §2.2.1, matrix elements are produced by the integrals along the real axis. Minding (2.10), we set

$$\mathcal{D}_{p,q}(z+\frac{1}{\hbar}u\circ v) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{(i\hbar)^{p+q}(\xi-iz+\frac{1}{2})_p(\xi-iz+\frac{1}{2})_q}} \zeta^p *\delta_*(z-\xi+\frac{1}{\hbar}u\circ v)*\hat{\zeta}^q d\xi.$$

Note first that Lemmas 2.2, 2.3 can be applied to this case. Thus, we see

$$\mathcal{D}_{p,q}(z+\frac{1}{\hbar}u\circ v)*\mathcal{D}_{r,s}(z'+\frac{1}{\hbar}u\circ v) = \delta_{q,r}\delta(z-z')\mathcal{D}_{p,s}(z+\frac{1}{\hbar}u\circ v)$$

3.1 Diagonal matrix calculations

Let $I_\circ(K) = (a, b)$ be the exchanging interval for a generic expression parameter K. $(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*\pm}$ is given by

$$(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*+} = \int_{-\infty}^{0} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt, \ (\mathrm{Re}\, z > -\frac{1}{2}), \quad (z+\frac{1}{i\hbar}u\circ v)^{-1}_{*-} = -\int_{0}^{\infty} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt \ (\mathrm{Re}\, z < \frac{1}{2}).$$

It is easy to see that in $Hol(\mathbb{C}^2)$

$$:(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*+}:_K = :e_*^{-a(z+\frac{1}{i\hbar}u\circ v)}:_K *_K :\int_{-\infty}^{a} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt:_K = :e_*^{-a(z+\frac{1}{i\hbar}u\circ v)}:_K *_K \left(\sum_{k=0}^{\infty} \frac{e^{a(z+n+\frac{1}{2})}}{z+n+\frac{1}{2}} E_{k,k}(K)\right),$$

but the next equality holds only if the right-multiplication operation $e_*^{-a(z+\frac{1}{i\hbar}u\circ v)}*$ is continuous. However, if one concerns only with coefficients of diagonal matrix elements, then

$$= \sum_{k=0}^{\infty} \frac{e^{a(z+k+\frac{1}{2})}}{z+k+\frac{1}{2}} :e_*^{-a(z+\frac{1}{i\hbar}u\circ v)}:_K *_K E_{k,k}(K) = \sum_{k=0}^{\infty} \frac{1}{z+k+\frac{1}{2}} E_{k,k}(K).$$

Thus, we denote this equality as

$$:(z+\frac{1}{i\hbar}u\circ v)^{-1}:_{E(K)mat} = \sum_{k=0}^{\infty} \frac{1}{z+k+\frac{1}{2}} E_{k,k}(K).$$

If $K \in \mathfrak{K}_+$, $\sum_{k=0}^{\infty} \frac{1}{z+n+\frac{1}{2}} E_{k,k}(K)$ converges in $Hol(\mathbb{C}^2)$ as partial fractions. Hence $:(z+\frac{1}{i\hbar}u\circ v)^{-1}:_{E(K)mat}$ may be viewed as the expression by the virtual expression parameter K such that $I_\circ(K) = (\infty, \infty)$. $E(K)mat$-expression makes sense in $Hol(\mathbb{C}^2)$, if $K \in \mathfrak{K}_+$.

If $a < 0 < b$, then we have a hybrid expression

$$:(z+\frac{1}{i\hbar}u\circ v)^{-1}_{*+}:_K =: \int_{-\infty}^{a} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt:_K +: \int_{a}^{0} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt:_K$$

$$= \sum_{k=0}^{\infty} \frac{e^{a(z+\frac{1}{2}+k)}}{z+\frac{1}{2}+k} E_{k,k}(K) + \sum_{n=-\infty}^{\infty} \frac{1-e^{a(z+n)}}{z+n} \tilde{D}_n(K).$$

However if we take the limit $a \to -\infty$, $b \to \infty$, then the second term diverges. Thus, the $D(K)mat$-expression $:(z+\frac{1}{i\hbar}u\circ v)^{-1}:_{D(K)mat}$ diverges. Similarly $:(z+\frac{1}{i\hbar}u\circ v)^{-1}:_{\overline{E}(K)mat}$ diverges, but we see that if $K \in \mathfrak{K}_-$, then recalling (2.1) we see that

$$:(z-\frac{1}{i\hbar}u\circ v)^{-1}:_{\overline{E}(K)mat} = \sum_{k=0}^{\infty} \frac{1}{z+k+\frac{1}{2}} \overline{E}_{k,k}(K)$$

converges in $Hol(\mathbb{C}^2)$ as partial fractions. $\overline{E}(K)mat$-expression makes sense in $Hol(\mathbb{C}^2)$, if $K \in \mathfrak{K}_-$.

Note There is no diagonal matrix expressions for $\delta_*(z+\frac{1}{i\hbar}u\circ v)$, as $\int_{\mathbb{R}} e^{s(z+n+\frac{1}{2})}ds$ diverges.

3.1.1 Further remarks for diagonal matrix calculations

Diagonal matrix calculus is the calculus based on the diagonal matrix expressions of $e_*^{(s+it)(z+\frac{1}{i\hbar}u\circ v)}$ and the termwise differentiations and integrations. Let $I_\circ(K) = (a,b)$ be the exchanging interval of generic ordered expression K. What is important is not an expression parameter, but the type E, $\bar{E}$ or D of matrix, Thus, this is the procedure of taking limit of expression parameter K so that $\lim I_\circ(K)$ tends to the whole line or $\emptyset$, that is, $E(K)mat$-expression (resp. $D(K)mat$-expression, $\overline{E}(K)mat$) is the diagonal matrix expression by the virtual expression parameter K such that $I_\circ(K) = (\infty,\infty)$ (resp. $I_\circ(K) = (-\infty,\infty)$, $I_\circ(K) = (-\infty,-\infty)$).

As diagonal matrices, we see the following:

Theorem 3.1 $e_*^{(s+it)\frac{1}{i\hbar}u\circ v}$ *is expressed by diagonal matrices*

$$
:e_*^{(s+it)(z+\frac{1}{i\hbar}u\circ v)}:_{E(K)mat} = \sum_{k=0}^{\infty} E_{k,k}(K)e^{(s+it)(z+k+\frac{1}{2})},
$$

$$
:e_*^{(s+it)(z+\frac{1}{i\hbar}u\circ v)}:_{D(K)mat} = \sum_{n=-\infty}^{\infty} \tilde{D}_n(K)e^{(s+it)(z+n)},
$$

$$
:e_*^{(s+it)(z-\frac{1}{i\hbar}u\circ v)}:_{\overline{E}(K)mat} = \sum_{k=0}^{\infty} \overline{E}_{k,k}(K)e^{(s+it)(z-k-\frac{1}{2})},
$$

where $E_{k,k}(K)$, $\overline{E}_{k,k}(K)$ equal to the ones given by (2.2), and $\tilde{D}_n(K) = D_{n,n}(K)$ if $K \in \mathfrak{K}_0$.

By this we see in particular

$$
:(z+\frac{1}{i\hbar}u\circ v):_{D(K)mat} = \sum_{n=-\infty}^{\infty} (z+n)D_{n,n}(K).
$$

On the other hand, we proved in [17] that in generic K-expression

$$
D_K^{-1}(z+\frac{1}{i\hbar}u\circ v) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n}D_{n,n}(K)
$$

converges in $Hol(\mathbb{C}^2)$ as partial fractions, and this is the inverse of $:(z+\frac{1}{i\hbar}u\circ v):_{D(K)mat}$ as diagonal matrix calculations.

Theorem 3.2 *In generic ordered expression, $\sum_{n=0}^{\infty} E_{n,n}(K)$, $\sum_{n=0}^{\infty} \overline{E}_{n,n}(K)$ and $\sum_{n=-\infty}^{\infty} D_{n,n}(K)$ represent 1, which will be denoted by $:1:_{Emat}$, $:1:_{\overline{E}mat}$, $:1:_{Dmat}$ respectively*

Hence every element of the Weyl algebra is expressed various ways depending on the expression parameter.

Note A diagonal matrix may be viewed as a "field" on a discrete set $\mathbb{Z}$ or $\mathbb{N}$. Componentwise/brockwise calculation will become a powerful tool to analyze peculiar nature of $*$-exponential functions and their integrals, just as particle physics jumps to the field theory to analyze creation/annihilation procedure.

4 Several ∗-special functions

As applications of calculus involving diagonal matrix elements, we define several ∗-special functions.

Ordinary beta function $B(x, y)$ is defined for $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$ by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Replacing t by e^s, we define in generic K-expression, ∗-beta functions by

$$(4.1) \qquad :B_*(\alpha \pm \frac{\tau}{i\hbar}u \circ v, y):_K = \int_{-\infty}^0 :e_*^{s(\alpha \pm \frac{\tau}{i\hbar}u \circ v)}:_K (1-e^s)^{y-1}ds, \ \operatorname{Re} \alpha > -\frac{|\tau|}{2}, \ \operatorname{Re} y > 0.$$

These converge uniformly on every compact domain of α to give $Hol(\mathbb{C}^2)$-valued holomorphic functions, but noting that $\int_{-\infty}^0 :e_*^{s(\alpha \pm \frac{1}{i\hbar}u \circ v)}:_K ds$ give two different inverses of $\alpha \pm \frac{\tau}{i\hbar}u \circ v$, we have to set

$$(4.2) \qquad B_*(\alpha + \frac{\tau}{i\hbar}u \circ v, 1) = (\alpha + \frac{\tau}{i\hbar}u \circ v)_{*+}^{-1}, \quad B_*(\alpha - \frac{\tau}{i\hbar}u \circ v, 1) = (\alpha - \frac{\tau}{i\hbar}u \circ v)_{*-}^{-1} = -(-\alpha + \frac{\tau}{i\hbar}u \circ v)_{*-}^{-1}.$$

4.1 Star-gamma functions

We define in generic K-expression as follows:

$$(4.3) \qquad :\Gamma_*(z \pm \frac{1}{i\hbar}u \circ v):_K = \int_{-\infty}^\infty e^{-e^s}:e_*^{s(z \pm \frac{1}{i\hbar}u \circ v)}:_K ds, \quad \operatorname{Re} z > -\frac{1}{2}$$

If $\operatorname{Re} z > -\frac{1}{2}$, this integral converges in a generic ordered expression to give an element of $Hol(\mathbb{C}^2)$. As the usual gamma function, integration by parts gives the identity

$$(4.4) \qquad \Gamma_*(z+1 \pm \frac{1}{i\hbar}u \circ v) = (z \pm \frac{1}{i\hbar}u \circ v) * \Gamma_*(z \pm \frac{1}{i\hbar}u \circ v).$$

Note here that $e^{-e^s}e^{\varepsilon s}$, $s \in \mathbb{R}$ is a rapidly decreasing function for every $\varepsilon > 0$. Its inverse Fourier transform is

$$\int_{\mathbb{R}} e^{-e^t}e^{\varepsilon t}e^{it\xi}đt = \frac{1}{\sqrt{2\pi}}\Gamma(i\xi + \varepsilon).$$

Moreover, e^{-e^s} is a tempered distribution. Hence its inverse Fourier transform is $\lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{2\pi}}\Gamma(i\xi + \varepsilon)$. Thus (3.2) gives

$$(4.5) \qquad :\Gamma_*(z \pm \frac{1}{i\hbar}u \circ v):_K = \frac{1}{2\pi}\lim_{\varepsilon \downarrow 0}\int_{\mathbb{R}}\Gamma(i\xi + \varepsilon):\delta_*(iz \pm \frac{1}{\hbar}u \circ v - \xi):_K d\xi$$

As $\Gamma(z)$ extends on $\mathbb{C}\backslash\{-\mathbb{N}\}$, recalling (2.5) one may consider

$$(4.6) \qquad \Gamma_*^C(z \pm \frac{1}{i\hbar}u \circ v) = \frac{1}{2\pi i}\int_C \Gamma(iw)\delta_*(iz \pm \frac{1}{\hbar}u \circ v - w)dw$$

to analyze the interaction of singularities of $\Gamma(iw)$ and $\delta_*(iz \pm \frac{1}{\hbar}u \circ v - w)$.

4.1.1 Analytic continuation of $:\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v):_K$ in $Hol(\mathbb{C}^2)$

As $\delta_*(z \pm \frac{1}{i\hbar}u\circ v)$ is analytically continued, the formula (4.5) gives also the analytic continuation of $\Gamma_*(z\pm\frac{\tau}{i\hbar}u\circ v)$, but as $*$-gamma function is defined by the integral form, it is easy to obtain the formula for the analytic continuation.

Proposition 4.1 $\Gamma_*(z \pm \frac{1}{i\hbar}u\circ v)$ extends to an $Hol(\mathbb{C}^2)$-valued holomorphic function on

$$z \in \mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$$

with simple poles.

Proof Split the integral into two parts $\int_{-\infty}^0 + \int_0^\infty$. It is easy to see that $\int_0^\infty e^{-e^t}e_*^{t(z\pm\frac{1}{i\hbar}u\circ v)}dt$ is entire. Thus we have only to show for the first term. For every z, choose an positive integer n such that $\mathrm{Re}(z + n) > 0$. Set

$$e^{-e^t} = 1-e^t + \frac{1}{2}e^{2t} + \cdots + \frac{(-1)^n}{n!}e^{nt} + e^{nt}F_n(t), \quad |F_n(t)| \leq C.$$

Then, it is easy to see $R_n(z\pm\frac{1}{i\hbar}u\circ v) = \int_{-\infty}^0 F_n(t)e^{nt}e_*^{t(z\pm\frac{1}{i\hbar}u\circ v)}dt$ converges, and hence

$$\int_{-\infty}^0 e^{-e^t}e_*^{t(z\pm\frac{1}{i\hbar}u\circ v)}dt$$

$$= (z\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1} - (1+z\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1} + \cdots + \frac{(-1)^n}{n!}(n+z\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1} + R_n(z\pm\frac{1}{i\hbar}u\circ v).$$

The result follows immediately. $\qquad\square$

4.1.2 Residues of $\Gamma_*(z \pm \frac{1}{i\hbar}u\circ v)$

The proof of Proposition 4.1 gives also the formula of residues. By Theorem 2.2, we have

$$\mathrm{Res}\Big(:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_K, -(n + \frac{1}{2})\Big)=\sum_{k=0}^n(-1)^k\frac{1}{(n-k)!}E_{k,k}(K)$$

$$\mathrm{Res}\Big(:\Gamma_*(z-\frac{1}{i\hbar}u\circ v):_K, -(n + \frac{1}{2})\Big)=\sum_{k=0}^n(-1)^k\frac{1}{(n-k)!}\overline{E}_{k,k}(K).$$

Using this we have

Proposition 4.2 In generic K-ordered expression,

$$\frac{1}{\Gamma(z+\frac{1}{2})}\Gamma_*(z+\frac{1}{i\hbar}u\circ v), \quad \frac{1}{\Gamma(z+\frac{1}{2})}\Gamma_*(z-\frac{1}{i\hbar}u\circ v)$$

are $Hol(\mathbb{C}^2)$-valued entire functions of z.

Note that all zero's of $\frac{1}{\Gamma(z+\frac{1}{2})}$ are eliminated by the singularities of $\Gamma(z\pm\frac{1}{i\hbar}u\circ v)$. Hence

$$\frac{1}{\Gamma(z+\frac{1}{2})}\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)$$

has no vanishing point. This suggests that the $*$-inverse of $\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)$ may be treated in this way.

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4.2 Diagonal matrix expressions of $\Gamma_*(z+\frac{1}{i\hbar}u\circ v)$

Let $I_\circ(K) = (a,b)$ is the exchanging interval of $:e_*^{z\frac{1}{i\hbar}u\circ v}:_K$. Applying (2.1) for $t=0$ in a generic K-expression, we have the hybrid matrix expression

$$:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_K = \int_{-\infty}^{\infty} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$= \int_{-\infty}^{a} e^{-e^s}\sum_{n=0}^{\infty} e^{s(z+(n+\frac{1}{2}))}E_{n,n}(K)ds + \int_a^b e^{-e^s}\sum_{n=-\infty}^{\infty} e^{s(z+n)}\tilde{D}_n(K)ds + \int_b^{\infty} e^{-e^s}\sum_{n=0}^{\infty} e^{s(z-(n+\frac{1}{2}))}\overline{E}_{n,n}(K)ds.$$

Using this, one may take diagonal matrix expressions, but here we give first a justification of the diagonal matrix calculations.

Note that for every σ

$$\int_{-\infty}^{\infty} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$= \int_{-\infty}^{a+\sigma} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds + \int_{a+\sigma}^{b+\sigma} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds + \int_{b+\sigma}^{\infty} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$= \int_{-\infty}^{a} e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds + \int_a^b e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds + \int_b^{\infty} e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

As the $*$-product $e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}*H_*$ is defined by the analytic solution of the evolution equation, we have

$$\int_{-\infty}^{a} e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds = :e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * \int_{-\infty}^{a} e^{-e^{s+\sigma}}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$\int_a^b e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds = :e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}* \int_a^b e^{-e^{s+\sigma}}e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$\int_b^{\infty} e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds = :e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}* \int_b^{\infty} e^{-e^{s+\sigma}}e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

We see

$$\lim_{\sigma\to\infty} \int_a^b e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds = 0, \quad \lim_{\sigma\to\infty} \int_b^{\infty} e^{-e^{s+\sigma}}:e_*^{(s+\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds = 0$$

One of the remarkable feature of $*$-gamma function is that $E(K)mat$-expression, and $\overline{E}(K)mat$-expression of $\Gamma_*(z\pm\frac{\tau}{i\hbar}u\circ v)$ are obtained for generic K-expression. First, to obtain $E(K)mat$-expression, note that for $\operatorname{Re} z > -\frac{1}{2}$

$$\lim_{\sigma\to\infty} \int_{-\infty}^{a-1+\sigma} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \int_{-\infty}^{\infty} e^{-e^s}:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds.$$

We see

$$:e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K = \sum_{n=0}^{\infty} e^{s(z+(n+\frac{1}{2}))}E_{n,n}(K), \quad s < a-1.$$

As this is the convergence of Taylor series at $w=0$, $w=e^t$, we see

Lemma 4.1 *The convergence is uniform on $s < a-1$.*

Termwise integration gives

$$\int_{-\infty}^{a-1} e^{-e^{s+\sigma}} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \sum_{n=0}^{\infty} \int_{-\infty}^{a-1} e^{-e^{s+\sigma}} e^{s(z+(n+\frac{1}{2}))} ds E_{n,n}(K).$$

in $Hol(\mathbb{C}^2)$, but the next one holds only in matrix

$$:e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * \int_{-\infty}^{a-1} e^{-e^{s+\sigma}} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \sum_{n=0}^{\infty} \int_{-\infty}^{a-1} e^{-e^{s+\sigma}} e^{s(z+(n+\frac{1}{2}))} ds :e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * E_{n,n}(K).$$

It is easy to see $:e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * E_{n,n}(K) = e^{\sigma(z+\frac{1}{2})} E_{n,n}(K)$.

By using the Lebesgue dominated convergence theorem, we see

$$\int_{-\infty}^{\infty} e^{-e^s} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-e^s} e^{s(z+(n+\frac{1}{2}))} ds E_{n,n}(K)$$

Hence, we have in generic K-expression,

$$:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_K = \sum_{n=0}^{\infty} \Gamma(z+(n+\frac{1}{2})) E_{n,n}(K)$$

but the r.h.s. converges only in diagonal matrices. Hence we denote this by

$$(4.7) \qquad :\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat} = \sum_{n=0}^{\infty} \Gamma(z+(n+\frac{1}{2})) E_{n,n}(K) = \Gamma(z+\frac{1}{2}) \sum_{n=0}^{\infty} (z+\frac{1}{2})_n E_{n,n}(K),$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$.

Recall that $E(K)mat$-expression is the expression by the virtual expression parameter such that $I_\circ(K) = (\infty, \infty)$.

$\overline{E}(K)mat$-expression is obtained by changing the orientation of the integration. For $\text{Re}\, z > -\frac{1}{2}$

$$\lim_{\sigma \to \infty} \int_{b+1-\sigma}^{\infty} e^{-e^s} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \int_{-\infty}^{\infty} e^{-e^s} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds.$$

$$\int_{b+1-\sigma}^{\infty} e^{-e^s} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds = \int_{b+1}^{\infty} e^{-e^{s-\sigma}} :e_*^{(s-\sigma)(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$= :e_*^{-\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * \int_{b+1}^{\infty} e^{-e^{s-\sigma}} :e_*^{s(z+\frac{1}{i\hbar}u\circ v)}:_K ds$$

$$= :e_*^{-\sigma(z+\frac{1}{i\hbar}u\circ v)}:_K * \int_{b+1}^{\infty} e^{-e^{s-\sigma}} \sum_{n=0}^{\infty} e^{s(z-n-\frac{1}{2})} ds \overline{E}_{n,n}(K)$$

$$= \sum_{n=0}^{\infty} \int_{b+1-\sigma}^{\infty} e^{-e^s} e^{s(z-n-\frac{1}{2})} ds \overline{E}_{n,n}(K).$$

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The last equality holds as diagonal matrices. Hence by the componentwise convergence we have

$$:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat} = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}e^{-e^s}e^{s(z-n-\frac{1}{2})}ds\overline{E}_{n,n}(K) = \sum_{n=0}^{\infty}\Gamma(z-n-\frac{1}{2})\overline{E}_{n,n}(K).$$

Since $\Gamma(z-n-\frac{1}{2})=(z+\frac{1}{2})_{-n}^{-1}\Gamma(z+\frac{1}{2})$ where $(a)_{-n}=(a-1)(a-2)\cdots(a-n)$, we have in generic K-expression,

$$(4.8) \qquad :\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_K = \sum_{n=0}^{\infty}\Gamma(z-(n+\frac{1}{2}))\overline{E}_{n,n}(K) = \Gamma(z+\frac{1}{2})\sum_{n=0}^{\infty}(z+\frac{1}{2})_{-n}^{-1}\overline{E}_{n,n}(K).$$

Recall again that $\overline{E}(K)mat$-expression is the expression by the virtual expression parameter such that $I_\circ(K) = (-\infty, -\infty)$.

Similarly, $D(K)mat$-expression $:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{D(K)mat}$ is

$$(4.9)\qquad \begin{aligned}:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{D(K)mat} &= \lim_{(a,b)\to(-\infty,\infty)}\int_a^b e^{-e^s}\sum_n e^{s(z+n)}ds \\ &= \sum_n\Gamma(z+n)D_{n,n}(K) = \Gamma(z)\sum_{n=-\infty}^{\infty}(z)_n^{sgn(n)1}D_{n,n}(K).\end{aligned}$$

Hence this can not converge in $Hol(\mathbb{C}^2)$, as $(z)_{n+1}\sim n!$ for $n > 0$.

Computations for $:\Gamma_*(z-\frac{1}{i\hbar}u\circ v):_{E(K)mat}$ is parallel to the above, but a little care is required, since the direction of s, which may be viewed as the time, is reversed. We summarize the results below:

$$(4.10)\qquad\begin{aligned}:\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat} &= \sum_{n=0}^{\infty}\Gamma(z+n+\frac{1}{2})E_{n,n}(K) = \Gamma(z+\frac{1}{2})\sum_{n=0}^{\infty}(z+\frac{1}{2})_n E_{n,n}(K)\\ :\Gamma_*(z+\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat} &= \sum_{n=0}^{\infty}\Gamma(z-n-\frac{1}{2})\overline{E}_{n,n}(K) = \Gamma(z+\frac{1}{2})\sum_{n=0}^{\infty}(z+\frac{1}{2})_{-n}^{-1}\overline{E}_{n,n}(K)\\ :\Gamma_*(z-\frac{1}{i\hbar}u\circ v):_{E(K)mat} &= \sum_{n=0}^{\infty}\Gamma(z-n-\frac{1}{2})E_{n,n}(K) = \Gamma(z+\frac{1}{2})\sum_{n=0}^{\infty}(z+\frac{1}{2})_{-n}^{-1}E_{n,n}(K)\\ :\Gamma_*(z-\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat} &= \sum_{n=0}^{\infty}\Gamma(z+n+\frac{1}{2})\overline{E}_{n,n}(K) = \Gamma(z+\frac{1}{2})\sum_{n=0}^{\infty}(z+\frac{1}{2})_n\overline{E}_{n,n}(K).\end{aligned}$$

4.2.1 The inverse $\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)^{-1}$

Diagonal matrix calculation is useful to find the formula of inverses. It is easy to see that the inverse of $\sum_n(z+\frac{1}{2})_n E_{n,n}(K)$ in diagonal matrices is given by $\sum_n(z+\frac{1}{2})_n^{-1}E_{n,n}(K)$.

By Theorem 2.1 we see the next

Theorem 4.1 *If* $K\in\mathfrak{K}_+$, *then*

$$:\Gamma_*^{-1}(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat} = \Gamma(z+\frac{1}{2})^{-1}\sum_n(z+\frac{1}{2})_n^{-1}E_{n,n}(K),$$

converge in $Hol(\mathbb{C}^2)$, and every component $\Gamma(z+\frac{1}{2})^{-1}(z+\frac{1}{2})_n^{-1}$ is an entire function. Similarly, if $K \in \mathfrak{K}_-$, then

$$:\Gamma_*^{-1}(z-\frac{1}{i\hbar}u \circ v):_{\overline{E(K)mat}} = \Gamma(z+\frac{1}{2})^{-1} \sum_n (z+\frac{1}{2})_n^{-1} \overline{E}_{n,n}(K)$$

converges in $Hol(\mathbb{C}^2)$ and every component $\Gamma(z+\frac{1}{2})^{-1}(z+\frac{1}{2})_n^{-1}$ is an entire function.

Proof Fix z so that $(z+\frac{1}{2})_n \neq 0$. Since $\sum_n(z\pm\frac{1}{2})_n^{-1}e^{in\theta}$ is viewed as the Fourier series of a smooth function $f(\theta)$ on S^1, one can apply Theorem 2.1. □

As the byproduct we have

Corollary 4.1 If $K \in \mathfrak{K}_+$, (resp. $\mathfrak{K}_-$), then

$$\Gamma(z+\frac{1}{2})^{-1} \sum_n (z+\frac{1}{2})_n^{-1} E_{n,n}(K), \quad \text{resp.} \quad \Gamma(z+\frac{1}{2})^{-1} \sum_n (z+\frac{1}{2})_n^{-1} \overline{E}_{n,n}(K)$$

is a genuine inverse of $\Gamma_*(z+\frac{1}{i\hbar}u \circ v)$ (resp. $\Gamma_*(z-\frac{1}{i\hbar}u \circ v)$).

Note that elements such as

$$\Gamma(z+\frac{1}{2})^{-1} \sum_n (z+\frac{1}{2})_n^{-1} E_{n,n}(K) + \Gamma(z+\frac{1}{2})^{-1} \sum_n (z+\frac{1}{2})_n^{-1} \overline{E}_{n,n}(K)$$

makes sense as diagonal matrices, which will be called a **hybrid** matrix expression. It is clear that

$$\sum_{n=0}^{\infty} \Gamma(z+n+\frac{1}{2})(E_{n,n}(K)+\overline{E}_{n,n}(K)) * \sum_{n=0}^{\infty} \Gamma(z+n+\frac{1}{2})^{-1}(E_{n,n}(K)+\overline{E}_{n,n}(K)) = \sum_{n=0}^{\infty}(E_{n,n}(K)+\overline{E}_{n,n}(K))$$

In the next section, we discuss the $*$-inverse of $\Gamma_*(z\pm\frac{1}{i\hbar}u \circ v)$ by using the infinite product formula.

5   The infinite product formula of $\Gamma_*(z\pm\frac{1}{i\hbar}u \circ v)$

Recall

$$:\Gamma_*(z+\frac{1}{i\hbar}u \circ v):_{E(K)mat} = \sum_{k=0}^{\infty} \Gamma(z+k+\frac{1}{2})E_{k,k}(K).$$

The infinite product formula of ordinary gamma function gives

$$:\Gamma_*(z+\frac{1}{i\hbar}u \circ v):_{E(K)mat} = \sum_{k=0}^{\infty} \left(e^{-\gamma(z+k+\frac{1}{2})}(z+k+\frac{1}{2})^{-1} \prod_{\ell=0}^{\infty} \left(1+\frac{1}{\ell}(z+k+\frac{1}{2})^{-1}e^{\frac{1}{\ell}(z+k+\frac{1}{2})}\right)\right) E_{k,k}(K).$$

It is a remarkable feature of diagonal matrix calculation that the multiplications commute with the summations allows:

$$:\Gamma_*(z+\frac{1}{i\hbar}u \circ v):_{E(K)mat} = \prod_{\ell=0}^{\infty} \left(\sum_{k=0}^{\infty} e^{-\gamma(z+k+\frac{1}{2})}(z+k+\frac{1}{2})^{-1}\left(1+\frac{1}{\ell}(z+k+\frac{1}{2})\right)^{-1}e^{\frac{1}{\ell}(z+k+\frac{1}{2})}E_{k,k}(K)\right)$$

$$= \prod_{\ell=0}^{\infty} \left(:e^{-\gamma(z+\frac{1}{i\hbar}u \circ v)}*(z+\frac{1}{i\hbar}u \circ v)_{*+}^{-1}*\left(1+\frac{1}{\ell}(z+\frac{1}{i\hbar}u \circ v)\right)_{*+}^{-1}e_*^{\frac{1}{\ell}(z+\frac{1}{i\hbar}u \circ v)}:_{Kmat}\right).$$

The next Lemma is crucial to obtain the infinite product formula in $Hol(\mathbb{C}^2)$.

Lemma 5.1 $B_*(z \pm \frac{1}{i\hbar}u \circ v, n+1) = n! \prod_{k=0}^{n} *(k+z \pm \frac{1}{i\hbar}u \circ v)_{*\pm}^{-1}$, $\operatorname{Re} z > -\frac{1}{2}$.

Proof The r.h.s. of the above equality will be denoted by $\dfrac{n!}{\{z \pm \frac{1}{i\hbar}u \circ v\}_{*n+1}^{(\pm)}}$.

The case $n = 0$ is given by (4.2). Suppose this is true for n. For the case $n+1$, the definition (4.3) and the induction assumption give

$$B_*(z \pm \frac{1}{\hbar i}u \circ v, n+2) = \int_{-\infty}^{0} e_*^{\tau(z \pm \frac{1}{i\hbar}u \circ v)}(1-e^{\tau})(1-e^{\tau})^n d\tau$$

$$= \frac{n!}{\{z \pm \frac{1}{\hbar i}u \circ v\}_{*n+1}^{(\pm)}} - \frac{n!}{\{1+z \pm \frac{1}{\hbar i}u \circ v\}_{*n+1}^{(\pm)}}.$$

By noting the standard resolvent identity

$$\frac{n!}{A*(A+1)*\cdots*(A+n)} - \frac{n!}{(A+1)(A+2)*\cdots*(A+n+1)} = \frac{(n+1)!}{A*(A+1)*\cdots*(A+n+1)}$$

we have that

$$B_*(z \pm \frac{1}{\hbar i}u \circ v, n+2) = \frac{(n+1)!}{\{z \pm \frac{1}{\hbar i}u \circ v\}_{*n+2}^{(\pm)}}.$$

Replace $n+2$ by $n+1$, we get the formula. $\qquad\square$

Since $*$-products of inverses are given by summations, we have also the following:

Proposition 5.1 $\dfrac{n!}{\{z \pm \frac{1}{\hbar i}u \circ v\}_{*n+1}^{(\pm)}}$ is analytically continued on $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$.

The infinite product formula for $*$-gamma function will be given by this formula. By Lemma 5.1, we see

$$\int_{-\infty}^{0} e_*^{\tau(z \pm \frac{1}{\hbar i}u \circ v)}(1 - e^{\tau})^n d\tau = \frac{n!}{\{z \pm \frac{1}{\hbar i}u \circ v\}_{*n+1}^{(\pm)}}, \quad \operatorname{Re} z > -\frac{1}{2}.$$

Replacing e^{τ} by $\frac{1}{n}e^{\tau'}$, namely setting $\tau = \tau' - \log n$ in the left hand side and multiplying $e_*^{(\log n)(z \pm \frac{1}{\hbar i}u \circ v)}$ to the both sides, we have in generic ordered expression

$$(5.1) \qquad \int_{-\infty}^{\log n} e_*^{\tau'(z \pm \frac{1}{i\hbar}u \circ v)}(1 - \frac{1}{n}e^{\tau'})^n d\tau' = \frac{n!}{\{z \pm \frac{1}{\hbar}u \circ v\}_{*n+1}^{(\pm)}} * e_*^{(\log n)(z \pm \frac{1}{\hbar}u \circ v)}.$$

Theorem 5.1 *In generic ordered expression, the left hand side converges with $n \to \infty$ uniformly on each compact set of z to*

$$\int_{-\infty}^{\infty} e_*^{\tau'(z \pm \frac{1}{i\hbar}u \circ v)} e^{-e^{\tau'}} d\tau' = \Gamma_*(z \pm \frac{1}{i\hbar}u \circ v)$$

in the space $\operatorname{Hol}(\mathbb{C}^2)$. In fact, the convergence is uniform on each domain $-\frac{1}{2} < \operatorname{Re} z < C$.

Proof It is easy to see that

$$\lim_{n\to\infty}\int_{-\infty}^{0} e_*^{\tau'(z\pm\frac{1}{i\hbar}u\circ v)}(1-\frac{1}{n}e^{\tau'})^n d\tau' = \int_{-\infty}^{0} e_*^{\tau'(z\pm\frac{1}{i\hbar}u\circ v)}e^{-e^{\tau'}}d\tau', \quad \mathrm{Re}\ z > -\frac{1}{2}.$$

Hence we have only to show for the integral $\int_0^{\log n} d\tau'$. For every positive integer ℓ, consider the nonnegative function $x^\ell(1-\frac{x}{n})^n$ on the interval $[0,n]$ for $n \gg 0$. At $x = \frac{n\ell}{n+\ell}$ this attains the maximum $(\frac{n\ell}{n+\ell})^\ell(1-\frac{\ell}{n+\ell})^n$. Since

$$\lim_{n\to\infty}(\frac{n\ell}{n+\ell})^\ell(1-\frac{\ell}{n+\ell})^n = \ell^\ell(e^{-\ell})^{-\ell} = \ell^\ell e^{\ell^2},$$

$x^\ell(1-\frac{x}{n})^n$ on the interval $[0,n]$ is dominated uniformly by $C'\ell^\ell e^{\ell^2}$, $(C' > 1)$. Hence one may assume

$$(1-\frac{e^{\tau'}}{n})^n < C'\ell^\ell e^{\ell^2}e^{-\ell\tau'}, \quad n \gg 0.$$

By using the Lebesgue dominated convergence theorem, we see that for $\mathrm{Re}z < \ell+\frac{1}{2}$,

$$\lim_{n\to\infty}\int_0^{\log n} e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}(1-\frac{e^{\tau'}}{n})^n d\tau' = \int_0^{\infty} e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'$$

in the space $Hol(\mathbb{C}^2)$. □

Note that the proof can be applied to the case stated below:

Proposition 5.2 Let $\{a_n\}_n$ be a converging series such that $\lim_n a_n=a$.

$$\lim_{n\to\infty} e_*^{a_n(z\pm\frac{1}{\hbar i}u\circ v)}*\int_{-\infty}^{\log n} e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau' = \int_{-\infty}^{\infty} e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'$$

$$= e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*\int_{-\infty}^{\infty} e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau', \quad \mathrm{Re}\ z > -\frac{1}{2}.$$

Proof For the second equality, note that

$$\frac{d}{da}\int_{-\infty}^{\infty} e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau' = \int_{-\infty}^{\infty}(z\pm\frac{1}{\hbar i}u\circ v)*e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'$$

$$= (z\pm\frac{1}{\hbar i}u\circ v)*\int_{-\infty}^{\infty} e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'.$$

As $e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}$ is real analytic w.r.t. a and the integral converges, we see that

$$\int_{-\infty}^{\infty}(z\pm\frac{1}{\hbar i}u\circ v)*e_*^{a(z\pm\frac{1}{\hbar i}u\circ v)}*e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'$$

is a real analytic solution of the evolution equation $\frac{d}{da}f_a = (z\pm\frac{1}{\hbar i}u\circ v)*f_a$ with the initial data $f_0 = \int_{-\infty}^{\infty} e_*^{\tau'(z\pm\frac{1}{\hbar i}u\circ v)}e^{-e^{\tau'}}d\tau'$. Hence we have the second equality. □

Since
$$\frac{n!}{\{z \pm \frac{1}{\hbar i} u \circ v\}_{*n+1}^{(\pm)}} = (z \pm \frac{1}{\hbar i} u \circ v)_{*\pm}^{-1} * \prod_{k=1}^{n} * (1 + \frac{1}{k}(z \pm \frac{1}{\hbar i} u \circ v))_{*\pm}^{-1},$$

Proposition 5.1 shows that the right hand side of (5.1) is changed on $\mathbb{C} \backslash \{-(\mathbb{N} + \frac{1}{2})\}$ into

$$e_*^{(\log n - (1 + \frac{1}{2} + \cdots + \frac{1}{n}))(z \pm \frac{1}{i\hbar} u \circ v)} * (z \pm \frac{1}{i\hbar} u \circ v)_{*\pm}^{-1} \prod_{k=1}^{n} * \left((1 + \frac{z \pm \frac{1}{i\hbar} u \circ v}{k})_{*\pm}^{-1} * e_*^{\frac{1}{k}(z \pm \frac{1}{i\hbar} u \circ v)} \right).$$

Note that $\lim_{n \to \infty} e_*^{(\log n - (1 + \frac{1}{2} + \cdots + \frac{1}{n}))(z \pm \frac{1}{i\hbar} u \circ v)} = e_*^{-\gamma(z \pm \frac{1}{\hbar i} u \circ v)}$ is obvious, where γ is Euler's constant. Thus, Proposition 5.2 gives the uniform convergence of

(5.2)
$$\Gamma_*(z + \frac{1}{i\hbar} u \circ v) = e_*^{-\gamma(z + \frac{1}{i\hbar} u \circ v)} * (z + \frac{1}{i\hbar} u \circ v)_{*+}^{-1} * \prod_{k=1}^{\infty} * \left((1 + \frac{1}{k}(z + \frac{1}{i\hbar} u \circ v))_{*+}^{-1} * e_*^{\frac{1}{k}(z + \frac{1}{i\hbar} u \circ v)} \right)$$

$$\Gamma_*(z - \frac{1}{i\hbar} u \circ v) = e_*^{-\gamma(z - \frac{1}{i\hbar} u \circ v)} * (z - \frac{1}{i\hbar} u \circ v)_{*-}^{-1} * \prod_{k=1}^{\infty} * \left((1 + \frac{1}{k}(z - \frac{1}{i\hbar} u \circ v))_{*-}^{-1} * e_*^{\frac{1}{k}(z - \frac{1}{i\hbar} u \circ v)} \right)$$

on each compact subset in the domain $\{z; \operatorname{Re} z > -1/2\}$. This is the infinite product formula on the domain $\{z; \operatorname{Re} z > -1/2\}$ in generic ordered expressions.

5.1 Analytic continuation of the infinite product formula

Here we show that the infinite product formula (5.2) holds on $\mathbb{C} \backslash \{-(\mathbb{N} + \frac{1}{2})\}$. In generic ordered expression, the double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{0} e^{-e^s} e_*^{(s+s')(z \pm \frac{1}{i\hbar} u \circ v)} ds ds'$$

converges to give an element of $Hol(\mathbb{C}^2)$. As $z \pm \frac{1}{i\hbar} u \circ v$ is a $*$-polynomial

$$(z \pm \frac{1}{i\hbar} u \circ v) * \int_{-\infty}^{\infty} \int_{-\infty}^{0} e^{-e^s} e_*^{(s+s')(z \pm \frac{1}{i\hbar} u \circ v)} ds ds' = \int_{-\infty}^{\infty} \int_{-\infty}^{0} e_*^{s'(z \pm \frac{1}{i\hbar} u \circ v)} * (z \pm \frac{1}{i\hbar} u \circ v) * e^{-e^s} e_*^{s(z \pm \frac{1}{i\hbar} u \circ v)} ds ds'$$

makes sense. As $(z \pm \frac{1}{i\hbar} u \circ v) * e^{-e^s} e_*^{s(z \pm \frac{1}{i\hbar} u \circ v)} = e^{-e^s} \frac{d}{ds} e_*^{s(z \pm \frac{1}{i\hbar} u \circ v)}$, integrating by ds first gives

$$\int_{-\infty}^{0} e_*^{s'(z \pm \frac{1}{i\hbar} u \circ v)} ds' * \int_{-\infty}^{\infty} e^{-e^s} e_*^{s(z+1 \pm \frac{1}{i\hbar} u \circ v)} ds = (z \pm \frac{1}{i\hbar} u \circ v)_{*+}^{-1} * \Gamma_*(z + 1 \pm \frac{1}{i\hbar} u \circ v)$$

by the integration by parts. As $e_*^{s'(z \pm \frac{1}{i\hbar} u \circ v)} * (z \pm \frac{1}{i\hbar} u \circ v) = \frac{d}{ds'} e_*^{s'(z \pm \frac{1}{i\hbar} u \circ v)}$, integrating by ds' first gives

$$\Gamma_*(z \pm \frac{1}{i\hbar} u \circ v) = (z \pm \frac{1}{i\hbar} u \circ v)_{*+}^{-1} * \Gamma_*(z + 1 \pm \frac{1}{i\hbar} u \circ v), \quad \operatorname{Re} z > -1/2.$$

Recall that $\Gamma_*(z \pm \frac{1}{i\hbar} u \circ v)$ and $(z \pm \frac{1}{i\hbar} u \circ v)_{*+}^{-1}$ are analytically continued on $\operatorname{Re} z > -\frac{3}{2}$, $z \neq -\frac{1}{2}$ and $\Gamma_*(z + 1 \pm \frac{1}{i\hbar} u \circ v)$ is analytic on $\operatorname{Re} z > -\frac{3}{2}$. By this we define

$$\Gamma_*(z \pm \frac{1}{i\hbar} u \circ v) = (z \pm \frac{1}{i\hbar} u \circ v)_{*+}^{-1} * \Gamma_*(z + 1 \pm \frac{1}{i\hbar} u \circ v), \quad \operatorname{Re} z > -3/2, z \neq -\frac{1}{2}.$$

Applying the infinite product formula (5.2) to the term $\Gamma_*(z+1\pm\frac{1}{i\hbar}u\circ v)$ by replacing z by $z+1$, and noting that

$$e^{-\gamma}\prod_{k=1}^{\infty}(1-\frac{1}{k})e^{\frac{1}{k}}=1,$$

we see that the formula (5.2) holds for $\operatorname{Re} z > -3/2, z \neq -\frac{1}{2}$. Repeating this procedure we see that the formula (5.2) holds on $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$.

Proposition 5.3 *In generic ordered expression,*

$$\lim_{N\to\infty} e_*^{\gamma(z\pm\frac{1}{i\hbar}u\circ v)}*(z\pm\frac{1}{i\hbar}u\circ v)*\prod_{k=1}^{N}*\Big(\big(1+\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)\big)*e_*^{-\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)}\Big)*\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)=1.$$

Namely, $\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)$ is invertible on $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\}$ in a certain weak sense.

Note that the proposition above does not necessarily imply the convergence of

$$\lim_{N\to\infty} e_*^{\gamma(z\pm\frac{1}{i\hbar}u\circ v)}*(z\pm\frac{1}{i\hbar}u\circ v)*\prod_{k=1}^{N}*\Big(\big(1+\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)\big)*e_*^{-\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)}\Big)$$

in $Hol(\mathbb{C}^2)$. In spite of this, it is still useful to generalize as follows:

Proposition 5.4 *If $\Gamma_*(z+\frac{1}{i\hbar}u\circ v)*H_*$ is defined by*

$$\lim_{N\to\infty} e_*^{-\gamma(z\pm\frac{1}{i\hbar}u\circ v)}*(z\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1}*\prod_{k=1}^{N}*\Big(\big(1+\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)\big)_{*\pm}^{-1}*e_*^{\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)}\Big)*H_*,$$

then

$$H_*=\lim_{N\to\infty} e_*^{\gamma(z\pm\frac{1}{i\hbar}u\circ v)}*(z\pm\frac{1}{i\hbar}u\circ v)*\prod_{k=1}^{N}*\Big(\big(1+\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)\big)*e_*^{-\frac{1}{k}(z\pm\frac{1}{i\hbar}u\circ v)}\Big)*\big(\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)*H_*\big)$$

5.2 About Euler's reflection formula

It is wellknown

$$\frac{\pi}{\sin\pi z}=z^{-1}\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})^{-1}=\Gamma(z)\Gamma(1-z).$$

By the diagonal matrix expression, we have

$$:\frac{1}{\pi}\sin_*\pi\big(z+\frac{1}{i\hbar}u\circ v\big):_{E(K)mat}=\sum_{k=0}^{\infty}\frac{1}{\pi}sin(\pi(z+k+\frac{1}{2}))E_{k,k}(K)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(z+k+\frac{1}{2})\Gamma(1-z-k-\frac{1}{2})}E_{k,k}(K).$$

By the calculation as diagonal matrices, the r.h.s. may be replaced by

$$\Big(\sum_{k=0}^{\infty}\frac{1}{\Gamma(z+k+\frac{1}{2})}E_{k,k}(K)\Big)*\Big(\sum_{k=0}^{\infty}\frac{1}{\Gamma(1-z-k-\frac{1}{2})}E_{k,k}(K)\Big).$$

By (4.10), we see the first component converges in $Hol(\mathbb{C}^2)$, but the second component diverges in $Hol(\mathbb{C}^2)$. Note also that $\sum_{k=0}^{\infty} \frac{1}{\Gamma(1-z-k-\frac{1}{2})} \overline{E}_{k,k}(K)$ converges.

Suggested by Theorem 4.1 and $E_{k,k}(K) * \overline{E}_{l,l}(K) = 0$, we have the following

Conjecture There is no expression parameter such that

$$\Gamma_*^{-1}\big(z+\frac{1}{i\hbar}u\circ v\big) * \Gamma_*^{-1}\big(1-z-\frac{1}{i\hbar}u\circ v\big)$$

is defined in $Hol(\mathbb{C}^2)$. This implies that Euler's reflection formula fails for $\sin_* \pi(z+\frac{1}{i\hbar}u\circ v)$.

6 Star-zeta function

As it is wellknown, the gamma function and the zeta function are deeply related. Ordinary zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \big(\frac{1}{n}\big)^s, \quad \mathrm{Re}\, s > 1.$$

Replacing $1/n$ by $e_*^{-(\log n)(z\pm\frac{1}{i\hbar}u\circ v)}$, we define the $*$-zeta function by

$$(6.1) \qquad\qquad \zeta_*\big(z\pm\frac{1}{i\hbar}u\circ v\big) = \sum_{n=1}^{\infty} e_*^{-(\log n)(z\pm\frac{1}{i\hbar}u\circ v)}.$$

Comparing this with the integral $\int_1^{\infty} e_*^{-(\log t)(z\pm\frac{1}{i\hbar}u\circ v)} dt$ in generic ordered expression and replacing $t = e^s$, we have a remarkable

Proposition 6.1 *In generic ordered expression, $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$ absolutely converges uniformly on every compact domain in $\{z; \mathrm{Re}\, z > 1/2\} \times \mathbb{C}^2$.*

Let $p_1 < p_2 < p_3 < \cdots$ be the series of all prime numbers; i.e. $p_1=2$, $p_2=3$, $p_3=5$, $\cdots$. Set

$$1 - p_n^{-(z\pm\frac{1}{i\hbar}u\circ v)*} = 1 - e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_n}, \quad \big(1 - e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_n}\big)_{*+}^{-1} = \sum_{k=0}^{\infty} e_*^{-k(z\pm\frac{1}{i\hbar}u\circ v)\log p_n}.$$

The latter converges in generic ordered expression. Thus, in generic ordered expression

$$\prod_{n=1}^{\ell} \big(1 - e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_n}\big)_*^{-1}$$

is welldefined to obtain

$$\sum e_*^{-k_1(z\pm\frac{1}{i\hbar}u\circ v)\log p_1} * e_*^{-k_2(z\pm\frac{1}{i\hbar}u\circ v)\log p_2} * \cdots * e_*^{-k_\ell(z\pm\frac{1}{i\hbar}u\circ v)\log p_\ell}$$

$$= \sum_{k_1,k_2,\ldots,k_\ell \geq 0} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)(\log p_1^{k_1} p_2^{k_2}\cdots p_\ell^{k_\ell})}.$$

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Thus, we have

$$(6.2) \qquad \lim_{\ell\to\infty}\prod_{n=1}^{\ell}(1-e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_\ell})_*^{-1} = \sum_{k_1,k_2,\ldots,k_\ell\geq 0} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)(\log p_1^{k_1} p_2^{k_2}\cdots p_\ell^{k_\ell})}.$$

Hence taking $\ell\to\infty$, the summation runs through all positive integers once for all by the unique factorization theorem for natural numbers. Thus, we have the Euler's infinite product formula

$$(6.3) \qquad \lim_{\ell\to\infty}\prod_{n=1}^{\ell}(1-e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_n})_*^{-1} = \sum_{n=1}^{\infty} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log n} = \zeta_*\big(z\pm\frac{1}{i\hbar}u\circ v\big).$$

On the other hand,

$$\prod_{n=1}^{\ell}(1-e_*^{-(z+\pm\frac{1}{i\hbar}u\circ v)\log p_n}) =$$

$$1-\sum_{n=1}^{\infty} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_n} + \sum_{1\leq i<j\leq\ell} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_i p_j} - \cdots +(-1)^{\ell}e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log p_1 p_2\cdots p_\ell}.$$

Note that $p_{i_1}p_{i_2}\cdots p_{i_k}$ which appear in the summation are much less than the positive integers. Hence, the r.h.s. absolutely converges uniformly on every compact region of $z\pm\frac{1}{i\hbar}u\circ v$ such that $\operatorname{Re} z > \frac{1}{2}$. Hence, we have

Proposition 6.2 *In generic K-expression, both $:\zeta_*(z\pm\frac{1}{i\hbar}u\circ v):_K$ and $:\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)_*^{-1}:_K$ exist on the domain $\operatorname{Re} z > \frac{1}{2}$.*

6.1 Generating function of numbers of partitions

Note that $*$-zeta function is still useful to treat prime numbers. Note that the mathematics is not based on the notion of cardinal numbers but on the notion of natural numbers axiomatized by the notion of ordinal numbers. Our ultimate theme is to know how mathematics recognize cardinal numbers and the "time".

Although this is not directly relevant to our purpose, a similar formula is proposed relating to the cardinal numbers.

Consider the function

$$f(x) = 1+\sum_{n=1}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1}, \quad |x| < 1.$$

Replacing x by $e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)}$, we define

$$f_*(e_*^{z\pm\frac{1}{i\hbar}u\circ v}) = \prod_{n=1}^{\infty} *(1-e_*^{-n(z\pm\frac{1}{i\hbar}u\circ v)})^{-1}$$

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Note that if $\operatorname{Re} z \geq 0$, then in generic ordered expression

$$(1-e_*^{-n(z\pm\frac{1}{i\hbar}u\circ v)})^{-1} = \sum_{k=0}^{\infty} e_*^{-k\,n(z\pm\frac{1}{i\hbar}u\circ v)}$$

converges absolutely and uniformly on every compact region of $z\pm\frac{1}{i\hbar}u\circ v$ such that $\operatorname{Re} z > 0$, and $f_*(e_*^{z\pm\frac{1}{i\hbar}u\circ v})$ plays the same role as a generating function of the number of partitions $p(n)$.

$$1+\sum_{n=1}^{\infty} p(n)e_*^{-n(z\pm\frac{1}{i\hbar}u\circ v)} = \prod_{\ell=1}^{\infty} * \sum_{k=0}^{\infty} e_*^{-k\ell(z\pm\frac{1}{i\hbar}u\circ v)}.$$

This means that the notion of cardinal number is independent of algebraic structure.

It is known by Postnikov that

$$p(n) = \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}n}\Big(1+O(\frac{\log n}{n^{1/4}})\Big).$$

Hence $f_*(e_*^{z\pm\frac{1}{i\hbar}u\circ v})$ converges for $\operatorname{Re} z > -\frac{1}{2}$. By the same reason as in $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$ we have

Proposition 6.3 *In generic ordered expression K, both $:f_*(z\pm\frac{1}{i\hbar}u\circ v):_K$ and $:f_*(e_*^{z\pm\frac{1}{i\hbar}u\circ v})^{-1}_*:_K$ exist on the domain $\operatorname{Re} z > 0$.*

Note There is no analytic continuation of the generating function $f_*(z+\frac{1}{i\hbar}u\circ v)$ of number of partitions. It is very likely that $\operatorname{Re} z = 0$ is the natural boundary.

6.2 Analytic continuation of $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$

First of all, note that the next equality holds for every $a \in \mathbb{R}$ and for every $*$-polynomial $p_*(u,v)$

$$p_*(u,v)*e_*^{a(z\pm\frac{1}{i\hbar}u\circ v)}*\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v) = \int_{-\infty}^{\infty} e^{-e^s} p_*(u,v)*e_*^{(s+a)(z\pm\frac{1}{i\hbar}u\circ v)}ds.$$

Now, set $e^s = ne^\sigma$ in the definition of $*$-gamma function, we have

$$\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v) = \int_{-\infty}^{\infty} e^{-e^s} e_*^{s(z\pm\frac{1}{i\hbar}u\circ v)}ds = \int_{-\infty}^{\infty} e^{-ne^\sigma} e_*^{(\sigma+\log n)(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma.$$

The exponential law $e_*^{(\sigma+\log n)(z\pm\frac{1}{i\hbar}u\circ v)} = e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}*e_*^{(\log n)(z\pm\frac{1}{i\hbar}u\circ v)}$ and the continuity of $e_*^{-(\log n)(z\pm\frac{1}{i\hbar}u\circ v)}*$ give

$$\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)*e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log n} = \int_{-\infty}^{\infty} e^{-ne^\sigma} e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma.$$

It follows

$$(6.4) \qquad \sum_{n=1}^{N} e_*^{-(z\pm\frac{1}{i\hbar}u\circ v)\log n} = \Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)^{-1}*\int_{-\infty}^{\infty}\sum_{n=1}^{N} e^{-ne^\sigma} e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma.$$

where $\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)^{-1}$ is defined by Theorem 4.1.

By taking the limit $N\to\infty$, the l.h.s. converges easily in $Hol(\mathbb{C}^2)$ to obtain for every $a\in\mathbb{C}$ and for every $*$-polynomial $p_*(u,v)$

$$(6.5) \qquad p_*(u,v)*e_*^{a(z\pm\frac{1}{i\hbar}u\circ v)}*\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$$
$$= p_*(u,v)*e_*^{a(z\pm\frac{1}{i\hbar}u\circ v)}*\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)^{-1}*\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma.$$

Consider the main part $\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$ of the r.h.s. of (6.5). By the definition of the Bernoulli numbers, we have

$$\frac{1}{e^{e^t}-1} = \frac{1}{e^t}-\frac{1}{2}+\sum_{n\geq 1}B_{2n}\frac{(-1)^{n-1}}{(2n)!}e^{(2n-1)t}, \quad B_{2n+1}=0, \; B_{2n}>0.$$

Considering domains $t>0$ and $t<0$ separately, we see $B(t)=\frac{1}{e^{e^t}-1}-e^{-t}$ is rapidly decreasing function.

By Proposition 5.4 and (6.5), we get

Proposition 6.4 $(z+\frac{1}{i\hbar}u\circ v)*\prod_{\ell=1}^{n}*(1+\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v)}*\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma$ *is holomorphic on the domain* $\mathbb{C}\backslash\{-(\mathbb{N}+n+\frac{1}{2})\cup\{\frac{1}{2}\}\}$. *Taking* $n\to\infty$ *we have*

$$\lim_{n\to\infty}e_*^{\gamma(z+\frac{1}{i\hbar}u\circ v))}*(z+\frac{1}{i\hbar}u\circ v)*\prod_{\ell=1}^{n}*(1+\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v)}*\Gamma_*(z+\frac{1}{i\hbar}u\circ v)*\zeta_*(z+\frac{1}{i\hbar}u\circ v)$$

$$= \lim_{n\to\infty}e_*^{\gamma(z+\frac{1}{i\hbar}u\circ v))}*(z+\frac{1}{i\hbar}u\circ v)*\prod_{\ell=1}^{n}*(1+\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v)}*\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma.$$

Since the first line converges to $\zeta_*(z+\frac{1}{i\hbar}u\circ v)$, *we see* $\zeta_*(z+\frac{1}{i\hbar}u\circ v)$ *is holomorphic on* $\mathbb{C}\backslash\{\frac{1}{2}\}$ *in generic ordered expression. Similarly,* $\zeta_*(z-\frac{1}{i\hbar}u\circ v)$ *is holomorphic on* $\mathbb{C}\backslash\{\frac{1}{2}\}$ *in generic ordered expression.*

By the proof of Lemma 6.2, we see that the residue of $\zeta_*(z+\frac{1}{i\hbar}u\circ v)$ is given by

$$\mathrm{Res}(\zeta_*(z+\frac{1}{i\hbar}u\circ v), z=\frac{1}{2}) = \mathrm{Res}((z+\frac{1}{i\hbar}u\circ v)_{*-}^{-1}, z=\frac{1}{2}) = \overline{\varpi}_{00}.$$

where $\overline{\varpi}_{00}$ in the r.h.s. is called the bar-vacuum.

However, in the proof of Proposition 6.4, we have only to eliminate the singularities of $\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma$ by multiplying odd factors

$$(z+\frac{1}{i\hbar}u\circ v)*\prod_{\ell=1}^{n}*(1+\frac{1}{2\ell+1}(z+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2\ell+1}(z+\frac{1}{i\hbar}u\circ v)}*.$$

Even factors $\prod_{\ell=1}^{n}(1+\frac{1}{2\ell}(z+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{\ell}(z+\frac{1}{i\hbar}u\circ v)}*$ are used only to get gamma function. Thus, corresponding to the trivial zero's of ordinary Riemann zeta function, we see that $\zeta_*(z+\frac{1}{i\hbar}u\circ v)$ has many factors: For every positive integer n,

$$\zeta_*(z+\frac{1}{i\hbar}u\circ v) = (z+2n+\frac{1}{i\hbar}u\circ v)*g_n(z+\frac{1}{i\hbar}u\circ v)$$

where $g_n(z+\frac{1}{i\hbar}u\circ v)$ is holomorphic on $\mathbb{C}\backslash\{\frac{1}{2}\}$. Thus, Proposition 6.4 does not imply that $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)_*^{-1}$ is holomorphic on $\mathbb{C}\backslash\{\frac{1}{2}\}$, but many singular points on negative real axis.

6.2.1 Diagonal matrix expression of $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$

First, we show

Lemma 6.1 *In a generic ordered expression*

$$p_*(u,v)*e_*^{a(z\pm\frac{1}{i\hbar}u\circ v)}*\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$$

is a Hol($\mathbb{C}^2$)-valued holomorphic function of z on the domain $\operatorname{Re}z>\frac{1}{2}$ for every $a\in\mathbb{C}$ and for every $*$-polynomial $p_*(u,v)$.

Proof It is clear that $\int_0^\infty\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$ is a Hol($\mathbb{C}^2$)-valued entire function of z.

For $\int_{-\infty}^0\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$, note that $e^{e^\sigma}-1>e^\sigma$ and $e_*^{\sigma\frac{1}{i\hbar}u\circ v}$ is of order $e^{-|\sigma|\frac{1}{2}}$ in generic ordered expressions. $\square$

The diagonal matrix expressions of $\int_0^\infty\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$ are

(6.6)
$$:\int_0^\infty\frac{1}{e^{e^s}-1}e_*^{s(z+\frac{1}{i\hbar}u\circ v)}ds:_{E(K)mat}=\sum_{k=0}^\infty\int_0^\infty\frac{1}{e^{e^s}-1}e_*^{s(z+n+\frac{1}{2})}ds E_{n,n}(K)$$

$$:\int_0^\infty\frac{1}{e^{e^s}-1}e_*^{s(z-\frac{1}{i\hbar}u\circ v)}ds:_{\overline{E}(K)mat}=\sum_{k=0}^\infty\int_0^\infty\frac{1}{e^{e^\sigma}-1}e_*^{s(z-n-\frac{1}{2})}ds\overline{E}_{n,n}(K)$$

and every component is an entire function of z.

Lemma 6.2 $\int_{-\infty}^\infty\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma$ *is analytically continued to the domain $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\cup\{\frac{1}{2}\}\}$. Similarly, $\int_{-\infty}^\infty\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z-\frac{1}{i\hbar}u\circ v)}d\sigma$ is analytically continued to the domain $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\cup\{\frac{1}{2}\}\}$.*
We denote these by $L_*(z+\frac{1}{i\hbar}u\circ v)$, $L_*(z-\frac{1}{i\hbar}u\circ v)$ respectively.

Proof It is enough to show for the integral $\int_{-\infty}^0\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$.

For every fixed z find an positive integer N such that $\operatorname{Re}z+N>\frac{1}{2}$, and set

$$\frac{1}{e^{e^t}-1}=\frac{1}{e^t}-\frac{1}{2}+\cdots+B_{2N}\frac{(-1)^{N-1}}{(2N)!}e^{(2N-1)t}+e^{(2N-1)t}R_n(t).$$

It is easy to see that

$$\int_{-\infty}^0 e^{(2N-1)t}R_n(t)*e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma$$

converges to give a holomorphic function defined on the domain $\operatorname{Re}(z+N)>\frac{1}{2}$. It follows by termwise integration

$$\int_{-\infty}^0\frac{1}{e^{e^t}-1}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt=\int_{-\infty}^0\Big(e^{-t}-\frac{1}{2}+\sum_{n\geq0}(-1)^{n-1}B_{2n}\frac{e^{(2n-1)t}}{(2n)!}\Big)e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

$$=((z-1)+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}-\frac{1}{2}(z+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}+\sum_{n\geq0}(-1)^{n-1}\frac{B_{2n}}{(2n)!}(z+2n-1+\frac{1}{i\hbar}u\circ v)_{*+}^{-1}.$$

Similarly,

$$\int_{-\infty}^{0}\frac{1}{e^{e^t}-1}e_*^{t(z-\frac{1}{i\hbar}u\circ v)}ds = \int_{-\infty}^{0}\Big(e^{-t}-\frac{1}{2}+\sum_{n\geq 0}(-1)^{n-1}B_{2n}\frac{e^{(2n-1)t}}{(2n)!}\Big)e_*^{t(z-\frac{1}{i\hbar}u\circ v)}dt$$

$$= ((z-1)-\frac{1}{i\hbar}u\circ v)_{*-}^{-1}-\frac{1}{2}(z-\frac{1}{i\hbar}u\circ v)_{*-}^{-1}+\sum_{n\geq 0}(-1)^{n-1}\frac{B_{2n}}{(2n)!}(z+2n-1-\frac{1}{i\hbar}u\circ v))_{*-}^{-1}$$

$$= -(-(z-1)+\frac{1}{i\hbar}u\circ v)_{*-}^{-1}+\frac{1}{2}(-z+\frac{1}{i\hbar}u\circ v)_{*-}^{-1}-\sum_{n\geq 0}(-1)^{n-1}\frac{B_{2n}}{(2n)!}(-(z+2n-1)+\frac{1}{i\hbar}u\circ v))_{*-}^{-1}$$

The result follows from the analytic continuations of $(z+2n-1\pm\frac{1}{i\hbar}u\circ v))_{*\pm}^{-1}$ given in [17]. Cf. Theorem 2.2 also). □

By Lemmas 6.1 and 6.2, we see that in a generic ordered expression,

$$\Gamma_*(z\pm\frac{1}{i\hbar}u\circ v)*\zeta_*(z\pm\frac{1}{i\hbar}u\circ v) = \int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z+\frac{1}{i\hbar}u\circ v)}d\sigma$$

is a $Hol(\mathbb{C}^2)$-valued holomorphic function of z defined on $\operatorname{Re}z > \frac{1}{2}$, and this is continued to the domain $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\cup\{\frac{1}{2}\}\}$ analytically.

The $E(K)mat$ and $\overline{E}(K)mat$-expressions of $\int_{-\infty}^{\infty}\frac{1}{e^{e^\sigma}-1}e_*^{\sigma(z\pm\frac{1}{i\hbar}u\circ v)}d\sigma$ are obtained by these formulas by splitting the integral into $\int_0^\infty+\int_{-\infty}^0$. We denote these as follows:

(6.7)
$$:L_*(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat} = \sum_{k=0}^{\infty}L(z+k+\frac{1}{2})E_{k,k}(K)$$

$$:L_*(z-\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat} = \sum_{k=0}^{\infty}L(z+k+\frac{1}{2})\overline{E}_{k,k}(K).$$

$L(z+k+\frac{1}{2})$ are holomorphic on $\mathbb{C}\backslash\{-(\mathbb{N}+\frac{1}{2})\cup\{\frac{1}{2}\}\}$.

Combining this with (4.10) as diagonal matrix calculations, we have

(6.8)
$$:\zeta_*(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat}=\sum_{k=0}^{\infty}\zeta(z+k+\frac{1}{2})E_{k,k}(K)=\sum_{k=0}^{\infty}\Gamma^{-1}(z+k+\frac{1}{2})L(z+k+\frac{1}{2})E_{k,k}(K)$$

$$:\zeta_*(z-\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat}=\sum_{k=0}^{\infty}\zeta(z+k+\frac{1}{2})\overline{E}_{k,k}(K)=\sum_{k=0}^{\infty}\Gamma^{-1}(z+k+\frac{1}{2})L(z+k+\frac{1}{2})\overline{E}_{k,k}(K).$$

Since all singular point of $L(z+k+\frac{1}{2})$ except $1/2$ are eliminated by $\Gamma^{-1}(z+k+\frac{1}{2})$, we have

Proposition 6.5 *Both* $:\zeta_*(z+\frac{1}{i\hbar}u\circ v):_{E(K)mat}$ *and* $:\zeta_*(z-\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat}$ *are holomorphic on* $\mathbb{C}\backslash\{1/2\}$.

In the next section, we prove this more generally.

By these we make the diagonal matrix expressions of their inverses:

(6.9)
$$:\zeta_*(z+\frac{1}{i\hbar}u\circ v)^{-1}:_{E(K)mat} = \sum_{k=0}^{\infty}\zeta(z+k+\frac{1}{2})^{-1}E_{k,k}(K)$$

$$:\zeta_*(z-\frac{1}{i\hbar}u\circ v)^{-1}:_{\overline{E}(K)mat} = \sum_{k=0}^{\infty}\zeta(z+k+\frac{1}{2})^{-1}\overline{E}_{k,k}(K).$$

Proposition 6.6 *If $\zeta(z_0+n+\frac{1}{2}) = 0$ for some z_0+n, then both*

$$:\zeta_*(z_0+\frac{1}{i\hbar}u\circ v)^{-1}:_{E(K)mat}, \quad :\zeta_*(z_0-\frac{1}{i\hbar}u\circ v)^{-1}:_{\overline{E(K)}mat}$$

must be singular.

7   Reflection property of $\zeta_*(s\pm\frac{1}{i\hbar}u\circ v)$

Now, setting $e^s = n^2\pi e^x$ in the definition of $*$-gamma function, the exponential law and the continuity of the $*$-multiplication $e_*^{(s+\frac{1}{i\hbar}u\circ v)(\log n+\frac{1}{2}\log \pi)}*$ give

$$\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)) = \int_{-\infty}^{\infty} e^{-n^2\pi e^x} e_*^{(2\log n+\log \pi+x)\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)} dx$$

$$=e_*^{(s\pm\frac{1}{i\hbar}u\circ v)(\log n+\frac{1}{2}\log \pi)}* \int_{-\infty}^{\infty} e^{-n^2\pi e^x} e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} dx, \quad \mathrm{Re}\, s > -\frac{1}{2}.$$

Hence, we have for every positive integer

$$\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2}\log \pi(s\pm\frac{1}{i\hbar}u\circ v)}*e_*^{-(\log n)(s\pm\frac{1}{i\hbar}u\circ v)} = \int_{-\infty}^{\infty} e^{-n^2\pi e^x} e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} dx, \quad \mathrm{Re}\, s > -\frac{1}{2}.$$

It follows by checking associativity that

$$\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)\log \pi}*\zeta_*(s\pm\frac{1}{i\hbar}u\circ v) = \lim_{N\to\infty}\int_{-\infty}^{\infty} \sum_{n=1}^{N} e^{-n^2\pi e^x} e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} dx.$$

Note that the term $\sum_{n=1}^{\infty} e^{-n^2\pi e^x}$ is rapidly decreasing in the positive direction w.r.t. x and bounded in the negative direction. $:e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}:_K$ is rapidly decreasing in the both directions in generic ordered expressions. Hence we see that the r.h.s. can be written as

$$(7.1) \qquad\qquad \Phi_*(s\pm\frac{1}{i\hbar}u\circ v) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} e^{-n^2\pi e^x} e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} dx$$

and this is holomorphic on the domain $\mathrm{Im}\, s > -\frac{1}{2}$, where $\sum_{n=1}^{\infty} e^{-n^2\pi e^x}$ in the integrand is the value at $w = 0$ of Jacobi's theta function $\theta_3(w, \tau)$ given in [14], that is

$$\sum_{n=1}^{\infty} e^{-n^2\pi e^x} = \frac{1}{2}\big(\theta_3(0, \pi e^x) - 1\big)$$

and recall here the famous theta relation $\theta_3(0, \tau) = \sqrt{\frac{\pi}{\tau}}\theta_3(0, \frac{\pi^2}{\tau})$ given in [14]. This gives in particular

$$(7.2) \qquad\qquad \theta_3(0, \pi e^x) = e^{-\frac{1}{2}x}\theta_3(0, \pi e^{-x}).$$

Proposition 7.1 *In generic ordered expression, $\Phi_*(s\pm\frac{1}{i\hbar}u\circ v)$ is a $Hol(\mathbb{C}^2)$-valued holomorphic function on $\mathrm{Re}\, s > -\frac{1}{2}$, and*

$$\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v))*\zeta_*(s\pm\frac{1}{i\hbar}u\circ v) = e_*^{\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)\log \pi}*\Phi_*(s\pm\frac{1}{i\hbar}u\circ v).$$

Split the integral (7.1) into $\int_0^\infty + \int_{-\infty}^0$, we have

$$\int_{-\infty}^\infty \sum_{n=1}^\infty e^{-n^2\pi e^x}e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx = (\int_0^\infty + \int_{-\infty}^0)\frac{1}{2}(\theta_3(0,\pi e^x)-1)e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx$$

$$= \int_0^\infty \frac{1}{2}(\theta_3(0,\pi e^x)-1)e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx + \int_{-\infty}^0 \frac{1}{2}\theta_3(0,\pi e^x)e_*^{\frac{1}{2}x(s+\frac{1}{i\hbar}u\circ v)}dx - \frac{1}{2}\int_{-\infty}^0 e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx$$

The last term make sense for $\mathrm{Re}\,s > -\frac{1}{2}$. Using (7.2) and changing variable x to $-x$, the second term becomes

$$\int_{-\infty}^0 \frac{1}{2}e^{-\frac{1}{2}x}\theta_3(0,\pi e^{-x})e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx = \int_0^\infty \frac{1}{2}(\theta_3(0,\pi e^x)-1)e_*^{\frac{1}{2}x(1-s\mp\frac{1}{i\hbar}u\circ v)}dx + \frac{1}{2}\int_0^\infty e_*^{\frac{1}{2}x(1-s\mp\frac{1}{i\hbar}u\circ v)}dx.$$

The last term makes sense for $\mathrm{Re}\,s > \frac{1}{2}$. Hence by assuming $\mathrm{Re}\,s > \frac{1}{2}$, we have

$$\int_{-\infty}^\infty \sum_{n=1}^\infty e^{-n^2\pi e^x}e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}dx = \int_0^\infty \frac{1}{2}(\theta_3(0,\pi e^x)-1)(e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} + e_*^{\frac{1}{2}x(1-s\mp\frac{1}{i\hbar}u\circ v)})dx$$

$$+ \int_0^\infty \frac{1}{2}e_*^{\frac{1}{2}x(1-s\mp\frac{1}{i\hbar}u\circ v)}dx - \int_{-\infty}^0 \frac{1}{2}e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)}.$$

Note that the first line is entire w.r.t. s. The integrals in the last line are changed into

$$\int_{-\infty}^0 \frac{1}{2}e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} = (s\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1}$$

and

$$\int_0^\infty \frac{1}{2}e_*^{\frac{1}{2}x(1-s\mp\frac{1}{i\hbar}u\circ v)}dx = \int_{-\infty}^0 e_*^{x(s-1\pm\frac{1}{i\hbar}u\circ v)}dx = (s-1\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1} = -(1-(s\pm\frac{1}{i\hbar}u\circ v))_{*\pm}^{-1}.$$

Note this is symmetric by the transformation $s\pm\frac{1}{i\hbar}u\circ v$ to $1-(s\pm\frac{1}{i\hbar}u\circ v)$.

These are analytically continued to the domain $\mathbb{C}\setminus\{\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\cdots\}$. Hence the formula holds on this domain, that is,

$$\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)\log\pi}\zeta_*(s\pm\frac{1}{i\hbar}u\circ v)$$

(7.3)
$$= \int_0^\infty \frac{1}{2}(\theta_3(0,\pi e^x)-1)(e_*^{\frac{1}{2}x(s\pm\frac{1}{i\hbar}u\circ v)} + e_*^{\frac{1}{2}x(1-(s\pm\frac{1}{i\hbar}u\circ v))})dx$$

$$-(1-(s\pm\frac{1}{i\hbar}u\circ v))_{*\pm}^{-1} - (s\pm\frac{1}{i\hbar}u\circ v)_{*\pm}^{-1}.$$

The similar proof of Proposition 6.4 gives

Proposition 7.2 *Singularities at $\{\frac{-1}{2},\frac{-3}{2},\cdots,\frac{-2n-1}{2}\}$ are eliminated in*

$$(s+\frac{1}{i\hbar}u\circ v)*\prod_{\ell=1}^n (1+\frac{1}{\ell}(s\pm\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{\ell}(s\pm\frac{1}{i\hbar}u\circ v)}*e_*^{\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)\log\pi}*\Phi_*(s\pm\frac{1}{i\hbar}u\circ v).$$

Note that (7.3) is symmetric under the transformation $s+\frac{1}{i\hbar}u\circ v \to 1-(s+\frac{1}{i\hbar}u\circ v)$. Hence we have in generic ordered expression

$$
(7.4) \quad
\begin{aligned}
\Gamma_*(\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v))&*e_*^{-\frac{1}{2}(s\pm\frac{1}{i\hbar}u\circ v)\log\pi}*\zeta_*(s\pm\frac{1}{i\hbar}u\circ v)\\
&=\Gamma_*(\frac{1}{2}(1-(s\pm\frac{1}{i\hbar}u\circ v)))*e_*^{-\frac{1}{2}(1-(s\pm\frac{1}{i\hbar}u\circ v))\log\pi}*\zeta_*(1-(s\pm\frac{1}{i\hbar}u\circ v)),
\end{aligned}
$$

on $\mathbb{C}\setminus\{\cdots,\frac{-3}{2},\frac{-1}{2},\frac{1}{2}\}$.

Note that (7.3) is not symmetric under the transformation $s\to 1-s$. Hence we consider

$$
(7.5) \quad
\begin{aligned}
F_*(s)&=\Gamma_*(\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)\log\pi}*\zeta_*(s+\frac{1}{i\hbar}u\circ v)\\
&+\Gamma_*(\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v))*e_*^{-\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v)\log\pi}*\zeta_*(s-\frac{1}{i\hbar}u\circ v)
\end{aligned}
$$

$$
(7.6) \quad
\begin{aligned}
G_*(s)&=\Gamma_*^{-1}(\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v))*e_*^{\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)\log\pi}*\zeta_*^{-1}(s+\frac{1}{i\hbar}u\circ v)\\
&+\Gamma_*^{-1}(\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v))*e_*^{\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v)\log\pi}*\zeta_*^{-1}(s-\frac{1}{i\hbar}u\circ v)
\end{aligned}
$$

It is obvious to have $F_*(s)=F_*(1-s)$, and $G_*(s)=G_*(1-s)$.

Consider now hybrid expressions

$$
:G_*(s):_{E\pm}\ =\ :\Gamma_*^{-1}(\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)):_{E(K)mat}+:\Gamma_*^{-1}(\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v)):_{\overline{E}(K)mat}\quad (\text{ Cf. } (4.10))
$$

$$
:e_*^{\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)\log\pi}:_{E\pm}\ =\ :e_*^{\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)\log\pi}:_{E(K)mat}+:e_*^{\frac{1}{2}(s-\frac{1}{i\hbar}u\circ v)\log\pi}:_{\overline{E}(K)mat}\quad (\text{Cf. Theorem 3.1})
$$

$$
:\zeta_*^{-1}(s+\frac{1}{i\hbar}u\circ v):_{E\pm}\ =\ :\zeta_*^{-1}(s+\frac{1}{i\hbar}u\circ v):_{E(K)mat}+:\zeta_*^{-1}(s-\frac{1}{i\hbar}u\circ v):_{\overline{E}(K)mat},\quad (\text{Cf. } (6.9)).
$$

Each of them has no singular point on $\mathrm{Re}\,s>1/2$. We have also

$$
G_*(s)\ =\ :G_*(s):_{E\pm}*:e_*^{\frac{1}{2}(s+\frac{1}{i\hbar}u\circ v)\log\pi}:_{E\pm}*:\zeta_*^{-1}(s+\frac{1}{i\hbar}u\circ v):_{E\pm}
$$

and similarly

$$
G_*(1-s)\ =\ :G_*(1-s):_{E\pm}*:e_*^{\frac{1}{2}(1-s-\frac{1}{i\hbar}u\circ v)\log\pi}:_{E\pm}*:\zeta_*^{-1}(1-s-\frac{1}{i\hbar}u\circ v):_{E\pm}.
$$

By Proposition 6.2 both $\zeta_*(z\pm\frac{1}{i\hbar}u\circ v)$ have genuine inverses for $\mathrm{Re}\,z>\frac{1}{2}$.

We now take the hybrid matrix expression of the identity $G_*(s)=G_*(1-s)$.

Note also that if a function $f(X)$ has no singular point on the domain $\mathrm{Re}X>\frac{1}{2}$, if $f(X)$ has the reflection symmetric property such as $f(X)=f(1-X)$, then the singular points of $f(X)$ possibly allows only on the line $\mathrm{Re}X=\frac{1}{2}$.

The next result might suggest useful information about Riemann conjecture.

Theorem 7.1 *Singular point of* $:\zeta_*(s+\frac{1}{i\hbar}u\circ v)^{-1}:_{E\pm}$ *on the strip* $0<\mathrm{Re}\,s<1$ *is possibly only on* $\mathrm{Re}\,s=\frac{1}{2}$.

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