# MODULI OF SHEAVES SUPPORTED ON CURVES OF GENUS TWO IN A QUADRIC SURFACE

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ABSTRACT. We study the moduli space of stable sheaves of Euler characteristic 1, supported on curves of arithmetic genus 2 contained in a smooth quadric surface. We show that this moduli space is rational. We give a classification of the stable sheaves involving locally free resolutions or extensions. We compute the Betti numbers by studying the variation of the moduli spaces of  $\alpha$ -semi-stable pairs.

#### 1. Introduction

Let  $\mathbb{P}^1$  be the complex projective line and let  $\mathcal{F}$  be a coherent algebraic sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  with support of dimension 1. We fix the polarization  $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . According to [1, Proposition 2], there are  $r, s, t \in \mathbb{Z}$  such that for any  $m, n \in \mathbb{Z}$  the Euler characteristic of the twisted sheaf  $\mathcal{F}(m, n)$  satisfies  $\chi(\mathcal{F}(m, n)) = rm + sn + t$ . The linear polynomial  $P_{\mathcal{F}}(m, n) = rm + sn + t$  is called the Hilbert polynomial of  $\mathcal{F}$  and the ratio  $p(\mathcal{F}) = t/(r+s)$  is called the slope of  $\mathcal{F}$  with respect to the fixed polarization. We recall that  $\mathcal{F}$  is semi-stable (respectively stable) with respect to the above polarization if it does not contain subsheaves with support of dimension zero and for any proper subsheaf  $\mathcal{E} \subset \mathcal{F}$  we have  $p(\mathcal{E}) \leq p(\mathcal{F})$  (respectively  $p(\mathcal{E}) < p(\mathcal{F})$ ). According to [19], for a given polynomial P, there is a coarse moduli space, denoted M(P), that is a projective variety, and that parametrizes S-equivalence classes of semi-stable sheaves on  $\mathbb{P}^1 \times \mathbb{P}^1$  with Hilbert polynomial P. Its dimension, as computed in [13, Proposition 2.3], is 2rs + 1. By the argument at [13, Theorem 3.1] M(P) is irreducible and by [13, Proposition 2.3] it is smooth at the points given by stable sheaves.

The first non-trivial examples of such moduli spaces are M(2m+2n+1) and M(2m+2n+2). They were studied in [1] which contains a classification of the semi-stable sheaves by means of locally free resolutions. The rationality of M(2m+2n+2) was proved in [5] by the wall-crossing method and in [17] by an elementary method.

The object of this paper is the study of  $\mathbf{M} = \mathrm{M}(3m+2n+1)$ . The points of  $\mathbf{M}$  are stable sheaves  $\mathcal{F}$  supported on curves of bidegree (2,3) contained in  $\mathbb{P}^1 \times \mathbb{P}^1$ , with  $\chi(\mathcal{F}) = 1$ . As noted above,  $\mathbf{M}$  is a smooth projective variety of dimension 13. Twisting by powers of the polarization provides isomorphisms  $\mathbf{M} \simeq \mathrm{M}(3m+2n+5t)$  for any  $t \in \mathbb{Z}$ .

For  $i, j \in \mathbb{Z}$  we use the abbreviation  $\mathcal{O}(i, j) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(i, j)$ . We fix vector spaces  $V_1$  and  $V_2$  over  $\mathbb{C}$  of dimension 2 and we make the identifications

$$\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V_1) \times \mathbb{P}(V_2), \qquad \mathrm{H}^0(\mathcal{O}(i,j)) = \mathrm{S}^i \, V_1^* \otimes \mathrm{S}^j \, V_2^*.$$

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We fix a basis  $\{x, y\}$  of  $V_1^*$  and a basis  $\{z, w\}$  of  $V_2^*$ . For a sheaf  $\mathcal{F}$  we denote by  $[\mathcal{F}]$  its S-equivalence class. If  $\mathcal{F}$  is stable, then  $[\mathcal{F}]$  is its isomorphism class.

**Theorem 1.1.** The variety  $\mathbf{M}$  is rational. We have a decomposition of  $\mathbf{M}$  into an open subvariety  $\mathbf{M}_0$ , a closed smooth irreducible subvariety  $\mathbf{M}_1$  of codimension 1, and a closed subvariety  $\mathbf{M}_2 \cup \mathbf{M}_3$  having two smooth irreducible components  $\mathbf{M}_2$ ,  $\mathbf{M}_3$  of codimension 2, respectively, 3. The subvarieties are defined as follows:  $\mathbf{M}_0 \subset \mathbf{M}$  is the subset of sheaves  $\mathcal{F}$  having a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-1, -2) \stackrel{\varphi}{\longrightarrow} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

where  $\varphi_{11}$  and  $\varphi_{12}$  define a subscheme of length 2 of  $\mathbb{P}^1 \times \mathbb{P}^1$ ;  $\mathbf{M}_1 \subset \mathbf{M}$  is the subset of sheaves  $\mathcal{F}$  having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) \stackrel{\varphi}{\longrightarrow} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

where  $\varphi_{11} \neq 0$ ,  $\varphi_{12} \neq 0$ ;  $\mathbf{M}_2$  is the set of twisted structure sheaves  $\mathcal{O}_C(0,1)$  for a curve  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (2,3);  $\mathbf{M}_3$  is the set of non-split extensions of  $\mathcal{O}_L$  by  $\mathcal{O}_Q$  for a line  $L \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (0,1) and a quartic  $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (2,2).

The subvariety  $\mathbf{M}_1$  is isomorphic to a  $\mathbb{P}^9$ -bundle over  $\mathbb{P}^1 \times \mathbb{P}^2$  and is the Brill-Noether locus of sheaves  $\mathcal{F}$  satisfying  $\mathrm{H}^0(\mathcal{F}(-1,1)) \neq 0$  (for  $\mathcal{F} \in \mathbf{M}_1$  we have  $\mathrm{H}^0(\mathcal{F}(-1,1)) \simeq \mathbb{C}$ );  $\mathbf{M}_2$  is isomorphic to  $\mathbb{P}^{11}$  and is the Brill-Noether locus of sheaves  $\mathcal{F}$  satisfying  $\mathrm{H}^1(\mathcal{F}) \neq 0$  (for  $\mathcal{F} \in \mathbf{M}_2$  we have  $\mathrm{H}^1(\mathcal{F}) \simeq \mathbb{C}$ );  $\mathbf{M}_3$  is isomorphic to a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^8 \times \mathbb{P}^1$ .

The proof of this theorem is distributed among the statements of Section 4.

As an application of our classification of sheaves we compute the Betti numbers of M. For a projective variety X we define the Poincaré polynomial

$$P(X)(\xi) = \sum_{i>0} \dim_{\mathbb{Q}} H^{i}(X, \mathbb{Q}) \xi^{i/2}.$$

The varieties occurring in this paper will have no odd cohomology, so the above will be a genuine polynomial expression.

**Theorem 1.2.** The integral homology groups of M have no torsion. The Poincaré polynomial of M is

$$\xi^{13} + 3\xi^{12} + 8\xi^{11} + 10\xi^{10} + 11\xi^9 + 11\xi^8 + 11\xi^7 + 11\xi^6 + 11\xi^5 + 11\xi^4 + 10\xi^3 + 8\xi^2 + 3\xi + 1.$$

The proof of this theorem takes up Section 5 and is based on the approach of Choi and Chung [3], where they study moduli spaces of  $\alpha$ -semi-stable pairs and their variation when the parameter  $\alpha$  changes. Thus, we show that  $\mathbf{M}$  is obtained from the relative Hilbert scheme of two points on the general curve of bidegree (2, 3) by performing one blowing up followed by two blowing down operations. The Betti numbers of  $\mathbf{M}$  have already been computed in [4, Section 9.2] in the context of physics. Our calculation agrees with the one in [4]. The Euler characteristic of  $\mathbf{M}$  is 110.

In Section 3 we prove that  $H^1(\mathcal{F}) = 0$  for  $\mathcal{F} \in \mathbf{M} \setminus \mathbf{M}_2$ , which is a crucial step in our classification of sheaves. In Section 2 we present our main technical tool: a spectral sequence converging to a coherent sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  reminiscent to the Beilinson spectral sequence on the projective plane.

#### 2. Preliminaries

According to [2, Lemma 1], for a given coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  there is a spectral sequence converging to  $\mathcal{F}$  in degree zero and to 0 in degrees different from zero. The sheaves  $E_1^{ij}$  on the first level  $E_1$  are defined as follows:

$$\begin{split} E_1^{ij} &= 0 \quad \text{for } i > 0 \text{ and } i < -2, \\ E_1^{0j} &= \mathrm{H}^j(\mathcal{F}) \otimes \mathcal{O}, \\ E_1^{-2,j} &= \mathrm{H}^j(\mathcal{F}(-1,-1)) \otimes \mathcal{O}(-1,-1). \end{split}$$

The sheaves  $E_1^{-1,j}$  fit into exact sequences

$$H^{j}(\mathcal{F}(0,-1)) \otimes \mathcal{O}(0,-1) \longrightarrow E_{1}^{-1,j} \longrightarrow H^{j}(\mathcal{F}(-1,0)) \otimes \mathcal{O}(-1,0).$$

For a sheaf  $\mathcal{F}$  with support of dimension 1, which will be our case, the relevant part of  $E_1$  is represented in the tableau

(1) 
$$H^1(\mathcal{F}(-1,-1)) \otimes \mathcal{O}(-1,-1) \xrightarrow{\varphi_1} E_1^{-1,1} \xrightarrow{\varphi_2} H^1(\mathcal{F}) \otimes \mathcal{O}$$

$$\mathrm{H}^{0}(\mathcal{F}(-1,-1))\otimes\mathcal{O}(-1,-1) \xrightarrow{\varphi_{3}} E_{1}^{-1,0} \xrightarrow{\varphi_{4}} \mathrm{H}^{0}(\mathcal{F})\otimes\mathcal{O}$$

where the middle sheaves are part of the exact sequences

$$(2) \qquad H^{0}(\mathcal{F}(0,-1))\otimes \mathcal{O}(0,-1) \longrightarrow E_{1}^{-1,0} \longrightarrow H^{0}(\mathcal{F}(-1,0))\otimes \mathcal{O}(-1,0),$$

$$(3) \qquad H^{1}(\mathcal{F}(0,-1))\otimes \mathcal{O}(0,-1) \longrightarrow E_{1}^{-1,1} \longrightarrow H^{1}(\mathcal{F}(-1,0))\otimes \mathcal{O}(-1,0).$$

The relevant part of the second level of the spectral sequence is represented in the tableau

$$\mathcal{K}er(\varphi_1) \underbrace{\qquad \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \qquad \mathcal{C}oker(\varphi_2)}_{\varphi_5}$$
 
$$\mathcal{K}er(\varphi_3) \qquad \mathcal{K}er(\varphi_4)/\mathcal{I}m(\varphi_3) \qquad \mathcal{C}oker(\varphi_4)$$

The spectral sequence degenerates at  $E_3 = E_{\infty}$ . The convergence of the spectral sequence implies that  $\varphi_2$  is surjective and that we have the exact sequence

$$(4) 0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \longrightarrow 0.$$

Let  $\mathcal{E}$  be a semi-stable sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  with  $P_{\mathcal{E}}(m,n) = rm + n + 1$ . According to [1, Proposition 11],  $\mathcal{E}$  has resolution

$$(5) 0 \longrightarrow \mathcal{O}(-1, -r) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Let  $\mathcal{E}$  be a semi-stable sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  with  $P_{\mathcal{E}}(m,n) = m + sn + 1$ . Then  $\mathcal{E}$  has resolution

$$(6) 0 \longrightarrow \mathcal{O}(-s, -1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

According to [1, Proposition 14], a semi-stable sheaf  $\mathcal{E}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  with Hilbert polynomial 2m+2n+1 has resolution

$$(7) 0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) \longrightarrow \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

For a sheaf  $\mathcal{F}$  of dimension 1, without zero-dimensional torsion, on  $\mathbb{P}^1 \times \mathbb{P}^1$  we define the *dual sheaf* 

$$\mathcal{F}^{\mathrm{d}} = \mathcal{E}xt^{1}_{\mathcal{O}_{\mathbb{D}^{1}\times\mathbb{D}^{1}}}(\mathcal{F},\omega_{\mathbb{P}^{1}\times\mathbb{P}^{1}}).$$

**Lemma 2.1.** The map  $[\mathcal{F}] \mapsto [\mathcal{F}^{D}]$  is well-defined and gives an isomorphism

$$M(rm + sn + t) \longrightarrow M(rm + sn - t).$$

*Proof.* Consider the Segre embedding  $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ . Then the dual of  $\mathcal{F}$  as a sheaf on  $\mathbb{P}^3$  is compatible with the dual of  $\mathcal{F}$  as a sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$$\mathcal{F}^{\mathrm{D}} \simeq \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\omega_{\mathbb{P}^3})|_{\mathbb{P}^1 \times \mathbb{P}^1}.$$

This allows us to apply [15, Theorem 13] to obtain the conclusion.

In particular,  $\mathbf{M} \simeq \mathrm{M}(3m+2n-1)$ . Note that the same argument applies for moduli spaces of one-dimensional sheaves on smooth projective varieties.

**Theorem 2.2.** We have a decomposition of M(3m+2n-1) into subsets  $\mathbf{M}_0^{\mathrm{D}}$ ,  $\mathbf{M}_1^{\mathrm{D}}$ ,  $\mathbf{M}_2^{\mathrm{D}} \cup \mathbf{M}_3^{\mathrm{D}}$ , where  $\mathbf{M}_i^{\mathrm{D}}$  is the image of  $\mathbf{M}_i$  under the above isomorphism. Thus,  $\mathbf{M}_0^{\mathrm{D}}$  is the subset of sheaves  $\mathcal{F}$  having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-2,-2) \oplus \mathcal{O}(-2,-1) \stackrel{\varphi}{\longrightarrow} 2\mathcal{O}(-1,0) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where  $\varphi_{12}$  and  $\varphi_{22}$  define a zero-dimensional subscheme of  $\mathbb{P}^1 \times \mathbb{P}^1$ ;  $\mathbf{M}_1^{\text{D}}$  is the subset of sheaves  $\mathcal{F}$  having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-2,-2) \oplus \mathcal{O}(-1,-1) \stackrel{\varphi}{\longrightarrow} \mathcal{O}(-1,1) \oplus \mathcal{O}(0,-1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where  $\varphi_{12} \neq 0$ ,  $\varphi_{22} \neq 0$ ;  $\mathbf{M}_2^D$  is the set of structure sheaves of curves of bidegree (2,3);  $\mathbf{M}_3^D$  is the set of non-split extensions of  $\mathcal{O}_Q$  by  $\mathcal{O}_L(-2,-1)$  with L a line of bidegree (0,1) and Q a quartic of bidegree (2,2).

### 3. Vanishing of Cohomology

The following lemma is analogous to [14, Lemma 6.7]. We will use the word *curve* to denote a subscheme defined by a polynomial equation.

**Lemma 3.1.** Let  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a curve and  $\mathcal{I} \subset \mathcal{O}_C$  an ideal sheaf. Then there is a curve  $C' \subset C$  such that the ideal sheaf of C' in C, denoted  $\mathcal{I}'$ , contains  $\mathcal{I}$ , and  $\mathcal{I}'/\mathcal{I}$  has support of dimension at most 0.

The following proposition is a strengthtening of [1, Lemma 9].

**Proposition 3.2.** Let  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a curve of bidegree (s,r). Then  $\mathcal{O}_C$  is semi-stable. If r > 0 and s > 0, then  $\mathcal{O}_C$  is stable.

*Proof.* Let  $\mathcal{I} \subset \mathcal{O}_C$  be a proper subsheaf and let  $\mathcal{I}'$  and C' be as in Lemma 3.1. Let t be the length of  $\mathcal{I}'/\mathcal{I}$  and let (s', r') be the bidegree of C'. The Hilbert polynomial of  $\mathcal{I}$  is given by

$$P_{\mathcal{I}} = P_{\mathcal{I}'} - t = P_{\mathcal{O}_C} - P_{\mathcal{O}_{C'}} - t$$
  
=  $rm + sn + r + s - rs - r'm - s'n - r' - s' + r's' - t$ .

Thus, the slopes of  $\mathcal{I}$  and  $\mathcal{O}_C$  are given by

$$p(\mathcal{I}) = 1 + \frac{r's' - rs - t}{r - r' + s - s'}, \qquad p(\mathcal{O}_C) = 1 - \frac{rs}{r + s}.$$

The inequality  $p(\mathcal{I}) \leq p(\mathcal{O}_C)$  follows from the inequality  $0 \leq rr'(s-s') + ss'(r-r')$ . If r > 0 and s > 0, then this inequality is strict because either r' < r or s' < s.  $\square$ 

**Proposition 3.3.** Let  $\mathcal{F}$  be a semi-stable sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  with Hilbert polynomial  $P_{\mathcal{F}}(m,n) = rm + sn + t$ . Let i and j be integers.

(i) If 
$$\max\{i, j\} < 1 - \frac{rs + t}{r + s}$$
, then  $H^0(\mathcal{F}(i, j)) = 0$ .

(ii) If 
$$\min\{i, j\} > -1 + \frac{rs - t}{r + s}$$
, then  $H^1(\mathcal{F}(i, j)) = 0$ .

*Proof.* Assume that  $H^0(\mathcal{F}(i,j)) \neq 0$ . Then there is a non-zero morphism  $\alpha \colon \mathcal{O}_D \to \mathcal{F}(i,j)$  for a curve  $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\mathcal{J} = \mathcal{K}er(\alpha)$ . By Lemma 3.1 there is a curve  $C \subset D$  such that the ideal sheaf  $\mathcal{I}$  of C in  $\mathcal{O}_D$  contains  $\mathcal{J}$  and  $\mathcal{I}/\mathcal{J}$  is supported on finitely many points. Since  $\mathcal{F}(i,j)$  has no zero-dimensional torsion,  $\alpha(\mathcal{I}/\mathcal{J}) = 0$ , hence  $\mathcal{J} = \mathcal{I}$ , and hence  $\alpha$  factors through an injective morphism  $\mathcal{O}_C \to \mathcal{F}(i,j)$ . From the semi-stability of  $\mathcal{F}$  we get the inequality

$$p(\mathcal{O}_C(-i, -j)) = 1 - \frac{r's' + r'i + s'j}{r' + s'} \le \frac{t}{r+s} = p(\mathcal{F}).$$

Combining this with the inequalities

$$-\frac{rs}{r+s} \leq -\frac{r's'}{r'+s'}, \qquad -\max\{i,j\} \leq -\frac{r'i+s'j}{r'+s'}$$

we obtain the inequality

$$1 - \frac{rs}{r+s} - \max\{i, j\} \le \frac{t}{r+s}.$$

This contradicts the hypothesis of (i). Part (ii) follows from (i) and Serre duality. We have

$$H^1(\mathcal{F}(i,j)) \simeq (H^0(\mathcal{F}^{D}(-i,-j)))^*$$

and, by Lemma 2.1,  $\mathcal{F}^{D}$  is semi-stable with Hilbert polynomial rm + sn - t. Thus, the right-hand-side vanishes if

$$\max\{-i, -j\} < 1 - \frac{rs - t}{r + s}.$$

Using this proposition we can give another proof to the fact shown at [1, Proposition 10] that there are no semi-stable sheaves on  $\mathbb{P}^1 \times \mathbb{P}^1$  with Hilbert polynomial rm+t, for  $r \geq 2$  and t not a multiple of r.

**Corollary 3.4.** The moduli spaces M(rm + t) are empty for  $r \ge 2$  and 0 < t < r.

*Proof.* Assume that  $\mathcal{F}$  is a semi-stable sheaf in one of these moduli spaces. From Proposition 3.3 (i) we get  $H^0(\mathcal{F}) = 0$ . From Proposition 3.3 (ii) we get  $H^1(\mathcal{F}) = 0$ . Thus,  $t = \chi(\mathcal{F}) = 0$ , which contradicts our choice of t.

**Proposition 3.5.** For  $\mathcal{F} \in \mathbf{M}$  we have  $H^0(\mathcal{F}(-1,-1)) = 0$ ,  $H^0(\mathcal{F}(-1,0)) = 0$ , and  $H^0(\mathcal{F}(0,-1)) \neq 0$  if and only if  $\mathcal{F} \simeq \mathcal{O}_C(0,1)$  for a curve  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (2,3).

*Proof.* The vanishing of  $H^1(\mathcal{F}(-1,-1))$  follows from Proposition 3.3 (i). Assume that  $H^0(\mathcal{F}(i,j)) \neq 0$ , where (i,j) = (-1,0) or (0,-1). As in the proof of Proposition 3.3, there is a curve C and an injective morphism  $\mathcal{O}_C \to \mathcal{F}(i,j)$ . In Table 1 below we have the possible bidegrees of C and the slopes of  $\mathcal{O}_C(-i,-j)$ .

The only case in which  $\mathcal{O}_C(-i,-j)$  does not violate the semi-stability of  $\mathcal{F}$  is when  $\deg(C)=(2,3)$  and (i,j)=(0,-1). We deduce that  $\mathrm{H}^0(\mathcal{F}(-1,0))=0$  and, if  $\mathrm{H}^0(\mathcal{F}(0,-1))\neq 0$ , then  $\mathcal{F}\simeq\mathcal{O}_C(0,1)$  for a curve  $C\subset\mathbb{P}^1\times\mathbb{P}^1$  of bidegree (2,3).

$\deg(C)$	$P_{\mathcal{O}_C}$	$p(\mathcal{O}_C(1,0))$	$p(\mathcal{O}_C(0,1))$		
(2,3)	3m + 2n - 1	2/5	1/5		
(2,2)	2m+2n	1/2	1/2		
(1,3)	3m + n + 1	1	1/2		
(2,1)	m + 2n + 1	2/3	1		
(1, 2)	2m + n + 1	1	2/3		
(0,3)	3m + 3	2	1		
(2,0)	2n + 2	1	2		
(1,1)	m + n + 1	1	1		
(0, 2)	2m + 2	2	1		
(1,0)	n+1	1	2		
(0,1)	m+1	2	1		

Table 1. Possibilities for C.

It remains to show that  $\mathcal{O}_C(0,1)$  is semi-stable. Let  $\mathcal{I} \subset \mathcal{O}_C$  be an ideal sheaf and let  $\mathcal{I}'$  and C' be as in Lemma 3.1. In Table 2 below we have the possible bidegrees of C' and the resulting slopes of  $\mathcal{I}'(0,1)$ .

Table 2. Possibilities for C'.

$\deg(C')$	$P_{\mathcal{O}_{C'}}$	$P_{\mathcal{I}'}$	$p(\mathcal{I}'(0,1))$
(2,2)	2m+2n	m-1	-1
(1,3)	3m + n + 1	n-2	-1
(2,1)	m + 2n + 1	2m-2	-1
(1,2)	2m + n + 1	m+n-2	-1/2
(0,3)	3m + 3	2n-4	-1
(2,0)	2n + 2	3m - 3	-1
(1,1)	m + n + 1	2m + n - 2	-1/3
(0,2)	2m + 2	m + 2n - 3	-1/3
(1,0)	n+1	3m + n - 2	-1/4
(0,1)	m+1	2m + 2n - 2	0

In all cases  $p(\mathcal{I}'(0,1)) < p(\mathcal{O}_C(0,1))$ . In conclusion,  $\mathcal{O}_C(0,1)$  is semi-stable.  $\square$ 

**Proposition 3.6.** Let  $\mathcal{F}$  give a point in  $\mathbf{M}$ . If  $H^0(\mathcal{F}(0,-1)) = 0$ , then  $H^1(\mathcal{F}) = 0$ .

*Proof.* Denote  $d = \dim H^1(\mathcal{F})$ . In view of Proposition 3.5 and exact sequence (2) we get  $E_1^{-1,0} = 0$ . Thus, the exact sequence (4) becomes

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} (d+1)\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \longrightarrow 0.$$

From Proposition 3.5 we get  $H^1(\mathcal{F}(-1,-1)) \simeq \mathbb{C}^4$ , hence we have the exact sequence

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \longrightarrow 4\mathcal{O}(-1, -1) \longrightarrow \mathcal{I}m(\varphi_1) \longrightarrow 0.$$

We also have the exact sequence

$$0 \longrightarrow \mathcal{K}er(\varphi_2) \longrightarrow E_1^{-1,1} \longrightarrow d\mathcal{O} \longrightarrow 0.$$

From these exact sequences we can compute the Hilbert polynomial of  $E_1^{-1,1}$ :

$$\begin{split} P_{E_1^{-1,1}} &= P_{\mathcal{K}er(\varphi_2)} + dP_{\mathcal{O}} = P_{\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)} + P_{\mathcal{I}m(\varphi_1)} + dP_{\mathcal{O}} \\ &= P_{\mathcal{F}} - (d+1)P_{\mathcal{O}} + P_{\mathcal{K}er(\varphi_1)} + 4P_{\mathcal{O}(-1,-1)} - P_{\mathcal{K}er(\varphi_1)} + dP_{\mathcal{O}} \\ &= P_{\mathcal{F}} - P_{\mathcal{O}} + 4P_{\mathcal{O}(-1,-1)} \\ &= 3mn + 2m + n. \end{split}$$

The exact sequence (3) becomes

$$\mathcal{O}(0,-1) \longrightarrow E_1^{-1,1} \longrightarrow 2\mathcal{O}(-1,0).$$

Since  $P_{E_1^{-1,1}} = P_{\mathcal{O}(0,-1)} + 2P_{\mathcal{O}(-1,0)}$  this sequence is also exact on the left and right, and, in fact, it is split exact. We deduce that  $E_1^{-1,1} \simeq \mathcal{O}(0,-1) \oplus 2\mathcal{O}(-1,0)$ . It follows that  $d \leq 2$  because there is, obviously, no surjective morphism

$$\varphi_2 \colon \mathcal{O}(0,-1) \oplus 2\mathcal{O}(-1,0) \longrightarrow d\mathcal{O}$$

for  $d \geq 3$ . Assume that d = 2. Then the maximal minors of  $\varphi_2$  have no common factor, otherwise  $\varphi_2$  would not be surjective. It follows that  $\mathcal{K}er(\varphi_2) \simeq \mathcal{O}(-2, -1)$ . We have

$$P_{Coker(\varphi_2)} = 2P_{\mathcal{O}} - P_{\mathcal{O}(0,-1)} - 2P_{\mathcal{O}(-1,0)} + P_{\mathcal{O}(-2,-1)} = 2$$

which contradicts the surjectivity of  $\varphi_2$ . Assume that d=1. If the restriction of  $\varphi_2$  to  $\mathcal{O}(0,-1)$  were zero, then  $\mathcal{K}er(\varphi_2)\simeq\mathcal{O}(0,-1)\oplus\mathcal{O}(-2,0)$ . This would yield a contradiction because  $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$  would contain  $\mathcal{O}(-2,0)$  as a direct summand, but there is no surjective morphism  $\mathcal{F}\to\mathcal{O}(-2,0)$ . Thus, we may write

$$\varphi_2 = \left[ \begin{array}{cccc} -1 \otimes z & x \otimes 1 & y \otimes 1 \end{array} \right],$$
 
$$\varphi_1 = \left[ \begin{array}{cccc} x \otimes 1 & y \otimes 1 & 0 & 0 \\ 1 \otimes z & 0 & 0 & 0 \\ 0 & 1 \otimes z & 0 & 0 \end{array} \right].$$

It follows that  $Ker(\varphi_1) \simeq 2\mathcal{O}(-1,-1)$ . Thus,  $Coker(\varphi_5)$  has Hilbert polynomial  $2P_{\mathcal{O}}-2P_{\mathcal{O}(-1,-1)}=2m+2n+2$ , hence it has slope 1/2, and hence it is a destabilizing subsheaf of  $\mathcal{F}$ . In conclusion, d=0.

#### 4. Classification of sheaves

Assume that  $\mathcal{F}$  gives a point in  $\mathbf{M}$  and that  $\mathrm{H}^0(\mathcal{F}(0,-1))=0$ . Then, as seen at Proposition 3.6,  $\mathrm{H}^1(\mathcal{F})=0$ , and, as seen in the proof of this proposition,  $E_1^{-1,1}\simeq \mathcal{O}(0,-1)\oplus 2\mathcal{O}(-1,0)$ . Thus, the exact sequence (4) becomes

(8) 
$$0 \longrightarrow \mathcal{K}er(\varphi_1) \stackrel{\varphi_5}{\longrightarrow} \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0,$$

where

$$\varphi_1: 4\mathcal{O}(-1,-1) \longrightarrow \mathcal{O}(0,-1) \oplus 2\mathcal{O}(-1,0).$$

**Lemma 4.1.** Assume that  $\mathcal{F}$  gives a point in  $\mathbf{M}$  and that  $H^0(\mathcal{F}(0,-1)) = 0$ . Assume that the maximal minors of  $\varphi_1$  have no common factor. Then  $\operatorname{Ker}(\varphi_1) \simeq \mathcal{O}(-2,-3)$  and  $\operatorname{Coker}(\varphi_1)$  is isomorphic to the structure sheaf of a zero-dimensional subscheme  $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$  of length 2. Moreover, Z is not contained in a line of bidegree (0,1). Thus, we have a non-split extension

$$(9) 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where C is a curve of bidegree (2,3) containing Z.

*Proof.* Let  $\zeta_j$ ,  $1 \leq j \leq 4$ , be the maximal minor of  $\varphi_1$  obtained by deleting column j, for a matrix representation of  $\varphi_1$ . It is well-known that the sequence

$$0 \longrightarrow \mathcal{O}(-2, -3) \xrightarrow{\zeta} 4\mathcal{O}(-1, -1) \xrightarrow{\varphi_1} \mathcal{O}(0, -1) \oplus 2\mathcal{O}(-1, 0),$$
$$\zeta = \begin{bmatrix} \zeta_1 & -\zeta_2 & \zeta_3 & -\zeta_4 \end{bmatrix}^{\mathrm{T}},$$

is exact. Let  $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$  be the subscheme given by the ideal  $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ . The Hilbert polynomial of  $\mathcal{O}_Z$  can be computed from the exact sequence

$$0 \longrightarrow \mathcal{O}(-2,-2) \oplus 2\mathcal{O}(-1,-3) \xrightarrow{\varphi_1^{\mathrm{T}}} 4\mathcal{O}(-1,-2) \xrightarrow{\zeta^{\mathrm{T}}} \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We get  $P_{\mathcal{O}_Z}=2$ , hence Z is zero-dimensional of length 2. From the short exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{H}om(\mathcal{O}, \mathcal{O}) \longrightarrow \mathcal{H}om(\mathcal{I}_Z, \mathcal{O})$$
$$\longrightarrow \mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{E}xt^1(\mathcal{O}, \mathcal{O}) \longrightarrow \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O})$$
$$\longrightarrow \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \longrightarrow \mathcal{E}xt^2(\mathcal{O}, \mathcal{O}).$$

The sheaves  $\mathcal{H}om(\mathcal{O}_Z, \mathcal{O})$ ,  $\mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O})$ ,  $\mathcal{E}xt^1(\mathcal{O}, \mathcal{O})$ ,  $\mathcal{E}xt^2(\mathcal{O}, \mathcal{O})$  are zero, hence we get the isomorphisms

$$\mathcal{H}om(\mathcal{I}_Z, \mathcal{O}) \simeq \mathcal{O}, \qquad \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}) \simeq \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \simeq \mathcal{O}_Z.$$

We apply the long  $\mathcal{E}xt(-,\mathcal{O})$ -sequence to the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus 2\mathcal{O}(-1, -3) \xrightarrow{\varphi_1^{\mathrm{T}}} 4\mathcal{O}(-1, -2) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

and we use the above isomorphisms to obtain the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow 4\mathcal{O}(1,2) \stackrel{\psi}{\longrightarrow} \mathcal{O}(2,2) \oplus 2\mathcal{O}(1,3) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

The morphism  $\psi$  is a twist of  $\varphi_1$ . Assume that Z were contained in a line of bidegree (0,1). Then we would have a commutative diagram

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus 2\mathcal{O}(-1, -3) \longrightarrow 4\mathcal{O}(-1, -2) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \parallel$$

$$0 \longrightarrow \mathcal{O}(-2, -1) \longrightarrow \mathcal{O}(-2, 0) \oplus \mathcal{O}(0, -1) \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

in which  $\alpha \neq 0$ . Thus  $\operatorname{rank}(\mathcal{K}er(\alpha)) = 3$ , hence  $\beta = 0$ , and hence  $\operatorname{Coker}(\beta) \simeq \mathcal{O}(-2,-1)$  contains  $\mathcal{O}(-2,0)$  as a direct summand. This is absurd.

The exact sequence (9) follows from (8) with  $\mathcal{O}_C = \mathcal{C}oker(\varphi_5)$ . From sequence (9), and since  $\mathcal{F}$  has no zero-dimensional torsion, we see that  $\mathcal{F}$  has schematic support C, hence Z is contained in C.

**Lemma 4.2.** Let  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a curve of bidegree (2,3) and let  $Z \subset C$  be a zero-dimensional subscheme of length 2. Let  $\mathcal{F}$  be an extension of  $\mathcal{O}_Z$  by  $\mathcal{O}_C$  that has no zero-dimensional torsion. Then  $\mathcal{F}$  is uniquely determined up to isomorphism. This means that if  $\mathcal{G}$  is another extension of  $\mathcal{O}_Z$  by  $\mathcal{O}_C$  that has no zero-dimensional torsion, then  $\mathcal{F} \simeq \mathcal{G}$ .

*Proof.* By Serre duality  $\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C) \simeq (\operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_Z))^*$ . From the short exact sequence

$$(10) 0 \longrightarrow \mathcal{O}(-2, -3) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_C, \mathcal{O}_Z) \simeq \operatorname{H}^0(\mathcal{O}_Z) \simeq \mathbb{C}^2 \longrightarrow \operatorname{Hom}(\mathcal{O}, \mathcal{O}_Z) \simeq \mathbb{C}^2$$
$$\longrightarrow \operatorname{Hom}(\mathcal{O}(-2, -3), \mathcal{O}_Z) \simeq \mathbb{C}^2 \longrightarrow \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_Z) \longrightarrow \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}_Z) = 0$$

We obtain  $\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C) \simeq \mathbb{C}^2$ .

Assume that  $Z = \{p, q\}$  for distinct points  $p, q \in C$ . We denote by  $\mathbb{C}_p$  and  $\mathbb{C}_q$  the structure sheaves of the subschemes  $\{p\}$ , respectively,  $\{q\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ . From sequence (10) we get the long exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_C, \mathbb{C}_p) \simeq \mathbb{C} \longrightarrow \operatorname{Hom}(\mathcal{O}, \mathbb{C}_p) \simeq \mathbb{C} \longrightarrow \operatorname{Hom}(\mathcal{O}(-2, -3), \mathbb{C}_p) \simeq \mathbb{C}$$
$$\longrightarrow \operatorname{Ext}^1(\mathcal{O}_C, \mathbb{C}_p) \simeq (\operatorname{Ext}^1(\mathbb{C}_p, \mathcal{O}_C))^* \longrightarrow \operatorname{Ext}^1(\mathcal{O}, \mathbb{C}_p) = 0.$$

Thus, there is a unique non-trivial extension of  $\mathbb{C}_p$  by  $\mathcal{O}_C$ , denoted by  $\mathcal{E}$ . From the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

we get the long exact sequence

$$0 = \operatorname{Hom}(\mathbb{C}_q, \mathbb{C}_p) \longrightarrow \operatorname{Ext}^1(\mathbb{C}_q, \mathcal{O}_C) \simeq \mathbb{C} \longrightarrow \operatorname{Ext}^1(\mathbb{C}_q, \mathcal{E}) \longrightarrow \operatorname{Ext}^1(\mathbb{C}_q, \mathbb{C}_p) = 0.$$

Thus, there is a unique non-trivial extension of  $\mathbb{C}_q$  by  $\mathcal{E}$ , hence  $\mathcal{F}$  is unique up to isomorphism.

We next consider the case when Z is a double point supported on  $p \in C$ . We construct a resolution of  $\mathcal{E}$  by combining resolution (10) with the resolution

$$0 \longrightarrow \mathcal{O}(-2, -3) \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \longrightarrow \mathcal{O}(-1, -2) \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

The map  $\mathcal{O}(-1,-2) \to \mathbb{C}_p$  lifts to  $\mathcal{E}$  because  $H^1(\mathcal{O}_C(1,2)) = 0$ . Applying the argument at the proof of [16, Proposition 2.3.2], which uses the fact that  $\operatorname{Ext}^1(\mathbb{C}_p,\mathcal{O}) = 0$ , we can show that the induced map  $\mathcal{O}(-2,-3) \to \mathcal{O}(-2,-3)$  is non-zero. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \xrightarrow{\psi} \mathcal{O}(-1, -2) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

in which  $\psi_{11} \neq 0$ ,  $\psi_{12} \neq 0$ ,  $\psi_{11}(p) = 0$ ,  $\psi_{12}(p) = 0$ . Moreover,  $\psi_{21}(p) = 0$  and  $\psi_{22}(p) = 0$  if and only if p is a singular point of C. We have the exact sequence

$$\operatorname{Hom}(\mathcal{O}(-1,-2) \oplus \mathcal{O}, \mathbb{C}_p) \simeq \mathbb{C}^2 \xrightarrow{\psi(p)} \operatorname{Hom}(\mathcal{O}(-2,-2) \oplus \mathcal{O}(-1,-3), \mathbb{C}_p) \simeq \mathbb{C}^2$$

$$\longrightarrow \operatorname{Ext}^1(\mathcal{E}, \mathbb{C}_p) \simeq (\operatorname{Ext}^1(\mathbb{C}_p, \mathcal{E}))^* \longrightarrow \operatorname{Ext}^1(\mathcal{O}(-1,-2) \oplus \mathcal{O}, \mathbb{C}_p) = 0.$$

We get a unique non-trivial extension of  $\mathbb{C}_p$  by  $\mathcal{E}$  if p is a regular point of C. In this case  $\mathcal{F}$  is unique up to isomorphism.

Assume now that p is a singular point of C. Then  $\psi(p) = 0$ , hence  $\operatorname{Ext}^1(\mathbb{C}_p, \mathcal{E}) \simeq \mathbb{C}^2$ . According to [11, Proposition 2.3.1], the subset  $U_Z \subset \mathbb{P}(\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C)) \simeq \mathbb{P}^1$  of extension sheaves having no zero-dimensional torsion is open. We construct a map  $v_Z \colon U_Z \to \mathbb{P}(\operatorname{Ext}^1(\mathbb{C}_p, \mathcal{E})) \simeq \mathbb{P}^1$  as follows. Let  $\mathcal{I}$  be the ideal sheaf of  $\{p\}$  in

Z. Note that  $\mathcal{I} \simeq \mathbb{C}_p$  as modules over  $\mathcal{O}$ . Given  $\mathcal{F} \in U_Z$  let  $\mathcal{A}$  be the pull-back in  $\mathcal{F}$  of  $\mathcal{I}$ . Then there is a unique isomorphism  $\mathcal{E} \to \mathcal{A}$  making the diagram commute

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{A} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

The composite map  $\mathcal{E} \to \mathcal{A} \to \mathcal{F}$  has cokernel  $\mathbb{C}_p$ , so  $\mathcal{F}$  is an extension of  $\mathbb{C}_p$  by  $\mathcal{E}$ . We claim that the image of  $v_Z$  is a point. If we can prove this claim, then it will follow that  $\mathcal{F}$  is uniquely determined up to isomorphism. Assume that the image of  $v_Z$  is an open subset of  $\mathbb{P}^1$ . The zero-dimensional schemes Z' of length 2 supported on p are parametrized by  $\mathbb{P}^1$ . Thus there is  $Z' \neq Z$  such that  $v_{Z'}(U_{Z'}) \cap v_Z(U_Z) \neq \emptyset$ . This means that we have extensions  $\mathcal{F} \in U_Z$ ,  $\mathcal{F}' \in U_{Z'}$ , and a commutative diagram with exact rows

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

$$\downarrow^{\simeq} \qquad \qquad \downarrow^{\simeq}$$

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}' \longrightarrow \mathbb{C}_p \longrightarrow 0$$

The isomorphism  $\mathcal{F} \to \mathcal{F}'$  fits into a commutative square

$$\begin{array}{ccc}
\mathcal{O}_C & \longrightarrow \mathcal{F} \\
\parallel & & \downarrow \simeq \\
\mathcal{O}_C & \longrightarrow \mathcal{F}'
\end{array}$$

We get an induced isomorphism of cokernels  $\mathcal{O}_Z \to \mathcal{O}_{Z'}$ , which contradicts our choice of Z'. In conclusion, the image of  $v_Z$  is a point.

The difficult case in the previous lemma is when Z is concentrated in one point. For this case we will give an alternate more general argument. The following lemma and its proof were provided by Jean-Marc Drézet, to whom the author is grateful.

**Lemma 4.3.** Let S be a smooth projective surface and  $C \subset S$  a Cohen-Macaulay curve. Let  $Z \subset C$  be a zero-dimensional subscheme of length 2 concentrated on a single point p, and  $\mathcal{L}$  a line bundle on C. Then there exists an extension

$$(11) 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where  $\mathcal{F}$  has no zero-dimensional torsion. The sheaf  $\mathcal{F}$  is unique up to isomorphism.

*Proof.* The extensions (11) on C and on S are the same. Indeed, by [7, Proposition 2.2.1] we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_Z, \mathcal{L}) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{L}) \longrightarrow \operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_C), \mathcal{L}).$$

The group on the right vanishes because  $\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_C)$  is supported on Z, yet  $\mathcal{L}$  has no zero-dimensional torsion. From Serre duality we have

(12) 
$$\operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{L}) \simeq \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S)^*.$$

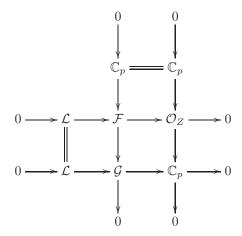
Again from [7, Proposition 2.2.1] we have the exact sequence

$$\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) \longrightarrow \operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_C), \mathcal{O}_Z \otimes \omega_S) \\ \longrightarrow \operatorname{Ext}^2_{\mathcal{O}_C}(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S).$$

The first and the last groups vanish, hence we obtain the functorial isomorphisms

$$\operatorname{Ext}_{\mathcal{O}_S}^1(\mathcal{L}, \mathcal{O}_Z \otimes \omega_S) \simeq \operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_C), \mathcal{O}_Z \otimes \omega_S)$$
$$\simeq \operatorname{Hom}(\mathcal{L}(-C), \mathcal{O}_Z \otimes \omega_S)$$
$$\simeq \operatorname{H}^0((\mathcal{L}^*(C) \otimes \omega_S)|_Z) \simeq \mathbb{C}^2.$$

Now consider an extension (11) which is non-split, and suppose that  $\mathcal{F}$  has a zero-dimensional subsheaf  $\mathcal{T}$ . Since  $\mathcal{L}$  is torsion-free on C the composition  $\mathcal{T} \to \mathcal{F} \to \mathcal{O}_Z$  is injective. There are only two non-zero subsheaves of  $\mathcal{O}_Z$ : the sheaf of sections vanishing at p, which is isomorphic to  $\mathbb{C}_p$ , and  $\mathcal{O}_Z$  itself. Since the extension is non-split, we have  $\mathcal{T} = \mathbb{C}_p$ . Let  $\mathcal{G} = \mathcal{F}/\mathcal{T}$ . We have a commutative diagram with exact rows and columns



Let  $\sigma \in \operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_Z, \mathcal{L})$  correspond to extension (11) and let  $\tau \in \operatorname{Ext}^1_{\mathcal{O}_C}(\mathbb{C}_p, \mathcal{L})$  correspond to the extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_n \longrightarrow 0.$$

Consider the morphism

$$\Phi \colon \operatorname{Ext}^1_{\mathcal{O}_G}(\mathbb{C}_p, \mathcal{L}) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_G}(\mathcal{O}_Z, \mathcal{L})$$

induced by the surjective morphism  $\mathcal{O}_Z \to \mathbb{C}_p$ . It is then easy to see that  $\Phi(\tau) = \sigma$  (see [6, Proposition 4.3.1]). It follows that for an extension (11) associated to  $\sigma$ , the sheaf  $\mathcal{F}$  has zero-dimensional torsion if and only if  $\sigma \in \operatorname{Im}(\Phi)$ .

According to (12) and to the above functorial isomorphisms,  $\Phi$  is the transpose of the canonical surjective morphism

$$\Psi \colon \operatorname{H}^{0}((\mathcal{L}^{*}(C) \otimes \omega_{S})|_{Z}) \longrightarrow \operatorname{H}^{0}((\mathcal{L}^{*}(C) \otimes \omega_{S})|_{p})$$

$$\parallel \qquad \qquad \parallel$$

$$\mathbb{C}^{2}$$

The kernel of  $\Psi$  is the set  $m_p \simeq \mathbb{C}$  of sections vanishing at p. Then  $\sigma \in \operatorname{Im}(\Phi)$  if and only if  $\sigma$  vanishes on  $m_p$ . The set of extensions  $\sigma$  that do not vanish on  $m_p$  is non-empty. This proves the existence part of the lemma. It is easy to check that the group of automorphisms of  $\mathcal{O}_Z$  acts transitively on the set of extensions  $\sigma$  that do not vanish on  $m_p$ . This proves the uniqueness part of the lemma.  $\square$ 

**Proposition 4.4.** Let  $\mathcal{F}$  be an extension as in (9), without zero-dimensional torsion, for a curve C of bidegree (2,3) and a subscheme  $Z \subset C$  that is the intersection of two curves of bidegree (1,1). Then  $\mathcal{F}$  gives a point in  $\mathbf{M}$ . Let  $\mathbf{M}_0 \subset \mathbf{M}$  be the subset of such sheaves  $\mathcal{F}$ . Then  $\mathbf{M}_0$  is open and it can be described as the set of sheaves  $\mathcal{G}$  having a resolution of the form

$$(13) 0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

where  $\varphi_{11}$  and  $\varphi_{12}$  define a zero-dimensional subscheme of  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Proof. Any  $Coker(\varphi)$  is an extension of  $\mathcal{O}_Z$  by  $\mathcal{O}_C$  without zero dimensional torsion, where  $Z = \{\varphi_{11} = 0, \varphi_{12} = 0\}$  and  $C = \{\det \varphi = 0\}$ , hence it is the unique extension of  $\mathcal{O}_Z$  by  $\mathcal{O}_C$  that has no zero-dimensional torsion. It remains to show that any sheaf  $\mathcal{G}$  having resolution (13) is semi-stable. Assume that  $\mathcal{G}$  had a destabilizing subsheaf  $\mathcal{E}$ . Without loss of generality we may take  $\mathcal{E}$  to be semi-stable. Since dim  $H^0(\mathcal{G}) = 1$ , we have  $\chi(\mathcal{E}) = 1$ . According to Corollary 3.4,  $\mathcal{E}$  cannot have Hilbert polynomial 2m + 1, 2n + 1, or 3m + 1. If  $P_{\mathcal{E}} = n + 1$ , then resolution (5) with r = 0 fits into the commutative diagram

$$0 \longrightarrow \mathcal{O}(-1,0) \xrightarrow{\psi} \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \downarrow$$

$$0 \longrightarrow 2\mathcal{O}(-1,-2) \xrightarrow{\varphi} \mathcal{O}(0,-1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

with  $\alpha \neq 0$ . Since  $\beta = 0$  we get  $\alpha \psi = 0$ , hence  $\psi = 0$ , which yields a contradiction. We obtain a contradiction in the same manner if  $P_{\mathcal{E}} = m+1, m+n+1, m+2n+1$ . Assume that  $P_{\mathcal{E}} = 2m+n+1$ . Then resolution (5) with r=2 is part of the commutative diagram

$$0 \longrightarrow \mathcal{O}(-1, -2) \xrightarrow{\psi} \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \downarrow$$

$$0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

Since  $\alpha_{11} = 0$  we obtain  $\varphi_{11}\beta_{11} + \varphi_{12}\beta_{21} = 0$ . This contradicts the fact that  $\varphi_{11}$  and  $\varphi_{12}$  are linearly independent. Assume that  $P_{\mathcal{E}} = 3m + n + 1$ . Then resolution (5) with r = 3 is the first line of the commutative diagram

$$0 \longrightarrow \mathcal{O}(-1, -3) \xrightarrow{\psi} \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \downarrow$$

$$0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

Write  $\beta_{11} = 1 \otimes l_1$ ,  $\beta_{21} = 1 \otimes l_2$ ,  $\varphi_{11} = x \otimes u_1 + y \otimes v_1$ ,  $\varphi_{12} = x \otimes u_2 + y \otimes v_2$ . From  $\alpha_{11} = 0$  we obtain

$$0 = \varphi_{11}(1 \otimes l_1) + \varphi_{12}(1 \otimes l_2),$$

$$0 = x \otimes (l_1u_1 + l_2u_2) + y \otimes (l_1v_1 + l_2v_2),$$

$$0 = l_1u_1 + l_2u_2, \quad 0 = l_1v_1 + l_2v_2,$$

$$u_1 = al_2, \quad u_2 = -al_1, \quad v_1 = bl_2, \quad v_2 = -bl_1,$$

$$\varphi_{11} = (ax + by) \otimes l_2, \quad \varphi_{12} = -(ax + by) \otimes l_1$$

for some  $a,b \in \mathbb{C}$ . This contradicts our hypothesis that  $\varphi_{11}$  and  $\varphi_{12}$  define a zero-dimensional subscheme of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Assume, finally, that  $P_{\mathcal{E}} = 2m + 2n + 1$ . Then resolution (7) fits into the commutative diagram

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) \xrightarrow{\psi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \downarrow$$

$$0 \longrightarrow 2\mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

We have  $\alpha_{22} \neq 0$  because the map  $\mathcal{E} \to \mathcal{G}$  is injective on global sections. It follows that  $\alpha$  is injective, otherwise  $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-1, -1)$ , but this cannot be a subsheaf of  $\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2)$ . It follows that  $\beta$  is injective, which is absurd.

Corollary 4.5. The variety M is rational.

*Proof.* Consider the open subset  $B \subset \mathbf{M}_0$  given by the condition that Z consist of two distinct points. Notice that B is a bundle with fiber  $\mathbb{P}^9$  and base an open subset of  $((\mathbb{P}^1 \times \mathbb{P}^1)^2 \setminus \Delta)/S_2$ . Here  $\Delta$  is the diagonal of the product of two copies of  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $S_2$  is the group of permutations of two elements.

**Proposition 4.6.** Let  $\mathcal{F}$  be an extension as in (9), that has no zero-dimensional torsion, for a curve C of bidegree (2,3) and a subscheme  $Z \subset C$  that is the intersection of two curves of bidegree (0,2), respectively, (1,0). Then  $\mathcal{F}$  gives a point in  $\mathbf{M}$ . Let  $\mathbf{M}_1 \subset \mathbf{M}$  be the subset of such sheaves  $\mathcal{F}$ . Then  $\mathbf{M}_1$  is irreducible of codimension 1 and it can be described as the set of sheaves  $\mathcal{G}$  having a resolution of the form

(14) 
$$0 \longrightarrow \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-3) \xrightarrow{\varphi} \mathcal{O}(-1,-1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$
  
where  $\varphi_{11} \neq 0$  and  $\varphi_{12} \neq 0$ .

*Proof.* We will show that any sheaf  $\mathcal{G}$  having resolution (14) has no destabilizing subsheaves. Assume that  $\mathcal{G}$  had a destabilizing subsheaf  $\mathcal{E}$ . Without loss of generality we may take  $\mathcal{E}$  to be semi-stable. Since dim  $H^0(\mathcal{G}) = 1$ , we have  $\chi(\mathcal{E}) = 1$ . According to Corollary 3.4,  $\mathcal{E}$  cannot have Hilbert polynomial 2m + 1, 2n + 1, or 3m + 1. If  $P_{\mathcal{E}} = n + 1$ , then resolution (5) with r = 0 fits into the commutative diagram

$$0 \longrightarrow \mathcal{O}(-1,0) \xrightarrow{\psi} \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \downarrow^{\alpha}$$

$$0 \longrightarrow \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-3) \xrightarrow{\varphi} \mathcal{O}(-1,-1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

with  $\alpha \neq 0$ . Since  $\beta = 0$  we get  $\alpha \psi = 0$ , hence  $\psi = 0$ , which yields a contradiction. We obtain a contradiction in the same manner if  $P_{\mathcal{E}} = m+1, m+n+1, 2m+n+1$ . Assume that  $P_{\mathcal{E}} = m+2n+1$ . Then resolution (6) with s=2 is part of the commutative diagram

$$0 \longrightarrow \mathcal{O}(-2, -1) \xrightarrow{\psi} \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \downarrow^{\alpha}$$

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

Since  $\alpha$  is injective on global sections,  $\alpha$  is injective, hence  $\beta$  is injective, too, and hence we may write

$$\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
 Then  $\varphi \beta = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \end{bmatrix} = \alpha \psi = \begin{bmatrix} 0 \\ \psi \end{bmatrix},$ 

hence  $\varphi_{11} = 0$ , which contradicts our hypothesis. We obtain a contradiction in the same manner if  $P_{\mathcal{E}} = 3m + n + 1$ . Assume, finally, that  $P_{\mathcal{E}} = 2m + 2n + 1$ . Then resolution (7) is the first line of the commutative diagram

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -2) \xrightarrow{\psi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \qquad \downarrow^{\alpha} \qquad \qquad \downarrow$$

$$0 \longrightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

Notice that  $\alpha$  and  $\alpha(1,1)$  are injective on global sections, hence  $\alpha$  is injective, and hence  $\beta$  is injective, which is absurd.

Let  $W_1$  be the set of morphisms  $\varphi$  occurring in resolution (14) and consider the algebraic group

$$G_1 = (\operatorname{Aut}(\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3)) \times \operatorname{Aut}(\mathcal{O}(-1, -1) \oplus \mathcal{O})) / \mathbb{C}^*$$

acting on  $W_1$  by conjugation.

**Proposition 4.7.** The variety  $\mathbf{M}_1$  is isomorphic to the geometric quotient  $W_1/G_1$ . Thus,  $\mathbf{M}_1$  is a  $\mathbb{P}^9$ -bundle over  $\mathbb{P}^1 \times \mathbb{P}^2$ , so it is smooth and closed in  $\mathbf{M}$ .

Proof. The canonical map  $W_1 \to \mathbf{M}_1$ ,  $\varphi \mapsto [\mathcal{C}oker(\varphi)]$ , has local sections, and its fibers are the  $G_1$ -orbits, hence it is a geometric quotient map. We construct the local sections as follows. Given  $[\mathcal{F}] \in \mathbf{M}_1$ , let C be the schematic support of  $\mathcal{F}$ , and let Z be the zero-dimensional scheme of length 2 given by the exact sequence (9). Then  $Z = L \cap (L_1 \cup L_2)$ , where L is a line of bidegree (1,0), and  $L_1$ ,  $L_2$  are lines, each of bidegree (0,1). Choose equations  $\varphi_{11} = 0$  of L,  $\varphi_{12} = 0$  of  $L_1 \cup L_2$ , and f = 0 of C. Then we can write  $f = \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21}$  for some  $\varphi_{21} \in S^2V_1^* \otimes V_2^*$ ,  $\varphi_{22} \in V_1^* \otimes S^3V_2^*$ . Map  $[\mathcal{F}]$  to the morphism represented by the matrix  $(\varphi_{ij})_{1 \leq i,j \leq 2}$ . This construction can be done for a local flat family in a neighborhood of  $[\mathcal{F}]$  in  $\mathbf{M}_1$ .

We now describe  $W_1/G_1$ . Let  $U \subset V_1^* \oplus S^2 V_2^*$  be the open subset

$$\{(\varphi_{11}, \varphi_{12}), \varphi_{11} \neq 0, \varphi_{12} \neq 0\}.$$

Let F be the trivial vector bundle on U with fiber  $(S^2 V_1^* \otimes V_2^*) \oplus (V_1^* \otimes S^3 V_2^*)$ . Consider the subbundle  $E \subset F$  which over the point  $(\varphi_{11}, \varphi_{12})$  has fiber  $(\varphi_{11}V_1^* \otimes V_2^*) \oplus (\varphi_{12}V_1^* \otimes V_2^*)$ . The quotient bundle G = F/E has rank 10 and is linearized for the canonical action of  $\mathbb{C}^* \times \mathbb{C}^* = \operatorname{Aut}(\mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -3))$  on U. Thus, G descends to a vector bundle H over  $U/\mathbb{C}^* \times \mathbb{C}^* = \mathbb{P}(V_1^*) \times \mathbb{P}(S^2 V_2^*) \simeq \mathbb{P}^1 \times \mathbb{P}^2$ . Clearly,  $\mathbb{P}(H) \simeq W_1/G_1$ .

**Proposition 4.8.** Assume that  $\mathcal{F}$  gives a point in  $\mathbf{M}$  and that  $H^0(\mathcal{F}(0,-1)) = 0$ . Assume that the maximal minors of  $\varphi_1$  have a common factor. Then  $\operatorname{Ker}(\varphi_1) \simeq \mathcal{O}(-2,-2)$  and  $\operatorname{Coker}(\varphi_1) \simeq \mathcal{O}_L$  for a line  $L \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (0,1). Thus, we have an extension

$$(15) 0 \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

for a quartic curve  $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (2,2). Conversely, any non-split extension of this form is semi-stable. We have  $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^1} \times \mathbb{P}^1}(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$ .

*Proof.* Let  $g = \gcd(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ , where  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  are defined as in the proof of Lemma 4.1. We have the exact sequence

$$0 \longrightarrow \mathcal{O}(i,j) \xrightarrow{\eta} 4\mathcal{O}(-1,-1) \xrightarrow{\varphi_1} \mathcal{O}(0,-1) \oplus 2\mathcal{O}(-1,0),$$
$$\eta = \begin{bmatrix} \frac{\zeta_1}{g} & -\frac{\zeta_2}{g} & \frac{\zeta_3}{g} & -\frac{\zeta_4}{g} \end{bmatrix}^{\mathrm{T}}.$$

The possibilities for the kernel of  $\varphi_1$  are given in Table 3 below.

Table 3. Kernel of  $\varphi_1$ .

	deg(g)	(i, j)	$P_{\mathcal{C}oker(\varphi_5)}$
ı	(1,0)	(-1, -3)	3m + n + 1
ı	(0,1)	(-2, -2)	2m + 2n
ı	(0,2)	(-2, -1)	m + 2n + 1
ı	(1, 1)	(-1, -2)	2m + n + 1

We see that the only case in which  $Coker(\varphi_5)$  does not destabilize  $\mathcal{F}$  is the case (i,j)=(-2,-2). Thus,  $Ker(\varphi_1)\simeq \mathcal{O}(-2,-2)$ . The cokernel of  $\varphi_1$  has no zero-dimensional torsion and has Hilbert polynomial m+1, hence it is of the form  $\mathcal{O}_L$  for a line L of bidegree (0,1). From sequence (8) we see that  $\mathcal{F}$  is an extension of  $\mathcal{O}_L$  by  $\mathcal{O}_Q$ .

Conversely, assume that  $\mathcal{F}$  is such an extension. By Proposition (3.2)  $\mathcal{O}_Q$  is stable. Thus, for any proper subsheaf  $\mathcal{E} \subset \mathcal{F}$  we have  $p(\mathcal{E} \cap \mathcal{O}_Q) < 0$  unless  $\mathcal{O}_Q \subset \mathcal{E}$ . Since, obviously,  $\mathcal{O}_L$  is stable, the image of  $\mathcal{E}$  in  $\mathcal{O}_L$  has slope at most 1. It follows that  $p(\mathcal{E}) < p(\mathcal{F})$ , hence  $\mathcal{F}$  is stable. From the short exact sequence

$$0 \longrightarrow \mathcal{O}(0,-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

we get the long exact sequence

$$0 = \operatorname{Hom}(\mathcal{O}_L, \mathcal{O}_Q) \longrightarrow \operatorname{H}^0(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow \operatorname{H}^0(\mathcal{O}_Q(0, 1)) \simeq \mathbb{C}^2$$
$$\longrightarrow \operatorname{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \longrightarrow \operatorname{H}^1(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow \operatorname{H}^1(\mathcal{O}_Q(0, 1)) = 0.$$

This proves that  $\operatorname{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$ .

Let  $\mathbf{M}_2 \subset \mathbf{M}$  be the subset of sheaves having resolution (??). Clearly,  $\mathbf{M}_2 \simeq \mathbb{P}^{11}$ . Let  $\mathbf{M}_3 \subset \mathbf{M}$  be the subset of extension sheaves  $\mathcal{F}$  as in (15). Clearly,  $\mathbf{M}_3$  is a bundle with base  $\mathbb{P}^8 \times \mathbb{P}^1$  and fiber  $\mathbb{P}^1$ . Thus,  $\mathbf{M}_3$  is closed of codimension 3. It intersects  $\mathbf{M}_2$  along a subvariety isomorphic to  $\mathbb{P}^8 \times \mathbb{P}^1$  consisting of twisted structure sheaves  $\mathcal{O}_C(0,1)$ , where  $C = Q \cup L$ . The subvarieties  $\mathbf{M}_0$ ,  $\mathbf{M}_1$ ,  $\mathbf{M}_2 \cup \mathbf{M}_3$  form a decomposition of  $\mathbf{M}$  and satisfy the properties from Theorem 1.1.

## 5. Variation of moduli of $\alpha$ -semi-stable pairs

Let X be a separated scheme of finite type over  $\mathbb{C}$ . An algebraic system on X is a triple  $\Lambda = (\Gamma, \sigma, \mathcal{F})$  consisting of an  $\mathcal{O}_X$ -module  $\mathcal{F}$ , a vector space  $\Gamma$  over  $\mathbb{C}$ , and a  $\mathbb{C}$ -linear map  $\sigma \colon \Gamma \to \mathrm{H}^0(\mathcal{F})$ . If  $\mathcal{F}$  is a coherent  $\mathcal{O}_X$ -module and  $\Gamma$  is finite dimensional, we say that  $\Lambda$  is a coherent system. A pair will be a coherent system in which  $\sigma$  is injective and dim  $\Gamma = 1$ . A morphism of algebraic systems  $(\gamma, \varphi) \colon (\Gamma, \sigma, \mathcal{F}) \to (\Gamma', \sigma', \mathcal{F}')$  consists of a  $\mathbb{C}$ -linear map  $\gamma \colon \Gamma \to \Gamma'$  together with

a morphism of  $\mathcal{O}_X$ -modules  $\varphi \colon \mathcal{F} \to \mathcal{F}'$ , which are compatible, in the sense that  $\mathrm{H}^0(\varphi)\sigma = \sigma'\gamma$ . These notions were introduced in [12] and [10] where appropriate semi-stability conditions of coherent systems were defined, which led in a natural manner to the construction of moduli spaces. The category of algebraic systems on X is abelian and, according to [10, Théorème 1.3], it has enough injectives. Thus, we can define the left derived functors of  $\mathrm{Hom}(\Lambda,-)$ , denoted  $\mathrm{Ext}^i(\Lambda,-)$ . Our basic tool for computing these extension spaces is [10, Corollaire 1.6], which we quote below.

**Proposition 5.1.** Let  $\Lambda = (\Gamma, \sigma, \mathcal{F})$  and  $\Lambda' = (\Gamma', \sigma', \mathcal{F}')$  be two algebraic systems on X with  $\sigma'$  injective. Then there is a long exact sequence

$$\begin{split} 0 &\longrightarrow \operatorname{Hom}(\Lambda, \Lambda') \longrightarrow \operatorname{Hom}(\mathcal{F}, \mathcal{F}') \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^0(\mathcal{F}')/\Gamma') \\ &\longrightarrow \operatorname{Ext}^1(\Lambda, \Lambda') \longrightarrow \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}') \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^1(\mathcal{F}')) \\ &\longrightarrow \operatorname{Ext}^2(\Lambda, \Lambda') \longrightarrow \operatorname{Ext}^2(\mathcal{F}, \mathcal{F}') \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^2(\mathcal{F}')). \end{split}$$

From now on we specialize to the case when  $X = \mathbb{P}^1 \times \mathbb{P}^1$  with fixed polarization  $\mathcal{O}(1,1)$ , and  $\mathcal{F}$  has dimension 1 with Hilbert polynomial  $P_{\mathcal{F}}(m,n) = rm + sn + t$ . Let  $\alpha$  be a positive rational number. We define the *slope* of a coherent system  $\Lambda = (\Gamma, \sigma, \mathcal{F})$  relative to  $\alpha$  and to the fixed polarization

$$p_{\alpha}(\Lambda) = \frac{\dim \Gamma}{r+s} \alpha + \frac{t}{r+s}.$$

We say that  $\Lambda$  is  $\alpha$ -semi-stable (respectively  $\alpha$ -stable) if  $\mathcal{F}$  has no zero-dimensional torsion,  $\sigma$  is injective, and for any proper coherent subsystem  $\Lambda' \subset \Lambda$  we have  $p_{\alpha}(\Lambda') \leq p_{\alpha}(\Lambda)$  (respectively  $p_{\alpha}(\Lambda') < p_{\alpha}(\Lambda)$ ). According to [10], for fixed polynomial P and  $\alpha \in \mathbb{Q}_{>0}$  there is a coarse moduli space  $\operatorname{Syst}_{X,\alpha}(P)$  parametrizing Sequivalence classes of  $\alpha$ -semi-stable coherent systems  $(\Gamma, \mathcal{F})$  such that  $P_{\mathcal{F}} = P$ . We have a decomposition of  $\operatorname{Syst}_{X,\alpha}(P)$  into disjoint components according to dim  $\Gamma$ . The component corresponding to the case dim  $\Gamma = 1$ , i.e. parametrizing  $\alpha$ -semi-stable pairs with fixed Hilbert polynomial P, will be denoted  $\operatorname{M}^{\alpha}(P)$ .

A value  $\alpha_0$  is said to be *regular* relative to P if it is contained in an interval  $(\alpha_1, \alpha_2)$  such that the set of  $\alpha$ -semi-stable pairs with Hilbert polynomial P remains unchanged as  $\alpha$  varies in  $(\alpha_1, \alpha_2)$ . If there is no such interval we say that  $\alpha_0$  is a wall relative to P. The following proposition is analogous to [3, Lemma 3.1].

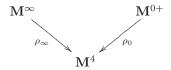
**Proposition 5.2.** Relative to P(m,n) = 3m + 2n + 1 we have only one wall at  $\alpha = 4$ .

*Proof.* According to the proof of [10, Théorème 4.2],  $\alpha$  is a wall if and only if there is a strictly  $\alpha$ -semi-stable pair  $\Lambda=(\Gamma,\mathcal{F})$ . There is a pair  $\Lambda'=(\Gamma',\mathcal{F}')\neq \Lambda$  which is a subpair of  $\Lambda$  or a quotient pair such that  $p_{\alpha}(\Lambda')=p_{\alpha}(\Lambda)$ . Write  $P_{\mathcal{F}'}(m,n)=rm+sn+t$  with  $r\leq 3,\ s\leq 2$ . We have the equation

(16) 
$$\frac{\alpha+t}{r+s} = \frac{\alpha+1}{5}.$$

Without loss of generality we may assume that  $\Gamma'$  generates  $\mathcal{F}'$  away, possibly, from finitely many points. Thus  $t \geq r+s-rs$ . The case when r=3, s=2 is unfeasible. Assume that r=2, s=2,  $t\geq 0$ . Equation (16) becomes  $\alpha=4-5t$ , which has solution  $\alpha=4$  when t=0. For all other choices of r and s we have  $t\geq 1$ , hence equation (16) has no positive solution.

We write  $\mathbf{M}^{\alpha} = \mathbf{M}^{\alpha}(3m+2n+1)$ . The moduli spaces  $\mathbf{M}^{\alpha}$  remain unchanged as  $\alpha$  varies in the interval (0,4) and will be denoted  $\mathbf{M}^{0+}$ . Likewise, for  $\alpha \in (4,\infty)$ ,  $\mathbf{M}^{\alpha}$  are all equal to a moduli space denoted  $\mathbf{M}^{\infty}$ . These moduli spaces are related by the flipping diagram



in which the maps  $\rho_{\infty}$  and  $\rho_0$  are induced by the inclusion of sets of  $\alpha$ -semi-stable pairs. In particular,  $\rho_{\infty}$  and  $\rho_0$  are birational.

The following proposition is a particular case of [18, Proposition B.8].

**Proposition 5.3.** The variety  $\mathbf{M}^{\infty}$  is isomorphic to the flag Hilbert scheme of zero-dimensional subschemes of length 2 contained in curves of bidegree (2,3) in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

In particular,  $\mathbf{M}^{\infty}$  is a bundle with base  $\mathrm{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2)$  and fiber  $\mathbb{P}^9$ , so it is smooth. This proposition gives another proof for the fact that  $\mathbf{M}$  is rational (Corollary 4.5).

Remark 5.4. From the proof of Proposition 5.2, we see that the S-equivalence type of a strictly  $\alpha$ -semi-stable pair in  $\mathbf{M}^4$  is of the form  $(\Gamma, \mathcal{E}) \oplus (0, \mathcal{O}_L)$ , where  $(\Gamma, \mathcal{E}) \in \mathbf{M}^{0+}(2m+2n)$  and  $L \subset \mathbb{P}^1 \times \mathbb{P}^1$  is a line of bidegree (0,1). As in the proof of Proposition 3.5,  $\mathcal{E}$  has a subsheaf isomorphic to the structure sheaf of a curve. By semi-stability, the curve must have bidegree (2,2). We see that  $\mathcal{E} \simeq \mathcal{O}_Q$  for a quartic curve  $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$  of bidegree (2,2). Thus,  $\mathbf{M}^{0+}(2m+2n) \simeq \mathbb{P}^8$ .

Let  $F^{\infty} \subset \mathbf{M}^{\infty}$  and  $F^{0} \subset \mathbf{M}^{0+}$  be the flipping loci, that is, the inverse images under  $\rho_{\infty}$ , respectively, under  $\rho_{0}$  of  $\mathbf{M}^{0+}(2m+2n) \times \mathbf{M}(m+1)$ . The fiber of  $F^{\infty}$  over  $(\Lambda_{1}, \Lambda_{2})$  is  $\mathbb{P}(\operatorname{Ext}^{1}(\Lambda_{1}, \Lambda_{2}))$ . The fiber of  $F^{0}$  over  $(\Lambda_{1}, \Lambda_{2})$  is  $\mathbb{P}(\operatorname{Ext}^{1}(\Lambda_{2}, \Lambda_{1}))$ .

**Remark 5.5.** The flipping locus  $F^{\infty}$  is a projective bundle with fiber  $\mathbb{P}^2$  and base  $M^{0+}(2m+2n)\times M(m+1)$ . The flipping locus  $F^0$  is a  $\mathbb{P}^1$ -bundle with the same base. Indeed, take  $\Lambda_1=(\Gamma,\mathcal{O}_Q)\in M^{0+}(2m+2n)$  and  $\Lambda_2=(0,\mathcal{O}_L)\in M(m+1)$ . Proposition 5.1 yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}(\Lambda_1, \Lambda_2) \longrightarrow \operatorname{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^0(\mathcal{O}_L)) \simeq \mathbb{C}$$
$$\longrightarrow \operatorname{Ext}^1(\Lambda_1, \Lambda_2) \longrightarrow \operatorname{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^1(\mathcal{O}_L)) = 0.$$

Any morphism  $\Lambda_1 \to \Lambda_2$  is zero because  $\Gamma = \mathrm{H}^0(\mathcal{O}_Q)$  generates  $\mathcal{O}_Q$ . If  $L \nsubseteq Q$ , then  $\mathrm{Hom}(\mathcal{O}_Q, \mathcal{O}_L) = 0$ ; if  $L \subset Q$ , then  $\mathrm{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}$ . From the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathcal{O}} \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \operatorname{H}^0(\mathcal{O}_L) \simeq \mathbb{C} \longrightarrow \operatorname{H}^0(\mathcal{O}_L(2, 2)) \simeq \mathbb{C}^3$$
$$\longrightarrow \operatorname{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \longrightarrow \operatorname{H}^1(\mathcal{O}_L) = 0.$$

Thus, if  $L \nsubseteq Q$ , then  $\operatorname{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}^2$ ; if  $L \subset Q$ , then  $\operatorname{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L) \simeq \mathbb{C}^3$ . In either case we get  $\operatorname{Ext}^1(\Lambda_1, \Lambda_2) \simeq \mathbb{C}^3$ .

We will now verify the isomorphism  $\operatorname{Ext}^1(\Lambda_2, \Lambda_1) \simeq \mathbb{C}^2$ . From Proposition 5.1 we have the exact sequence

$$0 = \operatorname{Hom}(0, \operatorname{H}^0(\mathcal{O}_Q)/\Gamma) \to \operatorname{Ext}^1(\Lambda_2, \Lambda_1) \to \operatorname{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \to \operatorname{Hom}(0, \operatorname{H}^1(\mathcal{O}_Q)) = 0$$

Thus, the middle arrow is an isomorphism. From Proposition 4.8 we know that  $\operatorname{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q) \simeq \mathbb{C}^2$ .

**Lemma 5.6.** For  $\Lambda \in F^0$  we have  $\operatorname{Ext}^2(\Lambda, \Lambda) = 0$ .

*Proof.* We have a non-split exact sequence

$$0 \longrightarrow \Lambda_1 \longrightarrow \Lambda \longrightarrow \Lambda_2 \longrightarrow 0$$

for some  $\Lambda_1 = (\Gamma, \mathcal{O}_Q) \in \mathcal{M}^{0+}(2m+2n)$  and  $\Lambda_2 = (0, \mathcal{O}_L) \in \mathcal{M}(m+1)$ . It is enough to show that  $\operatorname{Ext}^2(\Lambda_i, \Lambda_j) = 0$  for i, j = 1, 2. From Proposition 5.1 we have the exact sequence

$$0 = \operatorname{Hom}(\Gamma, \operatorname{H}^{1}(\mathcal{O}_{L})) \longrightarrow \operatorname{Ext}^{2}(\Lambda_{1}, \Lambda_{2}) \longrightarrow \operatorname{Ext}^{2}(\mathcal{O}_{Q}, \mathcal{O}_{L}) \simeq \operatorname{Hom}(\mathcal{O}_{L}, \mathcal{O}_{Q} \otimes \omega)^{*}.$$

The group on the right vanishes because  $\mathcal{O}_L$  is stable, by Proposition 3.2  $\mathcal{O}_Q \otimes \omega$  is stable and  $p(\mathcal{O}_L) > p(\mathcal{O}_Q \otimes \omega)$ . Thus,  $\operatorname{Ext}^2(\Lambda_1, \Lambda_2) = 0$ . The exact sequence

$$0 = \operatorname{Hom}(0, \operatorname{H}^{1}(\mathcal{O}_{Q})) \to \operatorname{Ext}^{2}(\Lambda_{2}, \Lambda_{1}) \longrightarrow \operatorname{Ext}^{2}(\mathcal{O}_{L}, \mathcal{O}_{Q}) \simeq \operatorname{Hom}(\mathcal{O}_{Q}, \mathcal{O}_{L} \otimes \omega)^{*} = 0$$

shows that  $\operatorname{Ext}^2(\Lambda_2, \Lambda_1) = 0$ . We have the exact sequence

$$0 = \operatorname{Hom}(\Gamma, \operatorname{H}^{0}(\mathcal{O}_{Q})/\Gamma)$$

$$\longrightarrow \operatorname{Ext}^{1}(\Lambda_{1}, \Lambda_{1}) \longrightarrow \operatorname{Ext}^{1}(\mathcal{O}_{Q}, \mathcal{O}_{Q}) \longrightarrow \operatorname{Hom}(\Gamma, \operatorname{H}^{1}(\mathcal{O}_{Q})) \simeq \mathbb{C}$$

$$\longrightarrow \operatorname{Ext}^{2}(\Lambda_{1}, \Lambda_{1}) \longrightarrow \operatorname{Ext}^{2}(\mathcal{O}_{Q}, \mathcal{O}_{Q}) \simeq \operatorname{Hom}(\mathcal{O}_{Q}, \mathcal{O}_{Q} \otimes \omega)^{*} = 0.$$

The space  $\operatorname{Ext}^1(\Lambda_1, \Lambda_1)$  is isomorphic to the tangent space of  $\operatorname{M}^{0+}(2m+2n) \simeq \mathbb{P}^8$  at  $\Lambda_1$ , so it is isomorphic to  $\mathbb{C}^8$ . From the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2, -2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

we get the long exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_Q, \mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow \operatorname{H}^0(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow \operatorname{H}^0(\mathcal{O}_Q(2, 2)) \simeq \mathbb{C}^8$$
$$\longrightarrow \operatorname{Ext}^1(\mathcal{O}_Q, \mathcal{O}_Q) \longrightarrow \operatorname{H}^1(\mathcal{O}_Q) \simeq \mathbb{C} \longrightarrow \operatorname{H}^1(\mathcal{O}_Q(2, 2)) = 0.$$

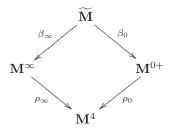
Thus  $\operatorname{Ext}^1(\mathcal{O}_Q,\mathcal{O}_Q)\simeq \mathbb{C}^9$ . We get the vanishing of  $\operatorname{Ext}^2(\Lambda_1,\Lambda_1)$ . Finally, from the exact sequence

$$0 = \operatorname{Hom}(0, \operatorname{H}^{1}(\mathcal{O}_{L})) \longrightarrow \operatorname{Ext}^{2}(\Lambda_{2}, \Lambda_{2}) \longrightarrow \operatorname{Ext}^{2}(\mathcal{O}_{L}, \mathcal{O}_{L}) \simeq \operatorname{Hom}(\mathcal{O}_{L}, \mathcal{O}_{L} \otimes \omega)^{*} = 0$$
we get the vanishing of  $\operatorname{Ext}^{2}(\Lambda_{2}, \Lambda_{2})$ .

The following theorem is analogous to [3, Theorem 3.3].

**Theorem 5.7.** Let  $\mathbf{M}^{\alpha}$  be the moduli space of  $\alpha$ -semi-stable pairs on  $\mathbb{P}^1 \times \mathbb{P}^1$  with Hilbert polynomial P(m,n) = 3m + 2n + 1. We have the following commutative

diagram expressing the variation of  $\mathbf{M}^{\alpha}$  as  $\alpha$  crosses the wall:



Here  $\beta_{\infty}$  is the blow-up with center  $F^{\infty}$  and  $\beta_0$  is the blow-down contracting the exceptional divisor  $\widetilde{F}$  in the direction of  $\mathbb{P}^2$ , where we regard  $\widetilde{F}$  as a  $\mathbb{P}^2 \times \mathbb{P}^1$ -bundle over  $M^{0+}(2m+2n) \times M(m+1)$ .

Proof. At [3, Theorem 3.3] a birational map  $\beta_0$  is constructed from the blow-up  $\widetilde{\mathbf{M}}$  of  $\mathbf{M}^{\infty}$  along  $F^{\infty}$  to  $\mathbf{M}^{0+}$ , which contracts  $\widetilde{F}$  in the  $\mathbb{P}^2$ -directions. Note that  $\beta_0$  gives an isomorphism on the complement of  $F^0$  and the preimages of points in  $F^0$  are isomorphic to  $\mathbb{P}^2$ . By Remark 5.5,  $F^0$  is smooth. We claim that  $\mathbf{M}^{0+}$  is also smooth. This can be verified using the smoothness criterion for moduli spaces of  $\alpha$ -semi-stable pairs: if  $\Lambda$  gives a stable point of  $\mathbf{M}^{0+}$  and  $\mathrm{Ext}^2(\Lambda,\Lambda)=0$ , then  $\Lambda$  gives a smooth point. It is enough to take  $\Lambda \in F^0$  and then we can apply Lemma 5.6. We can now apply the Universal Property of the blow-up [9, p. 604], to conclude that  $\beta_0$  is a blow-up with center  $F^0$  and exceptional divisor  $\widetilde{F}$ .

The following proposition is analogous to [3, Proposition 4.4]. We define the forgetful morphism  $\phi \colon \mathbf{M}^{0+} \to \mathbf{M}$  by  $\phi(\Gamma, \mathcal{F}) = [\mathcal{F}]$ .

**Proposition 5.8.** The forgetful morphism  $\phi \colon \mathbf{M}^{0+} \to \mathbf{M}$  is a blow-up of  $\mathbf{M}$  along  $\mathbf{M}_2$ .

Proof. We will give a simpler argument then the one found at [3, Proposition 4.4]. As seen in the proof of Theorem 5.7,  $\mathbf{M}^{0+}$  is smooth. The varieties  $\mathbf{M}$  and  $\mathbf{M}_2$  are also smooth. Away from  $\mathbf{M}_2$ ,  $\phi$  is an isomorphism because, by Theorem 1.1, for  $\mathcal{F} \in \mathbf{M} \setminus \mathbf{M}_2$  we have  $\mathbf{H}^0(\mathcal{F}) \simeq \mathbb{C}$ , hence we may identify  $\mathcal{F}$  with the  $\alpha$ -stable pair  $(\mathbf{H}^0(\mathcal{F}), \mathcal{F})$  for sufficiently small  $\alpha$ . For  $\mathcal{F} \in \mathbf{M}_2$ ,  $\phi^{-1}([\mathcal{F}]) = \mathbb{P}(\mathbf{H}^0(\mathcal{F})) \simeq \mathbb{P}^1$ . By the Universal Property of the blow-up [9, p. 604],  $\phi$  is a blow-up with center  $\mathbf{M}_2$ .

**Proof of Theorem 1.2.** The integral homology groups of M have no torsion because  $M^{\infty}$  enjoys this property and M is obtained from  $M^{\infty}$  by a sequence of blow-ups and blow-downs. By Theorem 5.7,

$$P(\mathbf{M}^{0+}) = P(\mathbf{M}^{\infty}) + (P(\mathbb{P}^1) - P(\mathbb{P}^2)) P(\mathbf{M}^{0+}(2m+2n) \times \mathbf{M}(m+1)).$$

By Proposition 5.3 and Remark 5.4,

$$P(\mathbf{M}^{0+}) = P(\mathbb{P}^9) P(\mathrm{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2)) + (P(\mathbb{P}^1) - P(\mathbb{P}^2)) P(\mathbb{P}^8) P(\mathbb{P}^1).$$

According to [8, Theorem 0.1],

$$P(\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(2))(\xi) = \xi^4 + 3\xi^3 + 6\xi^2 + 3\xi + 1.$$

In view of Proposition 5.8,

$$P(\mathbf{M}) = P(\mathbf{M}^{0+}) - \xi P(\mathbf{M}_2) = P(\mathbf{M}^{0+}) - \xi P(\mathbb{P}^{11}).$$

In conclusion.

$$P(\mathbf{M}) = \frac{\xi^{10} - 1}{\xi - 1} (\xi^4 + 3\xi^3 + 6\xi^2 + 3\xi + 1) - \xi^2 \frac{\xi^9 - 1}{\xi - 1} (\xi + 1) - \xi \frac{\xi^{12} - 1}{\xi - 1}.$$

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