HIGHER-ORDER WEIERSTRASS WEIGHTS OF BRANCH POINTS ON SUPERELLIPTIC CURVES

CALEB MCKINLEY SHOR

ABSTRACT. In this paper we consider the problem of calculating the higher-order Weierstrass weight of the branch points of a superelliptic curve C. For any q>1, we give an exact formula for the q-weight of an affine branch point. We also find a formula for the q-weight of a point at infinity in the case where n and d are relatively prime. With these formulas, for any fixed n, we obtain an asymptotic formula for the ratio of the q-weight of the branch points, denoted BW_q , to the total q-weight of points on the curve:

$$\liminf_{d \to \infty} \frac{BW_q}{g(g-1)^2(2q-1)^2} \ge \frac{n+1}{3(n-1)^2(2q-1)^2},$$

with equality when the limit is taken such that gcd(n, d) = 1.

1. Introduction

Let $q \in \mathbb{N}$. A q-Weierstrass point (or higher-order Weierstrass point) is a point P on a curve for which there exist holomorphic q-differentials that have higher than expected orders of vanishing at P. Each q-Weierstrass point has an associated q-weight, denoted $w^{(q)}(P)$, which measures how much higher than expected those orders of vanishing are. A curve of genus $g \geq 2$ has finitely many q-Weierstrass points.

Importantly, the q-weight of a point is invariant under automorphism. Thus, higher-order Weierstrass points are important in the study of automorphisms of algebraic curves. For instance, Lewittes showed in [6] that if an automorphism has at least five fixed points, then all of its fixed points are 1-Weierstrass points. Further, Mumford, in [8], has suggested that q-Weierstrass points on an algebraic curve are analogous to q-torsion points on an elliptic curve. For more on the history of Weierstrass points, we refer the reader to [2]. For background material of Weierstrass points specifically on superelliptic curves, see [12].

Let C be a curve of genus $g \geq 2$ of the form $y^n = f(x)$ where f(x) is a separable polynomial of degree $d > n \geq 2$. Such a curve is said to be superelliptic. In the cover $C \to \mathbb{P}^1$, the points above the roots of f(x) are branch points. If $n \nmid d$, the point (or points) above ∞ in the nonsingular model of C is also a branch point. One can show that each branch point is a q-Weierstrass point for all q; in the case where n = 2, the branch points are exactly the 1-Weierstrass points. Let B be an affine branch point on C and, if $n \nmid d$, P^{∞} a nonsingular branch point at infinity. In [16, Theorem 8], Towse calculated the 1-weight of the branch points (affine and

1

²⁰¹⁰ Mathematics Subject Classification. Primary 14H55, 11G30.

Key words and phrases. Weierstrass points, higher-order Weierstrass points, superelliptic curves, branch points, numerical semigroups.

at infinity) as a function of n and d. In the case of gcd(n, d) = 1, he found that

$$w^{(1)}(B) = \frac{g(n+1)(d-7)}{12} + (d-1)\sum_{j=1}^{n-1} \left\{ \frac{-dj}{n} \right\} j,$$

where $\{x\}$ denotes the fractional part of x, and

$$w^{(1)}(P^{\infty}) = \frac{(n^2 - 1)(d^2 - 1)}{24} - g.$$

Given that the total 1-weight of points on a curve of genus g is $g^3 - g$, he was able to calculate the fraction that the branch points' 1-weight (denoted BW) accounted for as

$$\lim_{d \to \infty} \frac{BW}{g^3 - g} \ge \frac{n+1}{3(n-1)^2},$$

with equality when the limit is taken over integers d such that gcd(n, d) = 1.

The goal of this paper is to extend Towse's results to higher-order Weierstrass weights of branch points on a superelliptic curve. To achieve this, we first produce a basis for the space of holomorphic q-differentials on a superelliptic curve C. A common approach to calculate the q-weight is to work with the Wronskian of this basis, a method first described by Hurwitz in [5]. However, we take a different approach, instead using results from numerical semigroups and non-representable numbers. In this way we obtain a formula for the q-weight of an affine branch point as a function of n, d, and q. The main result is Theorem 2. In particular, when $\gcd(n,d)=1$ we find for $q\geq 2$

$$w^{(q)}(B) = \frac{g(n+1)(d-7)}{12} + g + (d-1)\sum_{i=1}^{n-1} \left\{ -\frac{(d+1)q + dj}{n} \right\} j.$$

As for the points at infinity, with two examples, we show that if $\gcd(n,d) > 1$, one cannot get a formula for $w^{(q)}(P^{\infty})$ based only on n, d, and q. However, if $\gcd(n,d) = 1$ and $q \geq 2$, then in Theorem 3 we have

$$w^{(q)}(P^{\infty}) = \frac{(n^2 - 1)(d^2 - 1)}{24}.$$

Since the total q-weight of points on a curve of genus g is $g(g-1)^2(2q-1)^2$ for $q \geq 2$, we get an similar asymptotic result in Proposition 6 for the proportion of branch points' q-weight (denoted BW_q). We find

$$\lim_{d \to \infty} \frac{BW_q}{g(g-1)^2(2q-1)^2} \ge \frac{n+1}{3(n-1)^2(2q-1)^2},$$

with equality when the limit is taken over integers d such that gcd(n, d) = 1.

The paper is organized as follows. In Section 2, we provide some background material on calculating the q-Weierstrass weight of a point. We also include some notation and results for non-representable integers in numerical semigroups with two generators. In Section 3, we find a basis for the space of holomorphic q-differentials on a curve C by presenting a set of linearly independent holomorphic differentials and then counting them to make sure there are as many as the Riemann-Roch theorem predicts. In Section 4, we have our main results. We find a formula for the q-weight of an affine branch point, and we use that to derive a few corollaries for specific cases of n and d. We also note that, for given n and d, the q-weight of a branch point depends only on the value of q modulo n. We also give examples to

show that if gcd(n, d) > 1 the q-weight of a point at infinity cannot necessarily be determined just by knowing n and d. If gcd(n, d) = 1, however, we can calculate the weight. In both cases, we obtain some asymptotic results about the proportion of q-weight that the branch points contain.

2. Preliminaries and notation

2.1. q-Weierstrass points. In this paper, we will follow the approach given in [13, Section 2]. We describe the notation and major results for calculating weights of q-Weierstrass points here.

Let k be an algebraically closed field, C be a non-singular projective curve over k of genus $g \geq 2$, and k(C) its function field. For any $f \in k(C)$, let $\operatorname{div}(f)$ denote the divisor associated to f. For any divisor $D = \sum_{P} n_{P}P$ and any point P, let $\nu_{P}(D) = n_{P}$, and let $\operatorname{ord}_{P}(f) = \nu_{P}(\operatorname{div}(f))$.

For any $q \in \mathbb{N}$, let $H^0(C, (\Omega^1)^q)$ be the \mathbb{C} -vector space of holomorphic q-differentials on C, a space of dimension

$$d_q = \begin{cases} g & \text{if } q = 1, \\ (g-1)(2q-1) & \text{if } q \ge 2. \end{cases}$$

For P a degree 1 point on C, consider a basis $\{\psi_1, \ldots, \psi_{d_q}\}$ of $H^0(C, (\Omega^1)^q)$ where

$$\operatorname{ord}_P(\psi_1) < \operatorname{ord}_P(\psi_2) < \cdots < \operatorname{ord}_P(\psi_{d_a}).$$

The q-weight of P is

$$w^{(q)}(P) = \sum_{i=1}^{d_q} \operatorname{ord}_P(\psi_i) - \sum_{j=0}^{d_q-1} j.$$

We call the point P a q-Weierstrass point if $w^{(q)}(P) > 0$.

Proposition 1. [4, III.5.10] Let C be a curve of genus $g \ge 2$ and let $q \ge 1$. Then the total q-weight of points on C is

$$\sum_{P \in C} w^{(q)}(P) = (g-1)d_q(2q-1+d_q) = \begin{cases} g^3 - g & \text{if } q = 1, \\ g(g-1)^2(2q-1)^2 & \text{if } q \ge 2. \end{cases}$$

2.2. Non-representable numbers. For notation, let \mathbb{N}_0 be the set of non-negative integers. Let $a, b \in \mathbb{N}$ and consider the set

$$R(a,b) = \{ax + by : x, y \in \mathbb{N}_0\}.$$

Elements of R(a,b) are called (a,b)-representable numbers. The complement of R(a,b) in \mathbb{N}_0 , denoted NR(a,b), is the set of (a,b)-representable numbers. When there is no confusion, we will omit the (a,b) and simply refer to these numbers as representable or non-representable.

The problem of calculating the cardinality of NR(a, b) dates to the late 19th century in [15]. Clearly, if gcd(a, b) > 1 then NR(a, b) is an infinite set. It is straightforward to show that the converse is true too. For example, see [10, Theorem 1.0.1] for two proofs of the following result.

Lemma 1. For $a, b \in \mathbb{N}$, if gcd(a, b) = 1 then NR(a, b) is a finite set.

For the rest of this section, we will assume gcd(a, b) = 1. Thus, NR(a, b) is finite and so we can compute its cardinality.

Proposition 2. [15, Page 134] For $a, b \in \mathbb{N}$ with gcd(a, b) = 1,

$$|NR(a,b)| = \frac{(a-1)(b-1)}{2}.$$

This result is important in the theory of algebraic curves. Suppose a plane curve C is given by the affine equation

$$\alpha_{a,0}x^a + \alpha_{0,b}y^b + \sum_{i,j} \alpha_{i,j}x^iy^j = 0$$

for constants $\alpha_{i,j}$ with $\alpha_{a,0} \cdot \alpha_{0,b} \neq 0$ and where the summation is over non-negative i,j such that aj + bi < ab. Such a curve is called a $C_{a,b}$ curve. These curves can be seen as a generalization of elliptic and hyperelliptic curves in Weierstrass form. With the Riemann-Roch Theorem, if the affine part of the curve is non-singular then one can show that the genus of such a curve is exactly (a-1)(b-1)/2, the cardinality of NR(a,b). For details, see [7] or [14].

For the purposes of this paper, we will also need to know the sum of the elements of NR(a, b). This problem was solved in [1] using generating functions.

Proposition 3. For $a, b \in \mathbb{N}$ with gcd(a, b) = 1,

$$\sum_{n \in NR(a,b)} n = \frac{(a-1)(b-1)(2ab-a-b-1)}{12}.$$

This result was generalized to a formula for the sum of the mth powers of elements of NR(a,b). See [11] or [17] for details.

3. A basis of holomorphic q-differentials

In this section, we give a basis for the space of holomorphic q-differentials on a superelliptic curve C. The main result of this section is as follows:

Theorem. Let C be a curve of genus $g \ge 2$ given in affine coordinates by $y^n = f(x)$, for f(x) a separable polynomial of degree $d > n \ge 2$. For $q \ge 1$, let $H^0(C, (\Omega^1)^q)$ be the space of holomorphic q-differentials on C. Let

$$\mathfrak{B}_{n,d,q} = \left\{ x^i y^j \left(\frac{\mathrm{d}x}{y^{n-1}} \right)^q : 0 \le i, \ 0 \le j < n, \ ni + dj \le (2g-2)q \right\}.$$

Then $\mathfrak{B}_{n,d,q}$ is a basis for $H^0(C,(\Omega^1)^q)$.

In order to prove this, we need the following results (based on the work in [16]) and a useful lemma.

Let $G = \gcd(n, d)$. For g the genus of C, we have 2g - 2 = nd - n - d - G. Let $\{\alpha_1, \ldots, \alpha_d\}$ denote the d distinct roots of f(x) and let $B_i = (\alpha_i, 0)$ for $i = 1, \ldots, d$. For each non-root ω of f(x), let $P_1^{\omega}, \ldots, P_n^{\omega}$ denote the n points on C over $x = \omega$. And let $P_1^{\infty}, \ldots, P_G^{\infty}$ denote the G points over ∞ in the non-singular model of C. One then has the following principal divisors.

•
$$\operatorname{div}(y) = \sum_{j=1}^{d} B_j - \frac{d}{G} \sum_{m=1}^{G} P_m^{\infty},$$

•
$$\operatorname{div}(x - \alpha_i) = nB_i - \frac{n}{G} \sum_{m=1}^{G} P_m^{\infty},$$

•
$$\operatorname{div}(x - \omega) = \sum_{j=1}^{n} P_{j}^{\omega} - \frac{n}{G} \sum_{m=1}^{G} P_{m}^{\infty}.$$

•
$$\operatorname{div}(\mathrm{d}x) = (n-1)\sum_{j=1}^{d} B_j - \left(\frac{n}{G} + 1\right)\sum_{m=1}^{G} P_m^{\infty},$$

From these, we see that $\operatorname{div}((\operatorname{d} x/y^{n-1})^q) = \frac{(2g-2)q}{G} \sum_{m=1}^G P_m^{\infty}$. For integers i,j, let $f_{i,j} = x^i y^j (\operatorname{d} x/y^{n-1})^q$. We want to find conditions on i and j such that $f_{i,j}$ is a holomorphic q-differential. Note that $f_{i,j}$ can have poles only at the points above ∞ if $i, j \geq 0$. In that situation, we find

$$\operatorname{ord}_{P_m^{\infty}}(f_{i,j}) = \frac{(2g-2)q - (ni + dj)}{G}$$

for each pair (i,j) and each m. Hence $f_{i,j}$ is a holomorphic q-differential as long as $i \ge 0, j \ge 0, \text{ and } ni + dj \le (2g - 2)q.$

Lemma 2. Let $n, d, q \in \mathbb{Z}$ with $2 \le n < d$ and $q \ge 2$. As above, let $G = \gcd(n, d)$ and 2g-2=nd-n-d-G. For all but finitely many triples (n,d,q), one has

$$(2g-2)q - d(n-1) \ge 0.$$

The exceptional cases are $(n, d, q) \in \{(2, 5, 2), (2, 6, 2)\}.$

Proof. First, note that (2g-2)q - d(n-1) = (nd - n - d - G)q - d(n-1) =d(n-1)(q-1)-q(n+G). Thus, to show our desired inequality, it is equivalent to show

$$d \ge \left(\frac{q}{q-1}\right) \left(\frac{n+G}{n-1}\right).$$

For notation, let $h(q,n) = \left(\frac{q}{q-1}\right) \left(\frac{n+G}{n-1}\right)$. We aim to show $d \ge h(q,n)$.

For $q \geq 2$, the maximum value of q/(q-1), which occurs when q=2, is 2. For $n \geq 2$, the maximum value of (n+G)/(n-1) occurs when G is largest, so when G=n, and when n=2. The maximum value is 4. Thus $h(q,n) \leq 2 \cdot 4 = 8$ so $d \ge h(q, n)$ for all $d \ge 8$.

Now we consider cases of n. If $n \geq 4$, then G = n or G < n. If G = n, then n|d, so $d \geq 2n \geq 8$, so $d \geq h(q,n)$. If G < n, then $G \leq n/2$, so $h(q,n) \leq n$ $2(n+n/2)/(n-1) = 3+3/(n-1) \le 4$. Since d > n, we have $d > n \ge 4 \ge h(q,n)$, as desired.

If n=3, then G=1 or G=3. If G=1, then $h(q,3) \leq 2 \cdot 2 = 4$. In this case, since d > n = 3, we have $d \ge 4$, so $d \ge h(q,3)$. If G = 3, then $h(q,3) \le 2 \cdot 3 = 6$. In this case, $d \ge 2n = 6 \ge h(q, 3)$, as desired.

If n=2, then G=1 or G=2. Note that to have $g\geq 2$, we only consider $d\geq 5$. If G = 1 and q = 2, then $h(2, 2) = 2 \cdot 3 = 6$. Since G = 1, d is odd, so $d \ge h(2, 2)$ for all d except for d=5. If G=1 and $q\geq 3$, then $h(q,2)\leq (3/2)\cdot 3=9/2$, so $d \geq h(q,2)$ for all $d \geq 5$. If G = 2 and q = 2, then $h(2,2) = 2 \cdot 4 = 8$. Since G=2, d is even, so $d\geq h(2,2)$ for all d except for d=6. If G=2 and $q\geq 3$, then $h(q, 2) \le (3/2) \cdot 4$, so $d \ge h(q, 2)$ for all $d \ge 6$.

Thus, the only exceptional cases are (n, d, q) = (2, 5, 2) or (2, 6, 2). For all other triples, we find that $d \geq h(q, n)$, or, equivalently, that

$$(2g-2)q - d(n-1) \ge 0$$
,

as desired.

Theorem 1. Let C be a curve of genus $g \geq 2$ given in affine coordinates by $y^n = f(x)$, for f(x) a separable polynomial of degree d > n. For $q \ge 1$, let $H^0(C,(\Omega^1)^q)$ be the space of holomorphic q-differentials on C. Let

$$\mathfrak{B}_{n,d,q} = \left\{ x^i y^j \left(\frac{\mathrm{d}x}{y^{n-1}} \right)^q : 0 \le i, \ 0 \le j < n, \ ni + dj \le (2g-2)q \right\}.$$

Then $\mathfrak{B}_{n,d,q}$ is a basis for $H^0(C,(\Omega^1)^q)$.

Proof. The q=1 case, in a slightly different form, is proved in [16]. For completeness, we will first prove the $q \geq 2$ case here and then adapt our argument to cover the q=1 case.

Suppose $q \geq 2$. With the restriction that $0 \leq j < n$, we see that these holomorphic q-differentials are linearly independent. We therefore need to show that $|\mathfrak{B}_{n,d,q}| = d_q = (2q-1)(g-1)$.

We first consider the case where $(n, d, q) \notin \{(2, 5, 2), (2, 6, 2)\}$ and let $\mathfrak{B} = \mathfrak{B}_{n,d,q}$. Note that we require $i \geq 0$ and $ni + dj \leq (2q - 2)q$, so

$$0 \le i \le \left| \frac{(2g-2)q - dj}{n} \right|.$$

For each j = 0, ..., n-1, we have $(2g-2)q - dj \ge (2g-2)q - d(n-1)$, which is non-negative by Lemma 2. (This is why we handle the (2,5,2) and (2,6,2) cases separately.) Thus, to calculate the number of pairs (i,j) in \mathfrak{B} , we will let j go from 0 to n-1 and count the number of indices i that correspond to each j value. I.e.,

$$|\mathfrak{B}| = \sum_{j=0}^{n-1} \left(1 + \left\lfloor \frac{(2g-2)q - dj}{n} \right\rfloor \right).$$

Since $\lfloor x \rfloor = x - \{x\}$, we simplify the sum to get

$$|\mathfrak{B}| = n + (2g - 2)q - \frac{d(n-1)}{2} - \sum_{i=0}^{n-1} \left\{ \frac{(-d-G)q - dj}{n} \right\}.$$

Now, we consider cases of G. If G=n, then n|d, so n|((-d-G)q-dj), so each term in the summation is 0. Note that n-d(n-1)/2=-(nd-d-2n)/2=-(nd-n-d-G)/2=-(g-1). Then $|\mathfrak{B}|=(2g-2)q-(g-1)=(2q-1)(g-1)=d_q$, as desired.

Next, suppose $G \neq n$. Let n' = n/G and d' = d/G. Dividing the numerator and denominator by G, the summation equals $\sum_{j=0}^{n-1} \left\{ \frac{(-d'-1)q-d'j}{n'} \right\}$. Since $\gcd(n',d') = 1$, as j goes from 0 to n'-1 modulo n', the numerators are distinct modulo n' and therefore in every congruence class exactly once modulo n'. Since n/n' = G, this summation equals $G \sum_{k=0}^{n'-1} \frac{k}{n'} = G(n'-1)/2$. All together,

$$|\mathfrak{B}| = n + (2q - 2)q - d(n - 1)/2 - (n - G)/2.$$

I.e. $|\mathfrak{B}| = n + 2q(g-1) - (1/2)(nd - d + n - G) = 2q(g-1) - (1/2)(nd - d - n - G) = (2q-1)(g-1) = d_q$, as desired.

To complete the proof for $q \ge 2$, we consider the exceptional cases. Suppose (n, d, q) = (2, 5, 2), so q = 2 and $d_2 = 3$. Then

$$\mathfrak{B}_{2.5.2} = \left\{ x^i y^j (dx/y)^2 : i \ge 0, 0 \le j < 2, 2i + 5j \le 4. \right\}$$

Thus, $\mathfrak{B}_{2,5,2} = \{(\mathrm{d}x/y)^2, \, x(\mathrm{d}x/y)^2, \, x^2(\mathrm{d}x/y)^2\}$, so $|\mathfrak{B}_{2,5,2}| = 3 = d_2$, as desired. Suppose (n,d,q) = (2,6,2), so g = 2 and $d_2 = 3$. Then

$$\mathfrak{B}_{2.6.2} = \left\{ x^i y^j (dx/y)^2 : i \ge 0, 0 \le j < 2, 2i + 6j \le 4. \right\}$$

Thus, $\mathfrak{B}_{2,6,2}=\{(\mathrm{d}x/y)^2,\,x(\mathrm{d}x/y)^2,\,x^2(\mathrm{d}x/y)^2\}$, so $|\mathfrak{B}_{2,6,2}|=3=d_2$, as desired. Now, suppose q=1. Following the approach above, given j we need integers i such that

$$0 \le i \le \left| \frac{(2g-2) - dj}{n} \right|.$$

If $j \le n-2$, then $(2g-2)-dj \ge (2g-2)-d(n-2)=d-n-G \ge 0$ since $d \ge n$. If j=n-1, then (2g-2)-d(n-1)=-n-G<0, so there are no such i. Thus, our summation for $|\mathfrak{B}|$ ends at j=n-2 instead of j=n-1. Since we have a formula for the summation above, we can subtract the j=n-1 term out front to get

$$|\mathfrak{B}| = -\left(1 + \left\lfloor \frac{-n-G}{n} \right\rfloor\right) + \sum_{j=0}^{n-1} \left(1 + \left\lfloor \frac{(2g-2)-dj}{n} \right\rfloor\right).$$

Since q = 1 the summation equals g - 1, so $|\mathfrak{B}| = -(1 - 2) + (g - 1) = g = d_1$, as desired.

4. Weights of branch points

In this section, we use the bases we found in the previous section to calculate the q-weight of the affine branch points and, in the case that gcd(n, d) = 1, the point at infinity.

4.1. Weights of affine branch points. Suppose $q \geq 2$. For C given by $y^n = f(x)$ with f(x) separable of degree d, let α be a root of f(x). Then $B = (\alpha, 0)$ is an affine branch point of C. Note that we can replace x by $(x - \alpha)$ in our basis $\mathfrak{B}_{n,d,q}$ to produce a new basis $\mathfrak{B}_{n,d,q,\alpha}$. That is,

$$\mathfrak{B}_{n,d,q,\alpha} = \{ (x - \alpha)^i y^j (\mathrm{d} x / y^{n-1})^q : i \ge 0, \ 0 \le j < n, \ ni + dj \le (2g - 2)q \}$$
 is a basis for $H^0(C, (\Omega^1)^q)$.

Let $f_{i,j,\alpha} = (x - \alpha)^i y^j (\mathrm{d}x/y^{n-1})^q \in \mathfrak{B}_{n,d,q,\alpha}$. Then

$$\nu_B(f_{i,j,\alpha}) = ni + j.$$

Since $0 \le j < n$, these valuations are all different, and thus

$$w^{(q)}(B) = \sum_{(i,j)\in S} (ni+j) - \sum_{k=0}^{d_q-1} k,$$

where $S = \{(i, j) \in \mathbb{Z}^2 : i \geq 0, 0 \leq j < n, ni + dj \leq (2g - 2)q\}$. We rewrite this as $w^{(q)}(B) = W_1 - W_2 - W_3$ where

(1)
$$W_1 = \sum_{(i,j)\in S} (ni+dj), \quad W_2 = (d-1)\sum_{(i,j)\in S} j, \quad W_3 = \sum_{k=0}^{d_q-1} k.$$

We have

$$W_3 = \frac{(d_q - 1)(d_q)}{2} = \frac{1}{2} \left((2g - 2)^2 q^2 + (2g - 2)(1 - 2g)q + g(g - 1) \right).$$

We will evaluate W_1 and W_2 with the following propositions.

Proposition 4. Let $n, d, q \in \mathbb{N}$ such that n < d and $q \ge 2$. Then

$$W_1 = 2(g-1)^2 q^2 + (g-1)Gq + \frac{G^2 - 1 - (n-1)(d-1)(2nd - n - d - 1)}{12}.$$

We will first sketch the proof in the situation where gcd(n, d) = 1. Afterward, we will prove the theorem for any gcd.

When gcd(n,d) = G = 1, for $(i,j) \in S$, the terms ni + dj are distinct integers from 0 to (2g-2)q. From Proposition 2, since $(2g-2)q \ge nd - n - d$ (by Lemma 3 below), all of the (n-1)(d-1)/2 (n,d)-non-representable integers are in that interval. The sum of the non-representable integers, as is given in Proposition 3, is (n-1)(d-1)(2nd-n-d-1)/12. Thus, if gcd(n,d) = 1, we add up all of the integers from 0 to (2g-2)q and subtract off the non-representable integers to get $W_1 = (2g-2)q((2g-2)q+1)/2 - (n-1)(d-1)(2nd-n-d-1)/12$.

If gcd(n, d) > 1, then the terms ni + dj are no longer distinct, so we need to evaluate the sum more carefully.

Proof. Let $G = \gcd(n, d)$. First, we observe that $W_1 = G \sum_{(i,j) \in S} (n'i + d'j)$ for n' = n/G and d' = d/G. Note that $\gcd(n', d') = 1$. For k from 0 to G - 1, let

$$S_k = \{(i,j) : i \ge 0, \, kn' \le j < (k+1)n', \, ni + dj \le (2g-2)q\}.$$

In particular, S is the disjoint union of the sets S_k . Let $W_{1,k} = \sum_{(i,j) \in S_k} (n'i + d'j)$. Then

$$W_1 = G \sum_{k=0}^{G-1} W_{1,k}.$$

Letting j' = j - kn', we rewrite S_k as

$$S_k = \{(i, j' + kn') : i \ge 0, 0 \le j' < n', ni + dj' \le (2g - 2)q - n'dk\}$$

and dividing the last inequality through by G we obtain

$$S_k = \{(i, j' + kn') : i \ge 0, \ 0 \le j' < n', \ n'i + d'j' \le \frac{(2g - 2)}{G}q - n'd'k\}.$$

Let $m_k = \frac{(2g-2)}{G}q - n'd'k$, the upper bound in S_k . The following lemma will allow us to conclude that all of the (n', d')-non-representable integers are less than m_k .

Lemma 3. Let $m_k = \frac{(2g-2)}{G}q - n'd'k$. Then $m_k \ge n'd' - n' - d'$ for all $n, d, q \in \mathbb{N}$ with $0 \le k \le G - 1$, n < d, $g \ge 2$, and $q \ge 2$.

Proof. First, note that for $0 \le k \le G-1$, $m_k = \frac{(2g-2)}{G}q - n'd'k \ge \frac{(2g-2)}{G}q - n'd'(G-1)$. So we need to show $\frac{(2g-2)}{G}q - n'd'(G-1) \ge n'd' - n' - d'$, which is equivalent to showing $(2g-2)q \ge nd - n - d$. Since nd - n > nd - n - d, by Lemma 2, we have $(2g-2)q \ge nd - d > nd - n - d$ for all (n,d,q) combinations except (2,5,2) and (2,6,2).

We compute the exceptional cases separately. If (n,d,q)=(2,5,2), then g=2 and $(2g-2)q=4\geq 3=nd-n-d$. If (n,d,q)=(2,6,2), then g=2 and $(2g-2)q=4\geq 4=nd-n-d$. Thus, the bound holds for the exceptional cases as well.

For S_k , since we are considering $i \ge 0$ and $0 \le j' < n'$, and since $m_k \ge n'd' - n' - d'$ for all k, our ordered pairs (i, j' + kn') are in one-to-one correspondence with the (n', d')-representable numbers in the interval $[0, m_k]$. And since $m_k \ge n'd' - n' - d'$, all of the (n'-1)(d'-1)/2 (n', d')-non-representable numbers are in this interval as well. Thus S_k contains $|S_k| = m_k + 1 - (n'-1)(d'-1)/2$ ordered pairs.

Then

$$\begin{split} W_{1,k} &= \sum_{(i,j) \in S_k} (n'i + d'j) \\ &= \sum_{(i,j'+kn') \in S_k} (n'i + d'j' + n'd'k) \\ &= n'd'k \cdot |S_k| + \sum_{(i,j'+kn') \in S_k} (n'i + d'j') \,. \end{split}$$

The summation is the sum of the (n', d')-representable numbers from 0 to m_k . We calculate this by summing all of the integers from 0 to m_k and subtracting the (n', d')-non-representable integers, which all lie in this interval. Using Proposition 3, the summation is $m_k(m_k + 1)/2 - (n' - 1)(d' - 1)(2n'd' - n' - d' - 1)/12$. Thus,

$$W_{1,k} = n'd'k (m_k + 1 - (n'-1)(d'-1)/2)$$

+ $m_k(m_k + 1)/2 - (n'-1)(d'-1)(2n'd'-n'-d'-1)/12,$

so

$$W_1 = G \sum_{k=0}^{G-1} \left[n'd'k \left(m_k + 1 - \frac{(n'-1)(d'-1)}{2} \right) + \frac{m_k(m_k+1)}{2} - \frac{(n'-1)(d'-1)(2n'd'-n'-d'-1)}{12} \right].$$

To evaluate this sum, we need the following calculations which are straightforward to compute.

•
$$\sum_{k=0}^{G-1} m_k = (2g-2)q - \frac{n'd'G(G-1)}{2}$$
.

•
$$\sum_{k=0}^{G-1} m_k^2 = \frac{(2g-2)^2}{G} q^2 - (2g-2)(G-1)d'n'q + \frac{d'^2n'^2(G-1)G(2G-1)}{6}$$

•
$$\sum_{k=0}^{G-1} km_k = (g-1)(G-1)q - \frac{d'n'(G-1)G(2G-1)}{6}$$
.

Simplifying the resulting expression, we find

$$W_1 = \frac{(2g-2)^2}{2}q^2 + (g-1)Gq + \frac{G^2 - 1 - (n-1)(d-1)(2nd - n - d - 1)}{12}$$

which completes the proof of Proposition 4.

Proposition 5. Let $n, d, q \in \mathbb{N}$ such that n < d and $q \ge 2$. Let

$$D(a,b,c) = \sum_{i=0}^{c-1} \left\{ \frac{a+bj}{c} \right\} j.$$

Then

$$W_2 = (d-1)\left((n-1)\left((g-1)q + \frac{-2nd + 3n + d}{6}\right) - D(-(d+G)q, -d, n)\right).$$

Proof. We will use Lemma 2, so we first assume $(n,d,q) \notin \{(2,5,2),(2,6,2)\}$. For $W_2 = (d-1) \sum_{(i,j) \in S} j$, we have

$$W_2 = (d-1)\sum_{j=0}^{n-1}\sum_{i=0}^{I_j}j,$$

for $I_j = \left\lfloor \frac{(2g-2)q-dj}{n} \right\rfloor$. By Lemma 2, $I_j \ge 0$ so $W_2 = (d-1)\sum_{j=0}^{n-1} (I_j+1)j$. Since $\lfloor x \rfloor = x - \{x\}$,

$$W_2 = (d-1)\sum_{j=0}^{n-1} \left(\frac{(2g-2)q - dj}{n} - \left\{ \frac{(2g-2)q - dj}{n} \right\} + 1 \right) j.$$

Note that $\left\{\frac{(2g-2)q-dj}{n}\right\} = \left\{\frac{(nd-n-d-G)q-dj}{n}\right\} = \left\{\frac{(-d-G)q-dj}{n}\right\}$. Expanding out, we get

$$W_2 = (g-1)(d-1)(n-1)q + \frac{(d-1)(n-1)}{6}(3n - d(2n-1))$$
$$-(d-1)\sum_{i=0}^{n-1} \left\{ -\frac{(d+G)q + dj}{n} \right\} j,$$

which can be rearranged to give the desired result.

Finally, if $(n,d,q) \in \{(2,5,2),(2,6,2)\}$, then $S = \{(0,0),(1,0),(2,0)\}$, and so $W_2 = \sum_{(i,j) \in S} j = 0$. We get the same value if we plug each these (n,d,q) triples into the above formula for W_2 .

Remark. The summation D(a,b,c) is related to a Dedekind sum. There is no closed form for such sums, though there is a reciprocity law. For a general reference, see [9].

Finally, we can combine and simplify $W_1 - W_2 - W_3$. Note that the q^2 and q terms (other than in the summation) cancel. With further manipulation, we have our main result.

Theorem 2. Let C be given in affine coordinates by $y^n = f(x)$ for f(x) a separable polynomial of degree d > n. Let $G = \gcd(n, d)$, and let $q \in \mathbb{Z}$ with $q \geq 2$. For any root α of f(x), let $B = (\alpha, 0)$ be a branch point.

The q-weight of B is $w^{(q)}(B) =$

$$w^{(q)}(B) = \frac{1}{24} \left((n-1)(d-1)(n+1)(d-7) + 12g(G+1) + 5(G^2-1) \right) + (d-1) \cdot D(-(d+G)q, -d, n)$$

Note that, for given values of n and d, the q-weight of B depends only on the value of q modulo n.

We will give results for some combinations of n and d in the corollaries below. First, we consider the case where gcd(n, d) = 1.

Corollary 1. If gcd(n, d) = 1,

$$w^{(q)}(B) = \frac{g}{12}(n+1)(d-7) + g + (d-1) \cdot D(-(d+1)q, -d, n).$$

Fix n and d (with any gcd). If one varies q, then one sees the value of $w^{(q)}(B)$ depends only on the congruence class of q modulo n/G. Further, if $d \equiv -G \pmod n$, then the summation term simplifies to $\sum_{j=0}^{n-1} \left\{ \frac{Gj}{n} \right\} j$, for which there is a closed form.

Corollary 2. If $d \equiv -G \pmod{n}$, then $w^{(q)}(B)$ doesn't depend on q. In particular,

$$w^{(q)}(B) = \frac{1}{24} \left((n-1)(d-1)(n+1)(d-7) + 12g(G+1) + 5(G^2 - 1) + 2(d-1)(n-G)(3n+n'-2) \right).$$

Proof. The summation term is $\sum_{j=0}^{n-1} \left\{ \frac{Gj}{n} \right\} j$, $= \sum_{j=0}^{n-1} \left\{ \frac{j}{n'} \right\} j$. Each j can be written uniquely as j = j' + kn' for $0 \le k < G$ and $0 \le j' < n'$. Thus, the summation is $\sum_{k=0}^{G-1} \sum_{j'=0}^{n'-1} \left\{ \frac{j'}{n'} \right\} j = \sum_{k=0}^{G-1} \sum_{j'=0}^{n'-1} \left(\frac{j'^2}{n'} + j'k \right)$, which simplifies to (n-G)(3n+n'-2)/12.

Combining the two corollaries above, we obtain the following.

Corollary 3. If $d \equiv -1 \pmod{n}$, then

$$w^{(q)}(B) = \frac{g(n+1)(d+1)}{12} = \frac{(n^2 - 1)(d^2 - 1)}{24}$$

for all $q \geq 2$.

Corollary 4. If $n \mid d$, then

$$w^{(q)}(B) = \frac{(n^2 - 1)(d^2 - 2d)}{24}.$$

Proof. If $n \mid d$, then G = n and $n \mid ((d+G)q+dj)$ for all j, so the summation is zero. Since 2g-2=nd-n-d-n, we have $g=\frac{(d-2)(n-1)}{2}$. Plugging in, the result follows.

4.2. Weights of points at infinity. If $n \mid d$, then there are n points at infinity in the smooth model of C, so these points are not branch points. However, we can still investigate their q-weights. If gcd(n,d) > 1, then we need to know more about f(x) to determine $w^{(q)}(P_m^{\infty})$. We give a few examples to illustrate this. In [3], the authors consider curves of the form $y^2 = f(x) = x^6 + ax^4 + bx^2 + 1$,

In [3], the authors consider curves of the form $y^2 = f(x) = x^6 + ax^4 + bx^2 + 1$, where a, b are parameters and f(x) is separable. In the non-singular models of these curves, there are $G = \gcd(n, d) = 2$ points at infinity P_1^{∞} and P_2^{∞} . If $4b = a^2$, then $w^{(3)}(P_1^{\infty}) = w^{(3)}(P_2^{\infty}) = 2$. If $4b \neq a^2$, then $w^{(3)}(P_1^{\infty}) = w^{(3)}(P_2^{\infty}) = 0$.

In [13, Lemma 4 and Proposition 3], the authors consider hyperelliptic curves of genus 3 of the form $y^2 = f(x)$ where $\deg(f) = 8$. In the non-singular models of these curves, there are $G = \gcd(n,d) = 2$ points at infinity P_1^{∞} and P_2^{∞} . If C is given by $y^2 = x^8 + x^6 + 16x^4 + x^2 + 1$, then $w^{(2)}(P_1^{\infty}) = w^{(2)}(P_2^{\infty}) = 1$. If C is given by $y^2 = x^8 + x^4 + 1$, then $w^{(2)}(P_1^{\infty}) = w^{(2)}(P_2^{\infty}) = 3$.

Thus, simply knowing n and d is not enough to calculate the q-weight of the points at infinity. However, there are some cases where we can get a result.

First, if d = n + 1, then the lone point at infinity is a nonsingular branch point, so it will have the same q-weight as the affine branch points. By Corollary 3, since $d \equiv -1 \pmod{n}$, $w^{(q)}(B) = \frac{(n^2-1)(d^2-1)}{24}$ for $q \geq 2$, so we will have $w^{(q)}(P^{\infty}) =$

 $\frac{(n^2-1)(d^2-1)}{24}$ for $q \ge 2$ as well. This is a special case of the more general result when $\gcd(n,d)=1$.

Theorem 3. Suppose C is a curve of genus $g \ge 2$ given by the affine equation $y^n = f(x)$ for f(x) a separable polynomial of degree d where n < d and gcd(n, d) = 1. Let P_1^{∞} be the lone point at infinity in the non-singular model of C. Then

$$w^{(q)}(P_1^\infty) = \begin{cases} \frac{g(n+1)(d+1)}{12} - g = \frac{(n^2-1)(d^2-1)}{24} - g & \text{if } q = 1, \\ \frac{g(n+1)(d+1)}{12} = \frac{(n^2-1)(d^2-1)}{24} & \text{if } q \geq 2. \end{cases}$$

Proof. For q=1, the formula is given at the end of the proof of [16, Theorem 8]. For $q\geq 2$ and G=1, let $\mathfrak{B}_{n,d,q}$ be as in Section 3, and again let $S=\{(i,j)\in\mathbb{Z}^2:i\geq 0,0\leq j< n,ni+dj\leq (2g-2)q\}$. Then $f_{i,j}\in\mathfrak{B}_{n,d,q}$ if and only if $(i,j)\in S$. Recall that $\mathrm{ord}_{P_m^\infty}(f_{i,j})=(2g-2)q-(ni+dj)$. These orders of vanishing are unique, so

$$w^{(q)}(P_m^{\infty}) = \left(\sum_{(i,j)\in S} \operatorname{ord}_{P_m^{\infty}}(f_{i,j})\right) - \sum_{k=0}^{d_q-1} k.$$

Since $|S| = d_q$,

$$w^{(q)}(P_m^{\infty}) = d_q(2g - 2)q - \left(\sum_{(i,j) \in S} (ni + dj)\right) - \frac{(d_q - 1)d_q}{2}.$$

The summation, which we called W_1 in Equation 1, is evaluated in Proposition 4. Plugging this and d_q in, the expression simplifies to $w^{(q)}(P_m^{\infty}) = \frac{(n^2-1)(d^2-1)}{24}$.

4.3. **Branch weight.** In the case where gcd(n, d) = 1, we can calculate the total q-weight of the branch points (both affine and at infinity) for $q \geq 2$, which we denote BW_q .

Corollary 5. Suppose gcd(n,d) = 1, so $g = \frac{(n-1)(d-1)}{2}$. Then the total branch q-weight is given by $BW_q = d \cdot w^{(q)}(B) + w^{(q)}(P_1^{\infty}) =$

$$d\left(\frac{g}{12}(n+1)(d-7) + g + (d-1) \cdot D(-(d+1)q, -d, n)\right) + g\frac{(n+1)(d+1)}{12}.$$

Rewritten in terms of q, we get

$$BW_q = \frac{n+1}{3(n-1)^2} \left(g^3 - 2g^2(n-1) - g(n-1)^2 \right) + d(d-1) \cdot D(-(d+1)q, -d, n)$$

From Proposition 1, we know the total weight of the q-Weierstrass points, for $q \geq 2$, is $g(g-1)^2(2q-1)^2$. We can now calculate the proportion of q-weight of the branch points.

Proposition 6. Fix n and let $q \geq 2$. Then

$$\liminf_{d \to \infty} \frac{BW_q}{g(g-1)^2(2q-1)^2} \geq \frac{n+1}{3(n-1)^2(2q-1)^2}.$$

If we restrict to values of d that are relatively prime to n then

$$\lim_{d \to \infty, (n,d)=1} \frac{BW_q}{g(g-1)^2(2q-1)^2} = \frac{n+1}{3(n-1)^2(2q-1)^2}.$$

Proof. For general n and d, since we do not have an exact formula for the q-weight of the points at infinity, we can only say $BW_q \ge d \cdot w^{(q)}(B)$. Using the result from Theorem 2, since

$$\sum_{j=0}^{n-1} \left\{ -\frac{(d+G)q+dj}{n} \right\} j \le \sum_{j=0}^{n-1} \frac{n-1}{n} j = \frac{(n-1)^2}{2},$$

in terms of d, the dominant term of $d \cdot w^{(q)}(B)$ is $d^3 \frac{(n-1)(n+1)}{24}$. Since g is on the order of d(n-1)/2, the dominant term of the denominator is $d^3 \frac{(n-1)^3(2q-1)^2}{8}$. The result follows.

For gcd(n,d) = 1, the lone point at infinity has weight $\frac{(d^2-1)(n^2-1)}{24}$. Thus, the dominant term of BW_q is precisely $d^3\frac{(n-1)(n+1)}{24}$, and we thus have an equality if we take a limit involving integers d such that gcd(n,d) = 1.

References

- [1] Tom C. Brown and Peter Jau-Shyong Shiue. A remark related to the Frobenius problem. *Fibonacci Quart.*, 31(1):32–36, 1993.
- [2] Andrea Del Centina. Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s. A. Ann. Univ. Ferrara, 54(1):37–59, 2008.
- [3] Mohamed Farahat and Fumio Sakai. The 3-Weierstrass points on genus two curves with extra involutions. Saitama Math. J., 28:1–12, 2011.
- [4] H. M. Farkas and I. Kra. Riemann surfaces, volume 71 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1992.
- [5] A. Hurwitz. Uber algebraische Gebilde mit Eindeutigen Transformationen in sich. Mathematische Annalen, 41:403–442, 1893.
- [6] Joseph Lewittes. Automorphisms of compact Riemann surfaces. American Journal of Mathematics, 85(4):734-752, 1963.
- [7] Shinji Miura. Algebraic geometric codes on certain plane curves. Electronics and Communications in Japan (Part III: Fundamental Electronic Science), 76(12):1–13, 1993.
- [8] David Mumford. The red book of varieties and schemes, volume 1358 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, expanded edition, 1999.
- [9] Hans Rademacher and Emil Grosswald. Dedekind sums. The Mathematical Association of America, Washington, D.C., 1972.
- [10] J. L. Ramírez Alfonsín. The Diophantine Frobenius problem, volume 30 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2005.
- [11] Øystein J. Rødseth. A note on T. C. Brown and P. J.-S. Shiue's paper: "A remark related to the Frobenius problem" [Fibonacci Quart. 31 (1993), no. 1, 32–36; MR1202340 (93k:11018)]. Fibonacci Quart., 32(5):407–408, 1994.
- [12] T. Shaska and C. Shor. Weierstrass points of superelliptic curves. In L. Beshaj, T. Shaska, and E. Zhupa, editors, Advances on Superelliptic Curves and Their Applications, NATO Science for Peace and Security Series - D: Information and Communication Security. IOS Press, 2015.
- [13] Tony Shaska and Caleb Shor. 2-Weierstrass points of genus 3 hyperelliptic curves with extra involutions. *Communications in Algebra*. In press.
- [14] Caleb M. Shor. Genus calculations for towers of functions fields arising from equations of C_{ab} curves. Albanian J. Math., 5(1):31–40, 2011.
- [15] J. J. Sylvester. On subvariants, i.e. semi-invariants to binary quantics of an unlimited order. American Journal of Mathematics, 5(1):79–136, 1882.
- [16] Christopher Towse. Weierstrass points on cyclic covers of the projective line. Trans. Amer. Math. Soc., 348(8):3355–3378, 1996.
- [17] Hans J.H. Tuenter. The Frobenius problem, sums of powers of integers, and recurrences for the Bernoulli numbers. *Journal of Number Theory*, 117(2):376 – 386, 2006.

Dept. of Mathematics, Western New England University, Springfield, MA 01119 E-mail address: cshor@wne.edu