ON $(H, \widetilde{H})$-HARMONIC MAPS BETWEEN
PSEUDO-HERMITIAN MANIFOLDS*

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ABSTRACT. In this paper, we investigate critical maps of the horizontal energy functional $E_{H,\widetilde{H}}(f)$ for maps between two pseudo-Hermitian manifolds $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$. These critical maps are referred to as $(H, \widetilde{H})$-harmonic maps. We derive a CR Bochner formula for the horizontal energy density $|df_{H,\widetilde{H}}|^2$, and introduce a Paneitz type operator acting on maps to refine the Bochner formula. As a result, we are able to establish some Bochner type theorems for $(H, \widetilde{H})$-harmonic maps. We also introduce $(H, \widetilde{H})$-pluriharmonic, $(H, \widetilde{H})$-holomorphic maps between these manifolds, which provide us examples of $(H, \widetilde{H})$-harmonic maps. Moreover, a Lichnerowicz type result is established to show that foliated $(H, \widetilde{H})$-holomorphic maps are actually minimizers of $E_{H,\widetilde{H}}(f)$ in their foliated homotopy classes. We also prove some unique continuation results for characterizing either horizontally constant maps or foliated $(H, \widetilde{H})$-holomorphic maps. Furthermore, Eells-Sampson type existence results for $(H, \widetilde{H})$-harmonic maps are established if both manifolds are compact Sasakian and the target is regular with non-positive horizontal sectional curvature. Finally, we give a foliated rigidity result for $(H, \widetilde{H})$-harmonic maps and Siu type strong rigidity results for compact regular Sasakian manifolds with either strongly negative horizontal curvature or adequately negative horizontal curvature.

Introduction

A smooth map f between two Riemannian manifolds M and N is called harmonic if it is a critical point of the energy functional (cf. [EL])

$$E(f) = \frac{1}{2} \int_M |df|^2 dv_M.$$

Harmonic maps became a useful tool for studying complex structures of Kähler manifolds through the fundamental work of Siu [Si1,2]. In his generalization of Mostow's rigidity theorem for Hermitian symmetric spaces, Siu proved that a harmonic map of sufficiently high maximum rank of a compact Kähler manifold to a compact Kähler manifold with strongly negative curvature or a compact quotient of an irreducible bounded

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symmetric domain must be holomorphic or anti-holomorphic. It follows that if a compact Kähler manifold is homotopic to such a target Kähler manifold, then the homotopy equivalent map is homotopic to a biholomorphic or anti-biholomorphic map. The latter result is usually known as Siu's strong rigidity theorem. Although harmonic maps are successful in the study of geometric and topological structures of Kähler manifolds, they are not always effective in other settings. For example, it is known that harmonic maps are no longer in force for investigating general Hermitian manifolds, since holomorphic maps between these manifolds are not necessarily harmonic. In recent years, some generalized harmonic maps were introduced and investigated in various geometric backgrounds (cf. [JY], [Kok], [BD], [BDU], [KW], [Pe], [CZ]).

First, we recall the notion of transversally harmonic maps between Riemannian foliations. Let (M, g, F) and $(N, h, \widetilde{F})$ be two compact Riemann manifolds with Riemannian foliations F and $\widetilde{F}$ respectively, and $f : M \to N$ a smooth foliated map. Denote by $\upsilon(F)$ and $\upsilon(\widetilde{F})$ the normal bundles of the foliations F and $\widetilde{F}$ respectively. The differential df gives rise naturally to a smooth section $d_T f$ of $Hom(\upsilon(F), f^{-1}\upsilon(\widetilde{F}))$. Then we may define the transverse energy

$$E_T(f) = \frac{1}{2} \int_M |d_T f|^2 dv_M$$

where $d_T f : \upsilon(F) \to \upsilon(\widetilde{F})$ is the induced map of the differential map df, called the transversally differential map. A smooth foliated map $f : (M, g, F) \to (N, h, \widetilde{F})$ is called transversally harmonic if it is an extremal of $E_T(\cdot)$ for any variation of f by foliated maps (cf. [BD], [KW]). Suppose now that the foliations F and $\widetilde{F}$ are two Kählerian foliations with complex structures J and $\widetilde{J}$ on their normal spaces $\upsilon(F)$ and $\upsilon(\widetilde{F})$ respectively. A foliated map $f : M \to N$ is said to be transversally holomorphic if $d_T f \circ J = \widetilde{J} \circ d_T f$. It was proved in [BD] that any transversally holomorphic map $f : (M, g, F) \to (N, h, \widetilde{F})$ of Kählerian foliations with F harmonic is transversally harmonic.

Recall that a CR structure on an $(2m + 1)$-dimensional manifold M^{2m+1} is an $2m$-dimensional distribution $H(M)$ endowed with a formally integrable complex structure J. The manifold M with the pair $(H(M), J)$ is called a CR-manifold. A pseudo-Hermitian manifold, which is an odd-dimensional analogue of Hermitian manifolds, is a CR manifold M endowed with a pseudo-Hermitian structure θ. The pseudo-Hermitian structure θ determines uniquely a global nowhere zero vector field ξ and a Riemannian metric g_θ on M. The integral curves of ξ forms a foliation F_ξ, called the Reeb foliation. From [Ta], [We], we know that each pseudo-Hermitian manifold admits a unique canonical connection ∇ (the Tanaka-Webster connection), which is compatible with both the metric g_θ and the CR structure (see Proposition 1.1). However, this canonical connection always has nonvanishing torsion $T_\nabla(\cdot, \cdot)$, whose partial component $T_\nabla(\xi, \cdot)$ is an important pseudo-Hermitian invariant, called the pseudo-Hermitian torsion. Pseudo-Hermitian manifolds with vanishing pseudo-Hermitian torsion are referred to as Sasakian manifolds which play an important role in AdS/CFT correspondence stemming from string theory (cf. [MSY]). An equivalent characterization for a pseudo-Hermitian manifold to be Sasakian is that F_ξ is a Riemannian foliation. It is known that Sasakian geometry sits naturally in between two Kähler geometries. On the one hand, Sasakian manifolds

are the bases of metric cones which are Kähler. On the other hand, the Reeb folia-tion F_ξ of a Sasakian manifold is a Kählerian foliation (cf. [BG], [BGS]). Due to these two aspects, a Sasakian manifold can be viewed as an odd-dimensional analogue of a Kähler manifold. Consequently, it is natural to expect similar Siu type rigidity theo-rems for Sasakian manifolds. In [CZ], an interesting Siu type holomorphicity result was asserted for transversally harmonic maps between Sasakian manifolds when the target has strongly negative transverse curvature. However, Siu type strong rigidity theorems have not been established for Sasakian manifolds yet.

For a map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermitian manifolds, Petit [Pe] defined a natural horizontal energy functional

$$E_{H,\widetilde{H}}(f) = \frac{1}{2} \int_M |df_{H,\widetilde{H}}|^2 dv_\theta$$

where $|df_{H,\widetilde{H}}|^2$ denotes the horizontal energy density (cf. §3), and he called a critical map of $E_{H,\widetilde{H}}(f)$ a pseudoharmonic map. The main purpose of [Pe] is to derive Mok-Siu-Yeung type formulas for horizontal maps from compact contact locally sub-symmetric spaces into pseudo-Hermitian manifolds and obtain some rigidity theorem for the hor-izontal pseudoharmonic maps. Note that the Euler-Lagrange equation for $E_{H,\widetilde{H}}(f)$ derived in [Pe] contains an extra condition on the pull-back torsion (see (3.10)). Be-sides, the authors in [BDU] introduced another kind of pseudoharmonic maps from a pseudo-Hermitian manifold to a Riemannian manifold. To avoid the extra torsion con-dition in the Euler-Lagrange equation of [Pe] and any possible confusion with the notion of pseudo-harmonic maps in [BDU], we modify Petit's definition slightly by restricting the variational vector field to be horizontal and refer to the corresponding critical maps as $(H, \widetilde{H})$-harmonic maps. Although the energy functional $E_{H,\widetilde{H}}(f)$ is defined in a way similar to that of the transversal energy functional, we would like to point out the differ-ences between the notions of $(H, \widetilde{H})$-harmonic maps and transversally harmonic maps. First, the Reeb foliation of a pseudo-Hermitian manifold is not a Riemannian foliation in general; secondly, a $(H, \widetilde{H})$-harmonic map is not a priori required to be a foliated map; thirdly, a horizontal variational vector field is not necessarily foliated too. There-fore the starting points for their definitions are different, though they may coincide for foliated maps between Sasakian manifolds. Actually we will see that $(H, \widetilde{H})$-harmonic maps may display some geometric phenomenon which are invisible from transversally harmonic maps.

In this paper, we will study some basic geometric properties and problems for $(H, \widetilde{H})$-harmonic maps such as Bochner-type, Lichnerowicz-type and Eells-Sampson-type, Siu type holomorphicity results, etc. Our main aim is to utilize $(H, \widetilde{H})$-harmonic maps to establish Siu type strong rigidity theorems for Sasakian manifolds. The paper is organized as follows. Section 1 begins to recall some basic facts and notions of pseudo-Hermitian geometry, including some properties of the curvature tensor of a pseudo-Hermitian manifold. Next, we introduce the notions of strongly negative or strongly semi-negative horizontal curvature, including adequately negative horizontal curvature, for Sasakian manifolds. Some model Sasakian spaces with either strongly negative or adequately negative horizontal curvature are given. In Section 2, we introduce the

second fundamental form $\beta(\cdot,\cdot)$ with respect to the Tanaka-Webster connections for a map f between two pseudo-Hermitian manifolds, and derive the commutation relations of its covariant derivatives. In Section 3, we recall the definition of a $(H,\widetilde{H})$-harmonic map and the corresponding Euler-Lagrange equation (see (3.4))

$$(0.1) \qquad\qquad \tau_{H,\widetilde{H}}(f) = 0,$$

where $\tau_{H,\widetilde{H}}(f)$ is called the horizontal tension field of f. Then some relationship among $(H,\widetilde{H})$-harmonic maps, pseudo-harmonic maps and harmonic maps are discussed. It turns out that although $(H,\widetilde{H})$-harmonic maps have many nice properties related to the pseudo-Hermitian structures, the PDE system (0.1) is too degenerate and thus its solutions may not be regular enough to detect global geometric properties, including the strong rigidity, of pseudo-Hermitian manifolds. Actually transversally harmonic maps between Riemannian foliations also have similar drawbacks. In order to repair these drawbacks, we define a special kind of $(H,\widetilde{H})$-harmonic maps as follows. For a map $f : M \to N$ between two pseudo-Hermitian manifolds, we set $\tau_H(f) = \tau_{H,\widetilde{H}}(f) + \tau_{H,\widetilde{L}}(f)$, where $\tau_{H,\widetilde{L}}(f)$ is the vertical component of $tr_{g_\theta}(\beta|_H)$. For our purpose, we introduce a nonlinear subelliptic system of equations

$$(0.2) \qquad\qquad \tau_H(f) = 0,$$

imposed on the map f. Since (0.2) implies (0.1), a solution of (0.2) is referred to as a special $(H,\widetilde{H})$-harmonic map (see Definition 3.2). Special $(H,\widetilde{H})$-harmonic maps will play an important role in our studying of the strong rigidity for Sasakian manifolds. In Section 4, we derive a CR Bochner formula for the horizontal energy density $|df_{H,\widetilde{H}}|^2$, whose main difficulty in applications comes from a mixed term consisting of some contractions of $df_{H,\widetilde{H}}$ and $\beta(\cdot,\xi)$. In order to deal with this term, we introduce a Paneitz type operator acting on the map, which enables us to refine the Bochner formula. As a result, we are able to establish some Bochner type theorems for $(H,\widetilde{H})$-harmonic maps. In Section 5, we first define the notions of $(H,\widetilde{H})$-pluriharmonic maps, $(H,\widetilde{H})$-holomorphic maps and $(H,\widetilde{H})$-biholomorphisms. It turns out that foliated $(H,\widetilde{H})$-holomorphic maps are $(H,\widetilde{H})$-pluriharmonic, and $(H,\widetilde{H})$-pluriharmonic maps are foliated $(H,\widetilde{H})$-harmonic. Next, we give a unique continuation theorem which asserts that a foliated $(H,\widetilde{H})$-harmonic map between two Sasakian manifolds must be $(H,\widetilde{H})$-holomorphic on the whole manifold if it is $(H,\widetilde{H})$-holomorphic on an open subset. Some examples of $(H,\widetilde{H})$-holomorphic maps are also given. From [BGS], we know that for a given Sasakian structure $S = (\xi,\theta,J,g_\theta)$ on M, the Reeb vector field ξ polarizes the Sasakian manifold (M,S), and the space $S(\xi,J_\upsilon)$ of all Sasakian structures with the fixed Reeb vector field ξ and the fixed transverse holomorphic structure J_υ on $\upsilon(F_\xi)$ is an affine space. We show that $id_M : (M,S_1) \to (M,S_2)$ for any $S_1, S_2 \in S(\xi,J_\upsilon)$ is a foliated $(H,\widetilde{H})$-biholomorphism. In addition, we discuss the case when $id_M : (M,S_1) \to (M,S_2)$ is a special $(H,\widetilde{H})$-biholomorphism. In Section 6, we obtain a Lichnerowicz type result which asserts that the difference of horizontal partial energies for a foliated map is a smooth foliated homotopy invariant. As an application, we deduce

that a foliated $(H, \widetilde{H})$-holomorphic map between two pseudo-Hermitian manifolds is an absolute minimum of the horizontal energy $E_{H,\widetilde{H}}(f)$. In Section 7, we study the existence problem for (0.2) by looking at the following subelliptic heat flow

(0.3)
$$\begin{cases} \frac{\partial f_t}{\partial t} & = \tau_H(f_t) \\ f|_{t=0} & = h \end{cases}$$

where $h : M \to N$ is a smooth map. In order to show that a solution of this system exists for all $t > 0$ and converges to a solution of (0.2) as $t \to \infty$, we impose a non-positivity condition on the horizontal curvature of N. The main result of this section asserts that if $h : M \to N$ is a foliated map between two compact Sasakian manifolds and N is regular with non-positive horizontal sectional curvature, then there exists a foliated special $(H, \widetilde{H})$-harmonic map in the same foliated homotopy class as h. In Section 8, we first give a foliated rigidity result which states that if $f : M \to N$ is a $(H, \widetilde{H})$-harmonic map between two compact Sasakian manifolds and the target N has non-positive horizontal curvature, then f must be foliated. Next, we obtain a $(H, \widetilde{H})$-holomorphicity result which asserts that a $(H, \widetilde{H})$-harmonic map of sufficiently high maximum rank of a compact Sasakian manifold to a compact Sasakian manifold with either strongly negative horizontal curvature or adequately negative horizontal curvature must be $(H, \widetilde{H})$-holomorphic or $(H, \widetilde{H})$-antiholomorphic. Besides, we establish some foliated strong rigidity theorems for Sasakian manifolds with either strongly negative horizontal curvature or adequately negative horizontal curvature (see Theorem 8.12 and Corollary 8.13). In Appendix A, we introduce another natural generalized harmonic maps between pseudo-Hermitian manifolds, called pseudo-Hermitian harmonic maps. First, we give a continuation theorem about the foliated property for pseudo-Hermitian harmonic maps. Next, we obtain a rigidity result which asserts that if $f : M \to N$ is a pseudo-Hermitian harmonic map between two compact Sasakian manifolds, and N has non-positive horizontal curvature, then f is a foliated special $(H, \widetilde{H})$-harmonic map. The latter result shows the rationality for using special $(H, \widetilde{H})$-harmonic maps as a tool in our study of global geometric and topological properties of Sasakian manifolds. In Appendix B, we give explicit formulations for both (0.2) and (0.3) , which are helpful for us to understand the existence theory in Section 7. The method for these formulations is possibly useful in studying the existence of other generalized harmonic maps.

1. Pseudo-Hermitian Geometry

In this section, we collect some facts and notations concerning pseudohermitian structures on CR manifolds (cf. [DTo], [BG] for details).

Definition 1.1. Let M^{2m+1} be a real $(2m + 1)$-dimensional orientable C^∞ manifold. A CR structure on M is a complex rank m subbundle $H^{1,0}M$ of $TM \otimes C$ satisfying
(i) $H^{1,0}M \cap H^{0,1}M = \{0\}$ $(H^{0,1}M = \overline{H^{1,0}M})$;
(ii) $[\Gamma(H^{1,0}M), \Gamma(H^{1,0}M)] \subseteq \Gamma(H^{1,0}M)$.
The pair $(M, H^{1,0}M)$ is called a CR manifold of CR dimension m.

The complex subbundle $H^{1,0}M$ corresponds to a real subbundle of TM:

(1.1)
$$H(M) = Re\{H^{1,0}M \oplus H^{0,1}M\}$$

which is called the Levi distribution. The Levi distribution $H(M)$ admits a natural complex structure defined by $J(V + \overline{V}) = i(V - \overline{V})$ for any $V \in H^{1,0}M$. Equivalently, the CR structure may be described by the pair $(H(M), J)$.

Let E be the conormal bundle of $H(M)$ in T^*M, whose fiber at each point $x \in M$ is given by

$$(1.2) \qquad E_x = \{\omega \in T_x^*M : \ker \omega \supseteq H_x(M)\}.$$

Since M is assumed to be orientable, and the complex structure J induces an orientation on $H(M)$, it follows that the real line bundle E is orientable. Thus E admits globally defined nowhere vanishing sections.

Definition 1.2. A globally defined nowhere vanishing section $\theta \in \Gamma(E)$ is called a pseudo-Hermitian structure on M. The Levi-form L_θ associated with a pseudo-Hermitian structure θ is defined by

$$(1.3) \qquad L_\theta(X, Y) = d\theta(X, JY)$$

for any $X, Y \in H(M)$. If L_θ is positive definite for some θ, then $(M, H(M), J)$ is said to be strictly pseudoconvex.

When $(M, H(M), J)$ is strictly pseudoconvex, it is natural to orient E by declaring a section θ to be positive if L_θ is positive. Henceforth we will assume that $(M, H(M), J)$ is a strictly pseudoconvex CR manifold and θ is a positive pseudo-Hermitian structure. The quadruple $(M, H(M), J, \theta)$ is called a pseudo-Hermitian manifold.

On a pseudo-Hermitian manifold $(M, H(M), J, \theta)$, one may use basic linear algebra to derive that $\ker \theta_x = H_x(M)$ for each point $x \in M$, and there is a unique globally defined vector field ξ such that

$$(1.4) \qquad \theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$$

The vector field ξ is referred to as the Reeb vector field. The collection of all its integral curves forms an oriented one-dimensional foliation F_ξ on M, which is called the Reeb foliation in this paper. Consequently there is a splitting of the tangent bundle TM

$$(1.5) \qquad TM = H(M) \oplus L_\xi,$$

where L_ξ is the trivial line bundle generated by ξ. Let $\nu(F_\xi)$ be the vector bundle whose fiber at each point $p \in M$ is the quotient space T_pM/L_ξ, and let $\pi_\nu : TM \to \nu(F_\xi)$ be the natural projection. Clearly $\pi_\nu|_{H(M)} : H(M) \to \nu(F_\xi)$ is a vector bundle isomorphism.

Let $\pi_H : TM \to H(M)$ denote the natural projection morphism. Set $G_\theta = \pi_H^* L_\theta$, that is,

$$(1.6) \qquad G_\theta(X, Y) = L_\theta(\pi_H X, \pi_H Y)$$

for any $X, Y \in TM$. Let us extend J to a $(1,1)$-tensor field on M by requiring that

$$(1.7) \qquad J\xi = 0.$$

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Then the integrability condition (ii) in Definition 1.1 implies that G_θ is J-invariant. The Webster metric on $(M, H(M), J, \theta)$ is a Riemannian metric defined by

$$(1.8) \qquad g_\theta = \theta \otimes \theta + G_\theta.$$

It follows that

$$(1.9) \qquad \theta(X) = g_\theta(\xi, X), \quad d\theta(X, Y) = g_\theta(JX, Y)$$

for any $X, Y \in TM$. We find that (1.5) is actually an orthogonal decomposition of TM with respect to g_θ. In terms of terminology from foliation theory, $H(M)$ and L_ξ are also called the horizontal and vertical distributions respectively. Clearly $\theta \wedge (d\theta)^m$ is, up to a constant, the volume form of (M, g_θ).

On a pseudo-Hermitian manifold, we have the following canonical linear connection which preserves both the CR and the metric structures.

Proposition 1.1 ([Ta], [We]). *Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. Then there exists a unique linear connection ∇ such that*
(i) $\nabla_X \Gamma(H(M)) \subset \Gamma(H(M))$ for any $X \in \Gamma(TM)$;
(ii) $\nabla g_\theta = 0$, $\nabla J = 0$ (hence $\nabla \xi = \nabla \theta = 0$);
(iii) The torsion T_∇ of ∇ is pure, that is, for any $X, Y \in H(M)$, $T_\nabla(X, Y) = d\theta(X, Y)\xi$ and $T_\nabla(\xi, JX) + JT_\nabla(\xi, X) = 0$.

The connection ∇ in Proposition 1.1 is called the Tanaka-Webster connection. Note that the torsion of the Tanaka-Webster connection is always non-zero. The pseudo-Hermitian torsion, denoted by τ, is the TM-valued 1-form defined by $\tau(X) = T_\nabla(\xi, X)$. The anti-symmetry of T_∇ implies that $\tau(\xi) = 0$. Using (iii) of Proposition 1.1 and the definition of τ, the total torsion of the Tanaka-Webster connection may be expressed as

$$(1.10) \qquad T_\nabla(X, Y) = (\theta \wedge \tau)(X, Y) + d\theta(X, Y)\xi$$

for any $X, Y \in TM$. Set

$$(1.11) \qquad A(X, Y) = g_\theta(\tau X, Y)$$

for any $X, Y \in TM$. Then the properties of ∇ in Proposition 1.1 also imply that $\tau(H^{1,0}(M)) \subset H^{0,1}(M)$ and A is a trace-free symmetric tensor field.

Lemma 1.2 (cf. [DTo]). *The Levi-Civita connection ∇^θ of (M, g_θ) is related to the Tanaka-Webster connection by*

$$(1.12) \qquad \nabla^\theta = \nabla - (\frac{1}{2}d\theta + A)\xi + \tau \otimes \theta + \theta \odot J$$

where $(\theta \odot J)(X, Y) = \frac{1}{2}(\theta(X)JY + \theta(Y)JX)$ for any $X, Y \in TM$.

Remark 1.1. The Levi form in this paper is 2 times that one in [DTo]. Thus the coefficient of the term $\theta \odot J$ in (1.12) is different from that in Lemma 1.3 of [DTo].

Lemma 1.3. *Let $(M^{2m+1}, H(M), J, \theta)$ be a pseudo-Hermitian manifold with the associated Tanaka-Webster connection ∇. Let X and ρ be a vector field and a 1-form on M respectively. Then*

$$div(X) = \sum_{A=0}^{2m} g_\theta(\nabla_{e_A} X, e_A) \quad and \quad \delta(\rho) = -\sum_{A=0}^{2m} (\nabla_{e_A} \rho)(e_A)$$

where $\{e_A\}_{A=0,1,...,2m} = \{\xi, e_1, ..., e_{2m}\}$ is any orthonormal frame field on M. Here $div(\cdot)$ and $\delta(\cdot)$ denote the divergence and the codifferential respectively.

Proof. According to (1.12) and using the property $tr(A) = 0$, it is easy to verify that

(1.13)
$$\nabla_\xi^\theta \xi = 0, \quad \sum_{A=1}^{2m} \nabla_{e_A}^\theta e_A = \sum_{A=1}^{2m} \nabla_{e_A} e_A.$$

Since both ∇^θ and ∇ are metric connections, we may employ (1.13) and the definition of $divX$ to find

$$divX = \sum_{A=0}^{2m} g_\theta(\nabla_{e_A}^\theta X, e_A) = \sum_{A=0}^{2m} \{e_A g_\theta(X, e_A) - g_\theta(X, \nabla_{e_A}^\theta e_A)\}$$
$$= \sum_{A=0}^{2m} g_\theta(\nabla_{e_A} X, e_A).$$

Similarly the codifferential of the 1-form ρ can be computed as follows

$$\delta(\rho) = -\sum_{A=0}^{2m} (\nabla_{e_A}^\theta \rho)(e_A) = -\sum_{A=0}^{2m} \{e_A \rho(e_A) - \rho(\nabla_{e_A}^\theta e_A)\}$$
$$= -\sum_{A=0}^{2m} (\nabla_{e_A} \rho)(e_A)$$

in view of (1.13) again. $\square$

For a pseudo-Hermitian manifold $(M, H(M), J, \theta)$, the sub-Laplacian operator $\triangle_H$ on a function $u \in C^2(M)$ is defined by

$$\triangle_H u = div(\nabla_H u),$$

where $\nabla_H u = \pi_H(\nabla u)$ is the horizontal gradient of u. In terms of a local orthonormal frame field $\{e_A\}_{A=1}^{2m}$ of $H(M)$ on an open subset $U \subset M$, the sub-Laplacian can be expressed as

(1.14)
$$\triangle_H u = \sum_{A=1}^{2m} \{e_A(e_A u) - (\nabla_{e_A} e_A)u\}.$$

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The non-degeneracy of the Levi form L_θ on $H(M)$ implies that $\{e_A, [e_A, e_B]\}_{1 \le A, B \le 2m}$ spans the tangent space $T_p M$ at each point $p \in U$. From [Hö], we know that $\triangle_H$ is a hypoelliptic operator.

For simplicity, we will denote by $\langle \cdot, \cdot \rangle$ the real inner product induced by g_θ on various tensor bundles of M. Recall that the curvature tensor R of the Tanaka-Webster connection ∇ is defined by

$$(1.15) \qquad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(TM)$. Set $R(X, Y, Z, W) = \langle R(Z, W)Y, X \rangle$. Then R satisfies

$$(1.16) \qquad R(X, Y, Z, W) = -R(X, Y, W, Z) = -R(Y, X, Z, W),$$

where the second equality is because of $\nabla g_\theta = 0$. However, the symmetric property $R(X, Y, Z, W) = R(Z, W, X, Y)$ is no longer true for a general pseudo-Hermitian manifold due to the failure of the first Bianchi identity.

The curvature tensor of (M, ∇) induces a morphism $Q : \wedge^2 TM \to \wedge^2 TM$ which is determined by

$$(1.17) \qquad \langle Q(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W)$$

for any $X, Y, Z, W \in TM$. The complex extension of Q (resp. R) to a morphism from $\wedge^2 TM^C$ (resp. $\otimes^4 TM^C$) is still denoted by the same notation. For a horizontal 2 plane $\sigma = span_R\{X, Y\} \subset H(M)$, we define the horizontal sectional curvature of σ by

$$(1.18) \qquad K^H(\sigma) = \frac{\langle Q(X \wedge Y), X \wedge Y \rangle}{\langle X \wedge Y, X \wedge Y \rangle}.$$

In particular, the horizontal holomorphic sectional curvature of a horizontal holomorphic 2-plane $\sigma = span\{X, JX\} \subset H(M)$ is given by

$$(1.19) \qquad K^H_{hol}(\sigma) = \frac{\langle Q(X \wedge JX), X \wedge JX \rangle}{\langle X \wedge JX, X \wedge JX \rangle}.$$

Definition 1.3. A pseudo-Hermitian manifold $(M, H(M), J, \theta)$ is called a Sasakian manifold if its pseudo-Hermitian torsion τ is zero. If $(M, H(M), J, \theta)$ is Sasakian, then the quadruple $(\xi, \theta, J, g_\theta)$ is referred to as a Sasakian structure on M with underlying CR structure $(H(M), J|_{H(M)})$.

It turns out that if $(M, H(M), J, \theta)$ is a Sasakian manifold, then its curvature tensor satisfies the Bianchi identities. Consequently

$$(1.20) \qquad R(X, Y, Z, W) = R(Z, W, X, Y)$$

for any $X, Y, Z, W \in TM$. Since $\nabla \xi = 0$, it follows that if one of the vectors X, Y, Z and W is vertical, then

$$(1.21) \qquad R(X, Y, Z, W) = 0.$$

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Furthermore, in terms of $\nabla J = 0$ and the J-invariance of G_θ, we have

$$(1.22) \qquad R(JX, JY, Z, W) = R(X, Y, JZ, JW) = R(X, Y, Z, W)$$

for any $X, Y, Z, W \in H(M)$. When X, Y, Z and W vary in $H(M)$, $R(X, Y, Z, W)$ may be referred to as the horizontal curvature tensor, which will be denoted by R^H. Hence we discover that all curvature information of the Sasakian manifold is contained in its horizontal curvature tensor R^H which enjoys the same properties as the curvature tensor of a Kähler manifold.

For a Sasakian manifold $(M, H(M), J, \theta)$, it is known that its Reeb foliation F_ξ defines a Riemannian foliation (cf. [BG], [DT]). In addition, F_ξ is transversely holomorphic in the following sense. There is an open covering $\{U_\alpha\}$ of M together with a family of diffeomorphisms $\Phi_\alpha : U_\alpha \to (-1, 1) \times W_\alpha \subset R \times C^m$ and submersions

$$(1.23) \qquad \varphi_\alpha = \pi \circ \Phi_\alpha : U_\alpha \to W_\alpha \subset C^m$$

where $\pi : (-1, 1) \times W_\alpha \to W_\alpha$ is the natural projection, such that $\varphi_\alpha^{-1}(\varphi_\alpha(x))$ is just the integral curve of ξ in U_α passing through the point x, and when $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a biholomorphism. Such a triple $(U_\alpha, \Phi_\alpha; \varphi_\alpha)$ is called a foliated coordinate chart. For every point $x \in U_\alpha$ the differential $d\varphi_\alpha : H_x(M) \to T_{\varphi_\alpha(x)}W_\alpha$ is an isomorphism taking the complex structure J_x on $H_x(M)$ to that on $T_{\varphi_\alpha(x)}W_\alpha$.

Definition 1.4. Let M be a compact Sasakian manifold and let F_ξ be the Reeb foliation defined by ξ. Then the foliation F_ξ is said to be quasi-regular if there is a positive integer k such that each point has a foliated coordinate chart (U, φ) such that each leaf of F_ξ passes through U at most k times, otherwise it is called irregular. If F_ξ is quasi-regular with the integer $k = 1$, then the foliation is called regular.

It is known that the quasi-regular property is equivalent to the condition that all the leaves of the foliation are compact. In the quasi-regular case, the leaf space has the structure of a Kähler orbifold. In the regular case, the foliation is simple so that the Sasakian manifold can be realized as a S^1-bundle over a Kähler manifold (the Boothby-Wang fibration [BW]), and the natural projection of this fibration is actually a Riemannian submersion. In general, in the irregular case, the leaf space is not even Hausdorff.

Definition 1.5. A Sasakian manifold M^{2m+1} with the Tanaka-Webster connection ∇ is said to have strongly negative horizontal curvature (resp. strongly seminegative horizontal curvature) if

$$\langle Q(\zeta), \overline{\zeta} \rangle < 0 \quad (\text{resp.} \leq 0)$$

for any $\zeta = (Z \wedge W)^{(1,1)} \neq 0$, $Z, W \in HM^C$. Here $\overline{\zeta}$ is the complex conjugate of ζ. In addition, we say that the horizontal curvature tensor R^H of a Sasakian manifold M^{2m+1} is negative of order k if it is strongly seminegative and it enjoys the following property. If $A = (A_i^\alpha)$, $B = (B_i^\alpha)$ are any two $m \times k$ matrices ($1 \leq \alpha \leq m$, $1 \leq i \leq k$) with

$$rank \begin{pmatrix} A & B \\ B & A \end{pmatrix} = 2k$$

and if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\xi_{i\overline{j}}^{\alpha\overline{\beta}}\,\overline{\xi_{i\overline{j}}^{\delta\overline{\gamma}}} = 0$$

for all $1 \le i, j \le k$, where

$$\xi_{i\overline{j}}^{\alpha\overline{\beta}} = A_{\overline{i}}^{\alpha}\overline{B_j^{\beta}} - A_{\overline{j}}^{\alpha}\overline{B_i^{\beta}},$$

then either $A = 0$ or $B = 0$. The horizontal curvature tensor R^H is called adequately negative if it is negative of order m.

By the J-invariant property (1.22), we find that the curvature operator Q annihilates any 2-vector of type (2,0) or (0,2). Therefore strongly negativity (resp. strongly semi-negativity) of the horizontal curvature tensor implies negativity (resp. semi-negativity) of the horizontal sectional curvature.

Example 1.1.

(i) A Sasakian manifolds $(M^{2m+1}, H(M), J, \theta)$ with K_{hol}^H constant is called a Sasakian space form. For each real number λ, Webster [We] gave a model $M(\lambda)$ for the Sasakian space form with $K_{hol}^H = \lambda$. The horizontal curvature tensor of $M(\lambda)$ may be expressed as

(1.24)
$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = -\frac{\lambda}{2}(g_{\alpha\overline{\beta}}g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}}g_{\gamma\overline{\beta}})$$

with respect to any frame $\{\eta_\alpha\}$ of $H^{1,0}M$ at every point. Following the method for complex ball in [Si1], one may verify that if $\lambda < 0$, then $M(\lambda)$ has strongly negative horizontal curvature.

(ii) By a theorem of Kobayashi ([Ko]), we know that if B is a compact Hodge manifold with integral Kähler form, there exists a Sasakian manifold M, which is the total space of a Riemannian submersion over B. Suppose M^{2m+1} is a compact Sasakian manifold with a Riemannian submersion $\pi : M \to B$ over a compact Kähler manifold. Denote by ∇^θ and ∇^B the Levi-Civita connections of M and B. Recall that a basic vector field on M is one that is both horizontal and projectable. Suppose X, Y are basic vector fields on M. Denote by $\widetilde{X}, \widetilde{Y}$ the vector fields on B that are π-related to X and Y. A basic property on the connections of a Riemannian submersion (cf.[O'N], [GW]) gives that $d\pi[\pi_H(\nabla_X^\theta Y)] = (\nabla_{\widetilde{X}}\widetilde{Y}) \circ \pi$, where $\pi_H : TM \to H(M)$ is the natural projection. Since M is Sasakian, we know from Lemma 1.2 that $\pi_H(\nabla_X^\theta Y) = \nabla_X Y$. Thus $d\pi(\nabla_X Y) = (\nabla_{\widetilde{X}}\widetilde{Y}) \circ \pi$, which implies that

(1.25)
$$R(X, Y, Z, W) = R^B(d\pi(X), d\pi(Y), d\pi(Z), d\pi(W))$$

for any $X, Y, Z, W \in H(M)$, where R^B is the curvature tensor of ∇^B. Hence M has strongly curvature horizontal curvature (resp. strongly semi-negative horizontal curvature with negative order k) if and only if B has strongly negative curvature (resp. strongly semi-negative curvature with negative order k) in the sense of [Si1]. In terms of [Si1], we find that if B is a compact quotient of an irreducible symmetric bounded domain, then M has adequately negative horizontal curvature. Since the foliation of a Sasakian manifold is locally a Riemannian foliation over a Kähler manifold, the above

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discussion helps us to understand the general curvature properties of a Sasakian manifold from those of a Kähler manifold. For example, it is proved in [Si1] that if the curvature tensor of a Kähler manifold is strongly negative, then it is negative of order 2. Therefore we may conclude that if a Sasakian manifold has strongly negative horizontal curvature, then its horizontal curvature tensor is negative of order 2.

2. Second fundamental forms and their covariant derivatives

Let $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be two pseudo-Hermitian manifolds. Denote by ∇ and $\widetilde{\nabla}$ the Tanaka-Webster connections of M and N respectively. Let $f : M \to N$ be a smooth map. Then the bundle $T^*M \otimes f^{-1}TN$ has the induced connection $\nabla \otimes f^{-1}\widetilde{\nabla}$, where $f^{-1}\widetilde{\nabla}$ is the pull-back connection in $f^{-1}TN$. For simplicity, we will write $f^{-1}\widetilde{\nabla}$ as $\widetilde{\nabla}$ when the meaning is clear. The second fundamental form of f with respect to $(\nabla, \widetilde{\nabla})$ is defined by:

$$
\begin{aligned}
\beta(X, Y) &= [(\nabla \otimes f^{-1}\widetilde{\nabla})_Y df](X) \\
&= \widetilde{\nabla}_Y df(X) - df(\nabla_Y X)
\end{aligned}
$$

(2.1)

for $X, Y \in \Gamma(TM)$. In what follows, we shall use the summation convention for repeated indices.

Lemma 2.1. *Let $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a map. Then*

$$
\widetilde{\nabla}_X df(Y) - \widetilde{\nabla}_Y df(X) - df([X, Y]) = \widetilde{T}_{\widetilde{\nabla}}(df(X), df(Y))
$$

for any $X, Y \in \Gamma(TM)$, where $\widetilde{T}_{\widetilde{\nabla}}$ denotes the torsion of the Tanaka-Webster connection $\widetilde{\nabla}$ on N.

Proof. Set $S(X, Y) = \widetilde{\nabla}_X df(Y) - \widetilde{\nabla}_Y df(X) - df([X, Y])$. It is easy to show that S is $C^\infty(M)$-bilinear. Choose a local coordinate chart (x^A) around p and a local coordinate chart $(u^{\widetilde{A}})$ around $f(p)$. Then

$$
df\left(\frac{\partial}{\partial x^A}\right) = \frac{\partial f^{\widetilde{C}}}{\partial x^A} \frac{\partial}{\partial u^{\widetilde{C}}}
$$

(2.2)

where $f^{\widetilde{C}} = u^{\widetilde{C}} \circ f$. By the definition of $f^{-1}\widetilde{\nabla}$ and using (2.2) , we deduce that

$$
\begin{aligned}
S\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right) &= \widetilde{\nabla}_{\frac{\partial}{\partial x^A}} df\left(\frac{\partial}{\partial x^B}\right) - \widetilde{\nabla}_{\frac{\partial}{\partial x^B}} df\left(\frac{\partial}{\partial x^A}\right) \\
&= \widetilde{\nabla}_{\frac{\partial}{\partial x^A}}\left(\frac{\partial f^{\widetilde{C}}}{\partial x^B} \frac{\partial}{\partial u^{\widetilde{C}}}\right) - \widetilde{\nabla}_{\frac{\partial}{\partial x^B}}\left(\frac{\partial f^{\widetilde{C}}}{\partial x^A} \frac{\partial}{\partial u^{\widetilde{C}}}\right) \\
&= \frac{\partial f^{\widetilde{C}}}{\partial x^B} \widetilde{\nabla}_{\frac{\partial}{\partial x^A}} \frac{\partial}{\partial u^{\widetilde{C}}} - \frac{\partial f^{\widetilde{C}}}{\partial x^A} \widetilde{\nabla}_{\frac{\partial}{\partial x^B}} \frac{\partial}{\partial u^{\widetilde{C}}} \\
&= \frac{\partial f^{\widetilde{C}}}{\partial x^B} \frac{\partial f^{\widetilde{D}}}{\partial x^A}\left[\widetilde{\nabla}_{\frac{\partial}{\partial u^{\widetilde{D}}}} \frac{\partial}{\partial u^{\widetilde{C}}} - \widetilde{\nabla}_{\frac{\partial}{\partial u^{\widetilde{C}}}} \frac{\partial}{\partial u^{\widetilde{D}}}\right] \\
&= \widetilde{T}_{\widetilde{\nabla}}\left(df\left(\frac{\partial}{\partial x^A}\right), df\left(\frac{\partial}{\partial x^B}\right)\right).
\end{aligned}
$$

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Hence this lemma is proved. □

We will use the moving frame method to perform local computations on maps between pseudo-Hermitian manifolds. Let us now recall the structure equations of the Tanaka-Webster connection on a pseudo-Hermitian manifold. For the pseudo-Hermitian manifold $(M^{2m+1}, H(M), J, \theta)$, we choose a local orthonormal frame field $\{e_A\}_{A=0}^{2m} = \{\xi, e_1, ..., e_m, e_{m+1}, ..., e_{2m}\}$ with respect to g_θ such that

$$\{e_{m+1}, ..., e_{2m}\} = \{Je_1, ..., Je_m\}.$$

Set

(2.3)
$$\eta_j = \frac{1}{\sqrt{2}}(e_j - iJe_j), \quad \eta_{\bar{j}} = \frac{1}{\sqrt{2}}(e_j + iJe_j) \quad (j = 1, ..., m).$$

Then $\{\xi, \eta_j, \eta_{\bar{j}}\}$ forms a frame field of $TM \otimes C$. Let $\{\theta, \theta^j, \theta^{\bar{j}}\}$ be the dual frame field of $\{\xi, \eta_j, \eta_{\bar{j}}\}$. From Proposition 1.1, we have

(2.4)
$$\nabla_X \xi = 0, \quad \nabla_X \eta_j = \theta_j^i(X)\eta_i, \quad \nabla_X \eta_{\bar{j}} = \theta_{\bar{j}}^{\bar{i}}(X)\eta_{\bar{i}}$$

for any $X \in TM$, where $\{\theta_0^0 = \theta_0^i = \theta_0^{\bar{i}} = \theta_i^0 = \theta_{\bar{i}}^0 = 0, \theta_j^i, \theta_{\bar{j}}^{\bar{i}}\}$ are the connection 1-forms of ∇ with respect to the frame field $\{\xi, \eta_j, \eta_{\bar{j}}\}$. According to (iii) of Proposition 1.1, the pseudo-Hermitian torsion may be expressed as

(2.5)
$$\tau = A_k^{\bar{j}}\theta^k \otimes \eta_{\bar{j}} + A_{\bar{k}}^j\theta^{\bar{k}} \otimes \eta_j.$$

The symmetry of A implies that $A_{jk} = A_{kj} = A_k^{\bar{j}}$. Since ∇ preserves $H^{1,0}M$, we may write

(2.6)
$$R(\eta_k, \eta_{\bar{l}})\eta_j = R_{jk\bar{l}}^i\eta_i.$$

From [We], we know that $\{\theta, \theta^i, \theta^{\bar{i}}, \theta_j^i, \theta_{\bar{j}}^{\bar{i}}\}$ satisfies the following structure equations (cf. also §1.4 of [DTo]):

(2.7)
$$\begin{cases} d\theta & = \sqrt{-1}\theta^j \wedge \theta^{\bar{j}} \\ d\theta^i & = -\theta_j^i \wedge \theta^j + A_{\bar{j}}^i\theta \wedge \theta^{\bar{j}} \\ d\theta_j^i & = -\theta_k^i \wedge \theta_j^k + \Psi_j^i \end{cases}$$

where

(2.8)
$$\Psi_j^i = W_{jk}^i\theta^k \wedge \theta - W_{j\bar{k}}^i\theta^{\bar{k}} \wedge \theta + \sqrt{-1}\theta^i \wedge A_k^{\bar{j}}\theta^k$$
$$- \sqrt{-1}A_{\bar{k}}^i\theta^{\bar{k}} \wedge \theta^{\bar{j}} + R_{jk\bar{l}}^i\theta^k \wedge \theta^l$$

and

(2.9)
$$W_{kl}^j = A_{kl,\bar{j}}, \quad W_{k\bar{l}}^j = A_{\bar{l}\bar{j},k}.$$

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Let $\{\widetilde{\xi}, \widetilde{\eta}_\alpha, \widetilde{\eta}_{\overline{\alpha}}\}_{\alpha=1,\ldots,n}$ be a local frame field on the pseudo-Hermitian manifold N^{2n+1}, and let $\{\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}^{\overline{\alpha}}\}_{\alpha=1,\ldots,n}$ be its dual frame field. We will denote the connection 1-forms, torsion and curvature, etc., of the Tanaka-Webster connection $\widetilde{\nabla}$ on N by the same notations as in M, but with $\widetilde{}$ on them. Then similar structure equations for $\widetilde{\nabla}$ are valid in N too. Henceforth we shall make use of the following convention on the ranges of indices:

$$A, B, C, \ldots = 0, 1, \ldots, m, \overline{1}, \ldots, \overline{m};$$
$$i, j, k, \ldots = 1, \ldots, m, \quad \overline{i}, \overline{j}, \overline{k}, \ldots = \overline{1}, \ldots, \overline{m};$$
$$\widetilde{A}, \widetilde{B}, \widetilde{C}, \ldots = 0, 1, \ldots, n, \overline{1}, \ldots, \overline{n};$$
$$\alpha, \beta, \gamma, \ldots = 1, \ldots, n, \quad \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \ldots = \overline{1}, \ldots, \overline{n}.$$

As usual repeated indices are summed over the respective ranges.

For a map $f : M \to N$, we express its differential as

(2.10) $$df = f^{\widetilde{A}}_B \theta^B \otimes \widetilde{\eta}_{\widetilde{A}}.$$

Therefore

(2.11)
$$\begin{cases} f^*\widetilde{\theta} & = f^0_0 \theta + f^0_j \theta^j + f^0_{\overline{j}} \theta^{\overline{j}} \\ f^*\widetilde{\theta}^\alpha & = f^\alpha_0 \theta + f^\alpha_j \theta^j + f^\alpha_{\overline{j}} \theta^{\overline{j}} \\ f^*\widetilde{\theta}^{\overline{\alpha}} & = f^{\overline{\alpha}}_0 \theta + f^{\overline{\alpha}}_j \theta^j + f^{\overline{\alpha}}_{\overline{j}} \theta^{\overline{j}}. \end{cases}$$

By taking the exterior derivative of the first equation in (2.11) and making use of the structure equations in M and N, we get

(2.12)
$$Df^0_0 \wedge \theta + Df^0_j \wedge \theta^j + Df^0_{\overline{j}} \wedge \theta^{\overline{j}} + if^0_0 \theta^j \wedge \theta^{\overline{j}}$$
$$+ f^0_j \theta \wedge A^j_{\overline{k}} \theta^{\overline{k}} + f^0_{\overline{j}} \theta \wedge A^{\overline{j}}_k \theta^k - if^*\widetilde{\theta}^\alpha \wedge f^*\widetilde{\theta}^{\overline{\alpha}} = 0$$

where

(2.13)
$$Df^0_0 = df^0_0 = f^0_{00} \theta + f^0_{0j} \theta^j + f^0_{0\overline{j}} \theta^{\overline{j}}$$
$$Df^0_j = df^0_j - f^0_k \theta^k_j = f^0_{j0} \theta + f^0_{jl} \theta^l + f^0_{j\overline{l}} \theta^{\overline{l}}$$
$$Df^0_{\overline{j}} = df^0_{\overline{j}} - f^0_{\overline{k}} \theta^{\overline{k}}_{\overline{j}} = f^0_{\overline{j}0} \theta + f^0_{\overline{j}l} \theta^l + f^0_{\overline{j}\overline{l}} \theta^{\overline{l}}.$$

Then (2.12) gives

(2.14)
$$f^0_{j0} - f^0_{0j} + f^0_{\overline{k}} A^{\overline{k}}_j = i(f^\alpha_0 f^{\overline{\alpha}}_j - f^{\overline{\alpha}}_0 f^\alpha_j)$$
$$f^0_{\overline{j}0} - f^0_{0\overline{j}} + f^0_k A^k_{\overline{j}} = i(f^\alpha_0 f^{\overline{\alpha}}_{\overline{j}} - f^{\overline{\alpha}}_0 f^\alpha_{\overline{j}})$$
$$f^0_{jl} - f^0_{lj} = i(f^{\overline{\alpha}}_j f^\alpha_l - f^\alpha_j f^{\overline{\alpha}}_l)$$
$$f^0_{\overline{j}\overline{l}} - f^0_{\overline{l}\overline{j}} = i(f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{\overline{l}} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{\overline{l}})$$
$$f^0_{j\overline{l}} - f^0_{\overline{l}j} - if^0_0 \delta^l_j = i(f^{\overline{\alpha}}_j f^\alpha_{\overline{l}} - f^\alpha_j f^{\overline{\alpha}}_{\overline{l}}).$$

To simplify the notations, we will set $\widehat{\theta}^\alpha_\beta = f^*\widetilde{\theta}^\alpha_\beta$, $\widehat{A}^\alpha_\beta = f^*\widetilde{A}^\alpha_\beta$, $\widehat{\Psi}^\alpha_\beta = f^*\widetilde{\Psi}^\alpha_\beta$, etc. Similar computations for the second equation in (2.11) yield

(2.15)
$$Df^\alpha_0 \wedge \theta + Df^\alpha_j \wedge \theta^j + Df^\alpha_{\overline{j}} \wedge \theta^{\overline{j}} + if^\alpha_0 \theta^j \wedge \theta^{\overline{j}}$$
$$+ f^\alpha_j A^j_{\overline{k}} \theta \wedge \theta^{\overline{k}} + f^\alpha_{\overline{j}} A^{\overline{j}}_k \theta \wedge \theta^k = \widehat{A}^\alpha_\beta f^*\widetilde{\theta} \wedge f^*\widetilde{\theta}^{\overline{\beta}}$$

where

(2.16)
$$Df^\alpha_0 = df^\alpha_0 + f^\beta_0 \widehat{\theta}^\alpha_\beta = f^\alpha_{00}\theta + f^\alpha_{0j}\theta^j + f^\alpha_{0\overline{j}}\theta^{\overline{j}},$$
$$Df^\alpha_j = df^\alpha_j - f^\alpha_k \theta^k_j + f^\beta_j \widehat{\theta}^\alpha_\beta = f^\alpha_{j0}\theta + f^\alpha_{jl}\theta^l + f^\alpha_{j\overline{l}}\theta^{\overline{l}},$$
$$Df^\alpha_{\overline{j}} = df^\alpha_{\overline{j}} - f^\alpha_{\overline{k}}\theta^{\overline{k}}_{\overline{j}} + f^\beta_{\overline{j}}\widehat{\theta}^\alpha_\beta = f^\alpha_{\overline{j}0}\theta + f^\alpha_{\overline{j}l}\theta^l + f^\alpha_{\overline{j}\overline{l}}\theta^{\overline{l}}.$$

From (2.15), it follows that

(2.17)
$$f^\alpha_{0j} - f^\alpha_{j0} - f^\alpha_{\overline{k}}A^{\overline{k}}_j = \widehat{A}^\alpha_{\overline{\beta}}(f^{\overline{\beta}}_j f^{\overline{\beta}}_0 - f^0_0 f^{\overline{\beta}}_j)$$
$$f^\alpha_{0\overline{j}} - f^\alpha_{\overline{j}0} - f^\alpha_k A^k_{\overline{j}} = \widehat{A}^\alpha_{\overline{\beta}}(f^0_{\overline{j}} f^{\overline{\beta}}_0 - f^0_0 f^{\overline{\beta}}_{\overline{j}})$$
$$f^\alpha_{jl} - f^\alpha_{lj} = \widehat{A}^\alpha_{\overline{\beta}}(f^0_l f^{\overline{\beta}}_j - f^0_j f^{\overline{\beta}}_l)$$
$$f^\alpha_{\overline{j}l} - f^\alpha_{l\overline{j}} + if^\alpha_0 \delta^j_l = \widehat{A}^\alpha_{\overline{\beta}}(f^0_l f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\beta}}_l f^0_{\overline{j}})$$
$$f^\alpha_{\overline{j}\overline{l}} - f^\alpha_{\overline{l}\overline{j}} = \widehat{A}^\alpha_{\overline{\beta}}(f^0_{\overline{l}} f^{\overline{\beta}}_{\overline{j}} - f^0_{\overline{j}} f^{\overline{\beta}}_{\overline{l}}).$$

Likewise we may deduce those commutative relations of $f^{\overline{\alpha}}_{AB}$ from the third equation of (2.11) by taking its exterior derivative, or directly from (2.17) by taking bar of each index. Clearly the second fundament form β can be expressed as

(2.18)
$$\beta = f^{\widetilde{A}}_{BC}\theta^B \otimes \theta^C \otimes \widetilde{\eta}_{\widetilde{A}}.$$

Due to the torsions of the Tanaka-Webster connections, β is not symmetric. Although the non-symmetry of β causes a little trouble, we will see that it may also lead to some unexpected geometric consequences.

By taking the exterior derivative of the first equation of (2.13) and making use of the structure equations, we get

(2.19)
$$Df^0_{00} \wedge \theta + Df^0_{0j} \wedge \theta^j + Df^0_{0\overline{j}} \wedge \theta^{\overline{j}} + if^0_{00}\theta^j \wedge \theta^{\overline{j}}$$
$$+ f^0_{0j} A^j_{\overline{k}}\theta \wedge \theta^{\overline{k}} + f^0_{0\overline{j}} A^{\overline{j}}_k \theta \wedge \theta^k = 0$$

where

(2.20)
$$Df^0_{00} = df^0_{00} = f^0_{000}\theta + f^0_{00j}\theta^j + f^0_{00\overline{j}}\theta^{\overline{j}}$$
$$Df^0_{0j} = df^0_{0j} - f^0_{0k}\theta^k_j = f^0_{0j0}\theta + f^0_{0jl}\theta^l + f^0_{0j\overline{l}}\theta^{\overline{l}}$$
$$Df^0_{0\overline{j}} = df^0_{0\overline{j}} - f^0_{0\overline{k}}\theta^{\overline{k}}_{\overline{j}} = f^0_{0\overline{j}0}\theta + f^0_{0\overline{j}l}\theta^l + f^0_{0\overline{j}\overline{l}}\theta^{\overline{l}}.$$

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It follows from (2.19) that

$$f^0_{00j} - f^0_{0j0} = f^0_{0\overline{k}}A^{\overline{k}}_j$$
$$f^0_{00\overline{j}} - f^0_{0\overline{j}0} = f^0_{0k}A^k_{\overline{j}}$$
(2.21)
$$f^0_{0jl} = f^0_{0lj}$$
$$f^0_{0\overline{j}l} - f^0_{0l\overline{j}} + if^0_{00}\delta^j_l = 0$$
$$f^0_{0\overline{j}\overline{l}} = f^0_{0\overline{l}\overline{j}}.$$

Similar computations for the second equation of (2.13) give

(2.22)
$$Df^0_{j0} \wedge \theta + Df^0_{jl} \wedge \theta^l + Df^0_{j\overline{l}} \wedge \theta^{\overline{l}} + if^0_{j0}\theta^k \wedge \theta^{\overline{k}}$$
$$+ f^0_{jl}A^l_{\overline{k}}\theta \wedge \theta^{\overline{k}} + f^0_{j\overline{l}}A^{\overline{l}}_k\theta \wedge \theta^k = -f^0_k\Psi^k_j$$

where

(2.23)
$$Df^0_{j0} = df^0_{j0} - f^0_{k0}\theta^k_j = f^0_{j00}\theta + f^0_{j0l}\theta^l + f^0_{j0\overline{l}}\theta^{\overline{l}}$$
$$Df^0_{jl} = df^0_{jl} - f^0_{kl}\theta^k_j - f^0_{jk}\theta^k_l = f^0_{jl0}\theta + f^0_{jlk}\theta^k + f^0_{jl\overline{k}}\theta^{\overline{k}}$$
$$Df^0_{j\overline{l}} = df^0_{j\overline{l}} - f^0_{k\overline{l}}\theta^k_j - f^0_{j\overline{k}}\theta^{\overline{k}}_{\overline{l}} = f^0_{j\overline{l}0}\theta + f^0_{j\overline{l}k}\theta^k + f^0_{j\overline{l}\overline{k}}\theta^{\overline{k}}.$$

Then (2.22) implies

(2.24)
$$f^0_{j0l} - f^0_{jl0} = f^0_{j\overline{k}}A^{\overline{k}}_l - f^0_k W^k_{jl}$$
$$f^0_{j0\overline{l}} - f^0_{j\overline{l}0} = f^0_{jk}A^k_{\overline{l}} + f^0_k W^k_{j\overline{l}}$$
$$f^0_{jkl} - f^0_{jlk} = \sqrt{-1}f^0_k A^{\overline{j}}_l - \sqrt{-1}f^0_l A^{\overline{j}}_k$$
$$f^0_{jk\overline{l}} - f^0_{j\overline{l}k} - f^0_{j0}\delta^l_k = f^0_t R^t_{jk\overline{l}}$$
$$f^0_{j\overline{k}\overline{l}} - f^0_{j\overline{l}\overline{k}} = \sqrt{-1}\delta^{\overline{j}}_{\overline{k}}f^0_t A^t_{\overline{l}} - \sqrt{-1}\delta^{\overline{j}}_{\overline{l}}f^0_t A^t_{\overline{k}}.$$

The commutative relations for $f^0_{\overline{j}AB}$ may be derived similarly from the third equation in (2.13) or directly from (2.24) by taking bar of each index.

By taking the exterior derivative of the first equation of (2.16), we deduce that

(2.25)
$$Df^\alpha_{00} \wedge \theta + Df^\alpha_{0j} \wedge \theta^j + Df^\alpha_{0\overline{j}} \wedge \theta^{\overline{j}} + if^\alpha_{00}\theta^j \wedge \theta^{\overline{j}}$$
$$+ f^\alpha_{0j}A^j_{\overline{k}}\theta \wedge \theta^{\overline{k}} + f^\alpha_{0\overline{j}}A^{\overline{j}}_k\theta \wedge \theta^k = f^\beta_0\widehat{\Psi}^\alpha_\beta$$

where

(2.26)
$$Df^\alpha_{00} = df^\alpha_{00} + f^\beta_{00}\widehat{\theta}^\alpha_\beta = f^\alpha_{000}\theta + f^\alpha_{00j}\theta^j + f^\alpha_{00\overline{j}}\theta^{\overline{j}}$$
$$Df^\alpha_{0j} = df^\alpha_{0j} - f^\alpha_{0k}\theta^k_j + f^\beta_{0j}\widehat{\theta}^\alpha_\beta = f^\alpha_{0j0}\theta + f^\alpha_{0jl}\theta^l + f^\alpha_{0j\overline{l}}\theta^{\overline{l}}$$
$$Df^\alpha_{0\overline{j}} = df^\alpha_{0\overline{j}} - f^\alpha_{0\overline{k}}\theta^{\overline{k}}_{\overline{j}} + f^\alpha_{0\overline{j}}\widehat{\theta}^\alpha_\beta = f^\alpha_{0\overline{j}0}\theta + f^\alpha_{0\overline{j}l}\theta^l + f^\alpha_{0\overline{j}\overline{l}}\theta^{\overline{l}}.$$

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Consequently

$$(2.27) \quad \begin{aligned} f^\alpha_{00j} - f^\alpha_{0j0} - f^\alpha_{0\overline{k}}A^{\overline{k}}_j &= f^\beta_0 \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_j f^{\overline{\delta}}_0 - f^{\overline{\delta}}_j f^\gamma_0) + f^\beta_0 \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_j f^0_0 - f^0_j f^\gamma_0) \\ &- f^\beta_0 \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_j f^0_0 - f^0_j f^{\overline{\gamma}}_0) + if^\beta_0 \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_j f^\delta_0 - f^\delta_j f^\alpha_0) - if^\beta_0 \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_j f^{\overline{\beta}}_0 - f^{\overline{\beta}}_j f^{\overline{\gamma}}_0), \end{aligned}$$

$$(2.28) \quad \begin{aligned} f^\alpha_{00\overline{j}} - f^\alpha_{0\overline{j}0} - f^\alpha_{0k}A^k_{\overline{j}} &= f^\beta_0 \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{j}} f^{\overline{\delta}}_0 - f^{\overline{\delta}}_{\overline{j}} f^\gamma_0) + f^\beta_0 \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_{\overline{j}} f^0_0 - f^0_{\overline{j}} f^\gamma_0) \\ &- f^\beta_0 \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{j}} f^0_0 - f^0_{\overline{j}} f^{\overline{\gamma}}_0) + if^\beta_0 \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_{\overline{j}} f^\delta_0 - f^\delta_{\overline{j}} f^\alpha_0) - if^\beta_0 \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{j}} f^{\overline{\beta}}_0 - f^{\overline{\beta}}_{\overline{j}} f^{\overline{\gamma}}_0), \end{aligned}$$

$$(2.29) \quad \begin{aligned} f^\alpha_{0jl} - f^\alpha_{0lj} &= f^\beta_0 \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_l f^{\overline{\delta}}_j - f^\gamma_j f^{\overline{\delta}}_l) + f^\beta_0 \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_l f^0_j - f^\gamma_j f^0_l) \\ &- f^\beta_0 \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_l f^0_j - f^{\overline{\gamma}}_j f^0_l) + if^\beta_0 \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_l f^\delta_j - f^\delta_j f^\alpha_l) - if^\beta_0 \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_l f^{\overline{\beta}}_j - f^{\overline{\gamma}}_j f^{\overline{\beta}}_l), \end{aligned}$$

$$(2.30) \quad \begin{aligned} f^\alpha_{0\overline{j}\overline{l}} - f^\alpha_{0\overline{l}\overline{j}} &= f^\beta_0 \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{l}} f^{\overline{\delta}}_{\overline{j}} - f^\gamma_{\overline{j}} f^{\overline{\delta}}_{\overline{l}}) + f^\beta_0 \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_{\overline{l}} f^0_{\overline{j}} - f^\gamma_{\overline{j}} f^0_{\overline{l}}) \\ &- f^\beta_0 \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{l}} f^0_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^0_{\overline{l}}) + if^\beta_0 \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_{\overline{l}} f^\delta_{\overline{j}} - f^\alpha_{\overline{j}} f^\delta_{\overline{l}}) - if^\beta_0 \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{l}} f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^{\overline{\beta}}_{\overline{l}}), \end{aligned}$$

$$(2.31) \quad \begin{aligned} f^\alpha_{0\overline{j}l} - f^\alpha_{0l\overline{j}} + if^\alpha_{00}\delta^j_l &= f^\beta_0 \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_l f^{\overline{\delta}}_{\overline{j}} - f^{\overline{\delta}}_l f^\gamma_{\overline{j}}) + f^\beta_0 \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_l f^0_{\overline{j}} - f^\gamma_{\overline{j}} f^0_l) \\ &- f^\beta_0 \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_l f^0_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^0_l) + if^\beta_0 \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_l f^\delta_{\overline{j}} - f^\alpha_{\overline{j}} f^\delta_l) - if^\beta_0 \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_l f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}} f^{\overline{\beta}}_l). \end{aligned}$$

Applying the exterior derivative to the second equation of (2.16), we obtain

$$(2.32) \quad \begin{aligned} &Df^\alpha_{j0} \wedge \theta + Df^\alpha_{jl} \wedge \theta^l + Df^\alpha_{j\overline{l}} \wedge \theta^{\overline{l}} + if^\alpha_{j0}\theta^k \wedge \theta^{\overline{k}} \\ &+ f^\alpha_{jl}A^l_{\overline{k}}\theta \wedge \theta^{\overline{k}} + f^\alpha_{j\overline{l}}A^{\overline{l}}_k \theta \wedge \theta^k = -f^\alpha_k \Psi^k_j + f^\beta_j \widehat{\Psi}^\alpha_\beta \end{aligned}$$

where

$$(2.33) \quad \begin{aligned} Df^\alpha_{j0} &= df^\alpha_{j0} - f^\alpha_{k0}\theta^k_j + f^\beta_{j0}\widehat{\theta}^\alpha_\beta = f^\alpha_{j00}\theta + f^\alpha_{j0l}\theta^l + f^\alpha_{j0\overline{l}}\theta^{\overline{l}} \\ Df^\alpha_{jl} &= df^\alpha_{jl} - f^\alpha_{kl}\theta^k_j - f^\alpha_{jk}\theta^k_l + f^\beta_{jl}\widehat{\theta}^\alpha_\beta = f^\alpha_{jl0}\theta + f^\alpha_{jlk}\theta^k + f^\alpha_{jl\overline{k}}\theta^{\overline{k}} \\ Df^\alpha_{j\overline{l}} &= df^\alpha_{j\overline{l}} - f^\alpha_{k\overline{l}}\theta^k_j - f^\alpha_{j\overline{k}}\theta^{\overline{k}}_{\overline{l}} + f^\beta_{j\overline{l}}\widehat{\theta}^\alpha_\beta = f^\alpha_{j\overline{l}0}\theta + f^\alpha_{j\overline{l}k}\theta^k + f^\alpha_{j\overline{l}\overline{k}}\theta^{\overline{k}}. \end{aligned}$$

Let us substitute (2.33) into (2.32) to get

$$(2.34) \quad \begin{aligned} f^\alpha_{j0l} - f^\alpha_{jl0} - f^\alpha_{j\overline{k}}A^{\overline{k}}_l &= -f^\alpha_t W^t_{jl} + f^\beta_j \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_l f^{\overline{\delta}}_0 - f^\gamma_0 f^{\overline{\delta}}_l) \\ &+ f^\beta_j \widehat{W}^\alpha_{\beta\gamma}(f^\gamma_l f^0_0 - f^\gamma_0 f^0_l) - f^\beta_j \widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_l f^0_0 - f^{\overline{\gamma}}_0 f^0_l) \\ &+ if^\beta_j \widehat{A}^{\overline{\beta}}_\delta (f^\alpha_l f^\delta_0 - f^\alpha_0 f^\delta_l) - if^\beta_j \widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_l f^{\overline{\beta}}_0 - f^{\overline{\gamma}}_0 f^{\overline{\beta}}_l), \end{aligned}$$

$$(2.35) \quad \begin{aligned} f_{j0\bar{l}}^\alpha - f_{j\bar{l}0}^\alpha - f_{jk}^\alpha A_{\bar{l}}^k &= f_t^\alpha W_{j\bar{l}}^t + f_j^\beta \widehat{R}_{\beta\gamma\bar{\delta}}^\alpha (f_{\bar{l}}^\gamma f_0^{\bar{\delta}} - f_0^\gamma f_{\bar{l}}^{\bar{\delta}}) \\ &+ f_j^\beta \widehat{W}_{\beta\gamma}^\alpha (f_{\bar{l}}^\gamma f_0^0 - f_0^\gamma f_{\bar{l}}^0) - f_j^\beta \widehat{W}_{\beta\bar{\gamma}}^\alpha (f_{\bar{l}}^{\bar{\gamma}} f_0^0 - f_0^{\bar{\gamma}} f_{\bar{l}}^0) \\ &+ i f_j^\beta \widehat{A}_\delta^{\bar{\beta}} (f_{\bar{l}}^\alpha f_0^\delta - f_0^\alpha f_{\bar{l}}^\delta) - i f_j^\beta \widehat{A}_{\bar{\gamma}}^\alpha (f_{\bar{l}}^{\bar{\gamma}} f_0^{\bar{\beta}} - f_0^{\bar{\gamma}} f_{\bar{l}}^{\bar{\beta}}), \end{aligned}$$

$$(2.36) \quad \begin{aligned} f_{jlk}^\alpha - f_{jkl}^\alpha &= i(A_k^{\bar{j}} f_l^\alpha - f_k^\alpha A_l^{\bar{j}}) + f_j^\beta \widehat{R}_{\beta\gamma\bar{\delta}}^\alpha (f_k^\gamma f_l^{\bar{\delta}} - f_l^\gamma f_k^{\bar{\delta}}) \\ &+ f_j^\beta \widehat{W}_{\beta\gamma}^\alpha (f_k^\gamma f_l^0 - f_l^\gamma f_k^0) - f_j^\beta \widehat{W}_{\beta\bar{\gamma}}^\alpha (f_k^{\bar{\gamma}} f_l^0 - f_l^{\bar{\gamma}} f_k^0) \\ &+ i f_j^\beta \widehat{A}_\delta^{\bar{\beta}} (f_k^\alpha f_l^\delta - f_l^\alpha f_k^\delta) - i f_j^\beta \widehat{A}_{\bar{\gamma}}^\alpha (f_k^{\bar{\gamma}} f_l^{\bar{\beta}} - f_l^{\bar{\gamma}} f_k^{\bar{\beta}}), \end{aligned}$$

$$(2.37) \quad \begin{aligned} f_{j\bar{l}k}^\alpha - f_{jk\bar{l}}^\alpha + i f_{j0}^\alpha \delta_{\bar{l}}^{\bar{k}} &= -f_t^\alpha R_{jk\bar{l}}^t + f_j^\beta \widehat{R}_{\beta\gamma\bar{\delta}}^\alpha (f_k^\gamma f_{\bar{l}}^{\bar{\delta}} - f_{\bar{l}}^\gamma f_k^{\bar{\delta}}) \\ &+ f_j^\beta \widehat{W}_{\beta\gamma}^\alpha (f_k^\gamma f_{\bar{l}}^0 - f_{\bar{l}}^\gamma f_k^0) - f_j^\beta \widehat{W}_{\beta\bar{\gamma}}^\alpha (f_k^{\bar{\gamma}} f_{\bar{l}}^0 - f_{\bar{l}}^{\bar{\gamma}} f_k^0) \\ &+ i f_j^\beta \widehat{A}_\delta^{\bar{\beta}} (f_k^\alpha f_{\bar{l}}^\delta - f_{\bar{l}}^\alpha f_k^\delta) - i f_j^\beta \widehat{A}_{\bar{\gamma}}^\alpha (f_k^{\bar{\gamma}} f_{\bar{l}}^{\bar{\beta}} - f_{\bar{l}}^{\bar{\gamma}} f_k^{\bar{\beta}}), \end{aligned}$$

$$(2.38) \quad \begin{aligned} f_{j\bar{l}\bar{k}}^\alpha - f_{j\bar{k}\bar{l}}^\alpha &= i f_t^\alpha (A_{\bar{k}}^t \delta_{\bar{l}}^{\bar{j}} - A_{\bar{l}}^t \delta_{\bar{k}}^{\bar{j}}) + f_j^\beta \widehat{R}_{\beta\gamma\bar{\delta}}^\alpha (f_{\bar{k}}^\gamma f_{\bar{l}}^{\bar{\delta}} - f_{\bar{l}}^\gamma f_{\bar{k}}^{\bar{\delta}}) \\ &+ f_j^\beta \widehat{W}_{\beta\gamma}^\alpha (f_{\bar{k}}^\gamma f_{\bar{l}}^0 - f_{\bar{l}}^\gamma f_{\bar{k}}^0) - f_j^\beta \widehat{W}_{\beta\bar{\gamma}}^\alpha (f_{\bar{k}}^{\bar{\gamma}} f_{\bar{l}}^0 - f_{\bar{l}}^{\bar{\gamma}} f_{\bar{k}}^0) \\ &+ i f_j^\beta \widehat{A}_\delta^{\bar{\beta}} (f_{\bar{k}}^\alpha f_{\bar{l}}^\delta - f_{\bar{l}}^\alpha f_{\bar{k}}^\delta) - i f_j^\beta \widehat{A}_{\bar{\gamma}}^\alpha (f_{\bar{k}}^{\bar{\gamma}} f_{\bar{l}}^{\bar{\beta}} - f_{\bar{l}}^{\bar{\gamma}} f_{\bar{k}}^{\bar{\beta}}). \end{aligned}$$

Next, computing the exterior derivative of the third equation of (2.16), we derive that

$$(2.39) \quad \begin{aligned} Df_{\bar{j}0}^\alpha \wedge \theta + Df_{\bar{j}l}^\alpha \wedge \theta^l + Df_{\bar{j}\bar{l}}^\alpha \wedge \theta^{\bar{l}} + i f_{\bar{j}0}^\alpha \theta^k \wedge \theta^{\bar{k}} \\ + f_{\bar{j}l}^\alpha A_{\bar{k}}^l \theta \wedge \theta^{\bar{k}} + f_{\bar{j}\bar{l}}^\alpha A_k^{\bar{l}} \theta \wedge \theta^k = -f_{\bar{k}}^\alpha \Psi_{\bar{j}}^{\bar{k}} + f_{\bar{j}}^\beta \widehat{\Psi}_\beta^\alpha \end{aligned}$$

where

$$(2.40) \quad \begin{aligned} Df_{\bar{j}0}^\alpha &= df_{\bar{j}0}^\alpha - f_{\bar{k}0}^\alpha \theta_{\bar{j}}^{\bar{k}} + f_{\bar{j}0}^\beta \widehat{\theta}_\beta^\alpha = f_{\bar{j}00}^\alpha \theta + f_{\bar{j}0l}^\alpha \theta^l + f_{\bar{j}0\bar{l}}^\alpha \theta^{\bar{l}} \\ Df_{\bar{j}l}^\alpha &= df_{\bar{j}l}^\alpha - f_{\bar{k}l}^\alpha \theta_{\bar{j}}^{\bar{k}} - f_{\bar{j}k}^\alpha \theta_l^k + f_{\bar{j}l}^\beta \widehat{\theta}_\beta^\alpha = f_{\bar{j}l0}^\alpha \theta + f_{\bar{j}lk}^\alpha \theta^k + f_{\bar{j}l\bar{k}}^\alpha \theta^{\bar{k}} \\ Df_{\bar{j}\bar{l}}^\alpha &= df_{\bar{j}\bar{l}}^\alpha - f_{\bar{k}\bar{l}}^\alpha \theta_{\bar{j}}^{\bar{k}} - f_{\bar{j}\bar{k}}^\alpha \theta_{\bar{l}}^{\bar{k}} + f_{\bar{j}\bar{l}}^\beta \widehat{\theta}_\beta^\alpha = f_{\bar{j}\bar{l}0}^\alpha \theta + f_{\bar{j}\bar{l}k}^\alpha \theta^k + f_{\bar{j}\bar{l}\bar{k}}^\alpha \theta^{\bar{k}}. \end{aligned}$$

Therefore we have

$$(2.41) \quad \begin{aligned} f_{\bar{j}0l}^\alpha - f_{\bar{j}l0}^\alpha - f_{\bar{j}k}^\alpha A_l^{\bar{k}} &= f_{\bar{t}}^\alpha W_{\bar{j}l}^{\bar{t}} + f_{\bar{j}}^\beta \widehat{R}_{\beta\gamma\bar{\delta}}^\alpha (f_l^\gamma f_0^{\bar{\delta}} - f_0^\gamma f_l^{\bar{\delta}}) \\ &+ f_{\bar{j}}^\beta \widehat{W}_{\beta\gamma}^\alpha (f_l^\gamma f_0^0 - f_0^\gamma f_l^0) - f_{\bar{j}}^\beta \widehat{W}_{\beta\bar{\gamma}}^\alpha (f_l^{\bar{\gamma}} f_0^0 - f_0^{\bar{\gamma}} f_l^0) \\ &+ i f_{\bar{j}}^\beta \widehat{A}_\delta^{\bar{\beta}} (f_l^\alpha f_0^\delta - f_0^\alpha f_l^\delta) - i f_{\bar{j}}^\beta \widehat{A}_{\bar{\gamma}}^\alpha (f_l^{\bar{\gamma}} f_0^{\bar{\beta}} - f_0^{\bar{\gamma}} f_l^{\bar{\beta}}), \end{aligned}$$

$$(2.42) \quad \begin{aligned} f^\alpha_{\overline{j}0\overline{l}} - f^\alpha_{\overline{j}\overline{l}0} - f^\alpha_{\overline{j}k}A^k_{\overline{l}} &= -f^\alpha_{\overline{t}}W^{\overline{t}}_{\overline{j}\overline{l}} + f^\beta_{\overline{j}}\widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{l}}f^{\overline{\delta}}_0 - f^\gamma_0 f^{\overline{\delta}}_{\overline{l}}) \\ &\quad + f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\gamma}(f^\gamma_{\overline{l}}f^0_0 - f^\gamma_0 f^0_{\overline{l}}) - f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{l}}f^0_0 - f^{\overline{\gamma}}_0 f^0_{\overline{l}}) \\ &\quad + if^\beta_{\overline{j}}\widehat{A}^{\overline{\beta}}_\delta(f^\alpha_{\overline{l}}f^\delta_0 - f^\alpha_0 f^\delta_{\overline{l}}) - if^\beta_{\overline{j}}\widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{l}}f^{\overline{\beta}}_0 - f^{\overline{\gamma}}_0 f^{\overline{\beta}}_{\overline{l}}), \end{aligned}$$

$$(2.43) \quad \begin{aligned} f^\alpha_{\overline{j}lk} - f^\alpha_{\overline{j}kl} &= -if^\alpha_{\overline{t}}A^{\overline{t}}_k\delta^j_l + if^\alpha_{\overline{t}}A^{\overline{t}}_l\delta^j_k + f^\beta_{\overline{j}}\widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_k f^{\overline{\delta}}_l - f^\gamma_l f^{\overline{\delta}}_k) \\ &\quad + f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\gamma}(f^\gamma_k f^0_l - f^\gamma_l f^0_k) - f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_k f^0_l - f^{\overline{\gamma}}_l f^0_k) \\ &\quad + if^\beta_{\overline{j}}\widehat{A}^{\overline{\beta}}_\delta(f^\alpha_k f^\delta_l - f^\alpha_l f^\delta_k) - if^\beta_{\overline{j}}\widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_k f^{\overline{\beta}}_l - f^{\overline{\gamma}}_l f^{\overline{\beta}}_k), \end{aligned}$$

$$(2.44) \quad \begin{aligned} f^\alpha_{\overline{j}l\overline{k}} - f^\alpha_{\overline{j}k\overline{l}} + if^\alpha_{\overline{j}0}\delta^{\overline{k}}_l &= -f^\alpha_{\overline{t}}R^{\overline{t}}_{\overline{j}k\overline{l}} + f^\beta_{\overline{j}}\widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_k f^{\overline{\delta}}_{\overline{l}} - f^\gamma_{\overline{l}} f^{\overline{\delta}}_k) \\ &\quad + f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\gamma}(f^\gamma_k f^0_{\overline{l}} - f^\gamma_{\overline{l}} f^0_k) - f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_k f^0_{\overline{l}} - f^{\overline{\gamma}}_{\overline{l}} f^0_k) \\ &\quad + if^\beta_{\overline{j}}\widehat{A}^{\overline{\beta}}_\delta(f^\alpha_k f^\delta_{\overline{l}} - f^\alpha_{\overline{l}} f^\delta_k) - if^\beta_{\overline{j}}\widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_k f^{\overline{\beta}}_{\overline{l}} - f^{\overline{\gamma}}_{\overline{l}} f^{\overline{\beta}}_k), \end{aligned}$$

$$(2.45) \quad \begin{aligned} f^\alpha_{\overline{j}\,\overline{l}k} - f^\alpha_{\overline{j}k\overline{l}} &= if^\alpha_{\overline{k}}A^j_{\overline{l}} - if^\alpha_{\overline{l}}A^j_{\overline{k}} + f^\beta_{\overline{j}}\widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{k}} f^{\overline{\delta}}_{\overline{l}} - f^\gamma_{\overline{l}} f^{\overline{\delta}}_{\overline{k}}) \\ &\quad + f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\gamma}(f^\gamma_{\overline{k}} f^0_{\overline{l}} - f^\gamma_{\overline{l}} f^0_{\overline{k}}) - f^\beta_{\overline{j}}\widehat{W}^\alpha_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{k}} f^0_{\overline{l}} - f^{\overline{\gamma}}_{\overline{l}} f^0_{\overline{k}}) \\ &\quad + if^\beta_{\overline{j}}\widehat{A}^{\overline{\beta}}_\delta(f^\alpha_{\overline{k}} f^\delta_{\overline{l}} - f^\alpha_{\overline{l}} f^\delta_{\overline{k}}) - if^\beta_{\overline{j}}\widehat{A}^\alpha_{\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{k}} f^{\overline{\beta}}_{\overline{l}} - f^{\overline{\gamma}}_{\overline{l}} f^{\overline{\beta}}_{\overline{k}}). \end{aligned}$$

Similarly the commutative relations of $f^{\overline{\alpha}}_{ABC}$ can be deduced from (2.27)-(2.31), (2.34)-(2.38) and (2.41)-(2.45) by taking bar of each index.

3. $(H, \widetilde{H})$-harmonic maps

Let $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be two pseudo-Hermitian manifolds endowed with the Tanaka-Webster connections ∇ and $\widetilde{\nabla}$ respectively. Suppose Ψ is a section of $Hom(\otimes^k TM, f^{-1}TN)$. Let $\Psi_{H,\widetilde{H}}$ be a section of $Hom(\otimes^k H(M), f^{-1}\widetilde{H}(N))$ defined by

$$(3.1) \qquad \Psi_{H,\widetilde{H}}(X_1, .., X_k) = \pi_{\widetilde{H}} \circ \Psi(X_1, ..., X_k)$$

for any $X_1, ..., X_k \in HM$, where $\pi_{\widetilde{H}} : TN \to \widetilde{H}(N)$ is the natural projection morphism. For convenience, one may extend $\Psi_{H,\widetilde{H}}$ to a section of $Hom(\otimes^k TM, f^{-1}TN)$ by requiring that $\Psi_{H,\widetilde{H}}(Z_1, .., Z_k) = 0$ if one of $Z_1, ..., Z_k \in TM$ is vertical.

For any smooth map $f : M \to N$ between the pseudo-Hermitian manifolds, Petit ([Pe]) introduced the following horizontal energy functional

$$(3.2) \qquad E_{H,\widetilde{H}}(f) = \frac{1}{2} \int_M |df_{H,\widetilde{H}}|^2 dv_\theta$$

where $dv_\theta = \theta \wedge (d\theta)^m$ and then he derived its first variational formula. Since our notations are slightly different from those in [Pe], we will derive the first variational formula of $E_{H,\widetilde{H}}$ again for the convenience of the readers.

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Proposition 3.1 ([Pe]). *Let $\{f_t\}_{|t|<\varepsilon}$ be a family of maps with $f_0 = f$ and $\frac{\partial f_t}{\partial t}|_{t=0} = v \in \Gamma(f^{-1}TN)$. Then*

$$(3.3) \qquad \frac{d}{dt}E_{H,\widetilde{H}}(f_t)|_{t=0} = -\int_M \langle v, \tau_{H,\widetilde{H}}(f) - tr_{G_\theta}(f^*\widetilde{A})_H\widetilde{\xi}\rangle dv_\theta$$

where $\tau_{H,\widetilde{H}}(f)$ is the horizontal tension field given by

$$(3.4) \qquad \tau_{H,\widetilde{H}}(f) = tr_{G_\theta}\big(\beta_{H,\widetilde{H}} + [(f^*\widetilde{\theta}) \otimes (f^*\widetilde{\tau})]_{H,\widetilde{H}}\big).$$

Proof. Let $\Phi : M \times (-\varepsilon, \varepsilon) \to N$ be the map defined by $\Phi(x,t) = f_t(x)$. A vector $X \in T_xM$ may be identified with a vector $(X,0) \in T_{(x,t)}(M \times (-\varepsilon, \varepsilon))$. This identification gives the following distribution

$$H_{(x,t)}(M \times (-\varepsilon, \varepsilon)) = span\{(X,0) : X \in H(M)\}$$

on $M \times (-\varepsilon, \varepsilon)$. Set $d\Phi_{H,\widetilde{H}} = \pi_{\widetilde{H}}(d\Phi|_{H(M\times(-\varepsilon,\varepsilon))})$. When the meaning is clear, we often write $(X,0)$ as X for simplicity.

We denote by $\widetilde{\nabla}$ the pull-back connection in $\Phi^{-1}TN$. Choose a local orthonormal frame field $\{e_A\}_{A=1}^{2m}$ of $H(M)$. Applying Lemma 2.1 and (1.10), we obtain that

$$\pi_{\widetilde{H}}\big(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\Phi(e_A) - \widetilde{\nabla}_{e_A}d\Phi(\frac{\partial}{\partial t})\big)_{t=0}$$

$$= \pi_{\widetilde{H}}\big(\widetilde{T}_{\widetilde{\nabla}}(d\Phi(\frac{\partial}{\partial t}), d\Phi(e_A))\big)_{t=0}$$

$$= \pi_{\widetilde{H}}\big(\widetilde{\theta}(v)\widetilde{\tau}(d\Phi(e_A))_{t=0} - \widetilde{\theta}(d\Phi(e_A))_{t=0}\widetilde{\tau}(v) + d\widetilde{\theta}(v, d\Phi(e_A)_{t=0})\widetilde{\xi}\big)$$

$$= \widetilde{\theta}(v)\widetilde{\tau}(df(e_A)) - \widetilde{\theta}(df(e_A))\widetilde{\tau}(v)$$

that is,

$$(3.5) \qquad [\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\Phi_{H,\widetilde{H}}(e_A)]_{t=0} = \pi_{\widetilde{H}}(\widetilde{\nabla}_{e_A}v) + \widetilde{\theta}(v)\widetilde{\tau}(df(e_A)) - \widetilde{\theta}(df(e_A))\widetilde{\tau}(v).$$

By (3.5), we compute

$$\frac{1}{2}\frac{d}{dt}|df_{tH,\widetilde{H}}|^2 \,|_{t=0} = \sum_{A=1}^{2m} g_{\widetilde{\theta}}\big(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\Phi_{H,\widetilde{H}}(e_A), d\Phi_{H,\widetilde{H}}(e_A)\big)_{t=0}$$

$$= \sum_{A=1}^{2m}\{g_{\widetilde{\theta}}\big(\widetilde{\nabla}_{e_A}v, df_{H,\widetilde{H}}(e_A)\big) + \widetilde{\theta}(v)g_{\widetilde{\theta}}\big(\widetilde{\tau}(df(e_A)), df_{H,\widetilde{H}}(e_A)\big)$$

$$- \widetilde{\theta}(df(e_A))g_{\widetilde{\theta}}\big(\widetilde{\tau}(v), df_{H,\widetilde{H}}(e_A)\big)\}$$

$$(3.6) \qquad = \sum_{A=1}^{2m}\{e_A g_{\widetilde{\theta}}\big(v, df_{H,\widetilde{H}}(e_A)\big) - g_{\widetilde{\theta}}\big(v, df_{H,\widetilde{H}}(\nabla_{e_A}e_A)\big)\}$$

$$- \sum_{A=1}^{2m}\{g_{\widetilde{\theta}}\big(v, (\widetilde{\nabla}_{e_A}df_{H,\widetilde{H}})(e_A)\big) + (f^*\widetilde{\theta})(e_A)g_{\widetilde{\theta}}\big(\widetilde{\tau}(df_{H,\widetilde{H}}(e_A)), v\big)$$

$$- (f^*\widetilde{A})(e_A, e_A)g_{\widetilde{\theta}}(\widetilde{\xi}, v)\}.$$

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Set $\alpha(X) = g_{\widetilde{\theta}}(v, df_{H,\widetilde{H}}(X))$ for any $X \in TM$. Using Lemma 1.3, we deduce that

$$\delta(\alpha) = -(\nabla_T \alpha)(T) - \sum_{A=1}^{2m} (\nabla_{e_A} \alpha)(e_A)$$

(3.7)

$$= -\sum_{A=1}^{2m} [e_A g_{\widetilde{\theta}}(v, df_{H,\widetilde{H}}(e_A)) - g_{\widetilde{\theta}}(v, df_{H,\widetilde{H}}(\nabla_{e_A} e_A))].$$

It follows from (3.6), (3.7) and the divergence theorem that

(3.8) $$\frac{d}{dt} E_{H,\widetilde{H}}(f_t)|_{t=0} = -\sum_{A=1}^{2m} \int_M \{g_{\widetilde{\theta}}(v, (\widetilde{\nabla}_{e_A} df_{H,\widetilde{H}})(e_A) + (f^*\widetilde{\theta})(e_A)\widetilde{\tau}(df_{H,\widetilde{H}}(e_A)))$$
$$- (f^*\widetilde{A})(e_A, e_A)g_{\widetilde{\theta}}(\widetilde{\xi}, v)\}dv_\theta.$$

Note that $\widetilde{\tau}$ is a $\widetilde{H}(N)$-valued 1-form and $\widetilde{\tau}(\widetilde{\xi}) = 0$. Thus

(3.9) $$\sum_{A=1}^{2m} (f^*\widetilde{\theta})(e_A)\widetilde{\tau}(df_{H,\widetilde{H}}(e_A)) = tr_{G_\theta}(f^*\widetilde{\theta} \otimes f^*\widetilde{\tau})_H$$
$$= tr_{G_\theta}(f^*\widetilde{\theta} \otimes f^*\widetilde{\tau})_{H,\widetilde{H}}.$$

Therefore (3.8) and (3.9) complete the proof of this proposition. $\square$

According to Proposition 3.1, $f : M \to N$ is a critical point of $E_{H,\widetilde{H}}$ if and only if

(3.10) $$\tau_{H,\widetilde{H}}(f) = 0 \quad \text{and} \quad tr_{G_\theta}(f^*\widetilde{A})_H = 0.$$

The critical point is referred to as a pseudoharmonic map in [Pe]. However, there is another kind of critical maps, which is also called a pseudoharmonic map (see [BDU]). To avoid any possible confusion, we modify Petit's definition slightly to introduce the following

Definition 3.1. A map $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is said to be $(H, \widetilde{H})$-harmonic if $D_v E_{H,\widetilde{H}}(f) = 0$ for any $v \in \Gamma(f^{-1}\widetilde{H}(N))$.

By Proposition 3.1, we have

Corollary 3.2. Let $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a map. Then f is $(H, \widetilde{H})$-harmonic if and only if $\tau_{H,\widetilde{H}}(f) = 0$, that is,

(3.11) $$tr_{G_\theta}\{\beta_{H,\widetilde{H}}(f) + (f^*\widetilde{\theta} \otimes f^*\widetilde{\tau})_{H,\widetilde{H}}\} = 0.$$

Remark 3.1. We see from (3.10) that pseudoharmonic maps in the sense of [Pe2] require an extra condition on the pull-back pseudo-Hermitian torsion. If the target manifold is Sasakian, then (3.3) implies that $D_v E_{H,\widetilde{H}}(f) = 0$ automatically for any vertical variation

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field v along f. Consequently, Petit's pseudoharmonic maps coincide with ours in this special case.

In terms of the notations in §2, $\beta_{H,\widetilde{H}}$ and $\left(f^*\widetilde{\theta}\otimes f^*\widetilde{\tau}\right)_{H,\widetilde{H}}$ may be expressed as follows

$$
\begin{aligned}
(3.12)\qquad \beta_{H,\widetilde{H}} &= f^\alpha_{ij}\theta^i\otimes\theta^j\otimes\widetilde{\eta}_\alpha + f^\alpha_{i\overline{j}}\theta^i\otimes\theta^{\overline{j}}\otimes\widetilde{\eta}_\alpha + f^\alpha_{\overline{i}j}\theta^{\overline{i}}\otimes\theta^j\otimes\widetilde{\eta}_\alpha \\
&\quad + f^\alpha_{\overline{ij}}\theta^{\overline{i}}\otimes\theta^{\overline{j}}\otimes\widetilde{\eta}_\alpha + f^{\overline{\alpha}}_{ij}\theta^i\otimes\theta^j\otimes\widetilde{\eta}_{\overline{\alpha}} + f^{\overline{\alpha}}_{\overline{i}j}\theta^{\overline{i}}\otimes\theta^j\otimes\widetilde{\eta}_{\overline{\alpha}} \\
&\quad + f^{\overline{\alpha}}_{i\overline{j}}\theta^i\otimes\theta^{\overline{j}}\otimes\widetilde{\eta}_{\overline{\alpha}} + f^{\overline{\alpha}}_{\overline{ij}}\theta^{\overline{i}}\otimes\theta^{\overline{j}}\otimes\widetilde{\eta}_{\overline{\alpha}}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.13)\qquad \left(f^*\widetilde{\theta}\otimes f^*\widetilde{\tau}\right)_{H,\widetilde{H}} &= \widehat{A}^\alpha_\beta(f^0_j f^{\overline{\beta}}_k\theta^j\otimes\theta^k + f^0_j f^{\overline{\beta}}_{\overline{k}}\theta^j\otimes\theta^{\overline{k}})\otimes\widetilde{\eta}_\alpha \\
&\quad + \widehat{A}^\alpha_\beta(f^0_{\overline{j}} f^{\overline{\beta}}_k\theta^{\overline{j}}\otimes\theta^k + f^0_{\overline{j}} f^{\overline{\beta}}_{\overline{k}}\theta^{\overline{j}}\otimes\theta^{\overline{k}})\otimes\widetilde{\eta}_\alpha \\
&\quad + \widehat{A}^{\overline{\alpha}}_\beta(f^0_j f^{\beta}_k\theta^j\otimes\theta^k + f^0_j f^{\beta}_{\overline{k}}\theta^j\otimes\theta^{\overline{k}})\otimes\widetilde{\eta}_{\overline{\alpha}} \\
&\quad + \widehat{A}^{\overline{\alpha}}_\beta(f^0_{\overline{j}} f^{\beta}_k\theta^{\overline{j}}\otimes\theta^k + f^0_{\overline{j}} f^{\beta}_{\overline{k}}\theta^{\overline{j}}\otimes\theta^{\overline{k}})\otimes\widetilde{\eta}_{\overline{\alpha}}.
\end{aligned}
$$

Hence (3.11) is equivalent to

$$
(3.14)\qquad f^\alpha_{k\overline{k}} + f^\alpha_{\overline{k}k} + \widehat{A}^\alpha_\beta f^0_k f^{\overline{\beta}}_{\overline{k}} + \widehat{A}^\alpha_\beta f^0_{\overline{k}} f^{\overline{\beta}}_k = 0.
$$

Recall that a map $f : (M, H(M), J, \theta) \rightarrow (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is said to be horizontal if $df(H(M)) \subset \widetilde{H}(N)$ (cf. [Pe2]), or equivalently, $f^*\widetilde{\theta} = u\theta$ for some $u \in C^\infty(M)$. It follows from (3.14) that

Corollary 3.3. *Let $f : (M, H(M), J, \theta) \rightarrow (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a map. Suppose that either f is horizontal or N is Sasakian. Then f is $(H, \widetilde{H})$-harmonic if and only if*

$$
f^\alpha_{k\overline{k}} + f^\alpha_{\overline{k}k} = 0.
$$

We introduce the following special kind of $(H, \widetilde{H})$-harmonic map, which will be an important tool for establishing rigidity results in this paper.

Definition 3.2. Let $f : (M, H(M), J, \theta) \rightarrow (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a map between two pseudo-Hermitian manifolds. We say that f is a special $(H, \widetilde{H})$-harmonic map if it is a $(H, \widetilde{H})$-harmonic map with the following additional property

$$
(3.15)\qquad f^0_{k\overline{k}} + f^0_{\overline{k}k} = 0.
$$

Note that if f is horizontal, then $f^0_k = f^0_{\overline{k}} = 0$, and thus $f^0_{k\overline{k}} + f^0_{\overline{k}k} = 0$. As a result, the map f is a special $(H, \widetilde{H})$-harmonic map if it is both horizontal and $(H, \widetilde{H})$-harmonic. Nevertheless, a $(H, \widetilde{H})$-harmonic map is not necessarily horizontal (see Example 5.2). We will see that the special condition (3.15) can not only enhance the regularity of a

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$(H, \widetilde{H})$-harmonic map, but also remove superfluous data for parameterizing all foliated $(H, \widetilde{H})$-biholomorphisms between two pseudo-Hermitian manifolds.

Let (N, g) be a Riemannian manifold and let ∇^g denote its Levi-Civita connection. For a map $f : (M, H(M), J, \theta) \to (N, g)$ from a compact pseudo-Hermitian manifold to the Riemannian manifold (N, g), we may define a horizontal energy for f by

$$(3.16) \qquad E_H(f) = \frac{1}{2} \int_M \sum_{A=1}^{2m} \langle df(e_A), df(e_A) \rangle dv_\theta$$

where $\{e_A\}_{A=1}^{2m}$ is any orthonormal basis in $H(M)$. According to [BDU], a critical map of the energy E_H is called pseudoharmonic. Let us define the following second fundamental form (with respect to the data (∇, ∇^g))

$$(3.17) \qquad \beta^g(f)(X, Y) = \nabla^g_Y df(X) - df(\nabla_Y X)$$

for $X, Y \in TM$. Set

$$(3.18) \qquad \tau^g_H(f) = tr_{G_\theta} \beta^g(f) = \sum_{A=1}^{2m} \beta^g(f)(e_A, e_A).$$

For any variation f_t of f, we have (cf. [BDU], [DT])

$$(3.19) \qquad \frac{d}{dt} E_H(f_t)|_{t=0} = -\int_M \langle v, \tau^g_H(f) \rangle dv_\theta$$

where $v = (\partial f_t / \partial t)|_{t=0}$. Hence f is pseudoharmonic if and only if $\tau^g_H(f) = 0$. Set

$$(3.20) \qquad \tau^g(f) = tr_{g_\theta} \beta^g(f) = \sum_{A=0}^{2m} \beta^g(f)(e_A, e_A).$$

From (1.13), we see that $\tau^g(f)$ is the usual tension field. Thus $f : (M, g_\theta) \to (N, g)$ is harmonic if and only if $\tau^g(f) = 0$ (cf. [EL]).

Let $f : (M, H(M), J, \theta) \to (N, H(N), \widetilde{J}, \widetilde{\theta})$ be a map between two pseudo-Hermitian manifolds, and let $\nabla^{\widetilde{\theta}}$ denote the Levi-Civita connection of the Webster metric $g_{\widetilde{\theta}}$. Using Lemma 1.2, we deduce that

$$
\begin{aligned}
(3.21) \qquad \tau^{g_{\widetilde{\theta}}}_H(f) &= \sum_{A=1}^{2m} [\nabla^{\widetilde{\theta}}_{e_A} df(e_A) - df(\nabla_{e_A} e_A)] \\
&= \sum_{A=1}^{2m} [\widetilde{\nabla}_{e_A} df(e_A) - df(\nabla_{e_A} e_A)] - \sum_{A=1}^{2m} \widetilde{A}(df(e_A), df(e_A)) \widetilde{\xi} \\
&\quad + \sum_{A=1}^{2m} \widetilde{\theta}(df(e_A)) \widetilde{\tau}(df(e_A)) + \sum_{A=1}^{2m} \widetilde{\theta}(df(e_A)) \widetilde{J} df(e_A)
\end{aligned}
$$

From (3.21), we immediately get

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Proposition 3.4. *Let $f : (M, H(M), J, \theta) \to (N, H(N), \widetilde{J}, \widetilde{\theta})$ be a horizontal map. Suppose N is Sasakian. Then f is $(H, \widetilde{H})$-harmonic if and only if f is pseudoharmonic.*

Definition 3.3. A map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermitian manifolds is called a foliated map if it preserves the leaves of the Reeb foliations.

Clearly a map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is foliated if and only if $f_0^\alpha = 0$ for $\alpha = 1, ...m$ or equivalently, $f_0^{\overline{\alpha}} = 0$ for $\alpha = 1, ...m$.

Lemma 3.5. *Suppose a map $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ is foliated and horizontal. Then $df(\xi) = \lambda \widetilde{\xi}$ and $f^*\widetilde{\theta} = \lambda \theta$ for some constant λ.*

Proof. Since f is both foliated and horizontal, there exists a function λ such that $df(\xi) = \lambda \widetilde{\xi}$ and $f^*\widetilde{\theta} = \lambda \theta$. Consequently,

$$(3.22) \qquad f_0^0 = \lambda, \quad f_0^\alpha = f_0^{\overline{\alpha}} = 0,$$

and

$$(3.23) \qquad f_j^0 = f_{\overline{j}}^0 = 0.$$

From the first and second equations in (2.14), (3.22) and (3.23), we find

$$(3.24) \qquad f_{0j}^0 = f_{0\overline{j}}^0 = 0,$$

that is, $e_j(\lambda) = e_{\overline{j}}(\lambda) = 0$ $(j = 1, ..., m)$ in view of the first equation of (2.13). It follows that λ is constant. $\square$

The following result gives some relationship between $(H, \widetilde{H})$-harmonic maps and harmonic maps of pseudo-Hermitian manifolds.

Proposition 3.6. *Let $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a foliated and horizontal map between two pseudo-Hermitian manifolds. Suppose N is Sasakian. Then f is $(H, \widetilde{H})$-harmonic if and only if f is harmonic.*

Proof. Choose a local orthonormal frame field $\{\xi, e_A\}_{A=1,...,2m}$ on M. According to (1.13) and Lemma 3.5, we have

$$(3.25) \qquad \widetilde{\nabla}_\xi^\theta df(\xi) = \lambda^2 \widetilde{\nabla}_{\widetilde{\xi}}^\theta \widetilde{\xi} = 0.$$

Under the assumptions that f is horizontal and N is Sasakian, we apply Lemma 1.2 to deduce that

$$(3.26) \qquad \sum_{A=1}^{2m} \widetilde{\nabla}_{e_A}^\theta df(e_A) = \sum_{A=1}^{2m} \widetilde{\nabla}_{e_A} df(e_A) = \sum_{A=1}^{2m} \widetilde{\nabla}_{e_A} df_{H,\widetilde{H}}(e_A).$$

From (1.13), (3.25) and (3.26), we find that

$$\tau^{g_{\widetilde{\theta}}}(f) = \tau_{H,\widetilde{H}}(f)$$

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Therefore f is $(H, \widetilde{H})$-harmonic if and only if f is harmonic. $\square$

Remark 3.2. From Lemma 3.5 and Proposition 3.6, we realize that preserving both horizontal and vertical distributions is a too restrictive condition for a map between two pseudo-Hermitian manifolds (see also Example 5.2).

We know that a pseudo-Hermitian manifold N is a compact regular Sasakian manifold if and only if the foliation of N induces a Riemannian submersion $\pi : (N, g_{\widetilde{\theta}}) \to (B, g_B)$ over a compact Kähler manifold B.

Proposition 3.7. *Let $(M^{2m+1}, H(M), J, \theta)$ be a compact pseudo-Hermitian manifold and let N be a Sasakian manifold which can be realized as a Riemannian submersion $\pi : N \to B$ over a Kähler manifold B. Suppose $f : M \to N$ is a map from M to N and $\varphi = \pi \circ f$. Then $E_{H, \widetilde{H}}(f) = E_H(\varphi)$ and $d\pi(\tau_{H, \widetilde{H}}(f)) = \tau_H^{g_B}(\varphi)$. In particular, if f is foliated, then $E_{H, \widetilde{H}}(f) = E(\varphi)$ and $d\pi(\tau_{H, \widetilde{H}}(f)) = \tau^{g_B}(\varphi)$, where $E(\cdot)$ and $\tau^{g_B}(\cdot)$ denote the usual energy functional and tension field for maps between the Riemannian manifolds (M, g_θ) and (B, g_B).*

Proof. Since $\pi : N \to B$ is a Riemannian submersion, we have

$$
\begin{aligned}
E_{H, \widetilde{H}}(f) &= \frac{1}{2} \int_M \sum_{A=1}^{2m} \langle df_{H, \widetilde{H}}(e_A), df_{H, \widetilde{H}}(e_A) \rangle dv_\theta \\
&= \frac{1}{2} \int_M \sum_{A=1}^{2m} \langle d\pi \circ df(e_A), d\pi \circ df(e_A) \rangle dv_\theta \\
&= E_H(\varphi).
\end{aligned}
$$

(3.27)

Let f_t ($|t| < \varepsilon$) be any variation of f with $f_0 = f$ and $v = \frac{\partial f_t}{\partial t}|_{t=0}$. Set $\varphi_t = \pi(f_t)$ and $w = d\pi(v)$. Clearly φ_t is a variation of φ with $w = \frac{\partial \varphi_t}{\partial t}|_{t=0}$. Then (3.27) yields

$$
(3.28) \qquad \frac{d}{dt} E_{H, \widetilde{H}}(f_t)|_{t=0} = \frac{d}{dt} E_H(\varphi_t)|_{t=0}.
$$

Applying Proposition 3.1 and (3.19) to the left hand and right hand sides of (3.28) respectively, we get

$$
\int_M \langle w, d\pi(\tau_{H, \widetilde{H}}(f)) - \tau_H^{g_B}(\varphi) \rangle dv_\theta = 0.
$$

Since w can be arbitrary vector field on B, we have $d\pi(\tau_{H, \widetilde{H}}(f)) = \tau_H^{g_B}(\varphi)$.

Now we assume that f is foliated, that is, $df(\xi) = 0$. Hence $E_{H, \widetilde{H}}(f) = E(\varphi)$. For any vector field w on B, we may lift it as a basic vector field v on N. Let ψ_t be the one parameter of transformations generated by v. Since $\psi_t : N \to N$ is foliated for each t, we see that $\{f_t = \psi_t \circ f\}$ is a foliated variation of f, that is, each f_t is a foliated map and $f|_{t=0} = f$. Then (3.27) implies $E_{H, \widetilde{H}}(f_t) = E(\varphi_t)$, where $\varphi_t = \pi \circ f_t$. Consequently

$$
(3.29) \qquad \int_M \langle w, d\pi(\tau_{H, \widetilde{H}}(f)) - \tau^{g_B}(\varphi) \rangle dv_\theta = 0
$$

since the gradient of the energy functional E is $-\tau^{g_B}(\varphi)$. Since w is arbitrary, we deduce from (3.29) that $d\pi(\tau_{H, \widetilde{H}}(f)) = \tau^{g_B}(\varphi)$. $\square$

Corollary 3.8. *Let M, N, B, f and φ be as in Proposition 3.8. Then f is $(H, \widetilde{H})$-harmonic if and only if φ is pseudoharmonic. In particular, if f is foliated, then f is $(H, \widetilde{H})$-harmonic if and only if φ is harmonic.*

4. Bochner formula for the horizontal energy density $e_{H,\widetilde{H}}$

In this section, we will derive the formula of $\triangle_b e_{H,\widetilde{H}}$ for a map f between two pseudo-Hermitian manifolds. According to the notations in §2,

$$df_{H,\widetilde{H}}(\eta_j) = f_j^\alpha \widetilde{\eta}_\alpha + f_j^{\overline{\alpha}} \widetilde{\eta}_{\overline{\alpha}}, \quad df_{H,\widetilde{H}}(\eta_{\overline{j}}) = f_{\overline{j}}^\alpha \widetilde{\eta}_\alpha + f_{\overline{j}}^{\overline{\alpha}} \widetilde{\eta}_{\overline{\alpha}}$$

and thus

(4.1) $$e_{H,\widetilde{H}} = \frac{1}{2}|df_{H,\widetilde{H}}|^2 = f_j^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_j^{\overline{\alpha}} f_{\overline{j}}^{\alpha}.$$

The horizontal differential of $e_{H,\widetilde{H}}$ is given by

$$d_H e_{H,\widetilde{H}}$$

(4.2)
$$= (f_j^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_j^{\overline{\alpha}} f_{\overline{j}}^{\alpha})_k \theta^k + (f_j^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_j^{\overline{\alpha}} f_{\overline{j}}^{\alpha})_{\overline{k}} \theta^{\overline{k}}$$
$$= (f_{jk}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_j^\alpha f_{\overline{j}k}^{\overline{\alpha}} + f_{jk}^{\overline{\alpha}} f_{\overline{j}}^{\alpha} + f_j^{\overline{\alpha}} f_{\overline{j}k}^{\alpha})\theta^k + (f_{j\overline{k}}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_j^\alpha f_{\overline{j}\overline{k}}^{\overline{\alpha}} + f_{j\overline{k}}^{\overline{\alpha}} f_{\overline{j}}^{\alpha} + f_j^{\overline{\alpha}} f_{\overline{j}\overline{k}}^{\alpha})\theta^{\overline{k}}.$$

Consequently

(4.3)
$$\triangle_b e_{H,\widetilde{H}} = |\beta_{H,\widetilde{H}}|^2 + f_{\overline{j}}^{\overline{\alpha}} f_{jk\overline{k}}^{\alpha} + f_j^\alpha f_{\overline{j}k\overline{k}}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{jk\overline{k}}^{\alpha}$$
$$+ f_j^{\overline{\alpha}} f_{\overline{j}k\overline{k}}^{\alpha} + f_{\overline{j}}^{\overline{\alpha}} f_{j\overline{k}k}^{\alpha} + f_j^\alpha f_{\overline{j}\overline{k}k}^{\overline{\alpha}} + f_j^\alpha f_{\overline{j}\overline{k}k}^{\overline{\alpha}} + f_j^{\overline{\alpha}} f_{\overline{j}\overline{k}k}^{\alpha}.$$

Using (2.17), (2.37), (2.38), (2.43) and (2.44), we perform the following computations

(4.4)
$$f_{jk\overline{k}}^\alpha = [f_{kj}^\alpha + \widehat{A}_{\overline{\beta}}^\alpha (f_k^0 f_j^{\overline{\beta}} - f_k^{\overline{\beta}} f_j^0)]_{\overline{k}}$$
$$= f_{kj\overline{k}}^\alpha + \widehat{A}_{\overline{\beta},\overline{k}}^\alpha (f_k^0 f_j^{\overline{\beta}} - f_k^{\overline{\beta}} f_j^0) + \widehat{A}_{\overline{\beta}}^\alpha (f_k^0 f_j^{\overline{\beta}} - f_k^{\overline{\beta}} f_j^0)_{\overline{k}}$$
$$= f_{k\overline{k}j}^\alpha + i f_{k0}^\alpha \delta_{\overline{k}}^{\overline{j}} + f_t^\alpha R_{kj\overline{k}}^t - f_k^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha (f_j^\gamma f_{\overline{k}}^{\overline{\delta}} - f_{\overline{k}}^\gamma f_j^{\overline{\delta}})$$
$$- f_k^\beta \widehat{W}_{\beta\gamma}^\alpha (f_j^\gamma f_{\overline{k}}^0 - f_{\overline{k}}^\gamma f_j^0) + f_k^\beta \widehat{W}_{\beta\overline{\gamma}}^\alpha (f_j^{\overline{\gamma}} f_{\overline{k}}^0 - f_{\overline{k}}^{\overline{\gamma}} f_j^0)$$
$$- i f_k^\beta \widehat{A}_\delta^{\overline{\beta}} (f_j^\alpha f_{\overline{k}}^\delta - f_{\overline{k}}^\alpha f_j^\delta) + i f_k^\beta \widehat{A}_{\overline{\gamma}}^\alpha (f_j^{\overline{\gamma}} f_{\overline{k}}^{\overline{\beta}} - f_{\overline{k}}^{\overline{\gamma}} f_j^{\overline{\beta}})$$
$$+ \widehat{A}_{\overline{\beta},\overline{k}}^\alpha (f_k^0 f_j^{\overline{\beta}} - f_j^0 f_k^{\overline{\beta}}) + \widehat{A}_{\overline{\beta}}^\alpha (f_{k\overline{k}}^0 f_j^{\overline{\beta}} + f_k^0 f_{j\overline{k}}^{\overline{\beta}} - f_{k\overline{k}}^{\overline{\beta}} f_j^0 - f_k^{\overline{\beta}} f_{j\overline{k}}^0),$$

(4.5)
$$f_{j\overline{k}k}^\alpha = [f_{\overline{k}j}^\alpha + i f_0^\alpha \delta_j^k + \widehat{A}_{\overline{\beta}}^\alpha (f_j^{\overline{\beta}} f_{\overline{k}}^0 - f_j^0 f_{\overline{k}}^{\overline{\beta}})]_k$$
$$= f_{\overline{k}kj}^\alpha - i f_{\overline{t}}^\alpha A_k^{\overline{t}} \delta_j^k + i f_{\overline{t}}^\alpha A_j^{\overline{t}} \delta_k^k + f_{\overline{k}}^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha (f_k^\gamma f_j^{\overline{\delta}} - f_j^\gamma f_k^{\overline{\delta}})$$
$$+ f_{\overline{k}}^\beta \widehat{W}_{\beta\gamma}^\alpha (f_k^\gamma f_j^0 - f_j^\gamma f_k^0) - f_{\overline{k}}^\beta \widehat{W}_{\beta\overline{\gamma}}^\alpha (f_k^{\overline{\gamma}} f_j^0 - f_j^{\overline{\gamma}} f_k^0)$$
$$+ i f_{\overline{k}}^\beta \widehat{A}_\delta^{\overline{\beta}} (f_k^\alpha f_j^\delta - f_j^\alpha f_k^\delta) - i f_{\overline{k}}^\beta \widehat{A}_{\overline{\gamma}}^\alpha (f_k^{\overline{\gamma}} f_j^{\overline{\beta}} - f_j^{\overline{\gamma}} f_k^{\overline{\beta}}) + i f_{0k}^\alpha \delta_j^k$$
$$+ \widehat{A}_{\overline{\beta},k}^\alpha (f_j^{\overline{\beta}} f_{\overline{k}}^0 - f_j^0 f_{\overline{k}}^{\overline{\beta}}) + \widehat{A}_{\overline{\beta}}^\alpha (f_{jk}^{\overline{\beta}} f_{\overline{k}}^0 + f_j^{\overline{\beta}} f_{\overline{k}k}^0 - f_{jk}^0 f_{\overline{k}}^{\overline{\beta}} - f_j^0 f_{\overline{k}k}^{\overline{\beta}}),$$

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$$
\begin{aligned}
f^{\alpha}_{\overline{j}k\overline{k}} =& [f^{\alpha}_{k\overline{j}} - if^{\alpha}_0 \delta^j_k + \widehat{A}^{\alpha}_{\beta}(f^0_k f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\beta}}_k f^0_{\overline{j}})]_{\overline{k}} \\
=& f^{\alpha}_{k\overline{k}\overline{j}} + if^{\alpha}_t(A^t_{\overline{k}}\delta^{\overline{k}}_{\overline{j}} - A^t_{\overline{j}}\delta^{\overline{k}}_{\overline{k}}) + f^{\beta}_k \widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_{\overline{k}}f^{\overline{\delta}}_{\overline{j}} - f^{\gamma}_{\overline{j}}f^{\overline{\delta}}_{\overline{k}}) \\
& + f^{\beta}_k \widehat{W}^{\alpha}_{\beta\gamma}(f^{\gamma}_{\overline{k}}f^0_{\overline{j}} - f^{\gamma}_{\overline{j}}f^0_{\overline{k}}) - f^{\beta}_k \widehat{W}^{\alpha}_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{k}}f^0_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}}f^0_{\overline{k}}) \\
& + if^{\beta}_k \widehat{A}^{\overline{\beta}}_{\delta}(f^{\alpha}_{\overline{k}}f^{\delta}_{\overline{j}} - f^{\alpha}_{\overline{j}}f^{\delta}_{\overline{k}}) - if^{\beta}_k \widehat{A}^{\alpha}_{\overline{\gamma}}(f^{\overline{\gamma}}_{\overline{k}}f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}}f^{\overline{\beta}}_{\overline{k}}) - if^{\alpha}_{0\overline{k}}\delta^j_k \\
& + \widehat{A}^{\alpha}_{\beta,k}(f^0_k f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\beta}}_k f^0_{\overline{j}}) + \widehat{A}^{\alpha}_{\beta}(f^0_{k\overline{k}}f^{\overline{\beta}}_{\overline{j}} + f^0_k f^{\overline{\beta}}_{\overline{j}k} - f^{\overline{\beta}}_{k\overline{k}}f^0_{\overline{j}} - f^{\overline{\beta}}_k f^0_{\overline{j}k})
\end{aligned}
$$
(4.6)

and

$$
\begin{aligned}
f^{\alpha}_{\overline{j}kk} =& [f^{\alpha}_{\overline{k}j} + \widehat{A}^{\alpha}_{\beta}(f^0_k f^{\overline{\beta}}_{\overline{j}} - f^0_{\overline{j}}f^{\overline{\beta}}_k)]_k \\
=& f^{\alpha}_{\overline{k}k\overline{j}} - if^{\alpha}_{\overline{k}0}\delta^{\overline{k}}_{\overline{j}} - f^{\alpha}_{\overline{t}}R^{\overline{t}}_{\overline{k}k\overline{j}} + f^{\beta}_{\overline{k}}\widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_k f^{\overline{\delta}}_{\overline{j}} - f^{\gamma}_{\overline{j}}f^{\overline{\delta}}_k) \\
& + f^{\beta}_{\overline{k}}\widehat{W}^{\alpha}_{\beta\gamma}(f^{\gamma}_k f^0_{\overline{j}} - f^{\gamma}_{\overline{j}}f^0_k) - f^{\beta}_{\overline{k}}\widehat{W}^{\alpha}_{\beta\overline{\gamma}}(f^{\overline{\gamma}}_k f^0_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}}f^0_k) \\
& + if^{\beta}_{\overline{k}}\widehat{A}^{\overline{\beta}}_{\delta}(f^{\alpha}_k f^{\delta}_{\overline{j}} - f^{\alpha}_{\overline{j}}f^{\delta}_k) - if^{\beta}_{\overline{k}}\widehat{A}^{\alpha}_{\overline{\gamma}}(f^{\overline{\gamma}}_k f^{\overline{\beta}}_{\overline{j}} - f^{\overline{\gamma}}_{\overline{j}}f^{\overline{\beta}}_k) \\
& + \widehat{A}^{\alpha}_{\beta,k}(f^0_k f^{\overline{\beta}}_{\overline{j}} - f^0_{\overline{j}}f^{\overline{\beta}}_k) + \widehat{A}^{\alpha}_{\beta}(f^0_{\overline{k}k}f^{\overline{\beta}}_{\overline{j}} + f^0_k f^{\overline{\beta}}_{\overline{j}k} - f^0_{\overline{j}k}f^{\overline{\beta}}_k - f^0_{\overline{j}}f^{\overline{\beta}}_{\overline{k}k}).
\end{aligned}
$$
(4.7)

Clearly the conjugates of (4.4), (4.5), (4.6) and (4.7) yield the expressions of $f^{\overline{\alpha}}_{jk\overline{k}}$, $f^{\overline{\alpha}}_{j\overline{k}k}$, $f^{\overline{\alpha}}_{\overline{j}k\overline{k}}$ and $f^{\overline{\alpha}}_{\overline{j}kk}$.

In the remaining of this section, we assume that N is Sasakian. It follows from (4.3), (4.4), (4.5), (4.6) and (4.7) that

$$
\begin{aligned}
\triangle_H e_{H,\widetilde{H}} =& |\beta_{H,\widetilde{H}}|^2 + \langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle + 2i(f^{\overline{\alpha}}_{\overline{j}}f^{\alpha}_{j0} - f^{\alpha}_j f^{\overline{\alpha}}_{\overline{j}0}) \\
& + 2i(f^{\alpha}_{\overline{j}}f^{\overline{\alpha}}_{j0} - f^{\overline{\alpha}}_{\overline{j}}f^{\alpha}_{j0}) + 2mi(f^{\alpha}_{\overline{j}}f^{\overline{\alpha}}_{\overline{k}}A_{jk} - f^{\alpha}_j f^{\overline{\alpha}}_k A_{\overline{jk}}) \\
& + f^{\alpha}_t f^{\overline{\alpha}}_{\overline{j}}R^t_{kj\overline{k}} + f^{\overline{\alpha}}_{\overline{t}}f^{\alpha}_j R^{\overline{t}}_{\overline{k}j k} + f^{\overline{\alpha}}_t f^{\alpha}_{\overline{j}}R^t_{kj\overline{k}} + f^{\alpha}_{\overline{t}}f^{\overline{\alpha}}_j R^{\overline{t}}_{\overline{k}j k} \\
& - f^{\overline{\alpha}}_{\overline{j}}f^{\beta}_k \widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_j f^{\overline{\delta}}_{\overline{k}} - f^{\gamma}_{\overline{k}}f^{\overline{\delta}}_j) - f^{\alpha}_j f^{\overline{\beta}}_{\overline{k}}\widehat{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\delta}(f^{\overline{\gamma}}_{\overline{j}}f^{\delta}_k - f^{\overline{\gamma}}_k f^{\delta}_{\overline{j}}) \\
& - f^{\overline{\alpha}}_{\overline{j}}f^{\beta}_{\overline{k}}\widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_j f^{\overline{\delta}}_k - f^{\gamma}_k f^{\overline{\delta}}_j) - f^{\alpha}_j f^{\overline{\beta}}_k \widehat{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\delta}(f^{\overline{\gamma}}_{\overline{j}}f^{\delta}_{\overline{k}} - f^{\overline{\gamma}}_{\overline{k}}f^{\delta}_{\overline{j}}) \\
& - f^{\overline{\alpha}}_j f^{\beta}_k \widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_{\overline{j}}f^{\overline{\delta}}_{\overline{k}} - f^{\gamma}_{\overline{k}}f^{\overline{\delta}}_{\overline{j}}) - f^{\alpha}_{\overline{j}}f^{\overline{\beta}}_k \widehat{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\delta}(f^{\overline{\gamma}}_j f^{\delta}_k - f^{\overline{\gamma}}_k f^{\delta}_j) \\
& - f^{\overline{\alpha}}_j f^{\beta}_{\overline{k}}\widehat{R}^{\alpha}_{\beta\gamma\overline{\delta}}(f^{\gamma}_{\overline{j}}f^{\overline{\delta}}_k - f^{\gamma}_k f^{\overline{\delta}}_{\overline{j}}) - f^{\alpha}_{\overline{j}}f^{\overline{\beta}}_k \widehat{R}^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}\delta}(f^{\overline{\gamma}}_j f^{\delta}_{\overline{k}} - f^{\delta}_{\overline{k}}f^{\overline{\gamma}}_j)
\end{aligned}
$$
(4.8)

where

(4.9)
$$
|\beta_{H,\widetilde{H}}|^2 = 2(f^{\alpha}_{jk}f^{\overline{\alpha}}_{\overline{jk}} + f^{\alpha}_{j\overline{k}}f^{\overline{\alpha}}_{\overline{j}k} + f^{\overline{\alpha}}_{jk}f^{\alpha}_{\overline{jk}} + f^{\overline{\alpha}}_{j\overline{k}}f^{\alpha}_{\overline{j}k}),
$$

(4.10)
$$
\begin{aligned}
\langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle =& f^{\overline{\alpha}}_{\overline{j}}(f^{\alpha}_{k\overline{k}j} + f^{\alpha}_{\overline{k}kj}) + f^{\overline{\alpha}}_j(f^{\alpha}_{k\overline{k}\overline{j}} + f^{\alpha}_{\overline{k}k\overline{j}}) \\
& + f^{\alpha}_j(f^{\overline{\alpha}}_{k\overline{k}\overline{j}} + f^{\overline{\alpha}}_{\overline{k}k\overline{j}}) + f^{\alpha}_{\overline{j}}(f^{\overline{\alpha}}_{k\overline{k}j} + f^{\overline{\alpha}}_{\overline{k}kj}).
\end{aligned}
$$

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The pseudo-Hermitian Ricci curvature is given by (cf. [We], [DTo])

$$(4.11) \qquad\qquad R_{j\overline{k}} = R^t_{jt\overline{k}} = R_{\overline{t}jt\overline{k}}$$

which has the property

$$(4.12) \qquad\qquad R_{\overline{j}k} = R_{k\overline{j}}.$$

One may define the pseudo-Hermitian Ricci transformation $Ric_H : HM^C \to HM^C$ by (cf. [Ta], [DTo, page 57])

$$(4.13) \qquad\qquad Ric_H(\eta_j) = R_{j\overline{k}}\eta_k, \quad Ric_H(\eta_{\overline{j}}) = R_{\overline{j}k}\eta_{\overline{k}}.$$

From (4.11), (4.12) and (4.13), we find

$$
\begin{aligned}
(4.14) \qquad & f^\alpha_t f^{\overline{\alpha}}_{\overline{j}} R^t_{kj\overline{k}} + f^{\overline{\alpha}}_{\overline{t}} f^\alpha_j R^{\overline{t}}_{\overline{kj}k} + f^{\overline{\alpha}}_t f^\alpha_{\overline{j}} R^t_{kj\overline{k}} + f^\alpha_{\overline{t}} f^{\overline{\alpha}}_j R^{\overline{t}}_{\overline{kj}k} \\
&= f^\alpha_t f^{\overline{\alpha}}_{\overline{j}} R_{j\overline{t}} + f^{\overline{\alpha}}_{\overline{t}} f^\alpha_j R_{t\overline{j}} + f^{\overline{\alpha}}_t f^\alpha_{\overline{j}} R_{j\overline{t}} + f^\alpha_{\overline{t}} f^{\overline{\alpha}}_j R_{t\overline{j}} \\
&= 2\langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}})\rangle.
\end{aligned}
$$

In terms of (1.20) and (1.22), the curvature terms of N appearing in (4.8) can be expressed as

$$
\begin{aligned}
(4.15) \qquad
& -f^{\overline{\alpha}}_{\overline{j}} f^\beta_k \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_j f^{\overline{\delta}}_{\overline{k}} - f^\gamma_k f^{\overline{\delta}}_{\overline{j}}) - f^\alpha_j f^{\overline{\beta}}_{\overline{k}} \widehat{R}^{\overline{\alpha}}_{\overline{\beta\gamma}\delta}(f^{\overline{\gamma}}_{\overline{j}} f^\delta_k - f^{\overline{\gamma}}_{\overline{k}} f^\delta_j) \\
& -f^{\overline{\alpha}}_{\overline{j}} f^\beta_k \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_j f^{\overline{\delta}}_{\overline{k}} - f^\gamma_k f^{\overline{\delta}}_{\overline{j}}) - f^\alpha_j f^{\overline{\beta}}_{\overline{k}} \widehat{R}^{\overline{\alpha}}_{\overline{\beta\gamma}\delta}(f^{\overline{\gamma}}_{\overline{j}} f^\delta_{\overline{k}} - f^{\overline{\gamma}}_{\overline{k}} f^\delta_{\overline{j}}) \\
& -f^{\overline{\alpha}}_j f^\beta_k \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{j}} f^{\overline{\delta}}_{\overline{k}} - f^\gamma_k f^{\overline{\delta}}_{\overline{j}}) - f^\alpha_{\overline{j}} f^{\overline{\beta}}_{\overline{k}} \widehat{R}^{\overline{\alpha}}_{\overline{\beta\gamma}\delta}(f^{\overline{\gamma}}_j f^\delta_k - f^{\overline{\gamma}}_{\overline{k}} f^\delta_j) \\
& -f^{\overline{\alpha}}_j f^\beta_k \widehat{R}^\alpha_{\beta\gamma\overline{\delta}}(f^\gamma_{\overline{j}} f^{\overline{\delta}}_{\overline{k}} - f^\gamma_k f^{\overline{\delta}}_{\overline{j}}) - f^\alpha_{\overline{j}} f^{\overline{\beta}}_k \widehat{R}^{\overline{\alpha}}_{\overline{\beta\gamma}\delta}(f^{\overline{\gamma}}_j f^\delta_{\overline{k}} - f^{\overline{\gamma}}_{\overline{k}} f^\delta_j) \\
&= -\widetilde{R}\big(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big) \\
& \quad - \widetilde{R}\big(df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k)\big) \\
& \quad - \widetilde{R}\big(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)\big) \\
& \quad - \widetilde{R}\big(df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big) \\
&= -2\widetilde{R}\big(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big) \\
& \quad - 2\widetilde{R}\big(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)\big).
\end{aligned}
$$

From (4.8), (4.13), (4.14) and (4.15), we obtain

$$
\begin{aligned}
(4.16) \qquad
\triangle_H e_{H,\widetilde{H}} =& |\beta_{H,\widetilde{H}}|^2 + \langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle - 2i(f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{j0} + f^\alpha_j f^{\overline{\alpha}}_{\overline{j}0} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{j0} - f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{j0}) \\
& + 2mi(f^\alpha_{\overline{j}} f^{\overline{\alpha}}_k A_{jk} - f^\alpha_j f^{\overline{\alpha}}_{\overline{k}} A_{\overline{jk}}) + 2\langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}})\rangle \\
& - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})) \\
& - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)).
\end{aligned}
$$

The main difficulty in applications of (4.16) comes from the mixed term, that is, the third term on the right hand side of (4.16). It is known that the CR Paneitz operator is a useful tool to deal with such a term. The usual Paneitz operator is a fourth order differential operator defined as the divergence of a third order differential operator P acting on functions. One property of the Paneitz operator is its nonnegativity, which plays an important role in some rigidity problems in pseudo-Hermitian geometry. We will generalize the operator P to a differential operator, still denoted by P, acting on maps between pseudo-Hermitian manifolds as follows.

Definition 4.1. Let $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ be a map between two pseudo-Hermitian manifolds. The primitive Paneitz operator $P(f)$ is a third order differential operator given by

$$P(f) = (f^\alpha_{\bar{k}kj} + imA_{jk}f^\alpha_{\bar{k}})\theta^j \otimes \widetilde{\eta}_\alpha + (f^{\overline{\alpha}}_{\bar{k}kj} + imA_{jk}f^{\overline{\alpha}}_{\bar{k}})\theta^j \otimes \widetilde{\eta}_{\overline{\alpha}}$$
$$= P^\alpha_j(f)\theta^j \otimes \widetilde{\eta}_\alpha + P^{\overline{\alpha}}_j(f)\theta^j \otimes \widetilde{\eta}_{\overline{\alpha}}$$

where

$$P^\alpha_j(f) = f^\alpha_{\bar{k}kj} + imA_{jk}f^\alpha_{\bar{k}}, \quad P^{\overline{\alpha}}_j(f) = f^{\overline{\alpha}}_{\bar{k}kj} + imA_{jk}f^{\overline{\alpha}}_{\bar{k}}.$$

We use $\overline{P}(f)$ to denote the conjugate of P, that is,

$$\overline{P}(f) = \overline{P^\alpha_j(f)}\theta^{\overline{j}} \otimes \widetilde{\eta}_{\overline{\alpha}} + \overline{P^{\overline{\alpha}}_j(f)}\theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha.$$

Note that N is assumed to be Sasakian in this section. Using the first and second equations of (2.17), we deduce that

(4.17)
$$i(f^{\overline{\alpha}}_j f^\alpha_{\overline{j}0} + f^\alpha_j f^{\overline{\alpha}}_{\overline{j}0} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{j0} - f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{j0})$$
$$=i(f^{\overline{\alpha}}_j f^\alpha_{0\overline{j}} + f^\alpha_j f^{\overline{\alpha}}_{0\overline{j}} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{0j} - f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{0j}) - iA_{\overline{k}\overline{j}}(f^\alpha_k f^{\overline{\alpha}}_j + f^{\overline{\alpha}}_k f^\alpha_j)$$
$$+ iA_{kj}(f^\alpha_{\overline{k}} f^{\overline{\alpha}}_{\overline{j}} + f^{\overline{\alpha}}_{\overline{k}} f^\alpha_{\overline{j}})$$
$$=i(f^{\overline{\alpha}}_j f^\alpha_{0\overline{j}} + f^\alpha_j f^{\overline{\alpha}}_{0\overline{j}} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{0j} - f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{0j}) - 2i(A_{\overline{k}\overline{j}}f^\alpha_k f^{\overline{\alpha}}_j - A_{kj}f^\alpha_{\overline{k}} f^{\overline{\alpha}}_{\overline{j}})$$

The fourth equation in (2.17) yields that

(4.18)
$$(f^\alpha_{k\overline{k}} - f^\alpha_{\overline{k}k}) = mif^\alpha_0, \quad (f^{\overline{\alpha}}_{k\overline{k}} - f^{\overline{\alpha}}_{\overline{k}k}) = mif^{\overline{\alpha}}_0.$$

It follows from (4.17)and (4.18) that

(4.19)
$$-2i(f^{\overline{\alpha}}_j f^\alpha_{\overline{j}0} + f^\alpha_j f^{\overline{\alpha}}_{\overline{j}0} - f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{j0} - f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{j0})$$
$$=4i(A_{\overline{k}\overline{j}}f^\alpha_k f^{\overline{\alpha}}_j - A_{kj}f^\alpha_{\overline{k}} f^{\overline{\alpha}}_{\overline{j}}) + \frac{2}{m}(f^{\overline{\alpha}}_j f^\alpha_{k\overline{k}\overline{j}} + f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{k\overline{k}j} + f^\alpha_j f^{\overline{\alpha}}_{\overline{k}k\overline{j}} + f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{\overline{k}kj})$$
$$- \frac{2}{m}(f^{\overline{\alpha}}_j f^\alpha_{k\overline{k}\overline{j}} + f^\alpha_{\overline{j}} f^{\overline{\alpha}}_{\overline{k}kj} + f^\alpha_j f^{\overline{\alpha}}_{k\overline{k}\overline{j}} + f^{\overline{\alpha}}_{\overline{j}} f^\alpha_{\overline{k}kj}).$$

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By the definitions of $P(f)$ and $df_{H,\widetilde{H}}$, one has

$$
(4.20) \qquad \langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle = (f_{\bar{j}}^{\bar{\alpha}} f_{k\bar{k}\bar{j}}^{\alpha} + f_{\bar{j}}^{\alpha} f_{\bar{k}kj}^{\bar{\alpha}} + f_{j}^{\alpha} f_{\bar{k}\bar{k}\bar{j}}^{\bar{\alpha}} + f_{\bar{j}}^{\bar{\alpha}} f_{kkj}^{\alpha})
$$
$$
- 2mi(A_{\bar{k}\bar{j}} f_k^{\alpha} f_{\bar{j}}^{\bar{\alpha}} - A_{kj} f_{\bar{k}}^{\alpha} f_{\bar{j}}^{\bar{\alpha}}).
$$

Then (4.19) and (4.20) imply that

$$
(4.21) \qquad
\begin{aligned}
&- 2i(f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha} + f_{j}^{\alpha} f_{\bar{j}0}^{\bar{\alpha}} - f_{\bar{j}}^{\alpha} f_{j0}^{\bar{\alpha}} - f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha}) \\
&= -\frac{2}{m}\langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle + \frac{2}{m}(f_{\bar{j}}^{\bar{\alpha}} f_{\bar{k}k\bar{j}}^{\alpha} + f_{\bar{j}}^{\alpha} f_{k\bar{k}j}^{\bar{\alpha}} + f_{j}^{\alpha} f_{\bar{k}\bar{k}\bar{j}}^{\bar{\alpha}} + f_{\bar{j}}^{\bar{\alpha}} f_{k\bar{k}j}^{\alpha}).
\end{aligned}
$$

On the other hand, using (4.10) and (4.20), one has

$$
(4.22) \qquad
\begin{aligned}
\langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle &= (f_{\bar{j}}^{\bar{\alpha}} f_{\bar{k}k\bar{j}}^{\alpha} + f_{\bar{j}}^{\alpha} f_{k\bar{k}j}^{\bar{\alpha}} + f_{j}^{\alpha} f_{\bar{k}\bar{k}\bar{j}}^{\bar{\alpha}} + f_{\bar{j}}^{\bar{\alpha}} f_{k\bar{k}j}^{\alpha}) \\
&\quad + \langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle + 2mi(A_{\bar{k}\bar{j}} f_k^{\alpha} f_{\bar{j}}^{\bar{\alpha}} - A_{kj} f_{\bar{k}}^{\alpha} f_{\bar{j}}^{\bar{\alpha}}).
\end{aligned}
$$

It follows from (4.21) and (4.22) that

$$
(4.23) \qquad
\begin{aligned}
&- 2i(f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha} + f_{j}^{\alpha} f_{\bar{j}0}^{\bar{\alpha}} - f_{\bar{j}}^{\alpha} f_{j0}^{\bar{\alpha}} - f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha}) = \frac{2}{m}\langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle \\
&- \frac{4}{m}\langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle - 4i(A_{\bar{k}\bar{j}} f_k^{\alpha} f_{\bar{j}}^{\bar{\alpha}} - A_{kj} f_{\bar{k}}^{\alpha} f_{\bar{j}}^{\bar{\alpha}}).
\end{aligned}
$$

From (4.16), (4.23), we conclude that

Theorem 4.1. *Let* $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ *be a map between two pseudo-Hermitian manifolds. If N is Sasakian, then*

$$
\begin{aligned}
\triangle_b e_{H,\widetilde{H}} =& |\beta_{H,\widetilde{H}}|^2 + (1 + \frac{2}{m})\langle \widetilde{\nabla}\tau_{H,\widetilde{H}}, df_{H,\widetilde{H}}\rangle - \frac{4}{m}\langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle \\
& - (2m+4)i\sum (f_j^{\alpha} f_{\bar{k}}^{\bar{\alpha}} A_{\bar{j}k} - f_{\bar{j}}^{\alpha} f_{\bar{k}}^{\bar{\alpha}} A_{jk}) + 2\langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\bar{j}})\rangle \\
& - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\bar{j}}), df_{H,\widetilde{H}}(\eta_{\bar{k}})) \\
& - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\bar{k}}), df_{H,\widetilde{H}}(\eta_{\bar{j}}), df_{H,\widetilde{H}}(\eta_k))
\end{aligned}
$$

To apply the above Bochner formula, we want to investigate the sign of the integral $\int_M \langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle$.

Lemma 4.2. *Let* $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ *be a map from a compact pseudo-Hermitian manifold to a Sasakian manifold. Then*

$$
(4.24) \qquad
\begin{aligned}
&i\int_M (f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha} + f_{j}^{\alpha} f_{\bar{j}0}^{\bar{\alpha}} - f_{\bar{j}}^{\alpha} f_{j0}^{\bar{\alpha}} - f_{\bar{j}}^{\bar{\alpha}} f_{j0}^{\alpha})dv_{\theta} \\
&= 2m\int_M f_0^{\alpha} f_0^{\bar{\alpha}} dv_{\theta} - 2i\int_M (A_{\bar{k}\bar{j}} f_k^{\alpha} f_{\bar{j}}^{\bar{\alpha}} - A_{kj} f_{\bar{k}}^{\alpha} f_{\bar{j}}^{\bar{\alpha}})dv_{\theta}.
\end{aligned}
$$

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Proof. We introduce a global 1-form on M as follows

$$\psi = i\big((f_{\bar{j}}^{\overline{\alpha}} f_0^{\alpha} + f_j^{\alpha} f_0^{\overline{\alpha}})\theta^j - (f_{\overline{j}}^{\alpha} f_0^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_0^{\alpha})\theta^{\bar{j}}\big).$$

Using (2.17), we compute

$$\begin{aligned}
\delta\psi &= -i\big\{(f_{\bar{j}}^{\overline{\alpha}} f_0^{\alpha} + f_j^{\alpha} f_0^{\overline{\alpha}})_{\bar{j}} - (f_{\overline{j}}^{\alpha} f_0^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_0^{\alpha})_j\big\} \\
&= -i\big\{(f_{\bar{j}\bar{j}}^{\overline{\alpha}} f_0^{\alpha} + f_{j\bar{j}}^{\alpha} f_0^{\overline{\alpha}}) - (f_{\overline{j}j}^{\alpha} f_0^{\overline{\alpha}} + f_{\overline{j}j}^{\overline{\alpha}} f_0^{\alpha})\big\} + i\big\{(f_{\bar{j}}^{\overline{\alpha}} f_{0j}^{\alpha} + f_j^{\alpha} f_{0\bar{j}}^{\overline{\alpha}}) - (f_{\overline{j}}^{\alpha} f_{0j}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{0j}^{\alpha})\big\} \\
&= -i\big\{(f_{\bar{j}\bar{j}}^{\overline{\alpha}} - f_{\overline{j}j}^{\overline{\alpha}})f_0^{\alpha} + (f_{j\bar{j}}^{\alpha} - f_{\overline{j}j}^{\alpha})f_0^{\overline{\alpha}}\big\} + i\big\{(f_{\bar{j}}^{\overline{\alpha}} f_{0j}^{\alpha} + f_j^{\alpha} f_{0\bar{j}}^{\overline{\alpha}}) - (f_{\overline{j}}^{\alpha} f_{0j}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{0j}^{\alpha})\big\} \\
&= 2m f_0^{\alpha} f_0^{\overline{\alpha}} - i\big\{(f_{\bar{j}}^{\overline{\alpha}} f_{0j}^{\alpha} + f_j^{\alpha} f_{0\bar{j}}^{\overline{\alpha}}) - (f_{\overline{j}}^{\alpha} f_{0j}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{0j}^{\alpha})\big\}.
\end{aligned}$$

The divergence theorem implies that

$$(4.25) \qquad i \int_M \big\{(f_{\bar{j}}^{\overline{\alpha}} f_{0j}^{\alpha} + f_j^{\alpha} f_{0\bar{j}}^{\overline{\alpha}}) - (f_{\overline{j}}^{\alpha} f_{0j}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{0j}^{\alpha})\big\} dv_\theta = 2m \int_M f_0^{\alpha} f_0^{\overline{\alpha}} dv_\theta.$$

Then (4.24) is a consequence of (4.17) and (4.25). □

Lemma 4.3. *Let $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ be a map from a compact pseudo-Hermitian manifold to a Sasakian manifold. Then we have*
$$(4.26)$$
$$2 \int_M \langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\bar{j}})\rangle dv_\theta$$
$$= \int_M \big\{ mi(f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}0}^{\alpha} + f_j^{\alpha} f_{\bar{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\alpha} f_{\bar{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\bar{j}0}^{\alpha}) - 2(f_{jk}^{\alpha} f_{\bar{j}k}^{\overline{\alpha}} + f_{\bar{j}k}^{\alpha} f_{jk}^{\overline{\alpha}} - f_{j\bar{k}}^{\alpha} f_{\bar{j}k}^{\overline{\alpha}} - f_{\bar{j}k}^{\alpha} f_{j\bar{k}}^{\overline{\alpha}})\big\} dv_\theta$$
$$+ 2 \int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\bar{k}}), df_{H,\widetilde{H}}(\eta_{\bar{j}}), df_{H,\widetilde{H}}(\eta_k)) dv_\theta$$
$$- 2 \int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\bar{j}}), df_{H,\widetilde{H}}(\eta_{\bar{k}})) dv_\theta.$$

Proof. Taking $k = l$ in (2.44) and (2.37) respectively and summing over k from 1 to m, we get

$$(4.27) \quad im f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}0}^{\alpha} = -f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}kk}^{\alpha} + f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}k\bar{k}}^{\alpha} + f_{\bar{j}}^{\overline{\alpha}} f_{\bar{t}}^{\alpha} R_{t\bar{j}kk} + \widetilde{R}_{\alpha\beta\gamma\delta} f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}}^{\beta} f_k^{\gamma} f_{\bar{k}}^{\overline{\delta}} + \widetilde{R}_{\alpha\beta\delta\gamma} f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}}^{\beta} f_k^{\overline{\delta}} f_{\bar{k}}^{\gamma}$$

and

$$(4.28) \quad im f_j^{\alpha} f_{\bar{j}0}^{\overline{\alpha}} = -f_j^{\alpha} f_{\bar{j}kk}^{\overline{\alpha}} + f_j^{\alpha} f_{\bar{j}k\bar{k}}^{\overline{\alpha}} + f_j^{\alpha} f_{\bar{t}}^{\overline{\alpha}} R_{t\bar{j}kk} + \widetilde{R}_{\overline{\alpha}\overline{\beta}\delta\overline{\gamma}} f_j^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_k^{\delta} f_{\bar{k}}^{\overline{\gamma}} + \widetilde{R}_{\overline{\alpha}\overline{\beta}\gamma\delta} f_j^{\alpha} f_{\bar{j}}^{\overline{\beta}} f_k^{\overline{\gamma}} f_{\bar{k}}^{\delta}.$$

Consequently
$$(4.29)$$
$$im(f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}0}^{\alpha} + f_j^{\alpha} f_{\bar{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\alpha} f_{\bar{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\bar{j}0}^{\alpha})$$
$$= -f_j^{\alpha} f_{\bar{j}kk}^{\overline{\alpha}} - f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}kk}^{\alpha} - f_{\overline{j}}^{\alpha} f_{jk\bar{k}}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{jk\bar{k}}^{\alpha} + f_j^{\alpha} f_{\bar{j}k\bar{k}}^{\overline{\alpha}} + f_{\bar{j}}^{\overline{\alpha}} f_{\bar{j}k\bar{k}}^{\alpha} + f_{\bar{j}}^{\alpha} f_{\bar{j}kk}^{\overline{\alpha}} + f_{\overline{j}}^{\overline{\alpha}} f_{\bar{j}kk}^{\alpha}$$
$$+ 2\langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\bar{j}})\rangle + 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\bar{j}}), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\bar{k}})).$$

By Bianchi identity, we find

(4.30)
$$
\begin{aligned}
&\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{k}}))\\
&= -\,\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k))\\
&\quad + \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})).
\end{aligned}
$$

Thus we obtain (4.26) by applying integration by parts to (4.29) and using (4.30). $\square$

The following result generalizes the non-negativity of the usual Paneitz operator.

Theorem 4.4. *Let* $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, H(N), \widetilde{J}, \widetilde{\theta})$ *be a map from a compact pseudo-Hermitian manifold with* $m \geq 2$ *to a Sasakian manifold with strongly seminegative horizontal curvature. Then*

$$
-\int_M \langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}}\rangle dv_\theta \geq 0.
$$

In particular, if M *is also Sasakian, then the above integral is always nonnegative for* $m \geq 1$ *and every target Sasakian manifold without any curvature condition.*

Proof. First we assume that $m \geq 2$ and N is a Sasakian manifold with strongly seminegative horizontal curvature. Integrating (4.16) and using Lemma 4.3, we discover
(4.31)
$$
\begin{aligned}
&\int_M |\beta_{H,\widetilde{H}}|^2 dv_\theta - \int_M |\tau_{H,\widetilde{H}}|^2 dv_\theta + (m-2)i\int_M (f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha + f_{\overline{j}}^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha)dv_\theta\\
&+ 2mi\int_M (A_{jk}f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\alpha}} - A_{\overline{jk}}f_j^\alpha f_k^{\overline{\alpha}})dv_\theta - 2\int_M (f_{jk}^\alpha f_{\overline{jk}}^{\overline{\alpha}} + f_{\overline{jk}}^\alpha f_{jk}^{\overline{\alpha}} - f_{jk}^\alpha f_{\overline{jk}}^{\overline{\alpha}} - f_{\overline{jk}}^\alpha f_{jk}^{\overline{\alpha}})dv_\theta\\
&- 4\int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))dv_\theta = 0
\end{aligned}
$$

It follows from (4.9) and (4.31) that

(4.32)
$$
\begin{aligned}
&4\int_M (f_{j\overline{k}}^\alpha f_{\overline{jk}}^{\overline{\alpha}} + f_{\overline{jk}}^\alpha f_{j\overline{k}}^{\overline{\alpha}})dv_\theta - \int_M |\tau_{H,\widetilde{H}}|^2 dv_\theta + 2mi\int_M (A_{jk}f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\alpha}} - A_{\overline{jk}}f_j^\alpha f_k^{\overline{\alpha}})dv_\theta\\
&+ (m-2)i\int_M (f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha + f_j^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha)dv_\theta\\
&- 4\int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))dv_\theta = 0.
\end{aligned}
$$

The integral of (4.23) yields
(4.33)
$$
\begin{aligned}
&i\int_M (f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha + f_j^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^\alpha f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^\alpha)dv_\theta\\
&= \frac{1}{m}\int_M \{|\tau_{H,\widetilde{H}}|^2 + 2 < P(f) + \overline{P(f)}, df_{H,\widetilde{H}} > -2mi(A_{jk}f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\alpha}} - A_{\overline{jk}}f_j^\alpha f_k^{\overline{\alpha}})\}dv_\theta.
\end{aligned}
$$

We multiply (4.33) by $(m-1)$ and minus (4.24) to get

$$(m-2)i\int_M (f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^{\alpha} + f_j^{\alpha} f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\alpha} f_{j0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{j0}^{\alpha})dv_\theta$$

$$=\frac{m-1}{m}\int_M |\tau_{H,\widetilde{H}}|^2 dv_\theta - 2m\int_M f_0^{\alpha} f_0^{\overline{\alpha}} dv_\theta + \frac{2(m-1)}{m}\int_M \langle P(f)+\overline{P(f)}, df_{H,\widetilde{H}}\rangle dv_\theta$$

$$- 2mi\int_M (A_{jk}f_{\overline{j}}^{\alpha} f_{\overline{k}}^{\overline{\alpha}} - A_{\overline{jk}}f_j^{\alpha} f_k^{\overline{\alpha}})dv_\theta$$

that is,
(4.34)

$$\int_M \left\{ (m-2)i(f_{\overline{j}}^{\overline{\alpha}} f_{\overline{j}0}^{\alpha} + f_j^{\alpha} f_{\overline{j}0}^{\overline{\alpha}} - f_{\overline{j}}^{\alpha} f_{j0}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{j0}^{\alpha}) + 2mi(A_{jk}f_{\overline{j}}^{\alpha} f_{\overline{k}}^{\overline{\alpha}} - A_{\overline{jk}}f_j^{\alpha} f_k^{\overline{\alpha}}) \right\}dv_\theta$$

$$= \frac{m-1}{m}\int_M |\tau_{H,\widetilde{H}}|^2 dv_\theta - 2m\int_M f_0^{\alpha} f_0^{\overline{\alpha}} dv_\theta + \frac{2(m-1)}{m}\int_M \langle P(f)+\overline{P(f)}, df_{H,\widetilde{H}}\rangle dv_\theta.$$

Substituting (4.34) into (4.32) gives
(4.35)

$$\int_M \left\{ 4(f_{j\overline{k}}^{\alpha} f_{\overline{j}k}^{\overline{\alpha}} + f_{\overline{j}k}^{\alpha} f_{j\overline{k}}^{\overline{\alpha}}) - \frac{1}{m}|\tau_{H,\widetilde{H}}|^2 + \frac{2(m-1)}{m}\langle P(f)+\overline{P(f)}, df_{H,\widetilde{H}}\rangle \right\}dv_\theta$$

$$- 2m\int_M f_0^{\alpha} f_0^{\overline{\alpha}} dv_\theta - 4\int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))dv_\theta = 0.$$

Using the Cauchy-Schwarz inequality, the parallelogram law and the fourth equation of (2.17), we discover

(4.36)

$$f_{j\overline{k}}^{\alpha} f_{\overline{j}k}^{\overline{\alpha}} + f_{\overline{j}k}^{\alpha} f_{j\overline{k}}^{\overline{\alpha}} \geq f_{k\overline{k}}^{\alpha} f_{\overline{k}k}^{\overline{\alpha}} + f_{\overline{k}k}^{\alpha} f_{k\overline{k}}^{\overline{\alpha}}$$

$$\geq \frac{1}{m}\sum_\alpha \left(|\sum_k f_{k\overline{k}}^{\alpha}|^2 + |\sum_k f_{\overline{k}k}^{\alpha}|^2 \right)$$

$$= \frac{1}{2m}\sum_\alpha \left(|\sum_k (f_{k\overline{k}}^{\alpha} + f_{\overline{k}k}^{\alpha})|^2 + |\sum_k (f_{k\overline{k}}^{\alpha} - f_{\overline{k}k}^{\alpha})|^2 \right)$$

$$= \frac{1}{4m}|\tau_{H,\widetilde{H}}|^2 + \frac{m}{2}f_0^{\alpha} f_0^{\overline{\alpha}}$$

where $|\tau_{H,\widetilde{H}}|^2 = 2\sum_\alpha |\sum_k (f_{k\overline{k}}^{\alpha} + f_{\overline{k}k}^{\alpha})|^2$. It follows from (4.35), (4.36) and the curvature assumption on N that

$$- \int_M \langle P(f)+\overline{P(f)}, df_{H,\widetilde{H}}\rangle dv_\theta$$

$$\geq -\frac{2m}{m-1}\int_M \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))dv_\theta$$

$$\geq 0.$$

The first claim is proved.

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Assume now that M is a Sasakian manifold with $m \geq 1$ and N is an arbitrary Sasakian manifold . Using (4.10) and the integration by parts, we get from (4.22) that

$$-\int_M \langle P(f) + \overline{P(f)}, df_{H,\widetilde{H}} \rangle dv_\theta = -\int_M \left(f_{\overline{j}}^{\overline{\alpha}} f_{k\overline{k}j}^\alpha + f_{\overline{j}}^{\overline{\alpha}} f_{\overline{k}k\overline{j}}^\alpha + f_j^\alpha f_{\overline{k}k\overline{j}}^{\overline{\alpha}} + f_{\overline{j}}^\alpha f_{k\overline{k}j}^{\overline{\alpha}} \right) dv_\theta$$

$$= \int_M \left(f_{\overline{j}j}^{\overline{\alpha}} f_{k\overline{k}}^\alpha + f_{j\overline{j}}^{\overline{\alpha}} f_{\overline{k}k}^\alpha + f_{j\overline{j}}^\alpha f_{\overline{k}k}^{\overline{\alpha}} + f_{\overline{j}j}^\alpha f_{k\overline{k}}^{\overline{\alpha}} \right) dv_\theta$$

$$\geq 0.$$

This gives the second claim. $\square$

Definition 4.2. A map $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is called horizontally constant if it maps the domain manifold into a single leaf of the pseudo-Hermitian foliation on N.

Lemma 4.5. Let $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a map between two pseudo-Hermitian manifolds. Then f is horizontally constant if and only if $df_{H,\widetilde{H}} = 0$.

Proof. If f is horizontally constant, then $df(X)$ is tangent to the fiber of N for any $X \in TM$. Clearly we have $df_{H,\widetilde{H}} = 0$.

Conversely, we assume that $df_{H,\widetilde{H}} = 0$, which is equivalent to $f_j^\alpha = f_{\overline{j}}^\alpha = f_j^{\overline{\alpha}} = f_{\overline{j}}^{\overline{\alpha}} = 0$. Then the fourth equation in (2.17) yields $f_0^\alpha = f_0^{\overline{\alpha}} = 0$. Hence $\pi_{\widetilde{H}} df(X) = 0$ for any $X \in TM$. Suppose that $f(p) = q$ and $\widetilde{C}_q$ is the integral curve of $\widetilde{\xi}$ passing through the point q. For any point $p' \in M$, let $c(t)$ be a smooth curve joining p and p'. Obviously $f(c(t))$ is a smooth curve passing through p and $df(c'(t)) = \lambda(t)\widetilde{\xi}$ for some function $\lambda(t)$, which means that $f(c(t))$ is the reparametrization of the integral curve of $\widetilde{\xi}$. Therefore $f(c(t)) \subset \widetilde{C}_q$. In particular, $f(p') \in \widetilde{C}_q$. Since p' is arbitrary, we conclude that $f(M) \subset \widetilde{C}_q$. $\square$

It is easy to see that a horizontally constant map is foliated and $(H, \widetilde{H})$-harmonic. The following result is also obvious by intuition.

Lemma 4.6. Suppose $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ is a horizontal map. If f is horizontally constant, then f is constant.

Proof. Since f is horizontal and horizontally constant, we have $f_j^0 = f_{\overline{j}}^0 = 0$ and $f_j^\alpha = f_{\overline{j}}^\alpha = f_j^{\overline{\alpha}} = f_{\overline{j}}^{\overline{\alpha}} = 0$. Hence $df \circ i_H = 0$. Then the fifth equation of (2.14) implies $f_0^0 = 0$. This shows that $df(\xi) = 0$, since f is foliated. Therefore we conclude that f is constant. $\square$

Lemma 4.7. If N has non-positive horizontal sectional curvature, then

(4.37)
$$\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))$$
$$+ \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)) \leq 0.$$

Proof. Write $df_{H,\widetilde{H}}(\eta_j) = X_j + iY_j$ $(j = 1, ..., m)$. We find that
(4.38)
  the l.h.s. of (4.37)

$$= \widetilde{R}(X_j, X_k, X_j, X_k) - \widetilde{R}(X_j, X_k, Y_j, Y_k) + \widetilde{R}(X_j, Y_k, X_j, Y_k) + \widetilde{R}(X_j, Y_k, Y_j, X_k)$$
$$+ \widetilde{R}(Y_j, X_k, X_j, Y_k) + \widetilde{R}(Y_j, X_k, Y_j, X_k) - \widetilde{R}(Y_j, Y_k, X_j, X_k) + \widetilde{R}(Y_j, Y_k, Y_j, Y_k)$$
$$+ \widetilde{R}(X_j, X_k, X_j, X_k) + \widetilde{R}(X_j, X_k, Y_j, Y_k) + \widetilde{R}(X_j, Y_k, X_j, Y_k) - \widetilde{R}(X_j, Y_k, Y_j, X_k)$$
$$- \widetilde{R}(Y_j, X_k, X_j, Y_k) + \widetilde{R}(Y_j, X_k, Y_j, X_k) + \widetilde{R}(Y_j, Y_k, X_j, X_k) + \widetilde{R}(Y_j, Y_k, Y_j, Y_k)$$
$$= 2\{\widetilde{R}(X_j, X_k, X_j, X_k) + \widetilde{R}(X_j, Y_k, X_j, Y_k) + \widetilde{R}(Y_j, X_k, Y_j, X_k) + \widetilde{R}(Y_j, Y_k, Y_j, Y_k)\},$$

which is nonpositive by the assumption that $\widetilde{K}^H \leq 0$. $\square$

Now we want to give some consequences of the Bochner formula in Theorem 4.1.

Theorem 4.8. *Let $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ be a $(H, \widetilde{H})$-harmonic map from a compact pseudo-Hermitian manifold with CR dimension $m \geq 2$ to a Sasakian manifold with strongly semi-negative horizontal curvature. Let $\sigma_0(x)$ be the maximal eigenvalue of the symmetric matrix $(|A_{jk}|_x)_{m \times m}$ at $x \in M$. Suppose that*

(4.39)
$$Ric_H - (m + 2)\sigma_0 L_\theta \geq 0,$$

where L_θ is the Levi-form (see Definition 1.2). Then
   (i) $\beta_{H,\widetilde{H}} = 0$;
   (ii) If $Ric_H - (m + 2)\sigma_0 L_\theta > 0$ at a point in M, then f is horizontally constant;
   (iii) If N has negative horizontal sectional curvature, then f is either horizontally constant or of horizontal rank one.

Proof. At each point, let λ be the minimal eigenvalue of the Hermitian matrix $(R_{j\overline{k}})$. Therefore

$$\langle df_{H,\widetilde{H}}(Ric_H(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}}) \rangle = R_{j\overline{k}} f_k^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_k^{\overline{\alpha}} f_{\overline{j}}^\alpha R_{j\overline{k}}$$
(4.40)
$$\geq \lambda (f_k^\alpha f_{\overline{k}}^{\overline{\alpha}} + f_k^{\overline{\alpha}} f_{\overline{k}}^\alpha)$$
$$= \lambda \sum_{\alpha,k} \left(|f_k^\alpha|^2 + |f_k^{\overline{\alpha}}|^2 \right).$$

By the definition of σ_0, one has

$$|i(f_j^\alpha f_{\overline{k}}^{\overline{\alpha}} A_{\overline{jk}} - f_{\overline{j}}^\alpha f_k^{\overline{\alpha}} A_{jk})| \leq 2\tau_0 \sum_{\alpha,k} |f_k^\alpha f_{\overline{k}}^{\overline{\alpha}}|$$
(4.41)
$$\leq \tau_0 \sum_{\alpha,k} \left(|f_k^\alpha|^2 + |f_k^{\overline{\alpha}}| \right).$$

From Theorems 4.1, 4.4, (4.39), (4.40) and (4.41), we immediately get (i). Clearly, $\beta_{H,\widetilde{H}} = 0$ implies that $e_{H,\widetilde{H}} =$ const. Besides, we have

(4.42)
$$\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}}))$$
$$+ \widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)) = 0.$$

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Next assume that $Ric_H - (m+2)\tau_0 L_\theta > 0$ at a point p in M. This additional condition clearly implies that $f_k^\alpha(p) = f_k^{\overline{\alpha}}(p) = 0$. Therefore $e_{H,\widetilde{H}} = 0$, that is, $df_{H,\widetilde{H}} = 0$. It follows from Lemma 4.5 that f is horizontally constant. This proves (ii).

Now we consider the claim (iii). If $K^H < 0$, then (4.42) implies that $rank(df_{H,\widetilde{H}})$ is zero or one in view of (4.38). Since $e_{H,\widetilde{H}}$ is constant, the rank is constant. In the first case, f is horizontally constant; and in the second case, we say that f is of horizontal rank one. $\square$

Remark 4.1.
(a) On a pseudo-Hermitian manifold, the interchange of two covariant derivatives with respect to the Tanaka-Webster connection yields not only the curvature terms, but also the pseudo-Hermitian torsion term. Hence it seems natural that the conditions for Bochner-type results include both ingredients.
(b) We have already known that a horizontally constant map is foliated. So the maps in cases (ii) and (iii) are foliated. Using the Sasakian assumption on the target manifold, the fourth equation of (2.17) implies that f is also foliated for the case (i) of Theorem 4.8.

The following two results show that if the domain manifold in Theorem 4.8 is also Sasakian, then the condition $m \geq 2$ is not necessary, and the curvature condition on the target manifold may be slightly weakened.

Corollary 4.9. *Let $f : (M^{2m+1}, J, \theta) \to (N^{2n+1}, \widetilde{J}, \widetilde{\theta})$ be a $(H, \widetilde{H})$-harmonic map from a compact Sasakian manifold to a Sasakian manifold with non-positive horizontal sectional curvature. Suppose $Ric_H \geq 0$. Then*
   (i) $\beta_{H,\overline{H}} = 0$;
   (ii) If $Ric_H > 0$ at a point p in M, then f is horizontally constant;
   (iii) If N has negative horizontal sectional curvature, then f is either horizontally constant or of horizontal rank one.

Proof. Using Lemma 4.7 and the second claim in Theorem 4.4, the remaining arguments are similar to that for Theorem 4.8. $\square$

Corollary 4.10. *Let M, N and f be as in Corollary 4.9. If f is horizontal, then*
   (i) $\beta = 0$ (This property is called totally geodesic);
   (ii) If $Ric_H > 0$ at a point, then f is constant;
   (iii) If N has negative horizontal sectional curvature, then f is either constant or of horizontal rank one.

Proof. From Corollary 4.9 and Remark 4.1 (b), we know that f is a foliated map with $\beta_{H,\widetilde{H}} = 0$, and thus Lemma 3.6 implies that

$$(4.43) \qquad\qquad df(\xi) = \lambda\widetilde{\xi}, \quad f^*\widetilde{\theta} = \lambda\theta$$

for some constant λ. Clearly the first equation of (4.43) yields that

$$(4.44) \qquad\qquad \beta(\xi, X) = 0$$

for any $X \in TM$. Since f is horizontal, we have $f_{jl}^0 = f_{j\overline{l}}^0 = 0$. Therefore we conclude that $\beta = 0$.

The results for cases (ii) and (iii) follow immediately from Lemma 4.6 and Corollary 4.9. □

Remark 4.2. Corollary 4.10 improves a similar theorem of [Pe2] in two aspects. It not only slightly strengthens the corresponding results, but also weakens the curvature condition for the target manifold.

5. $(H, \widetilde{H})$-pluriharmonic and $(H, \widetilde{H})$-holomorphic maps

In this section, we first introduce two special kinds of $(H, \widetilde{H})$-harmonic maps: $(H, \widetilde{H})$-pluriharmonic maps and foliated $(H, \widetilde{H})$-holomorphic maps. Secondly, we give a unique continuation theorem which ensures that a $(H, \widetilde{H})$-harmonic map must be $(H, \widetilde{H})$-holomorphic on the whole manifold if it is $(H, \widetilde{H})$-holomorphic on an open subset. Clearly a similar unique continuation result holds true for $(H, \widetilde{H})$-antiholomorphicity. As a result, we easily deduce a unique continuation theorem for horizontally constant maps.

The $(H, \widetilde{H})$-harmonicity equation (3.11) suggests us to introduce the following

Definition 5.1. A map $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermitian manifolds is called a $(H, \widetilde{H})$-pluriharmonic map if it satisfies

$$(5.1) \qquad (\beta_{H,\widetilde{H}} + f^*\theta \otimes f^*\widetilde{\tau})^{(1,1)} = 0,$$

where the left hand side term of (5.1) denotes the restriction of $\beta_{H,\widetilde{H}} + f^*\theta \otimes f^*\widetilde{\tau}$ to $H^{1,1}(M)$. Here $H^{1,1}(M)$ denotes the $(1,1)$-part of $H(M)^C \otimes H(M)^C$ with respect to the complex structure on $H(M)$.

Clearly (5.1) implies that

$$tr_{G_\theta}(\beta_{H,\widetilde{H}} + f^*\theta \otimes f^*\widetilde{\tau}) = 0,$$

that is, a $(H, \widetilde{H})$-pluriharmonic map is automatically $(H, \widetilde{H})$-harmonic. It follows from (3.12), (3.13) and (5.1) that a map f is $(H, \widetilde{H})$-pluriharmonic if and only if

$$(5.2) \qquad f^\alpha_{j\overline{k}} + \widehat{A}^\alpha_{\overline{\beta}} f^0_j f^{\overline{\beta}}_{\overline{k}} = 0$$

and

$$(5.3) \qquad f^\alpha_{\overline{k}j} + \widehat{A}^\alpha_{\overline{\beta}} f^0_{\overline{k}} f^{\overline{\beta}}_j = 0$$

for $1 \le \alpha \le n$, $1 \le j, k \le m$ or equivalently, $f^{\overline{\alpha}}_{j\overline{k}} + \widehat{A}^{\overline{\alpha}}_{\beta} f^0_j f^\beta_{\overline{k}} = 0$ and $f^{\overline{\alpha}}_{\overline{k}j} + \widehat{A}^{\overline{\alpha}}_{\beta} f^0_{\overline{k}} f^\beta_j = 0$ for $1 \le \alpha \le n$, $1 \le j, k \le m$.

Proposition 5.1. *Suppose that* $f : (M, H(M), \theta, J) \to (N, \widetilde{H}(N), \widetilde{\theta}, \widetilde{J})$ *is a* $(H, \widetilde{H})$-*pluriharmonic map. Then* f *is a foliated* $(H, \widetilde{H})$-*harmonic map.*

Proof. We have already shown that f is $(H, \widetilde{H})$-harmonic. From (2.17), (5.2), (5.3), one may find that

$$f^\alpha_0 = f^{\overline{\alpha}}_0 = 0,$$

that is, f is foliated. □

Definition 5.2. A map $f : (M, H(M), \theta, J) \to (N, \widetilde{H}(N), \widetilde{\theta}, \widetilde{J})$ between two pseudo-hermitian manifolds is called $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) if it satisfies

$$\text{(5.4)} \qquad df_{H,\widetilde{H}} \circ J = \widetilde{J} \circ df_{H,\widetilde{H}} \quad (\text{resp. } df_{H,\widetilde{H}} \circ J = -\widetilde{J} \circ df_{H,\widetilde{H}})$$

Furthermore, if f is foliated, then it is called a foliated $(H, \widetilde{H})$-holomorphic map (resp. a foliated $(H, \widetilde{H})$-antiholomorphic map).

Remark 5.1. Clearly the composition of two foliated (resp. $(H, \widetilde{H})$-holomorphic) maps is still a foliated (resp. $(H, \widetilde{H})$-holomorphic) map. Note that the foliation of a pseudo-Hermitian manifold M is not transversally holomorphic in general, although there is a complex structure J on its horizontal distribution $H(M)$.

Suppose $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is a smooth map between two pseudo-Hermitian manifolds. The complexification of $df_{H,\widetilde{H}}$ determines various partial horizontal differentials by the compositions with the inclusions of $H^{1,0}(M)$ and $H^{0,1}(M)$ in $H(M)^C$ respectively and the projections of $\widetilde{H}(N)^C$ on $H^{1,0}(N)$ and $H^{0,1}(N)$ respectively. Thus we have the following bundle morphisms (cf. [Si1], [Do])

$$\text{(5.5)} \qquad \begin{aligned} \partial f_{H,\widetilde{H}} &: H^{1,0}(M) \to \widetilde{H}^{1,0}(N), \quad \overline{\partial} f_{H,\widetilde{H}} : H^{0,1}(M) \to \widetilde{H}^{1,0}(N), \\ \partial \overline{f_{H,\widetilde{H}}} &: H^{1,0}(M) \to \widetilde{H}^{0,1}(N), \quad \overline{\partial f_{H,\widetilde{H}}} : H^{0,1}(M) \to \widetilde{H}^{0,1}(N), \end{aligned}$$

which can be locally expressed as follows

$$\text{(5.6)} \qquad \begin{aligned} \partial f_{H,\widetilde{H}} &= f_j^\alpha \theta^j \otimes \widetilde{\eta}_\alpha, \quad \overline{\partial} f_{H,\widetilde{H}} = f_{\overline{j}}^\alpha \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha, \\ \partial \overline{f_{H,\widetilde{H}}} &= f_j^{\overline{\alpha}} \theta^j \otimes \widetilde{\eta}_{\overline{\alpha}}, \quad \overline{\partial f_{H,\widetilde{H}}} = f_{\overline{j}}^{\overline{\alpha}} \theta^{\overline{j}} \otimes \widetilde{\eta}_{\overline{\alpha}}. \end{aligned}$$

From (5.4), it is clear to see that $f : M \to N$ is $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) if and only if $\overline{\partial} f_{H,\widetilde{H}} = 0$ (resp. $\partial f_{H,\widetilde{H}} = 0$).

Proposition 5.2. *Suppose that $f : M \to N$ is either $(H, \widetilde{H})$-holomorphic or $(H, \widetilde{H})$-antiholomorphic. Then f is $(H, \widetilde{H})$-harmonic if and only if f is foliated.*

Proof. Without loss of generality, we assume that f is $(H, \widetilde{H})$-holomorphic. Then the $(H, \widetilde{H})$-holomorphicity of f means that $f_{\overline{k}}^\alpha = f_k^{\overline{\alpha}} = 0$, and thus $f_{\overline{k}j}^\alpha = 0$. Consequently, the $(H, \widetilde{H})$-harmonicity equation (3.14) becomes

$$\text{(5.7)} \qquad f_{k\overline{k}}^\alpha + \widehat{A}_{\overline{\beta}}^\alpha f_k^0 f_{\overline{k}}^{\overline{\beta}} = 0.$$

On the other hand, the fourth equation of (2.17) yields that

$$\text{(5.8)} \qquad f_{k\overline{k}}^\alpha + \widehat{A}_{\overline{\beta}}^\alpha f_k^0 f_{\overline{k}}^{\overline{\beta}} = m i f_0^\alpha$$

Therefore we conclude from (5.7) and (5.8) that f is $(H, \widetilde{H})$-harmonic if and only if it is foliated. $\square$

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Theorem 5.3. *Let $f : (M, H(M), \theta, J) \to (N, \widetilde{H}(N), \widetilde{\theta}, \widetilde{J})$ be either a foliated $(H, \widetilde{H})$-holomorphic map or a foliated $(H, \widetilde{H})$-antiholomorphic map. Then f is a $(H, \widetilde{H})$-pluriharmonic map.*

Proof. Without loss of generality, we may assume that f is $(H, \widetilde{H})$-holomorphic. Then we get $f_{\overline{j}}^{\alpha} = f_{\overline{j}}^{\overline{\alpha}} = f_{\overline{jl}}^{\alpha} = 0$ as in Proposition 5.2. It follows that

(5.9)
$$f_{\overline{jl}}^{\alpha} + \widehat{A}_{\overline{\beta}}^{\alpha} f_{\overline{j}}^{0} f_{l}^{\overline{\beta}} = 0.$$

Since f is foliated, we have $f_{0}^{\alpha} = 0$. Consequently, (2.17) and (5.9) give that

$$f_{l\overline{j}}^{\alpha} + \widehat{A}_{\overline{\beta}}^{\alpha} f_{l}^{0} f_{\overline{j}}^{\overline{\beta}} = 0.$$

This proves the $(H, \widetilde{H})$-pluriharmonicity of f. $\square$

Recall that a map $f : M \to N$ is called a CR map if f is horizontal and $df_{H, \widetilde{H}} \circ J = \widetilde{J} \circ df_{H, \widetilde{H}}$ ([DTo]). Thus CR maps provide us many examples of $(H, \widetilde{H})$-holomorphic maps. Since a CR map is not necessarily a foliated map, it is not $(H, \widetilde{H})$-harmonic in general. A map $f : M \to N$ between two pseudo-Hermitian manifolds is called a CR-holomorphic map (cf. [Dr], [IP], [Ur]) if

(5.10)
$$df \circ J = \widetilde{J} \circ df.$$

From [IP], we know that a CR-holomorphic map is a harmonic map between the two Riemannian manifolds (M, g_θ) and $(N, \widetilde{g}_{\widetilde{\theta}})$ in the usual sense. Clearly (5.10) implies that $\widetilde{J} \circ df(\xi) = 0$ and $df(H(M)) \subset \widetilde{H}(N)$. Hence a CR-holomorphic map is just a foliated CR map. In addition, a special kind of CR maps, called pseudo-Hermitian immersions in [Dr], also provide us a lot of examples for foliated $(H, \widetilde{H})$-holomorphic maps.

Definition 5.3. We call a diffeomorphism $f : (M^{2m+1}, J, \theta) \to (N^{2m+1}, \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermitian manifolds a $(H, \widetilde{H})$-biholomorphism if f and f^{-1} are $(H, \widetilde{H})$-holomorphic and $(\widetilde{H}, H)$-holomorphic respectively. Furthermore, if f is foliated, then it is called a foliated $(H, \widetilde{H})$-biholomorphism.

Example 5.1. Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. For any positive function u on M, we set $\widetilde{\theta} = u\theta$. Then

$$d\widetilde{\theta} = du \wedge \theta + u d\theta.$$

Write $\widetilde{\xi} = \lambda \xi + \widetilde{T}^{\alpha} \eta_{\alpha} + \widetilde{T}^{\overline{\alpha}} \eta_{\overline{\alpha}}$. By requiring that $i_{\widetilde{\xi}} \widetilde{\theta} = 1$ and $i_{\widetilde{\xi}} d\widetilde{\theta} = 0$, one gets

$$\lambda = u^{-1}, \quad \widetilde{T}^{\alpha} = iu^{-1}\eta_{\overline{\alpha}}(\log u), \quad \widetilde{T}^{\overline{\alpha}} = -iu^{-1}\eta_{\alpha}(\log u).$$

Obviously

$$id_M : (M, H(M), J, \theta) \to (M, H(M), J, \widetilde{\theta})$$

is a $(H, \widetilde{H})$-biholomorphism. Note that $\widetilde{H}(M) = H(M)$ in this example. If u is not constant, then $\widetilde{\xi} \nparallel \xi$, and thus id_M is not foliated. This provides us an example of non-foliated $(H, \widetilde{H})$-biholomorphisms. In this circumstance, we know from Proposition 5.2 that $id_M : (M, H(M), J, \theta) \to (M, H(M), J, \widetilde{\theta})$ is not $(H, \widetilde{H})$-harmonic. When u is constant, $id_M : (M, H(M), J, \theta) \to (M, H(M), J, \widetilde{\theta})$ is clearly a foliated $(H, \widetilde{H})$-biholomorphism.

Example 5.2. Let $S = (\xi, \theta, J, g_\theta)$ be a Sasakian structure on M. Then we have its Reeb foliation F_ξ and the associated quotient vector bundle $\nu(F_\xi)$. Let $\pi_\nu : TM \to \nu(F_\xi)$ ($X \mapsto [X]$ for any $X \in TM$) be the natural projection. The background structure $(\xi, \theta, J, g_\theta)$ induces a transversal complex structure J_ν on $\nu(F_\xi)$ by $J_\nu[X] = [JX]$. According to [BG], [BGS], we consider the following space of Sasakian structures with fixed Reeb vector field ξ and fixed transversal complex structure J_ν:

$$S(\xi, J_\nu) = \{\text{Sasakian structure } \widetilde{S} = (\widetilde{\xi}, \widetilde{\theta}, \widetilde{J}, g_{\widetilde{\theta}}) \text{ on } M \mid \widetilde{\xi} = \xi, \widetilde{J}_\nu = J_\nu\}$$

where $\widetilde{J}_\nu$ denotes the transversal complex structure on $\nu(F_\xi)$ induced by $\widetilde{J}$. For any $\widetilde{S} \in S(\xi, J_\nu)$, we assert that $id_M : (M, S) \to (M, \widetilde{S})$ is a foliated $(H, \widetilde{H})$-biholomorphism. To prove this assertion, we may decompose any $X \in TM$ as $X = a\xi + X_H = b\xi + X_{\widetilde{H}}$ with $X_H \in H(M) = \ker\theta$ and $X_{\widetilde{H}} \in \widetilde{H}(M) = \ker\widetilde{\theta}$ for some $a, b \in R$. Then $\pi_{\widetilde{H}}(X_H) = X_{\widetilde{H}}$, and (1.7) implies that $JX = JX_H$ and $\widetilde{J}X = \widetilde{J}X_{\widetilde{H}}$. By definition, $J_\nu[X] = [JX_H]$ and $\widetilde{J}_\nu[X] = [\widetilde{J}X_{\widetilde{H}}]$. Since $\pi_\nu : H(M) \to \nu(F_\xi)$ and $\pi_\nu : \widetilde{H}(M) \to \nu(F_\xi)$ are both vector bundle isomorphisms, we see that $\widetilde{J}_\nu = J_\nu$ if and only if $\pi_\nu \widetilde{J}X_{\widetilde{H}} = \pi_\nu JX_H$ for any $X \in TM$. On the other hand, $di_{H,\widetilde{H}} \circ JX = \widetilde{J}di_{H,\widetilde{H}}X$ if and only if $\pi_{\widetilde{H}}JX_H = \widetilde{J}\pi_{\widetilde{H}}(X_H)$ that is, $\pi_{\widetilde{H}}JX_H = \widetilde{J}X_{\widetilde{H}}$. Taking projection π_ν on both sides of the previous equality, we get the result.

Recall that a function u on a foliated manifold M is called basic if it is constant along the leaves. Denote by $C_B^\infty(M)$ the space of smooth basic functions on M. In terms of [BG1,2], we know that the space $S(\xi, J_\nu)$ is an affine space modeled on $(C_B^\infty(M)/R) \times (C_B^\infty(M)/R) \times H^1(M, Z)$. Indeed, if $S = (\xi, \theta, J, g_\theta)$ is a given Sasakian structure in $S(\xi, J_\nu)$, any other Sasakian structure $\widetilde{S} = (\widetilde{\xi}, \widetilde{\theta}, \widetilde{J}, g_{\widetilde{\theta}})$ in it is determined by real valued basic functions $\varphi, \psi \in C_B^\infty(M)$ up to a constant and $\alpha \in H^1(M, Z)$ a harmonic 1-form such that

$$(5.11) \qquad\qquad \widetilde{\theta} = \theta + d^c\varphi + \alpha + d\psi$$

where $d^c = \frac{\sqrt{-1}}{2}(\overline{\partial} - \partial)$. Thus one may denote the Sasakian structure $\widetilde{S}$ by $\widetilde{S}_{\varphi, \psi, \alpha}$ if S is fixed. In order to use the notations in §2, we write $f = id_M$ and express (5.11) as

$$(5.12) \qquad f^*\widetilde{\theta} = \theta + \frac{\sqrt{-1}}{2}\left(\varphi_{\overline{k}}\theta^{\overline{k}} - \varphi_k\theta^k\right) + \alpha_k\theta^k + \alpha_{\overline{k}}\theta^{\overline{k}} + \psi_k\theta^k + \psi_{\overline{k}}\theta^{\overline{k}}$$

where $\varphi_k = \eta_k(\varphi)$, $\psi_k = \eta_k(\psi)$ and $\alpha_k = \alpha(\eta_k)$. Then

$$f_k^0 = -\frac{\sqrt{-1}}{2}\varphi_k + \alpha_k + \psi_k, \ \ f_{\overline{k}}^0 = \frac{\sqrt{-1}}{2}\varphi_{\overline{k}} + \alpha_{\overline{k}} + \psi_{\overline{k}}.$$

It follows that

$$(5.13)$$
$$f_{k\overline{k}}^0 + f_{\overline{k}k}^0 = \frac{\sqrt{-1}}{2}\left(\varphi_{\overline{k}k} - \varphi_{k\overline{k}}\right) + \alpha_{k\overline{k}} + \alpha_{\overline{k}k} + \psi_{k\overline{k}} + \psi_{\overline{k}k}$$
$$= -\frac{m\sqrt{-1}}{2}\xi(\varphi) + \alpha_{k\overline{k}} + \alpha_{\overline{k}k} + \psi_{k\overline{k}} + \psi_{\overline{k}k}$$

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where we use the known formula $\varphi_{k\bar{l}} - \varphi_{\bar{l}k} = \sqrt{-1}\delta_{kl}\xi(\varphi)$ for a function in the second equality (see for example [Le], [CDRY]). Since φ is basic and α is harmonic, we find from (5.13) that

$$f^0_{k\overline{k}} + f^0_{\overline{k}k} = \triangle_H(\psi).$$

It follows that $id_M : (M, S) \to (M, \widetilde{S}_{\varphi, 0, \alpha})$ is a special foliated $(H, \widetilde{H})$-biholomorphism. In general, if ψ is not constant, then $id_M : (M, S) \to (M, \widetilde{S}_{\varphi, \psi, \alpha})$ is not special in the sense of Definition 3.2.

Theorem 5.4. *Suppose that M, N are Sasakian manifolds and $f : M \to N$ is a foliated $(H, \widetilde{H})$-harmonic map. Let U be a nonempty open subset of M. If f is $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) on U, then f is $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) on M.*

Proof. Without loss of generality, we assume that f is $(H, \widetilde{H})$-holomorphic on U. Although f satisfies a PDE system of 'subelliptic type', to the author's knowledge, the unique continuation theorem is still open for such kind of PDE systems. We will try to show this theorem by using the Aroszajin's continuation theorem for elliptic PDE systems and the moving frame method.

Let Ω be the largest connected open subset of M containing U such that $\overline{\partial} f_{H, \widetilde{H}}$ vanishes identically on Ω. Suppose Ω has a boundary point q. Let W be a connected open neighborhood of q in M such that

  i) there exists a frame field $\{\xi, \eta_1, ...\eta_m, \eta_{\overline{1}}, ..., \eta_{\overline{m}}\}$ of TM^C on some open neighborhood of the closure of W and

  ii) there exists a frame field $\{\widetilde{\xi}, \widetilde{\eta}_1, ..., \widetilde{\eta}_n, \widetilde{\eta}_{\overline{1}}, ..., \widetilde{\eta}_{\overline{m}}\}$ of TN^C on some open neighborhood of the closure of $f(W)$.

  The assumption that f is foliated means that

$$(5.11) \qquad\qquad\qquad f^\alpha_0 = f^{\overline{\alpha}}_0 = 0.$$

Since N is Sasakian, the $(H, \widetilde{H})$-harmonic equation for f becomes

$$(5.12) \qquad\qquad\qquad f^\alpha_{k\overline{k}} + f^\alpha_{\overline{k}k} = 0.$$

By definition of covariant derivatives, we have

$$(5.13) \qquad D\overline{\partial} f_{H, \widetilde{H}} = df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha$$

and

$$\begin{aligned}
D^2 \overline{\partial} f_{H, \widetilde{H}} =& \nabla df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + df^\alpha_{\overline{j}} \otimes \nabla \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha \\
(5.14) \qquad & + df^\alpha_{\overline{j}} \otimes \nabla \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla^2 \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha \\
& + df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \theta^{\overline{j}} \otimes \widetilde{\nabla}^2 \widetilde{\eta}_\alpha.
\end{aligned}$$

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Now we compute the trace Laplacian of the section $\overline{\partial} f_{H,\widetilde{H}}$ as follows:

$$\triangle \overline{\partial} f_{H,\widetilde{H}} = tr_{g_\theta} D^2 \overline{\partial} f_{H,\widetilde{H}}$$

$$= (f^\alpha_{\overline{j}k\overline{k}} + f^\alpha_{\overline{j}kk} + f^\alpha_{\overline{j}00}) \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha$$

(5.15)
$$= (\triangle_M f^\alpha_{\overline{j}}) \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + tr_{g_\theta} \{ df^\alpha_{\overline{j}} \otimes \nabla \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha$$

$$+ df^\alpha_{\overline{j}} \otimes \nabla \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla^2 \theta^{\overline{j}} \otimes \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha$$

$$+ df^\alpha_{\overline{j}} \otimes \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \nabla \theta^{\overline{j}} \otimes \widetilde{\nabla} \widetilde{\eta}_\alpha + f^\alpha_{\overline{j}} \theta^{\overline{j}} \otimes \widetilde{\nabla}^2 \widetilde{\eta}_\alpha \}$$

where $\triangle_M$ denotes the Laplace-Beltrami operator acting on functions. Since M and N are Sasakian, we derive from the second equation of (2.17) and (5.11) that

(5.16)
$$f^\alpha_{\overline{j}0} = 0$$

which yields

(5.17)
$$f^\alpha_{\overline{j}00} = 0.$$

Using (2.17), (2.38), (2.44) and (5.12), we discover

(5.18)
$$f^\alpha_{\overline{j}k\overline{k}} + f^\alpha_{\overline{j}kk} = f^\alpha_{k\overline{j}k} + f^\alpha_{\overline{k}\overline{j}k}$$

$$= f^\beta_k \widehat{R}^\alpha_{\beta\gamma\overline{\delta}} (f^\gamma_{\overline{k}} f^{\overline{\delta}}_{\overline{j}} - f^\gamma_{\overline{j}} f^{\overline{\delta}}_{\overline{k}}) - f^\alpha_{\overline{t}} R^{\overline{t}}_{\overline{k}k\overline{j}} + f^\beta_{\overline{k}} \widehat{R}^\alpha_{\beta\gamma\overline{\delta}} (f^\gamma_k f^{\overline{\delta}}_{\overline{j}} - f^\gamma_{\overline{j}} f^{\overline{\delta}}_k).$$

Consequently

(5.19)
$$|f^\alpha_{\overline{j}k\overline{k}} + f^\alpha_{\overline{j}kk} + f^\alpha_{\overline{j}00}| \le C \sum_{l,\beta} |f^\beta_{\overline{l}}|$$

on W for some positive number C. From (5.15) and (5.19), we find that there is a positive number C' such that

(5.20)
$$|\triangle_M (f^\alpha_{\overline{j}})| \le C' (\sum_{l,\beta} |\nabla f^\beta_{\overline{l}}| + \sum_{l,\alpha} |f^\alpha_{\overline{l}}|)$$

where ∇ denotes the gradient of the functions $\{f^\alpha_{\overline{j}}\}$. By applying the Aronszajn's unique continuation theorem (cf. [Ar], [PRS]) to the system of functions $Re\{f^\alpha_{\overline{j}}\}$, $Im\{f^\alpha_{\overline{j}}\}$ $(1 \le j \le m, 1 \le \alpha \le n)$ and to the elliptic operator $\triangle_M$, we conclude from the identical vanishing of $Re\{f^\alpha_{\overline{j}}\}$, $Im\{f^\alpha_{\overline{j}}\}$ on $W \cap \Omega$ that $Re\{f^\alpha_{\overline{j}}\}$, $Im\{f^\alpha_{\overline{j}}\}$ vanish identically on W. This contradicts the fact that q is a boundary point of Ω. Hence $\Omega = M$, which implies that $\overline{\partial} f_{H,\widetilde{H}} = 0$ on the whole domain manifold. $\square$

Remark 5.2. Note that we verify the structural assumptions of Aronszajn-Cordes in the proof of Theorem 5.4 by adopting the moving frame method, whose advantage is its operability. This method will be used again in the appendix.

Note that f is both $(H, \widetilde{H})$-holomorphic and $(H, \widetilde{H})$-antiholomorphic if and only if $df_{H,\widetilde{H}} = 0$, that is, f is horizontally constant.

Corollary 5.5. *Suppose that M, N are Sasakian manifolds and $f : M \to N$ is a foliated $(H, \widetilde{H})$-harmonic map. Let U be a nonempty open subset of M. If f is horizontally constant on U, then f is horizontally constant on M.*

Proof. Since f is horizontally constant on U, it is both $(H, \widetilde{H})$–holomorphic and $(H, \widetilde{H})$-antiholomorphic on U. It follows from Theorem 5.4 that f is both $(H, \widetilde{H})$-holomorphic and $(H, \widetilde{H})$-anitholomorphic on the whole M. Hence we conclude that f is horizontally constant on M. □

It would be interesting to know whether the Sasakian and foliated conditions in Theorem 5.4 or Corollary 5.5 can be removed or not. In terms of Proposition 5.2, one way to give an answer is to establish a unique continuation result for the foliated property. We would like to propose the following question:

Question. *Suppose $f : M \to N$ is a $(H, \widetilde{H})$-harmonic map between two pseudo-Hermitian manifolds or even Sasakian manifolds. If f is foliated on a nonempty open subset U of M, can we deduce that f is foliated on the whole M?*

Though a general Sasakian manifold is not a global Riemannian submersion over a Kähler manifold, the following result will help us to understand the general local picture and properties about $(H, \widetilde{H})$-holomorphic maps between Sasakian manifolds.

Proposition 5.6. *Suppose $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ are compact Sasakian manifolds which are the total spaces of Riemannian submersions $\pi : M \to B$ and $\widetilde{\pi} : N \to \widetilde{B}$ over compact Kähler manifolds B and $\widetilde{B}$ respectively. Suppose $f : M \to N$ is a foliated map which induces a map $h : B \to \widetilde{B}$ between the base manifolds. Then f is a foliated $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) map if and only if h is a holomorphic (resp. anti-holomorphic) map.*

Proof. Since f is foliated, we have $h \circ \pi = \widetilde{\pi} \circ f$, and thus $dh \circ d\pi = d\widetilde{\pi} \circ df$. Denote by J_1 and J_2 the complex structures of B and $\widetilde{B}$ respectively. Since $d\pi \circ J = J_1 \circ d\pi$ and $d\widetilde{\pi} \circ \widetilde{J} = J_2 \circ d\widetilde{\pi}$, we immediately have that f is $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) if and only if h is holomorphic (resp. antiholomorphic). □

Remark 5.2. Suppose now that M and N are two general Sasakian manifolds, which are not necessarily total spaces of Riemannian submersions, and $f : M \to N$ is a foliated map. Let p be any point in M and $q = f(p)$. We have foliated neighborhoods U_1 and $\widetilde{U}_2$ of p and q respectively, together with Riemannian submersions $\pi_1 : U_1 \to W_1$ and $\widetilde{\pi}_2 : \widetilde{U}_2 \to \widetilde{W}_2$ over two Kähler manifolds W_1 and $\widetilde{W}_2$. Assuming that $f(U_1) \subset \widetilde{U}_2$, then f induces locally a map $h_p : W_1 \to \widetilde{W}_2$. According to Proposition 5.6, we find that $f : M \to N$ is $(H, \widetilde{H})$-holomorphic (resp. $(H, \widetilde{H})$-antiholomorphic) if and only if the locally induced map h_p for each $p \in M$ is holomorphic (resp. anti-holomorphic).

Definition 5.3. A foliated map $f : (M, H(M), \theta, J) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermtian manifolds is called a horizontally one-to-one map if it induces a one-to-one map $h : M/F_\xi \to N/F_{\widetilde{\xi}}$ between the spaces of leaves.

Proposition 5.7. *Let M and N be Sasakian manifolds and let $f : M \to N$ be a foliated $(H, \widetilde{H})$-holomorphic map. If f is both one-to-one and horizontally one-to-one, then f^{-1} is a $(\widetilde{H}, H)$-holomorphic map, that is, f is a foliated $(H, \widetilde{H})$-biholomorphism.*

Proof. For any $p \in M$, we let $\pi_1 : U_1 \to W_1$, $\pi_2 : \widetilde{U}_2 \to \widetilde{W}_2$ and $h_p : W_1 \to \widetilde{W}_2$ be as in Remark 5.2. Then $h_p : W_1 \to \widetilde{W}_2$ is holomorphic. Since f is horizontally one-to-one, we see that $h_p : W_1 \to h_p(W_1)$ is one-to-one. By a known result for holomorphic maps (cf. Proposition 1.1.13 in [Hu]), we know that h_p is a local biholomorphic, that is, $h_p(W_1)$ is an open set of $\widetilde{W}_2$ and $h_p : W_1 \to h_p(W_1)$ is biholomorphic.

Set $\widetilde{V}_2 = f(U_1) \subset \widetilde{U}_2$. Clearly $f^{-1} : N \to M$ is also a foliated map with $f^{-1}(\widetilde{V}_2) = U_1$, and f^{-1} induces the holomorphic map $h_p^{-1} : \widetilde{h}_p(W_1) \to W_1$. Using Proposition 5.6, we find that f^{-1} is $(\widetilde{H}, H)$-holomorphic at $q = f(p)$. Since p is arbitrary, we conclude that f is a foliated $(H, \widetilde{H})$-biholomorphism. $\square$

6. Lichnerowicz type results

By definition, one gets from (5.6) that (cf. [Do])

$$
|\partial f_{H,\widetilde{H}}|^2 = \langle \partial f_{H,\widetilde{H}}(\eta_j), \overline{\partial f_{H,\widetilde{H}}(\eta_j)} \rangle
$$

$$
(6.1) \qquad = \frac{1}{4} \sum_{j=1}^{m} \{ \langle df_{H,\widetilde{H}}(e_j), df_{H,\widetilde{H}}(e_j) \rangle + \langle df_{H,\widetilde{H}}(Je_j), df_{H,\widetilde{H}}(Je_j) \rangle
$$

$$
+ 2 \langle df_{H,\widetilde{H}}(Je_j), \widetilde{J} df_{H,\widetilde{H}}(e_j) \rangle \}
$$

and

$$
|\overline{\partial} f_{H,\widetilde{H}}|^2 = \langle \overline{\partial} f_{H,\widetilde{H}}(\eta_{\bar{j}}), \overline{\overline{\partial} f_{H,\widetilde{H}}(\eta_{\bar{j}})} \rangle
$$

$$
(6.2) \qquad = \frac{1}{4} \sum_{j=1}^{m} \{ \langle df_{H,\widetilde{H}}(e_j), df_{H,\widetilde{H}}(e_j) \rangle + \langle df_{H,\widetilde{H}}(Je_j), df_{H,\widetilde{H}}(Je_j) \rangle
$$

$$
- 2 \langle df_{H,\widetilde{H}}(Je_j), \widetilde{J} df_{H,\widetilde{H}}(e_j) \rangle \}.
$$

Thus

$$
(6.3) \qquad \frac{1}{2} |df_{H,\widetilde{H}}|^2 = |\partial f_{H,\widetilde{H}}|^2 + |\overline{\partial} f_{H,\widetilde{H}}|^2.
$$

Define

$$
(6.4) \qquad E'_{H,\widetilde{H}}(f) = \int_M |\partial f_{H,\widetilde{H}}|^2 dv_\theta, \quad E''_{H,\widetilde{H}}(f) = \int_M |\overline{\partial} f_{H,\widetilde{H}}|^2 dv_\theta.
$$

Then $E_{H,\widetilde{H}}(f) = E'_{H,\widetilde{H}}(f) + E''_{H,\widetilde{H}}(f)$. Set

$$
(6.5) \qquad k_{H,\widetilde{H}}(f) = |\partial f_{H,\widetilde{H}}|^2 - |\overline{\partial} f_{H,\widetilde{H}}|^2, \quad K_{H,\widetilde{H}}(f) = E'_{H,\widetilde{H}}(f) - E''_{H,\widetilde{H}}(f).
$$

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Lemma 6.2. *Set $\omega^M = d\theta$ and $\omega^N = d\widetilde{\theta}$. Then*

$$k_{H,\widetilde{H}}(f) = \langle \omega^M, f^*\omega^N \rangle.$$

Proof. Choose an orthonormal frame $\{e_1, ..., e_m, Je_1, ..., Je_m\}$ of $H(M)$. Using (1.9), we deduce that

$$
\begin{aligned}
\langle \omega^M, f^*\omega^N \rangle &= \sum_{i<j} \{(f^*\omega^N)(e_i, e_j)\omega^M(e_i, e_j) + (f^*\omega^N)(Je_i, Je_j)\omega^M(Je_i, Je_j)\} \\
&\quad + \sum_{i,j} (f^*\omega^N)(e_i, Je_j)\omega^M(e_i, Je_j) \\
&= \sum_i \langle \widetilde{J}df(e_i), df(Je_i)\rangle \\
&= \sum_i \langle \widetilde{J}df_{H,\widetilde{H}}(e_i), df_{H,\widetilde{H}}(Je_i)\rangle.
\end{aligned}
$$

(6.6)

Consequently (6.1), (6.2) and (6.6) imply the lemma. $\square$

We need the following lemma:

Lemma 6.3 (Homotopy Lemma, cf. [Lic], [EL]). *Let $f_t : M \to N$ be a smooth family of maps between the smooth manifolds M and N, parameterized by the real number t, and let ω be a closed two-form on N. Then*

$$\frac{\partial}{\partial t}(f_t^*\omega) = d\big(f_t^* i(\frac{\partial f_t}{\partial t})\omega\big)$$

where $i(X)\omega$ denotes the interior product of the vector X with the two-form ω.

Lemma 6.4. *Let $f_t : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a family of smooth maps between two pseudo-Hermitian manifolds. Then*

$$\frac{d}{dt}K_{H,\widetilde{H}}(f_t) = m\int_M d\widetilde{\theta}(v_t, df_t(\xi))dv_\theta$$

where $v_t = \partial f_t/\partial t$.

Proof. Clearly Lemmas 6.2, 6.3 imply that

$$
\begin{aligned}
\frac{d}{dt}K_{H,\widetilde{H}}(f_t) &= \int_M \langle \frac{\partial}{\partial t}f_t^*\omega^N, \omega^M \rangle dv_\theta \\
&= \int_M \langle d\sigma_t, \omega^M \rangle dv_\theta \\
&= \int_M \langle \sigma_t, \delta\omega^M \rangle dv_\theta
\end{aligned}
$$

(6.7)

where $\sigma_t = f_t^* i(\frac{\partial f_t}{\partial t})\omega^N$. Choose an orthonormal frame field $\{e_A\}_{A=0}^{2m} = \{\xi, e_1, ..., e_{2m}\}$ on M. From Lemma 1.2, we get

(6.8)
$$\nabla_{e_A}^\theta X = \nabla_{e_A} X - (\frac{1}{2}d\theta(e_A, X) + A(e_A, X))\xi$$

45

and

$$(6.9) \qquad \nabla_{e_A}^\theta \xi = \tau(e_A) + \frac{1}{2} J e_A$$

for any $X \in \Gamma(H(M))$ and $1 \le A \le 2m$. Using (1.4), (1.13) and (6.8), we compute the codifferential $(\delta\omega^M)(X)$ for $X \in \Gamma(H(M))$ as follows:

$$
\begin{aligned}
(\delta\omega^M)(X) &= -\sum_{A=0}^{2m} (\nabla_{e_A}^\theta d\theta)(e_A, X) \\
&= -\sum_{A=0}^{2m} \{e_A[d\theta(e_A, X)] - d\theta(\nabla_{e_A}^\theta e_A, X) - d\theta(e_A, \nabla_{e_A}^\theta X)\} \\
&= -\sum_{A=1}^{2m} \{e_A[d\theta(e_A, X)] - d\theta(\nabla_{e_A} e_A, X)\} + \sum_{A=1}^{2m} d\theta(e_A, \nabla_{e_A} X) \\
&= -\sum_{A=1}^{2m} (\nabla_{e_A} d\theta)(e_A, X) \\
&= 0,
\end{aligned}
$$

(6.10)

due to the fact that $\nabla d\theta = 0$. Next

$$
\begin{aligned}
(\delta\omega^M)(\xi) &= \sum_{A=0}^{2m} d\theta(e_A, \nabla_{e_A}^\theta \xi) \\
&= \sum_{A=1}^{2m} d\theta(e_A, \tau(e_A) + \frac{1}{2} J e_A) \\
&= \sum_{A=1}^{2m} [g_\theta(J e_A, \tau(e_A)) + \frac{1}{2} g_\theta(J e_A, J e_A)] \\
&= m.
\end{aligned}
$$

(6.11)

It follows from (6.7), (6.10) and (6.11) that

$$
\begin{aligned}
\frac{d}{dt} K_{H,\widetilde{H}}(f_t) &= m \int_M \sigma_t(\xi) dv_\theta \\
&= m \int_M d\widetilde{\theta}(v_t, df_t(\xi)) dv_\theta.
\end{aligned}
$$

□

Definition 6.2. Let f_0 and f_1 be two maps between two pseudo-Hermitian manifolds M and N. We say that f_0 and f_1 are vertically homotopic if there exists a map $F : M \times [0,1] \to N$ such that $F(\cdot, 0) = f_0$, $F(\cdot, 1) = f_1$ and for each point $x \in M$, the tangent vector at each point along the curve $F(x, \cdot)$ is vertical.

Theorem 6.5. *Let $(M, H(M), J, \theta)$ and $(N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be two pseudoHermitian manifolds. Suppose that M is compact. Then $K_{H,\widetilde{H}}(f)$ is a smooth vertical homotopy invariant, that is, if f_t is any smooth 1-parameter vertical variation f_t of f, then $t \mapsto K_{H,\widetilde{H}}(f_t)$ is a constant map.*

Proof. Let f_0 and f_1 be two maps from M to N through a family of maps $f_t : M \to N$, $t \in [0, 1]$ with the property that $\partial f_t / \partial t$ is vertical. By Lemma 6.4 and (1.4), we get

$$\frac{d}{dt} K_{H,\widetilde{H}}(f_t) = 0.$$

Consequently $t \mapsto K_{H,\widetilde{H}}(f_t)$ is a constant map. $\square$

Remark 6.2. When N is Sasakian, we even have stronger results. Let f be a map from a pseudo-Hermitian manifold to a Sasakian manifold and $\{f_t\}_{|t|<\varepsilon}$ a vertical variation of f. Set $v_t = \frac{\partial f_t}{\partial t}$ and $\Phi(\cdot, t) = f_t(\cdot)$. Using (1.10), Lemma 2.1 and a direct computation, we may derive from (6.2) that

$$
\begin{aligned}
\frac{\partial}{\partial t} |\overline{\partial} f_{tH,\widetilde{H}}|^2 =& \frac{1}{2} \sum_{j=1}^{m} \langle \widetilde{\nabla}_{e_j} d\Phi_{H,\widetilde{H}}(\frac{\partial}{\partial t}), d\Phi_{H,\widetilde{H}}(e_j) \rangle + \langle \widetilde{\nabla}_{Je_j} d\Phi_{H,\widetilde{H}}(\frac{\partial}{\partial t}), d\Phi_{H,\widetilde{H}}(Je_j) \rangle \\
& - \langle \widetilde{\nabla}_{Je_j} d\Phi_{H,\widetilde{H}}(\frac{\partial}{\partial t}), \widetilde{J} d\Phi_{H,\widetilde{H}}(e_j) \rangle - \langle \Phi_{H,\widetilde{H}}(Je_j), \widetilde{J}\widetilde{\nabla}_{e_j} d\Phi_{H,\widetilde{H}}(\frac{\partial}{\partial t}) \rangle] \\
=& 0.
\end{aligned}
$$

Similarly we have $\frac{\partial}{\partial t} |\partial f_{tH,\widetilde{H}}|^2 = 0$. This shows that the horizontal partial energy densities are preserved under the vertical deformation. Consequently $E_{H,\widetilde{H}}(f_t)$, $E'_{H,\widetilde{H}}(f_t)$ and $E''_{H,\widetilde{H}}(f_t)$ are invariant under the vertical variation of f.

Theorem 6.6. *Let $(M, H(M), J, \theta)$ and $(N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be two pseudoHermitian manifolds and let $f : M \to N$ be a foliated map. Suppose that M is compact. Then $K_{H,\widetilde{H}}(f)$ is a smooth foliated homotopy invariant, that is, $t \mapsto K_{H,\widetilde{H}}(f_t)$ is a constant map for any smooth 1-parameter of foliated maps f_t with $f_0 = f$.*

Proof. Suppose f_t is a smooth 1-parameter of foliated maps with $f_0 = f$. Since f_t is foliated, $df_t(T)$ is vertical. Hence Lemma 6.4 yields

$$\frac{d}{dt} K_{H,\widetilde{H}}(f_t) = 0$$

that is, $t \mapsto K_{H,\widetilde{H}}(f_t)$ is constant. $\square$

Remark 6.3. Although pseudo-Hermitian foliations are not Kähler foliations in general, we would mention that the authors in [BD] proved a similar result for foliated maps between Kähler foliations.

Theorem 6.7. *Suppose* $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ *is either a* $(H, \widetilde{H})$-*holomorphic map or a* $(H, \widetilde{H})$-*antiholomorphic map. Then*

(i) f *is an absolute minimum of* $E_{H,\widetilde{H}}$ *among its vertical homotopy class;*

(ii) If f *is foliated, then it is also an absolute minimum of* $E_{H,\widetilde{H}}$ *among its foliated homotopy class.*

Proof. Without loss of generality, we assume that f is a $(H, \widetilde{H})$-holomorphic map. Let $\widetilde{f}$ be any smooth map in the vertical homotopy class of f. By Theorem 6.5, we have

$$(6.12) \qquad E'_{H,\widetilde{H}}(\widetilde{f}) - E''_{H,\widetilde{H}}(\widetilde{f}) = E'_{H,\widetilde{H}}(f) - E''_{H,\widetilde{H}}(f) = E'_{H,\widetilde{H}}(f)$$

Then $E'_{H,\widetilde{H}}(f) \leq E'_{H,\widetilde{H}}(\widetilde{f})$ and thus $E_{H,\widetilde{H}}(f) \leq E_{H,\widetilde{H}}(\widetilde{f})$. This proves that f is an absolute minimum of $E_{H,\widetilde{H}}$ among its vertical homotopy class. Similarly one may prove that f is an absolute minimum of $E_{H,\widetilde{H}}$ among its foliated homotopy class, provided that f is foliated. $\square$

Remark 6.4. If $f : M \to N$ is a foliated map, then any vertical variation f_t of f is clearly a foliated variation.

Corollary 6.8. *Let* $f : (M^{2m+1}, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ *be either a foliated* $(H, \widetilde{H})$-*holomorphic map or a foliated* $(H, \widetilde{H})$-*antiholomorphic map between two pseudo-Hermitian manifolds. Then*

(i) f *is a pseudo-harmonic map in the sense of [Pe2], that is,* f *is a critical point of* $E_{H,\widetilde{H}}(f_t)$ *for any variation* $\{f_t\}$ *with* $(\partial f_t/\partial t)|_{t=0} \in \Gamma(f^{-1}TN)$;

(ii) If f_t *is a foliated variation of* f, *then* $\frac{d^2}{dt^2}E_{H,\widetilde{H}}(f_t)|_{t=0} \geq 0$.

Proof. Without loss of generality, we assume that f is a foliated $(H, \widetilde{H})$-holomorphic map. From Proposition 5.2, one knows that f is pseudo-harmonic, that is, f is a critical point of $E_{H,\widetilde{H}}(f_t)$ for any horizontal variation $\{f_t\}$. From Theorem 6.7, it follows that f is also a critical point of $E_{H,\widetilde{H}}(f_t)$ for any vertical variation $\{f_t\}$. Hence f is a critical point of $E_{H,\widetilde{H}}(f_t)$ for any variation $\{f_t\}$. This proves (i). It is clear that (ii) follows directly from Theorem 6.7. $\square$

7. Existence of $(H, \widetilde{H})$-harmonic maps under $\widetilde{K}^H \leq 0$

We will introduce a subelliptic heat flow for maps between pseudo-Hermitian manifolds in order to find special $(H, \widetilde{H})$-harmonic maps between these manifolds. We always assume that both M and N are compact, and N is Sasakian in this section.

For a map $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ between two pseudo-Hermitian manifolds, besides the horizontal differential $df_{H,\widetilde{H}} : H(M) \to \widetilde{H}(N)$, we have the following partial differentials $df_{L,\widetilde{H}} : L \to \widetilde{H}(N)$, $df_{L,\widetilde{L}} : L \to \widetilde{L}$ and $df_{H,\widetilde{L}} : H(M) \to \widetilde{L}$ defined respectively by:

$$df_{L,\widetilde{H}} = \pi_{\widetilde{H}} \circ df \circ i_L,$$
$$df_{L,\widetilde{L}} = \pi_{\widetilde{L}} \circ df \circ i_L,$$
$$df_{H,\widetilde{L}} = \pi_{\widetilde{L}} \circ df \circ i_H,$$

where $i_L : L \to TM$, $i_H : H(M) \to TM$ are the inclusion morphisms, and $\pi_{\widetilde{H}} : TN \to \widetilde{H}(N)$, $\pi_{\widetilde{L}} : TN \to \widetilde{L}$ are the natural projection morphisms. The corresponding energy densities are given respectively by

$$
\begin{aligned}
e_{L,\widetilde{H}} &= \frac{1}{2}|df_{L,\widetilde{H}}|^2 = f_0^\alpha f_0^{\overline{\alpha}}, \\
e_{L,\widetilde{L}} &= \frac{1}{2}|df_{L,\widetilde{L}}|^2 = \frac{1}{2}(f_0^0)^2, \\
e_{H,\widetilde{L}} &= \frac{1}{2}|df_{H,\widetilde{L}}|^2 = f_j^0 f_{\overline{j}}^0.
\end{aligned}
$$

(7.1)

Let us introduce the following partial second fundamental forms:

$$
\begin{aligned}
\beta_H &= \beta\big(i_H(\cdot), i_H(\cdot)\big), \quad \beta_{H,\widetilde{H}} = \pi_{\widetilde{H}}(\beta_H), \quad \beta_{H,\widetilde{L}} = \pi_{\widetilde{L}}(\beta_H), \\
\beta_{L\times H} &= \beta\big(i_L(\cdot), i_H(\cdot)\big), \quad \beta_{H\times L} = \beta\big(i_H(\cdot), i_L(\cdot)\big), \\
\beta_{L\times H,\widetilde{H}} &= \pi_{\widetilde{H}}(\beta_{L\times H}), \quad \beta_{H\times L,\widetilde{H}} = \pi_{\widetilde{H}}(\beta_{H\times L}), \\
\beta_{L\times H,\widetilde{L}} &= \pi_{\widetilde{L}}(\beta_{L\times H}), \quad \beta_{H\times L,\widetilde{L}} = \pi_{\widetilde{L}}(\beta_{H\times L}).
\end{aligned}
$$

(7.2)

and set

(7.3)
$$
\tau_H = tr_{G_\theta}\beta_H, \quad \tau_{H,\widetilde{L}} = tr_{G_\theta}\beta_{H,\widetilde{L}}.
$$

Recalling that $\tau_{H,\widetilde{H}}(f) = tr_{G_\theta}\beta_{H,\widetilde{H}}$ (see the notations in §3), we have the following decomposition

(7.4)
$$
\tau_H(f) = \tau_{H,\widetilde{H}}(f) + \tau_{H,\widetilde{L}}(f).
$$

Hence $\tau_H(f) = 0$ if and only if $\tau_{H,\widetilde{H}}(f) = \tau_{H,\widetilde{L}}(f) = 0$, that is, f is a special $(H, \widetilde{H})$-harmonic map.

Now we consider the following evolution problem on $M \times [0, T)$:

(7.5)
$$
\begin{cases}
\frac{\partial f_t}{\partial t} &= \tau_H(f_t) \\
f|_{t=0} &= h
\end{cases}
$$

where $h : M \to N$ is a smooth map. Since the horizontal part of $\tau_H(f)$ is the gradient of the functional $E_{H,\widetilde{H}}$, the flow (7.5) has a partial variational structure. In the appendix, we show that, in terms of the Nash embedding of N into some Euclidean space R^K, the PDE system in (7.5) can be equivalently expressed as the following type of subelliptic parabolic system (see Theorem B4):

(7.6)
$$
(\triangle_H - \frac{\partial}{\partial t})u^a = P_{bc}^a(u)\langle \nabla_H u^b, \nabla_H u^c\rangle, \quad 1 \le a, b, c \le K
$$

for a map $u : M \times [0, T) \to R^K$, where $P_{bc}^a : B(N) \to R$ are functions on a tubular neighborhood of $N \subset R^K$.

We shall apply the regularity theory in [RS] to investigate solutions of (7.6). Let U be a relatively compact open subset of M, on which there is an orthonormal frame field $\{e_A\}_{A=1,...,2m}$ for $H(M)$. Set $X_0 = \frac{\partial}{\partial t}$, $X_A = e_A$ ($A = 1, ..., 2m$). Clearly $\{X_0, X_1, ..., X_{2m}\}$ together with their commutators of 2 order span the tangent space of $U \times (0, \infty)$ at any point. In terms of (1.14), we know from [Hö], [FS] that the operator $\triangle_H - \frac{\partial}{\partial t}$ is hypoelliptic on $M \times (0, \infty)$.

Let us recall briefly the function spaces adapted with the differential operator $\triangle_H - \frac{\partial}{\partial t}$. Let $U_T = U \times (0, T)$ for some $T > 0$. For a monomial $X_{A_1} \cdots X_{A_l}$ with $0 \leq A_s \leq 2m$, $s = 1, .., l$, its weight is defined as an integer $r_1 + 2r_2$, where r_1 is the number of X_j's that enter with j between 1 and $2m$, and r_2 is the number of X_0's. We also write $w(A_1, ..., A_l) = r_1 + 2r_2$. For any integer $k \geq 0$ and any p, $1 < p < \infty$, we define $S_k^p(U_T)$ to consist of all $u \in L^p(U_T)$ such that $(X_{i_1} X_{i_2} \cdots X_{i_l})u \in L^p(U_T)$ for all monomials of weight $\leq k$. For the norm, we take

$$\|u\|_{S_k^p(U_T)} = \sum_{w(A_1,...,A_l) \leq k} \|X_{A_1} \cdots X_{A_l} f\|_{L^p(U_T)},$$

that is, the sum is taken over all ordered monomials $X_{A_1} \cdots X_{A_l}$ of weight $\leq k$. Using a C^∞ partition of unity subordinate to a finite open cover $\{U_j\}$ of M, one may define the space $S_k^p(M \times (0, T))$.

For any two points $x, y \in M$, the Carnot-Carathéodory distance is defined by

$$d_C(x, y) = \inf\{L(\gamma) \mid \gamma : [0, T] \to M \text{ is a horizontal } C^1 \text{ curve with}$$
$$\gamma(0) = x, \gamma(T) = y\}$$

where $L(\gamma)$ denotes the length of γ defined by the Webster metric g_θ. The parabolic Carnot-Carathédory distance on $M \times (0, \infty)$ is defined by (cf. [BB])

$$d_P\big((x, t), (y, s)\big) = \sqrt{d_C(x, y) + |t - s|}.$$

We now define the parabolic Hölder spaces adapted to the operator $\triangle_H - \frac{\partial}{\partial t}$. Let $\Omega \subset U_T$ be any open subset. For any integer $k \geq 0$ and any $\alpha > 0$, let

$$C_P^{k,\alpha}(\Omega) = \left\{ u : \Omega \to R : \|u\|_{C_P^{k,\alpha}} < \infty \right\},$$

$$\|u\|_{C_P^{k,\alpha}(\Omega)} = \sum_{w(A_1,...,A_l) \leq k} \|X_{A_1} \cdots X_{A_l} u\|_{C_P^\alpha(\Omega)},$$

$$\|u\|_{C_P^\alpha(\Omega)} = |u|_{C_P^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)},$$

$$|u|_{C_P^\alpha(\Omega)} = \sup \left\{ \frac{|u(x,t) - u(y,s)|}{d_P\big((x,t),(y,s)\big)^\alpha} : (x,t), (y,s) \in \Omega, (x,t) \neq (y,s) \right\},$$

where $0 \leq A_s \leq 2m$, $1 \leq s \leq l$. Similarly one may use a C^∞ partition of unity to define the function space $C_P^{k,\alpha}(M \times [T_1, T_2])$ for any $[T_1, T_2] \subset (0, \infty)$. Let $d(x, y)$ be the Riemannian distance of x and y in (M, g_θ). For the relatively compact open set $U \subset M$, there exist positive constants c_1, c_2 depending on U such that (cf. [NSW])

$$c_1 d(x, y) \leq d_C(x, y) \leq c_2 d(x, y)^{1/2}$$

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for any $x, y \in U$. We have a natural Riemannian distance

$$\widehat{d}((x,t),(y,s)) = \sqrt{d(x,y) + |t-s|^2}$$

on $M \times (0, \infty)$. Using the distance $\widehat{d}$, one may define the usual Hölder space $C^{k,\alpha}(\Omega)$ for any open subset $\Omega \subset U_T$. Clearly there exist two positive constants C_1, C_2 such that

$$C_1 \widehat{d}((x,t),(y,s)) \leq d_P((x,t),(y,s)) \leq C_2 \widehat{d}((x,t),(y,s))^{\frac{1}{2}}$$

for any $x, y \in U$ and $0 \leq |t-s| << 1$. This implies the following relations between the parabolic Hörmander Hölder spaces and the usual Hölder spaces

$$(7.7) \qquad C^\alpha(\Omega) \subset C_P^\alpha(\Omega) \subset C^{\frac{\alpha}{2}}(\Omega), \quad C^{k,\alpha}(\Omega) \subset C_P^{k,\alpha}(\Omega), \quad C_P^{2k,\alpha}(\Omega) \subset C^{k,\frac{\alpha}{2}}(\Omega).$$

Proposition 7.1. (cf. [RS, Theorem 18], [BB, Theorem 1.1]) Let $U_T = U \times (0,T)$ $(T > 0)$ and let $\Omega \Subset U_T$ be a relatively compact open subset of U_T. Suppose u is locally in $L^p(U_T)$, and $(\triangle_H - \frac{\partial}{\partial t})u = v$.

a) If $v \in S_k^p(U_T)$, then $\chi u \in S_{k+2}^p(U_T)$ for any $\chi \in C_0^\infty(U_T)$. In particular, there exists a constant $c > 0$ such that

$$\|u\|_{S_{k+2}^p(\Omega)} \leq c \left(\|u\|_{L^p(U_T)} + \|v\|_{S_k^p(U_T)} \right).$$

b) If $v \in C_P^{k,\alpha}(U_T)$, then there exists a constant c such that

$$\|u\|_{C_P^{k+2,\alpha}(\Omega)} \leq c \left\{ \|v\|_{C_P^{k,\alpha}(U_T)} + \|u\|_{L^\infty(U_T)} \right\}.$$

Remark 7.1.
(i) It is known that if kp is large enough, then the Sobolev type space S_k^p is contained in some Hölder space (cf. [RS], [FS], [DT], [FGN]). For example, let $k = 2$ and $p > 2n+4$. If $u \in S_2^p(U_T)$, then for any $\chi \in C_0^\infty(U_T)$, we have $\chi u \in \Omega_P^{1,\alpha}(U_T)$ with $\alpha = 1 - \frac{2n+4}{p}$. In particular, for any relatively compact open subset Ω of U_T, there exists a positive constant c such that $\|u\|_{C_P^{1,\alpha}(\Omega)} \leq c\|u\|_{S_2^p(U_T)}$.
(ii) Combining (7.7) and Proposition 7.1(b), we have

$$\|u\|_{C^{l+1,\frac{\alpha}{2}}(\Omega)} \leq C\{\|v\|_{C_P^{2l,\alpha}(U_T)} + \|u\|_{L^\infty(U_T)}\}.$$

Since the linearization of (7.6) is a linear subelliptic parabolic system, the short time existence and uniqueness of solution to (7.6) follow from a standard argument. By Proposition 7.1 and a bootstrapping argument, one can always assume that the short-time solution u of (7.6) (or (7.5)) is smooth on $M \times [0,T]$ for some $T > 0$.

Lemma 7.2. *Let M be a compact pseudo-Hermitian manifold and let N be a Sasakian manifold. For any $0 < T \leq \infty$, if $f \in C^\infty(M \times [0,T); N)$ solves (7.5), then*

$$E_{H,\widetilde{H}}(f_t) + \int_0^t \int_M |\tau_{H,\widetilde{H}}(f_s)|^2 dv_\theta ds = E_{H,\widetilde{H}}(h)$$

for any $t \in [0,T)$. In particular, the energy $E_{H,\widetilde{H}}$ decays along the flow.

Proof. By Proposition 3.1, we get

$$\frac{dE_{H,\widetilde{H}}(f_s)}{ds} = -\int_M \langle \frac{\partial f}{\partial s}, \tau_{H,\widetilde{H}}(f_s) \rangle dv_\theta$$

Therefore (7.4) and (7.5) imply that

$$\frac{dE_{H,\widetilde{H}}(f_s)}{ds} = -\int_M |\tau_{H,\widetilde{H}}(f_s)|^2 dv_\theta$$

Integrating the above equality over $[0,t]$ then proves this lemma. $\square$

Let $f : M \times [0,T) \to N$ be a C^∞ solution of (7.5). In terms of Lemma 2.1, (1.10) and the assumption that N is Sasakian, we have

$$\begin{aligned}
\langle \widetilde{\nabla} \tau_{H,\widetilde{H}}, df_{H,\widetilde{H}} \rangle &= \langle \widetilde{\nabla}(\tau_{H,\widetilde{H}} + \tau_{H,\widetilde{L}}), df_{H,\widetilde{H}} \rangle \\
&= \sum_{A=1}^{2m} \langle \widetilde{\nabla}_{e_A} df(\frac{\partial}{\partial t}), df_{H,\widetilde{H}}(e_A) \rangle \\
&= \sum_{A=1}^{2m} \langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} df(e_A) + \widetilde{T}_{\widetilde{\nabla}}(df(e_A), df(\frac{\partial}{\partial t})), df_{H,\widetilde{H}}(e_A) \rangle \\
&= \frac{\partial}{\partial t} e_{H,\widetilde{H}}.
\end{aligned}$$

Using (2.17), we get from (4.16) that
(7.8)
$$\begin{aligned}
&(\triangle_H - \frac{\partial}{\partial t}) e_{H,\widetilde{H}} \\
&= |\beta_{H,\widetilde{H}}|^2 - 2i(f_{\overline{j}}^{\overline{\alpha}} f_{0j}^\alpha + f_j^\alpha f_{0\overline{j}}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{0\overline{j}}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{0j}^\alpha) + (2m-4)i(f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\alpha}} A_{jk} - f_j^\alpha f_k^{\overline{\alpha}} A_{\overline{jk}}) \\
&\quad + 2\langle df_{H,\widetilde{H}}(Ric(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}}) \rangle - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})) \\
&\quad - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)) \\
&= |\beta_{H,\widetilde{H}}|^2 + \beta_{L \times H,\widetilde{H}} * df_{H,\widetilde{H}} + A * (df_{H,\widetilde{H}}) * (df_{H,\widetilde{H}}) \\
&\quad + 2\langle df_{H,\widetilde{H}}(Ric(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}}) \rangle - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_k), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_{\overline{k}})) \\
&\quad - 2\widetilde{R}(df_{H,\widetilde{H}}(\eta_j), df_{H,\widetilde{H}}(\eta_{\overline{k}}), df_{H,\widetilde{H}}(\eta_{\overline{j}}), df_{H,\widetilde{H}}(\eta_k)),
\end{aligned}$$

where the notation $\Phi * \Psi$ denotes some contraction of two tensors Φ and Ψ.

Now we want to compute $(\triangle_H - \frac{\partial}{\partial t})e_{L,\widetilde{H}}$. In terms of (2.17), (2.35), (2.41), we deduce that

$$\triangle_H e_{L,\widetilde{H}} = (f_0^\alpha f_0^{\overline{\alpha}})_{k\overline{k}} + (f_0^\alpha f_0^{\overline{\alpha}})_{\overline{k}k}$$

$$= (f_{0k}^\alpha f_0^{\overline{\alpha}} + f_0^\alpha f_{0k}^{\overline{\alpha}})_{\overline{k}} + (f_{0\overline{k}}^\alpha f_0^{\overline{\alpha}} + f_0^\alpha f_{0\overline{k}}^{\overline{\alpha}})_k$$

$$= 2(f_{0k}^\alpha f_{0\overline{k}}^{\overline{\alpha}} + f_{0\overline{k}}^\alpha f_{0k}^{\overline{\alpha}}) + f_0^{\overline{\alpha}} f_{0k\overline{k}}^\alpha + f_0^{\overline{\alpha}} f_{0\overline{k}k}^\alpha + f_0^\alpha f_{0\overline{k}k}^{\overline{\alpha}} + f_0^\alpha f_{0k\overline{k}}^{\overline{\alpha}}$$

$$= 2(f_{0k}^\alpha f_{0\overline{k}}^{\overline{\alpha}} + f_{0\overline{k}}^\alpha f_{0k}^{\overline{\alpha}}) + f_0^{\overline{\alpha}} f_{\overline{k}k0}^\alpha + f_0^{\overline{\alpha}} f_{\overline{k}k0}^\alpha + f_0^\alpha f_{\overline{k}k0}^{\overline{\alpha}} + f_0^\alpha f_{\overline{k}k0}^{\overline{\alpha}}$$

$$+ f_0^{\overline{\alpha}} f_k^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha (f_{\overline{k}}^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_{\overline{k}}^{\overline{\delta}}) + f_0^{\overline{\alpha}} f_{\overline{k}}^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha (f_k^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_k^{\overline{\delta}})$$

$$+ f_0^\alpha f_{\overline{k}}^{\overline{\beta}} \widehat{R}_{\overline{\beta}\overline{\gamma}\delta}^{\overline{\alpha}} (f_k^{\overline{\gamma}} f_0^\delta - f_0^{\overline{\gamma}} f_k^\delta) + f_0^\alpha f_k^{\overline{\beta}} \widehat{R}_{\overline{\beta}\overline{\gamma}\delta}^{\overline{\alpha}} (f_{\overline{k}}^{\overline{\gamma}} f_0^\delta - f_0^{\overline{\gamma}} f_{\overline{k}}^\delta)$$

$$+ A_{\overline{k}}^j f_0^{\overline{\alpha}} f_{kj}^\alpha + W_{k\overline{k}}^t f_0^{\overline{\alpha}} f_t^\alpha + A_k^{\overline{j}} f_0^{\overline{\alpha}} f_{\overline{j}k}^\alpha + A_{k,\overline{k}}^{\overline{j}} f_0^{\overline{\alpha}} f_{\overline{j}}^\alpha$$

$$+ A_k^{\overline{j}} f_0^{\overline{\alpha}} f_{\overline{k}j}^\alpha + W_{\overline{k}k}^{\overline{t}} f_0^{\overline{\alpha}} f_{\overline{t}}^\alpha + A_{\overline{k}}^j f_0^{\overline{\alpha}} f_{jk}^\alpha + A_{\overline{k},k}^j f_0^{\overline{\alpha}} f_j^\alpha$$

$$+ A_k^{\overline{j}} f_0^\alpha f_{\overline{k}j}^{\overline{\alpha}} + W_{\overline{k}k}^{\overline{t}} f_0^\alpha f_{\overline{t}}^{\overline{\alpha}} + A_{\overline{k}}^j f_0^\alpha f_{jk}^{\overline{\alpha}} + A_{\overline{k},k}^j f_0^\alpha f_j^{\overline{\alpha}}$$

$$+ A_{\overline{k}}^j f_0^\alpha f_{kj}^{\overline{\alpha}} + W_{k\overline{k}}^t f_0^\alpha f_t^{\overline{\alpha}} + A_k^{\overline{j}} f_0^\alpha f_{\overline{j}k}^{\overline{\alpha}} + A_{k,\overline{k}}^{\overline{j}} f_0^\alpha f_{\overline{j}}^{\overline{\alpha}}.$$

Consequently
(7.9)
$$\triangle_H \, e_{L,\widetilde{H}}$$
$$= |\beta_{L\times H,\widetilde{H}}|^2 + \langle \widetilde{\nabla}_\xi \tau_H, df_{L,\widetilde{H}}(\xi) \rangle - 2\widetilde{R}\big(df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_k), df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big)$$
$$+ A * (df_{L,\widetilde{H}}) * \beta_{H,\widetilde{H}} + \nabla A * (df_{L,\widetilde{H}}) * (df_{H,\widetilde{H}}).$$

Using Lemma 2.1 and (7.4), we get

(7.10)
$$\widetilde{\nabla}_\xi \tau_H = \widetilde{\nabla}_\xi df\big(\frac{\partial}{\partial t}\big)$$
$$= \widetilde{\nabla}_{\frac{\partial}{\partial t}} df(\xi) + T_{\widetilde{\nabla}}\big(df(\xi), df\big(\frac{\partial}{\partial t}\big)\big).$$

From (7.9), (7.10), (1.10) and the assumption that N is Sasakian, we obtain

(7.11)
$$(\triangle_H - \frac{\partial}{\partial t})e_{L,\widetilde{H}} = |\beta_{L\times H,\widetilde{H}}|^2 - 2\widetilde{R}\big(df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_k), df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big)$$
$$+ A * (df_{L,\widetilde{H}}) * \beta_{H,\widetilde{H}} + \nabla A * (df_{L,\widetilde{H}}) * (df_{H,\widetilde{H}}).$$

Lemma 7.3. *If N has non-positive horizontal sectional curvature, then*

$$\widetilde{R}\big(df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_k), df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big) \leq 0.$$

Proof. Write $df_{H,\widetilde{H}}(\eta_k) = X_k + iY_k$ $(k = 1, ..., m)$. Since $\widetilde{K}^H \leq 0$, we find

$$\widetilde{R}\big(df_{L,\widetilde{H}}(\xi), X_k + iY_k, df_{L,\widetilde{H}}(\xi), X_k - iY_k\big)$$
$$= \widetilde{R}\big(df_{L,\widetilde{H}}(\xi), X_k, df_{L,\widetilde{H}}(\xi), X_k\big) + \widetilde{R}(df_{L,\widetilde{H}}(\xi), Y_k, df_{L,\widetilde{H}}(\xi), Y_k)$$
$$\leq 0.$$

$\square$

Lemma 7.4. *Let N be a Sasakian manifold with $\widetilde{K}^H \leq 0$ and let $f \in C^\infty(M \times [0,T), N)$ be a solution of (7.5). Set $e_{\widetilde{H}} = e_{H,\widetilde{H}} + e_{L,\widetilde{H}}$. Then*

$$(\triangle_H - \frac{\partial}{\partial t})e_{\widetilde{H}} \geq -Ce_{\widetilde{H}}.$$

Here C is a positive constant depending only on the pseudo-Hermitian Ricci curvature and the torsion of $(M, H(M), J, \theta)$.

Proof. Utilizing (7.8), (7.11), Lemma 7.3 and Cauchy-Schwarz inequality, we deduce that

$$
\begin{aligned}
&(\triangle_H - \frac{\partial}{\partial t})e_{\widetilde{H}} \\
&\geq |\beta_{H,\widetilde{H}}|^2 + |\beta_{L\times H,\widetilde{H}}|^2 + \beta_{L\times H,\widetilde{H}} * df_{H,\widetilde{H}} + A * (df_{H,\widetilde{H}}) * (df_{H,\widetilde{H}}) \\
&\quad + 2\langle df_{H,\widetilde{H}}(Ric(\eta_j)), df_{H,\widetilde{H}}(\eta_{\overline{j}})\rangle + A * (df_{L,\widetilde{H}}) * \beta_{H,\widetilde{H}} + \nabla A * (df_{L,\widetilde{H}}) * (df_{H,\widetilde{H}}) \\
&\geq (1 - \frac{1}{2}\varepsilon_1)|\beta_{H,\widetilde{H}}|^2 + (1 - \frac{1}{2}\varepsilon_1)|\beta_{L\times H,\widetilde{H}}|^2 - \frac{C_1}{\varepsilon_1}(e_{H,\widetilde{H}} + e_{L,\widetilde{H}}) \\
&\geq (1 - \frac{1}{2}\varepsilon_1)|\beta_{H,\widetilde{H}}|^2 + (1 - \frac{1}{2}\varepsilon_1)|\beta_{L\times H,\widetilde{H}}|^2 - \frac{C_1}{\varepsilon_1}e_{\widetilde{H}}
\end{aligned}
$$

for any $\varepsilon_1 > 0$, where C_1 is a positive constant depending only on Ric, A and ∇A. Taking $\varepsilon_1 = 2$ in the above inequality, we prove this lemma. $\square$

In order to estimate $e_{\widetilde{H}}$, let us recall Moser's Harnack inequality ([Mo]). For any $z_0 = (x_0, t_0) \in M \times (0,T)$, let $0 < \delta < inj(M)$ (the injectivity radius), $0 < \sigma < t_0$ and let $R(z_0, \delta, \sigma)$ be the following cylinder

$$R(z_0, \delta, \sigma) = \left\{(x, t) \in M \times [0, \infty) : d(x, x_0) < \delta, \ t_0 - \sigma < t < t_0\right\}$$

where d denotes the distance function of the Webster metric g_θ.

Lemma 7.5. *Let u be a non-negative smooth solution of*

$$(\triangle_H - \frac{\partial}{\partial t})u \geq 0$$

on M. Then

$$u(z_0) \leq C(m, \delta, \sigma) \int_{R(z_0, \delta, \sigma)} u(x, t) dv_\theta dt,$$

where C is a positive constant depending only on m, δ and σ.

Since M is compact, it follows from Lemma 7.5 that

(7.12)
$$u(z_0) \leq C(m, \sigma) \int_{t_0 - \sigma}^{t_0} \int_M u(x, t) dv_\theta dt$$

for any $z_0 = (x_0, t_0) \in M \times [\sigma, T)$.

Henceforth in this section, we assume that M is also Sasakian.

Lemma 7.6. *Let M be a compact Sasakian manifold and let N be a Sasakian manifold with $\widetilde{K}^H \leq 0$. Suppose $f \in C^\infty(M \times [0, T), N)$ is a solution of (7.5). Then the energy $E_{L,\widetilde{H}}(f(t))$ is decreasing in t. In particular, if the initial map h is foliated, then $f(t)$ is foliated for each $t \in [0, T)$.*

Proof. From (7.11), the divergence theorem and Lemma 7.3, we have

$$\frac{d}{dt}E_{L,\widetilde{H}}(f(t)) = \int_M \frac{\partial}{\partial t}(e_{L,\widetilde{H}}(f))dv_\theta$$

$$\leq \int_M \left\{ -|\beta_{L\times H,\widetilde{H}}|^2 + 2\widetilde{R}\big(df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_k), df_{L,\widetilde{H}}(\xi), df_{H,\widetilde{H}}(\eta_{\overline{k}})\big) \right\}dv_\theta$$

$$\leq 0.$$

$\square$

Lemma 7.7. *Let M be a compact Sasakian manifold and let N be a Sasakian manifold with $\widetilde{K}^H \leq 0$. Suppose $f \in C^\infty(M \times [0, T), N)$ $(0 < T \leq \infty)$ is a solution of (7.5). Then $e_{\widetilde{H}}(f)$ is uniformly bounded.*

Proof. From Lemma 7.4, we know that $e_{\widetilde{H}}$ satisfies

$$\Big(\triangle_H - \frac{\partial}{\partial t}\Big)e_{\widetilde{H}} \geq -Ce_{\widetilde{H}}$$

for some constant C. Let

$$F(x, t) := e^{-Ct}e_{\widetilde{H}}(f), \quad (x, t) \in M \times [0, T).$$

It follows that

$$\Big(\triangle_H - \frac{\partial}{\partial t}\Big)F(x, t) \geq 0.$$

Let $0 < \sigma < T$. Then for any $z_0 = (x_0, t_0) \in M \times [\sigma, T)$, (7.12) implies that

$$e_{\widetilde{H}}(f)(z_0) \leq C_1 e^{Ct_0} \int_{t_0-\sigma}^{t_0} \int_M e^{-Ct}e_{\widetilde{H}}(f)dv_\theta dt$$

$$\leq C_1 e^{C\sigma} \int_{t_0-\sigma}^{t_0} \int_M e_{\widetilde{H}}(f)dv_\theta dt$$

$$\leq C_2 \int_{t_0-\sigma}^{t_0} E_{\widetilde{H}}(f(t))dt$$

$$\leq C_2 E_{\widetilde{H}}(h)$$

since $E_{H,\widetilde{H}}(f(t))$ and $E_{L,\widetilde{H}}(f(t))$ are decreasing in t in view of Lemmas 7.2, 7.6. $\square$

Next we want to derive Bochner formulas for $e_{H,\widetilde{L}}(f_t)$ and $e_{L,\widetilde{L}}(f_t)$. According to (7.1) and the definition of $\triangle_H$, one has

(7.13)
$$\triangle_H e_{H,\widetilde{L}} = (f_j^0 f_{\overline{j}}^0)_{k\overline{k}} + (f_j^0 f_{\overline{j}}^0)_{\overline{k}k}$$

$$= 2(f_{jk}^0 f_{\overline{jk}}^0 + f_{\overline{jk}}^0 f_{\overline{j}k}^0) + f_{\overline{j}}^0 f_{jk\overline{k}}^0 + f_j^0 f_{\overline{j}k\overline{k}}^0 + f_{\overline{j}}^0 f_{jk\overline{k}}^0 + f_j^0 f_{\overline{j}k k}^0$$

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and

$$\triangle_H e_{L,\widetilde{L}} = \frac{1}{2}\big((f_0^0)_{k\overline{k}}^2 + (f_0^0)_{\overline{k}k}^2\big)$$

(7.14)

$$= 2f_{0k}^0 f_{0\overline{k}}^0 + f_0^0(f_{0k\overline{k}}^0 + f_{0\overline{k}k}^0)$$

Using (2.14) and (2.24), we deduce from (7.16) and (7.17) that

$$\triangle_H e_{H,\widetilde{L}} = 2(f_{jk}^0 f_{\overline{jk}}^0 + f_{j\overline{k}}^0 f_{\overline{j}k}^0) + f_{\overline{j}}^0 f_{k\overline{k}j}^0 + f_j^0 f_{k\overline{k}\overline{j}}^0 + f_{\overline{j}}^0 f_{\overline{k}kj}^0 + f_j^0 f_{\overline{k}k\overline{j}}^0$$

$$+ f_{\overline{j}}^0 f_{j0}^0 + f_j^0 f_{\overline{j}0}^0 + if_{\overline{j}}^0 f_{0j}^0 - if_j^0 f_{0\overline{j}}^0 + f_{\overline{j}}^0 f_t^0 R_{kj\overline{k}}^t + f_j^0 f_{\overline{t}}^0 R_{\overline{k}\overline{j}k}^{\overline{t}}$$

$$+ if_{\overline{j}}^0 \big(f_j^{\overline{\alpha}} f_k^\alpha - f_j^\alpha f_k^{\overline{\alpha}}\big)_{\overline{k}} - if_j^0 \big(f_{\overline{k}}^{\overline{\alpha}} f_{\overline{j}}^\alpha - f_{\overline{k}}^\alpha f_{\overline{j}}^{\overline{\alpha}}\big)_{\overline{k}}$$

(7.15)

$$+ if_{\overline{j}}^0 \big(f_k^\alpha f_j^{\overline{\alpha}} - f_k^{\overline{\alpha}} f_j^\alpha\big)_k - if_j^0 \big(f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\alpha}} - f_{\overline{j}}^{\overline{\alpha}} f_{\overline{k}}^\alpha\big)_k$$

$$= |\beta_{H\times H,\widetilde{L}}|^2 + \langle \widetilde{\nabla}\tau_{H,\widetilde{L}}, df_{H,\widetilde{L}}\rangle + \beta_{H\times L,\widetilde{L}} * df_{H,\widetilde{L}} + \beta_{L\times H,\widetilde{L}} * df_{H,\widetilde{L}}$$

$$+ \langle df_{H,\widetilde{L}}(Ric_H(\eta_j)), df_{H,\widetilde{L}}(\eta_{\overline{j}})\rangle + \langle df_{H,\widetilde{L}}(Ric_H(\eta_{\overline{j}})), df_{H,\widetilde{L}}(\eta_j)\rangle$$

$$+ \beta_{H,\widetilde{H}} * (df_{H,\widetilde{H}}) * (df_{H,\widetilde{L}})$$

and

(7.16)

$$\triangle_H e_{L,\widetilde{L}} = 2f_{0k}^0 f_{0\overline{k}}^0 + f_0^0 f_{\overline{k}k0}^0 + f_0^0 f_{\overline{k}k0}^0 + if_0^0\big((f_0^{\overline{\alpha}} f_{\overline{k}}^\alpha - f_0^\alpha f_{\overline{k}}^{\overline{\alpha}})_k - (f_0^\alpha f_{\overline{k}}^{\overline{\alpha}} - f_0^{\overline{\alpha}} f_{\overline{k}}^\alpha)_{\overline{k}}\big)$$

$$= |\beta_{L\times H,\widetilde{L}}|^2 + \langle \widetilde{\nabla}_\xi \tau_{H,\widetilde{L}}, df_{L,\widetilde{L}}(\xi)\rangle + \beta_{L\times H,\widetilde{H}} * df_{H,\widetilde{H}} * df_{L,\widetilde{L}}$$

$$+ \beta_{H,\widetilde{H}} * df_{L,\widetilde{L}} * df_{L,\widetilde{H}}.$$

In terms of (1.7), (1.9), (1.10), Lemma 2.1 and (7.5), we have

$$\langle \widetilde{\nabla}\tau_{H,\widetilde{L}}, df_{H,\widetilde{L}}\rangle = \frac{\partial}{\partial t} e_{H,\widetilde{L}} + \sum_{A=1}^{2m} \langle \widetilde{J} df_{H,\widetilde{H}}(e_A), \tau_{H,\widetilde{H}}(f)\rangle \widetilde{\theta}\big(df_{H,\widetilde{L}}(e_A)\big)$$

and

$$\langle \widetilde{\nabla}_\xi \tau_{H,\widetilde{L}}, df_{L,\widetilde{L}}(\xi)\rangle = \frac{\partial}{\partial t} e_{L,\widetilde{L}} + \langle \widetilde{J} df_{L,\widetilde{H}}(\xi), \tau_{H,\widetilde{H}}(f)\rangle \widetilde{\theta}\big(df_{L,\widetilde{L}}(\xi)\big).$$

In addition, (2.14) implies

$$\beta_{H\times L,\widetilde{L}} * df_{H,\widetilde{L}} = \beta_{L\times H,\widetilde{L}} * df_{H,\widetilde{L}} + df_{H,\widetilde{H}} * df_{L,\widetilde{H}} * df_{H,\widetilde{L}}.$$

Consequently

$$(\triangle_H - \frac{\partial}{\partial t})e_{H,\widetilde{L}} = |\beta_{H\times H,\widetilde{L}}|^2 + df_{H,\widetilde{H}} * \tau_{H,\widetilde{H}} * df_{H,\widetilde{L}} + \beta_{L\times H,\widetilde{L}} * df_{H,\widetilde{L}}$$

(7.17)

$$+ df_{H,\widetilde{H}} * df_{L,\widetilde{H}} * df_{H,\widetilde{L}} + \beta_{H,\widetilde{H}} * (df_{H,\widetilde{H}}) * (df_{H,\widetilde{L}})$$

$$+ \langle df_{H,\widetilde{L}}(Ric_H(\eta_j)), df_{H,\widetilde{L}}(\eta_{\overline{j}})\rangle + \langle df_{H,\widetilde{L}}(Ric_H(\eta_{\overline{j}})), df_{H,\widetilde{L}}(\eta_j)\rangle$$

and

(7.18)

$$(\triangle_H - \frac{\partial}{\partial t})e_{L,\widetilde{L}} = |\beta_{L\times H,\widetilde{L}}|^2 + df_{L,\widetilde{H}} * \tau_{H,\widetilde{H}} * df_{L,\widetilde{L}} + \beta_{L\times H,\widetilde{H}} * df_{H,\widetilde{H}} * df_{L,\widetilde{L}}$$

$$+ \beta_{H,\widetilde{H}} * df_{L,\widetilde{L}} * df_{L,\widetilde{H}}.$$

Clearly the usual energy density of the map f is given by

$$e(f) = e_{H,\widetilde{H}}(f) + e_{L,\widetilde{H}}(f) + e_{H,\widetilde{L}}(f) + e_{L,\widetilde{L}}(f)$$

and thus $E(f) = E_{H,\widetilde{H}}(f) + E_{L,\widetilde{H}}(f) + E_{H,\widetilde{L}}(f) + E_{L,\widetilde{L}}(f)$.

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Lemma 7.8. *Let M be a compact Sasakian manifold and N be a Sasakian manifold with $\widetilde{K}^H \leq 0$. Suppose $f \in C^\infty(M \times [0,T), N)$ is a solution of (7.5). Then*

$$e(f_t) \leq C(\sigma) \sup_{[t-\sigma,t]} E(f_t)$$

for $t \in [\sigma, T)$.

Proof. From the proof of Lemma 7.4, we have

$$
\begin{aligned}
(7.19) \quad (\triangle_H - \frac{\partial}{\partial t})\left(e_{H,\widetilde{H}}(f_t) + e_{L,\widetilde{H}}(f_t)\right) &\geq (1 - \frac{1}{2}\varepsilon_1)|\beta_{H,\widetilde{H}}|^2 + (1 - \frac{1}{2}\varepsilon_1)|\beta_{L\times H,\widetilde{H}}|^2 \\
&\quad - \frac{C_1}{\varepsilon_1}\left(e_{H,\widetilde{H}}(f_t) + e_{L,\widetilde{H}}(f_t)\right)
\end{aligned}
$$

Utilizing Lemma 7.7 and Cauchy-Schwarz inequality, we derive from (7.17) and (7.18) that

$$
\begin{aligned}
(7.20) \quad &(\triangle_H - \frac{\partial}{\partial t})\left(e_{H,\widetilde{L}}(f_t) + e_{L,\widetilde{L}}(f_t)\right) \\
&= |\beta_{H\times H,\widetilde{L}}|^2 + df_{H,\widetilde{H}} * \tau_{H,\widetilde{H}} * df_{H,\widetilde{L}} + \beta_{L\times H,\widetilde{L}} * df_{H,\widetilde{L}} + df_{H,\widetilde{H}} * df_{L,\widetilde{H}} * df_{H,\widetilde{L}} \\
&\quad + |\beta_{L\times H,\widetilde{L}}|^2 + \langle df_{H,\widetilde{L}}(Ric_H(\eta_j)), df_{H,\widetilde{L}}(\eta_{\bar{j}})\rangle + \langle df_{H,\widetilde{L}}(Ric_H(\eta_{\bar{j}})), df_{H,\widetilde{L}}(\eta_j)\rangle \\
&\quad + \beta_{H,\widetilde{H}} * (df_{H,\widetilde{H}}) * (df_{H,\widetilde{L}}) + df_{L,\widetilde{H}} * \tau_{H,\widetilde{H}} * df_{L,\widetilde{L}} + \beta_{L\times H,\widetilde{H}} * df_{H,\widetilde{H}} * df_{L,\widetilde{L}} \\
&\quad + \beta_{H,\widetilde{H}} * df_{L,\widetilde{L}} * df_{L,\widetilde{H}} \\
&\geq -\frac{1}{2}\varepsilon_2|\beta_{H,\widetilde{H}}|^2 + (1 - \frac{1}{2}\varepsilon_2)|\beta_{L\times H,\widetilde{L}}|^2 - \frac{1}{2}\varepsilon_2\left(e_{H,\widetilde{H}}(f_t) + e_{L,\widetilde{H}}(f_t)\right) \\
&\quad - \frac{1}{2}\varepsilon_2|\beta_{L\times H,\widetilde{H}}|^2 - \frac{C_2}{\varepsilon_2}\left(e_{H,\widetilde{L}}(f_t) + e_{L,\widetilde{L}}(f_t)\right).
\end{aligned}
$$

Taking $\varepsilon_1 = \varepsilon_2 = 1$, it follows from (7.19) and (7.20) that

$$
(7.21) \quad (\triangle_H - \frac{\partial}{\partial t})e(f_t) \geq -\widetilde{C}e(f_t)
$$

for some positive $\widetilde{C}$ depending only on M, N and h. Therefore this lemma follows immediately from (7.21) and Lemma 7.5. $\square$

Now we want to estimate the partial energies $E_{H,\widetilde{L}}(f_t)$ and $E_{L,\widetilde{L}}(f_t)$.

Lemma 7.9. *Let M, N and $f \in C^\infty(M \times [0,T), N)$ be as in Lemma 7.8. Then*

$$\frac{d}{ds}E_{H,\widetilde{L}}(f_t) = -\int_M |\tau_{H,L}(f_t)|^2 dv_\theta + \int_M \langle J\tau_{H,\widetilde{H}}(f_t), df_{H,\widetilde{H}}(e_A)\rangle \widetilde{\theta}\left(df_{H,\widetilde{L}}(e_A)\right) dv_\theta.$$

Furthermore, we have

$$E_{H,\widetilde{L}}(f_t) \leq C_2 t + C_3$$

for any $t \in [0,T)$ and some constants C_2, C_3 depending on M, N and h.

Proof. By definition, $E_{H,\widetilde{L}}(f_t)$ is given by

$$E_{H,\widetilde{L}}(f_t) = \frac{1}{2}\int_M \sum_{A=1}^{2m} \langle df_{H,\widetilde{L}}(e_A), df_{H,\widetilde{L}}(e_A)\rangle dv_\theta$$

where $\{e_A\}_{A=1}^{2m}$ is an orthonormal frame field of $H(M)$. Then, by using Lemma 2.1, (1.9), (1.10) and the divergence theorem, we derive that

(7.22)

$$\frac{d}{ds}E_{H,\widetilde{L}}(f_t)$$

$$= \frac{1}{2}\int_M \sum_{A=1}^{2m} \frac{\partial}{\partial t}\langle df_{H,\widetilde{L}}(e_A), df_{H,\widetilde{L}}(e_A)\rangle dv_\theta$$

$$= \int_M \sum_{A=1}^{2m} \langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} df(e_A), df_{H,\widetilde{L}}(e_A)\rangle dv_\theta$$

$$= \int_M \sum_{A=1}^{2m} \left\{\langle \widetilde{\nabla}_{e_A} df(\frac{\partial}{\partial t}), df_{H,\widetilde{L}}(e_A)\rangle + \langle \widetilde{T}_{\widetilde{\nabla}}\big(df(\frac{\partial}{\partial t}), df(e_A)\big), df_{H,\widetilde{L}}(e_A)\rangle\right\} dv_\theta$$

$$= \int_M \sum_{A=1}^{2m} \left\{\langle \widetilde{\nabla}_{e_A}\tau_H(f_t), df_{H,\widetilde{L}}(e_A)\rangle + \langle d\widetilde{\theta}(\tau_{H,\widetilde{H}}(f_t), df_{H,\widetilde{H}}(e_A))\widetilde{\xi}, df_{H,\widetilde{L}}(e_A)\rangle\right\} dv_\theta$$

$$= -\int_M |\tau_{H,L}(f_t)|^2 dv_\theta + \int_M \langle J\tau_{H,\widetilde{H}}(f_t), df_{H,\widetilde{H}}(e_A)\rangle\widetilde{\theta}(df_{H,\widetilde{L}}(e_A)) dv_\theta.$$

In terms of Lemma 7.7, (7.25) and Hölder's inequality, we have

$$\frac{d}{dt}E_{H,\widetilde{L}}(f_t) \leq C\int_M |\tau_{H,\widetilde{H}}(f_t)||df_{H,\widetilde{L}}|dv_\theta$$

$$\leq \sqrt{2}C\Big(\int_M |\tau_{H,\widetilde{H}}(f_t)|^2 dv_\theta\Big)^{1/2}\Big(\int_M e_{H,\widetilde{L}}(f_t)dv_\theta\Big)^{1/2}$$

which implies

$$\int_0^t \frac{dE_{H,\widetilde{L}}(f_s)}{2\sqrt{E_{H,\widetilde{L}}(f_s)}} \leq \frac{C}{\sqrt{2}}\int_0^t \Big(\int_M |\tau_{H,\widetilde{H}}(f_s)|^2 dv_\theta\Big)^{1/2}ds.$$

Consequently

(7.23) $$\sqrt{E_{H,\widetilde{L}}(f_t)} - \sqrt{E_{H,\widetilde{L}}(h)} \leq \frac{C}{\sqrt{2}}\Big(\int_0^t\int_M |\tau_{H,\widetilde{H}}(f_s)|^2 dv_\theta\Big)^{1/2}\sqrt{t}.$$

Thus Lemma 7.2 and (7.23) imply

$$\sqrt{E_{H,\widetilde{L}}(f_t)} \leq \frac{C}{\sqrt{2}}\sqrt{E_{H,\widetilde{H}}(h)}\sqrt{t} + \sqrt{E_{H,\widetilde{L}}(h)}.$$

$\square$

Lemma 7.10. *Let M, N and $f \in C^\infty(M \times [0, T), N)$ be as in Lemma 7.7. Suppose h is foliated. Then $e_{L,\widetilde{L}}(f_t)$ is uniformly bounded, and $E_{L,\widetilde{L}}(f_t)$ is decreasing in t.*

Proof. Since h is foliated, we see from Lemma 7.6 that f_t is foliated for each $t \in [0, T)$. As a result, (7.18) becomes

$$(7.24) \qquad (\triangle_H - \frac{\partial}{\partial t}) e_{L,\widetilde{L}}(f_t) = |\beta_{L \times H, \widetilde{L}}|^2 \geq 0.$$

By Maximum principle, we have $e_{L,\widetilde{L}}(f_t) \leq \sup_M e_{L,\widetilde{L}}(h)$. In terms of the divergence theorem, (7.24) implies that

$$\frac{d}{dt} \int_M e_{L,\widetilde{L}}(f_t) dv_\theta = \int_M \left(\triangle_H \, e_{L,\widetilde{L}}(f_t) - |\beta_{L \times H, \widetilde{L}}|^2 \right) dv_\theta \leq 0.$$

Hence $E_{L,\widetilde{L}}(f_t)$ is decreasing in t. $\square$

From Lemmas 7.2, 7.6, 7.8, 7.9 and 7.10, we immediately get the following global existence of (7.5).

Proposition 7.11. *Let M and N be compact Sasakian manifolds. Suppose N has non-positive horizontal curvature and the initial map $h : M \to N$ is foliated. Then the solution f of (7.5) exits for all $t \geq 0$.*

Now we are able to establish some existence results for $(H, \widetilde{H})$-harmonic maps when the target manifold N is a compact regular Sasakian manifold, that is, N can be realized as a Riemannian submersion $\pi : (N, g_{\widetilde{\theta}}) \to (B, g_B)$ over a compact Kähler manifold. Let $i(B)$ be the injectivity radius of B. We denote by $B_r(y)$ the geodesic ball centered at y with radius r in B. Hence, if $r \leq i(B)$, then any two points in $B_r(y)$ can be joined by a unique geodesic in $B_r(y)$.

Theorem 7.12. *Let M be a compact Sasakian manifold and let N be a compact Sasakian manifold with non-positive horizontal sectional curvature. Suppose $\pi : N \to B$ is a Riemannian submersion over a Kähler manifold B and $h : M \to N$ is a given foliated map. Then there exists a smooth foliated $(H, \widetilde{H})$-harmonic map in the same homotopy class as h.*

Proof. From Proposition 7.11, we know that there is a global solution $f : M \times [0, \infty) \to N$ of (7.5) with the initial map h. Set $\varphi_t = \pi \circ f_t$ and $\psi = \pi \circ h$. Since f_t is foliated for each $t \in [0, \infty)$ in view of Lemma 7.6, it follows from Proposition 3.8 that $\varphi : M \times [0, \infty) \to N$ satisfies the following harmonic heat flow

$$(7.25) \qquad \begin{cases} \frac{\partial \varphi}{\partial t} = \tau^{g_B}(\varphi_t) \\ \varphi|_{t=0} = \psi. \end{cases}$$

Observe from (1.25) that B has non-positive sectional curvature duo to the non-positive horizontal curvature condition on N. By Eells-Sampson theorem, the solution φ of (7.25) converges in $C^\infty(M, B)$ to a harmonic map φ_∞ as $t \to \infty$. Therefore there is a sufficiently large $T > 0$ such that if $t \geq T$, then $\varphi_t(x) \in B_{i(B)}(\varphi_\infty(x))$. In particular,

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$\varphi_T(x) \in B_{i(B)}(\varphi_\infty(x)))$ for each $x \in M$. Set $v(x) = \exp^{-1}_{\varphi_T(x)}(\varphi_\infty(x))$. Clearly $v \in \Gamma(\varphi^{-1}(TB))$. We may define a horizontal vector field $\widetilde{v}$ along f_T such that $\widetilde{v}(x) \in H_{f_T(x)}(N)$ and $d\pi() = v(x)$ for $x \in M$. Set

$$\widehat{f}(x) = \exp^{\perp}_{f_T(x)}(\widetilde{v}(x)), \quad x \in M,$$

where $\exp^{\perp}$ denotes the normal exponential map along the Reeb leaf $\pi^{-1}(\varphi_T(x))$. Note that $\varphi_T(x)$ and $\varphi_\infty(x)$ can be joined by a unique geodesic γ_x lying in $B_{i(B)}(\varphi_\infty(x))$. Actually $\widehat{f}(x)$ is the image of $f_T(x)$ under the honolomy diffeomorphism $h_{\gamma_x} : \pi^{-1}($ $\varphi_T(x)) \to \pi^{-1}(\varphi_\infty(x))$ associated to the geodesic γ_x from $\varphi_T(x)$ to $\varphi_\infty(x)$. Thus $\widehat{f}$ is a foliated map. Obviously the map $\widehat{f} : M \to N$ lies in the same homotopy class as h and satisfies

$$\pi \circ \widehat{f} = \varphi_\infty.$$

Thus Proposition 3.8 implies that $\widehat{f}$ is a $(H, \widetilde{H})$-harmonic map. $\square$

Remark 7.2. Let V be any vertical vector field on N and let ζ_s denote the one parameter transformation group generated by V. Then $\zeta_s \circ \widehat{f}$ is also a $(H, \widetilde{H})$-harmonic map in the same homotopy class as h. Hence we do not have the uniqueness for $(H, \widetilde{H})$-harmonic maps in a fixed homotopy class in general.

Lemma 7.13. *Let M, N and B be as in Theorem 7.12. Let $f : M \times [0, \infty) \to N$ be a solution of (7.5) with initial map h. Suppose $h : M \to N$ is a foliated $(H, \widetilde{H})$-harmonic map. Then $f_t : M \to N$ is a foliated $(H, \widetilde{H})$-harmonic map for each $t \in [0, \infty)$.*

Proof. Set $\varphi_t = \pi \circ f_t$ and $\psi = \pi \circ h$, where $\pi : N \to B$ is the Riemannian submersion. From the proof of Theorem 7.12, we know that φ_t satisfies the harmonic map heat flow (7.25). Since h is assumed to be a foliated $(H, \widetilde{H})$-harmonic map, Proposition 3.8 implies that $\psi : M \to B$ is a harmonic map, which may be also regarded as a solution of (7.25) independent of the time t. By the uniqueness for solutions of (7.25), we find that φ_t is harmonic for each t. It follows from Proposition 3.8 again that $f_t : M \to N$ is $(H, \widetilde{H})$-harmonic. $\square$

Theorem 7.14. *Let M and N be compact Sasakian manifolds and let $h : M \to N$ be a foliated map. Suppose N is regular with non-positive horizontal sectional curvature. Then there exists a foliated special $(H, \widetilde{H})$-harmonic map in the same foliated homotopy class as h.*

Proof. Without loss of generality, we may assume that h is a foliated $(H, \widetilde{H})$-harmonic map in view of Theorem 7.12. Suppose $f : M \times [0, \infty)$ is a solution of (7.25) with the initial map h. By Lemma 7.13 , each f_t is a foliated $(H, \widetilde{H})$-harmonic map. Then Lemma 7.9 gives

$$\frac{d}{dt} E_{H,\widetilde{L}}(f_t) = -\int_M |\tau_{H,\widetilde{L}}(f_t)|^2 dv_\theta.$$

This yields that $E_{H,\widetilde{L}}(f_t)$ is decreasing in t and

$$\text{(7.26)} \qquad \int_0^\infty \int_M |\tau_{H,\widetilde{L}}(f_t)|^2 dv_\theta dt < \infty.$$

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Consequently Lemmas 7.2, 7.6, 7.8, 7.10 imply that $e(f_t)$ is uniformly bounded. Applying Proposition 7.1 to the solution of (7.6), we find that all higher derivatives of f are uniformly bounded. On the other hand, (7.26) implies that there exists a sequence $\{t_k\}$ such that

$$(7.27) \qquad |\tau_{H,\widetilde{L}}(f_{t_k})|_{L^2(M)} \to 0 \quad \text{as } t_k \to \infty.$$

In terms of the Arzela-Ascoli theorem, by passing a subsequence $\{t_{k_l}\}$ of $\{t_k\}$, we conclude that $f(\cdot, t_{k_l})$ converges in $C^\infty(M, N)$ to a limit f_∞ (as $t_{k_l} \to \infty$), which satisfies both $\tau_{H,\widetilde{H}}(f_\infty) = 0$ and $\tau_{H,\widetilde{L}}(f_\infty) = 0$. Clearly f_∞ lies in the same foliated homotopy class as h. $\square$

8. Foliated rigidity and Siu-type strong rigidity results

First, we introduce the following

Definition 8.1. We say that a map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ has split horizontal second fundamental form if $\pi_{\widetilde{H}}(\beta(T, X)) = 0$ for any $X \in H(M)$, that is, $f_{0k}^\alpha = f_{0\overline{k}}^\alpha = 0$ for $k = 1, ..., m$.

Lemma 8.1. *Let* $f : M^{2m+1} \to N^{2n+1}$ *be a* $(H, \widetilde{H})$-*harmonic map with split horizontal second fundamental form. Suppose that M is compact and N is Sasakian. Then f is foliated.*

Proof. By (2.17), we have

$$(8.1) \qquad f_{k\overline{k}}^\alpha = f_{\overline{k}k}^\alpha + mi f_0^\alpha$$

It follows from Corollary 3.3 and (8.1) that

$$(8.2) \qquad f_{k\overline{k}}^\alpha = \frac{mi}{2} f_0^\alpha, \quad f_{\overline{k}k}^\alpha = -\frac{mi}{2} f_0^\alpha.$$

Let us define a global 1-form on M by $\widetilde{\rho} = -i(f_0^\alpha f_{\overline{k}}^{\overline{\alpha}} \theta^k - f_0^{\overline{\alpha}} f_k^\alpha \theta^{\overline{k}})$. Using Lemma 1.3, (8.2) and the assumption that $f_{0k}^\alpha = f_{0\overline{k}}^\alpha = 0$, we find that

$$
\begin{aligned}
0 &= \int_M \delta\widetilde{\rho} \\
&= i \int_M \{(f_0^\alpha f_{\overline{k}}^{\overline{\alpha}})_{\overline{k}} - (f_0^{\overline{\alpha}} f_k^\alpha)_k\} dv_\theta \\
&= i \int_M \{f_{0\overline{k}}^\alpha f_{\overline{k}}^{\overline{\alpha}} + f_0^\alpha f_{k\overline{k}}^{\overline{\alpha}} - f_{0k}^{\overline{\alpha}} f_k^\alpha - f_0^{\overline{\alpha}} f_{\overline{k}k}^\alpha\} dv_\theta \\
&= i \int_M \{\frac{mi}{2} f_0^\alpha f_0^{\overline{\alpha}} + \frac{mi}{2} f_0^{\overline{\alpha}} f_0^\alpha\} dv_\theta \\
&= -m \int_M \sum_\alpha |f_0^\alpha|^2 dv_\theta,
\end{aligned}
$$

which implies $f_0^\alpha = 0$, that is, f is foliated. $\square$

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Theorem 8.2. *Let $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be compact Sasakian manifolds and let $f : M \to N$ be a $(H, \widetilde{H})$-harmonic map. If $(N^{2n+1}, \widetilde{\nabla})$ has nonpositive horizontal sectional curvature, then f is foliated.*

Proof. First we define a global 1-form $\widetilde{\rho}_1$ by

$$(8.3) \qquad \widetilde{\rho}_1 = -\{(f_{0k}^\alpha f_0^{\overline{\alpha}} + f_{0k}^{\overline{\alpha}} f_0^\alpha)\theta^k + (f_{0\overline{k}}^{\overline{\alpha}} f_0^\alpha + f_{0\overline{k}}^\alpha f_0^{\overline{\alpha}})\theta^{\overline{k}}\}.$$

Using (1.20), (1.22), (2.17), (2.35) and (2.41), we deduce that

$$
\begin{aligned}
\delta\widetilde{\rho}_1 =&(f_{0k}^\alpha f_0^{\overline{\alpha}} + f_{0k}^{\overline{\alpha}} f_0^\alpha)_{\overline{k}} + (f_{0\overline{k}}^{\overline{\alpha}} f_0^\alpha + f_{0\overline{k}}^\alpha f_0^{\overline{\alpha}})_k \\
=&2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2 + f_0^{\overline{\alpha}}(f_{0k\overline{k}}^\alpha + f_{0\overline{k}k}^\alpha) + f_0^\alpha(f_{0k\overline{k}}^{\overline{\alpha}} + f_{0\overline{k}k}^{\overline{\alpha}}) \\
=&2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2 + f_0^{\overline{\alpha}}(f_{k0\overline{k}}^\alpha + f_{\overline{k}0k}^\alpha) + f_0^\alpha(f_{k0\overline{k}}^{\overline{\alpha}} + f_{\overline{k}0k}^{\overline{\alpha}}) \\
=&2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2 + f_0^{\overline{\alpha}}(f_{k\overline{k}0}^\alpha + f_{\overline{k}k0}^\alpha) + f_0^\alpha(f_{k\overline{k}0}^{\overline{\alpha}} + f_{\overline{k}k0}^{\overline{\alpha}}) \\
&+ f_0^{\overline{\alpha}} f_k^\beta \widehat{R}_{\beta\gamma\delta}^\alpha(f_{\overline{k}}^\gamma f_0^\delta - f_0^\gamma f_{\overline{k}}^\delta) + f_0^\alpha f_{\overline{k}}^{\overline{\beta}} \widehat{R}_{\overline{\beta}\overline{\gamma}\delta}^{\overline{\alpha}}(f_k^{\overline{\gamma}} f_0^\delta - f_0^{\overline{\gamma}} f_k^\delta) \\
&+ f_0^{\overline{\alpha}} f_{\overline{k}}^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha(f_k^\gamma f_0^\delta - f_0^\gamma f_k^\delta) + f_0^\alpha f_k^{\overline{\beta}} \widehat{R}_{\overline{\beta}\overline{\gamma}\delta}^{\overline{\alpha}}(f_{\overline{k}}^{\overline{\gamma}} f_0^\delta - f_0^{\overline{\gamma}} f_{\overline{k}}^\delta) \\
=&2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2 + f_0^{\overline{\alpha}}(f_{k\overline{k}0}^\alpha + f_{\overline{k}k0}^\alpha) + f_0^\alpha(f_{k\overline{k}0}^{\overline{\alpha}} + f_{\overline{k}k0}^{\overline{\alpha}}) \\
&- \widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k)}) \\
&- \widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}}), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}})})
\end{aligned}
$$
(8.4)

where $df_{L,\widetilde{H}}(\xi) = \pi_{\widetilde{H}}(df(\xi))$. It follows from Corollary 3.3 and (8.4) that

$$
\begin{aligned}
\delta\widetilde{\rho}_1 =&2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2 - \widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k)}) \\
&- \widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}}), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}})})
\end{aligned}
$$
(8.5)

Note that

$$
\begin{aligned}
&\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k)}) \\
&+\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}}), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}})}) \\
&=\frac{1}{2}\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k - iJe_k), df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k + iJe_k)) \\
&+\frac{1}{2}\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k + iJe_k), df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k - iJe_k)) \\
&=\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k), df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(e_k)) \\
&+\widetilde{Q}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(Je_k), df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(Je_k)) \\
&\leq 0
\end{aligned}
$$
(8.6)

in view of the curvature condition on N. Consequently

$$\delta\widetilde{\rho}_1 \geq 2|f_{0k}^\alpha|^2 + 2|f_{0\overline{k}}^\alpha|^2$$

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which gives that $f_{0k}^\alpha = f_{0\overline{k}}^\alpha = 0$ by the divergence theorem. In terms of Lemma 8.1, we see that f is foliated. $\square$

In terms of Remark 3.2 and Theorem 8.2, we find that $(H, \widetilde{H})$-harmonic maps seem to be more sensitive to the foliated structures than to the horizontal distributions, at least when the target manifolds are Sasakian manifolds with non-positive horizontal sectional curvature. However, we will see that under some further conditions on the maps and the target manifolds, these critical maps may be related to the CR structures in a horizontally projective way.

Theorem 8.3. *Let* $(M^{2m+1}, H(M), J, \theta)$ *and* $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ *be compact Sasakian manifolds and* $f : M \to N$ *be a* $(H, \widetilde{H})$-*harmonic map. If* $(N^{2n+1}, \widetilde{\nabla})$ *has strongly seminegative horizontal curvature, then* f *is* $(H, \widetilde{H})$-*pluriharmonic and*

$$(8.7) \qquad \widetilde{Q}(df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k)}) = 0$$

for any unitary frame $\{\eta_j\}$ *of* $H^{1,0}(M)$.

Proof. Let us define a global 1-form by

$$(8.8) \qquad \widetilde{\rho}_2 = -(f_{\overline{j}k}^\alpha f_{\overline{j}}^{\overline{\alpha}} \theta^k + f_{\overline{j}k}^{\overline{\alpha}} f_{\overline{j}}^\alpha \theta^{\overline{k}}).$$

According to (1.20), (1.22), (2.17) and (2.38), we compute that

$$
\begin{aligned}
\delta\widetilde{\rho}_2 =& (f_{\overline{j}k}^\alpha f_{\overline{j}}^{\overline{\alpha}})_{\overline{k}} + (f_{\overline{j}k}^{\overline{\alpha}} f_{\overline{j}}^\alpha)_k \\
=& f_{\overline{j}k}^\alpha f_{\overline{j}\overline{k}}^{\overline{\alpha}} + f_{\overline{j}\overline{k}}^{\overline{\alpha}} f_{\overline{j}k}^\alpha + f_{\overline{j}k\overline{k}}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_{\overline{j}\overline{k}k}^{\overline{\alpha}} f_{\overline{j}}^\alpha \\
=& 2|f_{\overline{j}k}^\alpha|^2 + (f_{k\overline{j}}^\alpha - if_0^\alpha \delta_k^j)_{\overline{k}} f_{\overline{j}}^{\overline{\alpha}} + (f_{\overline{k}j}^{\overline{\alpha}} + if_0^{\overline{\alpha}} \delta_{\overline{k}}^{\overline{j}})_k f_{\overline{j}}^\alpha \\
=& 2|f_{\overline{j}k}^\alpha|^2 + (f_{k\overline{j}\overline{k}}^\alpha - if_{0\overline{k}}^\alpha \delta_k^j) f_{\overline{j}}^{\overline{\alpha}} + (f_{\overline{k}jk}^{\overline{\alpha}} + if_{0k}^{\overline{\alpha}} \delta_{\overline{k}}^{\overline{j}}) f_{\overline{j}}^\alpha \\
=& 2|f_{\overline{j}k}^\alpha|^2 + i(f_{0j}^{\overline{\alpha}} f_{\overline{j}}^\alpha - f_{0\overline{j}}^\alpha f_{\overline{j}}^{\overline{\alpha}}) + f_{k\overline{j}\overline{k}}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_{\overline{k}jk}^{\overline{\alpha}} f_{\overline{j}}^\alpha \\
=& 2|f_{\overline{j}k}^\alpha|^2 + i(f_{0j}^{\overline{\alpha}} f_{\overline{j}}^\alpha - f_{0\overline{j}}^\alpha f_{\overline{j}}^{\overline{\alpha}}) + f_{\overline{k}k\overline{j}}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_{\overline{k}kj}^{\overline{\alpha}} f_{\overline{j}}^\alpha \\
& + f_{\overline{j}}^{\overline{\alpha}} f_k^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha (f_{\overline{k}}^\gamma f_{\overline{j}}^{\overline{\delta}} - f_{\overline{j}}^\gamma f_{\overline{k}}^{\overline{\delta}}) + f_{\overline{j}}^\alpha f_{\overline{k}}^{\overline{\beta}} \widehat{R}_{\overline{\beta}\overline{\gamma}\delta}^{\overline{\alpha}} (f_k^{\overline{\gamma}} f_{\overline{j}}^\delta - f_{\overline{j}}^{\overline{\gamma}} f_k^\delta) \\
=& 2|f_{\overline{j}k}^\alpha|^2 + i(f_{0j}^{\overline{\alpha}} f_{\overline{j}}^\alpha - f_{0\overline{j}}^\alpha f_{\overline{j}}^{\overline{\alpha}}) + f_{\overline{k}k\overline{j}}^\alpha f_{\overline{j}}^{\overline{\alpha}} + f_{\overline{k}kj}^{\overline{\alpha}} f_{\overline{j}}^\alpha \\
& - \widetilde{Q}(df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k)}).
\end{aligned}
\tag{8.9}
$$

Note that a Sasakian manifold with strongly semi-negative horizontal curvature has automatically non-positive horizontal sectional curvature. Then Theorem 8.2 yields that $f_0^\alpha = f_0^{\overline{\alpha}} = 0$, and thus

$$(8.10) \qquad f_{0j}^\alpha = f_{0\overline{j}}^\alpha = 0.$$

From the fourth equation of (2.17), we derive that

$$
\begin{aligned}
(8.11) \qquad 2f_{k\overline{k}}^\alpha &= f_{k\overline{k}}^\alpha + f_{\overline{k}k}^\alpha + mif_0^\alpha \\
&= mif_0^\alpha \\
&= 0.
\end{aligned}
$$

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It follows from (8.9), (8.10) and 8.11) that

$$
(8.12) \qquad \delta\widetilde{\rho}_2 = 2|f^\alpha_{\overline{j}k}|^2 - \widetilde{Q}(df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k)}).
$$

In terms of the curvature condition of $(N, \widetilde{\nabla})$ and the divergence theorem, we get from (8.12) that

$$
f^\alpha_{\overline{j}k} = 0
$$

and

$$
\widetilde{Q}(df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{H,\widetilde{H}}(\eta_j) \wedge df_{H,\widetilde{H}}(\eta_k)}) = 0
$$

for any $1 \le j, k \le m$. Since $f^\alpha_0 = 0$, (2.17) implies that $f^\alpha_{k\overline{j}} = f^\alpha_{\overline{j}k} = 0$. According to (5.2) and (5.3), f is $(H, \widetilde{H})$-pluriharmonic. Hence we complete the proof. $\square$

Theorem 8.4. *Suppose $(M^{2m+1}, H(M), J, \theta)(m \ge 2)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ are compact Sasakian manifolds and N has strongly negative horizontal curvature. Suppose $f : M \to N$ is a $(H, \widetilde{H})$-harmonic map with $\max_M \mathrm{rank}_R\{df_{H,\widetilde{H}}\} \ge 3$. Then f is either a foliated $(H, \widetilde{H})$-holomorphic map or a foliated $(H, \widetilde{H})$-antiholomorphic map.*

Proof. From Theorem 8.2, we know that f is foliated. The rank condition for f means that there exists a point $p \in M$ such that $\mathrm{rank}_R\{df_{H,\widetilde{H}}(p)\} \ge 3$. Consequently $\mathrm{rank}_R\{df_{H,\widetilde{H}}\} \ge 3$ in some connected open neighborhood U of p. Write $df_{H,\widetilde{H}}(\eta_{\overline{j}}) = f^\alpha_{\overline{j}}\widetilde{\eta}_\alpha + f^{\overline{\alpha}}_{\overline{j}}\widetilde{\eta}_{\overline{\alpha}}$. Then

$$
(8.13) \qquad \left(df_{H,\widetilde{H}}(\eta_{\overline{j}}) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}})\right)^{(1,1)} = (f^\alpha_{\overline{j}}f^{\overline{\beta}}_{\overline{k}} - f^\alpha_{\overline{k}}f^{\overline{\beta}}_{\overline{j}})\widetilde{\eta}_\alpha \wedge \widetilde{\eta}_{\overline{\beta}}.
$$

Since N has strongly negative horizontal curvature, it follows from (8.7) and (8.13) that

$$
(8.14) \qquad f^\alpha_{\overline{j}}f^{\overline{\beta}}_{\overline{k}} - f^\alpha_{\overline{k}}f^{\overline{\beta}}_{\overline{j}} = 0
$$

for $1 \le j, k \le m$ and $1 \le \alpha, \beta \le n$. We want to show that for every point $q \in U$, either $\partial f_{H,\widetilde{H}}(q) = 0$ or $\overline{\partial} f_{H,\widetilde{H}}(q) = 0$.

Without loss of generality, we assume that $\partial f_{H,\widetilde{H}}(q) \ne 0$. This means that $f^\gamma_k(q) \ne 0$ for some $1 \le k \le m$ and some $1 \le \gamma \le n$. Therefore (8.14) yields that

$$
(8.15) \qquad \{f^\alpha_{\overline{1}}(q), \cdots, f^\alpha_{\overline{m}}(q)\} = c^\alpha\{f^{\overline{\gamma}}_{\overline{1}}(q), \cdots, f^{\overline{\gamma}}_{\overline{m}}(q)\}
$$

for each $1 \le \alpha \le m$, where $c^\alpha = f^\alpha_{\overline{k}}(q)/f^{\overline{\gamma}}_{\overline{k}}(q)$. If $\overline{\partial} f_{H,\widetilde{H}}(q) \ne 0$ too, then $f^\delta_{\overline{j}}(q) \ne 0$ for some $1 \le j \le m$ and some $1 \le \delta \le n$. Using (8.14) again, we get

$$
(8.16) \qquad \{f^{\overline{\beta}}_{\overline{1}}(q), \cdots, f^{\overline{\beta}}_{\overline{m}}(q)\} = d^{\overline{\beta}}\{f^\delta_{\overline{1}}(q), \cdots, f^\delta_{\overline{m}}(q)\}
$$

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for each $1 \leq \beta \leq m$, where $d^{\overline{\beta}} = f_{\overline{j}}^{\overline{\beta}}(q)/f_{\overline{j}}^{\delta}(q)$. In terms of (8.15) and (8.16), we find

$$
\begin{aligned}
(8.17) \qquad df_{H,\widetilde{H}}(\eta_{\overline{i}}) &= f_{\overline{i}}^{\alpha} \eta_{\alpha} + f_{\overline{i}}^{\overline{\alpha}} \eta_{\overline{\alpha}} \\
&= f_{\overline{i}}^{\overline{\gamma}}(c^{\alpha} \eta_{\alpha}) + f_{\overline{i}}^{\delta}(d^{\overline{\alpha}} \eta_{\overline{\alpha}}) \\
&= f_{\overline{i}}^{\overline{\gamma}}(c^{\alpha} \eta_{\alpha}) + c^{\delta} f_{\overline{i}}^{\overline{\gamma}}(d^{\overline{\alpha}} \eta_{\overline{\alpha}}) \\
&= f_{\overline{i}}^{\overline{\gamma}}(c^{\alpha} \eta_{\alpha} + c^{\delta} d^{\overline{\alpha}} \eta_{\overline{\alpha}})
\end{aligned}
$$

at $q \in U$. From (8.17), we have

$$
\begin{aligned}
span_C \{df_{H,\widetilde{H}}(\eta_{\overline{i}})\}_q &= span_C \{c^{\alpha} \eta_{\alpha} + c^{\delta} d^{\overline{\alpha}} \eta_{\overline{\alpha}}\} \\
&\leq 1
\end{aligned}
$$

which implies that $rank_R \{df_{H,\widetilde{H}}(q)\} \leq 2$, contradicting $q \in U$. Thus, for each $q \in U$, either $\partial f_{H,\widetilde{H}}(q) = 0$ or $\overline{\partial} f_{H,\widetilde{H}}(q) = 0$.

Since $rank_R \{df_{H,\widetilde{H}} > 0\}$ in U, the two sets $U \cap \{\partial f_{H,\widetilde{H}} = 0\}$ and $U \cap \{\overline{\partial} f_{H,\widetilde{H}} = 0\}$ are disjoint closed subsets of U, and their union is the connected set U. It follows that either $\partial f_{H,\widetilde{H}} \equiv 0$ on U or $\overline{\partial} f_{H,\widetilde{H}} \equiv 0$ on U. From Theorem 5.4 and Theorem 8.2, we conclude that either $\partial f_{H,\widetilde{H}} \equiv 0$ on M or $\overline{\partial} f_{H,\widetilde{H}} \equiv 0$ on M. $\square$

Theorem 8.5. *Let $k \geq 2$. Suppose $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ are compact Sasakian manifolds and the horizontal curvature of N is negative of order k. Suppose $f : M \to N$ is a $(H, \widetilde{H})$-harmonic map and $\max_M \{df_{H,\widetilde{H}}\} \geq 2k$. Then f is either a foliated $(H, \widetilde{H})$-holomorphic map or a foliated $(H, \widetilde{H})$-antiholomorphic map.*

Proof. By a similar argument as that in Theorem 8.4, we may deduce the conclusion of this theorem from (8.7), (8.13) and Definition 8.2. $\square$

For a manifold M, we use $H_l(M, R)$ to denote its usual singular homology.

Corollary 8.6. *Suppose $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is a foliated $(H, \widetilde{H})$-harmonic map between compact Sasakian manifolds. Suppose the horizontal curvature of N is negative of order k and $k \geq 2$. If the induced map $f_* : H_l(M, R) \to H_l(N, R)$ is nonzero for some $l \geq 2k+1$, then f is either $(H, \widetilde{H})$-holomorphic or $(H, \widetilde{H})$-anti-holomorphic.*

Proof. The assumption that $f_* : H_l(M, R) \to H_l(N, R)$ is nonzero for some $l \geq 2k+1$ implies that $rank_R \{df\} \geq 2k + 1$ at some point p of M. Since f is foliated,

$$
rank_R \{\pi_{\widetilde{L}} \circ df\} + rank_R \{df_{H,\widetilde{H}}\} \geq rank_R \{df\}
$$

at each point of M. Hence $rank_R \{df_{H,\widetilde{H}}\}_p \geq 2k$. By Theorem 8.5, we find that f is either a foliated $(H, \widetilde{H})$-holomorphic or a foliated $(H, \widetilde{H})$-anti-holomorphic map. $\square$

Before investigating the strong rigidity of Sasakian manifolds with some kind of negative horizontal curvature, let us recall some basic notions and results for foliations,

especially for Riemannian foliations. Given a foliation F on a manifold M, a differential form $\omega \in \Omega^*(M)$ is called basic if for every vector field X tangent to the leaves, $i_X \omega = 0$ and $i_X d\omega = 0$, where i_X denotes the interior product with respect to X. In particular, a basic form of degree 0 is just a basic function. Clearly the exterior derivative of a basic form is again basic, so all basic forms $(\Omega_B^*(M), d_B)$ constitutes a subcomplex of the de Rham complex $(\Omega^*(M), d)$, where $d_B = d|_{\Omega_B^*(M)}$. The cohomology $H_B^*(M/F) = \ker d_B / \operatorname{Im} d_B$ of this subcomplex is called the basic cohomology of (M, F). In general, the basic cohomology groups are not always finite dimensional. It is known that Riemannian foliations on compact manifolds form a large class of foliations for which the basic cohomology groups are finite-dimensional (cf. [KHS], [KT], [PR]).

Let (M, F) and $(N, \widetilde{F})$ be two manifolds endowed with complete Riemannian foliations F and $\widetilde{F}$ respectively. Similar to Definition 3.3, we have the notion of continuous foliated map between (M, F) and $(N, \widetilde{F})$, that is, a continuous map from M to N mapping leaves of F into leaves of $\widetilde{F}$. A homotopy between M and N consisting of continuous (resp. smooth) foliated maps is called a continuous (resp. smooth) foliated homotopy, and the corresponding homotopy equivalence can be defined in a natural way. Actually we may work in smooth category due to the following result.

Lemma 8.7. *([LM]) Any continuous foliated map between complete Riemannian foliations is foliatedly homotopic to a C^∞ foliated map.*

In view of Lemma 8.7, two complete Riemannian foliated manifolds (M, F) and $(N, \widetilde{F})$ have the same foliated homotopy type in the C^∞ sense if and only if they have the same foliated homotopy type in the usual continuous sense. Another important property of basic cohomologies of Riemannian foliations is the homotopy invariance.

Lemma 8.8. *(cf. [Vi]) Let (M, F) and $(N, \widetilde{F})$ be two compact Riemannian foliated manifolds. Suppose $f, g : (M, F) \to (N, \widetilde{F})$ are foliated homotopic. Then $f^* = g^* : H_B^*(N/\widetilde{F}) \to H_B^*(M/F)$. In particular, if $f : (M, F) \to (N, \widetilde{F})$ is a foliated homotopy equivalence, then $f^* : H_B^*(N/\widetilde{F}) \to H_B^*(M/F)$ is an isomorphism.*

As mentioned in §1, the Reeb foliation of a Sasakian manifold is a Riemannian foliation of dimension 1. Suppose $(M^{2m+1}, H(M), J, \theta)$ is a compact Sasakian manifold with the Reeb foliation F_ξ. Then each basic cohomology group $H_B^k(M/F_\xi)$ $(k = 0, 1, ..., 2m)$ is finite dimensional. Using the second property in (1.4), it is easy to verify that $d\theta$ is a basic form, and thus so is $(d\theta)^k$ for $2 \leq k \leq m$.

Lemma 8.9. *Suppose $(M^{2m+1}, H(M), J, \theta)$ is a compact Sasakian manifold. Then $0 \neq [(d\theta)^k]_B \in H_B^{2k}(M/F)$ $(1 \leq k \leq m)$.*

Proof. We prove this lemma by contradiction. Suppose $(d\theta)^k = d\alpha$ for some $\alpha \in \Omega_B^{2k-1}(M)$. Then

$$(8.18) \qquad (d\theta)^m = d\alpha \wedge (d\theta)^{m-k} = d\big(\alpha \wedge (d\theta)^{m-k}\big).$$

Using (8.18) and the Stokes formula, the volume of M is given, up to a positive constant,

by

$$\int_M \theta \wedge (d\theta)^m = \int_M \theta \wedge d\big(\alpha \wedge (d\theta)^{m-k}\big)$$
$$= \int_M d\theta \wedge \alpha \wedge (d\theta)^{m-k}$$
$$= 0,$$

where the last equality follows from the fact that $d\theta \wedge \alpha \wedge (d\theta)^{m-k} = 0$, since it is a basic form of degree $2m+1$. Hence we get a contradiction. $\square$

Remark 8.1. The above result is actually true for a general compact pseudo-Hermitian manifold.

Recall that a smooth complex-valued function $f : M \to C$ on a CR manifold M is called a CR function if $Z(f) = 0$ for any $Z \in H^{0,1}(M)$.

Definition 8.2. Let M^{2m+1} be a Sasakian manifold. We say that a subset V of M is a foliated analytic subvariety if, for any point $p \in V$, there exists a foliated coordinate chart $(U, \Phi; \varphi)$ of p such that $V \cap U$ is the common zero locus of a finite collection of basic CR functions $f_1, ..., f_k$ on U. In particular, V is called a foliated analytic hypersurface if V is locally the zero locus of a single nonzero basic CR function f.

More explicitly, let $\varphi : U \to W \subset C^m$ be the submersion associated with the foliated coordinate chart $(U, \Phi; \varphi)$, that is, $\varphi = \pi \circ \Phi$ (see (1.23) in §1). Since the CR functions $f_1, .., f_k$ are constant along the leaves, there are holomorphic function $\widetilde{f}_1, ..., \widetilde{f}_k$ on W such that $f_i = \widetilde{f}_i \circ \pi$ ($i = 1, ..., k$). Set $\widetilde{V}_\varphi = \varphi(V \cap U)$. Thus $\widetilde{V}_\varphi$ is a complex analytic subvariety in W defined by the common zero locus of $\widetilde{f}_1, .., \widetilde{f}_k$.

A point $p \in V$ is called a smooth point of V if V is a submanifold of M near p. The locus of smooth points of V is denoted by V^*. A point $p \in V - V^*$ is called a singular point of V; the singular locus $V - V^*$ of V is denoted by V^s. Similarly we have the notions of smooth points and singular points for $\widetilde{V}_\varphi$. Let $\widetilde{V}_\varphi^*$ (resp. $\widetilde{V}_\varphi^s$) denote the locus of smooth points (resp. singular points) of $\widetilde{V}_\varphi$. Clearly $\widetilde{V}_\varphi^* = \varphi(V^* \cap U)$ and $\widetilde{V}_\varphi^s = \varphi(V^s \cap U)$ of $\widetilde{V}$. By the proposition on page 32 in [GH], we know that $\widetilde{V}_\varphi^*$ has finite volume in bounded regions. Consequently V^* has finite volume in bounded regions too. Therefore we may define the integral of a differential form ω on M over V to be the integral of ω over the smooth locus V^* of V.

We need the following Stokes' formula for foliated analytic subvarieties, which is a generalization of the usual Stokes' formula for analytic subvarieties.

Proposition 8.10. *Let M be a Sasakian manifold and let $V \subset M$ be a foliated analytic subvarieties of real dimension $2k+1$. Suppose α is a differential form of degree $2k$ with compact support in M. Then*

$$\int_V d\alpha = 0.$$

Proof. The question is local, it will be sufficient to show that for every point $p \in V$, there exists a neighborhood U of p such that for every $\alpha \in A_c^{2k}(U)$ (the space of differential

forms of degree $2k$ with compact support in U)

$$\int_V d\alpha = 0.$$

Suppose $(U, \Phi; \varphi)$ is a bounded foliated coordinate chart around p. Let $\varphi : U \to W$ be the submersion associated with $(U, \Phi; \varphi)$ and let $\widetilde{p} = \varphi(p) \in \widetilde{V}_\varphi \subset W$. By the local structure of an analytic subvariety in C^m (cf. [GH]), we may find a coordinate system $z = (z_1, ..., z_m)$ and a polycylinder $\triangle$ around $\widetilde{p}$ such that the projection $\widetilde{\pi} : (z_1, ..., z_m) \to (z_1, ..., z_k)$ expresses $\widetilde{V}_\varphi \cap \triangle$ as a branched cover of $\triangle' = \widetilde{\pi}(\triangle)$, branched over an analytic hypersurface $\Sigma \subset \triangle'$. Let T^ε be the ε-neighborhood of Σ in $\triangle'$ and

$$\widetilde{V}_\varphi^\varepsilon = (\widetilde{V}_\varphi \cap \triangle) - \widetilde{\pi}^{-1}(T^\varepsilon) \subset \widetilde{V}_\varphi^*.$$

Set $V^\varepsilon = \varphi^{-1}(\widetilde{V}_\varphi^\varepsilon)$. Clearly $\varphi^{-1}(\triangle) \subset U$ is a foliated neighborhood of p. For $\alpha \in A_c^{2k}(\varphi^{-1}(\triangle))$, we have

$$\int_V d\alpha = \int_{V \cap \varphi^{-1}(\triangle)} d\alpha$$
$$= \lim_{\varepsilon \to 0} \int_{V^\varepsilon} d\alpha$$
$$= \lim_{\varepsilon \to 0} \int_{\partial V^\varepsilon} \alpha$$
$$= \lim_{\varepsilon \to 0} \int_{\varphi^{-1}(\partial \widetilde{\pi}^{-1}(T^\varepsilon))} \alpha.$$

Thus to prove the result, we simply have to prove that $vol(\varphi^{-1}(\partial \widetilde{\pi}^{-1}(T^\varepsilon))) \to 0$ as $\varepsilon \to 0$ or equivalently $vol(\partial \widetilde{\pi}^{-1}(T^\varepsilon)) \to 0$ as $\varepsilon \to 0$. However, the latter one has already be shown on page 33 in [GH] . $\square$

We will use special $(H, \widetilde{H})$-harmonic maps to establish strong rigidity results for Sasakian manifolds.

Lemma 8.11. *Suppose* $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ *is a foliated special* $(H, \widetilde{H})$*-harmonic map between Sasakian manifolds. If M is compact, then* $df(\xi) = \lambda \widetilde{\xi}$ *with* λ *constant.*

Proof. Since f is foliated, $df(\xi) = \lambda \widetilde{\xi}$ for some smooth function λ on M, that is, $f_0^0 = \lambda$ and $f_0^\alpha = f_0^{\overline{\alpha}} = 0$. Then (2.14) yields that

(8.19) $$f_{0j}^0 = f_{j0}^0, \quad f_{0\overline{j}}^0 = f_{\overline{j}0}^0.$$

In terms of (2.24) and (8.19), we get

(8.20) $$f_{0j\overline{j}}^0 = f_{j0\overline{j}}^0 = f_{j\overline{j}0}^0, \quad f_{0\overline{j}j}^0 = f_{\overline{j}0j}^0 = f_{\overline{j}j0}^0.$$

Since f is special, we derive from (8.19) and (8.20) that

$$\triangle_b \lambda = f_{0j\overline{j}}^0 + f_{0\overline{j}j}^0 = f_{j\overline{j}0}^0 + f_{\overline{j}j0}^0$$
$$= 0.$$

Due to the compactness of M, it follows that λ is constant. $\square$

Theorem 8.12. *Let $f : (M, H(M), J, \theta) \to (N, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ be a foliated special $(H, \widetilde{H})$-harmonic map of Sasakian manifolds, both of CR dimension $m \geq 2$. Suppose the horizontal curvature of N is either strongly negative or adequately negative. Suppose f is of degree 1 and the map $f^* : H_B^{2m-2}(N) \to H_B^{2m-2}(M)$ induced by f is surjective. Then $f : M \to N$ is either $(H, \widetilde{H})$-biholomorphic or $(H, \widetilde{H})$-anti-biholomorphic.*

Proof. Let V be the set of points of M where f is not locally diffeomorphic. Suppose V is nonempty. We will derive a contradiction in the following steps.

Step 1. Since $\deg(f) = 1$, $V \neq M$ and f maps $M - f^{-1}(f(V))$ bijectively onto $N - f(V)$. By either Theorem 8.4 or Theorem 8.5, f is either $(H, \widetilde{H})$-holomorphic or $(H, \widetilde{H})$-antiholomorphic. Without loss of generality, we assume now that f is a foliated special $(H, \widetilde{H})$-holomorphic map. In terms of Lemma 8.11 and $V \neq M$, we see that $df(\xi) = \lambda \widetilde{\xi}$ for some nonzero constant λ. For any point $p \in M$, let $q = f(p) \in N$. Let $(U_1, \Phi_1; \varphi_1)$ and $(U_2, \Phi_2; \varphi_2)$ be foliated coordinate charts around p and q respectively, and let $\varphi_i : U_i \to W_i \subset \mathbb{C}^m$ be the submersion associated with $(U_i, \Phi_i; \varphi_i)$ $(i = 1, 2)$. Suppose $f(U_1) \subset U_2$. Then f induces a holomorphic map $\widetilde{f} : W_1 \to W_2$ such that $\varphi_2 \circ f = \widetilde{f} \circ \varphi_1$. Clearly $p \in V$ if and only if $\widetilde{f}$ is not diffeomorphic at $\varphi_1(p)$. Hence V is defined locally by the zero locus of $\det(\partial w^\alpha \circ \widetilde{f}/\partial z^i) \circ \varphi_1$, where (z^i) and (w^α) are holomorphic coordinate systems of W_1 and W_2 respectively. This shows that V is a foliated analytic hypersurface in M. Since f is a foliated $(H, \widetilde{H})$-holomorphic map, $f(V)$ is a foliated analytic subvariety in N. It is obvious that both $M - f^{-1}(f(V))$ and $N - f(V)$ are foliated open submanifolds of M and N respectively.

Step 2. We claim that f is a horizontally one-to-one map from $M - f^{-1}(f(V))$ to $N - f(V)$. Let $\widehat{f} : (M - f^{-1}(f(V)))/F_\xi \to (N - f(V))/\widetilde{F}_{\widetilde{\xi}}$ denote the induced map of $f|_{M-f^{-1}(f(V))}$. Suppose there are two leaves $L_1, L_2 \subset M - f^{-1}(f(V))$ and a leaf $\widetilde{L} \subset N - f(V)$ such that $f(L_1), f(L_2) \subset \widetilde{L}$. Let $\gamma_1(t), \gamma_2(t)$ $(t \in (-\infty, \infty))$ be the integral curves of ξ whose images are L_1 and L_2 respectively. In terms of the fact that $df(\xi) = \lambda \widetilde{\xi}$ with constant $\lambda \neq 0$, we see that both $f(\gamma_1)$ and $f(\gamma_2)$ are integral curves of $\lambda \widetilde{\xi}$ with possibly different initial points. As a result, their image must be $\widetilde{L}$, that is, $f(L_1) = f(L_2) = \widetilde{L}$. The injectivity of $f|_{M-f^{-1}(f(V))}$ implies that $L_1 = L_2$, that is, $\widehat{f}$ is injective. By the surjectivity of f from $M - f^{-1}(f(V))$ to $N - f(V)$, we conclude that $\widehat{f}$ is surjective, and thus $\widehat{f}$ is bijective. By Proposition 5.7, $f : M - f^{-1}(f(V)) \to N - f(V)$ is a foliated $(H, \widetilde{H})$-biholomorphism.

Step 3. Now we assert that $f(V)$ must be a foliated analytic subvariety with real codimension at least 4, that is, the transversal complex codimension is at least two. Otherwise, suppose $f(V)$ is also a foliated analytic hypersurface, we are going to prove that the critical points set V of f is actually removable. For any $p \in V$, let $(U_1, \Phi_1; \varphi_1)$ and $(U_2, \Phi_2; \varphi_2)$ be foliated coordinate charts around, respectively, p and $q = f(p)$ as in Step 1, such that $f(U_1) \subset U_2$ and $f(V) \cap U_2$ is defined by the zero locus of a single basic CR function. Set $V_{\varphi_1} = \varphi_1(V \cap U_1)$ and $[f(V)]_{\varphi_2} = \varphi_2(f(V) \cap U_2)$. From Steps 1 and 2, we know that the induced holomorphic map $\widetilde{f} : W_1 \to W_2$ is injective on $\widetilde{f} : W_1 - \widetilde{f}^{-1}([f(V)]_{\varphi_2})$. Clearly V_{φ_1} and $[f(V)]_{\varphi_2}$ are analytic hypersurfaces in W_1 and W_2 respectively, and $\widetilde{f}(V_{\varphi_1}) \subset [f(V)]_{\varphi_2}$. Then there exists $v \in V_{\varphi_1}$ such that v is an

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isolated point of $\widetilde{f}^{-1}(\widetilde{f}(v))$ and, by using a local coordinate chart (z^i) of W_1 at v and by applying the Riemann removable singularity theorem to $z^i \circ \widetilde{f}^{-1}$ on $W_2 - [f(V)]_{\varphi_2}$ for some open neighborhood of $\widetilde{f}(v)$ in W_2, we find that $\widetilde{f}$ is locally diffeomorphic at v. This implies that f is locally diffeomorphic at any point in $\varphi_1^{-1}(v)$, contradicting $\varphi_1^{-1}(v) \subset V$.

Step 4. From Lemma 8.9, we know that $[(d\theta)^{m-1}]_B \neq 0$. By the assumption that $f^* : H_B^{2m-2}(N) \to H_B^{2m-2}(M)$ is surjective, there exists an element $[\beta]_B \in H_B^{2m-2}(N)$ such that $f^*[\beta]_B = [(d\theta)^{m-1}]_B$, that is,

$$(8.21) \qquad f^*\beta = (d\theta)^{m-1} + d\alpha$$

for some $\alpha \in \Omega_B^{2m-3}(M)$. Since $df(\xi) = \lambda\widetilde{\xi}$, we may write $f^*\widetilde{\theta} = \lambda\theta + f_j^0\theta^j + f_{\overline{j}}^0\theta^{\overline{j}}$. Note that $f_j^0\theta^j + f_{\overline{j}}^0\theta^{\overline{j}}$ is a global 1-form on M. Since V is foliated and $i_\xi\{(f_j^0\theta^j + f_{\overline{j}}^0\theta^{\overline{j}}) \wedge f^*\beta\} = 0$, we have

$$(8.22) \qquad \int_V (f_j^0\theta^j + f_{\overline{j}}^0\theta^{\overline{j}}) \wedge f^*\beta = 0.$$

From (8.21), (8.22) and Proposition 8.12, we get

$$
\begin{aligned}
(8.23) \qquad \int_V f^*\widetilde{\theta} \wedge f^*\beta &= \lambda \int_V \theta \wedge f^*\beta + \int_V \left(f_j^0\theta^j + f_{\overline{j}}^0\theta^{\overline{j}}\right) \wedge f^*\beta \\
&= \lambda \int_V \theta \wedge (d\theta)^{m-1} + \int_V \theta \wedge d\alpha \\
&= \lambda \int_V \theta \wedge (d\theta)^{m-1} + \int_V d\theta \wedge \alpha.
\end{aligned}
$$

Clearly we have $i_\xi(d\theta \wedge \alpha) = 0$, which implies that

$$(8.24) \qquad \int_V d\theta \wedge \alpha = 0.$$

It follows from (8.23) and (8.24) that

$$(8.25) \qquad \int_V f^*\widetilde{\theta} \wedge f^*\beta = \lambda \int_V \theta \wedge (d\theta)^{m-1} > 0.$$

On the other hand, since $f(V)$ is of dimension less than $2m - 3$, one has

$$\int_V f^*\widetilde{\theta} \wedge f^*\beta = \int_{f(V)} \widetilde{\theta} \wedge \beta = 0$$

which contradicts to (8.25).

It follows from the above discussion that V must be empty. In terms of step 2, we may conclude that $f : M \to N$ is a foliated $(H, \widetilde{H})$-biholomorphism. $\square$

Remark 8.2. The argument for removing the critical points of $\widetilde{f}$ in Step 3 is inspired by the related argument in Theorem 8 of [Si1].

Corollary 8.13. *Let $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ $(n \geq 2)$ be a compact regular Sasakian manifold with either strongly negative or adequately negative horizontal curvature. Suppose $(M, H(M), J, \theta)$ is a compact Sasakian manifold with the same foliated homotopy type as N. Then $(M, H(M), J)$ is $(H, \widetilde{H})$-biholomorphic to either $(N, \widetilde{H}(N), \widetilde{J})$ or $(N, \widetilde{H}(N), -\widetilde{J})$.*

Proof. Since M is foliatedly homotopic to N, we have a foliated smooth homotopy equivalent map $h : M \to N$. Consequently $\deg(h) = 1$ and $h^* : H_B^{2m-2}(N) \to H_B^{2m-2}(M)$ is an isomorphism. By Theorem 7.14, there exists a foliated special $(H, \widetilde{H})$-harmonic map $f : M \to N$ which is foliatedly homotopic to h. In terms of Lemma 8.8, $\deg(f) = 1$ and $f^* : H_B^{2m-2}(N) \to H_B^{2m-2}(M)$ is an isomorphism too. This corollary then follows immediately from Theorem 8.12. $\square$

Remark 8.3. Note that the $(H, \widetilde{H})$-biholomorphism f between M and N in Corollary 8.13 is actually a vertically homothetic map, that is, $df(\xi) = \lambda \widetilde{\xi}$ with λ constant. In Example 1.1, we give some Sasakian manifolds with either strongly negative or adequately negative horizontal curvature. These Sasakian manifolds, which may be regarded as model spaces, appear also in the classification of contact sub-symmetric spaces by [BFG]. As applications, Corollary 8.13 exhibits the foliated strong rigidity of these model spaces.

Appendix

A. Pseudo-Hermitian harmonic maps

In this subsection, we introduce another natural generalized harmonic map between pseudo-Hermitian manifolds.

Definition A1. A map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$ is called a pseudo-Hermitian harmonic map if it satisfies

$$\text{(A1)} \qquad \tau(f) = tr_{g_\theta} \beta = 0,$$

that is

$$\text{(A2)} \qquad \begin{aligned} f_{00}^\alpha + f_{k\overline{k}}^\alpha + f_{\overline{k}k}^\alpha &= 0 \\ f_{00}^0 + f_{k\overline{k}}^0 + f_{\overline{k}k}^0 &= 0. \end{aligned}$$

Remark A1. Similar ideas for introducing generalized harmonic maps as Definition A1 were also mentioned in [DT] and [Kok].

Clearly if f is a foliated pseudo-Hermitian harmonic map, then f automatically satisfies the equation $f_{k\overline{k}}^\alpha + f_{\overline{k}k}^\alpha = 0$, that is, f is $(H, \widetilde{H})$-harmonic. Concerning the Question proposed in section 5, we establish a continuation result about the foliated property for pseudo-Hermitian harmonic maps.

Theorem A1. *Let M and N be Sasakian manifolds and let $f : M \to N$ be a pseudo-Hermitian harmonic map. Assume that U is a nonempty open subset of M. If f is foliated on U, then f is foliated on M.*

Proof. The argument is similar to that for Theorem 5.4. Choose any point $q \in \partial U$. Let W be a connected open neighborhood of q such that there exist a frame field $\{\xi, \eta_1, ..., \eta_m, \eta_{\overline{1}}, ..., \eta_{\overline{m}}\}$ on W and a frame field $\{\widetilde{\xi}, \widetilde{\eta}_1, ..., \widetilde{\eta}_n, \widetilde{\eta}_{\overline{1}}, ..., \widetilde{\eta}_{\overline{n}}\}$ on some open neighborhood of $f(W)$ respectively. Write

$$df_{L, \widetilde{H}^{1,0}} = f_0^\alpha \theta \otimes \widetilde{\eta}_\alpha.$$

Clearly $df_{L, \widetilde{H}^{1,0}} \in Hom(L, f^{-1}\widetilde{H}^{1,0})$. By definition, it is easy to derive the following

$$\triangle df_{T, \widetilde{H}^{1,0}} = tr_{g_\theta} D^2 df_{T, \widetilde{H}^{1,0}}$$

(A3)
$$= (f_{000}^\alpha + f_{0k\overline{k}}^\alpha + f_{0\overline{k}k}^\alpha)\theta \otimes \widetilde{\eta}_\alpha$$

$$= (\triangle_M f_0^\alpha)\theta \otimes \widetilde{\eta}_\alpha + tr_{g_\theta}\{df_0^\alpha \otimes \theta \otimes \widetilde{\nabla}\widetilde{\eta}_\alpha + f_0^\alpha \theta \otimes \widetilde{\nabla}^2\widetilde{\eta}_\alpha\}$$

where the property that $\nabla\theta = 0$ is used. From (2.17) and the Sasakian conditions of both M and N, we have

(A4)
$$f_{0k}^\alpha = f_{k0}^\alpha, \quad f_{0\overline{k}}^\alpha = f_{\overline{k}0}^\alpha.$$

In terms of (A4), (2.35) and (2.41), we derive that

$$f_{0k\overline{k}}^\alpha + f_{0\overline{k}k}^\alpha = f_{k0\overline{k}}^\alpha + f_{\overline{k}0k}^\alpha$$

(A5)
$$= f_{k\overline{k}0}^\alpha + f_k^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha(f_{\overline{k}}^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_{\overline{k}}^{\overline{\delta}}) + f_{\overline{k}k0}^\alpha + f_{\overline{k}}^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha(f_k^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_k^{\overline{\delta}})$$

$$= f_{k\overline{k}0}^\alpha + f_{\overline{k}k0}^\alpha + f_k^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha(f_{\overline{k}}^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_{\overline{k}}^{\overline{\delta}}) + f_{\overline{k}}^\beta \widehat{R}_{\beta\gamma\overline{\delta}}^\alpha(f_k^\gamma f_0^{\overline{\delta}} - f_0^\gamma f_k^{\overline{\delta}}).$$

It is easy to see from (A2) and (A5) that there exists a constant C such that

(A6)
$$|f_{000}^\alpha + f_{0k\overline{k}}^\alpha + f_{0\overline{k}k}^\alpha| \leq C \sum_\alpha |f_0^\alpha|$$

on a fixed open subset W of M. Consequently

(A7)
$$|\triangle_M f_0^\alpha| \leq C' \sum_\alpha \left(|f_0^\alpha| + |\nabla f_0^\alpha|\right),$$

that is, $\{f_0^\alpha\}$ satisfies the structural assumptions of Aronszajn-Cordes. Since $f_0^\alpha = 0$ on $W \cap U$, then $f_0^\alpha = 0$ on W, and thus we may conclude that f is foliated on M. $\square$

Remark A2. From the proof of Theorem A1, we see that the above continuation result about the foliated property still holds if f only satisfies $tr_{g_\theta}(\pi_{\widetilde{H}}\beta) = 0$.

For a smooth map $f : (M^{2m+1}, H(M), J, \theta) \to (N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$, we introduce the following two 1-forms:

(A8)
$$\widetilde{\rho}_3 = -(f_{00}^\alpha f_0^{\overline{\alpha}} + f_{00}^{\overline{\alpha}} f_0^\alpha)\theta + \widetilde{\rho}_1$$

where $\widetilde{\rho}_1$ is given by (8.3), and

(A9)
$$\widetilde{\rho}_4 = -(f_{00}^0 f_0^0 \theta + f_{0k}^0 f_0^0 \theta^k + +f_{0\overline{k}}^0 f_0^0 \theta^{\overline{k}}).$$

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Lemma A2. *Let $f : M \to N$ be a map between two Sasakian manifolds. Then*

$$\delta\widetilde{\rho}_3 = 2|f_{00}^{\alpha}|^2 + 2|f_{0k}^{\alpha}|^2 + 2|f_{0\overline{k}}^{\alpha}|^2 + (f_{000}^{\alpha} + f_{k\overline{k}0}^{\alpha} + f_{\overline{k}k0}^{\alpha})f_0^{\overline{\alpha}}$$

(A10) $\quad + (f_{000}^{\overline{\alpha}} + f_{k\overline{k}0}^{\overline{\alpha}} + f_{\overline{k}k0}^{\overline{\alpha}})f_0^{\alpha} - \widehat{R}(df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k), \overline{df_{L,\widetilde{H}}(\xi) \wedge df_{H,\widetilde{H}}(\eta_k)})$

$$- \widehat{R}(df_{L,\widetilde{H}}(T) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}}), \overline{df_{L,\widetilde{H}}(T) \wedge df_{H,\widetilde{H}}(\eta_{\overline{k}})}).$$

If f is foliated, then

(A11) $$\delta\widetilde{\rho}_4 = |f_{00}^0|^2 + |f_{0k}^0|^2 + |f_{0\overline{k}}^0|^2 + f_0^0(f_{000}^0 + f_{k\overline{k}0}^0 + f_{\overline{k}k0}^0).$$

Proof. From (8.4) and (A8), we immediately get (A10). Now suppose f is foliated. Then (2.14) yield

(A12) $$f_{0k}^0 = f_{k0}^0, \quad f_{0\overline{k}}^0 = f_{\overline{k}0}^0.$$

Using (A12), (2.24), we deduce from (A9) that

$$\begin{aligned}
\delta\widetilde{\rho}_4 &= |f_{00}^0|^2 + |f_{0k}^0|^2 + |f_{0\overline{k}}^0|^2 + f_0^0(f_{000}^0 + f_{0k\overline{k}}^0 + f_{0\overline{k}k}^0) \\
&= |f_{00}^0|^2 + |f_{0k}^0|^2 + |f_{0\overline{k}}^0|^2 + f_0^0(f_{000}^0 + f_{k0\overline{k}}^0 + f_{\overline{k}0k}^0) \\
&= |f_{00}^0|^2 + |f_{0k}^0|^2 + |f_{0\overline{k}}^0|^2 + f_0^0(f_{000}^0 + f_{k\overline{k}0}^0 + f_{\overline{k}k0}^0).
\end{aligned}$$

$\square$

Remark A3. We only use the Sasakian condition on M to derive (A11), so it is still valid if N is any pseudo-Hermitian manifold.

Theorem A3. *Let M and N be two compact Sasakian manifolds and let $f : M \to N$ be a pseudo-Hermitian harmonic map. If $(N^{2n+1}, \widetilde{\nabla})$ has non-positive horizontal sectional curvature, then f is a foliated special $(H, \widetilde{H})$-harmonic map with $df(\xi) = \lambda\widetilde{\xi}$ for some constant λ.*

Proof. Using Lemma A2 and the divergence theorem, we get

$$0 \geq \int_M \{2|f_{00}^{\alpha}|^2 + 2|f_{0k}^{\alpha}|^2 + 2|f_{0\overline{k}}^{\alpha}|^2\}dv_\theta,$$

and thus

(A13) $$f_{00}^{\alpha} = f_{0k}^{\alpha} = f_{0\overline{k}}^{\alpha} = 0.$$

Consequently, f is a $(H, \widetilde{H})$-harmonic map with split horizontal second fundamental form. Therefore Lemma 8.1 implies that f is foliated.

Since f is both foliated and pseudo-Hermitian harmonic, we know that (A11) holds and becomes

(A14) $$\delta\widetilde{\rho}_4 = |f_{00}^0|^2 + |f_{0k}^0|^2 + |f_{0\overline{k}}^0|^2.$$

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Applying the divergence theorem to (A14), we obtain

$$(A15) \qquad f_{00}^0 = f_{0k}^0 = f_{0\overline{k}}^0 = 0.$$

From (A2) and (A15), we find that f is special in the sense of Definition 3.2, and $df(\xi) = \lambda\widetilde{\xi}$ with λ constant. $\quad\square$

B. An explicit formulation for special $(H, \widetilde{H})$-harmonic maps

Now we want to give the explicit formulations for both the special $(H, \widetilde{H})$-harmonic map equation and its parabolic version, which are convenient for proving the existence theory of special $(H, \widetilde{H})$-harmonic maps between two pseudo-Hermitian manifolds $(M^{2m+1}, H(M), J, \theta)$ and $(N^{2n+1}, \widetilde{H}(N), \widetilde{J}, \widetilde{\theta})$. As in the theory of harmonic maps (cf. [Li]), one can always assume, in view of the Nash embedding theorem, that $I : (N, g_{\widetilde{\theta}}) \hookrightarrow (R^K, g_E)$ is an isometric embedding in some Euclidean space, where g_E denotes the standard Euclidean metric. Let ∇ and $\widetilde{\nabla}$ denote the Tanaka-Webster connections of M and N respectively, and let $\nabla^{\widetilde{\theta}}$ and D denote the Levi-Civita connections of $(N, g_{\widetilde{\theta}})$ and (R^K, g_E) respectively.

For a map $f : (M, \nabla) \to (N, \widetilde{\nabla})$ between the two manifolds, the second fundamental form of f with respect to $(\nabla, \widetilde{\nabla})$ is defined by

$$(B1) \qquad \beta(f; \nabla, \widetilde{\nabla})(X, Y) = \widetilde{\nabla}_Y df(X) - df(\nabla_Y X)$$

for any vector fields X, Y on M. Applying the composition formula for second fundamental forms (see Proposition 2.20 on page 16 of [EL]) to the maps $f : (M, \nabla) \to (N, \widetilde{\nabla})$ and $I : (N, \widetilde{\nabla}) \to (R^K, D)$, we have

$$(B2) \qquad \beta(I \circ f; \nabla, D)(\cdot, \cdot) = dI\big(\beta(f; \nabla, \widetilde{\nabla})(\cdot, \cdot)\big) + \beta(I; \widetilde{\nabla}, D)\big(df(\cdot), df(\cdot)\big).$$

Define a 2-tensor field S on N by

$$(B3) \qquad S(Z_1, Z_2) = \nabla^{\widetilde{\theta}}_{Z_2} Z_1 - \widetilde{\nabla}_{Z_2} Z_1$$

where Z_1, Z_2 are any vector fields on N. Therefore

$$(B4) \qquad \beta(I; \widetilde{\nabla}, D)(\cdot, \cdot) = \beta(I; \nabla^{\widetilde{\theta}}, D)(\cdot, \cdot) + dI\big(S(\cdot, \cdot)\big).$$

Note that $\beta(I; \nabla^{\widetilde{\theta}}, D)$ is the usual second fundamental form of the submanifold $I : (N, g_{\widetilde{\theta}}) \hookrightarrow (R^K, g_E)$. For simplicity, we shall identify N with $I(N)$, and write $I \circ f$ as u, which is a map from M to R^K. Set

$$(B5) \qquad \tau_H(u; \nabla, D) = tr_{g_\theta}\big(\beta(u; \nabla, D)|_H\big).$$

Lemma B1. *Suppose $f : M \to N$ is a map between pseudo-Hermitian manifolds and N is Sasakian. Suppose $I : N \hookrightarrow R^K$ is an isometric embedding. Set $u = I \circ f$. Then f is a special $(H, \widetilde{H})$-harmonic map if and only if*

$$\tau_H(u; \nabla, D) - tr_{g_\theta}\beta(I; \nabla^{\widetilde{\theta}}, D)(df_H, df_H) - tr_{g_\theta}dI\big(S(df_H, df_H)\big) = 0$$

where df_H denotes the restriction of df to $H(M)$.

Proof. From (B2), (B4) and (B5), we get

$$(B6) \quad \tau_H(u; \nabla, D) = dI\big(\tau_H(f)\big) + tr_{g_\theta}\beta(I; \nabla^{\widetilde{\theta}}, D)(df_H, df_H) + tr_{g_\theta}dI\big(S(df_H, df_H)\big)$$

where $\tau_H(f) = tr_{g_\theta}\big(\beta(f; \nabla, \widetilde{\nabla})|_H\big)$. We know from Corollary 3.3 that if N is Sasakian, then f is a special $(H, \widetilde{H})$-harmonic map if and only if $\tau_H(f) = 0$. Consequently this proposition follows immediately from (B6). $\square$

Suppose now that N is a compact Sasakian manifold. By compactness of N, there exists a tubular neighborhood $B(N)$ of N in R^K which can be realized as a submersion $\Pi : B(N) \to N$ over N. Actually the projection map Π is simply given by mapping any point in $B(N)$ to its closest point in N. Clearly its differential $d\Pi_y : T_y R^K \to T_y R^K$ when evaluate at a point $y \in N$ is given by the identity map when restricted to the tangent space TN of N and maps all the normal vectors to N to the zero vector. Since $\Pi \circ I = I : N \hookrightarrow R^K$, we have

$$\beta(I; \nabla^{\widetilde{\theta}}, D)(\cdot, \cdot) = d\Pi(\beta(I; \nabla^{\widetilde{\theta}}, D)(\cdot, \cdot)) + \beta(\Pi; D, D)(dI, dI)$$

and thus

$$(B7) \qquad\qquad \beta(I; \nabla^{\widetilde{\theta}}, D) = \beta(\Pi; D, D)(dI, dI).$$

The tensor field S may be extended to a tensor field $\widehat{S} = \Pi^*(dI \circ S)$ on $B(N)$, that is,

$$(B8) \qquad\qquad \widehat{S}(W_1, W_2) = dI\big(S(d\Pi(W_1), d\Pi(W_2))\big)$$

for any $W_1, W_2 \in TB(N)$. Let $\{y^a\}_{1 \le a \le K}$ be the natural Euclidean coordinate system of R^K. Set $u^a = y^a \circ u$, $\Pi^a = y^a \circ \Pi$, and write $\widehat{S}(\cdot, \cdot) = \widehat{S}^a(\cdot, \cdot)\frac{\partial}{\partial y^a}$. Choose an orthonormal frame field $\{e_A\}_{A=0,1,\dots,2m}$ around any point of M such that $span\{e_A\}_{1 \le A \le 2m} = H(M)$. By definition of the second fundamental form , we have

$$(B9) \qquad\qquad \tau_H(u; \nabla, D) = \triangle_H u^a \frac{\partial}{\partial y^a},$$

and

$$
\begin{aligned}
(B10) \qquad tr_{g_\theta}\beta(I; \nabla^{\widetilde{\theta}}, D)(df_H, df_H) &= tr_{g_\theta}\beta(\Pi; D, D)(du_H, du_H)\\
&= \sum_{A=1}^{2m} \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c}e_A(u^b)e_A(u^c)\frac{\partial}{\partial y^a}\\
&= \Pi^a_{bc}\langle \nabla_H u^b, \nabla_H u^c\rangle\frac{\partial}{\partial y^a}
\end{aligned}
$$

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where $\Pi^a_{bc} = \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c}$. In terms of (B8), we derive that

$$
tr_{g_\theta} dI\big(S(df_H, df_H)\big) = tr_{g_\theta} \widehat{S}^a(du_H, du_H) \frac{\partial}{\partial y^a}
$$

(B11)
$$
= \sum_{A=1}^{2m} \widehat{S}^a(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^c}) e_A(u^b) e_A(u^c) \frac{\partial}{\partial y^a}
$$

$$
= \widehat{S}^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle \frac{\partial}{\partial y^a}
$$

where $\widehat{S}^a_{bc} = \widehat{S}^a(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^c})$. It follows from (B6), (B9), (B10) and (B11) that

(B12) $\qquad dI(\tau_H(f)) = (\triangle_H u^a - \Pi^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle - \widehat{S}^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle) \frac{\partial}{\partial y^a}.$

In view of (B12), we obtain

Proposition B2. *Let M, N, f and u be as in Lemma B1. Then f is a special $(H, \widetilde{H})$-harmonic map if and only if*

$$
\triangle_H u^a - \Pi^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle - \widehat{S}^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle = 0
$$

where $\Pi^a_{bc} = \frac{\partial^2 \Pi^a}{\partial y^b \partial y^c}$ and $\widehat{S}^a_{bc} = \widehat{S}^a(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^c})$ $(1 \le a, b, c \le K)$ are smooth functions defined on $B(N) \subset R^K$.

In section 7, we study the existence problem of the special $(H, \widetilde{H})$-harmonic map equation $\tau_H(f) = 0$ by solving the corresponding subelliptic heat flow (7.5), that is,

$$
\begin{cases} \frac{\partial f_t}{\partial t} & = \tau_H(f_t) \\ f|_{t=0} & = h \end{cases}
$$

for some map $h : M \to N$. Inspired by the above explicit formulation, we will establish the fact that in order to solve (7.5), it suffices to solve the following system

(B13)
$$
\begin{cases} \frac{\partial u^a}{\partial t} & = \triangle_H u^a - \Pi^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle - \widehat{S}^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle, \\ u^a|_{t=0} & = h^a, \quad (1 \le a, b, c \le K), \end{cases}
$$

where $h^a = y^a \circ h$. Let us define a map $\rho : B(N) \to R^K$ by

$$
\rho(y) = y - \Pi(y), \quad y \in B(N).
$$

Obviously, $\rho(y)$ is normal to N and $\rho(y) = 0$ if and only if $y \in N$.

Lemma B3. Let $u(x,t) = (u^a(x,t))$ $((x,t) \in M \times [0, T_0))$ be a solution of (B13) with initial condition $h = (h^a) : M \to R^K$. Then the quantity

$$\int_M |\rho(u(x,t)|^2 dv_\theta$$

is a nonincreasing function of t. In particular, if $h(M) \subset N$, then $u(x,t) \in N$ for all $(x,t) \in M \times [0, T_0)$.

Proof. Since $\rho(y) = y - \Pi(y)$, we have

(B14) $$\rho_b^a = \delta_b^a - \Pi_b^a$$

and

(B15) $$\rho_{bc}^a = -\Pi_{bc}^a$$

where $\rho_b^a = \frac{\partial \rho^a}{\partial y^b}$ and $\rho_{bc}^a = \frac{\partial^2 \rho^a}{\partial y^b \partial y^c}$. By applying the composition law ([EL]) to the maps $u_t : (M, \nabla) \to (B(N), D)$ and $\rho : (B(N), D) \to (R^K, D)$, we have

(B16) $$\triangle_H \rho(u) = d\rho(\triangle_H u) + tr_{g_\theta} \beta(\rho; D, D)(du_H, du_H).$$

Using (B16), (B14), (B15) and (B13), we derive that

$$
\begin{aligned}
(\triangle_H \rho(u))^a &= \rho_b^a \triangle_H u^b + \rho_{bc}^a \langle \nabla_H u^b, \nabla_H u^c \rangle \\
&= \triangle_H u^a - \Pi_b^a \triangle_H u^b - \Pi_{bc}^a \langle \nabla_H u^b, \nabla_H u^c \rangle \\
\text{(B17)} \qquad &= \frac{\partial u^a}{\partial t} + \widehat{S}_{bc}^a \langle \nabla_H u^b, \nabla_H u^c \rangle - \Pi_b^a \triangle_H u^b \\
&= \rho_b^a \frac{\partial u^b}{\partial t} + \Pi_b^a (\frac{\partial u^b}{\partial t} - \triangle_H u^b) + \widehat{S}_{bc}^a \langle \nabla_H u^b, \nabla_H u^c \rangle
\end{aligned}
$$

Since $d\Pi(\frac{\partial u}{\partial t} - \triangle_H u)$ and $\widehat{S}(du_H, du_H)$ are tangent to N and $\rho(u)$ is normal to N, we find from (B17) that

(B18) $$\rho^a(u) (\triangle_H \rho(u))^a = \rho^a(u)\rho_b^a(u)\frac{\partial u^b}{\partial t}.$$

Using (B18), we deduce that

$$
\begin{aligned}
\frac{\partial}{\partial t} \int_M (\rho^a(u))^2 dv_\theta &= 2 \int_M \rho^a(u)\rho_b^a(u)\frac{\partial u^b}{\partial t} dv_\theta \\
&= 2 \int_M \rho^a(u) (\triangle_H \rho(u))^a dv_\theta \\
&= -2 \int_M |\nabla_H \rho(u)|^2 dv_\theta \\
&\leq 0
\end{aligned}
$$

which implies that $\int_M |\rho(u)|^2 dv_\theta$ is decreasing in t. $\square$

In terms of (B12) and Lemma B3, we conclude that

Theorem B4. *Let $h : M \to N \subset R^K$ be a smooth map given by $h = (h^1, ..., h^K)$ in the Euclidean coordinates. If $u : M \times [0, T_0) \to N \subset R^K$ is a solution of the following system*

$$\frac{\partial u^a}{\partial t} = \triangle_H u^a - \Pi^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle - \widehat{S}^a_{bc} \langle \nabla_H u^b, \nabla_H u^c \rangle, \quad 1 \le a \le K,$$

with initial condition $(u^a(x,0)) = (h^a(x))$ for all $x \in M$, then u solves the subelliptic heat flow

$$\frac{\partial u}{\partial t} = \tau_H(u)$$

with initial condition $u(x, 0) = h(x)$.

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