Affine geometry of equal-volume polygons in 3-space

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Abstract. Equal-volume polygons are obtained from adequate discretizations of curves in 3-space, contained or not in surfaces. In this paper we explore the similarities of these polygons with the affine arc-length parameterized smooth curves to develop a theory of discrete affine invariants. Besides obtaining discrete affine invariants, equal-volume polygons can also be used to estimate projective invariants of a planar curve. This theory has many potential applications, among them evaluation of the quality and computation of affine invariants of silhouette curves.

1. Introduction

We say that a smooth curve $\gamma(t)$ in 2-space is parameterized by *affine arc-length* if $[\gamma'(t), \gamma''(t)] = 1$, where $[\cdot, \cdot]$ denotes the determinant of 2 vectors. For polygons, the corresponding condition is that the area of the triangle determined by three consecutive vertices is constant. Planar polygons satisfying this last condition are called *equal-area* and the affine geometry of these polygons has been recently studied ([2],[11]). In this paper we generalize this study to polygons in 3-space by considering the concept of equal-volume polygons. Since we obtain discrete counterparts of known objects of the smooth theory, our results clearly belong to the field of Discrete Differential Geometry.

For a smooth curve ϕ contained in a surface M, we say that the parameterization $\phi(t)$ is *adapted to M* if

$$[\phi'(t), \phi''(t), \xi(t)] = 1, \tag{1.1}$$

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where $[\cdot, \cdot, \cdot]$ denotes the determinant of 3 vectors and ξ is the parallel Darboux vector field of $\phi \subset M$ ([3]). In centro-affine geometry, we consider curves ϕ in 3-space together with a distinguished origin O, and we say that $\phi(t)$ is parameterized by centro-affine arc-length with respect to O if

$$[\phi(t) - O, \phi'(t), \phi''(t)] = 1, \qquad (1.2)$$

([6]). A smooth curve $\Phi(t)$ in 3-space is parameterized by *affine arc-length* if

$$[\Phi'(t), \Phi''(t), \Phi'''(t)] = 1, \qquad (1.3)$$

([5],[9]). We observe that, in all these contexts, the basic condition is the constancy of some volume. In this paper, we describe polygons in 3-space whose corresponding volumes are constant, and we call them *equal-volume* polygons. We obtain affine invariant measures only for these equal-volume polygons, but we also describe a simple algorithm that, by re-sampling an arbitrary polygon, obtain an equal-volume one.

We say that a smooth curve ϕ contained in a surface M is *non-degenerate* if its osculating plane does not coincide with the tangent plane of M at any point. For such curves, there exists a vector field ξ tangent to M and transversal to ϕ such that its derivative ξ' is tangent to M. The direction defined by ξ is unique and is called the *Darboux direction* of $\phi \subset M$. Moreover, there exists a vector field ξ in the Darboux direction such that ξ' is tangent to the curve ϕ, i.e.,

$$\xi'(t) = -\sigma(t)\phi'(t), \qquad (1.4)$$

for some scalar function σ. This vector field is unique up to a multiplicative constant and is called the *parallel Darboux vector field*. It turns out that $\phi \subset M$ is a silhouette curve with respect to some point O if and only if σ is constant ([3]).

As a discrete model for curves contained in surfaces, consider a polyhedron M whose faces are planar quadrilaterals and let $\phi(i)$ be vertices of a polygon ϕ whose sides are connecting opposite edges of a face of M. The edges of M containing vertices $\phi(i)$ correspond to Darboux directions and we can choose a vector field $\xi(i)$ in this direction such that the difference $\xi(i+1) - \xi(i) = \xi'(i+\frac{1}{2})$ is parallel to the corresponding side of the polygon ϕ. This vector field is unique up to a multiplicative constant, and is called the *parallel Darboux vector field*. We can write a discrete counterpart of equation (1.4), namely

$$\xi'(i + \tfrac{1}{2}) = -\sigma(i + \tfrac{1}{2})\phi'(i + \tfrac{1}{2}), \qquad (1.5)$$

for some scalar function σ, where we are replacing derivatives by differences. We prove that ϕ is a silhouette polygon for the polyhedron M if and only if σ is constant. This result may be used as a measure of quality of a silhouette polygon.

A non-degenerate curve $\phi \subset M$ admits a parameterization satisfying equation (1.1), unique up to a translation. The plane $\mathcal{A} = \mathcal{A}(t)$ generated by $\{\phi(t), \phi''(t)\}$ is called the *affine normal plane*, while the envelope $\mathcal{B}$ of these affine normal planes is a developable surface $\mathcal{B}$ called the *affine focal set* of

the pair $\phi \subset M$ ([3],[4]). For silhouette curves relative to O, equation (1.1) reduces to equation (1.2).

We say that the polygon ϕ contained in the polyhedron M is *equal-volume* if

$$\left[\phi'(i-\tfrac{1}{2}), \phi'(i+\tfrac{1}{2}), \xi(i)\right] = 1, \tag{1.6}$$

for all i. Note that equation (1.6) is a discrete counterpart of equation (1.1). For such polygons, define the *affine normal plane* $\mathcal{A}(i)$ as the plane generated by $\{\phi(i), \phi''(i)\}$ and the *affine focal set* $\mathcal{B} = \mathcal{B}(\phi, M)$ as a discrete envelope of these affine normal planes. For silhouette polygons ϕ relative to O, equation (1.6) reduces to

$$[\phi(i-1) - O, \phi(i) - O, \phi(i+1) - O] = 1, \tag{1.7}$$

which is a discrete counterpart of equation (1.2).

The smooth curves $\phi \subset M$ whose affine focal set $\mathcal{B}$ reduces to a single line were characterized in [4]. Consider a smooth planar curve $\Gamma(t)$ parameterized by affine arc-length and denote by $z(t)$ the *affine distance* or *support function* of $\Gamma(t)$ with respect to some point $P \in \mathbb{R}^2$ ([1]). Then the affine focal set of the silhouette curve $\phi(t) = (\Gamma'(t), z(t))$ reduces to a single line and conversely, if $\mathcal{B}(\phi, M)$ is a single line, then ϕ is a silhouette curve obtained by this construction for some planar curve Γ and some $P \in \mathbb{R}^2$. We prove in this paper a discrete counterpart of this characterization for equal-volume polygons contained in a polyhedron.

Consider a smooth curve $\Phi(t)$ in 3-space parameterized by affine arc-length, i.e., satisfying equation (1.3). The planes through Φ parallel to $\{\Phi', \Phi'''\}$ are called *affine rectifying planes* and the envelope of the affine rectifying planes $RS(\Phi)$ is called the *intrinsic affine binormal developable* ([9]). The characterization of curves Φ such that $RS(\Phi)$ is cylindrical is easily obtained from the characterization of curves $\phi = \Phi'$ whose affine focal set is a single line ([4],[9]). A polygon $\Phi(i+\tfrac{1}{2})$ in 3-space is said to be *equal-volume* if

$$[\Phi'(i-1), \Phi'(i), \Phi'(i+1)] = 1, \tag{1.8}$$

for all i, which is equivalent to say that the difference polygon $\phi(i) = \Phi'(i)$ is equal-volume with respect to the origin. Although it is not clear how to obtain a discrete version of the intrinsic affine binormal developable, we can obtain interesting consequences of the discrete characterization of polygons ϕ whose affine focal set is a single line.

We can also apply the equal-volume model in a projective setting. Given a smooth planar curve $(\tilde{\phi}(t), 1)$, there exists a projectively equivalent curve $\phi(t)$ in 3-space satisfying equation (1.2) with O equal the origin. From this curve, we can define the projective length $pl(\tilde{\phi})$ (see [7]). For a planar polygon $(\tilde{\phi}(i), 1)$, we can also obtain a projectively equivalent equal-volume polygon $\phi(i)$ in 3-space and, from this polygon, we obtain two definitions for the projective length, $pl_1(\tilde{\phi})$ and $pl_2(\tilde{\phi})$, that unfortunately do not coincide. Nevertheless, we prove that if the polygon is obtained from a dense enough

sampling of a smooth curve, both the discrete projective length $pl_1(\tilde{\phi})$ and $pl_2(\tilde{\phi})$ are close to the projective length of the smooth curve.

The paper is organized as follows: In section 2 we review the smooth results for affine geometry of curves contained in surfaces, affine geometry of curves in 3-space and projective geometry of planar curves. In section 3 we calculate affine invariants of equal-volume polygons contained in polyhedra. In section 4, we apply the results of section 3 to compute affine invariants for equal-volume polygons in 3-space. In section 5 we discuss the projective length of a planar polygon.

2. Affine geometry of smooth curves in 3-space

2.1. Curves contained in surfaces

Let $\phi : I \to \mathbb{R}^3$ be a curve contained in a surface M and ξ a vector field tangent to M and transversal to ϕ. We shall assume that $\phi \subset M$ is non-degenerate, i.e., the osculating plane of ϕ does not coincide with the tangent plane of M at any point. Under this hypothesis, there exists a vector field $\xi(t)$, unique up to scalar (non-constant) multiple, such that $\xi'(t)$ is tangent to M, for any $t \in I$. The vector field ξ determines a unique direction tangent to M, which is called the *Darboux direction* along ϕ. In the Darboux direction, there exists a vector field $\xi(t)$, unique up to a constant multiple, such that $\xi'(t)$ is tangent to $\phi(t)$, for any $t \in I$. We call this vector field the *parallel Darboux vector field*. The parallel Darboux vector field satisfies equation (1.4), for some scalar function σ.

The envelope of tangent planes is the developable surface

$$x(t, u) = \phi(t) + u\xi(t).$$

This surface is called the *Osculating Tangent Developable Surface of M along ϕ* and will be denoted $\mathcal{E}$ ([3],[10]). The surface $\mathcal{E}$ is a cone if and only if σ is constant. In this case, the vertex of the cone is given by $O = \phi + \sigma^{-1}\xi$ and the curve ϕ is a silhouette curve from the point of view of O.

Under the non-degeneracy hypothesis, there exists a parameterization $\phi(t)$ of ϕ, unique up to a translation, such that equation (1.1) holds. The plane $\mathcal{A}(t)$ generated by $\{\xi(t), \phi''(t)\}$ is called the *affine normal plane* of $\phi \subset M$. Condition (1.1) is equivalent to $\phi'''(t)$ tangent to M. Thus we can write

$$\phi'''(t) = -\rho(t)\phi'(t) + \tau(t)\xi(t), \qquad (2.1)$$

for some scalar functions ρ and τ.

There exists a basis $\{\xi(t), \eta(t)\}$ of the affine normal plane $\mathcal{A}(t)$ with η parallel, i.e., η' tangent to ϕ. In fact, define the vector field η by $\eta = \phi'' + \lambda\xi$, where $\lambda' = -\tau$. Taking $\mu = \rho + \lambda\sigma$, we obtain the equation

$$\eta'(t) = -\mu(t)\phi'(t), \qquad (2.2)$$

which in particular says that η is parallel. The *affine focal set* $\mathcal{B}$, or *affine evolute*, is the envelope of affine normal planes. It is the developable surface generated by the lines passing through $O = \phi + \sigma^{-1}\xi$ and $Q = \phi + \mu^{-1}\eta$. The affine focal set reduces to a single line if and only if σ and μ are constant ([4]).

If ϕ is contained in a plane L, then $\tau(t) = 0$ for any $t \in I$ and conversely, if $\tau(t) = 0$ for any $t \in I$, then ϕ is planar. Denote by n a euclidean unitary normal to L and let ξ be the vector field in the Darboux direction such that $\xi \cdot n = 1$, where $\cdot$ denotes the usual inner product. Then ξ is a parallel Darboux vector field. In this case, the adapted parameter t corresponds to the affine arc-length parameter and $\rho(t)$ is the affine curvature of $\phi \subset L$. For planar curves, $\lambda = 0$ and so $\phi'' = \eta \subset L$ is parallel. Then the set $\mathcal{B} \cap L$ coincides with the affine evolute of the planar curve $\phi \subset L$ ([8]).

For silhouette curves, $\xi(t) = \phi(t) - O$ and so equation (1.1) becomes (1.2), i.e., $\phi(t)$ is parameterized by *centro-affine arc-length*. Assuming O equals the origin, equation (2.1) becomes

$$\phi'''(t) = -\rho(t)\phi'(t) + \tau(t)\phi(t). \tag{2.3}$$

Moreover $\mu = \rho - \lambda$, which implies $\mu' = \rho' + \tau$.

Curves whose affine focal set $\mathcal{B}$ reduces to a single line. The affine focal set reduces to a single line if and only if μ and σ are constant. Since σ is constant, ϕ is necessarily a silhouette curve from the point of view of O, that we shall assume to be the origin. The condition μ constant can be written as $\rho' + \tau = 0$. In this case equation (2.3) becomes $\phi''' = -(\rho\phi)'$, which is equivalent to

$$\phi''(t) = -\rho(t)\phi(t) + Q,$$

for some constant vector Q. Assuming that $Q = (0, 0, 1)$ and writing $\phi(t) = (\gamma(t), z(t))$, this equation becomes

$$\gamma''(t) = -\rho(t)\gamma(t); \quad z''(t) = -\rho(t)z(t) + 1. \tag{2.4}$$

Consider a convex planar curve $\Gamma(t)$. Assume that $\Gamma(t)$ is parameterized by affine arc-length, i.e., $[\Gamma'(t), \Gamma''(t)] = 1$, and let $\rho(t)$ denotes the affine curvature of Γ, i.e., $\Gamma'''(t) = -\rho(t)\Gamma'(t)$, $t \in I$. Let $\gamma(t) = \Gamma'(t)$ and denote by $z(t) = [\Gamma(t) - P, \gamma(t)]$ the *affine distance*, or *support function*, of Γ with respect to a point $P \in \mathbb{R}^2$ ([1]).

The following proposition was proved in [4]:

Proposition 2.1. *The affine focal set $\mathcal{B}$ of the curve $\phi(t) = (\gamma(t), z(t))$ is a single line, and conversely, any curve ϕ whose affine focal set is a single line can be obtained as above, for some planar curve Γ and some point $P \in \mathbb{R}^2$.*

2.2. Curves in 3-space

Consider now a curve Φ in the 3-space, without being contained in a given surface M. We say that a parameterization $\Phi(t)$ of Φ is by *affine arc-length* if

formula (1.3) holds. This condition implies that $\Phi''''(t)$ belongs to the plane generates by $\phi''(t)$ and $\phi'(t)$ and thus we obtain equation

$$\Phi''''(t) = -\rho(t)\Phi''(t) + \tau(t)\Phi'(t), \qquad (2.5)$$

for some scalar functions ρ and τ. Writing $\phi(t) = \Phi'(t)$, we observe that equation (2.5) reduces to (2.3).

The plane passing through $\Phi(t)$ and generated by $\{\Phi'(t), \Phi'''(t)\}$ is called *affine rectifying plane* and the envelope $RS(\Phi)$ of the affine rectifying planes is called the *intrinsic affine binormal developable* of Φ. It is proved in [9] that $RS(\Phi)$ is cylindrical if and only if $\rho' + \tau = 0$.

Curves with μ constant. The condition μ constant is equivalent to $\rho' + \tau = 0$. Consider a convex planar curve Γ parameterized by affine arc-length and let

$$Z(t) = \int_{t_0}^{t} [\Gamma(s) - P, \Gamma'(s)]\, ds.$$

Then $Z(t)$ represents the area of the planar region bounded by $\Gamma(s)$, $t_0 \leq s \leq t$, and the segments $P\Gamma(t_0)$ and $P\Gamma(t)$. The following proposition is a direct consequence of proposition 2.1:

Proposition 2.2. *For the curve $\Phi(t) = (\Gamma(t), Z(t))$, μ is constant, and conversely, any curve Φ in 3-space with μ constant is obtained by this construction, for some convex planar curve Γ and some point $P \in \mathbb{R}^2$.*

2.3. Projective invariants

Consider a parameterized planar curve $\tilde{\phi}(t)$, $t \in I$, without inflection points. Any curve of the form $\phi(t) = a(t)\tilde{\phi}(t)$ is projectively equivalent to $\tilde{\phi}(t)$ and is called a *representative* of $\tilde{\phi}(t)$. It turns out that there exists a representative $\phi(t)$ of $\tilde{\phi}(t)$ satisfying formula (1.2) with O equal to the origin. Then $\phi'''(t)$ belongs to the space generated by $\{\phi'(t), \phi(t)\}$ and equation (2.3) holds, for some scalar functions ρ and τ.

The quantity $\rho'(t) + 2\tau(t)$ is projectively invariant and $\rho'(t) + 2\tau(t) = 0$ if and only if ϕ is contained in a quadratic cone. In fact,

$$pl(\tilde{\phi}) = \int_{I} (\rho'(t) + 2\tau(t))^{1/3} dt \qquad (2.6)$$

is the projective length of $\tilde{\phi}$ (see [7]).

3. Affine geometry of equal-volume polygons contained in polyhedra

In this section, we obtain discrete counterparts of the results of section 2.1. The derivatives are replaced by differences, and so for a function $f : \{1, ..., N\} \rightarrow \mathbb{R}^k$, we denote

$$f'(i + \tfrac{1}{2}) = f(i + 1) - f(i), \quad f''(i) = f'(i + \tfrac{1}{2}) - f'(i - \tfrac{1}{2}),$$

and so on.

3.1. Basic model

Consider a polyhedron M whose faces are planar quadrilaterals and let ϕ be a polygonal line such that each of its sides are connecting opposite edges of a face of M. We shall denote by $\phi(i)$, $1 \leq i \leq N$, the vertices of such polygon and by $\xi(i)$ a vector in the direction of the edges of M containing $\phi(i)$. Edges of M that don't intersect ϕ are not important in our model (see Figure 1).
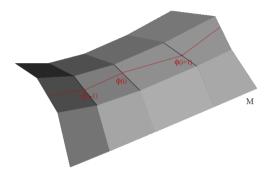

FIGURE 1. Polygonal line ϕ contained in polyhedron M.

We shall denote by $T_{i+1/2}M$ the face of M that contains the side $i+1/2$. By the planar quadrilaterals hypothesis, the vectors $\phi'(i + \frac{1}{2})$, $\xi(i)$ and $\xi(i+1)$ belong to $T_{i+1/2}M$, which is a discrete counterpart of the Darboux condition ξ' tangent to M. It is clear that there exists ξ, unique up to a multiplicative constant, such that $\xi'(i + 1/2)$ is parallel to $\phi'(i + 1/2)$. This vector field is the *parallel Darboux* vector field of $\phi \subset M$ and equation (1.5) holds, for some scalar function σ (see Figure 2).
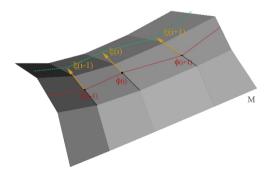

FIGURE 2. Parallel Darboux vectors field ξ. The segments connecting the endpoints of the vectors $\xi(i)$ are parallel to the sides of ϕ.

3.2. Osculating developable polyhedron

The line $x(i, u) = \phi(i) + u\xi(i)$, $u \in \mathbb{R}$, is the support line of the edge of the polyhedron M that contains ϕ. Thus $x(i, u)$ and $x(i + 1, u)$ are co-planar and denote by $O(i + \frac{1}{2})$ the intersection point of these lines. We have that

$$O(i + \tfrac{1}{2}) = \phi(i) + \sigma^{-1}(i + \tfrac{1}{2})\xi(i) = \phi(i + 1) + \sigma^{-1}(i + \tfrac{1}{2})\xi(i + 1). \quad (3.1)$$

The *osculating developable polyhedron* $\mathcal{E}$ is the polyhedron whose face $i + 1/2$ is the region of $T_{i+1/2}M$ bounded by $x(i, u)$ and $x(i + 1, u)$ and containing the side $i + 1/2$ of ϕ (see Figure 3).
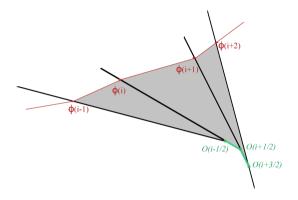

FIGURE 3. Osculating developable polyhedron $\mathcal{E}$

We have the following proposition:

Proposition 3.1. *The following statements are equivalent:*

1. *$\sigma(i + \frac{1}{2})$ does not depend on i.*
2. *The point $O(i + \frac{1}{2})$ does not depend on i.*
3. *$\mathcal{E}$ is a cone.*

Proof. Observe that

$$O(i + \tfrac{1}{2}) - O(i - \tfrac{1}{2}) = \left(\sigma^{-1}(i + \tfrac{1}{2}) - \sigma^{-1}(i + \tfrac{1}{2})\right)\xi(i) \quad (3.2)$$

Thus (1) is equivalent to (2). The equivalence between (2) and (3) is obvious.
$\square$

The polygons $\phi \subset M$ for which $\mathcal{E}$ reduces to a point O is the object of study of the centro-affine geometry. In this case, the polygon ϕ can be thought as a silhouette polygon of M from the point of view of O.

3.3. Equal-volume polygons

We say that the polygon ϕ contained in the polyhedron M is *equal-volume* if equation (1.6) holds.

Lemma 3.2. *The polygon $\phi \subset M$ is equal-volume if and only if*

$$\phi'''(i + \tfrac{1}{2}) \in T_{i+1/2}M.$$

Proof. Equation (1.6) is equivalent to

$$\left[\phi'(i + \tfrac{1}{2}), \phi'(i + \tfrac{3}{2}), \xi(i+1) \right] - \left[\phi'(i - \tfrac{1}{2}), \phi'(i + \tfrac{1}{2}), \xi(i) \right] = 0,$$

for each i, which is equivalent to

$$\left[\phi'(i + \tfrac{1}{2}), \phi'(i + \tfrac{3}{2}), \xi(i+1) - \xi(i) \right] - \left[\phi'(i + \tfrac{1}{2}), \phi''(i) - \phi''(i+1), \xi(i) \right] = 0.$$

By the parallel Darboux condition, the first parcel is zero and thus the above condition is equivalent to

$$\left[\phi'(i + \tfrac{1}{2}), \phi''(i) - \phi''(i+1), \xi(i) \right] = 0,$$

which is clearly equivalent to $\phi'''(i + 1/2)$ belongs to $T_{i+1/2}M$. $\qquad\square$

We shall assume along the paper that the polygon $\phi \subset M$ is equal-volume. By the above lemma we can write

$$\begin{cases} \phi'''(i + \tfrac{1}{2}) = -\rho_2(i)\phi'(i + \tfrac{1}{2}) + \tau(i + \tfrac{1}{2})\xi(i+1) \\ \phi'''(i + \tfrac{1}{2}) = -\rho_1(i+1)\phi'(i + \tfrac{1}{2}) + \tau(i + \tfrac{1}{2})\xi(i) \end{cases} \qquad (3.3)$$

for some scalar functions ρ_1, ρ_2 and τ satisfying the compatibility equation

$$-\tau(i + \tfrac{1}{2})\sigma(i + \tfrac{1}{2}) = \rho_2(i) - \rho_1(i+1). \qquad (3.4)$$

Equations (3.3) are discrete counterparts of equation (2.1).

Remark 3.3. Starting from a general polygon $\phi \subset M$, we may obtain an equal-volume polygon $\bar{\phi} \subset \bar{M}$ by the following inductive algorithm (see Figure 4):

1. Let $(\bar{\phi}(i), \bar{\xi}(i)) = (\phi(i), \xi(i))$, for $i = 1, 2, 3$.
2. Given the pair $(\bar{\phi}, \bar{\xi})$ at $i-1$, i and $i+1$, consider a plane parallel to $T_{i+1/2}\bar{M}$ through $\bar{\phi}(i-1)$ and let $\bar{\phi}(i+2)$ be the intersection of this plane with the polygonal line ϕ.
3. The direction of the vector $\bar{\xi}(i+2)$ is obtained by linear interpolation of $\xi(k)$ and $\xi(k+1)$, where $k + \tfrac{1}{2}$ is the index of the side of ϕ containing $\bar{\phi}(i+2)$. Thus $\bar{\xi}(i+2) \in T_{k+1/2}M$.
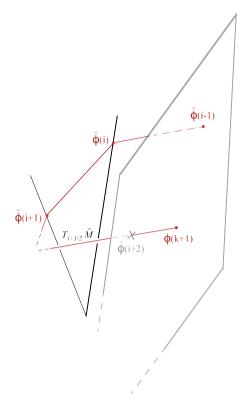
FIGURE 4. Algorithm to construct an equal-volume polygon

3.4. Discrete affine focal set

Take any λ satisfying $\lambda(i) - \lambda(i+1) = \tau(i + \frac{1}{2})$ and define

$$\eta(i) = \phi''(i) + \lambda(i)\xi(i). \tag{3.5}$$

Define also

$$\mu(i + \tfrac{1}{2}) = \rho_1(i+1) + \sigma(i + \tfrac{1}{2})\lambda(i+1) = \rho_2(i) + \sigma(i + \tfrac{1}{2})\lambda(i). \tag{3.6}$$

Lemma 3.4. *The following discrete counterpart of equation* (2.2) *holds:*

$$\eta'(i + \tfrac{1}{2}) = -\mu(i + \tfrac{1}{2})\phi'(i + \tfrac{1}{2}). \tag{3.7}$$

In particular, η is parallel.

Proof. We have that

$$\eta'(i + \tfrac{1}{2}) = \phi'''(i + \tfrac{1}{2}) + \lambda'(i + \tfrac{1}{2})\xi(i+1) + \lambda(i)\xi'(i + \tfrac{1}{2})$$

$$= -\rho_2(i)\phi'(i + \tfrac{1}{2}) + \lambda(i)\xi'(i + \tfrac{1}{2}) = -\left(\rho_2(i) + \sigma(i + \tfrac{1}{2})\lambda(i)\right)\phi'(i + \tfrac{1}{2}),$$

thus proving the lemma. □

Define

$$Q(i + \tfrac{1}{2}) = \phi(i) + \mu^{-1}(i + \tfrac{1}{2})\eta(i) = \phi(i+1) + \mu^{-1}(i + \tfrac{1}{2})\eta(i+1), \quad (3.8)$$

and denote by $l(i + \tfrac{1}{2})$ the line connecting $O(i + \tfrac{1}{2})$ and $Q(i + \tfrac{1}{2})$, where $O(i + \tfrac{1}{2})$ is defined by equation (3.1).

The *discrete affine focal set* $\mathcal{B}$ is the polyhedron with edges $l(i + \tfrac{1}{2})$, $i = 1, ..., N - 1$, and faces contained in $\mathcal{A}(i)$, $i = 2, ..., N - 1$, bounded by $l(i - \tfrac{1}{2})$ and $l(i + \tfrac{1}{2})$ containing the segments $Q(i - \tfrac{1}{2})Q(i + \tfrac{1}{2})$ and $O(i - \tfrac{1}{2})O(i + \tfrac{1}{2})$ (see Figure 5).
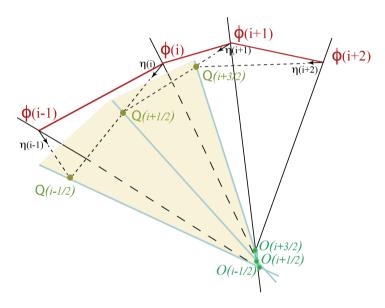

FIGURE 5. Discrete affine focal set

Proposition 3.5. *The following statements are equivalent:*

1. *σ and μ are constant.*
2. *The points O and Q are fixed.*
3. *The discrete affine focal set $\mathcal{B}$ reduces to a single line.*

Proof. By proposition 3.1, σ constant is equivalent to O fixed. From equation (3.8) we obtain

$$Q(i + \tfrac{1}{2}) - Q(i - \tfrac{1}{2}) = \left(\mu^{-1}(i + \tfrac{1}{2}) - \mu^{-1}(i - \tfrac{1}{2})\right)\eta(i), \quad (3.9)$$

which implies that μ is constant if and only if Q is fixed. Thus (1) and (2) are equivalent. It is obvious that (2) implies (3) and so it remains to prove that (3) implies (2). If O and Q were not both fixed, then equations (3.2) and (3.9) say that O or Q are not changing in the direction of $Q - O$. Thus $\mathcal{B}$ would not be a single line. $\qquad\square$

3.5. Planar polygons

Lemma 3.6. *A polygon ϕ is contained in a plane L if and only if $\tau = 0$.*

Proof. Observe that $\tau(i + \frac{1}{2}) = 0$ if and only if the points $\phi(i - 1)$, $\phi(i)$, $\phi(i + 1)$ and $\phi(i + 2)$ are co-planar. $\square$

Denote by n a euclidean unitary normal to L and let ξ be the vector field in the Darboux direction such that $\xi \cdot n = 1$, where $\cdot$ denotes the usual inner product. Then ξ is a parallel Darboux vector field. In this case equation (1.6) can be written as

$$\left[\phi'(i - \tfrac{1}{2}), \phi'(i + \tfrac{1}{2})\right] = 1,$$

where $[\cdot, \cdot]$ denotes determinant in the plane L. Thus $\phi \subset L$ is an equal-area polygon and ρ is its discrete affine curvature ([2],[11]). The set $\mathcal{B} \cap L$ is exactly the discrete affine evolute of the planar equal-area polygon $\phi \subset L$ ([2]) (see Figure 6).
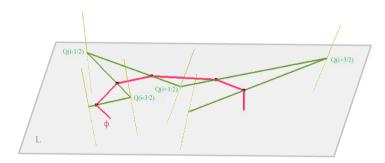

FIGURE 6. For a planar curve ϕ, the set $\mathcal{B} \cap L$ coincides with the discrete affine evolute of ϕ.

3.6. Silhouette polygons

Assume that $\phi \subset \mathbb{R}^3$ is a silhouette polygon from the point of view of O, that we assume to be the origin. In this case, equation (1.6) becomes equation (1.7). By Lemma 3.2, this condition is equivalent to $\phi'''(i + \frac{1}{2})$ belongs to the plane generated by $\phi(i)$ and $\phi(i + 1)$.

Since $\xi(i) = \phi(i)$, we have that $\sigma(i + \frac{1}{2}) = -1$, $i = 1, ..., N$. The Frenet equations (3.3) reduce to

$$\begin{cases} \phi'''(i + \tfrac{1}{2}) = -\rho_2(i)\phi'(i + \tfrac{1}{2}) + \tau(i + \tfrac{1}{2})\phi(i + 1) \\ \phi'''(i + \tfrac{1}{2}) = -\rho_1(i + 1)\phi'(i + \tfrac{1}{2}) + \tau(i + \tfrac{1}{2})\phi(i), \end{cases} \tag{3.10}$$

while equation (3.6) becomes

$$\mu(i + \tfrac{1}{2}) = \rho_1(i + 1) - \lambda(i + 1) = \rho_2(i) - \lambda(i). \tag{3.11}$$

We have also that
$$\mu'(i) = \rho_1'(i + \tfrac{1}{2}) + \tau(i + \tfrac{1}{2}) = \rho_2'(i - \tfrac{1}{2}) + \tau(i - \tfrac{1}{2}). \qquad (3.12)$$

3.7. Polygons whose discrete affine focal set reduces to a line

By proposition 3.5, $\mathcal{B}$ reduces to a single line if and only if μ and σ are constant. Since σ is constant, ϕ is a silhouette polygon. By formula (3.12), the condition μ constant is equivalent to $\rho_1' + \tau = \rho_2' + \tau = 0$.

Assume $\mu(i + \tfrac{1}{2}) = \mu_0$ constant. Then equation (3.7) implies that
$$\eta(i) = -\mu_0 \phi(i) + Q,$$
for some constant vector Q. Assume $Q = (0, 0, 1)$ and write $\phi(i) = (\gamma(i), z(i))$. Then, using equation (3.5) we obtain
$$\gamma''(i) + \lambda(i)\gamma(i) = -\mu_0\gamma(i), \quad z''(i) + \lambda(i)z(i) = -\mu_0 z(i) + 1,$$
and so
$$\gamma''(i) = -(\lambda(i) + \mu_0)\gamma(i), \quad z''(i) = -(\lambda(i) + \mu_0)z(i) + 1. \qquad (3.13)$$
Observe that
$$[\gamma(i), \gamma(i + 1)] - [\gamma(i - 1), \gamma(i)] = [\gamma(i), \gamma''(i)] = 0,$$
and so $[\gamma(i), \gamma(i+1)] = c$, for some constant c. By rescaling ϕ we may assume that $c = 1$.

Denote by $\Gamma(i + \tfrac{1}{2})$ a polygon such that $\Gamma'(i) = \gamma(i)$. Then
$$\Gamma'''(i) = -(\lambda(i) + \mu_0)\Gamma'(i),$$
and so Γ is an equal-area polygon with discrete affine curvature $\lambda(i) + \mu_0$. The *affine distance* or *support function* of Γ with respect to a point $P \in \mathbb{R}^2$ is given by
$$z(i) = \left[\Gamma(i + \tfrac{1}{2}) - P, \gamma(i)\right] = \left[\Gamma(i - \tfrac{1}{2}) - P, \gamma(i)\right] \qquad (3.14)$$
(see Figure 7, left).

Proposition 3.7. *The polygonal line $\phi(i) = (\gamma(i), z(i))$ (see Figure 7, right) satisfies equation (3.13), and conversely, any solution of the difference equation (3.13) is obtained by this construction, for some planar polygon $\Gamma(i + \tfrac{1}{2})$ and some point $P \in \mathbb{R}^2$.*

Proof. Observe first that
$$z'(i + \tfrac{1}{2}) = \left[\Gamma(i + \tfrac{1}{2}) - P, \gamma(i + 1)\right] - \left[\Gamma(i + \tfrac{1}{2}) - P, \gamma(i)\right]$$
$$= \left[\Gamma(i + \tfrac{1}{2}) - P, \gamma'(i + \tfrac{1}{2})\right].$$
Thus
$$z''(i) = \left[\Gamma(i + \tfrac{1}{2}) - P, \gamma'(i + \tfrac{1}{2})\right] - \left[\Gamma(i - \tfrac{1}{2}) - P, \gamma'(i - \tfrac{1}{2})\right]$$
$$= \left[\gamma(i), \gamma'(i + \tfrac{1}{2})\right] + \left[\Gamma(i - \tfrac{1}{2}) - P, \gamma''(i)\right]$$
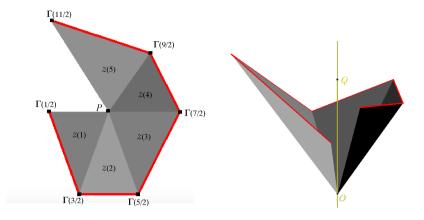$$= 1 - (\lambda(i) + \mu_0)z(i),$$

FIGURE 7. A planar equal-area polygon Γ and its support function z (left). The corresponding spatial curve ϕ and its affine focal set (right).

thus proving that $(\phi(i), z(i))$ satisfies equation (3.13). Since P has two degrees of freedom, this is the general solution of the second order difference equation (3.13). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

4. Polygons in 3-space

In this section, we obtain discrete counterparts of the results of section 2.2. Consider a polygon $\Phi(i + \frac{1}{2})$ in 3-space, without being contained in any polyhedron M. The polygon Φ is equal-volume, i.e., satisfies equation (1.8), if and only if the difference polygon $\phi(i) = \Phi'(i)$ is equal-volume with respect to the origin.

4.1. Frenet equations

For equal-volume polygons Φ, Frenet equations (3.10) are written as

$$\begin{cases} \Phi''''(i + \frac{1}{2}) = -\rho_2(i)\Phi''(i + \frac{1}{2}) + \tau(i + \frac{1}{2})\Phi'(i + 1) \\ \Phi''''(i + \frac{1}{2}) = -\rho_1(i + 1)\Phi''(i + \frac{1}{2}) + \tau(i + \frac{1}{2})\Phi'(i). \end{cases} \qquad (4.1)$$

Defining $\mu(i + \frac{1}{2})$ by equation (3.11), equation (3.12) still holds. It is not clear how to define a discrete version of the intrinsic affine binormal developable.

4.2. Polygons with μ constant

Consider an equal area planar polygon $\Gamma(i + \frac{1}{2})$ and let $Z(i + \frac{1}{2})$ be given by

$$Z(i + \tfrac{1}{2}) = \sum_{j=1}^{i} z(j),$$

where $z(i)$ is given by equation (3.14), for some point $P \in \mathbb{R}^2$. Then $Z(i+\frac{1}{2})$ represents the area of the planar region bounded by $\Gamma'(j)$, $j = 1...i$, and the segments $P\Gamma(\frac{1}{2})$ and $P\Gamma(i+\frac{1}{2})$ (see Figure 8). In this context, Proposition 3.7 can be written as follows:

Proposition 4.1. *The polygon $\Phi = (\Gamma, Z)$ has constant μ, and conversely, any equal-volume polygon Φ with constant μ is obtained by this construction, for some planar polygonal line Γ and some point $P \in \mathbb{R}^2$.*
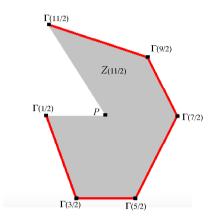

FIGURE 8. A planar equal-area polygon Γ and the area represented by $Z(\frac{11}{2})$.

5. Projective polygons

In this section, we obtain discrete counterparts of the results of section 2.3.

Consider a planar polygon $\tilde{\phi}(i)$, $i = 1, ..., N$. Assume that

$$\left[\tilde{\phi}'(i - \tfrac{1}{2}), \tilde{\phi}'(i + \tfrac{1}{2}) \right] = b(i) > 0. \tag{5.1}$$

5.1. Equal-volume representative

Any polygon ϕ in $\mathbb{R}^3$ of the form $\phi(i) = a(i) \left(\tilde{\phi}(i), 1 \right)$, $a(i) > 0$, is a projective representative of $\tilde{\phi}$.

Lemma 5.1. *There exists a projective representative ϕ of $\tilde{\phi}$ such that equation (1.7) holds with O equal to the origin.*

Proof. Observe first that

$$[\phi(i-1), \phi(i), \phi(i+1)] = a(i-1)a(i)a(i+1) \left[\tilde{\phi}'(i - \tfrac{1}{2}), \tilde{\phi}'(i + \tfrac{1}{2}) \right].$$

So we need to choose $a(i)$, $i = 1..., N$ such that

$$a(i-1)a(i)a(i+1)b(i) = c, \quad i = 2, ..., N-1, \qquad (5.2)$$

for some constant c. Since by the hypothesis (5.1) $b(i) > 0$, given $a(1) > 0$ and $a(2) > 0$ we can find unique $a(i) > 0$, $i = 3, ..., N$ such that (5.2) holds.   □

Assume that ϕ is a representative of $\tilde{\phi}$ such that equation (1.7) holds with O equal to the origin (Figure 9). Then, by lemma 3.2, $\phi'''(i + \frac{1}{2})$ belongs to the plane generated by $\{\phi(i), \phi(i+1)\}$. So we can use equations (3.10) to define $\rho_1(i)$, $\rho_2(i)$ and $\tau(i + \frac{1}{2})$.
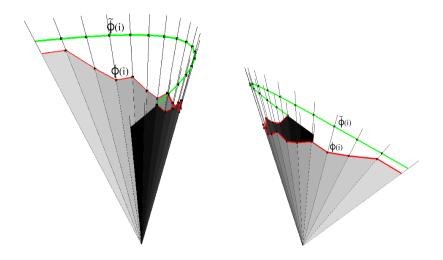

FIGURE 9. Two views of a planar projective polygon $\tilde{\phi}$ and its equal-volume representative ϕ.

5.2. Projective length

We would like to define the projective length of $\tilde{\phi}$ as

$$pl_1(\tilde{\phi}) = \sum_{j=2}^{N-2} \left(\rho_1'(i + \tfrac{1}{2}) + 2\tau(i + \tfrac{1}{2}) \right)^{1/3} \qquad (5.3)$$

or

$$pl_2(\tilde{\phi}) = \sum_{j=2}^{N-2} \left(\rho_2'(i + \tfrac{1}{2}) + 2\tau(i + \tfrac{1}{2}) \right)^{1/3} \qquad (5.4)$$

but unfortunately these two definitions do not coincide. Nevertheless, if the polygonal line is obtained from a dense enough sampling of a smooth curve, both of these formulas are close to projective length of the smooth curve given by equation (2.6). Denote by $O(h^k)$ any quantity such that $\lim_{h \to 0} \frac{O(h^k)}{h^{k-\epsilon}} = 0$, for any $\epsilon > 0$.

Lemma 5.2. *Assume that the polygonal line $\phi(i)$, $i = 1, ...N$, is obtained from $\phi(t)$, $0 \le t \le T$, by uniform sampling. Then, for $Nh = T$, $ih = t$, we have*

$$\rho_1'(i + \tfrac{1}{2}) + 2\tau(i + \tfrac{1}{2}) = (\rho'(t) + 2\tau(t))\, h^3 + O(h^4).$$

A similar result holds for ρ_2.

Proof. It is standard in numerical analysis that $\phi'(i + \tfrac{1}{2}) = h\phi'(t) + O(h^2)$ and $\phi'''(i + \tfrac{1}{2}) = \phi'''(t)h^3 + O(h^4)$. Thus equation (2.3) can be written as

$$\phi'''(i + \tfrac{1}{2}) = -\rho(t)h^2\phi'(i + \tfrac{1}{2}) + \tau(t)h^3\phi(i) + O(h^4).$$

We conclude that $\tau(i + \tfrac{1}{2}) = \tau(t)h^3 + O(h^4)$ and $\rho_1(i+1) = \rho(t)h^2 + c(t)h^3 + O(h^4)$. This last equation implies that $\rho_1'(i + \tfrac{1}{2}) = h^3\rho'(t) + O(h^4)$. Thus we conclude that

$$\rho_1'(i + \tfrac{1}{2}) + 2\tau(i + \tfrac{1}{2}) = (\rho'(t) + 2\tau(t))\, h^3 + O(h^4),$$

which proves the lemma. $\square$

From this lemma we can obtain the following convergence result:

Corollary 5.3. *The discrete projective lengths given by equations (5.3) and (5.4) converge to the smooth projective length given by (2.6) when $h \to 0$.*

Example 1. Consider

$$\tilde{\phi}(t) = (\exp(-t)\cos(t), \exp(-t)\sin(t), 1), \quad 0 \le t \le 2\pi.$$

Then $\tilde{\phi}$ is projectively equivalent to $\phi(t) = 2^{-1/3}\exp(2t/3)\tilde{\phi}(t)$, which satisfies equation (1.2) with O equal the origin. Straightforward calculations show that $\rho(t) = 2/3$, $\tau(t) = 20/27$ and

$$pl(\tilde{\phi}) = 2\pi \tfrac{\sqrt[3]{40}}{3} \approx 7.162519249.$$

We have done some experiments considering uniform samplings of this curve with N points. Table 1 presents the results for $N = 10, 100, 1000$. Observe that both $pl_1(\tilde{\phi})$ and $pl_2(\tilde{\phi})$ get closer to $pl(\tilde{\phi})$ as $h = \tfrac{2\pi}{N}$ decreases.

TABLE 1. Experimental results of example 1.

N	h	$pl_1(\tilde{\phi})$	$pl_2(\tilde{\phi})$
10	0.62831	4.26627	3.55522
100	0.06283	6.87572	6.80410
1000	0.00628	7.13407	7.12691

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