Binary linear codes with at most 4 weights

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Abstract

For the past decades, linear codes with few weights have been widely studied, since they have applications in space communications, data storage and cryptography. In this paper, a class of binary linear codes is constructed and their weight distribution is determined. Results show that they are at most 4-weight linear codes. Additionally, these codes can be used in secret sharing schemes.

Index Terms

Binary linear code, Weight distribution, Secret sharing.

I. Introduction and main results

Throughout this paper, let $q=2^m$ for a positive integer m. Denote $\mathbb{F}_q=\mathbb{F}_{2^m}$ the finite field with q elements and \mathbb{F}_q^* the multiplicative group of \mathbb{F}_q .

Let \mathbb{F}_2^n denote the vector space of all *n*-tuples over the binary field \mathbb{F}_2 . A binary code \mathcal{C} of length n is a subset of \mathbb{F}_2^n . Usually, the vectors in \mathcal{C} are called codewords of \mathcal{C} . For codewords \mathbf{x} and $\mathbf{y} \in \mathcal{C}$, the distance $d(\mathbf{x}, \mathbf{y})$ is referred as the number of coordinates in which \mathbf{x} and \mathbf{y} differ. The (Hamming) distance of a code \mathcal{C} is the smallest distance between distinct codewords and is an important invariant. An [n, k, d] binary linear code \mathcal{C} is defined as a k-dimensional subspace of \mathbb{F}_2^n with distance d.

For a codeword $\mathbf{c} \in \mathcal{C}$, the (Hamming) weight $wt(\mathbf{c})$ is the number of nonzero coordinate in \mathbf{c} . We use A_i to denote the number of codewords of weight i in \mathcal{C} . Then $(1,A_1,\cdots,A_n)$ is called the weight distribution of \mathcal{C} . And the weight enumerator is defined to be the polynomial $1+A_1x+A_2x^2+\cdots+A_nx^n$. If the number of nonzero A_i $(1 \le i \le n)$ equals t, then \mathcal{C} is called a t-weight code. Readers can refer to [22] for a general theory of linear codes.

The weight distribution is an important research topic in coding theory, as it contains crucial information to compute the probability of error correcting and detection. A great deal of researchers are devoted to construct and determine specific linear codes [6], [15], [25], [27]. The weight distribution of Reed–Solomon codes were determined by Blake [1] and Kith [23]. A survey of the hamming weights in irreducible cyclic codes was given by Ding and Yang in [16]. The weight distributions of reducible cyclic codes could be found in [13], [18], [19], [20], [24], [29]. Recently, Ding [9], [14] proposed a generic construction of linear codes as follows.

Let $D = \{d_1, d_2, \dots, d_n\} \subseteq \mathbb{F}_q^*$ and Tr denote the trace function from \mathbb{F}_q to \mathbb{F}_2 . Linear codes \mathcal{C}_D of length n can be constructed by

$$C_D = \{ (\operatorname{Tr}(xd_1), \operatorname{Tr}(xd_2), \dots, \operatorname{Tr}(xd_n)) : x \in \mathbb{F}_q \}.$$

Here D is called the defining set of C_D . The dimension of the code C_D have been presented in [17] and is equal to the dimension of the \mathbb{F}_2 -linear space of \mathbb{F}_q spanned by D. This method has been widely used

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by some researchers to acquire linear codes with few weights [8], [10], [11], [12], [28], [31], [32]. In this paper, we will present a class of binary linear codes with at most four weights.

For $a \in \mathbb{F}_2$, we set

$$D_a = \{ x \in \mathbb{F}_q^* : \operatorname{Tr}(x) = a \}. \tag{1}$$

Motivated by the research work in [10], a class of binary linear codes C_{D_a} is defined by

$$C_{D_a} = \left\{ \left(\operatorname{Tr}(xd^{2^h+1}) \right)_{d \in D_a} : x \in \mathbb{F}_q \right\}, \tag{2}$$

where $a \in \mathbb{F}_2$ and h < m is a positive factor of m. The weight distribution of the presented linear codes is settled and the main results are listed as follows.

TABLE I
THE WEIGHT DISTRIBUTION OF THE CODES OF THEOREM 1

| Weight w | Multiplicity A |
|---------------------------------|-----------------------------------|
| 0 | 1 |
| 2^{m-2} | $2^m - 1 - 2^{m-h}$ |
| $2^{m-2}-2^{\frac{m+h-4}{2}}$ | $2^{m-h-1} + 2^{\frac{m-h-2}{2}}$ |
| $2^{m-2} + 2^{\frac{m+h-4}{2}}$ | $2^{m-h-1}-2^{\frac{m-h-2}{2}}$ |

Theorem 1. Let m/h be odd. Then the code C_{D_0} defined in (2) is a $[2^{m-1}-1,m]$ binary linear code with weight distribution in Table I.

Theorem 2. Let m/h be odd. Then the code C_{D_1} defined in (2) is a $[2^{m-1}, m]$ binary linear code with weight distribution in Table II.

| Weight w | Multiplicity A |
|---------------------------------|---------------------------------|
| 0 | 1 |
| 2^{m-2} | $2^m - 1 - 2^{m-h}$ |
| $2^{m-2}-2^{\frac{m+h-4}{2}}$ | $2^{m-h-1}-2^{\frac{m-h-4}{2}}$ |
| $2^{m-2} + 2^{\frac{m+h-4}{2}}$ | $2^{m-h-1}+2^{\frac{m-h-4}{2}}$ |

 $\begin{tabular}{ll} TABLE III \\ THE WEIGHT DISTRIBUTION OF THE CODES OF THEOREM 3. \end{tabular}$

| Weight w | Multiplicity A |
|--|--|
| 0 | 1 |
| $2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-1}$ | $\frac{2^{m-2h-1}-1-(-1)\frac{e}{h}2^{e-h-1}}{2^h+1}$ |
| $2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-2}$ | $(2^h - 1)2^{m-2h}$ |
| 2^{m-2} | $2^{m-1} - (-1)^{\frac{e}{h}} (2^h - 1)(2^{m-2h-1} + 2^{e-h-1})$ |
| $2^{m-2} - (-1)^{\frac{e}{h}} 2^{e-1}$ | $\frac{2^{m+h-1} + 2^{m-2h-1} - 2^{m-1} - 2^h + (-1)^{\frac{\theta}{h}} \left(2^{e+h-1} + 2^{m-1} - 2^{m-2h-1}\right)}{2^h + 1}$ |

The above two theorems present the parameters of C_{D_a} (a=0,1) of (2) for the case that $m/h \equiv 1 \pmod{2}$. Next, we will assume m/h is even and m=2e. In this case, the parameters of C_{D_a} (a=0,1) of (2) are given in the following two theorems.

Theorem 3. Let m/h be even and m/h > 2. Then the code C_{D_0} defined in (2) is a $[2^{m-1} - 1, m]$ binary linear code with weight distribution in Table III.

TABLE IV THE WEIGHT DISTRIBUTION OF THE CODES OF THEOREM 4

| Weight w | Multiplicity A |
|--|--|
| 0 | 1 |
| $2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-1}$ | $\frac{2^{m-2h-1} + (-1)^{\frac{e}{h}} 2^{e-h-1}}{2^h + 1}$ |
| $2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-2}$ | $(2^h - 1)2^{m-2h}$ |
| 2^{m-2} | $2^{m-1}-1+(-1)^{\frac{e}{h}}2^{e-h-1}(2^h-1)-(2^h-1)2^{m-2h-1}$ |
| $2^{m-2} - (-1)^{\frac{e}{h}} 2^{e-1}$ | $\frac{(2^{e}-(-1)^{\frac{h}{h}})2^{e+h-1}}{2^{h}+1}$ |

Theorem 4. Let m/h be even and m/h > 2. Then the code C_{D_1} defined in (2) is a $[2^{m-1}, m]$ binary linear code with weight distribution in Table IV.

Let $D = \mathbb{F}_q^*$. If m/h is odd, then $\gcd(2^h + 1, 2^m - 1) = 1$ (Lemma 2.1, [5]) and it is straightforward to verify that \mathcal{C}_D of (2) is a constant binary linear code. If m/h is even and m > 2, the code \mathcal{C}_D of (2) is a 2-weight binary linear code, and the weight distribution of C_D is given in Theorem 5.

Theorem 5. Let m/h be even, m > 2 and $D = \mathbb{F}_q^*$. Then the code C_D defined in (2) is a $[2^m - 1, m]$ binary linear code with weight distribution in Table V.

TABLE V THE WEIGHT DISTRIBUTION OF THE CODES OF THEOREM 5.

| Weight w | Multiplicity A |
|--|----------------------------|
| 0 | 1 |
| $2^{m-1} - (-1)^{\frac{e}{h}} 2^{e-1}$ | $\frac{(2^m-1)2^h}{2^h+1}$ |
| $2^{m-1} + (-1)^{\frac{e}{h}} 2^{e+h-1}$ | $\frac{2^m-1}{2^h+1}$ |

If m/h is even, by Lemma 2.1 in [5], we know $gcd(2^h+1,2^m-1)=2^h+1$, i.e., $2^h+1\mid 2^m-1$. Hence $f(x) = x^{2^h+1}$ is a (2^h+1) -to-1 function over \mathbb{F}_q^* in the case that $m/h \equiv 0 \pmod{2}$. This implies that a binary code may be punctured from the code \mathcal{C}_D in Theorem 5. Let $\overline{D} = \{x^{2^h+1} : x \in \mathbb{F}_q^*\}$ and

Let
$$\overline{D} = \{x^{2^h+1} : x \in \mathbb{F}_q^*\}$$
 and

$$C_{\overline{D}} = \left\{ (\operatorname{Tr}(xd))_{d \in \overline{D}} : x \in \mathbb{F}_q \right\}. \tag{3}$$

Then the parameters of $C_{\overline{D}}$ of (3) can be easily derived from the code C_D in Theorem 5, and are given in the following corollary.

Corollary 6. Let m/h be even and m > 2. Then the code $C_{\overline{D}}$ defined in (3) is a $\left[\frac{2^m-1}{2^h+1}, m\right]$ binary linear code with weight distribution in Table VI.

TABLE VI THE WEIGHT DISTRIBUTION OF THE CODES OF COROLLARY 6

| Weight w | Multiplicity A |
|--|----------------------------|
| 0 | 1 |
| $\frac{2^{m-1} - (-1)^{\frac{e}{h}} 2^{e-1}}{2^h + 1}$ | $\frac{(2^m-1)2^h}{2^h+1}$ |
| $\frac{2^{m-1} + (-1)^{\frac{e}{h}} 2^{e+h-1}}{2^h + 1}$ | $\frac{2^m-1}{2^h+1}$ |

Example 1. Let (m,h) = (5,1). For a = 0, the code C_{D_0} in Theorem 1 has parameters [15,5,6] with weight distribution enumerator $1 + 10x^6 + 15x^8 + 6x^{10}$. For a = 1, the code C_{D_1} in Theorem 2 has parameters [16,5,8] with weight enumerator $1 + 6x^6 + 15x^8 + 10x^{10}$.

Example 2. Let (m,h)=(8,2). For a=0, the code \mathcal{C}_{D_0} in Theorem 3 has parameters [127,8,56] with weight distribution enumerator $1+108x^{56}+98x^{64}+48x^{80}+x^{96}$. For a=1, the code \mathcal{C}_{D_1} in Theorem 4 has parameters [128,8,56] with weight enumerator $1+96x^{56}+109x^{64}+48x^{80}+2x^{96}$.

Example 3. Let (m,h) = (6,1). Then the code C_D in Theorem 5 has parameters [63,6,24] with weight enumerator $1 + 21x^{24} + 42x^{36}$. The code C_D in Corollary 6 has parameters [21,6,8] with weight enumerators $1 + 21x^8 + 42x^{12}$.

II. PRELIMINARIES

In this section, we present some results on Weil sums, which will be needed in calculating the weight distribution of the codes defined in (2).

An additive character of \mathbb{F}_q is a group homomorphism χ from \mathbb{F}_q to unit circle of the complex plane. Each additive character can be defined as a mapping

$$\chi_b(c) = (-1)^{\operatorname{Tr}(bc)} \text{ for all } c \in \mathbb{F}_q,$$

with some $b \in \mathbb{F}_q$. For b = 0, the additive character χ_0 is called *trivial* and the other characters χ_b with $b \in \mathbb{F}_q^*$ are called *nontrivial*. For b = 1, the character χ_1 is called the *canonical additive character* of \mathbb{F}_q . And it is well-known that $\chi_b(x) = \chi_1(bx)$ for all $x \in \mathbb{F}_q$ [26].

Define the Weil sum

$$S_h(a,b) = \sum_{x \in \mathbb{F}_q} \chi_1 \left(ax^{2^h+1} + bx \right)$$

where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. In this paper, we restrict that h is a proper positive divisor of m. Generally, to evaluate an exponential sum over a finite field is a challenge task. At present, it has been determined only in certain cases [2], [3], [4], [5], [19], [21]. Among them is the following cases of $S_h(a,b)$.

Lemma 7 ([5], Theorem 4.1). If m/h is odd, then $\sum_{x \in \mathbb{F}_q} \chi_1\left(ax^{2^h+1}\right) = 0$ for each $a \in \mathbb{F}_q^*$.

Lemma 8 ([5], Theorem 4.2,). Let $b \in \mathbb{F}_q^*$ and suppose m/h is odd. Then $S_h(a,b) = S_h(1,bc^{-1})$, where $c \in \mathbb{F}_q^*$ is the unique element satisfying $c^{2^h+1} = a$. Further we have

$$S_h(1,b) = \begin{cases} 0, & \text{if } \operatorname{Tr}_h(b) \neq 1, \\ \pm 2^{\frac{m+h}{2}}, & \text{if } \operatorname{Tr}_h(b) = 1, \end{cases}$$

where and hereafter Tr_h is the trace function from \mathbb{F}_q to \mathbb{F}_{2^h} .

Lemma 9 ([5], Theorem 5.2). Let m/h be even and m = 2e for some integer e. Then

$$S_h(a,0) = \begin{cases} (-1)^{\frac{e}{h}} 2^e, & \text{if } a \neq g^{t(2^h+1)} \text{ for any integer } t, \\ -(-1)^{\frac{e}{h}} 2^{e+h}, & \text{if } a = g^{t(2^h+1)} \text{ for some integer } t, \end{cases}$$

where g is a generator of \mathbb{F}_q^* .

Lemma 10 ([5], Theorem 5.3). Let $b \in \mathbb{F}_q^*$ and suppose m/h is even so that m = 2e for some integer e. Let $f(x) = a^{2^h} x^{2^{2^h}} + ax \in \mathbb{F}_q[x]$. There are two cases.

1) If $a \neq g^{t(2^h+1)}$ for any integer t then f is a permutation polynomial of \mathbb{F}_q . Let x_0 be the unique element satisfying $f(x) = b^{2^h}$. Then

$$S_h(a,b) = (-1)^{\frac{e}{h}} 2^e \chi_1 \left(a x_0^{2^h + 1} \right).$$

2) If $a = g^{t(2^h+1)}$ then $S_h(a,b) = 0$ unless the equation $f(x) = b^{2^h}$ is solvable. If the equation is solvable, with solution x_0 say, then

$$S_h(a,b) = \begin{cases} -(-1)^{\frac{e}{h}} 2^{e+h} \chi_1\left(ax_0^{2^h+1}\right), & \text{if } \operatorname{Tr}_h(a) = 0, \\ (-1)^{\frac{e}{h}} 2^e \chi_1\left(ax_0^{2^h+1}\right), & \text{if } \operatorname{Tr}_h(a) \neq 0, \end{cases}$$

where Tr_h is the trace function from \mathbb{F}_q to \mathbb{F}_{2^h} .

Lemma 11 ([5], Theorem 3.1). Let g be a primitive element of \mathbb{F}_q . For any $a \in \mathbb{F}_q^*$ consider the equation $a^{2^h}x^{2^{2h}} + ax = 0$ over \mathbb{F}_q .

- 1) If m/h is odd then there are 2^h solutions to this equation for any choice of $a \in \mathbb{F}_q^*$.
- 2) If m/h is even then there are two possible cases. If $a = g^{t(2^h+1)}$ for some t, then there are 2^{2h} solutions to the equation. If $a \neq g^{t(2^h+1)}$ for any t then there exists one solution only, x = 0.

III. THE PROOFS OF THE MAIN RESULTS

We follow the notations fixed in Sect. 2. In this section, we will determine the length of the code C_{D_a} (a=0,1) of (2), and give a formula on the weight of a codeword \mathbf{c}_b $(b \in \mathbb{F}_q^*)$ in C_{D_a} (a=0,1) of (2). Then we give the proofs of Theorems 1 and 3.

By the definition of D_a (a = 0, 1) in (1), we know

$$|D_a| = \begin{cases} 2^{m-1} - 1, & \text{if } a = 0, \\ 2^{m-1}, & \text{if } a = 1. \end{cases}$$

Define $N(a,b) = \{x \in \mathbb{F}_q : \operatorname{Tr}(x) = a \text{ and } \operatorname{Tr}(bx^{2^h+1}) = 0\}$. We use $wt(c_b)$ to denote the Hamming weight of the codeword \mathbf{c}_b with $b \in \mathbb{F}_q^*$ of the code \mathcal{C}_{D_a} (a=0,1) defined in (2). It can be easily checked that

$$wt(\mathbf{c}_b) = 2^{m-1} - |N(a,b)|.$$
 (4)

In terms of exponential sums, for $b \in \mathbb{F}_q^*$, we have

$$|N(a,b)| = 2^{-2} \sum_{x \in \mathbb{F}_q} \left(\sum_{y \in F_2} (-1)^{y \operatorname{Tr}(x) - ya} \right) \left(\sum_{z \in F_2} (-1)^{z \operatorname{Tr}(bx^{2^h + 1})} \right)$$

$$= 2^{-2} \sum_{x \in \mathbb{F}_q} \left(1 + (-1)^{\operatorname{Tr}(x) - a} \right) \left(1 + (-1)^{\operatorname{Tr}(bx^{2^h + 1})} \right)$$

$$= 2^{m-2} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\operatorname{Tr}(bx^{2^h + 1})} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\operatorname{Tr}(x + bx^{2^h + 1}) - a}$$

$$= 2^{m-2} + 2^{-2} \left(S_h(b, 0) + (-1)^a S_h(b, 1) \right).$$
(5)

Based on the discussion above, the weight distribution of C_{D_a} of (2) can be determined by the value distribution of $S_h(b,c)$ with $b \in \mathbb{F}_q^*$ and $c \in \mathbb{F}_2$. Combining (5) and the lemmas in preliminaries, we are ready to compute the weight distribution of the codes C_{D_a} (a = 0,1) defined in (2).

Proof of Theorem 1. By Lemma 7, we have $S_h(b,0)=0$ for $b\in\mathbb{F}_q^*$. It follows from Lemma 8 that

$$S_h(b,1) = S_h(1,c^{-1}) = \begin{cases} 0, & \text{if } \operatorname{Tr}_h(c^{-1}) \neq 1, \\ \pm 2^{\frac{m+h}{2}}, & \text{if } \operatorname{Tr}_h(c^{-1}) = 1, \end{cases}$$

$$(6)$$

where $c^{2^h+1} = b$ and $c \in \mathbb{F}_q^*$. Together with equation (5), we get

$$|N(0,b)| \in \left\{2^{m-2}, 2^{m-2} - 2^{\frac{m+h-4}{2}}, 2^{m-2} + 2^{\frac{m+h-4}{2}}\right\}.$$

Hence,

$$\operatorname{wt}(c_b) = 2^{m-1} - |N(0,b)| \in \left\{2^{m-2}, 2^{m-2} \pm 2^{\frac{m+h-4}{2}}\right\}.$$

Suppose

$$w_1 = 2^{m-2} - 2^{\frac{m+h-4}{2}}, \ w_2 = 2^{m-2}, \ w_3 = 2^{m-2} + 2^{\frac{m+h-4}{2}}.$$

Note that $\gcd(2^h+1,2^m-1)=1$ if m/h is odd. When b ranges over \mathbb{F}_q^* , the element c $(c^{2^h+1}=b)$ takes on each element of \mathbb{F}_q^* exactly 1 time. Hence, for $b\in\mathbb{F}_q^*$ we obtain

$$\left|\left\{c \in \mathbb{F}_q : \operatorname{Tr}(c^{-1}) \neq 1, \ c^{2^h + 1} = b\right\}\right| = 2^m - 2^{m - h} - 1,$$

i.e., $A_{w_2} = 2^m - 2^{m-h} - 1$. The first two Pless Power Moments ([22], P.260) yield the following two equations:

$$\begin{cases}
A_{w_1} + A_{w_2} + A_{w_3} = 2^m - 1, \\
w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} = n 2^{m-1},
\end{cases}$$
(7)

where $n = 2^{m-1} - 1$. Solving the system of equations in (7) gets Theorem 1.

The proof of Theorem 2 is similar to that of Theorem 1 and we omit the details.

In the sequel, we assume $m/h \equiv 0 \pmod{2}$, m = 2e and g is a generator of \mathbb{F}_q^* . In order to give the proof of Theorem 3, the following auxiliary lemma is needed. This lemma can be found in equation (10) in [10].

Lemma 12. Let $T_0 = |\{x \in \mathbb{F}_q : \operatorname{Tr}(x^{2^h+1}) = 0\}|$ and $T_1 = |\{x \in \mathbb{F}_q : \operatorname{Tr}(x^{2^h+1}) = 1\}|$. If m/h is even, then $T_0 = 2^{m-1} - (-1)^{\frac{e}{h}} 2^{e+h-1}$ and $T_1 = 2^{m-1} + (-1)^{\frac{e}{h}} 2^{e+h-1}$.

Proof of Theorem 3. If $b \in \mathbb{F}_q^*$ and $b \neq (g^{2^h+1})^t$ for any integer t, then by Lemma 9, we have

$$S_h(b,0) = (-1)^{\frac{e}{h}} 2^e$$
,

and by Lemma 10, we get

$$S_h(b,1) = (-1)^{\frac{e}{h}} 2^e \chi_1(b x_0^{2^h+1}),$$

where $b^{2^h}x_0^{2^{2h}} + bx_0 = 1$.

If $b = (g^{2^h+1})^t$ for some integer t, it follows from Lemma 9 that

$$S_h(b,0) = -(-1)^{\frac{e}{h}} 2^{e+h}$$

Assume $c = g^t$, then $b = c^{2^h + 1}$ and by Lemma 8 $S_h(b, 1) = S_h(1, c^{-1})$. For the above $c \in \mathbb{F}_q^*$, let

$$f_c(x) = x^{2^{2h}} + x - (c^{-1})^{2^h}. (8)$$

If $f_c(x)$ has no root in \mathbb{F}_q^* , by Lemma 10, we obtain

$$S_h(b,1) = S_h(1,c^{-1}) = 0.$$

Note that $\operatorname{Tr}_h(1) = 0$, since m/h is even. If $f_c(x)$ has a root x_0 in \mathbb{F}_q^* , by Lemma 10, we get

$$S_h(b,1) = S_h(1,c^{-1}) = -(-1)^{\frac{e}{h}} 2^{e+h} \chi_1(x_0^{2^h+1}).$$

Together with (4) and (5), we know that for $b \in \mathbb{F}_a^*$,

$$wt(\mathbf{c}_b) \in \left\{2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-1}, 2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-2}, 2^{m-2}, 2^{m-2} - (-1)^{\frac{e}{h}} 2^{e-1}\right\}.$$

Define

$$w_1 = 2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-1}, \ w_2 = 2^{m-2} + (-1)^{\frac{e}{h}} 2^{e+h-2}, \ w_3 = 2^{m-2}, \ w_4 = 2^{m-2} - (-1)^{\frac{e}{h}} 2^{e-1}$$

The next step is to determine the number A_{w_i} of codewords with weight w_i . If $f_c(x) = 0$ (for some $c \in \mathbb{F}_q$) is solvable in \mathbb{F}_q , by Lemma 11, there are 2^{2h} solutions of this equation over \mathbb{F}_q . It can be easily checked that

$$\left\{x_0 \in \mathbb{F}_q : x_0^{2^{2h}} + x_0 = (c^{-1})^{2^h}, c \in \mathbb{F}_q\right\} = \mathbb{F}_q.$$

Hence we get

$$\left| \left\{ c \in \mathbb{F}_q^* : x^{2^{2h}} + x = (c^{-1})^{2^h} \text{ is solvable in } \mathbb{F}_q \right\} \right| = 2^{m-2h} - 1,$$

and

$$\left|\left\{c \in \mathbb{F}_q^* : x^{2^{2h}} + x = (c^{-1})^{2^h} \text{ has no root in } \mathbb{F}_q\right\}\right| = 2^m - 2^{m-2h}.$$

Since x^{2^h+1} is a (2^h+1) -to-1 function on \mathbb{F}_q , there are $\frac{2^m-2^{m-2h}}{2^h+1}$ b's $(b=c^{2^h+1}\in\mathbb{F}_q^*)$ such that $S_h(b,1)=0$, i.e., $A_{w_2}=\frac{2^m-2^{m-2h}}{2^h+1}$. It follows from Lemmas 10 and 12 that

$$\left| \left\{ c \in \mathbb{F}_q^* : S_h(1, c^{-1}) = (-1)^{\frac{e}{h}} 2^{e+h} \right\} \right| = \frac{2^{m-1} + (-1)^{\frac{e}{h}} 2^{e+h-1}}{2^{2h}}.$$

Then we have

$$\left|\left\{b \in \mathbb{F}_q^* : b = c^{2^h + 1} \text{ and } S_h(b, 1) = (-1)^{\frac{e}{h}} 2^{e + h}\right\}\right| = \frac{2^{m - 1} + (-1)^{\frac{e}{h}} 2^{e + h - 1}}{2^{2h} (2^h + 1)}.$$

Therefore, $A_{w_1} = \frac{2^{m-2h-1}-1-(-1)^{\frac{e}{h}}2^{e-h-1}}{2^h+1}$. By the Pless Power Moments ([22], p. 260) we obtain the following two equations:

$$\begin{cases}
A_{w_1} + A_{w_2} + A_{w_3} + A_{w_4} = 2^m - 1, \\
w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} + w_4 A_{w_4} = 2^{m-1} (2^m - 1).
\end{cases}$$
(9)

The solutions of (9) yield the weight distribution of Table III. The proof of Theorem 3 is completed. We omit the proof of Theorems 4 and 5, since it is similar to that of Theorem 3.

IV. CONCLUDING REMARKS

In this paper, we present a class of binary linear codes with no more than four weights. A number of linear codes with at most five-weight codes were discussed in [7], [8], [10], [11], [12], [31], [32].

It should be remarked that he parameters of the binary linear codes in Theorem 1 are the same as those in Theorem 1 in [10]. It is open whether the two class of codes are equivalent. The readers are invited to attack this problem.

Denote the minimum and maximum nonzero weight of a linear code C over \mathbb{F}_p by w_{\min} and w_{\max} , respectively. By the results in [30], if the code C satisfies the following inequality

$$\frac{w_{\min}}{w_{\max}} > \frac{p-1}{p},$$

then C can be employed to construct secret sharing schemes with interesting properties.

Let m > h + 2. Then for the codes in Theorems 1 and 2, we have

$$\frac{w_{\min}}{w_{\max}} = \frac{2^{m-2} - 2^{\frac{m+h-4}{2}}}{2^{m-2} + 2^{\frac{m+h-4}{2}}} > \frac{1}{2}.$$

If $(m,h) \neq (4,1)$ or (6,1), then for the codes in Theorems 3 and 4, it can be easily checked that

$$\frac{w_{\min}}{w_{\max}} > \frac{1}{2}$$
.

This conclusion is true also for Theorem 5 and Corollary 6.

Hence, the binary linear codes presented in this paper are suitable for constructing secret sharing schemes in many cases.

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