Existence and properties of ancient solutions of the Yamabe flow

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Abstract

Let $n \geq 3$ and $m = \frac{n-2}{n+2}$. We construct 5-parameters, 4-parameters, and 3-parameters ancient solutions of the equation $v_t = (v^m)_{xx} + v - v^m$, $v > 0$, in $\mathbb{R} \times (-\infty, T)$ for some $T \in \mathbb{R}$. This equation arises in the study of Yamabe flow. We obtain various properties of the ancient solutions of this equation including exact decay rate of ancient solutions as $|x| \to \infty$. We also prove that both the 3-parameters ancient solution and the 4-parameters ancient solution are singular limit solution of the 5-parameters ancient solutions. We also prove the uniqueness of the 4-parameters ancient solutions. As a consequence we prove that the 4-parameters ancient solutions that we construct coincide with the 4-parameters ancient solutions constructed by P. Daskalopoulos, M. del Pino, J. King, and N. Sesum in [DPKS2].

Key words: ancient solutions, Yamabe flow, decay rate, singular limit solution

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1 Introduction

As observed by P. Daskalopoulos, M. del Pino, J. King, M. Sáez, N. Sesum, and others [DPKS1], [DPKS2], [PS], the metric $g = u^{\frac{4}{n-2}} dy^2$ satisfies the Yamabe flow [B1], [B2],

$$\frac{\partial g}{\partial t} = -Rg \tag{1.1}$$

on $\mathbb{R}^n$, $n \geq 3$, for $0 < t < T$, where R is the scalar curvature of the metric g, if and only if u satisfies

$$(u^p)_t = \frac{n-1}{m}\Delta u, \quad u > 0, \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.2}$$

where

$$m = \frac{n-2}{n+2}, \quad p = \frac{n+2}{n-2},$$

and Δ is the Euclidean laplacian on $\mathbb{R}^n$. When u is radially symmetric, we can write $g = u^{\frac{4}{n-2}}dy^2 = w^{\frac{4}{n-2}}g_{cyl}$ where

$$w(x,t) = |y|^{\frac{n-2}{2}}u(y,t), \quad x = \log|y|,$$

and $g_{cyl} = dx^2 + g_{S^{n-1}}$ is the cylindrical metric on $S^{n-1}\times\mathbb{R}$, with $g_{S^{n-1}}$ being the standard metric on the unit sphere S^{n-1} in $\mathbb{R}^n$. Since

$$R = w^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\Delta_{g_{cyl}}w + R_{g_{cyl}}w\right),$$

where $\Delta_{g_{cyl}}w$ and $R_{g_{cyl}} = (n-1)(n-2)$ are the laplace operator and scalar curvature with respect to the cylindrical metric g_{cyl}, by (1.1) w satisfies

$$(w^{\frac{n+2}{n-2}})_t = \frac{n-1}{m}\left(w_{xx} - \frac{(n-2)^2}{4}w\right), \quad w > 0, \tag{1.3}$$

in $\mathbb{R}\times(0,T)$. Let $\tau = -\log(T-t)$ and

$$\widetilde{u}(x,\tau) = (T-t)^{-\frac{n-2}{4}}w(x,t).$$

Then by (1.3),

$$(\widetilde{u}^p)_\tau = \frac{n-1}{m}\left(\widetilde{u}_{xx} + \frac{n-2}{4(n-1)}\widetilde{u}^p - \frac{(n-2)^2}{4}\widetilde{u}\right), \quad \widetilde{u} > 0, \tag{1.4}$$

in $\mathbb{R}\times(-\log T,\infty)$. Let $\widetilde{g}(y,\tau) = \widetilde{u}(x,\tau)^{\frac{4}{n-2}}g_{cyl}$ with $x = \log|y|$. Then $\widetilde{g} = (T-t)^{-1}g$. By (1.1), $\widetilde{g}$ satisfies the normalized Yamabe flow,

$$\frac{\partial\widetilde{g}}{\partial\tau} = -(R_{\widetilde{g}} - 1)\widetilde{g}$$

in $\mathbb{R}^n\times(-\log T,\infty)$ where $R_{\widetilde{g}}$ is the scalar curvature of $\widetilde{g}$. Let

$$\hat{u}(x,\tau) = [(n-1)(n-2)]^{-\frac{n-2}{4}}\widetilde{u}\left(\frac{2x}{n-2}, \frac{4\tau}{n+2}\right) \quad \text{and} \quad v(x,\tau) = \hat{u}(x,\tau)^p. \tag{1.5}$$

Then (1.4) is equivalent to

$$(\hat{u}^p)_\tau = \hat{u}_{xx} + \hat{u}^p - \hat{u}, \quad \hat{u} > 0$$

or

$$v_\tau = (v^m)_{xx} + v - v^m, \quad v > 0. \tag{1.6}$$

Hence existence of ancient radially symmetric solutions of (1.1) with metric $g = u^{\frac{4}{n-2}} dy^2$ is equivalent to the existence of ancient solutions of (1.6) in $\mathbb{R} \times (-\infty, T)$ for some constant $T \in \mathbb{R}$.

Existence of 5-parameters and 4-parameters ancient solutions of (1.6) for some $T \in \mathbb{R}$ have been constructed by P. Daskalopoulos, M. del Pino, J. King and N. Sesum, in [DPKS1], [DPKS2]. In this paper we will construct new 5-parameters, 4-parameters, and 3-parameters ancient solutions of the equation (1.6). As a result of our construction of solutions we obtain exact decay rate of the ancient solutions of (1.6).

A natural question to ask is whether the 4-parameters ancient solutions of (1.6) that we construct is equal to the 4-parameters ancient solutions of (1.6) constructed by P. Daskalopoulos, M. del Pino, J. King, and N. Sesum in [DPKS2]. We answer this in the affirmative and prove that the 4-parameters ancient solutions that we construct coincide with the 4-parameters ancient solutions of (1.6) constructed by P. Daskalopoulos, M. del Pino, J. King, and N. Sesum in [DPKS2]. In particular under a mild decay condition on the 4-parameters ancient solutions of (1.6) we prove the uniqueness of the 4-parameters ancient solutions.

In the paper [HN] F. Hamel and N. Nadirashvili proved various properties of the ancient solutions of the equation

$$u_t = u_{xx} + f(u), u > 0, \quad \text{in } \mathbb{R} \times (-\infty, T) \tag{1.7}$$

for some $T \in \mathbb{R}$ where $f(s) > 0$ for $0 < s < 1$, $f(0) = f(1) = 0$, $f'(0) > 0 > f'(1)$ and $f'(s) \le f'(0)$ for any $s \in [0, 1]$. In this paper we will prove that many properties of the ancient solutions of (1.7) remains valid for the ancient solutions of (1.6). In particular the 4-parameters ancient solution is the singular limit solution of the 5-parameters ancient solutions and the 3-parameters ancient solution is the singular limit solution of the 4-parameters ancient solution, etc.

Let u be a radially symmetric solution of (1.2) and $\overline{u}(y, t) = u(y, t)^p$. Then $\overline{u}$ satisfies

$$\overline{u}_t = \frac{n-1}{m} \Delta \overline{u}^m \tag{1.8}$$

and

$$\overline{u}(y, t) = [(n-1)(n-2)(T-t)]^{\frac{1}{1-m}} |y|^{-\frac{2}{1-m}} v\left(\frac{n-2}{2} x, \frac{n+2}{4} \tau\right) \tag{1.9}$$

where $x = \log |y|$, $\tau = -\log(T - t)$, and v is given by (1.5). If $\overline{u}$ is a backward radially symmetric self-similar solution of (1.8) that is

$$\overline{u}(y, t) = (T - t)^{\alpha} f((T - t)^{\beta} |y|) \tag{1.10}$$

where f is a radially symmetric solution of

$$\Delta f^m + \alpha f + \beta x \cdot \nabla f = 0 \quad \text{in } \mathbb{R}^n \tag{1.11}$$

and $\beta = \frac{\lambda}{2m} > 0$, $\alpha = \frac{2\beta+1}{1-m}$, are some constants, then by the discussion in [DKS] the corresponding function $v(x, \tau) = v_\lambda(x, \tau) = v_\lambda(x - \lambda\tau)$ of (1.9) is a travelling wave solution of (1.6) in $\mathbb{R}\times\mathbb{R}$ with

$$v_\lambda(\log(|z|^{\frac{n-2}{2}})) = [(n-1)(n-2)]^{-\frac{1}{1-m}}|z|^{\frac{2}{1-m}}f(z) \quad \forall z\in\mathbb{R}^n$$

or equivalent

$$v_\lambda(x) = [(n-1)(n-2)]^{-\frac{1}{1-m}}e^{px}f\left(e^{\frac{2x}{n-2}}\right) \quad \forall x\in\mathbb{R}. \tag{1.12}$$

By Theorem 1.1 of [H2] for any $\alpha = \frac{2\beta+1}{1-m}$,

$$\beta = \frac{\lambda}{2m} \ge \frac{m}{n-2-nm} = \frac{1}{2} \quad \Leftrightarrow \quad \lambda \ge m = \frac{n-2}{n+2}$$

and $\mu>0$, there exists a unique radially symmetric solution f of (1.11) satisfying $f(0)=\mu$. By Theorem 1.2 of [H3] and (1.12) when

$$\beta = \frac{\lambda}{2m} \ge \frac{1}{n-2} \quad \Leftrightarrow \quad \lambda \ge \frac{2}{n+2},$$

the corresponding travelling wave solution $v_\lambda(x-\lambda\tau)$ of (1.6) in $\mathbb{R}\times\mathbb{R}$ satisfies

$$\lim_{x\to-\infty} v_\lambda(x) = 0, \quad \lim_{x\to\infty} v_\lambda(x) = 1 \tag{1.13}$$

and

$$e^{-px}v_\lambda(x) \approx C \quad \text{as } x\to-\infty \tag{1.14}$$

for some constant $C>0$. Then by the intermediate value theorem there exists $x_0\in\mathbb{R}$ such that $v_\lambda(x_0)=\frac{1}{2}$. By translation if necessary we may assume that $v_\lambda(0)=\frac{1}{2}$. Hence for any $\lambda\ge\max(\frac{n-2}{n+2},\frac{2}{n+2})$ there exists a travelling wave solution $v_\lambda(x,\tau)=v_\lambda(x-\lambda\tau)$ of (1.6) which satisfies $v_\lambda(0)=\frac{1}{2}$ and $v_\lambda(x)$ satisfies

$$(v^m)_{xx} + \lambda v_x + v - v^m = 0, \quad v>0, \quad \text{in } \mathbb{R} \tag{1.15}$$

and (1.12) for some radially symmetric solution f of (1.11). By the results of [DKS], [DPKS1] and[DPKS2], for any $\lambda>1$, there exist positive constants C_λ and γ_λ such that

$$v_\lambda(x) = 1 - C_\lambda e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}) \quad \text{and} \quad v_\lambda'(x) = C_\lambda\gamma_\lambda e^{-\gamma_\lambda x} + o(e^{-\gamma_\lambda x}) \quad \text{as } x\to\infty \tag{1.16}$$

where

$$\gamma_\lambda = \frac{\lambda p - \sqrt{\lambda^2p^2-4(p-1)}}{2}$$

is the smallest root of the equation

$$\gamma^2 - \lambda p\gamma + p - 1 = 0. \tag{1.17}$$

Note that for any $\lambda>1$, $\lambda'>1$, $h,h'\in\mathbb{R}$, both $v_{\lambda,h}(x,\tau) := v_\lambda(x-\lambda\tau+h)$ and $\overline{v}_{\lambda',h'}(x,\tau) := v_{\lambda'}(-x-\lambda'\tau+h')$ are travelling wave solutions of (1.6). By (1.14) and (1.16),

$$v_{\lambda,h}(x,\tau) = O(e^{p(x-\lambda\tau+h)}) \quad \text{as } x\to-\infty \quad \text{and} \quad \overline{v}_{\lambda',h'}(x,\tau) = O(e^{p(-x-\lambda'\tau+h')}) \quad \text{as } x\to\infty \tag{1.18}$$

and

$$\begin{cases} v_{\lambda,h}(x,\tau) = 1 - C_\lambda e^{-\gamma_\lambda(x-\lambda\tau+h)} + o(e^{-\gamma_\lambda(x-\lambda\tau+h)}) \quad \text{as} \quad x - \lambda\tau + h \to \infty \\ \overline{v}_{\lambda',h'}(x,\tau) = 1 - C_{\lambda'} e^{-\gamma_{\lambda'}(-x-\lambda'\tau+h')} + o(e^{-\gamma_{\lambda'}(-x-\lambda'\tau+h')}) \quad \text{as} \quad -x - \lambda'\tau + h' \to \infty. \end{cases} \tag{1.19}$$

Note that by (1.14) and (1.15) there exists a constant $C > 0$ such that

$$((v_\lambda^m)_x + \lambda v_\lambda)_x = v_\lambda^m - v_\lambda = Ce^x + o(e^x) \quad \text{as } x \to -\infty. \tag{1.20}$$

By (1.14) and the mean value theorem for any $i \in \mathbb{N}$ there exists a constant $x_i \in (-i-1, -i)$ such that

$$|v_\lambda'(x_i)| = |v_\lambda(-i-1) - v_\lambda(-i)| \le Ce^{-pi} \to 0 \quad \text{as } i \to \infty. \tag{1.21}$$

Integrating (1.20) over (x_i, x) and letting $i \to \infty$, by (1.14) and (1.21),

$$\begin{aligned} & (v_\lambda^m)_x + \lambda v_\lambda = Ce^x + o(e^x) \quad \text{as } x \to -\infty \\ \Rightarrow \quad & (v_\lambda^m)_x = Ce^x + o(e^x) \quad \text{as } x \to -\infty \\ \Rightarrow \quad & v_{\lambda,x}(x) = v_\lambda(x)^{1-m}(C'e^x + o(e^x)) \quad \text{as } x \to -\infty \\ \Rightarrow \quad & v_{\lambda,x}(x) = C''e^{px} + o(e^{px}) \quad \text{as } x \to -\infty \end{aligned} \tag{1.22}$$

for some constants $C' > 0$, $C'' > 0$. Note that by (1.16) and (1.22) $v_\lambda' \in L^\infty(\mathbb{R})$ for any $\lambda > 1$. Let $k_0 > 0$, $h_0, h_0' \in \mathbb{R}$. We choose $\tau_0' \in \mathbb{R}$ such that $k_0 e^{\frac{p-1}{p}\tau_0'} < 1/2$. Let

$$\xi_k(\tau) = (1 - ke^{\frac{p-1}{p}\tau})^{\frac{p}{p-1}} \quad \forall 0 < k \le k_0, \tau \le \tau_0'.$$

Then by direct computation ξ_k satisfies (cf. [DPKS2]),

$$\xi_k'(\tau) = \xi_k(\tau) - \xi_k(\tau)^m \quad \forall 0 < k \le k_0, \tau \le \tau_0'$$

and

$$\xi_k(\tau) = 1 - \frac{pk}{p-1} e^{\frac{p-1}{p}\tau} + o(e^{\frac{p-1}{p}\tau}) \quad \text{as } \tau \to -\infty. \tag{1.23}$$

Hence ξ_k is a solution of (1.6) in $\mathbb{R} \times (-\infty, \tau_0')$. For any $\lambda > 1$, $\lambda' > 1$, $h, h' \in \mathbb{R}$, $0 < k \le k_0$, let

$$\begin{aligned} & f_{\lambda,\lambda',h,h',k}(x,\tau) = (v_{\lambda,h}(x, f(\tau))^{1-p} + \overline{v}_{\lambda',h'}(x, f(\tau))^{1-p} + \xi_k(\tau)^{1-p} - 2)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}, \tau \le \tau_0', \\ & \overline{f}_{\lambda,\lambda',h,h',k}(x,\tau) = min(v_{\lambda,h}(x, f(\tau)), \overline{v}_{\lambda',h'}(x, f(\tau)), \xi_k(\tau)) \quad \forall x \in \mathbb{R}, \tau \le \tau_0', \\ & f_{\lambda,\lambda',h,h'}(x,\tau) = (v_{\lambda,h}(x, f(\tau))^{1-p} + \overline{v}_{\lambda',h'}(x, f(\tau))^{1-p} - 1)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}, \tau \in \mathbb{R}, \\ & \overline{f}_{\lambda,\lambda',h,h'}(x,\tau) = min(v_{\lambda,h}(x, f(\tau)), \overline{v}_{\lambda',h'}(x, f(\tau))) \quad \forall x \in \mathbb{R}, \tau \in \mathbb{R}, \\ & f_{\lambda,h,k}(x,\tau) = (v_{\lambda,h}(x, f(\tau))^{1-p} + \xi_k(\tau)^{1-p} - 1)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}, \tau \le \tau_0', \\ & \overline{f}_{\lambda,h,k}(x,\tau) = min(v_{\lambda,h}(x, f(\tau)), \xi_k(\tau)) \quad \forall x \in \mathbb{R}, \tau \le \tau_0', \end{aligned}$$

where

$$f(\tau) = \tau\left(1 + qe^{\frac{p-1}{p}\tau}\right) \quad \forall \tau \le \tau_0' \tag{1.24}$$

for some constant $q = q(p) > 0$. Note that

$$f_{\lambda,\lambda',h,h',k} \le \min(f_{\lambda,\lambda',h,h'}, f_{\lambda,h,k}) \quad \text{and} \quad \overline{f}_{\lambda,\lambda',h,h',k} \le \min(\overline{f}_{\lambda,\lambda',h,h'}, \overline{f}_{\lambda,h,k}) \quad \text{in } \mathbb{R}\times(-\infty,\tau_0')$$

for any $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and $0 < k \le k_0$. Note that for $\tau < 0$ sufficiently small the last two terms of (3.5) of [DPKS2] are negative only if the constant q there is negative instead of positive since $(\tau e^{\frac{p-1}{p}\tau})' < 0$ for $\tau < 0$ sufficiently small. Hence the constant q in (1.30) of [DPKS2] should be negative instead of positive in order for the function $w_{\lambda,\lambda',h,h',k}$ there to be a supersolution of (1.16) of [DPKS2]. Thus by the proof of [DPKS2] there exist constants $\tau_0 = \tau_0(h_0, h_0', k_0) < \tau_0'$ and $q = q(p) > 0$ such that $f_{\lambda,\lambda',h,h',k}$ and $f_{\lambda,\lambda',h,h'}$ are subsolutions of (1.6) in $\mathbb{R}\times(-\infty,\tau_0)$ for any $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0$ and $0 < k \le k_0$. By an argument similar to the proof of [DPKS2] we can choose the constants $\tau_0 = \tau_0(h_0, h_0', k_0) < \tau_0'$ and $q = q(p) > 0$ such that $f_{\lambda,h,k}$ is also a subsolution of (1.6) in $\mathbb{R}\times(-\infty,\tau_0)$ for any $\lambda > 1$, $h \ge h_0$ and $0 < k \le k_0$. By choosing $\tau_0 < \min(\tau_0', -p/(p-1))$ to be sufficiently small we also have $0 < f'(\tau) < 1$ for any $\tau \le \tau_0$. We will assume $\lambda > 1$ and v_λ is the solution of (1.15) which satisfies

$$v_\lambda(0) = 1/2, \quad \lim_{x\to-\infty} v_\lambda(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} v_\lambda(x) = 1$$

for the rest of the paper. In this paper we will prove the following main results.

Theorem 1.1. *There exists $\overline{\tau}_0 < \tau_0$ such that for any $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and $0 < k \le k_0$, there exists a solution $v = v_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ of (1.6) in $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ which satisfies*

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \le v_{\lambda,\lambda',h,h',k}(x,\tau) \le \overline{f}_{\lambda,\lambda',h,h',k}(x,\tau) \quad \forall x\in\mathbb{R}, \tau < \overline{\tau}_0 \tag{1.25}$$

and the following holds.

(i) For any $x \in \mathbb{R}$, $v(x,\tau)$ is a decreasing function of τ and $v(x,\tau) \to 1$ as $\tau \to -\infty$.

(ii) For any $\tau < \overline{\tau}_0$, $v(x,\tau) \to 0$ as $|x| \to \infty$ and there exists $x_0(\tau) \in \mathbb{R}$ such that $v_x(x_0(\tau),\tau) = 0$, $v_x(x,\tau) > 0$ if $x < x_0(\tau)$ and $v_x(x,\tau) < 0$ if $x > x_0(\tau)$. Furthermore if $\lambda = \lambda'$, then $x_0(\tau) = \frac{h'-h}{2}$ and $v(x,\tau)$ is symmetric with respect to $x_0 := \frac{h'-h}{2}$ for any $\tau < \overline{\tau}_0$.

(iii) $v(x_0(\tau),\tau) = \max_{x\in\mathbb{R}} v(x,\tau) \approx \xi_k(\tau)$ as $\tau \to -\infty$.

(iv) $v_{\lambda,\lambda',h,h',k}$ is increasing in h and h' and decreasing in $0 < k \le k_0$.

(v) As $\tau \to -\infty$, we have:

(a) If $c > \lambda$, then $v(x + c\tau,\tau) \to 0$ uniformly on $(-\infty, A]$ for any $A \in \mathbb{R}$.

(b) If $-\lambda' < c < \lambda$, then $v(x + c\tau,\tau) \to 1$ uniformly in any compact subset of $\mathbb{R}$.

(c) If $c < -\lambda'$, then $v(x + c\tau,\tau) \to 0$ uniformly on $[A,\infty)$ for any $A \in \mathbb{R}$.

(d) $v(x + \lambda\tau,\tau) \to v_\lambda(x + h)$ uniformly on $(-\infty, A]$ for any $A \in \mathbb{R}$.

(e) $v(x-\lambda'\tau,\tau)\to v_{\lambda'}(-x+h')$ *uniformly on* $[A,\infty)$ *for any* $A\in\mathbb{R}$.

Theorem 1.2. *There exists a constant* $\overline{\tau}_0<\tau_0$ *such that for any* $\lambda>1$, $\lambda'>1$, $h\ge h_0$, $h'\ge h'_0$, *there exists a unique solution* $v=v_{\lambda,\lambda',h,h'}\in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ *of* (1.6) *in* $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ *which satisfies*

$$\underline{f}_{\lambda,\lambda',h,h'}(x,\tau)\le v_{\lambda,\lambda',h,h'}(x,\tau)\le \overline{f}_{\lambda,\lambda',h,h'}(x,\tau)\quad \forall x\in\mathbb{R},\tau<\overline{\tau}_0. \tag{1.26}$$

and

$$v_{\lambda,\lambda',h,h'}(x,\tau)\ge v_{\lambda,\lambda',h,h',k}(x,\tau)\quad \forall x\in\mathbb{R},\tau<\overline{\tau}_0,0<k\le k_0 \tag{1.27}$$

where $v_{\lambda,\lambda',h,h',k}$ *is as constructed in Theorem 1.1. Moreover the following holds.*

(i) *For any* $x\in\mathbb{R}$, $v(x,\tau)$ *is a decreasing function of* τ *and* $v(x,\tau)\to 1$ *as* $\tau\to-\infty$.

(ii) *For any* $\tau<\overline{\tau}_0$, $v(x,\tau)\to 0$ *as* $|x|\to\infty$ *and there exists* $x_0(\tau)\in\mathbb{R}$ *such that* $v_x(x_0(\tau),\tau)=0$, $v_x(x,\tau)>0$ *if* $x<x_0(\tau)$ *and* $v_x(x,\tau)<0$ *if* $x>x_0(\tau)$. *Further more if* $\lambda=\lambda'$, *then* $x_0(\tau)=\frac{h'-h}{2}$ *and* $v(x,t)$ *is symmetric with respect to* $x_0:=\frac{h'-h}{2}$ *for any* $\tau<\overline{\tau}_0$.

(iii) $v(x_0(\tau),\tau)=\max_{x\in\mathbb{R}}v(x,\tau)\approx\max_{x\in\mathbb{R}}\overline{f}_{\lambda,\lambda',h,h'}(x,\tau)$ *as* $\tau\to-\infty$.

(iv) $v_{\lambda,\lambda',h,h'}$ *is increasing in* h *and* h'.

(v) *As* $\tau\to-\infty$, *we have:*

  (a) *If* $c>\lambda$, *then* $v(x+c\tau,\tau)\to 0$ *uniformly on* $(-\infty,A]$ *for any* $A\in\mathbb{R}$.
  (b) *If* $-\lambda'<c<\lambda$, *then* $v(x+c\tau,\tau)\to 1$ *uniformly in any compact subset of* $\mathbb{R}$.
  (c) *If* $c<-\lambda'$, *then* $v(x+c\tau,\tau)\to 0$ *uniformly on* $[A,\infty)$ *for any* $A\in\mathbb{R}$.
  (d) $v(x+\lambda\tau,\tau)\to v_\lambda(x+h)$ *uniformly on* $(-\infty,A]$ *for any* $A\in\mathbb{R}$.
  (e) $v(x-\lambda'\tau,\tau)\to v_{\lambda'}(-x+h')$ *uniformly on* $[A,\infty)$ *for any* $A\in\mathbb{R}$.

(vi) *As* $h\to\infty$, $v_{\lambda,\lambda',h,h'}$ *converges uniformly on* $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ *for any* $A\in\mathbb{R}$ *and* $\tau_1<\overline{\tau}_0$ *to* $\overline{v}_{\lambda',h'}(x,f(\tau))$. *Similarly as* $h'\to\infty$, $v_{\lambda,\lambda',h,h'}$ *converges uniformly on* $(-\infty,A]\times[\tau_1,\overline{\tau}_0]$ *for any* $A\in\mathbb{R}$ *and* $\tau_1<\overline{\tau}_0$ *to* $v_{\lambda,h}(x,f(\tau))$.

(vii) *As* $k\to 0$, *the solution* $v_{\lambda,\lambda',h,h',k}$ *of* (1.6) *in* $\mathbb{R}\times(-\infty,\tau_0)$ *given by Theorem 1.1 increases and converges uniformly in* $C^{2,1}(K)$ *for any every compact subset* K *of* $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ *to the unique solution* $v_{\lambda,\lambda',h,h'}$ *of* (1.6) *in* $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ *which satisfies* (1.26).

Theorem 1.3. *There exists a constant* $\overline{\tau}_0<\tau_0$ *such that for any* $\lambda>1$, $h\ge h_0$ *and* $0<k\le k_0$, *there exists a solution* $v=v_{\lambda,h,k}\in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ *of* (1.6) *in* $\mathbb{R}\times(-\infty,\tau_0)$ *which satisfies*

$$\underline{f}_{\lambda,h,k}(x,\tau)\le v_{\lambda,h,k}(x,\tau)\le\overline{f}_{\lambda,h,k}(x,\tau)\quad\forall x\in\mathbb{R},\tau<\overline{\tau}_0 \tag{1.28}$$

and

$$v_{\lambda,h,k}(x,\tau)\ge v_{\lambda,\lambda',h,h',k}(x,\tau)\quad\forall x\in\mathbb{R},\tau<\overline{\tau}_0,\lambda'>1,h'\ge h'_0. \tag{1.29}$$

where $v_{\lambda,\lambda',h,h',k}$ *is as constructed in Theorem 1.1. Moreover the following holds.*

(i) *For any* $x \in \mathbb{R}$, $v(x,\tau)$ *is a decreasing function of* τ *and* $v(x,\tau) \to 1$ *as* $\tau \to -\infty$.

(ii) *For any* $\tau < \overline{\tau}_0$, $v_x(x,\tau) > 0$ *for any* $x \in \mathbb{R}$.

(iii) *For any* $\tau < \overline{\tau}_0$, $v(x,t) \to 0$ *as* $x \to -\infty$ *and* $v(x,t) \to \xi_k(\tau)$ *as* $x \to \infty$.

(iv) $v_{\lambda,h,k}$ *is increasing in* h *and decreasing in* $0 < k \le k_0$.

(v) *As* $\tau \to -\infty$, *we have:*

> *(a)* *If* $c > \lambda$, *then* $v(x + c\tau, \tau) \to 0$ *uniformly on* $(-\infty, A]$ *for any* $A \in \mathbb{R}$.
>
> *(b)* *If* $c < \lambda$, *then* $v(x + c\tau, \tau) \approx \xi_k(\tau)$ *uniformly on* $[A, \infty)$ *for any* $A \in \mathbb{R}$.
>
> *(c)* $v(x + \lambda\tau, \tau) \to v_\lambda(x + h)$ *uniformly on any compact subset of* $\mathbb{R}$.

(vi) *As* $k \to 0$, $v_{\lambda,h,k}$ *converges uniformly on every compact subset of* $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ *to* $v_{\lambda,h}(x, f(\tau))$.

(vii) *As* $h \to \infty$, $v_{\lambda,h,k}$ *converges to* ξ_k *uniformly on* $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ *for any* $A \in \mathbb{R}$ *and* $\tau_1 < \overline{\tau}_0$.

(viii) *As* $h' \to \infty$, *the solution* $v_{\lambda,\lambda',h,h',k}$ *of* (1.6) *in* $\mathbb{R}\times(-\infty,\tau_0)$ *given by Theorem 1.1 increases and converges uniformly in* $C^{2,1}(K)$ *for any every compact subset* K *of* $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ *to a solution* v *of* (1.6) *in* $\mathbb{R}\times(-\infty,\tau_0)$ *which satisfies* (1.28) *and* (1.29) *with* $v_{\lambda,h,k}$ *there being replaced by* v.

Remark 1.4. *By (iii) and (v) of Theorem 1.1, (v) of Theorem 1.2 and (iii) and (v) of Theorem 1.3, for any* $\lambda_1 > 1$, $\lambda_2 > 1$, $\lambda_1' > 1$, $\lambda_2' > 1$, $h_1 \ge h_0$, $h_1' \ge h_0'$, $h_2 \ge h_0'$, $h_2' \ge h_0'$, $k_1, k_2 \in (0, k_0]$, *the following holds.*

(i) $v_{\lambda_1,\lambda_1',h_1,h_1',k_1} \ne v_{\lambda_2,\lambda_2',h_2,h_2',k_2}$ *if* $(\lambda_1,\lambda_1',h_1,h_1',k_1) \ne (\lambda_2,\lambda_2',h_2,h_2',k_2)$

(ii) $v_{\lambda_1,\lambda_1',h_1,h_1'} \ne v_{\lambda_2,\lambda_2',h_2,h_2'}$ *if* $(\lambda_1,\lambda_1',h_1,h_1') \ne (\lambda_2,\lambda_2',h_2,h_2')$

(iii) $v_{\lambda_1,h_1,k_1} \ne v_{\lambda_2,h_2,k_2}$ *if* $(\lambda_1,h_1,k_1) \ne (\lambda_2,h_2,k_2)$.

Remark 1.5. *By choosing* $\overline{\tau}_0$ *to be sufficiently small we may assume without loss of generality that the constant* $\overline{\tau}_0$ *is the same in Theorem 1.1, Theorem 1.2 and Theorem 1.3.*

Remark 1.6. *Existence of* 4*-parameters and* 5*-parameters solutions of* (1.6) *are also constructed in [DPKS1] and [DPKS2]. However properties (ii)–(v) of Theorem 1.1 for the* 5*-parameters solutions of* (1.6), *properties (ii)–(vii) of Theorem 1.2 for the* 4*-parameters solutions of* (1.6) *and the uniqueness of the* 4*-parameters solutions* $v_{\lambda,\lambda',h,h'}$ *in Theorem 1.2 are new results.*

The plan of the paper is as follows. In section 2 we will prove the comparison principle and local existence solutions of (1.6). In section 3 we will prove the existence of various ancient solutions and various properties of the ancient solutions of (1.6) stated in Theorem 1.1, Theorem 1.2 and Theorem 1.3.

We start with some definitions. For any $R > 0$ and $l \in \mathbb{N}$, let $B_R = \{x \in \mathbb{R}^l : |x| \le R\}$. For any $\tau_2 > \tau_1$, we say that v is a solution of (1.6) in $\mathbb{R}\times(\tau_1,\tau_2)$ ($B_R \times (\tau_1,\tau_2)$, respectively) if $0 < v \in C^2(\mathbb{R})$ ($0 < v \in C^2(B_R)$, respectively) is a classical solution of (1.6) in $\mathbb{R} \times (\tau_1, \tau_2)$ ($B_R \times (\tau_1, \tau_2)$, respectively). We say that v is a subsolution (supersolution, respectively) of (1.6) in $\mathbb{R} \times (\tau_1, \tau_2)$ if $0 < v \in C(\mathbb{R} \times (\tau_1, \tau_2)) \cap L^\infty(\mathbb{R} \times (\tau_1, \tau_2))$ satisfies

$$\int_{B_R} v(x,\tau_4)\eta(x,\tau_4)\,dx \le \int_{\tau_3}^{\tau_4}\int_{B_R} [v\eta_\tau + v^m\eta_{xx} + (v - v^m)\eta]\,dx\,d\tau - \int_{\tau_3}^{\tau_4}\int_{\partial B_R} v^m \frac{\partial \eta}{\partial n}\,d\sigma\,d\tau + \int_{B_R} v(x,\tau_3)\eta(x,\tau_3)\,dx \quad \forall \tau_1 < \tau_3 < \tau_4 < \tau_2$$

($\ge$, respectively) for any $R > 0$ and function $0 \le \eta \in C^{2,1}(\overline{B}_R \times [\tau_3, \tau_4])$ satisfying $\eta \equiv 0$ on $\partial B_R \times [\tau_3, \tau_4]$ where $\partial/\partial n$ is the exterior normal derivative with respect to the unit outward normal on ∂B_R. For any $0 \le v_0 \in L^1_{loc}(\mathbb{R})$, we say that a solution (subsolution, supersolution, respectively) v of (1.6) in $\mathbb{R} \times (\tau_1, \tau_2)$ has initial value v_0 at τ_1 if $v(\cdot, \tau) \to v_0$ in $L^1_{loc}(\mathbb{R})$ as $\tau \searrow \tau_1$.

For any $R > 0$, $0 \le v_0 \in L^\infty(-R, R)$, $0 \le g_0 \in L^\infty(\{\pm R\} \times (\tau_1, \tau_2))$ we say that v is a solution (subsolution, supersolution, respectively) of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R) \times (\tau_1,\tau_2) \\ v = g_0 & \text{on } \{\pm R\} \times [\tau_1,\tau_2) \\ v(x,\tau_1) = v_0(x) & \text{on } (-R,R) \end{cases} \tag{1.30}$$

if $0 < v \in C((-R, R) \times (\tau_1, \tau_2)) \cap L^\infty((-R, R) \times (\tau_1, \tau_2))$ satisfies

$$\int_{-R}^{R} v(x,\tau_3)\eta(x,\tau_3)\,dx = \int_{\tau_1}^{\tau_3}\int_{-R}^{R} [v\eta_\tau + v^m\eta_{xx} + (v - v^m)\eta]\,dx\,d\tau - \int_{\tau_1}^{\tau_3}\int_{\partial B_R} g_0^m \frac{\partial \eta}{\partial n}\,d\sigma\,d\tau + \int_{-R}^{R} v_0(x)\eta(x,\tau_1)\,dx \quad \forall \tau_1 < \tau_3 < \tau_2 \tag{1.31}$$

($\le, \ge$, respectively) for any function $0 \le \eta \in C^{2,1}(\overline{B}_R \times [\tau_1, \tau_3])$ satisfying $\eta \equiv 0$ on $\{\pm R\} \times [\tau_1, \tau_3]$ where $\partial/\partial n$ is the derivative with respect to the unit normal on ∂B_R.

For any $R > 0$, $0 \le v_0 \in L^\infty(-R, R)$, $g_0 \in L^\infty(\{\pm R\} \times [\tau_1, \tau_2))$ we say that v is a solution (subsolution, supersolution, respectively) of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R) \times (\tau_1,\tau_2) \\ \dfrac{\partial v^m}{\partial n} = g_0 & \text{on } \{\pm R\} \times [\tau_1,\tau_2) \\ v(x,\tau_1) = v_0(x) & \text{on } (-R,R) \end{cases} \tag{1.32}$$

if $0 < v \in C((-R, R) \times (\tau_1, \tau_2)) \cap L^\infty((-R, R) \times (\tau_1, \tau_2))$ satisfies

$$\int_{-R}^{R} v(x,\tau_3)\eta(x,\tau_3)\,dx = \int_{\tau_1}^{\tau_3}\int_{-R}^{R} [v\eta_\tau + v^m\eta_{xx} + (v - v^m)\eta]\,dx\,d\tau + \int_{\tau_1}^{\tau_3}\int_{\partial B_R} g_0\eta\,d\sigma\,d\tau + \int_{-R}^{R} v_0(x)\eta(x,\tau_1)\,dx \quad \forall \tau_1 < \tau_3 < \tau_2$$

($\leq, \geq$, respectively) for any function $0 \leq \eta \in C^{2,1}(\overline{B}_R \times [\tau_1, \tau_3])$ satisfying $\eta_x \equiv 0$ on $\{\pm R\} \times [\tau_1, \tau_3]$.

For any set $A \subset \mathbb{R}$, we let χ_A be the characteristic function of the set A. For any $a \in \mathbb{R}$, we let $a_+ = \max(a, 0)$ and $a_- = -\min(a, 0)$.

Remark 1.7. *Suppose v is a solution of (1.6) in $B_R \times (\tau_1, \tau_2)$. Then since $\inf_{B_{R'} \times [\tau_3, \tau_4]} v > 0$ for any $0 < R' < R$ and $\tau_1 < \tau_3 < \tau_4 < \tau_2$, the equation (1.6) for v is uniformly parabolic on every compact subset of $B_R \times (\tau_1, \tau_2)$. Hence by the standard Schauder estimates [LSU] and a bootrap argument $v \in C^\infty(B_R \times (\tau_1, \tau_2))$.*

2 Local existence and comparison principles

In this section we will prove various local existence and comparison principles for the solutions of (1.6).

Lemma 2.1. *For any $\lambda > 1$,*

$$v_\lambda'(x) > 0 \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Hence

$$0 < v_\lambda(x) < 1 \quad \forall x \in \mathbb{R}, \lambda > 1. \tag{2.2}$$

Proof: This result is used without proof in [DPKS1]. For the sake of completeness we will give a proof of this result here. By (1.12) and Lemma 3.1 of [H3], for any $\lambda > 1$, $x \in R$,

$$\begin{aligned}
& v_\lambda(x) = \frac{(r^2 f(r)^{1-m})^{\frac{n+2}{4}}}{[(n-1)(n-2)]^{\frac{1}{1-m}}} \\
\Rightarrow \quad & v_\lambda'(x) = \frac{(n+2)(r^2 f(r)^{1-m})^{\frac{n-6}{4}}}{4[(n-1)(n-2)]^{\frac{1}{1-m}}} (2rf(r)^{1-m} + (1-m)r^2 f(r)^{-m} f'(r)) \\
& = \frac{rf(r)^{1-m}(r^2 f(r)^{1-m})^{\frac{n-6}{4}}}{[(n-1)(n-2)]^{\frac{1}{1-m}}} \left(\frac{2}{1-m} + \frac{rf'(r)}{f(r)} \right) \\
& > 0
\end{aligned}$$

where $r = e^{\frac{2x}{n-2}}$. By (1.13) and (2.1) we get (2.2) and the lemma follows. □

Lemma 2.2. *Let $\tau_1 < \tau_2$, $M > 0$, and $v_1, v_2 \in C(\mathbb{R} \times (\tau_1, \tau_2)) \cap L^\infty(\mathbb{R} \times (\tau_1, \tau_2))$, be subsolution and supersolution of (1.6) in $\mathbb{R} \times (\tau_1, \tau_2)$ with initial values $v_{0,1}, v_{0,2} \in L^\infty(\mathbb{R})$, at τ_1 respectively such that $0 < v_i \leq M$ in $\mathbb{R} \times [\tau_1, \tau_2)$ for $i = 1, 2$ and for any constant $R > 0$, there exists a constant $C_R > 0$ such that*

$$v_i(x, \tau) \geq C_R \quad \forall |x| \leq R, \tau_1 < \tau < \tau_2, i = 1, 2. \tag{2.3}$$

Then

$$\int_{\mathbb{R}} (v_1(x, \tau) - v_2(x, \tau))_+ \, dx \leq \int_{\mathbb{R}} e^{(1-mM^{m-1})(\tau-\tau_1)} (v_{0,1}(x) - v_{0,2}(x))_+ \, dx \tag{2.4}$$

hold for any $\tau_1 \le \tau < \tau_2$ *if* $(v_{0,1} - v_{0,2})_+ \in L^1(\mathbb{R})$. *Hence if* $v_{0,1}(x) \le v_{0,2}(x)$ *a.e.* $x \in \mathbb{R}$, *then*

$$v_1(x,\tau) \le v_2(x,\tau) \quad \forall x \in \mathbb{R}, \tau_1 < \tau < \tau_2. \tag{2.5}$$

If v_1, v_2, *are also solutions of* (1.6) *in* $\mathbb{R}^n \times (\tau_1, \tau_2)$ *and* $v_{0,1} - v_{0,2} \in L^1(\mathbb{R})$, *then*

$$\int_{\mathbb{R}} |v_1(x,\tau) - v_2(x,\tau)|\, dx \le \int_{\mathbb{R}} e^{(1-mM^{m-1})(\tau-\tau_1)} |v_{0,1}(x) - v_{0,2}(x)|\, dx \tag{2.6}$$

holds for any $\tau_1 \le \tau < \tau_2$.

Proof: We will use a modification of the proof of Lemma 2.3 of [DK] and Theorem 2.1 of [PV] to prove the lemma. We first suppose that $(v_{0,1} - v_{0,2})_+ \in L^1(\mathbb{R})$. Let $\tau_1 < \tau_3 < \tau_2$, $R_0 > 0$, $\theta \in C_0^\infty(\mathbb{R})$ such that $0 \le \theta \le 1$ in $\mathbb{R}$ and supp $\theta \subset B_{R_0}$. Let

$$A = \begin{cases} \dfrac{v_1^m - v_2^m}{v_1 - v_2} & \text{if } v_1 \ne v_2 \\ m v_1^{m-1} & \text{if } v_1 = v_2 \end{cases}$$

Then

$$mM^{m-1} \le A(x,\tau) \le mC_R^{m-1} \quad \forall |x| \le R, \tau_1 \le \tau < \tau_2, R > 0.$$

We choose a sequence of functions $\{A_k\}_{k=1}^\infty \subset C^\infty(\mathbb{R} \times [\tau_1, \tau_2))$ such that A_k converges uniformly to A on every compact subset of $\mathbb{R} \times (\tau_1, \tau_2)$ as $k \to \infty$ and

$$mM^{m-1} \le A_k(x,\tau) \le mC_{R+1}^{m-1} \quad \forall |x| \le R, \tau_1 \le \tau \le \tau_2, k \in \mathbb{N}, R > 0. \tag{2.7}$$

For any $R > R_0 + 2$, $k \in \mathbb{N}$, let $\eta_{R,k} \in C^\infty(\overline{B}_R \times [\tau_1, \tau_3])$ be the solution of

$$\begin{cases} \eta_\tau + A_k \eta_{xx} + (1 - A_k)\eta = 0 & \text{in } (-R,R) \times (\tau_1, \tau_3) \\ \eta = 0 & \text{on } \{\pm R\} \times [\tau_1, \tau_3] \\ \eta(x, \tau_3) = \theta(x) & \forall x \in (-R, R). \end{cases} \tag{2.8}$$

By the maximum principle, $\eta_{R,k} \ge 0$ in $[-R,R] \times [\tau_1, \tau_3]$. Hence $\partial \eta_{R,k}/\partial n \le 0$ on $\{\pm R\} \times [\tau_1, \tau_3)$. Then by (2.8),

$$\begin{aligned} & (\eta_{R,k})_\tau + A_k (\eta_{R,k})_{xx} + (1 - mM^{m-1})\eta_{R,k} \ge 0 \quad \text{in } (-R,R) \times (\tau_1, \tau_3) \\ \Rightarrow \quad & (e^{(1-mM^{m-1})\tau} \eta_{R,k})_\tau + A_k (e^{(1-mM^{m-1})\tau} \eta_{R,k})_{xx} \ge 0 \quad \text{in } (-R,R) \times (\tau_1, \tau_3) \\ \Rightarrow \quad & 0 \le \eta_{R,k}(x,\tau) \le e^{(1-mM^{m-1})(\tau_3 - \tau)} \theta(x) \le e^{(1-mM^{m-1})(\tau_3 - \tau)} \quad \forall |x| \le R, \tau_1 \le \tau \le \tau_3. \end{aligned} \tag{2.9}$$

By (2.8),

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial \tau} \int_{\mathbb{R}} \eta_{R,k,x}^2\, dx &= \int_{\mathbb{R}} \eta_{R,k,x} (\eta_{R,k})_{x,\tau}\, dx = -\int_{\mathbb{R}} (\eta_{R,k})_{xx} \eta_{R,k,\tau}\, dx \\ &= \int_{\mathbb{R}} A_k \eta_{R,k,xx}^2\, dx + \int_{\mathbb{R}} (1 - A_k) \eta_{R,k} \eta_{R,k,xx}\, dx. \end{aligned}$$

Hence

$$
\begin{aligned}
\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}} A_k\eta_{R,k,xx}^2\,dx\,dt \le& \frac{1}{2}\int_{\mathbb{R}}\theta_x^2\,dx+\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}}(A_k-1)\eta_{R,k}\eta_{R,k,xx}\,dx\,dt\\
\le& \frac{1}{2}\int_{\mathbb{R}}\theta_x^2\,dx+\frac{1}{2}\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}}A_k\eta_{R,k,xx}^2\,dx\,dt+\frac{1}{2}\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}}\frac{(A_k-1)^2}{A_k^2}\eta_{R,k}^2\,dx\,dt.
\end{aligned}
\tag{2.10}
$$

Thus by (2.7), (2.9) and (2.10),

$$
\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}} A_k\eta_{R,k,xx}^2\,dx\,dt \le \int_{\mathbb{R}}\theta_x^2\,dx+\int_{\tau_1}^{\tau_3}\int_{\mathbb{R}}\frac{(A_k-1)^2}{A_k^2}\eta_{R,k}^2\,dx\,dt \le \int_{\mathbb{R}}\theta_x^2\,dx+C_R' \tag{2.11}
$$

for some constant $C_R'>0$. Since v_1, v_2, are subsolution and supersolution of (1.6) in $\mathbb{R}\times(\tau_1,\tau_2)$, by (2.9) and (2.11),

$$
\begin{aligned}
&\int_{B_{R_0}}(v_1-v_2)(x,\tau_3)\theta(x)\,dx\\
\le&\int_{\tau_1}^{\tau_3}\int_{B_R}(v_1-v_2)\left[\eta_{R,k,\tau}+A(\eta_{R,k})_{xx}+(1-A)\eta_{R,k}\right]dx\,dt+\int_{B_R}(v_{0,1}(x)-v_{0,2}(x))\eta_{R,k}(x,\tau_1)\,dx\\
&\quad+\int_{\tau_1}^{\tau_3}\int_{\partial B_R}(v_2^m-v_1^m)\frac{\partial\eta_{R,k}}{\partial n}\,d\sigma\,dt\\
\le&\int_{\tau_1}^{\tau_3}\int_{B_R}(v_1-v_2)\left[(A-A_k)(\eta_{R,k})_{xx}+(A_k-A)\eta_{R,k}\right]dx\,dt+\int_{B_R}(v_{0,1}(x)-v_{0,2}(x))\eta_{R,k}(x,\tau_1)\,dx\\
&\quad+\int_{\tau_1}^{\tau_3}\int_{\partial B_R}v_1^m\left|\frac{\partial\eta_{R,k}}{\partial n}\right|d\sigma\,dt\\
\le&2M\left(\int_{\tau_1}^{\tau_3}\int_{B_R}\frac{(A-A_k)^2}{A_k}\,dx\,dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\theta_x^2\,dx+C_R'\right)^{\frac{1}{2}}+C_1\int_{\tau_1}^{\tau_3}\int_{B_R}|A_k-A|\,dx\,dt\\
&\quad+\int_{B_R}(v_{0,1}(x)-v_{0,2}(x))\eta_{R,k}(x,\tau_1)\,dx+M^m\int_{\tau_1}^{\tau_3}\int_{\partial B_R}\left|\frac{\partial\eta_{R,k}}{\partial n}\right|d\sigma\,dt.
\end{aligned}
\tag{2.12}
$$

for some constant $C_1>0$ where $\partial/\partial n$ is the exterior derivative with respect to the unit outward normal on ∂B_R. We will now estimate the last term on the right hand side of (2.12). As observed in [DPKS1] the function

$$
\phi_0(x)=\left(\frac{k_n e^{\frac{2x}{n-2}}}{1+e^{\frac{4x}{n-2}}}\right)^{\frac{n+2}{2}},\quad k_n=\sqrt{\frac{4n}{n-2}},
$$

satisfies (1.15) with $\lambda=0$. Hence

$$
m\phi_{0,xx}=-\phi_0^{2-m}+\phi_0+m(1-m)\phi_0^{-1}\phi_{0,x}^2\quad\text{in }\mathbb{R}. \tag{2.13}
$$

Note that $\phi_0(x)$ is an even function and it is monotone decreasing in $x \geq 0$. Moreover $\phi_0(x) \leq C_2 e^{-p|x|}$ for any $x \in \mathbb{R}$ where $C_2 = k_n^{\frac{n+2}{2}}$ and

$$
\begin{aligned}
&\phi_{0,x}(x) = \frac{n+2}{n-2} \cdot \left(\frac{1 - e^{\frac{4x}{n-2}}}{1 + e^{\frac{4x}{n-2}}} \right) \phi_0(x) \quad \forall x \in \mathbb{R} \\
\Rightarrow \quad &\phi_{0,x}(x) \approx -p \operatorname{sign}(x) \phi_0(x) \quad \text{ as } |x| \to \infty \quad \text{ and } \quad |\phi_{0,x}(x)| \leq p\phi_0(x) \quad \forall x \in \mathbb{R}. \qquad (2.14)
\end{aligned}
$$

Let $\phi(x,\tau) = K_1 e^{\tau_3 - \tau} \phi_0(mx)$ where $K_1 = \phi_0(mR_0)^{-1}$. We claim that ϕ is a supersolution of (2.8). To prove the claim we observe that by (2.13) and (2.14),

$$
\begin{aligned}
\phi_{xx}(x,\tau) =& m^2 K_1 e^{\tau_3-\tau} \phi_{0,xx}(mx) \\
=& mK_1 e^{\tau_3-\tau} \left[-\phi_0(mx)^{2-m} + \phi_0(mx) + m(1-m)\phi_0(mx)^{-1} \phi_{0,x}(mx)^2 \right] \\
\leq& m\phi(x,\tau) \left[1 + m(1-m)\phi_0(mx)^{-2} \phi_{0,x}(mx)^2 \right] \\
\leq& \phi(x,\tau) \quad \forall x \in \mathbb{R}, \tau_1 \leq \tau \leq \tau_3.
\end{aligned}
$$

Hence

$$
\phi_\tau + A_k \phi_{xx} + (1-A_k)\phi \leq (-1 + A_k + (1-A_k))\,\phi = 0 \quad \forall x \in \mathbb{R}, \tau_1 \leq \tau \leq \tau_3. \qquad (2.15)
$$

Now

$$
\phi(x,\tau_3) = \frac{\phi_0(mx)}{\phi_0(mR_0)} \geq 1 \geq \theta(x) \quad \forall |x| \leq R_0. \qquad (2.16)
$$

By (2.15) and (2.16), ϕ is a supersolution of (2.8). Hence by the maximum principle,

$$
0 \leq \eta_{R,k}(x,\tau) \leq \phi(x,\tau) \leq C_2 K_1 e^{\tau_3-\tau} e^{-|x|} \quad \forall |x| \leq R, \tau_1 \leq \tau \leq \tau_3, k \in \mathbb{N}. \qquad (2.17)
$$

Let

$$
H(x,\tau) = K_1 e^{\tau_3-\tau} (R - |x|) \phi_0(m(R-1)).
$$

Then by (2.17),

$$
\begin{cases}
H_\tau + A_k H_{xx} + (1-A_k)H = -A_k H \leq 0 & \forall R-1 \leq |x| \leq R, \tau_1 \leq \tau \leq \tau_3 \\
H(x,\tau) = \phi(R-1,\tau) \geq \eta_{R,k}(x,\tau) & \forall |x| = R-1, \tau_1 \leq \tau \leq \tau_3 \\
H(x,\tau) \geq 0 = \eta_{R,k}(x,\tau) & \forall |x| = R, \tau_1 \leq \tau \leq \tau_3 \\
H(x,\tau_3) \geq 0 = \eta_{R,k}(x,\tau_3) & \forall R-1 \leq |x| \leq R.
\end{cases}
$$

Hence by the maximum principle in $(B_R \setminus B_{R-1}) \times (\tau_1, \tau_3)$,

$$
\begin{aligned}
&0 \leq \eta_{R,k}(x,\tau) \leq H(x,\tau) \quad \forall R-1 \leq |x| \leq R, \tau_1 \leq \tau \leq \tau_3, k \in \mathbb{N} \\
\Rightarrow \quad &0 \geq \frac{\partial \eta_{R,k}}{\partial n}(x,\tau) \geq \frac{\partial H}{\partial n}(x,\tau) \geq -e^{\tau_3-\tau} K_1 \phi_0(m(R-1)) \geq -C_3 e^{\tau_3-\tau} e^{-R} \qquad (2.18)
\end{aligned}
$$

for any $|x| = R > R_0 + 2$ and $\tau_1 \le \tau \le \tau_3$ where $C_3 = ek_n^{\frac{n+2}{2}}\phi_0(mR_0)^{-1} > 0$. Letting $k \to \infty$ in (2.12), by (2.9), (2.17) and (2.18), for any $R_0 > 0$, $R > R_0 + 2$ and $\theta \in C_0^\infty(B_{R_0})$ such that $0 \le \theta(x) \le 1$ for any $x \in \mathbb{R}$, we have for any $\tau_1 < \tau_3 < \tau_2$,

$$\begin{aligned} &\int_{B_{R_0}} (v_1(x,\tau_3) - v_2(x,\tau_3))\theta(x)\,dx \\ &\le \int_{B_R} e^{(1-mM^{m-1})(\tau-\tau_1)}(v_{0,1}(x) - v_{0,2}(x))_+\,dx + 2(\tau_3-\tau_1)C_3M^m e^{\tau_3-\tau_1}e^{-R} \\ \Rightarrow\quad &\int_{B_{R_0}} (v_1(x,\tau_3) - v_2(x,\tau_3))\theta(x)\,dx \le \int_{\mathbb{R}} e^{(1-mM^{m-1})(\tau-\tau_1)}(v_{0,1}(x) - v_{0,2}(x))_+\,dx \quad \text{as } R \to \infty. \end{aligned} \tag{2.19}$$

We now choose a sequence of functions $\{\theta_i\}_{I=1}^\infty \subset C_0^\infty(B_{R_0})$, $0 \le \theta_i \le 1$ for all $i \in \mathbb{N}$, such that $\theta_i(x) \nearrow (\text{ sign}(v_1(x) - v_2(x)))\chi_{B_{R_0}}(x)$ in B_{R_0} as $i \to \infty$. Putting $\theta = \theta_i$ and letting first $i \to \infty$ and then $R_0 \to \infty$ in (2.19), (2.4) follows. Hence if $v_{0,1}(x) \le v_{0,2}(x)$ a.e. $x \in \mathbb{R}$, then (2.5) holds. Similarly if v_1 and v_2 are also solutions of (1.6) in $\mathbb{R}^n \times (\tau_1,\tau_2)$ and $v_{0,1} - v_{0,2} \in L^1(\mathbb{R})$, then for any $\tau_1 \le \tau < \tau_2$,

$$\int_{\mathbb{R}} (v_1(x,\tau) - v_1(x,\tau))_-\,dx \le \int_{\mathbb{R}} e^{(1-mM^{m-1})(\tau-\tau_1)}(v_{0,1}(x) - v_{0,2}(x))_-\,dx. \tag{2.20}$$

By (2.4) and (2.20), (2.6) follows. □

Similarly we have the following two lemmas.

Lemma 2.3. *Let $R > 0$, $\tau_1 < \tau_2$, $v_{0,1}, v_{0,2} \in L^\infty(-R,R)$ be such that $v_{0,2} \ge v_{0,1} \ge 0$ in $(-R,R)$ and let $g_1, g_2 \in L^\infty(\{\pm R\}\times(\tau_1,\tau_2))$ be such that $g_2 \ge g_1 \ge 0$ on $\{\pm R\}\times(\tau_1,\tau_2)$. Let $v_1, v_2 \in C((-R,R)\times(\tau_1,\tau_2)) \cap L^\infty((-R,R)\times(\tau_1,\tau_2))$ be subsolution and supersolution of* (1.30) *in $(-R,R)\times(\tau_1,\tau_2)$ with $v_0 = v_{0,1}, v_{0,2}$ and $g_0 = g_1$, g_2 resepectively. Suppose v_1, v_2, satisfies* (2.3) *for some constant $C_R > 0$. Then*

$$v_1(x,\tau) \le v_2(x,\tau) \quad \forall |x| \le R, \tau_1 < \tau < \tau_2.$$

Lemma 2.4. *Let $R > 0$, $\tau_1 < \tau_2$, $v_{0,1}, v_{0,2} \in L^\infty(-R,R)$ be such that $v_{0,2} \ge v_{0,1} \ge 0$ in $(-R,R)$ and let $g_1, g_2 \in L^\infty(\{\pm R\}\times(\tau_1,\tau_2))$ be such that $g_2 \ge g_1 \ge 0$ on $\{\pm R\}\times(\tau_1,\tau_2)$. Let $v_1, v_2 \in C((-R,R)\times(\tau_1,\tau_2)) \cap L^\infty((-R,R)\times(\tau_1,\tau_2))$ be subsolution and supersolution of* (1.32) *in $(-R,R)\times(\tau_1,\tau_2)$ with $v_0 = v_{0,1}, v_{0,2}$ and $g_0 = g_1$, g_2 resepectively. Suppose v_1, v_2, satisfies* (2.3) *for some constant $C_R > 0$. Then*

$$v_1(x,\tau) \le v_2(x,\tau) \quad \forall |x| \le R, \tau_1 < \tau < \tau_2.$$

Lemma 2.5. *For any $\lambda > 1$, $\lambda' > 1$, $h, h' \in \mathbb{R}$ and $\tau \in \mathbb{R}$. Then there exists a constant $\overline{\tau}_0 < \tau_0$ such that the following holds.*

(i) (cf. [DPKS1]) For any $\tau \le \overline{\tau}_0$, there exists a unique constant $x(\tau) \in \mathbb{R}$ such that

$$\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) = \begin{cases} v_\lambda(x - \lambda f(\tau) + h) & \forall x \le x(\tau) \\ v_{\lambda'}(-x - \lambda' f(\tau) + h') & \forall x \ge x(\tau) \end{cases} \tag{2.21}$$

where

$$x(\tau) = \frac{\gamma_\lambda - \gamma_{\lambda'}}{p}\tau + \frac{1}{\gamma_\lambda + \gamma_{\lambda'}}\left(\log\frac{C_\lambda}{C_{\lambda'}} + h'\gamma_{\lambda'} - h\gamma_\lambda\right) + o(1) \tag{2.22}$$

and

$$\begin{aligned}\overline{f}_{\lambda,\lambda',h,h'}(x(\tau),\tau) =& \max_{x\in\mathbb{R}}\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) = v_{\lambda,h}(x(\tau),f(\tau)) = \overline{v}_{\lambda',h'}(x(\tau),f(\tau))\\ =& 1 - C_{\lambda,\lambda',h,h'}e^{d\tau} + o(e^{d\tau})\end{aligned} \tag{2.23}$$

for some constant $C_{\lambda,\lambda',h,h'} > 0$ *where*

$$d = \frac{\gamma_\lambda\gamma_{\lambda'} + p - 1}{p}. \tag{2.24}$$

(ii) *For any* $\tau \le \overline{\tau}_0$*, there exist unique constants* $y(\tau) < x(\tau) < z(\tau)$ *such that for any* $0 < k \le k_0$*,*

$$\overline{f}_{\lambda,\lambda',h,h',k}(x,\tau) = \begin{cases} v_\lambda(x - \lambda f(\tau) + h) & \forall x \le y(\tau)\\ \xi_k(\tau) & \forall y(\tau) \le x \le z(\tau)\\ v_{\lambda'}(-x - \lambda' f(\tau) + h') & \forall x \ge z(\tau).\end{cases} \tag{2.25}$$

Moreover,

$$\begin{cases} y(\tau) = \dfrac{\gamma_\lambda}{p}\tau + \dfrac{1}{\gamma_\lambda}\log\left(\dfrac{(p-1)C_\lambda}{k}\right) - h + o(1)\\ z(\tau) = -\dfrac{\gamma_{\lambda'}}{p}\tau - \dfrac{1}{\gamma_{\lambda'}}\log\left(\dfrac{(p-1)C_{\lambda'}}{k}\right) + h' + o(1)\end{cases} \quad \text{for } \tau \le \overline{\tau}_0. \tag{2.26}$$

Proof: Since $f(\tau) \approx \tau$ as $\tau \to -\infty$, (i) follows by an argument similar to the proof of Lemma 2.1 of [DPKS1] and we only need to prove (ii). For any $M > 0$ we let

$$x_1(\tau) = \frac{\gamma_\lambda}{p}\tau - M, \quad x_2(\tau) = \left(\frac{\gamma_\lambda - \gamma_{\lambda'}}{p}\right)\tau \quad \text{and} \quad x_3(\tau) = -\frac{\gamma_{\lambda'}}{p}\tau + M.$$

Then

$$x_1(\tau) < x_2(\tau) < x_3(\tau) \quad \forall \tau < 0.$$

Hence by (1.17),

$$\begin{cases} x_1(\tau) - \lambda f(\tau) + h = \dfrac{\gamma_\lambda - \lambda p}{p}\tau + h + o(1) = -\dfrac{(p-1)}{p\gamma_\lambda}\tau - M + h + o(1) \to \infty\\ x_2(\tau) - \lambda f(\tau) + h = \dfrac{\gamma_\lambda - \lambda p - \gamma_{\lambda'}}{p}\tau + h + o(1) = -\dfrac{(\gamma_\lambda\gamma_{\lambda'} + p - 1)}{p\gamma_\lambda}\tau + h + o(1) \to \infty\\ -x_3(\tau) - \lambda' f(\tau) + h' + o(1) = \dfrac{\gamma_{\lambda'} - \lambda' p}{p}\tau - M + h' + o(1) = -\dfrac{(p-1)}{p\gamma_{\lambda'}}\tau - M + h' + o(1) \to \infty\end{cases} \tag{2.27}$$

as $\tau \to -\infty$. Let $0 < k \le k_0$. By (1.19), (1.23) and (2.27), as $\tau \to -\infty$,

$$\begin{aligned} v_\lambda(x_2(\tau) - \lambda f(\tau) + h) =& 1 - C_\lambda e^{\frac{(\gamma_\lambda \gamma_{\lambda'} + p - 1)}{p}\tau - \gamma_\lambda h} + o\left(e^{\frac{(\gamma_\lambda \gamma_{\lambda'} + p - 1)}{p}\tau - \gamma_\lambda h}\right) \\ =& 1 - o\left(e^{\frac{p-1}{p}\tau}\right) \\ >& 1 - \frac{pk}{p-1} e^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right) \\ =& \xi_k(\tau) \end{aligned} \tag{2.28}$$

and similarly,

$$v_{\lambda'}(-x_2(\tau) - \lambda' f(\tau) + h') = 1 - C_{\lambda'} e^{\frac{(\gamma_\lambda \gamma_{\lambda'} + p - 1)}{p}\tau - \gamma_{\lambda'} h'} + o\left(e^{\frac{(\gamma_\lambda \gamma_{\lambda'} + p - 1)}{p}\tau - \gamma_{\lambda'} h'}\right) > \xi_k(\tau) \quad \text{as } \tau \to -\infty. \tag{2.29}$$

By Lemma 2.1, $v_\lambda(x - \lambda f(\tau) + h)$ is a strictly monotone increasing function from 0 to 1 and $v_{\lambda'}(-x - \lambda' f(\tau) + h')$ is a strictly monotone decreasing function from 1 to zero of $x \in \mathbb{R}$. Hence by (1.19), (1.23), (2.28), and (2.29), there exists a constant $\overline{\tau}_0 < \tau_0$ such that for any $\tau \le \overline{\tau}_0$ there exist unique constants $y(\tau) < x(\tau) < z(\tau)$ such that for any $0 < k \le k_0$ (2.25) holds and $y(\tau) < x_2(\tau) < z(\tau)$. By (1.19), (1.23) and (2.27), as $\tau \to -\infty$,

$$v_\lambda(x_1(\tau) - \lambda f(\tau) + h) = 1 - C_\lambda e^{\frac{(p-1)}{p}\tau + \gamma_\lambda(M-h)} + o\left(e^{\frac{(p-1)}{p}\tau + \gamma_\lambda(M-h)}\right) < 1 - \frac{pk}{p-1} e^{\frac{p-1}{p}\tau} + o(e^{\frac{p-1}{p}\tau}) = \xi_k(\tau)$$

if we choose M to be sufficiently large. Hence by choosing $\overline{\tau}_0$ sufficiently small and M sufficiently large,

$$x_1(\tau) < y(\tau) < x_2(\tau) \quad \forall \tau \le \overline{\tau}_0. \tag{2.30}$$

Then by (2.27) and (2.30), $y(\tau) - \lambda f(\tau) + h \to \infty$ as $\tau \to -\infty$. Hence for $\overline{\tau}_0$ sufficiently small and M sufficiently large,

$$\begin{aligned} v_{\lambda,h}(y(\tau), f(\tau)) =& 1 - C_\lambda e^{-\gamma_\lambda(y(\tau) - \lambda f(\tau) + h)} + o\left(e^{-\gamma_\lambda(y(\tau) - \lambda f(\tau) + h)}\right) \\ =& 1 - C_\lambda e^{-\gamma_\lambda(y(\tau) - \lambda\tau + h)} + o\left(e^{-\gamma_\lambda(y(\tau) - \lambda\tau + h)}\right) \\ =& 1 - \frac{pk}{p-1} e^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right) = \xi_k(\tau) \end{aligned} \tag{2.31}$$

holds for any $\tau \le \overline{\tau}_0$. Similarly by (1.19), (1.23), and (2.27), for M large and $\overline{\tau}_0$ sufficiently small, we have

$$\begin{aligned} & x_2(\tau) < z(\tau) < x_3(\tau) \quad \forall \tau \le \overline{\tau}_0 \\ \Rightarrow \quad & -z(\tau) - \lambda' f(\tau) + h' \to \infty \quad \text{as} \quad \tau \to -\infty. \end{aligned}$$

Hence for $\overline{\tau}_0$ sufficiently small and M sufficiently large,

$$\begin{aligned} \overline{v}_{\lambda',h'}(z(\tau), f(\tau)) =& 1 - C_{\lambda'} e^{-\gamma_{\lambda'}(-z(\tau) - \lambda' f(\tau) + h')} + o\left(e^{-\gamma_{\lambda'}(-z(\tau) - \lambda' f(\tau) + h')}\right) \\ =& 1 - C_{\lambda'} e^{-\gamma_{\lambda'}(-z(\tau) - \lambda'\tau + h')} + o\left(e^{-\gamma_{\lambda'}(-z(\tau) - \lambda'\tau + h')}\right) \\ =& 1 - \frac{pk}{p-1} e^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right) = \xi_k(\tau). \end{aligned} \tag{2.32}$$

holds for any $\tau \le \overline{\tau}_0$. By (2.31) and (2.32), (2.26) follows. $\square$

Lemma 2.6. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$, and let $\overline{\tau}_0$ be as in Lemma 2.5. Then for any $0 < k \le k_0$ and $a > -\overline{\tau}_0$, the function $\overline{f}_a(x) := \overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$ satisfies*

$$(v^m)'' + v - v^m \le 0 \quad \text{in distribution sense in } \mathbb{R}. \tag{2.33}$$

Proof: Since $v_{\lambda,h}$ satisfies (1.15), by Lemma 2.1,

$$(v^m_{\lambda,h})_{xx}(x,f(-a)) + v_{\lambda,h}(x,f(-a)) - v_{\lambda,h}(x,f(-a))^m = -\lambda v_\lambda'(x-\lambda f(-a)+h) < 0 \quad \text{in } \mathbb{R}. \tag{2.34}$$

Similarly

$$(\overline{v}^m_{\lambda',h'})_{xx}(x,f(-a)) + \overline{v}_{\lambda',h'}(x,f(-a)) - \overline{v}_{\lambda',h'}(x,f(-a))^m = -\lambda v_{\lambda'}'(-x-\lambda' f(-a)+h') < 0 \quad \text{in } \mathbb{R}. \tag{2.35}$$

Since $0 < \xi_k(\tau) < 1$ for all $\tau < \tau_0$,

$$\xi_{k,xx} + \xi_k - \xi_k^m = \xi_k - \xi_k^m < 0 \quad \text{in } \mathbb{R} \quad \forall \tau < \tau_0. \tag{2.36}$$

By (2.25), (2.34), (2.35) and (2.36),

$$(\overline{f}_a^m)_{xx}(x) + \overline{f}_a(x) - \overline{f}_a(x)^m < 0 \quad \forall x \in \mathbb{R} \setminus \{y(-a), z(-a)\}. \tag{2.37}$$

Let $0 \le \eta \in C_0^2(\mathbb{R})$. For any $0 < \varepsilon < (z(-a)-y(-a))/4$, let $\psi_\varepsilon \in C_0^2(\mathbb{R})$, $0 \le \psi_\varepsilon \le 1$, be such that $\psi_\varepsilon(x) = 1$ for any $x \in (y(-a)-\frac{\varepsilon}{2}, y(-a)+\frac{\varepsilon}{2}) \cup (z(-a)-\frac{\varepsilon}{2}, z(-a)+\frac{\varepsilon}{2})$, $\psi_\varepsilon(x) = 0$ for any $|x-y(-a)| \ge \varepsilon$ and $|x-z(-a)| \ge \varepsilon$, and $|\psi_\varepsilon'(x)| \le C/\varepsilon$ on $\mathbb{R}$ for some constant $C > 0$. Then we can write $\eta = \eta_{1,\varepsilon} + \eta_{2,\varepsilon}$ where $\eta_{1,\varepsilon} = \eta\psi_\varepsilon$ and $\eta_{2,\varepsilon} = \eta(1-\psi_\varepsilon)$. By (2.25) and (2.37),

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\overline{f}_a^m \eta'' + (\overline{f}_a - \overline{f}_a^m)\eta\right] dx \\
&= \int_{\mathbb{R}} \left[\overline{f}_a^m \eta_{1,\varepsilon}'' + (\overline{f}_a - \overline{f}_a^m)\eta_{1,\varepsilon}\right] dx + \int_{\mathbb{R}} \left[\overline{f}_a^m \eta_{2,\varepsilon}'' + (\overline{f}_a - \overline{f}_a^m)\eta_{2,\varepsilon}\right] dx \\
&= -\int_{\mathbb{R}} (\overline{f}_a^m)_x \eta\psi_\varepsilon' \, dx - \int_{\mathbb{R}} (\overline{f}_a^m)_x \eta'\psi_\varepsilon \, dx + \int_{\mathbb{R}} (\overline{f}_a - \overline{f}_a^m)\eta_{1,\varepsilon} \, dx + \int_{\mathbb{R}} \left[(\overline{f}_a^m)_{xx} + \overline{f}_a - \overline{f}_a^m\right]\eta_{2,\varepsilon} \, dx \\
&\le -\int_{y(-a)-\varepsilon}^{y(-a)} (\overline{f}_a^m)_x \eta\psi_\varepsilon' \, dx - \int_{z(-a)}^{z(-a)+\varepsilon} (\overline{f}_a^m)_x \eta\psi_\varepsilon' \, dx - \int_{\mathbb{R}} (\overline{f}_a^m)_x \eta'\psi_\varepsilon \, dx + \int_{\mathbb{R}} (\overline{f}_a - \overline{f}_a^m)\eta_{1,\varepsilon} \, dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.38}$$

Now

$$|I_3| + |I_4| \le C\varepsilon \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.39}$$

$$I_1 = -\int_{y(-a)-\varepsilon}^{y(-a)} (v_\lambda^m)'(x-\lambda f(-a)+h)\eta(x)\psi_\varepsilon'(x) \, dx \to -(v_\lambda^m)_x(y(-a)-\lambda f(-a)+h)\eta(y(-a)) \tag{2.40}$$

as $\varepsilon \to 0$ and

$$I_2 = \int_{z(-a)}^{z(-a)+\varepsilon} (v_{\lambda'}^m)_x(-x - \lambda' f(-a) + h')\eta\psi_\varepsilon'\, dx \to -(v_{\lambda'}^m)_x(-z(-a) - \lambda' f(-a) + h')\eta(z(-a)) \quad (2.41)$$

as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (2.38), by (2.39), (2.40) and (2.41), for any $0 \le \eta \in C_0^2(\mathbb{R})$,

$$\begin{aligned}&\int_{\mathbb{R}} \left[\overline{f}_a^m \eta'' + (\overline{f}_a - \overline{f}_a^m)\eta\right] dx \\ &= -(v_\lambda^m)_x(y(-a) - \lambda f(-a) + h)\eta(y(-a)) - (v_{\lambda'}^m)_x(-z(-a) - \lambda' f(-a) + h')\eta(z(-a)) \le 0.\end{aligned}$$

Hence $\overline{f}_a$ satisfies of (2.33) and the lemma follows. □

By a similar argument we have the following lemma.

Lemma 2.7. *Let $\lambda > 1$, $h \ge h_0$, and let $\overline{\tau}_0$ be as in Lemma 2.5. Then for any $0 < k \le k_0$ and $a > -\overline{\tau}_0$, the function $\overline{f}_{\lambda,h,k}(x,-a)$ satisfies* (2.33).

Lemma 2.8. *(cf. proof of Lemma 2.4 of [DPKS1]) Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and let $\overline{\tau}_0$ be as in Lemma 2.5. Then for any $a > -\overline{\tau}_0$, the function $\overline{f}_{\lambda,\lambda',h,h'}(x,-a)$ satisfies* (2.33).

Lemma 2.9. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$, $0 < k \le k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $R > 0$, $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(-R,R)$ be such that*

$$f_{\lambda,\lambda',h,h',k}(x,-a) \le v_0(x) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad a.e.\ x \in (-R,R). \tag{2.42}$$

Then there exists a unique solution $v_R \in C^{2,1}([-R,R] \times (-a,\overline{\tau}_0]) \cap L^\infty((-R,R) \times (-a,\overline{\tau}_0))$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R) \times (-a,\overline{\tau}_0) \\ v(x,\tau) = f_{\lambda,\lambda',h,h',k}(x,\tau) & \text{on } \{\pm R\} \times (-a,\overline{\tau}_0) \\ v(x,-a) = v_0(x) & \text{on } (-R,R) \end{cases} \tag{2.43}$$

which satisfies

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \le v_R(x,\tau) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \le R, -a < \tau < \overline{\tau}_0. \tag{2.44}$$

Proof: By Lemma 2.3 the solution of (2.43) is unique. Hence we only need to prove existence of solution of (2.43). We will use a modification of proof of Theorem 3.5 of [Hu] to prove the existence of solution of (2.43). We divide the proof into two cases.

Case 1: $v_0 \in C^\infty([-R,R])$ and there exists $0 < \delta < R$ such that $v_0(x) = f_{\lambda,\lambda',h,h',k}(x,-a)$ for all $R - \delta \le |x| \le R$.

Let

$$a_1 = \min_{\substack{|x| \le R \\ -a \le \tau \le \overline{\tau}_0}} f_{\lambda,\lambda',h,h',k}^m(x,\tau) \quad \text{and} \quad M_1 = \|v_0\|_{L^\infty(-R,R)}.$$

Then $0 < a_1 \le M_1^m < 1$. Let $0 < b(s) \in C^\infty(\mathbb{R})$ be a monotone increasing function such that $b(s) = (a_1/3)^{1-p}$ for all $s \le a_1/3$, $b(s) = (3M_1^m)^{1-p}$ for all $s \ge 3M_1^m$, and $b(s) = s^{1-p}$

for all $a_1/2 \le s \le 2M_1^m$. By standard parabolic theory [LSU] there exists a solution $u_R \in C^{2,1}([-R,R]\times[-a,\overline{\tau}_0])$ of

$$\begin{cases} pu_\tau = b(u)u_{xx} + u - ub(u) & \text{in } (-R,R)\times(-a,\overline{\tau}_0)\\ u(x,\tau) = f_{\lambda,\lambda',h,h',k}(x,\tau)^m & \text{on } \{\pm R\}\times(-a,\overline{\tau}_0)\\ u(x,-a) = v_0(x)^m & \text{on } (-R,R). \end{cases}$$

Since $u_R \in C^{2,1}([-R,R]\times[-a,\overline{\tau}_0])$, there exists $\varepsilon \in (0,\overline{\tau}_0+a)$ such that $a_1/2 \le u_R(x,\tau) \le 2M_1^m$ for any $|x| \le R$, $-a \le \tau \le -a+\varepsilon$. Hence $b(u_R(x,\tau)) = u_R(x,\tau)^{1-p}$ for any $|x| \le R$, $-a \le \tau \le -a+\varepsilon$. Thus u_R satisfies

$$pu_\tau = u^{1-p}u_{xx} + u - u^{2-p} \quad \text{in } (-R,R)\times(-a,-a+\varepsilon). \tag{2.45}$$

Let $v_R = u_R^p$. Then $v_R \in C^{2,1}([-R,R]\times[-a,-a+\varepsilon])$ satisfies

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R)\times(-a,-a+\varepsilon)\\ v(x,\tau) = f_{\lambda,\lambda',h,h',k}(x,\tau) & \text{on } \{\pm R\}\times(-a,-a+\varepsilon)\\ v(x,-a) = v_0(x) & \text{on } (-R,R). \end{cases} \tag{2.46}$$

Since M_1 is a supersolution of (2.46) and $f_{\lambda,\lambda',h,h',k}$ is a subsolution [DPKS2] of (2.46), by Lemma 2.3,

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \le v_R(x,\tau) \le M_1 \quad \forall |x| \le R, -a \le \tau \le -a+\varepsilon \tag{2.47}$$

$$\Rightarrow \quad a_1 \le u_R(x,\tau) \le M_1^m \quad \forall |x| \le R, -a \le \tau \le -a+\varepsilon \tag{2.48}$$

Let $(-a,T)$, $-a+\varepsilon \le T \le \overline{\tau}_0$, be the maximal time interval such that $v_R \in C^{2,1}([-R,R]\times[-a,T])$ satisfies

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R)\times(-a,T)\\ v(x,\tau) = f_{\lambda,\lambda',h,h',k}(x,\tau) & \text{on } \{\pm R\}\times(-a,T)\\ v(x,-a) = v_0(x) & \text{on } (-R,R) \end{cases} \tag{2.49}$$

and (2.47) in $[-R,R]\times[-a,T]$. Suppose $T < \overline{\tau}_0$. Since $u_R \in C(\overline{B}_R\times[-a,T])$ satisfies (2.48) in $[-R,R]\times[-a,T]$, there exists a constant $T_1 \in (T,\overline{\tau}_0)$ such that $a_1/2 \le u_R(x,\tau) \le 2M_1^m$ for any $|x| \le R$, $-a \le \tau \le T_1$. By repeating the above argument $v_R \in C^{2,1}([-R,R]\times[-a,T_1])$ satisfies (2.47) and (2.49) with $-a+\varepsilon$ and T being replaced by T_1. This contradicts the choice of T. Hence $T = \overline{\tau}_0$ and $v_R \in C^{2,1}([-R,R]\times[-a,T])$ satisfies (2.43) and (2.47) in $[-R,R]\times[-a,T]$ and (2.44) follows.

Case 2: $v_0 \in L^\infty(-R,R)$

By (2.42) we can choose a sequence of functions $\{v_{0,i}\}_{i=1}^\infty \subset C^\infty([-R,R])$ satisfying

$$f_{\lambda,\lambda',h,h',k}(x,-a) \le v_0(x) \le v_{0,i+1}(x) \le v_{0,i}(x) \le \|v_{0,i}\|_{L^\infty(-R,R)} < 1 \quad \forall x \in (-R,R), i \in \mathbb{N} \tag{2.50}$$

and

$$v_{0,i}(x) = f_{\lambda,\lambda',h,h',k}(x,-a) \quad \forall iR/(i+1) \le |x| \le R, i \in \mathbb{N}$$

with $v_{0,i}$ converges to v_0 in $L^1(-R,R)$ and $\|v_{0,i}\|_{L^\infty} \to \|v_0\|_{L^\infty}$ as $i \to \infty$. For each $i \in \mathbb{N}$, by case 1 and Lemma 2.3 there exists a solution $v_{R,i} \in C^{2,1}([-R,R] \times [-a,\overline{\tau}_0])$ of (2.43) with v_0 being replaced $v_{0,i}$ which satisfies

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \le v_{R,i+1}(x,\tau) \le v_{R,i}(x,\tau) \le \|v_{0,i}\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \le R, -a \le \tau < \overline{\tau}_0, i \in \mathbb{N}. \tag{2.51}$$

By (2.51) the equation (1.6) for $\{v_{R,i}\}_{i=1}^\infty$ is uniformly parabolic on $[-R,R] \times [-a,\overline{\tau}_0]$. Hence by the standard Schauder estimates [LSU] the sequence $\{v_{R,i}\}_{i=1}^\infty$ is equi-continuous in $C^{2,1}(K)$ for any compact set $K \subset [-R,R] \times (-a,\overline{\tau}_0]$. Thus the sequence $\{v_{R,i}\}_{i=1}^\infty$ will decrease uniformly in $C^{2,1}(K)$ to some function $v_R \in C^{2,1}([-R,R] \times (-a,\overline{\tau}_0])$ as $i \to \infty$ for any compact subset K of $[-R,R] \times (-a,\overline{\tau}_0]$. Putting $v = v_{R,i}$, $g_0 = v_{0,i}$, $v_0 = v_{0,i}$ in (1.31) and letting $i \to \infty$, we get that v_R satisfies (1.31) with $g_0 = v_0$. Hence v_R is a solution of (2.43). Letting $i \to \infty$ in (2.51), we get (2.44) and the lemma follows. □

By a similar argument we have the following two lemmas.

Lemma 2.10. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$, $R > 0$, and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(-R,R)$ be such that*

$$f_{\lambda,\lambda',h,h'}(x,-a) \le v_0(x) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \textit{a.e. } x \in (-R,R).$$

Then there exists a unique solution $v_R \in C^{2,1}([-R,R] \times (-a,\overline{\tau}_0]) \cap L^\infty((-R,R) \times (-a,\overline{\tau}_0))$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R) \times (-a,\overline{\tau}_0) \\ v(x,\tau) = f_{\lambda,\lambda',h,h'}(x,\tau) & \text{on } \{\pm R\} \times (-a,\overline{\tau}_0) \\ v(x,-a) = v_0(x) & \text{on } (-R,R) \end{cases}$$

which satisfies

$$f_{\lambda,\lambda',h,h'}(x,\tau) \le v_R(x,\tau) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \le R, -a < \tau < \overline{\tau}_0.$$

Lemma 2.11. *Let $\lambda > 1$, $h \ge h_0$, $0 < k \le k_0$, $R > 0$, and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(-R,R)$ be such that*

$$f_{\lambda,h,k}(x,-a) \le v_0(x) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \textit{a.e } x \in (-R,R).$$

Then there exists a unique solution $v_R \in C^{2,1}([-R,R] \times (-a,\overline{\tau}_0]) \cap L^\infty((-R,R) \times (-a,\overline{\tau}_0))$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R,R) \times (-a,\overline{\tau}_0) \\ v(x,\tau) = f_{\lambda,h,k}(x,\tau) & \text{on } \{\pm R\} \times (-a,\overline{\tau}_0) \\ v(x,-a) = v_0(x) & \text{on } (-R,R) \end{cases}$$

which satisfies

$$f_{\lambda,h,k}(x,\tau) \le v_R(x,\tau) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \le R, -a < \tau < \overline{\tau}_0.$$

Lemma 2.12. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$, $0 < k \le k_0$. Let $\overline{\tau}_0 < 0$ be as in Lemma 2.5 and $a > -\overline{\tau}_0$. Then there exists a constant $R_0 = R_0(a) > 0$ such that for any $R \ge R_0$ and $v_0 \in L^\infty(-R, R)$ satisfying* (2.42)*, there exists a unique solution $v_R \in C^{2,1}([-R, R] \times (-a, \overline{\tau}_0)) \cap L^\infty(B_R \times (-a, \overline{\tau}_0))$ of*

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R, R) \times (-a, \overline{\tau}_0) \\ (v^m)_x = (f^m_{\lambda,\lambda',h,h',k})_x & \text{on } \{\pm R\} \times (-a, \overline{\tau}_0) \\ v(x, -a) = v_0(x) & \text{on } (-R, R) \end{cases} \tag{2.52}$$

which satisfies (2.44)*. Moreover if there exists $x_0 \in (-R, R)$ such that*

$$\begin{cases} v_0(x) \text{ is monotone increasing on } [-R, x_0] \\ v_0(x) \text{ is monotone decreasing on } [x_0, R], \end{cases} \tag{2.53}$$

then for any $-a < \tau < \overline{\tau}_0$ there exists $x_R(\tau) \in (-R, R)$ such that

$$\begin{cases} v_{R,x}(x, \tau) > 0 > v_{R,x}(y, \tau) & \forall -R \le x < x_R(\tau) < y < R, -a < \tau < \overline{\tau}_0 \\ v_{R,x}(x_R(\tau), \tau) = 0 & \forall -a < \tau < \overline{\tau}_0. \end{cases} \tag{2.54}$$

Proof: By Lemma 2.4 the solution of (2.52) is unique. Hence we only need to prove existence of solution of (2.52). Let $y(\tau)$ and $z(\tau)$ be as in Lemma 2.5. Let

$$R_0 = R_0(a) > \max\left(\lambda\overline{\tau}_0 - h, , \lambda'\overline{\tau}_0 - h', \max_{-a \le \tau \le \overline{\tau}_0} (|y(\tau)|, |z(\tau)|)\right)$$

be a constant to be determined later and let $R \ge R_0$. Then

$$v_{\lambda,h}(R, \tau) \ge v_\lambda(0) = \frac{1}{2} \quad \text{and} \quad \overline{v}_{\lambda',h'}(-R, \tau) \ge v_{\lambda'}(0) = \frac{1}{2} \quad \forall \tau \le \overline{\tau}_0. \tag{2.55}$$

We divide the existence proof into two cases.

Case 1: $v_0 \in C^\infty([-R, R])$ and there exists a constant $\delta \in (0, R)$ such that $v_0(x) = f_{\lambda,\lambda',h,h',k}(x, -a)$ for any $R - \delta \le |x| \le R$.

By an argument similar to the proof of Lemma 2.9 there exists $\varepsilon \in (0, \overline{\tau}_0)$ such that there exists a unique solution $v_R \in C^{2,1}([-R, R] \times [-a, -a + \varepsilon])$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R, R) \times (-a, -a + \varepsilon) \\ (v^m)_x = (f^m_{\lambda,\lambda',h,h',k})_x & \text{on } \{\pm R\} \times (-a, -a + \varepsilon) \\ v(x, -a) = v_0(x) & \text{on } (-R, R). \end{cases} \tag{2.56}$$

By (1.14), (1.16), (1.18), (1.22) and (2.55),

$$\begin{aligned} &\frac{\partial}{\partial x} f_{\lambda,\lambda',h,h',k}(R, \tau) \\ =&\left[v_{\lambda,h}(R, f(\tau))^{-p} v_\lambda'(R - \lambda f(\tau) + h) - \overline{v}_{\lambda',h'}(R, f(\tau))^{-p} v_{\lambda'}'(-R - \lambda' f(\tau) + h')\right] f_{\lambda,\lambda',h,h',k}(R, \tau)^p \\ \le&\left[2^p \left(C_\lambda \gamma_\lambda e^{-\gamma_\lambda(R - \lambda f(\tau) + h)} + o(e^{-\gamma_\lambda(R - \lambda f(\tau) + h)})\right) - Ce^{(p^2 - p)(R + \lambda' f(\tau) - h')}\right] f_{\lambda,\lambda',h,h',k}(R, \tau)^p \\ <&0 \quad \forall -a \le \tau \le \overline{\tau}_0 \end{aligned} \tag{2.57}$$

if R_0 is sufficiently large where f is given by (1.24) and

$$\begin{aligned}
&\frac{\partial}{\partial x} f_{\lambda,\lambda',h,h',k}(-R,\tau)\\
=&\left[v_{\lambda,h}(-R,f(\tau))^{-p}v_\lambda'(-R-\lambda f(\tau)+h)-\overline{v}_{\lambda',h'}(-R,f(\tau))^{-p}v_{\lambda'}'(R-\lambda' f(\tau)+h')\right]f_{\lambda,\lambda',h,h',k}(-R,\tau)^p\\
\ge&\left[Ce^{(p^2-p)(R+\lambda f(\tau)-h)}-2^p\left(C_{\lambda'}\gamma_{\lambda'}e^{-\gamma_{\lambda'}(R-\lambda' f(\tau)+h')}+o(e^{-\gamma_{\lambda'}(R-\lambda' f(\tau)+h')})\right)\right]f_{\lambda,\lambda',h,h',k}(-R,\tau)^p\\
>&0 \quad \forall -a\le\tau\le\overline{\tau}_0
\end{aligned} \tag{2.58}$$

if R_0 is sufficiently large. We now choose R_0 sufficiently large such that both (2.57) and (2.58) hold. Then by (2.57) and (2.58), $M := \|v_0\|_{L^\infty(-R,R)} < 1$ is a supersolution of (2.56). On the other hand $f_{\lambda,\lambda',h,h',k}$ is a subsolution of (2.56). Hence by Lemma 2.4, v_R satisfies (2.44) in $\overline{B}_R\times[-a,-a+\varepsilon]$.

Let $(-a,T)$, $-a+\varepsilon\le T\le\overline{\tau}_0$, be the maximal time interval of existence of solution $v_R\in C^{2,1}([-R,R]\times[-a,T])$ of

$$\begin{cases} v_\tau=(v^m)_{xx}+v-v^m & \text{in } (-R,R)\times(-a,T)\\ (v^m)_x=(f^m_{\lambda,\lambda',h,h',k})_x & \text{on } \{\pm R\}\times(-a,T)\\ v(x,-a)=v_0(x) & \text{on } (-R,R)\end{cases} \tag{2.59}$$

which satisfies (2.44) in $[-R,R]\times[-a,T]$. Suppose $T<\overline{\tau}_0$. Then by (2.44) and an argument similar to the proof of Lemma 2.9 there exists a constant $T_1\in(T,\overline{\tau}_0)$ such that v_R can be extended to a solution of (2.59) in $(-R,R)\times(-a,T_1)$ that satisfies (2.44) in $[-R,R]\times[-a,T_1]$ and $v_R\in C^{2,1}([-R,R]\times[-a,T_1])$. This contradicts the choice of T. Hence $T=\overline{\tau}_0$.

Case 2: $v_0\in L^\infty(-R,R)$

We choose a sequence of function $\{v_{0,i}\}_{i=1}^\infty\subset C^\infty([-R,R])$ satisfying (2.50) and

$$v_{0,i}(x)=f_{\lambda,\lambda',h,h',k}(x,-a)\quad\forall(1-2^{-1-i})R\le|x|\le R, i\in\mathbb{N} \tag{2.60}$$

and $v_{0,i}$ converges to v_0 in $L^1(-R,R)$ and $\|v_{0,i}\|_{L^\infty(-R,R)}\to\|v_0\|_{L^\infty(-R,R)}$ as $i\to\infty$. By case 1 for any $i\in\mathbb{N}$ there exists a solution $v_{R,i}\in C^{2,1}([-R,R]\times[-a,\overline{\tau}_0])$ of (2.52) with v_0 being replaced by $v_{0,i}$ and $v_{R,i}$ satisfies (2.51). Then by (2.51) the equation (1.6) for the sequence $\{v_{R,i}\}_{i=1}^\infty$ is uniformly parabolic on $[-R,R]\times[-a,\overline{\tau}_0]$. Hence by the Schauder's estimates [LSU] the sequence $\{v_{R,i}\}_{i=1}^\infty$ is equi-Holder continuous in $C^{2,1}(K)$ for any compact subset K of $[-R,R]\times(-a,\overline{\tau}_0]$. Thus by the Ascoli Theorem and a diagonalization argument the sequence $\{v_{R,i}\}_{i=1}^\infty$ has a subsequence $\{v_{R,i_k}\}_{k=1}^\infty$ that converges in $C^{2,1}(K)$ to some function $v_R\in C^{2,1}([-R,R]\times(-a,\overline{\tau}_0])$ as $k\to\infty$ for any compact subset K of $[-R,R]\times(-a,\overline{\tau}_0]$. Since $v_{R,i}\in C^{2,1}([-R,R]\times(-a,\overline{\tau}_0])$ satisfies (2.52),

$$\begin{aligned}\int_{-R}^R v_{R,i}(x,\tau)\eta(x,\tau)\,dx=&\int_{-a}^\tau\int_{-R}^R\left[v_{R,i}\eta_\tau+v_{R,i}^m\eta_{xx}+(v_{R,i}-v_{R,i}^m)\eta\right]dx\,d\tau\\&+\int_{-a}^\tau\int_{\partial B_R}\eta\frac{\partial}{\partial n}f^m_{\lambda,\lambda',h,h,k}\,d\sigma\,d\tau+\int_{-R}^R v_{0,i}(x)\eta(x,\tau_1)\,dx\end{aligned} \tag{2.61}$$

holds for any $-a < \tau < \overline{\tau}_0$, $i \in \mathbb{N}$, and function $0 \le \eta \in C^{2,1}([-R,R] \times [-a,\tau])$ satisfying $\eta_x \equiv 0$ on $\{\pm R\} \times [-a,\tau]$. Putting $i = i_k$ in (2.61) and letting $k \to \infty$,

$$\int_{-R}^{R} v_R(x,\tau)\eta(x,\tau)\,dx = \int_{-a}^{\tau}\int_{-R}^{R}\left[v_R\eta_\tau + v_R^m\eta_{xx} + (v_R - v_R^m)\eta\right]dx\,d\tau$$
$$+ \int_{-a}^{\tau}\int_{\partial B_R}\eta\frac{\partial}{\partial n}f^m_{\lambda,\lambda',h,h,k}\,d\sigma\,d\tau + \int_{-R}^{R} v_0(x)\eta(x,\tau_1)\,dx$$

holds for any $-a < \tau < \overline{\tau}_0$, $i \in \mathbb{N}$, and function $0 \le \eta \in C^{2,1}([-R,R] \times [-a,\tau])$ satisfying $\eta_x \equiv 0$ on $\{\pm R\} \times [-a,\tau]$. Letting $i = i_k \to \infty$ in (2.51), we get (2.44). Hence by Lemma 2.4 $v_R \in C^{2,1}([-R,R] \times (-a,\overline{\tau}_0]) \cap L^\infty((-R,R) \times (-a,\overline{\tau}_0))$ is the unique solution of (2.52). Thus v_i converges in $C^{2,1}(K)$ to v_R as $i \to \infty$ for any compact subset K of $[-R,R] \times (-a,\overline{\tau}_0]$.

Suppose now v_0 satisfies (2.53) for some $x_0 \in (-R,R)$. We claim that for any $-a < \tau < \overline{\tau}_0$ there exists $x_R(\tau) \in (-R,R)$ such that (2.54) holds. We divide the proof of the claim into two cases.

Case A: $v_0 \in C^\infty([-R,R])$ and there exists a constant $\delta \in (0,R)$ such that $v_0(x) = f_{\lambda,\lambda',h,h',k}(x,-a)$ for any $R - \delta \le |x| \le R$ and

$$\begin{cases} v_0'(x) > 0 > v_0'(y) & \forall -R \le x \le x_0 - \varepsilon_1, x_0 + \varepsilon_1 \le x \le R \\ v_0''(x) < 0 & \forall x \in [x_0 - \varepsilon_1, x_0 + \varepsilon_1] \end{cases} \tag{2.62}$$

for some constants $x_0 \in (-R,R)$ and $\varepsilon_1 \in \left(0, \frac{1}{2}\min(R - x_0, R + x_0)\right)$. Let $v_R \in C^{2,1}(\overline{B}_R \times [-a,\overline{\tau}_0])$ be the solution of (2.52) given by case 1 above. By (2.52), $v_{R,x}$ satisfies

$$w_\tau = mv^{m-1}w_{xx} - 3m(1-m)v^{m-2}ww_x + m(1-m)(2-m)v^{m-3}w^3 + w - mv^{m-1}w \tag{2.63}$$

in $(-R,R) \times (-a,\overline{\tau}_0)$. Since $v_R \in C^{2,1}([-R,R] \times [-a,\overline{\tau}_0])$, by (2.62) there exists $T_2 \in (-a,\overline{\tau}_0)$ such that

$$\begin{cases} v_{R,x}(x,\tau) > 0 > v_{R,x}(y,\tau) & \forall -R \le x \le x_0 - \varepsilon_1, x_0 + \varepsilon_1 \le y \le R, -a \le \tau \le T_2 \\ v_{R,xx}(x,\tau) < 0 & \forall x_0 - \varepsilon_1 \le x \le x_0 + \varepsilon_1, -a \le \tau \le T_2. \end{cases} \tag{2.64}$$

By (2.64) for any $-a \le \tau \le T_2$ there exists a unique $x_R(\tau) \in (x_0 - \varepsilon_1, x_0 + \varepsilon_1)$ such that

$$\begin{cases} v_{R,x}(x,\tau) > 0 > v_{R,x}(y,\tau) & \forall -R \le x < x_R(\tau) < y \le R, -a < \tau \le T_2 \\ v_{R,x}(x_R(\tau),\tau) = 0 & \forall -a \le \tau \le T_2. \end{cases}$$

Let $(-a,T_2')$, $T_2 \le T_2' \le \overline{\tau}_0$, be the maximal interval such that for any $-a < \tau < T_2'$ there exists a unique $x_R(\tau) \in (-R,R)$ such that

$$\begin{cases} v_{R,x}(x,\tau) > 0 > v_{R,x}(y,\tau) & \forall -R \le x < x_R(\tau) < y \le R, -a < \tau < T_2' \\ v_{R,x}(x_R(\tau),\tau) = 0 & \forall -a < \tau < T_2'. \end{cases} \tag{2.65}$$

Suppose $T_2' < \overline{\tau}_0$. Then by compactness there exists a sequence $-a < \tau_i < T_2'$, $\tau_i \to T_2'$ as $i \to \infty$, such that $x_R(\tau_i)$ converges to some point $x_R(T_2') \in [-R,R]$ as $i \to \infty$. Since

$$v_{R,x}(x_R(T_2'),T_2') = \lim_{i\to\infty} v_{R,x}(x_R(\tau_i),\tau_i) = 0, \tag{2.66}$$

by (2.52), (2.57) and (2.58), $x_R(T_2') \in (-R+\varepsilon_2, R-\varepsilon_2)$ for some constant $\varepsilon_2 \in (0, R/2)$. Let

$$D = \{(x,\tau) : -R < x < x_R(\tau), -a < \tau \le T_2'\}$$

By (2.65),

$$v_{R,x}(x,\tau) \ge 0 \quad \forall (x,\tau) \in \overline{D}. \tag{2.67}$$

By (2.44) the equation (2.63) for $v_{R,x}$ is uniformly parabolic on $[-R,R]\times[-a,\overline{\tau}_0]$. Hence by (2.52), (2.58), (2.63), (2.67) and the strong maximum principle,

$$v_{R,x}(x,\tau) > 0 \quad \forall -R \le x < x_R(\tau), -a < \tau \le T_2'. \tag{2.68}$$

Similarly,

$$v_{R,x}(x,\tau) < 0 \quad \forall x_R(\tau) < x \le R, -a < \tau \le T_2'. \tag{2.69}$$

We next observe that since the equation (2.63) for $v_{R,x}$ is uniformly parabolic on $[-R,R]\times[-a,\overline{\tau}_0]$, the results of Lemma 2.4 of [H1] (cf. [A], [CP], [M]) remains valid for the solution $v_{R,x}$ of (2.63). Hence there exist constants $0 < \delta_1 < \min\left(\frac{R-x_R(T_2')}{2}, \frac{R+x_R(T_2')}{2}\right)$ and $0 < \delta_2 < \overline{\tau}_0 - T_2'$ such that for any $T_2' \le \tau \le T_2' + \delta_2$ there exists $x_R(\tau) \in \left(x_R(T_2') - \delta_1, x_R(T_2') + \delta_1\right)$ such that

$$\begin{cases} v_{R,x}(x,\tau) > 0 > v_{R,x}(y,\tau) & \forall x_R(T_2') - \delta_1 \le x < x_R(\tau) < y \le x_R(T_2') + \delta_1, T_2' \le \tau \le T_2' + \delta_2 \\ v_{R,x}(x_R(\tau),\tau) = 0 & \forall T_2' \le \tau \le T_2' + \delta_2. \end{cases} \tag{2.70}$$

Since $v_R \in C^{2,1}([-R,R]\times[-a,\overline{\tau}_0])$, by (2.68) and (2.69) there exists a constant $0 < \delta_3 < \delta_2$ such that

$$v_{R,x}(x,\tau) > 0 > v_{R,x}(y,\tau) \quad \forall -R \le x \le x_R(T_2') - \delta_1, x_R(T_2') + \delta_1 \le y \le R, T_2' \le \tau \le T_2' + \delta_3. \tag{2.71}$$

By (2.70) and (2.71), (2.65) holds with T_2' replaced by $T_2' + \delta_3$. This contradicts the choice of T_2'. Hence $T_2' = \overline{\tau}_0$. Hence for any $-a < \tau < \overline{\tau}_0$ there exists $x_R(\tau) \in (-R,R)$ such that (2.54) holds.

Case B: $v_0 \in L^\infty(-R,R)$

We choose a sequence of function $\{v_{0,i}\}_{i=1}^\infty \subset C^\infty([-R,R])$ satisfying (2.50), (2.60), $v_{0,i}$ converges to v_0 in $L^1(-R,R)$ and $\|v_{0,i}\|_{L^\infty(-R,R)} \to \|v_0\|_{L^\infty(-R,R)}$ as $i \to \infty$, and

$$\begin{cases} v_{0,i}'(x) > 0 > v_{0,i}'(y) & \forall -R \le x \le x_{0,i} - \delta_0, x_{0,i} + \delta_0 \le y \le R \\ v_{0,i}''(x) < 0 & \forall x \in [x_{0,i} - \delta_0, x_{0,i} + \delta_0] \end{cases} \tag{2.72}$$

holds for some sequences $\{x_{0,i}\}_{i=1}^\infty \subset (x_0 - \delta_0, x_0 + \delta_0)$ where $\delta_0 = min\left(\frac{R-x_0}{4}, \frac{R+x_0}{4}\right)$ such that $x_{0,i} \to x_0$ as $i \to \infty$. For any $i \in \mathbb{N}$ let $v_{R,i} \in C^{2,1}([-R,R]\times[-a,\overline{\tau}_0])$ be the solution of (2.52) with v_0 being replaced by $v_{0,i}$ such that $v_{R,i}$ satisfies (2.44). By case A, for any $i \in \mathbb{N}$ and $-a < \tau < \overline{\tau}_0$ there exists $x_{R,i}(\tau) \in (-R,R)$ such that

$$\begin{cases} v_{R,i,x}(x,\tau) > 0 > v_{R,i,x}(y,\tau) & \forall -R \le x < x_{R,i}(\tau) < y \le R, -a < \tau < \overline{\tau}_0 \\ v_{R,i,x}(x_{R,i}(\tau),\tau) = 0 & \forall -a < \tau < \overline{\tau}_0. \end{cases} \tag{2.73}$$

By case 2, v_i converges in $C^{2,1}(K)$ to the solution $v_R \in C^{2,1}([-R,R]\times(-a,\overline{\tau}_0])\cap L^\infty((-R,R)\times(-a,\overline{\tau}_0))$ of (2.52) as $i\to\infty$ for any compact subset K of $[-R,R]\times(-a,\overline{\tau}_0]$ and v_R satisfies (2.44). For any $-a<\tau<\overline{\tau}_0$, let

$$D_0(\tau)=\{x\in[-R,R]:v_{R,x}(x,\tau)=0\}. \tag{2.74}$$

Since $[-R,R]$ is compact, for any $-a<\tau<\overline{\tau}_0$ the sequence $\{x_{R,i}(\tau)\}_{i=1}^\infty$ has a convergence subsequence $\{x_{R,i_k}(\tau)\}_{k=1}^\infty$. Let $x_R(\tau)=\lim_{k\to\infty}x_{R,i_k}(\tau)$. Then by (2.73), $v_{R,x}(x_R(\tau),\tau)=0$. Hence $D_0(\tau)\neq\phi$.

Since $v_R\in C^{2,1}([-R,R]\times(-a,\overline{\tau}_0])$, by (2.52), (2.57) and(2.58), for any $-a<\tau<\overline{\tau}_0$ there exists a constant $\varepsilon_\tau\in(0,R)$ such that

$$\begin{aligned} & v_{R,x}(x,\tau)>0>v_{R,x}(y,\tau)\quad\forall -R\le x\le -R+\varepsilon_\tau, R-\varepsilon_\tau\le y\le R && (2.75)\\ \Rightarrow\quad & D_0(\tau)\subset[-R+\varepsilon_\tau,R-\varepsilon_\tau]. && (2.76)\end{aligned}$$

We claim that $D_0(\tau)$ is a singleton for any $-a<\tau<\overline{\tau}_0$. Suppose not. Then there exists $\tau_1\in(-a,\overline{\tau}_0)$ such that $D_0(\tau_1)$ is not a singleton. By the discussion on P.241 of [SGKM] and [CP] $v_{R,x}(x,\tau_1)$ is an analytic function of $x\in(-R,R)$. Hence either all the zeros of $v_{R,x}(x,\tau_1)$ are isolated zeros or

$$v_{R,x}(x,\tau_1)=0\quad\forall|x|\le R. \tag{2.77}$$

By (2.76), (2.77) is not possible. Hence all the zeros of $v_{R,x}(x,\tau_1)$ are isolated zeros. Thus $D_0(\tau_1)$ has a finite number of elements and we can write $D_0(\tau_1)=\{x_1,...,x_k\}$ for some $k\ge 2$ such that $-R<x_1<x_2<\cdots<x_k<R$. Let $x_0=-R$ and $x_{k+1}=R$. By (2.75),

$$v_{R,x}(x,\tau_1)>0>v_{R,x}(y,\tau_1)\quad\forall -R\le x<x_1, x_k<y\le R.$$

Hence there exists $k_0\in\{1,\dots,k\}$ such that

$$v_{R,x}(x,\tau_1)>0>v_{R,x}(y,\tau_1)\quad\forall x_{k_0-1}<x<x_{k_0}<y<x_{k_0+1}.$$

Without loss of generality we may assume that

$$v_{R,x}(x,\tau_1)>0>v_{R,x}(y,\tau_1)\quad\forall -R<x<x_1<y<x_2. \tag{2.78}$$

Let $\delta_1=\min\left(\frac{x_1+R}{4},\frac{x_2-x_1}{4}\right)$. Since $v_{R,i,x}(\cdot,\tau_1)$ converges uniformly to $v_R(\cdot,\tau_1)$ on $[-R,R]$ as $i\to\infty$, by (2.78) there exists $i_0\in\mathbb{N}$ such that

$$v_{R,i,x}(x_1-\delta_1,\tau_1)>0>v_{R,i,x}(x_1+\delta_1,\tau_1)\quad\forall i\ge i_0. \tag{2.79}$$

By (2.79) and the intermediate value theorem for any $i\ge i_0$, there exist $z_i\in(x_1-\delta_1,x_1+\delta_1)$ such that

$$v_{R,i,x}(z_i,\tau_1)=0\quad\forall i\ge i_0. \tag{2.80}$$

By compactness the sequence $\{z_i\}$ has a convergence subsequence which we may assume without loss of generality to be the sequence itself that converges to some point $z_0\in[x_1-\delta_1,x_1+\delta_1]$. Letting $i\to\infty$ in (2.80),

$$v_{R,x}(z_0,\tau_1)=0. \tag{2.81}$$

By (2.78) and (2.81), $z_0 = x_1$. By (2.73) and (2.80), $z_i = x_{R,i}(\tau_1)$ for all $i \ge i_0$. Putting $\tau = \tau_1$ in (2.73) and letting $i \to \infty$,

$$v_{R,x}(x, \tau_1) \ge 0 \ge v_{R,x}(y, \tau_1) \quad \forall -R \le x \le x_1 \le y \le R. \tag{2.82}$$

Since all the zeros of $v_{R,x}(x, \tau_1)$ are isolated zeros, by (2.83),

$$v_{R,x}(x, \tau_1) < 0 \quad \forall x_1 < x < x_3, x \ne x_2. \tag{2.83}$$

Let $\delta_2 = \min\left(\frac{x_2 - x_1}{4}, \frac{x_3 - x_2}{4}\right)$. Then by (2.83) there exists $\delta_3 \in (0, \tau_1 + a)$ such that

$$v_{R,x}(x_2 - \delta_2, \tau) < 0 \quad \text{and} \quad v_{R,x}(x_2 + \delta_1, \tau) < 0 \quad \forall \tau_1 - \delta_3 \le \tau \le \tau_1. \tag{2.84}$$

By (2.73) and an argument similar to the above one there exists a constant $\overline{x}_1 \in (-R, R)$ such that

$$v_{R,x}(x, \tau_1 - \delta_3) \ge 0 \ge v_{R,x}(y, \tau_1 - \delta_3) \quad \forall -R \le x \le \overline{x}_1 \le y \le R \tag{2.85}$$

By (2.84) and (2.85), $\overline{x}_1 \le x_2 - \delta_2$ and

$$v_{R,x}(x, \tau_1 - \delta_3) \le 0 \quad \forall x_2 - \delta_2 \le x \le R. \tag{2.86}$$

By (2.44) the equation (2.63) for $v_{R,x}$ is uniformly parabolic on $[-R, R] \times [-a, \overline{\tau}_0]$. Hence by (2.63), (2.84), (2.86) and the strong maximum principle in $(x_2 - \delta_2, x_2 + \delta_2) \times (\tau_1 - \delta_3, \tau_1)$,

$$v_{R,x}(x_2, \tau_1) < 0$$

and contradiction arises. Hence no such τ_1 exists and $D_0(\tau)$ is a singleton for any $-a < \tau < \overline{\tau}_0$. Hence $D_0(\tau) = \{x_R(\tau)\}$ and $x_R(\tau) = \lim_{i\to\infty} x_{R,i}(\tau)$. Letting $i \to \infty$ in (2.73),

$$v_{R,x}(x, \tau) \ge 0 \ge v_{R,x}(y, \tau) \quad \forall -R \le x \le x_R(\tau) \le y \le R, -a < \tau < \overline{\tau}_0 \tag{2.87}$$

Since for any $-a < \tau < \overline{\tau}_0$, $x_R(\tau)$ is an isolated zero of $v_R(\cdot, \tau)$, by (2.87) we get (2.54) and the lemma follows. □

By a similar argument we have the two following lemmas.

Lemma 2.13. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(-R, R)$ be such that*

$$f_{\lambda,\lambda',h,h'}(x, -a) \le v_0(x) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \text{a.e. } x \in (-R, R)$$

holds. Then there exists a constant $R_0 = R_0(a) > 0$ such that for any $R \ge R_0$ there exists a unique solution $v_R \in C^{2,1}([-R, R] \times (-a, \tau_0]) \cap L^\infty((-R, R) \times (-a, \overline{\tau}_0))$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R, R) \times (-a, \overline{\tau}_0) \\ (v^m)_x = (f^m_{\lambda,\lambda',h,h'})_x & \text{on } \{\pm R\} \times (-a, \overline{\tau}_0) \\ v(x, -a) = v_0(x) & \text{on } (-R, R) \end{cases}$$

which satisfies

$$f_{\lambda,\lambda',h,h'}(x, \tau) \le v_R(x, \tau) \le \|v_0\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \le R, -a < \tau < \overline{\tau}_0.$$

Moreover if there exists $x_0 \in (-R, R)$ such that (2.53) *holds, then for any $-a < \tau < \overline{\tau}_0$ there exists $x_R(\tau) \in (-R, R)$ such that* (2.54) *holds.*

Lemma 2.14. *Let $\lambda > 1$, $h \geq h_0$, $0 < k \leq k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(-R, R)$ be such that*

$$f_{\lambda,h,k}(x, -a) \leq v_0(x) \leq \|v_0\|_{L^\infty(-R,R)} < 1 \quad a.e.\ x \in (-R, R)$$

holds. Then there exists a constant $R_0 = R_0(a) > 0$ such that for any $R \geq R_0$ there exists a unique solution $v_R \in C^{2,1}([-R, R] \times (-a, \tau_0]) \cap L^\infty((-R, R) \times (-a, \overline{\tau}_0))$ of

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m & \text{in } (-R, R) \times (-a, \overline{\tau}_0) \\ (v^m)_x = (f^m_{\lambda,h,k})_x & \text{on } \{\pm R\} \times (-a, \overline{\tau}_0) \\ v(x, -a) = v_0(x) & \text{on } (-R, R) \end{cases}$$

which satisfies

$$f_{\lambda,h,k}(x, \tau) \leq v_R(x, \tau) \leq \|v_0\|_{L^\infty(-R,R)} < 1 \quad \forall |x| \leq R, -a < \tau < \overline{\tau}_0.$$

Moreover if v_0 is monotone increasing on $[-R, R]$, then

$$v_{R,x}(x, \tau) > 0 \quad \forall x \in [-R, R), -a < \tau < \overline{\tau}_0.$$

3 Existence and properties of ancient solutions

In this section we will prove the existence and various properties of the 5-parameters, 4-parameters, 3-parameters ancient solutions of (1.6). We will also prove the uniqueness of the 4-parameters ancient solution $v_{\lambda,\lambda',h,h'}$.

Lemma 3.1. *Let $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h_0'$, $0 < k \leq k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Then both $v_{\lambda,h}(x, f(\tau))$ and $\overline{v}_{\lambda,h}(x, f(\tau))$ are supersolutions of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$.*

Proof: Let $q(x, \rho) = v_{\lambda,h}(x, f(\rho))$ and $\tau = f(\rho)$ where f is given by (1.24). Since $\overline{\tau}_0 < -p/(p-1)$ and $(v_{\lambda,h})_\tau(x, \tau) = -\lambda v_\lambda'(x - \lambda\tau + h) < 0$ in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$,

$$\begin{aligned} q_\rho(x, \rho) =& (v_{\lambda,h})_\tau(x, f(\rho)) f'(\rho) = (v_{\lambda,h})_\tau(x, f(\rho)) \left(1 + q\left(1 + \frac{p-1}{p}\rho\right) e^{\frac{p-1}{p}\rho}\right) \\ \geq& (v_{\lambda,h})_\tau(x, f(\rho)) = (v_{\lambda,h}(x, f(\rho))^m)_{xx} + v_{\lambda,h}(x, f(\rho)) - v_{\lambda,h}(x, f(\rho))^m \quad \text{in } \mathbb{R} \times (-\infty, \overline{\tau}_0] \end{aligned}$$

Hence $v_{\lambda,h}(x, f(\tau))$ is a supersolution of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$. Similarly $\overline{v}_{\lambda,h}(x, f(\tau))$ is a supersolution of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$. □

Lemma 3.2. *Let $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h_0'$, $0 < k \leq k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(\mathbb{R})$ satisfies*

$$f_{\lambda,\lambda',h,h',k}(x, -a) \leq v_0(x) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x, -a) \quad a.e.\ x \in \mathbb{R} \tag{3.1}$$

where f is given by (1.24). *Then there exists a unique solution* $\overline{v} \in C^{2,1}(\mathbb{R} \times (-a, \overline{\tau}_0])$ *of*

$$\begin{cases} v_\tau = (v^m)_{xx} + v - v^m, \quad v > 0, \quad \text{in } \mathbb{R} \times (-a, \overline{\tau}_0) \\ v(x, -a) = v_0(x) \quad \text{on } \mathbb{R} \end{cases} \tag{3.2}$$

which satisfies

$$\begin{cases} f_{\lambda,\lambda',h,h',k}(x,\tau) \le \overline{v}(x,\tau) \le \|v_0\|_{L^\infty(\mathbb{R})} \quad \forall x \in \mathbb{R}, -a < \tau \le \overline{\tau}_0 \\ \overline{v}(x,\tau) \le \overline{f}_{\lambda,\lambda',h,h',k}(x,\tau) \quad \forall x \in \mathbb{R}, -a < \tau \le \overline{\tau}_0. \end{cases} \tag{3.3}$$

If there exists $x_0 \in \mathbb{R}$ *such that*

$$\begin{cases} v_0(x) \text{ is monotone increasing on } (-\infty, x_0] \\ v_0(x) \text{ is monotone decreasing on } [x_0, \infty) \end{cases} \tag{3.4}$$

holds, then for any $-a < \tau < \overline{\tau}_0$ *there exists* $x_a(\tau) \in \mathbb{R}$ *such that*

$$\begin{cases} \overline{v}_x(x,\tau) > 0 > \overline{v}_x(y,\tau) \quad \forall x < x_a(\tau) < y, -a < \tau < \overline{\tau}_0 \\ \overline{v}_x(x_a(\tau),\tau) = 0 \quad \forall -a < \tau < \overline{\tau}_0. \end{cases} \tag{3.5}$$

Moreover if $v_0(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$ *on* $\mathbb{R}$, *then* $\overline{v}(\cdot,\tau)$ *is also a monotone decreasing function of* $\tau \in [-a, \overline{\tau}_0]$ *and for any* $a < \tau < \overline{\tau}_0$ *there exists* $x_a(\tau) \in \mathbb{R}$ *such that* (3.5) *holds.*

Furthermore if $v_0(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$ *on* $\mathbb{R}$ *and* $\lambda = \lambda'$, *then* $x_a(\tau) = \frac{h'-h}{2}$ *and* $v(x,\tau)$ *is symmetric with respect to* $x_0 := \frac{h'-h}{2}$ *for any* $-a < \tau < \overline{\tau}_0$.

Proof: Uniqueness of solution of (3.2) follows from Lemma 2.2. Hence it remains to prove existence of solution of (3.2). By Lemma 2.9 for any $i \in \mathbb{N}$, there exists a unique solution $v_i \in C^{2,1}([-i,i] \times (-a, \overline{\tau}_0]) \cap L^\infty((-i,i) \times (-a, \overline{\tau}_0))$ of (2.43) with $R = i$ which satisfies (2.44) with $R = i$. By (2.44) the equation (1.6) for the sequence $\{v_i\}_{i=1}^\infty$ is uniformly parabolic on every compact subset of $\mathbb{R} \times (-a, \overline{\tau}_0]$. Hence by the parabolic Schauder estimates [LSU] the sequence $\{v_i\}_{i=1}^\infty$ is equi-Holder continuous in $C^{2,1}(K)$ for any compact subset $K \subset \mathbb{R} \times (-a, \overline{\tau}_0]$. Hence by the Ascoli Theorem and a diagonalization argument the sequence $\{v_i\}_{i=1}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R} \times (-a, \tau_0]$ to some function $\overline{v} \in C^{2,1}(\mathbb{R} \times (-a, \overline{\tau}_0])$ as $i \to \infty$. Then $\overline{v}$ satisfies (1.6) in $\mathbb{R} \times (-a, \tau_0]$. Putting $R = i$ in (2.44) and letting $i \to \infty$ we get

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \le \overline{v}(x,\tau) \le \|v_0\|_{L^\infty(\mathbb{R})} < 1 \quad \forall x \in \mathbb{R}, -a \le \tau < \overline{\tau}_0 \tag{3.6}$$

For any $\eta \in C_0^\infty(\mathbb{R})$ such that supp $\eta \subset (-i_0, i_0)$ for some $i_0 \in \mathbb{N}$, by (3.6),

$$
\begin{aligned}
\left|\int_{-i_0}^{i_0} v_i(x,\tau)\eta(x)\,dx - \int_{-i_0}^{i_0} v_0\eta\,dx\right| =&\left|\int_{-a}^{\overline{\tau}_0}\int_{-i_0}^{i_0} v_{i,\tau}(x,\tau)\eta(x)\,dx\,d\tau\right|\\
=&\left|\int_{-a}^{\tau}\int_{-i_0}^{i_0} (v_i^m\eta_{xx} + (v_i - v_i^m)\eta\,dx\,ds\right|\\
\le& C(\tau+a) \quad \forall i > i_0, -a < \tau < \overline{\tau}_0\\
\Rightarrow \quad \left|\int_{\mathbb{R}} \overline{v}(x,\tau)\eta(x)\,dx - \int_{\mathbb{R}} v_0\eta\,dx\right| \le& C(\tau+a) \quad \forall -a < \tau < \overline{\tau}_0 \quad \text{ as } i \to \infty\\
\Rightarrow \quad \lim_{\tau\to -a}\left|\int_{\mathbb{R}} \overline{v}(x,\tau)\eta(x)\,dx - \int_{\mathbb{R}} v_0\eta\,dx\right| =& 0.
\end{aligned}
$$

This together with (3.6) and the Lebesgue dominated convergence theorem implies that $\overline{v}(\cdot,\tau) \to v_0$ in $L^1_{loc}(\mathbb{R})$ as $\tau \to 0$. Hence $\overline{v}$ satisfies (3.2).

Since by Lemma 3.1 both $v_{\lambda,h}(x, f(\tau))$ and $\overline{v}_{\lambda,h}(x, f(\tau))$ are supersolutions of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$, by (3.1), (3.2) and Lemma 2.2, we get

$$
\overline{v}(x,\tau) \le v_{\lambda,h}(x, f(\tau)) \quad \text{ and } \quad \overline{v}(x,\tau) \le \overline{v}_{\lambda,h}(x, f(\tau)) \quad \forall x \in \mathbb{R}, -a < \tau \le \overline{\tau}_0. \tag{3.7}
$$

Similarly,

$$
\overline{v}(x,\tau) \le \xi_k(\tau) \quad \forall x \in \mathbb{R}, -a < \tau \le \overline{\tau}_0. \tag{3.8}
$$

By (3.6), (3.7) and (3.8) we get (3.3).

Let $R_0 > 0$ be as given by Lemma 2.12. By Lemma 2.12 for any $i > R_0$ there exists a unique solution $\overline{v}_i \in C^{2,1}([-i,i] \times (-a, \overline{\tau}_0]) \cap L^\infty((-i,i) \times (-a, \overline{\tau}_0))$ of (2.52) with $R = i$ which satisfies (2.44) with $R = i$. Then by a similar argument as before, $\overline{v}_i$ converges uniformly in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R} \times (-a, \overline{\tau}_0]$ to $\overline{v}$ as $i \to \infty$. If there exists $x_0 \in \mathbb{R}$ such that (3.4) holds, then by Lemma 2.12 for any $i > R_1 := \max(R_0, |x_0|)$ and $-a < \tau < \overline{\tau}_0$ there exists $x_i(\tau) \in (-i, i)$ such that

$$
\begin{cases}
\overline{v}_{i,x}(x,\tau) > 0 > \overline{v}_{i,x}(y,\tau) \quad \forall -i \le x < x_i(\tau) < y \le i, -a < \tau < \overline{\tau}_0\\
\overline{v}_{i,x}(x_i(\tau),\tau) = 0 \quad \forall -a < \tau < \overline{\tau}_0.
\end{cases} \tag{3.9}
$$

Then for any $-a < \tau < \overline{\tau}_0$ the sequence $\{x_i(\tau)\}_{i>R_1}$ has a subsequence $\{x_{i_l}(\tau)\}$ converging to some point $x_0(\tau) \in \mathbb{R} \cup \{\pm\infty\}$ as $l \to \infty$. If there exists $\tau_1 \in (-a, \overline{\tau}_0)$ such that $x_0(\tau_1) = \infty$, then by (3.9),

$$
\overline{v}_x(x,\tau_1) \ge 0 \quad \forall x \in \mathbb{R}. \tag{3.10}
$$

By (3.3),

$$
\overline{v}(x,\tau) \to 0 \quad \text{ as } |x| \to \infty \quad \forall -a < \tau < \overline{\tau}_0. \tag{3.11}
$$

Hence by (3.10) and (3.11),

$$
\overline{v}(x,\tau_1) = 0 \quad \forall x \in \mathbb{R} \tag{3.12}
$$

and contradiction arises since $v(x,\tau_1) > 0$ for any $x \in \mathbb{R}$. Hence $x_0(\tau) \neq \infty$ for any $\tau \in (-a,\overline{\tau}_0)$. Similarly $x_0(\tau) \neq -\infty$. Hence $x_0(\tau) \in \mathbb{R}$ for any $\tau \in (-a,\overline{\tau}_0)$. Let

$$D_0(\tau) = \{x \in \mathbb{R} : \overline{v}_x(x,\tau) = 0\} \quad \forall \tau \in (-a,\overline{\tau}_0).$$

Then $x_0(\tau) \in D_0(\tau)$ for any $-a < \tau < \overline{\tau}_0$. Let $-a < \tau_1 < \overline{\tau}_0$. By (3.3) the equation (2.63) for v_x is uniformly parabolic on any compact subset K of $\mathbb{R} \times [-a,\overline{\tau}_0]$. Hence by the discussion on P.241 of [SGKM] and [CP] $\overline{v}_x(x,\tau_1)$ is an analytic function of $x \in \mathbb{R}$. Hence either all the zeros of $\overline{v}_x(x,\tau_1)$ are isolated zeros or

$$\overline{v}_x(x,\tau_1) = 0 \quad \forall x \in \mathbb{R}. \tag{3.13}$$

If (3.13) holds, then by (3.11) we get (3.12) and contradiction arises. Hence (3.13) is not possible and all the zeros of $\overline{v}_x(x,\tau_1)$ are isolated zeros. Putting $i = i_l$ and letting $l \to \infty$ in (3.9),

$$\overline{v}_x(x,\tau_1) \ge 0 \ge \overline{v}_x(y,\tau_1) \quad \forall x \le x_0(\tau_1) \le y. \tag{3.14}$$

By (3.14) an argument similar to the proof of Lemma 2.12 $D_0(\tau_1) \cap (x_0(\tau_1),\infty) = \phi$ and $D_0(\tau_1) \cap (-\infty, x_0(\tau_1)) = \phi$. Hence $D_0(\tau_1) = \{x_0(\tau_1)\}$. Since $\overline{v}_x(x,\tau_1)$ is an analytic function of $x \in \mathbb{R}$, by (3.14),

$$\overline{v}_x(x,\tau_1) > 0 > \overline{v}_x(y,\tau_1) \quad \forall x < x_0(\tau_1) < y$$

and (3.5) follows.

We now let $v_0(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$. Since by Lemma 2.6 $\overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$ is a supersolution of (1.6), by (3.1) and Lemma 2.2,

$$\begin{aligned}
&\overline{v}(x,\tau) \le \overline{f}_{\lambda,\lambda',h,h',k}(x,-a) \quad \forall x \in \mathbb{R}, -a \le \tau < \overline{\tau}_0 \\
\Rightarrow\quad &\overline{v}(x,\tau_1+\tau) \le \overline{v}(x,-a+\tau) \quad \forall x \in \mathbb{R}, -a \le \tau_1 < \overline{\tau}_0, 0 \le \tau \le \overline{\tau}_0 - \tau_1.
\end{aligned}$$

Hence $\overline{v}(\cdot,\tau)$ is also a monotone decreasing function of $\tau \in [-a,\tau_0]$.

Let $y(\tau)$ and $z(\tau)$ be as in Lemma 2.5. By Lemma 2.5 $v_0(x) = v_{\lambda,h}(x,f(-a))$ is a monotone increasing function on $(-\infty, y(-a))$, $v_0(x) = \xi_k(-a)$ is a constant function on $[y(-a), z(-a)]$, and $v_0(x) = \overline{v}_{\lambda,h}(x,f(-a))$ is a monotone decreasing function on $[z(-a),\infty)$. Hence v_0 satisfies (3.4) with $x_0 = y(-a)$. Thus by the above argument for any $-a < \tau < \overline{\tau}_0$ there exists $x_a(\tau) \in \mathbb{R}$ such that (3.5) holds.

Suppose now $v_0(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-a)$ on $\mathbb{R}$ and $\lambda = \lambda'$. Let $x_0 := \frac{h'-h}{2}$. Then $v_0(x+x_0) = v_0(-x+x_0)$ for any $x \in \mathbb{R}$. Since both $\overline{v}(x+x_0,\tau)$ and $\overline{v}(-x+x_0,\tau)$ satisfies (3.2) and (3.3), by Lemma 2.2 and (3.5),

$$\begin{aligned}
&\overline{v}(x+x_0,\tau) = \overline{v}(-x+x_0,\tau) \quad \forall x \in \mathbb{R}, -a \le \tau \le \overline{\tau}_0 \\
\Rightarrow\quad &\overline{v}_x(x_0,\tau) = 0 \quad \forall -a \le \tau \le \overline{\tau}_0 \\
\Rightarrow\quad &x_a(\tau) = x_0 \quad \forall -a \le \tau \le \overline{\tau}_0.
\end{aligned}$$

□

By Lemma 2.10, Lemma 2.11, Lemma 2.13, Lemma 2.14, Lemma 3.1 and a similar argument we have the following two results.

Lemma 3.3. *Let $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h'_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(\mathbb{R})$ satisfies*

$$f_{\lambda,\lambda',h,h'}(x,-a) \leq v_0(x) \leq \overline{f}_{\lambda,\lambda',h,h'}(x,-a) \quad a.e.\ x \in \mathbb{R}$$

where f is given by (1.24). *Then there exists a unique solution $\overline{v} \in C^{2,1}(\mathbb{R} \times (-a, \overline{\tau}_0])$ of* (3.2) *which satisfies*

$$\begin{cases} f_{\lambda,\lambda',h,h'}(x,\tau) \leq \overline{v}(x,\tau) \leq \|v_0\|_{L^\infty(\mathbb{R})} & \forall x \in \mathbb{R}, -a < \tau \leq \overline{\tau}_0 \\ \overline{v}(x,\tau) \leq \overline{f}_{\lambda,\lambda',h,h'}(x,\tau) \quad \forall x \in \mathbb{R}, -a < \tau \leq \overline{\tau}_0. \end{cases}$$

If there exists $x_0 \in \mathbb{R}$ such that (3.4) *holds, then for any $-a < \tau < \overline{\tau}_0$ there exists $x_a(\tau) \in \mathbb{R}$ such that* (3.5) *holds.*

Moreover if $v_0(x) = \overline{f}_{\lambda,\lambda',h,h'}(x,-a)$, then $\overline{v}(\cdot,\tau)$ is a monotone decreasing function of $\tau \in [-a, \overline{\tau}_0]$ and for any $a < \tau < \overline{\tau}_0$ there exists $x_a(\tau) \in \mathbb{R}$ such that (3.5) *holds.*

Furthermore if $v_0(x) = \overline{f}_{\lambda,\lambda',h,h'}(x,-a)$ on $\mathbb{R}$ and $\lambda = \lambda'$, then $x_a(\tau) = \frac{h'-h}{2}$ and $v(x,\tau)$ is symmetric with respect to $x_0 := \frac{h'-h}{2}$ for any $-a < \tau < \overline{\tau}_0$.

Lemma 3.4. *Let $\lambda > 1$, $h \geq h_0$, $0 < k \leq k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $a > -\overline{\tau}_0$ and $v_0 \in L^\infty(\mathbb{R})$ satisfies*

$$f_{\lambda,h,k}(x,-a) \leq v_0(x) \leq \overline{f}_{\lambda,h,k}(x,-a) \quad a.e.\ x \in \mathbb{R}$$

where f is given by (1.24). *Then there exists a unique solution $\overline{v} \in C^{2,1}(\mathbb{R} \times (-a, \overline{\tau}_0])$ of* (3.2) *which satisfies*

$$\begin{cases} f_{\lambda,h,k}(x,\tau) \leq \overline{v}(x,\tau) \leq \|v_0\|_{L^\infty(\mathbb{R})} & \forall x \in \mathbb{R}, -a < \tau \leq \overline{\tau}_0 \\ \overline{v}(x,\tau) \leq \overline{f}_{\lambda,h,k}(x,\tau) \quad \forall x \in \mathbb{R}, -a < \tau \leq \overline{\tau}_0. \end{cases}$$

If v_0 is monotone increasing in $\mathbb{R}$, then

$$v_x(x,\tau) > 0 \quad \forall x \in \mathbb{R}, -a < \tau < \overline{\tau}_0.$$

Moreover if $v_0(x) = \overline{f}_{\lambda,h,k}(x,-a)$, then $\overline{v}(\cdot,\tau)$ is a monotone decreasing function of $\tau \in [-a, \overline{\tau}_0]$.

Theorem 3.5. *Let $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h'_0$, $0 < k \leq k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $j_0 \in \mathbb{N}$ be such that $j_0 > -\overline{\tau}_0$ and $\{v_{0,j}\}_{j \geq j_0} \subset L^\infty(\mathbb{R})$ be such that*

$$f_{\lambda,\lambda',h,h',k}(x,-j) \leq v_{0,j}(x) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x,-j) \quad a.e.\ x \in \mathbb{R} \quad \forall j \geq j_0.$$

Let $v_j \in C^{2,1}(\mathbb{R} \times (-j, \overline{\tau}_0])$ be the unique solution of (3.2) *in $\mathbb{R} \times (-j, \overline{\tau}_0)$ with $a = j$ and $v_0 = v_{0,j}$ given by Lemma 3.2. Then the sequence $\{v_j\}_{j \geq j_0}$ has a subsequence $\{v_{j_l}\}$ that converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to a solution $v = v_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ which satisfies* (1.25) *as $l \to \infty$.*

If for each $j \geq j_0$, there exists $x_{0,j} \in \mathbb{R}$ such that

$$\begin{cases} v_{0,j}(x) \text{ is monotone increasing on } (-\infty, x_{0,j}] \\ v_{0,j}(x) \text{ is monotone decreasing on } [x_{0,j}, \infty) \end{cases} \tag{3.15}$$

holds, then for any $\tau < \overline{\tau}_0$ there exists $x_0(\tau) \in \mathbb{R}$ such that

$$\begin{cases} v_x(x,\tau) > 0 > v_x(y,\tau) & \forall x < x_0(\tau) < y, \tau < \overline{\tau}_0 \\ v_x(x_0(\tau),\tau) = 0 & \forall \tau < \overline{\tau}_0. \end{cases} \tag{3.16}$$

If $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-j)$ for all $j \geq j_0$, the solution $v(x,\tau)$ is a monotone decreasing function of $\tau \in (-\infty, \overline{\tau}_0]$ and for any $\tau < \overline{\tau}_0$ there exists $x_0(\tau) \in \mathbb{R}$ such that (3.16) *holds. Moreover in this case, v_j will converge uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $v_{\lambda,\lambda',h,h',k}$ as $j \to \infty$.*

Furthermore if $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-j)$ on $\mathbb{R}$ and $\lambda = \lambda'$, then $x_0(\tau) = \frac{h'-h}{2}$ and $v(x,\tau)$ is symmetric with respect to $x_0 := \frac{h'-h}{2}$ for any $\tau < \overline{\tau}_0$.

Proof: By Lemma 3.2,

$$f_{\lambda,\lambda',h,h',k}(x,\tau) \leq v_j(x,\tau) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x,\tau) \quad \forall x \in \mathbb{R}, -j < \tau \leq \overline{\tau}_0, j \geq j_0. \tag{3.17}$$

By (3.17) the equation (1.6) for the sequence $\{v_j\}_{j \geq j_0}$ is uniformly parabolic on every compact subset of $\mathbb{R} \times (-\infty, \tau_0]$. Then by the parabolic Schauder estimates [LSU] the sequence $\{v_j\}_{j \geq j_0}$ is equi-Holder continuous in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$. Hence by the Ascoli Theorem and a diagonalization argument the sequence $\{v_j\}_{j \geq j_0}$ has a subsequence $\{v_{j_l}\}_{l=1}^{\infty}$ that converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to some solution $v = v_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ as $l \to \infty$. Letting $j = j_l \to \infty$ in (3.17), we get (1.25).

Suppose now for each $j \geq j_0$, there exists $x_{0,j} \in \mathbb{R}$ such that (3.15) holds. Then by Lemma 3.2 for any $j \geq j_0$ and $-j < \tau < \overline{\tau}_0$ there exists $x_j(\tau) \in \mathbb{R}$ such that

$$\begin{cases} v_{j,x}(x,\tau) > 0 > v_{j,x}(y,\tau) & \forall x < x_j(\tau) < y, -j < \tau < \overline{\tau}_0 \\ v_{j,x}(x_a(\tau),\tau) = 0 & \forall -j < \tau < \overline{\tau}_0. \end{cases} \tag{3.18}$$

Then by (3.17), (3.18) and an argument similar to the proof of Lemma 3.2, $x_0(\tau) := \lim_{j \to \infty} x_j(\tau) \in \mathbb{R}$ exists for any $\tau < \overline{\tau}_0$ and $x_0(\tau)$ satisfies (3.16).

If $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-j)$ for all $j \geq j_0$, then by Lemma 3.2 for any $j \geq j_0$ the solution $v_j(\cdot,\tau)$ is a monotone decreasing function of $\tau \in (-j, \overline{\tau}_0]$. Hence $v(\cdot,\tau)$ is a monotone decreasing function of $\tau \in (-\infty, \overline{\tau}_0]$. Let $y(\tau)$ and $z(\tau)$ be as in Lemma 2.5. By Lemma 2.5 $v_{0,j}(x) = v_{\lambda,h}(x, f(-j))$ is a monotone increasing function on $(-\infty, y(-j))$, $v_{0,j}(x) = \xi_k(-j)$ is a constant function on $[y(-j), z(-j)]$, and $v_{0,j}(x) = \overline{v}_{\lambda,h}(x, f(-j))$ is a monotone decreasing function on $[z(-j), \infty)$. Hence $v_{0,j}$ satisfies (3.15) with $x_{0,j} = y(-j)$. Thus by the above argument for any $\tau < \overline{\tau}_0$ there exists $x_0(\tau) \in \mathbb{R}$ such that (3.16) holds.

Suppose $\{v_{j_l'}\}_{l=1}^{\infty}$ is a another subsequence of $\{v_j\}_{j \geq j_0}$ which converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to some solution $\overline{v}_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ as $l \to \infty$ and $\overline{v}_{\lambda,\lambda',h,h',k}$ also satisfies (1.25). Then by (1.25) and Lemma 2.2,

$$\begin{aligned} & v_{\lambda,\lambda',h,h',k}(x,-j) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x,-j) = v_{0,j}(x) && \forall x \in \mathbb{R}, j \geq j_0 \\ \Rightarrow \quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq v_j(x,\tau) && \forall x \in \mathbb{R}, -j < \tau < \overline{\tau}_0, j \geq j_0 \\ \Rightarrow \quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq \overline{v}_{\lambda,\lambda',h,h',k}(x,\tau) && \forall x \in \mathbb{R}, \tau < \overline{\tau}_0 \quad \text{as } j = j_l' \to \infty. \end{aligned} \tag{3.19}$$

By interchanging the role of $v_{\lambda,\lambda',h,h',k}$ and $\overline{v}_{\lambda,\lambda',h,h',k}$ in (3.19),

$$\overline{v}_{\lambda,\lambda',h,h',k}(x,\tau) \le v_{\lambda,\lambda',h,h',k}(x,\tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0. \tag{3.20}$$

Hence by (3.19) and (3.20),

$$v_{\lambda,\lambda',h,h',k}(x,\tau) = \overline{v}_{\lambda,\lambda',h,h',k}(x,\tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0.$$

Thus v_j converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $v_{\lambda,\lambda',h,h',k}$ as $j \to \infty$.

Suppose now $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-j)$ on $\mathbb{R}$ for any $j \ge j_0$ and $\lambda = \lambda'$. Then by Lemma 3.2 v_j is symmetric with respect to $x_0 := \frac{h'-h}{2}$ and $x_j(\tau) = \frac{h'-h}{2} = x_0$ for any $-j < \tau < \overline{\tau}_0$. Hence $x_0(\tau) = lim_{j\to\infty} x_j(\tau) = \frac{h'-h}{2} = x_0$ and $v(x,\tau)$ is symmetric with respect to $x_0 := \frac{h'-h}{2}$ for any $\tau < \overline{\tau}_0$. □

By an argument similar to the proof of Theorem 3.5 but with Lemma 3.3 (Lemma 3.4 respectively) replacing Lemma 3.2 in the proof we obtain the following two theorems.

Theorem 3.6. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $j_0 \in \mathbb{N}$ be such that $j_0 > -\overline{\tau}_0$ and $\{\overline{v}_{0,j}\}_{j \ge j_0} \subset L^\infty(\mathbb{R})$ be such that*

$$f_{\lambda,\lambda',h,h'}(x,-j) \le \overline{v}_{0,j}(x) \le \overline{f}_{\lambda,\lambda',h,h'}(x,-j) \quad \forall x \in \mathbb{R}, j_0 \le j \in \mathbb{N}.$$

Let $\overline{v}_j$ be the unique solution of (3.2) *in $\mathbb{R} \times (-j, \overline{\tau}_0)$ with $a = j$ and $v_0 = \overline{v}_{0,j}$ given by Lemma 3.3. Then the sequence $\{\overline{v}_j\}_{j \ge j_0}$ has a subsequence $\{\overline{v}_{j_l}\}$ that converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to a solution $v = v_{\lambda,\lambda',h,h'} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ which satisfies* (1.26) *as $l \to \infty$.*

If for each $j \ge j_0$, there exists $x_{0,j} \in \mathbb{R}$ such that (3.15) *holds, then for any $\tau < \overline{\tau}_0$ there exists $x_0(\tau) \in \mathbb{R}$ such that* (3.16) *holds.*

If $\overline{v}_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h'}(x,-j)$ for all $j \ge j_0$, the solution $v(x,\tau)$ is a monotone decreasing function of $\tau \in (-\infty, \overline{\tau}_0]$ and for any $\tau < \overline{\tau}_0$ there exists $x_0(\tau) \in \mathbb{R}$ such that (3.16) *holds. Moreover in this case, v_j will converge uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $v_{\lambda,\lambda',h,h'}$ as $j \to \infty$.*

Furthermore if $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h'}(x,-j)$ on $\mathbb{R}$ and $\lambda = \lambda'$, then $x_0(\tau) = \frac{h'-h}{2}$ and $v(x,\tau)$ is symmetric with respect to $x_0 := \frac{h'-h}{2}$ for any $\tau < \overline{\tau}_0$.

Theorem 3.7. *Let $\lambda > 1$, $h \ge h_0$, $0 < k \le k_0$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let $j_0 \in \mathbb{N}$ be such that $j_0 > -\overline{\tau}_0$ and $\{\widetilde{v}_{0,j}\}_{j \ge j_0} \subset L^\infty(\mathbb{R})$ be such that*

$$f_{\lambda,h,k}(x,-j) \le \widetilde{v}_{0,j}(x) \le \overline{f}_{\lambda,h,k}(x,-j) \quad \forall x \in \mathbb{R}, j_0 \le j \in \mathbb{N}.$$

Let $\widetilde{v}_j$ be the unique solution of (3.2) *in $\mathbb{R} \times (-j, \overline{\tau}_0)$ with $a = j$ and $v_0 = \widetilde{v}_{0,j}$ given by Lemma 3.4. Then the sequence $\{\widetilde{v}_j\}_{j \ge j_0}$ has a subsequence $\{\widetilde{v}_{j_l}\}$ that converges uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to a solution $v = v_{\lambda,h,k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ which satisfies* (1.28) *as $l \to \infty$.*

If for each $j \geq j_0$, $v_{0,j}$ is monotone increasing on $\mathbb{R}$, then

$$v_x(x,\tau) > 0 \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0.$$

If $\widetilde{v}_{0,j}(x) = \overline{f}_{\lambda,h,k}(x,-j)$ for all $j \geq j_0$, the solution $v(x,\tau)$ is a monotone decreasing function of $\tau \in (-\infty, \overline{\tau}_0]$ and v_j will converge uniformly in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $v_{\lambda,h,k}$ as $j \to \infty$.

Remark 3.8. *Existence of solutions of* (3.2) *for $v_0(x) = \min(v_{\lambda,h}(x,\tau), \overline{v}_{\lambda',h'}(x,\tau))$ or $f_{\lambda,\lambda',h,h',k}(x,-a)$ are also given in [DPKS1] and [DPKS2] respectively. Existence of solution $v_{\lambda,\lambda',h,h',k}$ of Theorem 3.5 with $v_{0,j}(x) = f_{\lambda,\lambda',h,h',k}(x,-j)$ is also proved in [DPKS2] and existence of solution $v_{\lambda,\lambda',h,h'}$ of Theorem 3.6 with $\overline{v}_{0,j}(x) = \min(v_{\lambda,h}(x,-j), \overline{v}_{\lambda',h'}(x,-j))$ is also proved in [DPKS1].*

Remark 3.9. *Suppose $v_{\lambda,\lambda',h,h',k}$, $v_{\lambda,\lambda',h,h'}$ and $v_{\lambda,h,k}$ are the solutions of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ constructed in Theorem 3.5, Theorem 3.6 and Theorem 3.7 by setting $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h',k}(x,-j)$, $\overline{v}_{0,j}(x) = \overline{f}_{\lambda,\lambda',h,h'}(x,-j)$ and $\widetilde{v}_{0,j}(x) = \overline{f}_{\lambda,h,k}(x,-j)$ in Theorem 3.5, Theorem 3.6 and Theorem 3.7 respectively. Let $\overline{v}_j$, $\widetilde{v}_j$, be the unique solutions of* (3.2) *in $\mathbb{R} \times (-j, \overline{\tau}_0)$ with $a = j$ and $v_0 = \overline{v}_{0,j}, \widetilde{v}_{0,j}$, respectively given by Lemma 3.3 and Lemma 3.4. Let $j_0 \in \mathbb{N}$ be such that $j_0 > -\overline{\tau}_0$.*

By (1.25)*, Lemma 2.2, Theorem 3.5, Theorem 3.6 and Theorem 3.7, we have*

$$\begin{aligned}
& v_{\lambda,\lambda',h,h',k}(x,-j) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x,-j) \leq \overline{v}_{0,j}(x) \quad && \forall x \in \mathbb{R}, j > j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq \overline{v}_j(x,\tau) && \forall x \in \mathbb{R}, -j < \tau < \overline{\tau}_0, j > -j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq v_{\lambda,\lambda',h,h'}(x,\tau) && \forall x \in \mathbb{R}, \tau < \overline{\tau}_0 \quad \text{as } j \to \infty
\end{aligned}$$

and

$$\begin{aligned}
& v_{\lambda,\lambda',h,h',k}(x,-j) \leq \overline{f}_{\lambda,\lambda',h,h',k}(x,-j) \leq \widetilde{v}_{0,j}(x) \quad && \forall x \in \mathbb{R}, j > j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq \widetilde{v}_j(x,\tau) && \forall x \in \mathbb{R}, -j < \tau < \overline{\tau}_0, j > -j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h,h',k}(x,\tau) \leq v_{\lambda,h,k}(x,\tau) && \forall x \in \mathbb{R}, \tau < \overline{\tau}_0 \quad \text{as } j \to \infty.
\end{aligned}$$

Hence (1.27) *and* (1.29) *hold.*

Remark 3.10. *Let $\lambda > 1$, $\lambda' > 1$, $h_2 \geq h_1 \geq h_0$, $h_2' \geq h_2 \geq h_0$ and $0 < k_2 \leq k_1 \leq k_0$. Suppose $v_{\lambda,\lambda',h_1,h_1',k_1}$ and $v_{\lambda,\lambda',h_2,h_2',k_2}$ are the solutions of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ constructed in Theorem 3.5 by setting $v_{0,j}(x) = \overline{f}_{\lambda,\lambda',h_1,h_1',k_1}(x,-j), \overline{f}_{\lambda,\lambda',h_2,h_2',k_2}(x,-j)$, in Theorem 3.5 respectively. Let v_j be the unique solution of* (3.2) *in $\mathbb{R} \times (-j, \overline{\tau}_0)$ with $a = j$ and $v_0 = \overline{f}_{\lambda,\lambda',h_2,h_2',k_2}(x, f(-j))$ given by Lemma 3.2. Let $j_0 \in \mathbb{N}$ be such that $j_0 > -\overline{\tau}_0$. By* (1.25)*, Lemma 2.2 and Theorem 3.5,*

$$\begin{aligned}
& v_{\lambda,\lambda',h_1,h_1',k_1}(x,-j) \leq \overline{f}_{\lambda,\lambda',h_1,h_1',k_1}(x,-j) \leq \overline{f}_{\lambda,\lambda',h_2,h_2',k_2}(x,-j) \quad && \forall x \in \mathbb{R}, j > j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h_1,h_1',k_1}(x,\tau) \leq v_j(x,\tau) && \forall x \in \mathbb{R}, -j < \tau < \overline{\tau}_0, j > -j_0 \\
\Rightarrow\quad & v_{\lambda,\lambda',h_1,h_1',k_1}(x,\tau) \leq v_{\lambda,\lambda',h_2,h_2',k_2}(x,\tau) && \forall x \in \mathbb{R}, \tau < \overline{\tau}_0 \quad \text{as } j \to \infty.
\end{aligned}$$

and (iv) of Theorem 1.1 follows.

Theorem 3.11. *Let $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h_0'$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Let V_1, $V_2 \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$, be solutions of* (1.6) *in* $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ *which satisfies* (1.26). *Then* $V_1 = V_2$ *in* $\mathbb{R}\times(-\infty,\overline{\tau}_0]$.

Proof: Since both V_1 and V_2 satisfies (1.26),

$$|V_1 - V_2| \leq \overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau) \quad \text{in } \mathbb{R}\times(-\infty,\overline{\tau}_0]. \tag{3.21}$$

Let $\tau \leq \overline{\tau}_0$ and $x(\tau)$ be as in Lemma 2.5. We first claim that $\overline{f}_{\lambda,\lambda',h,h'}(\cdot,\tau) - f_{\lambda,\lambda',h,h'}(\cdot,\tau) \in L^1(\mathbb{R})$. To prove the claim we observe that by (1.18) and (2.21) for any $x \leq x(\tau)$,

$$\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) = v_{\lambda,h}(x,f(\tau)) = O(e^{p(x-\lambda f(\tau)+h)}) \quad \text{as} \quad x \to -\infty. \tag{3.22}$$

On the other hand by (1.18) and (1.19),

$$\begin{aligned} f_{\lambda,\lambda',h,h'}(x,\tau) =& v_{\lambda,h}(x,f(\tau))[1 + v_{\lambda,h}(x,f(\tau))^{p-1}(\overline{v}_{\lambda',h'}(x,f(\tau))^{1-p} - 1)]^{-\frac{1}{p-1}} \\ =& v_{\lambda,h}(x,f(\tau))\left(1 - \frac{1}{p-1}v_{\lambda,h}(x,f(\tau))^{p-1}\left(\overline{v}_{\lambda',h'}(x,f(\tau))^{1-p} - 1\right) + o(v_{\lambda,h}(x,f(\tau))^{p-1})\right) \end{aligned} \tag{3.23}$$

as $x \to -\infty$ where $f(\tau)$ is given by (1.24). By (1.26), (3.22) and (3.23),

$$0 \leq \overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau) = \frac{1}{p-1}v_{\lambda,h}(x,f(\tau))^p\left(\overline{v}_{\lambda',h'}(x,f(\tau))^{1-p} - 1\right) + o\left(v_{\lambda,h}(x,f(\tau))^p\right) \tag{3.24}$$

$$\leq O(v_{\lambda,h}(x,f(\tau))^p) = O(e^{p^2(x-\lambda f(\tau)+h)}) \quad \text{as } x \to -\infty. \tag{3.25}$$

Similarly,

$$\begin{aligned} 0 \leq \overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau) =& \frac{1}{p-1}\overline{v}_{\lambda',h'}(x,f(\tau))^p\left(v_{\lambda,h}(x,f(\tau))^{1-p} - 1\right) + o(\overline{v}_{\lambda',h'}(x,f(\tau))^p) \\ \leq& O(\overline{v}_{\lambda',h'}(x,f(\tau))^p) = O(e^{p^2(-x-\lambda' f(\tau)+h')}) \quad \text{as } x \to \infty. \end{aligned} \tag{3.26}$$

By (3.25) and (3.26) $\overline{f}_{\lambda,\lambda',h,h'}(\cdot,\tau) - f_{\lambda,\lambda',h,h'}(\cdot,\tau) \in L^1(\mathbb{R})$ and the claim follows. Then by Lemma 2.2,

$$\begin{aligned} \int_{\mathbb{R}} |V_1(x,\tau_1) - V_2(x,\tau_1)|\,dx \leq& \int_{\mathbb{R}} |\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau)|e^{(1-m)(\tau_1-\tau)}\,dx \quad \forall \tau < \tau_1 \leq \overline{\tau}_0 \\ =& I_1 + I_2 \end{aligned} \tag{3.27}$$

where

$$\begin{cases} I_1 = \displaystyle\int_{-\infty}^{x(\tau)} |\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau)|e^{(1-m)(\tau_1-\tau)}\,dx \\ I_2 = \displaystyle\int_{x(\tau)}^{\infty} |\overline{f}_{\lambda,\lambda',h,h'}(x,\tau) - f_{\lambda,\lambda',h,h'}(x,\tau)|e^{(1-m)(\tau_1-\tau)}\,dx. \end{cases} \tag{3.28}$$

We will prove that $I_i \to 0$ as $\tau \to -\infty$ for $i = 1, 2$. Since $\overline{v}_{\lambda',h'}(x, f(\tau))$ is a monotone decreasing function of $x \in \mathbb{R}$, by Lemma 2.5,

$$1 > \overline{v}_{\lambda',h'}(x, f(\tau)) \geq \overline{v}_{\lambda',h'}(x(\tau), f(\tau)) \to 1 \quad \text{uniformly on } (-\infty, x(\tau)] \quad \text{as } \tau \to -\infty.$$

Then (3.23) and (3.24) hold uniformly on $(-\infty, x(\tau)]$ as $\tau \to -\infty$. Now by (1.17) and (2.22) for any $x \leq x(\tau)$,

$$\begin{aligned} -x - \lambda' f(\tau) + h' \geq & - x(\tau) - \lambda' f(\tau) + h' = \left(\frac{\gamma_{\lambda'} - \lambda' p - \gamma_\lambda}{p} \right) \tau + O(1) \\ & = - \left(\frac{\gamma_\lambda \gamma_{\lambda'} + p - 1}{p \gamma_{\lambda'}} \right) \tau + O(1) \to \infty \end{aligned} \tag{3.29}$$

uniformly on $(-\infty, x(\tau)]$ as $\tau \to -\infty$. Hence by (1.19) and (3.29),

$$\begin{aligned} \overline{v}_{\lambda',h'}(x, f(\tau))^{1-p} - 1 = & \left(1 - C_{\lambda'} e^{-\gamma_{\lambda'}(-x - \lambda' f(\tau) + h')} + o(e^{-\gamma_{\lambda'}(-x-\lambda' f(\tau)+h')})\right)^{1-p} - 1 \\ = & (p-1) C_{\lambda'} e^{-\gamma_{\lambda'}(-x-\lambda' f(\tau)+h')} + o(e^{-\gamma_{\lambda'}(-x-\lambda' f(\tau)+h')}) \end{aligned} \tag{3.30}$$

uniformly on $(-\infty, x(\tau)]$ as $\tau \to -\infty$. By (3.24) and (3.30),

$$0 < \overline{f}_{\lambda,\lambda',h,h'}(x, \tau) - f_{\lambda,\lambda',h,h'}(x, \tau) \leq C e^{-\gamma_{\lambda'}(-x-\lambda' f(\tau)+h')} \leq C' e^{-\gamma_{\lambda'}(-x-\lambda'\tau+h')} \tag{3.31}$$

on $(-\infty, x(\tau)]$ as $\tau \to -\infty$ for some constants $C > 0$, $C' > 0$. Then by (1.17), (2.22), (3.28) and (3.31),

$$I_1 \leq C \int_{-\infty}^{x(\tau)} e^{-\gamma_{\lambda'}(-x-\lambda'\tau+h')} \cdot e^{-(1-m)\tau}\, dx \leq C' e^{\lambda'\gamma_{\lambda'}\tau - (1-m)\tau + \gamma_{\lambda'} x(\tau)} \leq C'' e^{\frac{\gamma_\lambda \gamma_{\lambda'}}{p}\tau} \to 0 \quad \text{as } \tau \to -\infty. \tag{3.32}$$

Similarly,

$$I_2 \to 0 \quad \text{as } \tau \to -\infty. \tag{3.33}$$

By (3.27), (3.32) and (3.33),

$$\begin{aligned} & \int_{\mathbb{R}} |V_1(x, \tau_1) - V_2(x, \tau_1)|\, dx = 0 \quad \forall \tau_1 \leq \overline{\tau}_0 \\ \Rightarrow \quad & V_1(x, \tau_1) = V_2(x, \tau_1) \qquad\qquad \forall x \in \mathbb{R}, \tau_1 \leq \overline{\tau}_0 \end{aligned}$$

and the theorem follows. □

Corollary 3.12. *Let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. Then the solution $v_{\lambda,\lambda',h,h'} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ which satisfies* (1.26) *given by Theorem 3.6 is a continuous function of $\lambda > 1$, $\lambda' > 1$, $h \geq h_0$, $h' \geq h_0'$.*

Proof: Let $\{(\lambda_i, \lambda_i', h_i, h_i')\}_{i=1}^{\infty} \subset (1, \infty) \times (1, \infty) \times [h_0, \infty) \times [h_0', \infty)$ be a sequence such that $(\lambda_i, \lambda_i', h_i, h_i') \to (\lambda_0, \lambda_0', h_0, h_0')$ as $i \to \infty$ for some $(\lambda_0, \lambda_0', h_0, h_0') \in (1, \infty) \times (1, \infty) \times [h_0, \infty) \times [h_0', \infty)$. For each $i \in \mathbb{N}$ let $v_i := v_{\lambda_i, \lambda_i', h_i, h_i'} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ be the corresponding unique solution of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ which satisfies (1.26) given by Theorem 3.6 and Theorem 3.11 which satisfies

$$f_{\lambda_i, \lambda_i', h_i, h_i'}(x, \tau) \le v_i(x, \tau) \le \overline{f}_{\lambda_i, \lambda_i', h_i, h_i'}(x, \tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0, i \in \mathbb{N}. \tag{3.34}$$

Since both $f_{\lambda_i, \lambda_i', h_i, h_i'}$ and $\overline{f}_{\lambda_i, \lambda_i', h_i, h_i'}$ converges uniformly on any compact subset K of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $f_{\lambda_0, \lambda_0', h_0, h_0'}$ and $\overline{f}_{\lambda_0, \lambda_0', h_0, h_0'}$ as $i \to \infty$, by (3.34) the equation (1.6) for the sequence $\{v_i\}_{i=1}^{\infty}$ is uniformly parabolic on every compact subset of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$. Then by Schauder's estimates [LSU] the sequence $\{v_i\}_{i=1}^{\infty}$ is equi-Holder continuous in $C^{2,1}(K)$ for any compact set $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$. Hence by the Ascoli Theorem and a diagonalization argument the sequence $\{v_i\}_{i=1}^{\infty}$ has a convergence subsequence $\{v_{i_k}\}_{k=1}^{\infty}$ that converges in $C^{2,1}(K)$ for any compact $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to a solution $v \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ as $k \to \infty$. Letting $i = i_k \to \infty$ in (3.34), v satisfies

$$f_{\lambda_0, \lambda_0', h_0, h_0'}(x, \tau) \le v(x, \tau) \le \overline{f}_{\lambda_0, \lambda_0', h_0, h_0'}(x, \tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0, i \in \mathbb{N}. \tag{3.35}$$

By (3.35) and the uniqueness Theorem 3.11, $v = v_{\lambda_0, \lambda_0', h_0, h_0'}$ in $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ where $v_{\lambda_0, \lambda_0', h_0, h_0'}$ is the unique solution of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ given by Theorem 3.6 and Theorem 3.11 which satisfies (3.35). Hence v_i converges in $C^{2,1}(K)$ for any compact $K \subset \mathbb{R} \times (-\infty, \overline{\tau}_0]$ to $v_{\lambda_0, \lambda_0', h_0, h_0'}$ as $i \to \infty$ and the corollary follows. □

Theorem 3.13. *Let $\lambda > 1$, $\lambda' > 1$, $h \ge h_0$, $h' \ge h_0'$ and let $\overline{\tau}_0 < 0$ be as in Lemma 2.5. For any $0 < k \le k_0$ let $v_{\lambda, \lambda', h, h', k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ be a solution of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ constructed in Theorem 3.5 which satisfies* (1.25). *Then $v_{\lambda, \lambda', h, h', k}$ increases and converges uniformly in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ to the unique solution $v_{\lambda, \lambda', h, h'} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of* (1.6) *in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ which satisfies* (1.26) *as $k \searrow 0$.*

Proof: By (1.25),

$$f_{\lambda, \lambda', h, h', k_0}(x, \tau) \le v_{\lambda, \lambda', h, h', k}(x, \tau) \le \overline{f}_{\lambda, \lambda', h, h'}(x, \tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0, 0 < k \le k_0.$$

Hence the equation (1.6) for the family of functions $\{v_{\lambda, \lambda', h, h', k}\}_{0 < k \le k_0}$ is uniformly parabolic on every compact subset K of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$. Then by Schauder's estimates [LSU] the family of functions $\{v_{\lambda, \lambda', h, h', k}\}_{0 < k \le k_0}$ is equi-Holder continuous in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$. Hence by Remark 3.10, the Ascoli Theorem and a diagonalization argument $v_{\lambda, \lambda', h, h', k}$ increases and converges uniformly in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R} \times (-\infty, \overline{\tau}_0]$ to a solution $v \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ as $k \searrow 0$. Letting $k \searrow 0$ in (1.25), v satisfies (1.26). By Theorem 3.11 $v = v_{\lambda, \lambda', h, h'}$ is the unique solution of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ which satisfies (1.26). □

We are now ready for the proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.1: Existence of solution $v = v_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R} \times (-\infty, \overline{\tau}_0])$ of (1.6) in $\mathbb{R} \times (-\infty, \overline{\tau}_0)$ satisfying (1.25) such that $v(x, \tau)$ is a decreasing function of $\tau < \overline{\tau}_0$ is proved in Theorem 3.5. Moreover by Theorem 3.5 and Remark 3.10 we can construct the solution $v = v_{\lambda,\lambda',h,h',k}$ such that v also satisfies (ii) and (iv) of Theorem 1.1.

Proof of (i) of Theorem 1.1: Since $v_{\lambda,h}(x, f(\tau)) \to 1$, $\overline{v}_{\lambda,h}(x, f(\tau)) \to 1$ and $\xi_k(\tau) \to 1$ for any $x \in \mathbb{R}$ as $\tau \to -\infty$, we have

$$f_{\lambda,\lambda',h,h',k}(x, \tau) \to 1 \quad \text{and} \quad \overline{f}_{\lambda,\lambda',h,h',k}(x, \tau) \to 1 \quad \forall x \in \mathbb{R} \quad \text{as } \tau \to -\infty.$$

Hence by (1.25) for any $x \in \mathbb{R}$, $v(x, \tau) \to 1$ as $\tau \to -\infty$ and (i) of Theorem 1.1 follows.

Proof of (iii) of Theorem 1.1: Let d be given by (2.24) and $x(\tau)$ be as in Lemma 2.5. Then $d > \frac{p-1}{p}$. Hence by (1.23) and (i) of Lemma 2.5, there exists constant $C_{\lambda,\lambda',h,h'} > 0$ such that

$$v_{\lambda,h}(x(\tau), f(\tau)) = \overline{v}_{\lambda',h'}(x(\tau), f(\tau)) = 1 - C_{\lambda,\lambda',h,h'} e^{d\tau} + o(e^{d\tau}) > \xi_k(\tau) \quad \text{as } \tau \to -\infty. \tag{3.36}$$

Since

$$1 - \xi_k(\tau)^{p-1} = 1 - \left(1 - ke^{\frac{p-1}{p}\tau}\right)^p = kpe^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right) \quad \text{as } \tau \to -\infty,$$

by (3.36),

$$\begin{aligned}
\max_{x\in\mathbb{R}} f_{\lambda,\lambda',h,h',k}(x, \tau) \ge & f_{\lambda,\lambda',h,h',k}(x(\tau), \tau)\\
\ge & (3\xi_k(\tau)^{1-p} - 2)^{-\frac{1}{p-1}} = \xi_k(\tau)\left(1 + 2(1 - \xi_k(\tau)^{p-1})\right)^{-\frac{1}{p-1}} && \text{as } \tau \to -\infty\\
= & \xi_k(\tau)\left(1 + 2\left(kpe^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right)\right)\right)^{-\frac{1}{p-1}} && \text{as } \tau \to -\infty\\
= & \xi_k(\tau)\left(1 - \frac{2kp}{p-1} e^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right)\right) && \text{as } \tau \to -\infty.
\end{aligned} \tag{3.37}$$

By (1.25) and (3.37),

$$\begin{aligned}
& \xi_k(\tau) \ge v(x_0(\tau), \tau) = \max_{x\in\mathbb{R}} v(x, \tau) \ge \xi_k(\tau)\left(1 - \frac{2kp}{p-1} e^{\frac{p-1}{p}\tau} + o\left(e^{\frac{p-1}{p}\tau}\right)\right) && \text{as } \tau \to -\infty\\
\Rightarrow \quad & |v(x_0(\tau), \tau) - \xi_k(\tau)| \le O\left(e^{\frac{p-1}{p}\tau}\right) && \text{as } \tau \to -\infty
\end{aligned}$$

and (iii) of Theorem 1.1 follows.

Proof of (v) of Theorem 1.1: If $c > \lambda$, then

$$v_{\lambda,h}(x+c\tau, f(\tau)) = v_\lambda(x+(c-\lambda-\lambda qe^{\frac{p-1}{p}\tau})\tau+h) \to 0 \quad \text{uniformly on } (-\infty, A] \quad \forall A \in \mathbb{R} \tag{3.38}$$

as $\tau \to -\infty$. By (1.25) and (3.38),

$$0 < v(x + c\tau, \tau) \le \overline{f}_{\lambda,\lambda',h,h',k}(x + c\tau, \tau) \le v_{\lambda,h}(x + c\tau, f(\tau)) \to 0 \quad \text{uniformly on } (-\infty, A]$$

for any $A \in \mathbb{R}$ as $\tau \to -\infty$ and (v)(a) of Theorem 1.1 follows.

If $-\lambda' < c < \lambda$, then

$$\begin{cases} v_{\lambda,h}(x + c\tau, f(\tau)) = v_\lambda(x + (c - \lambda - \lambda q e^{\frac{p-1}{p}\tau})\tau + h) \to 1 \\ \overline{v}_{\lambda',h'}(x + c\tau, f(\tau)) = v_{\lambda'}(-x - (c + \lambda' + \lambda' q e^{\frac{p-1}{p}\tau})\tau + h') \to 1 \end{cases} \tag{3.39}$$

uniformly on any compact subset of $\mathbb{R}$ as $\tau \to -\infty$. By (1.23), (1.25) and (3.39),

$$1 \ge v(x + c\tau, \tau) \ge f_{\lambda,\lambda',h,h',k}(x + c\tau, \tau) \to 1$$

uniformly on any compact subset of $\mathbb{R}$ as $\tau \to -\infty$ and (v)(b) of Theorem 1.1 follows.

If $c < -\lambda'$, then

$$\overline{v}_{\lambda',h'}(x + c\tau, f(\tau)) = v_{\lambda'}(-x - (c + \lambda' + \lambda' q e^{\frac{p-1}{p}\tau})\tau + h') \to 0 \quad \text{uniformly on } [A, \infty) \quad \forall A \in \mathbb{R} \tag{3.40}$$

as $\tau \to -\infty$. By (1.25) and (3.40),

$$0 < v(x + c\tau, \tau) \le \overline{v}_{\lambda',h'}(x + c\tau, f(\tau)) \to 0 \quad \text{uniformly on } [A, \infty)$$

for any $A \in \mathbb{R}$ as $\tau \to -\infty$ and (v)(c) of Theorem 1.1 follows.

If $c = \lambda$, then

$$\begin{aligned} |v_{\lambda,h}(x + c\tau, f(\tau)) - v_\lambda(x + h)| =& |v_\lambda(x + h - \lambda q\tau e^{\frac{p-1}{p}\tau}) - v_\lambda(x + h)| \\ \le& \|v_\lambda'\|_{L^\infty(\mathbb{R})}\lambda q|\tau| e^{\frac{p-1}{p}\tau} \to 0 \quad \text{uniformly on } \mathbb{R} \quad \text{as } \tau \to -\infty \end{aligned} \tag{3.41}$$

and

$$\overline{v}_{\lambda',h'}(x + c\tau, f(\tau)) = v_{\lambda'}(-x - (\lambda + \lambda' + \lambda' q e^{\frac{p-1}{p}\tau})\tau + h') \to 1 \quad \text{uniformly on } (-\infty, A] \tag{3.42}$$

for any $A \in \mathbb{R}$ as $\tau \to -\infty$. Since by (1.23) $\xi_k(\tau) \to 1$ as $\tau \to -\infty$, by (1.25), (3.41) and (3.42),

$$\overline{f}_{\lambda,\lambda',h,h',k}(x + c\tau, \tau) \le v_{\lambda,h}(x + c\tau, f(\tau)) \to v_\lambda(x + h) \quad \text{uniformly on } \mathbb{R} \quad \text{as } \tau \to -\infty \tag{3.43}$$

and

$$f_{\lambda,\lambda',h,h',k}(x + c\tau, \tau) \to v_\lambda(x + h) \quad \text{uniformly on } (-\infty, A] \quad \forall A \in \mathbb{R} \quad \text{as } \tau \to -\infty \tag{3.44}$$

By (1.25), (3.43) and (3.44), (v)(d) of Theorem 1.1 follows.

If $c = -\lambda'$, then

$$\begin{aligned} |\overline{v}_{\lambda',h'}(x + c\tau, f(\tau)) - v_{\lambda'}(-x + h')| =& |v_{\lambda'}(-x + h' - \lambda' q\tau e^{\frac{p-1}{p}\tau}) - v_{\lambda'}(-x + h')| \\ \le& \|v_{\lambda'}'\|_{L^\infty(\mathbb{R})}\lambda' q|\tau| e^{\frac{p-1}{p}\tau} \to 0 \quad \text{uniformly on } \mathbb{R} \quad \text{as } \tau \to -\infty \end{aligned} \tag{3.45}$$

and

$$v_{\lambda,h}(x+c\tau, f(\tau)) = v_\lambda(x-(\lambda+\lambda'+\lambda q e^{\frac{p-1}{p}\tau})\tau+h) \to 1 \quad \text{uniformly on } [A,\infty) \tag{3.46}$$

for any $A \in \mathbb{R}$ as $\tau \to -\infty$. By (1.23), (3.45) and (3.46),

$$f_{\lambda,\lambda',h,h',k}(x+c\tau,\tau) \to v_{\lambda'}(-x+h') \quad \text{uniformly on } [A,\infty) \quad \forall A \in \mathbb{R} \quad \text{as } \tau \to -\infty \tag{3.47}$$

and

$$\overline{f}_{\lambda,\lambda',h,h',k}(x+c\tau,\tau) \le \overline{v}_{\lambda',h'}(x+c\tau, f(\tau)) \to v_{\lambda'}(-x+h') \quad \text{uniformly on } \mathbb{R} \quad \text{as } \tau \to -\infty. \tag{3.48}$$

By (1.25), (3.47) and (3.47), (v)(e) of Theorem 1.1 follows.

□

Proof of Theorem 1.2: By Theorem 3.6, Remark 3.9, Theorem 3.11, Theorem 3.13 and an argument similar to the proof of Theorem 1.1, there exists a unique solution $v = v_{\lambda,\lambda',h,h'} \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ of (1.6) in $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ which satisfies (1.26), (1.27) and (i), (ii), (v) of Theorem 1.2. By an argument similar to the proof of Remark 3.10, (iv) of Theorem 1.2 holds. Note the (vii) of Theorem 1.2 is proved in Theorem 3.13.

Proof of (iii) of Theorem 1.2: Let $x(\tau)$ be as in Lemma 2.5 and $x_0(\tau)$ be given by (ii) of Theorem 1.2. By (1.17),

$$-x(\tau)-\lambda' f(\tau) = -\left(\frac{\gamma_\lambda\gamma_{\lambda'}+p-1}{p\gamma_{\lambda'}}\right)\tau + O(1) \to \infty \quad \text{as } \tau \to -\infty.$$

Hence by (1.19) there exists a constant $C_{\lambda'} > 0$ such that

$$\begin{aligned}
\overline{v}_{\lambda',h'}(x(\tau), f(\tau))^{1-p} - 1 =& \left(1 - C_{\lambda'} e^{-\gamma_{\lambda'}(-x(\tau)-\lambda' f(\tau)+h')} + o(e^{-\gamma_{\lambda'}(-x(\tau)-\lambda' f(\tau)+h')})\right)^{1-p} - 1\\
=& \left(1 - C_{\lambda'} e^{d\tau} + o(e^{d\tau})\right)^{1-p} - 1\\
=& (p-1)C_{\lambda'} e^{d\tau} + o(e^{d\tau}) \quad \text{as } \tau \to -\infty
\end{aligned} \tag{3.49}$$

where d is given by (2.24). By (1.26), (2.23) and (3.49),

$$\begin{aligned}
v_{\lambda,h}(x(\tau), f(\tau)) =& \max_{x\in\mathbb{R}} \overline{f}_{\lambda,\lambda',h,h'}(x,\tau)\\
&\ge v(x_0(\tau),\tau) = \max_{x\in\mathbb{R}} v(x,\tau) \ge v(x(\tau),\tau) \ge f_{\lambda,\lambda',h,h'}(x(\tau),\tau)\\
&= v_{\lambda,h}(x(\tau), f(\tau))\left(1 + v_{\lambda,h}(x(\tau), f(\tau))^{p-1}\left(\overline{v}_{\lambda',h'}(x(\tau), f(\tau))^{1-p} - 1\right)\right)^{-\frac{1}{p-1}}\\
&= v_{\lambda,h}(x(\tau), f(\tau))\left(1 + (p-1)C_{\lambda'} v_{\lambda,h}(x(\tau), f(\tau))^{p-1} e^{d\tau} + o(e^{d\tau})\right)^{-\frac{1}{p-1}}\\
&= v_{\lambda,h}(x(\tau), f(\tau))\left(1 - C_{\lambda'} v_{\lambda,h}(x(\tau), f(\tau))^{p-1} e^{d\tau} + o(e^{d\tau})\right) \quad \text{as } \tau \to -\infty.
\end{aligned}$$

Hence

$$\left|\max_{x\in\mathbb{R}} v(x,\tau) - v_{\lambda,h}(x(\tau), f(\tau))\right| \le (C_{\lambda'}+1)e^{d\tau} \le C e^{\frac{p-1}{p}\tau} \quad \text{as } \tau \to -\infty$$

and (iii) of Theorem 1.2 follows.

Proof of (vi) of Theorem 1.2: Since $v_{\lambda,h}(x, f(\tau))$ converges to 1 uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ as $h \to \infty$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$, both $f_{\lambda,\lambda',h,h'}(x,\tau)$ and $\overline{f}_{\lambda,\lambda',h,h'}(x,\tau)$ converges to $\overline{v}_{\lambda',h'}(x, f(\tau))$ uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$ as $h \to \infty$. Hence by (1.26) $v_{\lambda,\lambda',h,h'}$ converges to $\overline{v}_{\lambda',h'}(x, f(\tau))$ uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$ as $h \to \infty$. The proof of the other case $h' \to \infty$ then also follows by a similar argument.

□

Proof of Theorem 1.3: By Theorem 3.7, Remark 3.9 and an argument similar to the proof of Theorem 1.1, there exists a solution $v = v_{\lambda,h,k} \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ of (1.6) in $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ which satisfies (1.28), (1.29) and (i), (ii), (v) of Theorem 1.3. By choosing $v_{0,j} = \overline{f}_{\lambda,h,k}(x,-j)$ in the construction of the solution $v_{\lambda,h,k}$ in Theorem 3.7 and an argument similar to the proof of Remark 3.10, we can construct the solution $v_{\lambda,h,k}$ such that (iv) of Theorem 1.3 holds. Note that (iii) of Theorem 1.3 follows directly from (1.28).

Since ξ_k converges to 1 uniformly on $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ as $k \to 0$, both $f_{\lambda,h,k}$ and $\overline{f}_{\lambda,h,k}$ converges to $v_{\lambda,h}(x, f(\tau))$ uniformly on every compact subset of $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ as $k \to 0$. Hence by (1.28), (vi) of Theorem 1.3 follows.

Since $v_{\lambda,h}(x, f(\tau))$ converge to 1 uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ as $h \to \infty$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$, both $f_{\lambda,h,k}$ and $\overline{f}_{\lambda,h,k}$ converges to ξ_k uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ as $h \to \infty$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$. Hence by (1.28) $v_{\lambda,h,k}$ converges to ξ_k uniformly on $[A,\infty)\times[\tau_1,\overline{\tau}_0]$ as $h \to \infty$ for any $A \in \mathbb{R}$ and $\tau_1 < \overline{\tau}_0$ and (vii) of Theorem 1.3 follows.

Proof of (viii) of Theorem 1.3: Let $v_{\lambda,\lambda',h,h',k} \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ be a solution of (1.6) in $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ given by Theorem 1.1 that satisfies (1.25). Then by (1.25),

$$f_{\lambda,\lambda',h,h_0',k}(x,\tau) \le v_{\lambda,\lambda',h,h',k}(x,\tau) \le \overline{f}_{\lambda,h,k}(x,\tau) \quad \forall x \in \mathbb{R}, \tau < \overline{\tau}_0, h' \ge h_0'.$$

Hence the equation (1.6) for the family of ancient solutions $\{v_{\lambda,\lambda',h,h',k}\}_{h'\ge h_0'}$ is uniformly parabolic on every compact subset K of $\mathbb{R}\times(-\infty,\overline{\tau}_0]$. By the parabolic Schauder estimates [LSU] the family $\{v_{\lambda,\lambda',h,h',k}\}_{h'\ge h_0'}$ is uniformly equi-Holder continuous in $C^{2,1}(K)$ on every compact subset K of $\mathbb{R}\times(-\infty,\overline{\tau}_0]$. Then by the Ascoli Theorem and (iv) of Theorem 1.1, the sequence $\{v_{\lambda,\lambda',h,h',k}\}_{h'\ge h_0'}$ increases and converges in $C^{2,1}(K)$ for any compact subset K of $\mathbb{R}\times(-\infty,\overline{\tau}_0]$ as $h' \to \infty$ to a solution $v \in C^{2,1}(\mathbb{R}\times(-\infty,\overline{\tau}_0])$ of (1.6) in $\mathbb{R}\times(-\infty,\overline{\tau}_0)$ that satisfies (1.29) with $v_{\lambda,h,k}$ there being replaced by v. Letting $h' \to \infty$ in (1.25), v satisfies (1.28).

□

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