Autocorrelation and Lower Bound on the 2-Adic Complexity of LSB Sequence of p-ary m-Sequence

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Abstract

In modern stream cipher, there are many algorithms, such as ZUC, LTE encryption algorithm and LTE integrity algorithm, using bit-component sequences of p-ary m-sequences as the inputs of the algorithms. Therefore, analyzing their statistical properties (For example, autocorrelation, linear complexity and 2-adic complexity) of bit-component sequences of p-ary m-sequences is becoming an important research topic. In this paper, we first derive some autocorrelation properties of LSB (Least Significant Bit) sequences of p-ary m-sequences, i.e., we convert the problem of computing autocorrelations of LSB sequences of period p^n-1 for any positive $n\geq 2$ to the problem of determining autocorrelations of LSB sequences of period p-1. Then, based on these properties and computer calculation, we list some autocorrelation distributions of LSB sequences of p-ary msequences with order n for some small prime p's, such as p = 3, 5, 7, 11, 17, 31. Additionally, using their autocorrelation distributions and the method inspired by Hu, we give the lower bounds on the 2-adic complexities of these LSB sequences. Our results show that the main parts of all the lower bounds on the 2-adic complexities of these LSB sequences are larger than $\frac{N}{2}$, where N is the period of these sequences. Therefore, these bounds are large enough to resist the analysis of RAA (Rational Approximation Algorithm) for FCSR (Feedback with Carry Shift Register). Especially, for a Mersenne prime $p=2^k-1$, since all its bit-component sequences of a p-ary m-sequence are shift equivalent, our results hold for all its bit-component sequences.

Index Terms. p-ary m-sequence; LSB sequence; autocorrelation; 2-adic complexity

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1 INTRODUCTION

Pseudo-random sequences with good correlation and large linear complexity have widely applications in communication systems and cryptography. Due to their ideal correlation property and other good performance measures such as highly efficient implementation, maximal length linear feedback shift register (LFSR) sequences (i.e., m-sequence) have been widely used in designing stream ciphers. However, since the linear complexity of these sequences is relatively low under the analysis of Berlekamp-Massey Algorithm (BMA), m sequences can not be used by itself. Therefore constructing nonlinear sequence with desirable good properties is becoming a more and more important topic. As one class of promising nonlinear sequence generators, feedback with carry shift registers (FCSRs) were originally presented by Klapper and Goresky in 1997 [1]. As a consequence, they introduced the notion of 2-adic complexity $\phi_2(s)$ for a binary periodic sequence s, i.e., the length of the shortest FCSR which generates s. And one direct result of this notion is that an m-sequence with period $N = 2^n - 1$ has maximal 2-adic complexity if $2^N - 1$ is a prime. In fact, Tian and Qi [2] proved that all binary m-sequences have maximal 2-adic complexity. Similar to BMA of LFSRs, Klapper and Goresky also proposed an algorithm, called Rational Approximation Algorithm (RAA), to determine the 2-adic complexity of s.

From the perspective of cryptography security, it is obvious that a desirable sequence must have both high linear complexity and high 2-adic complexity, namely, greater than or equal to half of the period. However, although the linear complexity of many classes of sequences have been obtained (See [3]-[14]), there are only a handful papers on 2-adic complexity. After Tian and Qi have made a breakthrough about the 2-adic complexity of all binary m-sequence [2], Xiong et al. [15, 16] presented a new method of circulant matrix to compute the 2-adic complexities of binary sequences. They showed that all the known sequences with ideal 2-level autocorrelation have maximum 2-adic complexity. Moreover, several other classes of sequences with optimal autocorrelation have also maximum complexity. Recently, Hu [17] presented a simpler method to obtain the results of Xiong et al. [15], using detailed autocorrelation values.

Since m-sequences can not be used directly for stream ciphers due to their low linear complexity, many modern stream ciphers, such as ZUC, LTE encryption algorithm and LTE integrity algorithm, are designed by using bit-component sequences (see Definition 1) of p-ary m-sequences as their inputs [18, 19]. We remark that those bit-component sequences can be easily implemented. Earlier, Chan and Games [20] have shown that these sequences have high linear complexities. However, the 2-adic complexities of bit-component sequences of p-ary m-sequences are still not studied as far as we know.

In this paper, we study autocorrelation and 2-adic complexities of LSB sequences (See Definition 1) of p-ary m-sequences for any prime p. We will first present some autocorrelation properties of LSB sequences of p-ary m-sequences. Suppose that s is the LSB sequence of a p-ary m-sequence of

period p^n-1 for any $n\geq 2$. Through these autocorrelation properties, the problem of computing the autocorrelation value $AC_s(\tau)$ for $1\leq \tau \leq p^n-2$ can be simplified to the problem of computing the autocorrelation $AC_b(\tau')$ of LSB sequence b of period p-1 (See Definition 2) only for $1\leq \tau' \leq \frac{p-3}{4}$ (If $p\equiv 3 \mod 4$) or for $1\leq \tau' \leq \frac{p-5}{4}$ (If $p\equiv 1 \mod 4$). As a consequence, we give explicit formula of the autocorrelation distributions of LSB sequences of p-ary m-sequences for some small p=3,5,7,11,17,31. Another consequence of our result is to give lower bounds on the 2-adic complexities of these LSB sequences. Our results show that the main parts of the lower bounds on the 2-adic complexity have a unified form of $\frac{N}{2} + \frac{N}{p-1}$, which is larger than $\frac{N}{2}$, i.e., the 2-adic complexity is large enough to resist RAA for FCSRs. According to our discussion and this unified form, we also present an open problem about the lower bound on the 2-adic complexity of the LSB sequence of any prime p-ary m-sequence. Particularly, since all the bit-component sequences of a p-ary m-sequence are shift equivalent for a Mersenne prime p, our results are available for all its bit-component sequences. Here, our method of determining the lower bounds on the 2-adic complexity is inspired by Hu [17].

The rest of this paper is organized as follows. We introduce notations and some well-known results in Section 2. The autocorrelation properties of LSB sequences of p-ary m-sequences for any prime p, as well as the autocorrelation distributions of LSB sequences of p-ary m-sequences for some small prime p, such as p = 3, 5, 7, 11, 17, 31, are given in Section 3. In Section 4, the lower bounds on 2-adic complexities of these LSB sequences are derived.

2 Preliminaries

In this section, we will introduce some notations and some well-known results, which will be used throughout this paper unless specified.

Let N be a positive integer and $s = (s_0, s_1, \dots, s_{N-1})$ a binary sequence of period N. The autocorrelation of s is given by

$$AC_s(\tau) = \sum_{i=0}^{N-1} (-1)^{s_t + s_{t+\tau}}, \ \tau = 0, 1, 2, \dots, N-1.$$
 (1)

Let $S(x) = \sum_{i=0}^{N-1} s_i x^i \in \mathbb{Z}[x]$. Then we write

$$\frac{S(2)}{2^N - 1} = \frac{\sum_{i=0}^{N-1} s_i 2^i}{2^N - 1} = \frac{p}{q}, \ 0 \le p \le q, \ \gcd(p, q) = 1.$$
 (2)

And the 2-adic complexity $\Phi_2(s)$ of the sequence s is the integer $\lfloor \log_2 q \rfloor$, i.e.,

$$\Phi_2(s) = \lfloor \log_2 \frac{2^N - 1}{\gcd(2^N - 1, S(2))} \rfloor,\tag{3}$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x and gcd(x,y) is the greatest common divisor of x and y.

Let p be any prime, n a positive integer, and α a primitive element of \mathbb{F}_{p^n} . Then

$$a_t = \text{Tr}(\alpha^t), \ t = 0, 1, 2, \dots, p^n - 2$$
 (4)

is a *p*-ary *m*-sequence, where $\operatorname{Tr}(x) = x + x^p + x^{p^2} + \dots + x^{p^{n-1}}$ is the trace function from \mathbb{F}_{p^n} to \mathbb{F}_p . For each element a_t of $\{a_t\}_{t=0}^{p^n-2}$, we have the following 2-adic expansion

$$a_t = a_{t,0} + a_{t,1} \times 2 + a_{t,2} \times 2^2 + \dots + a_{t,k-1} \times 2^{k-1}, \ a_{t,i} \in \{0,1\}, \ i = 0, 1, \dots, k-1,$$

where $k = \lceil \log_2 p \rceil$ and $\lceil x \rceil$ is the least integer that is larger than or equal to x. Here, we identify the bit string $(a_{t,0}, a_{t,1}, a_{t,2}, \dots, a_{t,k-1})$ of length k with the element a_t and call the i-th element $a_{t,i-1}$ the i-th bit-component of a_t . But the element $0 \in \mathbb{F}_p$ is written as p, i.e., 0 is identified with $(p_0, p_1, p_2, \dots, p_{k-1})$, where the 2-adic expansion of p is $p_0 + p_1 \times 2 + \dots + p_{k-1} \times 2^{k-1}$ (This is to be in accordance with ZUC algorithm).

Definition 1 For a fixed $i \in \{1, 2, \dots, k\}$, the sequence $\{a_{t,i-1}\}_{t=0}^{p^n-2}$ is called the *i*-th bit-component sequence of $\{a_t\}_{t=0}^{p^n-2}$. Particularly, the bit-component sequence $\{a_{t,0}\}_{t=0}^{p^n-2}$ is called the Least Significant Bit sequence (the LSB sequence) of $\{a_t\}_{t=0}^{p^n-2}$ and we denote $\{s_t\}_{t=0}^{p^n-2} = \{a_{t,0}\}_{t=0}^{p^n-2}$ for convenience. In fact, it can also be expressed as

$$s_t = \begin{cases} \operatorname{Tr}(\alpha^t) \pmod{2}, & \text{if } \operatorname{Tr}(\alpha^t) \in \mathbb{F}_p^*, \\ 1, & \text{if } \operatorname{Tr}(\alpha^t) = 0. \end{cases}$$
 (5)

Definition 2 Suppose that $\beta = \alpha^{\frac{p^n-1}{p-1}}$, a primitive element of \mathbb{F}_p . Then the sequence $\{b_j\}_{j=0}^{p-2}$ of period p-1 is defined as $b_j = \beta^j \pmod{2}$.

Remark 1 Note that $\beta = \alpha$ for n = 1. In fact, one of contributions of this paper is that we convert the problem of computing the autocorrelation $AC_s(\tau)$ of $\{s_t\}_{t=0}^{p^n-2}$ for $\tau = 1, 2, \dots, p^n-2$ to the problem of computing the autocorrelation $AC_b(\tau')$ of $\{b_j\}_{j=0}^{p-2}$ for $\tau' = 1, 2, \dots, \frac{p-5}{4}$ if $p \equiv 1 \mod 4$ and $\tau' = 1, 2, \dots, \frac{p-3}{4}$ if $p \equiv 3 \mod 4$.

Definition 3 A function from \mathbb{F}_{p^n} to \mathbb{F}_p is said to be balanced if the element 0 appears one less time than each nonzero element in \mathbb{F}_p in the list $f(\alpha^0)$, $f(\alpha^1)$, \cdots , $f(\alpha^{p^n-2})$, where α is a primitive element of \mathbb{F}_{p^n} .

Definition 4 Let f(x) be a function on \mathbb{F}_{p^n} over \mathbb{F}_p . Then the function f(x) is called difference-balanced if f(xz) - f(x) is balanced for any $z \in \mathbb{F}_{p^n}$ but $z \neq 1$.

Remark 2 It is well known that the trace function $\operatorname{Tr}(x)$ from \mathbb{F}_{p^n} to \mathbb{F}_p is difference-balanced, which is in fact a linear function over \mathbb{F}_p .

3 Autocorrelation properties of LSB sequences of p-ary msequences

In this section, we will derive some autocorrelation properties of LSB sequences of p-ary m-sequence and give autocorrelation distributions of the LSB sequences for some small prime p, such as 3,5,7,11,17,31. We denote $N = p^n - 1$, $M = \frac{N}{p-1}$, $\{s_t\}_{t=0}^{N-1} = \{a_{t,0}\}_{t=0}^{N-1}$, and $\mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}$ unless specified.

Lemma 1 Let $n \ge 2$. Then, for $0 < \tau < N$ and $\tau \notin \{M\tau' | \tau' = 1, 2, \dots, p-2\}$, the autocorrelation value $AC_s(\tau)$ of $\{s_t\}_{t=0}^{N-1}$ is given by

$$AC_s(\tau) = p^{n-2} - 1.$$

Proof. Recall that the autocorrelation function of $\{s_t\}_{t=0}^{N-1}$ at τ is defined by

$$AC_s(\tau) = \sum_{t=0}^{N-1} (-1)^{s_t + s_{t+\tau}}, \ \tau = 1, 2, \dots, N-1.$$

For a fixed τ , we denote $D_{\tau} = \{t | s_t \neq s_{t+\tau}, \ t \in \mathbb{Z}_N \}$. Then we get

$$AC_s(\tau) = |Z_N \setminus D_{\tau}| - |D_{\tau}| = N - 2|D_{\tau}|. \tag{6}$$

By Eq. (5) in Definition 1, we know

$$|D_{\tau}| = |\{t|s_{t} \neq s_{t+\tau}, \ t \in \mathbb{Z}_{N}\}|$$

$$= |\{t|\operatorname{Tr}(\alpha^{t}) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{t+\tau}) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{t}) \equiv 1 \ (\operatorname{mod} \ 2), \ \operatorname{and} \ \operatorname{Tr}(\alpha^{t+\tau}) \equiv 0 \ (\operatorname{mod} \ 2), \ t \in \mathbb{Z}_{N}\}|$$

$$+ |\{t|\operatorname{Tr}(\alpha^{t}) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{t+\tau}) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{t}) \equiv 0 \ (\operatorname{mod} \ 2), \ \operatorname{and} \ \operatorname{Tr}(\alpha^{t+\tau}) \equiv 1 \ (\operatorname{mod} \ 2), \ t \in \mathbb{Z}_{N}\}|$$

$$+ |\{t|\operatorname{Tr}(\alpha^{t}) = 0, \ \operatorname{Tr}(\alpha^{t+\tau}) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(\alpha^{t+\tau}) = 0 \ (\operatorname{mod} \ 2), \ t \in \mathbb{Z}_{N}\}|$$

$$+ |\{t|\operatorname{Tr}(\alpha^{t+\tau}) = 0, \ \operatorname{Tr}(\alpha^{t}) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(\alpha^{t}) = 0 \ (\operatorname{mod} \ 2), \ t \in \mathbb{Z}_{N}\}|$$

$$= |\{x|\operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{\tau}x) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(x) \equiv 1 \ (\operatorname{mod} \ 2), \ \operatorname{and} \ \operatorname{Tr}(\alpha^{\tau}x) \equiv 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{Tr}(\alpha^{\tau}x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(\alpha^{\tau}x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(x) = 0, \ \operatorname{Tr}(\alpha^{\tau}x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(\alpha^{\tau}x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(\alpha^{\tau}x) = 0, \ \operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(\alpha^{\tau}x) = 0, \ \operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(\alpha^{\tau}x) = 0, \ \operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

$$+ |\{x|\operatorname{Tr}(\alpha^{\tau}x) = 0, \ \operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}, \ \operatorname{and} \ \operatorname{Tr}(x) = 0 \ (\operatorname{mod} \ 2), \ x \in \mathbb{F}_{p^{n}}^{*}\}|$$

In the following, we will determine the values of (7)-(10) respectively. From Definition 1, it is obvious that

$$s_t \neq s_{t+\tau} \Rightarrow \operatorname{Tr}(\alpha^t) - \operatorname{Tr}(\alpha^{t+\tau}) \neq 0 \Rightarrow \operatorname{Tr}(x) - \operatorname{Tr}(\alpha^\tau x) \neq 0$$
, where $x = \alpha^t$.

For a fixed t satisfying $s_t \neq s_{t+\tau}$, without loss of generality, we suppose $\operatorname{Tr}(\alpha^t) - \operatorname{Tr}(\alpha^{t+\tau}) = a \neq 0$, i.e., $\operatorname{Tr}(x) - \operatorname{Tr}(\alpha^\tau x) = a \neq 0$. By Remark 2 we know that the trace function $\operatorname{Tr}(x)$ is difference-balanced, namely, for each fixed $a \in \mathbb{F}_p^*$, the number of x's in $\mathbb{F}_{p^n}^*$ satisfying the equation $\operatorname{Tr}(x) - \operatorname{Tr}(\alpha^\tau x) = a$ is p^{n-1} . And the number of x's to the equation $\operatorname{Tr}(x) - \operatorname{Tr}(\alpha^\tau x) = a$ is exactly the sum of the numbers of x's to the following p equation systems

$$\begin{cases}
\operatorname{Tr}(x) = c + a, \\
\operatorname{Tr}(\alpha^{\tau} x) = c,
\end{cases}$$
(11)

where c runs through \mathbb{F}_p . Note that \mathbb{F}_{p^n} is an n-dimensional vector space over \mathbb{F}_p . Then, for each fixed $a \in \mathbb{F}_p^*$ and $c \in \mathbb{F}_p$, the above equation system is equivalent to a linear equation system over \mathbb{F}_p with n unknowns. when $\alpha^{\tau} \notin \mathbb{F}_p^*$, i.e., $\tau \notin \{M\tau'|\tau'=1,2,\cdots,p-1\}$, the vectors composed of coefficients on the left side of the equations are linearly independent, which implies that there are p^{n-2} solutions in \mathbb{F}_{p^n} to the equation system (11) for each $a \in \mathbb{F}_p^*$ and $c \in \mathbb{F}_p$. Therefore, we can determine the values of (7)-(10) by discussing the values of c + a and c.

Firstly, we will prove that the value of (7) is equal to

$$\begin{aligned} &|\{x|\operatorname{Tr}(x)\in\mathbb{F}_p^*,\ \operatorname{Tr}(\alpha^\tau x)\in\mathbb{F}_p^*,\ \operatorname{Tr}(x)\equiv 1\ (\mathrm{mod}\ 2),\ \mathrm{and}\ \operatorname{Tr}(\alpha^\tau x)\equiv 0\ (\mathrm{mod}\ 2),\ x\in\mathbb{F}_{p^n}^*\}|\\ &=&p^{n-2}\times|\{(c+a,c)|c+a\in\mathbb{F}_p^*,\ c\in\mathbb{F}_p^*,\ c+a\equiv 1\ (\mathrm{mod}\ 2),\ \mathrm{and}\ c=0\ (\mathrm{mod}\ 2),\mathrm{where}\ a\in\mathbb{F}_p^*\}|.\\ &=&p^{n-2}\times\frac{(p-1)^2}{4}.\end{aligned} \tag{12}$$

Note that the addition c+a is operated in \mathbb{F}_p and that p is odd. Then, for an even $c=2k\in\mathbb{F}_p^*$, $1\leq k\leq \frac{p-1}{2}$, $c+a\in\mathbb{F}_p^*$ is odd if and only if a is odd but c< c+a< p or a is even but $p< c+a\leq c+p-1$ (Here the comparison and the addition in c< c+a< p and $p< c+a\leq c+p-1$ are operated in integer set \mathbb{Z}). Furthermore, for odd a but 0< a< p-c=p-2k, the number of a's is $\frac{p-1}{2}-k$, and for even a but $p-c=p-2k< a\leq p-1$, the number of a's is k. Therefore, for a fixed c=2k, the number of pairs (c+a,c) satisfying the condition in the set of (7) is $\frac{p-1}{2}$. Note that the number of c's

is $\frac{p-1}{2}$. Hence Eq. (12) holds. Similarly, we can get the values of (8) is

$$\begin{aligned} &|\{x|\operatorname{Tr}(x)\in\mathbb{F}_p^*,\ \operatorname{Tr}(\alpha^\tau x)\in\mathbb{F}_p^*,\ \operatorname{Tr}(x)\equiv 0\ (\mathrm{mod}\ 2),\ \mathrm{and}\ \operatorname{Tr}(\alpha^\tau x)\equiv 1\ (\mathrm{mod}\ 2),\ x\in\mathbb{F}_{p^n}^*\}|\\ &=&p^{n-2}\times|\{(c+a,c)|c+a\in\mathbb{F}_p^*,\ c\in\mathbb{F}_p^*,\ c+a\equiv 0\ (\mathrm{mod}\ 2),\ \mathrm{and}\ c\equiv 1\ (\mathrm{mod}\ 2), \mathrm{where}\ a\in\mathbb{F}_p^*\}|.\\ &=&p^{n-2}\times\frac{(p-1)^2}{4}.\end{aligned} \tag{13}$$

And the values of (9) and (10) are

$$\begin{split} &|\{x|\text{Tr}(x)=0, \ \text{Tr}(\alpha^{\tau}x)\in \mathbb{F}_p^*, \ \text{and} \ \text{Tr}(\alpha^{\tau}x)=0 \ (\text{mod} \ 2), \ x\in \mathbb{F}_{p^n}^*\}|\\ &=&p^{n-2}\times |\{(c+a,c)|\ c+a=0, \ c\in \mathbb{F}_p^*, \ \text{and} \ c\equiv 0 \ (\text{mod} \ 2), \text{where} \ a\in \mathbb{F}_p^*, \}|\\ &=&p^{n-2}\times |\{(p-a,0)|a\in \mathbb{F}_p^* \ \text{and} \ a \ \text{is} \ \text{odd}\}|=p^{n-2}\times \frac{p-1}{2}, \end{split} \tag{14}$$

and

$$\begin{aligned} &|\{x|\text{Tr}(\alpha^{\tau}x)=0,\ \text{Tr}(x)\in\mathbb{F}_{p}^{*},\ \text{and}\ \text{Tr}(x)=0\ (\text{mod}\ 2),\ x\in\mathbb{F}_{p}^{*}\}|\\ &=p^{n-2}\times|\{(c+a,c)|\ c=0,\ c+a\in\mathbb{F}_{p}^{*},\ \text{and}\ c+a\equiv 0\ (\text{mod}\ 2), \text{where}\ a\in\mathbb{F}_{p}^{*},\}|\\ &=p^{n-2}\times|\{(0,a)|a\in\mathbb{F}_{p}^{*}\ \text{and}\ a\ \text{is even}\}|=p^{n-2}\times\frac{p-1}{2} \end{aligned} \tag{15}$$

respectively. With the above results of (12)-(15), we get

$$|D_{\tau}| = p^{n-2} \left(\frac{(p-1)^2}{4} + \frac{(p-1)^2}{4} + \frac{p-1}{2} + \frac{p-1}{2} \right) = \frac{p^n - p^{n-2}}{2}.$$

By Eq. (6), the autocorrelation is

$$AC_s(\tau) = N - 2 \times \frac{p^n - p^{n-2}}{2} = p^n - 1 - (p^n - p^{n-2}) = p^{n-2} - 1.$$

Lemma 2 For $\tau \in \{M\tau' | \tau' = 1, 2, \cdots, p-2\}$, the autocorrelation of the LSB sequence $\{s_t\}_{t=0}^{p^n-1}$ with period $p^n - 1$ satisfies the following relation

$$AC_s(\tau) = (AC_b(\tau') + 1) p^{n-1} - 1.$$

where the sequence $\{b_j\}_{j=0}^{p-2}$ is defined as in Definition 2 and $AC_b(\tau')$ is the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ at τ' .

Proof. First of all, note that $\alpha^{\tau} = \beta^{\tau'} \in \mathbb{F}_p^*$ for $\tau \in \{M\tau' | \tau' = 1, 2, \dots, p-2\}$, where α is the primitive element of \mathbb{F}_{p^n} introduced in (4), $\beta = \alpha^M$, and $M = \frac{p^n - 1}{p - 1}$. Then, for $x \in \mathbb{F}_{p^n}^*$ we know that $\operatorname{Tr}(\alpha^{\tau} x) = \operatorname{Tr}(\beta^{\tau'} x) = \beta^{\tau'} \operatorname{Tr}(x)$. Therefore,

$$\operatorname{Tr}(\alpha^{\tau} x) \in \mathbb{F}_{p}^{*} \Leftrightarrow \operatorname{Tr}(x) \in \mathbb{F}_{p}^{*}.$$
 (16)

By similar discussion to that in Lemma 1, we have

$$AC_s(\tau) = |Z_N \setminus D_{\tau}| - |D_{\tau}| = N - 2|D_{\tau}|,\tag{17}$$

where $D_{\tau} = \{t | s_t \neq s_{t+\tau}, t \in \mathbb{Z}_N \}$. And we can also get

$$s_t \neq s_{t+\tau} \Rightarrow \operatorname{Tr}(\alpha^t) - \operatorname{Tr}(\alpha^{t+\tau}) \neq 0 \Rightarrow \operatorname{Tr}(x) - \operatorname{Tr}(\alpha^\tau x) \neq 0$$
$$\Rightarrow \operatorname{Tr}(x) \in \mathbb{F}_n^* \text{ and } \operatorname{Tr}(\alpha^\tau x) \in \mathbb{F}_n^*, \text{ where } x = \alpha^t, \tag{18}$$

where (18) comes from (16). Then,

$$|D_{\tau}| = |\{t | s_t \neq s_{t+\tau}, \ t \in \mathbb{Z}_N\}|$$

$$= |\{x | \operatorname{Tr}(x) \in \mathbb{F}_p^*, \ \operatorname{Tr}(x) \not\equiv \beta^{\tau'} \operatorname{Tr}(x) \ (\text{mod } 2), \ x \in \mathbb{F}_{p^n}^*\}|$$

$$= p^{n-1} \times |\{(c, \beta^{\tau'} c) | c \in \mathbb{F}_p^*, \ c \not\equiv \beta^{\tau'} c \ (\text{mod } 2)\}|$$

$$(19)$$

$$=p^{n-1} \times |\{j|\beta^j \not\equiv \beta^{j+\tau'} \pmod{2}, \ j=0,1,2,\cdots,p-2\}| = p^{n-1} \times |D'_{\tau'}|, \tag{20}$$

where $D'_{\tau'}=\{j|\beta^j\not\equiv\beta^{j+\tau'}\pmod{2},\ j=0,1,2,\cdots,p-2\}$ and Eq. (19) holds because the equation ${\rm Tr}(x)=c$ has exact p^{n-1} solutions in $F^*_{p^n}$ for any fixed $c\in\mathbb{F}^*_p$. Hence, by Eq. (17), we have

$$AC_s(\tau) = N - 2|D_{\tau}| = (p^n - 1) - 2p^{n-1}|D'_{\tau'}| = (p - 2|D'_{\tau'}|)p^{n-1} - 1$$
(21)

Recall that the sequence $\{b_j\}_{j=0}^{p-2}$ in Definition 2 is defined by $b_j \equiv \beta^j \pmod{2}$ and that the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ at τ' is given by

$$AC_b(\tau') = \sum_{i=0}^{p-2} (-1)^{b_j - b_{j+\tau'}} = |Z_{p-1} \setminus D'_{\tau'}| - |D'_{\tau'}| = p - 1 - 2|D'_{\tau'}|.$$
(22)

The result follows.

Through the results of Lemmas 1 and 2, we have simplified the problem of computing the autocorrelation of the LSB sequence $\{s_t\}_{t=0}^{N-1}$ of period p^n-1 for any positive integer $n \geq 2$ to the problem of computing the autocorrelation of the LSB sequence $\{b_j\}_{j=0}^{p-2}$. Next, we will present the autocorrelation properties of the sequence $\{b_j\}_{j=0}^{p-2}$.

Lemma 3 With the symbols be the same as above, we have the following results.

- (1) For $1 \le \tau' \le \frac{p-3}{2}$, $AC_b(p-1-\tau') = AC_b(\tau')$.
- (2) For $p \equiv 1 \mod 4$ and $1 \le \tau' \le \frac{p-1}{4}$ or for $p \equiv 3 \mod 4$ and $1 \le \tau' \le \frac{p-3}{4}$, we get $AC_b(\frac{p-1}{2} \tau') = -AC_b(\tau')$. Particularly, when $p \equiv 1 \mod 4$, we have $AC_b(\frac{p-1}{4}) = 0$.
- (3) $AC_b(\frac{p-1}{2}) = -(p-1).$

Proof: (1) From the discussion in Lemma 2, for a fixed $1 \le \tau' \le p-2$, the autocorrelation value $AC_b(\tau')$ depends on $|D_{\tau'}|$, the number of c's in \mathbb{F}_p^* such that the pair $(c, \beta^{\tau'}c)$ has different LSB (See Eqs. (19)-(20)). For $1 \le \tau' \le \frac{p-3}{2}$, let $c' = \beta^{\tau'}c$. Then $(c, \beta^{\tau'}c) = (\beta^{-\tau'}\beta^{\tau'}c, c') = (\beta^{p-1-\tau'}c', c')$. Note that c' runs exactly through \mathbb{F}_p^* when c runs through \mathbb{F}_p^* , which implies $|D_{\tau'}| = |D_{p-1-\tau'}|$. Then, by Eq. (22), we get $AC_b(p-1-\tau') = AC_b(\tau')$.

(2) Similar to the above argument, let $c = \beta^l$ and $c' = \beta^{\tau'+l}$. Then we get $(c, \beta^{\tau'}c) = (\beta^l, \beta^{\tau'+l})$ and

$$(\beta^{\tau'}c, -c) = (\beta^{\tau'+l}, -\beta^l) = (\beta^{l+\tau'}, \beta^{\frac{p-1}{2}+l}) = (\beta^{l+\tau'}, \beta^{\frac{p-1}{2}-\tau'}\beta^{l+\tau'}) = (c', \beta^{\frac{p-1}{2}-\tau'}c').$$

which implies $|D_{\frac{p-1}{2}-\tau'}| = p - 1 - 2|D_{\tau'}|$. Again by Eq. (22), $AC_b(\frac{p-1}{2} - \tau') = -AC_b(\tau')$. Particularly, for $p \equiv 1 \mod 4$ and $\tau' = \frac{p-1}{4}$, we get $AC_b(\frac{p-1}{4}) = -AC_b(\frac{p-1}{4})$, which implies $AC_b(\frac{p-1}{4}) = 0$.

(3) Note that $\beta^{\frac{p-1}{2}} = -1$. For $c \in \mathbb{F}_p^*$, the pair (c, -c) has always different LSB.

In convenience, we will always use the following notations:

$$I = \{1, 2, 3, \dots, \frac{p-5}{4}\}$$
 for $p \equiv 1 \mod 4$,
 $I = \{1, 2, 3, \dots, \frac{p-3}{4}\}$ for $p \equiv 3 \mod 4$,
 $AC_b(I) = (AC_b(i))_{i \in I}$.

Combining all the results above, we can give the following Theorem 1.

Theorem 1 Let $n \geq 2$ be a positive integer, p an odd prime, $N = p^n - 1$, $M = \frac{N}{p-1}$, α a primitive element of \mathbb{F}_{p^n} , $\beta = \alpha^M$, and $\{a_t\}_{t=0}^{N-1}$, defined by $a_t = \text{Tr}(\alpha^t)$, a p-ary m-sequence of order n. Suppose $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of $\{a_t\}_{t=0}^{N-1}$ and $\{b_j\}_{j=0}^{p-1}$ is the sequence defined by β as in Definition 2.

| p | β | $AC_b(I)$ |
|----|----------|--|
| 3 | 2 | / (Since $I = \emptyset$) |
| 5 | 2,3 | / (Since $I = \emptyset$) |
| 7 | 3,5 | (2) |
| 11 | 2,6,7,8 | (-2, 2) |
| 13 | 2,6,7,11 | (0, -4) |
| 17 | 3 | (4, 0, -4) |
| 19 | 2 | (-2, 2, -2, -6) |
| 23 | 5 | (2, -2, 2, -2, -6) |
| 29 | 2 | (0, -4, 0, -4, 8, 4) |
| 31 | 3 | (10, 6, 2, -2, -6, -2, 2) |
| 37 | 2 | (0, -4, 0, 4, -8, 4, 0, -12) |
| 41 | 6 | (4, -8, 4, 0, -12, 0, 4, 0, 4) |
| 43 | 3 | (14, 2, -2, -6, -2, 2, 6, 2, -2, 2) |
| 47 | 5 | (10, -2, -14, -2, 2, 6, 2, -2, -6, -2, 2) |
| 53 | 2 | (0, -4, 0, -4, 8, 4, 0, -4, -16, 4, 0, -4) |
| 59 | 2 | (-2, 2, -2, -6, -2, 10, -2, 18, -2, 2, -10, 2, -2, 2) |
| 61 | 2 | (0, -4, 0, -4, 0, 20, 0, -12, 0, -4, -8, 4, 0, -4) |
| 67 | 2 | (-2, 2, -2, 2, -2, -22, 6, 2, -2, -6, -2, 10, -2, -6, 14, 2) |
| 71 | 7 | (10,6,2,-10,2,-2,-14,-2,-22,-2,2,-2,2,-2,2,6,-6) |
| 73 | 5 | (12, 0, -12, 0, 4, 24, 4, 0, -4, 0, 4, 8, 4, 0, -4, 0, 4) |
| 79 | 3 | (26, 6, 2, -2, -6, -2, -6, -2, 2, 6, 2, -2, -6, -10, 2, 14, 2, -2, 2) |
| 83 | 2 | (-2, 2, -2, -6, 6, -6, -2, 10, -2, 26, -2, 2, -2, -14, -2, 2, -2, 2, -2, 10) |
| 89 | 3 | (28,8,4,8,4,8,12,0,-4,0,4,0,-4,0,-4,0,4,16,4,0,-4) |
| 97 | 5 | (20, 0, -4, -8, 4, 0, 4, 0, -4, 8, 4, 0, 4, 0, 4, 0, -12, 0, 4, 0, -4, -32, -12) |

Table 1: Examples of $AC_b(I)$ for all odd primes less than 100

Then the autocorrelation of $\{s_t\}_{t=0}^{N-1}$ can be expressed as

$$AC_{s}(\tau) = \begin{cases} (1 + AC_{b}(\tau'))p^{n-1} - 1, & \text{if } \tau \in \{M\tau'|\tau' \in I\} \cup \{M(p-1-\tau')|\tau' \in I\}, \\ (1 - AC_{b}(\tau'))p^{n-1} - 1, & \text{if } \tau \in \{M(\frac{p-1}{2} - \tau')|\tau' \in I\} \cup \{M(\frac{p-1}{2} + \tau')|\tau' \in I\}, \\ p^{n-1} - 1, & \text{if } p \equiv 1 \mod 4 \text{ and } \tau = \frac{p^{n} - 1}{4}, \\ -(p-2)p^{n-1} - 1, & \text{if } \tau = \frac{p^{n} - 1}{2}, \\ p^{n-2} - 1, & \text{otherwise.} \end{cases}$$

$$(23)$$

Consequently, since $I = \emptyset$ for p = 3, 5, the corresponding autocorrelations $AC_s(\tau)$ for p = 3 and p = 5 can be given directly by

$$AC_s(\tau) = \begin{cases} -3^{n-1} - 1, & \text{if } \tau = M\\ 3^{n-2} - 1, & \text{otherwise.} \end{cases}$$
 (24)

and

$$AC_s(\tau) = \begin{cases} 5^{n-1} - 1, & \text{if } \tau = M \text{ or } 3M, \\ -3 \times 5^{n-1} - 1, & \text{if } \tau = 2M, \\ 5^{n-2} - 1, & \text{otherwise.} \end{cases}$$
 (25)

respectively.

Through Theorem 1, for any odd prime $p \geq 7$, the problem of determining the autocorrelation values of the LSB sequence $\{s_t\}_{t=0}^{N-1}$ of a p-ary m-sequence $\{a_t\}_{t=0}^{N-1}$ of period $p^n - 1$ has been converted to the problem of determining the autocorrelation of the sequence $\{b_j\}_{j=0}^{p-2}$ of period p-1. Not only that, for

the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ of period p-1, we have reduced this problem from a set $\{1,2,\cdots,p-2\}$ with relatively large size to a set $\{1,2,\cdots,\frac{p-5}{4}\}$ or $\{1,2,\cdots,\frac{p-3}{4}\}$ with relatively small size, which can in fact be determined by computer. And we also present the corresponding ordered array $AC_b(I)$ for all odd primes smaller than 100 in Table 1. It can be observed from these examples that all the autocorrelation satisfies $-\frac{p-1}{3} \leq AC_b(\tau') \leq \frac{p-1}{3}$ for $\tau' \in \{1,2,\cdots,p-2\}$ but $\tau' \neq \frac{p-1}{2}$. Finding out the complete and theoretical result of the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ will be an interesting research work and we also sincerely invite the reader to participate in this work.

It is well-known that sequences with cyclic shift equivalent property have the same autocorrelation and 2-adic complexity. In fact, for a Mersenne prime p, all the bit-component sequences of a p-ary m-sequence are equivalent to its LSB sequence. In the following Fact 1, we give a simple proof about this conclusion. Therefore, we know that our results in this paper are available for all the bit-component sequences of p-ary m-sequences for a Mersenne prime p.

Fact 1 Let k be a prime such that $2^k - 1$ is also a prime. Recall that $\{a_t\}_{t=0}^{p^n-2}$ is a p-ary m-sequence of order n and that $\{a_{t,i-1}\}_{t=0}^{p^n-2}$ is the i-th bit-component sequence of $\{a_t\}_{t=0}^{p^n-2}$. Then, for $2 \le i \le k$, the i-th bit-component sequence $\{a_{t,i-1}\}_{t=0}^{p^n-2}$ is a cyclic shift of the LSB sequence $\{a_{t,0}\}_{t=0}^{p^n-2}$.

Proof. For the element $2 \in \mathbb{F}_p$, there exists some $1 \le j_0 \le p-2$ and $\tau_0 = \frac{p^n-1}{p-1}j_0$ such that $2 = \alpha^{\tau_0}$. Note that the trace function Tr(x) from \mathbb{F}_{p^n} to \mathbb{F}_p is linear over \mathbb{F}_p . Then

$$2a_t = 2\operatorname{Tr}(\alpha^t) = \operatorname{Tr}(\alpha^{t+\tau_0}) = a_{t+\tau_0},$$

which shows that $\{2a_t\}$ is the left cyclic shift of $\{a_t\}$ by τ_0 . Correspondingly,

$$2a_t \mod p = a_{t,k-1} + a_{t,0} \times 2 + a_{t,1} \times 2^2 + \dots + a_{t,k-3} \times 2^{k-2} + a_{t,k-2} \times 2^{k-1},$$

that is, the binary string of $2a_t$ is the left cyclic shift of the binary string of a_t by 1. Therefore, we know that, for $1 \le i \le k$, the $((i \mod k)+1)$ -th bit-component sequence is the left cyclic shift of the *i*-th bit-component sequence by τ_0 , which implies that all the bit-component sequences of a *p*-ary *m*-sequence are cyclic shift equivalent.

In this paper, our another main aim is to find the lower bounds on the 2-adic complexities of the LSB sequences of p-ary m-sequences. Here, our method of determining the lower bounds on the 2-adic complexity is inspired by Hu [17], which will involve the autocorrelation of these sequences. Due to the complexity of the autocorrelation of $\{b_j\}_{j=0}^{p-2}$, we cannot give a uniform proof for the lower bounds on the 2-adic complexity of all the LSB sequences of m-sequences using this method in this paper. Therefore, we will take p = 3, 5, 7, 11, 17, 31 as examples to give the 2-adic complexity property of LSB sequences

of m-sequences. Of course, for other primes, the corresponding results can be obtained similarly. It needs to be explained that when different primitive element α of \mathbb{F}_{p^n} is taken, different $\beta \in \mathbb{F}_p$ might be resulted in, which correspondingly maybe give different order of the autocorrelation values of $\{b_j\}_{j=0}^{p-2}$. And on the face of the method of calculating the 2-adic complexity in this paper, it seems that different order of the autocorrelation values of $\{b_j\}_{j=0}^{p-2}$ might further result in different 2-adic complexity. Since the autocorrelation values of the LSB sequences for p=3,5 are straightforward results of Eq. (23), they have no different order. Although there are several primitive elements for p=7,11, the corresponding LSB sequences have still no different order autocorrelation values. By simple calculation, we can see that 17 is the smallest odd prime and 31 is the smallest Mersenne prime such that the autocorrelation values of the LSB sequences have different orders. This is the reason for us to take 17 and 31 as our final examples to list their autocorrelations and their lower bounds on the 2-adic complexities of the LSB sequences. However, from the process of proof, even though the autocorrelation values of the LSB sequences seem not to be much influenced.

Besides Table 1, we also give a detailed autocorrelation distribution of the LSB sequence for p = 7,11,17,31 in the following Corollary 1 so that we can more conveniently use them for determining the lower bound on the 2-adic complexity of these sequences.

Corollary 1 Let $n \geq 2$ be a positive integer. Then we have the following results.

(1) Let p = 7, $N = 7^n - 1$, $M = \frac{N}{7-1} = \frac{N}{6}$, and $\{a_t\}_{t=0}^{N-1}$ a 7-ary m-sequence of period N. Suppose that $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of $\{a_t\}_{t=0}^{N-1}$. Then, $I = \{1, \dots, \frac{p-3}{4}\} = \{1\}$, $AC_b(I) = (2)$, and for $0 < \tau < N$ the autocorrelation of $\{s_t\}_{t=0}^{N-1}$ is given by

$$AC_s(\tau) = \begin{cases} 3 \times 7^{n-1} - 1, & \text{if } \tau = M \text{ or } 5M, \\ -7^{n-1} - 1, & \text{if } \tau = 2M \text{ or } 4M, \\ -5 \times 7^{n-1} - 1, & \text{if } \tau = 3M, \\ 7^{n-2} - 1, & \text{otherwise.} \end{cases}$$
(26)

(2) Let p = 11, $N = 11^n - 1$, $M = \frac{N}{11-1} = \frac{N}{10}$, and $\{a_t\}_{t=0}^{N-1}$ a 11-ary m-sequence of period N. Suppose that $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of $\{a_t\}_{t=0}^{N-1}$. Then, $I = \{1, \dots, \frac{p-3}{4}\} = \{1, 2\}$, $AC_b(I) = (-2, 2)$, and for $0 < \tau < N$ the autocorrelation of $\{s_t\}_{t=0}^{N-1}$ is given by

$$AC_s(\tau) = \begin{cases} -11^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{1, 4, 6, 9\}, \\ 3 \times 11^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{2, 3, 7, 8\}, \\ -9 \times 11^{n-1} - 1, & \text{if } \tau = 5M, \\ 11^{n-2} - 1, & \text{otherwise.} \end{cases}$$

$$(27)$$

(3) Let p = 17, $N = 17^n - 1$, $M = \frac{N}{17-1} = \frac{N}{16}$, α is a primitive element of \mathbb{F}_{17^n} such that $\beta = \alpha^M = 3$, and $\{a_t\}_{t=0}^{N-1}$ a 17-ary m-sequence of period N determined by α . Suppose that $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of $\{a_t\}_{t=0}^{N-1}$. Then, $I = \{1, \dots, \frac{p-5}{4}\} = \{1, 2, 3\}$, $AC_b(I) = (4, 0, -4)$, and for $0 < \tau < N$, the autocorrelation of $\{s_t\}_{t=0}^{N-1}$ is given by

$$AC_{s}(\tau) = \begin{cases} 5 \times 17^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{1, 7, 9, 15\}, \\ 17^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{2, 4, 6, 10, 12, 14\}, \\ -3 \times 17^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{3, 5, 11, 13\}, \\ -15 \times 17^{n-1} - 1, & \text{if } \tau = 8M, \\ 17^{n-2} - 1, & \text{otherwise.} \end{cases}$$

$$(28)$$

(4) Let p = 31, $N = 31^n - 1$, $M = \frac{N}{31-1} = \frac{N}{30}$, α is a primitive element of \mathbb{F}_{31^n} such that $\beta = \alpha^M = 3$, and $\{a_t\}_{t=0}^{N-1}$ a 31-ary m-sequence of period N determined by α . Suppose that $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of $\{a_t\}_{t=0}^{N-1}$. Then, $I = \{1, \dots, \frac{p-3}{4}\} = \{1, 2, 3, 4, 5, 6, 7\}$, $AC_b(I) = \{10, 6, 2, -2, -6, -2, 2\}$, and for $0 < \tau < N$, the autocorrelation of $\{s_t\}_{t=0}^{N-1}$ is given by

$$AC_{s}(\tau) = \begin{cases} 11 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{1, 29\}, \\ 7 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{2, 10, 20, 28\}, \\ 3 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{3, 7, 9, 11, 19, 21, 23, 27\}, \\ -1 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{4, 6, 8, 12, 18, 22, 24, 26\}, \\ -5 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{5, 13, 17, 25\}, \\ -9 \times 31^{n-1} - 1, & \text{if } \tau = M\tau', \ \tau' \in \{14, 16\}, \\ -29 \times 31^{n-1} - 1, & \text{if } \tau = 15M, \\ 31^{n-2} - 1, & \text{otherwise.} \end{cases}$$
(29)

4 Lower bounds on 2-adic complexities of six classes of LSB sequences

In this section, we try to derive a lower bound on the 2-adic complexity of the LSB sequence of p-ary m-sequence for each odd prime p. Unfortunately, due to the complexity of the autocorrelation of the sequence $\{b_j\}_{j=0}^{p-2}$ and our limited research level, we can not give a unified proof about this problem. Therefore, we will take p=3,5,7,11,17,31 as examples to calculate the lower bound on the 2-adic complexities of these sequences. But the method in this section, not limited to these examples, can also be used to prove the lower bounds on the 2-adic complexity of the LSB sequence of other p-ary m-sequences. By observing these examples, we find out that these lower bounds on the 2-adic complexities of these sequences can be expressed as a unified form, so we give a conjecture on the lower bound on

the 2-adic complexity of this class of LSB sequences. Moreover, our experimental results show that, for all primes $p \leq 31$, this conjecture holds for $n \leq 7$. As for those more larger p's and n's, we can not verify it because of the limitations of our existing computer facilities.

Recall that $\{s_t\}_{t=0}^{N-1}$ is the LSB sequence of a *p*-ary *m*-sequence of order n, $N = p^n - 1$, $M = \frac{N}{p-1}$, and $S(x) = \sum_{t=0}^{N-1} s_t x^t$. Then, we describe the method of Hu [17] as the following Lemma 4 and give some other useful lemmas.

Lemma 4 Let $T(x) = \sum_{t=0}^{N-1} (-1)^{s_t} x^t \in \mathbb{Z}[x]$ and $AC_s(\tau)$ the autocorrelation value of the sequence $\{s_t\}_{t=0}^{N-1}$ at τ . Then

$$-2S(x)T(x^{-1}) \equiv N + \sum_{\tau=1}^{N-1} AC_s(\tau)x^{\tau} - T(x^{-1}) \left(\sum_{t=0}^{N-1} x^t\right) \bmod (x^N - 1).$$
 (30)

Proof. According to the definition of T(x), we have

$$T(x)T(x^{-1})$$

$$= \left(\sum_{i=0}^{N-1} (-1)^{s_i} x^i\right) \left(\sum_{j=0}^{N-1} (-1)^{s_j} x^{-j}\right) \equiv \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (-1)^{s_i + s_j} x^{(i-j) \mod N} \mod (x^N - 1)$$

$$\equiv N + \sum_{\tau=1}^{N-1} \sum_{j=0}^{N-1} (-1)^{s_{j+\tau} + s_j} x^{\tau} \mod (x^N - 1) \equiv N + \sum_{\tau=1}^{N-1} AC_s(\tau) x^{\tau} \mod (x^N - 1). \tag{31}$$

Furthermore, we have

$$T(x) = \sum_{i=0}^{N-1} (-1)^{s_i} x^i = \sum_{i=0}^{N-1} (1 - 2s_i) x^i = \sum_{t=0}^{N-1} x^t - 2 \times S(x).$$
 (32)

Combining Eqs. (31)-(32), we get the result.

Lemma 5 Let the notations be the same as above. Suppose that $n \ge 2$ is a positive integer and that I is the same as that in Theorem 1. Then we have

$$S(2)T(2^{-1}) \equiv \left(\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1)\right) p^{n-1} \bmod \left(2^{\frac{N}{2}} + 1\right),\tag{33}$$

$$S(2)T(2^{-1}) \equiv -\frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}(p - 1)p^{n-2} \bmod (2^{\frac{N}{2}} - 1).$$
(34)

where $AC_b(\tau')$ is the autocorrelation of $\{b_j\}_{j=0}^{p-2}$.

Proof. We only prove this result for the case of $p \equiv 3 \pmod{4}$ and the other case can be similarly proved. Substituting Eq. (23) in Theorem 1 into Eq. (30) in Lemma 4, we have

$$\begin{split} -2S(x)T(x^{-1}) &\equiv N + \sum_{\tau=1}^{N-1} AC_s(\tau)x^{\tau} - T(x^{-1}) \left(\sum_{t=0}^{N-1} x^t\right) \bmod (x^N - 1) \\ &\equiv N + \sum_{\tau \neq M\tau', \tau' = 1, 2, \cdots, p-2} (p^{n-2} - 1)x^{\tau} + \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 + AC_b(\tau'))p^{n-1} - 1 \right] x^{M\tau'} \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 - AC_b(\tau'))p^{n-1} - 1 \right] x^{M(\frac{p-1}{2} - \tau')} + \left[-(p - 2)p^{n-1} - 1 \right] x^{\frac{N}{2}} \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 - AC_b(\tau'))p^{n-1} - 1 \right] x^{M(p-1-\tau')} - T(x^{-1}) \left(\sum_{t=0}^{N-1} x^t \right) \bmod (x^N - 1) \\ &= N - (p^{n-2} - 1) + \sum_{\tau = 0}^{N-1} (p^{n-2} - 1)x^{\tau} + \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 + AC_b(\tau'))p^{n-1} - p^{n-2} \right] x^{M\tau'} \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 - AC_b(\tau'))p^{n-1} - p^{n-2} \right] x^{M(\frac{p-1}{2} - \tau')} + \left[-(p - 2)p^{n-1} - p^{n-2} \right] x^{\frac{N}{2}} \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 - AC_b(\tau'))p^{n-1} - p^{n-2} \right] x^{M(\frac{p-1}{2} + \tau')} \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} \left[(1 + AC_b(\tau'))p^{n-1} - p^{n-2} \right] x^{M(p-1-\tau')} - T(x^{-1}) \left(\sum_{t=0}^{N-1} x^t \right) \bmod (x^N - 1) \\ &= \left((p^2 - 1) + \sum_{\tau' = 1}^{\frac{p-3}{2}} ((1 + AC_b(\tau'))p - 1)x^{M\tau'} + \sum_{\tau' = 1}^{\frac{p-3}{2}} ((1 - AC_b(\tau'))p - 1)x^{M(\frac{p-1}{2} + \tau')} \right. \\ &+ \sum_{\tau' = 1}^{\frac{p-3}{2}} ((1 + AC_b(\tau'))p - 1)x^{M(p-1-\tau')} \right) p^{n-2} - \left(p^{n-2} - 1 + T(x^{-1}) \right) \left(\sum_{t=0}^{N-1} x^t \right) \bmod (x^N - 1). \end{split}$$

Notice that $2^{M \times \frac{p-1}{2}} = 2^{\frac{N}{2}}$. Substituting x for 2, then we have

$$-2S(2)T(2^{-1}) \equiv \left(2p(p-1) + \sum_{\tau'=1}^{\frac{p-3}{4}} 2pAC_b(\tau')2^{M\tau'} - \sum_{\tau'=1}^{\frac{p-3}{4}} 2pAC_b(\tau')2^{M(\frac{p-1}{2} - \tau')}\right) p^{n-2} \mod (2^{\frac{N}{2}} + 1)$$

$$\equiv 2\left((p-1) + \sum_{\tau'=1}^{\frac{p-3}{4}} AC_b(\tau')2^{M\tau'} - \sum_{\tau'=1}^{\frac{p-3}{4}} AC_b(\tau')2^{M(\frac{p-1}{2} - \tau')}\right) p^{n-1} \mod (2^{\frac{N}{2}} + 1)$$

$$\equiv -2\left(\sum_{\tau'=1}^{\frac{p-3}{4}} \left(AC_b(\tau')(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'})\right) - (p-1)\right) p^{n-1} \mod (2^{\frac{N}{2}} + 1),$$

and

$$\begin{split} -2S(2)T(2^{-1}) \equiv & 2\left(1 + \sum_{\tau'=1}^{\frac{p-3}{4}} 2^{M\tau'} + \sum_{\tau'=1}^{\frac{p-3}{4}} 2^{M(\frac{p-1}{2} - \tau')}\right) (p-1)p^{n-2} \bmod (2^{\frac{N}{2}} - 1) \\ \equiv & \frac{2(2^{\frac{N}{2}} - 1)}{2^M - 1} (p-1)p^{n-2} \bmod (2^{\frac{N}{2}} - 1). \end{split}$$

The results follow.

Lemma 6 Let p be an odd prime and n a positive integer. Then $p|2^{p^n-1}-1$, but $p^2|2^{p^n-1}-1$ \Leftrightarrow $p^2|2^{p-1}-1$. Furthermore, if $p=2^k-1$ is a Mersenne prime, then $p^2\nmid 2^{p-1}-1$, i.e., $p^2\nmid 2^{p^n-1}-1$. Particularly, for odd k, we have $p|2^{\frac{p^n-1}{2}}-1$, $p^2\nmid 2^{\frac{p^n-1}{2}}-1$, $p\nmid 2^{\frac{p^n-1}{2}}+1$.

Proof. Note that $p-1|p^n-1$, i.e., $2^{p-1}-1|2^{p^n-1}-1$. By Fermat Theorem we know that $p|2^{p-1}-1$. Then we have $p|2^{p^n-1}-1$. Moreover, by Euler Theorem, we have $2^{\phi(p^2)}=2^{p(p-1)}\equiv 1 \mod p^2$, where $\phi(\cdot)$ is Euler Function. And $p^n-1=(p-1)(p^{n-1}+p^{n-2}+\cdots+p+1)\equiv p-1 \mod (p(p-1))$, which implies that $2^{p^n-1}-1\equiv 2^{p-1}-1 \mod p^2$. Therefore, we have $p^2|2^{p^n-1}-1\Leftrightarrow p^2|2^{p-1}-1$.

Furthermore, let $p=2^k-1$ be a Mersenne prime. Then k=2 or k is an odd prime. If k=2, i.e., p=3, then we have $p^2 \nmid 2^{p-1}-1$, which implies $p^2 \nmid 2^{p^n-1}-1$ by the above discussion. Now suppose that k is an odd prime and that $p^2|2^{p-1}-1$. Then $\left(2^k-1\right)^2|\left(2^k-1\right)\left(2^{(\frac{p-1}{k}-1)k}+2^{(\frac{p-1}{k}-2)k}+\cdots+2^k+1\right)$, which implies

$$2^{k} - 1 | 2^{(\frac{p-1}{k} - 1)k} + 2^{(\frac{p-1}{k} - 2)k} + \dots + 2^{k} + 1.$$
 (35)

But we have $2^{(\frac{p-1}{k}-1)k} + 2^{(\frac{p-1}{k}-2)k} + \dots + 2^k + 1 \equiv \frac{p-1}{k} \equiv \frac{2(2^{k-1}-1)}{k} \mod (2^k-1)$ and $\gcd(2(2^{k-1}-1), 2^k-1) = 1$, i.e.,

$$\gcd(2^k - 1, 2^{(\frac{p-1}{k} - 1)k} + 2^{(\frac{p-1}{k} - 2)k} + \dots + 2^k + 1) = 1,$$

a contradiction to the Eq. (35). Particularly, if k is odd prime, then $k|2^{k-1}-1$ by Fermat Theorem. Note that $2^{k-1}-1=\frac{p-1}{2}$ and $\frac{p-1}{2}|\frac{p^n-1}{2}$. Then $k|\frac{p^n-1}{2}$ and $2^k-1|2^{\frac{p^n-1}{2}}-1$, i.e., $p|2^{\frac{p^n-1}{2}}-1$. But since $p^2 \nmid 2^N-1$, then $p^2 \nmid 2^{\frac{N}{2}}-1$ and $p \nmid 2^{\frac{N}{2}}+1$. The desired result follows.

Lemma 7 Let the notations be the same as above. Suppose that $n \ge 2$ is a positive integer and that I is the same as in Theorem 1. Then we have the following two results:

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \begin{cases} \gcd\left(p^{n-1}\left(\sum_{\tau' \in I} AC_b(\tau')(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}) - (p-1)\right), 2^{\frac{N}{2}} + 1\right), & \text{if } \operatorname{Ord}_p(2) \nmid \frac{p-1}{2} \\ \gcd\left(\sum_{\tau' \in I} AC_b(\tau')(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}) - (p-1), 2^{\frac{N}{2}} + 1\right), & \text{if } \operatorname{Ord}_p(2) \mid \frac{p-1}{2} \\ & \text{or } n \text{ is even} \end{cases}$$
(36)

and

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} \gcd\left((p-1)p^{n-2}, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \equiv 0 \text{ mod } \operatorname{Ord}_{p}(2) \text{ but } n \neq 2, \\ \gcd\left(p-1, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \not\equiv 0 \text{ mod } \operatorname{Ord}_{p}(2) \text{ or } n = 2, \end{cases}$$

$$(37)$$

where $AC_b(\tau')$ is the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ and $Ord_p(2)$ is the multiplicative order of 2 modular p. Particularly, for p=3 we have

$$\gcd(S(2)T(2^{-1}), \ 2^N - 1) = \begin{cases} 1, & \text{if } n = 2, \\ 3, & \text{if } n > 2, \end{cases}$$
 (38)

and for p = 5 we have

$$\gcd(S(2)T(2^{-1}), \ 2^N - 1) = \begin{cases} 2^M + 1, & \text{if } n \equiv 2 \bmod 4, \\ 5(2^M + 1), & \text{otherwise.} \end{cases}$$
(39)

(2) If $p = 2^k - 1$ is a Mersenne prime and k is odd, then

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) =
\begin{cases}
\gcd\left(p\left(\sum_{\tau' \in I} AC_b(\tau')(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}) - (p-1)\right), 2^{\frac{N}{2}} + 1\right), & \text{if } \operatorname{Ord}_p(2) \nmid \frac{p-1}{2} \\
\gcd\left(\sum_{\tau' \in I} AC_b(\tau')(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}) - (p-1), 2^{\frac{N}{2}} + 1\right), & \text{if } \operatorname{Ord}_p(2)|\frac{p-1}{2} \text{ or } \\
n \text{ is even}
\end{cases} (40)$$

and

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} \gcd\left(p - 1, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}p, & n \equiv 0 \text{ mod } \operatorname{Ord}_{p}(2) \text{ but } n \neq 2, \\ \gcd\left(p - 1, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \not\equiv 0 \text{ mod } \operatorname{Ord}_{p}(2) \text{ or } n = 2, \end{cases}$$

$$(41)$$

where $AC_b(\tau')$ is the autocorrelation of $\{b_j\}_{j=0}^{p-2}$ and $Ord_p(2)$ is the multiplicative order of 2 modular p.

Proof. (1) From Eqs. (33) and (34), we get

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \gcd\left(\left(\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1)\right) p^{n-1}, \ 2^{\frac{N}{2}} + 1\right)$$

$$\tag{42}$$

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \gcd\left(-\frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}(p - 1)p^{n - 2}, 2^{\frac{N}{2}} - 1\right)$$

$$= \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}\gcd\left((p - 1)p^{n - 2}, 2^{M} - 1\right)$$
(43)

Note that $2^{p^i} \equiv 2 \mod p$ for any nonnegative integer i by Fermat Theorem and

$$M = \frac{N}{p-1} = p^{n-1} + p^{n-2} + \dots + p+1$$
 and $\frac{N}{2} = \frac{(p-1)M}{2} = \frac{p-1}{2}(p^{n-1} + p^{n-2} + \dots + p+1).$

Then,

$$2^{M} = 2^{p^{n-1} + p^{n-2} + \dots + p + 1} \equiv 2^{n} \bmod p \text{ and } 2^{\frac{N}{2}} = (2^{M})^{\frac{p-1}{2}} \equiv 2^{\frac{n(p-1)}{2}} \bmod p. \tag{44}$$

Therefore, if $n \equiv 0 \mod \operatorname{Ord}_p(2)$ then $2^M - 1 \equiv 0 \mod p$, otherwise, $2^M - 1 \not\equiv 0 \mod p$, which results in Eq. (37). Furthermore, by Fermat Theorem, we have $2^{p-1} \equiv 1 \mod p$, and $2^{\frac{p-1}{2}} \equiv 1 \mod p$ if

 $\operatorname{Ord}_p(2)|\frac{p-1}{2}$ and $2^{\frac{p-1}{2}} \equiv -1 \mod p$ if $\operatorname{Ord}_p(2) \nmid \frac{p-1}{2}$. Therefore, we get

$$2^{\frac{N}{2}} + 1 \bmod p \equiv \begin{cases} 0, \text{ if } \operatorname{Ord}_p(2) \nmid \frac{p-1}{2} \text{ and } n \text{ is odd }, \\ 2, \text{ if } \operatorname{Ord}_p(2) \mid \frac{p-1}{2} \text{ or } n \text{ is even.} \end{cases}$$

which results in Eq. (36).

Particularly, for p=3, we have $I=\emptyset$ by Theorem 1, $\gcd(p-1,\ 2^{\frac{N}{2}}+1)=\gcd(2,\ 2^{\frac{N}{2}}+1)=1$, $N=2M,\ \frac{2^{\frac{N}{2}}-1}{2^M-1}=1$, $\gcd(p-1,\ 2^M-1)=\gcd(2,\ 2^M-1)=1$, $\gcd(2)=2$, and $\gcd(2)\nmid\frac{p-1}{2}$. Then, Eqs. (36) and (37) become

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \begin{cases} \gcd(3^{n-1}, 2^{\frac{N}{2}} + 1), & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even} \end{cases}$$
(45)

and

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} \gcd(3^{n-2}, 2^M - 1), & \text{if } n \equiv 0 \bmod 2 \text{ but } n \neq 2, \\ 1, & \text{if } n \not\equiv 0 \bmod 2 \text{ or } n = 2. \end{cases}$$

$$\tag{46}$$

Further, since $3^2 \nmid 2^{3-1}-1$, we have $3^2 \nmid 2^N-1$ from Lemma 6, which implies $3^2 \nmid 2^{\frac{N}{2}}+1$ and $3^2 \nmid 2^M-1$, i.e., $\gcd(3^{n-1}, \ 2^{\frac{N}{2}}+1)=3$ for odd n and $\gcd(3^{n-2}, \ 2^M-1)=3$ for even n but $n \neq 2$. Therefore, the Eq. (38) holds. Similarly, for p=5, we can also get Eq. (39).

(2) Note that, for a Mersenne prime $p = 2^k - 1$, we have $p^2 \nmid 2^{\frac{N}{2}} + 1$ and $p^2 \nmid 2^M - 1$. The rest of proof is similar to the above discussion.

Next, we give the lower bounds on the 2-adic complexities of the LSB sequences of ternary, 5-ary, 7-ary, 11-ary, 17-ary and 31-ary m-sequences respectively.

4.1 2-adic complexity of the LSB sequence for p=3

Now, we present the lower bound on the 2-adic complexity of the LSB sequence of ternary m-sequence.

Theorem 2 Let p=3, $n \geq 2$ a positive integer, $N=3^n-1$, $M=\frac{N}{p-1}=\frac{N}{2}$ or N=2M, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of any ternary m-sequence of order n. Then the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies $\Phi_2(s) \geq N-3$.

Proof. By the definition of 2-adic complexity, we know that the 2-adic complexity of $\{s_t\}_{t=0}^{3^n-2}$ satisfies

$$\Phi_2(s) = \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2), 2^N - 1)} \rfloor \ge \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2)T(2^{-1}), 2^N - 1)} \rfloor$$
$$\ge N - 1 - \lceil \log_2 \gcd(S(2)T(2^{-1}), 2^N - 1) \rceil.$$

By Eq. (38) in Lemma 7, we have $\log_2 \gcd(S(2)T(2^{-1}), 2^N - 1) \le \log_2 3 < 2$. The result follows.

4.2 2-adic complexity of the LSB sequence for p = 5

Now, we present the lower bound on the 2-adic complexity of the LSB sequence of 5-ary m-sequence.

Theorem 3 Let p = 5, $n \ge 2$ a positive integer, $N = 5^n - 1$, $M = \frac{N}{p-1} = \frac{N}{4}$ or N = 4M, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of any 5-ary m-sequence of order n. Then the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies

$$\Phi_2(s) \ge \begin{cases}
\frac{3N}{4} - 2, & \text{if } n \equiv 2 \mod 4, \\
\frac{3N}{4} - 5, & \text{otherwise.}
\end{cases}$$
(47)

Consequently, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is bounded by $\Phi_2(s) \geq \frac{3N}{4} - 5$.

Proof. The proof is similar to that of Theorem 2 except to using Eq. (39) in Lemma 7.

4.3 2-adic complexity of the LSB sequence for p = 7

In the following, we give the lower bound on the LSB sequence of 7-ary m-sequence.

Theorem 4 Let p = 7, $n \ge 2$ a positive integer, $N = 7^n - 1$, $M = \frac{N}{6}$ or N = 6M, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of any 7-ary m-sequence of order n. Then we have

$$\gcd\left(S(2)T(2^{-1}), 2^{N} - 1\right) = \begin{cases} 3(2^{2M} + 2^{M} + 1), & \text{if } n \equiv 0 \bmod 2 \text{ but } n \neq 0 \bmod 3, \\ 7(2^{2M} + 2^{M} + 1), & \text{if } n \equiv 0 \bmod 3 \text{ but } n \neq 0 \bmod 2, \\ 21(2^{2M} + 2^{M} + 1), & \text{if } n \equiv 0 \bmod 6, \\ 2^{2M} + 2^{M} + 1, & \text{otherwise.} \end{cases}$$
(48)

Therefore, the lower bound on the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is given by

$$\Phi_{2}(s) \geq \begin{cases}
\frac{2N}{3} - 4, & \text{if } n \equiv 0 \mod 2 \text{ but } n \neq 0 \mod 3, \\
\frac{2N}{3} - 5, & \text{if } n \equiv 0 \mod 3 \text{ but } n \neq 0 \mod 2, \\
\frac{2N}{3} - 7, & \text{if } n \equiv 0 \mod 6, \\
\frac{2N}{3} - 2, & \text{otherwise.}
\end{cases} \tag{49}$$

Consequently, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies $\Phi_2(s) \geq \frac{2N}{3} - 7$.

Proof. By the result (1) of Corollary 1, we know that $I = \{1\}$ and $AC_b(I) = (2)$ for p = 7. Then we get

$$\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1) = 2(2^{2M} - 2^M - 3).$$

Note that $7 = 2^3 - 1$ is a Mersenne prime and $Ord_7(2) = 3 = \frac{p-1}{2}$. Then, by Eq. (40), we have

$$\gcd\left(S(2)T(2^{-1}),2^{\frac{N}{2}}+1\right)=\gcd\left(2(2^{2M}-2^M-3),2^{3M}+1\right)=\gcd\left(2^{2M}-2^M-3,2^{3M}+1\right).$$

It is easy to see $2^{3M}+1=(2^M+1)(2^{2M}-2^M+1)$. On one hand, since $2^{2M}-2^M-3\equiv -1 \pmod{2^M+1}$, we get $\gcd(2^{2M}-2^M-3,2^M+1)=1$. On the other hand, since $2^{2M}-2^M-3\equiv -4 \pmod{2^{2M}-2^M+1}$, we get $\gcd(2^{2M}-2^M-3,2^{2M}-2^M+1)=1$. Thus we have proved

$$\gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1) = 1.$$
(50)

Moreover, by Eq. (41), we have

$$\gcd\left(S(2)T(2^{-1}),2^{\frac{N}{2}}-1\right) = \left\{ \begin{array}{l} 7 \times \frac{2^{3M}-1}{2^{M}-1} \times \gcd\left(3,2^{M}-1\right), \text{ if } n \equiv 0 \bmod 3, \\ \frac{2^{3M}-1}{2^{M}-1} \times \gcd\left(3,2^{M}-1\right), & \text{if } n \not\equiv 0 \bmod 3. \end{array} \right.$$

It can be easy to verify that $3|2^M-1 \Leftrightarrow 2|M \Leftrightarrow 2|n$. Therefore, we get

$$\gcd\left(S(2)T(2^{-1}),2^{\frac{N}{2}}-1\right) = \left\{ \begin{array}{ll} 21(2^{2M}+2^M+1), \text{ if } n \equiv 0 \bmod 6, \\ 7(2^{2M}+2^M+1), & \text{if } n \equiv 0 \bmod 3 \text{ but } n \not\equiv 0 \bmod 2, \\ 3(2^{2M}+2^M+1), & \text{if } n \equiv 0 \bmod 2 \text{ but } n \not\equiv 0 \bmod 3, \\ 2^{2M}+2^M+1, & \text{otherwise.} \end{array} \right.$$

Combining Eq. (50), we know that Eq. (48) holds. Note that $\log_2(2^N - 1) > N - 1$, $\log_2 3 < 2$, $\log_2 7 < 3$, $\log_2 21 < 5$, $\log_2(2^{2M} + 2^M + 1) < 2M + 1$, and N = 6M. Then, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies

$$\Phi_2(s) = \frac{2^N - 1}{\gcd\left(S(2)T(2^{-1}), 2^N - 1\right)} \ge \begin{cases} \frac{2N}{3} - 4, & \text{if } n \equiv 0 \bmod 2 \text{ but } n \neq 0 \bmod 3, \\ \frac{2N}{3} - 5, & \text{if } n \equiv 0 \bmod 3 \text{ but } n \neq 0 \bmod 2, \\ \frac{2N}{3} - 7, & \text{if } n \equiv 0 \bmod 6, \\ \frac{2N}{3} - 2, & \text{otherwise.} \end{cases}$$

The result follows.

4.4 2-adic complexity of the LSB sequence for p = 11

Now, we present the lower bound on the 2-adic complexity of the LSB sequence of 11-ary m-sequence.

Theorem 5 Let p=11, $n\geq 2$ a positive integer, $N=11^n-1$, $M=\frac{N}{p-1}=\frac{N}{10}$ or N=10M, and

 $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of any 11-ary m-sequence of order n. Then

$$\gcd\left(S(2)T(2^{-1}), 2^{N} - 1\right) = \begin{cases} 33(2^{4M} + 2^{3M} + 2^{2M} + 2^{M} + 1), & \text{if } n \text{ is odd,} \\ 5(2^{4M} + 2^{3M} + 2^{2M} + 2^{M} + 1), & \text{if } n \text{ is even but } n \not\equiv 0 \bmod 10, \\ 55(2^{4M} + 2^{3M} + 2^{2M} + 2^{M} + 1), & \text{if } n \equiv 0 \bmod 10. \end{cases}$$

$$(51)$$

Therefore, the lower bound on the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is given by

$$\Phi_2(s) \ge \begin{cases}
\frac{3N}{5} - 8, & \text{if } n \text{ is odd,} \\
\frac{3N}{5} - 5, & \text{if } n \text{ is even but } n \not\equiv 0 \mod 10, \\
\frac{3N}{5} - 8, & \text{if } n \equiv 0 \mod 10.
\end{cases}$$
(52)

Consequently, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies $\Phi_2(s) \geq \frac{3N}{5} - 8$.

Proof. By the result (2) in Corollary 1, we know that $I = \{1, 2\}$ and $AC_b(I) = (-2, 2)$ for p = 11. Then we get

$$\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1) = -2(2^{4M} - 2^{3M} + 2^{2M} - 2^M + 5).$$

Note that $Ord_{11}(2) = 10$, $\frac{p-1}{2} = 5$ and $10 \nmid 5$. Then, by Eq. (36), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \begin{cases} \gcd\left(11^{n-1}\left(2^{4M} - 2^{3M} + 2^{2M} - 2^{M} + 5\right), 2^{\frac{N}{2}} + 1\right), & \text{if } n \text{ is odd,} \\ \gcd\left(2^{4M} - 2^{3M} + 2^{2M} - 2^{M} + 5, 2^{\frac{N}{2}} + 1\right), & \text{if } n \text{ is even.} \end{cases}$$

$$(53)$$

We first determine the value of $\gcd\left(2^{4M}-2^{3M}+2^{2M}-2^M+5,2^{\frac{N}{2}}+1\right)$. It is easy to see $2^{\frac{N}{2}}+1=2^{5M}+1=(2^M+1)(2^{4M}-2^{3M}+2^{2M}-2^M+1)$. On one hand, we have $2^{4M}-2^{3M}+2^{2M}-2^M+5\equiv 9 \mod(2^M+1)$, further, $3|2^M+1\Leftrightarrow M$ is odd $\Leftrightarrow n$ is odd, and $9|2^M+1\Leftrightarrow n$ is odd and 3|M. But

$$M = \frac{11^{n} - 1}{10} = 11^{n-1} + \dots + 11 + 1 \equiv \sum_{i=0}^{n-1} (-1)^{i} \mod 3 = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$
 Then we get

$$\gcd\left(2^{4M} - 2^{3M} + 2^{2M} - 2^M + 5, 2^M + 1\right) = \begin{cases} 3, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$
 (54)

On the other hand, we have $2^{4M} - 2^{3M} + 2^{2M} - 2^M + 5 \equiv 4 \mod 2^{4M} - 2^{3M} + 2^{2M} - 2^M + 1$, which implies that

$$\gcd(2^{4M} - 2^{3M} + 2^{2M} - 2^M + 5, 2^{4M} - 2^{3M} + 2^{2M} - 2^M + 1) = 1.$$
(55)

Combining Eqs. (53)-(55), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \begin{cases} 3 \times \gcd\left(11^{n-1}, 2^{\frac{N}{2}} + 1\right), & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$
 (56)

Since M is odd for odd n, then $11|2^{5M}+1$ for odd n. Further, it can be straightly verified $11^2 \nmid 2^{10}-1$, which implies $11^2 \nmid 2^N-1$ by Lemma 6, then $11^2 \nmid 2^{5M}+1$. Thus we have $\gcd(11^{n-1}, 2^{5M}+1)=11$ for odd n. Then by Eq. (58), we obtain

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \begin{cases} 33, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$
 (57)

Now, we compute the value of $gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}}-1)$. By Eq. (37), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} \gcd\left(5 \times 11^{n-2}, 2^M - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^M - 1}, & n \equiv 0 \mod 10, \\ \gcd\left(5, 2^M - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^M - 1}, & n \not\equiv 0 \mod 10. \end{cases}$$

$$(58)$$

Note that $5|2^M - 1 \Leftrightarrow M \equiv 0 \mod 4$. But

$$M = 11^{n-1} + 11^{n-2} + \dots + 11 + 1 \equiv \sum_{i=0}^{n-1} (-1)^i \mod 4 = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Thus we have

$$5|2^M - 1 \Leftrightarrow n \equiv 0 \mod 2,\tag{59}$$

Moreover, by the proof of Lemma 7, we know $11|2^M - 1$ for $n \equiv 10$. It can be directly verified that $11^2 \nmid 2^{10} - 1 \Rightarrow 11^2 \nmid 2^N - 1$ from Lemma 6, which implies $11^2 \nmid 2^M - 1$. Then by Eqs. (58) and (59), we get

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} 55(2^{4M} + 2^{3M} + 2^{2M} + 2^M + 1), & \text{if } n \equiv 0 \bmod 10, \\ 5(2^{4M} + 2^{3M} + 2^{2M} + 2^M + 1), & \text{if } n \text{ is even but } n \not\equiv 10, \\ 2^{4M} + 2^{3M} + 2^{2M} + 2^M + 1, & \text{if } n \text{ is odd.} \end{cases}$$
(60)

Combining Eqs. (57) and (60), we know that Eq. (51) holds. Note that $\log_{\ell} 2^N - 1 > N - 1$, $\log_2 33 < 6$, $\log_2 5 < 3$, $\log_2 55 < 6$, $\log_2 (2^{4M} + 2^{3M} + 2^{2M} + 2^M + 1) < 4M + 1$, and N = 10M. Then, the 2-adic

complexity of $\{s_t\}_{t=0}^{N-2}$ satisfies

$$\begin{split} \Phi_2(s) &= \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2), 2^N - 1)} \rfloor \geq \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2)T(2^{-1}), 2^N - 1)} \rfloor \\ &\geq \left\{ \begin{array}{l} \frac{3N}{5} - 8, & \text{if n is odd,} \\ \frac{3N}{5} - 4, & \text{if n is even but $n \not\equiv 10$,} \\ \frac{3N}{5} - 8, & \text{if n is even but 10,} \end{array} \right. \end{split}$$

4.5 2-adic complexity of the LSB sequence for p = 17

Now, we present the lower bound on the 2-adic complexity of the LSB sequence of 17-ary m-sequence.

Theorem 6 Let p=17, $n \geq 2$ a positive integer, $N=17^n-1$, $M=\frac{N}{16}$ or N=16M, α a primitive element of \mathbb{F}_{17^n} such that $\beta=\alpha^M=3$, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of the 17-ary m-sequence defined by α . Then

$$\gcd\left(S(2)T(2^{-1}), 2^{N} - 1\right) = \begin{cases} 17 \times \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & \text{if } n \equiv 0 \mod 8, \\ \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & \text{if } n \not\equiv 0 \mod 8. \end{cases}$$

$$\tag{61}$$

Therefore, the lower bound on the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is given by

$$\Phi_2(s) \ge \begin{cases}
\frac{9N}{16} - 7, & \text{if } n \equiv 0 \mod 8, \\
\frac{9N}{16} - 2, & \text{if } n \not\equiv 0 \mod 8.
\end{cases}$$
(62)

Consequently, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies $\Phi_2(s) \geq \frac{9N}{16} - 7$.

Proof. By the result (3) in Corollary 1, we know that $I = \{1, 2, 3\}$ and $AC_b(I) = (4, 0, -4)$ for p = 17. Then we get

$$\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1) = 4(2^{7M} - 2^{5M} + 2^{3M} - 2^M - 4).$$

Note that $\operatorname{Ord}_{17}(2) = 8 = \frac{p-1}{2}$. Then, by Eq. (36), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \gcd\left(2^{7M} - 2^{5M} + 2^{3M} - 2^{M} - 4, 2^{8M} + 1\right). \tag{63}$$

By Euclid algorithm, we have

$$2^{8M} + 1 = 2^{M}(2^{7M} - 2^{5M} + 2^{3M} - 2^{M} - 4) + (2^{6M} - 2^{4M} + 2^{2M} + 4 \times 2^{M} + 1),$$

$$2^{7M} - 2^{5M} + 2^{3M} - 2^{M} - 4 = 2^{M}(2^{6M} - 2^{4M} + 2^{2M} + 4 \times 2^{M} + 1) - 4(2^{2M} + \frac{1}{2} \times 2^{M} + 1),$$

$$16(2^{6M} - 2^{4M} + 2^{2M} + 4 \times 2^{M} + 1) = (16 \times 2^{4M} - 8 \times 2^{3M} - 28 \times 2^{2M} + 22 \times 2^{M} + 33)(2^{2M} + \frac{1}{2} \times 2^{M} + 1) + (\frac{51}{2} \times 2^{M} - 17),$$

$$51(2^{2M} + \frac{1}{2} \times 2^{M} + 1) = (2 \times 2^{M} + 2)(\frac{51}{2} \times 2^{M} - 17) + (\frac{17}{2} \times 2^{M} + 85),$$

$$\frac{51}{2} \times 2^{M} - 17 = 3 \times (\frac{17}{2} \times 2^{M} + 85) - 16 \times 17,$$

Then, we know that $\gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}}+1)|\gcd(\frac{17}{2}\times 2^M+85, 16\times 17)$ and $\gcd(\frac{17}{2}\times 2^M+85, 16\times 17)=16\times 17$. But, it is easy to know $\gcd(2^{\frac{N}{2}}+1, 17\times 16)=\gcd(2^{\frac{N}{2}}+1, 17)=1$, where the latter equality is from Lemma 6. Therefore, we have

$$\gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1) = 1. \tag{64}$$

Now, by Eq. (37), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} \gcd\left(p^{n-2}, 2^M - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^M - 1}, & n \equiv 0 \mod 8, \\ \frac{2^{\frac{N}{2}} - 1}{2^M - 1}, & n \not\equiv 0 \mod 8, \end{cases}$$

$$(65)$$

Moreover, by the proof of Lemma 7, we know $17|2^M - 1$ for $n \equiv 0 \mod 8$. It can be directly verified that $17^2 \nmid 2^{16} - 1 \Rightarrow 17^2 \nmid 2^N - 1$ from Lemma 6, which implies $17^2 \nmid 2^M - 1$. Then by Eq. (65), we get

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} 17 \times \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \equiv 0 \mod 8, \\ \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \not\equiv 0 \mod 8, \end{cases}$$

$$\tag{66}$$

Combining Eqs. (64) and (66), we know that Eq. (61) holds. Note that $\log_2(2^N - 1) > N - 1$, $\log_2 17 < 5$, $\log_2(2^{7M} + 2^{6M} + 2^{5M} + 2^{4M} + 2^{3M} + 2^{2M} + 2^{M} + 1) < 7M + 1$, and N = 16M. Then, the 2-adic complexity of $\{s_t\}_{t=0}^{N-2}$ satisfies

$$\Phi_2(s) = \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2), 2^N - 1)} \rfloor \ge \lfloor \log_2 \frac{2^N - 1}{\gcd(S(2)T(2^{-1}), 2^N - 1)} \rfloor \ge \begin{cases} \frac{9N}{16} - 7, & \text{if } n \equiv 0 \mod 8, \\ \frac{9N}{16} - 2, & \text{if } n \not\equiv 0 \mod 8. \end{cases}$$

4.6 2-adic complexity of the LSB sequence for p = 31

Finally, we will give a lower bound on the bit-component sequence of 31-ary m-sequence, which needs the following result.

Lemma 8 Let p=31, $n \geq 2$ a positive integer, $N=31^n-1$, $M=\frac{N}{30}$ or N=30M, α a primitive element of \mathbb{F}_{31^n} such that $\beta=\alpha^M=3$, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of the 31-ary m-sequence defined by α . Then, we have

$$\gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1) = \begin{cases} 3, & n \text{ is odd but } n \neq 3 \text{ mod } 6, \\ 9, & n \equiv 3 \text{ mod } 6, \\ 1, & \text{otherwise.} \end{cases}$$
 (67)

and

$$\gcd(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1) = \begin{cases} 15 \times \frac{2^{N} - 1}{2^{M} - 1}, & \text{if } n \equiv 0 \text{ mod } 2 \text{ but } n \neq 0 \text{ mod } 5, \\ 31 \times \frac{2^{N} - 1}{2^{M} - 1}, & \text{if } n \equiv 0 \text{ mod } 5 \text{ but } n \neq 0 \text{ mod } 2, \\ 31 \times 15 \times \frac{2^{N} - 1}{2^{M} - 1}, & \text{if } n \equiv 0 \text{ mod } 10, \\ \frac{2^{N} - 1}{2^{M} - 1}, & \text{otherwise.} \end{cases}$$

$$(68)$$

Proof. By the result (4) in Corollary 1, we know that $I = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$AC_b(I) = (10, 6, 2, -2, -6, -2, 2)$$

for p = 31. Then, we have

$$\sum_{\tau' \in I} AC_b(\tau') \left(2^{M(\frac{p-1}{2} - \tau')} - 2^{M\tau'}\right) - (p-1)$$

$$= 2\left(5\gamma^{14} + 3\gamma^{13} + \gamma^{12} - \gamma^{11} - 3\gamma^{10} - \gamma^9 + \gamma^8 - \gamma^7 + \gamma^6 + 3\gamma^5 + \gamma^4 - \gamma^3 - 3\gamma^2 - 5\gamma - 15\right)$$

where $\gamma = 2^M$. In convenience, we denote

$$h(\gamma) = 5\gamma^{14} + 3\gamma^{13} + \gamma^{12} - \gamma^{11} - 3\gamma^{10} - \gamma^{9} + \gamma^{8} - \gamma^{7} + \gamma^{6} + 3\gamma^{5} + \gamma^{4} - \gamma^{3} - 3\gamma^{2} - 5\gamma - 15.$$

Note that $31 = 2^5 - 1$ is a Mersenne prime, $Ord_{31}(2) = 5$, $\frac{p-1}{2} = 15$ and 5|15. Then, by Eq. (40), we have

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} + 1\right) = \gcd\left(h(\gamma), 2^{\frac{N}{2}} + 1\right) = \gcd\left(h(\gamma), \gamma^{15} + 1\right). \tag{69}$$

It is easy to see that

$$\gamma^{15} + 1 = (\gamma^3 + 1)(\gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1)(\gamma^8 + \gamma^7 - \gamma^5 - \gamma^4 - \gamma^3 + \gamma + 1).$$

In the following, using Euclidean algorithm, we will prove

$$\gcd(h(\gamma), \gamma^8 + \gamma^7 - \gamma^5 - \gamma^4 - \gamma^3 + \gamma + 1) = 1,$$
(70)

$$\gcd(h(\gamma), \gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1) = 1,\tag{71}$$

$$\gcd(h(\gamma), \gamma^3 + 1) = \begin{cases} 3, & n \text{ is odd but } n \neq 3 \mod 6, \\ 9, & n \equiv 3 \mod 6, \\ 1, & \text{otherwise.} \end{cases}$$
 (72)

Firstly, we have

$$h(\gamma) = (5\gamma^{6} - 2\gamma^{5} + 3\gamma^{4} + \gamma^{3} - \gamma^{2} + 6\gamma - 3)(\gamma^{8} + \gamma^{7} - \gamma^{5} - \gamma^{4} - \gamma^{3} + \gamma + 1)$$

$$+4(\gamma^{6} + \gamma^{5} - \gamma^{3} - 2\gamma^{2} - 2\gamma - 3),$$

$$\gamma^{8} + \gamma^{7} - \gamma^{5} - \gamma^{4} - \gamma^{3} + \gamma + 1 = \gamma^{2}(\gamma^{6} + \gamma^{5} - \gamma^{3} - 2\gamma^{2} - 2\gamma - 3) + (\gamma^{4} + \gamma^{3} + 3\gamma^{2} + \gamma + 1),$$

$$\gamma^{6} + \gamma^{5} - \gamma^{3} - 2\gamma^{2} - 2\gamma - 3 = (\gamma^{2} - 3)(\gamma^{4} + \gamma^{3} + 3\gamma^{2} + \gamma + 1) + \gamma(\gamma^{2} + 6\gamma + 1),$$

$$\gamma^{4} + \gamma^{3} + 3\gamma^{2} + \gamma + 1 = (\gamma^{2} - 5\gamma + 32)(\gamma^{2} + 6\gamma + 1) - 31(6\gamma + 1),$$

$$6(\gamma^{2} + 6\gamma + 1) = (\gamma + 5)(6\gamma + 1) + (5\gamma + 1),$$

$$6\gamma + 1 = (5\gamma + 1) + \gamma,$$

$$(73)$$

where we need notice that $\gcd(31,2^{\frac{N}{2}}+1)=1$ in Eq. (73) by Lemma 6. Then we can see $\gcd(h(\gamma),\gamma^8+\gamma^7-\gamma^5-\gamma^4-\gamma^3+\gamma+1)=1$ because $\gcd(h(\gamma),\gamma^8+\gamma^7-\gamma^5-\gamma^4-\gamma^3+\gamma+1)|\gcd(5\gamma+1,\gamma)$ and $\gcd(5\gamma+1,\gamma)=1$.

Secondly, through straight computation, we can get $h(\gamma) \equiv 7\gamma^4 + \gamma^3 - \gamma^2 - 7\gamma - 21 \mod (\gamma^5 + 1)$. Note that $\gamma^5 + 1 = (\gamma + 1)(\gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1)$. Then we have $\gcd(h(\gamma), \gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1) = \gcd(7\gamma^4 + \gamma^3 - \gamma^2 - 7\gamma - 21, \gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1)$. Further,

$$\begin{aligned} 7\gamma^4 + \gamma^3 - \gamma^2 - 7\gamma - 21 &= 7(\gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1) + 4(2\gamma^3 - 2\gamma^2 - 7), \\ 2(\gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1) &= \gamma(2\gamma^3 - 2\gamma^2 - 7) + (2\gamma^2 + 5\gamma + 2), \\ 2\gamma^3 - 2\gamma^2 - 7 &= (\gamma - 4)(2\gamma^2 + 5\gamma + 2) + (\gamma^2 + 18\gamma + 1), \\ 2\gamma^2 + 5\gamma + 2 &= 2(\gamma^2 + 18\gamma + 1) - 31\gamma, \end{aligned}$$

which implies that $gcd(h(\gamma), \gamma^4 - \gamma^3 + \gamma^2 - \gamma + 1) = gcd(\gamma^2 + 18\gamma + 1, -31\gamma) = 1$.

Thirdly, it can be computed that $h(\gamma) \equiv \gamma^2 - \gamma - 11 \pmod{\gamma^3 + 1}$. Note that $\gamma^3 + 1 = (\gamma + 1)(\gamma^2 - \gamma + 1)$. Therefore, we will discuss the values of $\gcd(\gamma^2 - \gamma - 11, \gamma + 1)$ and $\gcd(\gamma^2 - \gamma - 11, \gamma^2 - \gamma + 1)$ respectively.

Since $\gamma^2 - \gamma - 11 \equiv -9 \mod (\gamma + 1)$, then $\gcd(h(\gamma), \gamma + 1) = \gcd(-9, \gamma + 1)$. Note that $3|\gamma + 1 \Leftrightarrow M$ is odd $\Leftrightarrow n$ is odd. Moreover, since $\operatorname{Ord}_9(2) = 6$, we know that $9|\gamma + 1 \Leftrightarrow M \equiv 3 \mod 6$. Further, $M = \frac{(30+1)^n-1}{30} = 30^{n-1} + C_n^1 \times 30^{n-2} + \dots + C_n^{n-2} \times 30 + n \equiv n \mod 6$. Then $9|\gamma + 1 \Leftrightarrow n \equiv 3 \mod 6$. Next, we know that $\gcd(\gamma^2 - \gamma - 11, \gamma^2 - \gamma + 1) = \gcd(\gamma^2 - \gamma - 11, 12) = \gcd(\gamma^2 - \gamma - 11, 3)$ and that $\gamma^2 - \gamma - 11 = (3+1)^M - (3-1)^M - 11 \equiv 1 - (-1)^M - 2 \mod 3$. Then $3|\gamma^2 - \gamma - 11 \Leftrightarrow M$ is odd $\Leftrightarrow n$ is odd. Combining the above discussion, we get

$$\gcd(h(\gamma), \gamma + 1) = \begin{cases} 3, & n \text{ is odd but } n \neq 3 \mod 6, \\ 9, & n \equiv 3 \mod 6, \\ 1, & \text{otherwise} \end{cases}$$
 (74)

and

$$\gcd(h(\gamma), \gamma^2 - \gamma + 1) = \begin{cases} 3, & n \text{ is odd,} \\ 1, & \text{otherwise.} \end{cases}$$
 (75)

Now, in order to determine the exact value of $\gcd(h(\gamma), \gamma^3 + 1)$, we need only find out the maximal integer l such that $3^l|\gamma^2 - \gamma - 11$ when n is odd. From the above argument, it is obvious that $3|\gamma^2 - \gamma - 11$ for odd n and M is odd for odd n. With out less of generality, let M = 6t + 1, 6t + 3 or 6t + 5 for some integer t, we will find $\gamma^2 - \gamma - 11 = 2^{2M} - 2^M - 11 \equiv 0 \mod 9$ if and only if M = 6t + 3 for some t. Furthermore, since $\operatorname{Ord}_{27}(2) = 18$, we will find $2^{2M} - 2^M - 11 \not\equiv 0 \mod 27$ for all the three cases if we take $t = 3t_1$, $3t_1 + 1$ or $3t_1 + 2$ respectively. Therefore, by the above discussion, we know that Eq. (67) holds. Thus, combining Eq. (70)-Eq. (72), we know that Eq. (67) holds.

Furthermore, by Eq. (41), we get

$$\gcd\left(S(2)T(2^{-1}), 2^{\frac{N}{2}} - 1\right) = \begin{cases} 31 \times \gcd\left(p - 1, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \equiv 0 \mod 5, \\ \gcd\left(p - 1, 2^{M} - 1\right) \frac{2^{\frac{N}{2}} - 1}{2^{M} - 1}, & n \not\equiv 0 \mod 5. \end{cases}$$
(76)

Note that $M = 31^{n-1} + 31^{n-2} + \dots + 31 + 1 \equiv \sum_{i=0}^{n-1} (-1)^i \mod 4$ and that

$$\gcd(p-1,2^M-1) = \gcd(2^4-1,2^M-1) = 2^{\gcd(4,M)} - 1 = \begin{cases} 2^4-1, \ 2|n, \\ 1, \ 2 \nmid n. \end{cases}$$

Then, Eq. (68) follows.

Now, we present the lower bound on the 2-adic complexity of the LSB sequence of 31-ary m-sequence.

Theorem 7 Let p=31, $n \geq 2$ a positive integer, $N=31^n-1$, $M=\frac{N}{30}$ or N=30M, α a primitive element of \mathbb{F}_{31^n} such that $\beta=\alpha^M=3$, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of the 31-ary m-sequence defined

by α . Then the lower bound on the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is given by

$$\Phi_{2}(s) \geq \begin{cases}
\frac{8N}{15} - 5, & \text{if } n \equiv 0 \mod 2 \text{ but } n \neq 0 \mod 5, \\
\frac{8N}{15} - 10, & \text{if } n \equiv 0 \mod 10, \\
\frac{8N}{15} - 3, & \text{if } n \text{ is odd, } n \neq 0 \mod 5, \text{ and } n \neq 3 \mod 6, \\
\frac{8N}{15} - 8, & \text{if } n \text{ is odd, } n \equiv 0 \mod 5, \text{ and } n \neq 3 \mod 6, \\
\frac{8N}{15} - 5, & \text{if } n \neq 0 \mod 5, \text{ and } n \equiv 3 \mod 6, \\
\frac{8N}{15} - 10, & \text{if } n \equiv 0 \mod 5, \text{ and } n \equiv 3 \mod 6,
\end{cases} \tag{77}$$

Consequently, the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ satisfies $\Phi_2(s) \geq \frac{8N}{15} - 10$.

Remark 3 In the process of computing the lower bounds on the 2-adic complexities of the above six classes of LSB sequences, we always suppose $n \ge 2$. In fact, it can be testified by simply calculation that all the lower bounds also hold for n = 1.

Remark 4 We have pointed out that different primitive element of \mathbb{F}_p maybe lead to different order of autocorrelation values of the sequence in Definition 2, which perhaps result in different lower bound on the 2-adic complexity. This point can also be observed from Lemma 7. In fact, we can take the sequence $\{b_j\}_{j=0}^{p-2}$ based on p=31 for example. It can be calculated by computer that $\Phi_2(b)=13$ for the primitive elements 11,13,21,22 of \mathbb{F}_{31} but $\Phi_2(b)=11$ for the primitive elements 3,12,17,24 of \mathbb{F}_{31} . However, we claim that the main part, i.e., the nonconstant part of the lower bound on the 2-adic complexity of the bit-component sequence of 31-ary m-sequence remains unchanged. Now, we explain this conclusion simply. Without loss of generality, let $\beta'=\beta^d$ be another primitive element of \mathbb{F}_{31} , where $\gcd(d,p-1)=1$. Suppose $\{b'_{j_1}\}_{j_1=0}^{p-2}$ is the sequence defined by β' through Definition 2. Then we know that $\{b'_{j_1}\}_{j_1=0}^{p-2}=\{b_{dj\pmod{p-1}}\}_{j=0}^{p-2}$, which implies that we need to determine $\gcd(h(\gamma^d), 2^{\frac{N}{2}}+1)$ to find out the lower bound on the 2-adic complexity of the bit-component sequence of 31-ary m-sequence defined by some primitive element α' of \mathbb{F}_{p^n} satisfying $(\alpha')^M=\beta'$. Here $h(\cdot)$ is the same as that in Lemma 8. Furthermore, we have $\gcd(h(\gamma^d), 2^{\frac{N}{2}}+1)|\gcd(h(\gamma^d), 2^{\frac{Nd}{2}}+1)$. By similar discussion to that in Lemma 8, it is not difficult to know that $\gcd(h(\gamma^d), 2^{\frac{Nd}{2}}+1)$ is also a constant.

Remark 5 In order to resist RAA, the 2-adic complexity of a binary sequence should be larger than half of its period. From the results of Theorems 2-7, it is obvious that, for $n \ge 2$, the lower bounds of the 2-adic complexity of the LSB sequences (all the bit-component sequences for a Mersenne prime) of ternary, 5-ary, 7-ary 11-ary, 17-ary and 31-ary m-sequences are large enough to achieve this requirement. In fact, it is not difficult to find from our discussions that all the main parts of these six lower bounds have a unified form, i.e., $\frac{N}{2} + \frac{N}{p-1}$, and we can also get a similar lower bound on the 2-adic complexity of

the LSB sequences of p-ary m-sequences through similar method for other odd prime, such as 13, 19, 23, 29 and so on. Therefore, we give the following conjecture.

Conjecture 1 Let p be any odd prime, n a positive integer, $N = p^n - 1$, and $\{s_t\}_{t=0}^{N-1}$ the LSB sequence of a p-ary m-sequence of order n. Then the 2-adic complexity $\Phi_2(s)$ of $\{s_t\}_{t=0}^{N-1}$ is lower bounded by $\frac{p+1}{2(p-1)}N - C_p$ which is larger than $\frac{N}{2}$ when $n \geq 2$, where the constant number C_p has nothing to do with n but only has relation to p.

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