# On the Automorphism Groups of the $Z_2Z_4$ -Linear 1-Perfect and Preparata-Like Codes

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**Abstract** We consider the symmetry group of a  $Z_2Z_4$ -linear code with parameters of a 1-perfect, extended 1-perfect, or Preparata-like code. We show that, provided the code length is greater than 16, this group consists only of symmetries that preserve the  $Z_2Z_4$  structure. We find the orders of the symmetry groups of the  $Z_2Z_4$ -linear (extended) 1-perfect codes.

**Keywords** additive codes  $\cdot$   $Z_2Z_4$ -linear codes  $\cdot$  1-perfect codes  $\cdot$  Preparatalike codes  $\cdot$  automorphism group  $\cdot$  symmetry group

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### 1 Introduction

Spaces considered in coding theory usually have both metrical and algebraic structures. From the point of view of the parameters of an error-correcting code, the metrical one is the most important, while the algebraic properties give an advantage in constructing codes, in developing coding and decoding algorithms, or in different applications. In some cases, there are some "rigid" connections between metrical and algebraic structures. For example, if the q-ary Hamming metric space with  $q = p^m = 2^1, 3^1, 2^2$  is considered as a vector space over the field GF(p), then any isometry of the space is necessarily an affine transformation. This is not the case for any prime power  $q \geq 5$ . However, the stabilizer of some codes in the group of space isometries consists of affine transformations only. So, informally, from the point of view of such codes, the algebraic structure is rigidly connected with the metrical one. For example,

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this was proved for the Hamming codes for an arbitrary q [12]. In the current paper, we prove a similar result for the  $Z_2Z_4$ -linear perfect, extended perfect, and Preparata-like codes with respect to a  $Z_2Z_4$  algebraic structure, which is, after the field structure, one of most important in coding theory. In contrast, the situation with the  $Z_2Z_4$ -linear Hadamard codes is different [17]: the automorphism group of such a code is larger than the group of automorphisms preserving the  $Z_2Z_4$ -linear structure.

In Sections 2–6, we give basic definitions and facts about the concepts discussed in the paper. The main result of the paper and important corollaries are formulated in Section 7. Section 8 contains a proof of the main result, which states that a code from the considered classes can admit only one  $Z_2Z_4$ -linear structure. As a direction for further research, it would be interesting to generalize this result to a more wide class of  $Z_2Z_4$ -linear codes. The study of automorphism groups is motivated by their role in decoding algorithms, see e.g. [3].

## $2 Z_2 Z_4$ -Linear Codes

A binary code  $C \subset \{0,1\}^n$  is called  $Z_2Z_4$ -linear if for some order-2 permutation (involution)  $\pi$  of the set  $\{0,1,\ldots,n-1\}$  of coordinates, C is closed with respect to the operation  $*_{\pi}$ , where

$$x *_{\pi} y \stackrel{\text{def}}{=} x + y + (x + \pi(x)) \cdot (y + \pi(y)),$$

+ and  $\cdot$  being the coordinatewise modulo-2 addition and multiplication respectively (here and elsewhere, the action of a permutation  $\sigma:\{0,1,\ldots,n-1\}\to \{0,1,\ldots,n-1\}$  on  $x=(x_0,x_1,\ldots,x_{n-1})\in\{0,1\}^n$  is defined as  $\sigma(x)\stackrel{\text{def}}{=}(x_{\sigma^{-1}(0)},x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n-1)})$ ). Given an involution  $\pi$ , we will say that coordinates i and j are adjacent if  $\pi(i)=j$ ; if  $\pi(i)=i$ , then i is self-adjacent. Clearly, the value of the result  $z=x*_{\pi}y$  in some coordinate i depends only on the values of x and y in the ith and  $\pi(i)$ th coordinates. So, considering the result of  $*_{\pi}$  in two different adjacent coordinates, we can determine the values by Table 1(a), while the case  $\pi(i)=i$  corresponds to Table 1(b).

Tables 1(a) and 1(b) are the value tables of groups isomorphic to  $Z_4$  and  $Z_2$ , respectively. Consequently,  $(\{0,1\}^n, *_{\pi})$  is isomorphic to the group  $Z_2^{\alpha} Z_4^{\beta}$ , where  $\alpha$  is the number of self-adjacent coordinates and  $\beta = (n - \alpha)/2$ . In this

Table 1 Values of  $*_{\pi}$  for two different adjacent coordinates and for a self-adjacent coordinate

				11					
	00 01	00	01	11	10			0	1
(a)	01	01	11	10	00	(b)	0	0	1
	11	11	10	<b>00</b> 01	01		1	1	0
	10	10	00	01	11				

case we will say that  $\pi$  is a  $Z_2Z_4$  structure of type  $(\alpha, \beta)$ . The binary codes closed with respect to  $*_{\pi}$  are known as  $Z_2Z_4$ -linear codes of type  $(\alpha, \beta)$ . The  $Z_2Z_4$ -linear codes with  $\alpha = n$  are called linear; with  $\alpha = 0$ ,  $Z_4$ -linear.

# $3 Z_2 Z_4$ -Additive Codes, Gray Map, Duality

In this section, we consider an alternative way to define  $Z_2Z_4$ -linear codes and related concepts. In the literature, this way is more popular than the definition given in Section 2; however, for presenting results of the current paper the last one is more convenient. The content of the section is not used in the formulation of the main result of the paper (Theorem 1) and its proof, but the concepts defined here are exploited in the proof of Corollary 2 (to derive the order of a  $Z_2Z_4$ -linear (extended) 1-perfect code from Theorem 1 and known facts) and in the formulation of Corollary 3.

A code  $C \subseteq \{0,1\}^{\alpha} \times \{0,1,2,3\}^{\beta}$  in the mixed  $Z_2 - Z_4$  alphabet is called additive  $(Z_2 Z_4 - additive)$  if it is closed with respect to the coordinatewise addition, modulo 2 in the first  $\alpha$  coordinates and modulo 4 in the last  $\beta$  coordinates. The one to one correspondence  $\Phi : \{0,1\}^{\alpha} \times \{0,1,2,3\}^{\beta} \to \{0,1\}^{\alpha+2\beta}$  known as the  $Gray\ map$  is defined as follows:

$$\Phi((x_0, \dots, x_{\alpha-1}, y_0, \dots, y_{\beta-1})) = (x_0, \dots, x_{\alpha-1}, \phi(y_0), \dots, \phi(y_{\beta-1})),$$

where  $\phi(0) = (0,0)$ ,  $\phi(1) = (0,1)$ ,  $\phi(2) = (1,1)$ ,  $\phi(3) = (1,0)$ . The following straightforward fact means that a  $Z_2Z_4$ -linear code can be defined as the image of a  $Z_2Z_4$ -additive code under the Gray map and a coordinate permutation.

**Proposition 1** A code  $C \subseteq \{0,1\}^{\alpha} \times \{0,1,2,3\}^{\beta}$  is additive if and only if its image  $\Phi(C)$  under the Gray map is closed under the operation  $*_{\pi}$ , where

$$\pi = (\alpha \ \alpha + 1)(\alpha + 2 \ \alpha + 3) \dots (\alpha + 2\beta - 2 \ \alpha + 2\beta - 1). \tag{1}$$

The inner product [x, y] of two words  $x = (x_0, \ldots, x_{\alpha-1}, x'_0, \ldots, x'_{\beta-1})$  and  $y = (y_0, \ldots, y_{\alpha-1}, y'_0, \ldots, y'_{\beta-1})$  from  $\{0, 1\}^{\alpha} \times \{0, 1, 2, 3\}^{\beta}$  is defined as

$$[x,y] \stackrel{\text{def}}{=} 2x_0y_0 + \ldots + 2x_{\alpha-1}y_{\alpha-1} + x_0'y_0' + \ldots + x_{\beta-1}'y_{\beta-1}' \mod 4.$$
 (2)

For a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}\subseteq\{0,1\}^{\alpha}\times\{0,1,2,3\}^{\beta}$ , its dual  $\mathcal{C}^{\perp}$  is defined as

$$\mathcal{C}^{\perp} \stackrel{\text{def}}{=} \{ x \in \{0, 1\}^{\alpha} \times \{0, 1, 2, 3\}^{\beta} \mid [x, y] = 0 \text{ for all } y \in \mathcal{C} \}.$$
 (3)

Readily,  $\mathcal{C}^{\perp}$  is also an additive code. Moreover,  $(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$ , see, e.g., [5].

#### 4 Symmetry Group

Let  $S_n$  be the set of permutations of  $\{0, 1, ..., n-1\}$ . The symmetry group of a code  $C \subset \{0, 1\}^n$  is defined as

$$\operatorname{Sym}(C) \stackrel{\text{def}}{=} \{ \sigma \in S_n \mid \sigma(x) \in C \text{ for all } x \in C \}.$$

Given a  $Z_2Z_4$ -structure  $\pi$ , we will also consider the group of  $Z_2Z_4$ -symmetries  $\operatorname{Sym}_{\pi}(C)$  as a subgroup of  $\operatorname{Sym}(C)$  consisting of symmetries  $\sigma$  that commute with the involution  $\pi$ :

$$\operatorname{Sym}_{\pi}(C) \stackrel{\text{def}}{=} \left\{ \sigma \in \operatorname{Sym}(C) \mid \sigma(\pi(i)) = \pi(\sigma(i)) \text{ for all } i \in \{0, 1, \dots, n-1\} \right\}.$$

In other words,  $\operatorname{Sym}_{\pi}(C)$  is the intersection of  $\operatorname{Sym}(C)$  with the automorphism group of the group  $(\{0,1\}^n, *_{\pi})$  (which is, by definition, the set of all permutations  $\sigma$  of  $\{0,1\}^n$  such that  $\sigma(x) *_{\pi} \sigma(y) = \sigma(x *_{\pi} y)$  for every x, y).

The group  $\operatorname{Sym}_{\pi}(C)$  has a natural treatment in terms of the preimage of C under the Gray map. Indeed, if  $C = \Phi(C)$  for some  $C \subseteq \{0,1\}^{\alpha} \times \{0,1,2,3\}^{\beta}$  and  $\pi$  is of form (1), then

$$\operatorname{Sym}_{\pi}(C) = \{ \Phi \sigma \Phi^{-1} \mid \sigma \in \operatorname{MAut}(C) \}, \tag{4}$$

where  $\mathrm{MAut}(\mathcal{C})$ , the monomial automorphism group of  $\mathcal{C}$ , is the stabilizer of  $\mathcal{C}$  in the group of monomial transformations of  $\{0,1\}^{\alpha} \times \{0,1,2,3\}^{\beta}$  (recall that a monomial transformation consists of a coordinate permutation followed by sign changes in some quaternary coordinates).

To prove one of the corollaries from the main theorem, we will need the following simple known fact.

**Proposition 2** For every  $Z_2Z_4$ -additive code C, it holds  $MAut(C) = MAut(C^{\perp})$ .

*Proof* At first, we see from (2) that  $[\sigma^{-1}(x), y] = [x, \sigma(y)]$  for every monomial transformation  $\sigma$ . This identity can be utilised to derive  $\sigma(\mathcal{C}^{\perp}) = (\sigma(\mathcal{C}))^{\perp}$  from (3):

$$\sigma(\mathcal{C}^{\perp}) = \{ \sigma(x) \mid [x, y] = 0 \ \forall y \in \mathcal{C} \} = \{ x' \mid [\sigma^{-1}(x'), y] = 0 \ \forall y \in \mathcal{C} \}$$
$$= \{ x' \mid [x', \sigma(y)] = 0 \ \forall y \in \mathcal{C} \} = \{ x' \mid [x', y'] = 0 \ \forall y' \in \sigma(\mathcal{C}) \} = (\sigma(\mathcal{C}))^{\perp}.$$

If  $\sigma \in \mathrm{MAut}(\mathcal{C})$ , then  $\sigma(\mathcal{C}) = \mathcal{C}$  and hence  $\sigma(\mathcal{C}^{\perp}) = \mathcal{C}^{\perp}$ . We conclude that  $\mathrm{MAut}(\mathcal{C}) \subseteq \mathrm{MAut}(\mathcal{C}^{\perp})$ . From  $(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$ , we get the inverse inclusion.

It is worth to mention the full automorphism group  $\operatorname{Aut}(C)$  of a binary code C, which is the stabilizer of the code in the group of isometries of the Hamming space. We only observe that for a  $Z_2Z_4$ -linear code C, this group is the product of  $\operatorname{Sym}(C)$  and the group of translations  $\{\operatorname{tr}_c \mid c \in C\}$ , where  $\operatorname{tr}_c(x) \stackrel{\text{def}}{=} c *_{\pi} x$ . As follows,  $|\operatorname{Aut}(C)| = |C| \cdot |\operatorname{Sym}(C)|$ .

#### 5 Perfect and Extended Perfect Codes

A binary code  $C \subset \{0,1\}^n$  is called 1-perfect (extended 1-perfect) if its cardinality is  $2^n/(n+1)$  (respectively,  $2^{n-1}/n$ ) and the distance between every two distinct codewords is at least 3 (respectively, 4), where the (Hamming) distance is defined as the number of positions in which the words differ. Note that the denominator (n+1) (respectively, n) coincides with the cardinality of a radius-1 ball (respectively, sphere) and must be a power of 2 for the existence of corresponding codes. So, a characterizing property of a 1-perfect code is that every binary word is at distance at most 1 from exactly one codeword. The codewords of an extended 1-perfect code have the same parity (i.e., the parity of the weight, the number of ones in the word), and every word of the other parity is at distance 1 from exactly one codeword.

There is a characterization of  $Z_2Z_4$ -linear 1-perfect and  $Z_2Z_4$ -linear extended 1-perfect binary codes, see [8] ( $Z_2Z_4$ -linear 1-perfect codes), [15,16] ( $Z_4$ -linear extended 1-perfect codes), and [6] (complete description). Recall that the rank of a binary code is the dimension of its linear closure over  $Z_2$ .

**Proposition 3** ([8,15,6]) (a) For any r and  $t \ge 4$  such that  $t/2 \le r \le t$ , there is exactly one  $Z_2Z_4$ -linear 1-perfect code (extended 1-perfect code) of type  $(2^r - 1, 2^{t-1} - 2^{r-1})$  (respectively,  $(2^r, 2^{t-1} - 2^{r-1})$ ), up to coordinate permutation.

- (b) For any  $t \geq 4$ , there are exactly  $\lfloor (t+1)/2 \rfloor$   $Z_2Z_4$ -linear extended 1-perfect codes of type  $(0,2^{t-1})$  (i.e.,  $Z_4$ -linear), up to coordinate permutation; all these codes have different ranks  $2^t r 1$ ,  $r = \lfloor t/2 \rfloor, \ldots, t 1$ , except for the case t = 4, r = 3, where the corresponding code is linear.
- (c) All codes from (a) and (b) are pairwise nonequivalent. There are no other  $Z_2Z_4$ -linear 1-perfect codes or  $Z_2Z_4$ -linear extended 1-perfect codes.

### 6 Preparata-Like Codes

A binary code  $C \subset \{0,1\}^n$  is called Preparata-like if its cardinality is  $2^n/n^2$  and the distance between every two distinct codewords is at least 6. Such codes exist if and only if n is a power of 4 [20]. The original Preparata code [20], and the generalizations [11], [1], [10] are not  $Z_4$ -linear if n > 16. A class of  $Z_4$ -linear Preparata-like codes was constructed in [13] for every  $n = 2^{t+1} \geq 16$ , t odd; codes nonequivalent to that from [13] were found in [9]. As was shown in [14, Theorem 5.11], there are many nonequivalent  $Z_4$ -linear Preparata-like codes of the same length (their number grows faster than any polynomial in n; however, there are some restrictions on  $n = 2^{t+1}$ : t is not a prime nor the product of two primes). We will use the following two facts, which make our results concerning Preparata-like codes simple corollaries from the results on extended 1-perfect codes.

**Proposition 4 ([21])** For every Preparata-like code P, there exists a unique extended 1-perfect code C including P.

**Proposition 5** ([7]) Assume that a Preparata-like code P is closed with respect to the operation  $*_{\pi}$ , where  $\pi$  is a  $Z_2Z_4$  structure. Then the extended 1-perfect code C including P is also closed with respect to  $*_{\pi}$ .

In [7], it was shown that any  $Z_2Z_4$ -linear Preparata-like code is necessarily  $Z_4$ -linear, i.e., the involution  $\pi$  has no fixed points.

Remark 1 In the current work, we consider the distance-6 Preparata-like codes, sometimes referred to as the extended Preparata-like codes. In one-to-one correspondence with such codes are the distance-5 Preparata-like codes, sometimes called the punctured Preparata-like codes (in fact, the original Preparata codes [20] were presented in terms of distance-5 codes). Reformulating Proposition 4, every punctured Preparata-like code is included in a unique 1-perfect code. Formally, we can include the punctured Preparata-like codes in the statement of Theorem 1 below, but this does not make any sense as there are no  $Z_2Z_4$ -linear codes among them, see [7].

#### 7 Results

Generally, a binary code can admit more than one  $Z_2Z_4$  structure. For example, the 1-perfect code {0000000, 0001011, 0010110, 0101100, 1011000, 0110001, 1100010, 1000101, 1110100, 1101001, 1010011, 0100111, 10011110, 0011101, 0111010, 1111111} (this is the cyclic Hamming code of length 7, see e.g. [18]) is closed with respect to  $*_{\pi}$  for 22 different involutions  $\pi$ , including Id, (01)(24), (02)(14), and (04)(12) (and all their cyclic shifts). From the characterisation of  $Z_2Z_4$ -linear 1-perfect codes, we see that a 1-perfect code of length at least 15 or an extended 1-perfect code of length more than 16 cannot admit two  $Z_2Z_4$  structures with different number of self-adjacent coordinates. The next theorem, which is the main result of the paper, states more.

**Theorem 1** Let C be a 1-perfect, extended 1-perfect, or Preparata-like code of length n > 16 closed with respect to both operations  $*_{\pi}$  and  $*_{\tau}$ , where  $\pi$  and  $\tau$  are  $Z_2Z_4$  structures. Then  $\pi = \tau$ .

The theorem will be proven in the next section. Here, we consider some important corollaries.

Corollary 1 Let C be a 1-perfect, extended 1-perfect, or Preparata-like code of length n > 16. If C is  $Z_2Z_4$ -linear, i.e, closed with respect to the operation  $*_{\pi}$ , for some  $Z_2Z_4$  structure  $\pi$ , then  $\operatorname{Sym}(C) = \operatorname{Sym}_{\pi}(C)$ .

*Proof* Seeking a contradiction, assume that  $\sigma$  is from  $\operatorname{Sym}(C)$  but not from  $\operatorname{Sym}_{\pi}(C)$ . Then the involution  $\tau \stackrel{\text{def}}{=} \sigma^{-1}\pi\sigma$  does not coincide with  $\pi$ . However, for any two codewords x and y.

$$x *_{\tau} y = x + y + (x + \tau(x)) \cdot (y + \tau(y))$$
  
=  $\sigma^{-1} (x' + y' + (x' + \pi(x')) \cdot (y' + \pi(y')))$ 

belongs to C (here  $x' = \sigma(x) \in C$  and  $y' = \sigma(y) \in C$ ). We have a contradiction with Theorem 1.

An exhaustive computer search shows that the statement of Corollary 1 holds also for the non- $Z_4$ -linear extended 1-perfect codes of lengths 16 (as follows, it is also true for the  $Z_2Z_4$ -linear 1-perfect codes of length 15, see the observation in the second paragraph of the next section). For the  $Z_4$ -linear extended 1-perfect codes of length 16, the situation is different. One of the two non-equivalent codes admits also the linear structure, and it is not difficult to find that the  $Z_4$ -linear structure is not preserved by all symmetries. The other code meets  $|\mathrm{Sym}(C)| = 3|\mathrm{Sym}_{\pi}(C)|$ . The order of the symmetry group of the unique Preparata-like code of length 16 is  $16 \cdot 15 \cdot 14 \cdot 12$  [2], see also [4].

Corollary 2 (a) If C' is a  $Z_2Z_4$ -linear 1-perfect code of type  $(2^r - 1, 2^{t-1} - 2^{r-1})$ ,  $t \geq 4$ ,  $\frac{t}{2} \leq r \leq t$ , then  $\operatorname{Sym}(C')$  is isomorphic to the automorphism group of the group  $Z_2^{\dot{\gamma}} \times Z_4^{\delta}$ ,  $\dot{\gamma} \stackrel{\text{def}}{=} 2r - t$ ,  $\delta \stackrel{\text{def}}{=} t - r$ , and has the cardinality

$$2^{\frac{1}{2}\dot{\gamma}^2 - \frac{1}{2}\dot{\gamma} + 2\dot{\gamma}\delta + \frac{3}{2}\delta^2 - \frac{1}{2}\delta} \prod_{i=1}^{\dot{\gamma}} (2^i - 1) \prod_{i=1}^{\delta} (2^i - 1).$$

(b) If C is a  $Z_2Z_4$ -linear extended 1-perfect code of type  $(2^r, 2^{t-1} - 2^{r-1})$ ,  $t \geq 4$ ,  $\frac{t}{2} \leq r \leq t$ , then  $\operatorname{Sym}(C)$  is isomorphic to a semidirect product of the automorphism group of the group  $Z_2^{\dot{\gamma}} \times Z_4^{\delta}$ ,  $\dot{\gamma} \stackrel{\text{def}}{=} 2r - t$ ,  $\delta \stackrel{\text{def}}{=} t - r$ , with the group  $Z_2^{\dot{\gamma} + \delta}$  of translations by an element of order less that 4 and has the cardinality

$$2^{\frac{1}{2}\dot{\gamma}^2 + \frac{1}{2}\dot{\gamma} + 2\dot{\gamma}\delta + \frac{3}{2}\delta^2 + \frac{1}{2}\delta} \prod_{i=1}^{\dot{\gamma}} (2^i - 1) \prod_{i=1}^{\delta} (2^i - 1).$$

(c) If C is a Z<sub>4</sub>-linear extended 1-perfect code of rank  $2^t-r-1$ , t>4,  $\frac{t-1}{2}\leq r\leq t-1$ , then  $\mathrm{Sym}(C)$  has the cardinality

$$2^{\frac{1}{2}\gamma^2 + \frac{3}{2}\gamma + 2\gamma\dot{\delta} + \frac{3}{2}\dot{\delta}^2 + \frac{5}{2}\dot{\delta} + 1} \prod_{i=1}^{\gamma} (2^i - 1) \prod_{i=1}^{\dot{\delta}} (2^i - 1),$$

where  $\gamma \stackrel{\text{def}}{=} 2r - t + 1$ ,  $\dot{\delta} \stackrel{\text{def}}{=} t - r - 1$ .

*Proof* (b,c) Without loss of generality, we can assume that the  $Z_2Z_4$  structure corresponding to the considered  $Z_2Z_4$ -linear code C has the form (1), where  $\alpha = 2^r$  in the case (b) and  $\alpha = 0$  in the case (c).

By Corollary 1,  $\operatorname{Sym}(C) = \operatorname{Sym}_{\pi}(C)$  (the case t = 4 is covered by the computational results mentioned above). Without loss of generality assume that  $\pi$  is of form (1). Denote  $\mathcal{C} \stackrel{\text{def}}{=} \Phi^{-1}(C)$ . According to (4), we have  $\operatorname{Sym}_{\pi}(C) \simeq \operatorname{MAut}(\mathcal{C})$ . By Proposition 2,  $\operatorname{MAut}(\mathcal{C}) = \operatorname{MAut}(\mathcal{C}^{\perp})$ . For a  $Z_2Z_4$ -linear extended 1-perfect code C, the related code  $\mathcal{C}^{\perp}$  is a so-called  $Z_2Z_4$ -additive Hadamard code, see [19]. The structure of the monomial automorphism group

of such codes was studied in [17], and statements (b) and (c) of the current corollary follow from [17, Theorem 3] and [17, Theorem 2], respectively.

(a) Since by appending a parity-check bit, every code C' from p.(a) results in a code C from p.(b), where the new coordinate is self-adjacent,  $\operatorname{Sym}_{\pi}(C')$  coincides with the stabilizer of a self-adjacent coordinate in  $\operatorname{Sym}_{\pi}(C)$ . As follows directly from the structure of  $\operatorname{MAut}(\mathcal{C}^{\perp})$  considered in [17, Section IV] and from the mentioned above connection between  $\operatorname{MAut}(\mathcal{C}^{\perp})$  and  $\operatorname{Sym}_{\pi}(C)$ , the last group contains a subgroup isomorphic to  $Z_2^r$  that acts transitively on the  $2^r$  self-adjacent coordinates (in the statement (b) of the current corollary, this subgroup is mentioned as the group of translations). By the orbit–stabilizer theorem, we have  $|\operatorname{Sym}_{\pi}(C')| = |\operatorname{Sym}_{\pi}(C)|/2^r$ .

Another interesting corollary from Theorem 1 was suggested by one of the reviewers. Recall that two  $Z_2Z_4$ -linear or  $Z_2Z_4$ -additive codes are equivalent if one of the codes can be obtained from the other by a monomial transformation, that is, by a coordinate permutation and, if necessary, sign changes in some coordinates.

Corollary 3 Let C and D be  $Z_2Z_4$ -additive codes such that  $C = \Phi(C)$  and  $D = \Phi(D)$  are 1-perfect, extended 1-perfect, or Preparata-like codes. If C and D are nonequivalent, then C and D are nonequivalent too.

Proof Both C and D are closed with respect to  $*_{\pi}$  where  $\pi$  is in the form (1). Assume that C and D are equivalent; i.e.,  $C = \sigma(D)$  for some coordinate permutation  $\sigma$ . Then, C is also closed with respect to  $*_{\sigma\pi\sigma^{-1}}$ . By Theorem 1, we have  $\pi = \sigma\pi\sigma^{-1}$ . This means that  $\sigma$  preserves the pairs of adjacent coordinates. It follows that  $\Phi^{-1}\sigma\Phi$  is a monomial transformation and the codes  $\mathcal{D}$  and  $C = \Phi^{-1}(C) = \Phi^{-1}(\sigma(D)) = \Phi^{-1}(\sigma(\Phi(D)))$  are equivalent.

For the  $Z_4$ -linear Preparata-like codes, this fact is new, as their class has not been completely characterized, in contrast to the case of (extended) 1-perfect codes. It should be remarked, however, that the proof of Theorem 10.3(ii,iv) in [9] stating the same as Corollary 3 for a partial class of  $Z_4$ -linear Preparata-like codes works for all  $Z_4$ -linear Preparata-like codes as well, taking into account the later result [7] that only the  $Z_4$ -linear extended perfect code of length  $2^t$  that has rank  $2^t - t$  can include a  $Z_4$ -linear Preparata-like code.

#### 8 Proof of Theorem 1

Proof We first note that by Propositions 4 and 5, the statement on the Preparata-like codes is straightforward from the one on the extended 1-perfect codes. Indeed, the only extended 1-perfect code including a given  $Z_2Z_4$ -linear Preparata-like code P must be  $Z_2Z_4$ -linear with the same  $Z_2Z_4$  structure as P.

At second, appending a parity-check bit to every codeword of a  $Z_2Z_4$ -linear 1-perfect code C' of type  $(\alpha, \beta)$  results in an extended 1-perfect code C of type  $(\alpha + 1, \beta)$ . Moreover any symmetry of C' is naturally extended to a symmetry

of C, which fixes the last (appended) coordinate. So, to prove the theorem, it is sufficient to consider the case of an extended 1-perfect code C.

Let C be a  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code with two  $\mathbb{Z}_2\mathbb{Z}_4$  structures,  $\pi$  and  $\tau$ . Seeking a contradiction, assume that for some coordinate i, we have  $\pi(i) \neq \tau(i)$ . Without loss of generality,  $\pi(i) \neq i$ . We will say that two coordinates j and j' are independent if  $j' \notin \{j, \pi(j), \tau(j)\}$  (equivalently,  $j \notin \{j', \pi(j'), \tau(j')\}$ ).

Suppose that C has two codewords  $v=(v_0,\ldots,v_{n-1})$  and  $u=(u_0,\ldots,u_{n-1})$  such that every nonzero coordinate of v is independent from every nonzero coordinate of u, with the only exception  $v_i=u_{\pi(i)}=1$ . Then  $v*_{\tau}u$  coincides with v+u (indeed, the situation in the middle bolded part of Table 1(a) never occurs in this sum; in fact, only the first row and the first column occur), while  $v*_{\pi}u$  differs from v+u in the coordinates i and  $\pi(i)$ . Since both  $v*_{\tau}u$  and  $v*_{\pi}u$  must belong to C, we have a contradiction with the code distance 4.

It remains to find such two codewords v, u. We restrict the search by the weight-4 codewords. Let  $i, i_2, i_3, i_4$  and  $\pi(i), j_2, j_3, j_4$  be the ones of v and the ones of u, respectively. It is easy to choose  $i_2, i_3, i_4$  independent from  $\pi(i)$  and to choose  $j_2$  independent from  $i, i_2, i_3,$  and  $i_4$ . For every choice of  $j_3$ , the fourth one  $j_4$  of u is defined uniquely. There are at least  $n-3\cdot 4-1$  ways to choose  $j_3$  independent from  $i, i_2, i_3,$  and  $i_4$ . In at least  $n-3\cdot 4-1-3\cdot 4$  of them, the resulting  $j_4$  is also independent from  $i, i_2, i_3,$  and  $i_4$ . Since  $n-3\cdot 4-1-3\cdot 4\geq 32-25>0$ , the result follows.

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#### References

- Baker, R.D., van Lint, J.H., Wilson, R.M.: On the Preparata and Goethals codes. IEEE Trans. Inf. Theory 29(3), 342–345 (1983). DOI 10.1109/TIT.1983.1056675
- Berlekamp, E.R.: Coding theory and the Mathieu groups. Inf. Control 18(1), 40–64 (1971). DOI 10.1016/S0019-9958(71)90295-6
- 3. Bernal, J.J., Borges, J., Fernández-Córdoba, C., Villanueva, M.: Permutation decoding of  $Z_2Z_4$ -linear codes. Des. Codes Cryptography **76**(2), 269–277 (2015). DOI 10.1007/s10623-014-9946-4
- 4. Bierbrauer, J.: Nordstrom–Robinson code and  $A_7$ -geometry. Finite Fields Appl. 13(1), 158–170 (2007). DOI 10.1016/j.ffa.2005.05.004
- 5. Borges, J., Fernández-Córdoba, C., Pujol, J., Rifa, J., Villanueva, M.:  $Z_2Z_4$ -linear codes: Generator matrices and duality. Des. Codes Cryptography  ${\bf 54}(2)$ , 167–179 (2010). DOI 10.1007/s10623-009-9316-9
- Borges, J., Phelps, K.T., Rifà, J.: The rank and kernel of extended 1-perfect Z<sub>4</sub>-linear and additive non-Z<sub>4</sub>-linear codes. IEEE Trans. Inf. Theory 49(8), 2028–2034 (2003). DOI 10.1109/TIT.2003.814490
- 7. Borges, J., Phelps, K.T., Rifà, J., Zinoviev, V.A.: On  $Z_4$ -linear Preparata-like and Kerdock-like codes. IEEE Trans. Inf. Theory  $\bf 49(11)$ , 3834–3843 (2003). DOI 10.1109/TIT.2003.819329

8. Borges, J., Rifà, J.: A characterization of 1-perfect additive codes. IEEE Trans. Inf. Theory 45(5), 1688–1697 (1999). DOI 10.1109/18.771247

- 9. Calderbank, A.R., Cameron, P.J., Kantor, W.M., Seidel, J.J.:  $Z_4$ -Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets  $\bf 75(2)$ ,  $\bf 436-480$  (1997). DOI  $\bf 10.1112/S0024611597000403$
- van Dam, E.R., Fon-Der-Flaass, D.: Uniformly packed codes and more distance regular graphs from crooked functions. J. Algebr. Comb. 12(2), 115–121 (2000). DOI 10.1023/ A:1026583725202
- 11. Dumer, I.I.: Some new uniformly packed codes. Proc. Moscow Inst. Physics and Technology. Ser. "Radiotekhnika i Elektronika", pp. 72–78 (1976). In Russian
- 12. Gorkunov, E.V.: The automorphism group of a q-ary Hamming code. Diskretn. Anal. Issled. Oper. 17(6), 50–66 (2010). In Russian
- Hammons Jr, A.R., Kumar, P.V., Calderbank, A.R., Sloane, N.J.A., Solé, P.: The Z4-linearity of Kerdock, Preparata, Goethals, and related codes. IEEE Trans. Inf. Theory 40(2), 301–319 (1994). DOI 10.1109/18.312154
- 14. Kantor, W.M., Williams, M.E.: Symplectic semifield planes and  $Z_4$ -linear codes. Trans. Am. Math. Soc. **356**(3), 895–938 (2004). DOI 10.1090/S0002-9947-03-03401-9
- Krotov, D.S.: Z<sub>4</sub>-linear perfect codes. Diskretn. Anal. Issled. Oper., Ser.1 7(4), 78–90 (2000). In Russian
- Krotov, D.S.: Z<sub>4</sub>-linear perfect codes. E-print 0710.0198, ArXiv.org. URL http://arxiv.org/abs/0710.0198. English translation
- 17. Krotov, D.S., Villanueva, M.: Classification of the  $Z_2Z_4$ -linear Hadamard codes and their automorphism groups. IEEE Trans. Inf. Theory  $\bf 61(2)$ , 887–894 (2015). DOI 10.1109/TIT.2014.2379644
- MacWilliams, F.J., Sloane, N.J.A.: The Theory of Error-Correcting Codes. Amsterdam, Netherlands: North Holland (1977)
- 19. Phelps, K.T., Rifà, J., Villanueva, M.: On the additive ( $Z_4$ -linear and non- $Z_4$ -linear) Hadamard codes: Rank and kernel. IEEE Trans. Inf. Theory  ${\bf 52}(1),\ 316-319$  (2006). DOI  $10.1109/{\rm TIT}.2005.860464$
- Preparata, F.P.: A class of optimum nonlinear double-error correcting codes. Inf. Control 13(4), 378–400 (1968). DOI 10.1016/S0019-9958(68)90874-7
- Zaitsev, G.V., Zinoviev, V.A., Semakov, N.V.: Interrelation of Preparata and Hamming codes and extension of Hamming codes to new double-error-correcting codes. In: P.N. Petrov, F. Csaki (eds.) Proc. 2nd Int. Symp. Information Theory, Tsahkadsor, Armenia, USSR, 1971, pp. 257–264. Akademiai Kiado, Budapest, Hungary (1973)