Speed selection and stability of wavefronts for delayed monostable reaction-diffusion equations

Abraham Solar · Sergei Trofimchuk

Abstract We study the asymptotic stability of traveling fronts and front's velocity selection problem for the time-delayed monostable equation (*) $u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), x \in \mathbb{R}, t > 0$, considered with Lipschitz continuous reaction term $g: \mathbb{R}_+ \to \mathbb{R}_+$. We are also assuming that g is $C^{1,\alpha}$ -smooth in some neighbourhood of the equilibria 0 and $\kappa > 0$ to (*). In difference with the previous works, we do not impose any convexity or subtangency condition on the graph of g so that equation (*) can possess pushed traveling fronts. Our first main result says that the non-critical wavefronts of (*) with monotone g are globally nonlinearly stable. In the special and easier case when the Lipschitz constant for g coincides with g'(0), we present a series of results concerning the exponential [asymptotic] stability of non-critical [respectively, critical] fronts for monostable model (*). As an application, we present a criterion of the absolute global stability of non-critical wavefronts to the diffusive Nicholson's blowflies equation.

Keywords Monostable equation \cdot Reaction-diffusion equation \cdot Delay \cdot Superand sub-solutions \cdot Wavefront \cdot Asymptotic stability \cdot Speed selection

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Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile E-mail: asolar.solar@gmail.com

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile

E-mail: trofimch@inst-mat.utalca.cl

A. Solar

S. Trofimchuk (corresponding author)

1 Introduction and main results

Set $\Pi_0 := [-h, 0] \times \mathbb{R} \subset \mathbb{R}^2$ and consider the family \mathcal{F} of continuous and uniformly bounded functions $w_0(s, x)$, $w_0 : \Pi_0 \to \mathbb{R}_+$, exponentially decaying (uniformly in s) as $x \to -\infty$ and separated from 0 (uniformly in s) as $x \to +\infty$. In particular, we assume that each $w_0 \in \mathcal{F}$ satisfies

$$(IC1) \quad 0 \le w_0(s,x) \le |w_0|_{\infty} := \sup_{(s,x) \in \Pi_0} w_0(s,x) < \infty, \quad (s,x) \in \Pi_0;$$

$$(IC2) \quad \liminf_{x \to +\infty} \min_{s \in [-h,0]} w_0(s,x) > 0.$$

Everywhere in the sequel, we will also assume that each element $w_0(s,x)$ of \mathcal{F} is locally Hölder continuous in $x \in \mathbb{R}$, uniformly with respect to $s \in [-h, 0]$.

Our goal in this work is to indicate subclasses of initial functions $w_0 \in \mathcal{F}$ for monostable reaction-diffusion equations with monotone delayed reaction

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + g(u(t-h,x)), \ t > 0, \ x \in \mathbb{R},$$
 (1)

$$u(s,x) = w_0(s,x), \ s \in [-h,0], \ x \in \mathbb{R},$$
 (2)

which yield solutions $u = u(t, x, w_0)$ converging, as $t \to +\infty$, to appropriate traveling fronts $u = \phi(x + ct, w_0)$, c > 0, of (1), (2). By definition, the front profile $\phi : \mathbb{R} \to \mathbb{R}_+$ is a positive bounded smooth function such that the limits $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$ exist. Here we are assuming that the continuous nonlinearity $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the monostability condition

(H) the equation g(x)=x has exactly two nonnegative solutions: 0 and $\kappa>0$. Moreover, g is C^1 -smooth in some δ_0 -neighborhood of the equilibria where g'(0)>1, $g'(\kappa)<1$, and it also satisfies the Lipshitz condition $|g(u)-g(v)|\leq L_g|u-v|$, $u,v\geq 0$. In addition, there are C>0, $\theta\in(0,1]$, such that $|g'(u)-g'(0)|\leq Cu^\theta$ for $u\in(0,\delta_0]$. To simplify the notation, we will extend g linearly and C^1 -smoothly on $(-\infty,0]$.

From [10] we know that above conditions imposed on w_0 are sufficient for the existence of a unique classical solution $u = u(t, x, w_0) : [-h, +\infty) \times \mathbb{R} \to \mathbb{R}_+$ to (1), (2) (i.e. of a continuous and bounded (at least, on finite time intervals) function u having continuous derivatives u_t, u_x, u_{xx} in $\Omega = (0, +\infty) \times \mathbb{R}$ and satisfying (1) in Ω as well as (2) in $[-h, 0] \times \mathbb{R}$). We will show that, similarly to $w_0(s, x)$ and $\phi(x + cs, w_0)$, the function $w_{(t)}(s, x) = u(t + s, x, w_0)$, $(s, x) \in \Pi_0$, will also belong to the class \mathcal{F} , for each fixed t > 0.

In this way, the concept of 'speed selection' reflects the evident fact that the properties of w_0 may determine the speed of propagation of the initial 'concentration' (of something) $w_0(s,x)$ from the right side of the x-axis \mathbb{R} (where w_0 is separated from 0) to the left side of \mathbb{R} (where w_0 vanishes). Moreover, in the non-delayed case (when h=0) it is well known [35] that, given a converging solution $u(t,x,w_0) \leadsto \phi(x+ct,w_0)$, the speed of propagation c 'choosen' by $u(t,x,w_0)$ depends mainly only on the asymptotic behavior of $w_0(s,x)$ at $x=-\infty$. It is clear also that the speed selection problem is closely related to the front stability question: indeed, if some wavefront $u=\phi(x+c_0t)$ is stable (in an appropriate metric phase space), then each initial datum $w_0(s,x)$ close to $\phi(x+c_0s)$ yields a 'concentration' distribution u(t,x) propagating to the left of $\mathbb R$ with the same velocity c_0 . Below we will give precise mathematical formulations for the above informal discussion.

The studies of wavefront stability in monostable monotone delayed model (1) (including its non-local and discrete Laplacian versions) were initiated in 2004-2005 by Mei $et\ al.\ [29]$ and Ma and Zou [21]. Their research was influenced by a series of previous results about a) the existence of monotone wavefronts [20, 46]; b) the stability of wavefronts in delayed bistable equations [31,38] and discrete monostable equations [5]. Over the last decade, the wave stability problem for equation (1) has attracted attention of many other mathematicians so that it would be difficult to mention all interesting findings in this area. We believe, however, that the strongest results concerning the wavefront stability in the $monotone\ Mackey-Glass\ type\ reaction-diffusion\ equation\ (1)\ can\ be\ found\ in\ [19,26,27,28,45]\ (see also [6,7,14,16,42,47]\ and\ references\ therein\ for\ the\ case\ of\ unimodal\ birth\ function\ g). In our work, rather then writing statements of the aforementioned results from [19,26,27,28,45], we prefer to discuss their relations with our two main theorems announced below.$

Now, two different approaches were employed in the cited works: a weighted energy approach [29,26,27,28] and the super- and sub-solution method [21,45]. The stability of monotone wavefronts to (1) was always proved under rather strong smoothness (C^2 -smoothness) and shape conditions on g. In particular, hypotheses imposed on q were always sufficient to assure the inequality q'(x) < q'(0) for all $x \in [0, \kappa]$ (cf. [43, Subsection 1.2]). The latter condition, however, excludes a subclass of equations (1) possessing so called pushed minimal traveling fronts [35, 43]. Since pushed wavefronts are quite interesting from both applied [11,33] and mathematical [4,12,15,17,34,35,41,43] points of view, their existence, uniqueness and stability properties in the case of delayed monotone model (1) were recently considered in [17, 40, 43]. Particularly, the existence of the minimal speed of front propagation c_* was proved in [17,43] (if g is neither monotone nor subtangential at 0, the existence of c_* is an important open problem). It should be also observed that, in general, either analytical determination or numerical approximation of the exact value of c_* is a quite difficult task [3,15,35,43]. By [17,40], c_* coincides with the asymptotic speed of propagation (this important concept was proposed by Aronson and Weinberger [2] in 1977). Next, the stability of pushed wavefronts to (1) was also investigated in [40].

In the present work, we continue our studies in [40], by analysing stability of other (i.e. not necessarily minimal) wavefronts $u = \phi(x + ct)$, $c \ge c_*$, to equation (1). One of the main difference with the previous works consists in generally nonconvex and non-smooth nature of the monotone birth function g: for instance, in our first results below, we do not even require the subtangency condition

$$g(x) \le g'(0)x, \ x \ge 0.$$

Before announcing our first theorem, we recall [43] that the condition $c \geq c_*$ implies that the characteristic equation at the trivial steady state

$$\chi_0(\lambda) := \lambda^2 - c\lambda - 1 + g'(0)e^{-\lambda ch} = 0$$

has exactly two real roots $\lambda_1 = \lambda_1(c) \le \lambda_2 = \lambda_2(c)$ (counting multiplicity), both of them are positive. Note also that $-\lambda_1(c)$, $\lambda_2(c)$ are increasing functions of c.

Next, for a non-negative λ , the norm $|f|_{\lambda}$ of function $f: \mathbb{R} \to \mathbb{R}$ is defined as

$$|f|_{\lambda} = \max\{\sup_{t \le 0} e^{-\lambda t} |f(t)|, \sup_{t \ge 0} |f(t)|\}.$$

If we set $\eta_{\lambda}(t) = \min\{e^{\lambda t}, 1\}$ then clearly

$$|f|_{\lambda} = \sup_{t \in \mathbb{R}} |f(t)|/\eta_{\lambda}(t).$$

The main result of this paper is the following

Theorem 1 Assume that the initial function w_0 satisfies the hypotheses (IC1), (IC2) and that, for some A > 0 and $c > c_*$, it holds

$$\lim_{x \to -\infty} w_0(s, x)e^{-\lambda_1(c)(x+cs)} = A$$

uniformly on $s \in [-h, 0]$. If, in addition, the birth function g is strictly increasing and satisfies (\mathbf{H}) , then the solution of (1), (2) satisfies

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \frac{|\phi(x + ct + a) - u(t, x)|}{\eta_{\lambda_1}(x + ct)} = 0,$$
(3)

where $a = (\lambda_1(c))^{-1} \ln A$ and the front profile ϕ (existing in virtue of the assumption $c > c_*$) is normalised by $\lim_{x \to -\infty} e^{-\lambda_1(c)x} \phi(x) = 1$.

Theorem 1 allows to answer the velocity selection question for solutions with initial data possessing exponential decay at $-\infty$. Indeed, suppose that, for some $\lambda > 0$, it holds

$$\lim_{x \to -\infty} w_0(s, x)e^{-\lambda x} = A(s) > 0, \text{ uniformly in } s \in [-h, 0].$$
 (4)

Then define $c(\lambda)$ by the formula $c(\lambda) = \mu/\lambda$, where μ is the unique positive root of the equation

$$\lambda^2 - \mu - 1 + g'(0)e^{-\mu h} = 0.$$

It is easy to see that $c(\lambda) \geq c_{\#}$, where $c_{\#} = c_{\#}(g'(0), h)$ is the so-called critical speed (a uniquely determined value of c for which the characteristic function $\chi_0(\lambda)$ has a double positive zero). Set $\lambda_* := \lambda_1(c_*)$. We claim that

$$c_{\lambda} := \begin{cases} c(\lambda), & \text{if } \lambda < \lambda_*, \\ c_*, & \text{if } \lambda \ge \lambda_*, \end{cases}$$

is the speed of propagation selected by solutions with initial data satisfying (4). More precisely, the following assertion holds.

Corollary 1 Assume that the initial function w_0 satisfies the hypotheses (IC1), (IC2) and (4). Suppose first that $\lambda > \lambda_*$ and $c_* > c_\#$, then the solution of (1), (2) satisfies

$$\lim_{t\to +\infty} \sup_{x\in\mathbb{R}} \frac{|\phi_*(x+c_*t)-u(t,x)|}{\eta_\nu(x+c_*t)} = 0$$

for each fixed $\nu \in (\lambda_*, \lambda)$. Here ϕ_* denotes the profile of appropriately shifted unique minimal (pushed) front to equation (1).

Next, let $\lambda < \lambda_*$ (so that $c(\lambda) = c_{\lambda}$) and $c_* \geq c_{\#}$. Set

$$a_{-} := \frac{1}{\lambda} \ln \left[\min_{s \in [-h,0]} A(s) e^{-\mu s} \right] \le a_{+} := \frac{1}{\lambda} \ln \left[\max_{s \in [-h,0]} A(s) e^{-\mu s} \right].$$

Then for every $\epsilon > 0$ there exists $T_1(\epsilon) > 0$ such that

$$(1 - \epsilon)\phi_{\lambda}(x + c_{\lambda}t + a_{-}) \le u(t, x) \le (1 + \epsilon)\phi_{\lambda}(x + c_{\lambda}t + a_{+}), \quad t \ge T_{1}(\epsilon), \ x \in \mathbb{R}.$$

Here ϕ_{λ} denotes the profile of the unique wavefront to equation (1) propagating with the velocity $c(\lambda)$ and satisfying $\lim_{x\to-\infty} e^{-\lambda x}\phi_{\lambda}(x)=1$.

Now, if $\lambda = \lambda_*$ and $c_* > c_\#$, then there exists $a' \in \mathbb{R}$ such that for every $\epsilon > 0$ and positive $\nu < \lambda_* < M < \lambda_2(c_*)$ it holds

$$\phi_*(x + c_*t + a') - \epsilon \eta_M(x + c_*t) \le u(t, x) \le (1 + \epsilon)\phi_\nu(x + c_\nu t), \ t \ge T_2, \ x \in \mathbb{R}, \ (5)$$

for an appropriate $T_2 = T_2(\epsilon) > 0$. Furthermore, in such a case, u(t,x) can not converge, uniformly on \mathbb{R} , to a wavefront solution of equation (1).

Finally, if $L_g = g'(0)$ (so that $c_* = c_\#$) and $\lambda \ge \lambda_*$, then there exists $b' \in \mathbb{R}$ such that for every $\epsilon > 0$

$$0 \le u(t,x) \le (1+\epsilon)\phi_*(x+c_*t+b'), \quad t \ge T_3, \ x \in \mathbb{R},$$
 (6)

whenever $T_3 = T_3(\epsilon) > 0$ is sufficiently large.

It is worth to note that there is an important difference between the speed selection results obtained in the non-delayed and delayed cases. Indeed, if h=0 and $\lambda<\lambda_*$ then $a_-=a_+$ and therefore u(t,x) converges to a single wavefront $\phi_\lambda(x+c_\lambda t+a_\pm)$ propagating with the velocity $c_\lambda=\lambda+(g'(0)-1)/\lambda$. In the delayed case, however, we only can say that u(t,x) evolves between two shifted traveling fronts, both of them moving with the same velocity c_λ . Observe also that, since $\mu=\mu(h)$ is a decreasing function of h, the inclusion of delay in problems modeled by (1) slows down the propagation of 'concentrations' having the same initial distribution which satisfies (4).

Remark 1 Consider again the final statement of Corollary 1. Under conditions assumed in it (at least when additionally $\lambda > \lambda_*$), it is natural to expect [35] the so-called *convergence in form* of u(t,x) to the minimal wavefront: that is

$$\sup_{x \in \mathbb{R}} |u(t,x) - \phi_*(x + c(t))| \to 0, \text{ as } t \to +\infty,$$

for an appropriate function c(t). Then (6) implies that the function $c(t) - c_* t$ is bounded from above: in other words, in such a case, the concentration u(t,x) should propagate behind the minimal front. A more detailed analysis of this phenomenon for some delayed reaction-difusion models will be given in the forthcoming work by the authors.

Another immediate consequence of Theorem 1 is the following assertion concerning the global asymptotic stability (without asymptotic phase) of wavefronts:

Corollary 2 Let g and w_0 satisfy the assumptions **(H)** and (IC1), (IC2). If g is strictly increasing and

$$\sup_{s \in [-h,0]} |\phi(\cdot + cs) - w_0(s,\cdot)|_{\mu} < \infty$$
 (7)

for some $c > c_*$ and $\mu > \lambda_1(c)$, then the solution of (1), (2) satisfies

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\frac{|\phi(x+ct)-u(t,x)|}{\eta_{\lambda_1}(x+ct)}=0.$$

Clearly, the statement of Theorem 1 (or Corollary 2) implies the uniqueness (up to a translation) of non-critical traveling fronts propagating with the same velocity c and having the same order of exponential decay at $-\infty$, cf. e.g. [21, Theorem 1.1], [45, Corollary 4.9]. In any event, the uniqueness of each front (including critical one) to the monotone model (1) was established in [43, Theorem 1.2] by means of the Berestycki-Nirenberg method of the sliding solutions. In the case when g is non-monotone, the wave uniqueness was investigated in [1], by applying a suitable L^2 -variant of the bootstrap argument suggested by Mallet-Paret in [23]. We recall here that, in the case of a unimodal birth-function g, equation (1) can possess non-monotone wavefronts (either slowly oscillating or eventually monotone). This fact was deduced in [7,14,42] from the seminal results [22,23,24,25] by Mallet-Paret and Sell.

It is instructive to compare Theorem 1 and Corollary 2 with the corresponding results from the above mentioned works [26,27,28,29,45] (restricting them to the particular family of the Mackey-Glass type diffusive equations (1)). It is easy to check that Theorem 1 amplifies Theorem 4.1 from [45] which was proved under more restrictive smoothness and geometric conditions on g and w_0 . (Theorem 2A below also extends the mentioned result by Wang $et\ al.$ for the critical case $c=c_\#$). In particular, the assumptions of [45] contain the inequality $g'(x) \leq g'(0), x \geq 0$, which excludes from consideration the pushed waves, see [43, Subsection 1.2] for more detail. The approach of [45] is a version of the super- and subsolutions method proposed in [5] and then further developed in [21]. The proofs given in the present paper are also based on the squeezing technique and the Phragmèn-Lindelöf principle for reaction-diffusion equations. Hence, we are also using adequate super- and sub-solutions (which generally are not C^1 -smooth and are simpler than those considered in [5,21,45]. In particular, the latter fact allows to shorten the proofs).

Another important approach to the wave stability problem in (1) is a weighted energy method developed by Mei et al. [26,27,28,29]. See also Kyrychko et al. [16], Lv and Wang [19], Wu et al. [47]. This method is based on rather technical weighted energy estimations and generally requires better properties from g and w_0 . For instance, it was assumed in [19,28] that $g''(x) \leq 0$, $x \geq 0$, and that the weighted initial perturbation $\delta(s,x) = (\phi(x+cs) - w_0(s,x))/\eta_{\mu}(x)$ belongs to the Sobolev space $H^1(\mathbb{R})$ for some $\mu > \lambda_1$ and for each fixed $s \in [-h,0]$. It was also assumed in [19,28] that $\delta: [-h,0] \to H^1(\mathbb{R})$ is a continuous function that implies immediately the fulfilment of (7), in virtue of the corresponding embedding theorem. Therefore Corollary 2 can be also used in such a situation. However, in difference with Corollary 2, the weighted energy method allows to prove the exponential stability of non-critical traveling fronts. Consequently, it gives the same convergence rates as the Sattinger functional analytical approach [36] gives in the case of non-delayed version of (1). We recall that the latter approach is based on the spectral analysis of equation (1) linearised along a wavefront. Thus a certain disadvantage of Theorem 1 as well as [5, Theorem 2], [21, Theorem 5.1], [45, Theorem 4.1] is that they do not give any estimation of the rate of convergence in (3). In this regard, it is a remarkable fact that super- and sub-solutions used in this work are also suitable to provide rather short proofs of the exponential stability [asymptotical stability] of non-critical [respectively, critical] wavefronts in equation (1) considered with the monotone birth function g satisfying relatively weak restrictions (**H**) and $L_g = g'(0)$. For example, $L_g = g'(0)$ if g is differentiable on \mathbb{R}_+ where $g'(x) \leq g'(0)$.

Theorem 2 In addition to (\mathbf{H}) , suppose that g is strictly increasing and $L_g = g'(0)$. If the initial function w_0 satisfies the assumptions (IC1), (IC2), then the solution u(t,x) of (1), (2) satisfies the following.

A. If $c \geq c_{\#}$ and

$$\lim_{z \to -\infty} w_0(s, x) / \phi(x) = 1,$$

uniformly on $s \in [-h, 0]$, then

$$|u(t,\cdot)/\phi(\cdot + ct) - 1|_0 = o(1), \ t \to +\infty.$$
 (8)

B. If $c > c_{\#}$ and $\lambda \in (\lambda_1(c), \lambda_2(c))$ then

$$\sup_{s \in [-h,0]} |\phi(\cdot + cs) - w_0(s,\cdot)|_{\lambda} < \infty \tag{9}$$

implies that

$$\sup_{x \in \mathbb{R}} \frac{|u(t,x) - \phi(x + ct)|}{\eta_{\lambda}(x + ct)} \le Ce^{-\gamma t}, \ t \ge 0,$$

for some C > 0 and $\gamma > 0$.

To the best of our knowledge, the description of front convergence in the form (8) was proposed by Chen and Guo [5]. Clearly, this kind of convergence is equivalent to the weighted convergence expressed by (3) (if $c > c_{\#}$) and it is stronger than the uniform convergence

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \to 0, \ t \to +\infty.$$

The stability results stated in Theorem 2 have the global character in the sense that none smallness restriction is imposed on the norm (9) of perturbation $\phi(x+cs)-w_0(s,x)$. Remarkably, in the case where we do not assume anymore that g is monotone, our approach still allows us to prove the local stability of fronts. Even more, we are also able to present some global stability results. In this way, our next main theorem and its corollary can be regarded as a further development of [18, Theorem 2.1] and [47, Theorems 2.4 and 2.6]. Before formulating the corresponding assertions, let us recall that the hypothesis

(UM) Let (H) be satisfied and suppose that $L_g = g'(0)$ and g is bounded implies the existence of a unique normalised (at $-\infty$) positive semi-wavewfront $u(t,x) = \phi_c(x+ct)$ to equation (1) for each $c \geq c_\#$, see e.g. [1,13]. We recall here that the definition of a semi-wavewfront is similar to the definition of a wavefront: the only part that is changing is the boundary condition $\phi_c(+\infty) = \kappa$ which should be replaced with $\liminf_{x\to +\infty} \phi_c(x) > 0$.

Theorem 3 Assume (UM) and let the initial function w_0 satisfy (IC1). Consider $c > c_{\#}$, $\lambda \in (\lambda_1(c), \lambda_2(c))$ and set $\xi(x, \lambda) = e^{\lambda x}$. Then the following holds.

A. The inequality (9) implies that the solution u(t,x) of (1), (2) converges to the semi-wavefront $\phi_c(x+ct)$: more precisely, there are positive C, γ such that

$$\sup_{x \in \mathbb{R}} \frac{|u(t,x) - \phi(x + ct)|}{\xi(x + ct, \lambda)} \le Ce^{-\gamma t}, \ t \ge 0.$$

B. Let, in addition, |g'(u)| < 1 on some interval $[\kappa - \rho, \kappa + \rho]$, $\rho > 0$. If, for some $b \ge 0$, the initial function w_0 and the semi-wavefront profile ϕ_c satisfy

$$\kappa - \rho/4 \le w_0(s, x), \phi(x + cs) < \kappa + \rho/4 \text{ for all } x \ge b - ch, \ s \in [-h, 0],$$

$$|w_0(s,x) - \phi(x+cs)| \le 0.5\rho e^{\lambda(x+cs-b)}, \ x \le b, \ s \in [-h,0],$$

then ϕ is actually a wavefront (i.e. $\phi(+\infty) = \kappa$) and the solution u(t,x) of (1), (2) satisfies

$$\sup_{x \in \mathbb{R}} \frac{|u(t,x) - \phi(x+ct)|}{\eta_{\lambda}(x+ct)} \le 0.5\rho e^{-\gamma t}, \ t \ge 0, \tag{10}$$

for some $\gamma > 0$.

Corollary 3 Let g satisfy (UM) and let g be a unimodal function, with a unique point $x_m \in (0, \kappa)$ of local extremum (maximum). Suppose further that |g'(x)| < 1 for all $x \in [g(g(x_m)), g(x_m)]$. Additionally, assume that the initial function w_0 satisfy (IC1) and (IC2) and consider $c > c_\#$, $\lambda \in (\lambda_1(c), \lambda_2(c))$. Then inequality (9) implies that the solution u(t, x) of (1), (2) uniformly converges to the wavefront $\phi(x + ct)$. More precisely, there are positive ρ, γ such that (10) holds.

Let us illustrate Corollary 3 by considering the well-known diffusive version of the Nicholson's blowflies equation

$$u_t(t,x) = u_{xx}(t,x) - \delta u(t,x) + pu(t-\tau,x)e^{-u(t-\tau,x)}.$$
 (11)

By rescaling space-time coordinates, we transform this equation into the form (1) with $g(x) = (p/\delta)xe^{-x}$ and $h = \tau\delta$. In the last decade, the wavefront solutions of equation (11) have been investigated by many authors, e.g. see [6,7,14,18,19, [20, 26, 28, 29, 45, 47]. If the positive parameters p, δ are such that $1 < p/\delta \le e$, then g is monotone and satisfies the hypothesis (H) with $L_g = g'(0)$ and $\kappa =$ $\ln(p/\delta)$. In such a case, Theorem 2 guarantees the global stability of all wavefronts, including the minimal one (these wavefronts are necessarily monotone). For the first time, such a global stability result was established by Mei et al. in [28]. Now, if $e < p/\delta < e^2$, the restriction of g on $[0, \kappa]$ is not monotone anymore. Nevertheless, we still have that $L_g = g'(0)$ while the inequality |g'(x)| < 1 holds for all $x \in [g(g(x_m)), g(x_m)]$, with $x_m = 1$. Therefore, for each $p/\delta \in (e, e^2)$, Corollary 3 assures the global exponential stability of all non-critical wavefronts to equation (11). Note that profiles of these wavefronts are not necessarily monotone and they can either slowly oscillate around κ or be non-monotone but eventually monotone at $+\infty$, cf. [7,14,18]. Observe also that the upper estimation e^2 for p/δ is optimal [14,18]. Under the same restriction $p/\delta \in (e,e^2)$, the local stability of wavefronts to (11) was investigated in [18,47].

To sum up: the main aim of the present work is to establish the stability properties of monostable wavefronts to the time-delayed reaction-diffusion model (1) with generally non-convex and non-smooth birth function g. We are going to achieve this goal by developing suitable ideas and methods from [2,9,37,40,44].

Finally, let us say a few words about the organization of the paper. In Sections 2 and 4 we prove several auxiliary comparison and stability results. Then Theorem 1 (with Corollary 1), Theorem 2 and Theorem 3 (with Corollary 3) are proved in Sections 5, 3 and 6, respectively.

2 Super- and sub-solutions: definition and properties

The stability analysis of a wavefront $u = \phi(x + ct)$ is usually realised in the comoving coordinate frame z = x + ct so that w(t, z) := u(t, z - ct) = u(t, x). Clearly, w satisfies the equation

$$w_t(t,z) = w_{zz}(t,z) - cw_z(t,z) - w(t,z) + g(w(t-h,z-ch)),$$
(12)

while the front profile $\phi(z)$ is a solution of the stationary equation

$$0 = \phi''(z) - c\phi'z - \phi(z) + g(\phi(z - ch)). \tag{13}$$

In order to study the front solutions of (12), (13), different versions of the method of super- and sub- solutions were successfully applied in [20,37,43,46] (in the case of stationary equations similar to (13)) and in [5,21,37,40,45] (in the case of non-stationary equations similar to (12)). An efficacious construction of these solutions is the key to the success of this approach. In particular, the studies of front's stability in [21,45] had used C^3 -smooth super- and sub-solutions previously introduced by Chen and Guo in [5, Lemma 3.7]. It is well known that, by cautiously weakening smoothness restrictions, we can improve the overall quality of super- and sub- solutions, cf. [9,20,34,37,43,45,46]. In this paper, inspired by the latter references, we propose to work with somewhat more handy C^1 -smooth super- and sub-solutions:

Definition 1 Continuous function $w_+:[-h,+\infty)\times\mathbb{R}\to\mathbb{R}$ is called a supersolution for (12), if, for some $z_*\in\mathbb{R}$, this function is $C^{1,2}$ -smooth in the domains $[-h,+\infty)\times(-\infty,z_*]$ and $[-h,+\infty)\times[z_*,+\infty)$ and, for every t>0,

$$\mathcal{N}w_{+}(t,z) \ge 0, \ z \ne z_{*}, \text{ while } (w_{+})_{z}(t,z_{*}-) > (w_{+})_{z}(t,z_{*}+),$$
 (14)

where the nonlinear operator \mathcal{N} is defined by

$$\mathcal{N}w(t,z) := w_t(t,z) - w_{zz}(t,z) + cw_z(t,z) + w(t,z) - g(w(t-h,z-ch)).$$

The definition of a sub-solution w_{-} is similar, with the inequalities reversed in (14).

The following comparison result is a rather standard one. However, since sub- and super-solutions considered in this paper have discontinuous spatial derivates and, in addition, equation (12) contains shifted arguments, we give its proof for the completeness of our exposition. See also [9,34,37,45].

Lemma 1 Assume (**H**) and the monotonicity of g. Let w_+, w_- be a pair of superand sub-solutions for equation (12) such that $|w_{\pm}(t,z)| \leq Ce^{D|z|}$, $t \geq -h$, $z \in \mathbb{R}$, for some C, D > 0 as well as

$$w_{-}(s,z) \le w_{0}(s,z) \le w_{+}(s,z), \text{ for all } s \in [-h,0], z \in \mathbb{R}.$$

Then the solution w(s,z) of equation (12) with the initial datum w_0 satisfies

$$w_{-}(t,z) < w(t,z) < w_{+}(t,z)$$
 for all $t > -h$, $z \in \mathbb{R}$.

Proof In view of the assumed conditions, we have that

$$\pm (g(w_{\pm}(t-h,z-ch)) - g(w(t-h,z-ch))) \ge 0, \quad t \in [0,h], \ z \in \mathbb{R}.$$

Therefore, for all $t \in (0, h]$, the function $\delta(t, z) := \pm (w(t, z) - w_{\pm}(t, z))$ satisfies the inequalities

$$\delta(0,z) \leq 0, \ |\delta(t,z)| \leq 2Ce^{D|z|}, \quad \delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) = \pm (\mathcal{N}w_{\pm}(t,z) - \mathcal{N}w(t,z) + g(w_{\pm}(t-h,z-ch)) - g(w(t-h,z-ch))) = \pm \mathcal{N}w_{\pm}(t,z) \pm (g(w_{\pm}(t-h,z-ch)) - g(w(t-h,z-ch))) \geq 0, \ z \in \mathbb{R} \setminus \{z_*\};$$

$$\frac{\partial \delta(t, z_{*}+)}{\partial z} - \frac{\partial \delta(t, z_{*}-)}{\partial z} = \pm \left(\frac{\partial w_{\pm}(t, z_{*}-)}{\partial z} - \frac{\partial w_{\pm}(t, z_{*}+)}{\partial z} \right) > 0.$$
 (15)

We claim that $\delta(t,z) \leq 0$ for all $t \in [0,h]$, $z \in \mathbb{R}$. Indeed, otherwise there exists $r_0 > 0$ such that $\delta(t,z)$ restricted to any rectangle $\Pi_r = [-r,r] \times [0,h]$ with $r > r_0$, reaches its maximal positive value $M_r > 0$ at at some point $(t',z') \in \Pi_r$.

We claim that (t',z') belongs to the parabolic boundary $\partial \Pi_r$ of Π_r . Indeed, suppose on the contrary, that $\delta(t,z)$ reaches its maximal positive value at some point (t',z') of $\Pi_r \setminus \partial \Pi_r$. Then clearly $z' \neq z_*$ because of (15). Suppose, for instance that $z' > z_*$. Then $\delta(t,z)$ considered on the subrectangle $\Pi = [z_*,r] \times [0,h]$ reaches its maximal positive value M_r at the point $(t',z') \in \Pi \setminus \partial \Pi$. Then the classical results [32, Chapter 3, Theorems 5,7] show that $\delta(t,z) \equiv M_r > 0$ in Π , a contradiction.

Hence, the usual maximum principle holds for each Π_r , $r \geq r_0$, so that we can appeal to the proof of the Phragmèn-Lindelöf principle from [32] (see Theorem 10 in Chapter 3 of this book), in order to conclude that $\delta(t,z) \leq 0$ for all $t \in [0,h], z \in \mathbb{R}$.

But then we can again repeat the above argument on the intervals [h, 2h], [2h, 3h], ... establishing that the inequality $w_{-}(t, z) \leq w(t, z) \leq w_{+}(t, z)$, $z \in \mathbb{R}$, holds for all t > -h.

To the best of our knowledge, the following important property of super- (sub-) solutions was first used by Aronson and Weinberger in [2]. See also [37, Proposition 2.9].

Corollary 4 Assume (H) and the monotonicity of g. Let $w_+(z)$ be an exponentially bounded super-solution for equation (12) and consider the solution $w^+(t,z), t \ge 0$, of the initial value problem $w^+(s,z) = w_+(z)$ for (12). Then $w^+(t_1,z) \ge w^+(t_2,z)$ for each $t_1 \le t_2$, $z \in \mathbb{R}$. A similar result is valid in the case of exponentially bounded sub-solutions $w_-(z)$ which do not depend on t: if $w^-(t,z)$ solves the initial value problem $w^-(s,z) = w_-(z)$ for (12), then $w^-(t_1,z) \le w^-(t_2,z)$ for each $t_1 \le t_2$, $z \in \mathbb{R}$.

Proof We prove only the first statement of the corollary (for super-solution w_+), the case of sub-solution $w_-(z)$ being completely analogous.

By Lemma 1, $w^+(t,z) \leq w_+(z)$ for each $t \geq 0$. Hence, fixing some positive l and considering the initial value problems $u(s,z) = w^+(s+l,z)$, $v(s,z) = w_+(z)$, $s \in [-h,0]$, $z \in \mathbb{R}$, for equation (12), we find that $u(t,z) = w^+(t+l,z) \leq v(t,z) = w^+(t,z)$, t > 0, $z \in \mathbb{R}$.

3 Proof of Theorem 2 and Corollary 1

In this section, we take some $c \geq c_{\#}$ and assume the conditions of Theorem 2. This result will follow from Theorem 4 proved below. Everywhere in the section we denote by w(t,z) solution of equation (12) satisfying the initial value condition $w(s,z) = w_0(s,z), (s,z) \in \Pi_0$.

It is easy to see that, given $q^* > 0, \ q_* \in (0, \kappa)$, there are $\delta^* < \delta_0, \ \gamma^* > 0$ such that

$$g(u) - g(u - qe^{\gamma h}) \le q(1 - 2\gamma), (u, q, \gamma) \in \Pi_{-} = [\kappa - \delta^*, \kappa] \times [0, q_*] \times [0, \gamma^*];$$
(16)

$$g(u) - g(u + qe^{\gamma h}) \ge -q(1 - 2\gamma), (u, q, \gamma) \in \Pi_{+} = [\kappa - \delta^{*}, \kappa] \times [0, q^{*}] \times [0, \gamma^{*}].$$
(17)

Indeed, it suffices to note that the continuous functions

$$G_{-}(u,q,\gamma) := \begin{cases} 1 + (g(u - e^{\gamma h}q) - g(u))/q, & (u,q,\gamma) \in \Pi_{-}; \\ 1 - e^{\gamma h}g'(u), & u \in [\kappa - \delta^*, \kappa], & q = 0, & \gamma \in [0, \gamma^*], \end{cases}$$

$$G_{+}(u,q,\gamma) := \begin{cases} 1 - (g(u + e^{\gamma h}q) - g(u))/q, & (u,q,\gamma) \in \Pi_{+}; \\ 1 - e^{\gamma h}g'(u), & u \in [\kappa - \delta^{*},\kappa], & q = 0, & \gamma \in [0,\gamma^{*}], \end{cases}$$

are positive on Π_{\pm} provided that γ^*, δ^* are sufficiently small.

From now on, we fix $\gamma \in [0, \gamma_*)$, $\delta \in (0, \delta^*)$ such that (16) and (17) hold and

$$-\gamma + c\lambda - \lambda^2 + 1 - g'(0)e^{\gamma h}e^{-\lambda ch} \ge 0.$$

It is easy to see that $\gamma = 0$ for $\lambda = \lambda_1(c)$ while γ can be chosen positive if $\lambda \in (\lambda_1(c), \lambda_2(c))$. Consider b determined by the equation $\phi(b - ch) = \kappa - \delta^*/2$. Without loss of generality we can assume that b > 0.

Lemma 2 Suppose that $L_g = g'(0)$ in (**H**). Let $\gamma \geq 0$ be as defined above. If either $c > c_\#$ with $\lambda \in (\lambda_1(c), \lambda_2(c))$ or $c \geq c_\#$ with $\lambda = \lambda_1(c)$, then

$$w_0(s,z) \le \phi(z) + q\eta_\lambda(z-b), \quad z \in \mathbb{R}, \quad s \in [-h,0],$$

with $q \in (0, q^*]$ implies

$$w(t,z) \le \phi(z) + qe^{-\gamma t} \eta_{\lambda}(z-b), \quad z \in \mathbb{R}, \quad t \ge -h.$$

Similarly, the inequality

$$\phi(z) - q\eta_{\lambda}(z-b) \le w_0(s,z), \quad z \in \mathbb{R}, \quad s \in [-h,0],$$

with some $0 < q \le q_*$ implies

$$\phi(z) - qe^{-\gamma t}\eta_{\lambda}(z-b) \le w(t,z), \quad z \in \mathbb{R}, \quad t \ge -h.$$

Each conclusion of the lemma holds without any upper restriction on the size of q if we replace $\eta_{\lambda}(z-b)$ with $\xi(z,\lambda) = \exp{(\lambda z)}$.

Proof Set $w_{\pm}(t,z) = \phi(z) \pm q e^{-\gamma t} \eta_{\lambda}(z-b)$. Then, for t > 0 and $z \in \mathbb{R} \setminus \{b\}$, after a direct calculation we find that

$$\mathcal{N}w_{\pm}(t,z) = \pm qe^{-\gamma t} [-\gamma \eta_{\lambda}(z-b) + c\eta_{\lambda}'(z-b) - \eta_{\lambda}''(z-b) + \eta_{\lambda}(z-b)] + g(\phi(z-ch)) - g(w_{\pm}(t-h,z-ch)).$$

It is clear that for z < b (if we are considering $\eta_{\lambda}(z - b)$) as well as for all $z \in \mathbb{R}$ (if we are using $\xi(z, \lambda)$ instead of $\eta_{\lambda}(z - b)$), it holds that

$$\pm \mathcal{N}w_{\pm}(t,z) \ge qe^{-\gamma t}e^{\lambda(z-b)}[-\gamma + c\lambda - \lambda^2 + 1 - g'(0)e^{\gamma h}e^{-\lambda ch}] \ge 0.$$

If z > b and $q \in (0, q^*]$, then (17) implies

$$\mathcal{N}w_{+}(t,z) \ge qe^{-\gamma t}[-\gamma + 1 - (1-2\gamma)] = \gamma qe^{-\gamma t} > 0.$$

Similarly, if z > b and $q \in (0, q_*]$, we obtain from (16) that

$$-\mathcal{N}w_{-}(t,z) \ge qe^{-\gamma t}[-\gamma + 1 - (1-2\gamma)] = \gamma qe^{-\gamma t} > 0.$$

Next, since

$$\pm \left(\frac{\partial w_{\pm}(t, b+)}{\partial z} - \frac{\partial w_{\pm}(t, b-)}{\partial z} \right) = -q\lambda e^{-\gamma t} < 0,$$

we conclude that $w_{\pm}(t, z)$ is a pair of super- and sub-solutions for equation (12). Finally, an application of Lemma 1 completes the proof.

Lemma 2 implies that front solutions of equation (1) are locally stable:

Corollary 5 Let the triple $(c, \lambda, \gamma) \in [c_{\#}, +\infty) \times [\lambda_1(c), \lambda_2(c)) \times \mathbb{R}_+$ be as in Lemma 2 and suppose that

$$\sup_{s \in [-h,0]} |\phi(\cdot) - w_0(s,\cdot)|_{\lambda} < \rho e^{-\lambda b}$$

for some $\rho < \kappa$. Then

$$|\phi(\cdot) - w(t, \cdot)|_{\lambda} < \rho e^{-\gamma t}, \ t \ge 0.$$

Proof The statement of the corollary is an immediate consequence of Lemma 2, since, due to our assumptions, for all $z \in \mathbb{R}, \ s \in [-h, 0],$

$$\phi(z) - \rho \eta_{\lambda}(z - b) \le \phi(z) - \rho e^{-\lambda b} \eta_{\lambda}(z) \le w_0(s, z) \le$$
$$\phi(z) + \rho e^{-\lambda b} \eta_{\lambda}(z) \le \phi(z) + \rho \eta_{\lambda}(z - b).$$

We note that assumption (IC1) allows consideration of initial functions w_0 which can be equal to 0 on compact subsets of Π_0 . This fact complicates the construction of adequate sub-solutions. In the next assertion we show that, without restricting generality, the positivity of w_0 can assumed in our proofs.

Corollary 6 Suppose that $L_g = g'(0)$ in (\mathbf{H}) and that $w_0(s,z)$, $(s,z) \in \Pi_0$, satisfies the assumptions (IC1), (IC2). Then the following holds.

A. If $c \geq c_{\#}$ and

$$\lim_{z \to -\infty} w_0(s, z) / \phi(z) = 1, \tag{18}$$

uniformly on $s \in [-h,0]$, then w(2h+s,z) > 0, $(s,z) \in \Pi_0$, also satisfies the assumptions (IC1), (IC2) and $\lim_{z\to-\infty} w(t,z)/\phi(z) = 1$ uniformly with respect to $t \in [0,+\infty)$.

B. Suppose that $c > c_{\#}$, $\lambda \in (\lambda_1(c), \lambda_2(c))$ together with

$$q_0 := \sup_{s \in [-h,0]} |\phi(\cdot) - w_0(s,\cdot)|_{\lambda} < \infty.$$
 (19)

Then w(2h+s,z) > 0, $(s,z) \in \Pi_0$, also satisfies the assumptions (IC1), (IC2) and, for each $t \geq 0$,

$$\sup_{s \in [-h,0]} |\phi(\cdot) - w(s+t,\cdot)|_{\lambda} < \infty.$$

Proof The positivity of w(2h+s,z) for $(s,z) \in \Pi_0$, is obvious. Next, the fulfilment of separation condition (IC2) for w(2h+s,z) can be proved similarly to [40, Proposition 1.2] (alternatively, the reader can use Duhamel's formula). Next, since $w \equiv 0$ and $w \equiv \max\{\kappa, |w_0|_{\infty}\}$ are, respectively, sub- and super-solutions of equation (12), the condition (IC1) is also fulfilled. Finally, the proofs of the persistence of properties (18) and (19) are given below.

A. Set $\lambda_c = \lambda_1$ if $c = c_{\#}$ or fix some $\lambda_c \in (\lambda_1(c), \lambda_2(c))$ if $c > c_{\#}$. It follows from (18) that for every $s \in \mathbb{R}$, it holds

$$\lim_{z \to -\infty} w_0(s, z) / \phi(z + s) = e^{\lambda_1 s}$$

uniformly on $s \in [-h, 0]$. Therefore, for each small $\delta > 0$ there exists a large $q = q(\delta, w_0) > 0$ such that

$$\phi(z-\delta) - q\xi(z,\lambda_c) \le w_0(s,z) \le \phi(z+\delta) + q\xi(z,\lambda_c), \ (s,z) \in \Pi_0.$$
 (20)

Then Lemma 2 assures that

$$\phi(z-\delta) - q\xi(z,\lambda_c) \le w(t,z) \le \phi(z+\delta) + q\xi(z,\lambda_c), \ t \ge 0, \ z \in \mathbb{R},$$

so that, for all $t \geq 0$ and $z \in \mathbb{R}$, it holds

$$\mathfrak{l}(z,\delta):=\frac{\phi(z-\delta)}{\phi(z)}-1-q\frac{\xi(z,\lambda_c)}{\phi(z)}\leq \frac{w(t,z)}{\phi(z)}-1\leq \mathfrak{r}(z,\delta):=\frac{\phi(z+\delta)}{\phi(z)}-1+q\frac{\xi(z,\lambda_c)}{\phi(z)}.$$

Now, since

$$\lim_{z \to -\infty} \mathfrak{l}(z, \delta) = e^{-\lambda_1 \delta} - 1, \quad \lim_{z \to -\infty} \mathfrak{r}(z, \delta) = e^{\lambda_1 \delta} - 1,$$

for each $\epsilon > 0$ we can indicate $\delta = \delta(\epsilon)$ and z_{ϵ} such that

$$-\epsilon \le \frac{w(t,z)}{\phi(z)} - 1 \le \epsilon$$
 for all $t \ge 0$ and $z \le z_{\epsilon}$.

B. We have that

$$\phi(z) - q_0 \xi(z, \lambda) \le w_0(s, z) \le \phi(z) + q_0 \xi(z, \lambda), \ (s, z) \in \Pi_0,$$

so that the last conclusion of the corollary follows from Lemma 2.

Remark 2 Corollary 6A shows that asymptotic relation (18) is a time invariant of w(t, z). In the next section, Lemma 4 gives an amplified version of this result.

Theorem 4 In addition to (H), suppose that g is stictly increasing and $L_g = g'(0)$. If the initial function w_0 satisfies the assumptions (IC1), (IC2), then the solution w(t,z) of the initial value problem $w(s,z) = w_0(s,z), (s,z) \in \Pi_0$, for (12) satisfies the following conclusions.

A. Take $c \geq c_{\#}$ and assume (18). Then

$$|w(t,\cdot)/\phi(\cdot) - 1|_0 = o(1), \ t \to +\infty.$$

B. If $c > c_{\#}$ and $\lambda \in (\lambda_1(c), \lambda_2(c))$ then (19) implies

$$|\phi(\cdot) - w(t, \cdot)|_{\lambda} \le Ce^{-\gamma t}, \ t \ge 0,$$

for some positive C, γ (in fact, $\gamma > 0$ can be chosen as in Lemma 2).

Proof In virtue of Corollary 6, without loss of generality, we can assume that $w_0(s,z) > 0$ on Π_0 .

A. As in the proof of Corollary 6A, set $\lambda_c = \lambda_1$ if $c = c_\#$ or take some $\lambda_c \in (\lambda_1(c), \lambda_2(c))$ if $c > c_\#$. We know from Lemma 2 that the functions $\phi(z) \pm q\xi(z, \lambda_c)$ constitute a pair of super- and sub-solutions for equation (12) for each positive q. The main drawback of these solutions is their unboundedness. Hence, first we show how to correct this deficiency of $\phi(z) \pm q\xi(z, \lambda_c)$.

So, fix $\delta > 0$ and take $q = q(\delta, w_0) > 0$ large enough to meet (20). Let $(-\infty, p)$ be the maximal interval where the function $\phi(z - \delta) - q\xi(z, \lambda_c)$ is positive. Then, for sufficiently small $\epsilon \in (0, \kappa)$, the equation

$$\phi(z - \delta) - q\xi(z, \lambda_c) = \epsilon$$

has exactly two solutions $z_1(\epsilon) < z_2(\epsilon)$ on $(-\infty, p)$. It holds that $z_1(0+) = -\infty$, $z_2(0+) = p$ and therefore we can find $\epsilon > 0$ such that $z_2(\epsilon) - z_1(\epsilon) > ch$ and

$$\inf\{w_0(s,z): z \ge z_1(\epsilon), \ s \in [-h,0]\} > \epsilon.$$

It is easy to see that the functions

$$w_{-}(z) := \begin{cases} \phi(z - \delta) - q\xi(z, \lambda_c), & z \le z_2(\epsilon), \\ \epsilon, & z_2(\epsilon) \le z, \end{cases}$$

and

$$w_{+}(z) := \min\{\kappa + |w_0|_{\infty}, \phi(z+\delta) + q\xi(z,\lambda_c)\}\$$

satisfy

$$w_{-}(z) \le w_{0}(s, z) \le w_{+}(z), (s, z) \in \Pi_{0},$$

and that they are, respectively, a sub-solution and a super-solution for equation (12). Thus Corollary 4 implies that

$$w_{-}(z) \le w^{-}(t,z) \le w(t,z) \le w^{+}(t,z) \le w_{+}(z),$$
 (21)

where $w^{\pm}(t,z)$ denote the solutions of (12) satisfying the initial conditions $w^{\pm}(s,z) = w_{\pm}(z), z \in \mathbb{R}, s \in [-h,0]$. From Corollary 4 we also obtain that $w^{\pm}(t,z)$ converge (uniformly on compact subsets of \mathbb{R}) to some functions $\phi^{\pm}(z)$ such that

$$w^{-}(z) \le \phi^{-}(z) \le \phi^{+}(z) \le w^{+}(z).$$

It is well known (see e.g. [40, Lemma 2.8]) that ϕ^{\pm} satisfy the profile equation (13). Since ϕ^{\pm} are positive and bounded, $\phi(-\infty) = 0$ and $\liminf_{z \to +\infty} \phi(z) > 0$, we conclude from [43, Proposition 2 and Theorem 1.2] that $\phi^{\pm}(z) = \phi(z \pm \delta_{\pm})$, $z \in \mathbb{R}$ for some $-\delta \leq \delta_{-} \leq \delta_{+} \leq \delta$.

Furthermore, we claim that

$$w^* := \limsup_{t \to +\infty} |w^+(t,\cdot)|_{\infty} \le \kappa, \quad w_* := \lim_{(T,Z) \to +\infty} \inf_{z \ge Z, t \ge T} w^-(t,z) = \kappa.$$

Clearly, $w_* \leq w^*$. To prove that $w^* \leq \kappa$, it suffices to observe that the homogeneous solution $w_g(t)$, $t \geq 0$, of equation (12) defined as the solution of the initial value problem

$$w'(t) = -w(t) + g(w(t - ch)), \ w_g(s) = |w_0|_{\infty} + \kappa, \ s \in [-h, 0],$$

dominates w^+ (i.e. $w^+(t,z) \leq w_g(t)$ for all $z \in \mathbb{R}, t \geq -h$) in view of Lemma 1 and converges to κ .

Next, suppose that $w_* < \kappa$ and take Z, T so large and $\delta_1 > \varepsilon_1 > 0$ so small that

- (i) $w^{-}(t,z) > w_* \delta_1$ for all $z \ge Z ch$, $t \ge T h$;
- (ii) $w^-(t,z) > \kappa \varepsilon_1$ for all $t \geq T h$, and $z \in [Z ch, Z]$;
- (iii) homogeneous solution $w_h(t)$, $t \ge 0$, of equation (12) defined as the solution of the initial value problem

$$w'(t) = -w(t) + g(w(t - ch)), \ w_h(s) = w_* - \delta_1, \ s \in [-h, 0],$$

satisfies the inequalities

$$w_h(t) \le (w_* + \kappa)/2, \ t \in [-h, a_1],$$

 $(w_* + \kappa)/2 \le w_h(t \le w_h(a_2) = \kappa - \varepsilon_1, \ t \in [a_1, a_2],$

for sufficiently large $a_2 > a_1 + h > h$ (observe here that from [39, Corollary 2.2, p. 82] we know that $w_h(t)$ converges monotonically to κ). Therefore, for each $T_1 \geq T$ and all $t \in (T_1, T_1 + h]$, $z \geq Z$, the function $\delta(t, z) = w_h(t - T_1) - w^-(t, z)$ satisfies the inequalities

$$|\delta(t,z)| \le \kappa$$
, $\delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z) = g(w^-(t-h,z-ch)) - g(w_h(t-T_1-h)) > 0.$

In addition, we have that

$$\delta(T_1, z) < 0, z > Z, \quad \delta(t, Z) < 0, \ t \in [T_1, T_1 + a_2].$$

In consequence, by the Phragmèn-Lindelöf principle,

$$\delta(t,z) = w_h(t-T_1) - w^-(t,z) \le 0$$
, for all $t \in [T_1, T_1 + h], z \ge Z$.

It is clear that, using step by step integration method, we can repeat the above procedure till the maximal moment t_* before which the inequality $g(w^-(t-h,z-ch)) \ge g(w_h(t-T_1-h))$ for $z \ge Z$ is preserved. Therefore

$$w_h(t-T_1) \le w^-(t,z)$$
 for all $t \in [T_1, T_1 + a_2], z \ge Z$,

so that

$$(w_* + \kappa)/2 \le w^-(t, z)$$
 for all $t \in [T_1 + a_1, T_1 + a_2], z \ge Z$.

However, since $T_1 \geq T$ is an arbitrarily chosen number, we conclude that $w_* \geq (w_* + \kappa)/2$, contradicting to our initial assumption that $w_* < \kappa$. Hence $w^{\pm}(t, z) \rightarrow \phi^{\pm}(z)$ as $t \to +\infty$ uniformly on \mathbb{R} . In virtue of (21), we obtain

$$\limsup_{t \to +\infty} |w(t,\cdot)/\phi(\cdot) - 1|_0 \le e^{\lambda_1 \delta} - 1,$$

for each small δ . This completes the proof of the first part of Theorem 4.

B. We deduce from (19) that

$$\phi(z) - q_0 e^{\lambda b} \xi(z - b, \lambda) \le w_0(s, z) \le \phi(z) + q_0 e^{\lambda b} \xi(z - b, \lambda), \quad z \in \mathbb{R}, \ s \in [-h, 0].$$

As a consequence, Lemma 2 guarantees that, for some positive γ and all $z \in \mathbb{R}$, $t \ge -h$,

$$\phi(z) - q_0 e^{\lambda b} e^{-\gamma t} \xi(z - b, \lambda) \le w(t, z) \le \phi(z) + q_0 e^{\lambda b} e^{-\gamma t} \xi(z - b, \lambda).$$

From the part A of this theorem, we also know that $\lim_{t\to+\infty} w(t,z) = \phi(z)$ uniformly on \mathbb{R} . Therefore there exist a large $T_1 > 0$ and positive $q_2 < \min\{q^*, q_*\}$ such that, for all $z \in \mathbb{R}$, $t \geq T_1 - h$,

$$\phi(z) - q_2 \eta_{\lambda}(z - b) \le w(t, z) \le \phi(z) + q_2 \eta_{\lambda}(z - b).$$

Again applying Lemma 2, we obtain that

$$\phi(z) - q_2 e^{-\gamma(t-T_1)} \eta_{\lambda}(z-b) \le w(t,z) \le \phi(z) + q_2 e^{-\gamma(t-T_1)} \eta_{\lambda}(z-b) \quad t > T_1, \ z \in \mathbb{R}.$$

Thus

$$|\phi(z) - w(t, z)|_{\lambda} \le (q_2 e^{\gamma T_1}) e^{-\gamma t}, \ t \ge T_1,$$

that proves the second statement of the theorem.

4 Stability lemma and invariance of the leading asymptotic term

In this section, we are presenting two additional results. First we demonstrate a quite general local stability lemma which will be used later in the proof of Theorem 1. Below we take $q_*, q^*, \delta^*, \gamma^*, b > 0$ as at the beginning of Section 3.

Lemma 3 Assume that $c > c_*$ and write, for short, $\eta_1(z) = \min\{1, e^{\lambda_1(c)z}\}$ instead of $\eta_{\lambda_1}(z)$. Then

$$w_{\pm}(t,z) := \phi(z \pm \epsilon_{\pm}(t)) \pm q e^{-\gamma t} \eta_1(z), \ q \in (0, \min\{q^*, q_*\}],$$

are super- and sub-solutions for appropriately chosen functions

$$\epsilon_+(t) := \frac{\alpha q}{\gamma} (e^{\gamma h} - e^{-\gamma t}) > 0, \quad \epsilon_-(t) := -\frac{\alpha q}{\gamma} e^{-\gamma t} < 0, \quad t > -h.$$

The parameters $\alpha > 0$ and $\gamma \in (0, \gamma^*)$ are fixed later in the proof and depend only on g, ϕ, c, h .

Proof Set $z_* = 0$ and observe that the smoothness conditions of Definition 1 and the second inequality in (14) are satisfied in view of

$$\pm \left(\frac{\partial w_{\pm}(t,0+)}{\partial z} - \frac{\partial w_{\pm}(t,0-)}{\partial z} \right) = -q\lambda_1(c)e^{-\gamma t} < 0.$$

In order to establish the first inequality of (14), we proceed with the following direct calculation:

$$\pm \mathcal{N}w_{\pm}(t,z) := \epsilon'_{\pm}(t)\phi'(z \pm \epsilon_{\pm}(t)) - \gamma q e^{-\gamma t} \eta_{1}(z) \mp \phi''(z \pm \epsilon_{\pm}(t)) - q e^{-\gamma t} \eta''_{1}(z)$$

$$\pm c \phi'(z \pm \epsilon_{\pm}(t)) + c q e^{-\gamma t} \eta'_{1}(z) \pm \phi(z \pm \epsilon_{\pm}(t)) + q e^{-\gamma t} \eta_{1}(z) \mp g(w_{\pm}(t - h, z - ch)) \ge$$

$$\alpha q e^{-\gamma t} \phi'(z \pm \epsilon_{\pm}(t)) - \gamma q e^{-\gamma t} \eta_{1}(z) + c q e^{-\gamma t} \eta'_{1}(z) + q e^{-\gamma t} \eta_{1}(z) - q e^{-\gamma t} \eta''_{1}(z)$$

$$\pm \left(g(\phi(z - ch \pm \epsilon_{\pm}(t))) - g(\phi(z - ch \pm \epsilon_{\pm}(t)) \pm q e^{-\gamma(t - h)} \eta_{1}(z - ch))\right), \ z \ne 0.$$

Here we are using the fact that g, ϕ, ϵ_{\pm} are increasing functions.

From now on, we fix positive number

$$\gamma < \min\{\gamma^*, (g'(0) - 1)e^{-\lambda_1 ch} \min\{1, \lambda_1^{-1}\}\}$$

and d, α defined by

$$d := \inf_{z < b} \phi'(z) / \eta_1(z) > 0 \quad \text{and} \quad \alpha := d^{-1} e^{\gamma h} L_g.$$
 (22)

Note that α , d, γ depend only on g, ϕ , c, h.

We claim that $\pm \mathcal{N}w_{\pm}(t,z) \geq 0$ for all $z \neq 0$, $t \geq 0$ and $q \in (0, \min\{q^*, q_*\}]$. Indeed, suppose first that $z \pm \epsilon_{\pm}(t) \leq b$. Then we find that

$$0 \ge \pm \left(g(\phi(z - ch \pm \epsilon_{\pm}(t))) - g(\phi(z - ch \pm \epsilon_{\pm}(t)) \pm qe^{-\gamma(t-h)}\eta_{1}(z - ch)) \right) \ge$$

$$-L_{g}qe^{-\gamma(t-h)}\eta_{1}(z - ch), \quad \pm \mathcal{N}w_{\pm}(t, z) \ge$$

$$qe^{-\gamma t} \left\{ \eta_{1}(z \pm \epsilon_{\pm}(t))d\alpha + ([1 - \gamma]\eta_{1}(z) + c\eta'_{1}(z) - \eta''_{1}(z) - e^{\gamma h}L_{g}\eta_{1}(z - ch)) \right\}$$

$$\ge qe^{-\gamma t} \left(\eta_{1}(z \pm \epsilon_{\pm}(t))d\alpha - e^{\gamma h}L_{g}\eta_{1}(z - ch) \right) > 0.$$

Similarly, if $z \pm \epsilon_{\pm}(t) \ge b$, then invoking (16) and (17) we obtain, for all $t \ge 0$, that

$$0 \ge \pm \left(g(\phi(z - ch \pm \epsilon_{\pm}(t))) - g(\phi(z - ch \pm \epsilon_{\pm}(t))) \pm qe^{-\gamma(t-h)} \eta_1(z - ch) \right) \ge$$

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$$-qe^{-\gamma t}\eta_{1}(z-ch)(1-2\gamma), \quad \pm \mathcal{N}w_{\pm}(t,z) \geq$$

$$qe^{-\gamma t}\left([1-\gamma]\eta_{1}(z)+c\eta_{1}'(z)-\eta_{1}''(z)-(1-2\gamma)\eta_{1}(z-ch)\right) \geq$$

$$qe^{-\gamma t}\left\{e^{\lambda_{1}z}[1-\gamma+c\lambda_{1}-\lambda_{1}^{2}-e^{-\lambda_{1}ch}(1-2\gamma)], z<0\atop \gamma, z>0\right\} > 0.$$

The proof of Lemma 3 is completed.

Corollary 7 Let $c > c_*$ and $\gamma > 0$ be as in Lemma 3 and α be as in (22). Then there exists positive number $C = C(g, \phi)$ such that for each non-negative initial function w_0 satisfying

$$\phi(z) - q_- \eta_1(z) \le w_0(s, z) \le \phi(z) + q_+ \eta_1(z), \quad z \in \mathbb{R}, \quad s \in [-h, 0],$$

for some $0 < q_{\pm} \le \varsigma_0 := \min\{\gamma, \min\{q_*, q^*\} \exp(-\lambda_1 \alpha e^{\gamma h})\}$, it holds

$$\phi(z - Cq_{-}) - Cq_{-}e^{-\gamma t}\eta_{1}(z) \le w(t, z) \le \phi(z + Cq_{+}) + Cq_{+}e^{-\gamma t}\eta_{1}(z), \quad (23)$$

for all $z \in \mathbb{R}, t \geq -h$.

Proof The right hand side inequality in (23) is a direct consequence of Lemmas 1 and 3 in view of the estimations

$$w_0(s,z) \le \phi(z) + q_+\eta_1(z) \le \phi(z + \epsilon_+(s)) + q_+e^{-\gamma s}\eta_1(z), \ (z,s) \in \Pi_0.$$

Since $\epsilon_+(t)$ increases on \mathbb{R} , this proves this part of inequality (23) with $C = C_1 := \alpha e^{\gamma h}/\gamma$.

In order to prove the left hand side inequality in (23), observe that

$$w_0(s, z - \epsilon_-(-h)) \ge \phi(z - \epsilon_-(s)) - q_- e^{-\lambda_1 \epsilon_-(-h)} e^{-\gamma s} \eta_1(z) \ge$$
$$\phi(z - \epsilon_-(s)) - \min\{q_*, q^*\} e^{-\gamma s} \eta_1(z), \quad (z, s) \in \Pi_0.$$

This implies that, for all $t \geq -h$, $z \in \mathbb{R}$, it holds

$$w(t,z) \ge \phi(z - \epsilon_{+}(t)) - q_{-}e^{-\lambda_{1}\epsilon_{-}(-h)}e^{-\gamma t}\eta_{1}(z) \ge$$

 $\phi(z - C_{1}q_{-}) - C_{2}q_{-}e^{-\gamma t}\eta_{1}(z), C_{2} := \exp\left(\lambda_{1}\alpha e^{\gamma h}\right).$

Setting $C = \max\{C_1, C_2\}$, we complete the proof of Corollary 7.

Corollary 8 For every $\epsilon > 0$ there exists $\varsigma(\epsilon) > 0$ such that

$$|\phi(\cdot) - w(s, \cdot)|_{\lambda}, < \varsigma(\epsilon), \quad s \in [-h, 0],$$

implies $|\phi(\cdot) - w(t, \cdot)|_{\lambda_1} < \epsilon$ for all $t \ge 0$.

Proof It suffices to take

$$\varsigma(\epsilon) = \min \left\{ \varsigma_0, \frac{\epsilon}{C(1 + e^{\lambda_1 C \varsigma_0} \sup_{z \in \mathbb{R}} \left[\phi'(z) / \eta_1(z) \right])} \right\},\,$$

where $C = \max\{C_1, C_2\}$ was defined in the proof of Corollary 7 and to apply Corollary 7.

The second main result of this section assures the invariance of the main asymptotic term at $-\infty$ of solutions with 'good' initial data. It sheds some new light on the conclusions of Corollary 6A.

Lemma 4 Suppose that the birth function g is bounded and that there exists g'(0) > 1. If the initial fragment u(s,z) of a bounded solution u(t,z) to equation (1) is such that, for some positive eigenvalue $\lambda_j(c)$, j = 1, 2, it holds that $u(s, x - cs)e^{-\lambda_j(c)x} \to 1$, $x \to -\infty$, for each $s \in [-h, 0]$. Then also it holds that $u(t, x - ct)e^{-\lambda_j(c)x} \to 1$, $x \to -\infty$, for each $t \ge 0$.

Proof Due to a step by step argument, it is sufficient to consider the situations when $t \in [0,h]$. Set $U(t,x) := e^t u(t,x)$, then $U(s,x-cs)e^{-\lambda_j(c)x} \to e^s$, $x \to -\infty$, and

$$U_t(t,x) = U_{xx}(t,x) + e^t g(e^{-t+h}U(t-h,x)), \quad t > 0, \ x \in \mathbb{R}.$$

Hence, by Duhamel's formula (see e.g. [10, Theorem 12, p. 25]),

$$U(t,x) = \Gamma(t,\cdot) * U(0,\cdot) + \int_0^t \Gamma(t-s,\cdot) * e^s g(e^{-s+h}U(s-h,\cdot))ds,$$
where
$$\Gamma(t,x) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}, \ t > 0, \ x \in \mathbb{R},$$

is the fundamental solution and $\Gamma(t,\cdot)*U(s,\cdot)$ denotes the convolution on $\mathbb R$ with respect to the missing space variable.

By Lebesgue's dominated convergence theorem, for each $s \in [-h, 0], t > 0$,

$$\lim_{x \to -\infty} e^{-\lambda_j x} \Gamma(t, \cdot) * U(s, \cdot) =$$

$$\frac{1}{2\sqrt{t\pi}}\int_{\mathbb{R}}e^{-\frac{1}{4t}[(y+2t\lambda_j)^2-4t^2\lambda_j^2]}\lim_{x\to-\infty}e^{-\lambda_j(x-y)}U(s,x-y)dy=e^{\lambda_j^2t+\lambda_jcs+s}.$$

Consequently, for $t \in (0, h]$, we have that

$$\lim_{x \to -\infty} e^{-\lambda_j x} U(t, x) = e^{\lambda_j^2 t} + \int_0^t \lim_{x \to -\infty} e^{-\lambda_j x} \Gamma(t - s, \cdot) * g(e^{-s + h} w(s - h, \cdot)) e^s ds$$
$$= e^{\lambda^2 t} + g'(0) e^{-\lambda c h} e^{\lambda^2 t} \int_0^t e^{(-\lambda^2 + \lambda c + 1)s} ds = e^{(1 + \lambda c)t}.$$

Finally, we obtain the relation $\lim_{x\to-\infty}e^{-\lambda_j x}u(t,x)=e^{\lambda_j ct}$ for each $t\in(0,h]$ which completes the proof of the lemma.

Remark 3 An obvious modification of the proof of Lemma 4 yields the following assertion: Assume that the birth function $g: \mathbb{R}_+ \to \mathbb{R}_+$, g(0) = 0, is bounded and Lipschitz continuous. Suppose also that the initial fragments $u_k(s, z)$, k = 1, 2, of bounded solutions $u_k(t, z)$ to equation (1) satisfy, for some positive μ , the relation

$$(u_1 - u_2)(s, x - cs)e^{-\mu x} \to 0, \ x \to -\infty, \ s \in [-h, 0].$$

Then
$$(u_1 - u_2)(t, x - ct)e^{-\mu x} \to 0$$
, $x \to -\infty$, for each $t \ge 0$.

This result provides a short and elementary justification for a delicate aspect of getting *a priori* estimates for a weighted energy method developed by Mei *et al.* [26,27,28,29]. Indeed, an important initial fragment of the derivation of these estimates includes elimination of the boundary term

$$(u - \phi)(t, x - ct)e^{-\mu x}|_{x = -\infty} = (w(t, z) - \phi(z))e^{-\mu z}e^{\mu ct}|_{z = -\infty} = 0.$$

For instance, see [19, p. 855], [28, formulas (3.9)-(3.11)] or [18, p. 1067].

5 Proof of Theorem 1

We start by establishing the following result.

Lemma 5 Assume that the initial function $w_0(s, z) \ge 0$ is uniformly bounded on the strip $[-h, 0] \times \mathbb{R}$ (say, by some K > 0) and satisfies the hypothesis (IC2) and, for some $c > c_*$, it holds

$$\lim_{z \to -\infty} w_0(s, z)e^{-\lambda_1(c)z} = 1$$

uniformly on $s \in [-h, 0]$. Then for each $\varsigma > 0$ there exists $L \in \mathbb{R}$ and $\psi \in C^2(\mathbb{R})$ such that $\psi(z) = (1 + \varsigma + o(1))e^{\lambda_1(c)z}$, $z \to -\infty$, and $\psi'(z) > 0$ for $z \in \mathbb{R}$, $\psi(L) = K$, $\psi(+\infty) = +\infty$, $w_0(s, z) < \psi(z)$, $z \le L$, $s \in [-h, 0]$, and

$$\psi''(z) - c\psi'(z) - \psi(z) + g(\psi(z - ch)) < 0, \ z \le L.$$

Proof Since $c > c_*$, the linearisation of equation (13) about 0 has exactly two real simple eigenvalues $\lambda_1(c) < \lambda_2(c)$. In particular, the linearised equation has a positive solution $(\phi(t), \phi'(t)) = (1, \lambda_2(c)))e^{\lambda_2(c)t}$. Moreover, the eigenvalue $\lambda_2 = \lambda_2(c)$ is dominant (i.e. $\Re \lambda_j(c) < \Re \lambda_2$ for all other eigenvalues $\lambda_j(c), j \neq 2$). As a consequence, equation (13) has a solution $\psi_2(t)$ with the following asymptotic behaviour at $-\infty$:

$$(\psi_2(t), \psi_2'(t)) = (1, \lambda_2)e^{\lambda_2 t} + O(e^{(\lambda_2 + \epsilon)t}), \ t \to -\infty, \ \epsilon > 0$$

(see e.g. [8, Theorem 2.1] for more detail).

In this way, there exists a maximal open non-empty interval (0,T), $T \in \mathbb{R} \cup \{+\infty\}$, such that $\psi_2(t) > 0$, $\psi_2'(t) > 0$ for all $t \in (0,T)$.

We claim that $\psi_2(T) > \kappa$ and $T = +\infty$. First, it should be noted that $\psi_2(T) \neq$ κ since otherwise we obtain a) if T is finite then $\psi_2(T) = \kappa > g(\psi_2(T-ch))$, $\psi_2'(T) = 0, \psi_2''(T) \le 0$, contradicting (13); b) if $T = +\infty$ then $\psi_2(t)$ is a monotone heteroclinic connection between 0 and κ , different from ψ_1 . Here $\psi_1(t)$ denotes the unique monotone wavefront to (13) normalised by the condition $\psi_1(t)e^{-\lambda_1 t} =$ $1+o(1), t\to -\infty$. This contradicts the uniqueness of the wavefront ψ_1 established in [43]. Next, suppose that $\psi_2(T) < \kappa$ and consider the difference $\theta_a(t) = \psi_1(t)$ $\psi_2(t+a), t \in \mathbb{R}$, for some fixed $a \in \mathbb{R}$. Since ψ_1 is a strictly monotone heteroclinic connection between 0 and κ , there exists a unique $S \in \mathbb{R}$ such that $\psi_1(S) = \psi_2(T)$. Now, taking into account the inequality $\lambda_1 < \lambda_2$, we obtain that, for each fixed a, the function $\theta_a(t)$ is positive in some maximal interval $(-\infty, \sigma(a))$. If we choose b = T - S then $\theta_b(S) = 0$, $\theta_b'(S) > 0$ and therefore $\sigma(b) = \sigma(T - S) < S$, $\theta_b(\sigma(b)) = 0$. On the other hand, $\theta_{a_1}(t) > 0$, $t \in [\sigma(b), S]$, for some large negative $a_1 \leq b$. Note also that $\theta_a(t) > \theta_b(t) > 0$, $t \leq \sigma(b)$ if a < b. In consequence, there exists $d \in (a_1, b]$ such that $\theta_d(\sigma(d)) = \theta'_d(\sigma(d)) = 0 \leq \theta''_d(\sigma(d))$ and $\theta_d(\sigma(d)) = 0$ $ch) = \psi_1(\sigma(d) - ch) - \psi_2(d + \sigma(d) - ch) > 0$. However, this yields the following contradiction:

$$0 = \theta_d''(\sigma(d)) - c\theta_d'(\sigma(d)) - \theta_d(\sigma(d)) + g(\psi_1(\sigma(d) - ch)) - g(\psi_2(d + \sigma(d) - ch)) > 0$$

because g is strictly increasing.

Finally, if $T < +\infty$ and $\psi_2(T) > \kappa$, then $g(\psi_2(T - ch)) < g(\psi_2(T)) < \psi_2(T)$. Since, in addition, $\psi_2''(T) \le \psi_2'(0) = 0$, the following contradiction

$$0 = \psi_2''(T) - c\psi_2'(T) - \psi_2(T) + g(\psi_2(T - ch)) < 0$$

proves the above claim

Next, we consider, for $\epsilon \in [0,1]$, θ, δ_0 as in **(H)** and $\mu \in (\lambda_1(c), \lambda_2(c))$, $\mu < (1+\theta)\lambda_1(c)$, the function

$$\psi(t,\epsilon) = \psi_2(t) + \epsilon(e^{\lambda_1 t} + e^{\mu t}).$$

It is clear that $\psi(t,\epsilon) \leq Ce^{\lambda_1 t}$, $t \leq 0$, for some C > 1 which does not depend on $\epsilon \in [0,1]$.

With $\chi_0(z) = z^2 - cz - 1 + g'(0)e^{-zch}$, we have that $\chi_0(\mu) < 0$ and

$$\mathcal{D}\psi := \psi''(t,\epsilon) - c\psi'(t,\epsilon) - \psi(t,\epsilon) + g(\psi(t-ch,\epsilon)) =$$

$$\epsilon \chi_0(\mu) e^{\mu t} + g(\psi(t - ch, \epsilon)) - g(\psi(t - ch, 0)) - g'(0) \epsilon(e^{\lambda_1(t - ch)} + e^{\mu(t - ch)t}).$$

Let $T_0 < 0$ be such that $\psi(t - ch, \epsilon) \le \delta_0 := \psi(T_0, 1)$ for all $t \le T_0$, $\epsilon \in [0, 1]$. Then, for some $P(t, \epsilon) \in [\psi(t - ch, 0), \psi(t - ch, \epsilon)]$, it holds that

$$|g(\psi(t-ch,\epsilon)) - g(\psi(t-ch,0)) - g'(0)\epsilon(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t})| = |g'(P(t,\epsilon)) - g'(0)|\epsilon(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t}) \le \epsilon(\psi(t-ch,\epsilon))^{\theta}|(e^{\lambda_1(t-ch)} + e^{\mu(t-ch)t}) \le 2C\epsilon e^{(1+\theta)\lambda_1 t}, \ t < T_0.$$

Thus, for a sufficiently large negative $T_1 < T_0$,

$$\mathcal{D}\psi \le \epsilon e^{\mu t} (\chi(\mu) + 2Ce^{[(1+\theta)\lambda_1 - \mu]t}) < 0$$

for all $\epsilon \in (0,1]$, $t \leq T_1$. As a consequence, if we define $\psi_{\epsilon}(t)$ by

$$\psi_{\epsilon}(t) := \begin{cases} \psi(t, \epsilon), \ 0 \le t \le T_1, \\ y(t, \epsilon), \ T_1 \le t, \end{cases}$$

where $y = y(t, \epsilon)$, $t \ge T_1$, solves the initial value problem $y(s, \epsilon) = \psi(s, \epsilon)$, $s \in [T_1 - ch, T_1]$, $y'(T_1, \epsilon) = \psi'(T_1, \epsilon)$ for the equation

$$y''(z) - cy'(z) - y(z) + g(y(z - ch)) = \mathcal{D}\psi(T_1, \epsilon) < 0,$$

then $\psi_{\epsilon} \in C^2(\mathbb{R})$ and $\mathcal{D}\psi_{\epsilon}(t) < 0$. Define T_K as the unique solution of the equation $\psi_2(T_K) = K$, then due to the smooth dependence of the initial function and $\mathcal{D}\psi(T_1,\epsilon), \mathcal{D}\psi(T_1,0) = 0$, on the parameter ϵ ,

$$(y(t,\epsilon), y'(t,\epsilon)) \to (\psi_2(t), \psi_2'(t)), \quad \epsilon \to 0+,$$

uniformly for $t \in [T_0, T_K]$.

Finally, due to the assumptions imposed on w_0 , there exists $T_2 < T_1$ such that

$$w_0(t,s) \le (1+\varsigma)e^{\lambda_1(c)z} < \psi_2(T_1), \ t \le T_2, \ s \in [-h,0].$$

For $\epsilon \in (0,1]$, set $p_{\epsilon} = \lambda_1^{-1}(c) \ln[(1+\varsigma)/\epsilon]$ and $\tilde{\psi}(t) := \psi_{\epsilon}(t+p_{\epsilon})$. Obviously, $\tilde{\psi}(t) > (1+\varsigma)e^{\lambda_1(c)z}$, $t \leq T_1 - p_{\epsilon}$, $\tilde{\psi}(t) > \psi_2(T_1)$, $t \in [T_1 - p_{\epsilon}, T_K - p_{\epsilon}]$ and $\tilde{\psi}(t) = (1+\varsigma+o(1))e^{\lambda_1(c)z}$, $t \to +\infty$. Since $\tilde{\psi}(T_K - p_{\epsilon}) = y(T_K, \epsilon) > K$, we obtain that

$$w_0(s,z) \le \tilde{\psi}(z), \quad s \in [-h,0].$$

whenever $T_K < T_2 + p_{\epsilon}$.

Next, for the solution w(t, z) of the initial value problem $w(s, z) = w_0(s, z), (s, z) \in [-h, 0] \times \mathbb{R}$, we define its ω -limit set by

$$\Omega(w_0) = \{w_* \in C^{1,2}([-h,0] \times \mathbb{R}) : \text{there exists some } t_k \to +\infty \text{ such that } t_k \to +\infty \}$$

$$\lim_{k \to \infty} w(t_k + s, z) = w_*(s, z) \text{ uniformly on compact subsets of } [-h, 0] \times \mathbb{R}.$$

Note that the set $\Omega(w_0)$ is non-empty, compact and invariant with respect to the flow generated by equation (12), e.g. see [40, Lemma 2.8].

Theorem 5 Assume that the initial function $w_0(s, z) \ge 0$ satisfies the hypotheses (IC1), (IC2) and that, for some A > 0 and $c > c_*$, it holds

$$\lim_{z \to -\infty} w_0(s, z)e^{-\lambda_1(c)z} = A$$

uniformly on $s \in [-h,0]$. Choose a shifted copy of the wavefront profile ϕ normalised by the boundary condition $\lim_{z\to-\infty}e^{-\lambda_1(c)z}\phi(z)=1$. Then

$$\lim_{t \to \infty} |\phi(\cdot + a) - w(t, \cdot)|_{\lambda_1} = 0,$$

where w(t,z) solves the initial value problem $w(s,z) = w_0(s,z), (s,z) \in \Pi_0$, for (12) and $a = (\lambda_1(c))^{-1} \ln A$.

Proof Without loss of generality, we may assume that A=1 (otherwise we can take a shifted copy of w_0). Fix an arbitrary $\varsigma>0$ and let L,K and ψ satisfy all the conclusions of Lemma 5. Then we have that $w_0(s,z)\leq \psi_+(z),\,(s,z)\in [-h,0]\times \mathbb{R}$, where

$$\psi_{+}(z) := \begin{cases} \psi(z), \ 0 \le z \le L, \\ K, \quad L \le z. \end{cases}$$

Since $K > g(K) \ge g(\psi_+(z-ch))$ and $\psi'_+(L-) > 0 = \psi'_+(L+)$, we conclude that $\psi_+(z)$ is a super-solution for equation (12). In view of Lemma 1, we also find that

$$w(t,z) \le \psi_+(z), \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}.$$

On the other hand, it is easy to see (e.g., cf. [44, p. 478]) that there exists a strictly increasing C^1 -function $\hat{g}: \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the hypothesis (**H**) and such that $g'(0) = \hat{g}'(0) \geq \hat{g}'(x)$, $g(x) \geq \hat{g}(x)$ for all $x \in [0, \kappa]$. Let $\hat{w}(t, z)$, $t > 0, z \in \mathbb{R}$, solve the initial value problem

$$w_t(t,z) = w_{zz}(t,z) - cw_z(t,z) - w(t,z) + \hat{g}(w(t-h,z-ch)),$$

$$w(s,z) = w_0(s,z), \ s \in [-h,0], \ z \in \mathbb{R},$$
(24)

then clearly w(t,z) is a super-solution for (24) and therefore Lemma 1 implies that $\hat{w}(t,z) \leq w(t,z)$ for all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}$. Furthermore, Theorem 4A assures that $\lim_{t \to +\infty} |\hat{w}(t,\cdot) - \hat{\phi}(\cdot)|_{\lambda_1} = 0$ for the wavefront $\hat{\phi}$ of equation (24) normalised by $\lim_{z \to -\infty} e^{-\lambda_1(c)z} \hat{\phi}(z) = 1$.

Next, let $w_u(t,z)$, $t>0, z\in\mathbb{R}$, denote the solution of the initial value problem $w_u(s,z)=\psi_+(z),\ s\in[-h,0],\ z\in\mathbb{R}$, for equation (12). Then Corollary 4 implies that

$$\hat{w}(t,z) \le w(t,z) \le w_u(t,z) \le \psi_+(z), \ (t,z) \in \mathbb{R}_+ \times \mathbb{R}. \tag{25}$$

Therefore it holds, for some $a_1 \in [0, \lambda^{-1}(c) \ln(1+\varsigma)]$ and for all $w_l \in \Omega(w_0)$, that

$$\hat{\phi}(z) \le w_l(s, z) \le \phi(z + a_1), \ z \in \mathbb{R}, \ s \in [-h, 0],$$
 (26)

where

$$1 = \lim_{z \to -\infty} \hat{\phi}(z)e^{-\lambda_1 z} \le \lim_{z \to -\infty} \phi(z + a_1)e^{-\lambda_1 z} \le \lim_{z \to -\infty} \psi_+(z)e^{-\lambda_1 z} = 1 + \varsigma.$$

Next, since $\hat{\phi}(z)$ is a sub-solution for equation (12), we find analogously that, for some $a_0 \in [0, a_1]$ and for all $w_{ll} \in \Omega(w_l) \subset \Omega(w_0)$,

$$\phi(z+a_0) \le w_{ll}(s,z) \le \phi(z+a_1), \ z \in \mathbb{R}, \ s \in [-h,0],$$

where

$$1 \le \lim_{z \to -\infty} \phi(z + a_0)e^{-\lambda_1 z} \le \lim_{z \to -\infty} \phi(z + a_1)e^{-\lambda_1 z} \le 1 + \varsigma.$$

Since the latter relation holds for every $\varsigma > 0$, we conclude that actually $a_0 = 0$ and $\{\phi(\cdot)\} = \Omega(w_l) \subset \Omega(w_0)$. Furthermore, as a consequence of (26), $\lim_{z \to -\infty} e^{-\lambda z} w_l(s,z) = 1$ uniformly in $s \in [-h,0]$.

Hence, for each $\varsigma>0$ there are $Z_1(\varsigma),\ T_\varsigma>0$ such that, for all $t\geq T_\varsigma,$ $z\leq Z_1(\varsigma),$ it holds

$$-2\varsigma \le e^{-\lambda_1 z} (\hat{w}(t, z) - \hat{\phi}(z)) - e^{-\lambda_1 z} (\phi(z) - \hat{\phi}(z)) \le e^{-\lambda_1 z} (w(t, z) - \phi(z)) \le e^{-\lambda_1 z} (\psi_+(z) - \phi(z)) < 2\varsigma.$$
(27)

In addition, $\{\phi(\cdot)\}\in\Omega(w_0)$ implies that there exits a sequence $t_n\to+\infty$ that $w(t_n+s,z)\to\phi(z)$ on compact subsets of Π_0 . This fact, together with (25) and (27), implies that

$$\sup_{s \in [-h,0]} |\phi(\cdot) - w(t_n + s, \cdot)|_{\lambda_1} \le 2\varsigma$$

for all sufficiently large n. Finally, an application of Corollary 8 completes the proof.

Below, we use Theorem 1 in order to analyse behavior of solutions whose initial data satisfy the hypotheses (IC1), (IC2) and (4).

Proof of Corollary 1:

Case I: $\lambda > \lambda_*$. The statement of the corollary is an immediate consequence of [40, Theorem 1.4].

<u>Case II:</u> $\lambda < \lambda_*$. Clearly, $\lambda = \lambda_1(c(\lambda))$. Set $A_- = \min_{s \in [-h,0]} A(s)e^{-\mu s}$. Then for each $A_1 < A_-$, the initial datum

$$w_1(s,x) := \min\{A_1 e^{\lambda(x+cs)}, w_0(s,x)\}$$

meets all the conditions of Theorem 1. Consequently, for each $\delta > 0$ there exists $T_{\delta} > 0$ such that solution $u_1(t,x)$ of the initial value problem $u_1(s,x) = w_1(s,x)$, $(s,x) \in H_0$, to equation (1) satisfies

$$\phi(x+ct+a_1)-\delta\eta_{\lambda}(x+ct)\leq u_1(t,x), \text{ for all } x\in\mathbb{R},\ t>T_{\delta}$$

with $a_1 = \lambda^{-1} \ln A_1$. Now, the functions ϕ and η_{λ} are equivalent at $-\infty$ so that, to each given $\epsilon > 0$ we can find A_1 close to A_- and $\delta > 0$ close to 0 such that

$$(1-\epsilon)\phi(x+ct+a_-) \le u_1(t,x) \le u(t,x), \quad x \in \mathbb{R}, \ t > T_{\delta}.$$

The upper estimation can be established in a similar way by comparing u(t,x) with solution $u_2(t,x)$ of (1) satisfying the initial condition

$$w_2(s,x) = \max\{A_2 e^{\lambda(x+cs)}, w_0(s,x)\}, \quad (s,x) \in \Pi_0,$$

with $A_2 > A_+ = \max_{s \in [-h,0]} A(s)e^{-\mu s}$.

<u>Case III:</u> $\lambda = \lambda_*$. In order to establish inequalities (5), we proceed in the same manner as in Case II by taking the initial functions

$$\tilde{w}_1(s,x) := \min\{A_1 e^{M(x+cs)}, w_0(s,x)\}, \ \tilde{w}_2(s,x) = \max\{A_2 e^{\nu(x+cs)}, w_0(s,x)\},$$

where $\nu < \lambda_* < M < \lambda_2(c_*)$, instead of $w_1(s,x)$ and $w_2(s,x)$. In addition, while proving the left side inequality of (5), we have also to use [40, Theorem 1.4] instead of Theorem 1 (cf. Case I above).

Now, inequalities (5) also imply that the only wavefront to which u(t,x) can converge (as $t \to +\infty$) is some translation $\phi_*(x+c_*t+b)$ of the critical wavefront $\phi_*(x+c_*t)$. However, this is not possible in view of the following argument. Take some $A_1 < A_-$ and some strictly increasing $\hat{g} \leq g$ satisfying (**H**) with $L_{\hat{g}} = g'(0)$. Set

$$w_*(s,x) = \min\{A_1 e^{\lambda_*(x+cs)}, w_0(s,x)\}.$$

Then by the comparison principle, solution $w_*(t,x)$ of the initial value problem

$$w_t(t,x) = w_{zz}(t,x) - w(t,x) + \hat{g}(w(t-h,x)), \ w(s,x) = w_*(s,x), \ (s,x) \in \Pi_0,$$

satisfies $w_*(t,x) \leq u(t,x)$ for all $t \geq 0$, $x \in \mathbb{R}$. On the other hand, by invoking Theorem 2, we find that $w_*(t,x)$ converges uniformly to some wavefront $\hat{\phi}_*(x+c_*t)$ of the modified equation. Keeping $z = x + c_*t$ fixed and passing to the limit in $w_*(t,x) \leq u(t,x)$ (as $t \to +\infty$) for each fixed z, we find that $\hat{\phi}_*(z) \leq \phi_*(z+b)$ for all $z \in \mathbb{R}$. However, this is not possible since $\phi_*(z+b)$ decays at $-\infty$ faster than $\hat{\phi}_*(z)$.

Finally, in order to prove inequality (6), it suffices to consider the initial function

$$\tilde{w}_3(s,x) = \max\{-xe^{\lambda_*(x+cs)}, w_0(s,x)\}, (s,x) \in \Pi_0,$$

instead of $w_2(s, x)$. Then we proceed can similarly to the proof of inequalities (5) by applying Theorem 2A.

6 Proof of Theorem 3 and Corollary 3

Let the triple $(c, \lambda_c, \gamma) \in [c_\#, +\infty) \times [\lambda_1(c), \lambda_2(c)) \times \mathbb{R}_+$ be as in Lemma 2 (i.e. $\lambda_c = \lambda_1, \gamma = 0$ if $c = c_\#$ and $\gamma > 0, \lambda_c \in (\lambda_1(c), \lambda_2(c))$ if $c > c_\#$). Theorem 3 and Corollary 3 follow from the next three assertions.

Lemma 6 Assume (UM) and let the initial function w_0 satisfy (IC1). Consider $c \ge c_\#$ and let $\phi(z)$ denote a positive semi-wavewfront to equation (13). Then the inequalities

$$\phi(z) - qe^{-\gamma s}\xi(z-b,\lambda_c) \le w_0(s,z) \le \phi(z) + qe^{-\gamma s}\xi(z-b,\lambda_c), (s,z) \in \Pi_0,$$

(where $q > 0, b \in \mathbb{R}$ are some fixed numbers) imply that the solution w(t, z) of the initial value problem $w(s, z) = w_0(s, z), (s, z) \in \Pi_0$, for (12) satisfies

$$\phi(z) - qe^{-\gamma t}\xi(z-b,\lambda_c) \le w(t,z) \le \phi(z) + qe^{-\gamma t}\xi(z-b,\lambda_c), \ t \ge 0, \ z \in \mathbb{R}.$$
 (28)

Proof Set
$$\delta_{\pm}(t,z) = \pm (w(t,z) - (\phi(z) \pm qe^{-\gamma t}\xi(z-b,\lambda_c)))$$
 and
$$(\mathcal{L}\delta)(t,z) := \delta_{zz}(t,z) - \delta_t(t,z) - c\delta_z(t,z) - \delta(t,z).$$

Then

$$(\mathcal{L}\delta_{\pm})(t,z) = \mp (g(w(t-h,z-ch)) - g(\phi(z-ch))) + qe^{\lambda_c(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma].$$

Therefore we obtain, for all $z \in \mathbb{R}$, $t \in (0, h]$,

$$(\mathcal{L}\delta_{\pm})(t,z) \ge qe^{\lambda_c(z-b)}e^{-\gamma t}[-\lambda^2 + c\lambda + 1 - \gamma - g'(0)e^{\gamma h}e^{-ch\lambda_c}] \ge 0.$$

Since, in addition, $\delta_{\pm}(0,z) \leq 0$ and $\delta(t,z)$ is exponentially bounded, an application of the Phragmèn-Lindelöf principle yields $\delta_{\pm}(t,z) \leq 0$ for all $t \in [0,h]$. Finally, step by step procedure completes the proof of the inequality $\delta_{\pm}(t,z) \leq 0$ for all $t \geq 0$.

Lemma 7 Let all the conditions of Lemma 6 be satisfied and $c > c_{\#}$. Assume, in addition, that |g'(u)| < 1 on some interval $[\kappa - \rho, \kappa + \rho]$, $\rho > 0$. If, for some $b \ge 0$, the initial function w_0 and the semi-wavefront profile ϕ_c satisfy

$$\kappa - \rho/4 \le w_0(s, z), \phi(z) \le \kappa + \rho/4 \text{ for all } z \ge b - ch, \ s \in [-h, 0],$$

$$|w_0(s,z) - \phi(z)| \le 0.5\rho e^{\lambda_c(z-b)}, \ z \le b, \ s \in [-h,0],$$
 (29)

then ϕ is actually a wavefront (i.e. $\phi(+\infty) = \kappa$) and the solution w(t,z) of the initial value problem $w(s,z) = w_0(s,z), (s,z) \in \Pi_0$, for (12) satisfies

$$|w(t,\cdot) - \phi(\cdot)|_{\lambda_c} \le 0.5\rho e^{-\gamma t}, \ t \ge 0, \tag{30}$$

for some $\gamma > 0$.

Proof Suppose that $\gamma > 0$ is sufficiently small to satisfy the inequality $m_g := \max\{|g'(u)| : u \in [\kappa - \rho, \kappa + \rho]\}e^{\gamma h} < 1 - \gamma$. Clearly, for all $(s, z) \in \Pi_0$, it holds that

$$\delta_{-}(s,z) := \phi(z) - 0.5\rho e^{-\gamma s} \eta_{\lambda_c}(z-b) - w_0(s,z) \le 0,$$

$$\delta_{+}(s,z) := w_0(s,z) - \phi(z) - 0.5\rho e^{-\gamma s} \eta_{\lambda_c}(z-b) \le 0.$$

Then Lemma 6 implies that (28) holds with $q = 0.5\rho$. From the proof of Lemma 6 we know that $(\mathcal{L}\delta_{\pm})(t,z) \geq 0$ for all $t \in (0,h]$ and z < b. Next, for each $t \in (0,h]$ and z > b, we find, by applying the Lagrange mean value theorem, that

$$(\mathcal{L}\delta_{\pm})(t,z) = \mp (g(w(t-h,z-ch)) - g(\phi(z-ch))) + 0.5\rho e^{-\gamma t}[1-\gamma] =$$

$$\mp (g'(\zeta)(w(t-h,z-ch)-\phi(z-ch))) + 0.5\rho e^{-\gamma t}[1-\gamma] \ge 0.5\rho e^{-\gamma t}(-m_g+1-\gamma) > 0.$$

Here $\zeta = \zeta(t, z)$ denotes some point in $[\kappa - \rho, \kappa + \rho]$.

Note also that $\delta_{\pm}(0,z) \leq 0$, $\delta_{\pm}(t,z)$ are uniformly bounded on $[0,h] \times \mathbb{R}$ and inequality (15) is satisfied for $\delta_{\pm}(t,z)$ with $z_* = b$. Thus, arguing as in the proof of Lemma 1, we conclude that $\delta_{\pm}(t,z) \leq 0$ for all $t \in [0,h]$, $z \in \mathbb{R}$. This estimation shows that actually inequality (30) is fulfilled for all $t \in [0,h]$. Now, we can apply step by step procedure in order to obtain $\delta_{\pm}(t,z) \leq 0$ for all $t \geq 0$, $z \in \mathbb{R}$.

Finally, since $g: [\kappa - \rho, \kappa + \rho] \to [\kappa - \rho, \kappa + \rho] =: \mathcal{I}$ is well defined and

$$\kappa - \rho \le m := \liminf_{z \to +\infty} \phi(z) \le M := \limsup_{z \to +\infty} \phi(z) \le \kappa + \rho,$$

it follows from [13, Remark 12] that $g([m, M]) \supseteq [m, M]$. On the other hand, g is a contraction on \mathcal{I} so that $M = m = \kappa$.

Lemma 8 Let g(x) and $w_0(t,z)$ meet all the assumptions of Corollary 3. Then inequality (19) implies that the solution w(t,z) of the initial value problem $w(s,z) = w_0(s,z), (s,z) \in \Pi_0$, for (12) satisfies (30) for some positive ρ, γ .

Proof Henceforth, we fix small $\epsilon > 0$, $\kappa_+ > g(x_m)$ close to $g(x_m)$ and $\kappa_- < g(\kappa_+)$ close to $g(g(x_m))$ such that |g'(x)| < 1 for all $x \in [\kappa_- - \epsilon, \kappa_+ + \epsilon]$. The latter inequality and the unimodality of g implies that κ is a global attractor of the map $g: (0, g(x_m)] \to (0, g(x_m)]$. Therefore each semi-wavefront ϕ_c to equation (13) actually is a wavefront (i.e. $\phi_c(+\infty) = \kappa$, e.g. see [13, Theorem 18]). It is easy to see that there exist strictly increasing functions $g_+, g_- : \mathbb{R}_+ \to \mathbb{R}_+$ possessing the following properties:

- (i) $g_{-}(x) \leq g(x) \leq g_{+}(x), x \in [0, \kappa_{+}];$
- (ii) $g_{-}(x) = g(x) = g_{+}(x)$ for all x from some neighbourhood of 0;
- (iii) g_{\pm} satisfies **(H)** with κ_{\pm} and $L_{g_{\pm}} = g'(0)$.

Let $w_{\pm}(t,z)$ denote the solution of the initial value problem

$$w_t(t,z) = w_{zz}(t,z) - cw_z(t,z) - w(t,z) + g_{\pm}(w(t-h,z-ch)),$$

$$w_{\pm}(s,z) = w_0(s,z), \ (s,z) \in \Pi_0,$$

and let ϕ_{\pm} be wavefront solutions of the stationary equations

$$0 = y''(z) - cy'(z) - y(z) + g_{\pm}(y(z - ch)).$$

normalised by the condition $\lim_{z\to-\infty} \phi_{\pm}(z)/\phi(z) = 1$ (this is possible in view of (ii)). Then Theorem 2A (applied to $w_{\pm}(t,z)$) and the comparison principle guarantee that there exist large b>0 and T>h such that, for all $t\geq T-h$, $z\geq b-ch$, it holds

$$\kappa_{-} - \epsilon < w_{-}(t, z) \le w(t, z) \le w_{+}(t, z) < \kappa_{+} + \epsilon,$$

 $\kappa_{-} - \epsilon < \phi(z) < \kappa_{+} + \epsilon.$

In addition, by Lemma 6, we also can assume that

$$(\mathcal{L}\delta_{\pm})(t,z) \ge 0, \ \delta_{\pm}(t,z) \le 0, \ t \ge T - h, \ z \le b,$$

where $\delta_{\pm}(t,z)$ are defined by

$$\delta_{-}(t,z) := \phi(z) - (\kappa - \kappa_{-} + 2\epsilon)e^{-\gamma(t-T)}\eta_{\lambda}(z-b) - w(t,z),$$

$$\delta_{+}(t,z) := w(t,z) - \phi(z) - (\kappa_{+} - \kappa + 2\epsilon)e^{-\gamma(t-T)}\eta_{\lambda}(z-b).$$

Thus $\delta_{\pm}(t,z) \leq 0$, $(t,z) \in [T-h,T] \times \mathbb{R}$, so that, arguing as in the proof of Lemma 7, we obtain

$$(\mathcal{L}\delta_{\pm})(t,z) \ge (|\kappa_{\pm} - \kappa| + 2\epsilon)e^{-\gamma(t-T)}(-m_g + 1 - \gamma) > 0, \quad t \ge T, \ z > b,$$

together with $\delta_{\pm}(t,z) \leq 0$ for all $t \geq T - h, z \in \mathbb{R}$. This completes the proof of Lemma 8.

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