Algebraic spectral gaps

Amaury Mouchet

Laboratoire de Mathématiques et Physique Théorique

Université François Rabelais de Tours

CNRS (UMR 6083), Fédération Denis Poisson

Parc de Grandmont 37200 Tours, France.

`mouchet@phys.univ-tours.fr`

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Abstract

For the one-dimensional Schrödinger equation, some real intervals with no eigenvalues (the spectral gaps) may be obtained rather systematically with a method proposed by H. Giacomini and A. Mouchet in 2007. The present article provides some alternative formulation of this method, suggests some possible generalizations and extensively discusses the higher-dimensional case.

H. Giacomini et A. Mouchet ont proposé en 2007 une méthode permettant d'obtenir des trous spectraux d'un opérateur de Schrödinger à une dimension, c'est-à-dire des intervalles ne contenant aucune valeur propre. Le présent article introduit une formulation différente de cette méthode, suggère des généralisations possibles et traite de façon exhaustive le cas de plusieurs dimensions.

Introduction

A spectral gap (or eigengap) of a self-adjoint operator is a closed real interval to which no eigenvalues belongs. In [2] we have presented a systematic method for finding gaps in the discrete part of the spectrum of a one-dimensional non-magnetic Schrödinger equation with a potential $V(x)$. When V is a polynomial half-bounded from below, the boundaries of the gaps are given by the real zeroes of a family of polynomials whose degree D may be arbitrary large. The construction of these polynomials is provided by an explicit and straightforward algorithm. For still not understood reasons, it happens that in every case we have considered, our method works surprisingly well when compared to numerical computations: when increasing D the more and more numerous intervals we compute resolve the spectrum from below (i.e. the lowest eigenvalues are separated by at least one gap) and the infima of each interval seem to converge quickly to the eigenvalues. For the moment, we have no clue to understand these two phenomena and our method comes without any estimation of the distance between the gaps and the spectrum.

Being as local as possible (no computation of integrals is required), our method differs strongly in spirit from other spectral approximations like the Rayleigh-Ritz variational methods or the Rayleigh-Schrödinger perturbation methods. Since generically the spectrum cannot be determined exactly, finding such gaps may offer a valuable piece of spectral information, complementary to the information obtained by other methods.

One natural issue is to try to extend our method to multi-dimensional systems. Then, the nature of the spectrum depends on the integrability properties of the system (among the vast literature on this subject see for instance [3]). Qualitatively the statistical distribution of the eigenvalues exhibits different correlations according to the nature of its symmetries. For instance, for non-integrable systems, the two point correlation function of the discrete spectrum exhibits a so-called "level repulsion" because, unlike what occurs in integrable cases, the

probability of finding two successive eigenvalues whose distance is s vanishes when s tends to zero. Therefore we expect that this dichotomy between integrable and non-integrable cases should somehow appear in any general method for finding spectral gaps. However, most unfortunately, we have not been able to generalise our strategy in higher dimensions. Although there was *a priori* no obstacle in sight to such an attempt, it happened that the origin of the obstruction came from very subtle arguments that are deeply hidden. One aim of this note is to explain (in section 2) this negative result with the hope that it may help to find out some way to bypass the pitfalls or, at least, to help avoiding the same tracks.

Our second aim is more optimistic but still rely on speculative grounds. After recalling the main ingredients of our method in section 1.1, I will introduce a systematic algebraic approach which at first sight seems to rephrase in a more elaborate way what we have done in [2]. However by associating with the Schrödinger equation the closed algebra of differential operators that will be introduced in §§ 1.2, 1.3, we can easily guess a fruitful strategy to deal with spectral problems associated with more general (1d) linear equations — for instance of order larger than two — or even with non-linear equation like the Gross-Pitaevskii equation (g is a real coupling constant)

$$\varphi'' = (V - E)\varphi + g\varphi^3 \ . \tag{1}$$

1 The one dimensional case: an algebraic approach

1.1 The principle of the method

The stationary one-dimensional non-magnetic Schrödinger equation can be written as follows

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = 2(V(x) - E)\varphi \ . \tag{2}$$

where we will take the real potential V to be smoothly defined on $\mathbf{R}$. The real E will be an eigenvalue whenever the real function φ is square integrable on $\mathbf{R}$. The key idea of our method is to construct, for a given integer $N \geqslant 1$, a real function $J_N(\varphi', \varphi, x, E)$ (the prime stands for the derivative $\mathrm{d}/\mathrm{d}x$) such that

$$\text{(i)} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\Big(J_N\big(\varphi'(x), \varphi(x), x, E\big)\Big) = \big(\varphi(x)\big)^N F_N(x, E) \ ,$$

$$\text{(ii)} \qquad \lim_{|x| \to +\infty} J\big(\varphi'(x), \varphi(x), x, E\big) = 0.$$

The real function F_N of both the spatial coordinate x and the energy E is φ-*independent* and is obtained from the potential V and its derivatives. For instance we will find the following expressions:

$$F_1 = -a_0'' + 2(V - E)\, a_0 \ , \tag{3a}$$

$$F_2 = \frac{1}{2}\, a_0''' - 4(V - E)\, a_0' - 2V'a_0 \ , \tag{3b}$$

$$F_3 = -\frac{1}{6}\, a_0^{(iv)} + \frac{10}{3}\, (V - E)\, a_0'' + \frac{10}{3}\, V'a_0' + \big(V'' - 6\, (V - E)^2\big)\, a_0 \ , \tag{3c}$$

$$F_4 = \frac{1}{24}\, a_0^{(v)} - \frac{5}{3}\, (V - E)\, a_0''' - \frac{5}{2}\, V'a_0'' + \left(-\frac{3}{2}\, V'' + \frac{32}{3}\, (V - E)^2\right) a_0'$$

$$+ \left(-\frac{1}{3}\, V''' + \frac{32}{3}\, V'(V - E)\right) a_0 \ . \tag{3d}$$

where a_0 is any smooth function such that

$$\lim_{|x| \to \infty} a_0(x)|\varphi|^N = 0 \tag{4}$$

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which is not very restrictive since we know using semi-classical arguments that φ itself is exponentially decreasing at infinity [1, chap.10]:

$$\varphi(x) \underset{|x|\to\infty}{\sim} \mathrm{e}^{-\int^{|x|}\sqrt{2(V(x)-E)}\,\mathrm{d}x} \tag{5}$$

Condition (i) is the cornerstone of our method and, before justifying how it can be obtained (we will see that condition (ii) is not so restrictive), let us first explain how gaps in the spectrum may be obtained.

When the conditions (i) and (ii) are simultaneously fulfilled, an immediate consequence is that the integral $\int_{-\infty}^{+\infty} \left(\varphi(x)\right)^N F_N(x,E)\,\mathrm{d}x$ vanishes. This implies that, if E is truly an eigenenergy, the function $x \mapsto \left(\varphi(x)\right)^N F_N(x,E)$ should change its sign. If N is even, we obtain a φ-independent condition: *for any fixed energy $x \mapsto F_N(x,E)$ must change its sign on the real axis.* For such a one-dimensional problem, and for a given N, we still can choose F_N in a wide continuous set of smooth functions on the real axis because we have a lot of freedom in choosing a_0. A forbidden value of E (*i.e.* E cannot be an eigenenergy) is obtained if we are able to chose a_0 such that F_N remains positive on the whole x-axis. Once this property is achieved, it remains stable under small perturbations within the set of a_0's, for instance by varying the control parameters λ on which a_0 may depend, and we obtain a whole interval where no eigenenergy can exist. More precisely, if we introduce explicitly the λ-dependence in F_N, the boundaries of the gaps will necessary be given by some solutions of the system of equations

$$F_N(x,E,\lambda) = 0 \; ; \tag{6a}$$

$$\partial_x F_N(x,E,\lambda) = 0 \; ; \tag{6b}$$

$$\partial_\lambda F_N(x,E,\lambda) = 0 \; . \tag{6c}$$

Using the implicit function theorem where

$$\begin{vmatrix} \partial_E F_N & \partial_{xE}^2 F_N \\ \partial_x F_N & \partial_{xx}^2 F_N \end{vmatrix} = |\partial_E F_N|\,|\partial_{xx}^2 F_N| \neq 0 \; , \tag{6d}$$

the first two equations define implicitly $x(\lambda)$ and $E(\lambda)$; these are the conditions for a bifurcation in the zeroes of $x \mapsto F_N(x,E,\lambda)$ to occur. On one side of the bifurcation $x \mapsto F_N(x,E,\lambda)$ has locally a constant sign (and therefore the corresponding value $E(\lambda)$ is forbidden) whereas on the other side $x \mapsto F_N(x,E,\lambda)$ locally changes its sign and $E(\lambda)$ cannot be ruled out from the spectrum. To put it differently, in the (x,E,λ), the set of zeroes of F_N becomes tangent to the x-space. Then for each value of λ, $E(\lambda)$ is a candidate for being the boundary of a spectral gap. If this is the case, we can reach an extremal value provided $0 = \partial E/\partial\lambda = -\partial_\lambda F_N/\partial_E F_N$ that leads to the equation (6c).

See [2] for an effective implementation of this method and for applications. In the following, we will remain at a more formal level and let us start by defining some notations.

1.2 Algebraic construction of condition (i) and classification of the possible F_N's

Denote by $\hat{\mathcal{P}}_N$, the vector space of smooth applications from $\mathbf{R}$ to $\mathbf{R}^{N+1}$. Any element a of $\hat{\mathcal{P}}_N$ may be represented by a vector field $x \mapsto \left(a_n(x)\right)_{n\in\{0,\dots,N\}} = \left(a_0(x),\dots,a_N(x)\right)$ and can be associated in a one-to-one correspondence with the homogeneous polynomial of degree N in the two variables Φ and Ψ:

$$P_a(\Psi,\Phi,x) \overset{\text{def}}{=} \sum_{n=0}^{N} a_n(x)\Psi^{N-n}\Phi^n \; ; \tag{7}$$

then, it may be used to construct the real function on $\mathbf{R}$ defined by

$$P_a\left(\varphi'(x),\varphi(x),x\right) \overset{\text{def}}{=} \sum_{n=0}^{N} a_n(x)\left(\varphi'(x)\right)^{N-n}\left(\varphi(x)\right)^n \; . \tag{8}$$

For computations we will distinguish the "total" derivative D of a function $P\big(\varphi'(x), \varphi(x), x\big)$ from its partial derivatives $\partial_{\varphi'}$, ∂_φ and ∂_x:

$$\mathsf{D}P\big(\varphi'(x), \varphi(x), x\big) \overset{\text{def}}{=} v(x)\varphi(x)\partial_{\varphi'}P\big(\varphi'(x), \varphi(x), x\big) + \varphi'(x)\partial_\varphi P\big(\varphi'(x), \varphi(x), x\big) + \partial_x P\big(\varphi'(x), \varphi(x), x\big) \quad (9)$$

where the substitution $\varphi'' = v\varphi$ has been made since we suppose that φ fulfills (2). For simplicity we have left implicit the E-dependence in

$$v(x) \overset{\text{def}}{=} 2\big(V(x) - E\big). \quad (10)$$

From its very definition, it is obvious that the set $\mathcal{P}_N$ of homogeneous polynomials of degree N in $\varphi'(x)$ and $\varphi(x)$ is stable under D and, moreover, D is represented by a linear operator $\hat{\mathsf{D}}$ in $\hat{\mathcal{P}}_N$:

$$\hat{\mathsf{D}}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} \partial_x & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ Nv & \partial_x & 2 & 0 & 0 & \ddots & & & \vdots \\ 0 & (N-1)v & \partial_x & 3 & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & (N-n+1)v & \partial_x & n+1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 & 3v & \partial_x & N-1 & 0 \\ \vdots & & & \ddots & 0 & 0 & 2v & \partial_x & N \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & v & \partial_x \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix}. \quad (11)$$

For each $n \in \{0, \cdots, N\}$, denote by $\hat{\mathcal{Q}}_n$ the subspace of $\hat{\mathcal{P}}_N$ defined by $a_n = 0$ and $\hat{\bar{\mathcal{Q}}}_n$ its complementary defined by the direct sum decomposition $\hat{\mathcal{P}}_N = \hat{\mathcal{Q}}_n \oplus \hat{\bar{\mathcal{Q}}}_n$. Looking for all the J_N that fulfill condition (i) can therefore be interpreted as the determination in $\hat{\mathcal{P}}_N$ of the preimage $\hat{\mathsf{D}}^{-1}\hat{\bar{\mathcal{Q}}}_N$. In [2] we have shown how to straightforwardly compute J_N and obtain F_N but let us propose a strategy based on a more algebraic formalism that may be useful as a warming up for higher dimensions.

From any a, we can systematically reduce the degree in φ' of P_a if we work up to a total derivative. Indeed, for any monomial caracterised by $n \in \{0, \cdots, N-1\}$, we use the identity

$$a_n\varphi'^{N-n}\varphi^n = a_n\varphi'^{N-n-1}D\left(\frac{1}{n+1}\varphi^{n+1}\right) ; \quad (12)$$

$$= -\frac{1}{n+1}a_n'\varphi'^{N-n-1}\varphi^{n+1} - \frac{N-n-1}{n+1}va_n\varphi'^{N-n-2}\varphi^{n+2} + D\left(\frac{1}{n+1}a_n\varphi'^{N-n-1}\varphi^{n+1}\right), \quad (13)$$

where, again, we have substituted φ'' by $v\varphi$. The operation that transforms $a_n\varphi'^{N-n}\varphi^n$ to the two first terms

in (13) may be linearly represented in $\hat{\mathcal{P}}_N$ by the reduction operator defined by

$$
\hat{\mathsf{R}}
\begin{pmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-1} \\ a_N
\end{pmatrix}
\overset{\text{def}}{=}
\begin{pmatrix}
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-\partial_x & 0 & 0 & \ddots & & & & \vdots \\
(1-N)v & -\frac{1}{2}\partial_x & \ddots & \ddots & & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & 0 & \frac{n-1-N}{n-1}v & -\frac{1}{n}\partial_x & 0 & 0 & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & 0 & \frac{-2}{N-2}v & -\frac{1}{N-1}\partial_x & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N-1}v & -\frac{1}{N}\partial_x & 1
\end{pmatrix}
\begin{pmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \\ a_{N-1} \\ a_N
\end{pmatrix} . \tag{14}
$$

For $N = 1$, we have $\hat{\mathsf{R}} = \left(\begin{smallmatrix} 0 & 0 \\ -\partial_x & 1 \end{smallmatrix} \right)$. By construction we have $a - \hat{\mathsf{R}}a = 0$ for any vector $a = (0, \ldots, 0, a_N) \in \hat{\tilde{\mathcal{Q}}}_N$; by (13), we have $a - \hat{\mathsf{R}}a \in \operatorname{Im}\hat{D}$ for any vector $a \in \hat{\tilde{\mathcal{Q}}}_n$ with $n \in \{0, \cdots, N-1\}$; therefore, by linearity, we have $a - \hat{\mathsf{R}}a \in \operatorname{Im}\hat{D}$ for any $a \in \hat{\mathcal{P}}_N$. Moreover, since $\hat{\mathsf{R}}$ is a lower triangular matrix with all diagonal terms but one being zero, all the components of $\hat{\mathsf{R}}^N a$ vanish but the last one $A_N[a] \overset{\text{def}}{=} (\hat{\mathsf{R}}^N a)_N$ which is a differential operator on a of order N. For instance we have

$$A_1[a] = -a_0' + a_1 ; \tag{15a}$$

$$A_2[a] = \frac{1}{2}a_0'' - va_0 - \frac{1}{2}a_1' + a_2 ; \tag{15b}$$

$$A_3[a] = -\frac{1}{6}a_0''' + \frac{7}{6}va_0' + \frac{2}{3}v'a_0 + \frac{1}{6}a_1'' - \frac{1}{2}va_1 - \frac{1}{3}a_2' + a_3 ; \tag{15c}$$

$$A_4[a] = \frac{1}{24}a_0^{(iv)} - \frac{2}{3}va_0'' - \frac{3}{4}v'a_0' - (\frac{1}{4}v'' - v^2)a_0 - \frac{1}{24}a_1''' + \frac{5}{12}va_1' + \frac{1}{4}v'a_1 + \frac{1}{12}a_2'' - \frac{1}{3}va_2 - \frac{1}{4}a_3' + a_4 . \tag{15d}$$

Now if we use the decomposition

$$1 = \hat{\mathsf{R}}^N + (1 - \hat{\mathsf{R}}) \sum_{n=0}^{N-1} \hat{\mathsf{R}}^n , \tag{16}$$

it can be seen immediately that any a can be uniquely decomposed in $\hat{\mathsf{R}}^N a \in \hat{\tilde{\mathcal{Q}}}_N$ plus a vector in $\operatorname{Im}\hat{D}$. Translating this decomposition into the language of functions and taking for a the vector associated to $\mathsf{D}K_N\big(\varphi'(x), \varphi(x), x\big)$ where K_N is any homogeneous polynomial of degree N in $\big(\varphi'(x), \varphi(x)\big)$, we have shown that there always exists a homogeneous polynomial $\tilde{K}_N\big(\varphi'(x), \varphi(x), x\big)$ of degree N in $\big(\varphi'(x), \varphi(x)\big)$, such that

$$\mathsf{D}K_N\big(\varphi'(x), \varphi(x)\big) = F_N(x)\big(\varphi(x)\big)^N - \mathsf{D}\tilde{K}_N\big(\varphi'(x), \varphi(x)\big) \tag{17}$$

and therefore, in order to recover (i), it is is sufficient to choose $J_N = K_N + \tilde{K}_N$. The function $F_N(x)$ is independent on φ and φ' and is just given by the action of the linear operator A_N on the coefficients of $\mathsf{D}K_N$.

A priori, we can start with any set of trial functions $(a_n)_{n \in \{1,\ldots,N\}}$ to build up our K_N, then compute F_N by computing the Nth power of $\hat{R}$. Before we try to control the sign of F_N for even N, the only restriction so far on the a's is to preserve (ii): a_n should not increase faster than $\varphi'^{N-n}\varphi^n$ at $|x| \to \infty$. However we will

now show that our freedom is in fact restricted to the choice of one test function only. In other words, many different choices of $(a_n)_n$ will lead to the same F_N and therefore will not help to gain any piece of information (in particular those leading to an identically vanishing F_N). To put it very qualitatively, $\bar{\mathcal{Q}}_N$ is a very thin subspace in $\mathcal{P}_N$ (of co-dimension N if seen as a vector space on smooth real functions) and the kernel of D is too small (given by the solutions of a linear ordinary differential equation of order N) for $D^{-1}\mathcal{Q}_N$ to decrease its codimension. To understand that, let us introduce the projector $\hat{\Pi}_n$ on the nth component of a. Our previous construction of condition (i) can therefore be re-written

$$\hat{D}a = \hat{R}^N \hat{D}a \ . \tag{18}$$

where a is the element of $\hat{\mathcal{P}}_N$ associated with the function J_N. In section 1.3 we will show directly for small N that

$$\hat{R}^N \hat{D}(1 - \hat{\Pi}_0) = 0 \tag{19}$$

which has the following consequence: adding to a any vector $b = (b_n)_n$ whose $b_0 = 0$ will not affect the left hand side of (18) from which F_N is computed. Therefore F_N depends only on one function, namely a_0. All the others can be canceled without loss of generality. Actually, if we start with $a = (a_0, 0, \ldots, 0)$, we have $\tilde{a} \stackrel{\text{def}}{=} \hat{D}a = (a_0', Nva_0, 0, \ldots, 0) = (\tilde{a}_0, \tilde{a}_1, 0, \ldots, 0)$ and substituting $\tilde{a}$ with a in (15) leads straightforwardly to (3). If we start with $a = (0, a_1, 0, \ldots, 0)$, we have $\tilde{a} \stackrel{\text{def}}{=} \hat{D}a = (a_1, a_1', (N-1)va_1, 0, \ldots, 0) = (\tilde{a}_0, \tilde{a}_1, 0, \ldots, 0)$ and (19) can be (tediously) checked in the special cases $N = 2$, 3 and 4 : by substituting $\tilde{a}$ with a in (15), Λ_N identically vanishes. The same remains true for $a = (0, 0, a_2, 0, \ldots, 0)$, $a = (0, 0, 0, a_3, 0, \ldots, 0)$, etc. The next section provides a systematic way of proving the last results and § 3.1 gives a more general argument.

1.3 The closed algebra

Let us prove (19) by introducing an algebra of operators that may be represented by their action on $\hat{\mathcal{P}}_N$ or on the functions in $\mathcal{P}_N$. The operator $S^\downarrow \stackrel{\text{def}}{=} \varphi\partial_{\varphi'}$ when multiplied by v allow to formalise the substitution of φ'' by $v\varphi$. We also define $S^\uparrow \stackrel{\text{def}}{=} \varphi'\partial_\varphi$. The arrows recall that these operators raise and lower the component of a:

$$\hat{S}^\downarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ Na_0 \\ (N-1)a_1 \\ \vdots \\ 3a_{N-3} \\ 2a_{N-2} \\ a_{N-1} \end{pmatrix} \quad ; \quad \hat{S}^\uparrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ (N-1)a_{N-1} \\ Na_N \\ 0 \end{pmatrix} \ . \tag{20}$$

If we introduce the diagonal matrices $\hat{\Lambda}_\searrow \stackrel{\text{def}}{=} \text{diag}(0, 1, \ldots, N)$, $\hat{\Lambda}_\swarrow \stackrel{\text{def}}{=} \text{diag}(N, \ldots, 1, 0)$ and the raising and lowering operators $\hat{S}^\pm$ defined by the following action on any a

$$\hat{S}^- \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ \vdots \\ a_{N-3} \\ a_{N-2} \\ a_{N-1} \end{pmatrix} \quad ; \quad \hat{S}^+ \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \\ a_N \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{N-1} \\ a_N \\ 0 \end{pmatrix} \quad ; \tag{21}$$

We have $\hat{S}^\downarrow = \hat{S}^- \hat{\Lambda}_\swarrow$, $\hat{S}^\uparrow = \hat{S}^+ \hat{\Lambda}_\searrow$. Within this formalism, the operator (11) is

$$\hat{D} = \hat{S}^\uparrow + \hat{\partial}_x + v\hat{S}^\downarrow = \hat{S}^+ \hat{\Lambda}_\searrow + \hat{\partial}_x + v\hat{S}^- \hat{\Lambda}_\swarrow \ , \tag{22}$$

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with $\hat{\partial}_x \stackrel{\text{def}}{=} \operatorname{diag}(\partial_x, \ldots, \partial_x)$. To implement the commutation rules between $\hat{S}^{\pm}$ and any diagonal matrix $\hat{\Lambda} = \operatorname{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$, we define the diagonal matrices

$$\hat{\Lambda}^{(1)} \stackrel{\text{def}}{=} \operatorname{diag}(\lambda_1 \lambda_2, \ldots, \lambda_N, \lambda_0) ; \quad \hat{\Lambda}^{(-1)} \stackrel{\text{def}}{=} \operatorname{diag}(\lambda_N, \lambda_0 \lambda_1, \ldots, \lambda_{N-1}) , \tag{23}$$

and for any strictly positive integer k, we define $\hat{\Lambda}^{(k)} \stackrel{\text{def}}{=} (\cdots (\hat{\Lambda}^{(1)})^{(1)} \cdots)^{(1)}$, $\hat{\Lambda}^{(-k)} \stackrel{\text{def}}{=} (\cdots (\hat{\Lambda}^{(-1)})^{(-1)} \cdots)^{(-1)}$ where the superscript occurs k times in the right hand side. We also take $\hat{\Lambda}^{(0)} \stackrel{\text{def}}{=} \hat{\Lambda}$. Therefore the nth entry of $\hat{\Lambda}^{(k)}$ for any integer k is $(\hat{\Lambda}^{(k)})_{n,n} = \lambda_{n+k \bmod N+1}$. Then we have

$$\hat{S}^{\pm} \hat{\Lambda} = \hat{\Lambda}^{(\pm 1)} \hat{S}^{\pm} . \tag{24}$$

In particular if we set $\hat{\Pi}_{-1} = \hat{\Pi}_{N+1} = 0$, for any $n \in \{0, \ldots, N\}$,

$$\hat{S}^{\pm} \hat{\Pi}_n = \hat{\Pi}_{n \mp 1} \hat{S}^{\pm} . \tag{25}$$

We can also express the reduction operator $\hat{R}$ in terms of the shift operators $\hat{S}$. From (14), we have

$$\hat{R} = \hat{\Pi}_N + \hat{A} + \hat{B} = \hat{\Pi}_N + \hat{C} . \tag{26}$$

with

$$\hat{A} \quad \stackrel{\text{def}}{=} \quad -\hat{S}^- \hat{\Lambda}_a \hat{\partial}_x ; \tag{27a}$$

$$\hat{B} \quad \stackrel{\text{def}}{=} \quad -(\hat{S}^-)^2 \hat{\Lambda}_b v ; \tag{27b}$$

and

$$\hat{\Lambda}_a \quad \stackrel{\text{def}}{=} \quad \operatorname{diag}\left(1, \ldots, \frac{1}{n+1}, \ldots, \frac{1}{N+1}\right) ; \tag{28a}$$

$$\hat{\Lambda}_b \quad \stackrel{\text{def}}{=} \quad \operatorname{diag}\left(N-1, \ldots, \frac{N-n-1}{n+1}, \ldots, \frac{1}{N-1}, 0, \frac{-1}{N+1}\right) . \tag{28b}$$

From (25) we have $\hat{S}^- \hat{\Pi}_N = 0$ and therefore $\hat{C} \hat{\Pi}_N = 0$. Besides, $\hat{C}^N$ has a unique non-zero element, namely $(\hat{C}^N)_{N,0}$ and then $\hat{\Pi}_N \hat{C}^N = \hat{C}^N$. Therefore, the expansion of $\hat{R}^N = (\hat{\Pi}_N + \hat{C})^N$ leads to

$$\hat{R}^N \quad = \quad \hat{\Pi}_N \left(1 + C + \cdots + C^N\right) ; \tag{29}$$

$$= \quad \hat{\Pi}_N \sum_{\substack{\text{words } w=(l,m) \\ 0 \leqslant |l|+|m| \leqslant N}} \hat{A}^{l_1} \hat{B}^{m_1} \hat{A}^{l_2} \hat{B}^{m_2} \cdots . \tag{30}$$

The last sum involves all the distinct words w that can be made with the two "letters" $\hat{A}$ and $\hat{B}$ whose length is $|l| + |m|$ between 0 and N (the zero-length word is the identity). Each word w is uniquely defined by two multiple indices, i.e. by two sequences $l = (l_1, l_2, \ldots)$ and $m = (m_1, m_2, \ldots)$ of positive integers that all vanish above a finite rank. We define $|l| \stackrel{\text{def}}{=} l_1 + l_2 + \cdots$ and $|m| \stackrel{\text{def}}{=} m_1 + m_2 + \cdots$. Eventually, the identity (19) holds if we can cancel, for each $n \in \{1, \ldots N\}$, the product

$$\hat{R}^N \hat{D} \hat{\Pi}_n = \hat{\Pi}_N \Big(\sum_{\substack{\text{words } w=(l,m) \\ 0 \leqslant |l|+|m| \leqslant N}} \hat{A}^{l_1} \hat{B}^{m_1} \hat{A}^{l_2} \hat{B}^{m_2} \cdots \Big) \big(\hat{\partial}_x + v \hat{S}^- \hat{\Lambda}_{\nwarrow} + \hat{S}^+ \hat{\Lambda}_{\searrow} \big) \hat{\Pi}_n . \tag{31}$$

To compute this sum, we start by rearranging each word in order to shift, with the help of (24), all the $\hat{S}^-$'s involved in $\hat{A}$ and $\hat{B}$ to the right:

$$\hat{A}^{l_1} \hat{B}^{m_1} \hat{A}^{l_2} \hat{B}^{m_2} \cdots = \hat{\Lambda}_w (\hat{S}^-)^{|l|+2|m|} \tag{32}$$

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with the diagonal matrix

$$\hat{\Lambda}_w = (-1)^{|l|+|m|} \underbrace{\hat{\Lambda}_a^{(-1)} \cdots \hat{\Lambda}_a^{(-l_1)}}_{l_1 \text{ factors}} \hat{\partial}_x^{l_1} \underbrace{\hat{\Lambda}_b^{(-l_1-2)} \cdots \hat{\Lambda}_b^{(-l_1-2m_1)}}_{m_1 \text{ factors}} v^{m_1}$$

$$\underbrace{\hat{\Lambda}_a^{(-l_1-2m_1-1)} \cdots \hat{\Lambda}_a^{(-l_1-2m_1-l_2)}}_{l_2 \text{ factors}} \hat{\partial}_x^{l_2} \underbrace{\hat{\Lambda}_b^{(-l_1-2m_1-l_2-2)} \cdots \hat{\Lambda}_b^{(-l_1-2m_1-l_2-2m_2)}}_{m_2 \text{ factors}} v^{m_2} \cdots \quad (33)$$

Now each word contributes to (31) via three terms

$$\hat{\Lambda}_w \hat{\Pi}_N (\hat{S}^-)^{|l|+2|m|-1} \hat{\Pi}_n \hat{S}^- \hat{S}^+ \hat{\Lambda}_\searrow = \hat{\Lambda}_w (\hat{S}^-)^{|l|+2|m|-1} \hat{\Pi}_n \delta_{N-|l|-2|m|+1,n} \hat{\Lambda}_\searrow ; \quad (34a)$$

$$\hat{\Lambda}_w \hat{\Pi}_N (\hat{S}^-)^{|l|+2|m|} \hat{\Pi}_n \hat{\partial}_x = \hat{\Lambda}_w (\hat{S}^-)^{|l|+2|m|} \hat{\Pi}_n \delta_{N-|l|-2|m|,n} \hat{\partial}_x ; \quad (34b)$$

$$\hat{\Lambda}_w \hat{\Pi}_N (\hat{S}^-)^{|l|+2|m|+1} \hat{\Pi}_n v \hat{\Lambda}_\nwarrow = \hat{\Lambda}_w (\hat{S}^-)^{|l|+2|m|+1} \hat{\Pi}_n \delta_{N-|l|-2|m|-1,n} v \hat{\Lambda}_\nwarrow . \quad (34c)$$

The passage from the left to the right hand sides has been obtained by moving all the projectors to the right with (25). The Kronecker symbols come from $\hat{\Pi}_r \hat{\Pi}_n = \delta_{r,n} \hat{\Pi}_n$. In (34a), the diagonal matrix $\hat{S}^- \hat{S}^+ = 1 - \hat{\Pi}_0$ can be replaced by 1 for $n \neq 0$. The left hand sides of (34) contain the same matrix $(\hat{S}^-)^{N-n} \hat{\Pi}_n$ whose all elements vanish but one: $((\hat{S}^-)^{N-n} \hat{\Pi}_n)_{N,n} = 1$. When acting on a vector a, only a_n gets involved together with the last diagonal element $\Lambda_{w,N} \stackrel{\text{def}}{=} (\hat{\Lambda}_w)_{N,N}$ of $\hat{\Lambda}_w$. Using $(\hat{\Lambda}_\nwarrow)_{n,n} = N - n$ and $(\hat{\Lambda}_\searrow)_{n,n} = n$, we eventually obtain a vector whose $\{0, 1, \ldots, N-1\}$ components vanish. The last one being

$$\left(\hat{R}^N \hat{D} \hat{\Pi}_n a \right)_N = \sum_{\substack{\text{words } w=(l,m) \\ 0 \leqslant |l|+|m| \leqslant N}} \Lambda_{w,N} \Big(\delta_{N-|l|-2|m|+1,n} n\, a_n + \delta_{N-|l|-2|m|,n} \partial_x a_n + \delta_{N-|l|-2|m|-1,n} (N-n)\, v\, a_n \Big). \quad (35)$$

The presence of δ's forces the relevant words we some on keep $|l| + 2|m| \leqslant N - 1$ since $n \geqslant 1$. Now we have

$$\Lambda_{w,N} = \alpha_{w,N} \partial_x^{l_1} v^{m_1} \partial_x^{l_2} v^{m_2} \cdots \quad (36)$$

with the numerical coefficient

$$\alpha_{w,N} = (-1)^{|l|+|m|} \frac{(N-l_1)!}{N!} \frac{(l_1+2m_1-1)!!}{(l_1-1)!!} \frac{(N-l_1-2m_1-1)!!}{(N-l_1-1)!!}$$

$$\frac{(N-l_1-2m_1-l_2)!}{N-l_1-2m_1!} \frac{(l_1+2m_1+l_2+2m_2-1)!!}{(l_1+2m_1+l_2-1)!!} \frac{(N-l_1-2m_1-l_2-2m_2-1)!!}{(N-l_1-2m_1-l_2-1)!!} \cdots \quad (37)$$

is obtained because $(\hat{\Lambda}_a^{(-k)})_{N,N} = 1/(N-k+1)$ and $(\hat{\Lambda}_b^{(-k)})_{N,N} = (k-1)/(N-k+1)$ for $0 \geqslant k \geq N$. We have also introduced the usual notation for any integer n

$$n!! \stackrel{\text{def}}{=} \begin{cases} 2^p\, p! & \text{if } n = 2p \text{ with } p \text{ integer}; \\ \dfrac{(2p+1)!}{2^p\, p!} & \text{if } n = 2p+1 \text{ with } p \text{ integer}. \end{cases} \quad (38)$$

I have not been able to prove diectly that the right hand side of (35) vanishes for an arbitrary N and for any $n \in \{1, \ldots, N\}$ but since we are now left with an explicit expression for a linear differential operator acting on a_n, it can be checked, possibly with the help of symbolic computations, for small specific N. It is rather straightforward to prove that $\left(\hat{R}^N \hat{D} \hat{\Pi}_n a \right)_N$ vanishes for $n = N$ and $n = N - 1$ with arbitrary N and that it also vanishes for $N \leqslant 4$ and any $n \in \{1, \ldots, N\}$.

1.4 Open questions for possible generalizations remaining with $d = 1$

It would be interesting to see if the tools introduced above may be adapted for higher order linear differential equations. More challenging, and this will constitute the object future research, we may obtain some interesting information on the solutions of some non-linear equations like (1). The price to pay is that the non-linear term prevents us to keep working in one $\hat{\mathcal{P}}_N$ space only; then, we should work in the whole $\bigoplus_N \hat{\mathcal{P}}_N$ and probably extend condition (i) by factorising a strictly positive polynomial in φ rather than considering only φ^N.

2 Higher dimensional cases

One natural generalization to dimension d is to consider the Schrödinger equation

$$\Delta_d\, \varphi = \sum_{\mu=1}^{d} \partial_\mu^2 \varphi = 2\big(V - E\big)\varphi = v\varphi \tag{39}$$

where $x = (x^\mu)_\mu \in \mathbf{R}^d$ (we keep definition (10)). In the following, Greek indices always label the dimension and we will follow the usual convention of letting implicit the sum from 1 to d over repeated Greek indices unless the opposite is specified. The partial derivative with respect to the μth coordinate is denoted by $\partial_\mu \overset{\text{def}}{=} \partial/\partial x^\mu$.

We will work within the space $\hat{\mathcal{P}}_{N,d} \overset{\text{def}}{=} \bigotimes_{N=1}^{d} \hat{\mathcal{P}}_N$ of smooth real functions $(a_n)_n$ where $n = (n_1, \ldots, n_d)$ is a multi-index with $n_\mu \in \{0, \ldots, N\}$ from which we can construct the set $\mathcal{P}_{N,d}$ of functions built from homogeneous polynomials, namely having the form

$$P_a\big(\partial\varphi(x), \varphi(x), x\big) \overset{\text{def}}{=} \sum_n a_n (\partial\varphi)^{N-n} \varphi^{N(1-d)+|n|}\Big|_x \tag{40}$$

where $|n| = n_1 + \cdots + n_d$, $(\partial\varphi)^{N-n}$ stands for $(\partial_1\varphi)^{N-n_1}(\partial_2\varphi)^{N-n_2}\cdots(\partial_d\varphi)^{N-n_d}$. To apply the same reasoning that led to gaps in the spectrum, condition (i) will be extended in d dimension by looking for a current $J_N = (J_N^\mu)_\mu$ whose divergence can be factorised by a φ-independent function times a positive function. More precisely, each J_N^μ is associated with an element of $\hat{\mathcal{P}}_{N,d}$ and is constructed in order to fulfill

$$(\mathrm{i})_d \qquad D_\mu\Big(J_N^\mu\big(\partial\varphi(x), \varphi(x), x, E\big)\Big) = \varphi^N F_N(x, E)\ . \tag{41}$$

Then by integrating it on the whole $\mathbf{R}^d$, provided that

$$(\mathrm{ii})_d \qquad \int_{\mathcal{V}} D_\mu J_N^\mu\, \mathrm{d}^d x = \int_{\partial\mathcal{V}} J_N^\mu\, \mathrm{d}^{d-1}\sigma_\mu \to 0 \tag{42}$$

where $\mathcal{V}$ is a closed radius whose typical length R tend to infinity ($\mathrm{d}^{d-1}\sigma_\mu$ is the measure on its boundary $\partial\mathcal{V}$ whose surface growths algebraically with R, therefore any exponential decrease of J_N will guarantee $(\mathrm{ii})_d$), the condition that $F_N(x, E)$ should change its sign for even N will hopefully lead to some constraints on E. The total derivative is defined as the linear operator in $\mathcal{P}_{N,d}$

$$D_\mu\Big(P_a\big(\partial\varphi(x), \varphi(x), x\big)\Big) \overset{\text{def}}{=} \frac{\partial P_a}{\partial(\partial_\nu\varphi)}\, \partial_{\nu,\mu}^2 \varphi + \frac{\partial P_a}{\partial\varphi}\, \partial_\mu\varphi + \partial_\mu P_a\ . \tag{43}$$

When $d > 1$, we cannot get rid of the second derivatives of φ as easily as for $d = 1$ because (39) provides us with only one substitution rule[1]: it is only when grouped into a Laplacian, that the substitution $\sum_{\mu=1}^{d} \partial_\mu^2 \varphi = v\varphi$ can

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[1]Without further information on V. In the non-generic case of a separable potential, there are in fact d independent substitution rules. The substitution rules may also implement some symmetries if there are any, like in the case of integrable systems.

be done. If we start looking for a J_N from a generic a, grouping the second derivative in $D_\mu J_N^\mu$ into Laplacians will eventually impose some relations on the $(a_n)_n$. Following what we have explained in the previous section, we will however systematically work up to a total derivative. We can also extend the factorisation in (i)$_d$ to other functions φ and its derivative whose sign is fixed. Rather than φ^N, we still can apply the argument if we manage to obtain

$$\text{(i)}'_d \qquad D_\mu \Big(J_N^\mu \big(\partial\varphi(x), \varphi(x), x, E \big) \Big) = \Big(B_{N/2} \big(\partial\varphi(x), \varphi(x), x, E \big) \Big)^2 F_N(x, E) . \qquad (44)$$

for even N where $B_{N/2}$ is an homogeneous polynomial in $(\partial\varphi, \varphi)$ of degree $N/2$.

2.1 Attempt for $N = 2$

Let us tentatively start with

$$J_2^\mu = g_0^{\mu\nu\rho} \, \partial_\nu \varphi \partial_\rho \varphi + g_1^{\mu\nu} \, \partial_\nu \varphi + g_2^\mu \qquad (45)$$

where the g's are smooth functions of φ, x and E. We will start with d functions of type g_2, d^2 functions of type g_1 and $d^2(d+1)/2$ functions of type g_0 such that, without loss of generality,

$$g_0^{\mu\nu\rho} = g_0^{\mu\rho\nu} . \qquad (46)$$

All the terms in $D_\mu J_2^\mu$ involving a second derivative in φ can be collected in

$$(2g_0^{\mu\nu\rho} \, \partial_\rho\varphi + g_1^{\mu\nu}) \partial_{\mu\nu}^2 \varphi . \qquad (47)$$

To construct a Laplacian, we must impose the parenthesis to be anti-symmetric when $\mu \neq \nu$:

$$g_0^{\mu\nu\rho} = -g_0^{\nu\mu\rho} \qquad (\mu \neq \nu) ; \qquad (48)$$

$$g_1^{\mu\nu} = -g_1^{\nu\mu} \qquad (\mu \neq \nu) , \qquad (49)$$

and independent of μ when $\nu = \mu$, that is there are d functions h_0^ρ and one function h_1 such that

$$g_0^{\mu\mu\rho} = h_0^\rho \qquad (\text{no summation on } \mu) \qquad (50)$$

and

$$g_1^{\mu\mu} = h_1 \qquad (\text{no summation on } \mu) . \qquad (51)$$

Combining (46) with (48) leads to $g_0^{\mu\nu\rho} = g_0^{\mu\rho\nu} = -g_0^{\rho\mu\nu} = -g_0^{\rho\nu\mu} = g_0^{\nu\rho\mu} = g_0^{\nu\mu\rho} = -g_0^{\mu\nu\rho}$ and therefore $g_0^{\mu\nu\rho} = 0$ when (μ, ν, ρ) are pairwise distinct. Collecting in $D_\mu J_2^\mu$ the cubic terms in $\partial\varphi$ leads to

$$(\partial_\varphi g_0^{\mu\nu\rho}) \, \partial_\mu\varphi \, \partial_\nu \, \varphi \, \partial_\rho\varphi = (\partial_\varphi h_0^\rho) \, \partial_\rho\varphi \sum_{\mu=1}^{d} (\partial_\mu\varphi)^2 . \qquad (52)$$

Therefore we will take

$$\partial_\varphi h_0^\rho = 0 . \qquad (53)$$

Quadratic terms in $\partial\varphi$ appearing in $D_\mu J_2^\mu$ are

$$(\partial_\rho g_0^{\rho\mu\nu} + \partial_\varphi g_1^{\mu\nu}) \, \partial_\mu\varphi \, \partial_\nu \, \varphi = \sum_{\mu=1}^{d} \sum_{\nu=1}^{d} \partial_\mu h_0^\nu \, \partial_\mu\varphi \, \partial_\nu \, \varphi + \sum_{\mu=1}^{d} \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{d} \partial_\nu h_0^\mu \, \partial_\mu\varphi \, \partial_\nu \, \varphi + \sum_{\mu=1}^{d} \left[(\partial_\mu\varphi)^2 (\partial_\varphi h_1 - \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{d} \partial_\nu h_0^\nu) \right] .$$
$$\qquad (54)$$

Canceling each independent term requires

$$\partial_\mu h_0^\nu + \partial_\nu h_0^\mu = 0 \qquad (\mu \neq \nu) \tag{55}$$

and

$$\partial_\mu h_0^\mu - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{d} \partial_\nu h_0^\nu + \partial_\varphi h_1 = 0 \qquad \text{(no summation on } \mu) . \tag{56}$$

The last equation appears as a linear system of d equations that can be rewritten with the help of the $d \times d$ matrix $2 - A_d$ where $(A_d)_{\mu,\nu} = 1$. Then, $\det(2 - A_d) = (-2)^{d-1}(d-2)$ and therefore when $d \neq 2$, it can be inverted and leads to

$$\partial_\mu h_0^\mu = \frac{1}{d-2} \partial_\varphi h_1 \qquad \text{(no summation on } \mu, \ d \neq 2) . \tag{57}$$

Now h_0^μ does not depend on φ, by (53), and then $\partial_\varphi h_1$ neither, hence there exist two φ independent functions $\tilde{h}_1$ and $\tilde{\tilde{h}}_1$ such that

$$h_1 = \tilde{h}_1 \varphi + \tilde{\tilde{h}}_1 . \tag{58}$$

The relation (57) implies (with now an implicit summation on μ)

$$\tilde{h}_1 = \frac{d-2}{d} \partial_\mu h_0^\mu . \tag{59}$$

In the special case $d = 2$, (56) leads to

$$\partial_1 h_0^1 = \partial_2 h_0^2 \tag{60}$$

and

$$\partial_\varphi h_1 = 0 \qquad . \tag{61}$$

Then we can keep (58) together with (59) even for $d = 2$.

Collecting all the previous relations we get

$$D_\mu J_2^\mu = \left(2h_0^\mu \Delta_d \varphi + \partial_\nu \, g_1^{\nu\mu} + \partial_\varphi g_2^\mu\right)\partial_\mu \varphi + h_1 \Delta_d \varphi + \partial_\mu g_2^\mu \tag{62}$$

where of course, we can use the substitution (39). Without loss of generality we can take $g_1^{\nu\mu} = 0$ for $\mu \neq \nu$ by possibly redefining

$$g_2^\mu(\varphi, x, E) \mapsto g_2^\mu(\varphi, x, E) - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{d} \int_0^\varphi \partial_\nu \, g_1^{\nu\mu}(\varphi', x, E)\mathrm{d}\varphi' \tag{63}$$

since $\partial_\mu g_2^\mu \mapsto \partial_\mu g_2^\mu - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{d} \int_0^\varphi \partial_\mu \partial_\nu g_1^{\nu\mu}(\varphi', x, E)\mathrm{d}\varphi' = \partial_\mu g_2^\mu$ because the integrand cancels by (49). To cancel the parenthesis in (62) we must take

$$\partial_\varphi g_2^\mu = -\partial_\mu h_1 - 2h_0^\mu v \, \varphi \tag{64}$$

The dependence in φ appears only through (58) and we can immediately integrate the last relation

$$g_2^\mu = -\left(\frac{1}{2}\partial_\mu \, \tilde{h}_1 + h_0^\mu v\right) \varphi^2 - (\partial_\mu \, \tilde{\tilde{h}}_1)\varphi + \tilde{g}_2^\mu \tag{65}$$

where $\tilde{g}_2$ is φ-independent. With this expression, (62) becomes

$$D_\mu J_2^\mu = -\varphi^2 \left[\frac{1}{2}\Delta_d \tilde{h}_1 - v\tilde{h}_1 + \partial_\mu(vh_0^\mu)\right] - \varphi[\Delta_d \tilde{\tilde{h}}_1 - v\tilde{\tilde{h}}_1] + \partial_\mu \tilde{g}_2^\mu . \tag{66}$$

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The last term is irrelevant because it is a total divergence and can be reabsorbed in the definition of J_2^μ. The second term does not contribute also since by integration by part it can be converted to $\tilde{\tilde{h}}_1[\Delta_d - v]\varphi$ which vanishes. with the use of (59), the first term can be further simplified in order to keep h_0^μ only.

To sum up, condition $(i)_d$ can be obtained for $N = 2$ with

$$F_2 = \frac{2-d}{2d}\,\Delta_d\partial_\mu h_0^\mu - \frac{2}{d}\,v\,\partial_\mu h_0^\mu - h_0^\mu\partial_\mu v \tag{67}$$

with h_0^μ being any d smooth functions such that

$$\partial_1 h_0^1 = \partial_2 h_0^2 = \cdots = \partial_d h_0^d\ ; \tag{68a}$$
$$\partial_\mu h_0^\nu = -\partial_\nu h_0^\mu \qquad (\mu \neq \nu)\ . \tag{68b}$$

Our freedom of choosing J_2 has therefore being reduced first because eliminating the second derivatives of φ through its Laplacian impose severe constraints and second because, as in the $d = 1$ case, many different initial choices lead to the same F_2; in others words the linear application from a to F has a non-zero kernel.

For $d = 1$, we have one test function $h_0 = g_0 = a_0$ at our disposal but without any restriction on its derivative and we can easily checked that, for $d = 1$, (67) is (3b).

For $d = 2$, we have $F_2 = -\partial_\mu(vh_0^\mu)$ and surprisingly (68) appears to be the Cauchy-Riemann conditions for $h_0^1 + ih_0^2$ to be analytic. Anyway, if we compute

$$\int_{\mathcal{S}} F_2\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\mathcal{S}} v\big(\begin{smallmatrix}h_0^1\\h_0^2\end{smallmatrix}\big)\cdot\mathrm{d}\vec{\sigma} \tag{69}$$

on a surface $\mathcal{S} \in \mathbf{R}^2$ whose boundary ∂S coincide with the energy level $V(x, y) = E$, that is $v = 0$, the integrand of the right hand side vanishes ($\mathrm{d}\vec{\sigma}$ is an infinitesimal 2d-vector normal to the curve ∂S pointing outwards say). Therefore whatever choice we make for h_0, for any energy E belonging to the image of V (where we know the spectrum lies), F_2 changes its sign and no information can be obtained further.

For $d \geqslant 3$, the constraints (68) are so strong that they limit the choice of h_0's to polynomials in x of degree at most 3. Indeed all the third derivatives of h_0 must cancel (up to equation (73) included, any pair of distinct Greek letters denote any pair of distinct values of indices and no summation over repeated indices is left implicit[2]). First,

$$\partial^2_{\mu\nu}h_0^\sigma = -\partial^2_{\sigma\nu}h_0^\mu = \partial^2_{\sigma\mu}h_0^\nu = -\partial^2_{\nu\mu}h_0^\sigma \tag{70}$$

and therefore $\partial^2_{\mu\nu}h_0^\sigma = 0$; then $\partial^3_{\mu\nu\rho}h_0^\sigma = 0$

$$\partial^3_{\mu\nu\rho}h_0^\sigma = 0; \quad \partial^3_{\mu\nu\rho}h_0^\mu = 0; \quad \partial^3_{\mu\mu\nu}h_0^\rho = 0. \tag{71}$$

Furthermore,

$$\partial^3_{\mu\mu\mu}h_0^\nu = -\partial^3_{\mu\mu\nu}h_0^\mu = -\partial^3_{\mu\rho\nu}h_0^\rho = 0 \tag{72}$$

since for $d \geq 3$ we can always find an index ρ distinct from both μ and ν; eventually we have

$$\partial^3_{\mu\mu\mu}h_0^\mu = \partial^3_{\mu\mu\nu}h_0^\nu = -\partial^3_{\mu\nu\nu}h_0^\mu = -\partial^3_{\rho\nu\nu}h_0^\rho = \partial^3_{\rho\rho\nu}h_0^\nu = \partial^3_{\rho\rho\mu}h_0^\mu = -\partial^3_{\rho\mu\mu}h_0^\rho = -\partial^3_{\mu\mu\mu}h_0^\mu \tag{73}$$

and all these third derivatives actually vanish as well. Now $h_0(x)$ being a polynomial of degree at most two in x, the constraints (68) on its coefficients leads to the general form

$$h_0^\mu(x) = h_0^\mu(0) + kx^\mu + A^\mu{}_\nu x^\nu - \frac{1}{2}\,l^\mu x^2 + x^\mu\,l\cdot x \tag{74}$$

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[2]I am grateful to Oleg Lisovyy[4] for providing the following arguments that concisely and rigourously proved my first guess of (74).

where A is a $d \times d$ constant antisymmetric real matrix, k a real constant, l a real constant d-vector ; the Cartesian product $l \cdot x = l^\mu x^\mu$ is used. Then, from (67) we get:

$$F_2(x, E, \lambda) = 4\big(E - V(x)\big)(k + l \cdot x) - \big(h_0^\mu(0) + kx^\mu + A^\mu{}_\nu x^\nu - \frac{1}{2} l^\mu x^2 + x^\mu \, l \cdot x\big)\partial_\mu V(x) \; . \qquad (75)$$

Unlike what occurs for $d = 1$ where we are free to construct F_2 from a whole set of test functions $x \mapsto a_0(x)$, for $d \geq 3$ we are left with only $(d^2 + 3d + 2)/2$ free x-independent parameters, namely $\lambda = \big(h_0^\mu(0), k, A^\mu{}_\nu, l^\mu\big)$.

Now the boundaries of the gaps must belong to the solutions of (6). The linearity of F_2 in λ simplify considerably the computations. The conditions (6c) imply (6a) and are equivalent to

$$E = V(x) \; ; \quad \partial_x V = 0 \; . \qquad (76)$$

Condition (6b) is guaranteed if we choose for instance

$$h_0^\mu(0) = -kx_c^\mu + A^\mu{}_\nu x^\nu + \frac{1}{2} l^\mu x_c^2 + x_c^\mu \, l \cdot \qquad (77)$$

and condition (6d) is generically fulfilled.

Therefore, with our method, possible candidates for the gap boundaries are the critical points x_c of V which is not a surprise from a semi-classical point of view. With this method we cannot expect to find more interesting and more relevant piece of information. Actually, some inequalities concerning the global spectrum may be obtained if we are to maintain the sign of (75), specially once a specific V is given; but our ambition was, as we have shown in [2] for $d = 1$, to obtain some local information in the very core of the spectrum.

3 Ending remarks

3.1 Simplified starting point

For $N = 2$ and any d we have shown directly that without loss of generality we could have started with no term in φ in J_2^μ, that is with a current such that

$$\partial_\varphi J_2^\mu\big(\partial\varphi(x), \varphi(x), x, E\big) = 0 \; . \qquad (78)$$

Indeed, the h_0^μ are φ independent, see (53), and we could have taken $\tilde{h}_1 = 0$ with no consequence on the result (67). For $d = 1$ and any N, we have proven this result through (19): we obtain all the possible F_N's even if we restrict our self to $a = (a_0, 0, \cdots)$. This result can be also obtained for any d and any N in another way. First remark that if we start with $J_N^\mu\big(\partial\varphi(x), \varphi(x), x, E\big)$ the second derivatives in

$$D_\mu J_N^\mu = \partial_\mu J_N^\mu + \partial_\varphi J_N^\mu \, \partial_\mu\varphi + \partial_{\partial_\nu\varphi} J_N^\mu \, \partial_{\mu\nu}\varphi \qquad (79)$$

can be eliminated with the help of (39) if and only if

$$\partial_{\partial_\nu\varphi} J_N^\mu = L\delta^\mu_\nu + W^\mu{}_\nu \qquad (80)$$

where δ is the Kronecker symbol, $W\big(\partial\varphi(x), \varphi(x), x, E\big)$ an antisymmetric $d \times d$ matrix and $L\big(\partial\varphi(x), \varphi(x), x, E\big)$ a function. Then after the substitution of (39), an integration by part can be made and we have

$$D_\mu J_N^\mu = \partial_\mu J_N^\mu + Lv\varphi - D_\mu(\partial_\varphi J_N^\mu)\varphi + D_\mu(\partial_\varphi J_N^\mu) \; . \qquad (81)$$

By computing $D_\mu(\partial_\varphi J_N^\mu)$ in the same way, and iterating the process up to infinity, we find that $D_\mu J_N^\mu$ can be written like

$$D_\mu J_N^\mu = v\varphi \sum_{n=0}^\infty \frac{(-\varphi)^n}{n!} \partial_\varphi^n L + \sum_{n=0}^\infty \frac{(-\varphi)^n}{n!} \partial_\varphi^n \partial_\mu J_N^\mu + D_\mu\left(J_N^\mu \varphi \sum_{n=0}^\infty \frac{(-\varphi)^n}{n!} \partial_\varphi^n J_N^\mu\right) \; ; . \qquad (82)$$

Up to a total derivative, starting from any J_N, we therefore are always led to

$$D_\mu\left(\Pi_0 J_N^\mu\right) = v\varphi\Pi_0 L + \Pi_0 \partial_\mu J_N^\mu \tag{83}$$

where Π_0 in the evaluation at $\varphi = 0$ that can be expressed as

$$\Pi_0 = \sum_{n=0}^\infty \frac{(-\varphi)^n}{n!}\, \partial_\varphi^n \tag{84}$$

for any analytic function of φ. Therefore even if we start with a J_N whose component takes the general form (40), working up to divergence terms, we will be led to the same identity as if we had started with all the a_n such that $N(1-d) + |n| > 0$ being zero (this is consistent with (19)). In $N = 2$, all the g's in (45) could have been taken independent of φ from the very beginning.

3.2 Remark about condition $(\mathrm{i})'_d$

For $d \neq 2$ and $N = 2$, condition $(\mathrm{i})'_d$ with

$$B_1 = b_1^\mu \partial_\mu \varphi + b_0 \varphi \tag{85}$$

where $\left(b_1 = (b_1^\mu)_\mu, b_0\right)$ are $d+1$ smooth real functions of x, does not provide F_2 explicitly but rather leads to a linear differential equation for it. For instance, when $d = 1$ we get

$$\frac{1}{2}\, b_1^2 F_2'' + (2b_1 b_1' - b_1 b_0)F_2' + \left[(b_1 b_1')' - vb_1^2 - (b_1 b_0)' + b_0^2\right]F_2 = \frac{1}{2}a_0''' - 2va_0' - v'a_0 \tag{86}$$

which reduces to (3b) when we make the simplest choice $b_0 = 1$ and $b_1 = 0$. For $d \geq 1$, by reproducing the same line of reasoning as in § 2.1, we get a second order linear partial differential equation for F_2

$$\frac{1}{2}\, (b_1)^2 \Delta_d F_2 + \left(\frac{2}{d}\, b_1 \cdot \partial_\mu b_1 - b_1^\mu b_0\right)\partial_\mu F_2 + \left[\frac{1}{2d}\Delta_d (b_1)^2 - \frac{1}{d}\, v(b_1)^2 - \partial_\mu(b_1^\mu b_0) + b_0^2\right]F_2$$
$$= \frac{2-d}{2d}\, \Delta_d \partial_\mu h_0^\mu - \frac{2}{d}\, v\, \partial_\mu h_0^\mu - h_0^\mu \partial_\mu v\ . \tag{87}$$

instead of (67) obtained for $b_0 = 1$ and $b_1 = 0$. The constraints on h_0^μ now involve the b's and are entangled with F_2:

$$\partial_1 h_0^1 - \frac{1}{2}\, (b_1^1)^2 F_2 = \partial_2 h_0^2 - \frac{1}{2}\, (b_1^2)^2 F_2 = \cdots = \partial_d h_0^d - \frac{1}{2}\, (b_1^d)^2 F_2\ ; \tag{88a}$$

$$\partial_\mu h_0^\nu + \partial_\nu h_0^\mu = b_1^\mu b_1^\nu F_2 \qquad (\mu \neq \nu)\ . \tag{88b}$$

For $d \geq 1$, the only way to get rid of F_2 from (88a) is to take b_1^μ independent of μ, and eventually $b_1 = 0$ if we want (88b) not to involve F_2 either. Then, since F_2 and $b_0^2 F_2$ have the same sign, (87) take us back to (67) that is to case $(\mathrm{i})_d$ where $b_0 = 1$.

Even for $d = 1$ and a specific v, I did not exploit further these possibilities, but the choice of b's for which (86) can be solved explicitly is rather limited (not to speak of the control of the sign of its solutions). In any case of course, we ought to work with simpler differential equations than the Schrödinger equation itself !

3.3 Attempt for $d = 2, N = 4$

The second remark concerns an attempt to obtain $(\mathrm{i})_d$ for $d = 2$ and $N = 4$. Using the same ideas as in the case $N = 2$ and with

$$J_4^\mu = \sum_{n=0}^4 g_{4-m,m}^\mu\, (\partial_1 \varphi)^{4-n}\, (\partial_2 \varphi)^n \tag{89}$$

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with, according to the last remark, taking g as φ-independent. The constraints (80) imposed by the elimination of the second derivatives of φ in $D_\mu J_4^\mu$ lead to two independent functions instead of the ten g's. But if we go further to eliminate the cubic terms in $\partial\varphi$, these functions must vanish identically and no non zero g can be found this way. Extending $(i)_d$ to $(i)'_d$ with

$$B_2 = b_{20}\,(\partial_1\varphi)^2 + b_{11}\,\partial_1\varphi\,\partial_2\,\varphi + b_{02}\,(\partial_2\varphi)^2 + b_{10}\,\varphi\,\partial_1\varphi + b_{01}\,\varphi\,\partial_2\varphi + b_{00}\varphi^2 \tag{90}$$

where the b's are smooth function of x, leads to the same conclusion.

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