A Graph Representation for Two-Dimensional Finite Type Constrained Systems

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Abstract—The demand of two-dimensional source coding and constrained coding has been getting higher these days, but compared to the one-dimensional case, many problems have remained open as the analysis is cumbersome. A main reason for that would be because there are no graph representations discovered so far. In this paper, we focus on a two-dimensional finite type constrained system, a set of two-dimensional blocks characterized by a finite number of two-dimensional constraints, and propose its graph representation. We then show how to generate an element of the two-dimensional finite type constrained system from the graph representation.

I. Introduction

In one-dimensional source coding, a graph representation (e.g., labelled directed graph for a given string) or a finite-state source (e.g., Markov source) is utilized as a probabilistic model. The graph representation is useful not only for data compression in practical but also for analysis of the data compression [1]. Moreover, in one-dimensional channel coding, constrained coding is utilized for reducing the likelihood of errors by removing data sequences that can be easily affected by the predictable noise. The study of one-dimensional constrained coding is based on the study of one-dimensional constrained systems, which can be represented by labelled directed graphs. Indeed, many important results are derived by analyzing the characteristics of those graphs [2].

On the other hand, the study of two or higher dimensional cases is cumbersome, and many important problems (*e.g.*, probabilistic models, the capacity, the existence of infinite arrays not containing forbidden patterns) are still open or known to be unsolvable in finite steps in general [3], [4]. The main reason for this would be because there are no explicit way to present such high-dimensional source and constrained coding using finite graphs [4]. Conversely, if some graph representation is proposed, then such a representation can be a breakthrough to approach open problems.

In this paper, for a given finite set \mathcal{F} of two-dimensional forbidden blocks (forbidden set), we focus on the two-dimensional Finite Type Constrained System (2D-FTCS) which is a set of blocks that do not contain any forbidden block in \mathcal{F} as subblock. We construct a labelled directed graph based on forbidden blocks and then show how to generate an arbitrary block in the 2D-FTCS from the graph. Thus, the existence of such a graph presentation can be used to answer the existence of blocks in the 2D-FTCS, and preferably, to explicitly compute its capacity.

The rest of the paper is organized as follows. We first give in Section II basic notations and definitions. In Section III, we propose a labelled graph representation for a 2D-FTCS and prove that the graph representation generates a block if and only if the block is an element of the 2D-FTCS. We then show in Section IV a process to generate an arbitrary block in the 2D-FTCS from the graph. We terminate this paper with conclusion and future works in Section V.

II. BASIC NOTATIONS AND DEFINITIONS

A. Alphabet and Block

Let Σ be an alphabet, a finite set of symbols. We denote by $\Sigma^{(m,n)}$ the set of $m \times n$ finite blocks $\boldsymbol{b} = (b_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ over Σ , where $b_{i,j} \in \Sigma$ is the element of \boldsymbol{b} at (i,j)-coordinate. For simplicity, we always assume that blocks are finite. Furthermore, define $\Sigma^{(*,*)} = \cup_{m,n \geq 0} \Sigma^{(m,n)}$, where $\Sigma^{(m,n)}$ consists only of the empty block $\boldsymbol{\lambda}$ when at least one of m and n is 0. Given a block $\boldsymbol{b} \in \Sigma^{(*,*)}$, denote by $|\boldsymbol{b}|_r$ and $|\boldsymbol{b}|_c$ the length of row (the beight) and the length of column (the width), respectively. For example, when $\Sigma = \{0,1\}$, Fig. 1 illustrates $\boldsymbol{b} \in \Sigma^{(3,3)}$ where $|\boldsymbol{b}|_r = |\boldsymbol{b}|_c = 3$.

0	0	0
0	0	1
1	0	0

	0		0	0]							
U	U		U	0		0	0	0		0	0	1
0	0	,	0	1	,		-		,		-	
1	0		0	0		0	0	1		1	0	0
1	U	1	0	0								

Fig. 1. A 3×3 Fig. 2. $\pi_c(b)$, $\sigma_c(b)$, $\pi_r(b)$, and $\sigma_r(b)$ of b in Fig. 1. block b.

B. Subblock, Prefix, Suffix and Concatenation

Throughout this subsection, we always let $\boldsymbol{b}=(b_{i,j})\in \Sigma^{(m,n)}$ and $\boldsymbol{b}'=(b'_{i',j'})\in \Sigma^{(m',n')}$ be two blocks of size $m\times n$ and $m'\times n'$, respectively.

Block b' is called a subblock of b when b' appears within b; that is, $b'_{i',j'} = b_{k+i',l+j'}$ for some fixed non-negative integers k,l and any integers $1 \le i' \le m', 1 \le j' \le n'$. In particular, subblock b' is called a prefix (resp., suffix) of b when k = l = 0 (resp. when k = m - m' and l = n - n'). Among prefixes and suffixes, we often focus on the prefix and the suffix of size $m \times (n-1)$, and the prefix and the suffix of size $(m-1) \times n$ which we denote by $\pi_c(b)$, $\sigma_c(b)$, $\pi_r(b)$, $\sigma_r(b)$, respectively. For example, for b in Fig. 1, Fig. 2 shows $\pi_c(b)$, $\sigma_c(b)$, $\pi_r(b)$, and $\sigma_r(b)$ from the left-hand side.

For blocks b and b', when m=m', we can consider the block $[b,b']_c \in \Sigma^{(m,n+n')}$ generated by column-wisely concatenating b and b'. Similarly, when n=n', we can

consider the block $[b, b']_r \in \Sigma^{(m+m',n)}$ generated by rowwisely concatenating b and b'.

C. Two-Dimensional Finite Type Constrained Systems

Let h, w be some fixed non-negative integers. Given a finite set $\mathcal{F} \subset \Sigma^{(h,w)}$, we define a *Two-Dimensional Finite Type Constrained System (2D-FTCS)* $\mathcal{S}_{\mathcal{F}}$ to be a subset of $\Sigma^{(*,*)}$ which can be characterized by \mathcal{F} . More precisely, for a 2D-FTCS $\mathcal{S}_{\mathcal{F}}$, a block \boldsymbol{b} is an element of $\mathcal{S}_{\mathcal{F}}$ if and only if \boldsymbol{b} does not contain any block $\boldsymbol{f} \in \mathcal{F}$ as subblock. We call \mathcal{F} a *forbidden set* and an element $\boldsymbol{f} \in \mathcal{F}$ a *forbidden block*.

We note that for one-dimensional case, the definition of an 1D-FTCS in general starts with the definition of a constrained system for which the existence of a graph representation (called *presentation*) matters. We do not refer it at this moment, but we will discuss it in latter sections which shows that out definition is a natural extension of the definition of a 1D-FTCS.

When a forbidden set $\mathcal{F} \subset \Sigma^{(h,w)}$ is given, we define the allowed set $\mathcal{A}_{\mathcal{F}} \subset \Sigma^{(h,w)}$ for \mathcal{F} as $\mathcal{A}_{\mathcal{F}} := \Sigma^{(h,w)} \setminus \mathcal{F}$. We further define the modified 2D-FTCS $\mathcal{S}_{\mathcal{F}}^{(*,*)}$ from a 2D-FTCS $\mathcal{S}_{\mathcal{F}}$ so that

$$\mathcal{S}_{\mathcal{F}}^{(*,*)} := \mathcal{S}_{\mathcal{F}} \cap \left(\bigcup_{m \ge h, n \ge w} \Sigma^{(m,n)} \right) \tag{1}$$

For example, consider the 2D-FTCS $\mathcal{S}_{\mathcal{F}}$ characterized by \mathcal{F} in Fig. 3, which is well-known as the hard-square constraint [4], forbidding the appearance of $[1,1]_r$ and $[1,1]_c$. Fig. 4 is the allowed set $\mathcal{A}_{\mathcal{F}}$ for \mathcal{F} . Observe that block \boldsymbol{b} in Fig. 1 is an element of $\mathcal{S}_{\mathcal{F}}$.

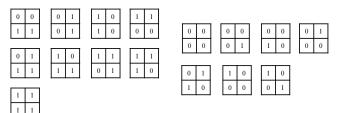


Fig. 4. Blocks in the allowed set $\mathcal{A}_{\mathcal{F}}$ for

Fig. 3. Forbidden blocks in \mathcal{F} .

III. A GRAPH REPRESENTATION FOR 2D-FTCSS

The main contribution of this paper is to propose a graph representation for 2D-FTCS $\mathcal{S}_{\mathcal{F}}$. We only focus on graph representations for $\mathcal{S}_{\mathcal{F}}^{(*,*)}$, but observe that it is enough since $\mathcal{S}_{\mathcal{F}} \setminus \mathcal{S}_{\mathcal{F}}^{(*,*)}$ is a finite set.

A. Notations and Definitions

Let $G=(V,E,\ell)$ be a labelled directed graph consisting of vertex set V, labelled edge set $E\subset V\times V$ and an edge labelling $\ell:E\to \Sigma^{(*,*)}$.

Now suppose a 2D-FTCS $\mathcal{S}=\mathcal{S}_{\mathcal{F}}$ characterized by finite forbidden set $\mathcal{F}\subset \Sigma^{(h,w)}$ is given. We construct labelled directed graphs $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ from \mathcal{S} as follows.

¹The definition of 2D-FTCSs can vary, depending on authors. Our definition covers more general case compared with the definition in [5].

Definition 1 Given a 2D-FTCS $S = S_F$ characterized by $F \subset \Sigma^{(h,w)}$, the row-wise presentation $G_r(S) = (V_r, E_r, \ell_r)$ of S is a labelled directed graph satisfying

- V_r is the allowed set $A_{\mathcal{F}}$ for \mathcal{F} ; and
- draw a blue (row-wise) edge from u to v labelled $a (= \ell_r(u, v)) \in \Sigma^{(1,w)}$ (i.e., $u \xrightarrow{a} v$) if and only if $\sigma_c(u) = \pi_c(v)$ and the h-th row of v is a.

Similarly, the column-wise presentation $G_c(S) = (V_c, E_c, \ell_c)$ of S is a labelled directed graph satisfying

- V_c is the allowed set $A_{\mathcal{F}}$ for \mathcal{F} ; and
- draw a red (column-wise) edge from u to v labelled $a (= \ell_c(u, v)) \in \Sigma^{(h,1)}$ (i.e., $u \xrightarrow{a} v$) if and only if $\sigma_r(u) = \pi_r(v)$ and the w-th column of v is a.

For simplicity, we omit edge labels. Furthermore, we identify a vertex $v \in \mathcal{A}_{\mathcal{F}}$ with its *identifier* g(v); a positive integer defined via bijection $g: \mathcal{A}_{\mathcal{F}} \to \{1, 2, \dots, ||\mathcal{A}_{\mathcal{F}}||\}$, where $||\mathcal{A}_{\mathcal{F}}||$ is the cardinality of $\mathcal{A}_{\mathcal{F}}$. For example, Fig. 5 and Fig. 6 show $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ for $\mathcal{A}_{\mathcal{F}}$ in Fig. 4, respectively, and the number in a circle represents its identifier.

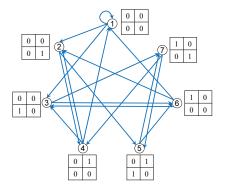


Fig. 5. The row-wise presentation $G_r(S)$.

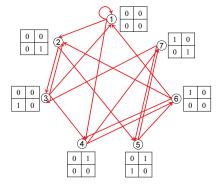


Fig. 6. The column-wise presentation $G_c(\mathcal{S})$.

Now take an arbitrary block x of height $|x|_r = h' > h$ and width w. We say a path $\eta: u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_{h'-h}$ in $G_r(\mathcal{S}) = (V_r, E_r, \ell_r)$ generates the block x if concatenating $\ell_r(u_i, u_{i+1})$ in the order of $i = 0, 1, \ldots, h' - h - 1$ after u_0 (the head block) in row-wise generates x. In other words, path η generates block x if

$$x = [u_0, \ell_r(u_0, u_1), \ell_c(u_1, u_2), \dots, \ell_r(u_{h'-h-1}, u_{h'-h})]_r.$$

If such a path η exists, we say $G_r(S)$ generates block x. By convention, $G_r(\mathcal{S})$ generates any $h \times w$ block $x \in \mathcal{A}_{\mathcal{F}}$, assuming path η in $G_r(S)$ consisting only of x.

Similarly, for a block y of height h and width $|y|_c$ (= w') > w, we say a path τ : $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{w'-w}$ in $G_c(\mathcal{S}) = (V_c, E_c, \ell_c)$ generates the block y if concatenating $\ell_c(\boldsymbol{v}_i, \boldsymbol{v}_{i+1})$ in the order of $j = 0, 1, \dots, w' - w - 1$ after the head block v_0 in column-wise generates y. In other words, path τ generates block u if

$$y = [v_0, \ell_c(v_0, v_1), \ell_c(v_1, v_2), \dots, \ell_c(v_{w'-w-1}, v_{w'-w})]_c.$$

If such a path τ exists, we say $G_c(S)$ generates block y. By convention, $G_c(S)$ generates any $h \times w$ block $y \in A_F$, assuming path τ in $G_c(S)$ consisting only of y.

Remark 1 From the definitions of $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$, it is straightforward to check that $G_r(\mathcal{S})$ generates a block x of width w if and only if each $h \times w$ subblock of x is in $\mathcal{A}_{\mathcal{F}}$, which is equivalent to say $x \in \mathcal{S}_{\mathcal{F}}^{(*,w)} := \mathcal{S}_{\mathcal{F}} \cap \left(\cup_{m \geq h} \Sigma^{(m,w)} \right)$. Similarly, $G_c(S)$ generates a block y of height h if and only if each $h \times w$ subblock of y is in $\mathcal{A}_{\mathcal{F}}$, which is equivalent to say $y \in \mathcal{S}_{\mathcal{F}}^{(h,*)} := \mathcal{S}_{\mathcal{F}} \cap \left(\cup_{n \geq w} \Sigma^{(h,n)} \right)$.

We can naturally extend the notion of generating a certain size of block to the notion of generating any block b of height $|\boldsymbol{b}|_r \geq h$ and width $|\boldsymbol{b}|_r \geq w$ as follows.

Definition 2 Given a block b of height $|b|_r \geq h$ and width $|\mathbf{b}|_r \geq w$, we say $G_r(\mathcal{S})$ generates \mathbf{b} if $G_r(\mathcal{S})$ can generate any $|b|_r \times w$ subblock of b, and $G_c(\mathcal{S})$ generates b if $G_c(\mathcal{S})$ can generate any $h \times |\mathbf{b}|_c$ subblock of \mathbf{b} .

From Definition 2, if $G_r(S)$ and $G_c(S)$ generates **b**, then any $h \times w$ subblock in **b** is an element of the allowed set, and hence, a vertex in $G_r(S)$ and $G_c(S)$.

B. Relationship between $G_r(S)$ and $G_c(S)$

Now take any block **b** such that $G_r(S)$ and $G_c(S)$ can generate. For a $h \times w$ subblock v in b whose right-bottom coordinate is (i,j) $(h \le i \le |\mathbf{b}|_r, w \le j \le |\mathbf{b}|_c)$, let s(i,j) be the identifier g(v) of the vertex v. Since any such subblock vis a vertex in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$, for given s(i-1,j-1) and s(i, j), where $k < i \le |\mathbf{b}|_r, l < j \le |\mathbf{b}|_c$, there must exist two paths from s(i-1, j-1) to s(i, j)

$$\begin{array}{ll} \textbf{1)} & s(i-1,j-1) {\longrightarrow} s(i-1,j) {\longrightarrow} s(i,j) \\ \textbf{2)} & s(i-1,j-1) {\longrightarrow} s(i,j-1) {\longrightarrow} s(i,j) \end{array}$$

as shown in Fig. 7. For example, consider b in Fig. 1, and $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ in Figs. 5 and 6. For s(2,2)=1 and s(3,3) = 4, there are two paths; $1 \rightarrow 2 (= s(2,3)) \rightarrow 4$ and $1 \rightarrow 3 (= s(3,2)) \rightarrow 4.$

For $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ in Figs. 5 and 6, Fig. 8 shows all the combinations of paths among four adjacent vertices in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$. Four numbers at left-top, right-top, left-bottom and right-bottom in a rectangle are represented by s(i-1, j-1), s(i-1,j), s(i,j-1), and s(i,j), respectively. For example, the first rectangle in the third row in Fig. 8, s(i-1, j-1), s(i-1,j), s(i,j-1) and s(i,j) are given by 2, 5, 4, and 6, respectively. In this example, there are total 63 combinations of paths among four adjacent vertices in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$.

Fig. 9 illustrates two combinations of paths among four adjacent vertices, (1,2,1,4) and (2,5,4,6), in $G_c(\mathcal{S})$ where

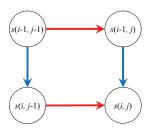


Fig. 7. Paths among four adjacent vertices in $G_r(S)$ and $G_c(S)$.

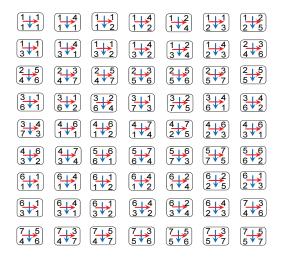


Fig. 8. An example of all the combinations of paths among four adjacent vertices in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$.

(a, b, c, d) represents a = s(i - 1, j - 1), b = s(i - 1, j), c =s(i, j-1), and d = s(i, j). Light green and orange rectangles represent (1, 2, 1, 4) and (2, 5, 4, 6), respectively.

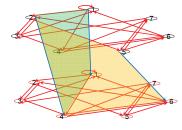


Fig. 9. Two combinations of paths among four adjacent vertices, (1, 2, 1, 4)and (2, 5, 4, 6), in $G_c(S)$.

C. Graph Representation and its Paths

We are now in a position of combining the row-wise presentation $G_r(\mathcal{S})$ and the column-wise presentation $G_c(\mathcal{S})$ to obtain a graph representation G(S).

Definition 3 We define a graph representation G(S) = $(V_{\mathcal{S}}, E_{\mathcal{S}}, \ell_{\mathcal{S}})$ of a 2D-FTCS $\mathcal{S} = \mathcal{S}_{\mathcal{F}}$ so that

- $V_{\mathcal{S}} = V_r = V_c = \mathcal{A}_{\mathcal{F}}.$
- $E_{\mathcal{S}} = E_r \cup E_c$.
 - $\ell_{\mathcal{S}}(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} \ell_r(\boldsymbol{u}, \boldsymbol{v}) & (\text{if } (\boldsymbol{u}, \boldsymbol{v}) \in E_r) \\ \ell_c(\boldsymbol{u}, \boldsymbol{v}) & (\text{if } (\boldsymbol{u}, \boldsymbol{v}) \in E_c) \end{cases}$

Thus, G(S) is easily obtained from $G_r(S)$ and $G_c(S)$ by combining them, so $G_r(S)$ and $G_c(S)$ are subgraphs of G(S). We define that G(S) generates \boldsymbol{b} as follows.

Definition 4 Let $S = S_F$ be a 2D-FTCS characterized by $F \subset \Sigma^{(h,w)}$. We say graph representation G(S) generates block b if

- any $|b|_r \times w$ subblock b' of b is generated by a path in G(S) consisting only of blue edges (and hence, b' is generated by a subgraph $G_r(S)$); and
- any $h \times |\mathbf{b}|_c$ subblock $\hat{\mathbf{b}}$ of \mathbf{b} is generated by a path in $G(\mathcal{S})$ consisting only of red edges (and hence, $\hat{\mathbf{b}}$ is generated by a subgraph $G_c(\mathcal{S})$).

Remark 2 For paths used to generate b, it is important to observe that they maintain the following conditions in terms of s(i, j).

Case 1: i = h or j = w.

Case 2: i > h and j > w.

<u>Case 1:</u> When i=h, a prefix of ${\bf b}$ of size $h\times |{\bf b}|_c$ is generated by $G_c(\mathcal{S})$. In this case, since i-1< h, s(i-1,j-1) does not exist. Hence, a path in $G_c(\mathcal{S})$ is not restricted by paths among four adjacent vertices in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ shown in Fig. 7. Moreover, for each $w\leq j\leq |{\bf b}|_c$, s(h,j) turns out to be the head block of the path τ in $G_r(\mathcal{S})$ generating a $|{\bf b}|_r\times w$ subblock of ${\bf b}$ whose right-bottom coordinate is $(|{\bf b}|_r,j)$.

Similarly, when j=w, a prefix of \mathbf{b} of size $|\mathbf{b}|_r \times w$ is generated by $G_r(\mathcal{S})$. In this case, since j-1 < w, s(i-1,j-1) does not exist. Hence, a path in $G_r(\mathcal{S})$ is not restricted by paths among four adjacent vertices in $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$ shown in Fig. 7. Moreover, for each $h \leq i \leq |\mathbf{b}|_r$, s(i,w) turns out to be the head block of the path η in $G_c(\mathcal{S})$ generating a $h \times |\mathbf{b}|_c$ subblock of \mathbf{b} whose right-bottom coordinate is $(i, |\mathbf{b}|_c)$.

<u>Case 2:</u> From paths among four adjacent vertices in $G_r(S)$ and $G_c(S)$ shown in Fig. 7, s(i-1,j-1), s(i-1,j), and s(i,j-1) are required to determine a path to s(i,j). Moreover, there must exist two paths from s(i-1,j-1) to s(i,j), those are $s(i-1,j-1) \rightarrow s(i-1,j) \rightarrow s(i,j)$ and $s(i-1,j-1) \rightarrow s(i,j-1) \rightarrow s(i,j)$.

We prove Theorem 1 which is a main result of the paper.

Theorem 1 G(S) generates b if and only if $b \in S_F^{(*,*)}$.

Proof: We first prove that if $G(\mathcal{S})$ generates \boldsymbol{b} then $\boldsymbol{b} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$. If $G(\mathcal{S})$ generates \boldsymbol{b} , then any $h \times w$ subblock of \boldsymbol{b} is a vertex in $G(\mathcal{S})$, and therefore, an element of $\mathcal{A}_{\mathcal{F}}$. Thus, \boldsymbol{b} does not contain any forbidden block as a subblock, and hence, $\boldsymbol{b} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$.

We next prove that if $\boldsymbol{b} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$ then $G(\mathcal{S})$ generates \boldsymbol{b} . We assume by contradiction that $G(\mathcal{S})$ cannot generate $\boldsymbol{u} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$. Since $\boldsymbol{u} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$, any prefix of \boldsymbol{u} of size $h \times |\boldsymbol{u}|_c$ and any prefix of \boldsymbol{u} of size $|\boldsymbol{u}|_r \times w$ are generated by subgraphs $G_c(\mathcal{S})$ and $G_r(\mathcal{S})$, respectively. Therefore, we have $h < |\boldsymbol{u}|_r$ and $w < |\boldsymbol{u}|_c$ (since otherwise, $G(\mathcal{S})$ can generate \boldsymbol{u}), and there exists a prefix \boldsymbol{v} of \boldsymbol{u} such that

$$G(S)$$
 does not generate v , (2)

$$G(S)$$
 generates $\pi_r(\mathbf{v})$, (3)

$$G(S)$$
 generates $\pi_c(\mathbf{v})$.

Observe that (3) and (4) imply $|v|_r > h$ and $|v|_c > w$, and therefore, $v \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$ holds.

Let (i,j) be the right-bottom coordinate of v in u, so the right-bottom coordinate of $\pi_r(v)$ and $\pi_c(v)$ are given by (i-1,j) and (i,j-1), respectively. From (3), the suffix of $\pi_r(v)$ of size $h \times |v|_c$ is an element of $\mathcal{S}_{\mathcal{F}}^{(h,*)}$, so path $s(i-1,j-1) \rightarrow s(i-1,j)$ exists. Similarly, from (4), the suffix of $\pi_c(v)$ of size $|v|_r \times w$ is an element of $\mathcal{S}_{\mathcal{F}}^{(*,w)}$, so path $s(i-1,j-1) \rightarrow s(i,j-1)$ exists. Moreover, since $v \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$, the suffix of v of size $h \times |v|_c$ is an element of $\mathcal{S}_{\mathcal{F}}^{(h,*)}$ so path $s(i,j-1) \rightarrow s(i,j)$ exists. Similarly, the suffix v of size v of s

Therefore, there exist two paths $s(i-1,j-1) \rightarrow s(i-1,j) \rightarrow s(i,j)$ and $s(i-1,j-1) \rightarrow s(i,j-1) \rightarrow s(i,j)$ shown in Fig. 7. Hence, G(S) generates v, which contradicts the assumption (2) as desired.

We note here a very important remark that the arguments we have done so far hold even though we generate $G_r(S)$ or $G_r(S)$ by allowing their vertex set $\mathcal{A}_{\mathcal{F}}$ to be a multi-set; that is, two or more same blocks can be used as distinct vertices in $G_r(S)$ or $G_r(S)$. That is a key point to generate blocks from G(S) which will be discussed in the next section.

IV. How to Generate Blocks from G(S)

A. Preliminaries for G(S)

From Theorem 1, for $\boldsymbol{b} \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$, a subblock \boldsymbol{y} of size $h \times |\boldsymbol{b}|_c$ is generated by a subgraph $G_c(\mathcal{S})$. So its $h \times w$ prefix \boldsymbol{v} is the head block of the path τ in the $G_c(\mathcal{S})$ generating \boldsymbol{y} . Since there are $||\mathcal{A}_{\mathcal{F}}||$ vertices in $G_c(\mathcal{S})$, we can consider $||\mathcal{A}_{\mathcal{F}}||$ different subgraphs $G_c(\mathcal{S})$'s with respect to the head block \boldsymbol{v} , and define the $g(\boldsymbol{v})$ -class of $G_c(\mathcal{S})$ to be the subgraph $G_c(\mathcal{S})$ each path in which has \boldsymbol{v} as its head block.

Now suppose that for the block v above, its right-bottom coordinate in b is (i-1,w) (i>h), and hence, the right-bottom coordinate of y in b is $(i-1,|b|_c)$. If v' is the $h\times w$ subblock of b whose right-bottom coordinate is (i,w), then v' is the head block of the path τ' generating $h\times |b|_c$ subblock y' of b whose right-bottom coordinate is $(i,|b|_c)$. In other words, the g(v')-class of $G_c(\mathcal{S})$ generates y'. In such a case, we can always find blue edges from the g(v)-class of $G_c(\mathcal{S})$ to the g(v')-class of $G_c(\mathcal{S})$ satisfying condition in Fig. 7, by considering s(i-1,j-1), s(i-1,j) and s(i,j-1), s(i,j) to be vertices in g(v)-class of $G_c(\mathcal{S})$ and vertices in g(v')-class of $G_c(\mathcal{S})$, respectively (except for the case that some vertices have no attached blue edges). We imply such a connection by drawing a blue edge from the g(v)-class of $G_c(\mathcal{S})$ to the g(v')-class of $G_c(\mathcal{S})$.

Fig. 10 illustrates the $g(\boldsymbol{v})$ -classes of $G_c(\mathcal{S})$ and the connections between them in $G(\mathcal{S})$ when $\mathcal{A}_{\mathcal{F}}$ in Fig. 4 is given. The subgraph $G_c(\mathcal{S})$ in a red circle having the number $k(=g(\boldsymbol{v}))$, $1 \leq k \leq 7 = ||\mathcal{A}_{\mathcal{F}}||$, in black circle represents the k-class of $G_c(\mathcal{S})$. Note that it is of course possible to follow the similar argument for $G_r(\mathcal{S})$ and define $||\mathcal{A}_{\mathcal{F}}||$ different $g(\boldsymbol{u})$ -class of $G_r(\mathcal{S})$'s, by considering a subblock \boldsymbol{x} of size $|\boldsymbol{b}|_r \times w$ and the prefix \boldsymbol{u} of \boldsymbol{x} of height h.

B. Process to Generate Blocks in $\mathcal{S}_{\mathcal{F}}^{(*,*)}$

We are now in a position to show how to generate a block $b \in \mathcal{S}_{\mathcal{F}}^{(*,*)}$ by means of the g(v)-classes of $G_c(\mathcal{S})$ and their

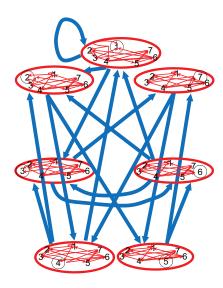


Fig. 10. The q(s)-classes of $G_c(S)$ and their connections in G(S)

connections in G(S). We explain, as an example, a case of generating a 3×5 block \boldsymbol{b} using Fig. 10 to avoid ambiguity. Fig. 11 shows a process to generate \boldsymbol{b} , where newly generated blocks and identifiers are written by bold numbers in each step.

In the first step, pick an arbitrary vertex v in $\mathcal{A}_{\mathcal{F}}$. Say suppose that v is the first block in Fig. 4 with identifier 1. The vertex of identifier 1(=s(2,2)) is the head block for both $G_r(\mathcal{S})$ and $G_c(\mathcal{S})$. Therefore, the 1-class of $G_r(\mathcal{S})$ generates a prefix of b of width w, and the 1-class of $G_c(\mathcal{S})$ generates a prefix of b of height b.

In the second and third steps, s(2,3)=2 and s(3,2)=3 are generated by the 1-class of $G_c(\mathcal{S})$ and $G_r(\mathcal{S})$, respectively. Since s(2(=h),3) and s(3,2(=w))=3, the second and the third step satisfy Case 1 in Remark 2. Moreover, a vertex of identifier 3(=s(3,2)) is the head block for $G_c(\mathcal{S})$, that is the 3-class of $G_c(\mathcal{S})$, which generates a subblock of the second and third rows. The fourth, seventh, and eighth steps satisfy paths among four adjacent vertices shown in Fig. 7 since Case 2 in Remark 2 is satisfied in these steps. For example, in the forth step, (1,2,3,4) at the fifth rectangle in the second row in Fig. 8 is utilized to determine s(3,3)(=4) In the seventh and eighth steps, (2,5,4,7) and (5,6,7,3) are utilized to determine s(3,4)(=7) and s(3,5)(=3), respectively. The fifth and sixth steps satisfy Case 1 in Remark 2 so that they are implemented in the 1-class of $G_c(\mathcal{S})$.

Strong points of our method are

- 1) we can freely select any vertex in $A_{\mathcal{F}}$ in the first step;
- we can freely select a step (so a process is not unique);
- 3) it is possible to make the size of the desired block b bigger in the middle of a process

as long as each step satisfies Case 1 or Case 2 in Remark 2. Thus, we can say that our method is easy to apply and flexible.

V. CONCLUSION

In this paper, we proposed a graph representation for the 2D-FTCS $S = S_F$ (or $S_F^{(*,*)}$ to be more precise) characterized by

Step	Blocks	Identifiers of Vertices
1	0 0 0	Row (Col.) num. 2 3 4 5 2 1 3 4
2	0 0 0 0 0 1	Row (Col.) num. 2 3 4 5 2 1 2 3 4
3	0 0 0 0 0 1 1 0	Row (Col.) num. 2 3 4 5
4	0 0 0 0 0 1 1 0 0	Row (Col.) num. 2 3 4 5 2 1 2 3 3 4 4
5	0 0 0 1 0 0 1 0 1 0 0	Row (Col.) num. 2 3 4 5 2 1 2 5 3 3 4
6	0 0 0 1 0 0 0 1 0 0 1 0 0	Row (Col.) num. 2 3 4 5 6 3 3 4 4
7	0 0 0 1 0 0 0 1 0 0 1 0 0 1	Row (Col.) num. 2 3 4 5 6 3 3 4 7
8	0 0 0 1 0 0 0 1 0 0 1 0 0 1 0	Row (Col.) num. 2 3 4 5 6 3 3 4 7 3 4 7

Fig. 11. A process to generate a block

a finite set of forbidden set \mathcal{F} . More precisely, we proposed a labelled directed graph $G(\mathcal{S})$ obtained from the row-wise and the column-wise presentations, and then proved that the $G(\mathcal{S})$ generates a block if and only if the block is an element of $\mathcal{S}_{\mathcal{F}}^{(*,*)}$. We further explained how to indeed generate an arbitrary block in $\mathcal{S}_{\mathcal{F}}^{(*,*)}$ from $G(\mathcal{S})$. Moreover, the proposed graph representation can be easily extended to three or higher dimensional forbidden set by adding a labelled directed graph such as $G_r(\mathcal{S})$ for an adding axis. As a future work, we hope to apply the the proposed graph for further analysis in two-dimensional case, such as a 2D antidictionary coding which utilizes a subset of 2D antidictionary [6], [7] and the computation of the capacities of 2D-FTCSs.

REFERENCES

- A. Moffat and A. Turpin, Compression and coding algorithms. Kluwer Academic Publishers, 2002.
- [2] D. Lind and B.Marcus, An introduction to symbolic dynamics and coding. Cambridge University Press, 1995.
- [3] D. Lind, "Multi-dimensional symbolic dynamics," in *Proc. of Symposia in Applied Mathematics*, pp. 61–80, January 2002.
- [4] J. Jusesen and S. Forchhammer, Two-Dimensional Information Theory and Coding. Cambridge University Press, 2010.
- [5] E. Louidor, T.-L. Poo, P. Chaichanavong, and B. Marcus, "Maximum insertion rate and capacity of multidimensional constraints," in *Proc. of IEEE 2008 Int. Symp. Inf. Theory*, pp. 1458–1462, July 2008.
- [6] T. Ota and H. Morita, "On a two-dimensional antidictionary construction using suffix tries," in *Proc. of the 2015 IEEE International Symposium on Information Theory*, pp. 2974–2978, June 2015.
 [7] T. Ota and H. Morita, "On a two-dimensional antidictionary coding," in
- [7] T. Ota and H. Morita, "On a two-dimensional antidictionary coding," in Proc. of the 2014 International Symposium on Information Theory and its Applications, pp. 289–293, October 2014.