# Subdiffusive Brownian ratchets rocked by a periodic force

# Igor Goychuk \*

Institute of Physics, University of Augsburg, Universitätsstr. 1, D-86135 Augsburg, Germany

#### **Abstract**

This work puts forward a generalization of the well-known rocking Markovian Brownian ratchets to the realm of antipersistent non-Markovian subdiffusion in viscoelastic media. A periodically forced subdiffusion in a parity-broken ratchet potential is considered within the non-Markovian generalized Langevin equation (GLE) description with a power-law memory kernel  $\eta(t) \propto t^{-\alpha}$  ( $0 < \alpha < 1$ ). It is shown that subdiffusive rectification currents, defined through the mean displacement and subvelocity  $v_{\alpha}$ ,  $\langle \delta x(t) \rangle \sim v_{\alpha} t^{\alpha} / \Gamma(1+\alpha)$ , emerge asymptotically due to the breaking of the detailed balance symmetry by driving. The asymptotic exponent is  $\alpha$ , the same as for free subdiffusion,  $\langle \delta x^2(t) \rangle \propto t^{\alpha}$ . However, a transient to this regime with some time-dependent  $\alpha_{\rm eff}(t)$  gradually decaying in time,  $\alpha \leq \alpha_{\rm eff}(t) \leq 1$ , can be very slow depending on the barrier height and the driving field strength. In striking contrast to its normal diffusion counterpart, the anomalous rectification current is absent asymptotically in the limit of adiabatic driving with frequency  $\Omega \to 0$ , displaying a resonance like dependence on the driving frequency. However, an anomalous current inversion occurs for a sufficiently fast driving, like in the normal diffusion case. In the lowest order of the driving field, such a rectification current presents a quadratic response effect. Beyond perturbation regime it exhibits a broad maximum versus the driving field strength. Moreover, anomalous current exhibits a maximum versus the potential amplitude.

Keywords: anomalous diffusion, viscoelasticity, 1/f noise, Brownian ratchets

# 1. Introduction

The phenomenon of directed current in unbiased on average periodic potentials, such as one in Fig. 1, due to a violation of the thermal detailed balance symmetry, - the so-called ratchet effect [1-6] - produced a huge literature (see, e.g. Refs. [7, 8] for recent reviews and further references). It was studied at length, inspired in particular by some biological molecular motors (e.g. one-headed kinesins C351) [9-12]. At the same time, the fascinating topic of anomalous diffusion [13–17] and, in particular, subdiffusion, where the position variance grows sublinearly in time, i.e.  $\langle \delta x^2(t) \rangle \sim t^{\alpha}$  with  $0 < \alpha < 1$ , gained dramatically in the importance and interest. This interest is promoted not in the last line by the emerging evidence for subdiffusion in such complex media as actin filament networks, other protein solutions, interiors of biological cells [18–30], as well

Whether thermal ratchets based on anomalous subdiffusion are possible or not presents a highly nontrivial challenge, even on the level of basic and somewhat oversimplified models. In particular, a periodically forced continuous time random walk (CTRW) subdiffusion and the associated fractional Fokker-Planck dynamics [39-42] turned out to be not sensitive to the action of time-periodic fields in the long time limit with an exception of enhancement of unbiased subdiffusion [41–43]. This fact seems to rule out the very possibility of rocking (i.e. fluctuating tilt [1, 2]) subdiffusive ratchets based on the CTRW mechanism [13-15, 44, 45], and possibly also on the quenched disorder [13, 46], leaving, however, a door for the flashing (i.e. pulsating potential [4, 6]) subdiffusive ratchets based on these mechanisms open [47].

as in the conformational dynamics of protein macromolecules [31–38]. At the moment, subdiffusion is experimentally firmly established in different media with different  $\alpha$ 's in the range of 0.25 – 0.9. It is thus natural that these two paradigms-breaking(-and-forming) research lines are currently crossing.

<sup>\*</sup>Tel.:+49-821-598-3235, Fax:+49-821-598-3222

Email addresses: igor.goychuk@physik.uni-augsburg.de (Igor Goychuk ), www.physik.uni-augsburg.de/~igor (Igor Goychuk )

In biological applications, the subdiffusion scenario based on the alternative to CTRW, fractional Brownian motion (FBM) [48, 49] and associated generalized Langevin equation (GLE) descriptions [36, 50–53] can, however, be more relevant [29, 54], in particular, because of such a subdiffusion is ergodic [54, 55]. It does not require a quasi-infinite mean residence time in any tiny spatial domain, or trap [54], and in contrast with the CTRW subdiffusion [56] does not suffer from such nonergodic features as spontaneous immobilization of some particles for the time of observation [45]. Given a small number of functionally specific (sub)diffusing macromolecules, a sudden "non-ergodic" standstill could have fatal consequences for the cell functioning.

However, what is the physical origin of GLE subdiffusion? In complex environments, such as interior of biological cells densely stuffed with different macromolecules, including actin filaments, either mobile, or building up the cytoskeleton networks, static, as in eucaryots, or dynamically flickering, as in bacteria, it can be due to long range negative correlations in the diffusing particle displacements (and velocity alternations) caused by crowding. A particle moving with an instant velocity v(t) in certain direction invokes besides a viscous drag also a quasi-elastic response of environment opposing the motion. The total dissipative force can be written in linear approximation as  $F_{v-el}(t)$  =  $-\int_{-\infty}^{t} \eta(t-t')v(t')dt'$  with a memory friction kernel  $\eta(t)$ . For an oscillatory motion,  $v(t) \sim \hat{v}(\omega) \exp(-i\omega t) + c.c$ , with frequency  $\omega$ , the friction force becomes frequencydependent, with amplitude  $\hat{F}_{v-el}(\omega) = -\hat{\eta}(\omega)\hat{v}(\omega)$ , where  $\hat{\eta}(\omega) = \int_0^\infty \eta(t) \exp(i\omega t) dt$  is the frequency-dependent friction <sup>1</sup>. The subdiffusive behavior can emerge for some special forms of  $\eta(t)$  (see below) when this friction diverges at zero-frequency, i.e.  $\hat{\eta}(0) =$  $\int_0^\infty \eta(t)dt \to \infty$ , or also on some sufficiently long transient time scale, when  $\hat{\eta}(0)$  is large but finite, which is a more realistic assumption. Furthermore, the fluctuationdissipation theorem (FDT) dictates that at thermal equilibrium the corresponding energy loss due to dissipation must always be compensated on average by an energy gain due to unbiased (on average) random force of environment  $\xi(t)$ . For a particle of mass m initially localized  $(v(t) = 0 \text{ for } t < t_0 = 0) \text{ this leads to GLE description}$ [58–63],

$$m\ddot{x} + \int_0^t \eta(t - t')\dot{x}(t')dt' + \frac{\partial V(x, t)}{\partial x} = \xi(t) , \quad (1)$$

where the memory kernel and the autocorrelation function of noise are related by the fluctuation-dissipation relation (FDR) [58]

$$\langle \xi(t)\xi(t')\rangle = k_B T \eta(|t-t'|) \tag{2}$$

at temperature T. The thermal random force  $\xi(t)$  has necessarily Gaussian statistics within the *linear* friction approximation considered here [64], i.e. within *linear* FDT, but *not* necessarily otherwise [65].

In the following, V(x,t) = U(x) - xf(t), includes some periodic spatial potential U(x) = U(x+L) with period L. It is modulated in time by an external force f(t). In the numerical simulations below it will be considered harmonic,  $f(t) = A \cos(\Omega t)$ , with amplitude A and angular frequency  $\Omega$ . Apart from above phenomenological justification, the GLE description can be derived from the Hamiltonian dynamics [58, 66], i.e. from the first principles. All this makes the GLE approach to anomalous diffusion and relaxation processes ever more attractive. For spatially and temporally unbiased (on average) force fields  $f(x,t) = -\partial V(x,t)/\partial x$ , a rectification current (i.e. ratchet effect) can emerge if only the symmetry of detailed balance is destroyed by an external driving [7] which provides also an energy supply to drive the rectified dissipative motion in certain direction.

A popular model of viscoelasticity with

$$\eta(t) = \eta_{\alpha} t^{-\alpha} / \Gamma(1 - \alpha) , \qquad (3)$$

where  $0 < \alpha < 1$  (the factor  $\Gamma(1-\alpha)$  is to relate our description with others<sup>2</sup>), was introduced by Gemant <sup>3</sup>[57]. It allows to rationalize the Cole-Cole dielectric response of harmonically bound,  $U(x) = kx^2/2$ , overdamped particles in the inertialess limit,  $m \to 0$  [68, 69]. Moreover, this model corresponds to the so-called sub-Ohmic spectral bath density  $J(\omega) \propto \eta_\alpha \omega^\alpha$  in the language of the Hamiltonian system-bath description leading to GLE upon integration over the initially canonically distributed bath variables at temperature T. The corresponding relation between  $\eta(t)$  and  $J(\omega)$  is  $\eta(t) = (2/\pi) \int_0^\infty d\omega J(\omega) \cos(\omega t)/\omega$  [62].

<sup>&</sup>lt;sup>1</sup>It is proportional to a frequency-dependent viscosity,  $\hat{\zeta}(\omega)$ ,  $\hat{\eta}(\omega) \propto \hat{\zeta}(\omega)$ , which was introduced in the theory of viscoelasticity by A. Gemant [57] along with the formalism of fractional derivatives.

<sup>&</sup>lt;sup>2</sup>Then the frictional term of GLE can be abbreviated as  $\eta_{\alpha} \frac{d^{\alpha}x(t)}{dt^{\alpha}}$  making use of the definition of fractional Caputo derivative of the fractional order  $\alpha$  [67]. For this particular kernel, the corresponding GLE is named sometimes *fractional*.

<sup>&</sup>lt;sup>3</sup>More precisely, he introduced a more general model for the complex viscosity which in a particular case yields  $\hat{\zeta}(\omega) = \zeta_{\alpha} \tau_c^{1-\alpha}/[1+(-i\omega\tau_c)^{1-\alpha}]$ . The corresponding  $\eta(t) \propto \zeta(t)$  matches Eq. (3) for  $t \ll \tau_c$ , while for  $t \gg \tau_c$  its asymptotics is  $\eta(t) \propto t^{\alpha-2}$ . In Ref. [68], Cole and Cole have remarked that the choice  $\hat{\zeta}(\omega) = \zeta_{\alpha}/(-i\omega)^{1-\alpha}$  following to Gemant [57] (our notations are different) leads to their now famous form of the dielectrical response function. This corresponds to the limit  $\tau_c \to \infty$ .

Within this model the variance of free subdiffusion, or subdiffusion biased by a constant force evolves in time asymptotically (for  $m \to 0$ , exactly if to assume initial velocities thermally distributed) as [50, 51]

$$\langle \delta x^2(t) \rangle \sim 2D_{\alpha}^{(0)} t^{\alpha} / \Gamma(1+\alpha),$$
 (4)

where  $D_{\alpha}^{(0)}$  is the subdiffusion coefficient, which is related to the anomalous friction coefficient  $\eta_{\alpha}$  and temperature T by the generalized Einstein-Stokes relation,  $D_{\alpha}^{(0)} = k_B T / \eta_{\alpha}$ .

The strict power law kernel represents, however, rather a theoretical abstraction. All the realistic power law kernels have cutoffs. A particular functional form with (upper) incomplete gamma-function,

$$\eta(t) = \alpha \eta_{\alpha} \tau_c^{-\alpha} \Gamma(-\alpha, t/\tau_c) / \Gamma(1 - \alpha), \tag{5}$$

corresponds to the Davidson-Cole dielectrical susceptibility of the harmonically bound particles [69] which is also typical for complex fluids, gels and glasses. For  $t \ll \tau_c$ , this kernel coincides with one in Eq. (3). However, for  $t > \tau_c$  it has an exponential cutoff which makes the zero-frequency friction  $\hat{\eta}(0) = \alpha \eta_{\alpha} \tau_{c}^{1-\alpha}$  finite. In such a more realistic case, subdiffusion occurs on the time scale  $\tau_b < t \ll \tau_c$  and turns over into normal diffusion for  $t > \tau_c$ .  $\tau_c$  can be, however, large enough (e.g. seconds to minutes, as in the interior of biological cells) for the subdiffusion-limited reactions to become important, in particular because of a finite size of biological cells since macromolecules can subdiffuse over the cell volume within some time less than  $\tau_c$ . Initially diffusion is ballistic on the time scale  $0 < t < \tau_b$  due to inertial effects, where ballistic time  $\tau_b$  depends on the details of memory kernel and mass m. It physically corresponds to the relaxation time scale of the momentum (on this time scale the ballistic superdiffusion is persistent). Interestingly enough, a short-time superdiffusion was already experimentally observed in viscoelastic fluids [70].

The noise  $\xi(t)$  corresponding to the model in Eq. (3) is mathematically the fractional Gaussian noise (fGn) by Mandelbrot and van Ness [48]. It provides an important instance of the so-called 1/f noises which encompass noises with a low-frequency power law feature in their power spectrum,  $S(\omega) \propto 1/\omega^{\gamma}, 0 < \gamma < 2[93]$ , here  $\gamma = 1 - \alpha$ . Moreover, in the absence of potential the considered model reproduces in the limit  $m \to 0$  the fractional Brownian motion (FBM) with anti-persistence [48, 49]. The motion in periodic potentials is also antipersistent and subdiffusive [54]. Moreover, it has been recently noticed that for  $\alpha < \alpha_c \approx 0.41$  the inertial effects are generally important [71]. With their inclusion, this model explains also a generalized

Rocard dielectric response [72] of harmonically bound particles ( $\tau_c \to \infty$ ) which can account also for the fast  $\beta$ -relaxation due to the cage effect. A corresponding generalization of the Davidson-Cole succeptibility to include inertial effects can also be readily given (as well as for any other memory kernel and model of anomalous dielectric relaxation and "glassy" behavior).

For particular viscoelastic media, the corresponding memory kernels  $\eta(t)$  can be derived from the rheological measurements [18, 19, 21, 28]. This is a phenomenological viewpoint. Moreover, one can derive many memory kernels from a theoretical approach, e.g. from the mode coupling theory [61] or from the theory of viscoelasticitity of semiflexible polymer entangled networks [73]. The later one yields e.g. a power law kernel with  $\alpha=0.75$  which agrees with some experiments [19, 28], and possesses also a long-time memory cutoff naturally related to the polymer network properties.

Moreover, such a GLE subdiffusion with  $\alpha = 0.5$ has been used to mimic [74] a potential-free singlefile subdiffusion of hard core Brownian particles due to a geometric crowding effect (i.e., a particle cannot move farther than some typical distance until another, road-blocking particle moves) [75-77]. The latter one has been experimentally realized and observed with colloidal particles [78, 79]. Even if the presence of a biasing force acting on all particles clearly destroys this correspondence (since all the particles drift normally in the same direction), one can imagine a situation where an external field acts only on some, e.g., electrically charged guest particles. Then one can expect that this correspondence might hold further and the behavior of the charged tracer particles (e.g. a globular proteins in a suspension of neutral colloidal particles confined in a quasi-one-dimensional geometry) in some electrical potential V(x, t) externally imposed can yet be mimicked by GLE in agreement with [74, 80]. Similar colloidal systems [78, 79] can be considered as plausible candidates for an experimental realization of subdiffusive ratchets with  $\alpha = 0.5$  which we consider in the following for the purpose of illustration. The qualitatively same effects were also found for  $\alpha = 0.75$  and are expected for other values of  $\alpha$  as well.

The emergence of subdiffusive ratchet effect is highly nontrivial. It has been recently shown numerically for sinusoidal potential [54] that such a viscoelastic subdiffusion is asymptotically not sensitive to the amplitude of the periodic potential, in agreement with [81] <sup>4</sup>. This fact might seem to rule out the very possibil-

<sup>&</sup>lt;sup>4</sup>The model in Ref. [81] is quantum-mechanical and it includes

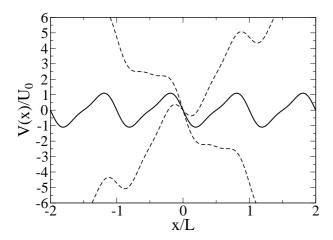


Figure 1: Considered ratchet potential, Eq. (6) without bias and under the critical forward tilt (potential barriers disappear for the motion to the right), as well as under the opposite tilt of the same magnitude. Take a note that the tilt in the opposite direction does not result in a barrierless total potential. This intuitively selects (think about an overdamped particle sliding in the tilted potential) the right direction for the net current in the case of a slow time-periodic forcing, symmetric and unbiased on average. For a smaller rocking force amplitude, the potential barriers in the right sliding direction are typically smaller.

ity of such a rocking ratchet as well. On the contrary, we show in this work that such subdiffusive Brownian ratchets are possible and moreover they display a resonance-like dependence on the driving frequency  $\Omega$ . The rectification current vanishes in the limit of adiabatic rocking, in a clear contrast with the normal diffusion rocking ratchets, where it is maximal [7, 8]. This behavior somewhat resembles pulsing potential ratchets of normal diffusion. The physical mechanisms are, however, different. In particular, our ratchet also exhibits the phenomenon of (sub)current inversion at sufficiently fast driving, similar to its normal diffusion counterpart [2], whereas the subdiffusion coefficient remains rather robust and weakly sensitive to the details of potential and driving, being close to that of free subdiffusion. With a further increase of driving frequency the rectification current gradually vanishes. We shall demonstrate below numerically and explain how this puzzling non-adiabatically rocking, anomalous subdiffusive ratchet mechanism operates.

tunneling effects. In the quantum case, this result is also supported by the so-called duality transformation mapping between the dissipative tight-binding and full potential problems with different bath spectral densities [62]. However, our results show that this puzzling effect is of purely classical origin. It is invoked by the anti-persistent nature of GLE subdiffusion.

#### 2. Theory

For arbitrary potentials, including a typical ratchet potential [2]

$$U(x) = -U_0[\sin(2\pi x/L) + (1/4)\sin(4\pi x/L)]$$
 (6)

with period L and height  $U_0$ , which we consider henceforth, the exact non-Markovian Fokker-Planck equation (NMFPE) which corresponds to the considered GLE model is not known. The cases of linear and parabolic U(x) present the only known exceptions [82–87] which cannot much help in the context of ratchet problem. This, in particular, makes a rigorous analytical study of subdiffusion in nontrivial potentials a highly notrivial problem without exact analytical solutions.

We approach it numerically by embedding the non-Markovian subdiffusive dynamics as a Markovian stochastic dynamics into the phase-space of higher dimensionality following to Ref. [54]. Namely, the considered power law kernel is approximated by a sum of *N*-exponentials,

$$\eta(t) = \sum_{i=0}^{N-1} \eta_i \exp(-\nu_i t),$$
 (7)

with  $v_i = v_0/b^i$  and  $\eta_i = (\eta_\alpha/\Gamma(1-\alpha))C_\alpha(b)v_0^\alpha/b^{i\alpha}$  scaled hierarchically using a dilation parameter b>1. In the theory of anomalous relaxation similar expansions are well-known [13, 88]. In the present context, the approach corresponds to expansion of fractional Gaussian noise into a sum of uncorrelated Ornstein-Uhlenbeck (OU) noises,  $\xi(t) = \sum_{i=0}^{N-1} \zeta_i(t)$ , with autocorrelation functions,  $\langle \zeta_i(t)\zeta_j(t')\rangle = k_BT\eta_i\delta_{ij}\exp(-v_i|t-t'|)$ . Then, the GLE (1) with the memory kernel in (7) can be obtained by eliminating the auxiliary variables  $u_i(t)$  from the following Markovian stochastic dynamics in the D=N+2 dimensional phase space  $(x,v,u_0,...,u_{N-1})$ :

$$\dot{x} = v,$$

$$m\dot{v} = -\frac{\partial V(x,t)}{\partial x} + \sum_{i=0}^{N-1} u_i(t),$$

$$\dot{u}_i = -\eta_i v - v_i u_i + \sqrt{2v_i \eta_i k_B T} \xi_i(t),$$
(8)

where  $\xi_i(t)$  are independent unbiased white Gaussian noise sources,  $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$ . To enforce the FDR (2) for all times, the initial  $u_i(0)$  are independently distributed with the standard deviations  $\sigma_i = \sqrt{k_B T \eta_i}$  and zero mean [51, 54, 90]. The idea of such a Markovian embedding of non-Markovian GLE dynamics is pretty old [89] and the embedding is not unique [51, 90]. However, our scheme is the simplest one which serves

the purpose and it leads to an insightful and simple interpretation of the anomalous rate processes in terms of slowly fluctuating rates, with the rms amplitude of rate fluctuations which is gradually dying out upon increasing the potential barriers. For sufficiently high potential barriers (e.g. exceeding 12  $k_BT$  for  $\alpha = 0.5$  and about 9  $k_BT$  for  $\alpha = 0.75$  [54]) the rate description for the escape processes is restored. Then the escape rate becomes excellently described by the celebrated non-Markovian rate theory [60, 86, 91, 92].

Moreover, independently of the FBM connection, the model in Eq. (8) can be considered as a physically plausible viscoelastic model in its own rights as any power law dependence observed in the nature can be approximated by a finite sum of exponentials and such an expansion is a standard methodology e.g. in spectroscopy (approximation of spectra by a sum of Lorentzians) and in modeling of ion channel kinetics in biology. The inverse of  $v_0$  corresponds to the fastest time scale of the physical noise, or the high-frequency cutoff introduced into the (thus approximated) fGn, and  $v_{N-1} = v_0/b^{N-1}$ corresponds to the low-frequency cutoff. Such or similar cutoffs are always present in any realistic physical setup [93]. Even if on the time scale  $t > b^{N-1}/v_0 = \tau_c$ the subdiffusion will turn over into the normal diffusion, this time can be extremally large and practically not reachable, as in our simulations (or sufficiently large for subdiffusion to become physically relevant, as in real experiments). As a rule of thumb, a power law extending over N time-decades can be reasonably well approximated by a sum of about N exponentials choosing the dilation parameter b = 10. The number N can thus be surprisingly small. For example, an experimental power law extending over about 5 time decades was nicely fitted to a sum of 6 exponentials in Ref. [94] in a more flexible way, i.e. not assuming a precise scaling <sup>5</sup>. So, choosing  $v_0 = 10^3$  (arbitrary units) and N = 16, one can fit with  $C_{1/2}(10) = 1.3$  the power law kernel for  $\alpha = 0.5 \text{ from } t = 10^{-3} \text{ till } t = 10^{11}, \text{ i.e. over 14th time}$ decades [54]. The "error" (if to think of the model of fractional Brownian motion with inertia as something really fundamental which is not so) introduced by such an approximation is within 1% only [54]. Moreover, in any realistic experimental setup with some observed viscoelastic memory kernel exhibiting normally different complex patterns, see e.g. in Refs. [18, 28], one can

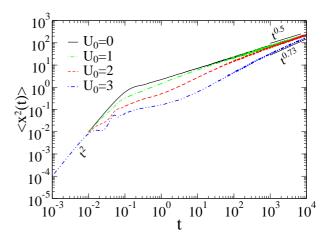


Figure 2: (Color online) Stochastic simulations with embedding dimension D=18 for different potential amplitudes (in units of  $k_BT$ ). The initial velocities are thermally distributed and initial diffusion is always ballistic. It changes into subdiffusion which is asymptotically not sensitive to the barrier height. The transient to this asymptotic regime can be however very slow, depending on the barrier height. For  $U_0=3$ , it is still not completed,  $\alpha_{\rm eff}(t)\approx 0.73$  for  $10^3 < t < 10^4$ . It will but decrease to  $\alpha=0.5$ , because of the subdiffusion in periodic potential cannot be faster than free subdiffusion, even if we cannot arrive at this regime numerically within two weeks of simulation. Stochastic Heun algorithm is used with time step  $\Delta t=10^{-4}$ ,  $\alpha=0.5$ , r=10, and  $n=10^4$  trajectories for the ensemble averaging.

do another expansion, not assuming any precise scaling, but rather simply fitting the experiment. Our flexible methodology will work anyway and the embedding dimension D=N+2 can typically be even smaller. For example, for all observed cases of biological subdiffusion (typically several time decades) the choice of N=6 is sufficient (will be published elsewhere). We are interested here but in a faithful Markovian embedding of the FBM type of subdiffusion (which formally requires infinite dimensions) and integrate GLE using embedding dimension D=18 (N=16) which provides excellent approximation.

## 3. Results

It is convenient to scale the coordinate x in units of L, and the time in units of  $\tau = [L^2\eta_\alpha/(k_BT)]^{1/\alpha}$ , so that the particle freely subdiffuses over a distance of the order L in time  $\tau$ . More precisely, in this scaling  $D_\alpha^{(0)} = 1$  for the strict power law memory kernel at any temperature. The potential amplitude parameter  $U_0$  is scaled in the units of  $k_BT$  and the driving force in the units of  $k_BT/L$ . The frequency is scaled in the units of  $\tau^{-1}$ . Furthermore, the influence of the inertia effects is captured by the parameter  $r = \tau/\tau_b$ , where  $\tau_b = L/v_T$  is the ballistic time for vanishing friction and  $v_T = \sqrt{k_BT/m}$  is the rms thermal

<sup>&</sup>lt;sup>5</sup>Of course, fitting a power law by a sum of exponentials does not answer the question about which physics is behind the power law scaling. However, it offers the way for an economical Markovian embedding of the observed non-Markovian dynamics. Moreover, the sum of exponentials can reflect some real hierarchical, tier like structured relaxation with a finite number of tier levels [16].

velocity. High damping corresponds to  $r \gg 1^6$ . We used the stochastic Heun algorithm [95] to integrate the system of stochastic differential equations (8). It is of the second order of weak convergency in the integration time step  $\Delta t$  for our particular case of additive noise. The Mersenne Twister pseudorandom number generator was used to produce the uniformly distributed random numbers which were transformed into Gaussian random numbers in accordance with the Box-Muller algorithm. With the time step  $\Delta t = 10^{-4}$  and  $n = 10^4$  trajectories used for the ensemble averaging, the typical accuracy of our simulations is within margin of several percents [54], as tested by comparison with the exact analytical results available for the potential-free subdiffusion and for parabolic potentials. This is a very good quality for stochastic numerics. It must be also noted that the use of double precision floating-point arithmetics cannot be avoided to arrive at convergent results.

# 3.1. Static ratchet potential

The influence of static ratchet potential U(x) on the subdifussion is illustrated in Fig. 2. After a short ballistic stage (within a potential well), the diffusion can look (depending on the potential barrier height) initially closer to normal. This is because of a finite mean residence time in a potential well exists and the escape kinetics, being asymptotically stretchedexponential, tends gradually to the normal exponential kinetics with an increase of the potential barrier height [54]. However, its slows down and asymptotically, independently of the barrier height (which is about  $2.2U_0$ for static potential), reaches the boardline of free subdiffusion which clearly cannot be crossed. This typical behavior can be characterized by a time-dependent exponent  $\alpha_{\text{eff}}(t)$  defined from the slope of  $\langle \delta x^2(t) \rangle$  curve in the double-logarithmic coordinates:  $\alpha_{\text{eff}} \rightarrow \alpha$  with  $t \to \infty$ , independently of  $U_0$ . Such a behavior reflects the physical nature of viscoelastic subdiffusion which is due to the long-range correlations in the particle's coordinate increments [54], being in a sharp contrast with the semi-Markov CTRW subdiffusion, where such correlations are absent in principle. These are not the rare events of the escape from potential wells which determine asymptotically the temporal pace of diffusion,

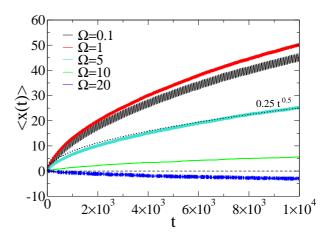


Figure 3: (Color online) Mean particle position as function of time for different driving frequencies,  $U_0=3$ , A=4. Other parameters are the same as in Fig. 2. The subvelocity  $v_\alpha$  is obtained by fitting the numerical dependences by  $v_\alpha t^{0.5}/\Gamma(1+\alpha)$  within the last window of length  $\Delta T=10^3$  in simulations (as it is shown for  $\Omega=5$ ). Notice that this provides only an estimate since still  $\alpha_{\rm eff}(t)\approx 0.58$ . The smaller is  $U_0$ , the more reliable is this estimate, since  $\alpha_{\rm eff}$  is better relaxed to the asymptotic value  $\alpha=0.5$ .

but the anti-persistent nature of viscoelastic subdiffusion which finally wins and limits diffusion in the periodic potentials by the free subdiffusion limit which is finally attended regardless the potential height. The transient to this astounding asymptotical behavior is, however, extremally slow and for this reason may be not achieved in practice. Even if the presence of periodic potential does not influence subdiffusion asymptotically (and therefore the adiabatically rocking subdiffusive ratchets are simply impossible), it does profoundly influence the whole time course of diffusion. This fact, which seems to go beyond any analytical treatment, is at the heart of anomalous ratchet effect. This physical picture implies that the subdiffusive rectified current has a resonance-like dependence against the driving frequency  $\Omega$ , because of in the limit  $\Omega \to \infty$  the ratchet effect must asymptotically vanish as well.

## 3.2. Rocking ratchets

To verify these qualitative considerations, we have tilted periodically the ratchet potential in Eq. (6) forth and back by  $f(t) = A\cos(\Omega t)$ . The amplitude A is scaled in the units of  $k_BT/L$ . For a given  $U_0$  (in units of  $k_BT$ ) there are two critical values of A for maximal force f(t): (i)  $A_1 = (3\pi/2)U_0 \approx 4.71U_0$  (when the potential barrier vanishes for the forward maximal tilt) and (ii)  $A_2 = 2A_1$  (same for the backward tilt). Moreover, for A between  $A_3 = \pi U_0$  and  $A_1$ , the potential U(x) has two small barriers within each spatial period of V(x) for the for-

<sup>&</sup>lt;sup>6</sup>Another useful scaling is: time in  $\tau_{\nu} = (m/\eta_{\alpha})^{1/(2-\alpha)}$  and energy in  $E_0 = L^2 \eta_{\alpha}^{2/(2-\alpha)}/m^{\alpha/(2-\alpha)}$ . The temperature is scaled then as  $\tilde{T} = k_B T/E_0$ . The connection between these two scalings is provided by:  $\tau_{\nu} = \tau r^{-2/(2-\alpha)}$  and  $\tilde{T} = r^{-2\alpha/(2-\alpha)}$ . One can easily rescale results at the fixed temperature (like in this paper) from one scaling to another one. Other scalings are also possible.

ward tilt. Generally, for a subcritical driving the potential barrier towards the direction of tilt is smaller for the forward tilt than for the backward tilt of equal amplitude. Therefore, intuitively the rectification current should flow in the positive direction for  $U_0 > 0$ , cf. Fig. 1, and this is indeed the fact, cf. Fig. 3, for a sufficiently slow driving. However, for the normal diffusion ratchets the current inversion is possible for a sufficiently high driving frequency  $\Omega$  [2]. A similar counter-intuitive inversion was found also in present case, see in Figs. 3,4.

Subdiffusive currents we define through the subvelocity  $^7$ 

$$v_{\alpha} = \Gamma(1 + \alpha) \lim_{t \to \infty} \frac{\langle x(t) \rangle}{t^{\alpha}}.$$
 (9)

Likewise, the subdiffusion coefficient in the potential is defined as

$$D_{\alpha} = \frac{1}{2}\Gamma(1+\alpha)\lim_{t\to\infty} \frac{\langle \delta x^2(t)\rangle}{t^{\alpha}},\tag{10}$$

so that it should coincide with  $D_{\alpha}^{(0)}$  in the absence of potential. Furthermore, the quality of the anomalous rectification can be characterized by a generalized Peclet number which we define as  $Pe_{\alpha} = v_{\alpha}L/D_{\alpha}$  by analogy with the normal diffusion case [96].

It must be noted that practically we are dealing with some time-dependent  $\alpha_{\rm eff}(t)$  in our numerics which relaxes very slow (depending on  $U_0$ ) to the asymptotic value  $\alpha$ . This asymptotic value is not easy to obtain for high  $U_0$ . To arrive at the end point  $t = 10^4$  in our simulations for one set of parameters and to have numerically reliable convergent results one has to propagate dynamics for about two weeks ( $\Delta t = 10^{-4}, n = 10^{4}$ ) on a single node of our modern Linux cluster using a highly optimized for the "number crunching" FORTRAN Intel compiler. We derive the corresponding values of  $v_{\alpha}$ and  $D_{\alpha}$  by fitting the dependencies  $\langle x(t) \rangle$  and  $\langle \delta x^2(t) \rangle$ with the at<sup>0.5</sup> dependence (extracting the corresponding a) within the last time window of length  $\Delta \mathcal{T} = 10^3$  in the simulations. For  $U_0 \le 2$  this gives reliable results, and for  $U_0 = 3$  the numerically derived values are less reliable (see but a fitting in Fig. 3 for  $\Omega = 5$  to get an idea on the quality of approximation). This is because of the corresponding  $\alpha_{\rm eff}(t)$  is still in fact about 0.58 within this time window, if to consider  $\alpha_{\rm eff}$  as an independent fitting parameter, i.e. it still did not relax completely to  $\alpha$ . This value is but essentially lower and closer to

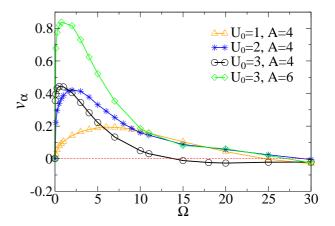


Figure 4: (Color online) Anomalous current (subvelocity  $v_{\alpha}$ ) as function of the driving frequency for different  $U_0$  and A. Rectification vanishes for both adiabatically slow and for a very fast driving. A current inversion is observed for sufficiently large  $\Omega$  in all cases.

 $\alpha$  than the corresponding  $\alpha_{\rm eff} \approx 0.73$  of the non-driven subdiffusion in the same potential, cf. Fig. 2. Generally, periodic driving accelerates the ultraslow relaxation of  $\alpha_{\rm eff}(t)$  to  $\alpha$ . However, for  $U_0=3$  it is, strictly speaking, still not quite achieved. We can compare only asymptotic transport coefficients in different potentials (a comparison makes just a little sense for some snapshot values  $\alpha_{\rm eff}(t)$ ). For larger  $U_0$  (e.g.  $U_0=4$ ), it becomes practically impossible to estimate these quantities. For this reason, such cases were not feasible for a quantitative study, even though the anomalous ratchet effect exists of course also. This striking feature of a time-dependent  $\alpha_{\rm eff}(t)$  must be kept in mind when to study subdiffusive ratchets experimentally.

The absence of asymptotic ratchet effect in adiabatic driving limit (a transient one yet exists!) is in a sharp contrast with the normal diffusion rocking ratchets where the current is maximized for  $\Omega \to 0$  [2, 7]. An inspection of single particle trajectories (not shown) indicates that the optimal frequency in our case corresponds to a synchronization of the potential tilts and the particle motion between potential wells being a kind of stochastic resonance [97]. In this respect, our anomalous rocking ratchet resembles more normal flashing ratchet than normal rocking ratchet. Indeed, if during a driving (half-)period the particles can move over many lattice periods the potential profile asymptotically does not matter and the current is suppressed. For the frequencies much larger than this inverse characteristic time the current is again suppressed (here per an analogy with normal diffusion ratchets: too frequent force alternations hamper the motion). This explains the occurrence of the resonance like feature. The maximal

 $<sup>^{7}</sup>$ The limiting procedure should be understood here in a physical sense, i.e. t is large but still much smaller than the memory cutoff  $\tau_c$ . Otherwise, the proper mathematical limit will yield infinity for a memory kernel with cutoff.

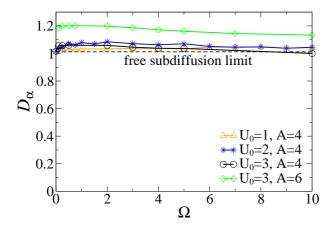


Figure 5: (Color online) Subdiffusion coefficient as function of driving frequency for different  $U_0$  and A.

amplitude of the rectification current first increases with increasing the barriers height, see in Fig. 4, where the current is maximal for the fixed A at the highest potential barrier being optimized for  $\Omega$ . It will however become obviously suppressed with a further increase of  $U_0$ . There are optimal potential amplitudes depending on driving strength and frequency, just like in the case of normal ratchets. The unexpected analogy to the flashing normal ratchets is, however, not complete as the (sub-)current inversion for sufficiently high frequency  $\Omega$  shows. This feature is similar to one in rocking normal ratchets and it unfortunately cannot be explained in simple intuitive terms.

The anomalous diffusion coefficient does not display such a profound dependence on frequency as  $v_{\alpha}$ . It becomes somewhat increased (less than few percent for A=4 and no more than 20% for A=6) as compare with the free subdiffusion coefficient (whose numerical value  $D_{\alpha}\approx 1.012$  agrees well with the theoretical value of  $D_{\alpha}^{(0)}=1$  in the used scaling). However, this effect is relatively small, compare Fig. 4 and Fig. 5. This remarkable feature is of the same origin as insensitivity of the GLE subdiffusion to the height of periodic potential. It is in a sharp contrast with the normal diffusion case. For this reason, the dependence of the generalized Peclet number on frequency just resembles the behavior of the absolute value of subcurrent in Fig. 4.

The dependence of rectification current on the driving field amplitude A is shown in Fig. 5 for  $U_0 = 2$  and  $\Omega = 1$ . It displays the same quadratic nonlinear dependence on driving amplitude for a small driving strength as in normal ratchets [2, 7]. Indeed, the rec-

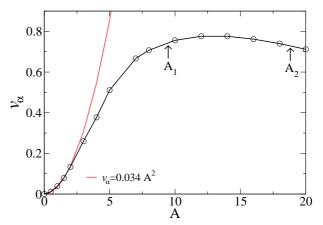


Figure 6: (Color online) Anomalous current as function of driving amplitude. Quadratic response fit is also displayed for comparison.  $U_0 = 2$ ,  $\Omega = 1$ .

tification current cannot be present within the linear response (due to the Onsager symmetry of kinetic coefficients). It first emerges as the lowest, second order nonlinear effect which is not forbidden by symmetry considerations. Moreover, beyond perturbation theory the rectification current displays a very broad maximum for  $A_1 < A < A_2$ , in accord with intuition.

## 4. Discussion and Summary

In this work we put forward a model of subdiffusive Brownian ratchets within the GLE description. Notably, a strict power law kernel is not required to reproduce subdiffusion on a very long time scale (the theoretical asymptotics of normal diffusion for  $t>10^{11}$  cannot be even reached numerically on the available computers what makes it practically irrelevant in our simulations), and the Markovian embedding dimension can be surprisingly small, allowing to accurately approximate nonlinear subdiffusive dynamics driven by Gaussian  $1/f^{1-\alpha}$  noise. Moreover, power law memory kernels with cutoffs are in fact more physical than strict power laws and their expansion over exponentials can reflect the corresponding relaxation spectrum of viscoelastic response of the surrounding medium.

The way we did this Markovian embedding, i.e. approximating the fractional Gaussian noise by a sum of Ornstein-Uhlenbeck processes is also insightful. In particular, it allows to justify the view of the escape dynamics out of a potential well as a rate process with non-Markovian fluctuating rate, when the potential barrier exceeds  $k_BT$  [54]. Such an escape dynamics is asymptotically stretched exponential, but it tends gradually to

a single-exponential with the increase of the potential barrier. Then the rate fluctuations gradually vanish and for sufficiently high barriers (exceeding e.g. 12  $k_BT$ for  $\alpha = 0.5$ , and about 9  $k_BT$  for  $\alpha = 0.75$ ) the escape dynamics is excellently approximated by the celebrated non-Markovian rate expression [60, 86, 91, 92], as it was shown by us recently [54]. However, even for such high potential barriers, when the escape dynamics out of a potential well becomes practically exponential, the diffusion in a periodic potential is not normal, but anomalously slow. This is because of viscoelastic subdiffusion is based not on the anomaly of the residence time distributions (i.e. divergent, or extremally large mean residence time, like in the case of CTRW subdiffusion and akin mechanisms), but on the long-time anticorrelations in the particle displacements (and velocity alternations). This fact in combination with ergodicity makes GLE subdiffusion physically much more appealing scenario, especially in biological applications where it can be combined with the quenched disorder due to the medium's inhomogeneity (not necessarily leading alone to the emergence of subdiffusion).

The subdiffusion in periodic potentials is described by some time-dependent  $\alpha_{\text{eff}}$  which relaxes very slowly to  $\alpha$ . The time pace of this relaxation depends very essential on the potential amplitude [54], even if asymptotically this subdiffusion is not sensitive to the presence of potential [81] because of the antipersistent, sluggish nature of this subdiffusion finally wins over thermally assisted hops between potential wells. The latter ones can take place quite frequently and do not provide a transport limiting step asymptotically (a huge difference with the CTRW approach!). This circumstance makes asymptotic ratchet effect impossible in the limit of adiabatically slow driving with vanishing driving frequency,  $\Omega \to 0$ . A somewhat similar suppression of the response to adiabatically slow driving was observed also in the non-Markovian stochastic resonance [98, 99]. It seems to be a general feature of the non-Markovian dynamics with infinite memory which culminates in the death of linear (and not only!) response in the case of subdiffusive CTRW and the associated fractional Fokker-Planck dynamics [41-43]. However, for a finite  $\Omega$  the asymptotic anomalous current response of subdiffusive ratchet dynamics is present. In the lowest order of driving strength A it emerges as a nonlinear quadratic response to driving which violates the symmetry of thermal detailed balance in the absence of driving, for the unbiased potential. This is similar to the out of equilibrium physics of normal diffusion ratchets. The rectification current is optimized not only for a driving strength between two critical values  $A_1 = 3\pi U_0/(2L)$ 

and  $A_2 = 2A_1$  for the consider potential, but also versus  $\Omega$  and  $U_0$  for the fixed temperature T. This is because of the rectification current vanishes in all these limits:  $\Omega \to 0, \Omega \to \infty, U_0 \to 0, U_0 \to \infty$ . The visual inspection of the single particle trajectories shows that the maximum of subvelocity versus  $\Omega$  corresponds to an optimal synchronization of the particle motions between potential wells and the potential tilts. This mechanism is somewhat similar to the current optimization for flashing, or pulsing normal diffusion ratchets [8]. However, the physics is different. In our case, the effect results because of the potential is not felt strongly by particles traveling large distances between the potential alternations. Moreover, for sufficiently large frequencies particles subdiffuse in the counter-intuitive direction, i.e. the anomalous current inversion occurs, cf. Figs. 3, 4. Like in the case of normal diffusion rocking ratchets [2], this effect does not have a simple intuitive explanation.

To conclude, this work put forward a generalization of the pioneering contributions [1, 2] to the realm of subdiffusive Brownian ratchets in viscoelastic media rocked by a periodic force. The author is confident that the bulk of future research on subdiffusive Brownian ratchets is ahead because of their surprising and counter-intuitive features which call for experimental verification.

#### References

- [1] M. O. Magnasco, Phys. Rev. Lett. 71 (1993) 1477.
- [2] R. Bartussek, P. Hänggi, and J. G. Kissner, Europhys. Lett. 28 (1994) 459.
- [3] C. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. 72 (1994) 2984.
- [4] A. Ajdari and J. Prost, C.R. Acad. Sci. Paris, Ser. II 315 (1992) 1635.
- [5] R.D. Astumian and M. Bier, Phys. Rev. Lett. 72 (1994) 1766.
- [6] J. Prost, J. Chauwin, L. Peliti, A. Ajdari, Phys. Rev. Lett. 72 (1994) 2652.
- [7] P. Reimann, Phys. Rep. 361 (2002) 57.
- [8] P. Hänggi and F. Marchesoni, Rev. Mod. Phys. 81 (2009) 387.
- [9] F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. 69 (1997) 1269.
- [10] Y. Okada and N. Hirokawa, Science 283 (1999) 1152.
- [11] Y. Inoue, A. H. Iwane, T. Miyai, E. Muto, and T. Yanagida, Biophys. J. 81 (2001) 2838.
- [12] P. Nelson, Biological Physics: Energy, Information, Life (Freeman, New York, 2004).
- [13] B. D. Hughes, Random Walks and Random Environments (Clarendon Press, Oxford, 1995).
- [14] H. Scher, E.W. Montroll, Phys. Rev. B 12 (1975) 2455.
- [15] M. Shlesinger, J. Stat. Phys. 10 (1974) 421.
- [16] J. P. Bouchaud and A. Georges, Phys. Rep. 195 (1990) 127.
- [17] S. Havlin and D. Ben-Avraham, Adv. Phys. 51 (2002) 187.
- [18] T. G. Mason and D. A. Weitz, Phys. Rev. Lett. 74 (1995) 1250.
- [19] F. Amblard, A. C. Maggs, B. Yurke, A. N. Pargellis, and S. Leibler, Phys. Rev. Lett. 77 (1996) 4470.

- [20] M. J. Saxton and K. Jacobson, Annu. Rev. Biophys. Biomol. Struct. 26 (1997) 373.
- [21] H. Qian, Biophys. J. 79 (2000) 137.
- [22] G. Seisenberger, M.U. Ried, T. Endreß, H. Büning, M. Hallek, and C. Bräuchle, Science 294 (2001) 1929.
- [23] A. Caspi, R. Granek, and M. Elbaum, Phys. Rev. E 66 (2002) 011916.
- [24] I. M. Tolic-Norrelykke, E.-L. Munteanu, G. Thon, L. Oddershede, and K. Berg-Sorensen, Phys. Rev. Lett. 93 (2004) 078102.
- [25] M. Weiss, M. Elsner, F. Kartberg, and T. Nilsson, Biophys. J. 87 (2004) 3518.
- [26] D. S. Banks and C. Fradin, Biophys. J. 89 (2005) 2960.
- [27] I. Golding and E. C. Cox, Phys. Rev. Lett. 96 (2006) 098102.
- [28] D. Mizuno, C. Tardin, C.F. Schmidt, and F. C. MacKintosh, Science 315 (2007) 370.
- [29] J. Szymanski and M. Weiss, Phys. Rev. Lett. 103 (2009) 038102.
- [30] W. Pan et al., Phys. Rev. Lett. 102 (2009) 058101.
- [31] T. Y. Shen, K. Tai, and J. A. McCammon, Phys. Rev. E 63 (2001) 041902.
- [32] A. E. Bizzarri and S. Cannistraro, J. Phys. Chem. B 106 (2002) 6617.
- [33] H. Yang, et al., Science 302 (2003) 262.
- [34] I. Goychuk and P. Hänggi, Phys. Rev. E 70 (2004) 051915.
- [35] G. R. Kneller and K. Hinsen, J. Chem. Phys. 121 (2004) 10278.
- [36] S.C. Kou, X.S. Xie, Phys. Rev. Lett. 93 (2004) 180603.
- [37] R. Granek and J. Klafter, Phys. Rev. Lett. 95 (2005) 098106.
- [38] G. Luo, I. Andricioaei, X. S. Xie, and M. Karplus, J. Phys. Chem. B 110 (2006) 9363.
   [30] P. Motzler, F. Parkei, and I. Klefter, Phys. Rev. Lett. 82 (1000).
- [39] R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. 82 (1999) 3563.
- [40] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1.
- [41] I. M. Sokolov and J. Klafter, Phys. Rev. Lett. 97 (2006) 140602
- [42] E. Heinsalu, M. Patriarca, I. Goychuk, and P. Hänggi, Phys. Rev. Lett. 99 (2007) 120602.
- [43] E. Heinsalu, M. Patriarca, I. Goychuk, and P. Hänggi, Phys. Rev. E 79 (2007) 041137.
- [44] I. Goychuk, E. Heinsalu, M. Patriarca, G. Schmid, and P. Hänggi, Phys. Rev. E 73 (2006) 020101(R).
- [45] E. Heinsalu, M. Patriarca, I. Goychuk, G. Schmid, and P. Hänggi, Phys. Rev. E 73 (2006) 046133.
- [46] J. Klafter and R. Silbey, Phys. Rev. Lett. 44 (1980) 55.
- [47] T. Harms and R. Lipowsky, Phys. Rev. Lett. 79 (1997) 2895.
- [48] B. B. Mandelbrot and J. W. van Ness, SIAM Rev. 10 (1968) 422.
- [49] J. Feder, Fractals (Plenum Press, New York, 1988).
- [50] K.G. Wang and M. Tokuyama, Physica A 265 (1999) 341.
- [51] R. Kupferman, J. Stat. Phys. 114 (2004) 291.
- [52] S. Chaudhury and B. J. Cherayil, J. Chem. Phys. 125 (2006) 024904
- [53] I. Goychuk and P. Hänggi, Phys. Rev. Lett. 99 (2007) 200601.
- [54] I. Goychuk, Phys. Rev. E 80 (2009) 046125.
- [55] W. Deng and E. Barkai, Phys. Rev. E 79 (2009) 011112.
- [56] G. Bel and E. Barkai, Phys. Rev. Lett. 94 (2005) 240602.
- [57] A. Gemant, Physics 7 (1936) 311.
- [58] R. Kubo, Rep. Prog. Phys. 29 (1966) 255.
- [59] R. Zwanzig, J. Stat. Phys. 9 (1973) 215 (1973)
- [60] P. Hänggi, P. Talkner, M. Borkovec, Rev. Mod. Phys. 62 (1990) 251.
- [61] R. Zwanzig, Nonequilibrium Statistical Mechanics (Oxford University Press, New York, 2001).
- [62] U. Weiss, Quantum Dissipative Systems, 2nd ed. (World Scientific, Singapore, 1999).

- [63] A. Nitzan, Chemical Dynamics in Condensed Phases: Relaxation, Transfer, and Reactions in Condensed Molecular Systems (Oxford University Press, Oxford, 2006).
- [64] P. Reimann, Chem. Phys. 268 (2001) 337.
- [65] A. Dubkov, P. Hänggi, and I. Goychuk, J. Stat. Mech. Theor. Exp. (2009) P01034.
- [66] N. N. Bogolyubov, On Some Statistical Methods in Mathematical Physics (Academy of Science of the Ukrainian SSR, Kiev, 1945), pp. 115-137, in Russian.
- [67] R. Gorenflo, F. Mainardi, in: Fractals and Fractional Calculus in Continuum Mechanics, edited by A. Carpinteri, F. Mainardi (Springer, Wien, 1997), pp. 223-276.
- [68] K. S. Cole and R. H. Cole, J. Chem. Phys. 9 (1941) 341.
- [69] I. Goychuk, Phys. Rev. E 76 (2007) 040102(R).
- [70] M. Atakhorrami, G. H. Koenderink, C. F. Schmidt, and F. C. MacKintosh, Phys. Rev. Lett. 95 (2005) 208302.
- [71] S. Burov and E. Barkai, Phys. Rev. Lett. 100 (2008) 070601.
- [72] S. Burov and E. Barkai, Phys. Rev. E 78 (2008) 031112.
- [73] F. Gittes and F. C. MacKintosh, Phys. Rev. E 58 (1998) 1241(R).
- [74] A. Taloni and M. A. Lomholt, Phys. Rev. E 78 (2008) 051116.
- [75] D. W. Jepsen, J. Math. Phys. 6 (1965) 405.
- [76] D. G. Levitt, Phys. Rev. A 8 (1973) 3050.
- [77] A. Taloni anf F. Marchesoni, Phys. Rev. Lett. 96 (2006) 020601.
- [78] Q.-H. Wei, C. Bechinger, and P. Leiderer, Science 287 (2000) 625
- [79] B. Lin, M. Meron, B. Cui, S. A. Rice, and H. Diamant, Phys. Rev. Lett. 94 (2005) 216001.
- [80] L. Lizana, T. Ambjörnsson, A. Taloni, E. Barkai, and M. A. Lomholt, arXiv:0909.0881[cond-mat.stat-mech] (2009).
- [81] Y.-C. Chen and J. L. Lebowitz, Phys. Rev. B 46 (1992) 10743.
- [82] S. A. Adelman, J. Chem. Phys. 64 (1976) 124.
- [83] P. Hänggi and H. Thomas, Z. Physik B 26 (1977) 85
- [84] P. Hänggi, H. Thomas, H. Grabert, and P. Talkner, J. Stat. Phys. 18 (1978) 155.
- [85] P. Hänggi, Z. Physik B 31 (1978) 407.
- [86] P. Hänggi and F. Mojtabai, Phys. Rev. A 26 (1982) 1168.
- [87] J. T. Hynes, J. Phys. Chem. 90 (1986) 3701.
- [88] R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. 53 (1984) 958.
- [89] F. Marchesoni and P. Grigolini, J. Chem. Phys. 78 (1983) 6287.
- [90] P. Siegle, I. Goychuk, P. Talkner, and P. Hänggi, Phys. Rev. E 81 (2010) 011136.
- [91] R. F. Grote and J. T. Hynes, J. Chem. Phys. 73 (1980) 2715.
- [92] E. Pollak, J. Chem. Phys. 85 (1986) 865.
- [93] M. B. Weissman, Rev. Mod. Phys. 60 (1988) 537.
- [94] M. S. P. Sansom et al., Biophys. J. 56 (1989) 1229.
- [95] T. C. Gard, Introduction to Stochastic Differential Equations (Dekker, New York, 1988).
- [96] B. Lindner, M. Kostur, and L. Schimansky-Geier, Fluct. Noise Lett. 1 (2001) R25.
- [97] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. 70 (1998) 223.
- [98] I. Goychuk and P. Hänggi, Phys. Rev. Lett. 91 (2003) 070601.
- [99] I. Goychuk and P. Hänggi, Phys. Rev. E 69 (2004) 021104.