ZERO KINEMATIC VISCOSITY-MAGNETIC DIFFUSION LIMIT OF THE INCOMPRESSIBLE VISCOUS MAGNETOHYDRODYNAMIC EQUATIONS WITH NAVIER BOUNDARY CONDITIONS

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ABSTRACT. We investigate the zero kinematic viscosity-magnetic diffusion limit of the incompressible viscous magnetohydrodynamic equations with Navier boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. We obtain the uniform regularity of solutions with respect to the kinematic viscosity coefficient and the magnetic diffusivity coefficient. These solutions are uniformly bounded in a conormal Sobolev space and $W^{1,\infty}(\Omega)$ which allow us to take the zero kinematic viscosity-magnetic diffusion limit. Moreover, we also get the rates of convergence.

1. INTRODUCTION

We consider the following incompressible viscous magnetohydrodynamic (MHD) equations ([6,8])

$$\partial_t v^\epsilon - \epsilon \Delta v^\epsilon + v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2}\nabla(|v^\epsilon|^2 - |H^\epsilon|^2) + \nabla p^\epsilon = 0, \qquad (1.1)$$

$$\partial_t H^\epsilon - \epsilon \Delta H^\epsilon + v^\epsilon \cdot \nabla H^\epsilon - H^\epsilon \cdot \nabla v^\epsilon = 0, \qquad (1.2)$$

$$\operatorname{div} v^\epsilon = \operatorname{div} H^\epsilon = 0 \qquad (1.3)$$

in $(0,T) \times \Omega$, where Ω is a smooth bounded domain of $\mathbb{R}^3$. The unknowns v^ϵ and H^ϵ represent the fluid velocity and the magnetic field, respectively. The pressure p^ϵ can be recovered from v^ϵ and H^ϵ via an explicit Caldern-Zygmund singular integral operator ([7]).

We add to v^ϵ and H^ϵ the following initial and boundary conditions

$$v^\epsilon \cdot n = 0, \quad (Sv^\epsilon \cdot n)_\tau = -\zeta v^\epsilon_\tau \quad \text{on } \partial\Omega, \qquad (1.4)$$

$$H^\epsilon \cdot n = 0, \quad (SH^\epsilon \cdot n)_\tau = -\zeta H^\epsilon_\tau \quad \text{on } \partial\Omega, \qquad (1.5)$$

$$(v^\epsilon, H^\epsilon)|_{t=0} = (v_0, H_0) \quad \text{in } \quad \Omega, \qquad (1.6)$$

where n stands for the outward unit normal vector to Ω, ζ is a coefficient measuring the tendency of the fluid to slip on the boundary, S is the strain tensor defined by

$$Su = \frac{1}{2}(\nabla u + \nabla u^t),$$

where ∇u^t denotes the transpose of the matrix ∇u, and u_τ stands for the tangential part of u on $\partial\Omega$, i.e.

$$u_\tau = u - (u \cdot n)n.$$

This kind boundary condition (1.4) was introduced by Navier in [17] to show that the velocity is propositional to the tangential part of the stress. It allows the fluid slip along the boundary and are often used to model rough boundaries. The Navier boundary condition (1.4) can be generalized to the following form ([10])

$$u \cdot n = 0, \ (Su \cdot n)_\tau + Au = 0, \tag{1.7}$$

where A is a $(1,1)$-type tensor on the boundary $\partial\Omega$. When $A = \zeta\,\mathrm{Id}$ (here Id denotes the identity matrix), (1.7) is reduced to the standard Navier boundary condition. For smooth functions, we can get the form of the vorticity

$$u \cdot n = 0, \ n \times \omega = [Bu]_\tau \quad \text{on} \quad \partial\Omega, \tag{1.8}$$

where $\omega = \nabla \times u$ is the vorticity and $B = 2(A - S(n))$ ([24]).

In this paper we are interested in the existence of strong solution to the problem (1.1)-(1.6) with uniform bounds on an interval of time independent of ϵ and taking the limit $\epsilon \to 0$ to obtain the ideal incompressible MHD equations, i.e.

$$\partial_t v + v \cdot \nabla v - H \cdot \nabla H - \frac{1}{2}\nabla(|v|^2 - |H|^2) + \nabla p = 0, \tag{1.9}$$

$$\partial_t H + v \cdot \nabla H - H \cdot \nabla v = 0, \tag{1.10}$$

$$\mathrm{div}\, v = \mathrm{div}\, H = 0 \tag{1.11}$$

with the following slip boundary conditions:

$$v \cdot n = H \cdot n = 0. \tag{1.12}$$

When taking $H^\epsilon = 0$ in the system (1.1)-(1.3), it is reduced to the classical incompressible Navier-Stokes equations and there are many literature on the vanishing viscosity limit of it. In the case that there is no boundary, a uniform time of existence and the vanishing viscosity limit have been obtained, see [13, 15, 20]. When the boundary appear, it is usually difficult to do higher order energy estimates near boundary because of the appearing of the boundary layer [18]. In particular, for the incompressible Navier-Stokes equations with no-slip boundary condition, the vanishing viscosity limit of it is wildly open except when the initial data is analytic [21, 22] or the initial vorticity is located away from the boundary in the two-dimensional half plane [14]. On the other hand, considering the incompressible Navier-Stokes system with Navier boundary conditions, more results are

available, see, for example, [2–5, 12, 26]. Xiao and Xin [26] investigate the vanishing viscosity limit to incompressible Navier-Stokes equation with the boundary conditions

$$u \cdot n = 0, \ n \times \omega = 0 \quad \text{on} \quad \partial\Omega. \tag{1.13}$$

Because the main part in the boundary layer vanishes (i.e. $V = 0$ in (1.14) below), they can obtain the local existence of strong solution with some uniform bounds in $H^3(\Omega)$ and the vanishing viscosity limit. Their approaches overcame the compatibility issues of the nonlinear terms with (1.13). The authors in [2] got uniform estimates in $W^{k,p}(\Omega)$ with $k \geq 3$ and $p \geq 2$. The main reason is that the boundary integrals vanishes on flat portions of the boundary, see also [3,4]. Later, the results in [2, 26] was generalized by Berselli and Spirito [5] to a general bounded domain under certain restrictions on the initial data. In order to analysis the effect of the boundary layer in a general bounded domain, Iftimie and Sueur [12] constructed the boundary layer for the incompressible Navier-Stokes equations with the Navier boundary condition (1.4) in the form

$$u^\epsilon(t, x) = u^\epsilon(t, x) + \sqrt{\epsilon}V\left(t, x, \frac{\phi(x)}{\sqrt{\epsilon}}\right) + O(\epsilon), \tag{1.14}$$

where the function V vanishes for x outside a small neighborhood of $\partial\Omega$ and $\phi(x)$ is the distance between x and $\partial\Omega$ for x in a neighborhood of $\partial\Omega$. The layers constructed in [12] are of width $O(\sqrt{\epsilon})$ like the Prandtl layer [18], but are of amplitude $O(\sqrt{\epsilon})$ (The Prandtl layer is of width $O(\sqrt{\epsilon})$ and of amplitude $O(1)$). So it is impossible to obtain the $H^3(\Omega)$ or $W^{2,p}(\Omega)$ (p large enough) uniform estimates for the incompressible Navier-Stokes equations. Recently, Masmoudi and Rousset [16] considered the the vanishing viscosity limit for the incompressible Navier-Stokes equation with the boundary condition (1.4) in anisotropic conormal Sobolev spaces which can eliminate the effects of normal derivatives near the boundary. They obtained uniform regularity and the convergence of the viscous solutions to the inviscid ones by compactness argument. Recently, some results in [16] was extended to the compressible isentropic Navier-Stokes equations with Navier boundary conditions [19, 23]. Moreover, based on the results in [16], the rates of convergence were obtained by Gie and Kelliher [10] and Xiao and Xin [24], respectively.

In [27], Xiao, Xin and Wu studied the inviscid limit for the system (1.1)-(1.3) with the boundary conditions

$$\begin{cases} v^\epsilon \cdot n = 0, & n \times \omega_v^\epsilon = 0 \quad \text{on} \quad \partial\Omega, \\ H^\epsilon \cdot n = 0, & n \times \omega_H^\epsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.15}$$

where they used the approaches similar to that in [26] and formulated the boundary value in a suitable functional setting so that the stokes operator is well behaved and the nonlinear terms fall into the desired functional spaces. These facts allow them

to get the uniform regularity for the viscous incompressible MHD system through the Galerkin approximation and a priori energy estimates.

Here we investigate the inviscid limit for the system (1.1)-(1.3) with the Navier boundary conditions (1.4)-(1.5) in a $3D$ bounded domain in the framework of anisotropic conormal Sobolev spaces. Due to the strong coupling between v^ϵ and H^ϵ, a priori estimates become more complicated than that in [16] on the incompressible Navier-Stokes equations. We obtain uniform regularity of the solutions and, with this well-posedness theory, pursue the vanishing viscosity limit to the problem (1.1)-(1.6). Moreover, we also obtain some rates of convergence for v^ϵ and H^ϵ. Hence our results can be regarded as generalizations of those in [10, 16, 24] to incompressible MHD eqautions.

Our first result of this paper reads as follows.

Theorem 1.1. *Let m be an integer satisfying $m > 6$ and Ω be a C^{m+2} domain. Assume that the initial data (v_0, H_0) satisfy*

$$(v_0, H_0) \in \mathcal{E}^m, \ (\nabla v_0, \nabla H_0) \in W_{co}^{1,\infty}(\Omega),$$

$$\nabla \cdot v_0 = \nabla \cdot H_0 = 0, \ v_0 \cdot n|_{\partial\Omega} = H_0 \cdot n|_{\partial\Omega} = 0.$$

Then, there exist $T_0 > 0$ and $\widetilde{C}$, independent of $\epsilon \in (0, 1]$ and $|\zeta| \leq 1$, such that there exists a unique solution of the problem (1.1)-(1.6) satisfying

$$(v^\epsilon, H^\epsilon) \in C([0, T_0], \mathcal{E}^m)$$

and

$$\sup_{t \in [0,T_0]} \left\{ \|(v^\epsilon, H^\epsilon)(t)\|_m + \|(\nabla v^\epsilon, \nabla H^\epsilon)(t)\|_{m-1} + \|(\nabla v^\epsilon, \nabla H^\epsilon)(t)\|_{1,\infty} \right\}$$

$$+ \epsilon \int_0^{T_0} (\|\nabla^2 v^\epsilon(t)\|_{m-1}^2 + \|\nabla^2 H^\epsilon(t)\|_{m-1}^2) dt \leq \widetilde{C}, \tag{1.16}$$

Here $\mathcal{E}^m := \{u \,|\, u \in H_{co}^m(\Omega), \nabla u \in H_{co}^{m-1}(\Omega)\}$ and the meanings of $W_{co}^{1,\infty}(\Omega)$, $H_{co}^m(\Omega)$, $\|\cdot\|_m$ and $\|\cdot\|_{m,\infty}$ will be explained in detail in next section.

Remark 1.1. *When the Navier boundary conditions (1.4) and (1.5) are replaced by the following*

$$\begin{cases} v^\epsilon \cdot n = 0, & n \times \omega_v^\epsilon = [Bv^\epsilon]_\tau \quad on \quad \partial\Omega, \\ H^\epsilon \cdot n = 0, & n \times \omega_H^\epsilon = [BH^\epsilon]_\tau \quad on \quad \partial\Omega, \end{cases} \tag{1.17}$$

we can also obtain the same results as those in Theorem 1.1, where $B = 2(A - S(n))$ and A is a $(1, 1)$-type tensor on the boundary $\partial\Omega$.

Remark 1.2. *Theorem 1.1 still holds if we replace the boundary conditions (1.4) and (1.5) by the slightly generalized one:*

$$\begin{cases} v^\epsilon \cdot n = 0, & (Sv^\epsilon \cdot n)_\tau = -\zeta_1 v^\epsilon \quad on \quad \partial\Omega, \\ H^\epsilon \cdot n = 0, & (SH^\epsilon \cdot n)_\tau = -\zeta_1 H^\epsilon \quad on \quad \partial\Omega, \end{cases} \tag{1.18}$$

where ζ_1 and ζ_2 are two different constants.

We now give some comments on the proof of Theorem 1.1. The main steps of the proof are similar to those in [16] in some sense. However, due to the strong coupling between v^ϵ and H^ϵ, we need to overcome some new difficulties and to face more complicated energy estimates. First, we get a conormal energy estimates in H_{co}^m (see the definition in next section) for (v^ϵ, H^ϵ). Here, we define $P_1^\epsilon + P_2^\epsilon := p^\epsilon - \frac{1}{2}(|v^\epsilon|^2 - |H^\epsilon|^2)$, where P_1^ϵ and P_2^ϵ satisfy corresponding boundary value problems (see (3.8) and (3.9) below), respectively. By doing this decomposition, we can avoid higher order terms which are out of control. In the second step, we estimate $\|(\partial_n v^\epsilon, \partial_n H^\epsilon)\|_{m-1}$. Due to the incompressible conditions (1.3), both $\partial_n v^\epsilon \cdot n$ and $\partial_n H^\epsilon \cdot n$ can be easily controlled by the H_{co}^m norm of (v^ϵ, H^ϵ). Thanks to the the the Nvier-slip boundary conditions, it is convenient to study $\eta_v^\epsilon = (Sv^\epsilon n + \zeta v^\epsilon)_\tau$ and $\eta_H^\epsilon = (SH^\epsilon n + \zeta H^\epsilon)_\tau$. We find that η_v^ϵ and η_H^ϵ satisfy equations with homogeneous Dirichlet boundary conditions, and we shall thus prove a control of $\|(\eta_v^\epsilon, \eta_H^\epsilon)\|_{m-1}$ by performing energy estimates on the equations solved by $(\eta_v^\epsilon, \eta_H^\epsilon)$. The third step is to estimate P_1^ϵ and P_2^ϵ. Note that they satisfy nonhomogeneous elliptic equations with Neumann boundary conditions. By using the regularity theory of elliptic equations with Neumann boundary conditions, we get the estimates on the pressure terms. Finally, we need to estimate $\|\nabla v^\epsilon\|_{1,\infty}$ and $\|\nabla H^\epsilon\|_{1,\infty}$. Similar to the second step, we find equivalent quantities $\overline{\eta}_v^\epsilon$ and $\overline{\eta}_H^\epsilon$. However, due to the strong coupling between $\overline{\eta}_v^\epsilon$ and $\overline{\eta}_H^\epsilon$, we cannot deal with the system on $\overline{\eta}_v^\epsilon$ and $\overline{\eta}_H^\epsilon$ directly as that in [16]. Instead, we need further to introduce another two quantities $\eta_1 := \overline{\eta}_v^\epsilon + \overline{\eta}_H^\epsilon$ and $\eta_2 := \overline{\eta}_v^\epsilon - \overline{\eta}_H^\epsilon$. We then estimate η_1 and η_2, respectively.

Based on Theorem 1.1 and Remark 1.1, we justify the vanishing viscosity limit as follows:

Theorem 1.2. *Assume that (v_0, H_0) belong to $H^3(\Omega)$ and satisfy the same assumptions as in Theorem 1.1. Let (v, H) be the smooth solution of (1.9)-(1.12) with the initial data $(v, H)|_{t=0} = (v_0, H_0)$ on $[0, T_1]$. Let (v^ϵ, H^ϵ) be the solution of (1.1)-(1.3) with the boundary condition (1.17) and the initial data $(v^\epsilon, H^\epsilon)|_{t=0} = (v_0, H_0)$. Then there exists a $T_2 = \min\{T_0, T_1\} > 0$ such that*

$$\|v^\epsilon - v\|_{L^2}^2 + \|H^\epsilon - H\|_{L^2}^2$$
$$+ \epsilon \int_0^t (\|(v^\epsilon - v)(s)\|_{H^1}^2 + \|(H^\epsilon - H)(s)\|_{H^1}^2)\, ds \leq C\epsilon^{\frac{3}{2}} \quad on \quad [0, T_2], \quad (1.19)$$

$$\|v^\epsilon - v\|_{H^1}^2 + \|H^\epsilon - H\|_{H^1}^2$$
$$+ \epsilon \int_0^t (\|(v^\epsilon - v)(s)\|_{H^2}^2 + \|(H^\epsilon - H)(s)\|_{H^2}^2)\, ds \leq C\epsilon^{\frac{1}{2}} \quad on \quad [0, T_2] \quad (1.20)$$

for ϵ small enough. Consequently,

$$\|v^\epsilon - v\|_{W^{1,p}}^p + \|H^\epsilon - H\|_{W^{1,p}}^p \leq C\epsilon^{\frac{1}{2}} \quad on \quad [0, T_2] \quad (1.21)$$

for $2 \leq p < \infty$ and ϵ small enough, and

$$\|v^\epsilon - v\|_{L^\infty([0,T_2]\times\Omega)} + \|H^\epsilon - H\|_{L^\infty([0,T_2]\times\Omega)} \leq C\epsilon^{\frac{3}{10}}. \qquad (1.22)$$

We now outline the proof of Theorem 1.2. Our approaches are similar to those in [24], but due to the strong coupling between magnetic field and velocity field, we meet some new difficulties. We first give the rates of the convergence in $L^\infty(0, T_2; L^2(\Omega))$ and $L^\infty([0, T_2] \times \Omega)$ by using an elementary energy estimate for the difference of the solutions between the incompressible viscous MHD equations and the ideal incompressible MHD equations and the Gagliardo-Nirenberg interpolation inequality. Next, because we find that it is very difficult to estimate some boundary terms caused by multiplying (4.1) by $\Delta(v^\epsilon - v)$ and (4.2) by $\Delta(H^\epsilon - H)$ directly in the proof of the rate of the convergence in $L^\infty(0, T_2; H^1(\Omega))$, we turn to consider the Stokes problem (4.12)-(4.14). Indeed, we can get $\|u\|_{H^2} \leq \|P\Delta u\| + \|u\|$ for $u \in W_B$, where W_B is defined in Lemma 4.2 and P is Leray projector. Finally, we replace $\Delta(v^\epsilon - v)$ and $\Delta(H^\epsilon - H)$ by $P\Delta(v^\epsilon - v)$ and $P\Delta(H^\epsilon - H)$ to do prove the rates of the convergence in $L^\infty(0, T_2; H^1(\Omega))$ and $L^\infty(0, T_2; W^{1,p}(\Omega))$.

This paper is organized as follows. In the following section, we give some assumptions on the domain and the definitions on conormal Sobolev spaces, and present some inequalities. In Section 3, we prove a priori energy estimates and give the proof of Theorem 1.1. Finally, we prove Theorem 1.2 in Section 4. Throughout the paper, we shall denote by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{1,\infty}}$ the usual Sobolev norms in Ω and $\|\cdot\|$ for the standard L^2 norm. The letter C is a positive number which may change from line to line, but independent of $\epsilon \in (0, 1]$ and $|\zeta| \leq 1$.

2. PRELIMINARIES

We first state the assumptions on the bounded domain $\Omega \subset \mathbb{R}^3$ and then introduce some norms. We assume that Ω has a covering such that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k, \qquad (2.1)$$

where $\overline{\Omega_0} \subset \Omega$ and in each Ω_k there exists a function ψ_k such that

$$\Omega \cup \Omega_k = \{ x = (x_1, x_2, x_3) \,|\, x_3 > \psi_k(x_1, x_2) \} \cup \Omega_k,$$
$$\partial\Omega \cup \Omega_k = \{ x = (x_1, x_2, x_3) \,|\, x_3 = \psi_k(x_1, x_2) \} \cup \Omega_k.$$

We say that Ω is C^m if the functions ψ_k are C^m functions.

To define the conormal Sobolev spaces, we consider $(Z_k)_{1\leq k\leq N}$, a finite set of generators of vector fields that are tangent to $\partial\Omega$, and set

$$H_{co}^m(\Omega) := \left\{ f \in L^2(\Omega) \,\middle|\, Z^I f \in L^2(\Omega) \text{ for } |I| \leq m, \ m \in \mathbb{N} \right\}, \qquad (2.2)$$

where $I = (k_1, ..., k_m)$, $Z^I := Z_{k_1} \cdots Z_{k_m}$. We define the norm of $H_{co}^m(\Omega)$:

$$\|f\|_m^2 := \sum_{|I| \leq m} \|Z^I f\|_{L^2}^2.$$

We say a vector field, u, is in $H_{co}^m(\Omega)$ if each of its components is in $H_{co}^m(\Omega)$ and

$$\|u\|_m^2 := \sum_{i=1}^3 \sum_{|I| \leq m} \|Z^I u_i\|_{L^2}^2$$

is finite. In the same way, we set

$$\|f\|_{m,\infty} := \sum_{|I| \leq m} \|Z^I f\|_{L^\infty},$$

$$\|\nabla Z^m u\|^2 := \sum_{|I| \leq m} \|\nabla Z^I u\|_{L^2}^2,$$

and we say that $f \in W_{co}^{m,\infty}(\Omega)$ if $\|f\|_{m,\infty}$ is finite. By using above covering of Ω, we can assume that each vector field is supported in one of $\{\Omega_i\}_{i=0}^n$. Also, we note that the $\|\cdot\|_m$ norm yields a control of the standard H^m norm in Ω_0, whereas if $\Omega_i \cap \partial\Omega \neq \emptyset$, there is no control of the normal derivatives.

Since $\partial\Omega$ is given locally by $x_3 = \psi(x_1, x_2)$ (we omit the subscript k for notational convenience), it is convenient to use the coordinates:

$$\Psi : (y, z) \mapsto (y, \psi(y) + z) = x. \tag{2.3}$$

A local basis is thus given by the vector fields $(\partial_{y^1}, \partial_{y^1}, \partial_z)$ where ∂_{y^1} and ∂_{y^2} are tangent to $\partial\Omega$ on the boundary and in general ∂_z is usually not a normal vector field. We sometimes use the notation ∂_{y^3} for ∂_z. By using this parametrization, we can take suitable vector fields compactly supported in Ω_i in the definition of the $\|\cdot\|_m$ norms:

$$Z_i = \partial_{y^i} = \partial_i + \partial_i\psi\partial_z, \ i = 1, 2, \quad Z_3 = \varphi(z)\partial_z,$$

where $\varphi(z) = \frac{z}{1+z}$ is a smooth and supported function in $(0, +\infty)$ and satisfies

$$\varphi(0) = 0, \ \varphi'(0) > 0, \ \varphi(z) > 0 \ \text{ for } \ z > 0.$$

In this paper, we shall still denote by $\partial_i, i = 1, \ 2, \ 3$ or ∇ the derivatives with respect to the standard coordinates of $\mathbb{R}^3$. The coordinates of a vector field u in the basis $(\partial_{y^1}, \partial_{y^1}, \partial_z)$ will be denote by u^i, thus

$$u = u^1\partial_{y^1} + u^2\partial_{y^2} + u^3\partial_z.$$

We denote by u_i the coordinates in the standard basis of $\mathbb{R}^3$, i.e.

$$u = u_1\partial_1 + u_2\partial_2 + u_3\partial_3.$$

Denote by n the unit outward normal vector which is given locally by

$$n(x) = n(\Psi(y,z)) = \frac{1}{\sqrt{1+|\nabla\psi(y)|^2}} \begin{pmatrix} \partial_1\psi(y) \\ \partial_2\psi(y) \\ -1 \end{pmatrix}$$

and by Π the orthogonal projection

$$\Pi(x) = \Pi(\Psi(y,z))u = u - [u \cdot n(\Psi(y,z))]n(\Psi(y,z)$$

which gives the orthogonal projector onto the tangent space of the boundary. Note that n and Π are defined in the whole Ω_k and do not depend on z. By using these notations, the Navier boundary conditions (1.4) and (1.5) read:

$$v^\epsilon \cdot n = 0, \quad \Pi\partial_n v^\epsilon = \theta(v^\epsilon) - 2\zeta\Pi v^\epsilon, \tag{2.4}$$

$$H^\epsilon \cdot n = 0, \quad \Pi\partial_n H^\epsilon = \theta(H^\epsilon) - 2\zeta\Pi H^\epsilon, \tag{2.5}$$

where θ is the shape operator (second fundamental form) of the boundary, $\theta(v^\epsilon) := \Pi((\nabla n)v^\epsilon)$ and $\theta(H^\epsilon) := \Pi((\nabla n)H^\epsilon)$.

First, we introduce a well-known inequality.

Lemma 2.1 ([1,26]). *For $u \in H^s(\Omega)$ $(s \geq 1)$, we have*

$$\|u\|_{H^s(\Omega)} \leq C \left(\|\nabla \times u\|_{H^{s-1}(\Omega)} + \|\nabla \cdot u\|_{H^{s-1}(\Omega)} + \|u\|_{H^{s-1}(\Omega)} + |u \cdot n|_{H^{s-\frac{1}{2}}(\partial\Omega)}\right).$$

Next, we introduce the Korn's inequality which play an important role in energy estimates below.

Lemma 2.2 (Korn's inequality [9]). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^3$. There exists a constant $C > 0$ depending only on Ω such that*

$$\|u\|_{H^1(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|S(u)\|_{L^2(\Omega)}\right), \quad \forall\, u \in (H^1(\Omega))^3.$$

Third, we also need the following anistropic Sobolev embedding and trace estimates.

Lemma 2.3 ([16,23]). *Let $m_1 \geq 0$ and $m_2 \geq 0$ be integers, $u \in H_{co}^{m_1}(\Omega) \cap H_{co}^{m_2}(\Omega)$ and $\nabla u \in H_{co}^{m_2}(\Omega)$. Then we have*

$$\|u\|_{L^\infty(\Omega)}^2 \leq C \left(\|\nabla u\|_{m_2} + \|u\|_{m_2}\right)\|u\|_{m_1}, \quad m_1 + m_2 \geq 3,$$

$$|u|_{H^s(\partial\Omega)}^2 \leq C \left(\|\nabla u\|_{m_2} + \|u\|_{m_2}\right)\|u\|_{m_1}, \quad m_1 + m_2 \geq 2s \geq 0.$$

Fourth, we introduce the following Gagliardo-Nirenberg-Moser inequality which will be used frequently.

Lemma 2.4 ([11]). *Let $u, v \in L^\infty(\Omega) \cap H_{co}^k(\Omega)$, we have*

$$\|Z^{\alpha_1}uZ^{\alpha_2}v\| \leq C \left(\|u\|_{L^\infty(\Omega)}\|v\|_k + \|v\|_{L^\infty(\Omega)}\|u\|_k\right), \quad |\alpha_1| + |\alpha_2| = k.$$

Finally, the following decomposition on H^s contributes to the proof of the convergence rate in H^1.

Lemma 2.5 ([25]). *For $H^s(\Omega)\,(s \geq 0)$, we have*

$$H^s(\Omega) = \nabla \times (FH \cap H^{s+1}(\Omega)) \oplus (HG \cap H^s(\Omega)) \oplus (GG \cap H^s(\Omega)),$$

where

$$FH = \left\{ u \,|\, u = \nabla \times \varphi,\ \varphi \in H^1(\Omega),\ \nabla \cdot \varphi = 0,\ n \times \varphi = 0\ \text{on}\ \partial\Omega \right\},$$

$$HG = \left\{ u \,|\, u = \nabla\varphi,\ \Delta\varphi = 0,\ \varphi = c_i\ \text{on}\ \Gamma_i,\ \bigcup_i \Gamma_i = \partial\Omega \right\},$$

$$GG = \left\{ u \,|\, u = \nabla\varphi,\ \varphi \in H_0^1(\Omega) \right\}.$$

3. A priori estimates and proof of Theorem 1.1

The main aim of this section is to prove the following a priori estimates which is the crucial step in the proof of Theorem 1.1.

Theorem 3.1. *For $m > 6$ and a C^m domain Ω, there exists a constant $C > 0$, independent of $\epsilon \in (0,1]$ and $|\zeta| \leq 1$, such that for any sufficiently smooth solution defined on $[0,T]$ of the problem (1.1)-(1.6) in Ω, we have*

$$N_m(t) \leq C\left(N_m(0) + (1 + t + \epsilon^3 t^2) \int_0^t (N_m^2(s) + N_m(s))\,ds \right), \quad \forall\, t \in [0,T], \quad (3.1)$$

where

$$N_m(t) := \|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|\nabla v^\epsilon\|_{1,\infty}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2 + \|\nabla H^\epsilon\|_{1,\infty}^2. \tag{3.2}$$

Since the proof of Theorem 3.1 is quite complicated and lengthy, we divided the proof into the following subsections.

3.1. **Conormal Energy Estimates.** In this subsection, we first give the basic L^2 energy estimates.

Lemma 3.1. *For a smooth solution of the problem (1.1)-(1.6), we have*

$$\frac{1}{2}\frac{d}{dt}(\|v^\epsilon(t)\|^2 + \|H^\epsilon(t)\|^2) + 2\epsilon(\|Sv^\epsilon\|^2 + \|SH^\epsilon\|^2)$$

$$+ 2\epsilon\zeta \int_{\partial\Omega} \left(|\,v_\tau^\epsilon\,|^2 + |\,H_\tau^\epsilon\,|^2 \right) = 0 \tag{3.3}$$

for every $\epsilon \in (0,1]$ and $|\zeta| \leq 1$.

Proof. Multiplying (1.1) and (1.2) by v^ϵ and H^ϵ respectively, using the boundary condition, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|v^\epsilon\|^2 + \|H^\epsilon\|^2) - \epsilon(\Delta v^\epsilon, v^\epsilon) - \epsilon(\Delta H^\epsilon, H^\epsilon)$$

$$- (H^\epsilon \cdot \nabla H^\epsilon, v^\epsilon) - (H^\epsilon \cdot \nabla v^\epsilon, H^\epsilon) = 0, \tag{3.4}$$

where $(\cdot, \cdot)$ stands for the L^2 scalar product. By integrating by parts and using the boundary conditions, we get

$$(H^\epsilon \cdot \nabla H^\epsilon, v^\epsilon) + (H^\epsilon \cdot \nabla v^\epsilon, H^\epsilon) = 0.$$

Now, let us treat the terms with the viscous coefficient ϵ in (3.4). Thanks to integrations by parts and the boundary condition (1.4), we have

$$(\epsilon \Delta v^\epsilon, v^\epsilon) = 2\epsilon(\nabla \cdot Sv^\epsilon, v^\epsilon) = -2\epsilon \|Sv^\epsilon\|^2 + 2\epsilon \int_{\partial\Omega} ((Sv^\epsilon) \cdot n) \cdot v^\epsilon$$

$$= -2\epsilon \|Sv^\epsilon\|^2 - 2\epsilon\zeta \int_{\partial\Omega} |v_\tau^\epsilon|^2. \tag{3.5}$$

Similarly, we have

$$(\epsilon \Delta H^\epsilon, H^\epsilon) = -2\epsilon \|SH^\epsilon\|^2 - 2\epsilon\zeta \int_{\partial\Omega} |H_\tau^\epsilon|^2. \tag{3.6}$$

Putting (3.5) and (3.6) into (3.4), we then obtain (3.3). $\qquad\qquad\square$

Now, we turn to the higher order energy estimates.

Lemma 3.2. *For every* $m \geq 0$, *a smooth solution of the problem* (1.1)-(1.6) *satisfies the estimate*

$$\frac{d}{dt}(\|v^\epsilon(t)\|_m^2 + \|H^\epsilon(t)\|_m^2) + \epsilon(\|\nabla v^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_m^2)$$

$$\leq C\left(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}}\right)(\|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2)$$

$$+ C\|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + C\epsilon^{-1}\|\nabla P_2^\epsilon\|_{m-1}^2, \tag{3.7}$$

where the pressure $P^\epsilon := p^\epsilon - \frac{1}{2}(|v^\epsilon|^2 - |H^\epsilon|^2) := P_1^\epsilon + P_2^\epsilon$. *Here,* P_1^ϵ *is the "Euler" part of the pressure which solves*

$$\begin{cases} \Delta P_1^\epsilon = -\nabla \cdot (v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) & in \quad \Omega, \\ \partial_n P_1^\epsilon = -(v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) \cdot n & on \quad \partial\Omega \end{cases} \tag{3.8}$$

and P_2^ϵ *is the "Navier-Stokes" part of the pressure which solves*

$$\begin{cases} \Delta P_2^\epsilon = 0 & in \quad \Omega, \\ \partial_n P_2^\epsilon = \epsilon \Delta v^\epsilon \cdot n & on \quad \partial\Omega. \end{cases} \tag{3.9}$$

Proof. The estimate for $m = 0$ has been given in Lemma 3.1. Now we assume Lemma 3.2 have been proved for $|\alpha| \leq m - 1$ and prove that it holds for $|\alpha| = m$. We apply Z^α to (1.1)-(1.2) for $|\alpha| = m$ to obtain

$$\partial_t Z^\alpha v^\epsilon + v^\epsilon \cdot \nabla Z^\alpha v^\epsilon - H^\epsilon \cdot \nabla Z^\alpha H^\epsilon + Z^\alpha \nabla P^\epsilon = \epsilon Z^\alpha \Delta v^\epsilon + \mathcal{C}_1,$$

$$\partial_t Z^\alpha H^\epsilon + v^\epsilon \cdot \nabla Z^\alpha H^\epsilon - H^\epsilon \cdot \nabla Z^\alpha v^\epsilon = \epsilon Z^\alpha \Delta H^\epsilon + \mathcal{C}_2,$$

where

$$\mathcal{C}_1 := -[Z^\alpha, v^\epsilon \cdot \nabla]v^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]H^\epsilon,$$

$$\mathcal{C}_2 := -[Z^\alpha, v^\epsilon \cdot \nabla]H^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]v^\epsilon.$$

Consequently, we get from the standard energy estimate that

$$\frac{1}{2}\frac{d}{dt}(\|Z^\alpha v^\epsilon\|^2 + \|Z^\alpha H^\epsilon\|^2) = \epsilon(Z^\alpha \Delta v^\epsilon, Z^\alpha v^\epsilon) + \epsilon(Z^\alpha \Delta H^\epsilon, Z^\alpha H^\epsilon)$$
$$+ (\mathcal{C}_1, Z^\alpha v^\epsilon) + (\mathcal{C}_2, Z^\alpha H^\epsilon) - (Z^\alpha \nabla P^\epsilon, Z^\alpha v^\epsilon). \tag{3.10}$$

First, by Lemma 2.4, we obtain

$$|(\mathcal{C}_1, Z^\alpha v^\epsilon) + (\mathcal{C}_2, Z^\alpha H^\epsilon)| \leq C\left(\|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}}\right)$$
$$(\|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2). \tag{3.11}$$

Next, we estimate the terms with the viscosity coefficient ϵ. We have

$$\epsilon \int_\Omega Z^\alpha \Delta v^\epsilon \cdot Z^\alpha v^\epsilon = 2\epsilon \int_\Omega (\nabla \cdot Z^\alpha S v^\epsilon) \cdot Z^\alpha v^\epsilon + 2\epsilon \int_\Omega ([Z^\alpha, \nabla \cdot] S v^\epsilon) \cdot Z^\alpha v^\epsilon. \tag{3.12}$$

Now, by integrating by parts, we get from the first term on the right hand side of (3.12) that

$$\epsilon \int_\Omega (\nabla \cdot Z^\alpha S v^\epsilon) \cdot Z^\alpha v^\epsilon = -\epsilon \int_\Omega Z^\alpha S v^\epsilon \cdot \nabla Z^\alpha v^\epsilon + \epsilon \int_{\partial\Omega} ((Z^\alpha S v^\epsilon) \cdot n) \cdot Z^\alpha v^\epsilon$$
$$= -\epsilon \|S(Z^\alpha v^\epsilon)\|^2 - \epsilon \int_\Omega [Z^\alpha, S] v^\epsilon \cdot \nabla Z^\alpha v^\epsilon$$
$$+ \epsilon \int_{\partial\Omega} ((Z^\alpha S v^\epsilon) \cdot n) \cdot Z^\alpha v^\epsilon. \tag{3.13}$$

Thanks to Lemma 2.2, there exists a $c_0 > 0$ such that

$$\epsilon \int_\Omega (\nabla \cdot Z^\alpha S v^\epsilon) \cdot Z^\alpha v^\epsilon \leq -c_0 \epsilon \|\nabla(Z^\alpha v^\epsilon)\|^2 + C\|v^\epsilon\|_m^2 + C\epsilon \|\nabla Z^\alpha v^\epsilon\| \|\nabla v^\epsilon\|_{m-1}$$
$$+ \epsilon \int_{\partial\Omega} ((Z^\alpha S v^\epsilon) \cdot n) \cdot Z^\alpha v^\epsilon. \tag{3.14}$$

It remains to estimate the boundary term of (3.14). Before we treat the boundary term, we have the following observations. Due to the Navier boundary condition (2.4), we get

$$|\Pi \partial_n v^\epsilon|_{H^m(\partial\Omega)} \leq |\theta(v^\epsilon)|_{H^m(\partial\Omega)} + 2\zeta |\Pi v^\epsilon|_{H^m(\partial\Omega)} \leq C |v^\epsilon|_{H^m(\partial\Omega)}. \tag{3.15}$$

To estimate the normal part of $\partial_n v^\epsilon$, we can use the divergence free condition to write

$$\nabla \cdot v^\epsilon = \partial_n v^\epsilon \cdot n + (\Pi \partial_{y_1} v^\epsilon)^1 + (\Pi \partial_{y_2} v^\epsilon)^2. \tag{3.16}$$

Hence, we easily get

$$|\partial_n v^\epsilon \cdot n|_{H^{m-1}(\partial\Omega)} \leq C |v^\epsilon|_{H^m(\partial\Omega)}. \tag{3.17}$$

From (3.15) and (3.17), we have

$$|\nabla v^\epsilon|_{H^{m-1}(\partial\Omega)} \leq C |v^\epsilon|_{H^m(\partial\Omega)}. \tag{3.18}$$

Thanks to $v^\epsilon \cdot n = 0$ on the boundary, we immediately obtain that

$$|(Z^\alpha v^\epsilon) \cdot n|_{H^1(\partial\Omega)} \leq C |v^\epsilon|_{H^m(\partial\Omega)}, \quad |\alpha| = m. \tag{3.19}$$

Now we return to deal with the boundary term of (3.14) as follows

$$\int_{\partial\Omega}((Z^\alpha Sv^\epsilon)\cdot n)\cdot Z^\alpha v^\epsilon = \int_{\partial\Omega} Z^\alpha(\Pi(Sv^\epsilon\cdot n))\cdot \Pi Z^\alpha v^\epsilon$$
$$+ \int_{\partial\Omega} Z^\alpha(\partial_n v^\epsilon\cdot n)Z^\alpha v^\epsilon\cdot n + \mathcal{C}_b^v,$$

where

$$\mathcal{C}_b^v = \int_{\partial\Omega}[\Pi, Z^\alpha](Sv^\epsilon\cdot n)\cdot \Pi Z^\alpha v^\epsilon + \int_{\partial\Omega}[n, Z^\alpha](Sv^\epsilon\cdot n)Z^\alpha v^\epsilon\cdot n.$$

Due to (3.18) and (1.4), we can easily obtain that

$$|C_b^v| \le C\,|\nabla v^\epsilon|_{H^{m-1}(\partial\Omega)}|v^\epsilon|_{H^m(\partial\Omega)} \le C\,|v^\epsilon|_{H^m(\partial\Omega)}^2, \tag{3.20}$$

$$\Big|\int_{\partial\Omega} Z^\alpha(\Pi(Sv^\epsilon\cdot n))\cdot \Pi Z^\alpha v^\epsilon\Big| \le C\,|v^\epsilon|_{H^m(\partial\Omega)}^2. \tag{3.21}$$

By integrating by parts along the boundary, we have that

$$\Big|\int_{\partial\Omega} Z^\alpha(\partial_n v^\epsilon\cdot n)Z^\alpha v^\epsilon\cdot n\Big| \le C\,|\partial_n v^\epsilon\cdot n|_{H^{m-1}(\partial\Omega)}|Z^\alpha v^\epsilon\cdot n|_{H^1(\partial\Omega)} \le C\,|v^\epsilon|_{H^m(\partial\Omega)}^2. \tag{3.22}$$

Hence, we get from (3.13), (3.14), and (3.20)-(3.22) that

$$\epsilon\int_\Omega(\nabla\cdot Z^\alpha Sv^\epsilon)\cdot Z^\alpha v^\epsilon \le C\left(\|v^\epsilon\|_m^2 + \epsilon\|\nabla Z^m v^\epsilon\|\|\nabla v^\epsilon\|_{m-1} + \epsilon|v^\epsilon|_{H^m(\partial\Omega)}^2\right)$$
$$- c_0\epsilon\|\nabla(Z^\alpha v^\epsilon)\|^2. \tag{3.23}$$

Next, we deal with the second term of the right hand side of (3.12), i.e. $\epsilon\int_\Omega([Z^\alpha,\nabla\cdot]Sv^\epsilon)\cdot Z^\alpha v^\epsilon$. We can expand it as a sum of terms under the form

$$\epsilon\int_\Omega\beta_k\partial_k(Z^{\tilde\alpha}Sv^\epsilon)\cdot Z^\alpha v^\epsilon, \quad |\tilde\alpha|\le m-1.$$

By using integrations by parts and (3.18), we have

$$\epsilon\Big|\int_\Omega\beta_k\partial_k(Z^{\tilde\alpha}Sv^\epsilon)\cdot Z^\alpha v^\epsilon\Big| \le C\,\epsilon(\|\nabla Z^{m-1}v^\epsilon\|\|\nabla Z^m v^\epsilon\| + \|v^\epsilon\|_m^2 + |v^\epsilon|_{H^m(\partial\Omega)}^2). \tag{3.24}$$

Consequently, from (3.23) and (3.24), we get

$$\epsilon\Big|\int_\Omega Z^\alpha\Delta v^\epsilon\cdot Z^\alpha v^\epsilon\Big| \le C\left\{\|v^\epsilon\|_m^2 + \epsilon\|\nabla Z^m v^\epsilon\|\|\nabla v^\epsilon\|_{m-1} + \epsilon|v^\epsilon|_{H^m(\partial\Omega)}^2\right.$$
$$\left. + \epsilon\|\nabla Z^m v^\epsilon\|\|\nabla Z^{m-1}v^\epsilon\|_{m-1}\right\} - c_0\epsilon\|\nabla(Z^\alpha v^\epsilon)\|^2. \tag{3.25}$$

Similarly, for the term $\epsilon(Z^\alpha\Delta H^\epsilon\cdot Z^\alpha H^\epsilon)$ in the right hand side of (3.10), we have

$$\epsilon\Big|\int_\Omega Z^\alpha\Delta H^\epsilon\cdot Z^\alpha H^\epsilon\Big| \le C\left\{\|H^\epsilon\|_m^2 + \epsilon\|\nabla Z^m H^\epsilon\|\|\nabla H^\epsilon\|_{m-1} + \epsilon|H^\epsilon|_{H^m(\partial\Omega)}^2\right.$$
$$\left. + \epsilon\|\nabla Z^m H^\epsilon\|\|\nabla Z^{m-1}H^\epsilon\|_{m-1}\right\} - c_0\epsilon\|\nabla(Z^\alpha H^\epsilon)\|^2. \tag{3.26}$$

Finally, we estimate the term involving the pressure P^ϵ in (3.10). We have

$$\left|\int_\Omega Z^\alpha \nabla P^\epsilon \cdot Z^\alpha v^\epsilon\right| \le \|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + \left|\int_\Omega Z^\alpha \nabla P_2^\epsilon \cdot Z^\alpha v^\epsilon\right|$$

$$\le \|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + C\|\nabla P_2^\epsilon\|_{m-1}\|v^\epsilon\|_m$$

$$+ \left|\int_\Omega \nabla Z^\alpha P_2^\epsilon \cdot Z^\alpha v^\epsilon\right|. \tag{3.27}$$

Now, we focus on the last term of (3.27). By integrating by parts, we obtain

$$\left|\int_\Omega \nabla Z^\alpha P_2^\epsilon \cdot Z^\alpha v^\epsilon\right| \le C\,\|\nabla P_2^\epsilon\|_{m-1}\|\nabla Z^\alpha v^\epsilon\| + \left|\int_{\partial\Omega} Z^\alpha P_2^\epsilon Z^\alpha v^\epsilon \cdot n\right|.$$

To estimate the boundary term, we note that when $m = 1$, (3.7) can be obtained easily. Here, we assume that $m \ge 2$. By integrating by parts along the boundary, we get

$$\left|\int_{\partial\Omega} Z^\alpha P_2^\epsilon Z^\alpha v^\epsilon \cdot n\right| \le C\,|Z^{\widetilde\alpha} P_2^\epsilon|_{L^2(\partial\Omega)}|Z^\alpha v^\epsilon \cdot n|_{H^1(\partial\Omega)},$$

where $|\widetilde\alpha| = m - 1$. By using (3.19) and Lemma 2.3, we have

$$\left|\int_\Omega Z^\alpha \nabla P^\epsilon \cdot Z^\alpha v^\epsilon\right| \le \|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + C\,\|\nabla P_2^\epsilon\|_{m-1}\|v^\epsilon\|_m$$

$$+ C\,\|\nabla P_2^\epsilon\|_{m-1}\|\nabla Z^\alpha v^\epsilon\| + \epsilon^{-1}\|\nabla P_2^\epsilon\|_{m-1}^2$$

$$+ \epsilon(\|\nabla v^\epsilon\|_m\|v^\epsilon\|_m + \|v^\epsilon\|_m^2). \tag{3.28}$$

Consequently, from (3.11), (3.25)-(3.27) and (3.28), we have

$$\frac{1}{2}\frac{d}{dt}(\|Z^\alpha v^\epsilon\|^2 + \|Z^\alpha H^\epsilon\|^2) + c_0\epsilon\|\nabla(Z^\alpha v^\epsilon)\|^2 + c_0\epsilon\|\nabla(Z^\alpha v^\epsilon)\|^2$$

$$\le C\,(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2)$$

$$+ C\left\{\epsilon\|\nabla Z^m v^\epsilon\|\|\nabla v^\epsilon\|_{m-1} + \epsilon|v^\epsilon|_{H^m(\partial\Omega)}^2 + \epsilon\|\nabla Z^m v^\epsilon\|\|\nabla Z^{m-1}v^\epsilon\|_{m-1}\right.$$

$$+ \epsilon\|\nabla Z^m H^\epsilon\|\|\nabla H^\epsilon\|_{m-1} + \epsilon|H^\epsilon|_{H^m(\partial\Omega)}^2 + \epsilon\|\nabla Z^m H^\epsilon\|\|\nabla Z^{m-1}H^\epsilon\|_{m-1}$$

$$+ \|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + \|\nabla P_2^\epsilon\|_{m-1}\|v^\epsilon\|_m + \|\nabla P_2^\epsilon\|_{m-1}\|\nabla Z^m v^\epsilon\|$$

$$+ \epsilon^{-1}\|\nabla P_2^\epsilon\|_{m-1}^2 + \epsilon(\|\nabla v^\epsilon\|_m\|v^\epsilon\|_m + \|v^\epsilon\|_m^2)\Big\}.$$

Next, by using Lemma 2.3, Young's inequality, the assumptions with respect to $|\alpha| \le m - 1$, we have

$$\frac{1}{2}\frac{d}{dt}(\|v^\epsilon\|_m^2 + \|H^\epsilon\|_m^2) + c_0\epsilon\|\nabla v^\epsilon\|_{m-1}^2 + c_0\epsilon\|\nabla v^\epsilon\|_{m-1}^2$$

$$\le C(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2)$$

$$+ C(\|\nabla^2 P_1^\epsilon\|_{m-1}\|v^\epsilon\|_m + \epsilon^{-1}\|\nabla P_2^\epsilon\|_{m-1}^2).$$

This ends the proof of Lemma 3.2. $\qquad\square$

3.2. **Normal Derivative Estimates.** In this subsection, we provide the estimates for $\|\nabla v^\epsilon\|_{m-1}$ and $\|\nabla H^\epsilon\|_{m-1}$. Noticing that

$$\|\chi\partial_{y^i} v^\epsilon\|_{m-1} \leq C \|v^\epsilon\|_m, \quad \|\chi\partial_{y^i} H^\epsilon\|_{m-1} \leq C \|H^\epsilon\|_m, \quad i=1,2,$$

it suffices to estimate $\|\chi\partial_n v^\epsilon\|_{m-1}$ and $\|\chi\partial_n H^\epsilon\|_{m-1}$, where χ is compactly supported in one of the Ω_i and with value one in a vicinity of the boundary. We shall thus use the local coordinates (2.3).

Due to (3.16), we immediately obtain that

$$\|\chi\partial_n v^\epsilon \cdot n\|_{m-1} \leq C \|v^\epsilon\|_m, \quad \|\chi\partial_n H^\epsilon \cdot n\|_{m-1} \leq C \|H^\epsilon\|_m. \tag{3.29}$$

Thus, it remains to estimate $\|\chi\Pi\partial_n v^\epsilon\|_{m-1}$ and $\|\chi\Pi\partial_n H^\epsilon\|_{m-1}$. We define

$$\eta_v^\epsilon := \chi\Pi((\nabla v^\epsilon + (\nabla v^\epsilon)^t)n) + 2\zeta\chi\Pi v^\epsilon,$$

$$\eta_H^\epsilon := \chi\Pi((\nabla H^\epsilon + (\nabla H^\epsilon)^t)n) + 2\zeta\chi\Pi H^\epsilon.$$

In view of the Navier boundary conditions (1.4) and (1.5), we have

$$\eta_v^\epsilon = 0, \quad \eta_H^\epsilon = 0 \quad \text{on} \quad \partial\Omega.$$

Moreover, since η_v^ϵ and η_H^ϵ have another forms in the vicinity of the boundary $\partial\Omega$:

$$\eta_v^\epsilon = \chi\Pi\partial_n v^\epsilon + \chi\Pi(\nabla(v^\epsilon \cdot n) - \nabla n \cdot v^\epsilon - v^\epsilon \times (\nabla \times n) + 2\zeta v^\epsilon), \tag{3.30}$$

$$\eta_H^\epsilon = \chi\Pi\partial_n H^\epsilon + \chi\Pi(\nabla(H^\epsilon \cdot n) - \nabla n \cdot H^\epsilon - H^\epsilon \times (\nabla \times n) + 2\zeta H^\epsilon), \tag{3.31}$$

we easily get that

$$\|\chi\Pi\partial_n v^\epsilon\|_{m-1} \leq C \left(\|\eta_v^\epsilon\|_{m-1} + \|v^\epsilon\|_m + \|\partial_n v^\epsilon \cdot n\|_{m-1}\right)$$

$$\leq C \left(\|\eta_v^\epsilon\|_{m-1} + \|v^\epsilon\|_m\right),$$

$$\|\chi\Pi\partial_n H^\epsilon\|_{m-1} \leq C \left(\|\eta_H^\epsilon\|_{m-1} + \|H^\epsilon\|_m + \|\partial_n H^\epsilon \cdot n\|_{m-1}\right)$$

$$\leq C \left(\|\eta_H^\epsilon\|_{m-1} + \|H^\epsilon\|_m\right).$$

Hence, it remains to estimate $\|\eta_v^\epsilon\|_{m-1}$ and $\|\eta_H^\epsilon\|_{m-1}$.

We have the following conormal estimates for η_v^ϵ and η_H^ϵ.

Lemma 3.3. *For every $m \geq 1$, we have*

$$\frac{1}{2}\frac{d}{dt}(\|\eta_v^\epsilon\|_{m-1}^2 + \|\eta_H^\epsilon\|_{m-1}^2) + \epsilon(\|\nabla\eta_v^\epsilon\|_{m-1}^2 + \|\nabla\eta_H^\epsilon\|_{m-1}^2)$$

$$\leq C\left(1 + \|v^\epsilon\|_{2,\infty} + \|\nabla v^\epsilon\|_{1,\infty} + \|H^\epsilon\|_{2,\infty} + \|\nabla H^\epsilon\|_{1,\infty}\right)$$

$$\times \left(\|\eta_v^\epsilon\|_{m-1}^2 + \|\eta_H^\epsilon\|_{m-1}^2 + \|v^\epsilon\|_m^2 + \|H^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|\nabla H^\epsilon\|_{m-1}^2\right)$$

$$+ C\left((\|\eta_v^\epsilon\|_{m-1} + \|v^\epsilon\|_m)(\|\nabla^2 P_1^\epsilon\|_{m-1} + \|\nabla P^\epsilon\|_{m-1}) + \epsilon^{-1}\|\nabla P_2^\epsilon\|_{m-1}^2\right). \tag{3.32}$$

Proof. Setting $M_v = \nabla v^\epsilon$ and $M_H = \nabla H^\epsilon$, we get from (1.1)-(1.2) that

$$\partial_t M_v - \epsilon\Delta M_v + v^\epsilon \cdot \nabla M_v - H^\epsilon \cdot \nabla M_H = (M_H)^2 - (M_v)^2 - \nabla^2 P^\epsilon,$$

$$\partial_t M_H - \epsilon\Delta M_H + v^\epsilon \cdot \nabla M_H - H^\epsilon \cdot \nabla M_v = M_v M_H - M_H M_v.$$

Hence, η_v^ϵ and η_H^ϵ solve the equations

$$\partial_t \eta_v^\epsilon - \epsilon \Delta \eta_v^\epsilon + v^\epsilon \cdot \nabla \eta_v^\epsilon - H^\epsilon \cdot \nabla \eta_H^\epsilon = F_v^b + F_v^\chi + F_v^\kappa - 2\chi\Pi(\nabla^2 P^\epsilon n), \quad (3.33)$$

$$\partial_t \eta_H^\epsilon - \epsilon \Delta \eta_H^\epsilon + v^\epsilon \cdot \nabla \eta_H^\epsilon - H^\epsilon \cdot \nabla \eta_v^\epsilon = F_H^b + F_H^\chi + F_H^\kappa, \quad (3.34)$$

where

$$F_v^b = -\chi\Pi((\nabla v^\epsilon)^2 + ((\nabla v^\epsilon)^t)^2 - (\nabla H^\epsilon)^2 - ((\nabla H^\epsilon)^t)^2)n - 2\zeta\chi\Pi\nabla P^\epsilon,$$

$$F_v^\chi = -\epsilon\Delta\chi(\Pi 2Sv^\epsilon n + 2\zeta\Pi v^\epsilon) - 2\epsilon\nabla\chi\cdot\nabla(\Pi 2Sv^\epsilon n + 2\zeta\Pi v^\epsilon)$$
$$\qquad + (v^\epsilon\cdot\nabla\chi)\Pi(2Sv^\epsilon n + 2\zeta v^\epsilon) - (H^\epsilon\cdot\nabla\chi)\Pi(2SH^\epsilon n + 2\zeta H^\epsilon),$$

$$F_v^\kappa = \chi(v^\epsilon\cdot\nabla\Pi)(2Sv^\epsilon n + 2\zeta v^\epsilon) + \chi\Pi(2Sv^\epsilon(v^\epsilon\cdot\nabla)n) - \epsilon\chi(\Delta\Pi)(2Sv^\epsilon n + 2\zeta v^\epsilon)$$
$$\qquad - 2\epsilon\chi\nabla\Pi\cdot\nabla(2Sv^\epsilon n + 2\zeta v^\epsilon) - \epsilon\chi\Pi(2Sv^\epsilon\Delta n + 2\nabla Sv^\epsilon\cdot\nabla n)$$
$$\qquad - \chi(H\cdot\nabla\Pi)(2SH^\epsilon n + 2\zeta H^\epsilon) - \chi\Pi(2SH^\epsilon(H\cdot\nabla)n),$$

$$F_H^b = -\chi\Pi(M_H M_v + M_v^t M_H^t - M_v M_H - M_H^t M_v^t)n,$$

$$F_H^\chi = -\epsilon\Delta\chi(\Pi 2SH^\epsilon n + 2\zeta H^\epsilon) - 2\epsilon\nabla\chi\cdot\nabla(\Pi 2SH^\epsilon n + 2\zeta H^\epsilon)$$
$$\qquad + (v^\epsilon\cdot\nabla\chi)\Pi(2SH^\epsilon n + 2\zeta\Pi H^\epsilon) - (H^\epsilon\cdot\nabla\chi)\Pi(2Sv^\epsilon n + 2\zeta\Pi v^\epsilon),$$

$$F_H^\kappa = \chi(v^\epsilon\cdot\nabla\Pi)(2SH^\epsilon n + 2\zeta H^\epsilon) + \chi\Pi(2SH^\epsilon(v^\epsilon\cdot\nabla)n) - \epsilon\chi(\Delta\Pi)(2SH^\epsilon n$$
$$\qquad + 2\zeta H^\epsilon) - 2\epsilon\chi\nabla\Pi\cdot\nabla(2SH^\epsilon n + 2\zeta H^\epsilon) - \epsilon\chi\Pi(2SH^\epsilon\Delta n + 2\nabla SH^\epsilon\cdot\nabla n)$$
$$\qquad - \chi(H\cdot\nabla\Pi)(2Sv^\epsilon n + 2\zeta v^\epsilon) - \chi\Pi(2Sv^\epsilon(H\cdot\nabla)n).$$

Let us start with the case of $m = 1$. By using the standard L^2 energy estimate, we get

$$\frac{1}{2}\frac{d}{dt}(\|\eta_v^\epsilon\|^2 + \|\eta_H^\epsilon\|^2) + \epsilon(\|\nabla\eta_v^\epsilon\|^2 + \|\nabla\eta_H^\epsilon\|^2)$$
$$= (F_v^b + F_v^\chi + F_v^\kappa, \eta_v^\epsilon) + (F_H^b + F_H^\chi + F_H^\kappa, \eta_H^\epsilon) - 2(\chi\Pi(\nabla^2 P^\epsilon n), \eta_v^\epsilon). \quad (3.35)$$

Now we estimate the right-hand side terms of (3.35). We easily arrive at

$$\|F_v^b\|_{m-1} + \|F_H^b\|_{m-1} \leq C(\|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|\nabla v^\epsilon\|_{m-1} + \|\nabla H^\epsilon\|_{m-1})$$
$$+ C\|\nabla P^\epsilon\|_{m-1}, \quad (3.36)$$

$$\|F_v^\kappa\|_{m-1} + \|F_H^\kappa\|_{m-1} \leq C\epsilon(\|\chi\nabla^2 v^\epsilon\|_{m-1} + \|\chi\nabla^2 H^\epsilon\|_{m-1} + \|\nabla H^\epsilon\|_{m-1}$$
$$+ \|\nabla v^\epsilon\|_{m-1} + \|v^\epsilon\|_m + \|H^\epsilon\|_m)$$
$$+ C(\|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_{m-1} + \|H^\epsilon\|_{m-1}$$
$$+ \|\nabla v^\epsilon\|_{m-1} + \|\nabla H^\epsilon\|_{m-1}). \quad (3.37)$$

Next, since F_v^χ and F_H^χ are supported away from the boundary, we can control any derivatives by the norm $\|\cdot\|_m$. We immediately get

$$\|F_v^\chi\|_{m-1} + \|F_H^\chi\|_{m-1} \leq C\epsilon(\|\nabla v^\epsilon\|_m + \|\nabla H^\epsilon\|_m)$$
$$+ C(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m + \|H^\epsilon\|_m).$$
$$(3.38)$$

Finally, we estimate $(\chi\Pi(\nabla^2 P^\epsilon n), \eta_v^\epsilon)$. Noting that $P^\epsilon = P_1^\epsilon + P_2^\epsilon$, we get

$$|(\chi\Pi(\nabla^2 P^\epsilon n), \eta_v^\epsilon)| \le \|\nabla^2 P_1^\epsilon\|\|\eta_v^\epsilon\| + \Big|\int_\Omega \chi\Pi(\nabla^2 P_2^\epsilon n) \cdot \eta_v^\epsilon\Big|. \qquad (3.39)$$

Since $\eta_v^\epsilon = 0$ on the boundary, we can integrate by the parts the last term in (3.39) to obtain

$$\Big|\int_\Omega \chi\Pi(\nabla^2 P_2^\epsilon n) \cdot \eta_v^\epsilon\Big| \le C\|\nabla P_2^\epsilon\|(\|\nabla\eta_v^\epsilon\| + \|\eta_v^\epsilon\|). \qquad (3.40)$$

Consequently, from (3.36)-(3.38), (3.39), (3.40), we have

$$\frac{1}{2}\frac{d}{dt}(\|\eta_v^\epsilon\|^2 + \|\eta_H^\epsilon\|^2) + \epsilon(\|\nabla\eta_v^\epsilon\|^2 + \|\nabla\eta_H^\epsilon\|^2)$$
$$\le C\,\epsilon(\|\chi\nabla^2 v^\epsilon\| + \|\chi\nabla^2 H^\epsilon\| + \|\nabla v^\epsilon\|_1 + \|\nabla H^\epsilon\|_1)(\|\eta_v^\epsilon\| + \|\eta_H^\epsilon\|)$$
$$+ C\,(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_1^2 + \|H^\epsilon\|_1^2 + \|\nabla v^\epsilon\|^2 + \|\nabla H^\epsilon\|^2$$
$$+ \|\eta_v^\epsilon\|^2 + \|\eta_H^\epsilon\|^2) + \|\nabla P_2^\epsilon\|(\|\nabla\eta_v^\epsilon\| + \|\eta_v^\epsilon\|) + \|\nabla^2 P_1^\epsilon\|\|\eta_v^\epsilon\| + \|\nabla P^\epsilon\|\|\eta_v^\epsilon\|.$$
$$(3.41)$$

Due to (3.29) and (3.30), we get that

$$\epsilon\|\chi\nabla^2 v^\epsilon\|_{m-1} \le C\,\epsilon(\|\nabla\eta_v^\epsilon\|_{m-1} + \|\nabla v^\epsilon\|_m + \|v^\epsilon\|_m). \qquad (3.42)$$

Similarly, we get

$$\epsilon\|\chi\nabla^2 H^\epsilon\|_{m-1} \le C\,\epsilon(\|\nabla\eta_H^\epsilon\|_{m-1} + \|\nabla H^\epsilon\|_m + \|H^\epsilon\|_m). \qquad (3.43)$$

By using (3.42), (3.43) and Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}(\|\eta_v^\epsilon\|^2 + \|\eta_H^\epsilon\|^2) + \epsilon(\|\nabla\eta_v^\epsilon\|^2 + \|\nabla\eta_H^\epsilon\|^2)$$
$$\le C\big\{\epsilon(\|\nabla v^\epsilon\|_1 + \|\nabla H^\epsilon\|_1)(\|\eta_v^\epsilon\| + \|\eta_H^\epsilon\|)$$
$$+ (1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_1^2 + \|H^\epsilon\|_1^2 + \|\nabla v^\epsilon\|^2 + \|\nabla H^\epsilon\|^2$$
$$+ \|\eta_v^\epsilon\|^2 + \|\eta_H^\epsilon\|^2) + \epsilon^{-1}\|\nabla P_2^\epsilon\|^2 + \|\eta_v^\epsilon\|(\|\nabla P^\epsilon\| + \|\nabla^2 P_1^\epsilon\|)\big\}. \qquad (3.44)$$

Since $\epsilon(\|\nabla v^\epsilon\|_1 + \|\nabla H^\epsilon\|_1)$ has been estimated in Lemma 3.2, this yields (3.32) for the case of $m = 1$.

Now we assume that Lemma 3.3 is true for $|\alpha| \le m - 2$ and let us consider the situation of $|\alpha| = m - 1$. By applying Z^α to (3.33)-(3.34), we have

$$\partial_t Z^\alpha \eta_v^\epsilon - \epsilon Z^\alpha \Delta\eta_v^\epsilon + v^\epsilon \cdot \nabla Z^\alpha \eta_v^\epsilon - H^\epsilon \cdot \nabla Z^\alpha \eta_H^\epsilon$$
$$= Z^\alpha F_v^b + Z^\alpha F_v^\chi + Z^\alpha F_v^\kappa - Z^\alpha(\chi\Pi(\nabla^2 P^\epsilon n)) + \mathcal{C}_3, \qquad (3.45)$$
$$\partial_t Z^\alpha \eta_H^\epsilon - \epsilon Z^\alpha \Delta\eta_H^\epsilon + v^\epsilon \cdot \nabla Z^\alpha \eta_H^\epsilon - H^\epsilon \cdot \nabla Z^\alpha \eta_v^\epsilon$$
$$= Z^\alpha F_H^b + Z^\alpha F_H^\chi + Z^\alpha F_H^\kappa + \mathcal{C}_4, \qquad (3.46)$$

where

$$\mathcal{C}_3 := -[Z^\alpha, v^\epsilon \cdot \nabla]\eta_v^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]\eta_H^\epsilon,$$
$$\mathcal{C}_4 := -[Z^\alpha, v^\epsilon \cdot \nabla]\eta_H^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]\eta_v^\epsilon.$$

From the standard energy estimate, we get

$$\frac{1}{2}\frac{d}{dt}(\|Z^\alpha\eta_v^\epsilon\|^2 + \|Z^\alpha\eta_H^\epsilon\|^2)$$
$$\leq \epsilon\,(Z^\alpha\Delta\eta_v^\epsilon, Z^\alpha\eta_v^\epsilon) + \epsilon(Z^\alpha\Delta\eta_H^\epsilon, Z^\alpha\eta_H^\epsilon)$$
$$+ (C_1, Z^\alpha\eta_v^\epsilon) + (C_2, Z^\alpha\eta_H^\epsilon) - 2(Z^\alpha(\chi\Pi(\nabla^2 P^\epsilon n)), Z^\alpha\eta_v^\epsilon)$$
$$+ (Z^\alpha F_v^b + Z^\alpha F_v^\chi + Z^\alpha F_v^\kappa, Z^\alpha\eta_v^\epsilon) + (Z^\alpha F_H^b + Z^\alpha F_H^\chi + Z^\alpha F_H^\kappa, Z^\alpha\eta_H^\epsilon).$$
$$(3.47)$$

First, let us estimate $\epsilon(Z^\alpha\Delta\eta_v^\epsilon, Z^\alpha\eta_v^\epsilon)$ and $\epsilon(Z^\alpha\Delta\eta_H^\epsilon, Z^\alpha\eta_H^\epsilon)$. We observe that

$$\int_\Omega Z^\alpha\partial_{ii}\eta_v^\epsilon \cdot Z^\alpha\eta_v^\epsilon = -\int_\Omega |\partial_i Z^\alpha\eta_v^\epsilon|^2 - \int_\Omega [Z^\alpha, \partial_i]\eta_v^\epsilon \cdot \partial_i Z^\alpha\eta_v^\epsilon$$
$$+ \int_\Omega [Z^\alpha, \partial_i]\partial_i\eta_v^\epsilon \cdot Z^\alpha\eta_v^\epsilon, \qquad (3.48)$$

where $i = 1, 2, 3$. To estimate the last two terms on the right hand side of (3.48), we use the structure of the commutator $[Z^\alpha, \partial_i]$ and the expansion $\partial_i = \beta^1\partial_{y^1} + \beta^2\partial_{y^2} + \beta^3\partial_{y^3}$ in the local basis. We have the following expansion

$$[Z^\alpha, \partial_i]\eta_v^\epsilon = \sum_{\gamma, |\gamma|\leq|\alpha|-1} c_\gamma\partial_z Z^\gamma\eta_v^\epsilon + \sum_{\beta, |\beta|\leq|\alpha|} c_\beta Z^\beta\eta_v^\epsilon.$$

This yields the estimates

$$\left|\int_\Omega [Z^\alpha, \partial_i]\eta_v^\epsilon \cdot \partial_i Z^\alpha\eta_v^\epsilon\right| \leq C\,\|\nabla Z^{m-1}\eta_v^\epsilon\|(\|\nabla\eta_v^\epsilon\|_{m-2} + \|\eta_v^\epsilon\|_{m-1}), \qquad (3.49)$$

$$\left|\int_\Omega [Z^\alpha, \partial_i]\partial_i\eta_v^\epsilon \cdot Z^\alpha\eta_v^\epsilon\right| \leq C\,\|\nabla\eta_v^\epsilon\|_{m-1}(\|\nabla\eta_v^\epsilon\|_{m-2} + \|\eta_v^\epsilon\|_{m-1}). \qquad (3.50)$$

Taking the same argument as above, we have

$$\left|\int_\Omega [Z^\alpha, \partial_i]\eta_H^\epsilon \cdot \partial_i Z^\alpha\eta_H^\epsilon\right| \leq C\,\|\nabla Z^{m-1}\eta_H^\epsilon\|(\|\nabla\eta_H^\epsilon\|_{m-2} + \|\eta_H^\epsilon\|_{m-1}), \qquad (3.51)$$

$$\left|\int_\Omega [Z^\alpha, \partial_i]\partial_i\eta_H^\epsilon \cdot Z^\alpha\eta_H^\epsilon\right| \leq C\,\|\nabla\eta_H^\epsilon\|_{m-1}(\|\nabla\eta_H^\epsilon\|_{m-2} + \|\eta_H^\epsilon\|_{m-1}). \qquad (3.52)$$

Consequently, we get from (3.47), (3.49)-(3.52) and Young's inequality that

$$\frac{1}{2}\frac{d}{dt}(\|Z^\alpha\eta_v^\epsilon\|^2 + \|Z^\alpha\eta_H^\epsilon\|^2) + \frac{\epsilon}{2}(\|\nabla Z^{m-1}\eta_v^\epsilon\|^2 + \|\nabla Z^{m-1}\eta_H^\epsilon\|^2)$$
$$\leq C\epsilon\,(\|\nabla\eta_H^\epsilon\|_{m-1}^2 + \|\eta_H^\epsilon\|_{m-1}^2 + \|\nabla\eta_v^\epsilon\|_{m-1}^2 + \|\eta_v^\epsilon\|_{m-1}^2)$$
$$+ (\|F_v^b\|_{m-1} + \|F_v^\chi\|_{m-1} + \|F_v^\kappa\|_{m-1})\|\eta_v^\epsilon\|_{m-1} + \|C_3\|\|\eta_v^\epsilon\|_{m-1}$$
$$+ (\|F_H^b\|_{m-1} + \|F_H^\chi\|_{m-1} + \|F_H^\kappa\|_{m-1})\|\eta_H^\epsilon\|_{m-1} + \|C_4\|\|\eta_H^\epsilon\|_{m-1}$$
$$- 2(Z^\alpha(\chi\Pi(\nabla^2 P^\epsilon n)), Z^\alpha\eta_v^\epsilon). \qquad (3.53)$$

Second, we get from (3.36)-(3.38), (3.42), and (3.43) that

$$\|F_v^b\|_{m-1} + \|F_v^\chi\|_{m-1} + \|F_v^\kappa\|_{m-1}$$
$$\leq C\,(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} + \|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1})$$

$$+ \epsilon\, C\, \|\nabla v^\epsilon\|_m + \epsilon\, C\, \|\nabla \eta_v^\epsilon\|_{m-1} + \|\nabla P^\epsilon\|_{m-1}, \tag{3.54}$$

$$\|F_H^b\|_{m-1} + \|F_H^\chi\|_{m-1} + \|F_H^\kappa\|_{m-1}$$
$$\leq C\,(1 + \|v^\epsilon\|_{W^{1,\infty}} + \|H^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} + \|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1})$$
$$+ \epsilon\, C\, \|\nabla H^\epsilon\|_m + \epsilon\, C\, \|\nabla \eta_H^\epsilon\|_{m-1}. \tag{3.55}$$

Next, we estimate $\|\mathcal{C}_3\|$ and $\|\mathcal{C}_4\|$. In the local coordinates, we observe

$$f \cdot \nabla g = f_1 \partial_{y^1} g + f_2 \partial_{y^2} g + f \cdot N \partial_z g.$$

Hence

$$[Z^\alpha, v^\epsilon \cdot \nabla]\eta_v^\epsilon$$
$$= \sum_{i=1,2} \sum_{|\beta|\geq 1, |\beta|+|\gamma|\leq|\alpha|} Z^\beta v_i^\epsilon Z^\gamma Z_i \eta_v^\epsilon + \sum_{|\beta|\geq 1, |\beta|+|\gamma|\leq|\alpha|} Z^\beta (v_3^\epsilon \cdot N) Z^\gamma \partial_z \eta_v^\epsilon$$
$$= \sum_{i=1,2} \sum_{|\beta|\geq 1, |\beta|+|\gamma|\leq|\alpha|} Z^\beta v_i^\epsilon Z^\gamma Z_i \eta_v^\epsilon + \sum_{|\widetilde{\beta}|\geq 1, |\widetilde{\beta}|+|\widetilde{\gamma}|\leq|\alpha|} Z^{\widetilde{\beta}} (\frac{v_3^\epsilon \cdot N}{\varphi(z)}) Z^{\widetilde{\gamma}} Z_3 \eta_v^\epsilon. \tag{3.56}$$

We can do similar caculations for other terms in $\mathcal{C}_3$ and $\mathcal{C}_4$. Consequently, from (1.4), (1.5) and Lemma 2.4, we get

$$\|\mathcal{C}_3\| \leq C\,(\|v^\epsilon\|_{2,\infty} + \|v^\epsilon\|_{w^{1,\infty}} + \|Z\eta_v^\epsilon\|_{L^\infty})(\|\eta_v^\epsilon\|_{m-1} + \|v^\epsilon\|_m)$$
$$+ C\,(\|H^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{w^{1,\infty}} + \|Z\eta_H^\epsilon\|_{L^\infty})(\|\eta_H^\epsilon\|_{m-1} + \|H^\epsilon\|_m), \tag{3.57}$$
$$\|\mathcal{C}_4\| \leq C\,(\|v^\epsilon\|_{2,\infty} + \|v^\epsilon\|_{w^{1,\infty}} + \|Z\eta_H^\epsilon\|_{L^\infty})(\|\eta_H^\epsilon\|_{m-1} + \|v^\epsilon\|_m)$$
$$+ C\,(\|H^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{w^{1,\infty}} + \|Z\eta_v^\epsilon\|_{L^\infty})(\|\eta_v^\epsilon\|_{m-1} + \|H^\epsilon\|_m). \tag{3.58}$$

Final, it remains to deal with the terms involving the pressure P^ϵ. As above, we use the split $P^\epsilon = P_1^\epsilon + P_2^\epsilon$ and we integrate by parts the terms involving P_2^ϵ. We have

$$|(Z^\alpha(\chi\Pi(\nabla^2 P^\epsilon n)), Z^\alpha \eta_v^\epsilon)| \leq C\,\big(\|\nabla^2 P_1^\epsilon\|_{m-1}\|\eta_v^\epsilon\|_{m-1}$$
$$+ \|\nabla P_2^\epsilon\|_{m-1}(\|\nabla Z^{m-1}\eta_v^\epsilon\| + \|\eta_v^\epsilon\|_{m-1})\big). \tag{3.59}$$

By combining (3.53), (3.54), (3.55), (3.57), (3.58), (3.59) and using the induction assumption and Young's inequality, we complete the proof of Lemma 3.3. $\qquad\square$

3.3. **Pressure Estimates.** It remains to estimate the pressure terms and the L^∞ norms, the aim of this subsection is to give the pressure estimates.

Lemma 3.4. *For every $m \geq 2$, we have the following estimates:*

$$\|\nabla P_1^\epsilon\|_{m-1} + \|\nabla^2 P_1^\epsilon\|_{m-1} \leq C\,(1 + \|v^\epsilon\|_{W^{1,\infty}})(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1})$$
$$+ C\,(1 + \|H^\epsilon\|_{W^{1,\infty}})(\|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1}), \tag{3.60}$$
$$\|\nabla P_2^\epsilon\|_{m-1} \leq C\,\epsilon\,(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1}). \tag{3.61}$$

Proof. Recall that $P^\epsilon = P_1^\epsilon + P_2^\epsilon$ and P_1^ϵ, P_2^ϵ are defined in (3.8) and (3.9), respectively. From the standard elliptic regularity results with Neumann boundary conditions, we obtain that

$$\|\nabla P_1^\epsilon\|_{m-1} + \|\nabla^2 P_1^\epsilon\|_{m-1}$$
$$\leq C \left(\|\nabla v \cdot \nabla v - \nabla H \cdot \nabla H\|_{m-1} + \|v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon\| \right.$$
$$\left. + \left| (v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) \cdot n \right|_{H^{m-\frac{1}{2}}(\partial\Omega)} \right).$$

Due to $v^\epsilon \cdot n = 0$, $H^\epsilon \cdot n = 0$ and Lemma 2.3, we get that

$$\left| (v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) \cdot n \right|_{H^{m-\frac{1}{2}}(\partial\Omega)}) \leq C \left(\|\nabla(v \otimes v)\|_{m-1} + \|v \otimes v\|_m \right.$$
$$\left. + \|\nabla(H \otimes H)\|_{m-1} + \|H \otimes H\|_m \right).$$

Using Lemma 2.4, we get (3.60).

It remains to estimate P_2^ϵ. By using the standard elliptic regularity results with Neumann boundary conditions again, we obtain

$$\|\nabla P_2^\epsilon\|_{m-1} \leq C \epsilon \left| \Delta v^\epsilon \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}.$$

Since

$$\Delta v^\epsilon \cdot n = 2 \Big(\nabla \cdot (S v^\epsilon n) - \sum_j (S v^\epsilon \partial_j n)_j \Big),$$

we can get

$$\left| \Delta v^\epsilon \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \nabla \cdot (S v^\epsilon n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C \left| \nabla v^\epsilon \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}.$$

Due to (2.4) and (3.16), we can further arrive at

$$\left| \Delta v^\epsilon \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \nabla \cdot (S v^\epsilon n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C \left| v^\epsilon \right|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

Let us estimate $\left| \nabla \cdot (S v^\epsilon n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}$. We can use (3.16) to obtain

$$\left| \nabla \cdot (S v^\epsilon n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \partial_n (S v^\epsilon n) \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}$$
$$+ C \left(\left| \Pi(S v^\epsilon n) \right|_{H^{m-\frac{1}{2}}(\partial\Omega)} + \left| \nabla v^\epsilon \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \right).$$

Also, due to (2.4), (3.16) and the Navier boundary conditions, we get

$$\left| \nabla \cdot (S v^\epsilon n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \partial_n (S v^\epsilon n) \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} + \left| v^\epsilon \right|_{H^{m-\frac{1}{2}}(\partial\Omega)}. \qquad (3.62)$$

The first term of the right-hand side of (3.62) have the following estimates

$$\left| \partial_n (S v^\epsilon n) \cdot n \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \partial_n (\partial_n v^\epsilon \cdot n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C \left| \nabla v^\epsilon \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}$$
$$\leq C \left| \partial_n (\partial_n v^\epsilon \cdot n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C \left| v^\epsilon \right|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

By taking the normal derivative of (3.16) and using (2.4), we obtain

$$\left| \partial_n (\partial_n v^\epsilon \cdot n) \right|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C \left| \Pi \partial_n v^\epsilon \right|_{H^{m-\frac{1}{2}}(\partial\Omega)} + C \left| \nabla v^\epsilon \right|_{H^{m-\frac{3}{2}}(\partial\Omega)}$$
$$\leq C \left| v^\epsilon \right|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

Consequently, we have

$$|\Delta v^\epsilon \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C\, |v^\epsilon|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

By Lemma 2.3, we finally get (3.61) which complete the proof of Lemma 3.4.   □

We can get from Lemmas 3.2-3.4 that

$$\|v^\epsilon\|_m^2 + \|H^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|\nabla H^\epsilon\|_{m-1}^2 + \epsilon \int_0^t (\|\nabla^2 v^\epsilon\|_{m-1} + \|\nabla^2 H^\epsilon\|_{m-1})$$

$$\leq C\,(\|v^\epsilon(0)\|_m^2 + \|H^\epsilon(0)\|_m^2 + \|\nabla v^\epsilon(0)\|_{m-1}^2 + \|\nabla H^\epsilon(0)\|_{m-1}^2)$$

$$+ C \int_0^t (1 + \|v^\epsilon\|_{2,\infty} + \|\nabla v^\epsilon\|_{1,\infty} + \|H^\epsilon\|_{2,\infty} + \|\nabla H^\epsilon\|_{1,\infty})$$

$$\times (\|v^\epsilon\|_m^2 + \|H^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|\nabla H^\epsilon\|_{m-1}^2). \tag{3.63}$$

3.4. L^∞ **estimates.** In order to close the estimates in (3.63), we need to give the L^∞ estimates on ∇v^ϵ and ∇H^ϵ. We have

Lemma 3.5. *For $m_0 > 1$, we have the following estimates:*

$$\|v^\epsilon\|_{2,\infty} \leq C(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1}) \leq C N_m(t)^{\frac{1}{2}} \quad m \geq m_0 + 3, \tag{3.64}$$

$$\|H^\epsilon\|_{2,\infty} \leq C(\|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1}) \leq C N_m(t)^{\frac{1}{2}} \quad m \geq m_0 + 3, \tag{3.65}$$

$$\|v^\epsilon\|_{W^{1,\infty}} \leq C(\|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} + \|\partial_z v^\epsilon\|_{L^\infty}) \leq C N_m(t)^{\frac{1}{2}} \quad m \geq m_0 + 2, \tag{3.66}$$

$$\|H^\epsilon\|_{W^{1,\infty}} \leq C(\|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1} + \|\partial_z H^\epsilon\|_{L^\infty}) \leq C N_m(t)^{\frac{1}{2}} \quad m \geq m_0 + 2, \tag{3.67}$$

where $N_m(t)$ is defined in (3.2).

Proof. By using lemma 2.3, we can obtain (3.64)-(3.65), and (3.66)-(3.67) are obvious.   □

Lemma 3.6. *For $m > 6$, we have the following estimate:*

$$\|\nabla v^\epsilon\|_{1,\infty}^2 + \|\nabla H^\epsilon\|_{1,\infty}^2 \leq C\Big(N_m(0) + (1 + t + \epsilon^3 t^2) \int_0^t (N_m(s) + N_m(s)^2) ds\Big).$$

Proof. We observe that, away from the boundary, the following estimates hold:

$$\|\beta_i \nabla v^\epsilon\|_{1,\infty} + \|\beta_i \nabla H^\epsilon\|_{1,\infty} \leq C\,(\|v^\epsilon\|_m + \|H^\epsilon\|_m), \quad m \geq 4,$$

where $\{\beta_i\}$ is a partition of unity subordinated to the covering (2.1). In order to estimate the near boundary parts, we adopt the ideas in the Proposition 21 of [16]. Here, we use a local parametrization in the vicinity of the boundary given by a normal geodesic system:

$$\Psi^n(y, z) = \begin{pmatrix} y \\ \psi(y) \end{pmatrix} - z n(y),$$

where

$$n(y) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix}.$$

Now, we can extend n and Π in the interior by setting

$$n(\Psi^n(y, z)) = n(y), \quad \Pi(\Psi^n(y, z)) = \Pi(y).$$

We observe $\partial_z = \partial_n$ and

$$\left(\partial_{y^i} \right)\Big|_{\Psi^n(y,z)} \cdot \left(\partial_z \right)\Big|_{\Psi^n(y,z)} = 0.$$

Hence, the Riemann metric g has the following form

$$g(y, z) = \begin{pmatrix} \widetilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, the Laplacian in this coordinate system reads:

$$\Delta f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\widetilde{g}} f,$$

where $|g|$ is the determinant of the matrix g and $\Delta_{\widetilde{g}}$ is defined by

$$\Delta_{\widetilde{g}} f = \frac{1}{|\widetilde{g}|^{\frac{1}{2}}} \sum_{1 \le i,j \le 2} \partial_{y^i} (\widetilde{g}^{ij} |\widetilde{g}|^{\frac{1}{2}} \partial_{y^j} f). \tag{3.68}$$

Here, $\{\widetilde{g}^{ij}\}$ is the inverse matrix to g and (3.68) only involves tangential derivatives.

With these preparation, we now turn to estimate the near boundary parts. Due to (3.16), (3.64) and (3.65), we have

$$\|\chi \nabla v^\epsilon\|_{1,\infty} \le C \left(\|\chi \Pi \partial_n v^\epsilon\|_{1,\infty} + \|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} \right), \tag{3.69}$$

$$\|\chi \nabla H^\epsilon\|_{1,\infty} \le C \left(\|\chi \Pi \partial_n H^\epsilon\|_{1,\infty} + \|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1} \right). \tag{3.70}$$

Hence, we need to estimate $\|\chi \Pi \partial_n v^\epsilon\|_{1,\infty}$ and $\|\chi \Pi \partial_n H^\epsilon\|_{1,\infty}$. To this end, we first introduce the vorticity

$$\omega_v^\epsilon = \nabla \times v^\epsilon, \quad \omega_H^\epsilon = \nabla \times H^\epsilon.$$

We find that

$$\Pi(\omega_v^\epsilon \times n) = \Pi(\nabla v^\epsilon - (\nabla v^\epsilon)^t) n$$
$$= \Pi(\partial_n v^\epsilon - \nabla(v^\epsilon \cdot n) + v^\epsilon \cdot \nabla n + v^\epsilon \times (\nabla \times n)). \tag{3.71}$$

Consequently, we have

$$\|\chi \Pi \partial_n v^\epsilon\|_{1,\infty} \le C \left(\|\chi \Pi(\omega_v^\epsilon \times n)\|_{1,\infty} + \|v^\epsilon\|_{2,\infty} \right). \tag{3.72}$$

By using (3.64) again, we get

$$\|\chi \nabla v^\epsilon\|_{1,\infty} \le C \left(\|\chi \Pi(\omega_v^\epsilon \times n)\|_{1,\infty} + \|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} \right). \tag{3.73}$$

Similar to v^ϵ, we have the following estimates for H^ϵ,

$$\|\chi\nabla H^\epsilon\|_{1,\infty} \le C\left(\|\chi\Pi(\omega_H^\epsilon \times n)\|_{1,\infty} + \|H^\epsilon\|_m + \|\nabla H^\epsilon\|_{m-1}\right). \tag{3.74}$$

Below we estimate $\|\chi\Pi(\omega_v^\epsilon \times n)\|_{1,\infty}$ and $\|\chi\Pi(\omega_H^\epsilon \times n)\|_{1,\infty}$. We know that ω_v^ϵ and ω_H^ϵ satisfy

$$\partial_t\omega_v^\epsilon - \epsilon\Delta\omega_v^\epsilon + v^\epsilon \cdot \nabla\omega_v^\epsilon - H^\epsilon \cdot \nabla\omega_H^\epsilon + \omega_H^\epsilon \cdot \nabla H^\epsilon - \omega_v^\epsilon \cdot \nabla v^\epsilon = 0,$$

$$\partial_t\omega_H^\epsilon - \epsilon\Delta\omega_H^\epsilon + v^\epsilon \cdot \nabla\omega_H^\epsilon - H^\epsilon \cdot \nabla\omega_v^\epsilon + [\nabla\times, v^\epsilon \cdot \nabla]H^\epsilon - [\nabla\times, H^\epsilon \cdot \nabla]v^\epsilon = 0.$$

By setting

$$\widetilde{\omega}_v^\epsilon(y,z) := \omega_v^\epsilon(\Psi^n(y,z)), \qquad \widetilde{v}^\epsilon(y,z) := v^\epsilon(\Psi^n(y,z)),$$

$$\widetilde{\omega}_H^\epsilon(y,z) := \omega_H^\epsilon(\Psi^n(y,z)), \quad \widetilde{H}^\epsilon(y,z) := H^\epsilon(\Psi^n(y,z)),$$

we have

$$\partial_t\widetilde{\omega}_v^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{\omega}_v^\epsilon + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{\omega}_v^\epsilon + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{\omega}_v^\epsilon - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{\omega}_H^\epsilon - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{\omega}_H^\epsilon$$
$$- \widetilde{H}^\epsilon \cdot n\partial_z\widetilde{\omega}_H^\epsilon = \epsilon(\partial_{zz}\widetilde{\omega}_v^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{\omega}_v^\epsilon + \Delta_{\widetilde{g}}\widetilde{\omega}_v^\epsilon) + \overline{F}^v,$$

$$\partial_t\widetilde{\omega}_H^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{\omega}_H^\epsilon + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{\omega}_H^\epsilon + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{\omega}_H^\epsilon - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{\omega}_v^\epsilon - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{\omega}_v^\epsilon$$
$$- \widetilde{H}^\epsilon \cdot n\partial_z\widetilde{\omega}_v^\epsilon = \epsilon(\partial_{zz}\widetilde{\omega}_H^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{\omega}_H^\epsilon + \Delta_{\widetilde{g}}\widetilde{\omega}_H^\epsilon) + \overline{F}^H,$$

$$\partial_t\widetilde{v}^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{v} + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{v} + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{v} - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{H} - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{H} - \widetilde{H}^\epsilon \cdot n\partial_z\widetilde{H}$$
$$= \epsilon(\partial_{zz}\widetilde{v}^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{v}^\epsilon + \Delta_{\widetilde{g}}\widetilde{v}^\epsilon) - (\nabla P^\epsilon) \circ \Psi^n,$$

$$\partial_t\widetilde{H}^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{H} + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{H} + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{H} - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{v} - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{v} - \widetilde{H}^\epsilon \cdot n\partial_z\widetilde{v}$$
$$= \epsilon(\partial_{zz}\widetilde{H}^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{H}^\epsilon + \Delta_{\widetilde{g}}\widetilde{H}^\epsilon),$$

where

$$\overline{F}^v := \omega_v^\epsilon \cdot \nabla v^\epsilon - \omega_H^\epsilon \cdot \nabla H^\epsilon, \quad \overline{F}^H := [\nabla\times, H^\epsilon \cdot \nabla]v^\epsilon - [\nabla\times, v^\epsilon \cdot \nabla]H^\epsilon.$$

By using (2.4) and (3.71) on the boundary, we have

$$\Pi(\widetilde{\omega}_v^\epsilon \times n) = 2\Pi(\widetilde{v}^\epsilon \cdot \nabla n - \zeta\widetilde{v}^\epsilon), \quad \Pi(\widetilde{\omega}_H^\epsilon \times n) = 2\Pi(\widetilde{H}^\epsilon \cdot \nabla n - \zeta\widetilde{H}^\epsilon), \ z = 0.$$

Consequently, we introduce the following quantities

$$\widetilde{\eta}_v^\epsilon(y,z) := \chi\Pi(\widetilde{\omega}_v^\epsilon \times n - 2\widetilde{v}^\epsilon \cdot \nabla n + 2\zeta\widetilde{v}^\epsilon),$$

$$\widetilde{\eta}_H^\epsilon(y,z) := \chi\Pi(\widetilde{\omega}_H^\epsilon \times n - 2\widetilde{H}^\epsilon \cdot \nabla n + 2\zeta\widetilde{H}^\epsilon).$$

Noting that $\widetilde{\eta}_v^\epsilon(y,0) = 0$ and $\widetilde{\eta}_H^\epsilon(y,0) = 0$, we easily get

$$\partial_t\widetilde{\eta}_v^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{\eta}_v^\epsilon + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{\eta}_v^\epsilon + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{\eta}_v^\epsilon - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{\eta}_H^\epsilon - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{\eta}_H^\epsilon$$
$$- \widetilde{H}^\epsilon \cdot n\partial_z\widetilde{\eta}_H^\epsilon = \epsilon(\partial_{zz}\widetilde{\eta}_v^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{\eta}_v^\epsilon) + \chi\Pi\overline{F}^v \times n + \overline{F}_v^v + \overline{F}_v^\chi + \overline{F}_v^\kappa, \tag{3.75}$$
$$\partial_t\widetilde{\eta}_H^\epsilon + (\widetilde{v}^\epsilon)^1\partial_{y^1}\widetilde{\eta}_H^\epsilon + (\widetilde{v}^\epsilon)^2\partial_{y^2}\widetilde{\eta}_H^\epsilon + \widetilde{v}^\epsilon \cdot n\partial_z\widetilde{\eta}_H^\epsilon - (\widetilde{H}^\epsilon)^1\partial_{y^1}\widetilde{\eta}_v^\epsilon - (\widetilde{H}^\epsilon)^2\partial_{y^2}\widetilde{\eta}_v^\epsilon$$

$$-\widetilde{H}^\epsilon \cdot n\partial_z\widetilde{\eta}_v^\epsilon = \epsilon(\partial_{zz}\widetilde{\eta}_H^\epsilon + \frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{\eta}_H^\epsilon) + \overline{F}_H^\chi + \overline{F}_H^\kappa + \chi\Pi\overline{F}^H \times n, \qquad (3.76)$$

where

$$\overline{F}_v^v = 2\chi\Pi(\nabla P^\epsilon \cdot \nabla n - \zeta\nabla P^\epsilon) \circ \Psi^n,$$

$$\overline{F}_v^\chi = (((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\chi)\Pi(\widetilde{\omega}_v^\epsilon \times n - 2\widetilde{v}^\epsilon \cdot \nabla n + 2\zeta\widetilde{v}^\epsilon)$$
$$- (((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\chi)\Pi(\widetilde{\omega}_H^\epsilon \times n - 2\widetilde{H}^\epsilon \cdot \nabla n + 2\zeta\widetilde{H}^\epsilon)$$
$$- \epsilon(\partial_{zz}\chi + 2\partial_z\chi\partial_z + \frac{1}{2}\partial_z(\ln|g|)\partial_z\chi)\Pi(\widetilde{\omega}_v^\epsilon \times n - 2\widetilde{v}^\epsilon \cdot \nabla n + 2\zeta\widetilde{v}^\epsilon),$$

$$\overline{F}_v^\kappa = \chi(((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})\Pi)(\widetilde{\omega}_v^\epsilon \times n - 2\widetilde{v}^\epsilon \cdot \nabla n + 2\zeta\widetilde{v}^\epsilon) + \epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{\omega}_v^\epsilon \times n)$$
$$- \chi(((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2})\Pi)(\widetilde{\omega}_H^\epsilon \times n - 2\widetilde{H}^\epsilon \cdot \nabla n + 2\zeta\widetilde{H}^\epsilon)$$
$$+ \chi\Pi((((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2})\nabla n)\widetilde{H}^\epsilon) - \chi\Pi(\widetilde{\omega}_H^\epsilon \times ((\widetilde{H}^\epsilon)^1\partial_{y^1}$$
$$+ (\widetilde{H}^\epsilon)^2\partial_{y^2})n) - 2\epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{v}^\epsilon\nabla n) - \chi\Pi((((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})\nabla n)\widetilde{v}^\epsilon)$$
$$+ \chi\Pi(\widetilde{\omega}_v^\epsilon \times ((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})n) + 2\zeta\epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{v}^\epsilon),$$

$$\overline{F}_H^\chi = (((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\chi)\Pi(\widetilde{\omega}_H^\epsilon \times n - 2\widetilde{H}^\epsilon \cdot \nabla n + 2\zeta\widetilde{H}^\epsilon)$$
$$- (((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\chi)\Pi(\widetilde{\omega}_v^\epsilon \times n - 2\widetilde{v}^\epsilon \cdot \nabla n + 2\zeta\widetilde{v}^\epsilon)$$
$$- \epsilon(\partial_{zz}\chi + 2\partial_z\chi\partial_z + \frac{1}{2}\partial_z(\ln|g|)\partial_z\chi)\Pi(\widetilde{\omega}_H^\epsilon \times n - 2\widetilde{H}^\epsilon \cdot \nabla n + 2\zeta\widetilde{H}^\epsilon),$$

$$\overline{F}_H^\kappa = \chi(((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})\Pi)(\widetilde{\omega}_H^\epsilon \times n - \widetilde{H}^\epsilon \cdot \nabla n + \zeta\widetilde{H}^\epsilon) + 2\zeta\epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{H}^\epsilon)$$
$$- \chi(((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2})\Pi)(\widetilde{\omega}_v^\epsilon \times n - \widetilde{v}^\epsilon \cdot \nabla n + \zeta\widetilde{H}^\epsilon) + \epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{\omega}_H^\epsilon \times n)$$
$$+ \chi\Pi((((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2})\nabla n)\widetilde{v}^\epsilon) - \chi\Pi(\widetilde{\omega}_v^\epsilon \times ((\widetilde{H}^\epsilon)^1\partial_{y^1} + (\widetilde{H}^\epsilon)^2\partial_{y^2})n)$$
$$- 2\epsilon\chi\Pi(\Delta_{\widetilde{g}}\widetilde{H}^\epsilon\nabla n) - \chi\Pi((((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})\nabla n)\widetilde{H}^\epsilon)$$
$$+ \chi\Pi(\widetilde{\omega}_v^\epsilon \times ((\widetilde{v}^\epsilon)^1\partial_{y^1} + (\widetilde{v}^\epsilon)^2\partial_{y^2})n).$$

We know that Π and n do not dependent the normal variable. Due to $\Delta_{\widetilde{g}}$ only involving the tangential derivatives and the derivatives of χ compactly supported away from the boundary, we easily obtain that

$$\|\overline{F}_v^v\|_{1,\infty} \le C\,(\|\Pi\nabla P^\epsilon\|_{1,\infty}, \qquad (3.77)$$

$$\|\overline{F}_\chi^v\|_{1,\infty} \le C\,(\|v^\epsilon\|_{1,\infty}\|v^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{1,\infty}\|H^\epsilon\|_{2,\infty} + \epsilon\|v^\epsilon\|_{3,\infty}), \qquad (3.78)$$

$$\|\overline{F}_v^\kappa\|_{1,\infty} \le C\,(\|v^\epsilon\|_{1,\infty}\|\nabla v^\epsilon\|_{1,\infty} + \|H^\epsilon\|_{1,\infty}\|\nabla H^\epsilon\|_{1,\infty} + \|v^\epsilon\|_{1,\infty}^2 + \|H^\epsilon\|_{1,\infty}^2$$
$$+ \epsilon\|v^\epsilon\|_{3,\infty} + \epsilon\|\nabla v^\epsilon\|_{3,\infty}), \qquad (3.79)$$

$$\|\overline{F}_\chi^H\|_{1,\infty} \le C\,(\|v^\epsilon\|_{1,\infty}\|H^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{1,\infty}\|v^\epsilon\|_{2,\infty} + \epsilon\|H^\epsilon\|_{3,\infty}), \qquad (3.80)$$

$$\|\overline{F}_H^\kappa\|_{1,\infty} \le C\,(\|v^\epsilon\|_{1,\infty}\|\nabla H^\epsilon\|_{1,\infty} + \|H^\epsilon\|_{1,\infty}\|\nabla v^\epsilon\|_{1,\infty} + \|v^\epsilon\|_{1,\infty}^2 + \|H^\epsilon\|_{1,\infty}^2$$
$$+ \epsilon\|H^\epsilon\|_{3,\infty} + \epsilon\|\nabla H^\epsilon\|_{3,\infty}). \qquad (3.81)$$

A crucial estimate towards the proof of Lemma 3.6 is the following:

Lemma 3.7 ([16]). *Let ρ is a smooth solution of*

$$\partial_t \rho + u \cdot \nabla \rho = \epsilon \partial_{zz} \rho + f, \quad z > 0, \quad \rho(t, y, 0) = 0,$$

where u satisfies the divergence free condition and $u \cdot n$ vanishes on the boundary. Assume that ρ and f are compactly supported with respect to z. Then, we have the estimate:

$$\|\rho\|_{1,\infty} \leq C\|\rho(0)\|_{1,\infty} + C \int_0^t \big\{ (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) \\ \times (\|\rho\|_{1,\infty} + \|\rho\|_{m_0+3}) + \|f\|_{1,\infty} \big\} \quad for \quad m_0 > 2.$$

In order to use Lemma 3.7, we shall eliminate $\partial_z(\ln|g|)\partial_z \widetilde{\eta}_v^\epsilon$ in (3.75) and $\partial_z(\ln|g|)\partial_z \widetilde{\eta}_H^\epsilon$ in (3.76), respectively. We set

$$\widetilde{\eta}_v^\epsilon = \frac{1}{|g|^{\frac{1}{4}}}\overline{\eta}_v^\epsilon = \gamma\overline{\eta}_v^\epsilon, \quad \widetilde{\eta}_H^\epsilon = \frac{1}{|g|^{\frac{1}{4}}}\overline{\eta}_H^\epsilon = \gamma\overline{\eta}_H^\epsilon.$$

We note that

$$\|\widetilde{\eta}_v^\epsilon\|_{1,\infty} \sim \|\overline{\eta}_v^\epsilon\|_{1,\infty}, \quad \|\widetilde{\eta}_H^\epsilon\|_{1,\infty} \sim \|\overline{\eta}_H^\epsilon\|_{1,\infty} \tag{3.82}$$

and $\overline{\eta}_v^\epsilon$ and $\overline{\eta}_H^\epsilon$ solve the equations

$$\partial_t \overline{\eta}_v^\epsilon + ((\widetilde{v}^\epsilon)^1 \partial_{y^1} + (\widetilde{v}^\epsilon)^2 \partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\overline{\eta}_v^\epsilon - ((\widetilde{H}^\epsilon)^1 \partial_{y^1} + (\widetilde{H}^\epsilon)^2 \partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\overline{\eta}_H^\epsilon$$

$$-\epsilon\partial_{zz}\overline{\eta}_v^\epsilon = \frac{1}{\gamma}(\chi\Pi\overline{F}^v \times n + \overline{F}_v^v + \overline{F}_v^\chi + \overline{F}_v^\kappa + \epsilon\partial_{zz}\gamma\overline{\eta}_v^\epsilon + \frac{\epsilon}{2}\partial_z(\ln|g|)\partial_z\gamma\overline{\eta}_v^\epsilon$$

$$- (\widetilde{v}^\epsilon \cdot \nabla\gamma)\overline{\eta}_v^\epsilon + (\widetilde{H}^\epsilon \cdot \nabla\gamma)\overline{\eta}_H^\epsilon) := S_1, \tag{3.83}$$

$$\partial_t \overline{\eta}_H^\epsilon + ((\widetilde{v}^\epsilon)^1 \partial_{y^1} + (\widetilde{v}^\epsilon)^2 \partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\overline{\eta}_H^\epsilon - ((\widetilde{H}^\epsilon)^1 \partial_{y^1} + (\widetilde{H}^\epsilon)^2 \partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\overline{\eta}_v^\epsilon$$

$$-\epsilon\partial_{zz}\overline{\eta}_H^\epsilon = \frac{1}{\gamma}(\chi\Pi\overline{F}^H \times n + \overline{F}_H^\chi + \overline{F}_H^\kappa + \epsilon\partial_{zz}\gamma\overline{\eta}_H^\epsilon + \frac{\epsilon}{2}\partial_z(\ln|g|)\partial_z\gamma\overline{\eta}_H^\epsilon$$

$$- (\widetilde{v}^\epsilon \cdot \nabla\gamma)\overline{\eta}_v^\epsilon + (\widetilde{H}^\epsilon \cdot \nabla\gamma)\overline{\eta}_H^\epsilon) := S_2. \tag{3.84}$$

Finally, we set

$$\eta_1 := \overline{\eta}_v^\epsilon + \overline{\eta}_H^\epsilon, \quad \eta_2 := \overline{\eta}_v^\epsilon - \overline{\eta}_H^\epsilon$$

and easily find

$$\partial_t \eta_1 + ((\widetilde{v}^\epsilon)^1 \partial_{y^1} + (\widetilde{v}^\epsilon)^2 \partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\eta_1$$

$$- ((\widetilde{H}^\epsilon)^1 \partial_{y^1} + (\widetilde{H}^\epsilon)^2 \partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\eta_1 - \epsilon\partial_{zz}\eta_1 = S_1 + S_2, \tag{3.85}$$

$$\partial_t \eta_2 + ((\widetilde{v}^\epsilon)^1 \partial_{y^1} + (\widetilde{v}^\epsilon)^2 \partial_{y^2} + \widetilde{v}^\epsilon \cdot n\partial_z)\eta_2$$

$$- ((\widetilde{H}^\epsilon)^1 \partial_{y^1} + (\widetilde{H}^\epsilon)^2 \partial_{y^2} + \widetilde{H}^\epsilon \cdot n\partial_z)\eta_2 - \epsilon\partial_{zz}\eta_1 = S_1 - S_2. \tag{3.86}$$

By applying Lemma 3.7 to (3.85), we directly obtain

$$\|\eta_1\|_{1,\infty} \leq C \|\eta_1(0)\|_{1,\infty} + \int_0^t \big\{ (\|v^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{2,\infty} + \|\nabla v^\epsilon\|_{1,\infty} + \|\nabla H^\epsilon\|_{1,\infty})$$

$$\times \left(\|\eta_1\|_{1,\infty} + \|\eta_1\|_{m_0+3} \right) + \|S_1\|_{1,\infty} + \|S_2\|_{1,\infty} \} \quad \text{for} \quad m_0 > 2.$$

From (3.64)-(3.67) and (3.77)-(3.81), we get

$$\|\eta_1\|_{1,\infty} \le C \|\eta_1(0)\|_{1,\infty} + \int_0^t \big\{ \big(\|v^\epsilon\|_{2,\infty} + \|H^\epsilon\|_{2,\infty} + \|\nabla v^\epsilon\|_{1,\infty} + \|\nabla H^\epsilon\|_{1,\infty} \big)$$

$$\times \big(\|\eta_1\|_{1,\infty} + \|\eta_1\|_{m_0+3} + \|\eta_2\|_{1,\infty} + \|\eta_2\|_{m_0+3} + N_m^{\frac{1}{2}} \big) + N_m$$

$$+ N_m^{\frac{1}{2}} + \epsilon (\|\nabla v^\epsilon\|_{3,\infty} + \|\nabla H^\epsilon\|_{3,\infty}) + \|\Pi \nabla P^\epsilon\|_{1,\infty}$$

$$+ \|\Pi(\nabla P^\epsilon \cdot \nabla n)\|_{1,\infty} \big\} \quad \text{for} \quad m_0 > 2. \tag{3.87}$$

Due to Lemmas 2.3 and 3.4, we have

$$\|\Pi \nabla P^\epsilon\|_{1,\infty} \le C(N_m^{\frac{1}{2}}(t) + N_m(t)) \quad \text{for} \quad m \ge 4. \tag{3.88}$$

Now, we deal with the terms with the coefficient ϵ. From Lemma 2.3, we get

$$\left(\epsilon \int_0^t (\|\nabla v^\epsilon\|_{3,\infty} + \|\nabla H^\epsilon)\|_{3,\infty}) \right)^2$$

$$\le C\epsilon^2 \left(\int_0^t \|\nabla^2 v^\epsilon\|_{m-1}^{\frac{1}{2}} + \|\nabla^2 H^\epsilon)\|_{m-1}^{\frac{1}{2}}) N_m^{\frac{1}{4}} \right)^2 + C\epsilon^2 t \int_0^t N_m$$

$$\le C\epsilon^2 t \left(\int_0^t (\|\nabla^2 v^\epsilon\|_{m-1}^2 + \|\nabla^2 H^\epsilon\|_{m-1}^2) \right)^{\frac{1}{2}} \left(\int_0^t N_m \right)^{\frac{1}{2}} + C\epsilon^2 t \int_0^t N_m$$

$$\le C\epsilon \int_0^t (\|\nabla^2 v^\epsilon\|_{m-1}^2 + \|\nabla^2 H^\epsilon\|_{m-1}^2) + C(\epsilon^2 t + \epsilon^3 t^2) \int_0^t N_m \tag{3.89}$$

for $m \ge m_0 + 4$. Consequently, we get from (3.63), (3.64)-(3.67) and (3.87)-(3.89) that

$$\|\eta_1\|_{1,\infty}^2 \le CN(0) + C(1 + t + \epsilon^3 t^2) \int_0^t (N_m^2 + N_m).$$

Similarly, we also get

$$\|\eta_2\|_{1,\infty}^2 \le CN(0) + C(1 + t + \epsilon^4 t^2) \int_0^t (N_m^2 + N_m).$$

Therefore, we complete the proof of Lemma 3.6. $\qquad \square$

3.5. **Proof of Theorem 3.1.** Based on Lemma 3.5, Lemma 3.6 and (3.63), we can easily prove Theorem 3.1. We omit the details here.

3.6. **Proof of Theorem 1.1.** By smoothing the initial data and using the a priori estimates obtained in Theorem 3.1 and the strong compactness argument, we can prove Theorem 1.1 in the same spirit of [16]. Hence we omit it here.

4. Proof of Theorem 1.2

In this section, we shall establish the convergence with a rate for the solution (v^ϵ, H^ϵ) to (v, H). We start with the rate of convergence in L^2.

Lemma 4.1. *Under the assumptions in the Theorem 1.2, we have*

$$\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2 + \epsilon \int_o^t (\|v^\epsilon - v\|_{H^1}^2 + \|H^\epsilon - H\|_{H^1}^2) \leq C\epsilon^{\frac{3}{2}} \quad on \quad [0, T_2],$$

where ϵ small enough and $T_2 = \min\{T_0, T_1\}$. Consequently, we have

$$\|v^\epsilon - v\|_{L^\infty([0,T_2]\times\Omega)} + \|H^\epsilon - H\|_{L^\infty([0,T_2]\times\Omega)} \leq C\epsilon^{\frac{3}{10}}.$$

Proof. We note that $v^\epsilon - v$ and $H^\epsilon - H$ satisfy

$$\partial_t(v^\epsilon - v) - \epsilon\Delta(v^\epsilon - v) + \Phi_1 + \nabla(p^\epsilon - p) = \epsilon\Delta v \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\partial_t(H^\epsilon - H) - \epsilon\Delta(H^\epsilon - H) + \Phi_2 = \epsilon\Delta H \quad \text{in} \quad \Omega, \tag{4.2}$$

$$\nabla \cdot v^\epsilon = 0 , \ \nabla \cdot H^\epsilon = 0 \quad \text{in} \quad \Omega, \tag{4.3}$$

$$(v^\epsilon - v) \cdot n = 0, \ n \times (\omega_v^\epsilon - \omega_v) = [B(v^\epsilon - v) + Bv]_\tau - n \times \omega_v \quad \text{on} \quad \partial\Omega, \tag{4.4}$$

$$(H^\epsilon - H) \cdot n = 0, \ n \times (\omega_H^\epsilon - \omega_H) = [B(H^\epsilon - H) + BH]_\tau - n \times \omega_H \quad \text{on} \quad \partial\Omega, \tag{4.5}$$

where $\omega_v^\epsilon = \nabla \times v^\epsilon, \ \omega_H^\epsilon = \nabla \times H^\epsilon, \ \omega_v = \nabla \times v, \ \omega_H = \nabla \times H$, and

$$\Phi_1 := v \cdot \nabla(v^\epsilon - v) + (v^\epsilon - v) \cdot \nabla v + (v^\epsilon - v) \cdot \nabla(v^\epsilon - v)$$
$$- H \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla H - (H^\epsilon - H) \cdot \nabla(H^\epsilon - H)$$
$$+ \frac{1}{2}\nabla(|H^\epsilon|^2 - |H|^2) - \frac{1}{2}\nabla(|v^\epsilon|^2 - |v|^2),$$

$$\Phi_2 := (v^\epsilon - v) \cdot \nabla H + (v^\epsilon - v) \cdot \nabla(H^\epsilon - H) + v \cdot \nabla(H^\epsilon - H)$$
$$- (H^\epsilon - H) \cdot \nabla v - (H^\epsilon - H) \cdot \nabla(v^\epsilon - v) - H \cdot \nabla(v^\epsilon - v).$$

Doing basic L^2-estimate, we obtain the following identity:

$$\frac{1}{2}\frac{d}{dt}(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2) + \epsilon(\|\nabla \times (v^\epsilon - v)\|^2 + \|\nabla \times (H^\epsilon - H)\|^2)$$
$$+ (\Phi_1, v^\epsilon - v) + (\Phi_2, H^\epsilon - H) + B_1 + B_2 = (\epsilon\Delta v, v^\epsilon - v) + (\epsilon\Delta H, H^\epsilon - H),$$

where

$$B_1 := \epsilon \int_{\partial\Omega} n \times (\omega_v^\epsilon - \omega_v)(v^\epsilon - v)$$
$$= \epsilon \int_{\partial\Omega} (B(v^\epsilon - v) + Bv - n \times \omega_v)(v^\epsilon - v),$$

$$B_2 := \epsilon \int_{\partial\Omega} n \times (\omega_H^\epsilon - \omega_H)(H^\epsilon - H)$$
$$= \epsilon \int_{\partial\Omega} (B(H^\epsilon - H) + BH - n \times \omega_H)(H^\epsilon - H).$$

First, we easily note that

$$|(\epsilon\Delta v, v^\epsilon - v)| + |(\epsilon\Delta H, H^\epsilon - H)| \leq C(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2) + \epsilon^2. \qquad (4.6)$$

Next, we deal with the boundary terms B_1 and B_2. For B_1, we have

$$B_1 = \epsilon \int_{\partial\Omega} (B(v^\epsilon - v) + Bv - n \times \omega_v)(v^\epsilon - v)$$

$$\leq C\epsilon \int_{\partial\Omega} (|v^\epsilon - v|^2 + |v^\epsilon - v|).$$

Due to the trace theorem:

$$|u|_{L^1(\partial\Omega)} \leq C\,|u|_{L^2(\partial\Omega)} \leq C\|u\|_{H^{\frac{1}{2}}} \qquad (4.7)$$

and the interpolation inequality:

$$\|u\|_{H^{\frac{1}{2}}(\Omega)} \leq C\,\|u\|^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}, \qquad (4.8)$$

we further obtain that

$$B_1 \leq C\epsilon\left(\|v^\epsilon - v\|\|\omega_v^\epsilon - \omega_v\| + |v^\epsilon - v|_{L^1(\partial\Omega)}\right)$$

$$\leq 2\delta\epsilon\|\omega_v^\epsilon - \omega_v\|^2 + C_\delta\|v^\epsilon - v\|^2 + \epsilon^{\frac{3}{2}}. \qquad (4.9)$$

Similarly, we also get that

$$B_2 \leq 2\delta\epsilon\|\omega_H^\epsilon - \omega_H\|^2 + C_\delta\|H^\epsilon - H\|^2 + \epsilon^{\frac{3}{2}}. \qquad (4.10)$$

Finally, we deal with $(\Phi_1, v^\epsilon - v)$ and $(\Phi_2, H^\epsilon - H)$. We have

$$|(\Phi_1, v^\epsilon - v) + (\Phi_2, H^\epsilon - H)|$$

$$= \Big|(v^\epsilon - v, v \cdot \nabla(v^\epsilon - v) + (v^\epsilon - v) \cdot \nabla v + (v^\epsilon - v) \cdot \nabla(v^\epsilon - v) - \frac{1}{2}\nabla(|v^\epsilon|^2 - |v|^2)$$

$$- H \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla H - (H^\epsilon - H) \cdot \nabla(H^\epsilon - H)$$

$$+ \frac{1}{2}\nabla(|H^\epsilon|^2 - |H|^2)) + (H^\epsilon - H, (v^\epsilon - v) \cdot \nabla H + (v^\epsilon - v) \cdot \nabla(H^\epsilon - H)$$

$$+ v \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla v - (H^\epsilon - H) \cdot \nabla(v^\epsilon - v) - H \cdot \nabla(v^\epsilon - v))\Big|.$$

We note that

$$(\frac{1}{2}\nabla(|v^\epsilon|^2 - |v|^2) - \frac{1}{2}\nabla(|H^\epsilon|^2 - |H|^2), v^\epsilon - v) = 0,$$

$$(v^\epsilon - v, v \cdot \nabla(v^\epsilon - v)) = 0, \quad (v^\epsilon - v, (v^\epsilon - v) \cdot \nabla(v^\epsilon - v)) = 0,$$

$$(H^\epsilon - H, v \cdot \nabla(H^\epsilon - H)) = 0, \quad (H^\epsilon - H, (v^\epsilon - v) \cdot \nabla(H^\epsilon - H)) = 0,$$

$$((H^\epsilon - H) \cdot \nabla(v^\epsilon - v), H^\epsilon - H) + ((H^\epsilon - H) \cdot \nabla H^\epsilon - H, v^\epsilon - v) = 0,$$

$$(H^\epsilon - H, H \cdot \nabla(v^\epsilon - v)) + (v^\epsilon - v, H \cdot \nabla(H^\epsilon - H)) = 0.$$

Consequently, one has

$$|(\Phi_1, v^\epsilon - v) + (\Phi_2, H^\epsilon - H)| \leq C(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2). \qquad (4.11)$$

From (4.6), (4.9), (4.10) and (4.11), we get

$$\frac{1}{2}\frac{d}{dt}(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2) + \epsilon(\|\nabla \times (v^\epsilon - v)\|^2 + \|\nabla \times (H^\epsilon - H)\|^2)$$
$$\leq C\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2 + \epsilon^{\frac{3}{2}}.$$

Then, by using Gronwall's inequality, we arrive at

$$\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2 + \epsilon \int_0^t (\|v^\epsilon - v\|_{H^1}^2 + \|H^\epsilon - H\|_{H^1}^2) \leq C\epsilon^{\frac{3}{2}}.$$

Consequently, by using the Gagliardo-Nirenberg interpolation inequality, we have

$$\|v^\epsilon - v\|_{L^\infty} + \|H^\epsilon - H\|_{L^\infty} \leq C(\|v^\epsilon - v\|^{\frac{2}{5}}\|v^\epsilon - v\|_{W^{1,\infty}}^{\frac{3}{5}}$$
$$+ \|H^\epsilon - H\|^{\frac{2}{5}}\|H^\epsilon - H\|_{W^{1,\infty}}^{\frac{3}{5}}) \leq C\epsilon^{\frac{3}{10}}.$$

$$\square$$

Before we go to prove the rate of the convergence in H^1, we have the following observation.

Lemma 4.2. *We have*

$$\|u\|_{H^2} \leq \|P\Delta u\| + \|u\|, \quad \forall u \in W_B,$$

where

$$W_B = \{u \in H^2(\Omega) \,\big|\, \nabla \cdot u = 0 \ in \ \Omega, \ u \cdot n = 0, \ n \times (\nabla \times u) = [Bu]_\tau \ on \ \partial\Omega\}.$$

Proof. We consider the following boundary value problem:

$$\gamma I - \Delta u + \nabla p = f \quad in \quad \Omega, \tag{4.12}$$

$$\nabla \cdot u = 0 \quad in \quad \Omega, \tag{4.13}$$

$$u \cdot n = 0, \quad n \times (\nabla \times u) = [Bu]_\tau \quad on \quad \Omega, \tag{4.14}$$

where γ is a large enough positive constant. Define a bilinear form as

$$\mathcal{B}(u, \phi) = \gamma(u, \phi) + (\nabla \times u, \nabla \times \phi) + \int_{\partial\Omega} Au \cdot \phi \tag{4.15}$$

with the domain $D(\mathcal{B}) = \{u \in H^1(\Omega) \,\big|\, \nabla \cdot u = 0 \ in \ \Omega, \ u \cdot n = 0 \ on \ \partial\Omega\}$.

It is clear that $\mathcal{B}(u, \phi)$ with domain $D(\mathcal{B})$ is a positive densely defined closed bilinear form. Let $\mathcal{O}$ be the self-extension of $\mathcal{B}(u, \phi)$. We find that $W_B \subset D(\mathcal{O})$ and $\mathcal{O}u = \gamma u + P(-\Delta u)$ for any $u \in D(\mathcal{O})$. Let $u \in W_B$ and $\mathcal{O}u = f$. It follows from (4.15) and Lemma 2.1 that

$$\|u\|_{H^1} \leq C\|f\|. \tag{4.16}$$

Now, let $n(x)$ and $B(x)$ be the internal smooth extensions of the normal vector n and B in (4.14). Based on Lemma 2.5, we have

$$B(x)u \times n(x) = \nabla \times k + \nabla h + \nabla g,$$

where $k \in FH \cap H^2$, $\nabla h \in HG$ and $\nabla g \in GG$. We find

$$\Delta g = \nabla \cdot (B(x)u \times n(x)) \quad \text{in} \quad \Omega,$$

$$g = 0 \quad \text{on} \quad \partial\Omega.$$

From the elliptic regularity theory, we obtain

$$\|\nabla g\|_{H^1} \leq C\|u\|_{H^1}.$$

Since HG is finite dimensional, the following inequality holds

$$\|\nabla h\|_{H^1} \leq C\|u\|.$$

Further, it follows from Lemma 2.1 and Poincaré type inequality in Lemma 3.3 of [25] that

$$\|k\|_{H^2} \leq C\|\nabla \times k\|_{H^1} \leq C\|u\|_{H^1} \leq C\|f\|. \tag{4.17}$$

Integrating by parts and noting that $n \times \nabla h = 0$, $n \times \nabla g = 0$ on the boundary, we have

$$\int_{\Omega} (\nabla \times k) \cdot (\nabla \times \phi) + \int_{\partial\Omega} n \times (Bu \times n) \cdot \phi = (-\Delta k, \phi)$$

for any $\phi \in H^1$. We observe that $n \times (Bu \times n) = Bu$, so we have

$$\int_{\Omega} (\nabla \times (u - k)) \cdot (\nabla \times \phi) = (P_{FH}(f - u + \Delta k), \phi), \quad \forall \phi \in H^1 \cap FH,$$

where P_{FH} denotes the projection on FH. Further, due to $\nabla \times u = \nabla \times P_{FH}(u)$, we get

$$\int_{\Omega} (\nabla \times (P_{FH}(u) - k)) \cdot (\nabla \times \phi) = (P_{FH}(f - \gamma u + \Delta k), \phi), \quad \forall \phi \in H^1 \cap FH$$

From Theorem 3.1 in [25], we obtain

$$\|P_{FH}(u) - k\|_{H^2} \leq C(\|f\| + \|\Delta k\| + \|u\|). \tag{4.18}$$

Since HH is finite dimensional, the following inequality holds

$$\|P_{HH}(u)\|_{H^2} \leq C\|u\|, \tag{4.19}$$

where

$$HH = \left\{ u \in L^2(\Omega) \,\middle|\, \nabla \cdot u = 0, \ \nabla \times u = 0 \ \text{in} \ \Omega, \ u \cdot n = 0 \ \text{on} \ \partial\Omega \right\},$$

$$\mathbb{H} = HH \oplus FH.$$

We get from (4.16), (4.17), (4.18) and (4.19) that

$$\|u\|_{H^2} \leq C\|f\|.$$

Consequently, we complete the proof of Lemma 4.2. $\qquad\qquad\qquad\qquad\qquad \square$

Now we turn to prove the rate of convergence in $H^1(\Omega)$.

Lemma 4.3. *Under the assumptions in Theorem 1.2, we have*

$$\|v^\epsilon - v\|_{H^1}^2 + \|H^\epsilon - H\|_{H^1}^2$$

$$+ \epsilon \int_0^t (\|v^\epsilon - v\|_{H^2}^2 + \|H^\epsilon - H\|_{H^2}^2) \le C\epsilon^{\frac{1}{2}} \quad on \quad [0, T_2], \qquad (4.20)$$

where ϵ small enough and $T_2 = \min\{T_0, T_1\}$. Also, we have

$$\|v^\epsilon - v\|_{W^{1,p}}^p + \|H^\epsilon - H\|_{W^{1,p}}^p \le C\epsilon^{\frac{1}{2}} \quad on \quad [0, T_2]$$

for $2 \le p < \infty$.

Proof. We note

$$\partial_t(v^\epsilon - v) \cdot n = 0, \quad \partial_t(H^\epsilon - H) \cdot n = 0.$$

It follows from (4.1)-(4.5) that

$$\frac{1}{2}\frac{d}{dt}(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2) + \epsilon(\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2)$$

$$= (\Phi_1, P\Delta(v^\epsilon - v)) + (\Phi_2, P\Delta(H^\epsilon - H)) + B_1 + B_2 - (\epsilon\Delta v, P\Delta(v^\epsilon - v))$$

$$- (\epsilon\Delta H, P\Delta(H^\epsilon - H)),$$

where Φ_1 and Φ_2 are as same as these in Lemma 4.1, but B_1 and B_2 have different forms:

$$B_1 := \int_{\partial\Omega} \partial_t(v^\epsilon - v) \cdot (n \times (\omega_v^\epsilon - \omega_v)), \quad B_2 := \int_{\partial\Omega} \partial_t(H^\epsilon - H) \cdot (n \times (\omega_H^\epsilon - \omega_H)).$$

Now, let us deal with these two boundary terms as follows

$$B_1 + B_2$$

$$= \int_{\partial\Omega} \partial_t(v^\epsilon - v) \cdot (B(v^\epsilon - v) + Bv - n \times \omega_v)$$

$$+ \int_{\partial\Omega} \partial_t(H^\epsilon - H) \cdot (B(H^\epsilon - H) + BH - n \times \omega_H)$$

$$= \frac{1}{2}\frac{d}{dt}\Big(\int_{\partial\Omega} B(v^\epsilon - v) \cdot (v^\epsilon - v) + 2\int_{\partial\Omega} (v^\epsilon - v) \cdot (Bv - n \times \omega_v)\Big) - \widetilde{B}_1$$

$$+ \frac{1}{2}\frac{d}{dt}\Big(\int_{\partial\Omega} B(H^\epsilon - H) \cdot (H^\epsilon - H) + 2\int_{\partial\Omega} (H^\epsilon - H) \cdot (BH - n \times \omega_H)\Big) - \widetilde{B}_2,$$

$$(4.21)$$

where

$$\widetilde{B}_1 := \int_{\partial\Omega} (v^\epsilon - v) \cdot \partial_t(Bv - n \times \omega_v), \quad \widetilde{B}_2 := \int_{\partial\Omega} (H^\epsilon - H) \cdot \partial_t(BH - n \times \omega_H). \quad (4.22)$$

It follows from Lemma 4.1, (4.7) and (4.8) that

$$|\widetilde{B}_1 + \widetilde{B}_2| \le C\Big(\int_{\partial\Omega} |v^\epsilon - v|^2\Big)^{\frac{1}{2}} + C\Big(\int_{\partial\Omega} |H^\epsilon - H|^2\Big)^{\frac{1}{2}}$$

$$\le \delta(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2) + C\epsilon^{\frac{1}{2}}. \qquad (4.23)$$

We easily get

$$
|(\epsilon\Delta v, P\Delta(v^\epsilon - v)) + (\epsilon\Delta H, P\Delta(H^\epsilon - H))|
$$
$$
\leq \frac{\epsilon}{2}(\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2) + C\,\epsilon \qquad (4.24)
$$

and

$$
-(\Phi_1, P\Delta(v^\epsilon - v)) - (\Phi_2, P\Delta(H^\epsilon - H))
$$
$$
= (P\Phi_1, -\Delta(v^\epsilon - v)) + (P\Phi_2, -\Delta(H^\epsilon - H))
$$
$$
= (\nabla\times\Phi_1, \omega_v^\epsilon - \omega_v) + \int_{\partial\Omega} n\times(\omega_v^\epsilon - \omega_v)\cdot P\Phi_1
$$
$$
+ (\nabla\times\Phi_2, \omega_H^\epsilon - \omega_H) + \int_{\partial\Omega} n\times(\omega_H^\epsilon - \omega_H)\cdot P\Phi_2
$$
$$
= (\nabla\times\Phi_1, \omega_v^\epsilon - \omega_v) + \int_{\partial\Omega} (B(v^\epsilon - v) + Bv - n\times\omega_v)\cdot P\Phi_1
$$
$$
+ (\nabla\times\Phi_2, \omega_H^\epsilon - \omega_H) + \int_{\partial\Omega} (B(H^\epsilon - H) + BH - n\times\omega_H)\cdot P\Phi_2. \qquad (4.25)
$$

From (4.21), (4.23), (4.24), and (4.25), we arrive at

$$
\frac{1}{2}\frac{d}{dt}E + \frac{\epsilon}{2}(\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2)
$$
$$
\leq I_1 + I_2 + I_3 + C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}), \qquad (4.26)
$$

where

$$
E := \|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2
$$
$$
- \int_{\partial\Omega} B(v^\epsilon - v)\cdot(v^\epsilon - v) - 2\int_{\partial\Omega}(v^\epsilon - v)\cdot(Bv - n\times\omega_v)
$$
$$
- \int_{\partial\Omega} B(H^\epsilon - H)\cdot(H^\epsilon - H) - 2\int_{\partial\Omega}(H^\epsilon - H)\cdot(BH - n\times\omega_H),
$$
$$
I_1 := |(\nabla\times\Phi_1, \omega_v^\epsilon - \omega_v) + (\nabla\times\Phi_2, \omega_H^\epsilon - \omega_H)|,
$$
$$
I_2 := \left|\int_{\partial\Omega} B(v^\epsilon - v)\cdot P\Phi_1 + \int_{\partial\Omega} B(H^\epsilon - H)\cdot P\Phi_2\right|,
$$
$$
I_3 := \left|\int_{\partial\Omega} (Bv - n\times\omega_v)\cdot P\Phi_1 + \int_{\partial\Omega} (BH - n\times\omega_H)\cdot P\Phi_2\right|.
$$

Now we estimate the terms I_1, I_2 and I_3 in turn. The term I_1 can be estimated easily by using Sobolev inequalities and the obtained uniform bounds for v^ϵ and H^ϵ in Theorem 1.1. We have

$$
I_1 = |(\nabla\times\Phi_1, \omega_v^\epsilon - \omega_v) + (\nabla\times\Phi_2, \omega_H^\epsilon - \omega_H)| \leq I_{11} + I_{12},
$$

where

$$
I_{11} = \big|(v\cdot\nabla(\omega_v^\epsilon - \omega_v) + (v^\epsilon - v)\cdot\nabla\omega_v + (v^\epsilon - v)\cdot\nabla(\omega_v^\epsilon - \omega_v)
$$
$$
- H\cdot\nabla(\omega_H^\epsilon - \omega_H) - (H^\epsilon - H)\cdot\nabla\omega_H - (H^\epsilon - H)\cdot\nabla(\omega_H^\epsilon - \omega_H), \omega_v^\epsilon - \omega_v)
$$

$$+ ((v^\epsilon - v) \cdot \nabla \omega_H + (v^\epsilon - v) \cdot \nabla(\omega_H^\epsilon - \omega_H) + v \cdot \nabla(\omega_H^\epsilon - \omega_H)$$

$$- (H^\epsilon - H) \cdot \nabla \omega_v - (H^\epsilon - H) \cdot \nabla(\omega_v^\epsilon - \omega_v) - H \cdot \nabla(\omega_v^\epsilon - \omega_v), \omega_H^\epsilon - \omega_H)\big|,$$

$$I_{12} = \big|([\nabla \times, v \cdot \nabla](v^\epsilon - v) + [\nabla \times, (v^\epsilon - v) \cdot \nabla]v + [\nabla \times, (v^\epsilon - v) \cdot \nabla](v^\epsilon - v)$$

$$- [\nabla \times, H \cdot \nabla](H^\epsilon - H) - [\nabla \times, (H^\epsilon - H) \cdot \nabla]H$$

$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla](H^\epsilon - H), \omega_v^\epsilon - \omega_v)$$

$$+ ([\nabla \times, (v^\epsilon - v) \cdot \nabla]H + [\nabla \times, (v^\epsilon - v) \cdot \nabla](H^\epsilon - H) + [\nabla \times, v \cdot \nabla](H^\epsilon - H)$$

$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla]v - [\nabla \times, (H^\epsilon - H) \cdot \nabla](v^\epsilon - v)$$

$$- [\nabla \times, H \cdot \nabla](v^\epsilon - v), \omega_H^\epsilon - \omega_H)\big|.$$

We observe that

$$(v \cdot \nabla(\omega_v^\epsilon - \omega_v), \omega_v^\epsilon - \omega_v) = 0, \qquad ((v^\epsilon - v) \cdot \nabla(\omega_v^\epsilon - \omega_v), \omega_v^\epsilon - \omega_v) = 0,$$

$$(v \cdot \nabla(\omega_H^\epsilon - \omega_H), \omega_H^\epsilon - \omega_H) = 0, \quad ((v^\epsilon - v) \cdot \nabla(\omega_H^\epsilon - \omega_H), \omega_H^\epsilon - \omega_H) = 0,$$

$$(H \cdot \nabla(\omega_H^\epsilon - \omega_H) + (H^\epsilon - H) \cdot \nabla(\omega_H^\epsilon - \omega_H), \omega_v^\epsilon - \omega_v)$$

$$+ ((H^\epsilon - H) \cdot \nabla(\omega_v^\epsilon - \omega_v) + H \cdot \nabla(\omega_v^\epsilon - \omega_v), \omega_H^\epsilon - \omega_H) = 0.$$

Hence

$$I_1 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2). \tag{4.27}$$

Next, we estimate the term I_2. We note that

$$P\Phi = \Phi + \nabla\phi$$

holds for any function $\Phi \in L^2(\Omega)$, so we need to estimate the scalar function ϕ which is difficult to estimate on the boundary. In order to overcome this difficulty, we need to transform it to an estimate on Ω. First, we should extend n and B to the interior of Ω as follows:

$$n(x) = \varphi(r(x))\nabla(r(x)), \quad B(x) = \varphi(r(x))B(\Pi x),$$

where

$$r(x) = \min_{y \in \partial\Omega} d(x, y), \quad \Pi x = y_x \in \partial\Omega$$

such that

$$r(x) := d(x, y_x)$$

is well-defined in $\Omega_\sigma = \{x \in \Omega, r(x) \leq 2\sigma\}$ for some $\sigma > 0$ and $\varphi(s) \in C_c^\infty[0, 2\sigma)$ satisfying

$$\varphi(s) = 1 \quad \text{in} \quad [0, \sigma].$$

Then, we can obtain that

$$I_2 = \Big| \int_{\partial\Omega} B(v^\epsilon - v) \cdot P\Phi_1 + \int_{\partial\Omega} B(H^\epsilon - H) \cdot P\Phi_2 \Big|$$

$$= \Big| \int_{\partial\Omega} \big((n \times B(v^\epsilon - v) \cdot (n \times P\Phi_1) + (n \times B(H^\epsilon - H)) \cdot (n \times P\Phi_2)\big) \Big|$$

$$= \big|(n \times B(v^\epsilon - v), \nabla \times \Phi_1) + (n \times B(H^\epsilon - H), \nabla \times \Phi_2)$$
$$- (\nabla \times (n \times B(v^\epsilon - v)), P\Phi_1) - (\nabla \times (n \times B(H^\epsilon - H)), P\Phi_2)\big|. \quad (4.28)$$

We easily get that

$$|(\nabla \times (n \times B(v^\epsilon - v)), P\Phi_1) + (\nabla \times (n \times B(H^\epsilon - H)), P\Phi_2)|$$
$$\leq \|n \times B(v^\epsilon - v)\| \|P\Phi_1\| + \|n \times B(H^\epsilon - H)\| \|P\Phi_2\|$$
$$\leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2). \quad (4.29)$$

Now, we turn to estimate the remaining terms in (4.28):

$$|(n \times B(v^\epsilon - v), \nabla \times \Phi_1) + (n \times B(H^\epsilon - H), \nabla \times \Phi_2)| \leq I_{21} + I_{22},$$

where

$$I_{21} = \big|(v \cdot \nabla(\omega_v^\epsilon - \omega_v) + (v^\epsilon - v) \cdot \nabla \omega_v + (v^\epsilon - v) \cdot \nabla(\omega_v^\epsilon - \omega_v)$$
$$- H \cdot \nabla(\omega_H^\epsilon - \omega_H) - (H^\epsilon - H) \cdot \nabla \omega_H$$
$$- (H^\epsilon - H) \cdot \nabla(\omega_H^\epsilon - \omega_H), n \times B(v^\epsilon - v))$$
$$+ ((v^\epsilon - v) \cdot \nabla \omega_H + (v^\epsilon - v) \cdot \nabla(\omega_H^\epsilon - \omega_H) + v \cdot \nabla(\omega_H^\epsilon - \omega_H)$$
$$- (H^\epsilon - H) \cdot \nabla \omega_v - (H^\epsilon - H) \cdot \nabla(\omega_v^\epsilon - \omega_v)$$
$$- H \cdot \nabla(\omega_v^\epsilon - \omega_v), n \times B(H^\epsilon - H))\big|,$$
$$I_{22} = \big|([\nabla \times, v \cdot \nabla](v^\epsilon - v) + [\nabla \times, (v^\epsilon - v) \cdot \nabla]v + [\nabla \times, (v^\epsilon - v) \cdot \nabla](v^\epsilon - v)$$
$$- [\nabla \times, H \cdot \nabla](H^\epsilon - H) - [\nabla \times, (H^\epsilon - H) \cdot \nabla]H$$
$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla](H^\epsilon - H), n \times B(v^\epsilon - v))$$
$$+ ([\nabla \times, (v^\epsilon - v) \cdot \nabla]H + [\nabla \times, (v^\epsilon - v) \cdot \nabla](H^\epsilon - H)$$
$$+ [\nabla \times, v \cdot \nabla](H^\epsilon - H) - [\nabla \times, (H^\epsilon - H) \cdot \nabla]v$$
$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla](v^\epsilon - v) - [\nabla \times, H \cdot \nabla](v^\epsilon - v), n \times B(H^\epsilon - H))\big|.$$

By using Hölder's inequality and Sobolev inequality, we obtain

$$|(n \times B(v^\epsilon - v), \nabla \times \Phi_1) + (n \times B(H^\epsilon - H), \nabla \times \Phi_2)| \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2).$$
$$(4.30)$$

Based on (4.29) and (4.30), we have

$$I_2 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2). \quad (4.31)$$

Finally, we need to estimate the term I_3, i.e.

$$\Big| \int_{\partial \Omega} (Bv - n \times \omega_v) \cdot P\Phi_1 + \int_{\partial \Omega} (BH - n \times \omega_H) \cdot P\Phi_2 \Big|.$$

We observe that the estimate is trivial if the ideal MHD satisfies the same boundary condition as that the MHD does. However, $[Bv]_\tau - n \times \omega_v$ and $[BH]_\tau - n \times \omega_H$

may be not equal to zero. As a result, the boundary layer may occur, so we will
experience more complicate estimates. Similar to the above, we get

$$I_3 = \Big| \int_{\partial\Omega} (Bv - n \times \omega_v) \cdot P\Phi_1 + \int_{\partial\Omega} (BH - n \times \omega_H) \cdot P\Phi_2 \Big|$$

$$= \big| \int_{\partial\Omega} (n \times (Bv - n \times \omega_v)) \cdot (n \times P\Phi_1) + \int_{\partial\Omega} (n \times (BH - n \times \omega_H)) \cdot (n \times P\Phi_2) \big|$$

$$= \big| (n \times (Bv - n \times \omega_v), \nabla \times \Phi_1) + (n \times (BH - n \times \omega_H), \nabla \times \Phi_2)$$

$$\quad - (\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1) - (\nabla \times (n \times (BH - n \times \omega_H)), P\Phi_2) \big|$$

$$\leq I_{31} + I_{32},$$

where

$$I_{31} = |(n \times (Bv - n \times \omega_v), \nabla \times \Phi_1) + (n \times (BH - n \times \omega_H), \nabla \times \Phi_2)|,$$

$$I_{32} = |(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1) + (\nabla \times (n \times (BH - n \times \omega_H)), P\Phi_2)|.$$

We first deal with the term I_{31} and note that

$$I_{31} \leq L_1 + L_2 + L_3 + L_4 + L_5 + L_6,$$

where

$$L_1 = |(n \times (Bv - n \times \omega_v), \nabla \times (v \cdot \nabla(v^\epsilon - v)) - \nabla \times (H \cdot \nabla(H^\epsilon - H)))|,$$

$$L_2 = |(n \times (Bv - n \times \omega_v), \nabla \times ((v^\epsilon - v) \cdot \nabla v) - \nabla \times ((H^\epsilon - H) \cdot \nabla H))|,$$

$$L_3 = |(n \times (Bv - n \times \omega_v), \nabla \times ((v^\epsilon - v) \cdot \nabla(v^\epsilon - v))$$

$$\quad - \nabla \times ((H^\epsilon - H) \cdot \nabla(H^\epsilon - H)))|,$$

$$L_4 = |(n \times (BH - n \times \omega_H), \nabla \times (v \cdot \nabla(H^\epsilon - H)) - \nabla \times (H \cdot \nabla(v^\epsilon - v)))|,$$

$$L_5 = |(n \times (BH - n \times \omega_H), \nabla \times ((v^\epsilon - v) \cdot \nabla H) - \nabla \times ((H^\epsilon - H) \cdot \nabla v))|,$$

$$L_6 = |(n \times (BH - n \times \omega_H), \nabla \times ((H^\epsilon - H) \cdot \nabla(v^\epsilon - v))$$

$$\quad - \nabla \times ((v^\epsilon - v) \cdot \nabla(H^\epsilon - H)))|.$$

We have

$$L_1 = |(n \times (Bv - n \times \omega_v), \nabla \times (v \cdot \nabla(v^\epsilon - v)) - \nabla \times (H \cdot \nabla(H^\epsilon - H)))|$$

$$= |(n \times (Bv - n \times \omega_v), v \cdot \nabla(\omega_v^\epsilon - \omega_v) - H \cdot \nabla(\omega_H^\epsilon - \omega_H)$$

$$\quad + [\nabla\times, v \cdot \nabla](v^\epsilon - v) - [\nabla\times, H \cdot \nabla](H^\epsilon - H))|.$$

Here, we first deal with the terms which contain higher derivatives and get that

$$|(n \times (Bv - n \times \omega_v), v \cdot \nabla(\omega_v^\epsilon - \omega_v) - H \cdot \nabla(\omega_H^\epsilon - \omega_H)|$$

$$= |(v \cdot \nabla(n \times (Bv - n \times \omega_v)), \omega_v^\epsilon - \omega_v) - (H \cdot \nabla(n \times (Bv - n \times \omega_v)), \omega_H^\epsilon - \omega_H)|$$

$$\leq \Big| \int_{\partial\Omega} n \times (v^\epsilon - v) \cdot (v \cdot \nabla(n \times (Bv - n \times \omega_v)))$$

$$\quad + (\nabla \times (v \cdot \nabla(n \times (Bv - n \times \omega_v))), v^\epsilon - v) \Big|$$

$$+ \Big| \int_{\partial\Omega} n \times (H^\epsilon - H) \cdot (H \cdot \nabla(n \times (Bv - n \times \omega_v)))$$

$$+ (\nabla \times (H \cdot \nabla(n \times (Bv - n \times \omega_v))), H^\epsilon - H)) \Big|$$

$$\leq C \left(|v^\epsilon - v|_{L^2(\partial\Omega)} + |H^\epsilon - H|_{L^2(\partial\Omega)} + \|v^\epsilon - v\| + \|H^\epsilon - H\| \right)$$

$$\leq C \left(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}} \right).$$

We also note that each component of $[\nabla\times, v \cdot \nabla](v^\epsilon - v) - [\nabla\times, H \cdot \nabla](H^\epsilon - H)$ is a combination of such terms $\partial_i v \cdot \nabla(v^\epsilon - v)_j$ and $\partial_k H \cdot \nabla(H^\epsilon - H)_l$. Without loss of generality, we consider the term

$$((n \times (Bv - n \times \omega_v))_m, \partial_i v \cdot \nabla(v^\epsilon - v)_j - \partial_k H \cdot \nabla(H^\epsilon - H)_l).$$

Since $\nabla \cdot \partial_i v = 0$ and $\nabla \cdot \partial_k H = 0$, we have

$$|((n \times (Bv - n \times \omega_v))_m, \partial_i v \cdot \nabla(v^\epsilon - v)_j - \partial_k H \cdot \nabla(H^\epsilon - H)_l)|$$

$$= \Big| (\partial_i v, \nabla((v^\epsilon - v)_j (n \times (Bv - n \times \omega_v))_m))$$

$$- (\partial_i v \cdot \nabla(n \times (Bv - n \times \omega_v))_m, (v^\epsilon - v)_j)$$

$$- (\partial_k H, \nabla((H^\epsilon - H)_l (n \times (Bv - n \times \omega_v))_m))$$

$$+ (\partial_k H \cdot \nabla(n \times (Bv - n \times \omega_v))_m, (H^\epsilon - H)_l) \Big|$$

$$= \Big| \int_{\partial\Omega} (v^\epsilon - v)_j (n \times (Bv - n \times \omega_v))_m \partial_i v \cdot n$$

$$- \int_{\partial\Omega} (H^\epsilon - H)_l (n \times (Bv - n \times \omega_v))_m \partial_k H \cdot n$$

$$- (\partial_i v \cdot \nabla(n \times (Bv - n \times \omega_v))_m, (v^\epsilon - v)_j)$$

$$+ (\partial_i H \cdot \nabla(n \times (Bv - n \times \omega_v))_m, (H^\epsilon - H)_l) \Big|$$

$$\leq C \left(|v^\epsilon - v|_{L^1(\Omega)} + |H^\epsilon - H|_{L^1(\Omega)} + \|v^\epsilon - v\| + \|H^\epsilon - H\| \right)$$

$$\leq C \left(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}} \right).$$

Hence, we obtain

$$L_1 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}).$$

Compared to L_1, both L_2 and L_3 can be easily estimated. In fact, we have

$$L_2 = |(n \times (Bv - n \times \omega_v), \nabla \times ((v^\epsilon - v) \cdot \nabla v) - \nabla \times ((H^\epsilon - H) \cdot \nabla H))|$$

$$= \Big| \int_{\partial\Omega} (n \times (Bv - n \times \omega_v))(n \times ((v^\epsilon - v) \cdot \nabla v) - n \times ((H^\epsilon - H) \cdot \nabla H))$$

$$+ (\nabla \times (n \times (Bv - n \times \omega_v)), (v^\epsilon - v) \cdot \nabla v - (H^\epsilon - H) \cdot \nabla H) \Big|$$

$$\leq C(|v^\epsilon - v|_{L^1(\partial\Omega)} + |H^\epsilon - H|_{L^1(\partial\Omega)} + \|v^\epsilon - v\| + \|H^\epsilon - H\|)$$

$$\leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}),$$

$$L_3 = \Big| (n \times (Bv - n \times \omega_v), \nabla \times ((v^\epsilon - v) \cdot \nabla(v^\epsilon - v))$$

$$- \nabla \times ((H^\epsilon - H) \cdot \nabla (H^\epsilon - H)))\big|$$

$$= \big| (n \times (Bv - n \times \omega_v), [\nabla \times, (v^\epsilon - v) \cdot \nabla](v^\epsilon - v)$$

$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla](H^\epsilon - H))$$

$$+ (n \times (Bv - n \times \omega_v), (v^\epsilon - v) \cdot \nabla(\omega_v^\epsilon - \omega_v) - (H^\epsilon - H) \cdot \nabla(\omega_H^\epsilon - \omega_H))\big|$$

$$= \big| (n \times (Bv - n \times \omega_v), [\nabla \times, (v^\epsilon - v) \cdot \nabla](v^\epsilon - v)$$

$$- [\nabla \times, (H^\epsilon - H) \cdot \nabla](H^\epsilon - H))$$

$$- ((v^\epsilon - v) \cdot \nabla(n \times (Bv - n \times \omega_v)), \omega_v^\epsilon - \omega_v)$$

$$+ ((H^\epsilon - H) \cdot \nabla(n \times (Bv - n \times \omega_v), \omega_H^\epsilon - \omega_H)\big|$$

$$\leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2).$$

We find that L_4, L_5 and L_6 have similar structures to L_1, L_2 and L_3 respectively, so we can get

$$L_4 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}),$$

$$L_5 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}),$$

$$L_6 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2).$$

From the estimates of $L_i \, (i = 1, \cdots, 6)$, we get

$$I_{31} \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}).$$

Now, it remains to estimate the term I_{32}, i.e.

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1) + (\nabla \times (n \times (BH - n \times \omega_H)), P\Phi_2)|.$$

First, we consider $(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1)$. Because it involves Leray projection, some terms which contain higher derivatives of $v^\epsilon - v$ or $H^\epsilon - H$ can not be estimated easily. We have the observations

$$v \cdot \nabla(v^\epsilon - v) - (v^\epsilon - v) \cdot \nabla v = \nabla \times ((v^\epsilon - v) \times v), \tag{4.32}$$

$$H \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla H = \nabla \times ((H^\epsilon - H) \times H). \tag{4.33}$$

Since $(v^\epsilon - v) \cdot n = 0$, $v \cdot n = 0$, $(H^\epsilon - H) \cdot n = 0$ and $H \cdot n = 0$, it means that

$$(v^\epsilon - v) \times v = \lambda_1 n, \quad (H^\epsilon - H) \times H = \lambda_2 n. \tag{4.34}$$

Due to (4.32)-(4.34), we easily obtain

$$\nabla \times ((v^\epsilon - v) \times v) \in \mathbb{H}, \quad \nabla \times ((H^\epsilon - H) \times H) \in \mathbb{H},$$

where $\mathbb{H}$ is Leray projection space. Thus we have the following equality

$$P\Phi_1 = v \cdot \nabla(v^\epsilon - v) - (v^\epsilon - v) \cdot \nabla v + P\Phi_v^1$$

$$- (H \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla H) - P\Phi_H^1,$$

where

$$P\Phi_v^1 = P[2(v^\epsilon - v) \cdot \nabla v + (v^\epsilon - v) \cdot \nabla(v^\epsilon - v)],$$

$$P\Phi_H^1 = P[2(H^\epsilon - H) \cdot \nabla H + (H^\epsilon - H) \cdot \nabla(H^\epsilon - H)].$$

Hence, we have

$$(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1)$$
$$=(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_v^1) - (\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_H^1)$$
$$+ (\nabla \times (n \times (Bv - n \times \omega_v)), v \cdot \nabla(v^\epsilon - v) - (v^\epsilon - v) \cdot \nabla v$$
$$- (H \cdot \nabla(H^\epsilon - H) - (H^\epsilon - H) \cdot \nabla H)).$$

First, we have

$$\|P\Phi_v^1\| \le C\left(\|v\|_{W^{1,\infty}} + \|v^\epsilon - v\|_{W^{1,\infty}}\right)\|v^\epsilon - v\|, \tag{4.35}$$

$$\|P\Phi_H^1\| \le C\left(\|H\|_{W^{1,\infty}} + \|H^\epsilon - H\|_{W^{1,\infty}}\right)\|H^\epsilon - H\|, \tag{4.36}$$

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_v^1) - (\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_H^1)|$$
$$\le C\|v\|_{H^2}(\|P\Phi_v^1\| + \|P\Phi_H^1\|). \tag{4.37}$$

From (4.35)-(4.37) and Lemma 4.1, we get

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_v^1) - (\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_H^1)| \le C\epsilon^{\frac{3}{4}}.$$

Next, note that

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), v \cdot \nabla(v^\epsilon - v)|$$
$$=|(v \cdot \nabla(\nabla \times (n \times (Bv - n \times \omega_v))), v^\epsilon - v)| \le C\|v\|_{H^3}\|v^\epsilon - v\| \le C\epsilon^{\frac{3}{4}}.$$

Similarly, we obtain

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), H \cdot \nabla(H^\epsilon - H)|$$
$$=|(H \cdot \nabla(\nabla \times (n \times (Bv - n \times \omega_v))), H^\epsilon - H)| \le C\|v\|_{H^3}\|H^\epsilon - H\| \le C\epsilon^{\frac{3}{4}}.$$

At the same time, we get directly that

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), (v^\epsilon - v) \cdot \nabla v + (H^\epsilon - H) \cdot \nabla H))|$$
$$\le C(\|v^\epsilon - v\| + \|H^\epsilon - H\|) \le C\epsilon^{\frac{3}{4}}.$$

Therefore,

$$|(\nabla \times (n \times (Bv - n \times \omega_v)), P\Phi_1)| \le C\epsilon^{\frac{3}{4}}.$$

By using the same methods as above, we observe

$$P\Phi_2 = \Phi_2.$$

Hence, we get

$$|(\nabla \times (n \times (BH - n \times \omega_H)), P\Phi_2)| = |(\nabla \times (n \times (BH - n \times \omega_H)), \Phi_2)| \le C\epsilon^{\frac{3}{4}}.$$

Finally, we have

$$I_{32} \le C\epsilon^{\frac{3}{4}}.$$

Thus, we conclude that

$$I_3 \leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}). \tag{4.38}$$

In conclusion, it follows from (4.27), (4.31) and (4.38) that

$$\frac{1}{2}\frac{d}{dt}E + \frac{\epsilon}{2}(\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2)$$

$$\leq C(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \epsilon^{\frac{1}{2}}).$$

Now, we need to deal with the left terms in the above inequality. Let us recall that

$$E = \|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2$$

$$- \int_{\partial\Omega} B(v^\epsilon - v) \cdot (v^\epsilon - v) - 2\int_{\partial\Omega}(v^\epsilon - v) \cdot (Bv - n \times \omega_v)$$

$$- \int_{\partial\Omega} B(H^\epsilon - H) \cdot (H^\epsilon - H) - 2\int_{\partial\Omega}(H^\epsilon - H) \cdot (BH - n \times \omega_H).$$

We note that

$$\left| \int_{\partial\Omega} B(v^\epsilon - v) \cdot (v^\epsilon - v) + \int_{\partial\Omega} B(H^\epsilon - H) \cdot (H^\epsilon - H) \right|$$

$$\leq C\left(|v^\epsilon - v|^2_{L^2(\partial\Omega)} + |H^\epsilon - H|^2_{L^2(\partial\Omega)} \right)$$

$$\leq \delta\left(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 \right) + C_\delta(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2),$$

$$\left| 2\int_{\partial\Omega}(v^\epsilon - v) \cdot (Bv - n \times \omega_v) + 2\int_{\partial\Omega}(H^\epsilon - H) \cdot (BH - n \times \omega_H) \right|$$

$$\leq C\left(|v^\epsilon - v|_{L^1(\partial\Omega)} + |H^\epsilon - H|_{L^1(\partial\Omega)} \right)$$

$$\leq \delta\left(\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 \right) + C_\delta(\|v^\epsilon - v\|^2 + \|H^\epsilon - H\|^2)$$

for some δ small enough. Consequently, we get

$$\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 + \frac{\epsilon}{2}\int_0^t (\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2)$$

$$\leq C\int_0^t (\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2) + C\epsilon^{\frac{1}{2}}.$$

By using Gronwall's inequality, we have

$$\|\omega_v^\epsilon - \omega_v\|^2 + \|\omega_H^\epsilon - \omega_H\|^2 \leq C\epsilon^{\frac{1}{2}} \quad \text{on} \quad [0, T_2]. \tag{4.39}$$

Thus

$$\|v^\epsilon - v\|^2_{H^1} + \|H^\epsilon - H\|^2_{H^1}$$

$$+ \epsilon\int_0^t (\|P\Delta(v^\epsilon - v)\|^2 + \|P\Delta(H^\epsilon - H)\|^2) \leq C\epsilon^{\frac{1}{2}}.$$

From Lemmas 4.1 and 4.2, we get

$$\epsilon\int_0^t (\|v^\epsilon - v\|^2_{H^2} + \|H^\epsilon - H\|^2_{H^2}) \leq C\epsilon^{\frac{1}{2}}. \tag{4.40}$$

Note that the following inequality holds

$$\|\nabla(u^\epsilon - u)\|_{L^p}^p \leq C\|\nabla(u^\epsilon - u)\|_{L^\infty}^{p-2}\|\nabla(u^\epsilon - u)\|^2. \tag{4.41}$$

Hence, we obtain

$$\|\nabla(v^\epsilon - v)\|_{L^p}^p + \|\nabla(H^\epsilon - H)\|_{L^p}^p \leq C\epsilon^{\frac{1}{2}}. \tag{4.42}$$

This completes the proof of Lemma 4.3. $\square$

From Lemmas 4.1 and 4.3, we easily get Theorem 1.2.

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