EXISTENCE AND APPLICATIONS OF RICCI FLOWS VIA PSEUDOLOCALITY

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ABSTRACT. We prove the short-time existence of Ricci flows on complete manifolds with scalar curvature bounded below uniformly, Ricci curvature bounded below by a negative quadratic function, and with almost Euclidean isoperimetric inequality holds locally. In particular, this result applies to manifolds with both Ricci curvature and injectivity radius bounded from below. We also study the short-time behaviour of these solutions which may have unbounded curvature at the initial time, and provide some applications. A key tool is Perelman's pseudolocality theorem.

1. INTRODUCTION

Since its introduction by Hamilton [15], the Ricci flow has been extensively studied and very fruitful in yielding deep results, although there are still many questions to be answered. One of the many open problems about the Ricci flow is the existence of solutions on noncompact manifolds. In fact, existence of the flow on general complete manifolds is expected to be not true, and we would like to restrict our attention to manifolds with curvature conditions relevant to potential applications. When the sectional curvature is uniformly bounded, it follows from the work of Shi [25] that the Ricci flow has a unique solution. When the curvature is unbounded, progress has been made by many authors under various assumptions, for an incomplete list, see Cabezas-Rivas and Wilking [2], Chau, Li and Tam [4], Giesen and Topping [12][13], Hochard [17], Simon [26][27], Topping [28], Xu [29], and the author of this article [16]. In this article, we study the existence of Ricci flows on complete noncompact manifolds with a set of assumptions motivated by Perelman's pseudolocality theorem. We focus on dimension $n \geq 3$ since the 2-dimensional case has been settled in [28] and [12], which also make use of pseudolocality. We also explore the properties of these solutions and point out some applications.

It is well-known that for any regular domain Ω in the Euclidean space $\mathbb{R}^n$, we have the *Euclidean isoperimetric inequality*

$$|\partial\Omega|^n \geq I_n|\Omega|^{n-1},$$

where $I_n = \frac{|\partial B_{\mathbb{R}^n}(0,1)|^n}{|B_{\mathbb{R}^n}(0,1)|^{n-1}}$ is the optimal constant. On a Riemannian manifold (M^n, g), let $B_g(p, r)$ be a geodesic ball with radius r centered at a point p. We say the $\delta-$*almost Euclidean isoperimetric inequality* holds in $B_g(p, r)$ with respect to the Riemannian metric g, if

$$(Area_g(\partial\Omega))^n \geq (1 - \delta)I_n(Vol_g(\Omega))^{n-1}$$

for any regular domain $\Omega \subset B_g(p, r)$. In [23], Perelman proved an interior curvature estimate for compact Ricci flows known as the pseudolocality theorem. The

complete noncompact case has been verified by Chau, Tam and Yu [7]. See also [11] for a detailed treatment.

Theorem 1.1 (Perelman's pseudolocality). *For every n and $A > 0$, there exist $\delta_0 > 0$ and $\epsilon_0 > 0$ depending only on A and n with the following property: Suppose $(M^n, g(t)), t \in [0, (\epsilon r)^2]$ is a complete solution of the Ricci flow with bounded curvature, where $0 < \epsilon \leq \epsilon_0$ and $r > 0$. Let x_0 be a point in M. If there is a scalar curvature lower bound*

$$S(x, 0) \geq -r^{-2} \quad for \quad any \quad x \in B_{g(0)}(x_0, r),$$

and if the δ_0-almost Euclidean isoperimetric inequality holds in $B_{g(0)}(x_0, r)$ with respect to the initial metric $g(0)$, then we have

$$|Rm|(x, t) \leq \frac{A}{t} + \frac{1}{(\epsilon_0 r)^2}$$

for $x \in B_{g(t)}(x_0, \epsilon_0 r)$ and $t \in (0, (\epsilon r)^2]$.

The validity of the $\delta-$almost Euclidean isoperimetric inequality under a fixed radius is actually a strong condition, roughly speaking it rules out too much positive curvature. Nevertheless, it does not require any point-wise curvature upper bound. An example is given by the neighbourhood of a rounded out flat cone point, with cone angle close to 2π so it is C^0-close to a Euclidean disk, while the sectional curvature can be made arbitrarily large and positive, and the injectivity radius can be arbitrarily small.

On the other hand, bounded curvature and a volume lower bound for all unit geodesic balls can imply the validity of the $\delta-$almost Euclidean isoperimetric inequality. By the work of Anderson and Cheeger [1], a lower bound of both the Ricci curvature and the injectivity radius implies a lower bound of the $W^{1,p}-$harmonic radius, hence also implies this condition.

Perelman's pseudolocality is not really a local result since it assumes a complete Ricci flow with bounded curvature, the completeness is necessary. Nevertheless, we can apply it to prove the short-time existence of Ricci flow solutions, with possibly unbounded curvature at the initial time. The key step is a conformal transformation which turns a compact domain into a complete Riemannian manifold, while keeping the scalar curvature lower bound and the isoperimetric inequality.

Theorem 1.2. *For any $n > 0$, $A > 0$, $k \geq 0$ and $L \geq 0$, there exists constants $\delta_1 > 0$ and ϵ_1 depending only on n and A, such that if (M^n, g) is a complete Riemannian manifold satisfying*

(i) $\liminf_{d(p,x) \to \infty} \frac{Ric(x)}{d(p,x)^2} \geq -L$, where $d(p, x)$ is the geodesic distance function from a fixed point $p \in M$,

(ii) scalar curvature $S \geq -k$,

(iii) the δ_1-almost Euclidean isoperimetric inequality holds in any geodesic ball with radius $r > 0$,

then (M, g) admits a complete solution of the Ricci flow with curvature bound

$$|Rm(x, t)| \leq \frac{A}{t} + \frac{1}{(\epsilon_1 \bar{r})^2},$$

for all $x \in M$ and $0 < t \leq (\epsilon_1 \bar{r})^2$, where $\bar{r} = \min\{r, \sqrt{1/k}, \sqrt{1/Lk}\}$. Moreover, $g(t)$ is $\kappa(n)-$noncollapsed under the scale $\sqrt{t}$, for any $0 < t \leq (\epsilon_1 \bar{r})^2$.

Recall that we say a Riemannian manifold (M, g) is $\kappa-$noncollapsed under the scale ρ if

$$\frac{Vol_g B_g(r)}{r^n} \geq \kappa, \quad for \quad all \quad 0 < r < \rho.$$

When the initial manifold has bounded curvature, the metrics under a smooth Ricci flow solution are uniformly equivalent to the initial metric, so there is no worry about drastic distance distortion or volume collapsing within a short time. This is no longer clear when the curvature is only bounded by a non-integrable function $\frac{A}{t}$. However, the solutions from Theorem 1.2 are noncollapsed for any positive time close enough to 0, hence enjoy the compactness property on positive time intervals in the sense of Hamilton [14].

As an interesting special case, we obtain the following corollary from the above theorem and [1].

Corollary 1.3. *Let (M^n, g) be a complete Riemannian manifold. Suppose there are constants l and $\iota > 0$, such that the Ricci curvature and injectivity radius are bounded from below uniformly*

$$Ric(x) \geq -l \quad and \quad inj_x \geq \iota,$$

then there is a complete smooth Ricci flow solution with (M^n, g) as initial data, moreover, this solution is noncollapsed in the sense of Proposition 3.2.

Remark 1.4. It has been proved in [17] that in dimension 3, the injectivity radius lower bound can be weakened to volume noncollapsing.

We would like to list some direct applications of our existence result. First recall that a curvature lower bound and a C^0 (almost) optimal condition can imply rigidity of Riemannian manifolds. For example, consider a complete Riemannian manifold (M^n, g) with nonnegative Ricci curvature, it follows directly from volume comparison theorem that if the volume growth rate is (exactly) Euclidean, then (M^n, g) is isometric to the Euclidean space $\mathbb{R}^n$; if the volume growth is almost optimal, Cheeger and Colding showed in [8] that the manifold is diffeomorphic to $\mathbb{R}^n$. Our first corollary is in the same spirit.

Corollary 1.5. *Let (M^n, g) be a complete Riemannian manifold. Suppose*
  (i) $\liminf_{d(p,x) \to \infty} \frac{Ric(x)}{d(p,x)^2} > -\infty$,
  (ii) the scalar curvature $S \geq 0$,
  (iii) the optimal Euclidean isoperimetric inequality holds on M.
  Then (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.

Ricci curvature lower bound and the validity of isoperimetric inequalities does not imply injectivity lower bound, however we can show that it rules out nontrivial topology locally under a certain scale.

Corollary 1.6. *For any n and k, There exist constants δ and η depending only on n and k, such that if the geodesic ball $B(p, 1)$ is relatively compact in a Riemmannian manifold (M, g), $Ric \geq -kg$ on $B(p, 1)$, and the $\delta-$almost Euclidean isoperimetric inequality holds in $B(p, 1)$, then $B(p, \eta)$ is diffeomorphic to a Euclidean ball of dimension n.*

The next application is related to Yau's uniformization conjecture for complete Kähler manifolds with positive holomorphic bisectional curvature, which predicts

that these manifolds are biholomorphic to $\mathbb{C}^n$. This conjecture has generated a lot of research, and there are many partial confirmations, one can refer to [5] for a survey, and to [21] for the most recent progress. By [6], Yau's conjecture is true under the additional assumption of maximal volume growth and bounded curvature. The bounded curvature condition can be relaxed in certain situations as shown in [18] using Ricci flows. It has been proved in [18] that the nonnegativity of holomorphic bisectional curvature is preserved under a smooth Ricci flow solution with curvature bounded by $\frac{A}{t}$, as long as A is sufficiently small. Since nonnegative holomorphic bisectional curvature implies nonnegative Ricci curvature, Theorem 1.2 is applicable. We can prove that maximal volume growth persists under the Ricci flow solution provided by Theorem 1.2. Therefore we can verify Yau's uniformization conjecture in the following special case:

Corollary 1.7. *For any n, there is $\delta(n) > 0$, such that if (M, g) is a complete $2n-$dimensional Kähler manifold with nonnegative holomorphic bisectional curvature and maximal volume growth, and suppose the $\delta(n)-$almost Euclidean isoperimetric inequality holds in every unit geodesic ball, then (M, g) is biholomorphic to $\mathbb{C}^n$.*

Remark 1.8. After the posting of the first draft of this work on arXiv, there has been many important progress on related problems. We are obliged to list some (but not all) recent results by other authors to keep the information up to date:

i) Yau's uniformization conjecture in the maximal volume growth case was proved first by Liu in an updated version of [21]. Later, Lee and Tam [20] gave another proof using Ricci flow, in particular, they proved the short-time existence of Kähler Ricci flow on noncollapsed Kähler manifolds with bisectional curvature bounded from below.

ii) By a recent work of Cavalletti and Mondino [3], almost Euclidean isoperimetric inequality can be implied by almost Euclidean volume growth when there is a Ricci lower bound, hence the assumption in our Corollary 1.3 and 1.6 can be weakened.

2. Proof of existence

Perelman's pseudolocality is applied to give the following estimate of the lifespan of Ricci flow solutions.

Lemma 2.1. *Let δ_0 and ϵ_0 be the constants in Theorem 1.1 (where $A > 0$ can be arbitrarily chosen). Let $(M^n, g(t)), t \in [0, T)$ be a complete Ricci flow solution with bounded curvature, where T is the maximal existence time. Suppose there is $r > 0$, such that $S(g(0)) \geq -\frac{1}{r^2}$ on M and the δ_0-almost Euclidean isoperimetric inequality holds in $B_{g(0)}(x, r)$ for all $x \in M$, then $T \geq \epsilon_0^2 r^2$.*

Proof. Since T is the maximal existence time, if $T < 0$ then
$$\limsup_{t \to T^-} \sup_M |Rm| = \infty.$$
However, if $T < (\epsilon_0 r)^2$, by Theorem 1.1, we have
$$|Rm|(x, t) \leq \frac{A}{t} + \frac{1}{\epsilon_0^2 r^2}$$

for all $t \in (0, T)$ and all $x \in M$, which leads to a contradiction. $\qquad\square$

Another ingredient in the proof of Theorem 1.2 is the following construction of a distance-like function with controlled Laplacian. Here we state a scaling invariant version.

Lemma 2.2 (Schoen-Yau [24])**.** *Let (M, g) be a complete Riemannian manifold with dimension n and $Ric \geq -l$ for some $l > 0$. Then there is a distance-like function $\gamma : M \to \infty$, such that*

$$(2.1) \qquad \lambda d(x, p) \leq \gamma(x) \leq \Lambda d(x, p),$$

$$|\nabla \gamma| \leq C_0 \quad and \quad |\Delta \gamma| \leq C_0 \sqrt{l},$$

when $d(x, p) \geq C_0/\sqrt{l}$, where constants λ, Λ, C_0 depends only on n and a lower bound of $Vol_g B_g(p, 1/\sqrt{l})/l^{n/2}$.

Remark 2.3. It is evident from the proof in [24] that Lemma 2.2 works on $B_g(p, \rho)$ for ρ large enough, as long as $B_g(p, \rho + 1/\sqrt{l})$ is relatively compact in (M, g), even when (M, g) is incomplete.

In the following, we denote the level sets of γ by

$$U_\rho = \gamma^{-1}([0, \rho))$$

for any $\rho > 0$.

In the next lemma we will construct a good conformal metric on any given level set of γ. The function $f(s)$ in the proof below has been used in the work of Hochard [17], also implicitly in Topping's [28], to conformally transform a compact domain into a complete manifold with bounded curvature. Note that $f(s)$ is essentially the conformal factor for a scaled hyperbolic metric on an Euclidean ball, and recall that the Euclidean isoperimetric inequality holds on hyperbolic spaces. Therefore we would like to construct a conformal factor by combining $f(s)$ with γ, then we can verify that the scalar curvature lower bound and the almost Euclidean isoperimetric inequality are roughly preserved.

Lemma 2.4. *For any n and δ_1, there are constants $C(n, \delta_1)$ and $c(n, \delta_1)$ with the following properties. Given γ defined in Lemma 2.2 (recall that $Ric \geq -l$), suppose on U_ρ we have $S \geq -k$ and the δ_1-almost Euclidean isoperimetric inequality holds in any $B(x, r) \subset U_\rho$. Then for any $\rho > 2c(n, \delta_1)C_0 r\sqrt{l + 1}$, there is a conformal metric h on U_ρ with the following properties:*

(i) (U_ρ, h) is a complete Riemannian manifold with uniformly bounded curvature, and $h \equiv g$ on $U_{\rho - c(n, \delta_1)C_0 r\sqrt{l+1}}$.

(ii) the scalar curvature of h is bounded from below by

$$S_h \geq -k - C(n, \delta_1) \max\{\frac{1}{r}, \frac{1}{r^2}\};$$

(iii) the $2\delta_1-$almost Euclidean isoperimetric inequality holds in any geodesic ball with radius $r/8$ with respect to h.

Proof. For $0 < \kappa < 1$ to be determined later, define $f : [0, 1) \to [0, \infty)$ by

$$f(s) = \begin{cases} 0, & 0 \leq s \leq 1 - \kappa; \\ -\ln\left(1 - (\frac{s - 1 + \kappa}{\kappa})^2\right), & 1 - \kappa < s < 1. \end{cases}$$

By simple computation, for any $1 - \kappa < s < 1$ we have

$$0 < \frac{df}{ds} = \frac{2(s-1+\kappa)}{\kappa^2 - (s-1+\kappa)^2} \leq \frac{2\kappa}{\kappa^2 - (s-1+\kappa)^2},$$

$$0 < \frac{d^2 f}{ds^2} = \frac{2(\kappa^2 + (s-1+\kappa)^2)}{(\kappa^2 - (s-1+\kappa)^2)^2} \leq \frac{4\kappa^2}{(\kappa^2 - (s-1+\kappa)^2)^2}.$$

Let $f(x) = f(\rho^{-1}\gamma(x))$ for $x \in U_\rho$. Define the conformal metric $h = e^{2f}g$. Then $h = g$ on $U_{(1-\kappa)\rho}$. The completeness of h is easy to check.

The scalar curvature of h is given by the well-known formula

$$S_h = e^{-2f}\left(S_g - \frac{4(n-1)}{n-2}e^{-(n-2)f/2}\Delta_g e^{(n-2)f/2}\right)$$

for $n \geq 3$. By simple calculation we have

$$S_h = e^{-2f}\left[S_g - \frac{4(n-1)}{n-2}\left((\frac{n-2}{2\rho})^2 |f'|^2 |\nabla\gamma|^2 + \frac{n-2}{2\rho^2}f''|\nabla\gamma|^2 + \frac{n-2}{2\rho}f'\Delta\gamma\right)\right].$$

Then by Lemma 2.2 and the fact that $f'e^{-2f} \leq 2/\kappa$, $|f'|^2 e^{-2f} \leq 4/\kappa^2$ and $f''e^{-2f} \leq 4/\kappa^2$, we have

$$(2.2) \qquad S_h \geq e^{-2f} \geq -k - \frac{4(n-1)^2 C_0^2}{\rho^2 \kappa^2} - \frac{4(n-1)C_0\sqrt{l}}{\rho\kappa}.$$

For $n = 2$ the calculation is similar.

Next we need to verify that h has uniformly bounded sectional curvature. By the compactness of $\bar{U}_\rho$ we can find a number $K > 0$ such that $|Rm_g| \leq K$ and $|\nabla^2 \rho| \leq K$ on $\bar{U}_\rho$. Hence

$$|\nabla f(\rho^{-1}\gamma(x))|^2 e^{-2f} \leq \frac{4C_0^2}{\kappa^2 \rho^2},$$

$$|\nabla^2 f(\rho^{-1}\gamma(x))|e^{-2f} \leq \left(\frac{C_0^2}{\kappa^2\rho} + \frac{2K}{\kappa^3}\right)\frac{1}{\rho}.$$

Recall the formula for sectional curvature under the conformal change is

$$K_{ij}^h = e^{-2f}\left(K_{ij}^g - \sum_{k \neq i,j} |\nabla_k f|^2 + \nabla_i\nabla_i f + \nabla_j\nabla_j f\right),$$

when calculated in an orthonormal frame. Therefore the metric h have sectional curvature uniformly bounded by a constant $C(n, K, \kappa, \rho)$.

Finally we need to estimate the isoperimetric constant for the conformal metric h. In this step we will determine the value of κ.

Let's fix a point $y \in U_\rho$, suppose $\gamma(y) = d\rho$ for some $d \in [0, 1)$. Let $0 < \alpha < \min\{d, 1-d\}$ be a constant whose value will be determined later. Since the function f is monotonically nondecreasing, on the set $U_{(d+\alpha)\rho}/U_{(d-\alpha)\rho}$ we have

$$e^{2f(d-\alpha)}g \leq h \leq e^{2f(d+\alpha)}g.$$

For any $0 < \tau < \rho/(C_0\sqrt{l+1})$, we can always choose α large enough such that $B_h(y, \tau) \subset B_g(y, \alpha\rho/(C_0\sqrt{l+1}))$, which is a subset of $U_{(d+\alpha)\rho}/U_{(d-\alpha)\rho}$ due to the gradient bound $|\nabla\gamma| \leq C_0$. To apply the isoperimetric inequality for g, we would like to choose α small enough such that $B_g(y, \alpha\rho/(C_0\sqrt{l+1})) \subset B_g(y, r)$. Therefore we would like to estimate the smallest possible α.

For any $x \in B_h(y, \tau)$ we have $d_g(x, y) \leq \tau e^{-f(d-\alpha)}$. So it is sufficient to choose α such that

$$\tau = \frac{\alpha e^{f(d-\alpha)} \rho}{C_0 \sqrt{l+1}}.$$

More precisely, let's define

$$\phi(\alpha) := \alpha e^{f(d-\alpha)} = \begin{cases} \frac{\alpha}{1-\left(\frac{d-1+\kappa-\alpha}{\kappa}\right)^2}, & \alpha < d-1+\kappa; \\ \alpha, & \alpha \geq d-1+\kappa. \end{cases}$$

It turns out that $\phi(\alpha)$ is a nondecreasing function with range $[0, 1)$. For simplicity let's denote $\tilde{\tau} = C_0 \sqrt{l+1} \tau / \rho$. by direct computation

$$\alpha_{\tilde{\tau}} := \phi^{-1}(\tilde{\tau}) = \begin{cases} \frac{-(\kappa^2 - 2\tilde{\tau}\beta) + \sqrt{(\kappa^2 - 2\tilde{\tau}\beta)^2 + 4\tilde{\tau}^2(\kappa^2 - \beta^2)}}{2\tilde{\tau}}, & \tilde{\tau} < \beta; \\ \tilde{\tau}, & \tilde{\tau} \geq \beta; \end{cases}$$

where

$$\beta = d - 1 + \kappa.$$

To simplify, let $\tilde{\tau} = a\kappa$, the value of a will be determined later. Then $\alpha_{\tilde{\tau}}$ can be written as

$$\alpha_{\tilde{\tau}} = \begin{cases} \frac{2a(\kappa^2 - \beta^2)}{(\kappa - 2a\beta) + \sqrt{(\kappa - 2a\beta)^2 + 4a^2(\kappa^2 - \beta^2)}}, & \alpha_{\tilde{\tau}} < \beta \\ a\kappa, & \alpha_{\tilde{\tau}} \geq \beta. \end{cases}$$

Let's take

$$0 < a < \frac{1}{4}.$$

When $\alpha_{\tilde{\tau}} < \beta$, we have

$$\alpha_{\tilde{\tau}} < \frac{2a(\kappa^2 - \beta^2)}{\kappa} < 4a(\kappa - \beta).$$

If we further take $a < \frac{1}{8}$, then

$$\alpha_{\tilde{\tau}} + \beta < \frac{\kappa + \beta}{2}.$$

Claim: For any $\epsilon > 0$, there is an a depending only on ϵ (and independent of d), such that

$$|f(d + \alpha_{\tilde{\tau}}) - f(d - \alpha_{\tilde{\tau}})| < \epsilon.$$

When $\alpha_{\tilde{\tau}} < \beta$, by the definition of f we have

$$e^{f(d+\alpha_{\tilde{\tau}}) - f(d-\alpha_{\tilde{\tau}})} = \frac{\kappa^2 - (\beta - \alpha_{\tilde{\tau}})^2}{\kappa^2 - (\beta + \alpha_{\tilde{\tau}})^2} = 1 + \frac{4\alpha_{\tilde{\tau}}\beta}{\kappa^2 - (\beta + \alpha_{\tilde{\tau}})^2};$$

$$e^{f(d-\alpha_{\tilde{\tau}}) - f(d+\alpha_{\tilde{\tau}})} = \frac{\kappa^2 - (\beta + \alpha_{\tilde{\tau}})^2}{\kappa^2 - (\beta - \alpha_{\tilde{\tau}})^2} = 1 - \frac{4\alpha_{\tilde{\tau}}\beta}{\kappa^2 - (\beta - \alpha_{\tilde{\tau}})^2};$$

Using $\alpha_{\tilde{\tau}} < 4a(\kappa - \beta)$ and $\beta + \alpha_{\tilde{\tau}} < \frac{\kappa + \beta}{2}$, we can estimate

$$0 < \frac{4\alpha_{\tilde{\tau}}\beta}{\kappa^2 - (\beta + \alpha_{\tilde{\tau}})^2} < 32a;$$

$$0 < \frac{4\alpha_{\tilde{\tau}}\beta}{\kappa^2 - (\beta - \alpha_{\tilde{\tau}})^2} < 16a.$$

When $\alpha_{\tilde{\tau}} \geq \beta$ (equivalently $d - \alpha_{\tilde{\tau}} \leq 1 - \kappa$), we have

$$1 \leq e^{f(d+\alpha_{\tilde{\tau}})-f(d-\alpha_{\tilde{\tau}})} \leq \frac{\kappa^2}{\kappa^2 - (\beta+\alpha_{\tilde{\tau}})^2} < \frac{1}{(1-4a^2)}.$$

$$1 \geq e^{f(d-\alpha_{\tilde{\tau}})-f(d+\alpha_{\tilde{\tau}})} \geq \frac{\kappa^2 - (\beta+\alpha_{\tilde{\tau}})^2}{\kappa^2} > 1 - \frac{4\alpha_{\tilde{\tau}}^2}{\kappa^2} = 1 - 4a^2.$$

Then it is clear that we can take a small enough and only depending on ϵ, thus the claim is proved.

Since $\alpha_{\tilde{\tau}} < 4a\kappa$ by the above analysis, we can take

$$\kappa = \frac{C_0 r \sqrt{l+1}}{8a\rho}$$

when ρ is large enough (so that $\kappa < 1$), then $B_h(y,\tau) \subset B_g(y,r)$. Let Ω be a connected domain with smooth boundary in $B_h(y,\tau)$,

$$Area_h(\partial\Omega) \geq e^{(n-1)f(d-\alpha_{\tilde{\tau}})} Area_g(\partial\Omega),$$

$$Vol_h(\Omega) \leq e^{nf(d+\alpha_{\tilde{\tau}})} Vol_g(\Omega).$$

By the isoperimetric assumption in the lemma,

$$(Vol_h(\Omega))^{n-1} \leq e^{n(n-1)(f(d+\alpha_{\tilde{\tau}})-f(d-\alpha_{\tilde{\tau}}))}(1-\delta_1)I_n(Area_h(\partial\Omega))^n.$$

Therefore we can find a number a depending on n, δ_1, and determines the above constants κ and τ, such that the almost Euclidean isoperimetric inequality

$$(2.3) \qquad (Vol_h(\Omega))^{n-1} \leq (1-2\delta_1)I_n(Area_h(\partial\Omega))^n$$

holds for any regular domain Ω in $B_h(y,\tau)$. Tracking the calculation we see that $\tau = \frac{r}{8}$, which is independent of y.

Now we have verified all desired properties. Let's point out that $f(s)$ is not smooth at $s = 1 - \kappa$. However, from the proof we can see that a smooth approximation of f will work as long as it is sufficiently close to f in C^0 norm. This can be done by standard mollifying method. $\qquad \square$

Proof of Theorem 1.2. By the assumptions, when R is sufficiently large we have $Ric \geq -LR^2$ on $B_g(p,R)$. The volume ratio $Vol_g B_g(p,s)/s^n$ is controlled from below by the isoperimetric inequality when $s < r$. By Lemma 2.2 (and the remark following it) we can construct a good distance-like function γ on $B_g(p,R)$, with constants λ, Λ and C_0 depending only on n (by requiring WLOG $\delta_1 < 1/2$). Since $\lambda d_g(p,x) < \gamma(x) < \Lambda d_g(p,x)$, we can choose $\rho = \lambda R$, such that the level set $U_\rho \subset B_g(p,R)$.

To use Lemma 2.4, we have to check that $\rho > 2c(n,\delta_1)C_0 r\sqrt{L}R$. This can be done by choosing $r \leq c_1/\sqrt{L}$ for some constant c_1 small enough depending on n and δ_1.

Therefore we have a complete metric h on U_ρ with bounded curvature, by Shi [25], a complete solution with bounded curvature exists. Then we can use Lemma 2.1 to get a lower bound of the lifespan of Shi's solution, which is independent of R. Therefore we can take $R \to \infty$, and the solutions converge subsequentially to a complete solution on M. The initial metric of this limit solution is g since $h = g$ on $B_g(p, 2^{-1}\Lambda^{-1}\lambda R)$ for each R large enough. The choice of $\bar{r} = \min\{r, \sqrt{1/k}, \sqrt{1/Lk}\}$ can be justified by scaling arguments.

Since the curvature is not bounded, this convergence does not follow directly from the compactness of Ricci flows. However, we can apply the interior curvature

estimate of B.L.Chen ([10] Theorem 3.1), and the modified Shi's estimates ([22] Theorem 11) to guarantee that the convergence is smooth and uniform on any compact set.

The non-collapsing claim comes from applying Lemma 3.1 to every bounded curvature solution in the converging sequence. $\qquad\square$

Now let's restate Corollary 1.3 more precisely.

Corollary 2.5. *Let (M^n, g) be a Riemannian manifold, suppose $Ric(x) \geq -l$ and $inj_x \geq \iota > 0$ for all $x \in M$, then for any $\epsilon > 0$, the Ricci flow has a short-time solution $(M, g(t))$ with $g(0) = g$, satisfying properties:*
(i) $|Rm|(t) \leq \frac{1}{t}$;
(ii) $Vol_{g(t)} B_{g(t)}(x, \sqrt{t}) \geq (1 - \epsilon)\omega_n(\sqrt{t})^n$;
where $t \in [0, T]$, T only depends on n, l, ι and e.

Proof. By [1], for any $x > 0$ and $\epsilon > 0$, (M, g) has $W^{1,p}$-harmonic radius centered at x bounded from below by a uniform constant $r_h(n, l, \iota, \epsilon)$, in particular, g in this harmonic coordinate satisfies

$$(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}.$$

Clearly the δ_1-almost Euclidean isoperimetric inequality can be verified in $B(x, r_h)$ when we choose ϵ small enough. Since r_h is independent of x, the assumptions of Theorem 1.2 and Proposition 3.2 are satisfied, and the corollary follows. $\qquad\square$

3. Short-time analysis of the solutions

Although the Ricci flow solutions provided by Theorem 1.2 are smoothly continuous to the initial data, the modulus of continuity a-priori depends on the initial sectional curvature bounds which are not assumed. In order to guarantee these solutions have some compactness property in the sense of [14], we would need uniform non-collapsing estimates depending only on the assumed conditions. In fact, a non-collapsing result is directly implied by the proof of the pseudolocality theorem as pointed out by Perelman in the compact case [23], see also [19]. The complete noncompact case can be proved by the same argument with the help of heat kernel estimates established in [7], we sketch a proof following [19].

Lemma 3.1 (Perelman). *There is a constant $\kappa(n)$ depending only on the dimension n, such that under the same assumptions of Theorem 1.1, we have*

$$Vol_{g(t)} B_{g(t)}(x, \sqrt{t}) \geq \kappa(n)t^{\frac{n}{2}}$$

for all $t \in (0, \epsilon_0^2 r^2]$ and $x \in B_{g(t)}(x_0, \epsilon_0 r)$.

Sketch of proof. For simplicity let's take $A = 1$ in the pseudolocality theorem and assume WLOG that $|Rm|(t) \leq \frac{1}{t}$.

For any $\epsilon > 0$, let $H(y, t)$ be the conjugate heat kernel centered at $(x, 2\epsilon^2)$, i.e.

$$(\frac{\partial}{\partial t} + \Delta_{g(t)} - S(t))H = 0, \quad t \in (0, 2\epsilon^2),$$

$$\lim_{t \to 2\epsilon^2} H = \delta_x.$$

Define the function f by the relation $H = (4\pi(2\epsilon^2 - t))^{-n/2}e^{-f}$. Choose a time dependent cut-off function h as in [23], recall that h satisfies $(\frac{\partial}{\partial t} - \Delta_{g(t)})h \leq \frac{10}{b(n)\epsilon^2}h$,

where $b(n)$ can be taken to be $400n$ for example. Since $\lim_{t \to 2\epsilon^2} \int hH = 1$, by examining $\frac{\partial}{\partial t} \int hH$ we can derive

$$\int hH d\mu(\epsilon^2) \geq c(n).$$

Since h is compactly supported in $B_{g(\epsilon^2)}(x, b(n)\epsilon)$, we have

$$c(n) \leq C(n) \frac{Vol_{g(\epsilon^2)} B_{g(\epsilon^2)}(x, \epsilon)}{\epsilon^n} e^{-f_m},$$

where $f_m = \inf_{B_{g(\epsilon^2)}(x, b(n)\epsilon)} f(\epsilon^2)$, and we have used volume comparison theorem and the curvature bound implicitly. Clearly $f_m \to -\infty$ when the volume ratio goes to 0.

By the Li-Yau type Harnack estimate for H in [7], and the curvature bound $|Rm|(t) \leq \frac{2}{\epsilon^2}$ for $t \in [\epsilon^2/2, \epsilon^2]$, we can show that

$$\int (\frac{\epsilon^2}{2} |\nabla f|^2 + f - n) hH d\mu(\frac{\epsilon^2}{2}) \leq C(n) + f_m,$$

which will be strictly negative when $f_m < -C(n) - 1$. This shows after integration by parts that the local entropy at time $\frac{\epsilon^2}{2}$ is bounded from above by a negative number. Then a contradiction follows the same argument as in the proof of the pseudolocality theorem.

$\square$

If we assume that the initial Ricci curvature is bounded from below, we can prove the following variant by a different method, where the non-collapsing constant can be made arbitrarily close to the Euclidean value.

Proposition 3.2. *For any n, k, Q and $\epsilon > 0$, there are constants δ_1 and τ depending only on n, k, Q and ϵ, with the following property. Let $g(t)$ be a complete Ricci flow solution on $M^n \times [0, \tau r^2]$ obtained from Theorem 1.2, where $r > 0$. Suppose*
   (i) $Ric(g(0)) \geq -\frac{k}{r^2}$ on $B_{g(0)}(p, r)$,
   (ii) The δ_1-almost Euclidean isoperimetric inequality holds in $B_{g(0)}(p, r)$.
Then
$$Vol_{g(t)} B_{g(t)}(p, Q\sqrt{t}) \geq (1 - \epsilon) \omega_n Q^n t^{\frac{n}{2}}$$
for all $0 < t \leq \tau r^2$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Stronger noncollapsing result can be proved when there is better control of the Ricci lower bound along the flow. In fact, when the dimension $n = 3$, with the help of the Hamilton-Ivey estimate, it has been proved in [17] that the volume is non-collapsed under a uniform scale.

In the rest of this section we will prove Proposition 3.2, and analyse distance distortion under the Ricci flow solutions of Theorem 1.2.

Since the solutions are obtained as a limit of bounded curvature solutions, Proposition 3.3 follows from the lemma below. The proof explores a similar idea as in the metric lemma of [17] which depends on the continuity of measure under Gromov-Hausdorff convergence proved by Cheeger and Colding [8], and we also need Perelman's pseudolocality.

Lemma 3.3. *For any n, k, Q and $\epsilon > 0$, there are constants δ_1 and τ depending only on n, k, Q and ϵ, with the following property. Let $g(t)$ a complete Ricci flow solution with bounded curvature on $M^n \times [0, \tau]$. Suppose*

(i) $Ric(g(0)) \geq -k$ on $B_{g(0)}(p, 1)$,

(ii) The δ_1-almost Euclidean isoperimetric inequality holds in $B_{g(0)}(p, 1)$.

Then
$$Vol_{g(t)} B_{g(t)}(p, Q\sqrt{t}) \geq (1 - \epsilon)\omega_n Q^n t^{\frac{n}{2}}$$
for all $0 < t \leq \tau$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. We only need to show the proof for $Q = 1$. Suppose the claim is not true, then for some n, k, ϵ, there is a sequence of $\delta_i \to 0$, a sequence of $t_i \to 0$, and a sequence of pointed Ricci flow solutions $(M_i, g_i(t), p_i), i = 1, 2, ...$ with $Ric(g_i(0)) \geq -kg_i(0)$ and the δ_i-almost Euclidean isoperimetric inequality holds on $B_{g_i(0)}(p_i, 1)$. In addition, since these solutions are smooth, we can choose t_i such that
$$Vol_{g_i(t)} B_{g_i(t)}(p_i, \sqrt{t}) > (1 - \epsilon)\omega_n t^{\frac{n}{2}}$$
for any $0 < t < t_i$, while
$$Vol_{g_i(t_i)} B_{g(t_i)}(p_i, \sqrt{t_i}) = (1 - \epsilon)\omega_n t_i^{\frac{n}{2}}.$$

Let $A_j \to 0$ be a sequence of positive numbers decreasing to 0. For each A_j, let $\bar{\delta}_j > 0$ and $\epsilon_j > 0$ be the claimed constants in Theorem 1.1. Since $\delta_i \to 0$, for each j, we have $\delta_i < \bar{\delta}_j$ when i is large enough. Thus Theorem 1.1 yields curvature estimates
$$|Rm|(g_i(x, t)) \leq \frac{A_j}{t} + \frac{1}{(\epsilon_j r_j)^2}, \quad x \in B_{g_i(t)}(p_i, \frac{1}{2}), \quad t \in (0, (\epsilon_j r_j)^2],$$
when i is large enough. Since $t_i \to 0$, we can find a subsequence of $\{t_i\}$, which we denote as $\{t_j\}$ for simplicity, such that $t_j < A_j(\epsilon_j r_j)^2$, hence
$$|Rm|(g_j(x, t)) \leq \frac{2A_j}{t}, \quad x \in B_{g_j(t)}(p_j, \frac{1}{2}), \quad t \in (0, t_j], \quad j = 1, 2, ...$$

Now we dilate this sequence of solutions to normalize their existence time intervals on which we have the above curvature control. Define
$$\tilde{g}_j(t) = t_j^{-1} g(t_j t), \quad t \in [0, 1], \quad j = 1, 2, ...$$
We have
$$|Rm|(\tilde{g}_j(x, t)) \leq \frac{2A_j}{t}, \quad x \in B_{\tilde{g}_j(t)}(p_j, \frac{1}{2\sqrt{t_j}}),$$
$$Vol_{\tilde{g}_j(t)} B_{\tilde{g}_j(t)}(p_j, \sqrt{t}) > (1 - \epsilon)\omega_n t^{\frac{n}{2}}$$
for $t \in (0, 1)$, and
$$Vol_{\tilde{g}_j(1)} B_{\tilde{g}_j(t)}(p_j, 1) = (1 - \epsilon)\omega_n.$$

A subsequence of the solutions $\{B_{\tilde{g}_j(t)}(p_j, \frac{1}{2\sqrt{t_j}}), \tilde{g}_j(t), t \in (0, 1], p_j\}$ converge smoothly in the pointed Cheeger-Gromov sense to a Ricci flow solution $(M_\infty, \tilde{g}_\infty, t \in (0, 1], p_\infty)$. Moveover, the convergence is uniform on $\Omega \times [s, 1]$ for any compact domain $\Omega \subset M_\infty$ and $s > 0$. The limit solution is complete since the radius $\frac{1}{2\sqrt{t_j}} \to \infty$ as $j \to \infty$. And the limit metric $g_\infty(t)$ is flat for any $t \in (0, 1]$ since $A_j \to 0$. Therefore, each time slice of the limit solution $(M_\infty, g_\infty(t))$ is the same quotient of the Euclidean space $\mathbb{R}^n$, so we can drop the time variable and denote it as $(M_\infty, g_\infty, p_\infty)$. Clearly it has
$$Vol_{g_\infty} B_{g_\infty}(p_\infty, 1) = (1 - \epsilon)\omega_n,$$

which implies that the smooth manifold (M_∞, g_∞) is a nontrivial quotient of $\mathbb{R}^n$, so there is some constant C_∞, such that

$$Vol_{g_\infty} B_{g_\infty}(p_\infty, \rho) \leq C_\infty \rho^{n-1}$$

when ρ is large.

The initial data $\{B_{\tilde{g}_j(0)}(p_j, \frac{1}{2\sqrt{t_j}}), g(0), p_j\}$ converges subsequentially in the the pointed Gromov-Hausdorff sense to a metric space (X, d_X, p_X). Since this sequence is not collapsed, by [8], the Riemannian measure converges to the $n-$dimensional Hausdorff measure on X. Thus the δ_j-almost Euclidean isoperimetric inequalities with $\delta_j \to 0$ imply that

$$\mathcal{H}^n_X(B_X(p_X, \rho)) = \omega_n \rho^n$$

for all $\rho > 0$.

For any $\rho > 0$, we can apply Lemma 3.5 and Lemma 3.7 when j is large enough to obtain

$$(3.1) \quad c_1(n) d_{\tilde{g}_j(t)}(x, y) - C_2(n) \leq d_{\tilde{g}_j(0)}(x, y) \leq d_{\tilde{g}_j(t)}(x, y) + 2(n-1)(2A_j + 1)\sqrt{t}$$

for $x, y \in B_{\tilde{g}_j(t)}(p_j, \rho)$ and $t \in [0, 1]$.

Claim: There exists a surjective map $f : M_\infty \to X$ with $f(p_\infty) = p_X$, which satisfies

$$c_1 d_{g_\infty}(y_1, y_2) - C_2 \leq d_X(f(y_1), f(y_2)) \leq d_{g_\infty}(y_1, y_2) \quad all \quad y_1, y_2 \in M_\infty.$$

Proof of Claim: For simplicity, we define X_j to be $B_{\tilde{g}_j(0)}(p_j, \frac{1/4 - r_j}{\sqrt{t_j}})$ equipped with the metric $\tilde{g}_j(0)$; and define $Y_j(t)$ to be $B_{\tilde{g}_j(t)}(p_j, \frac{1/4 - r_j}{\sqrt{t_j}})$ with $\tilde{g}_j(t)$. And we will omit the isometries involved in the pointed convergence.

Let $\mathcal{D} = \{y_i\}$ be a countable dense subset of M_∞. Let's first fix a $t \in (0, 1]$. For each y_i, let $\{y_i^j\}$ be its lifts to the converging sequence $Y_j(t)$. Equation (3.1) guarantees that $\{y_i^j\}$ is a bounded sequence in X_j when j is large enough, hence we can pass to a subsequence of $\{X_j\}$ (which clearly still converges to the same X), so that $\{y_i^j\}$ converge to some point in $x_i \in X$. We define $f(y_i) = x_i$. By passing to a diagonal sequence we can define f on $\mathcal{D}$. It's clear that we can define $f(p_\infty) = p_X$.

To make f surjective, let $\mathcal{E} = \{x_k\}$ be a countable dense subset of X, we will construct f so that $\mathcal{E}$ is in the image. For each x_k, let $\{x_k^j\}$ be its lifts in the converging sequence X_j. (3.1) guarantees that $\{x_k^j\}$ is a bounded sequence in $Y_j(t)$ for j large enough. By passing to a subsequence of $\{Y_j(t)\}$ we can have $\{x_k^j\}$ converge to some point $y_k \in M_\infty$. We add y_k to $\mathcal{D}$ and define $f(y_k) = x_k$, clearly this agrees with the definition in the previous paragraph.

Equation (3.1) implies that

$$d_X(f(y_1), f(y_2)) \leq d_{g_\infty}(y_1, y_2) + 2(n-1)\sqrt{t} \quad for \quad any \quad y_1, y_2 \in \mathcal{D}.$$

Since $\{Y_j(t)\}$ converges to the same (M_∞, g_∞) for all $t \in (0, 1]$, we can let $t \to 0$ and pass to a subsequence again to define f, such that

$$d_X(f(y_1), f(y_2)) \leq d_{g_\infty}(y_1, y_2) \quad for \quad any \quad y_1, y_2 \in \mathcal{D}.$$

Now, f is defined on a dense subset $\mathcal{D}$, it's uniformly Lipschitz continuous, and its image contains a dense subset $\mathcal{E}$, thus f can be extended to a surjection from M_∞ to X. The first inequality in (3.1) naturally passes to the limit. So the claim is proved.

By the definition of Hausdorff measure we have

$$\mathcal{H}_X^n(\Omega) \leq \mathcal{H}_{M_\infty}^n(f^{-1}(\Omega)), \quad \Omega \subset X.$$

Since $f^{-1}(B_X(p_X, \rho)) \subset B_{g_\infty}(p_\infty, c_1^{-1}(\rho + C_2))$, and the Hausdorff measure coincide with the Riemannian measure on a Riemmannian manifold, we have

$$\mathcal{H}_X^n(B_X(p_X, \rho)) \leq \mathcal{H}_{M_\infty}^n(f^{-1}(B_X(p_X, \rho))) \leq \mathcal{H}_{M_\infty}^n(B_{g_\infty}(p_\infty, c_1^{-1}(\rho + C_2))),$$

which, by previous analysis, implies that

$$\omega_n \rho^n \leq C_\infty c_1^{-n-1}(\rho + C_2)^{n-1}$$

for all $\rho > 0$, clearly this is a contradiction. $\qquad\qquad\square$

The following lemma of Perelman [23] is frequently used in the study of Ricci flows.

Lemma 3.4 (Perelman). *Let $(M^n, g(t))$, $t \in [0, T)$, be a complete Ricci flow solution, for any $t_0 \in [0, T)$ and $x_0, x_1 \in M$, suppose $Ric(x, t_0) \leq (n-1)K$ for all $x \in B_{g(t_0)}(x_0, r_0) \cup B_{g(t_0)}(x_1, r_0)$ for some $K \geq 0$ and $r_0 > 0$. Then*

$$\frac{\partial}{\partial t}|_{t=t_0} d_{g(t_0)}(x_0, x_1) \geq -2(n-1)\left(Kr_0 + \frac{1}{r_0}\right).$$

When the curvature is bounded by a function $\frac{A}{t}$, the above lemma implies the following distance distortion estimate from below:

Lemma 3.5. *Let $(M^n, g(t))$, $t \in [0, T)$ be a complete Ricci flow solution. Suppose there is a constant A, $|Ric|(x, t) \leq \frac{A}{t}$ for any $t \in (0, T)$ and $x \in B_{g(t)}(p, r + \sqrt{t})$. Then for any $x, y \in B_{g(t)}(p, r - 4(n-1)(A+1)\sqrt{t})$, we have*

$$d_{g(t)}(x, y) \geq d_{g(0)}(x, y) - 4(n-1)(A+1)\sqrt{t}.$$

Proof. Suppose ρ is the largest number such that $B_{g(t)}(p, \rho) \subset B_{g(s)}(p, r)$ for all $s \in [0, t]$. For any $x, y \in B_{g(t)}(p, \rho)$ and any $s \in [0, t]$, we can take $r_0 = \sqrt{s}$ in Lemma 3.4 which yields

$$\frac{\partial}{\partial t}|_{t=s} d_{g(t)}(x, y) \geq -\frac{2(n-1)(A+1)}{\sqrt{s}}.$$

Hence

$$d_{g(t)}(x, y) \geq d_{g(s)}(x, y) - 4(n-1)(A+1)(\sqrt{t} - \sqrt{s}),$$

which implies $B_{g(t)}(p, r - 4(n-1)(A+1)(\sqrt{t} - \sqrt{s})) \subset B_{g(s)}(p, r)$ for all $s \in [0, t]$. Therefore $\rho \geq r - 4(n-1)(A+1)\sqrt{t}$, and we have the claimed result. $\qquad\square$

Our strategy to obtain a distance distortion upper estimate is through controlling volume inflation of large domains, and controlling volume of small geodesic balls from below. Since the volume element under the Ricci flow satisfies

$$\frac{\partial}{\partial t} d\mu = -Sd\mu,$$

the following local estimate of scalar curvature lower bound proved by B.L.Chen [10] can help us control the volume. (Note that when the curvature is bounded by $\frac{A}{t}$, we can choose a time dependent cut-off function as in the proof of Lemma 3.1, then the original proof of Proposition 2.1 in [10] still works.)

Lemma 3.6 (B.L.Chen). *Let $(M^n, g(t))$, $t \in [0, T)$ be a Ricci flow solution. Suppose for some $r \geq 2\sqrt{T}$, $B_{g(t)}(p, 2r)$ is relatively compact in M for each t, $|Ric|(g(t)) \leq \frac{A}{t}$ on $B_{g(t)}(p, 2r)$, and $S(g(0)) \geq -k$ on $B_{g(0)}(p, 2r)$, then*

$$S(x, t) \geq -\max\{k, \frac{C(n, A)}{r^2}\},$$

where $x \in B_{g(t)}(p, r)$, $t \in [0, T)$, and $C(n, A)$ is some constant depending only on n and A.

Lemma 3.7. *For any $n, k, A, v_0 > 0$, there is a constant $\Lambda(n, A)$, with the following property. Let $(M^n, g(t))$, $t \in [0, T)$ be a Ricci flow solution, $r > (1 + 2B)\sqrt{T}$ where $B = 4(n-1)(A+1)$, and $\tilde{r} = \Lambda e^{T \max\{k, 1/r^2\}} v_0^{-1} r$. Suppose $B_{g(t)}(p, 2\tilde{r} + \sqrt{t})$ is relatively compact in M for each $t \in [0, T]$, and*
  (i) $Ric(g(0)) \geq -kg(0)$ on $B_{g(0)}(p, 2r)$,
  (ii) $|Ric|(x, t) \leq \frac{A}{t}$ for all $x \in B_{g(t)}(p, 2\tilde{r})$, $t \in (0, T)$;
  (iii) $Vol_{g(t)} B_{g(t)}(x, \sqrt{t}) \geq v_0 t^{\frac{n}{2}}$ for all $x \in B_{g(t)}(p, 2\tilde{r})$ and $t \in (0, T]$.
Then for any $x, y \in B_{g(0)}(p, r)$, and for any $t \in [0, T]$, we have

$$d_{g(t)}(x, y) \leq \frac{\Lambda e^{T \max\{k, 1/r^2\}}}{v_0} \sqrt{T} \quad when \quad d_{g(0)}(x, y) < 2B\sqrt{T},$$

$$d_{g(t)}(x, y) \leq \frac{\Lambda e^{T \max\{k, 1/r^2\}}}{v_0} d_{g(0)}(x, y) \quad when \quad d_{g(0)}(x, y) \geq 2B\sqrt{T}.$$

Proof. Let's define $\tilde{r}(t) = \Lambda e^{t \max\{k, 1/r^2\}} v_0^{-1} r$ for positive t. (Note that Λ is undefined yet, but for the moment we can choose it to be any large number, for example 100, and later choose the value of Λ to be greater than 100.)

Let $T_1 \leq T$ be the largest number such that $B_{g(0)}(p, r) \subset B_{g(t)}(p, 2\tilde{r}(T_1))$ for all $t \in [0, T_1]$.

We first examine the dilation of geodesic balls with the fixed radius $B\sqrt{T_1}$. For any $x \in B_{g(0)}(p, r - (1 + B)\sqrt{T_1})$, suppose the diameter of $B_{g(0)}(x, B\sqrt{T_1})$ with respect to $g(T_1)$ is $D > 0$, then the $\sqrt{T_1}$–neighborhood of $B_{g(0)}(x, B\sqrt{T_1})$ with respect to $g(T_1)$ contains at least $[\frac{D}{2\sqrt{T_1}}]$ disjoint geodesic balls with radius $\sqrt{T_1}$ with respect to $g(T_1)$; on the other hand this neighborhood is contained in $B_{g(0)}(x, (1 + 2B)\sqrt{T_1})$ by Lemma 3.5. So by assumption (iii) we have

$$Vol_{g(T_1)} B_{g(0)}(x, (1 + 2B)\sqrt{T_1}) \geq [\frac{D}{2\sqrt{T_1}}] v_0 T_1^{\frac{n}{2}},$$

by Lemma 3.6 we have

$$\frac{Vol_{g(T_1)} B_{g(0)}(x, (1 + 2B)\sqrt{T_1})}{Vol_{g(0)} B_{g(0)}(x, (1 + 2B)\sqrt{T_1})} \leq e^{T_1 \max\{(n-1)k, C(n, A)/r^2\}}.$$

Hence

$$D \leq \frac{C(n, A) e^{T_1 \max\{k, 1/r^2\}}}{v_0} \sqrt{T_1},$$

where we have used the Bishop-Gromov volume comparison theorem.

Now for any $x, y \in B_{g(0)}(p, r)$. Suppose $d_{g(0)}(x, y) = l$, then the shortest geodesic connecting x and y can be covered by $[\frac{l}{2B\sqrt{T_1}}] + 1$ geodesic balls with radius $B\sqrt{T_1}$,

all with respect to $g(0)$. Then the above analysis shows that

$$d_{g(T_1)}(x, y) \leq \left(\frac{l}{2B\sqrt{T_1}} + 1\right) \frac{C(n, A)e^{T_1 \max\{k, 1/r^2\}}}{v_0} \sqrt{T_1}.$$

We can apply Lemma 3.5 on the time interval $[t, T_1]$ to compare $d_{g(t)}(x, y)$ and $d_{g(T_1)}(x, y)$. Now we can choose an appropriate Λ and get the claimed result with T_1 instead of T.

If $T_1 < T$, then the maximality of T_1 implies that $B_{g(0)}(p, r)$ has to touch the boundary of $B_{g(T_1)}(p, 2\tilde{r}(T_1))$. However, this will not happen since the above estimate implies $B_{g(0)}(p, r) \subset B_{g(T_1)}(p, \tilde{r}(T_1))$. Therefore $T_1 = T$ and the proof is finished.

$\square$

4. APPLICATIONS

In this section, let's restate (for readers' convenience) the applications in the introduction and provide their proofs.

If the optimal Euclidean isoperimetric inequality holds on the entire manifold where the Ricci curvature is nonnegative, then this manifold is isometric to $\mathbb{R}^n$. We can now relax the condition to nonnegative scalar curvature when the Ricci curvature is bounded from below by any negative sub-quadratic function.

Corollary 4.1. *Let (M^n, g) be a complete Riemannian manifold. Suppose*
(i) *$\liminf_{d(p,x) \to \infty} \frac{Ric(x)}{d(p,x)^2} \geq -L > -\infty$ for some fixed point p,*
(ii) *the scalar curvature $S(x) \geq 0$ for all $x \in M$,*
(iii) *the optimal Euclidean isoperimetric inequality holds on M.*
Then (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.

Proof. For any $A > 0$ and $r > 1$. Theorem 1.2 provides a complete Ricci flow solution on (M, g) with

$$|Rm|(g(t)) \leq \frac{A}{t} + \frac{1}{c(n, A)r^2}, \quad t \in (0, c(n, A)r^2).$$

Note that since $k = 0$ in this case, the time interval of existence no longer depends on k or L.

Under the assumption of the corollary, we can take $r \to \infty$, hence

$$|Rm|(g(t)) \leq \frac{A}{t}, \quad t \in (0, \infty).$$

Take a sequence of positive numbers $A_i \to 0$ as $i \to \infty$, and denote the corresponding sequence of Ricci flow solutions as $g_i(t)$, $t \in [0, \infty)$. We have the curvature upper bounds

$$|Rm|(g_i(t)) \leq \frac{A_i}{t} \to 0 \quad as \quad i \to \infty \quad for \quad t > 0.$$

Then a subsequence will converge in the pointed Cheeger-Gromov sense to a complete smooth solution $g(t)$ for the same reason as explained in the proof of Theorem 1.2. This limit solution $g(t)$ is flat for any $t > 0$, thus the initial metric must be flat. The isoperimetric inequality implies maximal volume growth, hence the initial manifold is isometric to $\mathbb{R}^n$. $\square$

We also prove a local result concerning the topology of almost isoperimetrically Euclidean geodesic balls with Ricci curvature bounded from below.

Corollary 4.2. *For any n and k, There exist constants δ and η depending only on n and k, such that if the geodesic ball $B(p,1)$ is relatively compact in a Riemannian manifold (M,g), $Ric \geq -k$ on $B(p,1)$, and the $\delta-$almost Euclidean isoperimetric inequality holds in $B(p,1)$, then $B(p,\eta)$ is diffeomorphic to a Euclidean ball of dimension n.*

Proof. Suppose the claim is not true, then there is a sequence of manifolds $\{(M_i,g_i)\}$ with geodesic balls $\{B_i(p_i,1)\}$ satisfying the assumptions with $\delta_i \to 0$, but the largest geodesic ball centered at p_i that is diffeomorphic to a Euclidean ball is $B_i(p_i,\epsilon_i)$, and $\epsilon_i \to 0$ as $i \to \infty$.

Define $\tilde{g}_i = \epsilon_i^{-2}g_i$, then $B_{\tilde{g}_i}(p_i,\epsilon_i^{-1}) = B_i(p_i,1)$ has radius $\epsilon_i^{-1} \to \infty$, $Ric(\tilde{g}_i) \geq -\epsilon_i^2 k\tilde{g}_i$, and the δ_i-almost isoperimetric inequality still holds.

Choose a sequence $A_j \to 0$. For each A_j, we can find i large enough, such that we can apply Lemma 2.4 to conformally transform $B_{\tilde{g}_i}(1,\epsilon_i^{-1})$ to a complete manifold (M_i,h_i) while keeping $B_{\tilde{g}_i}(p_i,\epsilon_i^{-1}/2)$ unchanged, and we can apply Theorem 1.2 to produce a complete solution of Ricci flow $(M_i,h_i(t))$ for $t \in [0,1]$ with

$$|Rm|(h_i(t)) \leq \frac{A_j}{t}, \quad t \in (0,1].$$

Moreover, $h_i(t)$ is $\omega_n/2$-noncollapsed under the scale $Q\sqrt{t}$ near the point p_i for all $t \in (0,\tau]$, where τ is from Lemma 3.3.

We can extract a subsequence $\{M_j,h_j(\tau),p_j\}$ which converges smoothly in the pointed Cheeger-Gromov sense to $(M_\infty,h_\infty,p_\infty)$, which is a quotient of $\mathbb{R}^n$, and

$$Vol_{g_\infty}B_{h_\infty}(p_\infty,Q) \geq \frac{\omega_n}{2}Q^n.$$

Thus there exists a number $a(n) \in (0,1)$, such that $B_{g_\infty}(p_\infty,aQ)$ is diffeomorphic to an $n-$d Euclidean ball. This topological information can be lifted to $B_{h_i(\tau)}(p_i,aQ)$ when i is large enough.

On the other hand, Lemma 3.7 implies that there is a constant $\Lambda(n)$, such that $B_{h_i(0)}(p_i,2) \subset B_{h_i(\tau)}(p_i,\Lambda)$. Therefore, if we choose $Q \geq \Lambda/a$, then we have a contradiction with the assumption that $B_{h_i(0)}(p_i,1+\epsilon)$ is not diffeomorphic to an n-d Euclidean ball for any $\epsilon > 0$.

$\qquad\square$

Next we prove Corollary 1.7. Though there are many partial confirmations, Yau's uniformization conjecture is still open in general. Corollary 1.7 provides another positive evidence, let's restate it here:

Corollary 4.3. *For any n, there is $\delta(n) > 0$, such that if (M,g) be a complete $2n-$dimensional Kähler manifold with nonnegative holomorphic bisectional curvature and maximal volume growth, and the $\delta(n)-$almost Euclidean isoperimetric inequality holds in every geodesic ball with radius r, then (M,g) is biholomorphic to $\mathbb{C}^n$.*

Proof. It has been proved in [18] that under the curvature bound $|Rm(g(t))| \leq \frac{A}{t}$ with A sufficiently small, the Kähler condition and the nonnegativity of the holomorphic bisectional curvature are preserved. We only need to check that the maximal volume growth is preserved under the Ricci flow solutions constructed in Theorem 1.2, which is done by the following Lemma 4.4. Then we have a complete Kähler metric with bounded nonnegative holomorphic bisectional curvature and maximal volume growth, which is biholomorphic to $\mathbb{C}^n$ by results in [6]. $\qquad\square$

In [18], the authors applied the Ricci flow solution of [26], which is uniformly C^0 close to the initial metric, hence the persistence of maximal volume growth is immediate.

For the Ricci flow solutions in this article, suppose the initial Ricci curvature is bounded from below, then the persistence of maximal volume growth can be proved using results of Section 3. However, since nonnegative holomorphic bisectional curvature implies nonnegative Ricci curvature, there is an easy proof and the non-collapsing scale can be made uniform in this situation, see the following lemma.

Lemma 4.4. *Let $(M, g(t))$ is a Ricci flow solution with $Ric(g(t)) \geq 0$ and $|Rm|(g(t)) \leq \frac{A}{t}$, and the $\delta-$almost isoperimetric inequality holds in $B_{g(0)}(x, r)$ for all $x \in M$. Suppose*

$$Vol_{g(0)} B_{g(0)}(p, s) \geq vs^m$$

as $s \to \infty$. Then there is a $\tau > 0$ depending only on n and A, such that

$$Vol_{g(t)} B_{g(t)}(x, s) \geq c(n)vs^m,$$

for $t \in [0, \tau]$, as $s \to \infty$.

Proof. The isoperimetric condition implies that (WLOG assume $\delta << 1$)

$$Vol_{g(0)} B_{g(0)}(x, s) \geq (1 - \delta)\omega_{2n} s^{2n}.$$

Claim: there is a $\tau > 0$ such that for any $0 < t < \tau$, such that for any x and $0 < s < r$

$$Vol_{g(t)} B_{g(\tau_i)}(x, s) \geq \frac{\omega_{2n}}{2} s^{2n}.$$

Proof of Claim. If the claim is not true, then there is a sequence of times $\tau_i \to 0$, a sequence of points x_i such that

$$Vol_{g(\tau_i)} B_{g(\tau_i)}(x_i, r) = \frac{\omega_{2n}}{2} r^{2n},$$

and

$$Vol_{g(\tau_i)} B_{g(\tau_i)}(x_i, s) \geq \frac{\omega_{2n}}{2} s^{2n}$$

for any $0 < s < r$. The sequence of pointed geodesic balls with the Riemannian distance and measure $(B_{g(0)}(x_i, r), d_{g(0)}, d\mu_{g(0)})$ converges in the Gromov-Hausdorff sense to a metric measure space $(X, d_X, d\mu_X)$, where $d\mu_X$ is the Hausdorff measure. Here the convergence of the measures follows from [8]. On this limit space we have

$$Vol_X B_X(x_\infty, s) \geq (1 - \delta)\omega_{2n}.$$

On the other hand, the sequence $(B_{g(\tau_i)}(x_i, r), d_{g(\tau_i)}, d\mu_{g(\tau_i)})$ converges to a metric measure space $(Y, d_Y, d\mu_Y)$, where $d\mu_Y$ is the Hausdorff measure. We have

$$Vol_Y B_Y(x_\infty, r) = \frac{\omega_{2n}}{2} r^{2n}.$$

However, by the curvature conditions and Lemma 3.5, we have

$$d_{g(0)}(x, y) \geq d_{g(t)}(x, y) \geq d_{g(0)}(x, y) - C(n)A\sqrt{t},$$

for any $x, y \in M$ and $0 < t < T$. The continuous surjection $f : Y \to X$ constructed in the proof of Lemma 3.3 is an isometry in this situation. Hence we have the same n-dimensional Hausdorff measure on X and Y, which leads to a contradiction and finishes the proof of the claim.

Now let's assume that the initial manifold has the following volume growth rate

$$Vol_{g(0)} B_{g(0)}(x, s) \geq vs^m.$$

Let $N(r,s)$ be the maximal number of disjoint $r-$geodesic balls with respect to $g(0)$ in $B_{g(0)}(x,s)$, then there are $N(r,s)$ geodesic balls with radius $2r$ that cover $B_{g(0)}(x,s)$. Hence

$$N(r,s) \geq \frac{vs^m}{\omega_{2n}(2r)^{2n}}.$$

Take $t < \tau$ and small enough so that $d_{g(t)} \geq d_{g(0)} - r/2$, then

$$B_{g(t)}(y,r/2) \subset B_{g(0)}(y,r),$$

hence there are at least $N(r,s)$ disjoint geodesic balls with radius $r/2$ w.r.t $g(t)$ in $B_{g(0)}(x,s) \subset B_{g(t)}(x,s)$. Therefore

$$Vol_{g(t)} B_{g(t)}(x,s) \geq N(r,s)\frac{\omega_{2n}}{2}\frac{r}{2}^{2n} \geq c(n)vs^m.$$

$\square$

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