A TOPOLOGICAL LOWER BOUND FOR THE ENERGY OF A UNIT VECTOR FIELD ON A CLOSED HYPERSURFACE OF THE EUCLIDEAN SPACE. THE 3-DIMENSIONAL CASE

FABIANO G. B. BRITO, ANDRÉ O. GOMES, AND ADRIANA V. NICOLI

ABSTRACT. In this short note we prove that the degree of the Gauss map ν of a closed 3-dimensional hypersurface of the Euclidean space is a lower bound for the total bending functional \mathcal{B} , introduced by G. Wiegmink. Consequently, the energy functional E introduced by C. M. Wood admits a topological lower bound.

1. Introduction

The energy of a unit vector field \vec{v} on a Riemannian compact manifold M is defined by Wood in [4], as the energy of the map $\vec{v}: M \to T_1M$, where T_1M is the unit tangent bundle equipped with the Sasaki metric,

(1)
$$E(\vec{v}) = \frac{1}{2} \int_{M} ||\nabla \vec{v}||^2 + \frac{m}{2} \text{vol}(M).$$

In [2], Wiegmink defines a quantitative measure for the extent to which a unit vector field fails to be parallel with respect to the Levi-Civita connection ∇ of a Riemannian manifold M. This measure is the total bending functional,

(2)
$$\mathcal{B}(\vec{v}) = \frac{1}{(m-1)\operatorname{vol}(\mathbb{S}^m)} \int_M ||\nabla \vec{v}||^2.$$

The energy of \vec{v} may be written in terms of the total bending,

(3)
$$E(\vec{v}) = \frac{(m-1)\operatorname{vol}(\mathbb{S}^m)}{2}\mathcal{B}(\vec{v}) + \frac{m}{2}\operatorname{vol}(M).$$

On the other hand, Brito showed that Hopf flows are absolute minima of the functional \mathcal{B} in \mathbb{S}^3 :

Theorem 1 (Brito, [5]). Hopf vector fields are the unique vector fields on \mathbb{S}^3 to minimize \mathcal{B} .

²⁰¹⁰ Mathematics Subject Classification. 57R25, 47H11, 58E20.

Gluck and Ziller proved that Hopf flows are also the unit vector fields of minimum volume, with respect to the following definition of volume,

$$\operatorname{vol}(\vec{v}) = \int_{M} \sqrt{\det(I + (\nabla \vec{v})(\nabla \vec{v})^{t})}.$$

Theorem 2 (Gluck Ziller, [3]). The unit vector fields of minimum volume on \mathbb{S}^3 are precisely the Hopf vector fields, and no others.

Reznikov compared this functional to the topology of an Euclidean hypersurface. Let M be a smooth closed oriented immersed hypersurface in \mathbb{R}^{n+1} with the induced metric, and let $S = \sup_{x \in M} ||S_x|| = \sup_{x \in M} |\lambda_i(x)|$, where S_x is the second fundamental operator in T_xM , and $\lambda_i(x)$ are the principal curvatures.

Theorem 3 (Reznikov, [6]). For any unit vector field \vec{v} on M we have

$$\operatorname{vol}\vec{v} - \operatorname{vol}(M) \geqslant \frac{\operatorname{vol}\mathbb{S}^n}{\mathcal{S}} |\operatorname{deg}(\nu)|,$$

where $deg(\nu)$ is the degree of the Gauss map $\nu: M \to \mathbb{S}^n$.

Recently, Brito et al, in [1], discovered a list of curvature integrals for an Euclidean (n+1)dimensional closed hypersurface M,

(4)
$$\int_{M} \eta_{k} = \begin{cases} \deg(\nu) \binom{n/2}{k/2} \operatorname{vol}(\mathbb{S}^{n+1}), & \text{if } k \text{ and } n \text{ are even,} \\ 0, & \text{if } k \text{ or } n \text{ is odd,} \end{cases}$$

where the functions η_k depend on a unit vector field and on the extrinsic geometry of M.

In this note, we take a 3-dimensional closed Euclidean hipersurface M and relate the total bending (consequently the energy) of a unit vector field \vec{v} to the topology of the manifold M, and we prove that

$$\deg(\nu) \leqslant 2 \, \widetilde{S} \, \mathcal{B}(\vec{v}),$$

where $\widetilde{S} := \max_{x \in M} \{||S(\vec{v})_x||\}$, and S is the Weingarten operator of M.

2. Total Bending and Energy of \vec{v}

Let M be a 3-dimensional closed manifold, immersed in the Euclidean space. We may assume that M is oriented, so the normal map $\nu: M \to \mathbb{S}^3$, $\nu(x) = N(x)$, is well defined, where N is a unitary normal field. Let ∇ and $\langle \cdot, \cdot \rangle$ be the Levi-Civita connection and the induced

Riemannian metric on M, respectively. Let $\vec{v}: M \to TM$ be a smooth unit vector field on M, and take an orthonormal basis $\{e_1, e_2, e_3 := \vec{v}\}$ at each point $x \in M$. Define the following notation: for $1 \leq A, B \leq 3$, $h_{AB} = \langle S(e_A), e_B \rangle$; for $1 \leq i, j \leq 2$, $a_{ij} = \langle \nabla_{e_i} \vec{v}, e_j \rangle$, $v_i = \langle \nabla_{\vec{v}} \vec{v}, e_i \rangle$.

From the equation 4, we know

(5)
$$\int_{M} \eta_{2} = \deg(\nu) \operatorname{vol}(\mathbb{S}^{3}),$$

where $deg(\nu)$ is the degree of ν and

(6)
$$\eta_2 = \begin{vmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ h_{31} & h_{32} & h_{33} \end{vmatrix}.$$

For more details concerning the functions η_k , see [1]. Set $\sigma_1 = \begin{vmatrix} a_{12} & v_1 \\ a_{22} & v_2 \end{vmatrix}$, $\sigma_2 = \begin{vmatrix} a_{11} & v_1 \\ a_{21} & v_2 \end{vmatrix}$ and

$$\sigma_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
. In this case, we have

$$\deg(\nu)\operatorname{vol}(\mathbb{S}^{3}) = \int_{M} \eta_{2} = \int_{M} (h_{31}\sigma_{1} - h_{32}\sigma_{2} + h_{33}\sigma_{3})$$

$$\leqslant \left| \int_{M} (h_{31}\sigma_{1} - h_{32}\sigma_{2} + h_{33}\sigma_{3}) \right|$$

$$\leqslant \int_{M} (|h_{31}||\sigma_{1}| + |h_{32}||\sigma_{2}| + |h_{33}||\sigma_{3}|)$$

$$\leqslant \frac{1}{2} \int_{M} \left(|h_{31}|(a_{12}^{2} + a_{22}^{2} + v_{1}^{2} + v_{2}^{2}) + |h_{32}|(a_{11}^{2} + a_{21}^{2} + v_{1}^{2} + v_{2}^{2}) + |h_{33}|(\sum_{i,j=1}^{2} a_{ij}^{2}) \right).$$

Note that $||S(\vec{v})|| = \sqrt{h_{31}^2 + h_{32}^2 + h_{33}^2}$, which implies that $||S(\vec{v})|| \ge |h_{3A}|$ for $1 \le A \le 3$. Then

(7)
$$\deg(\nu)\operatorname{vol}(\mathbb{S}^3) \leqslant \int_M ||S(\vec{v})|| \left(\sum_{i,j=1}^2 a_{ij} + \sum_{i=1}^2 v_i\right).$$

By our definition $\widetilde{S} = \max_{x \in M} \{||S(\vec{v})_x||\}$, so

$$\deg(\nu)\operatorname{vol}(\mathbb{S}^3) \leqslant \widetilde{S} \int_M \left(\sum_{i,j=1}^2 a_{ij} + \sum_{i=1}^2 v_i\right).$$

A TOPOLOGICAL LOWER BOUND FOR THE ENERGY OF A UNIT VECTOR FIELD ON A CLOSED HYPERSURFACE OF

On the other hand, by equation 2, the total bending of \vec{v} may be written as

(8)
$$\mathcal{B}(\vec{v}) = \frac{1}{2\text{vol}(\mathbb{S}^3)} \int_M ||\nabla \vec{v}||^2.$$

Finally

(9)
$$\deg(\nu) \leqslant 2\widetilde{S} \,\mathcal{B}(\vec{v}).$$

By equations 1 and 3, we deduce the following lower bound for the energy functional:

$$E(\vec{v}) = \frac{1}{2} \int_{M} ||\nabla(\vec{v})||^{2} + \frac{3}{2} \operatorname{vol}(M)$$

$$= \mathcal{B}(\vec{v}) \operatorname{vol}(\mathbb{S}^{3}) + \frac{3}{2} \operatorname{vol}(M)$$

$$E(\vec{v}) \geqslant \frac{1}{2\widetilde{S}} \operatorname{vol}(\mathbb{S}^{3}) \operatorname{deg}(\nu) + \frac{3}{2} \operatorname{vol}(M)$$

Now consider $M = \mathbb{S}^3$. We are interested in computing the energy of a Hopf vector field v_H . By the equation 5

$$\deg(\nu)\operatorname{vol}(\mathbb{S}^3) = \int_{\mathbb{S}^3} \eta_2 = \int_{\mathbb{S}^3} \sigma_3,$$

since $h_{32} = h_{31} = 0$ and $h_{33} = 1$. When the canonical immersion of \mathbb{S}^3 in \mathbb{R}^4 is considered, we have that $\deg(\nu) = 1$. On the other hand, by definition of E,

$$\int_{\mathbb{S}^3} \sigma_3 = E(v_H) - \frac{3}{2} \text{vol}(\mathbb{S}^3),$$

since $a_{11} = a_{22} = 0$ for any Hopf vector field v_H . Therefore,

$$E(v_H) = \frac{5}{2} \text{vol}(\mathbb{S}^3),$$

as we expected.

3. Further research

We intend to extend these preliminary results in two main directions:

- (1) Considering the energy of unit vector fields on arbitrary n-dimensional hypersurfaces of \mathbb{R}^{n+1} .
- (2) Restricting our attention to totally geodesic flows with integrable normal bundle. This is equivalent of studying the energy of Riemannian foliations of closed hypersurfaces of \mathbb{R}^3 .

References

- [1] F. Brito, I. Gonçalves, Degree of the Gauss map and curvature integrals for closed hypersurfaces, preprint: arXiv:1609.04670[math.DG] (2016).
- [2] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, Math. Ann. 303 (1995) 325-344.
- [3] H. Gluck and W. Ziller, On the volume of a unit field on the three-sphere, *Comment Math. Helv.* **61** (1986) 177-192.
- [4] C. M. Wood, On the energy of a unit vector field, Geom. Dedicata 64 (1997) 319-330.
- [5] F. G. B. Brito, Total bending of flows with mean curvature correction, *Diff. Geom. and its App.* **12** (2000) 157-163.
- [6] A. G. Reznikov, Lower bounds on volumes of vector fields, Arch. Math. 58 (1992) 509-513.

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09.210-170 Santo André, Brazil

 $E ext{-}mail\ address: fabiano@ime.usp.br}$

DPTO. DE MATEMÁTICA, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, R. DO MATÃO 1010, SÃO PAULO-SP 05508-900, BRAZIL.

E-mail address: gomes@ime.usp.br

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09.210-170 Santo André, Brazil

E-mail address: dri.nicoli@gmail.com