On the problem of calculation of correlation functions in the six-vertex model with domain wall boundary conditions

F. Colomo[1] and A. G. Pronko[2]

ABSTRACT. The problem of calculation of correlation functions in the six-vertex model with domain wall boundary conditions is addressed by considering a particular nonlocal correlation function, called row configuration probability. This correlation function can be used as building block for computing various (both local and nonlocal) correlation functions in the model. The row configuration probability is calculated using the quantum inverse scattering method; the final result is given in terms of a multiple integral. The connection with the emptiness formation probability, another nonlocal correlation function which was computed elsewhere using similar methods, is also discussed.

1. Introduction

One of the most fundamental problems in the theory of integrable models is the exact calculation of correlation functions [1]. In recent years, there has been an increasing interest (motivated by various mathematical and physical applications) in obtaining exact results for correlation functions of statistical mechanics models defined on finite lattices and with fixed boundary conditions. Because of the lack of translational invariance, the systematic computation of correlation functions for these models represents a difficult problem.

An important example of such a model is the six-vertex model with domain wall boundary conditions [2]. The partition function of the model on the finite lattice is given exactly in terms of certain determinant [3,4]. This formula, known as Izergin-Korepin determinant formula, turned out a powerful tool in proving important combinatorial results. The current interest in the model is mostly motivated by occurrence of the phase separation phenomena (see [5,6] and references therein).

Some progress in the calculation of the correlation functions of the six-vertex model with domain wall boundary conditions has been achieved when correlations are considered near the boundaries [7–10]. An example of correlation function which can be computed away from the boundary is the so-called emptiness formation probability [11]. Some generalisations have been considered recently in [12]. However the problem of a systematic treatment of correlation functions, especially when correlations are considered away from the boundaries, is still far from being solved.

To address this problem, in the present paper we introduce a particular nonlocal correlation function, called row configuration probability. This correlation

[1]INFN, SEZIONE DI FIRENZE, VIA G. SANSONE 1, 50019 SESTO FIORENTINO (FI), ITALY
[2]SAINT PETERSBURG DEPARTMENT OF V. A. STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, FONTANKA 27, 191023 SAINT PETERSBURG, RUSSIA
E-mails: colomo@fi.infn.it, agp@pdmi.ras.ru.

function describes the probability of observing a given configuration of arrows on the vertical edges located between two consecutive horizontal lines of the square lattice. The row configuration probability can be used as building a block to compute other (both local and non-local) correlation functions. In particular, it is closely related to the emptiness formation probability.

The row configuration probability besides being interesting for evaluation of other correlation functions, is also interesting on its own right: there are analogue in the context of phase separation for dimers models [13], and of enumerative combinatorics [14].

To compute the row configuration probability we consider the inhomogeneous version of the model and formulate it in the framework of the quantum inverse scattering method (QISM) [15] (for a survey, see [1]). We use the fact that the row configuration probability can be represented as a product of two factors. For computing the first factor we use a side result of paper [16], while for the second one we use the technique developed in papers [8, 10, 11]. For the homogeneous model both factors are represented in terms of multiple integrals.

To demonstrate how these results can be used for computing other correlation functions, we discuss here the connection with the emptiness formation probability. The computation is based on performing certain sums and integrals, and making use of identities involving antisymmetrisation of multi-variable functions.

The paper is organized as follows. In the next section after recalling basic facts about the model we set up the considered problem in terms of QISM objects. The derivation of the row configuration probability is given in sections 3 and 4. The relation with the emptiness formation probability is discussed in section 5.

2. Six-vertex model, domain wall boundary conditions, and row configuration probability

We consider the six-vertex model on a square lattice formed by intersection of N horizontal and N vertical lines (an $N \times N$ lattice), with special fixed boundary conditions called domain wall boundary conditions. Recall that the six-vertex model is a model in which local states are arrows pointing along edges of the lattices; the allowed arrow configurations are subject to the 'ice-rule': each vertex should have the same number of incoming and outgoing arrows. The Boltzmann weights are assigned to the six possible vertex configurations of arrows allowed by the ice-rule, and in the model invariant under reversal of all arrows there are three different Boltzmann weights, usually denoted a, b, and c. The domain wall boundary conditions mean that all arrows on the left and right boundaries are outgoing while all arrows on the top and bottom boundaries are incoming, see figure 1.

To use QISM in calculations we consider the inhomogeneous version of the model, in which the weights of the vertex being at the intersection of αth vertical line (enumerated from the right) and kth horizontal line (enumerated from the top) are $a_{\alpha,k} = a(\lambda_\alpha, \nu_k)$, $b_{\alpha,k} = b(\lambda_\alpha, \nu_k)$, and $c_{\alpha,k} = c$, where

$$a(\lambda, \nu) = \sin(\lambda - \nu + \eta), \qquad b(\lambda, \nu) = \sin(\lambda - \nu - \eta), \qquad c = \sin 2\eta. \qquad (2.1)$$

The parameters $\lambda_1, \ldots, \lambda_N$ are assumed to be all different; the same is assumed about $\nu_1, \ldots, \nu_N$. The parameter

$$\Delta = \frac{a_{\alpha,k}^2 + b_{\alpha,k}^2 - c_{\alpha,k}^2}{2a_{\alpha,k}b_{\alpha,k}} = \cos 2\eta \qquad (2.2)$$
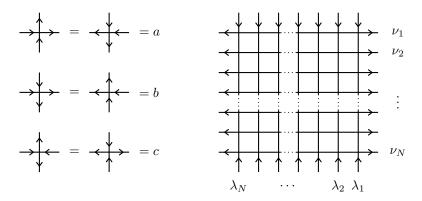
2

FIGURE 1. The six vertices and their weights (left), and N-by-N square lattice with domain wall boundary conditions (right).

takes the same value for all vertices, that ensures integrability [17]. The partition function is defined as follows

$$Z_N = \sum_{\mathcal{C}} \prod_{\alpha,k=1}^{N} w_{\alpha,k}(C), \tag{2.3}$$

where $w_{\alpha,k}(C)$ takes values $w_{\alpha,k}(C) = a_{\alpha k}, b_{\alpha k}, c_{\alpha k}$, depending on the configuration C. Clearly, $Z_N = Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N)$ where $\lambda_1, \ldots, \lambda_N$ and $\nu_1, \ldots, \nu_N$ can be regarded as 'variables'; parameter η has the meaning of a 'coupling constant' and it is often omitted in the notations. After QISM calculations, the homogeneous model quantities (e.g., partition function) can be obtained from the inhomogeneous ones upon setting $\lambda_\alpha = \lambda$ ($\alpha = 1, \ldots, N$) and $\nu_k = \nu$ ($k = 1, \ldots, N$), where, with no loss of generality, one can further put $\nu = 0$, see (2.1). We shall refer to this procedure as homogeneous limit.

We now define the main objects of QISM in relation to the model. First, let us consider vector space $\mathbb{C}^2$ and denote its basis vectors as the spin-up and spin-down states

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.4}$$

To each horizontal and vertical line of the lattice we associate vector space $\mathbb{C}^2$. We also use the convention that upward and right arrows correspond to the 'spin up' state while downward and left arrows correspond to the 'spin down' state.

Next, to each vertex being intersection of the αth vertical line and the kth horizontal line we associate the operator $L_{\alpha,k}(\lambda_\alpha, \nu_k)$ which acts nontrivially in the direct product of two vector spaces $\mathbb{C}^2$: in the 'horizontal' space $\mathcal{H}_k = \mathbb{C}^2$ (associated with the kth horizontal line) and in the 'vertical' space $\mathcal{V}_\alpha = \mathbb{C}^2$ (associated with the αth vertical line). Referring to the scattering matrix picture, the arrow states on the top and right edges of the vertex can be regarded as 'in' indices of the L-operator while those on the bottom and left edges as 'out' ones, that gives

$$L_{\alpha,k}(\lambda_\alpha, \nu_k) = a_{\alpha,k} \frac{1 + \tau_\alpha^z \sigma_k^z}{2} + b_{\alpha,k} \frac{1 - \tau_\alpha^z \sigma_k^z}{2} + c_{\alpha,k}(\tau_\alpha^- \sigma_k^+ + \tau_\alpha^+ \sigma_k^-). \tag{2.5}$$

Here τ_α^l and σ_k^l ($l = +, -, z$) denote operators acting as Pauli matrices in $\mathcal{V}_\alpha$ and $\mathcal{H}_k$, respectively, and identically elsewhere.

3

An ordered product of L-operators along a vertical (or horizontal) line of the lattice corresponds in QISM to a monodromy matrix. To construct, e.g., the vertical line monodromy matrix, it is useful to think of the L-operator as a 2×2 matrix acting in space $\mathcal{V}_\alpha$, with the operator entries acting in the space $\mathcal{H}_{1,\ldots,N} = \otimes_{k=1}^N \mathcal{H}_k$, i.e.,

$$L_{\alpha,k}(\lambda,\nu) = \begin{pmatrix} a(\lambda,\nu)\dfrac{1+\sigma_k^z}{2} + b(\lambda,\nu)\dfrac{1-\sigma_k^z}{2} & c \cdot \sigma_k^- \\ c \cdot \sigma_k^+ & b(\lambda,\nu)\dfrac{1+\sigma_k^z}{2} + a(\lambda,\nu)\dfrac{1-\sigma_k^z}{2} \end{pmatrix}_{[\mathcal{V}_\alpha]}.$$
(2.6)

Here the subscript indicates that this is a matrix in $\mathcal{V}_\alpha$. The ordered product along the αth vertical line is the 'vertical' monodromy matrix:

$$\begin{aligned} T_\alpha^{\mathrm{V}}(\lambda_\alpha) &= L_{\alpha,N}(\lambda_\alpha,\nu_N)\cdots L_{\alpha,2}(\lambda_\alpha,\nu_2)L_{\alpha,1}(\lambda_\alpha,\nu_1) \\ &= \begin{pmatrix} A_{1,\ldots,N}^{\mathrm{V}}(\lambda_\alpha) & B_{1,\ldots,N}^{\mathrm{V}}(\lambda_\alpha) \\ C_{1,\ldots,N}^{\mathrm{V}}(\lambda_\alpha) & D_{1,\ldots,N}^{\mathrm{V}}(\lambda_\alpha) \end{pmatrix}_{[\mathcal{V}_\alpha]}. \end{aligned}$$
(2.7)

The operators $A_{1,\ldots,N}^{\mathrm{V}}(\lambda) = A_{1,\ldots,N}^{\mathrm{V}}(\lambda;\nu_1,\ldots,\nu_N)$, etc, act in $\mathcal{H}_{1,\ldots,N}$ and they are independent of α. Each of these operators corresponds to a vertical line of the lattice, with the top and bottom vertical arrows fixed.

Similarly, one can consider the 'horizontal' monodromy matrices,

$$\begin{aligned} T_k^{\mathrm{H}}(\nu_k) &= L_{N,k}(\lambda_N,\nu_k)\cdots L_{2,k}(\lambda_2,\nu_k)L_{1,k}(\lambda_1,\nu_k) \\ &= \begin{pmatrix} A_{1,\ldots,N}^{\mathrm{H}}(\nu_k) & B_{1,\ldots,N}^{\mathrm{H}}(\nu_k) \\ C_{1,\ldots,N}^{\mathrm{H}}(\nu_k) & D_{1,\ldots,N}^{\mathrm{H}}(\nu_k) \end{pmatrix}_{[\mathcal{H}_k]}, \end{aligned}$$
(2.8)

where operators $A_{1,\ldots,N}^{\mathrm{H}}(\nu) = A_{1,\ldots,N}^{\mathrm{H}}(\nu;\lambda_1,\ldots,\lambda_N)$, etc, act in $\mathcal{V}_{1,\ldots,N} := \otimes_{\alpha=1}^N \mathcal{V}_\alpha$. Each of these operators correspond to a horizontal line of the lattice, with the rightmost and leftmost horizontal arrows fixed.

The importance of the monodromy matrix operator entries is that they obey a quadratic algebra, called the algebra of monodromy matrix or Yang-Baxter algebra [1]. The algebra involves in total 16 commutation relations, and in the following we will need some of these commutation relations, namely

$$B(\lambda)\,B(\lambda') = B(\lambda')\,B(\lambda), \qquad C(\lambda)\,C(\lambda') = C(\lambda')\,C(\lambda), \tag{2.9}$$

and

$$A(\lambda)\,B(\lambda') = f(\lambda,\lambda')\,B(\lambda')\,A(\lambda) + g(\lambda',\lambda)\,B(\lambda)\,A(\lambda'), \tag{2.10}$$

where $A(\lambda) = A_{1,\ldots,N}^{\mathrm{V}}(\lambda)$, etc, and functions $f(\lambda',\lambda)$ and $g(\lambda',\lambda)$ are

$$f(\lambda',\lambda) = \frac{\sin(\lambda - \lambda' + 2\eta)}{\sin(\lambda - \lambda')}, \qquad g(\lambda',\lambda) = \frac{\sin 2\eta}{\sin(\lambda - \lambda')}. \tag{2.11}$$

Exactly the same relations are also valid for $A(\nu) = A_{1,\ldots,N}^{\mathrm{H}}(\nu)$, etc (after replacing $\lambda \mapsto \nu$ and $\lambda' \mapsto \nu'$ in (2.9) and (2.10)). For the full list of commutation relations, as well as for details of their derivation, we refer to chapter VIII of book [1].

Now we are ready to formulate the model in the framework of QISM. Denoting by $|\uparrow_k^{\mathrm{V}}\rangle$ and $|\downarrow_k^{\mathrm{V}}\rangle$ the basis vectors of space $\mathcal{H}_k$, let us introduce states

$$\left|\Uparrow_{1,\ldots,N}^{\mathrm{V}}\right\rangle := \underset{k=1}{\overset{N}{\otimes}} \left|\uparrow_k^{\mathrm{V}}\right\rangle, \qquad \left|\Downarrow_{1,\ldots,N}^{\mathrm{V}}\right\rangle := \underset{k=1}{\overset{N}{\otimes}} \left|\downarrow_k^{\mathrm{V}}\right\rangle. \tag{2.12}$$

4

These states are the 'all spins up' and 'all spins down' states in the space $\mathcal{H}_{1,\ldots,N}$, respectively. Taking into account that the domain wall boundary conditions select for αth vertical line the operator $B^{\mathrm{V}}_{1,\ldots,N}(\lambda_\alpha)$, we can write the partition function as the matrix element:

$$Z_N(\lambda_1,\ldots,\lambda_N;\nu_1,\ldots,\nu_N) = \langle \Downarrow^{\mathrm{V}}_{1,\ldots,N}| \prod_{\alpha=1}^{N} B^{\mathrm{V}}_{1,\ldots,N}(\lambda_\alpha) |\Uparrow^{\mathrm{V}}_{1,\ldots,N}\rangle. \qquad (2.13)$$

We also recall that $B^{\mathrm{V}}_{1,\ldots,N}(\lambda) = B^{\mathrm{V}}_{1,\ldots,N}(\lambda;\nu_1,\ldots,\nu_N)$.

Essentially in the same way, one can construct the partition function considering operators associated with the horizontal lines. Denoting by $|\uparrow^{\mathrm{H}}_\alpha\rangle$ and $|\downarrow^{\mathrm{H}}_\alpha\rangle$ the basis vectors of $\mathcal{V}_\alpha$, we can introduce states

$$|\Uparrow^{\mathrm{H}}_{1,\ldots,N}\rangle := \mathop{\otimes}_{\alpha=1}^{N} |\uparrow^{\mathrm{H}}_\alpha\rangle, \qquad |\Downarrow^{\mathrm{H}}_{1,\ldots,N}\rangle := \mathop{\otimes}_{\alpha=1}^{N} |\downarrow^{\mathrm{H}}_\alpha\rangle, \qquad (2.14)$$

which are the 'all spins up' and 'all spins down' states of space $\mathcal{V}_{1,\ldots,N}$. The partition function reads:

$$Z_N(\lambda_1,\ldots,\lambda_N;\nu_1,\ldots,\nu_N) = \langle \Uparrow^{\mathrm{H}}_{1,\ldots,N}| \prod_{k=1}^{N} C^{\mathrm{H}}_{1,\ldots,N}(\nu_k) |\Downarrow^{\mathrm{H}}_{1,\ldots,N}\rangle, \qquad (2.15)$$

and we recall that $C^{\mathrm{H}}_{1,\ldots,N}(\nu) = C^{\mathrm{H}}_{1,\ldots,N}(\nu;\lambda_1,\ldots,\lambda_N)$.

The partition function is known to be given by Izergin-Korepin determinant formula (see [2–4])

$$Z_N = \frac{\prod_{\alpha=1}^{N}\prod_{k=1}^{N} a(\lambda_\alpha,\nu_k)b(\lambda_\alpha,\nu_k)}{\prod_{1\leqslant\alpha<\beta\leqslant N} d(\lambda_\beta,\lambda_\alpha)\prod_{1\leqslant j<k\leqslant N} d(\nu_j,\nu_k)} \det_{1\leqslant\alpha,k\leqslant N}\{\varphi(\lambda_\alpha,\nu_k)\} \qquad (2.16)$$

where $d(\lambda,\lambda') := \sin(\lambda - \lambda')$ and

$$\varphi(\lambda,\nu) = \frac{c}{a(\lambda,\nu)b(\lambda,\nu)}, \qquad (2.17)$$

while $a(\lambda,\nu)$, $b(\lambda,\nu)$ and c are defined in (2.1). For the original proof of (2.16) see [4]; an alternative derivation of this formula can be found in [8, 11].

In the homogenous limit, i.e., when $\lambda_1 = \cdots = \lambda_N = \lambda$ and $\nu_1 = \cdots = \nu_N = 0$, expression (2.16) becomes

$$Z_N(\lambda,\ldots,\lambda;0,\ldots,0) = \frac{[\sin(\lambda-\eta)\sin(\lambda+\eta)]^{N^2}}{\prod_{n=1}^{N-1}(n!)^2} \det_{1\leqslant\alpha,k\leqslant N}\{\partial_\lambda^{\alpha+k-2}\varphi(\lambda)\} \quad (2.18)$$

where $\varphi(\lambda) := \varphi(\lambda,0)$. Below we often use simplified notations for the homogeneous model quantities, e.g., writing Z_N for $Z_N(\lambda,\ldots,\lambda;0,\ldots,0)$, and so on.

Let us now turn to the row configuration probability. To define this quantity it is useful to mention first that in the six-vertex model with domain wall boundary conditions all configurations are such that on the sth row (i.e., on the N vertical edges between the sth and the $(s+1)$th horizontal lines, counted from the top, in our conventions) there are exactly s arrows pointing up. It is therefore natural to study the probability of observing a given configuration of arrows on a given row, or row configuration probability, for short. Namely, we denote by $H_{N,s}^{(r_1,\ldots,r_s)}$ the probability that the s up-arrows of the sth row are exactly at the positions $r_1,\ldots,r_s$ (counted from the right), see figure 2.

Since the row configuration probability describes generic configurations of the model, it can be used as a building block to compute other correlation functions.
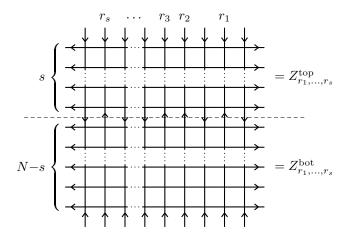
FIGURE 2. Row configuration with s up arrows between sth and $(s+1)$th horizontal lines from the top, and the resulting partition functions for the top and bottom portions of the original lattice.

For example, by properly summing over positions of up-arrows, one can recover the so-called the emptiness formation probability studied in [11]. This connection is discussed in section 5.

To compute the row configuration probability, we separate the original $N \times N$ lattice into two smaller lattices: an upper lattice, with s horizontal and N vertical lines, and a lower lattice, with $N-s$ horizontal and N vertical lines. We shall denote $Z^{\text{top}}_{r_1,\ldots,r_s}$ and $Z^{\text{bot}}_{r_1,\ldots,r_s}$ the partition functions of the six-vertex model on the upper and lower sublattices, respectively (see figure 2). The row configuration probability is essentially given as a product of the partition functions of the six-vertex model on these two smaller lattices,

$$H^{(r_1,\ldots,r_s)}_{N,s} = \frac{Z^{\text{top}}_{r_1,\ldots,r_s} Z^{\text{bot}}_{r_1,\ldots,r_s}}{Z_N}. \tag{2.19}$$

Our main goal in the present paper is therefore the derivation of some useful representations for the partition functions $Z^{\text{top}}_{r_1,\ldots,r_s}$ and $Z^{\text{bot}}_{r_1,\ldots,r_s}$. Specifically, we provide multiple (s-fold) integral representations for these quantities.

In terms of QISM objects, the partition functions on the upper, $s \times N$, sublattice can be written similarly to representation (2.15), as follows:

$$Z^{\text{top}}_{r_1,\ldots,r_s} = \left\langle \Downarrow^{\text{H}}_{1,\ldots,N} \right| \tau^-_{r_s} \cdots \tau^-_{r_2} \tau^-_{r_1} \prod_{k=1}^{s} C^{\text{H}}_{1,\ldots,N}(\nu_k) \left| \Downarrow^{\text{H}}_{1,\ldots,N} \right\rangle, \tag{2.20}$$

where τ^-_j $(j = 1, \ldots, N)$, as above, denote Pauli matrices acting in spaces $\mathcal{V}_j$.

To write the partition function of the lower, $(N-s) \times N$, sublattice as a matrix element, let us define vectors:

$$\left| \Uparrow^{\text{V}}_{s+1,\ldots,N} \right\rangle = \mathop{\otimes}_{k=s+1}^{N} \left| \uparrow^{\text{V}}_k \right\rangle, \qquad \left| \Downarrow^{\text{V}}_{s+1,\ldots,N} \right\rangle = \mathop{\otimes}_{k=s+1}^{N} \left| \downarrow^{\text{V}}_k \right\rangle. \tag{2.21}$$

6

These are 'all spins up' and 'all spin down' states of the space $\mathcal{H}_{s+1,\ldots,N}$. Correspondingly, let us consider the matrix elements of the 'truncated' vertical monodromy matrix given as the product $L_{\alpha,N}(\lambda_\alpha, \nu_N) \cdots L_{\alpha,s+1}(\lambda_\alpha, \nu_{s+1})$. These matrix elements are operators $A^{\mathrm{V}}_{s+1,\ldots,N}(\lambda_\alpha) = A^{\mathrm{V}}_{s+1,\ldots,N}(\lambda_\alpha; \nu_{s+1}, \ldots, \nu_N)$, etc, acting in $\mathcal{H}_{s+1,\ldots,N}$. The partition function $Z^{\mathrm{bot}}_{r_1,\ldots,r_s}$ can be written as

$$Z^{\mathrm{bot}}_{r_1,\ldots,r_s} = \langle \Downarrow^{\mathrm{V}}_{s+1,\ldots,N}| \prod_{\alpha=r_s+1}^{N} B(\lambda_\alpha) \cdot A(\lambda_{r_s}) \cdot \prod_{\alpha=r_{s-1}+1}^{r_s-1} B(\lambda_\alpha)$$

$$\times \cdots \times A(\lambda_{r_2}) \prod_{\alpha=r_1+1}^{r_2-1} B(\lambda_\alpha) \cdot A(\lambda_{r_1}) \cdot \prod_{\alpha=1}^{r_1-1} B(\lambda_\alpha) |\Uparrow^{\mathrm{V}}_{s+1,\ldots,N}\rangle, \quad (2.22)$$

where $A(\lambda) := A^{\mathrm{V}}_{s+1,\ldots,N}(\lambda)$ and $B(\lambda) := B^{\mathrm{V}}_{s+1,\ldots,N}(\lambda)$.

3. Calculation of $Z^{\mathrm{top}}_{r_1,\ldots,r_s}$

The matrix element in (2.20) can be formally evaluated (as a function of $\lambda_1, \ldots, \lambda_N$ and $\nu_1, \ldots, \nu_s$) using the equivalence of the algebraic and coordinate versions of the Bethe Ansatz. This equivalence was first explicitly proved, as a side result, in [16] (see appendix D of that paper); see also book [1], Chapter VII.

For simplicity, we start directly from the case where parameters $\lambda_1, \ldots, \lambda_N$ are already taken to the same value λ, but the parameters $\nu_1, \ldots, \nu_s$ are left arbitrary (and not equal to each other). Equation (D.4) of reference [16] in such a case implies

$$Z^{\mathrm{top}}_{r_1,\ldots,r_s} = c^s \prod_{k=1}^{s} [a(\lambda, \nu_k)]^{N-1} \prod_{1 \leqslant j < k \leqslant s} \frac{1}{t_k - t_j}$$

$$\times \sum_{P \in \Omega_s} (-1)^{[P]} \prod_{j=1}^{s} t_{P_j}^{r_j-1} \prod_{1 \leqslant j < k \leqslant s} (t_{P_j} t_{P_k} - 2\Delta t_{P_j} + 1), \quad (3.1)$$

where

$$t_k := \frac{b(\lambda, \nu_k)}{a(\lambda, \nu_k)}, \quad (3.2)$$

and the sum is taken over elements of the symmetric group Ω_s, i.e., permutations $P : 1, \ldots, s \mapsto P_1, \ldots, P_s$, with $[P]$ denoting the parity of P. Clearly, the expression standing in the second line in (3.1) is exactly the s-particle coordinate Bethe Ansatz trial wave-function.

To study the homogeneous limit of (3.1) in the remaining set of parameters, we first transform slightly this expression. Let us set $t_k = t + x_k$ ($k = 1, \ldots, s$) where t is an arbitrary parameter and the new parameters $x_1, \ldots, x_s$ are all different. Using the fact that for a function $f(x)$, regular near point $x = t$, one can always write $f(t + x) = \exp(x\partial_z)f(t + z)|_{z=0}$, we bring (3.1) to the form

$$Z^{\mathrm{top}}_{r_1,\ldots,r_s} = c^s \prod_{k=1}^{s} [a(\lambda, \nu_k)]^{N-1} \prod_{1 \leqslant j < k \leqslant s} \frac{1}{x_k - x_j} \det_{1 \leqslant j,k \leqslant s} \left\{ \exp\left(x_j \partial_{z_k} \right) \right\}$$

$$\times \prod_{j=1}^{s} (t + z_j)^{r_j-1} \prod_{1 \leqslant j < k \leqslant s} [(t + z_j)(t + z_k) - 2\Delta(t + z_j) + 1] \Bigg|_{z_1 = \cdots = z_s = 0}$$

$$(3.3)$$

which simply represents an equivalent way to write (3.1).

Let us now consider the homogeneous limit in the parameters $\nu_1, \ldots, \nu_s$. Since t is arbitrary, we can perform this limit such that $t_1 = \cdots = t_k = t$ in the limit, and put $t = b/a$, where a and b are the homogeneous model weights, see (2.1). We thus have to consider (3.1) at $x_1 = \cdots = x_s = 0$. The limit in (3.3) can be done using the relation

$$\left.\frac{\det_{1 \leqslant j,k \leqslant s}\{\exp(x_j \partial_{z_k})\}}{\prod_{1 \leqslant j < k \leqslant s}(x_k - x_j)}\right|_{x_1 = \cdots = x_s = 0} = \det_{1 \leqslant j,k \leqslant s}\left\{\frac{1}{(j-1)!}\partial_{z_k}^{j-1}\right\}. \qquad (3.4)$$

Reexpressing the values of derivatives at $z_1 = \cdots = z_s = 0$ as residues, we obtain a multiple integral representation

$$Z_{r_1,\ldots,r_s}^{\text{top}} = c^s a^{s(N-1)} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s (t+z_j)^{r_j - 1} \det_{1 \leqslant j,k \leqslant s}\left\{z_k^{-j}\right\}$$

$$\times \prod_{1 \leqslant j < k \leqslant s} [(t+z_j)(t+z_k) - 2\Delta(t+z_j) + 1]\frac{\mathrm{d}^s z}{(2\mathrm{i}\pi)^s}. \qquad (3.5)$$

Here C_0 is a small, simple, closed, positively-oriented contour enclosing point $z = 0$. Evaluating the Vandermonde determinant and making the change $z_k \mapsto w_k = (z_k + t)/t$, we finally obtain:

$$Z_{r_1,\ldots,r_s}^{\text{top}} = c^s a^{s(N-1)} \prod_{j=1}^s t^{r_j - j} \oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^s \frac{w_j^{r_j - 1}}{(w_j - 1)^s}$$

$$\times \prod_{1 \leqslant j < k \leqslant s} [(w_j - w_k)(t^2 w_j w_k - 2\Delta t w_j + 1)]\frac{\mathrm{d}^s w}{(2\pi\mathrm{i})^s}. \qquad (3.6)$$

Here C_1 denotes a small, simple, closed, positively-oriented contour enclosing point $z = 1$.

Formulae (3.1) and (3.6) can also been derived by other methods (i.e., without using the equivalence of the algebraic and coordinate Bethe Ansatz), e.g., starting with vertical monodromy matrix formulation of $Z_{r_1,\ldots,r_s}^{\text{top}}$, analogous to (2.22) for $Z_{r_1,\ldots,r_s}^{\text{bot}}$, and next using the technique of paper [18] to evaluate the matrix element[1]. We also mention that formula (3.3), in a different form and for special values of t and Δ, has been found in the context of enumerative combinatorics [19].

4. Calculation of $Z_{r_1,\ldots,r_s}^{\text{bot}}$

Taking into account commutativity of B-operators, see (2.9), and using relation (2.10), we can obtain, in the usual spirit of the algebraic Bethe Ansatz calculation (for details see, e.g., [1]), the relation:

$$A(\lambda_r)\prod_{\beta=1}^{r-1} B(\lambda_\beta) = \sum_{\alpha=1}^r \frac{g(\lambda_\alpha, \lambda_r)}{f(\lambda_\alpha, \lambda_r)} \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^r f(\lambda_\alpha, \lambda_\beta) \prod_{\substack{\beta=1 \\ \beta \neq \alpha}}^r B(\lambda_\beta)A(\lambda_\alpha). \qquad (4.1)$$

Using this commutation relation and taking into account that

$$A_{s+1,\ldots,N}^{\text{V}}(\lambda)\left|\Uparrow_{s+1,\ldots,N}^{\text{V}}\right\rangle = \prod_{k=s+1}^N a(\lambda, \nu_k)\left|\Uparrow_{s+1,\ldots,N}^{\text{V}}\right\rangle, \qquad (4.2)$$

---

[1]We are indebted to P. Zinn-Justin for explaining us this alternative derivation.

and also using (2.13), we obtain

$$Z^{\text{bot}}_{r_1,\ldots,r_s} = \sum_{\alpha_1=1}^{r_1} \sum_{\substack{\alpha_2=1 \\ \alpha_2 \neq \alpha_1}}^{r_2} \cdots \sum_{\substack{\alpha_s=1 \\ \alpha_s \neq \alpha_1,\ldots,\alpha_{s-1}}}^{r_s} \prod_{j=1}^{s} \prod_{k=s+1}^{N} a(\lambda_{\alpha_j}, \nu_k) \prod_{j=1}^{s} \frac{g(\lambda_{\alpha_j}, \lambda_{r_j})}{f(\lambda_{\alpha_j}, \lambda_{r_j})}$$

$$\times \prod_{\substack{\beta_1=1 \\ \beta_1 \neq \alpha_1}}^{r_1} f(\lambda_{\alpha_1}, \lambda_{\beta_1}) \prod_{\substack{\beta_2=1 \\ \beta_2 \neq \alpha_1, \alpha_2}}^{r_2} f(\lambda_{\alpha_2}, \lambda_{\beta_2}) \cdots \prod_{\substack{\beta_s=1 \\ \beta_s \neq \alpha_1,\ldots,\alpha_s}}^{r_s} f(\lambda_{\alpha_s}, \lambda_{\beta_s})$$

$$\times Z_{N-s}(\lambda_{\bar{\alpha}_1}, \ldots, \lambda_{\bar{\alpha}_{N-s}}; \nu_{s+1}, \ldots, \nu_N), \quad (4.3)$$

where $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_{N-s}\} := \{1, \ldots, N\} \backslash \{\alpha_1, \ldots, \alpha_s\}$.

To proceed further, it is convenient to introduce function:

$$v_r(\lambda) = \frac{\prod_{\alpha=r+1}^{N} d(\lambda_\alpha, \lambda) \prod_{\alpha=1}^{r-1} e(\lambda_\alpha, \lambda)}{\prod_{k=s+1}^{N} b(\lambda, \nu_k)}, \quad (4.4)$$

where $d(\lambda, \lambda') := \sin(\lambda - \lambda')$ and $e(\lambda, \lambda') := \sin(\lambda - \lambda' + 2\eta)$. Expressing functions $f(\lambda, \lambda')$ and $g(\lambda, \lambda')$ appearing in (4.3) in terms of functions $d(\lambda, \lambda')$ and $e(\lambda, \lambda')$ and substituting the Izergin-Korepin determinant formula, see (2.16), for the partition function standing in (4.3), we arrive at the expression:

$$Z^{\text{bot}}_{r_1,\ldots,r_s} = \frac{\prod_{\alpha=1}^{N} \prod_{k=s+1}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)}{\prod_{1 \leqslant \alpha < \beta \leqslant N} d(\lambda_\beta, \lambda_\alpha) \prod_{s+1 \leqslant j < k \leqslant N} d(\nu_j, \nu_k)}$$

$$\times \sum_{\alpha_1} \sum_{\substack{\alpha_2=1 \\ \alpha_2 \neq \alpha_1}}^{r_2} \cdots \sum_{\substack{\alpha_s=1 \\ \alpha_s \neq \alpha_1,\ldots,\alpha_{s-1}}}^{r_s} (-1)^{\sum_{j=1}^{s}(\alpha_j - 1) - \sum_{1 \leqslant j < k \leqslant s} \chi(\alpha_k, \alpha_j)}$$

$$\times \prod_{j=1}^{s} v_{r_j}(\lambda_{\alpha_j}) \prod_{1 \leqslant j < k \leqslant s} \frac{1}{e(\lambda_{\alpha_j}, \lambda_{\alpha_k})} \det_{1 \leqslant j, k \leqslant N-s} \{\varphi(\lambda_{\bar{\alpha}_j}, \nu_{s+k})\}. \quad (4.5)$$

Here $\chi(\beta, \alpha) = 1$ if $\beta > \alpha$, and $\chi(\beta, \alpha) = 0$ otherwise.

Clearly, the multiple sum in (4.5) reminds the Laplace expansion of some $N \times N$ determinant. This is also in agreement with the fact that since $v_r(\lambda_\alpha) = 0$ ($\alpha = r+1, \ldots, N$) all summations in (4.5) can be extended till the value N. To write down such a determinant formula, let us set $\lambda_\alpha = \lambda + \xi_\alpha$ ($\alpha = 1, \ldots, N$), where λ is some arbitrary parameter, and parameters $\xi_1, \ldots, \xi_N$ are all different. Using again the fact that for a function $f(\xi)$, regular near point $\xi = \lambda$, we can write $f(\lambda + \xi) = \exp(\xi \partial_\varepsilon) f(\lambda + \varepsilon)|_{\varepsilon=0}$, we can bring (4.5) to the form

$$Z^{\text{bot}}_{r_1,\ldots,r_s} = \frac{\prod_{\alpha=1}^{N} \prod_{k=s+1}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)}{\prod_{1 \leqslant \alpha < \beta \leqslant N} d(\lambda_\beta, \lambda_\alpha) \prod_{s+1 \leqslant j < k \leqslant N} d(\nu_j, \nu_k)}$$

$$\times \begin{vmatrix} \exp(\xi_1 \partial_{\varepsilon_1}) & \cdots & \exp(\xi_1 \partial_{\varepsilon_s}) & \varphi(\lambda_1, \nu_{s+1}) & \cdots & \varphi(\lambda_1, \nu_N) \\ \exp(\xi_2 \partial_{\varepsilon_1}) & \cdots & \exp(\xi_2 \partial_{\varepsilon_s}) & \varphi(\lambda_2, \nu_{s+1}) & \cdots & \varphi(\lambda_2, \nu_N) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \exp(\xi_N \partial_{\varepsilon_1}) & \cdots & \exp(\xi_N \partial_{\varepsilon_s}) & \varphi(\lambda_N, \nu_{s+1}) & \cdots & \varphi(\lambda_N, \nu_N) \end{vmatrix}$$

$$\times \prod_{j=1}^{s} v_{r_j}(\lambda + \varepsilon_j) \prod_{1 \leqslant j < k \leqslant s} \frac{1}{e(\lambda + \varepsilon_j, \lambda + \varepsilon_k)} \Bigg|_{\varepsilon_1 = \ldots = \varepsilon_s = 0}. \quad (4.6)$$

We stress that this expression is valid for the inhomogeneous model.

Let us now perform the homogeneous limit. We regard λ as the parameter of the weights of the homogeneous model, so that parameters $\xi_1, \ldots, \xi_N$ and $\nu_{s+1}, \ldots, \nu_N$ are sent to zero in the limit. The procedure can be done along the lines of [4] and it is explained in full detail in [11]. As a result, we obtain the expression

$$
Z_{r_1,\ldots,r_s}^{\text{bot}} = \frac{(ab)^{N(N-s)}}{\prod_{j=1}^{N-s-1} j! \prod_{k=1}^{N-1} k!}
\begin{vmatrix}
\varphi(\lambda) & \cdots & \partial_\lambda^{N-s-1}\varphi(\lambda) & 1 & \cdots & 1 \\
\partial_\lambda \varphi(\lambda) & \cdots & \partial_\lambda^{N-s}\varphi(\lambda) & \partial_{\varepsilon_1} & \cdots & \partial_{\varepsilon_s} \\
\hdotsfor{6} \\
\partial_\lambda^{N-1}\varphi(\lambda) & \cdots & \partial_\lambda^{2N-s-2}\varphi(\lambda) & \partial_{\varepsilon_1}^{N-1} & \cdots & \partial_{\varepsilon_s}^{N-1}
\end{vmatrix}
$$

$$
\times \left. \prod_{j=1}^{s} \frac{(\sin \varepsilon_j)^{N-r_j}[\sin(\varepsilon_j - 2\eta)]^{r_j - 1}}{[\sin(\varepsilon_j + \lambda - \eta)]^{N-s}} \prod_{1 \leqslant j < k \leqslant s} \frac{1}{\sin(\varepsilon_j - \varepsilon_k + 2\eta)} \right|_{\varepsilon_1 = \ldots = \varepsilon_s = 0},
$$

$$(4.7)$$

where in writing the determinant we have changed the order of columns, in comparison with (4.6).

In order to represent (4.7) in terms of a multiple integral, we first transform the $N \times N$ determinant representation (4.7) to an $s \times s$ one, given in terms of certain set of orthogonal polynomials. The construction is based on the following general facts from the theory of orthogonal polynomials (see, e.g., [20]). Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of orthogonal polynomials,

$$
\int P_n(x) P_m(x) \mu(x)\, dx = h_n \delta_{nm}, \tag{4.8}
$$

where the integration domain is assumed over the real axis and we choose normalisation such that $P_n(x) = x^n + \ldots$, and let c_n denote the nth moment of the weight $\mu(x)$,

$$
c_n = \int x^n \mu(x)\, dx \qquad (n = 0, 1, \ldots). \tag{4.9}
$$

Then $\det_{1 \leqslant j, k \leqslant N}\{c_{j+k-2}\} = h_0 h_1 \cdots h_{N-1}$ and, more generally, for $s = 1, \ldots, N$, the following formula is valid:

$$
\begin{vmatrix}
c_0 & c_1 & \cdots & c_{N-s-1} & 1 & 1 & \cdots & 1 \\
c_1 & c_2 & \cdots & c_{N-s} & x_1 & x_2 & \cdots & x_s \\
\hdotsfor{8} \\
c_{N-1} & c_N & \cdots & c_{2N-s-2} & x_1^{N-1} & x_2^{N-1} & \cdots & x_s^{N-1}
\end{vmatrix}
$$

$$
= h_0 h_1 \cdots h_{N-s-1} \det_{1 \leqslant j, k \leqslant s}\{P_{N-s+j-1}(x_k)\}. \tag{4.10}
$$

In our case $c_n := \partial_\lambda^n \varphi(\lambda)$, and the integration measure $\mu(x)dx$ can be found through the Laplace transform for function $\varphi(\lambda)$; for explicit expressions, see [21].

As in [11], we denote

$$
K_n(x) = \frac{n!\, \varphi^{n+1}}{h_n}\, P_n(x), \tag{4.11}
$$

where $\varphi := \varphi(\lambda)$, and h_n is as in (4.8). We also introduce functions

$$
\omega(\epsilon) := \frac{a}{b} \frac{\sin \varepsilon}{\sin(\varepsilon - 2\eta)}, \qquad \tilde{\omega}(\epsilon) := \frac{b}{a} \frac{\sin \varepsilon}{\sin(\varepsilon + 2\eta)} \tag{4.12}
$$

which, in particular, satisfy the relation

$$
\frac{b}{c} \frac{\sin(\varepsilon - 2\eta)}{\sin(\varepsilon + \lambda - \eta)} = \frac{1}{\omega(\varepsilon) - 1}, \tag{4.13}
$$

where the short notations for the weights $a := \sin(\lambda + \eta)$, $b := \sin(\lambda - \eta)$, and $c := \sin 2\eta$ are used. Taking into account that $\varphi = c/ab$, and using the relation

$$\frac{\sin(\varepsilon_1 + \lambda + \eta)\sin(\varepsilon_2 + \lambda - \eta)}{\sin(\varepsilon_1 - \varepsilon_2 + 2\eta)} = \frac{1}{\varphi}\frac{(1 - \tilde{\omega}(\varepsilon_1))(\omega(\varepsilon_2) - 1)}{\tilde{\omega}(\varepsilon_1)\omega(\varepsilon_2) - 1}, \tag{4.14}$$

after applying (4.10) to (4.7), we obtain:

$$Z_{r_1,\ldots,r_s}^{\mathrm{bot}} = \frac{Z_N}{a^{\frac{s(2N-s+1)}{2}}b^{\frac{s(s-3)}{2}}c^s}\left(\frac{a}{b}\right)^{r_1+\cdots+r_s}\det_{1\leqslant j,k\leqslant s}\left\{K_{N-s+j-1}(\partial_{\varepsilon_k})\right\}$$

$$\times \prod_{j=1}^{s}\frac{[\omega(\varepsilon_j)]^{N-r_j-s+j}[\tilde{\omega}(\varepsilon_j)]^{s-j}}{[\omega(\varepsilon_j) - 1]^{N-s}}\prod_{1\leqslant j<k\leqslant s}\frac{1}{\tilde{\omega}(\varepsilon_j)\omega(\varepsilon_k) - 1}\bigg|_{\varepsilon_1=\ldots=\varepsilon_s=0}. \tag{4.15}$$

In deriving of this formula, we have also used (2.18) to express a proper factor as the partition function Z_N.

Now we are ready to write representation (4.15) as a multiple integral. We follow the procedure developed in [11]. The key relation here, valid for an arbitrary function $f(z)$ regular at the origin, is

$$K_{N-1}(\partial_\varepsilon)\,f(\omega(\varepsilon))\Big|_{\varepsilon=0} = \frac{1}{2\pi\mathrm{i}}\oint_{C_0}\frac{(z-1)^{N-1}}{z^N}h_N(z)f(z)\,\mathrm{d}z. \tag{4.16}$$

Here C_0, as above, is a small, simple, closed, positively-oriented contour enclosing point $z = 0$, and $h_N(z)$ (not to be confused with h_n in (4.8)) is the generating function for the one-point boundary correlation function, $h_N(z) = \sum_{r=1}^{N}H_N^{(r)}z^{r-1}$, where

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon)\frac{[\omega(\varepsilon)]^{N-r}}{[\omega(\varepsilon) - 1]^{N-1}}\bigg|_{\epsilon=0}. \tag{4.17}$$

This function can be viewed as the $s = 1$ case of the row configuration probability, $H_N^{(r)} := H_{N,1}^{(r)}$. Indeed, in this case the partition function of the upper sublattice is simply $Z_r^{\mathrm{top}} = a^{N-r}b^{r-1}c$, while Z_r^{bot} can be found from (4.15), thus reproducing (4.17).

To write down the resulting multiple integral representation for $Z_{r_1,\ldots,r_s}^{\mathrm{bot}}$, we introduce functions

$$h_{N,s}(z_1,\ldots,z_s) = \frac{\det_{1\leqslant j,k\leqslant s}\{z_k^{s-j}(z_k-1)^{j-1}h_{N-s+j}(z_k)\}}{\prod_{1\leqslant j<k\leqslant s}(z_k - z_j)}, \tag{4.18}$$

which can be viewed as multi-variable generalisations of $h_N(z)$ (for a detailed discussion of its properties, see [11]). Noticing that

$$\tilde{\omega}(\epsilon) = \frac{t^2\omega(\varepsilon)}{2\Delta t\omega(\varepsilon) - 1}, \tag{4.19}$$

where $t = b/a$ and $\Delta = (a^2 + b^2 - c^2)/2ab$, we can readily rewrite the orthogonal polynomial representation (4.15) in virtue of (4.16) as follows:

$$Z_{r_1,\ldots,r_s}^{\mathrm{bot}} = Z_N\frac{\prod_{j=1}^{s}t^{j-r_j}}{a^{s(N-1)}c^s}\oint_{C_0}\cdots\oint_{C_0}\prod_{j=1}^{s}\frac{1}{z_j^{r_j}}\prod_{1\leqslant j<k\leqslant s}\frac{z_k - z_j}{t^2z_jz_k - 2\Delta tz_j + 1}$$

$$\times h_{N,s}(z_1,\ldots,z_s)\frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}. \tag{4.20}$$

This formula is the desired representation for $Z^{\text{bot}}_{r_1,\ldots,r_s}$, valid for the homogeneous model.

5. Emptiness formation probability

An important example of correlation function which can be built from the row configuration probability is the emptiness formation probability. As in [11], we denote by $F_N^{(r,s)}$ the probability of observing all arrows on the first s horizontal edges (counted, as usual, from the top) located between r-th and $(r+1)$-th vertical lines (counted, as usual, from the right) to be all pointing left. Equivalently, due to both the domain wall boundary conditions and the ice-rule, we can define it as the probability of observing the last $N-r$ arrows between the sth and $(s+1)$th horizontal lines to be all pointing down, and hence (see also figure 2) we have the relation:

$$F_N^{(r,s)} = \sum_{1\leqslant r_1 < r_2 < \cdots < r_s \leqslant r} H_{N,s}^{(r_1,\ldots,r_s)}. \tag{5.1}$$

Our aim here is to address how this summation can be done for the row configuration probability, given by (2.19), (3.6) and (4.20), to reproduce the multiple integral representations for the emptiness formation probability obtained in [11].

Let us first recall the results of paper [11]. The following two multiple integral representations have been obtained:

$$
\begin{aligned}
F_N^{(r,s)} &= (-1)^s \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t)z_j + 1]^{s-j}}{z_j^r(z_j-1)^{s-j+1}} \prod_{1\leqslant j<k\leqslant s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} \\
&\quad \times h_{N,s}(z_1,\ldots,z_s)\frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s} \\
&= \frac{(-1)^s Z_s}{s! a^{s(s-1)} c^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t)z_j + 1]^{s-1}}{z_j^r(z_j-1)^{s}} \prod_{\substack{j,k=1 \\ j\neq k}}^{s} \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} \\
&\quad \times h_{N,s}(z_1,\ldots,z_s) h_{s,s}(u(z_1),\ldots,u(z_s))\frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}, \tag{5.2}
\end{aligned}
$$

where

$$u(z) := -\frac{z-1}{(t^2 - 2\Delta t)z + 1}. \tag{5.3}$$

The two representations in (5.2) are related by a symmetrization of the integrand, which is due to the following relation

$$
\begin{aligned}
&\underset{z_1,\ldots,z_s}{\text{Asym}}\left[\prod_{1\leqslant j<k\leqslant s} \frac{[(t^2 - 2\Delta t)z_j + 1](t^2 z_j z_k - 2\Delta t z_k + 1)}{(z_j - 1)}\right] \\
&\quad = \frac{Z_s}{s! a^{s(s-1)} c^s} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t)z_j + 1]^{s-1}}{(z_j-1)^{s-1}} \prod_{1\leqslant j<k\leqslant s}(z_k - z_j)\, h_{s,s}(u(z_1),\ldots,u(z_s)). \end{aligned}
$$
$$\tag{5.4}$$

Here $\underset{z_1,\ldots,z_s}{\text{Asym}} f(z_1,\ldots,z_s) := \frac{1}{s!}\sum_P (-1)^{[P]} f(z_{P_1},\ldots,z_{P_s})$, and the sum is taken over permutations $P : 1,\ldots,s \mapsto P_1,\ldots,P_s$, with $[P]$ denoting the parity of P. For details on the proof of relation (5.4), see [11].

Let us discuss the representation for emptiness formation probability obtained from (2.19), (3.6) and (4.20), according to relation (5.1). Direct substitution gives

$$
F_N^{(r,s)} = \oint_{C_1} \cdots \oint_{C_1} \frac{\mathrm{d}^s w}{(2\pi\mathrm{i})^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{(w_j-1)^s} \sum_{1 \leqslant r_1 < r_2 < \cdots < r_s \leqslant r} \prod_{j=1}^{s} \frac{w_j^{r_j-1}}{z_j^{r_j}}
$$
$$
\times \prod_{1 \leqslant j < k \leqslant s} \frac{(w_j-w_k)(t^2 w_j w_k - 2\Delta t w_j + 1)(z_k-z_j)}{t^2 z_j z_k - 2\Delta t z_j + 1} h_{N,s}(z_1,\ldots,z_s) \frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}
$$
$$(5.5)$$

so performing here the multiple sum and integrating over a set of variables (e.g., over $w_1,\ldots,w_s$) we should reproduce, modulo symmetrization of the integrand, the s-fold integral representations (5.2).

To address this problem, let us first consider the evaluation of the multiple sum in (5.5). Observing that the integral over z_j vanish for $r_j \leqslant 0$, because in this case the integrand is regular at $z_j = 0$, we can replace the sum in (5.5) over values $1 \leqslant r_1 < r_2 \cdots < r_s \leqslant r$ with a sum over values $-\infty < r_1 < r_2 \cdots < r_s \leqslant r$. Then, denoting $X_j = z_j/w_j$, the summation can be done using the identity

$$
\sum_{-\infty < r_1 < r_2 < \cdots < r_s \leqslant r} \prod_{j=1}^{s} \frac{1}{X_j^{r_j}} = \prod_{j=1}^{s} \frac{1}{X_j^{r-s+j}\left(1 - \prod_{l=1}^{j} X_l\right)}, \qquad (5.6)
$$

which can be easily verified by expanding the denominators in the right hand side in Taylor series. As a result, we find that (5.5) simplifies to expression:

$$
F_N^{(r,s)} = \oint_{C_1} \cdots \oint_{C_1} \frac{\mathrm{d}^s w}{(2\pi\mathrm{i})^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{w_j^r}{(w_j-1)^s z_j^{r-s+j}\left(\prod_{l=1}^{j} w_l - \prod_{l=1}^{j} z_l\right)}
$$
$$
\times \prod_{1 \leqslant j < k \leqslant s} \frac{(w_j-w_k)(t^2 w_j w_k - 2\Delta t w_j + 1)(z_k-z_j)}{t^2 z_j z_k - 2\Delta t z_j + 1} h_{N,s}(z_1,\ldots,z_s) \frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}
$$
$$(5.7)$$

and we are left with performing an s-fold integration.

We shall integrate over variables $w_1,\ldots,w_s$ in (5.7). Let us consider the equivalent integral where the integrand is symmetrized with respect to permutations of these variables. Define function

$$
\Phi_s(w_1,\ldots,w_s; z_1,\ldots,z_s) = \prod_{1 \leqslant j < k \leqslant s} \frac{1}{w_k - w_j}
$$
$$
\times \operatorname*{Asym}_{w_1,\ldots,w_s} \left[\frac{\prod_{1 \leqslant j < k \leqslant s}(t^2 w_j w_k - 2\Delta t w_j + 1)}{\prod_{j=1}^{s}\left(\prod_{l=1}^{j} w_l - \prod_{l=1}^{j} z_l\right)} \right]. \quad (5.8)
$$

Integration over the w_j's is done with the result

$$
\oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^{s} \frac{w_j^r}{(w_j-1)^s} \prod_{1 \leqslant j < k \leqslant s} (w_j-w_k)^2 \, \Phi_s(w_1,\ldots,w_s; z_1,\ldots,z_s) \frac{\mathrm{d}^s w}{(2\pi\mathrm{i})^s}
$$
$$
= (-1)^{s(s-1)/2} s! \, \Phi_s(1,\ldots,1; z_1,\ldots,z_s), \quad (5.9)
$$

which can be easily found by noticing that in evaluating the residues one has to differentiate only the factor $\prod_{j<k}(w_j-w_k)^2$.

13

Finally, the desired result for the emptiness formation probability amounts in proving the identity:

$$
s! \operatorname*{Asym}_{z_1,\ldots,z_s} \left[\Phi_s(1,\ldots,1;z_1,\ldots,z_s) \prod_{1\leqslant j<k\leqslant s} [z_j(t^2 z_j z_k - 2\Delta t z_k + 1)] \right]
$$

$$
= \frac{(-1)^{s(s+1)/2}}{\prod_{j=1}^{s}(z_j-1)} \operatorname*{Asym}_{z_1,\ldots,z_s} \left[\prod_{1\leqslant j<k\leqslant s} \frac{[(t^2-2\Delta t)z_j+1](t^2 z_j z_k - 2\Delta t z_k + 1)}{(z_j-1)} \right].
$$

(5.10)

This identity has to be used together with identity (5.4) to reproduce (5.2). We find identity (5.10) rather difficult to prove directly, and presently we have only been able to verify it through computer-aided calculations for small values of s. We note that rather similar identities have been discussed in [22–24].

In conclusion, in this paper we have introduced and calculated a nonlocal correlation function of the six-vertex model with domain wall boundary conditions, the row configuration probability. It is given as a product of two factors which can be treated as the partition functions on upper and lower sublattices of the original lattice (see figure 2). We have represented these partition functions in terms of multiple integrals, see (3.6) and (4.20). The row configuration probability can be used for computing other correlation functions, provided that sums like those appearing in (5.1) can be evaluated. To illustrate this, we have considered the problem of reproducing the known result for the emptiness formation probability. We have shown that in this case the problem boils down to identity (5.10). A direct proof of this identity, in addition to the indirect one following from the known equality of (5.2) and (5.5), could be useful for the evaluation of other correlation functions.

Acknowledgments

References

[1] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge, 1993.

[2] V. E. Korepin, *Calculations of norms of Bethe wave functions*, Commun. Math. Phys. **86** (1982), 391–418.

[3] A. G. Izergin, *Partition function of the six-vertex model in the finite volume*, Sov. Phys. Dokl. **32** (1987), 878–879.

[4] A. G. Izergin, D. A. Coker, and V. E. Korepin, *Determinant formula for the six-vertex model*, J. Phys. A **25** (1992), 4315–4334.

[5] F. Colomo and A. G. Pronko, *The limit shape of large alternating-sign matrices*, SIAM J. Discrete Math. **24** (2010), 1558–1571, arXiv:0803.2697.

[6] F. Colomo and A. G. Pronko, *The arctic curve of the domain-wall six-vertex model*, J. Stat. Phys. **138** (2010), 662–700, arXiv:0907.1264.

[7] N. M. Bogoliubov, A. V. Kitaev, and M. B. Zvonarev, *Boundary polarization in the six-vertex model*, Phys. Rev. E **65** (2002), 026126, arXiv:`cond-mat/0107146`.

[8] N. M. Bogoliubov, A. G. Pronko, and M. B. Zvonarev, *Boundary correlation functions of the six-vertex model*, J. Phys. A **35** (2002), 5525–5541, arXiv:`math-ph/0203025`.

[9] O. Foda and I. Preston, *On the correlation functions of the domain wall six vertex model*, J. Stat. Mech. **0411** (2004), P001, arXiv:`math-ph/0409067`.

[10] F. Colomo and A. G. Pronko, *On two-point boundary correlations in the six-vertex model with domain wall boundary conditions*, J. Stat. Mech. Theory Exp. (2005), P05010, arXiv: `math-ph/0503049`.

[11] F. Colomo and A. G. Pronko, *Emptiness formation probability in the domain-wall six-vertex model*, Nucl. Phys. B **798** [**FS**] (2008), 340–362, arXiv:`0712.1524`.

[12] K. Motegi, *Boundary correlation functions of the six and nineteen vertex models with domain wall boundary conditions*, arXiv:`1101.0187`.

[13] P. Di Francesco and N. Reshetikhin, *Asymptotic shapes with free boundaries* (2009), arXiv: `0908.1630`.

[14] I. Fischer and D. Romik, *More refined enumerations of alternating sign matrices*, Adv. Math. **222** (2009), no. 6, 2004–2035, arXiv:`0903.5073`.

[15] L. A. Takhtadjan and L. D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russ. Math. Surveys **34** (1979), no. 5, 11–68.

[16] A. G. Izergin, V. E. Korepin, and N. Yu. Reshetikhin, *Correlation functions in a one-dimensional Bose gas*, J. Phys. A **20** (1987), 4799–4822.

[17] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, San Diego, CA, 1982.

[18] N. Kitanine, J.-M. Maillet, and V. Terras, *Form factors of the XXZ Heisenberg spin-1/2 finite chain*, Nucl. Phys. B **554** (1999), 647–678, arXiv:`math-ph/9907019`.

[19] I. Fischer, *The number of monotone triangles with prescribed bottom row*, Adv. in Appl. Math. **37** (2006), no. 2, 249–267, arXiv:`math/0501102`.

[20] G. Szegö, *Orthogonal polinomials*, 4th ed., American Colloquium Publications, vol. XXIII, American Mathematical Society, Providence, RI, 1975.

[21] P. Zinn-Justin, *Six-vertex model with domain wall boundary conditions and one-matrix model*, Phys. Rev. E **62** (2000), 3411–3418, arXiv:`math-ph/0005008`.

[22] P. Zinn-Justin and P. Di Francesco, *Quantum Knizhnik-Zamolodchikov equation, totally symmetric self-complementary plane partitions, and alternating sign matrices*, Theor. Math. Phys. **154** (2008), no. 3, 331–348, arXiv:`math-ph/0703015`.

[23] D. Zeilberger, *Proof of a conjecture of Philippe Di Francesco and Paul Zinn-Justin related to the qKZ equation and to Dave Robbins' two favorite combinatorial objects*, 2007, `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/diFrancesco.html`.

[24] C. A. Tracy and H. Widom, *Integral formulas for the asymmetric simple exclusion process*, Commun. Math Phys. **279** (2008), 815–844, arXiv:`0704.2633`.