Inverse problems for Jacobi operators III: Mass-spring perturbations of semi-infinite systems

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Abstract

Consider an infinite linear mass-spring system and a modification of it obtained by changing the first mass and spring of the system. We give results on the interplay of the spectra of such systems and on the reconstruction of the system from its spectrum and the one of the modified system. Furthermore, we provide necessary and sufficient conditions for two sequences to be the spectra of the mass-spring system and the perturbed one.

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1. Introduction

Let $l_{\mathrm{fin}}(\mathbb{N})$ be the linear space of complex sequences with a finite number of non-zero elements. In the Hilbert space $l_2(\mathbb{N})$, consider the operator J_0 defined for every $f = \{f_k\}_{k=1}^\infty$ in $l_{\mathrm{fin}}(\mathbb{N})$ by

$$(J_0 f)_1 := q_1 f_1 + b_1 f_2\,, \tag{1.1}$$
$$(J_0 f)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1}\,, \quad k \in \mathbb{N} \setminus \{1\}, \tag{1.2}$$

where $q_n \in \mathbb{R}$ and $b_n > 0$ for any $n \in \mathbb{N}$. The operator J_0 is symmetric and has deficiency indices $(1,1)$ or $(0,0)$ [1, Chap. 4, Sec. 1.2]. Fix a self-adjoint extension of J_0 and denote it by J. Thus, either $J \supsetneq \overline{J_0}$ or $J = \overline{J_0}$. According to the definition of the matrix representation for an unbounded symmetric operator [2, Sec. 47], $\overline{J_0}$ is the operator whose matrix representation with respect to the canonical basis $\{\delta_n\}_{n=1}^\infty$ in $l_2(\mathbb{N})$ is

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}. \tag{1.3}$$

Along with J, we consider the operator

$$\begin{aligned} \widetilde{J} = J + &\left[q_1(\theta^2 - 1) + \theta^2 h\right] \langle \delta_1, \cdot \rangle \, \delta_1 \\ &+ b_1(\theta - 1)(\langle \delta_1, \cdot \rangle \, \delta_2 + \langle \delta_2, \cdot \rangle \, \delta_1)\,, \quad \theta > 0\,, \quad h \in \mathbb{R}\,, \end{aligned} \tag{1.4}$$

which is a self-adjoint extension of the operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is

$$\begin{pmatrix} \theta^2(q_1 + h) & \theta b_1 & 0 & 0 & \cdots \\ \theta b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & & \vdots & & \ddots & \ddots \end{pmatrix}. \tag{1.5}$$

Note that $\widetilde{J}$ is obtained from J by a particular kind of rank-two perturbation.

Under the assumption that J has discrete spectrum (as explained in Section 2, when J_0 has deficiency indices $(1,1)$, this is always the case), this work treats the inverse spectral problem of reconstructing, from the spectra of J and $\widetilde{J}$, the matrix (1.3) and the "boundary condition at infinity" defining the self-adjoint extension J if necessary (i.e. if J_0 is not essentially self-adjoint, cf. [8, Sec. 2]). To solve this inverse problem, one should elucidate the distribution of the perturbed spectrum

relative to the unperturbed one and determine the necessary input data for recovering the matrix. An important point to note is that this work provides necessary and sufficient conditions for two sequences to be the spectra of J and $\widetilde{J}$. Also, we discuss (the lack of) uniqueness of the reconstruction.

Although the two spectra inverse problem for the rank-one perturbation family of Jacobi operators has been thoroughly studied (see for instance [9, 12, 16, 20] and [5, 6, 10, 11] for the case of finite matrices), there is scarce literature dealing with inverse problems of other kind of perturbations (cf. [8]).

The motivation for this work is the inverse spectral problem studied in [15] and [7] which is in its turn related with the physical problem of measuring micro-masses with the help of micro-cantilevers [18, 19]. Micro-cantilevers are modeled by spring-mass systems whose masses and spring constants are determined by the mechanical parameters of the micro-cantilevers.

In this work we consider the semi-infinite mass-spring system given in Fig. 1. with masses $\{m_j\}_{j=1}^{\infty}$ and spring constants $\{k_j\}_{j=1}^{\infty}$. This system is modeled by the Jacobi matrix (1.3) with

$$q_j = -\frac{k_{j+1} + k_j}{m_j}\,, \qquad b_j = \frac{k_{j+1}}{\sqrt{m_j m_{j+1}}}\,, \qquad j \in \mathbb{N}\,.$$

In [11, 14] it is explained how to deduce these formulae. Since J is considered to have discrete spectrum, the movement of the system is a superposition of harmonic oscillations whose frequencies are the square roots of the modules of the eigenvalues. The modified mass-spring system corresponding to the perturbed operator $\widetilde{J}$ is
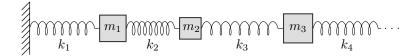

Figure 1: Semi-infinite mass-spring system

obtained by changing the first mass by $\Delta m = m_1(\theta^{-2} - 1)$ and the first spring by $\Delta k = -hm_1$ (see Fig. 2). Here we also consider negative values of Δm and Δk which correspond to $\theta > 1$ and $h < 0$, respectively. Note that the perturbation
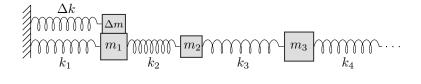

Figure 2: Perturbed semi-infinite mass-spring system

involved here is the result of the combined effect of a rank-one perturbation (studied

thoroughly in [16]) and the particular rank-two perturbation studied in [8]. However, most of the results obtained here cannot be found from the results in [16] and [8], and require their own proof. Moreover, it turns out that one can single-out classes of isospectral operators within the two parameter perturbation family considered in this work that were not studied before.

The paper is organized as follows. In Section 2 we fix the notation, lay down a convention for enumerating sequences and recall some results of the inverse spectral theory for Jacobi operators. Section 3 gives a detailed spectral analysis of the family of perturbed Jacobi operators. The solution of the two spectra inverse problem for J and $\widetilde{J}$ is given in Section 4. This section also discusses the non-uniqueness of the reconstruction and gives some characterization of isospectral operators in the perturbation family under consideration.

2. A review on inverse spectral theory for Jacobi operators

Let us denote by $\sigma(J)$ the spectrum of J and consider the spectral resolution of the identity E for J given by the spectral theorem. Then the spectral function ρ of J is defined by

$$\rho(t) := \langle \delta_1, E(t)\delta_1 \rangle \,. \tag{2.1}$$

All the moments of ρ exist [1, Thm. 4.1.3], that is, for all $k \in \mathbb{N} \cup \{0\}$,

$$s_k = \int_{\mathbb{R}} t^k d\rho(t) \in \mathbb{R} \,.$$

Moreover, since J turns out to be simple with δ_1 being a cyclic vector, the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \rho)$ (defined on the maximal domain) is unitarily equivalent to J.

Alongside the spectral function we consider the corresponding Weyl m-function given by

$$m(\zeta) := \langle \delta_1, (J - \zeta I)^{-1}\delta_1 \rangle = \int_{\mathbb{R}} \frac{d\rho(t)}{t - \zeta} \,, \qquad \zeta \notin \sigma(J) \,. \tag{2.2}$$

Because of the inverse Stieltjes transform one uniquely recovers ρ from m, so ρ and m are in one-to-one correspondence.

The Weyl m-function has the following asymptotic behavior

$$m(\zeta) = -\frac{1}{\zeta} - \frac{q_1}{\zeta^2} - \frac{b_1^2 + q_1^2}{\zeta^3} + O(\zeta^{-4}) \,, \tag{2.3}$$

as $\zeta \to \infty$ with $\operatorname{Im}\zeta \geq \epsilon$, $\epsilon > 0$ (see [10, Eq. 1.5] and [16, Eq. 2.10]).

The inverse spectral theory for the Jacobi operator J is centered on the fact that the Weyl m-function (or, equivalently, ρ) uniquely determines the matrix (1.3) and the boundary condition at infinity that defines the self-adjoint extension if necessary. Indeed, for recovering the matrix one may use a method based on a discrete Riccati

equation (see [10, Eq. 2.15], [20, Eq. 2.23]) or the method of orthonormalization of the polynomial sequence $\{t^k\}_{k=0}^\infty$ in $L_2(\mathbb{R}, \rho)$ [3, Chap. 7, Sec. 1.5]. If (1.3) is the matrix representation of a non-self-adjoint operator, then the condition at infinity may be found by the method exposed in [16, Sec. 2].

In this work we restrict our considerations to the case of $\sigma(J)$ being discrete, viz., $\sigma_{\text{ess}}(J) = \emptyset$. It is well known that this is always the case when J_0 is not essentially self-adjoint [17, Thm. 4.11], [21, Lem. 2.19]. The discreteness of $\sigma(J)$ implies that (2.1) can be written as follows

$$\rho(t) = \sum_{\lambda_k < t} \frac{1}{\alpha_k}\,, \tag{2.4}$$

where the coefficients $\{\alpha_k\}_k$ are called the normalizing constants. From (2.2) and (2.4) it follows that

$$m(\zeta) = \sum_k \frac{1}{\alpha_k(\lambda_k - \zeta)}\,. \tag{2.5}$$

The function m is meromorphic, and, since it is also Herglotz, its zeros and poles interlace, i.e., between two contiguous zeros there is only one pole and between two contiguous poles there is only one zero (see the proof of [13, Chap. 7, Thm. 1]).

Now, in the subspace $\delta_1^\perp$ of $l_2(\mathbb{N})$, consider the operator J_{T} which is the restriction of J to $\text{dom}(J) \cap \delta_1^\perp$. Note that J_T is a self-adjoint extension of the operator whose matrix representation with respect to the basis $\{\delta_k\}_{k=2}^\infty$ of the space $\delta_1^\perp$ is (1.3) with the first column and row removed. The following proposition is well known (see for instance [16]).

Proposition 2.1. *Under the assumption that $\sigma(J)$ is discrete, $\sigma(J)$ and $\sigma(J_{\text{T}})$ interlace. Moreover $\sigma(J)$ coincides with the set of poles of the function m and $\sigma(J_{\text{T}})$ is the set of its zeros.*

Proof. Clearly, one should only establish that the zeros and poles of m are as stated in the proposition. But this is a straightforward conclusion from the definition of the Weyl m-function and the formula

$$b_1^2 m_{\text{T}}(\zeta) = q_1 - \zeta - \frac{1}{m(\zeta)}\,, \tag{2.6}$$

where m_{T} is the Weyl m-function corresponding to J_{T}. Equation (2.6) is a particular case of [10, Eq. 2.15] or [20, Eq. 2.23]. $\qquad\square$

(C1) Convention for enumerating a sequence. Let S be an infinite countable set of real numbers without finite points of accumulation and M an infinite subset of consecutive integers such that there is a strictly increasing function $f : M \to S$ such that $f^{-1}(0) = 0$. We write $S = \{\lambda_k\}_{k \in M}$, where $\lambda_k = f(k)$. Note that M is semi-bounded from above (below) if and only if the same holds for S and that in $\{\lambda_k\}_{k \in M}$ only λ_0 is allowed to be zero.

Remark 1. Clearly, if two real sequences S, S' without finite accumulation points interlace, then one always can find M and functions $f : M \to S$ and $f' : M \to S'$ with the properties given in our convention (C1) such that, for any $k \in M$, either

$$\lambda_k < \lambda'_k < \lambda_{k+1} \quad \text{or} \quad \lambda'_k < \lambda_k < \lambda'_{k+1} \,,$$

where $\lambda_k = f(k)$ and $\lambda'_k = f'(k)$. If S is not semi-bounded, then both possibilities hold simultaneously.

The proof of the following proposition can be found in [8, Lem. 4.1] and [16, Sec. 4] and the starting point for it is [13, Chap. 7, Thm. 1].

Proposition 2.2. *Let J have discrete spectrum and assume that $\sigma(J) = \{\lambda_k\}_{k\in M}$, and $\sigma(J_\mathrm{T}) = \{\eta_k\}_{k\in M}$. Then*

$$m(\zeta) = C\frac{\zeta - \eta_0}{\zeta - \lambda_0} \prod_{\substack{k \in M \\ k \neq 0}} \left(1 - \frac{\zeta}{\eta_k}\right) \left(1 - \frac{\zeta}{\lambda_k}\right)^{-1} , \tag{2.7}$$

Moreover, $C < 0$ and

$$\eta_k < \lambda_k < \eta_{k+1} \,, \quad \forall k \in M \,, \tag{2.8}$$

if $\sigma(J)$ is semi-bounded from above, while, $C > 0$ and

$$\lambda_k < \eta_k < \lambda_{k+1} \,, \quad \forall k \in M \,, \tag{2.9}$$

otherwise.

3. Direct spectral analysis for J and $\widetilde{J}$

Let J and $\widetilde{J}$ be the operators defined in the Introduction. Since $J_\mathrm{T} = \widetilde{J}_\mathrm{T}$, where $\widetilde{J}_\mathrm{T}$ is the operator in the space $\delta_1^\perp$ obtained by restricting $\widetilde{J}$ to $\mathrm{dom}(\widetilde{J}) \cap \delta_1^\perp$, one obtains from (2.6) that

$$\theta^2 \left(\zeta + \frac{1}{m(\zeta)} + h\right) = \zeta + \frac{1}{\widetilde{m}(\zeta)} \,, \tag{3.1}$$

where $\widetilde{m}$ is the Weyl m-function corresponding to $\widetilde{J}$. Let us define the function

$$\mathfrak{m}(\zeta) := \frac{m(\zeta)}{\widetilde{m}(\zeta)} \tag{3.2}$$

Immediately from (3.1) one proves the following proposition. Prior to stating it, in order to simplify the writing of some expressions, let us introduce a constant that

will be used recurrently throughout the paper.

$$\gamma := \frac{\theta^2 h}{1 - \theta^2} \,. \tag{3.3}$$

Proposition 3.1. *Consider the Jacobi operator J and the operator $\widetilde{J}$ as given in (1.4) with $\theta \neq 1$. If J has discrete spectrum, then*

i) *the set of poles of $\mathfrak{m}$ is a subset of $\sigma(J)$ and the set of zeros is contained in $\sigma(\widetilde{J})$,*

ii) *$\gamma \in \sigma(J)$ if and only if $\gamma \in \sigma(\widetilde{J})$,*

iii) *the sets $\sigma(J)$ and $\sigma(\widetilde{J})$ can intersect only at γ.*

The following alternative expression for $\mathfrak{m}$:

$$\mathfrak{m}(\zeta) = (\theta^2 - 1)\,(\zeta - \gamma)\,m(\zeta) + \theta^2 \,, \tag{3.4}$$

which is obtained by combining (3.1) and (3.2), is the main ingredient in the proof of the following proposition.

Proposition 3.2. *Consider the Jacobi operator J and the operator $\widetilde{J}$ as given in (1.4) with $\theta \neq 1$. If J has discrete spectrum, then the spectra $\sigma(J)$, $\sigma(\widetilde{J})$ interlace in the intervals $(\gamma, +\infty)$ and $(-\infty, \gamma)$. Moreover, $\sigma(\widetilde{J})$ in the interval $(\gamma, +\infty)$, respectively $(-\infty, \gamma)$, is shifted with respect to $\sigma(J)$ to the left, respectively right, when $\theta < 1$ and to the right, respectively left, when $\theta > 1$.*

Remark 2. The set $\sigma(J) \cap (\gamma, +\infty)$, respectively $\sigma(J) \cap (-\infty, \gamma)$, may be empty and, then, there is no spectrum of $\widetilde{J}$ in $(\gamma, +\infty)$, respectively $(-\infty, \gamma)$. If λ is the only element in $\sigma(J) \cap (\gamma, +\infty)$, respectively $\sigma(J) \cap (-\infty, \gamma)$, then there is exactly one element of $\sigma(\widetilde{J})$ in $(\gamma, +\infty)$, respectively $(-\infty, \gamma)$.

Proof. Let us first prove that between two contiguous eigenvalues of J there is exactly one eigenvalue of $\widetilde{J}$. Assume that $\theta > 1$ and consider two contiguous eigenvalues $\lambda, \widehat{\lambda}$ of J such that $\gamma < \lambda < \widehat{\lambda}$. Then, by (2.5) and (3.4), one has

$$\lim_{\substack{t \to \widehat{\lambda}^- \\ t \in \mathbb{R}}} \mathfrak{m}(t) = +\infty \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} \mathfrak{m}(t) = -\infty \,.$$

The function $\mathfrak{m} \upharpoonright_{\mathbb{R}}$, should cross the 0-axis in $(\lambda, \widehat{\lambda})$ an odd number of times. Actually, it crosses the 0-axis only once. Indeed, if one assumes that $\mathfrak{m} \upharpoonright_{\mathbb{R}}$ crosses the 0-axis three or more times as in Fig. 3 (a), then, in view of Propositions 2.1 and 3.1, there would be at least two elements of $\sigma(J_{\mathrm{T}})$ in $(\lambda, \widehat{\lambda})$. Note that one crossing of the 0-axis and a tangential touch of it as in Fig. 3 (b) and (c) is also impossible since the poles of $\widetilde{m}$ are simple. Analogously, between two contiguous eigenvalues
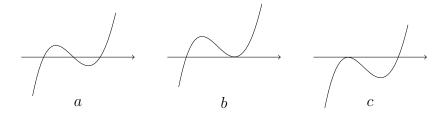
6

Figure 3: Impossible crossings of the 0-axis by $\mathfrak{m}$

of $\widetilde{J}$, $1/\mathfrak{m} \restriction_\mathbb{R}$ crosses the 0-axis exactly once. Thus, by means of Proposition 3.1, the interlacing of $\sigma(J)$ and $\sigma(\widetilde{J})$ in $(\gamma, +\infty)$ has been proven.

When $\theta < 1$, one has

$$\lim_{\substack{t \to \widehat{\lambda}^- \\ t \in \mathbb{R}}} \mathfrak{m}(t) = -\infty \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} \mathfrak{m}(t) = +\infty \,.$$

and by the same reasoning used above the interlacing of the spectra in $(\gamma, +\infty)$ is established. The interlacing in $(-\infty, \gamma)$ is proven analogously.

Let us now prove the second assertion of the proposition. To this end suppose first that $\gamma \notin \sigma(J)$ and observe that, under this assumption, (3.4) implies that

$$\mathfrak{m}(\gamma) = \theta^2 \,. \tag{3.5}$$

Let us now assume that the contiguous eigenvalues $\lambda, \widehat{\lambda}$ of J are such that

$$\lambda < \gamma < \widehat{\lambda} \,.$$

Under the premise that $\theta > 1$, we have

$$\lim_{\substack{t \to \widehat{\lambda}^- \\ t \in \mathbb{R}}} \mathfrak{m}(t) = +\infty \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} \mathfrak{m}(t) = +\infty \,. \tag{3.6}$$

In view of (3.5) and (3.6), if $\mathfrak{m} \restriction_\mathbb{R}$ crosses the 0-axis one time in the interval (λ, γ), it should cross it in (λ, γ) at least twice. The same is true for the interval $(\gamma, \widehat{\lambda})$. Note that $\mathfrak{m} \restriction_\mathbb{R}$ cannot tangentially touch the 0-axis due to the simplicity of its zeros. So, the assumption that $\mathfrak{m} \restriction_\mathbb{R}$ crosses the 0-axis, from what has already been proven above, would imply that in (λ, γ), respectively $(\gamma, \widehat{\lambda})$, there is at least one eigenvalue of J, which contradicts the fact that λ and $\widehat{\lambda}$ are contiguous. Thus, there is no crossing of the 0-axis by $\mathfrak{m} \restriction_\mathbb{R}$ in the interval $(\lambda, \widehat{\lambda})$, which means the absence of eigenvalues of $\widetilde{J}$ in $(\lambda, \widehat{\lambda})$. If now $\theta < 1$, instead of (3.6), one has

$$\lim_{\substack{t \to \widehat{\lambda}^- \\ t \in \mathbb{R}}} \mathfrak{m}(t) = -\infty \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} \mathfrak{m}(t) = -\infty \,.$$

7

From this asymptotic behavior, together with (3.5) and a similar reasoning as the one given above, it follows that $\mathfrak{m} \upharpoonright_{\mathbb{R}}$ crosses the 0-axis exactly once in (λ, γ) and once in $(\gamma, \widehat{\lambda})$.

The case when γ is in $\sigma(J)$ is treated analogously. Here one only has to take into account two things: firstly that now

$$\mathfrak{m}(\gamma) = \theta^2 + (\theta^2 - 1) \operatorname*{Res}_{\zeta = \gamma} m(\zeta) \tag{3.7}$$

and secondly, that, since $-[\operatorname{Res}_{\zeta=\gamma} m(\zeta)]^{-1}$ is the normalizing constant of J corresponding to the eigenvalue γ (see (2.5)), one has

$$\mathfrak{m}(\gamma) > 0$$

either when $\theta > 1$ or $\theta < 1$. $\qquad\square$

Remark 3. Although, the case $\theta = 1$ reduces to an additive rank-one perturbation, the well known interlacing property (see for instance the proof of [16, Thm. 3.3]) cannot be obtained from Proposition 3.2 by a limiting procedure since the limit of $\gamma(\theta)$ when $\theta \to 1$ does not exist (see (3.3)).

Remark 4. Let the positive number $\theta \neq 1$ and $h \in \mathbb{R}$. It is straightforward to verify that, for $\sigma(J)$ and $\sigma(\widetilde{J})$, there exist a set M and functions $f : M \to \sigma(J)$ and $\widetilde{f} : M \to \sigma(\widetilde{J})$, with the properties given in our convention (C1) for enumerating sequences, such that the following conditions hold under the assumption that $\lambda_k = f(k)$ and $\mu_k = \widetilde{f}(k)$:

$$\lambda_k < \mu_k < \lambda_{k+1} \quad \text{in } (\gamma, +\infty), \qquad \lambda_{k-1} < \mu_k < \lambda_k \quad \text{in } (-\infty, \gamma), \tag{3.8}$$

when $\theta > 1$, and

$$\mu_k < \lambda_k < \mu_{k+1} \quad \text{in } (\gamma, +\infty), \qquad \mu_{k-1} < \lambda_k < \mu_k \quad \text{in } (-\infty, \gamma), \tag{3.9}$$

if $\theta < 1$. Here, implicitly, the intersection of $\sigma(J)$ with the semi-infinite intervals is not empty, but we are also considering the case when the intersection with one of the semi-infinite intervals is empty (see Remark 2). Also, we are not excluding the case when γ is in $\sigma(J)$ for which there is $k_0 \in M$ such that $\lambda_{k_0} = \mu_{k_0} = \gamma$.

Proposition 3.3. *Suppose that $h \in \mathbb{R}$ is such that if $\theta = 1$ then $h \neq 0$. Let J have discrete spectrum and assume that $\sigma(J) = \{\lambda_k\}_{k \in M}$ and $\sigma(\widetilde{J}) = \{\mu_k\}_{k \in M}$, where the sequences have been arranged according to Remark 4 if $\theta \neq 1$ and according to Remark 1 otherwise. Then*

$$\sum_{k \in M} (\mu_k - \lambda_k) = h + q_1(\theta^2 - 1) \tag{3.10}$$

Proof. Consider the sequence $\{\eta_k\}_{k \in M}$ being the spectrum of $\widehat{J}$, where

$$\widehat{J} := J + h \langle \delta_1, \cdot \rangle \delta_1 \,.$$

In the proof of [16, Thm. 3.4] it is shown that

$$\sum_{k \in M} (\eta_k - \lambda_k) = h \,,$$

where $\eta_k > \lambda_k$ for all $k \in M$ when $h > 0$ and $\eta_k \leq \lambda_k$ for all $k \in M$ otherwise. On the other hand, by [8, Prop. 4.1], one has

$$\sum_{k \in M} (\mu_k - \eta_k) = q_1(\theta^2 - 1) \,,$$

where the enumeration obeys [8, Remark 5] if $\theta \neq 1$.

Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of M, such that $M_n \subset M_{n+1}$ and $\cup_n M_n = M$. Then the assertion follows from the linearity of the limit

$$\lim_{n \to \infty} \left[\sum_{k \in M_n} (\mu_k - \eta_k) + \sum_{k \in M_n} (\eta_k - \lambda_k) \right] \,,$$

as soon as one notices that the enumeration has been done according to Remark 4 when $\theta \neq 1$. $\square$

Proposition 3.4. *Suppose that $h \in \mathbb{R}$ is such that if $\theta = 1$ then $h \neq 0$. Let the Jacobi operator J have discrete spectrum and assume that $\sigma(J) = \{\lambda_k\}_{k \in M}$ and $\sigma(\widetilde{J}) = \{\mu_k\}_{k \in M}$, where $\widetilde{J}$ is given by (1.4), and the sequences have been arranged according to Remark 4 if $\theta \neq 1$ and according to Remark 1 otherwise. Then,*

$$\mathfrak{m}(\zeta) = \prod_{k \in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k} \,.$$

Proof. When $\theta = 1$ the assertion follows from the proof of [16, Thm. 3.4] If $\theta \neq 1$, the proof repeats the one of [8, Prop. 4.2], so we omit some details that the reader can reestablish from [8, Prop. 4.2] if necessary.

From Proposition 2.2 and (3.2) it follows that

$$\mathfrak{m}(\zeta) = C \frac{\zeta - \mu_0}{\zeta - \lambda_0} \prod_{\substack{k \in M \\ k \neq 0}} \left(1 - \frac{\zeta}{\mu_k} \right) \left(1 - \frac{\zeta}{\lambda_k} \right)^{-1} \,.$$

By Proposition 3.3, one actually has

$$\mathfrak{m}(\zeta) = C\frac{\zeta - \mu_0}{\zeta - \lambda_0} \prod_{\substack{k \in M \\ k \neq 0}} \frac{\lambda_k}{\mu_k} \prod_{\substack{k \in M \\ k \neq 0}} \frac{\zeta - \mu_k}{\zeta - \lambda_k} \tag{3.11}$$

Indeed, (3.10) implies the convergence of the products in (3.11).

Now, the assertion of the proposition follows from

$$\lim_{\substack{\zeta \to \infty \\ \operatorname{Im}\zeta \geq \epsilon > 0}} \mathfrak{m}(\zeta) = 1 \quad \text{and} \quad \lim_{\substack{\zeta \to \infty \\ \operatorname{Im}\zeta \geq \epsilon}} \prod_{k \in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k} = 1 \,. \tag{3.12}$$

The first limit is obtained from (2.3) and (3.4). The second one is a consequence of the uniform convergence of

$$\prod_{k \in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k}$$

in compacts of $\mathbb{C} \setminus \mathbb{R}$, which, in its turn, can be proven on the basis of (3.10). $\qquad\square$

4. Inverse spectral analysis for J and $\widetilde{J}$

In this section we give results on reconstruction of the operator J from its spectrum and the one of $\widetilde{J}$. Additionally, we provide necessary and sufficient conditions for two sequences to be the spectra of the operators J and $\widetilde{J}$. Finally, we discuss isospectral operators within the perturbed family of Jacobi operators.

Theorem 4.1. *Let the Jacobi operator J have discrete spectrum and $\widetilde{J}$ be as in (1.4) with $\theta \neq 1$. If γ is not in $\sigma(J)$, then the sets $\sigma(J)$, $\sigma(\widetilde{J})$, and the constant γ uniquely determine the matrix (1.3), the parameters θ and h, and the boundary condition at infinity if necessary (i. e. if J_0 turns out to be non-essentially self-adjoint).*

Proof. In view of what has been said in Section 2, it suffices to show that the input data uniquely determine the Weyl m-function of J, and the parameters θ and h.

On the basis of Proposition 3.4, one construct $\mathfrak{m}$ from the sets $\sigma(J)$ and $\sigma(\widetilde{J})$. Then, since $\gamma \notin \sigma(J)$, it follows from (3.4) that $\mathfrak{m}(\gamma) = \theta^2$. Now, the constants γ and θ allow to find h. Finally, by means of (3.4), one determines the function m. $\qquad\square$

Theorem 4.2. *Let the Jacobi operator J have discrete spectrum and $\widetilde{J}$ be as in (1.4) with $\theta \neq 1$. Assuming that γ is in $\sigma(J)$, suppose that one is given the sets $\sigma(J)$, $\sigma(\widetilde{J})$ and one of the following constants*

(a) θ,   (b) the normalizing constant corresponding to γ,   (c) h,

then one recovers uniquely the matrix (1.3), the constant h in case (a), θ and h in case (b), θ in case (c), and the boundary condition at infinity if necessary (i. e. if J_0 turns out to be non-essentially self-adjoint).

10

Proof. The proof is similar to the one of Theorem 4.1. The sets $\sigma(J)$ and $\sigma(\widetilde{J})$ determine $\mathfrak{m}$ and, then, one should obtain from it the function m using either the constant θ or the normalizing constant corresponding to γ. From Proposition 3.1 it follows that

$$\sigma(J) \cap \sigma(\widetilde{J}) = \{\gamma\}.$$

Thus θ or h determine θ and h. On the other hand, from (3.4) and taking into account that $\gamma \in \sigma(J)$, we obtain

$$\mathfrak{m}(\gamma) = \theta^2 - \alpha^{-1}(\theta^2 - 1), \tag{4.1}$$

where α is the normalizing constant corresponding to the eigenvalue γ. $\qquad\square$

Suppose now that we are required to enumerate the sequences $\sigma(J)$ and $\sigma(\widetilde{J})$ according to Remark 4, but no information is given about the constant γ other than it is not in $\sigma(J)$. Clearly, one does not need this number for accomplishing this task, as is stated in the following remark.

Remark 5. Assuming that J has discrete spectrum, let $S = \sigma(J)$, $\widetilde{S} = \sigma(\widetilde{J})$ be disjoint, and take any $\theta \neq 1$ and $h \in \mathbb{R}$. It follows from Proposition 3.2 that one can find a set M and functions $f : M \to S$, $\widetilde{f} : M \to \widetilde{S}$, with the properties given in our convention for enumerating sequences (C1), such that there exists a unique $k_0 \in M$ for which the following conditions hold under the assumption that $\lambda_k = f(k)$ and $\mu_k = \widetilde{f}(k)$ for $k \in M$:

a) $\widetilde{S} \cap (\lambda_{k_0-1}, \lambda_{k_0}) = \emptyset$,

b) $\lambda_k < \mu_k < \lambda_{k+1}$, $\forall k \geq k_0$,

c) $\lambda_{k-1} < \mu_k < \lambda_k$, $\forall k < k_0$,

if $\theta > 1$, and

a') $\widetilde{S} \cap (\lambda_{k_0-1}, \lambda_{k_0}) = \{\mu_{k_0-1}, \mu_{k_0}\}$,

b') $\lambda_k < \mu_{k+1} < \lambda_{k+1}$, $\forall k \geq k_0$,

c') $\lambda_{k-1} < \mu_{k-1} < \lambda_k$, $\forall k < k_0$.

if $\theta < 1$

Before we state the necessary and sufficient conditions for two sequences to be the spectra of a Jacobi operator J and its perturbation $\widetilde{J}$, let us introduce the following parameterized sequence. Suppose that two sequences $\{\lambda_k\}_{k\in M}$ and $\{\mu_k\}_{k\in M}$ are given and enumerated by the set M as convened before. Whenever the series

$$\sum_{k \in M} (\mu_k - \lambda_k)$$

11

converges, the sequence

$$\tau_n(\omega) := \frac{(\mu_n - \lambda_n) \displaystyle\prod_{\substack{k \in M \\ k \neq n}} \dfrac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}}{(\lambda_n - \omega) \left(\displaystyle\prod_{k \in M} \dfrac{\omega - \mu_k}{\omega - \lambda_k} - 1 \right)}, \quad \forall n \in M. \tag{4.2}$$

is well defined for any $\omega \in \mathbb{R}$.

Theorem 4.3. *Let S and $\widetilde{S}$ be two disjoint infinite real sequences without finite points of accumulation. There exist $\theta > 1$, $h \in \mathbb{R}$, and a matrix (1.3) such that $S = \sigma(J) \not\ni \gamma$ and $\widetilde{S} = \sigma(\widetilde{J})$ if and only if the following conditions hold:*

i) *There exist a set M and functions $h : M \to S$, $\widetilde{h} : M \to \widetilde{S}$ with the properties given in our convention for enumerating sequences (C1) such that one can find a unique $k_0 \in M$ for which a),b),c) of Remark 5 take place with $\lambda_k = h(k)$ and $\mu_k = \widetilde{h}(k)$.*

ii) *The series $\sum_{k \in M} (\mu_k - \lambda_k)$ is convergent.*

iii) *There exists $\widehat{\omega} \in (\lambda_{k_0-1}, \lambda_{k_0})$ such that*

a) *For $m = 0, 1, 2, \ldots$, the series*

$$\sum_{k \in M} \lambda_k^{2m} \tau_k(\widehat{\omega}) \quad converges.$$

b) *If a sequence of complex numbers $\{\beta_k\}_{k \in M}$ is such that the series*

$$\sum_{k \in M} |\beta_k|^2 \tau_k(\widehat{\omega}) \quad converges$$

and, for $m = 0, 1, 2, \ldots$,

$$\sum_{k \in M} \beta_k \lambda_k^m \tau_k(\widehat{\omega}) = 0,$$

then $\beta_k = 0$ for all $k \in M$.

Proof. Due to Propositions 3.2 and 3.3, for proving the necessity of the conditions, it only remains to show the existence of $\widehat{\omega}$ in $(\lambda_{k_0-1}, \lambda_{k_0})$ such that $\tau_n(\widehat{\omega}) = \alpha_n^{-1}$ for all $n \in M$. Indeed *iiia*) and *iiib*) will follow from the fact that all moments of the spectral measure (2.4) exist and that the polynomials are dense in $L_2(\mathbb{R}, \rho)$.

Clearly, $\gamma \in (\lambda_{k_0-1}, \lambda_{k_0})$, so let $\widehat{\omega} = \gamma$. Then, from (2.5),(3.4), and Proposition 3.4, it follows that

$$
\begin{aligned}
\alpha_n^{-1} &= \frac{1}{\theta^2 - 1} \lim_{\zeta \to \lambda_n} \frac{\lambda_n - \zeta}{\zeta - \gamma} \mathfrak{m}(\zeta) \\
&= \frac{\mu_n - \lambda_n}{(\lambda_n - \gamma)(\theta^2 - 1)} \prod_{\substack{k \in M \\ k \neq n}} \frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k} .
\end{aligned}
\tag{4.3}
$$

Hence, taking into account (3.5), one verifies that $\tau_n(\widehat{\omega}) = \alpha_n^{-1}$.

We now prove that conditions i), ii), $iiia$), and $iiib$) are sufficient.

The condition i) implies that

$$
\frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k} > 0 \qquad \forall k \in M , \ k \neq n
$$

On the other hand, by ii) one can define the number

$$
\vartheta = + \sqrt{\prod_{k \in M} \frac{\widehat{\omega} - \mu_k}{\widehat{\omega} - \lambda_k}}
\tag{4.4}
$$

which is clearly strictly greater than 1 since if $\widehat{\omega} \in (\lambda_{k_0-1}, \lambda_{k_0})$, then $|\widehat{\omega} - \mu_k| > |\widehat{\omega} - \lambda_k|$ for all $k \in M$. Thus,

$$
\frac{\mu_n - \lambda_n}{(\lambda_n - \widehat{\omega})(\vartheta^2 - 1)} > 0 \qquad \forall n \in M
$$

Hence, for all $n \in M$, $\tau_n(\widehat{\omega}) > 0$, so define the function

$$
\rho(t) := \sum_{\lambda_k < t} \tau_k(\widehat{\omega}) .
\tag{4.5}
$$

It follows from $iiia$) that the moments of the measure corresponding to ρ are finite.

Now, on the basis of i) and ii), define the meromorphic functions

$$
\check{\mathfrak{m}}(\zeta) := \prod_{k \in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k}
$$

and

$$
\check{m}(\zeta) := \frac{\check{\mathfrak{m}}(\zeta) - \vartheta^2}{(\zeta - \widehat{\omega})(\vartheta^2 - 1)} .
\tag{4.6}
$$

Thus, taking into account (4.2), one has

$$
\operatorname*{Res}_{\zeta = \lambda_n} \check{m}(\zeta) = (\vartheta^2 - 1)^{-1} \lim_{\zeta \to \lambda_n} \frac{\zeta - \lambda_n}{\zeta - \widehat{\omega}} \check{\mathfrak{m}}(\zeta) = -\tau_n(\widehat{\omega}) .
\tag{4.7}
$$

13

Therefore, on the basis of the second equality in (3.12),

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\ge\epsilon>0}} \check{m}(\zeta) = \left(\vartheta^2-1\right)^{-1} \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\ge\epsilon>0}} \frac{\check{\mathfrak{m}}(\zeta)}{\zeta-\widehat{\omega}} = 0 \tag{4.8}$$

By (4.7) and (4.8), [13, Chap. VII, Sec.1 Theorem 2] implies that

$$\check{m}(\zeta) = \sum_{k\in M} \frac{\tau_k(\widehat{\omega})}{\lambda_k-\zeta}\,. \tag{4.9}$$

On the other hand, using again the first equality in (3.12), one obtains

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\ge\epsilon>0}} \zeta\check{m}(\zeta) = \left(\vartheta^2-1\right)^{-1} \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\ge\epsilon>0}} \frac{1}{1-\widehat{\omega}/\zeta}\left(\check{\mathfrak{m}}(\zeta)-\vartheta^2\right) = -1\,.$$

But

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\ge\epsilon>0}} \zeta\check{m}(\zeta) = -\sum_{k\in M} \tau_k(\widehat{\omega})\,,$$

so it has been proven that, for the function given in (4.5),

$$\int_{\mathbb{R}} d\rho(t) = 1\,.$$

Thus the measure corresponding to ρ is appropriately normalized and, because of *iiia)*, all the moments exist, so in $L_2(\mathbb{R},\rho)$ apply the Gram-Schmidt procedure of orthonormalization to the sequence $\{t^k\}_{k=0}^\infty$ to obtain a Jacobi matrix as was explained in the Section 2. Consider the operator J_0 with domain $l_{\mathrm{fin}}(\mathbb{N})$ generated by this Jacobi matrix as explained in the Introduction. Now, as a consequence of condition *iiib)*, which means that the polynomials are dense in $L_2(\mathbb{R},\rho)$, ρ corresponds to the resolution of the identity of a self-adjoint extension J of J_0 [17, Prop. 4.15].

Finally, consider

$$\widetilde{J} = J + \left[q_1(\theta^2-1)+\theta^2 h\right]\langle\delta_1,\cdot\rangle\,\delta_1 + b_1(\theta-1)(\langle\delta_1,\cdot\rangle\,\delta_2 + \langle\delta_2,\cdot\rangle\,\delta_1)\,, \tag{4.10}$$

where

$$\theta = \vartheta\,, \qquad h = \widehat{\omega}\frac{1-\vartheta^2}{\vartheta^2}\,.$$

By construction the sequence $\{\lambda_k\}_{k\in M}$ is the spectrum of J. For the proof to be complete it only remains to show that $\{\mu_k\}_{k\in M}$ is the spectrum of $\widetilde{J}$. For the function given in (3.2), taking into account (2.5) and (3.4), one has

$$\mathfrak{m}(\zeta) = \theta^2 + (\zeta-\widehat{\omega})\left(\theta^2-1\right)\sum_{k\in M} \frac{1}{\alpha_k(\lambda_k-\zeta)}\,.$$

On the other hand, from (4.6) and (4.9), it follows that

$$\check{\mathfrak{m}}(\zeta) = \vartheta^2 + (\zeta - \widehat{\omega}) \left(\vartheta^2 - 1\right) \sum_{k \in M} \frac{\tau_k(\widehat{\omega})}{\lambda_k - \zeta} \, .$$

But $\theta = \vartheta$ and we have already proven that $\alpha_k^{-1} = \tau_k(\widehat{\omega})$ for $k \in M$. Thus $\mathfrak{m} = \check{\mathfrak{m}}$, meaning that the zeros of $\mathfrak{m}$ are given by the sequence $\{\mu_k\}_{k \in M}$. $\qquad\square$

Remark 6. In accordance with Theorem 4.1, the proof of Theorem 4.3 shows that the sequences S, $\widetilde{S}$, and the parameter $\widehat{\omega}$ satisfying *i)*, *ii)*, and *iii)*, uniquely determine the perturbation parameters θ and h, and the matrix (1.3) with the boundary condition at infinity if necessary. Thus, S, $\widetilde{S}$, and $\widehat{\omega}$ amount to the complete input data for solving uniquely the inverse spectral problem.

Remark 7. Clearly, the assertion of Theorem 4.3 holds true if one substitutes $\theta > 1$ by $\theta < 1$, conditions a), b), c) by a'), b'), c'), and $\widehat{\omega} \in (\lambda_{k_0-1}, \lambda_{k_0})$ by $\widehat{\omega} \in (\mu_{k_0-1}, \mu_{k_0})$.

Proposition 4.1. *Let S and $\widetilde{S}$ be two infinite real sequences without finite points of accumulation that satisfy i) and ii) of Theorem 4.3. Suppose that there is $\widehat{\omega} \in (\lambda_{k_0-1}, \lambda_{k_0})$ so that the sequence $\{\tau_n(\widehat{\omega})\}_{n \in M}$ satisfies iiia) and iiib) of Theorem 4.3, then $\{\tau_n(\omega)\}_{n \in M}$ also satisfies iiia) and iiib) for all $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$.*

Proof. Let

$$\rho_\omega(t) := \sum_{\lambda_k < t} \tau_k(\omega) \, .$$

As in the proof of Theorem 4.3 one verifies that if $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$, ρ_ω is a positive non-decreasing function and that

$$\int_{\mathbb{R}} d\rho_\omega(t) = 1 \, , \qquad \forall \omega \in (\lambda_{k_0-1}, \lambda_{k_0}) \, .$$

By hypothesis all the moments of the measure $\rho_{\widehat{\omega}}$ are finite and the polynomials are dense in $L_2(\mathbb{R}, \rho_{\widehat{\omega}})$. For the proposition to be proven, one needs to show that this implies that all the moments of the measure ρ_ω are finite and the polynomials are dense in $L_2(\mathbb{R}, \rho_\omega)$ for all $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$. But, since the support of the measure is the same for all $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$, this implication will indeed take place if for any fixed $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$ there are positive constants C_1, C_2 such that

$$C_1 \tau_n(\widehat{\omega}) \le \tau_n(\omega) \le C_2 \tau_n(\widehat{\omega}) \, , \quad \forall n \in M \, . \tag{4.11}$$

Fix $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$. From (4.2), it follows that

$$\tau_n(\omega) = C \frac{\lambda_n - \widehat{\omega}}{\lambda_n - \omega} \tau_n(\widehat{\omega}) \, , \tag{4.12}$$

where $C = \left| \prod_{k \in M} \frac{\omega - \mu_k}{\omega - \lambda_k} - 1 \right|^{-1} \left| \prod_{k \in M} \frac{\widehat{\omega} - \mu_k}{\widehat{\omega} - \lambda_k} - 1 \right|$. By elementary estimates of $\left| \frac{\lambda_n - \widehat{\omega}}{\lambda_n - \omega} \right|$, one verifies from (4.12) that if

$$C_1 := \frac{\min\{|\lambda_{k_0} - \widehat{\omega}|, |\lambda_{k_0-1} - \widehat{\omega}|\}}{\max\{|\lambda_{k_0} - \omega|, |\lambda_{k_0-1} - \omega|\}} \qquad C_2 := 1 + \frac{\max\{|\lambda_{k_0} - \widehat{\omega}|, |\lambda_{k_0-1} - \widehat{\omega}|\}}{\min\{|\lambda_{k_0} - \omega|, |\lambda_{k_0-1} - \omega|\}},$$

then (4.11) holds. $\qquad\square$

Remark 8. As in Remark 7, the assertion of Proposition 4.1 holds true if one assumes that i) is satisfied with a'), b'),c') instead of a), b), c) and substitute the interval $(\lambda_{k_0-1}, \lambda_{k_0})$ by (μ_{k_0-1}, μ_{k_0}).

Theorem 4.4. *Let $\theta \neq 1$ and assume that the disjoint sets $\sigma(J)$ and $\sigma(\widetilde{J})$ are enumerated according to Remark 5 with a), b), c) if $\theta > 1$, and with a'), b'), c') otherwise. Then, for any $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$ when $\theta > 1$ and for any $\omega \in (\mu_{k_0-1}, \mu_{k_0})$ when $\theta < 1$, there is a matrix*

$$\begin{pmatrix} q_1' & b_1' & 0 & 0 & \cdots \\ b_1' & q_2' & b_2' & 0 & \cdots \\ 0 & b_2' & q_3' & b_3' & \\ 0 & 0 & b_3' & q_4' & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \qquad (4.13)$$

where $q_n' \in \mathbb{R}$ and $b_n' > 0$ for all $n \in \mathbb{N}$, and a self-adjoint extension J' of the operator whose matrix representation is (4.13), such that $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J}') = \sigma(\widetilde{J})$, where

$$\widetilde{J}' := J' + [q_1'((\theta')^2 - 1) + (\theta')^2 h'] \langle \delta_1, \cdot \rangle \delta_1 + b_1'(\theta' - 1)(\langle \delta_1, \cdot \rangle \delta_2 + \langle \delta_2, \cdot \rangle \delta_1) \quad (4.14)$$

with

$$\theta' := +\sqrt{\mathfrak{m}(\omega)}, \qquad h' := \omega \frac{1 - \mathfrak{m}(\omega)}{\mathfrak{m}(\omega)}.$$

Proof. We prove the assertion for $\theta > 1$. The other case is completely analogous, one only has to take into account Remarks 7 and 8. By Theorem 4.3, it follows that $\sigma(J)$ and $\sigma(\widetilde{J})$ satisfy i), ii), iiia), and iiib). Then, from Proposition 4.1, iiia) and iiib) are satisfied for any $\omega \in (\lambda_{k_0-1}, \lambda_{k_0})$. Now, again by Theorem 4.3, there are operators J' and $\widetilde{J}'$ such that their spectra coincide with $\sigma(J)$ and $\sigma(\widetilde{J})$. $\qquad\square$

Lemma 4.1. *Let $\theta \neq 1$ and assume that the disjoint sets $\sigma(J)$ and $\sigma(\widetilde{J})$ are enumerated according to Remark 5 with a), b), c) if $\theta > 1$, and with a'), b'), c') otherwise. Then, the equation*

$$\mathfrak{m} \restriction_{(\lambda_{k_0-1}, \lambda_{k_0})} (s) = \theta^2 \qquad (4.15)$$

has only the solutions $s = \gamma$ and $s = \widehat{\gamma}$, where $\widehat{\gamma}$ is the only point in $\sigma(J_{\mathrm{T}}) \cap (\lambda_{k_0-1}, \lambda_{k_0})$. Moreover, if $\gamma = \widehat{\gamma}$, then γ is a local extremum of $\mathfrak{m} \restriction_{(\lambda_{k_0-1}, \lambda_{k_0})}$.

16

Proof. First notice that γ is in $(\lambda_{k_0-1}, \lambda_{k_0})$ if $\theta > 1$ and in (μ_{k_0-1}, μ_{k_0}) otherwise. By Proposition 2.1 the set $\sigma(J_{\mathrm{T}}) \cap (\lambda_{k_0-1}, \lambda_{k_0})$, has only one element. If $\theta < 1$, since $J_{\mathrm{T}} = \widetilde{J}_{\mathrm{T}}$, this only element is actually in (μ_{k_0-1}, μ_{k_0}). Moreover, when $\theta < 1$, by what was said in the proof of Proposition 3.2, $m \upharpoonright_{\mathbb{R}}$ takes negative values outside (μ_{k_0-1}, μ_{k_0}). Now, from (3.4), the solutions of (4.15) are the zeros of $(\zeta - \gamma)m(\zeta)$ which are γ and $\widehat{\gamma}$. Clearly, if $\gamma = \widehat{\gamma}$, the function $(\zeta - \gamma)m(\zeta)$ has a zero of multiplicity two which implies the second assertion. $\qquad\square$

Lemma 4.2. *Let $\theta \neq 1$ and assume that the disjoint sets $\sigma(J)$ and $\sigma(\widetilde{J})$ are enumerated according to Remark 5 with a), b), c) if $\theta > 1$, and with a'), b'), c') otherwise. Then, the function $\mathfrak{m} \upharpoonright_{(\lambda_{k_0-1}, \lambda_{k_0})}$ has only one local extremum in $(\lambda_{k_0-1}, \lambda_{k_0})$ when $\theta > 1$, and in (μ_{k_0-1}, μ_{k_0}) when $\theta < 1$, which turns out to be a global minimum greater than 1 if $\theta > 1$, and a global maximum less that 1 if $\theta < 1$.*

Proof. Suppose that $\theta > 1$ and that $\mathfrak{m} \upharpoonright_{(\lambda_{k_0-1}, \lambda_{k_0})}$ has more than one local extremum. Then one verifies that there are three different points $\omega_1, \omega_2, \omega_3$ in $(\lambda_{k_0-1}, \lambda_{k_0})$ such that

$$\mathfrak{m}(\omega_1) = \mathfrak{m}(\omega_2) = \mathfrak{m}(\omega_3).$$

By Theorem 4.4, for ω_1 there are Jacobi operators J' and $\widetilde{J}'$ such that $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J}') = \sigma(\widetilde{J})$. Let $\mathfrak{n}$ be the quotient of the Weyl m-function of J' and the Weyl m-function of $\widetilde{J}'$. By Proposition 3.4, $\mathfrak{m} = \mathfrak{n}$. Hence, on the basis of Theorem 4.4, it follows that

$$\mathfrak{n}(\omega_1) = \mathfrak{n}(\omega_2) = \mathfrak{n}(\omega_3) = (\theta')^2. \tag{4.16}$$

On the other hand, Lemma 4.1 tells us that the equation

$$\mathfrak{n} \upharpoonright_{(\lambda_{k_0-1}, \lambda_{k_0})} (s) = (\theta')^2,$$

where $\theta' = +\sqrt{\mathfrak{m}(\omega_1)}$, has only the solutions ω_1 and the only element of $\sigma(J'_{\mathrm{T}})$ in $(\lambda_{k_0-1}, \lambda_{k_0})$. This is in contradiction with (4.16).

Thus there is only one extremum of $\mathfrak{m} \upharpoonright_{(\lambda_{k_0-1}, \lambda_{k_0})}$ when $\theta > 1$. The same reasoning given above, but replacing all appearances of the interval $(\lambda_{k_0-1}, \lambda_{k_0})$ by (μ_{k_0-1}, μ_{k_0}), works for the case $\theta < 1$.

Now, on the basis of the behavior of $\mathfrak{m}$ in the interval $(\lambda_{k_0-1}, \lambda_{k_0})$ if $\theta > 1$, and in (μ_{k_0-1}, μ_{k_0}) if $\theta < 1$, given in the proof of Proposition 3.2, one completes the proof. $\qquad\square$

Theorem 4.5. *Under the assumptions of Lemma 4.1, if $\gamma \neq \widehat{\gamma}$, then there are exactly two different matrices (1.3) and (4.13) such that $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J}') = \sigma(\widetilde{J})$ with $\theta = \theta'$. If $\gamma = \widehat{\gamma}$, then for all operators $J' \neq J$ for which $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J}') = \sigma(\widetilde{J})$ it turns out that $\theta \neq \theta'$.*

Proof. Due to Theorem 4.4 and Lemmas 4.1 and 4.2, the proof is straightforward. $\qquad\square$

Remark 9. Clearly, the condition $\gamma = \widehat{\gamma}$ is equivalent to γ being equal to the minimum of $\mathfrak{m} \restriction_{(\lambda_{k_0-1}, \lambda_{k_0})}$ if $\theta > 1$, and being equal to the maximum of $\mathfrak{m} \restriction_{(\mu_{k_0-1}, \mu_{k_0})}$ if $\theta < 1$.

Let us now reformulate and summarize some of our results in terms of the mass-spring systems mentioned in the Introduction.

Suppose that one knows the spectrum of the Jacobi operator corresponding to the mass-spring system given in Fig. 1, and then, after carrying out a mass-spring perturbation on the system as illustrated in Fig. 2, one is given the new spectrum, which does not intersect with the first one. Clearly, by the spectra alone, one determines if Δm is positive or negative (see Proposition 3.2). For definiteness, suppose that $\Delta m > 0$. If no more information is given, then for any value of the ratio of masses $\theta \in (0, \max_{t \in (\mu_{k_0-1}, \mu_{k_0})} \mathfrak{m}(t)]$ there are mass-spring systems corresponding to Figs. 1 and 2 having the measured spectra (see Theorem 4.4). However, when one knows the ratio of masses θ then, in general, there are only two mass-spring systems corresponding to Fig. 1 that comply with the conditions after the corresponding perturbation (see Theorem 4.5). Moreover, if

$$\theta = \max_{t \in (\mu_{k_0-1}, \mu_{k_0})} \mathfrak{m}(t),$$

there is only one system with the required properties (see Theorem 4.5).

Let us now turn to the case when $\gamma \in \sigma(J)$ or, equivalently, when the spectra of J and $\widetilde{J}$ intersect. Thus, according to Remark 4, consider the sequences $\{\lambda_k\}_{k \in M}$ and $\{\mu_k\}_{k \in M}$ such that $\lambda_{k_0} = \mu_{k_0} = \gamma$. If

$$\sum_{k \in M} (\mu_k - \lambda_k)$$

converges, then, for any $\omega \in \mathbb{R}$ and $n \in M$, one defines

$$v_n(\omega) := \begin{cases} \dfrac{(\mu_n - \lambda_n)}{(\lambda_n - \gamma)(\omega - 1)} \displaystyle\prod_{\substack{k \in M \\ k \neq n}} \dfrac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}, & n \neq k_0 \\[2em] (\omega - 1)^{-1} \left(\omega - \displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \dfrac{\gamma - \mu_k}{\gamma - \lambda_k} \right) & n = k_0 \end{cases} \tag{4.17}$$

Theorem 4.6. *Let S and $\widetilde{S}$ be two infinite real sequences without finite points of accumulation such that $S \cap \widetilde{S} = \{\gamma\}$. There exist $\theta > 1$, $h \in \mathbb{R}$, and a matrix (1.3) such that $S = \sigma(J)$ and $\widetilde{S} = \sigma(\widetilde{J})$ if and only if the following conditions hold:*

I) There exist a set M and functions $h : M \to S$, $\widetilde{h} : M \to \widetilde{S}$ with the properties given in Remark 4 such that (3.8) holds and there is a $k_0 \in M$ such that $\lambda_{k_0} = \mu_{k_0} = \gamma$.

II) The series $\sum_{k \in M}(\mu_k - \lambda_k)$ is convergent.

III) There exists $\widehat{\omega} > \prod_{\substack{k \in M \\ k \neq k_0}} \dfrac{\gamma - \mu_k}{\gamma - \lambda_k}$ such that

    a) For $m = 0, 1, 2, \ldots$, the series

$$\sum_{k \in M} \lambda_k^{2m} v_k(\widehat{\omega}) \quad \text{converges.}$$

    b) If a sequence of complex numbers $\{\beta_k\}_{k \in M}$ is such that the series

$$\sum_{k \in M} |\beta_k|^2 \, v_k(\widehat{\omega}) \quad \text{converges}$$

    and, for $m = 0, 1, 2, \ldots$,

$$\sum_{k \in M} \beta_k \lambda_k^m v_k(\widehat{\omega}) = 0 \,,$$

    then $\beta_k = 0$ for all $k \in M$.

Proof. For proving the necessity of the conditions, in view of Propositions 3.2 and 3.3, one only needs to show the existence of $\widehat{\omega}$ strictly greater than $\mathfrak{m}(\gamma)$ such that $v_k(\widehat{\omega}) = \alpha_k^{-1}$ for all $k \in M$. From (4.1) and the properties of the normalizing constants, it follows that

$$1 < \mathfrak{m}(\gamma) < \theta^2 \,. \tag{4.18}$$

Let $\widehat{\omega} = \theta^2$, then (4.3) yields $v_k(\widehat{\omega}) = \alpha_k^{-1}$ for $k \in M$, $k \neq k_0$. Moreover, (4.1) implies that $v_{k_0}(\widehat{\omega}) = \alpha_{k_0}^{-1}$.

Let us now prove that *I)*, *II)*, *IIIa)*, and *IIIb)* are sufficient.

It follows from (3.8) that $|\gamma - \mu_k| > |\gamma - \lambda_k|$ for any $k \in M \setminus \{k_0\}$. Since $\gamma - \mu_k$ and $\gamma - \lambda_k$ have the same sign,

$$\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \mu_k}{\gamma - \lambda_k} > 1 \,.$$

Thus $\widehat{\omega} > 1$ and $v_{k_0}(\widehat{\omega}) > 0$. Now fix $n \in \mathbb{N}$, $n \neq k_0$. By *I)* one has

$$\frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k} > 0 \qquad \forall k \in M \,, \; k \neq n \,.$$

Since $\mu_n - \lambda_n$ and $\lambda_n - \gamma$ are positive or negative simultaneously, we conclude that

$$v_n(\widehat{\omega}) > 0 \,, \qquad \forall n \in M \,.$$

19

Define the function

$$\rho(t) := \sum_{\lambda_k < t} v_k(\widehat{\omega}) \,.$$

It follows from *IIIa)* that all the moments of the measure corresponding to ρ are finite.

On the basis of *I)* and *II)*, define the meromorphic functions

$$\check{\mathbf{m}}(\zeta) := \prod_{\substack{k \in M \\ k \neq k_0}} \frac{\zeta - \mu_k}{\zeta - \lambda_k} \tag{4.19}$$

and

$$\check{m}(\zeta) := \frac{\check{\mathbf{m}}(\zeta) - \widehat{\omega}}{(\zeta - \gamma)(\widehat{\omega} - 1)} \,. \tag{4.20}$$

As it was shown in the proof of Theorem 4.3, one verifies that

$$\operatorname*{Res}_{\zeta = \lambda_n} \check{m}(\zeta) = -v_n(\widehat{\omega}) \,, \qquad n \neq k_0 \,.$$

It is also straightforward to show that

$$\operatorname*{Res}_{\zeta = \gamma} \check{m}(\zeta) = \frac{\check{\mathbf{m}}(\gamma) - \widehat{\omega}}{\widehat{\omega} - 1} \,.$$

Thus, since the function $\check{m}(\zeta)$ vanishes as $\zeta \to \infty$ along curves in the upper complex half plane, according to [13, Chap. VII, Sec.1 Theorem 2], one can write

$$\check{m}(\zeta) = \sum_{k \in M} \frac{v_k(\widehat{\omega})}{\lambda_k - \zeta} \,. \tag{4.21}$$

From (4.21) and the fact that $\lim_{\substack{\zeta \to \infty \\ \operatorname{Im} \zeta \geq \epsilon > 0}} \zeta \check{m}(\zeta) = -1$, it follows that

$$\sum_{k \in M} v_k(\widehat{\omega}) = 1 \quad \text{or, equivalently,} \quad \int_{\mathbb{R}} d\rho(t) = 1 \,.$$

On the other hand, by *IIIa)*, all the moments of ρ exist. Hence, using the method explained in Section 2, one obtains a Jacobi matrix and the operator J_0 generated by it (see the Introduction). Condition *IIIb)* implies that ρ is the spectral function of a self-adjoint extension J of J_0 [17, Prop. 4.15]. Now, consider (4.10), where now

$$\theta = +\sqrt{\widehat{\omega}}\,, \qquad h = \gamma \left(\frac{1}{\widehat{\omega}} - 1 \right) \,.$$

By construction the sequence $\{\lambda_k\}_{k \in M}$ is the spectrum of J. For the proof to be complete it only remains to show that $\{\mu_k\}_{k \in M}$ is the spectrum of $\widetilde{J}$. For the

function given in (3.2), taking into account (2.5) and (3.4), one has

$$\mathfrak{m}(\zeta) = \theta^2 + (\zeta - \gamma)\left(\theta^2 - 1\right)\sum_{k \in M}\frac{1}{\alpha_k(\lambda_k - \zeta)}\,.$$

In view of (4.20) and (4.21), one has

$$\check{\mathfrak{m}}(\zeta) = \widehat{\omega} + (\zeta - \gamma)\left(\widehat{\omega} - 1\right)\sum_{k \in M}\frac{\upsilon_k(\widehat{\omega})}{\lambda_k - \zeta}\,.$$

But, since $\theta = \widehat{\omega}$ and the fact that $\alpha_k^{-1} = \upsilon_k(\widehat{\omega})$ for $k \in M$, it follows that $\mathfrak{m} = \check{\mathfrak{m}}$. In its turn, this means that the zeros of $\mathfrak{m}$ are given by the sequence $\{\mu_k\}_{k \in M}$. $\square$

Remark 10. By repeating the reasoning of the proof of Theorem 4.6, it is straightforward to verify that Theorem 4.6 remains true if one substitutes $\theta > 1$ by $\theta < 1$, (3.8) by (3.9) in *I)*, and

$$\widehat{\omega} > \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k} \quad \text{by} \quad \widehat{\omega} < \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k}\,.$$

Proposition 4.2. *Let S and $\widetilde{S}$ be two infinite real sequences without finite points of accumulation such that $S \cap \widetilde{S} = \{\gamma\}$ and I) and II) of Theorem 4.6 hold. Suppose that there is*

$$\widehat{\omega} > \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k}$$

so that the sequence $\{\upsilon_n(\widehat{\omega})\}_{n \in M}$ satisfies IIIa) and IIIb) of Theorem 4.6, then $\{\upsilon_n(\omega)\}_{n \in M}$ also satisfies IIIa) and IIIb) for all

$$\omega > \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k}$$

Proof. For proving the claim one repeats the reasoning of the proof of Proposition 4.1. Here we observe that, for $n \in M$, $n \neq k_0$,

$$\upsilon_n(\omega) = C\upsilon_n(\widehat{\omega})\,,$$

where $C = \frac{\widehat{\omega}-1}{\omega-1}$. $\square$

Remark 11. If, in Proposition 4.2, one substitutes (3.8) by (3.9) in *I)* and

$$\widehat{\omega} > \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k}, \qquad \omega > \prod_{\substack{k \in M \\ k \neq k_0}}\frac{\gamma - \mu_k}{\gamma - \lambda_k}$$

21

by

$$\widehat{\omega} < \prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \mu_k}{\gamma - \lambda_k}, \qquad \omega < \prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \mu_k}{\gamma - \lambda_k},$$

then the new assertion holds true.

By repeating the proof of Theorem 4.4 with a minor modification one arrives at the following theorem.

Theorem 4.7. *Let $\theta \neq 1$ and assume that the intersecting sets $\sigma(J)$ and $\sigma(\widetilde{J})$ are enumerated according to Remark 4 with (3.8) if $\theta > 1$ and (3.9) if $\theta < 1$. Then, for any $\omega > \mathfrak{m}(\gamma)$ when $\theta > 1$ and for any $\omega < \mathfrak{m}(\gamma)$ when $\theta < 1$, there is a matrix (4.13) and a self-adjoint extension J' of the operator whose matrix representation is (4.13), such that $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J}') = \sigma(\widetilde{J})$, where $\widetilde{J}'$ is given by (4.14) with*

$$\theta' := +\sqrt{\omega}, \qquad h' := \gamma \left(\frac{1}{\omega} - 1 \right).$$

Let us now comment on the last results in terms of the perturbed mass-spring systems.

Assume that the spectra of the mass-spring system given in Fig. 1 and Fig. 2 are given and they intersect. By Proposition 3.2, these input data determine the sign of Δm. Let us suppose that $\Delta m > 0$. Due to Theorem 4.7, for any value of the ratio of masses $\theta < \mathfrak{m}(\gamma)$ there are mass-spring systems corresponding to Figs. 1 and 2 having the measured spectra. The knowledge of the ratio of masses completely determines the mass-spring systems.

We have given above the ratio of masses as a parameter of the system when the spectra intersect (see Theorems 4.6 and 4.2 where ω and $\widehat{\omega}$ play the role of the ratio of masses). This is a "natural" choice because the parameter used in the case when the spectra are disjoint, namely γ, is now given with the spectra. There is also another choice for the parameter: the spring constant h. Below we briefly discuss this parameterization where now the role of the spring constant is played by ω and $\widehat{\omega}$. We begin by defining

$$\widetilde{v}_n(\omega) := \begin{cases} \dfrac{(\mu_n - \lambda_n)(\omega + \gamma)}{(\gamma - \lambda_n)\omega} \displaystyle\prod_{\substack{k \in M \\ k \neq n}} \frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}, & n \in M, \quad n \neq k_0 \\[4ex] \dfrac{\omega + \gamma}{\omega} \displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \mu_k}{\gamma - \lambda_k} - \frac{\gamma}{\omega} & n = k_0 \end{cases} \tag{4.22}$$

Theorem 4.8. *Let S and $\widetilde{S}$ be two infinite real sequences without finite points of accumulation such that $S \cap \widetilde{S} = \{\gamma\}$. There exist $\theta > 1$, $h \in \mathbb{R}$, and a matrix (1.3) such that $S = \sigma(J)$ and $\widetilde{S} = \sigma(\widetilde{J})$ if and only if the conditions I and II of Theorem 4.6 hold along with*

22

III') There exists a real number $\widehat{\omega}$ satisfying

$$\widehat{\omega} \begin{cases} = 0 & \text{if } \gamma = 0 \\[2mm] < \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma > 0 \\[4mm] > \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma < 0 \end{cases}$$

such that

a) For $m = 0, 1, 2, \ldots$, the series

$$\sum_{k \in M} \lambda_k^{2m} \widetilde{v}_k(\widehat{\omega}) \quad \text{converges.}$$

b) If a sequence of complex numbers $\{\beta_k\}_{k \in M}$ is such that the series

$$\sum_{k \in M} |\beta_k|^2 \, \widetilde{v}_k(\widehat{\omega}) \quad \text{converges}$$

and, for $m = 0, 1, 2, \ldots$,

$$\sum_{k \in M} \beta_k \lambda_k^m \widetilde{v}_k(\widehat{\omega}) = 0 \,,$$

then $\beta_k = 0$ for all $k \in M$.

Proof. The proof is similar to the one of Theorem 4.6 and we restrict ourselves to the case when $\gamma > 0$. The other cases are proven analogously. Thus, for the necessity of the conditions to be proven, one only should establish that there is

$$\widehat{\omega} < \gamma \left(\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right)$$

such that $\widetilde{v}_k(\widehat{\omega}) = \alpha_k^{-1}$ for all $k \in M$. On the basis of (3.3) and (4.18), one has

$$1 < \mathfrak{m}(\gamma) < \frac{\gamma}{\gamma + h} \,.$$

Note that $\gamma + h \neq 0$. Since γ and $\gamma + h$ have always the same sign and we are

assuming that $\gamma > 0$, the inequality

$$h < \gamma \left(\frac{1}{\mathfrak{m}(\gamma)} - 1 \right)$$

holds. So let $\widehat{\omega} = h$, then (4.3) yields $\widetilde{v}_k(\widehat{\omega}) = \alpha_k^{-1}$ for $k \in M$, $k \neq k_0$. Moreover, (4.1) implies that $\widetilde{v}_{k_0}(\widehat{\omega}) = \alpha_{k_0}^{-1}$.

Let us now prove that *I)*, *II)*, *IIIa)*, and *IIIb)* are sufficient. Reasoning as before, one verifies that

$$\widetilde{v}_n(\widehat{\omega}) > 0 \, , \forall n \in M \, .$$

Now, instead of (4.20) one defines

$$\check{m}(\zeta) := \frac{(\widehat{\omega} - \gamma)\check{\mathfrak{m}}(\zeta) - \gamma}{(\zeta - \gamma)(2\gamma - \widehat{\omega})} \, ,$$

where $\check{\mathfrak{m}}$ is given in (4.19). Then it is shown that

$$\operatorname*{Res}_{\zeta = \lambda_n} \check{m}(\zeta) = -\widetilde{v}_n(\widehat{\omega}) \quad \forall n \in M \, . \tag{4.23}$$

Having defined

$$\rho(t) := \sum_{\lambda_k < t} \widetilde{v}_k(\widehat{\omega}) \, ,$$

the asymptotic behavior of $\zeta \check{m}(\zeta)$ and (4.23) imply that $\int_{\mathbb{R}} d\rho(t) = 1$. Furthermore, by *IIIa)*, all the moments exist, so one constructs the operator J_0 as was done before and, by *IIIb)* ρ corresponds to a self-adjoint extension J of J_0. Let us now consider (4.10) with

$$\theta = +\sqrt{\frac{\gamma}{\widehat{\omega} + \gamma}} \, , \qquad h = \widehat{\omega} \, .$$

Clearly, $\sigma(J) = \{\lambda_k\}_{k \in M}$. Hence it only remains to show that $\sigma(\widetilde{J}) = \{\mu_k\}_{k \in M}$. By (2.5) and (3.4), one has

$$\mathfrak{m}(\zeta) = \theta^2 + (\zeta - \gamma) \left(\theta^2 - 1 \right) \sum_{k \in M} \frac{1}{\alpha_k(\lambda_k - \zeta)} \, .$$

On the other hand (4.20) and (4.21) imply that

$$\check{\mathfrak{m}}(\zeta) = \frac{\gamma}{\widehat{\omega} + \gamma} + (\gamma - \zeta)\frac{\widehat{\omega}}{\widehat{\omega} + \gamma} \sum_{k \in M} \frac{\widetilde{v}_k(\widehat{\omega})}{\lambda_k - \zeta} \, .$$

Since $\theta^2 = \gamma/(\widehat{\omega} + \gamma)$, we conclude that $\mathfrak{m} = \check{\mathfrak{m}}$. In its turn, this means that the zeros of $\mathfrak{m}$ are given by the sequence $\{\mu_k\}_{k \in M}$. $\qquad\square$

Remark 12. Theorem 4.8 holds true after substituting $\theta > 1$ by $\theta < 1$ and instead

24

of

$$
\widehat{\omega}
\begin{cases}
= 0 & \text{if } \gamma = 0 \\[2ex]
< \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma > 0 \\[3ex]
> \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma < 0
\end{cases}
$$

one writes

$$
\widehat{\omega}
\begin{cases}
= 0 & \text{if } \gamma = 0 \\[2ex]
> \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma > 0 \\[3ex]
< \gamma \left(\displaystyle\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right) & \text{if } \gamma < 0
\end{cases}
$$

The proof of this claim proceeds in exactly the same way as the proof of Theorem 4.8.

Since assertions analogous to Proposition 4.2 and Remark 11 hold when one considers the sequence (4.22) instead of (4.17), the proof of the following statement can be done by repeating, with just minor modifications, the proof of Theorem 4.7.

Theorem 4.9. *Let $\theta \neq 1$ and assume that the intersecting sets $\sigma(J)$ and $\sigma(\widetilde{J})$ are enumerated according to Remark 4 with (3.8) if $\theta > 1$ and (3.9) if $\theta < 1$. Assume that $\gamma > 0$, then, for any*

$$
\omega < \gamma \left(\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right)
$$

when $\theta > 1$, and for any

$$
\omega > \gamma \left(\prod_{\substack{k \in M \\ k \neq k_0}} \frac{\gamma - \lambda_k}{\gamma - \mu_k} - 1 \right)
$$

when $\theta < 1$, there is a matrix (4.13) and a self-adjoint extension J' of the operator whose matrix representation is (4.13) such that $\sigma(J') = \sigma(J)$ and $\sigma(\widetilde{J'}) = \sigma(\widetilde{J})$,

where $\widetilde{J}'$ is given by (4.14) with

$$\theta' := +\sqrt{\frac{\gamma}{\omega + \gamma}}\,, \qquad h' := \omega\,.$$

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