THE REGULARITY OF SONIC CURVES FOR THE TWO-DIMENSIONAL RIEMANN PROBLEMS OF THE NONLINEAR WAVE SYSTEM OF CHAPLYGIN GAS

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ABSTRACT. We study the regularity of sonic curves to a two-dimensional Riemann problem for the nonlinear wave system of Chaplygin gas, which is an essential step for the global existence of solutions to the two-dimensional Riemann problems. As a result, we establish the global existence of *uniformly* smooth solutions in the semi-hyperbolic patches up to the sonic boundary, where the degeneracy of hyperbolicity occurs. Furthermore, we show the C^1 regularity of sonic curves.

1. INTRODUCTION

The purpose of this paper is to provide the regularity of sonic curves arising from a two-dimensional Riemann problem governed by a self-similar nonlinear wave equations. Let density be ρ, velocity (u, v), the pressure $\tilde{p}$ be given as a function of ρ. From the well-known 2-D compressible Euler system for isentropic flow, when the flow is irrotational and the nonlinear velocity terms are ignored, we can derive the following nonlinear wave system

$$\begin{aligned}
\rho_t + m_x + n_y &= 0, \\
m_t + \tilde{p}_x &= 0, \\
n_t + \tilde{p}_y &= 0,
\end{aligned} \qquad (1.1)$$

where $(m, n) = (u\rho, v\rho)$ are momenta. We refer the readers to [4, 14, 15] for more information on this system. In this paper we are interested in the 2-D Riemann problem of system (1.1) with Chaplygin gas equation of state $\tilde{p} = -1/\rho$. The Chaplygin gas, which can be used to depict some dark-energy models in cosmology, has been widely studied. Recently, some interesting and important results have been obtained, especially for the Riemann problem. We refer to [7, 8, 18, 23, 28] and the references cited therein for the related results.

For decades, there have been wide and intensive developments in the Riemann problems of conservation laws, in particular the 2-D compressible Euler system and its simple models [21,22]. Such models are the unsteady transonic small disturbance (UTSD) equations [1–3], the pressure gradient system [11, 17, 19, 26, 31], the potential flow [5, 6, 10, 16], and the nonlinear wave equations [4,14,15,27] and so on. For Riemann problems in two-dimensional flow, those governing equations become quasilinear and mixed types. The type of the flow in the far-field is hyperbolic, while near the origin the type is mixed. The sonic lines,
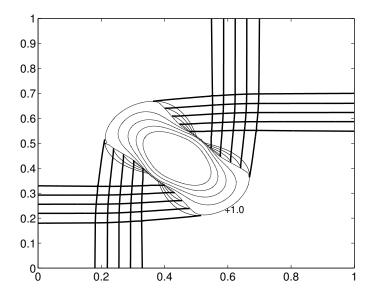

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FIGURE 1. Four semi-hyperbolic patches, marked by the short and light curves and overlapped by some bold ones, in the interaction of two forward and two backward rarefaction waves for the Euler system with a gas constant $\gamma = 1.4$. The five closed curves at the center are contour curves of the pseudo-Mach number which mark the subsonic region; bold curves originating from the boundaries are characteristics; the short and light curves in the semi-hyperbolic regions are also characteristics. The solution has two axes of symmetry (Courtesy of Glimm et. al. [13]).

located between the elliptic part and the hyperbolic part, are important to obtain the global existence of solutions to the mixed type problem.

In this paper we consider the presence of the semi-hyperbolic patches with a transonic shock and a sonic curve in a simple wave environment. The regions identified by the existence of a family of characteristics that start on sonic curves and end on transonic shock curves are called semi-hyperbolic patches. The study on this region was initiated in [26] via the pressure gradient system and is being continued in [20, 29, 30]. These regions are different from the typical fully hyperbolic regions. As in the result of [13], in several numerical calculations, we frequently encounter this kind of regions in several configurations in the two-dimensional Riemann problems of the Euler system as well as its simplified models (Figure 1). One of the examples for semi-hyperbolic patches happens when air is accelerated in a planar tunnel over a small inward bulge on one of the walls [9]. A global solution to the boundary value problem of the Riemann problem of a hyperbolic system can be patched together by pieces along characteristic lines, sonic curves, shock waves or other natural boundaries. We believe that it is helpful and essential to understand these patches of solutions locally before resolving the Riemann problems globally.

In [12], Hu and Wang have studied the semi-hyperbolic patches of solutions to the 2-D nonlinear wave system for Chaplygin gas state. They have constructed the global existence

of solutions by solving a Goursat-type boundary value problem which has a sonic curve as a degenerate boundary, and verified that the sonic curve is Lipschitz continuous. However, the smoothness of the sonic curve and the exact behavior of solutions near the sonic curve are still open. The main difficulty is the degeneracy of hyperbolicity near the sonic lines. In several numerical simulations, we can see that the demarcation line between hyperbolic regions and elliptic regions is decomposed of sonic curves and a transonic shock. Also the pressure seems to change smoothly across the sonic curves. Hence, we mainly focus on the study of the solutions' precise behaviors near sonic curves, and expect better regularity of sonic curves. As a result, we prove that the semi-hyperbolic patches are uniformly smooth up to the sonic curves, and the sonic curves are C^1 continuous.

In the self-similar coordinate $(\xi, \eta) = (x/t, y/t)$, the nonlinear wave system (1.1) becomes

$$-\xi \tilde{p}_\xi - \eta \tilde{p}_\eta + \tilde{p}^2 m_\xi + \tilde{p}^2 n_\eta = 0,$$
$$-\xi m_\xi - \eta m_\eta + \tilde{p}_\xi = 0,$$
$$-\xi \eta_\xi - \eta n_\eta + \tilde{p}_\eta = 0,$$

which yields a second order partial differential equation of $\tilde{p}$

$$(\tilde{p}^2 - \xi^2)\tilde{p}_{\xi\xi} - 2\xi\eta\tilde{p}_{\xi\eta} + (\tilde{p}^2 - \eta^2)\tilde{p}_{\eta\eta} + \frac{2}{\tilde{p}}(\xi\tilde{p}_\xi + \eta\tilde{p}_\eta)^2 - 2(\xi\tilde{p}_\xi + \eta\tilde{p}_\eta) = 0. \qquad (1.2)$$

The two characteristics $\eta = \eta(\xi)$ are defined by

$$\frac{d\eta}{d\xi} = \frac{\xi\eta \pm \sqrt{\tilde{p}^2(\xi^2 + \eta^2 - \tilde{p}^2)}}{\xi^2 - \tilde{p}^2} \qquad (1.3)$$

when $\xi^2 + \eta^2 > \tilde{p}^2$. In the polar coordinate system $(r, \theta) = (\sqrt{\xi^2 + \eta^2}, \arctan(\eta/\xi))$, the equation (1.2) becomes

$$(p^2 - r^2)p_{rr} + \frac{p^2}{r^2}p_{\theta\theta} + \frac{p^2}{r}p_r + \frac{2r^2}{p}p_r^2 - 2rp_r = 0, \qquad (1.4)$$

where $p(r, \theta)$ is a simple notation of $\tilde{p}(r\cos\theta, r\sin\theta)$. Equation (1.4) is elliptic in the region of $r^2 - p^2 < 0$, and degenerates when $r + p = 0$. In particular, in the area of $r^2 - p^2 > 0$, it is hyperbolic, and two characteristics can be defined by

$$\frac{dr}{d\theta} = \pm\lambda^{-1} = \pm\sqrt{\frac{r^2(r^2 - p^2)}{p^2}}.$$

In the hyperbolic region, equation (1.4) can be decoupled to

$$\partial^+\partial^- p = Q(\partial^+ p - \partial^- p)\partial^- p, \quad \partial^-\partial^+ p = Q(\partial^- p - \partial^+ p)\partial^+ p, \qquad (1.5)$$

where

$$Q := \frac{r^2}{2p(r^2 - p^2)},$$

and $\partial^\pm := \partial_\theta \pm \lambda^{-1}\partial_r$ are directional derivatives along the positive and negative characteristics, respectively. That is, we have a new system

$$\begin{pmatrix} \partial^+ p \\ \partial^- p \\ p \end{pmatrix}_\theta + \begin{pmatrix} -\lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial^+ p \\ \partial^- p \\ p \end{pmatrix}_r = \begin{pmatrix} Q(\partial^- p - \partial^+ p)\partial^+ p \\ Q(\partial^+ p - \partial^- p)\partial^- p \\ \frac{1}{2}(\partial^+ p + \partial^- p) \end{pmatrix}. \qquad (1.6)$$
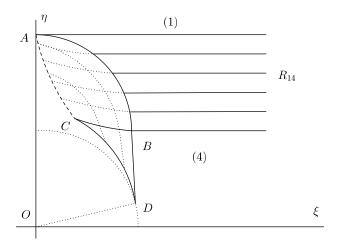
FIGURE 2. **The semi-hyperbolic region** ABC. Here (1) and (4) denote the constant states of $p = p_1$ and $p = p_4$, respectively, and $R_{14} = -\eta$ is a planar wave. Arc CD is an envelope and the region of $OACD$ is a subsonic region. We are interested in the solutions in a curvilinear triangle ABC and the regularity of the sonic curve AC for the further research in the whole region of $OABD$.

Using this decomposition, the existence of solutions to (1.6) can be obtained in the hyperbolic region even if we cannot get the Riemann invariants $[12, 20, 26]$. Letting

$$R := \partial^+ p, \quad S := \partial^- p,$$

we have

$$R_\theta - \lambda^{-1} R_r = Q(S - R)R, \quad S_\theta + \lambda^{-1} S_r = Q(R - S)S.$$

At this point, let us give a short description of the problem and known results. In the self-similar coordinates, two constant states (p_1, m_1, n_1) and (p_4, m_4, n_4) are given with $p_1 < p_4 < 0$, and a planar wave

$$R_{14} : \begin{cases} p(\xi, \eta) = -\eta, \\ n(\xi, \eta) = \ln \frac{p_4}{p(\xi,\eta)}, \\ m(\xi, \eta) = m_1 = m_4 = 0, \end{cases}$$

connects these two constant states in the domain $\{\xi > 0, \eta > 0\}$. Let the point $(0, -p_1)$ be A, which is a sonic point satisfying $\xi^2 + \eta^2 = p^2$. We draw a positive characteristic curve defined by (1.3) passing A in the planar wave, and intersecting the bottom boundary of the planar wave at B. Next we draw a negative characteristic defined by (1.3) from B toward the subsonic region, and let C be the ending point of the negative characteristic on the sonic curve. Upon this configuration in Figure 2, if we assume that the negative characteristic BC is given, then the region ABC bounded by two characteristics AB and BC forms a semi-hyperbolic region, and AC is a sonic curve. Here, the curve BC serves as a spine to support the whole patch.

Let $W := (R, S, p)$, and we also denote the arcs AB and BC by Γ_+ and Γ_-, respectively. For given boundary conditions on $\Gamma_\pm$, the global existence and a priori C^1 estimate of W in the domain ABC have been obtained in [12] as follows:

Theorem 1.1. (**Global existence** [12]). *Assume that two constant states (p_1, m_1, n_1) and (p_4, m_4, n_4) are given with $p_1 < p_4 < 0$ and a planar wave connects these two constant states in the domain $\xi > 0$. For a given negative characteristic curve Γ_-, there exists a classical solution W to the Goursat boundary value problem of (1.6) in the whole domain ABC with a Lipschitz continuous sonic boundary AC if p_4 is close enough to p_1. Furthermore, $\partial^+ p$ and $\partial^- p$ are strictly negative if the boundary data of $\partial^+ p, \partial^- p$ on $\Gamma_\pm$ are strictly negative, and $\partial^+ p = \partial^- p$ on the sonic curve. And $\partial^+ p, \partial^- p$ are uniformly bounded in the whole semi-hyperbolic region ABC.*

In fact, we do not know exactly how far we can extend the C^1 solution in Theorem 1.1 towards the sonic boundary. In this paper, we improve the results of [12] in the sense that the solutions in the semi-hyperbolic patches are *uniformly* smooth up to the sonic curve, where the degeneracy of hyperbolicity occurs, and that the sonic curve is C^1 continuous. In fact, the smoothness of sonic curve is a necessary requirement for the global existence of solutions in the region of $OABD$ in Figure 2 for the future research. The main theorem is as follows.

Theorem 1.2. (**Regularity**) *For the Goursat problem with prescribed characteristic boundaries Γ_+ and Γ_-, there exists a global uniformly smooth solution of (1.6) in the domain ABC under the same conditions as in Theorem 1.1. Moreover, the sonic line AC is C^1 continuous, and the two derivatives $\partial^+ p$ and $\partial^- p$ approach a common value on the sonic curve with a rate of $O(\sqrt{r + p})$.*

To overcome the difficulty of degeneracy, we will employ a new different coordinate system (r, t) with $t = \sqrt{r + p(r, \theta)}$ in Section 2. Under the new coordinates, we derive new systems for (R, S). The proof of the main theorem is based heavily on the uniform bounds of $t^\delta R_r$ and $t^\delta S_r$ for any $1 < \delta < 2$. In the following section, we will prove the above main theorem using the bootstrap method to get the uniform estimates as in [19, 29]. Especially upon the method of [29], we show that a global smooth solution in [12] is extended up to the sonic boundary by obtaining the convergence rate of $\partial^+ p$ and $\partial^- p$ near the sonic curve. With those estimates and the convergence rate, we are able to prove $\partial^+ p$ and $\partial^- p$ are uniformly continuous on the sonic curve, and furthermore, the sonic curve is C^1 continuous.

For recent related progresses on the classical solution near the sonic lines for 2-D the pressure gradient systems and Euler systems, we would like to mention the work of Zhang and Zheng [30] first. In [30], for any given smooth curve as a sonic line with any specified non-tangential derivative of the speed of sound, they have constructed classical solutions from the sonic line towards the supersonic side by establishing a convergent iteration scheme. On the contrary, in [29] Q. Wang and Y. Zheng solved a Goursat problem and considered the regularity of solutions from the supersonic domain towards the sonic curve. They considered the problem from a different aspect but also hoped it can be used to build a transonic solution that bridges between a semi-hyperbolic patch to an elliptic domain. For the two-dimensional Euler system, M. Li and Y. Zheng [20] studied the semi-hyperbolic

patches of solutions. In a forthcoming paper [25], we will consider the regularity of sonic curves for the Euler system.

2. Main Results

2.1. **Derivation of New Systems.** We introduce the new coordinates (r, t), where

$$t = \sqrt{r + p(r, \theta)}$$

and r is a radius-component in the polar coordinates. Then $\{t = 0\}$ flattens the sonic boundary, and

$$Q = \frac{-r^2}{2t^2(r - t^2)(2r - t^2)}, \quad \lambda^{-1} = \frac{rt\sqrt{2r - t^2}}{r - t^2}.$$

We note that λ^{-1} goes to zero as $t \to 0$.

The sonic line is defined by the limit of the following level curves $\{(r, \theta) : r + p(r, \theta) = \varepsilon\}$. Let such level curves be $\theta = \theta_\varepsilon(r)$. Then

$$\theta_\varepsilon'(r) = -\frac{p_r + 1}{p_\theta}.$$

Here, p_θ is negative and bounded in the whole domain. However,

$$p_r = \frac{R - S}{2\lambda^{-1}},$$

where $R - S$ and λ^{-1} vanish simultaneously as $t \to 0$. It seems necessary to know the rate of vanishing of these two components, more precisely the vanishing rate of $R - S$ with respect to t, in order to know the regularity of the sonic line.

In the coordinate system (r, t), we have

$$\begin{cases} R_t - \dfrac{2t\lambda^{-1}}{S - \lambda^{-1}}R_r = \dfrac{2t}{S - \lambda^{-1}}Q(S - R)R, \\ S_t + \dfrac{2t\lambda^{-1}}{R + \lambda^{-1}}S_r = \dfrac{2t}{R + \lambda^{-1}}Q(R - S)S. \end{cases} \tag{2.1}$$

Since λ^{-1} goes to zero as $t \to 0$, we note that $S - \lambda^{-1} \neq 0$ and $R + \lambda^{-1} \neq 0$ in the (r, t)-plane when t is sufficiently small. Letting

$$\Lambda_+ := \frac{2t\lambda^{-1}}{R + \lambda^{-1}}, \quad \Lambda_- := -\frac{2t\lambda^{-1}}{S - \lambda^{-1}},$$

we have

$$\left(\frac{1}{R}\right)_t = \frac{2t^2 Q}{1 - S^{-1}\lambda^{-1}}\left(\frac{1}{S} - \frac{1}{R}\right)\frac{1}{t} + \frac{2t\lambda^{-1}S^{-1}}{1 - S^{-1}\lambda^{-1}}\left(\frac{1}{R}\right)_r,$$

$$\left(\frac{1}{S}\right)_t = \frac{2t^2 Q}{1 + R^{-1}\lambda^{-1}}\left(\frac{1}{R} - \frac{1}{S}\right)\frac{1}{t} - \frac{2t\lambda^{-1}R^{-1}}{1 + R^{-1}\lambda^{-1}}\left(\frac{1}{S}\right)_r.$$

Defining

$$U := \frac{1}{R} + \frac{1}{S}, \quad V := \frac{1}{S} - \frac{1}{R},$$

where $V = 0$ on the sonic line, we have

$$U_t = \frac{2tQUV\lambda^{-1}}{(1 - S^{-1}\lambda^{-1})(1 + R^{-1}\lambda^{-1})} + \frac{2t\lambda^{-1}S^{-1}}{1 - S^{-1}\lambda^{-1}}\left(\frac{1}{R}\right)_r - \frac{2t\lambda^{-1}R^{-1}}{1 + R^{-1}\lambda^{-1}}\left(\frac{1}{S}\right)_r,$$

and

$$V_t = \frac{2tQV(-2+\lambda^{-1}V)}{(1-S^{-1}\lambda^{-1})(1+R^{-1}\lambda^{-1})} - \frac{2t\lambda^{-1}S^{-1}}{1-S^{-1}\lambda^{-1}}\left(\frac{1}{R}\right)_r - \frac{2t\lambda^{-1}R^{-1}}{1+R^{-1}\lambda^{-1}}\left(\frac{1}{S}\right)_r. \qquad (2.2)$$

Furthermore, we define

$$G := \partial_+R - \partial_-R, \quad H := \partial_+S - \partial_-S,$$

where $\partial_\pm := \partial_t \pm \Lambda_\pm \partial_r$. Then

$$\partial_-G = \frac{\partial_-\Lambda_+ - \partial_+\Lambda_-}{\Lambda_+ - \Lambda_-}G + (\partial_+\partial_-R - \partial_-\partial_-R),$$

$$\partial_+H = \frac{\partial_-\Lambda_+ - \partial_+\Lambda_-}{\Lambda_+ - \Lambda_-}H + (\partial_+\partial_+S - \partial_-\partial_+S).$$

Here,

$$\Lambda_+ - \Lambda_- = 2t\lambda^{-1}\frac{R+S}{(R+\lambda^{-1})(S-\lambda^{-1})},$$

and by direct calculations,

$$\frac{\partial_-\Lambda_+ - \partial_+\Lambda_-}{\Lambda_+ - \Lambda_-} = \frac{2}{t} + h(r,t),$$

where

$$h(r,t) := \frac{t(3r-t^2)}{(r-t^2)(2r-t^2)} + \frac{r^3(3R-3S+4\lambda^{-1}) + 2t\lambda^{-1}(t^5+r^2t-3rt^3)}{(R+\lambda^{-1})(S-\lambda^{-1})(r-t^2)^2\sqrt{2r-t^2}}.$$

Note that $h \to 0$ as $t \to 0$. Furthermore,

$$\partial_+\partial_-R - \partial_-\partial_-R = (\Lambda_+ - \Lambda_-)(\partial_-R)_r := tf_1(r,t)R_r + tf_2(r,t)S_r + t^2f_3(r,t), \qquad (2.3)$$

where

$$f_1(r,t) := \frac{r^2(2R-S)}{(S-\lambda^{-1})(r-t^2)(2r-t^2)}E,$$

$$f_2(r,t) := \frac{-R}{S-\lambda^{-1}}\left(1 + \frac{R-S}{S-\lambda^{-1}}\right)\frac{r^2}{(r-t^2)(2r-t^2)}E,$$

$$f_3(r,t) := \frac{rR(R-S)}{(S-\lambda^{-1})(r-t^2)^2(2r-t^2)^{3/2}}\left\{\frac{-3r^2t^2+rt^4+r^3}{(S-\lambda^{-1})(r-t^2)} - \frac{t(3r-2t^2)}{\sqrt{2r-t^2}}\right\}E,$$

and

$$E(r,t) := 2r\frac{\sqrt{2r-t^2}}{r-t^2}\left(\frac{1}{R+\lambda^{-1}} + \frac{1}{S-\lambda^{-1}}\right).$$

We note that functions E, f_j for $j = 1, 2, 3$ are all bounded in bounded regions. Also we can obtain by direct calculations that

$$\partial_+\partial_+S - \partial_-\partial_+S := tg_1(r,t)R_r + tg_2(r,t)S_r + t^2g_3(r,t), \qquad (2.4)$$

where

$$g_1(r,t) := \frac{-S}{R + \lambda^{-1}}\left(1 + \frac{S - R}{R + \lambda^{-1}}\right)\frac{r^2}{(r - t^2)(2r - t^2)}E,$$

$$g_2(r,t) := \frac{r^2(2S - R)}{(R + \lambda^{-1})(r - t^2)(2r - t^2)}E,$$

$$g_3(r,t) := \frac{rS(S - R)}{(R + \lambda^{-1})(r - t^2)^2(2r - t^2)^{3/2}}\left\{\frac{3r^2t^2 - rt^4 - r^3}{(R + \lambda^{-1})(r - t^2)} - \frac{t(3r - 2t^2)}{\sqrt{2r - t^2}}\right\}E.$$

We note that functions g_j for $j = 1, 2, 3$ are all bounded in bounded regions, too.

2.2. **Regularity of Solutions near Sonic Curves.** Since we are concerned with the regularity of semi-hyperbolic patches near the sonic line, for any fixed point $(\bar{r}, 0)$ on the sonic line AC, we shall take a new point $B'(\bar{r}, t_{B'})$, where $t_{B'}$ is positive and small so that B' remains in the domain ABC (Figure 3). Then, through the point B' we can draw the positive and negative characteristic curves $r_+(B')$ and $r_-(B')$ until they intersect the sonic line AC in two points A' and C', respectively. Let us denote the region ABC by Ω and $A'B'C'$ by $\mathcal{D}$.

Since R, S are uniformly bounded and negative in the domain Ω and $R = S$ on the sonic line, we can obtain $h(r,t), f_j(r,t), g_j(r,t)(j = 1, 2, 3)$ are also uniformly bounded in the small subdomain $\mathcal{D}$. Let

$$K_1 = \max_{(r,t)\in\mathcal{D}}\left\{|h(r,t)|, |f_3(r,t)|, |g_3(r,t)|\right\}, \tag{2.5}$$

$$K_2 = \max_{(r,t)\in\mathcal{D}}\left\{|f_j(r,t)|, |g_j(r,t)|, j = 1, 2\right\}, \tag{2.6}$$

and $K_3 = \min\limits_{(r,t)\in\mathcal{D}}|E|$. We can see that K_1 tends to zero and K_2, K_3 have positive bounds as $t_{B'} \to 0^+$. If $t_{B'} = 0$, the domain $\mathcal{D}$ degenerates into a point on the sonic line, and two constants K_2 and K_3 satisfy that $2K_2 = |E(\bar{r}, 0)| = K_3$ with $K_3 > 2K_2\delta^{-1}$ for any $\delta \in (1, 2)$. Thus, we can let $K_3 > 2\delta^{-1}K_2e^{2K_1t_{B'}}$ and $K_1 < K_2$ by taking suitable small $t_{B'}$.

Let $\Omega(\bar{r}, 0)$ denote a bounded domain surrounded by a positive characteristic and a negative characteristic starting from $(\bar{r}, 0)$ and the characteristics $B'C'$ and $B'A'$. For any $(r,t) \in \Omega(\bar{r}, 0)$, let a and b be intersections of the negative characteristic and the positive characteristic through (r,t) with the boundaries $B'C'$ and $B'A'$, respectively. Let

$$M_0 = \max\left\{\max_{\Omega(\bar{r},0)}\frac{|H(t_a)|}{K_2t_a^{2-\delta}e^{K_1t_a}} + 1, \quad \max_{\Omega(\bar{r},0)}\frac{|G(t_b)|}{K_2t_b^{2-\delta}e^{K_1t_b}} + 1\right\},$$

where t_a and t_b are t-coordinates of points a and b, respectively. As discussed before, M_0 is well-defined and uniform in the domain $\Omega(\bar{r}, 0)$, but depending on δ. We can prove the following result.

Lemma 2.1. *For any* $(r,t) \in \Omega(\bar{r}, 0)$, *and any* $\delta \in (1, 2)$, *there hold*

$$|t^\delta R_r| \le M, \quad |t^\delta S_r| \le M, \tag{2.7}$$

where $M > 0$ *is depending on* δ. *That is,* $t^\delta R_r$ *and* $t^\delta S_r$ *are uniformly bounded in the domain* $\Omega(\bar{r}, 0)$.
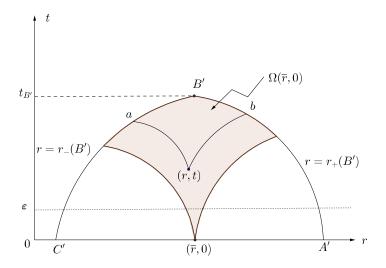
FIGURE 3.   The regions of $A'B'C'$ and $\Omega(\bar{r}, 0)$

Proof. For a fixed $\varepsilon \in (0, t_{B'})$, let

$$\Omega_\varepsilon := \left\{ (r, t) \mid \varepsilon \leq t \leq t_{B'},\, r_-(B') \leq r \leq r_+(B') \right\} \cap \Omega(\bar{r}, 0).$$

Defining $M_\varepsilon = \max\limits_{\Omega_\varepsilon}\{|t^\delta R_r|, |t^\delta S_r|\}$, we have

$$t^\delta |R_r| \leq M_\varepsilon, \quad t^\delta |S_r| \leq M_\varepsilon, \tag{2.8}$$

in Ω_ε with M_ε depending on ε. If for any $\varepsilon \in (0, t_{B'})$, we have $M_\varepsilon \leq M_0$, then (2.8) hold in the whole domain $\Omega(\bar{r}, 0)$. If not, then there exists $\varepsilon_0 \in (0, t_{B'})$ making $M_{\varepsilon_0} > M_0$. So we let

$$M_{\varepsilon_0} = \max\limits_{\Omega_{\varepsilon_0}}\{|t^\delta R_r|, |t^\delta S_r|\}.$$

For $(r_{\varepsilon_0}, \varepsilon_0) \in \Omega(\bar{r}, 0)$, we denote the two characteristics passing through $(r_{\varepsilon_0}, \varepsilon_0)$ by r_+^a and r_-^b, where a, b are the intersection points with $B'C'$ and $B'A'$ respectively. Here $a = a_{\varepsilon_0}$ and $b = b_{\varepsilon_0}$ are dependent on ε_0 and for convenience we omit the subscript ε_0. From the equation

$$\partial_- G = \frac{\partial_- \Lambda_+ - \partial_+ \Lambda_-}{\Lambda_+ - \Lambda_-} G + (\partial_+ \partial_- R - \partial_- \partial_- R),$$

we have

$$\partial_- \left(G(r, t) \exp \int_{t_b}^t -\left(\frac{2}{\tau} + h(r_-(\tau), \tau) \right) d\tau \right)$$

$$= (\partial_+ \partial_- R - \partial_- \partial_- R) \exp \int_{t_b}^t -\left(\frac{2}{\tau} + h(r_-(\tau), \tau) \right) d\tau.$$

Integrating this along the negative characteristic passing through $(r_{\varepsilon_0}, \varepsilon_0)$, we get

$$G(r,t)\left(\frac{t_b}{t}\right)^2 \exp \int_{t_b}^t \left(-h(r_-(\tau),\tau)\right)d\tau$$

$$= G(r,t_b) - \int_t^{t_b} (\partial_+\partial_- R - \partial_-\partial_- R)\left(\frac{t_b}{\tau}\right)^2 \exp \int_{t_b}^\tau \left(-h(r_-(s),s)ds\right) d\tau.$$

Thus, from (2.3) and (2.5) we obtain

$$|G(r,t)|(\frac{t_b}{t})^2 \exp \int_{t_b}^t \left(-h(r_-(\tau),\tau)\right)d\tau$$

$$\leq |G(r,t_b)| + \int_t^{t_b} \left(\frac{t_b}{\tau}\right)^2 \exp(K_1(t_b-\tau))|\partial_+\partial_- R - \partial_-\partial_- R|d\tau$$

$$\leq |G(r,t_b)| + t_b^2 e^{K_1 t_b} \int_t^{t_b} \frac{1}{\tau^2}\left(2K_2 M_{\varepsilon_0}\tau^{1-\delta} + K_1\tau^2\right)d\tau$$

$$= |G(r,t_b)| + t_b^2 e^{K_1 t_b}\left[\frac{2K_2 M_{\varepsilon_0}}{\delta}(t^{-\delta} - t_b^{-\delta}) + K_1(t_b-t)\right]$$

$$< \frac{2K_2 M_{\varepsilon_0} t_b^2 e^{K_1 t_b}}{\delta t^\delta},$$

if $|G(r,t_b)| + K_1(t_b-t)t_b^2 e^{K_1 t_b} - 2\delta^{-1}K_2 M_{\varepsilon_0} e^{K_1 t_b} t_b^{2-\delta} < 0$, which is true by the definition of M_0 and the fact of $M_{\varepsilon_0} > M_0$. Thus we obtain

$$|G(r,t)| < t^{2-\delta}\frac{2K_2 M_{\varepsilon_0} e^{K_1 t_b}}{\delta} \exp \int_{t_b}^t h(r_-(\tau),\tau)d\tau$$

$$\leq 2\delta^{-1}t^{2-\delta}K_2 M_{\varepsilon_0} e^{2K_1 t_b}.$$

On the other hand,

$$G(r,t) = (\Lambda_+ - \Lambda_-)R_r = 2t\lambda^{-1}\left(\frac{1}{R+\lambda^{-1}} + \frac{1}{S-\lambda^{-1}}\right)R_r(r,t).$$

Thus we obtain

$$|R_r|(t=\varepsilon_0) < \frac{1}{t^\delta} \cdot \frac{K_2 M_{\varepsilon_0} e^{2K_1 t_b}}{\delta \cdot |\frac{1}{R+\lambda^{-1}} + \frac{1}{S-\lambda^{-1}}|} \cdot \frac{r-t^2}{r\sqrt{2r-t^2}}$$

$$\leq \frac{1}{t^\delta} \cdot \frac{2K_2 M_{\varepsilon_0} e^{2K_1 t_b}}{\delta K_3} \leq \frac{M_{\varepsilon_0}}{t^\delta}.$$

So we have a strict inequality such that

$$|R_r| < M_{\varepsilon_0}/\varepsilon_0^\delta \tag{2.9}$$

on the line segment $t = \varepsilon_0$. Similarly, integration of the equation

$$\partial_+ H = \frac{\partial_-\Lambda_+ - \partial_+\Lambda_-}{\Lambda_+ - \Lambda_-}H + (\partial_+\partial_+ S - \partial_-\partial_+ S)$$

along the positive characteristic makes

$$|H(r,t)| < t^{2-\delta}\frac{2K_2 M_{\varepsilon_0} e^{2K_1 t_a}}{\delta}.$$

Also by the definition of H, we get the following estimate

$$|S_r|_{t=\varepsilon_0} < M_{\varepsilon_0}/\varepsilon_0^\delta. \tag{2.10}$$

According to (2.9) and (2.10), $|t^\delta R_r|$ and $|t^\delta S_r|$ do not have the maximum values on the line segment $t = \varepsilon_0$, that is, the maximum happens on $\varepsilon_0 < t \le t_{B'}$. Hence we conclude that this argument also holds in a larger domain $\Omega_{\varepsilon'}$ with $\varepsilon' < \varepsilon_0$. Repeating this process, we can extend the domain larger and larger until it reaches the whole domain $\Omega(\bar{r}, 0)$. Thus, we find a constant M depending on δ only and satisfying (2.7). The proof is completed. $\qquad\square$

Next, for any $(\bar{r}, 0) \in A'C'$, we can take a small open interval $(\bar{r} - r_0, \bar{r} + r_0) \subset A'C'$. Suppose the negative and positive characteristics passing through points $P(\bar{r} - r_0, 0)$ and $Q(\bar{r} + r_0, 0)$ intersect the boundaries $B'C'$ and $B'A'$ at P' and Q', respectively. Then we have the same estimates in the region of $B'P'PQQ'$ in the following lemma. The proof of it is very similar to that of the previous lemma, so let us omit the proof.

Lemma 2.2. *For any $\delta \in (1, 2)$, there exists a constant M only depending on the interval $(\bar{r} - r_0, \bar{r} + r_0)$ and δ such that*

$$|t^\delta R_r| \le M, \quad |t^\delta S_r| \le M$$

hold in the domain $B'P'PQQ'$.

Now we can show that $|V(r, t)|/t$ is uniformly bounded in the whole domain.

Lemma 2.3. *$|V(r, t)|/t$ is uniformly bounded in the domain ABC.*

Proof. Let $(\bar{r}, 0) \in AC$. From (2.2) we have

$$V_t = \frac{2tQV(-2 + \lambda^{-1}V)}{(1 - S^{-1}\lambda^{-1})(1 + R^{-1}\lambda^{-1})} - \frac{2t\lambda^{-1}S^{-1}}{1 - S^{-1}\lambda^{-1}}\left(\frac{1}{R}\right)_r - \frac{2t\lambda^{-1}R^{-1}}{1 + R^{-1}\lambda^{-1}}\left(\frac{1}{S}\right)_r$$

$$= \frac{r^2(2 - \lambda^{-1}V)}{(r - t^2)(2r - t^2)(1 - S^{-1}\lambda^{-1})(1 + R^{-1}\lambda^{-1})} \cdot \frac{V}{t}$$

$$+ t^{2-\delta}\frac{2r\sqrt{2r - t^2}}{(r - t^2)RS}\left[\frac{t^\delta R_r}{(1 - S^{-1}\lambda^{-1})R} + \frac{t^\delta S_r}{(1 + R^{-1}\lambda^{-1})S}\right]$$

$$=: l_1(r, t)\frac{V}{t} + l_2(r, t)t^{2-\delta}. \tag{2.11}$$

It is obvious that $\lim_{t \to 0^+} l_1(r, t) = 1$, and $l_2(r, t)$ is bounded in the domain $\mathcal{D}$. Moreover, we note that

$$\frac{l_1(r, t) - 1}{t} = \frac{3rt^2 - t^4 + r^2\lambda^{-1}V - 3rt^2\lambda^{-1}V + \lambda^{-1}t^4V + (2r^2 - 3rt^2 + t^4)\lambda^{-2}R^{-1}S^{-1}}{t(1 - S^{-1}\lambda^{-1})(1 + R^{-1}\lambda^{-1})(r - t^2)(2r - t^2)}$$

also tends to zero as $t \to 0^+$ since $V \to 0$ as $t \to 0^+$. Let

$$K_4 = \max_{\mathcal{D}}\left|\frac{l_1(r, t) - 1}{t}\right|, \quad K_5 = \max_{\mathcal{D}}|l_2(r, t)|.$$

Now we rewrite (2.11) in the form of

$$\partial_t\left(V\exp\left(\int_t^{t_{B'}}\frac{l_1(r, \tau)}{\tau}d\tau\right)\right) = l_2(r, t)t^{2-\delta}\exp\left(\int_t^{t_{B'}}\frac{l_1(r, \tau)}{\tau}d\tau\right),$$

that is,

$$\partial_t \left(\frac{V}{t} \exp \left(\int_t^{t_{B'}} \frac{l_1(r,\tau) - 1}{\tau} d\tau \right) \right) = l_2(r,t) t^{1-\delta} \exp \left(\int_t^{t_{B'}} \frac{l_1(r,\tau) - 1}{\tau} d\tau \right). \qquad (2.12)$$

Integration of the above equation from t to $t_{B'}$ yields that

$$\frac{V}{t}(r,t_{B'}) - \frac{V}{t}(r,t) \exp \left(\int_t^{t_{B'}} \frac{l_1(r,\tau) - 1}{\tau} \right) = \int_t^{t_{B'}} l_2(r,\tau) \tau^{1-\delta} \exp \left(\int_\tau^{t_{B'}} \frac{l_1(r,s) - 1}{s} ds \right) d\tau.$$

Thus

$$e^{-K_4 t_{B'}} \left| \frac{V}{t}(r,t) \right| < \left| \frac{V}{t}(r,t) \right| \exp \left(\int_t^{t_{B'}} \frac{l_1(r,\tau) - 1}{\tau} d\tau \right)$$

$$\leq \left| \frac{V}{t} \right| (r,t_{B'}) + \int_t^{t_{B'}} |l_2(r,\tau) \tau^{1-\delta}| \exp \left(\int_\tau^{t_{B'}} \left| \frac{l_1(r,s) - 1}{s} \right| ds \right) d\tau$$

$$\leq \left| \frac{V}{t} \right| (r,t_{B'}) + e^{K_4 t_{B'}} \frac{K_5 t_{B'}^{2-\delta}}{2 - \delta},$$

that is,

$$\left| \frac{V}{t}(r,t) \right| < e^{K_4 t_{B'}} \left\{ \max_{(r,t_{B'}) \in ABC} \left| \frac{V}{t} \right| (r,t_{B'}) + e^{K_4 t_{B'}} \frac{K_5 t_{B'}^{2-\delta}}{2 - \delta} \right\} := \widehat{M}. \qquad (2.13)$$

Because of the arbitrariness of t and the continuity of V, we obtain that $|V(r,t)| \leq \widehat{M} t$ holds for any $t \in [0, t_{B'}]$. $\square$

Thus we see that the two derivatives $\partial^+ p$ and $\partial^- p$ approach a common value on the sonic curve with a rate of $O(\sqrt{r + p})$. Furthermore, we have the following result.

Lemma 2.4. $R, S,$ and $V(r,t)/t$ is uniformly continuous in the domain ABC including the sonic line.

Proof. Let us choose two sonic points $P_1(r_1, 0), P_2(r_2, 0) \in AC$. Then we draw a positive characteristic γ_+ passing through P_2 and a negative characteristic γ_- passing through P_1. Let $Q(r_b, t_b) \in \Omega$ be an intersection point of γ_+ and γ_-. We note that Q approaches a sonic point as $|r_2 - r_1| \to 0$. From (2.1), we see that

$$\partial_- R = \frac{2t^2}{S - \lambda^{-1}} QR \cdot \frac{S - R}{t}, \quad \partial_+ S = \frac{2t^2}{R + \lambda^{-1}} QS \cdot \frac{R - S}{t}. \qquad (2.14)$$

Since $V(r,t)/t$ is uniformly bounded, we have $|\partial_- R|, |\partial_+ S| \leq L$ for some constant L.

Now integrating the first equation of (2.14) along γ_- from P_1 to Q gives

$$|R(r_b, t_b) - R(r_1, 0)| \leq \int_0^{t_b} \left| \frac{2t^2 QR}{S - \lambda^{-1}} \cdot \frac{S - R}{t} \right| dt \leq L t_b.$$

Similary the second equation of (2.14) along γ_+ from P_2 to Q gives

$$|S(r_b, t_b) - S(r_2, 0)| \leq L t_b.$$

Moreover, since $R = S$ on AC and R, S are continuous on Ω, $R(r_j, 0) = S(r_j, 0)$ for $j = 1, 2$, and $|R(r_b, t_b) - S(r_2, 0)| \to 0$ as $t_b \to 0$. Then

$$|R(r_1, 0) - R(r_2, 0)| \leq |R(r_1, 0) - R(r_b, t_b)| + |S(r_2, 0) - S(r_b, t_b)| + |R(r_b, t_b) - S(r_b, t_b)|$$

$$\leq w(t_b),$$

where $w(t_b) := 2Lt_b + |R(r_b, t_b) - S(r_b, t_b)|$. Similarly, $|S(r_1, 0) - S(r_2, 0)| \leq w(t_b)$. Since $w(t_b) \to 0$ as $|r_2 - r_1| \to 0$, which implies that R and S are both continuous on the sonic line AC, we prove that R and S are both uniformly continuous in the whole domain ABC.

Let $\varepsilon > 0$ be given and $t_b \in (0, t_{B'})$ with $t_{B'} \ll 1$ will be determined later. From (2.12), we have

$$\partial_t \Big(\frac{V}{t} \exp \Big(\int_t^{t_b} \frac{l_1(r, \tau) - 1}{\tau} d\tau \Big) \Big) = l_2(r, t) t^{1-\delta} \exp \Big(\int_t^{t_b} \frac{l_1(r, \tau) - 1}{\tau} d\tau \Big),$$

which yields

$$\frac{V}{t}(r, t_b) - \frac{V}{t}(r, 0) \exp \Big(\int_0^{t_b} \frac{l_1 - 1}{\tau} d\tau \Big) = \int_0^{t_b} l_2(r, \tau) \tau^{1-\delta} \exp \Big(\int_\tau^{t_b} \frac{l_1(r, s) - 1}{s} ds \Big) d\tau.$$

Then

$$\Big| \frac{V}{t}(r, t_b) - \frac{V}{t}(r, 0) \Big| \leq \Big| \frac{V}{t}(r, 0) \Big| \Big(\exp \int_0^{t_b} \Big| \frac{l_1 - 1}{\tau} \Big| d\tau - 1 \Big) + \int_0^{t_b} |l_2 \tau^{1-\alpha}| \exp \Big(\int_\tau^{t_b} \frac{l_1 - 1}{s} ds \Big)$$

$$\leq 2\widehat{M} K_4 t_b + \frac{K_5 t_b^{2-\delta} e^{K_4 t_b}}{2 - \delta}$$

$$\leq K_0 t_b^{2-\delta}$$

for some constant K_0. Now we take sufficiently small t_b so that $K_0 t_b^{2-\delta} < \varepsilon/4$. For this fixed t_b, since V/t is continuous inside Ω, we can take $d > 0$ such that, if $|r_2 - r_1| \leq d$, then we have

$$\Big| \frac{V}{t}(r_1, t_b) - \frac{V}{t}(r_2, t_b) \Big| < \frac{\varepsilon}{4}.$$

Thus for any $|r_1 - r_2| \leq d$, we have

$$\Big| \frac{V}{t}(r_1, 0) - \frac{V}{t}(r_2, 0) \Big| \leq \Big| \frac{V}{t}(r_1, 0) - \frac{V}{t}(r_1, t_b) \Big| + \Big| \frac{V}{t}(r_1, t_b) - \frac{V}{t}(r_2, t_b) \Big|$$

$$+ \Big| \frac{V}{t}(r_2, t_b) - \frac{V}{t}(r_2, 0) \Big|$$

$$< 2K_0 t_b^{2-\delta} + \frac{\varepsilon}{4} < \varepsilon.$$

Therefore, V/t is uniformly continuous in the whole domain. $\square$

Proof of the Main Theorem: We consider the sonic curve as the limit of the level curves $\theta = \theta_\varepsilon(r)$ in the (r, θ) plane:

$$r + p(r, \theta) = \varepsilon.$$

In the semi-hyperbolic regions, R, S are both strictly negative and uniformly bounded, and thus

$$p_\theta = \frac{R + S}{2}$$

is negative and bounded. Therefore, each $\theta_\varepsilon'(r) = -(1 + p_r)/p_\theta$ is well defined. Next we have

$$p_r = \frac{R - S}{2\lambda^{-1}} = \frac{R - S}{t} \cdot \frac{r - t^2}{2r\sqrt{2r - t^2}}.$$

According to Lemma 2.3, p_r is uniformly bounded. Therefore $|\theta_\varepsilon'(r)|$ is also uniformly bounded in the whole domain, including the sonic line AC. Moreover, since p_r and p_θ are

uniformly continuous in the whole domain, $\theta'_\varepsilon(r)$ is uniformly continuous as well. Hence $\theta'(r)$ is continuous on the sonic line. That is, the sonic line is C^1 continuous.

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