On distance, geodesic and arc transitivity of graphs

Alice Devillers, Wei Jin*, Cai Heng Li and Cheryl E. Praeger‡

Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics,

The University of Western Australia, Crawley, WA 6009, Australia

Abstract

We compare three transitivity properties of finite graphs, namely, for a positive integer s, s-distance transitivity, s-geodesic transitivity and s-arc transitivity. It is known that if a finite graph is s-arc transitive but not $(s+1)$-arc transitive then $s \leq 7$ and $s \neq 6$. We show that there are infinitely many geodesic transitive graphs with this property for each of these values of s, and that these graphs can have arbitrarily large diameter if and only if $1 \leq s \leq 3$. Moreover, for a prime p we prove that there exists a graph of valency p that is 2-geodesic transitive but not 2-arc transitive if and only if $p \equiv 1 \pmod 4$, and for each such prime there is a unique graph with this property: it is an antipodal double cover of the complete graph K_{p+1} and is geodesic transitive with automorphism group $PSL(2,p) \times Z_2$.

Keywords: Graphs; distance-transitivity; geodesic-transitivity; arc-transitivity

1 Introduction

The study of finite distance-transitive graphs goes back to Higman's paper [9] in which "groups of maximal diameter" were introduced. These are permutation groups which act distance transitively on some graph. The family

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‡E-mail addresses: alice.devillers@uwa.edu.au (A.Devillers), weijin@maths.uwa.edu.au (W.Jin), cai.heng.li@uwa.edu.au (C.H.Li) and cheryl.praeger@uwa.edu.au (C.E.Praeger).

of distance transitive graphs includes many interesting and important graphs, such as the Johnson graphs, Hamming graphs, Odd graphs, Paley graphs and certain antipodal double covers of complete graphs that are discussed in this paper.

We examine graphs with various symmetry properties which are stronger than arc-transitivity. The weakest of these properties is s-distance transitivity where s is at most the diameter of the graph (see Section 2 for a precise definition), in which, all pairs of vertices at a given distance at most s are equivalent under graph automorphisms. If s is equal to the diameter, the graph is said to be *distance transitive*.

For a fixed integer $s \geq 2$, two stronger concepts than s-distance transitivity are important for our work. The first is s-geodesic transitivity for s at most the diameter, in which for each integer $t \leq s$, all ordered t-paths $(v_0, v_1, \cdots, v_t)$ for which v_0, v_t are at distance t, are equivalent under graph automorphisms. Such t-paths are called *t-geodesics*. The second property is s-arc transitivity, in which for each integer $t \leq s$, all ordered t-walks $(v_0, v_1, \cdots, v_t)$, with $v_{i-1} \neq v_{i+1}$, for each $i = 1, 2, \cdots, t-1$, are equivalent. It is straightforward to verify that, for s at most the diameter, s-arc transitivity implies s-geodesic transitivity, which in turn implies s-distance transitivity. The purpose of this paper is to provide some insight into the differences between these conditions, especially for $s = 2$.

A graph is called *geodesic transitive* if it is s-geodesic transitive for s equal to the diameter. Geodesic transitive graphs are in particular 1-arc transitive, and may or may not be s-arc transitive for some $s > 1$. Our first result specifies the possible arc transitivities for geodesic transitive graphs. We note that (see [20]) if a finite graph is s-arc transitive but not $(s+1)$-arc transitive then $s \in \{1, 2, 3, 4, 5, 7\}$.

Theorem 1.1 *For each $s \in \{1, 2, 3, 4, 5, 7\}$, there are infinitely many geodesic transitive graphs that are s-arc transitive but not $(s+1)$-arc transitive. Moreover, there exist geodesic transitive graphs that are s-arc transitive but not $(s+1)$-arc transitive with arbitrarily large diameter if and only if $s \in \{1, 2, 3\}$.*

Theorem 1.1 is proved in Section 3 by analysing some well-known families of distance transitive graphs, namely, Johnson graphs, Hamming graphs, Odd graphs and classical generalized polygons. In fact if $s \geq 4$, then all graphs with the property of Theorem 1.1 have diameter at most 8 and are known explicitly, see Proposition 3.6.

Our second result is the main result of the paper, proved in Subsection 5.1. It classifies explicitly all 2-geodesic transitive graphs of prime valency that are not 2-arc transitive.

Theorem 1.2 *Let Γ be a connected 2-geodesic transitive but not 2-arc transitive graph of prime valency p. Then Γ is a nonbipartite antipodal double cover of the complete graph K_{p+1}, where $p \equiv 1 \pmod 4$. Further, Γ is geodesic transitive and Γ is unique up to isomorphism with automorphism group $PSL(2,p) \times Z_2$.*

This family of graphs arose also in [7, 10, 17] in different contexts, and they are distance transitive of diameter 3. We prove in Lemma 5.4 that each graph in the family is geodesic transitive. In Subsection 5.2, we construct these graphs as in [10] and prove that they satisfy the hypotheses of Theorem 1.2. It would be interesting to know if a similar classification is possible for non-prime valencies. This is the subject of further research by the second author.

We complete our comparison of these transitivity properties by producing 2-distance transitive graphs that are not 2-geodesic transitive. We give just one infinite family of examples, namely the Paley graphs $P(q)$ where $q \geq 13$ is a prime power and $q \equiv 1 \pmod 4$ (see Section 4 for a definition). These graphs $P(q)$ have diameter 2 and are well-known to be distance transitive. The information about these graphs is important for the proof of Theorem 1.1.

Theorem 1.3 *Let $q \equiv 1 \pmod 4$ be a prime power. Then the Paley graph $P(q)$ is distance transitive for all q, but $P(q)$ is geodesic transitive if and only if $q = 5$ or 9.*

Remark 1.4 The Paley graphs $P(q)$ with $q > 9$ seem to be the first family of diameter 2 graphs observed to be distance transitive but not geodesic transitive. Since all diameter 2 distance transitive graphs are known, it would be interesting to know which of them are geodesic transitive.

These results give some insight into the relationship between s-distance transitivity, s-geodesic transitivity and s-arc transitivity for $s = 2$. It would be interesting to understand the relationship between these properties for larger values of s.

2 Preliminaries

All graphs of this paper are finite, undirected simple graphs. We first give some definitions which will be used throughout the paper. Let Γ be a graph. We use $V\Gamma, E\Gamma$, and $\mathrm{Aut}\Gamma$ to denote its *vertex set*, *edge set* and *full automorphism group*, respectively. The size of $V\Gamma$ is called the *order* of the graph. The graph Γ is said to be *vertex transitive* (or *edge transitive*) if the action of $\mathrm{Aut}\Gamma$ on $V\Gamma$ (or $E\Gamma$) is transitive.

For (not necessarily distinct) vertices u and v in $V\Gamma$, a *walk* from u to v is a finite sequence of vertices $(v_0, v_1, \cdots, v_n)$ such that $v_0 = u$, $v_n = v$ and $(v_i, v_{i+1}) \in E\Gamma$ for all i with $0 \le i < n$, and n is called the *length* of the walk. If $v_i \ne v_j$ for $0 \le i < j \le n$, the walk is called a *path* from u to v. The smallest value for n such that there is a path of length n from u to v is called the *distance* from u to v and is denoted $d_\Gamma(u, v)$. The *diameter* $\mathrm{diam}(\Gamma)$ of a connected graph Γ is the maximum of $d_\Gamma(u, v)$ over all $u, v \in V\Gamma$.

Let $G \le \mathrm{Aut}\Gamma$ and $s \le \mathrm{diam}(\Gamma)$. We say that Γ is (G, s)*-distance transitive* if, for any two pairs of vertices (u_1, v_1), (u_2, v_2) with the same distance $t \le s$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$.

For a positive integer s, an s-arc of Γ is a walk $(v_0, v_1, \cdots, v_s)$ of length s in Γ such that $v_{j-1} \ne v_{j+1}$ for $1 \le j \le s - 1$. Moreover, a 1-arc is called an *arc*. Suppose $G \le \mathrm{Aut}\Gamma$. Then Γ is said to be (G, s)*-arc transitive* if, for any two t-arcs α and β where $t \le s$, there exists $g \in G$ such that $\alpha^g = \beta$. A remarkable result of Tutte about (G, s)-arc transitive graphs with valency three shows that $s \le 5$, see [18, 19]. About twenty years later, relying on the classification of finite simple groups, Weiss in [20] proved that there are no $(G, 8)$-arc transitive graphs with valency at least three. For more work on (G, s)-arc transitive graphs see [6, 14, 15].

Let u, v be distinct vertices of Γ. Then a path of shortest length from u to v is called a *geodesic* from u to v, or sometimes an i*-geodesic* if $d_\Gamma(u, v) = i$. Moreover, 1-geodesics are arcs. If Γ is connected, for each $i \in \{1, \cdots, \mathrm{diam}(\Gamma)\}$, we set $geod_i(\Gamma) = \{$all i-geodesics of $\Gamma\}$. Let $G \le \mathrm{Aut}\Gamma$ and $s \le \mathrm{diam}(\Gamma)$. Then Γ is said to be (G, s)*-geodesic transitive* if, for each $i = 1, 2, \cdots, s$, G is transitive on $\mathrm{geod}_i(\Gamma)$. When $s = \mathrm{diam}(\Gamma)$, Γ is said to be G*-geodesic transitive*. Moreover, if we do not wish to specify the group we will say that Γ is s*-geodesic transitive* or *geodesic transitive* respectively, and similarly for the other properties.

The following are some examples of geodesic transitive graphs.

Example 2.1 (i) *For any* $n \ge 1$*, both the complete graph* K_n *and the complete bipartite graph* $K_{n,n}$ *are geodesic transitive.*

(ii) *Let* $\Gamma = \mathrm{K}_{m[b]}$ *be a complete multipartite graph with* $m \ge 3$ *parts of size* $b \ge 2$*. Then* $A := \mathrm{Aut}\Gamma = S_b \wr S_m$ *is transitive on* $V\Gamma$ *and on the set of arcs* $\mathrm{geod}_1(\Gamma)$*. Let* (u, v) *be an arc of* Γ*. Then* $|\Gamma_2(u) \cap \Gamma(v)| = b - 1$ *and* $A_{u,v}$ *induces* S_{b-1} *on* $\Gamma_2(u) \cap \Gamma(v)$*, so* Γ *is* 2*-geodesic transitive. Since the diameter of* Γ *is* 2*, it follows that* Γ *is geodesic transitive.*

We also give three infinite families of geodesic transitive graphs with arbitrarily large diameter in Section 3.

If a graph Γ is (G,s)-arc transitive and $s \leq \mathrm{diam}(\Gamma)$, then s-geodesics and s-arcs are same, and Γ is (G,s)-geodesic transitive. However, Γ can be (G,s)-geodesic transitive but not (G,s)-arc transitive. The girth of Γ, denoted by $girth(\Gamma)$, is the length of the shortest cycle in Γ.

Lemma 2.2 *Suppose that a graph Γ is (G,s)-geodesic transitive for some $G \leq \mathrm{Aut}\Gamma$ with $2 \leq s \leq \mathrm{diam}(\Gamma)$. Then Γ is (G,s)-arc transitive if and only if* $\mathrm{girth}(\Gamma) \geq 2s$.

Proof. Note that each i-geodesic is an i-arc, for $1 \leq i \leq \mathrm{diam}(\Gamma)$. Thus Γ is (G,s)-arc transitive if and only if each s-arc is an s-geodesic, and this is true if and only if $\mathrm{girth}(\Gamma) \geq 2s$. $\Box$

An example of graphs which do not have the property of Lemma 2.2 for $s = 2$ are the complete multipartite graphs $\mathrm{K}_{m[b]}$ with $m \geq 3$ parts of size $b \geq 2$, which have girth 3, are 2-geodesic transitive (see Example 2.1 (ii)), but are not 2-arc transitive (by Lemma 2.2).

In our study of Paley graphs we use the concept of a Cayley graph. For a finite group G, and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \,|\, g \in G, s \in S\}$. The group $R(G) = \{\rho_x | x \in G\}$, where $\rho_x : g \mapsto gx$, is a subgroup of the automorphism group of $\mathrm{Cay}(G,S)$ and acts regularly on the vertex set, that is to say, $R(G)$ is transitive and only the identity ρ_{1_G} fixes a vertex. It follows that $\mathrm{Cay}(G,S)$ is vertex transitive.

The following is a criterion for a connected graph to be a Cayley graph.

Lemma 2.3 ([1, Lemma 16.3]) *Let Γ be a connected graph. Then a subgroup H of* $\mathrm{Aut}\Gamma$ *acts regularly on the vertices if and only if Γ is isomorphic to a Cayley graph* $\mathrm{Cay}(H,S)$ *for some set S which generates H.*

For a graph Γ, the *k-distance graph* Γ_k of Γ is the graph with vertex set $V\Gamma$, such that two vertices are adjacent if and only if they are at distance k in Γ. If $d = \mathrm{diam}(\Gamma) \geq 2$, and Γ_d is a disjoint union of complete graphs, then Γ is said to be an *antipodal graph*.

Suppose that Γ is an antipodal distance-transitive graph of diameter d. Then we may partition its vertices into sets, called *fibres*, such that any two distinct vertices in the same fibre are at distance d and two vertices in different fibres are at distance less than d. Godsil, Liebler and Praeger gave a complete classification of antipodal distance transitive covers of complete graphs.

The following lemma follows directly from the Main Theorem of [7]. We shall apply it to characterise the examples in Theorem 1.2.

Lemma 2.4 ([7]) *Suppose that G is a distance transitive automorphism group of a finite nonbipartite graph Γ. Suppose further that Γ is antipodal with fibres of size 2 and with antipodal quotient the complete graph K_n. Then either $\Gamma = K_{n[2]}$ of diameter 2, or Γ has diameter 3 and one of the following holds.*

I. Γ *appears in* [17] *and is one of*

(a) $n = 2^{2m-1} \pm 2^{m-1}, G \leq 2 \times Sp(2m, 2)$ *for some* $m \geq 3$.

(b) $n = 2^{2a+1} + 1$, $G \leq 2 \times \mathrm{Aut}(R(q))$ *for some* $a \geq 1$.

(c) $n = 176$, $HiS \leq G \leq 2 \times HiS$, *or* $n = 276$, $Co_3 \leq G \leq 2 \times Co_3$.

(d) $n = q^3 + 1$, $G \leq 2 \times P\Gamma U(3, q^2)$ *for some* $q > 3$.

(e) $n = q + 1$, $G \leq 2 \times P\Sigma L(2, q)$ *for some* $q \equiv 1 \pmod 4$.

II. Γ *is a graph appearing in Example* 3.6 *of* [7], *and* $n = q^{2d}$ *with* q *even.*

Partitions and quotient graphs: Let G be a group of permutations acting on a set Ω. A *G-invariant partition* of Ω is a partition $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$ such that for each $g \in G$, and each $B_i \in \mathcal{B}$, the image $B_i^g \in \mathcal{B}$. The parts of Ω are often called *blocks* of G on Ω. For a G-invariant partition $\mathcal{B}$ of Ω, we have two smaller transitive permutation groups, namely the group $G^{\mathcal{B}}$ of permutations of $\mathcal{B}$ induced by G; and the group $G_{B_i}^{B_i}$ induced on B_i by G_{B_i} where $B_i \in \mathcal{B}$.

Let Γ be a graph, and $G \leq \mathrm{Aut}\Gamma$. Suppose $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$ is a G-invariant partition of $V\Gamma$. The *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ such that $\{B_i, B_j\}$ is an edge of $\Gamma_{\mathcal{B}}$ if and only if there exist $x \in B_i, y \in B_j$ such that $\{x, y\} \in E\Gamma$. We say that $\Gamma_{\mathcal{B}}$ is *nontrivial* if $1 < |\mathcal{B}| < |V\Gamma|$. The graph Γ is said to be a *cover* of $\Gamma_{\mathcal{B}}$ if for each edge $\{B_i, B_j\}$ of $\Gamma_{\mathcal{B}}$ and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$.

3 Proof of Theorem 1.1

In this section, we first describe three families of geodesic transitive graphs, each with unbounded diameter and valency, namely the Johnson graphs, Hamming graphs and Odd graphs. Graphs in these families are s-arc transitive but not $(s+1)$-arc transitive for various $s \leq 3$. In the last subsection, we give the proof of Theorem 1.1.

In the following discussion, for integers i, j, we define $[i, j] = \{n \in Z \mid i \leq n \leq j\}$. Note that $[i, i] = \{i\}$ and $[i, j] = \emptyset$ when $i > j$.

3.1 Johnson graphs

Let $\Omega = [1, n]$ where $n \geq 3$, and let $1 \leq k \leq [\frac{n}{2}]$ where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then the *Johnson graph* $J(n, k)$ is the graph whose vertex set is the set of all k-subsets of Ω, and two vertices u and v are adjacent if and only if $|u \cap v| = k - 1$. Let $\Gamma = J(n,k)$. By [3, Section 9.1], Γ has the following properties: Γ has diameter k, valency $k(n-k)$, $\mathrm{Aut}\Gamma \cong S_n \times Z_2$ when $n = 2k \geq 4$, otherwise $\mathrm{Aut}\Gamma \cong S_n$, Γ is distance transitive, and for any two vertices u and v,

$$u \in \Gamma_j(v) \text{ where } j \leq k \text{ if and only if } |u \cap v| = k - j. \qquad (J*)$$

Note that for $k = 1$, $J(n,k) \cong K_n$ which has diameter 1. So in the following discussion, we assume that $k \geq 2$ and $n \geq 4$.

Proposition 3.1 *Let $\Gamma = J(n,k)$ where $2 \leq k \leq [\frac{n}{2}]$ and $n \geq 4$. Then Γ has girth 3, is geodesic transitive, but not 2-arc transitive.*

Proof. Since $k \geq 2$ and $n \geq 4$, it follows that $u_1 = [1,k]$, $u_2 = \{1\} \cup [3, k+1]$ and $u_3 = [2, k+1]$ are three vertices of Γ. By $(J*)$, they are pairwise adjacent, so Γ has girth 3. Hence Γ is not 2-arc transitive. We will prove that Γ is geodesic transitive. Since Γ is distance transitive, it follows that Γ is 1-geodesic transitive.

Now suppose that Γ is $(j-1)$-geodesic transitive where $j \in [2,k]$. Let $\mathcal{V} = (v_0, v_1, \cdots, v_{j-1}, v_j)$ where $v_i = [1, k-i] \cup [k+1, k+i]$ for each $i \in [0, j]$. Then by $(J*)$, $\mathcal{V}$ is a j-geodesic. Let $\mathcal{U}$ be any other j-geodesic. Then since Γ is $(j-1)$-geodesic transitive, there exists $\alpha \in \mathrm{Aut}\Gamma$ such that $\mathcal{U}^\alpha = (v_0, v_1, \cdots, v_{j-1}, u_j)$ which is also a j-geodesic, and in particular $u_j \in \Gamma_j(v_0)$, so $|v_0 \cap u_j| = k - j$ by $(J*)$. Since v_{j-1} and u_j are adjacent, $|v_{j-1} \cap u_j| = k-1$. Hence there exist a unique x such that $\{x\} = v_{j-1} \setminus u_j$ and a unique y such that $\{y\} = u_j \setminus v_{j-1}$. First, if $x \geq k+1$, then $[1, k-(j-1)] \subseteq u_j$, so $|v_0 \cap u_j| \geq k-(j-1)$ which contradicts $|v_0 \cap u_j| = k - j$. Thus $x \in [1, k-(j-1)]$, and hence $[k+1, k+(j-1)] \subseteq u_j$. Second, if $y \leq k$, then $y \in [k-(j-2), k]$. It follows that $v_0 \cap u_j$ contains $([1, k-(j-1)] \cup \{y\}) \setminus \{x\}$, a set of size $k-(j-1)$, which also contradicts $|v_0 \cap u_j| = k - j$. Thus $y > k$, and hence $y \in [k+j, n]$. Therefore, $u_j = ([1, k-(j-1)] \setminus \{x\}) \cup [k+1, k+(j-1)] \cup \{y\}$. Let $A = \mathrm{Aut}\Gamma$. Since $[1, k-(j-1)] \subseteq v_0 \cap v_1 \cap \cdots \cap v_{j-1}$ and $[k+j, n] \subseteq \Omega \setminus (v_0 \cup v_1 \cup \cdots \cup v_{j-1})$, and since $Sym(\Omega) \leq A$, it follows that $Sym([1, k-(j-1)]) \times Sym([k+j, n]) \leq A_{v_0, v_1, \cdots, v_{j-1}}$. Hence there exists $\beta \in A_{v_0, v_1, \cdots, v_{j-1}}$ such that $x^\beta = k - (j-1)$ and $y^\beta = k + j$, and so $(v_0, v_1, v_2, \cdots, u_j)^\beta = (v_0, v_1, v_2, \cdots, v_{j-1}, v_j) = \mathcal{V}$, that is $\mathcal{U}^{\alpha\beta} = \mathcal{V}$. This completes the induction. Thus Γ is geodesic transitive.
$\square$

3.2 Hamming graphs

Let $n \geq 2$ and let d be a positive integer. Then the *Hamming graph* $\mathrm{H}(d,n)$ has vertex set $Z_n^d = Z_n \times Z_n \times \cdots \times Z_n$, seen as a module on the ring $Z_n = [0, n-1]$, and two vertices u, v are adjacent if and only if $u - v$ has exactly one non-zero entry. For a vertex $u \in V\mathrm{H}(d, n)$, we denote by $|u|$ the number of its non-zero entries. Then by [3, Section 9.2], $\mathrm{H}(d, n)$ has diameter d, valency $d(n-1)$, is distance transitive, $\mathrm{AutH}(d, n) \cong S_n \wr S_d$, and for two vertices u, v,

$$u \in \Gamma_k(v) \text{ where } k \leq d \text{ if and only if } |u - v| = k. \qquad (H*)$$

Proposition 3.2 *Let $\Gamma = \mathrm{H}(d, n)$ with $d \geq 2$, $n \geq 2$. Then Γ is geodesic transitive. Moreover, if $n = 2$ and $d \geq 3$ then Γ has girth 4 and is 2-arc transitive but not 3-arc transitive, while if $n \geq 3$ then Γ has girth 3 and is not 2-arc transitive.*

Proof. Since Γ is distance transitive, it follows that Γ is 1-geodesic transitive.

Now, suppose that Γ is $(j-1)$-geodesic transitive where $j \in [2, d]$. Let $\mathcal{V} = (v_0, v_1, \cdots, v_j)$ where for each $i \in [0, j]$, $v_i = (1, \cdots, 1, 0, \cdots, 0)$, the first i entries are equal to 1 and the last $(d-i)$ entries are equal to 0. Then by $(H*)$, $\mathcal{V}$ is a j-geodesic.

Suppose that $\mathcal{U}$ is any other j-geodesic of Γ. Since Γ is $(j-1)$-geodesic transitive, there exists $\alpha \in \mathrm{Aut}\Gamma$ such that $\mathcal{U}^\alpha = (v_0, v_1, \cdots, v_{j-1}, u_j)$ for some u_j. Suppose that the last $d-(j-1)$ entries of u_j are 0. Since v_{j-1}, u_j are adjacent, $|u_j - v_{j-1}| = 1$. Hence one of the first $j-1$ entries of u_j is equal to x and the rest are 1 for some $x \neq 1$, while the last $d-(j-1)$ entries are 0. Thus $|u_j - v_0| = j-2$ or $j-1$ according as x is 0 or not. However by $(H*)$, $|u_j - v_0| = j$ which is a contradiction. Thus, u_j has at least one non-zero entry in the last $d-(j-1)$ entries. Further, since the last $d-(j-1)$ entries of v_{j-1} are 0 and $|u_j - v_{j-1}| = 1$, it follows that for some $x \in Z_n \setminus \{0\}$, the first $(j-1)$ entries of u_j are 1, and x is the unique non-zero entry in the last $d-(j-1)$ entries. Moreover, since for each $i \in [0, j-1]$, the last $d-(j-1)$ entries of v_i are 0, it follows that $Sym(Z_n \setminus \{0\}) \wr Sym([j, d]) \leq A_{v_0, \cdots, v_{j-1}}$. Thus there exists $\beta \in A_{v_0, \cdots, v_{j-1}}$ such that $u_j^\beta = (1, \cdots, 1, 1, 0, \cdots, 0) = v_j$. Therefore, $\mathcal{U}^{\alpha\beta} = \mathcal{V}$, and hence Γ is j-geodesic transitive. By induction, Γ is geodesic transitive.

If $n = 2$, then for each vertex u, and for any two vertices v, w of $\Gamma(u)$, $|v - w| = 2$, that is v, w are not adjacent, so the girth of Γ is not 3. Further, $u_1 = (0, 0, \cdots, 0)$, $u_2 = (1, 0, \cdots, 0)$, $u_3 = (0, 1, 0, \cdots, 0)$ and $u_4 = (1, 1, 0 \cdots, 0)$ are four vertices of Γ such that $(u_1, u_2, u_4, u_3, u_1)$ is a 4-cycle, so the girth is 4. Now 2-arcs and 2-geodesics are the same, and since $\mathrm{diam}(\Gamma) = d \geq 3$, it follows that Γ is 2-arc transitive but not 3-arc transitive. If $n \geq 3$, then (u_1, u_2, u_3, u_1) is a triangle where $u_1 = (0, 0, \cdots, 0)$, $u_2 = (1, 0, 0, \cdots, 0)$ and $u_3 = (2, 0, 0, \cdots, 0)$, so Γ has girth 3, that is, Γ is arc-transitive but not 2-arc transitive. □

3.3 Odd graphs

Let $\Omega = [1, 2k+1]$ where $k \geq 1$. Then the *Odd graph* O_{k+1} is the graph whose vertex set is the set of all k-subsets of Ω, and two vertices are adjacent if and only if they are disjoint. By [3, Section 9.1], O_{k+1} is distance transitive, its diameter is k, valency is $k+1$, and $\mathrm{Aut}O_{k+1} \cong S_{2k+1}$. By [2], for two vertices u, v of O_{k+1},

$u \in \Gamma_i(v)$ if and only if $|u \cap v| = j$ when $i = 2j+1$; $|u \cap v| = k-j$ when $i = 2j$. $\quad (O*)$

Note that if $k = 1$, then $O_2 \cong C_3$ is s-arc transitive for all $s \geq 1$. So we will assume that $k \geq 2$.

Proposition 3.3 *Let $\Gamma = O_{k+1}$ with $k \geq 2$. Then Γ is geodesic transitive, and is 3-arc transitive but not 4-arc transitive.*

Proof. Since Γ is distance transitive, it follows that Γ is 1-geodesic transitive.

Now, suppose that Γ is $(j-1)$-geodesic transitive where $j \in [2, k]$. Let $\mathcal{V} = (v_0, v_1, \cdots, v_j)$ where $v_{2i} = [1, k-i] \cup [2k-i+2, 2k+1]$ and $v_{2i+1} = [k-i+1, 2k-i]$ for $i \geq 0$. Then by $(O*)$, $\mathcal{V}$ is a j-geodesic. Suppose that $\mathcal{U}$ is any other j-geodesic of Γ. Then since Γ is $(j-1)$-geodesic transitive, there exists $\alpha \in A := \mathrm{Aut}\Gamma$ such that $\mathcal{U}^\alpha = (v_0, v_1, \cdots, v_{j-1}, u_j)$ for some u_j.

First, suppose that $j = 2l$ is even. Let $\Delta_1 = [1, k-l+1]$, $\Delta_2 = [k-l+2, k]$, $\Delta_3 = [k+1, 2k-l+1]$ and $\Delta_4 = [2k-l+2, 2k+1]$. Then $\Omega = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, $v_{j-1} = \Delta_2 \cup \Delta_3$ and $v_j = (\Delta_1 \cup \Delta_4) \setminus \{k-l+1\}$. Since v_{j-1} and u_j are adjacent, it follows that $v_{j-1} \cap u_j = \emptyset$, and hence $u_j \subseteq \Delta_1 \cup \Delta_4$. Since $u_j \in \Gamma_j(v_0)$ and j is even, by $(O*)$, $|v_0 \cap u_j| = k-l$. Hence $|u_j \setminus v_0| = l$. Since $\Delta_1 \subseteq v_0$, it follows that $u_j \setminus v_0 \subseteq \Delta_4$. Since $|u_j \setminus v_0| = l = |\Delta_4|$, it follows that $u_j \setminus v_0 = \Delta_4$, so $\Delta_4 \subseteq u_j$. Hence $u_j = (\Delta_1 \cup \Delta_4) \setminus \{x\}$ for some $x \in \Delta_1$. Since for each $m \in [0, j-1]$, $\Delta_1 \subseteq v_m$ when m is even, and $\Delta_1 \cap v_m = \emptyset$ when m is odd, it follows that $Sym(\Delta_1) \leq A_{v_0, v_1, \cdots, v_{j-1}}$. There exists $\beta \in Sym(\Delta_1)$ such that $x^\beta = k-l+1$. Hence $u_j^\beta = v_j$ and so $\mathcal{U}^{\alpha\beta} = \mathcal{V}$.

Second, suppose that $j = 2l+1$ is odd. We let $\Delta_1 = [1, k-l]$, $\Delta_2 = [k-l+1, k]$, $\Delta_3 = [k+1, 2k-l+1]$ and $\Delta_4 = [2k-l+2, 2k+1]$. Then $\Omega = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, $v_{j-1} = \Delta_1 \cup \Delta_4$ and $v_j = (\Delta_2 \cup \Delta_3) \setminus \{2k-l+1\}$, so $u_j \subseteq \Delta_2 \cup \Delta_3$. Since $v_0 \cap \Delta_3 = \emptyset$, it follows that $v_0 \cap u_j \subseteq \Delta_2$. Further since j is odd, by $(O*)$, $|v_0 \cap u_j| = l = |\Delta_2|$. Hence $v_0 \cap u_j = \Delta_2$. Thus $u_j = (\Delta_2 \cup \Delta_3) \setminus \{x\}$ for some $x \in \Delta_3$. Since for each $m \in [0, j-1]$, $\Delta_3 \subseteq v_m$ when m is odd, and $\Delta_3 \cap v_m = \emptyset$ when m is even, it follows that $Sym(\Delta_3) \leq A_{v_0, v_1, \cdots, v_{j-1}}$. There exists $\beta \in Sym(\Delta_3)$ such that $x^\beta = 2k-l+1$, and hence $u_j^\beta = v_j$, and $\mathcal{U}^{\alpha\beta} = \mathcal{V}$. Thus Γ is j-geodesic transitive. Therefore, by induction Γ is geodesic transitive.

If $k = 2$, then Γ is the Petersen graph which has girth 5 and is 3-arc transitive but not 4-arc transitive. If $k \geq 3$, then by [3, Section 9.1], Γ has girth 6, that is, 3-arcs and 3-geodesics are the same. This together with geodesic transitivity show that Γ is 3-arc transitive. Let $k = 3$ and $v_0 = \{1, 2, 3\}$, $v_1 = \{4, 5, 6\}$, $v_2 = \{1, 2, 7\}$, $v_3 = \{3, 4, 5\}$, $v_4 = \{1, 2, 6\}$ and $v_5 = \{1, 6, 7\}$. Then $\mathcal{W}_1 = (v_0, v_1, v_2, v_3, v_4)$ and $\mathcal{W}_2 = (v_0, v_1, v_2, v_3, v_5)$ are two 4-arcs, $d_\Gamma(v_0, v_4) = 2$ and $d_\Gamma(v_0, v_5) = 3$. So there is no automorphism mapping $\mathcal{W}_1$ to $\mathcal{W}_2$, and hence Γ is not 4-arc transitive. If $k \geq 4$, then $\mathrm{diam}(\Gamma) = k \geq 4$ and some 4-arcs lie in 6-cycles and so are not 4-geodesics. Hence Γ is 3-arc transitive but not 4-arc transitive. □

3.4 Proof of Theorem 1.1

First we collect information about the geodesic transitivity of several 4-arc transitive graphs.

Lemma 3.4 *The Biggs-Smith graph and the Foster graph have valency and diameter as in Table 1, and are geodesic transitive. Moreover, for s as in Table 1, these graphs are s-arc transitive but not $(s + 1)$-arc transitive.*

Proof. Let $(\Gamma, s) \in \{(Biggs{-}Smith\ graph, 4), (Foster\ graph, 5)\}$. Then by [3, p.221] and [21, Theorem 1.1], Γ has valency and diameter as in Table 1, and for s as in Table 1, Γ is s-arc transitive but not $(s + 1)$-arc transitive. Thus Γ is s-geodesic transitive. Let $d = \mathrm{diam}(\Gamma)$ and $(v_0, v_1, \cdots, v_d)$ be a d-geodesic. Then $d = s + 3$. By [3, p.221], $|\Gamma(v_j) \cap \Gamma_{j+1}(v_0)| = 1$ for every $j = d - 3, d - 2, d - 1$, and it follows that Γ is geodesic transitive. □

As in [8, p.84] we define a *generalized polygon*, or more precisely, a *generalized d-gon*, as a bipartite graph with diameter d and girth $2d$. The generalized polygons related to the Lie type groups $A_2(q), B_2(q)$ and $G_2(q)$ (q is prime power) are *classical generalized polygons*, and are denoted by $\Delta_{3,q}$, $\Delta_{4,q}$ and $\Delta_{6,q}$, respectively. They are regular of valency $q + 1$.

Lemma 3.5 *The only distance transitive generalized polygons of valency at least 3 that are s-arc transitive but not $(s+1)$-arc transitive, for some $s \geq 4$, are $\Delta_{s-1,q}$ where $(s, q) \in S = \{(4, q), (5, 2^m), (7, 3^m)|$ q is a prime power and m is a positive integer $\}$. Moreover, all these graphs are geodesic transitive.*

Proof. Let Γ be a distance transitive generalized polygon of valency at least 3 that is s-arc transitive but not $(s + 1)$-arc transitive for some $s \geq 4$. Let g be its girth. Suppose that $s < \frac{g-2}{2}$. Then for any two vertices u, v at distance $s + 1$, there exists a unique $(s + 1)$-arc between them. Since Γ is distance transitive, it follows that Γ is $(s+1)$-arc transitive, which contradicts our assumption. Thus $s \geq \frac{g-2}{2}$. Since Γ is a generalized polygon, g is even. By [21, Theorem 1.1], $s \in \{\frac{g+2}{2}, \frac{g}{2}, \frac{g-2}{2}\}$. If $s = \frac{g+2}{2}$ or $\frac{g}{2}$, then [21, Theorem 1.1] shows that Γ is one of $\Delta_{s-1,q}$ where $(s,q) \in S$. Let $A = \mathrm{Aut}\Gamma$. Since Γ is distance transitive, A_u is transitive on $\Gamma_{s+1}(u)$ for each vertex u. Thus, if $s = \frac{g-2}{2}$, by [21, Theorem 1.1], Γ is the Biggs-Smith graph, which is not a generalized polygon. Moreover, since each $\Delta_{s-1,q}$ has diameter $s-1$, it follows that all these graphs are geodesic transitive. $\Box$

These lemmas allow us to specify precisely the geodesic transitive graphs which are 4-arc transitive.

Proposition 3.6 *Let Γ be a regular graph of valency at least 3. Then Γ is geodesic transitive and s-arc transitive but not $(s+1)$-arc transitive for some $s \geq 4$ if and only if Γ is in one of the lines of Table 1. Table 1 also gives the valency, integer s and diameter for each graph. In particular,* $\mathrm{diam}(\Gamma) \leq 8$.

Table 1: Geodesic and s-arc transitive graphs that not $(s+1)$-arc transitive

Graph Γ	Valency	s	Diameter
Foster graph	3	5	8
Biggs-Smith graph	3	4	7
$\Delta_{3,q}$, q is a prime power	$q+1$	4	3
$\Delta_{4,q}$, $q=2^m$, m is a positive integer	$q+1$	5	4
$\Delta_{6,q}$, $q=3^m$, m is a positive integer	$q+1$	7	6

Proof. By Corollary 1.2 of [21], if Γ is distance transitive, s-arc transitive but not $(s+1)$-arc transitive for some $s \geq 4$, then Γ is either the Foster graph or the Biggs-Smith graph, or a generalized polygon. The result now follows from Lemmas 3.4 and 3.5. $\Box$

Proof of Theorem 1.1. It follows from Propositions 3.1, 3.2, 3.3 and 3.6 that, for each $s \in \{1,2,3,4,5,7\}$, there are infinitely many geodesic transitive graphs that are s-arc transitive but not $(s+1)$-arc transitive. Further, for each $s \in \{1,2,3\}$, Propositions 3.1, 3.2 and 3.3 show that there are such graphs with arbitrarily large diameter. Finally, by Proposition 3.6, this is not the case for $s \in \{4,5,7\}$. Therefore geodesic transitive graphs that are s-arc transitive but not $(s+1)$-arc transitive with arbitrarily large diameter occur only for $s \in \{1,2,3\}$. $\Box$

4 Paley graphs

In this section, we discuss a special family of connected Cayley graphs, namely the Paley graphs, which were first defined by Paley in 1933, see [13]. We prove that the Paley graph $P(q)$ is distance transitive but not geodesic transitive whenever $q \geq 13$.

Let $q = p^e$ be a prime power such that $q \equiv 1 \pmod 4$. Let F_q be a finite field of order q. The *Paley graph* $P(q)$ is the graph with vertex set F_q, and two distinct vertices u, v are adjacent if and only if $u - v$ is a nonzero square in F_q. The congruence condition on q implies that -1 is a square in F_q, and hence $P(q)$ is an undirected graph.

Note that the field F_q has $\frac{q-1}{2}$ elements which are nonzero squares, so $P(q)$ has valency $\frac{q-1}{2}$. Moreover, $P(q)$ is a Cayley graph for the additive group $G = F_q^+ \cong Z_p^e$. Let w be a primitive element of F_q. Then $S = \{w^2, w^4, \cdots, w^{q-1} = 1\}$ is the set of nonzero squares of F_q, and $P(q) = \mathrm{Cay}(G, S)$. Define $\tau : F_q \mapsto F_q, x \mapsto x^p$. Then τ is an automorphism of the field F_q, called the *Frobenius automorphism*, and $\mathrm{Aut}F_q = \langle \tau \rangle \cong Z_e$. By [4] (or see [11]), $\mathrm{Aut}P(q) = (G : \langle w^2 \rangle).\langle \tau \rangle \cong (Z_p^e : Z_{\frac{q-1}{2}}).Z_e \leq \mathrm{A\Gamma L(1, q)}$.

Let S' be the set of all nonsquare elements of G. Then $|S'| = \frac{q-1}{2}$. Define the Cayley graph $\Sigma = \mathrm{Cay}(G, S')$ where two vertices u, v are adjacent if and only if $u - v \in S'$. Then Σ is the complement of the Paley graph $P(q)$. Further, multiplication by w induces an isomorphism $\Sigma \cong P(q)$, see [16].

Now, we cite a property of Paley graphs.

Lemma 4.1 ([8, p.221]) *Let $\Gamma = P(q)$, where q is a prime power such that $q \equiv 1 \pmod 4$. Let u, v be distinct vertices of Γ. If u, v are adjacent, then $|\Gamma(u) \cap \Gamma(v)| = \frac{q-5}{4}$; if u, v are not adjacent, then $|\Gamma(u) \cap \Gamma(v)| = \frac{q-1}{4}$.*

4.1 Proof of Theorem 1.3

Let F_q, G and w be as above, and let $G^* = \langle w \rangle$ be the multiplicative group of F_q. As discussed above, $P(q) = \mathrm{Cay}(G, S)$ where $S = \{w^2, w^4, \cdots, w^{q-1} = 1\}$, and $\mathrm{Aut}P(q) = (G : \langle w^2 \rangle).\langle \tau \rangle \cong (Z_p^e : Z_{\frac{q-1}{2}}).Z_e$.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\Gamma = P(q)$ and $A = \mathrm{Aut}\Gamma$. Let $u = 0 \in G$. Then $A_u = \langle w^2 \rangle.\langle \tau \rangle$ has orbits $\{0\}$, S and $S' = G \setminus (\{0\} \cup S)$ on vertices. Now $S = \Gamma(u)$ and as Γ is connected, $\Gamma_2(u)$ must be the other orbit S'. In particular, Γ has diameter 2 and is distance transitive and arc transitive.

Suppose that $v \in \Gamma(u)$. By Lemma 4.1, $|\Gamma(u) \cap \Gamma(v)| = \frac{q-5}{4}$. Thus, $|\Gamma_2(u) \cap \Gamma(v)| = |\Gamma(v)| - |\Gamma(u) \cap \Gamma(v)| - 1 = \frac{q-1}{4}$. If A is transitive on the 2-geodesics of Γ then $A_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$. In particular $\frac{q-1}{4} = |\Gamma_2(u) \cap \Gamma(v)|$ divides $|A_{uv}| = e$ and hence $q = 5$ or 9. We consider $P(5)$ and $P(9)$.

If $q = 5$, then $P(q) \cong C_5$, so $P(q)$ is geodesic transitive.

Now, suppose that $q = 9 = 3^2$. The field F_9 is $\{a + bx \mid a, b \in Z_3\}$ under polynomial addition and multiplication modulo $f(x) = x^2 + 1$. The set S is $\{1, 2, x, 2x\}$, and $\Gamma_2(u) = \{x+1, x+2, 2x+1, 2x+2\}$. Let $v = 1 \in S$. Then $\Gamma_2(u) \cap \Gamma(v) = \{x+1, 2x+1\}$. Since $A_{uv} = \langle \tau \rangle$ and $(x+1)^\tau = x^3+1 = 2x+1$, it follows that A_{uv} is transitive on $\Gamma_2(u) \cap \Gamma(v)$, and hence Γ is $(A, 2)$-geodesic transitive. Since $\mathrm{diam}(\Gamma) = 2$, $\Gamma = P(9)$ is geodesic transitive. $\Box$

4.2 Arc-transitive graphs of odd prime order

In this subsection we characterise the Paley graphs $P(p)$, for primes p, as arc-transitive graphs of given prime order and given valency. This result is used in our proof of Theorem 1.2.

Proposition 4.2 *Let Γ be an arc-transitive graph of prime order p and valency $\frac{p-1}{2}$. Then $p \equiv 1 \pmod 4$, $\mathrm{Aut}\Gamma \cong Z_p : Z_{\frac{p-1}{2}}$, and $\Gamma \cong P(p)$.*

The proof uses the following famous result of Burnside.

Lemma 4.3 ([5, Theorem 3.5B]) *Suppose that G is a primitive permutation group of prime degree p. Then G is either 2-transitive, or solvable and $G \leq AGL(1, p)$.*

Proof of Proposition 4.2. Since Γ has valency $\frac{p-1}{2}$, p is an odd prime. Since Γ is undirected and arc-transitive, it follows that Γ has $p(\frac{p-1}{2})/2$ edges. This implies that $p \equiv 1 \pmod 4$.

Let $A = \mathrm{Aut}\Gamma$. Since A is transitive on $V\Gamma$ and p is a prime, A is primitive on $V\Gamma$. Since Γ is neither complete nor empty, it follows by Lemma 4.3 that $A < AGL(1, p) = Z_p : Z_{p-1}$. Again by vertex transitivity, $Z_p \leq A$. Thus, $A \cong Z_p : Z_m$ where $Z_m < Z_{p-1}$.

Since Z_p is regular on $V\Gamma$, it follows from Lemma 2.3 that Γ is a Cayley graph for Z_p. Thus $\Gamma = \mathrm{Cay}(G, S)$ where $G \cong Z_p$, $S \subseteq G \setminus \{0\}$, $S = S^{-1}$ and $|S| = \frac{p-1}{2}$. Let $v \in V\Gamma$ be the vertex corresponding to $0 \in G$. Then $A_v = Z_m$ acts semiregularly on $G \setminus \{v\}$ with orbits of size m. Since Γ is arc-transitive, A_v acts transitively on S, so $m = |S| = \frac{p-1}{2}$. Thus $A \cong Z_p : Z_{\frac{p-1}{2}}$.

Now we may identify G with F_p^+ and v with 0. Then A_v is the unique subgroup of order $\frac{p-1}{2}$ of $F_p^* = \langle w \rangle$, that is, $A_v = \langle w^2 \rangle$. The A_v-orbits in F_p are $\{0\}$, $S_1 = \{w^2, w^4, \cdots, w^{p-1}\}$ and $S_2 = \{w, w^3, \cdots, w^{p-2}\}$, and so $S = S_1$ or S_2, and $\Gamma = P(p)$ or its complement respectively. In either case, $\Gamma \cong P(p)$. $\Box$

5 Graphs of prime valency that are 2-geodesic transitive but not 2-arc transitive

We prove Theorem 1.2 in Subsection 5.1, that is, we give a classification of connected 2-geodesic transitive graphs of prime valency which are not 2-arc transitive. Note that the assumption of 2-geodesic transitivity implies that the graph is not complete. Since the identification of the examples is made by reference to deep classification results, we give in Subsection 5.2 an explicit construction of these graphs as coset graphs and verify the properties claimed in Theorem 1.2.

5.1 Proof of Theorem 1.2

We will prove Theorem 1.2 in a series of lemmas. Throughout this subsection we assume that Γ is a connected 2-geodesic transitive but not 2-arc transitive graph of prime valency p and that $A = \mathrm{Aut}\Gamma$. The first Lemma 5.1 determines some intersection parameters.

Lemma 5.1 *Let (v, u, w) be a 2-geodesic of Γ. Then $p \equiv 1 \pmod 4$, $|\Gamma(v) \cap \Gamma(u)| = |\Gamma_2(v) \cap \Gamma(u)| = \frac{p-1}{2}$ and $|\Gamma(v) \cap \Gamma(w)|$ divides $\frac{p-1}{2}$. Moreover, $A_v^{\Gamma(v)} \cong Z_p : Z_{\frac{p-1}{2}}$, $A_{v,u}^{\Gamma(v)} \cong Z_{\frac{p-1}{2}}$ and $A_{v,u}$ is transitive on $\Gamma(v) \cap \Gamma(u)$.*

Proof. Since Γ is 2-geodesic transitive but not 2-arc transitive, it follows that Γ is not a cycle. In particular, p is an odd prime. Let $|\Gamma(v) \cap \Gamma(u)| = x$ and $|\Gamma_2(v) \cap \Gamma(u)| = y$. Then $x + y = |\Gamma(u) \setminus \{v\}| = p - 1$. Since Γ is 2-geodesic transitive but not 2-arc transitive, it follows that $\mathrm{girth}(\Gamma) = 3$, so $x \geq 1$. Since the induced subgraph $[\Gamma(v)]$ is an undirected regular graph with $\frac{px}{2}$ edges, and since p is odd, it follows that x is even. This together with $x + y = p - 1$ and the fact that $p - 1$ is even, implies that y is also even.

Since Γ is arc-transitive, $A_v^{\Gamma(v)}$ is transitive on $\Gamma(v)$. Since p is a prime, $A_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$. By Lemma 4.3, either $A_v^{\Gamma(v)}$ is 2-transitive, or $A_v^{\Gamma(v)}$ is solvable and $A_v^{\Gamma(v)} \leq AGL(1, p)$. Since Γ is not complete, it follows that $[\Gamma(v)]$ is not a complete graph. Also since $\mathrm{girth}(\Gamma) = 3$, $[\Gamma(v)]$ is not an empty graph and so $A_v^{\Gamma(v)}$ is not 2-transitive. Hence $A_v^{\Gamma(v)} < AGL(1, p)$. Thus $A_v^{\Gamma(v)} \cong Z_p : Z_m$, where $m|(p-1)$ and $m < p - 1$. Hence $m \leq \frac{p-1}{2}$.

Since Γ is vertex transitive, it follows that $A_u^{\Gamma(u)} \cong Z_p : Z_m$, and hence $A_{u,v}^{\Gamma(u)} \cong Z_m$ is semiregular on $\Gamma(u) \setminus \{v\}$ with orbits of size m. Since Γ is 2-geodesic transitive, $A_{u,v}^{\Gamma(u)}$ is transitive on $\Gamma_2(v) \cap \Gamma(u)$, and hence $y = |\Gamma_2(v) \cap \Gamma(u)| = m$, so $x = p - 1 - m = m(\frac{p-1}{m} - 1) \geq m$, and x is divisible by m.

Now again by arc transitivity, $|\Gamma(u) \cap \Gamma(w)| = |\Gamma(u) \cap \Gamma(v)| = x$. Since $|\Gamma_2(v) \cap \Gamma(u)| = m$, it follows that $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| \leq m - 1$. Since $\Gamma(w) \cap \Gamma(u) = (\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)) \cup (\Gamma(w) \cap \Gamma(u) \cap \Gamma_2(v))$, it follows that

$$x \leq |\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)| + (m-1). \qquad (*)$$

Let $z = |\Gamma(v) \cap \Gamma(w)|$ and $n = |\Gamma_2(v)|$. Since Γ is 2-geodesic transitive, z, n are independent of v, w and, counting edges between $\Gamma(v)$ and $\Gamma_2(v)$ we have $pm = nz$. Now $z \leq |\Gamma(v)| = p$. Suppose first that $z = p$. Then $m = n$ and $\Gamma(v) = \Gamma(w)$, and so for distinct $w_1, w_2 \in \Gamma_2(v)$, $d_\Gamma(w_1, w_2) = 2$. Since Γ is 2-geodesic transitive, it follows that $\Gamma(v) = \Gamma(v')$ whenever $d_\Gamma(v, v') = 2$. Thus $\mathrm{diam}(\Gamma) = 2$, $V\Gamma = \{v\} \cup \Gamma(v) \cup \Gamma_2(v)$ and $|V\Gamma| = 1 + p + m$. Let $\Delta = \{v\} \cup \Gamma_2(v)$. Then for distinct $v_1, v_1' \in \Delta$, $d_\Gamma(v_1, v_1') = 2$; for any $v_1'' \in V\Gamma \setminus \Delta$, v_1, v_1'' are adjacent. Thus, for any $v_1 \in \Delta$, $\Delta = \{v_1\} \cup \Gamma_2(v_1)$. It follows that Δ is a block of imprimitivity for A of size $m + 1$. Hence $(m+1)|(p+m+1)$, so $(m+1)|p$. Since $m|(p-1)$, it follows that $m + 1 = p$ which contradicts the inequality $m \leq \frac{p-1}{2}$.

Thus $z < p$, and so z divides m. Since $|\Gamma(w) \cap \Gamma(u) \cap \Gamma(v)| \leq z$, it follows from $(*)$ that $x \leq z + (m-1) \leq 2m - 1 < 2m$. Since x is divisible by m and $x \geq m$ we have $x = m$. Thus $2m = x + y = p - 1$, so $x = y = m = \frac{p-1}{2}$, and since x is even, $p \equiv 1 \pmod 4$. Also $x = m$ implies that $A_{v,u}$ is transitive on $\Gamma(v) \cap \Gamma(u)$. Finally, since $nz = pm = p(\frac{p-1}{2})$ and $z < p$, it follows that z divides $\frac{p-1}{2}$. $\square$

Lemma 5.2 *For $v \in V\Gamma$, the stabiliser $A_v \cong Z_p : Z_{\frac{p-1}{2}}$.*

Proof. Suppose that (v, u) is an arc of Γ. Then by Lemma 5.1, $A_v^{\Gamma(v)} \cong Z_p : Z_{\frac{p-1}{2}}$, and $A_{v,u}^{\Gamma(v)} \cong Z_{\frac{p-1}{2}}$ is regular on $\Gamma(v) \cap \Gamma(u)$. Let E be the kernel of the action of A_v on $\Gamma(v)$. Let $u' \in \Gamma(v) \cap \Gamma(u)$ and $x \in E$. Then $x \in A_{v,u,u'}$. Since $A_{u,v}^{\Gamma(u)} \cong Z_{\frac{p-1}{2}}$ is semiregular on $\Gamma(u) \setminus \{v\}$, it follows that x fixes all vertices of $\Gamma(u)$. Since x also fixes all vertices of $\Gamma(v)$, this argument for each $u \in \Gamma(v)$ shows that x fixes all vertices of $\Gamma_2(v)$. Since Γ is connected, x fixes all vertices of Γ, hence $x = 1$. Thus $E = 1$, so $A_v \cong Z_p : Z_{\frac{p-1}{2}}$. $\square$

Lemma 5.3 *Let (v, u, w) be a 2-geodesic of Γ. Then $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$, $|\Gamma_2(v) \cap \Gamma(w) \cap \Gamma(u)| = \frac{p-1}{4}$, $|\Gamma_2(v)| = p$, and $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$.*

Proof. Let $z = |\Gamma(v) \cap \Gamma(w)|$ and $n = |\Gamma_2(v)|$. By Lemma 5.1, $|\Gamma(u) \cap \Gamma_2(v)| = \frac{p-1}{2}$ and $z|\frac{p-1}{2}$. Counting the edges between $\Gamma(v)$ and $\Gamma_2(v)$ gives $\frac{p-1}{2}p = nz$. By Lemma 5.2, $A_{v,u} = Z_{\frac{p-1}{2}}$, and by Lemma 5.1, $A_{v,u}$ is transitive on $\Gamma(v) \cap \Gamma(u)$, so $[\Gamma(u)]$ is A_u-arc transitive. Since p is a prime, it follows by Lemma 4.2 that $[\Gamma(u)]$ is a Paley graph $P(p)$. Since $v, w \in \Gamma(u)$ are not adjacent, by Lemma 4.1, $|\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{4}$, hence $z \geq \frac{p-1}{4} + 1$. Since $z | \frac{p-1}{2}$, it follows that $z = \frac{p-1}{2}$. Hence $n = p$. Thus, $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$ and $|\Gamma_2(v)| = p$.

By Lemma 5.1, we have $|\Gamma(v) \cap \Gamma(u)| = \frac{p-1}{2}$. Since Γ is arc transitive, it follows that $|\Gamma(v_1) \cap \Gamma(v_2)| = \frac{p-1}{2}$ for every arc (v_1, v_2). Thus, $|\Gamma(u) \cap \Gamma(w)| = \frac{p-1}{2}$. Since $\Gamma(u) \cap \Gamma(w) = (\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)) \cup (\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w))$ and $|\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{4}$, it follows that $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = \frac{p-1}{2} - \frac{p-1}{4} = \frac{p-1}{4}$. Since $A_v = Z_p : Z_{\frac{p-1}{2}}$, it follows that $A_{v,w} = Z_{\frac{p-1}{2}}$ and $A_{v,w}$ is semiregular on $\Gamma_2(v) \setminus \{w\}$ with orbits of size $\frac{p-1}{2}$. Since $\Gamma_2(v) \cap \Gamma(w) \subseteq \Gamma(w) \setminus \Gamma(v)$ (of size $\frac{p-1}{2}$) and since $|\Gamma_2(v) \cap \Gamma(w) \cap \Gamma(u)| = \frac{p-1}{4} > 0$, it follows that $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$. $\Box$

Lemma 5.4 *Let v be a vertex of Γ. Then $|\Gamma_3(v)| = 1$ and $\mathrm{diam}(\Gamma) = 3$. Further, Γ is geodesic transitive.*

Proof. Suppose that (v, u, w) is a 2-geodesic of Γ. Then by Lemma 5.3, $|\Gamma(v) \cap \Gamma(w)| = \frac{p-1}{2}$ and $|\Gamma_2(v) \cap \Gamma(w)| = \frac{p-1}{2}$. Hence $|\Gamma_3(v) \cap \Gamma(w)| = p - |\Gamma(v) \cap \Gamma(w)| - |\Gamma_2(v) \cap \Gamma(w)| = 1$. Since Γ is 2-geodesic transitive, it follows that $|\Gamma_3(v) \cap \Gamma(w_1)| = 1$ for all $w_1 \in \Gamma_2(v)$. Further Γ is 3-geodesic transitive.

Let $\Gamma_3(v) \cap \Gamma(w) = \{v'\}$, $n = |\Gamma_3(v)|$ and $i = |\Gamma_2(v) \cap \Gamma(v')|$. Counting edges between $\Gamma_2(v)$ and $\Gamma_3(v)$, we have $p = ni$. Since $[\Gamma(w)]$ is a Paley graph and $u, v' \in \Gamma(w)$ are not adjacent, it follows from Lemma 4.1 that $|\Gamma(u) \cap \Gamma(w) \cap \Gamma(v')| = \frac{p-1}{4}$. Since $\Gamma(u) \cap \Gamma_2(v)$ contains these $\frac{p-1}{4}$ vertices as well as w, we have $i \geq \frac{p+3}{4} > 1$. Thus $i = p$ and $n = 1$, that is, $|\Gamma_3(v)| = 1$. Since $|\Gamma_2(v) \cap \Gamma(v')| = p$ and $|\Gamma_2(v)| = p$, it follows that $\Gamma_2(v) = \Gamma(v')$, and so $\mathrm{diam}(\Gamma) = 3$. Therefore Γ is geodesic transitive. $\Box$

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Since the graph Γ is 2-geodesic transitive but not 2-arc transitive, it follows that $\mathrm{girth}(\Gamma) = 3$, and hence Γ is nonbipartite. Let $v \in V\Gamma$. Then it follows from Lemmas 5.1 to 5.4 that $p \equiv 1 \pmod 4$, $|\Gamma_2(v)| = p$, $|\Gamma_3(v)| = 1$ and $\mathrm{diam}(\Gamma) = 3$. Thus, $V\Gamma = \{v\} \cup \Gamma(v) \cup \Gamma_2(v) \cup \{v'\}$, where $\Gamma_3(v) = \{v'\}$, $\Gamma(v) = \Gamma_2(v')$ and $\Gamma_2(v) = \Gamma(v')$. Since Γ is vertex transitive, these properties hold for all vertices of Γ. Thus, Γ is an antipodal graph. By Lemma 5.4, Γ is geodesic transitive, and hence distance transitive.

Let $\mathcal{B} = \{\Delta_1, \Delta_2, \cdots, \Delta_{p+1}\}$ where $\Delta_i = \{u_i, u_i'\}$ such that $d_\Gamma(u_i, u_i') = 3$. Then each Δ_i is a block for $\mathrm{Aut}\Gamma$ of size 2 on $V\Gamma$. Further, for each $j \neq i$, u_i

is adjacent to exactly one vertex of Δ_j, and u_i' is adjacent to the other. The quotient graph Σ such that $V\Sigma = \mathcal{B}$, and two vertices Δ_i, Δ_j are adjacent if and only if $\{\Delta_i, \Delta_j\}$ contains an edge of Γ, is therefore a complete graph $\Sigma \cong K_{p+1}$ and Γ is a cover of Σ.

Therefore, we know that Γ is a nonbipartite antipodal distance transitive cover with fibres of size 2 of the complete graph K_n, where $n = p+1$, $p \equiv 1 \pmod 4$, and $\mathrm{diam}(\Gamma) = 3$, so it is one of the graphs listed in I or II of Lemma 2.4. Since $p \equiv 1 \pmod 4$, it follows that $n \equiv 2 \pmod 4$. However, for I (a), (c), and II, $n \equiv 0 \pmod 4$, so Γ is not a graph in one of these cases. For I (b) and (d), $n-1$ is not a prime, so Γ is not in one of these cases either. Thus Γ is the graph in I (e) of Lemma 2.4 with $q = p$ prime. Hence Γ is unique up to isomorphism and $A = \mathrm{Aut}(\Gamma) \leq PSL(2,p) \times Z_2$. By Lemma 5.2, $A_v = Z_p : Z_{\frac{p-1}{2}}$ for every $v \in V\Gamma$, and since Γ is vertex transitive, it follows that $|A| = p(p+1)(p-1) = |PSL(2,p) \times Z_2|$. Thus $A = PSL(2,p) \times Z_2$. $\Box$

5.2 Construction

In Subsection 5.1, we proved Theorem 1.2 using results of [7] to classify the connected 2-geodesic transitive graphs of prime valency p which are not 2-arc transitive. Here we identify the examples explicitly. The unique example of valency 5 is the icosahedron, and we assume from now on that $p > 5$ and $p \equiv 1 \pmod 4$. Taylor gave a construction of this family of graphs from regular two-graphs, see [3, p.14] and [17]. Here we present a direct construction of these graphs as coset graphs, fleshing out the construction given by the third author in the proof of [10, Theorem 1.1] in order to prove the additional properties we need for Theorem 1.2.

For a finite group G, a core-free proper subgroup H, and an element $g \in G$ such that $G = \langle H, g\rangle$ and $g^2 \in H$, the *coset graph* $\mathrm{Cos}(G, H, HgH)$ is the graph with vertex set $\{Hx | x \in G\}$, and two vertices Hx, Hy adjacent if and only if $yx^{-1} \in HgH$. It is a connected, undirected, and G-arc transitive graph of valency $|H : H \cap H^g|$, see [12].

Construction 5.5 Let $G = PSL(2,p)$ where $p > 5$ is a prime and $p \equiv 1 \pmod 4$. Choose $a \in G$ such that $o(a) = p$. Then $N_G(\langle a\rangle) = \langle a\rangle : \langle b\rangle \cong Z_p : Z_{\frac{p-1}{2}}$ for some $b \in G$, $o(b) = \frac{p-1}{2}$. Further, there exists an involution $g \in G$ such that $N_G(\langle b^2\rangle) = \langle b\rangle : \langle g\rangle \cong D_{p-1}$. Let $H = \langle a\rangle : \langle b^2\rangle$ and $\Gamma = \mathrm{Cos}(G, H, HgH)$.

First, in the following lemma, we show that the coset graph in Construction 5.5 is unique up to isomorphism for each p. We repeatedly use the fact that each $\sigma \in \mathrm{Aut}G$ induces an isomorphism from $\mathrm{Cos}(G, H, HgH)$ to $\mathrm{Cos}(G, H^\sigma, H^\sigma g^\sigma H^\sigma)$, and in particular, we use this fact for the conjugation action by elements of G.

Lemma 5.6 *For each fixed prime $p > 5$ and $p \equiv 1 \pmod 4$, up to isomorphism, the graph Γ in Construction 5.5 is independent of the choices of H and g.*

Proof. Let $G = PSL(2, p)$ where $p > 5$ is a prime and $p \equiv 1 \pmod 4$. Let elements a_i, b_i, g_i and subgroup H_i be chosen as in Construction 5.5 for $i \in \{1, 2\}$. Let $X = PGL(2, p) \cong \mathrm{Aut}(G)$.

Since all subgroups of G of order p are conjugate there exists $x \in G$ such that $\langle a_2\rangle^x = \langle a_1\rangle$, so we may assume that $\langle a_1\rangle = \langle a_2\rangle = K$, say. Let $Y = N_X(K)$. Then $Y = K : \langle y\rangle$ where $o(y) = p - 1$, and $H_1 = K : \langle b_1^2\rangle$ and $H_2 = K : \langle b_2^2\rangle$ are equal to the unique subgroup of Y of order $\frac{p(p-1)}{4}$, that is, $H_1 = H_2 = K : \langle y^4\rangle = H$, say. Next, since all subgroups of Y of order $\frac{p-1}{4}$ are conjugate, there exist $x_1, x_2 \in Y$ such that $\langle b_1^2\rangle^{x_1} = \langle b_2^2\rangle^{x_2} = \langle y^4\rangle$. Since each x_i normalises H we may assume in addition that $\langle b_1^2\rangle = \langle b_2^2\rangle = \langle y^4\rangle < \langle y\rangle$. Thus g_1, g_2 are non-central involutions in $N_G(\langle y^4\rangle) \cong D_{p-1}$, an index 2 subgroup of $N_X(\langle y^4\rangle) = \langle y\rangle : \langle z\rangle \cong D_{2(p-1)}$. The set of non-central involutions in $N_G(\langle y^4\rangle)$ form a conjugacy class of $N_X(\langle y^4\rangle)$ of size $\frac{p-1}{2}$ and consists of the elements $y^{2i}z$, for $0 \le i < \frac{p-1}{2}$. The group $\langle y\rangle$ acts transitively on this set of involutions by conjugation (and normalises H). Hence, for some $u \in \langle y\rangle$, $H^u = H$ and $g_2^u = g_1$. $\Box$

Now we show that the coset graph Γ in Construction 5.5 is 2-geodesic transitive but not 2-arc transitive of prime valency p. We first state some properties of Γ which can be found in [10, Theorem 1.1] and its proof.

Remark 5.7 Let $\Gamma = \mathrm{Cos}(G, H, HgH)$ as in Construction 5.5. Then $G = \langle H, g\rangle$, Γ is connected and G-arc transitive of valency p, $\mathrm{Aut}\Gamma \cong G \times Z_2$, $|V\Gamma| = |G : H| = 2p + 2$. Further, $\mathrm{diam}(\Gamma) = \mathrm{girth}(\Gamma) = 3$, so Γ is not 2-arc transitive.

Again, by the proof of [10, Theorem 1.1], the action of $\mathrm{Aut}\Gamma$ on $V\Gamma$ has a unique system of imprimitivity $\mathcal{B} = \{\Delta_1, \Delta_2, \cdots, \Delta_{p+1}\}$, with $\Delta_i = \{v_i, v_i'\}$ of size 2, and the kernel of the action of $\mathrm{Aut}\Gamma$ on $\mathcal{B}$ has order 2. Moreover, v_i is not adjacent to v_i', and for each $j \ne i$, v_i is adjacent to exactly one point of Δ_j and v_i' is adjacent to the other. Thus, $\Gamma(v_1) \cap \Gamma(v_1') = \emptyset$, $V\Gamma = \{v_1\} \cup \Gamma(v_1) \cup \{v_1'\} \cup \Gamma(v_1')$, and Γ is a nonbipartite double cover of K_{p+1}.

Lemma 5.8 *The graph $\Gamma = \mathrm{Cos}(G, H, HgH)$ in Construction 5.5 is 2-geodesic transitive but not 2-arc transitive.*

Proof. Let $A := \mathrm{Aut}\Gamma$, $v_1 \in V\Gamma$ and $u \in \Gamma(v_1)$. Let E be the kernel of the A-action on $\mathcal{B}$ and $\overline{A} = A/E$. Then by the proof of [10, Theorem 1.1], $E \cong Z_2 \lhd A$, $A = G \times E$, $\overline{A} \cong G = PSL(2,p)$ and $\overline{A}_{\Delta_1} \cong A_{v_1}$. Since $A \cong G \times Z_2$, it follows that $|A_{v_1}| = \frac{p(p-1)}{2}$, and by Lemma 2.4 of [10], $A_{v_1} \cong Z_p : Z_{\frac{p-1}{2}}$, which has a unique permutation action of degree p, up to permutational isomorphism. Since Γ is A-arc-transitive, A_{v_1} is transitive on $\Gamma(v_1)$ and hence on $\mathcal{B} \setminus \{\Delta_1\}$, and therefore also on $\Gamma(v_1')$, all of degree p. Thus the A_{v_1}-orbits in $V\Gamma$ are $\{v_1\}, \Gamma(v_1), \Gamma(v_1')$ and $\{v_1'\}$, and it follows that $\Gamma(v_1') = \Gamma_2(v_1)$. Moreover, $A_{v_1,u} \cong Z_{\frac{p-1}{2}}$ has orbit lengths $1, \frac{p-1}{2}, \frac{p-1}{2}$ in $\Gamma(v_1)$, and hence has the same orbit lengths in $\Gamma_2(v_1)$, and also in $\Gamma(u)$ (since $A_{v_1,u}$ is the point stabiliser of A_u acting on $\Gamma(u)$). Since $\Gamma(v_1) \cap \Gamma(u) \neq \emptyset$, it follows that the $A_{v_1,u}$-orbits in $\Gamma(u)$ are $\{v_1\}, \Gamma(v_1) \cap \Gamma(u)$, and $\Gamma_2(v_1) \cap \Gamma(u)$. It follows that Γ is $(A,2)$-geodesic transitive. Since $\mathrm{girth}(\Gamma) = 3$, Γ is not 2-arc transitive. $\square$

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