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An exact analytical solution for generalized growth models driven by a Markovian dichotomic noise

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Abstract. - Logistic growth models are recurrent in biology, epidemiology, market models, and neural and social networks. They find important applications in many other fields including laser modelling. In numerous realistic cases the growth rate undergoes stochastic fluctuations and we consider a growth model with a stochastic growth rate modelled via an asymmetric Markovian dichotomic noise. We find an exact analytical solution for the probability distribution providing a powerful tool with applications ranging from biology to astrophysics and laser physics.

Introduction. – In this letter we focus our attention on a growth model that was first used to describe the statistical behavior of a population of individual species; for example, human population growth. This is an area of interest several centuries old and is perhaps one of the oldest branches of biology to be studied quantitatively. The first model for human population growth was proposed by Malthus in 1798. In 1838 Verhulst [1] corrected this first model taking into account both the limitation of the growth of a population due to the competition between individuals, and the limitation on the density of the population that the environment can support. An example of this would be the limitation on the amount of food that the system is capable of producing. The proposed equation is known as the logistic equation, $\dot{x} = a_0 x(1 - x)$.

There are many examples of assemblies that consist of a number of elements that interact through cooperative or competitive mechanisms. Some important examples include species that share a given environment, such as animals that live in the seas and rivers of our planet; the components of the central nervous system of a living being; transmission of diseases caused by different types of viruses; interacting vortices in a turbulent fluid; coupled

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reactions between different chemical elements that make up our atmosphere; interactions between galaxies; competition between different political parties; business companies; and negotiating treaties of exchange between different countries.

In the last few decades generalizations of the Malthus-Verhulst model have been applied to lasers physics [2,3] and have been widely considered in scientific literature. In Ref. [4], the reader can find a large list of references that use this type of model for many other physical processes such as the saturation growth process of a population [5–8], which is considered one of the most successful models in the field of population dynamics. In addition the Malthus-Verhulst model has found applications the social sciences [9, 10], autocatalytic chemical reactions [11, 12], biological and biochemical systems [7], grain growth in polycrystalline materials [13], cell growth in foam [14], and as an effective model for the description of the populations of photons in a single mode laser [7, 15–17]. The Lotka-Volterra model that was introduced early last century [18] is a useful model to describe the interaction between two species.

Mathematical models can be constructed either intuitively or from first principles to describe the phenomenon of competition and cooperation of many of the afore mentioned assemblies. This leads one to propose balance equations generally coupled and nonlinear. They contain some parameters that must be determined empirically or calculated from auxiliary equations. When the number of interacting variables is large, the number of balance equations is too large and therefore very difficult to solve. An example of this is classical mechanics applied to many-body systems. One does not generally know all the initial conditions and therefore it is necessary to develop statistical methods for multiple coupled rate equations that describe the behavior of the system far from equilibrium. Some important aspects of any assembly of elements that can be studied using statistical methods is its inherent stability, that is its stability with respect to small changes in growth rate and the introduction of new elements.

The generalized Malthus-Verhulst model. – Our starting point is the generic stochastic differential equation for a growth model driven by a Markovian dichotomic noise

$$\dot{x} = [a_0(t) + a_1\xi(t)] \, x \frac{(1 - x^\mu)}{\mu} \tag{1}$$

where the deterministic growth rate $a_0(t)$ is perturbed by the Markovian dichotomic noise $\xi(t)$, in which a_1 and μ, with $\mu \geq 0$, are free parameters. Our main objective is to calculate the exact probability distribution for this generic model.

The state space of the noise $\xi(t)$ consists only of two levels, $(\Delta_1, -\Delta_2)$. This noise is called asymmetric Markovian dichotomic noise and it is also known as random telegraph noise. The temporal evolution of the conditional probability $P(\xi, t \mid \xi_0, t_0)$ that completely characterizes the process is given by the following master equation [5, 12]

$$\frac{d}{dt}\left(\begin{array}{c} P(\Delta_1, t \mid \xi_0, t_0) \\ P(-\Delta_2, t \mid \xi_0, t_0) \end{array} \right) = \left(\begin{array}{cc} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{array} \right) \left(\begin{array}{c} P(\Delta_1, t \mid \xi_0, t_0) \\ P(-\Delta_2, t \mid \xi_0, t_0) \end{array} \right) \tag{2}$$

where λ_1 and λ_2 are the probabilities by unit time of switching between states Δ_1 and Δ_2, so that $\tau_j = 1/\lambda_j$ are the mean sojourn times in these states. The stationary solution of Eq. (2) can be obtained by setting

$$P(\xi, \infty \mid \xi_0, t_0) = \frac{1}{\gamma} \left(\lambda_2 \delta_{\Delta_1, \xi} + \lambda_1 \delta_{-\Delta_2, \xi} \right) \tag{3}$$

where $\gamma = \lambda_1 + \lambda_2$. If the Markovian dichotomic noise has Eq. (3) as the initial condition, then $\xi(t)$ is a stationary process. From Eq. (2) it follows that the mean value is

$$\langle \xi(t) \rangle = \frac{\lambda_2 \Delta_1 - \lambda_1 \Delta_2}{\gamma}. \tag{4}$$

For the sake of simplicity, we require that the mean value $\langle \xi(t) \rangle$ vanish. This means that

$$\lambda_2 \Delta_1 = \lambda_1 \Delta_2 = \omega_0. \tag{5}$$

The correlation function is

$$\langle \xi(t) \, \xi(t') \rangle = \frac{\lambda_1 \lambda_2}{\gamma^2} \left(\Delta_1 + \Delta_2 \right)^2 e^{-\gamma |t-t'|}. \tag{6}$$

Higher order correlation functions are more complicated. However, since the correlation function given by Eq. (6) is indistinguishable from the Ornstein-Uhlenbeck process, the dichotomic noise found wide applications in building the models [5]. Furthermore, by an appropriate procedure of limit the dichotomic noise converges at the Gaussian white noise as the Ornstein-Uhlenbeck does, and it also converges at the white shot noise [19]. Experimental evidences of the dichotomic noise have been found frequently in the litterature [5, 20–22].

In Ref. [4] the author uses the direct method, which consists of formally integrating the stochastic differential equation and then taking the mean value over all realizations of the stochastic process. This method allows analytical treatment of the moments $\langle x^n(t) \rangle$ for different types of noise, in particular for Gaussian white noise and white shot noise. In Ref. [23] the same author uses the inverse Mellin transform to calculate the stationary probability distribution. This last procedure appears to be difficult for two reasons. The first reason is that the mathematical problem of finding a distribution knowing its moments generally does not have a unique solution. The literature refers to this as the classical problem of moments [24]. The second reason is that indeed it is a very hard task to find an analytical inverse Mellin transform. For example if we consider the case with $\mu = 1$ in Eq. (1) and with $\xi(t)$ an Ornstein-Uhlenbeck process, the moments can be expressed as the following integral [25]

$$\langle x^n(t) \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz \frac{e^{-z^2}}{\left[1 + \left(\frac{1-x_0}{x_0} \right) e^{-\left(a_0 t + 2a_1 z \sqrt{t} \right)} \right]^n} \, , \qquad n = 1, 2, 3, \ldots. \tag{7}$$

To find the inverse Mellin transform the parameter n has to be considered a real parameter. This fact makes it very difficult to perform the inversion.

Using a relatively simple procedure the authors in Ref. [26] provide the exact probability distribution for a model like Eq. (1) when the noise $\xi(t)$ is given by the Ornstein-Uhlenbeck process. A further generalization can be found in Ref. [27] where several cases of white non-Gaussian noise are examined.

For $\mu = 0$, Eq. (1) reduces to the Gompertz model [8], $\dot{x} = [a_0(t) + a_1 \xi(t)] \, x \ln x$, and for $\mu = 1$ it becomes the logistic equation driven by the dichotomic noise, namely

$$\dot{x} = (a_0(t) + a_1 \xi(t)) \, x(1 - x). \tag{8}$$

Stochastic effects on Eq. (8) have been considered frequently in the literature. Refs. [5–8, 18] contain several applications and developments of this model. In Refs. [28–31] the transient behavior has been investigated when the system is driven by the same type of perturbation and the relaxation time of the system is calculated as a function of noise intensity. In Ref. [25] the results of Ref. [29] are extended to the case in which a_0 is perturbed by a colored Gaussian noise and confirmed by an analogical experiment, as well as by numerical simulations. To analyze a cancer cell population the authors of Ref. [32] consider the model $\dot{x} = a \, x - b \, x^2 + x \, \xi_1(t) + \xi_2(t)$ where $\xi_1(t)$ and $\xi_2(t)$ are correlated Gaussian white noises. They write the corresponding Fokker-Planck equation and analyze the behavior of the stationary probability density.

The exact probability distribution. – The model described by Eq. (1) using the Stratonovich approach, can be reduced to an elementary differential equation by means of the transformation

$$y = \ln \left(\frac{1 - x^\mu}{\mu x^\mu} \right) \tag{9}$$

which leads to the equation

$$\dot{y} = -a_0 - a_1 \xi(t). \tag{10}$$

We emphasize that for the points $x = 0$ and $x = 1$, the transformation Eq. (9) does not hold. The behavior of the system in these points has to be analyzed through a limit procedure. Following Ref. [33], we can write the stochastic Liouville equation for the density function $\rho(y, t; \xi)$ of a set of realizations of Eq. (10) as

$$\frac{\partial}{\partial t} \rho(y, t; \xi) = a_0 \frac{\partial}{\partial y} \rho(y, t; \xi) + a_1 \frac{\partial}{\partial y} \xi(t) \rho(y, t; \xi). \tag{11}$$

Taking the mean value over all realizations of $\xi(t)$ we obtain

$$\frac{\partial p}{\partial t} = a_0 \frac{\partial p}{\partial y} + a_1 \frac{\partial p_1}{\partial y} \tag{12}$$

where $p \equiv p(y, t) = \langle \rho(y, t; \xi) \rangle$ and $p_1 \equiv p_1(y, t) = \langle \xi(t) \rho(y, t; \xi) \rangle$. Next, using the well-known Shapiro-Loginov formula for differentiation of exponentially correlated stochastic functions [34], we obtain the following differential equation for the function $p_1(y, t)$

$$\frac{\partial p_1}{\partial t} = -\Lambda p_1 + a_0 \frac{\partial p_1}{\partial y} + a_1 \frac{\partial}{\partial y} \langle \xi^2(t) \rho(y, t; \xi) \rangle \tag{13}$$

where $\Lambda = \lambda_1 + \lambda_2$. Following Ref. [35] we have $\xi^2(t) = \Delta^2 + \Delta_0 \xi(t)$, where $\Delta^2 = \Delta_1 \Delta_2$ and $\Delta_0 = \Delta_1 - \Delta_2$, transforms Eq. (13) into

$$\frac{\partial p_1}{\partial t} = -\Lambda p_1 + (a_0 + a_1 \Delta_0) \frac{\partial p_1}{\partial y} + a_1 \Delta^2 \frac{\partial p}{\partial y}. \tag{14}$$

Taking the time derivative of Eq. (12) and combining it with Eq. (14) we finally obtain

$$\frac{\partial^2 p}{\partial t^2} - (2a_0 + a_1 \Delta_0) \frac{\partial^2 p}{\partial t \partial y} + \left(a_0^2 + a_0 a_1 \Delta_0 - a_1^2 \Delta^2 \right) \frac{\partial^2 p}{\partial y^2} - a_0 \Lambda \frac{\partial p}{\partial y} + \Lambda \frac{\partial p}{\partial t} = 0. \tag{15}$$

The Shapiro-Loginov formula has a hypothesis stating the statistical independence between $\xi(t)$ and $\rho(y, t; \xi)$ at the initial time $t = 0$. Therefore at the time $t = 0$, we have $p_1(y, 0) = \langle \xi(t) \rho(y, t; \xi) \rangle |_{t=0} = 0$. Consequently the initial conditions for Eq. (15) are

$$p(y, t)|_{t=0} = \delta (y - y_0), \quad \frac{\partial}{\partial t} p(y, t)|_{t=0} = a_0 \delta' (y - y_0). \tag{16}$$

The following change of variables

$$t = \tau, \quad y = z - a_0 \tau \tag{17}$$

further simplifies Eq. (15), so that we end up with

$$\left[\frac{\partial^2}{\partial \tau^2} - a_1 \Delta_0 \frac{\partial^2}{\partial \tau \partial z} - a_1^2 \Delta^2 \frac{\partial^2}{\partial z^2} + \Lambda \frac{\partial}{\partial \tau} \right] P(z, \tau) = 0 \tag{18}$$

where $P(z, t) = p(y, t)$. A formal solution of Eq. (18), satisfying the initial conditions (16) and vanishing for $z \to \pm \infty$, is given by

$$P(z,\tau) = e^{-\frac{\Lambda}{2}\tau} \int\limits_{-\infty}^{\infty} e^{\imath k r} \left[\cos \frac{\lambda}{2}\tau + \frac{\Lambda - \imath a_1 \Delta_0 k}{\lambda} \sin \frac{\lambda}{2}\tau \right] \frac{dk}{2\pi} \qquad (19)$$

where

$$r = \left(\frac{a_1 \Delta_0}{2}\tau + z - z_0 \right), \quad \lambda = \sqrt{(\Delta_1 + \Delta_2)^2 \, a_1^2 k^2 + 2\imath a_1 \Delta_0 \Lambda k - \Lambda^2}. \qquad (20)$$

Evaluating the integral we obtain

$$\begin{aligned}
P(z,\tau) &= \frac{e^{-\frac{\Lambda}{2}\tau + ur}}{2} \theta\left(v\tau - |r|\right) \left[\frac{\left(v\tau + a_1 \Delta_0 \frac{r}{2v}\right) b}{\sqrt{v^2\tau^2 - r^2}} I_1 \left(b\sqrt{v^2\tau^2 - r^2} \right) + \right. \\
&\quad + \left. \frac{\Lambda - a_1 \Delta_0 u}{2v} I_0 \left(b\sqrt{v^2\tau^2 - r^2} \right) \right] + \\
&\quad + \frac{e^{-\frac{\Lambda}{2}\tau + ur}}{2} \left[\left(1 + \frac{a_1 \Delta_0}{2v} \right) \delta\left(v\tau - r\right) + \left(1 - \frac{a_1 \Delta_0}{2v} \right) \delta\left(v\tau + r\right) \right], \qquad (21)
\end{aligned}$$

where r, as function of z, is defined by Eq. (20), $I_n(q)$ are the modified Bessel's functions, $\theta(q)$ is the step function, $\delta(q)$ is the Dirac's delta. Furthermore, we defined in a compact manner

$$u = \frac{a_1 \Lambda \Delta_0}{4v^2}, \quad b = \frac{\sqrt{\Delta_1 \Delta_2}\Lambda \, |a_1|}{2v^2}, \quad v = |a_1| \frac{\Delta_1 + \Delta_2}{2}.$$

With the help of Eqs. (9) and (17) we may write the solution as function of the original variables x, t as

$$p(x,t) = \frac{\mu}{x\left(1 - x^\mu\right)} P \left[\ln\left(\frac{1 - x^\mu}{\mu x^\mu} \right) + a_0 t, t \right]. \qquad (22)$$

Numerical Analysis. – In the following we numerically implement the dynamic equation (10) driving the diffusion of the variable y. An ensemble of N "trajectories" of the variable $\xi(t)$, the dichotomic noise in (10), is produced via a generator of random numbers with poisson distribution which returns the time intervals in which the stochastic dichotomous variable $\xi(t)$ retains either of its two values. For each trajectory of $\xi(t)$, a trajectory of the variable y is obtained, integrating Eq. (10). The probability for a given value of y at time t is then calculated as a simple average over the ensemble of N trajectories so obtained. A subsequent conversion to the x-space through the transformation (9) allows us to obtain the probability density $P(x,t)$. The dichotomous process $\xi(t)$ is assumed in a stationary condition at time $t = 0$, i.e. at such a time $N\lambda_1/(\lambda_1 + \lambda_2)$ trajectories are taken with initial value Δ_1 for the variable $\xi(t)$ and $N\lambda_2/(\lambda_1 + \lambda_2)$ with the value $-\Delta_2$. The numerical results are compared to the analytical expression given by Eqs. (21) and (22) in Fig. (1) which shows a perfect agreement.

Analysis of the results. – Thanks to calculations performed in the previous sections we have at our disposal the exact solution for any value of the parameter μ of physical interest. A detailed study of $P(x,t)$ is beyond the purpose of this letter. We shall limit ourselves to rediscover some results known in the literature for the case $\mu = 1$, which is the Malthus-Verhulst model, and for $\mu = 0$, which is the Gompertz model.

We note a first result of solution (18), specifically of Eq. (21). Due to the asymmetric case under consideration, there exists a particular choice of parameters such that a coefficient of the two deltas vanishes. To fix the ideas let us set $a_1 > 0$, then we can make one of the two delta coefficients vanish if $a_1 \Delta_0 = 2v$, that is to say $\Delta_2 = 0$. Note that $\Delta_2 = 0$ does
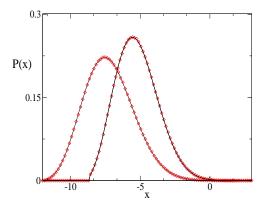
Fig. 1: The probability distribution at times $t = 5.7$ (right curve) and $t = 7.7$ (left curve) with parameters $\lambda_1 = 1.2, \lambda_2 = 2.8, a_0 = 1.0, a_1 = 1.0, \mu = 1.0, \Delta_1 = 0.6, \Delta_2 = 1.4$. The diamonds are the result of the numerical simulation, the continuous curves are the analytic solutions as given by Eqs. (21) and (22). The dirac deltas have been subtracted in both cases.

not imply a vanishing value of the parameter v that represents the propagation speed of the peaks.

Taking appropriate limits on the parameters Λ, Δ_1, and Δ_2 we can rediscover different well known stochastic processes. Following Ref. [19], we consider the limit

$$\lambda_1 = \lambda_2 = \lambda \to \infty, \quad \Delta_1 = \Delta_2 = \Delta \to \infty$$

which corresponds to the Gaussian white noise. Keeping constant the ratio Δ^2/λ we obtain

$$
\begin{aligned}
P(z,\tau) &\approx \frac{e^{-\lambda\tau}}{2} \theta\left(v\tau - |r|\right) \left[b I_1\left(bv\tau - \frac{br^2}{2v\tau}\right) + \frac{\lambda}{v} I_0\left(bv\tau - \frac{br^2}{2v\tau}\right) \right] \\
&+ \frac{e^{-\lambda\tau}}{2} \left[\left(1 + \frac{a_1\Delta_0}{2v}\right) \delta\left(v\tau - r\right) + \left(1 - \frac{a_1\Delta_0}{2v}\right) \delta\left(v\tau + r\right) \right].
\end{aligned}
\tag{23}
$$

Using the asymptotic expression for the Bessel's functions that is $I_n(x) \approx \exp[x]/\sqrt{2\pi x}$ for $x \to \infty$, we finally obtain

$$P(z,\tau) \approx \frac{1}{\sqrt{2\pi D\tau}} \exp\left[-\frac{r^2}{2D\tau}\right] \tag{24}$$

where by definition $D = a_1^2 \Delta^2/\lambda$ and we neglect the two exponentially damped deltas.

Still following Ref. [19], to obtain the white shot noise limit we have to take the symmetric dichotomous noise limit

$$\lambda_1 = \lambda_2 = \lambda \to \infty, \quad \Delta_1 = \Delta_2 = \Delta \to \infty, \quad \frac{\Delta}{\lambda} = \gamma$$

where the γ parameter is called non-Gaussianity parameter. The relation with the diffusion coefficient D is given by $D = \gamma^2\lambda$. In the above limit $bv = \lambda \to \infty$ so that we again end up with Eq. (23) because the argument of Bessel's functions becomes infinite. Consequently we rediscover Eq. (24)

As the last result we consider the limit for $\mu \to 0$ which leads us to the Gompertz model. The transformation (9) becomes

$$y = \ln[-\ln x]. \tag{25}$$

For brevity we consider the symmetric case so that $\Delta_0 = 0$. Using Eq. (22) we find that the asymptotic solution is

$$P(x,t) \approx -\frac{1}{\sqrt{2\pi Dt}}\frac{1}{x\ln x}\exp\left[-\frac{(\ln[-\ln x]+a_0 t)^2}{2Dt}\right]. \tag{26}$$

An analysis of this solution shows that $P(x,t)$, given by Eq. (26), always has a minimum located near the origin and a maximum located near $x = 1$. Finally $P(x,t)$ diverges at $x = 0$ as

$$P(x,t) \sim \frac{1}{x}\frac{1}{(-\ln x)^{\frac{a_0}{D}+1}}.$$
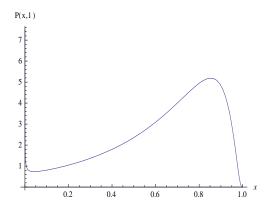
These results are graphed in Fig. 2.

Fig. 2: The probability distribution $P(x,t)$ for the Gompertz model at times $t = 1$. The value of the parameters are $a_0 = 1$ and $D = 1$.

Concluding remarks. – In this letter we considered a growth model with a stochastic growth rate modelled with a Markovian dichotomic noise. We found an exact solution for the generic parameters μ, Δ_1 and Δ_2 which makes it possible to apply the result to a variety of models ranging from biology [8, 18] to astrophysics [36] and laser physics. For example our model fits the the dichotomic noise condition found in ion channel experiments [38–40]. Patch-clamp experiments detect a fluctuating dichotomous current which is related to the number of ions flowing in the channel. Yet, in single mode lasers, the growth model with $\mu = 2$ and a constant growth rate, describes the time evolution for the Nth mode of the electric field in the laser cavity [2, 3]. The model in this letter makes a case for dichotomic fluctuations around a resonance frequency, providing opportunities for additional applications. An interpretation of the Malthus-Verhulst model which describes the photon population in a laser cavity, corresponding to the case $\mu = 1$, was introduced by McNeil and Walls [37]. We found these to be interesting cases and chose to examine the Malthus-Verhulst model, which has a very wide range of applications such as dynamics of populations, chemical reactions, and photon population in a laser cavity [11, 37]. In addition we studied the Gompertz model, corresponding to $\mu = 0$, when slightly modified, is used to describe the tumor growth dynamics. We confirmed our results with a numerical simulation showing perfect agreement with the analytical formulas. In this letter we focused only on two models,

but the obtained results may applied to other systems that will be the subject of future research.

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