Non-existence of positive weak solutions for some nonlinear (p, q)-Laplacian systems

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Abstract

In this work we deal with the class of nonlinear (**p, q**)-Laplacian system of the form

$$\left.\begin{array}{rll} -\Delta_p u = \mu\rho_1(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \nu\rho_2(x)g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\}$$

where Δ_p with $p > 1$ denotes the p-Laplacian defined by $\Delta_p u \equiv div[|\nabla u|^{p-2}\nabla u]$, μ, ν are positive parameters, $\rho_1(x)$, $\rho_2(x)$ are weight functions, $f, g : [0, \infty) \to \mathbb{R}$ are continuous functions and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Non-existence results of positive weak solutions are established under some certian conditions on f, g when $\mu\nu$ is small.

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1 Introduction:

In this paper we first consider a non-existence result of positive weak solutions for the following nonlinear system

$$\left.\begin{array}{rll} -\Delta_p u = \lambda a_1(x)v^{p-1} - b_1(x)v^{\alpha-1} - c_1(x) & \text{in } \Omega, \\ -\Delta_q v = \lambda a_2(x)u^{q-1} - b_2(x)u^{\beta-1} - c_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.1}$$

where Δ_p with $p > 1$ denotes the weighted p-Laplacian defined by $\Delta_p u \equiv div[|\nabla u|^{p-2}\nabla u]$, $a_i(x)$, $b_i(x)$ and $c_i(x), i = 1, 2$ are weight functions, α and β are positive constants and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

We first show that if $\lambda < \max(\lambda_p, \lambda_q)$, where λ_p, λ_q is the first eigenvalue of $-\Delta_p, -\Delta_q$ respectively, then system (1.1) has no positive weak solutions.

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Next we consider the nonlinear system

$$
\left.
\begin{aligned}
-\Delta_p u &= \mu\rho_1(x)f(v) && \text{in } \Omega, \\
-\Delta_q v &= \nu\rho_2(x)g(u) && \text{in } \Omega, \\
u = v &= 0 && \text{on } \partial\Omega.
\end{aligned}
\right\}
\tag{1.2}
$$

where μ, ν are positive parameters, $\rho_1(x)$, $\rho_2(x)$ are weight functions and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let $f, g : [0, \infty) \to \mathbb{R}$ are continuous functions. Also, assume that there exist positive numbers K_i and $M_i, i = 1, 2$ such that

$$
f(v) \leq K_1 v^{p-1} - M_1, \quad \text{for all } v \geq 0
\tag{1.3}
$$

and

$$
g(u) \leq K_2 u^{q-1} - M_2, \quad \text{for all } u \geq 0.
\tag{1.4}
$$

We discuss a non-existence result for system (1.2) when $\mu\nu$ is small.

Problems of the form (1.1) and (1.2) arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mappings (see [13]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids. The p-Laplacian also appears in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, (see [5]), glacial sliding ($p \in (1; \frac{4}{3}]$, see [10] or flow through porous media ($p = \frac{3}{2}$, see [11]). For existence and non-existence results of positive weak solutions for systems involving the weighted p-Laplacian, see ([2, 3, 6, 7, 8, 9, 12]).

This paper is organized as follows: In section 2, we introduce some technical results and notations, which are established in [4]. In section 3, we prove the non-existence of positive weak solutions for system (1.1) and (1.2).

2 Technical Results

Let us introduce the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, defined as the completion of $C^\infty(\Omega)$ with respect to the norm (see [4])

$$
\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega |u|^p + \int_\Omega |\nabla u|^p \right]^{\frac{1}{p}} < \infty.
\tag{2.1}
$$

Since we are dealing with the Dirichlet problem, we define the space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm

$$
\|u\|_{W_0^{1,p}(\Omega)} = \left[\int_\Omega |\nabla u|^p \right]^{\frac{1}{p}} < \infty,
\tag{2.2}
$$

which is equivalent to the norm given by (2.1). Both spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are well defined reflexive Banach Spaces.

Now, we introduce some technical results concerning the eigenvalue problem

$$
\left.
\begin{aligned}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
\right\}
\tag{2.3}
$$

We will say $\lambda \in R$ is an eigenvalue of (2.3) if there exists $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx = \lambda \int_\Omega a(x) u^{p-2} u\phi dx, \tag{2.4}$$

holds for $\phi \in W_0^{1,p}(\Omega)$. Then u is called an eigenfunction corresponding to the eigenvalue λ.

Lemma 1 *There exists the first eigenvalue $\lambda_p > 0$ and precisely one corresponding eigenfunction $\phi_p \geq 0$ a.e. in Ω of the eigenvalue problem (2.3). Moreover, it is characterized by*

$$\lambda_p = \frac{\int_\Omega |\nabla \phi_p|^p}{\int_\Omega a(x)|\phi_p|^p} = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega a(x)|u|^p} \leq \frac{\int_\Omega |\nabla u|^p}{\int_\Omega a(x)|u|^p} = \lambda.$$

Definition 1 *A pair of non-negative functions $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ are called a weak solution of (1.2) if they satisfy*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \zeta dx = \mu \int_\Omega \rho_1(x) f(v) \zeta dx,$$

$$\int_\Omega |\nabla v|^{q-2} \nabla v \nabla \eta dx = \nu \int_\Omega \rho_2(x) g(u) \eta dx,$$

for all test functions $\zeta \in W_0^{1,p}(P, \Omega), \eta \in W_0^{1,q}(\Omega)$.

3 Non-existence Results

In this section we state our main results. Throught this section, we assume q be such that $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2 *For $\lambda \leq \lambda^*$, system (1.1) has no positive weak solution.*

Proof. Assume that there exist a positive solution $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of (1.1). Multiplying the first equation of (1.1) by u, we have

$$\int_\Omega |\nabla u|^p dx = \int_\Omega [\lambda a_1(x) v^{p-1} - b_1(x) v^{\alpha-1} - c_1(x)] u dx \tag{3.1}$$

$$< \int_\Omega [\lambda a_1(x) v^{p-1} - c_1(x)] u dx.$$

But, from the characterization of the first eigenvalue, we have

$$\lambda_p \int_\Omega a(x)|u|^p \leq \int_\Omega |\nabla u|^p. \tag{3.2}$$

Combining (1) and (3.2), we have

$$\lambda_p \int_\Omega a_1(x) u^p < \int_\Omega \lambda a_1(x) v^{p-1} u dx - \int_\Omega c_1(x) u dx. \tag{3.3}$$

Similarly, from the second equation of (1.1), we obtain

$$\lambda_q \int_\Omega a_2(x) v^q < \int_\Omega \lambda a_2(x) u^{q-1} v dx - \int_\Omega c_2(x) v dx. \tag{3.4}$$

3

Adding (3.3) and (3.4), we get

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q \quad < \quad \int_\Omega \lambda a_1(x)v^{p-1}udx + \int_\Omega \lambda a_2(x)u^{q-1}vdx$$

$$- \int_\Omega c_1(x)udx - \int_\Omega c_2(x)vdx$$

$$< \quad \int_\Omega \lambda a_1(x)v^{p-1}udx + \int_\Omega \lambda a_2(x)u^{q-1}vdx$$

Applying the Young inequality on the right hand side of the above equation, we have

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)[\frac{u^p}{p} + \frac{v^p}{q}]dx + \int_\Omega \lambda a_2(x)[\frac{v^q}{q} + \frac{u^q}{p}]dx \qquad (3.5)$$

Now, we discuss the following two cases:

Case I, if $u \le v$ for all x, then (3.5) becomes

$$\lambda_p \int_\Omega a_1(x)v^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)v^p dx + \int_\Omega \lambda a_2(x)v^q dx.$$

Hence,

$$(\lambda_p - \lambda) \int_\Omega a_1(x)v^p + (\lambda_q - \lambda) \int_\Omega a_2(x)v^q < 0$$

which is a contradiction if $\lambda \le \max(\lambda_p, \lambda_q) = \lambda^*$.

Case II, if $u \ge v$ for all x, then (3.5) becomes

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)u^q < \int_\Omega \lambda a_1(x)u^p dx + \int_\Omega \lambda a_2(x)u^q dx.$$

Hence,

$$(\lambda_p - \lambda) \int_\Omega a_1(x)u^p + (\lambda_q - \lambda) \int_\Omega a_2(x)u^q < 0$$

which is a contradiction if $\lambda \le \max(\lambda_p, \lambda_q) = \lambda^*$. The proof complete.

Now we consider the main result for system (2.2):

Theorem 3 *Let (1.3) and (1.4) hold. Then system (1.2) has no positive weak solution if $\mu\nu \le \frac{\lambda_1^2}{K_1 K_2}$.*

Proof. Suppose $u > 0$ and $v > 0$ be such that (u, v) is a solution of (2.2). We prove our theorem by arriving at a contradiction. Multiplying the first equation in (2.2) by a positive eigenfunction say ϕ_p corresponding to λ_p, we obtain

$$- \int_\Omega \Delta_p u \, \phi_p dx = \mu \int_\Omega \rho_1(x)f(v)\phi_p dx,$$

and hence using (2.1) and (1.3), we have

$$\lambda_p \int_\Omega \rho_1(x)u^{p-1}\phi_p dx \le \mu \int_\Omega \rho_1(x)[K_1 v^{q-1} - M_1]\phi_p dx. \qquad (3.6)$$

4

Similarly using the second equation in (2.2) and (1.4) we obtain

$$\lambda_q \int\limits_{\Omega} \rho_2(x)v^{q-1}\phi_q dx \leq \nu \int\limits_{\Omega} \rho_2(x)[K_2 u^{p-1} - M_2]\phi_q dx. \tag{3.7}$$

From (3.7), we have

$$v^{q-1} \leq \frac{\nu}{\lambda_q}[K_2 u^{p-1} - M_2] \tag{3.8}$$

Combining (3.6) and (3.8) we obtain

$$[\lambda_p - \mu\nu\frac{K_1 K_2}{\lambda_q}] \int\limits_{\Omega} \rho_1(x)u^{p-1}\phi_p \leq -\mu \int\limits_{\Omega} \rho_1(x)[\frac{\nu K_1 M_2}{\lambda_q} + M_1]\phi_p < 0.$$

Hence system (1.2) has no positive weak solution if $\mu\nu \leq \frac{\lambda_p \lambda_q}{K_1 K_2}$.

Remark 4 *If f, g be such that*

$$f(v) \geq K_1 v^{q-1} + M_1, \quad \text{for all} \ \ v \geq 0, \tag{3.9}$$

and

$$g(u) \geq K_2 u^{p-1} + M_2, \quad \text{for all} \ \ u \geq 0, \tag{3.10}$$

then we have the following theorem:

Theorem 5 *Let (3.9) and (3.10) hold. Then system (1.2) has no positive weak solution if $\mu\nu \geq \frac{\lambda_p \lambda_q}{K_1 K_2}$.*

Proof. The proof proceeds in the same way as for Theorem 6.

Remark 6 *When $p = q$, $m_1(x) = m_2(x) = m, m = a, b, c$ is constant , and $\alpha = \beta$, we have some results for (1.1) in [1].*

Remark 7 *When $p = q$ and $\rho_1(x) = \rho_2(x) = 1$, we have some results for (1.2) in [1] .*

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