Alternative Proofs on the Indices of Cacti and Unicyclic Graphs with n Vertices

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September 22, 2017

Abstract

Let H_n be the cactus obtained from the star $K_{1,n-1}$ by adding $\left\lfloor \frac{n-1}{2} \right\rfloor$ independent edges between pairs of pendant vertices. Let $K^+_{1,n-1}$ be the unicyclic graph obtained from the star $K_{1,n-1}$ by appending one edge. In this paper we give alternative proofs of the following results: Among all cacti with n vertices, H_n is the unique cactus whose spectral radius is maximal, and among all unicyclic graphs with n vertices, $K^+_{1,n-1}$ is the unique unicyclic graph whose spectral radius is maximal. We also prove that among all odd-cycle graphs with n vertices, H_n is the unique odd-cycle graph whose spectral radius is maximal.

1 Introduction

Let G be a simple graph with vertex set $\{v_1, v_2, ..., v_n\}$. The adjacency matrix of G, $A(G) = [a_{ij}]$ is defined to be the $n \times n$ matrix such that $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Since $A(G)$ is symmetric, all of its eigenvalues are real. The spectral radius of G, $\rho(G)$, is the largest eigenvalue of $A(G)$ and it is also called the *index* of G.

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When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem [5, p 181], $\rho(G)$ is a simple eigenvalue of $A(G)$ and there is a unique positive unit eigenvector corresponding to $\rho(G)$. This eigenvector is called the *Perron vector* of G.

A *pendant* vertex is a vertex of degree 1. We call an edge a *pendant edge* if it is a bridge connecting a pendant vertex. Lets denote the degree of a vertex v by $d(v)$. Let $\Delta(G)$ denote the highest degree of all vertices of G. We denote the set of all vertices adjacent to v by $N(v)$.
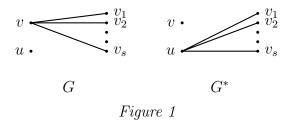
A path is called an *odd-path* if its length (i.e., the number of its edges) is odd. Otherwise it is called an *even-path*. A cycle is called an *odd-cycle* if its length (i.e., the number of its edges) is odd. Otherwise it is called an *even-cycle*. A graph is called an *odd-cycle graph* if each of its cycles is an *odd-cycle*.

A graph is called a *cactus* if its cycles have at most one common vertex. Let H_n be the cactus obtained from the star $K_{1,n-1}$ by adding $\lfloor \frac{n-1}{2} \rfloor$ independent edges between pairs of pendant vertices (see Fig. 2). Let $\mathcal{C}(n)$ be the set of all cacti of order n (i.e., with n vertices).

A connected graph with a unique cycle is called a *unicyclic graph*. So a unicyclic graph can be seen as a tree with an extra edge. By $K_{1,n-1}^+$ we denote the unicyclic graph obtained from the star $K_{1,n-1}$ by appending one edge (see Fig. 3). Let $\mathcal{U}(n)$ be the set of all unicyclic graphs of order n.

2 Main Results

Theorem 2.1. [2, *Thm.1*] *Let u, v be two vertices of the connected graph G. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d(v))$ are some vertices of $N(v) \backslash N(u)$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G^*) > \rho(G)$.*

$$G \qquad\qquad G^*$$

Figure 1

Using mainly this theorem we will give an alternative proof of the following theorem:

Theorem 2.2. [3, *Thm. 3.1*] *Let* $G \in \mathcal{C}(n)$. *Then* $\rho(G) \leq \rho(H_n)$, *equality holds if and only if* $G \cong H_n$.

$$v_n \; v_{n-1} \quad v_5 \quad v_4 \quad v_3 \quad v_2 \qquad\qquad v_{n-1}v_{n-2} \quad v_5 \quad v_4 \quad v_3 \quad v_2$$

$$v_1 \qquad\qquad\qquad\qquad v_n \quad v_1$$

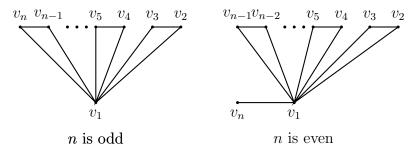n is odd $\qquad\qquad\qquad\qquad$ n is even

Figure 2: H_n

First we prove the above theorem for all connected cacti with maximal number of edges as following.

Theorem 2.3. *Let* $G \in \mathcal{C}(n)$ *be connected with maximal number of edges. Then* $\rho(G) \leq \rho(H_n)$, *equality holds if and only if* $G \cong H_n$.

Before proving the theorem we will record the following propositions regarding connected cactus G of order n with maximal number of edges:

Proposition 2.4. *Let* G *be a connected cactus* G *of order* n *with maximal number of edges. Then*
(a) All cycles of G *are triangles with at most one edge not in some triangle except when* $G \cong C_4$.[1, *Lemma 6.7*]
(b) If $n \leq 5$ *and* $G \not\cong C_4$ *then* $G \cong H_n$.
(c) Let $t(G) :=$ *the number of vertices of* G *of degree* ≥ 3. *If* $n \geq 6$ *then* $t(G) = 1$ *if and only if* $G \cong H_n$.
(d) If $t(G) > 1$ *then there are always two adjacent vertices of degree* ≥ 3.[*from (a)*]

Lemma 2.5. *Let* G *be a connected cactus of order* n *with maximal number of edges. Let* u *and* v *be two adjacent vertices of* G *of degree* ≥ 3 *such that* $N(v) \backslash \{u \cup N(u)\} = \{v_1, v_2, \ldots, v_s\}$. *Let* G_1 *be the graph obtained from* G *by deleting the edges* vv_i *and adding the edges* uv_i $(1 \leq i \leq s)$. *Then* G_1 *is also a connected cactus of order* n *with maximal number of edges.*

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Proof. Since u and v are adjacent and G is connected then G_1 is connected. G_1 and G have the same number of edges since in making of G_1 the numbers of deleted edges and added edges are same. So it suffices to show that G_1 is a cactus. By Proposition 2.4(a), all cycles of G are triangles with at most one edge not in some triangle. If s is even $v_1, v_2, \ldots, v_s$ form exactly $\frac{s}{2}$ triangles at v having no other common vertex. These $\frac{s}{2}$ triangles at v corresponds $\frac{s}{2}$ branches of induced subgraphs of G which are also connected cacti with maximal number of edges having a unique common vertex v. When we delete vv_i and add uv_i ($1 \leq i \leq s$), v becomes a vertex of degree ≤ 2 and u is added with $\frac{s}{2}$ branches of connected cacti with maximal number of edges having a unique common vertex u. So G_1 is a connected cactus with maximal number of edges. If s is odd $v_1, v_2, \ldots, v_s$ form one edge and exactly $\frac{s-1}{2}$ triangles at v having no other common vertex. Then by similar arguments G_1 is a connected cactus with maximal number of edges. $\qquad\blacksquare$

Proof of Theorem 2.3. Suppose G is a connected cactus of order n with maximal number of edges such that $\rho(G) \geq \rho(G')$ for all connected cactus G' of order n with maximal number of edges. If $G \cong H_n$ then there is nothing to prove. Let $G \not\cong H_n$. Now $G \not\cong C_4$ since $\rho(H_4) > \rho(C_4)$. Then by Proposition 2.4(c), $t(G) > 1$. Now by Proposition 2.4(d), suppose u and v are two adjacent vertices of G of degree ≥ 3.

Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G, where x_i corresponds to the vertex v_i ($1 \leq i \leq n$). Suppose $x_u \geq x_v$. By Proposition 2.4(a), G has no 4-cycle. So u and v can have at most one common adjacent vertex. Since v has degree at least 3, $N(v) \backslash \{u \cup N(u)\} \neq \phi$. Let $v_1, v_2, \ldots, v_s$ be all vertices of $N(v) \backslash \{u \cup N(u)\}$. Let G_1 be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i ($1 \leq i \leq s$). By Lemma 2.5, G_1 is also a connected cactus of order n with maximal number of edges. Now by Theorem 2.1, $\rho(G_1) > \rho(G)$ which is a contradiction to the fact that $\rho(G) \geq \rho(G')$ for all edge maximal connected cactus G' of order n. $\qquad\blacksquare$

Corollary 2.6. *Let G be a connected cactus of order n with maximal number of edges such that $t(G) > 1$. Then there exists a connected cactus G_1 of order n with maximal number of edges, not necessarily isomorphic to H_n, such that $t(G_1) = t(G) - 1$ and $\rho(G_1) > \rho(G)$.*

Proof. From the proof of Lemma 2.5 it is clear that when we delete vv_i and add uv_i ($1 \leq i \leq s$) in G, v becomes a vertex in G_1 of degree ≤ 2. So

$t(G_1) = t(G) - 1$. Now by the above proof $\rho(G_1) > \rho(G)$. $\qquad\square$

Let $G - kl$ denote the graph G without the edge kl.

Lemma 2.7. [1, *Lemma* 6.4] $\rho(G) \geq \rho(G - kl)$ *for any edge* kl *of* G, *with strict inequality when* G *is connected.*

By the above lemma it suffices to prove Theorem 2.2 for edge maximal connected cacti. Since we already proved Theorem 2.2 for connected cacti with maximal number of edges in Theorem 2.3, then to prove Theorem 2.2 it suffices to prove the following theorem.

Theorem 2.8. *Let* $G \in \mathcal{C}(n)$ *be edge maximal connected. Then there is a connected cactus* G^* *with maximal number of edges such that* $\rho(G) \leq \rho(G^*)$.

Proof. If G is a connected cactus with maximal number of edges there is nothing to prove. Suppose G is an edge maximal connected cactus without maximal number of edges. Let C be a cycle of G and e be an edge at a vertex of C but not in C. Since G is edge maximal then one of the following is true.
(a) e is in a cycle $C' \neq C$ in G.
(b) e is a bridge between two cycles C and $C' \neq C$ in G.
(c) e is a pendant edge in G.
Since G is edge maximal there are no two consecutive edges which are not in any circle. Because if uv and vw are such two, then we can add a new edge uw while the new graph is still a connected cactus. Now we construct G^* from G using the following steps.
Step 1. Now let C_k, $k \geq 4$ be a cycle in G. Let u and v be two adjacent vertices in C_k. Since $k \geq 4$, suppose u is adjacent to $x(\neq v)$ and v is adjacent to $y(\neq u)$. Then $N(u)\backslash\{v \cup N(v)\} = \{x\}$ and $N(v)\backslash\{u \cup N(u)\} = \{y\}$ in C_k. Let $x = (x_1, x_2, ..., x_n)^T$ be the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Suppose $x_u \geq x_v$. Then deleting the edge vy and adding the edge uy in G we will get a graph G' which is same as G except in which C_k becomes C_{k-1} joined with the edge uv. Now by Theorem 2.1, $\rho(G') > \rho(G)$. Repeating this process in every C_k, $k \geq 4$ in G we get a connected cactus G_1 of order n in which cycles are triangles and $\rho(G_1) > \rho(G)$. If G_1 has at most one edge not in any triangle then by Proposition 2.4(a), G_1 is a connected cactus with maximal number of edges. Then we are done.
Step 2. Suppose G_1 has at least two edges not in any triangle. If uv and vw are such two, then we can add a new edge uw producing an extra

triangle. Repeating this in all possible cases we get a connected cactus G_2 of order n in which cycles are triangles and by Lemma 2.7, $\rho(G_2) \geq \rho(G_1)$, equality holds if and only if there are no two consecutive edges that are not in any cycle. Similarly if G_2 has at most one edge not in any triangle then by Proposition 2.4(a), G_1 is a connected cactus with maximal number of edges. Then we are done.

Step 3. Suppose G_2 has at least two edges not in any triangle. By construction of G_2 there are no two consecutive edges which are not in any triangle. Let uv be an edge that is not in any triangle such that uv is a bridge between two triangles in G_2. Then obviously $N(u)\backslash\{v \cup N(v)\} \neq \phi$ and $N(v)\backslash\{u \cup N(u)\} \neq \phi$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G_2, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Suppose $x_u \geq x_v$. Let $N(v)\backslash\{u \cup N(u)\} = \{v_1, v_2, \cdots, v_s\}$. Let G_2^* be the graph obtained from G_2 by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. Then G_2^* is a connected cactus of order n in which cycles are triangles with the pendant edge uv and by Theorem 2.1, $\rho(G_2^*) > \rho(G_2)$. Repeating this process for all bridges between triangles we can get a connected cactus G_3 of order n in which cycles are triangles and by Theorem 2.1, $\rho(G_3) > \rho(G_2)$. If G_3 has at most one edge not in any triangle then by Proposition 2.4(a), G_3 is a connected cactus with maximal number of edges. Then we are done.

Step 4. Suppose G_3 has at least two edges not in any triangle. By construction of G_3, all the edges of G_3 which are not in any triangle are pendant edges. Let ux and vy be two pendant edges of G_3. If $x = y$, add the edge uv to G_3 which increases number of triangle and also spectral radius of G_3 by Lemma 2.7. We will do this in all possible cases and get a connected cactus G_3^* in which cycles are triangles and by Lemma 2.7, $\rho(G_3^*) > \rho(G_3)$. Now let $x \neq y$. Then $N(u)\backslash\{v \cup N(v)\} = \{x\}$ and $N(v)\backslash\{u \cup N(u)\} = \{y\}$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G_3^*, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Suppose $x_u \geq x_v$. Then deleting the edge vy and adding the edge uy in G we will get a graph G_4 of order n. Then G_4 is a connected cactus of order n in which cycles are triangles and by Theorem 2.1, $\rho(G_4) > \rho(G_3)$. Repeating this process for all pendant edges we can get a connected cactus G_5 of order n in which cycles are triangles and by Theorem 2.1, $\rho(G_5) > \rho(G_4)$.

Step 5. By construction of G_5, all the pendant edges of G_5, except at most one, form pairs having a common vertex. Now joining corresponding two pendant vertices of each pair we can form a new triangle for each such pair. Then we get a connected cactus G^* of order n in which cycles are tri-

angles and by Lemma 2.7, $\rho(G^*) > \rho(G_5)$. By construction of G^*, it has at most one pendant edge. Then $\rho(G^*) \geq \rho(G)$ and by Proposition 2.4(a), G^* is a connected cactus of order n with maximal number of edges. $\square$

Now lets prove a corollary of Theorem 2.2 as following.

Corollary 2.9. *For all odd-cycle graph G, $\rho(G) \leq \rho(H_n)$, equality holds if and only if $G \cong H_n$.*

Before proving this corollary we will prove the following lemma.

Lemma 2.10. *Every odd-cycle graph is a cactus.*

Proof. Let G be an odd-cycle graph. Suppose G is not a cactus. Then G have two odd cycles, say C and C' such that they have at least two common vertices. Let $v_1, v_2, \ldots, v_k$ be all common vertices of C and C'. So these vertices divide each of C and C' into a series of consecutive paths, say $P^1, P^2, \ldots, P^k$ for C and $P^{1'}, P^{2'}, \ldots, P^{k'}$ for C' where P^i is the path from v_i to v_{i+1} in C and $P^{i'}$ is the path from v_i to v_{i+1} in C' ($1 \leq i \leq k$), assuming $v_{k+1} = v_1$. Since $C \neq C'$, then $P^i \neq P^{i'}$ for some i. If P^i and $P^{i'}$ both are even-paths or odd-paths then $P^i \cup P^{i'}$ is an even-cycle in G - a contradiction. Otherwise suppose P^i is an even-path and $P^{i'}$ is an odd-path. Let P be a path from v_i to v_{i+1} obtained from C by deleting nonpendant vertices of P^i and corresponding incident edges. Since C is an odd-cycle and P^i is an even-path, then P is an odd-path from v_i to v_{i+1} in G. Now since odd-paths P and $P^{i'}$ are disjoint except at the end points v_i and v_{i+1}, then $P \cup P^{i'}$ is an even-cycle in G - a contradiction. $\square$

Proof of Corollary 2.9. From the above lemma every odd-cycle graph is a cactus. So a connected odd-cycle graph is a connected cactus. Now by Theorem 2.2 for all connected cactus G, $\rho(G) \leq \rho(H_n)$, equality holds if and only if $G \cong H_n$. Since H_n is a connected odd-cycle graph the corollary follows. $\square$

Theorem 2.11. [4] *Let $G \in \mathcal{U}(n)$. Then $\rho(G) \leq \rho(K_{1,n-1}^+)$, equality holds if and only if $G \cong K_{1,n-1}^+$.*
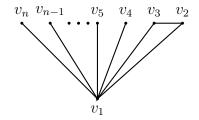
Figure 3: $K_{1,n-1}^+$

Using mainly Theorem 2.1 we will give an alternative proof of the above theorem. Before proving the theorem we will record the following propositions regarding unicyclic graph G of order n :

Proposition 2.12. *(a) Let $G \in \mathcal{U}(n)$. Two adjacent vertices in G have at most one common vertex. Two vertices in G have one common vertex if and only if they are in a triangle.*
(b) Let $G \in \mathcal{U}(n)$. If $n = 3$ then $G \cong K_{1,n-1}^+$.
(c) Let $G \in \mathcal{U}(n)$. $G \cong K_{1,n-1}^+$ if and only if $\Delta(G) = n - 1$.
(d) Let G be a graph of order n. Then G is unicyclic if and only if G is connected having exactly n edges.

Lemma 2.13. *Let $G \in \mathcal{U}(n)$ and $G \not\cong K_{1,n-1}^+$. Then there are two adjacent vertices u and v in G such that $N(u)\backslash\{v \cup N(v)\} \neq \phi$ and $N(v)\backslash\{u \cup N(u)\} \neq \phi$.*

Proof. Since $G \not\cong K_{1,n-1}^+$ then by Proposition 2.12(b), $n \geq 4$. If $G \cong C_n$, $n \geq 4$ then $d(v) = 2$ for every vertex v of G. Let u and v be two adjacent vertices in $G \cong C_n$, $n \geq 4$. Suppose u is adjacent to $x(\neq v)$ and v is adjacent to $y(\neq u)$. Then $N(u)\backslash\{v \cup N(v)\} = \{x\}$ and $N(v)\backslash\{u \cup N(u)\} = \{y\}$. Suppose $G \not\cong C_n$, $n \geq 4$. Then there is a vertex v in G such that $d(v) \geq 3$. Since $G \not\cong K_{1,n-1}^+$ by Proposition 2.12(c), $d(v) \leq \Delta(G) < n - 1$. So v is not adjacent to at least one vertex in G. Let w be one such. Suppose P is a shortest path between v and w. Take the vertex adjacent to v in P as u. Now by Proposition 2.12(a), v and u have at most one common vertex, say x. Since $d(v) \geq 3$, v is adjacent to a vertex, say y that is different from u and x. Then $y \in N(v)\backslash\{u \cup N(u)\}$. Let z be the other adjacent vertex of u in P. If z is not adjacent to v then $z \in N(u)\backslash\{v \cup N(v)\}$. Otherwise we have a path from v to w of length shorter than that of P, a contradiction. $\square$

Lemma 2.14. *Let $G \in \mathcal{U}(n)$ and $G \not\cong K_{1,n-1}^+$. Let u and v be two adjacent vertices in G such that $N(u)\backslash\{v \cup N(v)\} \neq \phi$ and $N(v)\backslash\{u \cup N(u)\} \neq \phi$.*

1. Let $N(u)\backslash\{v \cup N(v)\} = \{u_1, u_2, \ldots, u_t\}$. Let G_1 be the graph obtained from G by deleting the edges uu_i and adding the edges vu_i $(1 \leq i \leq t)$. Then G_1 is also a unicyclic graph of order n.

2. Let $N(v)\backslash\{u \cup N(u)\} = \{v_1, v_2, \ldots, v_s\}$. Let G_1 be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. Then G_1 is also a unicyclic graph of order n.

Proof. 1. Since u and v are adjacent and G is connected, then G_1 is also connected. Also it is clear that G and G_1 have same number of edges which is n. So G_1 is a connected graph of order n having exactly n edges. Then by Proposition 2.12(d), G_1 is unicyclic.

2. It follows from similar arguments. $\square$

Proof of Theorem 2.11. Let $G \in \mathcal{U}(n)$ such that $\rho(G) \geq \rho(G')$ for all $G' \in \mathcal{U}(n)$. If $G \cong K_{1,n-1}^+$ there is nothing to prove. Let $G \not\cong K_{1,n-1}^+$. By Lemma 2.13, there are two adjacent vertices u and v in G such that $N(u)\backslash\{v \cup N(v)\} \neq \phi$ and $N(v)\backslash\{u \cup N(u)\} \neq \phi$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$.

Case 1. $x_u \geq x_v$. Let $v_1, v_2, \ldots, v_s$ be all vertices of $N(v)\backslash\{u \cup N(u)\}$. Let G_1 be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. By Lemma 2.14, G_1 is also a unicyclic graph of order n. Now by Theorem 2.1, $\rho(G_1) > \rho(G)$.

Case 2. $x_v \geq x_u$. Let $u_1, u_2, \ldots, u_t$ be all vertices of $N(u)\backslash\{v \cup N(v)\}$. Let G_1 be the graph obtained from G by deleting the edges uu_i and adding the edges vu_i $(1 \leq i \leq t)$. By Lemma 2.14, G_1 is also a unicyclic graph of order n. Now by Theorem 2.1, $\rho(G_1) > \rho(G)$.

In either case $\rho(G_1) > \rho(G)$ which is a contradiction to the fact that $\rho(G) \geq \rho(G')$ for all $G' \in \mathcal{U}(n)$. $\square$

Acknowledgements

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