# Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations\*

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Abstract We study the boundary behaviour of the solutions of (E)  $-\Delta_p u + |\nabla u|^q = 0$  in a domain  $\Omega \subset \mathbb{R}^N$ , when  $N \geq p > q > p-1$ . We show the existence of a critical exponent  $q_* < p$  such that if  $p-1 < q < q_*$  there exist positive solutions of (E) with an isolated singularity on  $\partial\Omega$  and that these solutions belong to two different classes of singular solutions. If  $q_* \leq q < p$  no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

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#### 1 Introduction

Let  $N \geq p > 1$ , q > p - 1 and  $\Omega \subset \mathbb{R}^N$  (N > 1) be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$ . In this article we study the boundary behavior at 0 of nonnegative functions  $u \in C^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$  which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \quad \text{in } \Omega, \tag{1.1}$$

where  $\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u)$ . The two main questions we consider are as follows:

**Q-1**- Existence of positive solutions of (1.1).

Q-2- Description of positive solutions with an isolated boundary singularity at 0.

When p=2 a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \tag{1.2}$$

in  $\Omega$  is provided by Nguyen-Phuoc and Véron [11]. In particular they prove the following series of results in the range of values 1 < q < 2.

1- Any signed solution of (1.3) verifies the estimates

$$|\nabla u(x)| \le c_{N,q} (d(x))^{-\frac{1}{q-1}} \qquad \forall x \in \Omega, \tag{1.3}$$

where  $d(x) = \text{dist}\,(x,\partial\Omega)$ . As a consequence, if  $u \in C(\overline{\Omega} \setminus \{0\})$  is a solution which vanishes on  $\partial\Omega \setminus \{0\}$ , it satisfies

$$|u(x)| \le c_{q,\Omega} d(x)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega.$$
(1.4)

**2-** If  $\frac{N+1}{N} \le q < 2$  any positive solution of (1.3) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$  is identically 0. An isolated boundary point is a removable singularity for (1.2).

**3-** If  $1 < q < \frac{N+1}{N}$  and k > 0 there exists a unique positive solution  $u := u_k$  of (1.2) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies  $u(x) \sim c_N k P^{\Omega}(x,0)$  as  $x \to 0$ , where  $P^{\Omega}$  is the Poisson kernel in  $\Omega \times \partial \Omega$ .

**4-** If  $1 < q < \frac{N+1}{N}$  there exists a unique positive solution u of (1.2) in the half-space  $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$  under the form  $u(x) = |x|^{-\frac{2-q}{q-1}} \omega(|x|^{-1}x)$  which vanishes on  $\partial \mathbb{R}^N_+ \setminus \{0\}$ . The function  $\omega$  is the unique positive solution of

$$-\Delta'\omega + \left( \left( \frac{2-q}{q-1} \right)^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{q}{2}} - \lambda_{N,q}\omega = 0 \quad \text{in } S_+^{N-1},$$

$$\omega = 0 \quad \text{in } \partial S_+^{N-1},$$

$$(1.5)$$

where  $S^{N-1}$  is the unit sphere of  $\mathbb{R}^N$ ,  $\partial S_+^{N-1} = \partial \mathbb{R}_+^N \cap S^{N-1}$ ,  $\Delta'$  the Laplace-Beltrami operator and  $\lambda_{N,q} > 0$  an explicit constant.

5- If  $1 < q < \frac{N+1}{N}$  and u is a positive solution of (1.3) in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus \{0\}$  and vanishes on  $\partial \Omega \setminus \{0\}$  the following dichotomy occurs:

(i) either  $u(x) \sim |x|^{-\frac{2-q}{q-1}} \omega(|x|^{-1}x)$  as  $x \to 0$ ,

(ii) or  $u(x) \sim kc_N P^{\Omega}(x,0)$  as  $x \to 0$  for some  $k \ge 0$ .

The aim of this article is to extend to the quasilinear case 1 the above mentioned results. The following pointwise gradient estimate valid for any signed solution <math>u of (1.1) has been proved in [3]: if  $0 there exists a constant <math>c_{N,p,q} > 0$  such that

$$|\nabla u(x)| \le c_{N,p,q} (d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega.$$
 (1.6)

As a consequence, any solution  $u \in C^1(\overline{\Omega} \setminus \{0\})$  satisfies

$$|u(x)| \le c_{p,q,\Omega} d(x) |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega.$$

$$(1.7)$$

Concerning boundary singularities, the situation is much more complicated than in the case p=2 and the threshold of critical exponent less explicit. We first consider the problem in  $\mathbb{R}_+^N$ . Assuming  $p-1 < q \leq p$ , separable solutions of (1.1) in  $\mathbb{R}_+^N$  vanishing on  $\partial \mathbb{R}_+^N \setminus \{0\}$  can be looked for in spherical coordinates  $(r,\sigma) \in \mathbb{R}_+^* \times S^{N-1}$  (we denote  $\mathbb{R}_+^* = (0,\infty)$ ) under the form

$$u(x) = u(r, \sigma) = r^{-\beta_q} \omega(\sigma), \quad r > 0, \ \sigma \in S_+^{N-1} := \{ S^{N-1} \cap \mathbb{R}_+^N \}.$$
 (1.8)

Then  $\omega$  is solution of the following problem

$$-div'\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1}$$

$$\omega = 0 \quad \text{on } \partial S_+^{N-1},$$

$$(1.9)$$

where

$$\beta_q = \frac{p-q}{q+1-p} \text{ and } \Lambda_{\beta_q} = \beta_q(p-1) + p - N,$$
 (1.10)

and  $\nabla'$  is the covariant derivative on  $S^{N-1}$  identified to the tangential gradient thanks to the canonical isometrical imbedding of  $S^{N-1}$  into  $\mathbb{R}^N$ , and div' the divergence operator acting on vector fields on  $S^{N-1}$ .

The existence of a positive solution to this problem cannot be separated from the problem of existence of *separable p-harmonic functions* which are *p*-harmonic in  $\mathbb{R}^N_+$  which vanish on  $\partial \mathbb{R}^N_+ \setminus \{0\}$  and have the form  $\Psi(x) = \Psi(r, \sigma) = r^{-\beta} \psi(\sigma)$  for some real number  $\beta$ . Necessarily such a  $\psi$  must satisfy

$$-div'\left(\left(\beta^{2}\psi^{2}+|\nabla'\psi|^{2}\right)^{\frac{p-2}{2}}\nabla'\psi\right)-\beta\Lambda_{\beta}\left(\beta^{2}\psi^{2}+|\nabla'\psi|^{2}\right)^{\frac{p-2}{2}}\psi=0 \quad \text{in } S_{+}^{N-1}$$

$$\psi=0 \quad \text{on } \partial S_{+}^{N-1},$$

$$(1.11)$$

where  $\Lambda_{\beta}=\beta(p-1)+p-N$ . We will refer to (1.11) as the spherical p-harmonic eigenvalue problem. The study of this problem has been initiated in the 2-dim case by Krol [8] ( $\beta<0$ ) and Kichenassamy and Véron [9] ( $\beta>0$ ). In this case  $\omega$  satisfies a completely integrable second order differential equation. In the case where  $S_+^{N-1}$  is replaced by a smooth domain  $S\subset S^{N-1}$  with  $N\geq 3$ , Tolksdorf [14] proved the existence of a unique couple  $(\tilde{\beta}_s,\tilde{\psi}_s)$  where  $\tilde{\beta}_s<0$  and  $\tilde{\psi}_s$  has constant sign and is defined up to an homothety. Recently Porretta and Véron [12] gave a simpler and more general proof of the existence of two couples  $(\tilde{\beta}_s,\tilde{\psi}_s)$  and  $(\beta_{*s},\psi_{*s})$  where  $\beta_{*s}>0$  and  $\tilde{\psi}_s$  and  $\psi_{*s}$  are positive solutions of (1.11) with

 $\beta = \tilde{\beta}_s$  and  $\beta = \beta_{*s}$  respectively and are unique up to a multiplication by a real number. When p=2 this problem is an eigenvalue problem for the Laplace-Beltrami operator on a subdomain of  $S^{N-1}$ . If  $S = S_+^{N-1}$ ,  $\tilde{\beta}_s$  and  $\beta_{*s}$  are respectively denoted by  $\tilde{\beta}$  and  $\beta_*$  and accordingly  $\tilde{\psi}_s$  and  $\psi_{*s}$  by  $\tilde{\psi}$  and  $\psi_*$ . Since  $x \mapsto x_N$  is p-harmonic,  $\tilde{\beta} = -1$ . Except in the cases N=2 where it is the positive root of some algebraic equation of degree 2, p=2 where it is N-1 and p=N where it is 1, the value of  $\beta_*$  is unknown besides the straightforward estimate  $\beta_* \geq \max\{1, \frac{N-p}{p-1}\}$ . Using the fact that  $\psi_*$  depends only on the azimuthal variable and satisfies a differential equation, we prove in Appendix II the following new estimate:

**Theorem A** Let 1 .

- (i) If  $2 \le p \le N$ , then  $\beta_* \le \frac{N-1}{p-1}$  with equality only if p = 2 or N.
- (ii) If  $1 \le p < 2$ , then  $\beta_* > \frac{N-1}{p-1}$ .

The p-harmonic function  $\Psi_*(x) = \Psi_*(r,\sigma) = r^{-\beta_*}\psi_*(\sigma)$  endows the role of a Poisson kernel. To this exponent  $\beta_*$  is associated the critical value  $q_*$  of q defined by  $\beta_* = \beta_q$ , or equivalently

$$q_* := \frac{\beta_*(p-1) + p}{\beta_* + 1} = p - \frac{\beta_*}{\beta_* + 1}.$$
 (1.12)

The following result characterizes strong singularities.

**Theorem B** Let 0 , then

- (i) If  $p-1 < q < q_*$  problem (1.9) admits a unique positive solution  $\omega_*$ .
- (ii) If  $q_* \leq q < p$  problem (1.9) admits no positive solution.

This critical exponent corresponds to the threshold of criticality for boundary isolated singularities.

**Theorem C** Assume  $q_* \le q . If <math>u \in C^1(\overline{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$ , it is identical zero.

As in the case p=2, there exist positive solutions (1.1) in  $\Omega$  with weak boundary singularities which are characterized by their blow-up near the singularity. By opposition to the case p=2 where existence is obtained by use of a weak formulation of the boundary value problem, combined with uniform integrability of the absorption term thanks to Poisson kernel estimates (see [11]), this approach cannot be performed in the case  $p \neq 2$ ; the obtention of solutions with weak singularities necessitates a very long and delicate construction of subsolutions and supersolutions. Furthermore, when  $p \neq N$ , the construction is done only if  $\Omega$  is locally an hyperplane near 0.

In the sequel we denote by  $B_R(a)$  the open ball of center a and radius R>0 and  $B_R=B_R(0)$ . We also set  $B_R^+(a):=\mathbb{R}_+^N\cap B_R(a)$ ,  $B_R^+:=\mathbb{R}_+^N\cap B_R$ ,  $B_R^-(a):=\mathbb{R}_-^N\cap B_R(a)$  and  $B_R^-:=\mathbb{R}_-^N\cap B_R$ , where  $\mathbb{R}_-^N:=\{x=(x',x_N):x'\in\mathbb{R}^{N-1},x_N<0\}$ . If  $\Omega$  is an open domain and R>0, we put  $\Omega_R=\Omega\cap B_R$ 

**Theorem D** Let  $\Omega \subset \mathbb{R}^N_+$  be a bounded domain such that  $0 \in \partial \Omega$ . Assume there exists  $\delta > 0$  such that  $\Omega_{\delta} = B_{\delta}^+$  and 0 . Then for any <math>k > 0 there exists a unique  $u := u_k \in C^1(\overline{\Omega} \setminus \{0\})$ , solution of (1.1) in  $\Omega$ , vanishing on  $\partial \Omega \setminus \{0\}$  and such that

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_{+}^{N-1}}} |x|^{\beta_*} u_k(x) = k \psi_*(\sigma). \tag{1.13}$$

Furthermore  $\lim_{k\to\infty} u_k = u_\infty$  and

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_{+}^{N-1}}} |x|^{\beta_q} u_{\infty}(x) = \psi_{*}(\sigma). \tag{1.14}$$

When p=N, then  $q_*=N-\frac{1}{2}$ ; in such a range of values we use the conformal invariance of  $\Delta_N$  and prove that the previous result holds if  $\Omega$  is any  $C^2$  domain. Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

**Theorem E** Let  $\Omega$  be a bounded domain such that  $0 \in \partial \Omega$ . Assume there exists  $\delta > 0$  such that  $\Omega_{\delta} = B_{\delta}^+$  and  $0 . If <math>u \in C^1(\overline{\Omega} \setminus \{0\})$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$ , then

(i) either there exists  $k \ge 0$  such that

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_{+}^{N-1}}} |x|^{\beta_*} u(x) = k \psi_*(\sigma); \tag{1.15}$$

(ii) or

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_{+}^{N-1}}} |x|^{\beta_{q}} u(x) = \psi_{*}(\sigma). \tag{1.16}$$

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# 2 A priori estimates

#### 2.1 The gradient estimates and its applications

We recall the following estimate and its consequences which are proved in [3].

**Proposition 2.1.** Assume q > p-1 and u is a  $C^1$  solution of (1.1) in a domain  $\Omega$ . Then

$$|\nabla u(x)| \le c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega.$$
(2.1)

The first application is a pointwise upper bound for solutions with isolated singularities.

**Corollary 2.2.** Assume q > p - 1 > 0,  $R^* > 0$  and  $\Omega$  is a domain containing 0 such that  $d(0) \ge 2R^*$ . Then for any  $x \in B_{R^*} \setminus \{0\}$ , and  $0 < R \le R^*$ , any  $u \in C^1(\Omega \setminus \{0\})$  solution of (1.1) in  $\Omega \setminus \{0\}$ ) satisfies

$$|u(x)| \le c_{N,p,q} \left| |x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\},$$
 (2.2)

if  $p \neq q$ , and

$$|u(x)| \le c_{N,p} \left( \ln R - \ln |x| \right) + \max\{|u(z)| : |z| = R\},\tag{2.3}$$

if p = q.

The second application corresponds to solutions with boundary blow-up. For  $\delta > 0$  small enough we set  $\Omega_{\delta} := \{z \in \Omega : d(z) < \delta\}$ .

**Corollary 2.3.** Assume q > p-1 > 0,  $\Omega$  is a bounded domain with a  $C^2$  boundary. Then there exists  $\delta_1 > 0$  which depends only on  $\Omega$  such that any  $u \in C^1(\Omega)$  solution of (1.1) in  $\Omega$  satisfies

$$|u(x)| \le c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1}$$
 (2.4)

if  $p \neq q$ , and

$$|u(x)| \le c_{N,p,q} (\ln \delta_1 - \ln d(x)) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1}$$
 (2.5)

if p = q.

*Remark.* As a consequence of (2.4) there holds for p > q > p - 1

$$u(x) \le (c_{N,p,q} + K \max\{|u(z)| : d(z) \ge \delta_1\}) (d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega$$
 (2.6)

where  $K = (\operatorname{diam}(\Omega))^{\frac{p-q}{q+1-p}}$ , with the standard modification if p = q.

As a variant of Corollary 2.3 the following upper estimate of solutions in an exterior domain will be used in the sequel.

**Corollary 2.4.** Assume q > p - 1 > 0, R > 0 and  $u \in C^1(B_{R_0}^c)$  is any solution of (1.1) in  $B_{R_0}^c$ . Then for any  $R > R_0$  there holds

$$|u(x)| \le c_{N,p,q} \left| (|x| - R_0)^{\frac{q-p}{q+1-p}} - (R - R_0)^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c$$
 (2.7)

if  $p \neq q$  and

$$|u(x)| \le c_{N,p,q} \left( \ln(|x| - R_0) - \ln(R - R_0) \right) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B_R^c$$
 (2.8)

if p = q.

*Proof.* The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z)dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where  $z = \frac{R}{|x|}x$ . Since by (2.1)

$$|\nabla u(tx + (1-t)z)| \le C_{N,n,q}(t|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}},$$

(2.7) and (2.8) follow by integration.

#### 2.2 Boundary a priori estimates

The next result is the extension to isolated boundary singularities of a previous regularity estimate dealing with singularity in a domain proved in [3, Lemma 3.10].

**Lemma 2.5.** Assume p-1 < q < p,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial \Omega$ . Let  $u \in C^1(\overline{\Omega} \setminus \{0\})$  be a solution of (1.1) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies

$$|u(x)| \le \phi(|x|) \qquad \forall x \in \Omega, \tag{2.9}$$

where  $\phi: \mathbb{R}_+^* \mapsto \mathbb{R}_+$  is continuous, nonincreasing and satisfies

$$\phi(rs) \le \gamma \phi(r)\phi(s)$$
 and  $r^{\frac{p-q}{q+1-p}}\phi(r) \le c,$  (2.10)

for some  $\gamma, c > 0$  and any r, s > 0. There exist  $\alpha \in (0,1)$  and  $c_1 = c_1(p,q,\Omega) > 0$  such that

(i) 
$$|\nabla u(x)| \le c_1 \phi(|x|) |x|^{-1}$$
  $\forall x \in \Omega,$   
(ii)  $|\nabla u(x) - \nabla u(y)| \le c_1 \phi(|x|) |x|^{-1-\alpha} |x-y|^{\alpha}$   $\forall x, y \in \Omega, |x| \le |y|.$  (2.11)

*Furthermore* 

$$u(x) \le c_1 \phi(|x|) \frac{d(x)}{|x|} \qquad \forall x \in \Omega.$$
 (2.12)

*Proof.* For  $\ell>0$ , we set  $\Omega^\ell:=\frac{1}{\ell}\Omega$ . If  $\ell\in(0,1]$  the curvature of  $\partial\Omega^\ell$  remains uniformly bounded. As in [5, p 622], there exists  $0<\delta_0\leq 1$  and an involutive diffeomorphism  $\psi$  from  $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$  into  $\overline{B}_{\delta_0}\cap(\Omega^{\delta_0})^c$  which is the identity on  $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$  and such that  $D\psi(\xi)$  is the symmetry with respect to the tangent plane  $T_\xi\partial\Omega$  for any  $\xi\in\partial\Omega\cap\overline{B}_{\delta_0}$ . We extend any function v defined in  $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$  and vanishing on  $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$  into a function  $\tilde{v}$  defined in  $\overline{B}_{\delta_0}$  by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0} \\ -v \circ \psi(x) & \text{if } x \in \overline{B}_{\delta_0} \cap (\Omega^{\delta_0})^c, \end{cases}$$
 (2.13)

If  $v \in C^1(\overline{B}_{\delta_0} \cap \overline{\Omega}^{\delta_0})$  is a solution of (1.1) in  $B_{\delta_0} \cap \Omega^{\delta_0}$  which vanishes on  $\partial \Omega^{\delta_0} \cap \overline{B}_{\delta_0}$ ,  $\tilde{v}$  satisfies

$$-\sum_{j} \frac{\partial}{\partial x_{j}} \tilde{A}_{j}(x, \nabla \tilde{v}) + B(x, \nabla \tilde{v}) = 0 \quad \text{in } B_{\delta_{0}}.$$
 (2.14)

As in [5, (2.37)] the  $A_i$  and B satisfy the following estimates

(i) 
$$\tilde{A}_{j}(x,0) = 0$$
(ii) 
$$\sum_{i,j} \frac{\partial}{\partial \eta_{i}} \tilde{A}_{j}(x,\eta) \xi_{i} \xi_{j} \geq C_{1} |\eta|^{p-1} |\xi|^{2}$$
(iii) 
$$\sum_{i,j} \left| \frac{\partial}{\partial \eta_{j}} \tilde{A}_{j}(x,\eta) \right| \leq C_{2} |\eta|^{p-2},$$
(2.15)

and

$$|B(x,\eta)| \le C_3 (1+|\eta|)^p, \tag{2.16}$$

where the  $C_j$  are positive constants. These estimates are the ones needed to apply Tolksdorf's result [15, Th 1,2]. There exists a constant C, such that for any ball  $\overline{B}_{3R} \subset \overline{B}_{\delta_0}$ , there holds

$$\|\nabla \tilde{v}\|_{L^{\infty}(B_R)} \le C,\tag{2.17}$$

where C depends on the constants  $C_k$  (k=1,2,3), N, p and  $\|\tilde{v}\|_{L^{\infty}(B_{3R})}$ . We define

$$\Phi_{\ell}[u](y) := u_{\ell} = \frac{1}{\phi(\ell)} u(\ell y) \qquad \forall y \in \Omega^{\ell}. \tag{2.18}$$

Then

$$|u_{\ell}(y)| \le \frac{\phi(\ell|y|)}{\phi(\ell)} \le \gamma\phi(|y|) \qquad \forall y \in \Omega^{\ell}$$
 (2.19)

and

$$-\Delta_p u_\ell + (\ell^{\beta_q} \phi(\ell))^{q+1-p} |\nabla u_\ell|^q = 0 \quad \text{in } \Omega^\ell.$$
 (2.20)

Using formula (2.13) we extend  $u_{\ell}$  into a function  $\tilde{u}_{\ell}$  which satisfies

$$-\sum_{j} \frac{\partial}{\partial y_{j}} \tilde{A}_{j}(y, \nabla \tilde{u}_{\ell}) + (\ell^{\beta_{q}} \phi(\ell))^{q+1-p} B(y, \nabla \tilde{u}_{\ell}) = 0 \quad \text{in } B_{\delta_{0}}.$$
 (2.21)

For  $0<|x|<\delta_0$  there exists  $\ell\in(0,2)$  such that  $\frac{\delta_0\ell}{2}\leq |x|\leq \delta_0\ell$ . Then  $y\mapsto \tilde{u}_\ell(y)$  with  $y=\frac{x}{\ell}$  satisfies (2.21) in  $B_{\delta_0}$  and  $|\tilde{u}_\ell(y)|\leq \gamma_*\phi(|y|)$  since  $\psi$  is a diffeomorphism and  $D\psi(\xi)\in O(N)$  for any  $\xi\in\partial\Omega\cap B_{\delta_0}$ . The function  $\tilde{u}_\ell$  remains bounded on any ball  $B_{3R}(z)\subset\Gamma:=\{y\in\mathbb{R}^N:\frac{\delta_0}{2}<|y|<\delta_0\}$ , therefore  $|\nabla \tilde{u}_\ell(y)|\leq c$  for any  $y\in B_R(z)$ , for some constant c>0. This implies

$$|\nabla u(x)| \le c\gamma_* \delta_0 \phi(\frac{2}{\delta_0}) \phi(|x|) |x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}, \tag{2.22}$$

which is (2.11)-(i). Moreover, by standard regularity estimates [10], there exists  $\alpha \in (0,1)$  such that  $|\nabla \tilde{u}_{\ell}(y) - \nabla \tilde{u}_{\ell}(y')| \le c |y - y'|^{\alpha}$  for all y and y' belonging to  $B_R(z)$ . This implies (2.11)-(ii).

Next we prove (2.12). Let  $0 < \delta_1 \le \delta_0$  such that at any boundary point z there exist two closed balls of radius  $\delta_1$  tangent to  $\partial\Omega$  at z and which are included in  $\Omega \cup \{z\}$  and in  $\overline{\Omega}^c \cup \{z\}$  respectively ( $\delta_1$  corresponds to the maximal radius of the interior and exterior sphere condition). Let  $x \in \Omega$  such that  $d(x) \le \delta_1$  (this is not a loss of generality) and  $z_x$  be the projection of x on  $\partial\Omega$ . We first assume that x does not belong to the cone  $\Sigma_{\frac{\pi}{4}}$  with vertex 0, axis  $-\mathbf{n}_0$ , where  $\mathbf{n}_0$  is the normal outward unit vector at 0, and angle  $\frac{\pi}{4}$ . Consider the path  $\zeta$  from  $z_x$  to x defined by  $\zeta(t) = tx + (1-t)z_x$  with  $0 \le t \le 1$ . Then

$$u(x) = \int_0^1 \frac{d}{dt} u \circ \zeta(t) dt = \int_0^1 \langle \nabla u \circ \zeta(t), x - z_x \rangle dt$$
 (2.23)

Thus, by the Cauchy-Schwarz inequality, using (2.11),

$$|u(x)| \le c_1 d(x) \int_0^1 \frac{\phi(|\zeta(t)|)}{|\zeta(t)|} dt.$$
 (2.24)

Since  $x \notin \Sigma_{\frac{\pi}{4}}$ ,  $\zeta(t) \notin \Sigma_{\frac{\pi}{4}}$  and there exists  $c_2 > 0$  depending on  $\Omega$  such that  $c_2^{-1} |x| \le |\zeta(t)| \le c_2 |x|$  for all  $0 \le t \le 1$ . Therefore  $\phi(|\zeta(t)|) \le \phi(c_2 |x|) \le \gamma \phi(c_2) \phi(|x|)$  by (2.10). This implies

$$|u(x)| \le \gamma c_1 c_2 \phi(c_2) \frac{d(x)\phi(|x|)}{|x|}$$
 (2.25)

by (2.12) whenever  $x \notin \Sigma_{\frac{\pi}{4}}$ . When  $x \in \Sigma_{\frac{\pi}{4}}$  then  $d(x) \le |x| \le c_3 d(x)$  where  $c_3 > 0$  depends on the curvature of  $\partial\Omega$ . Then (2.9) combined with (2.10) implies the claim.

**Lemma 2.6.** Assume  $p-1 < q \le p$ ,  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial \Omega$  and  $R_0 = \max\{|z| : z \in \Omega\}$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) which vanishes on  $\partial \Omega \setminus \{0\}$ , it satisfies

$$u(x) \le \begin{cases} c_2 \left( |x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}} \right) & \text{if } q (2.26)$$

for all  $x \in \Omega$ , where  $c_2 = c_2(p,q) > 0$ .

*Proof.* For  $\epsilon > 0$  we denote by  $P_{\epsilon} : \mathbb{R} \mapsto \mathbb{R}_+$  the function defined by

$$P_{\epsilon}(r) = \begin{cases} 0 & \text{if } 0 \le r \le \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \ge 2\epsilon, \end{cases}$$
 (2.27)

and by  $u_{\epsilon}$  the extension of  $P_{\epsilon}(u)$  by zero outside  $\Omega$ . There exists  $R_0$  such that  $\Omega \subset B_{R_0}$ . Since  $0 \leq P_{\epsilon}(r) \leq |r|$  and  $P_{\epsilon}$  is convex,  $u_{\epsilon} \in C(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N \setminus \{0\})$  and

$$-\Delta_p u_{\epsilon} + |\nabla u_{\epsilon}|^q \le 0 \qquad \text{in } \mathbb{R}^N.$$

Let  $R > R_0$ . If p - 1 < q < p

$$U_{\epsilon,R}(|x|) = c_2 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (R - \epsilon)^{\frac{q-p}{q+1-p}} \right) \quad \text{in } B_R \setminus B_{\epsilon}, \tag{2.28}$$

with  $c_2=(p-q)^{-1}(q+p-1)^{\frac{q-p}{q+1-p}}$ . Then  $-\Delta_p U_{\epsilon,R}+|\nabla U_{\epsilon,R}|^q\geq 0$ . Since  $u_\epsilon$  vanishes on  $\partial B_R$  and is finite on  $\partial B_\epsilon$ , it follows  $u_\epsilon\leq U_{\epsilon,R}$ . Letting successively  $\epsilon\to 0$  and  $R\to R_0$  yields to (2.26). If q=p we take

$$U_{\epsilon,R}(|x|) = (p-1)\ln\left(\frac{R-\epsilon}{|x|-\epsilon}\right) \quad \text{in } B_R \setminus B_{\epsilon}, \tag{2.29}$$

which turns out to be a supersolution of (1.1); the end of the proof is similar.

As a consequence of Lemma 2.5 and Lemma 2.6, we obtain.

**Corollary 2.7.** Let  $p, q \Omega$  and u be as in Lemma 2.6. Then there exists a constant  $c_3 = c_3(p, q, \Omega) > 0$  such that

$$|\nabla u(x)| \le c_3 |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega$$
 (2.30)

and

$$u(x) \le c_3 d(x) |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \overline{\Omega} \setminus \{0\}. \tag{2.31}$$

*Remark.* If  $\Omega$  is locally flat near 0, then estimates (2.30) and (2.31) are valid without any sign assumption on u. More precisely, if  $\partial\Omega \cap B_{\delta_0} = T_0\partial\Omega \cap B_{\delta_0}$  we can perform the reflection of u through the tangent plane  $T_0\partial\Omega$  to  $\partial\Omega$  at 0 and the new function  $\tilde{u}$  is a solution of (1.1) in  $B_{\delta_0} \setminus \{0\}$ . By Proposition 2.1, it satisfies

$$|\nabla \tilde{u}(x)| \le c_{N,p,q} |x|^{-\frac{1}{q+1-p}} \qquad \forall x \in B_{\frac{\delta_0}{2}} \setminus \{0\}.$$
(2.32)

Integrating this relation as in [3], we derive that for any  $x \in B_{\frac{\delta_0}{2}} \cap \Omega$ , there holds

$$|u(x)| \le \begin{cases} c_{N,p,q} \left( |x|^{-\beta_q} - \left(\frac{\delta_0}{2}\right)^{-\beta_q} \right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p \ne q \\ c_{N,p} \ln\left(\frac{\delta_0}{2|x|}\right) + \max\{|u(z)| : |z| = \frac{\delta_0}{2}\} & \text{if } p = q. \end{cases}$$
(2.33)

In the next result we allow the boundary singular set to be a compact set.

**Proposition 2.8.** Let p-1 < q < p and  $\delta_1$  as above. There exist  $r^* \in (0, \delta_1]$  and  $c_4 = c_4(N, p, q) > 0$  such that for any nonempty compact set  $K \subset \partial \Omega$ ,  $K \neq \partial \Omega$  and any positive solution  $u \in C(\overline{\Omega} \setminus K) \cap C^1(\Omega)$  of (1.1) which vanishes on  $\partial \Omega \setminus K$ , there holds

$$u(x) \le c_4 d(x) (d_K(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \partial \Omega \text{ s.t. } d(x) \le r^*, \tag{2.34}$$

where  $d_K(x) = \text{dist}(x, K)$ .

Proof. Step 1: Tangential estimates. Let  $x \in \Omega$  such that  $d(x) \leq \delta_1$ . We denote by  $\sigma(x)$  the projection of x onto  $\partial\Omega$ , unique since  $d(x) \leq \delta_1$ . Let  $r, r', \tau > 0$  such that  $\frac{3}{4}r < r' < \frac{7}{8}r$  and  $0 < \tau \leq \frac{r'}{2}$  and put  $\omega_{\tau,x} = \sigma(x) + \tau \mathbf{n}_{\sigma(x)}$ . Since  $\partial\Omega$  is  $C^2$ , there exists  $0 < r^* \leq \delta_1$  depending on  $\Omega$  such that  $d_K(\omega_{\tau,x}) > \frac{7}{8}r$  whenever  $d(x) \leq r^*$ . Let a > 0 and b > 0 to be specified later on; we define  $\tilde{v}(s) = a(r'-s)^{\frac{q-p}{q+1-p}} - b$  and  $v(y) = \tilde{v}(|y-\omega_{\tau,x}|)$  in [0,r') and  $B_{r'}(\omega_{\tau,x})$  respectively. Then

$$|\tilde{v}'|^{p-2} \left( |\tilde{v}'|^{q+2-p} - (p-1)\tilde{v}'' - \frac{N-1}{s}\tilde{v}' \right) = a^{p-1} \left( \frac{p-q}{q+1-p} \right)^{p-1} (r'-s)^{-\frac{q}{q+1-p}} X(s)$$

where

$$X(s) = \left(a\frac{p-q}{q+1-p}\right)^{q+1-p} - \frac{p-1}{q+1-p} - \frac{(N-1)(r'-s)}{s}.$$

For any  $\tau \in (0, r')$  there exists a > 0 such that

$$\left(a\frac{p-q}{q+1-p}\right)^{q+1-p} \ge \frac{p-1}{q+1-p} + \frac{(N-1)(r'-s)}{s} \qquad \forall \tau \le s \le r'.$$

This implies

$$-\Delta_p v + |\nabla v|^q \ge 0 \qquad \text{in } B_{r'}(\omega_{\tau,x}) \setminus B_{\tau}(\omega_{\tau,x}). \tag{2.35}$$

Next we take  $b=a(r'-\tau)^{\frac{q-p}{q+1-p}}$ , thus v=0 on  $\partial B_{\tau}(\omega_{\tau,x})$ . Clearly  $B_{\tau}(\omega_{\tau,x})\subset\overline{\Omega}^c$  since  $\tau<\delta_1$ . Therefore  $v\geq 0=u$  on  $\partial\Omega\cap B_{r'}(\omega_{\tau,x})$  and  $u\leq v=\infty$  on  $\Omega\cap\partial B_{r'}(\omega_{\tau,x})$ . By the comparison principle,  $v\geq u$  in  $\Omega\cap B_{r'}(\omega_{\tau,x})$ . In particular

$$u(x) \le v(x) \le a(r' - \tau - d(x))^{\frac{q-p}{q+1-p}} - a(r' - \tau)^{\frac{q-p}{q+1-p}}.$$

We take now  $\tau = \frac{r'}{2}$  and  $d(x) \leq \frac{r}{4}$  and we derive by the mean value theorem

$$u(x) \le c_4' r'^{-\frac{1}{q+1-p}} d(x) = c_4' d(x) (d_K(x))^{-\frac{1}{q+1-p}}, \tag{2.36}$$

with  $c_4' = c_4'(p,q) > 0$  Letting  $r' \to \frac{7}{8}r$ , we get (2.12).

Step 2: Global estimates. If  $d(x) \ge \frac{1}{4} d_K(x)$ , there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \ge 2^{-\frac{2}{q+1-p}}(d(x))^{\frac{q-p}{q+1-p}}$$

Combining this inequality with (2.6) and obtain (2.34).

*Remark.* Under the assumption of Proposition 2.8, it follows from the maximum principle that u is upper bounded in the set  $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$  by the solution w of

$$-\Delta_{p}w + |\nabla w|^{q} = 0 \qquad \text{in } \Omega_{r^{*}}$$

$$w = c_{4}d(x)(d_{K}(x))^{-\frac{1}{q+1-p}} \qquad \text{in } \partial\Omega_{r^{*}},$$
(2.37)

and w itself is bounded by  $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}.$ 

Next we prove a boundary Harnack inequality. We recall that  $\delta_1$  has been introduced at Corollary 2.3, and that the interior and exterior sphere conditions hold in the set  $\{x \in \mathbb{R}^N : d(x) \leq \delta_1\}$ .

**Theorem 2.9.** Let q > p-1 and  $0 \in \partial\Omega$ . Then there exists  $c_5 = c_5(N, p, q, \Omega) > 0$  such that for any positive solution  $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ , there holds

$$\frac{u(y)}{c_5 d(y)} \le \frac{u(x)}{d(x)} \le c_5 \frac{u(y)}{d(y)}$$
 (2.38)

for all  $x, y \in B_{\frac{2\delta_1}{2}} \cap \Omega$  such that  $\frac{1}{2}|x| \leq |y| \leq 2|x|$ .

For proving Theorem 2.9 we need some intermediate lemmas. First we recall the following result from [1].

**Lemma 2.10.** Assume that  $a \in \partial\Omega$ ,  $0 < r < \delta_1$  and h > 1 is an integer. There exists an integer  $N_0$ , depending only on  $\delta_1$ , such that for any points x and y in  $\Omega \cap B_{\frac{3r}{2}}(a)$  verifying  $\min\{d(x), d(y)\} \ge r/2^h$ , there exists a connected chain of balls  $B_1, ..., B_j$  with  $j \le N_0 h$  such that

$$x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \text{ for } 1 \leq i \leq j-1$$
  
and  $2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j.$  (2.39)

The next result is a standard Harnack inequality.

**Lemma 2.11.** Assume  $a \in (\partial \Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \le |a|/4$ . Let  $u \in C(\Omega \cup ((\partial \Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial \Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a positive constant  $c_6 > 1$  depending on N, p, q and  $\delta_1$  such that

$$u(x) \le c_6^h u(y), \tag{2.40}$$

for every  $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$  such that  $\min\{d(x), d(y)\} \geq r/2^h$  for some  $h \in \mathbb{N}$ .

*Proof.* For  $\ell > 0$ , we define  $T_{\ell}[u]$  by

$$T_{\ell}[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x),$$
 (2.41)

and we notice that if u satisfies (1.1) in  $\Omega$ , then  $T_{\ell}[u]$  satisfies the same equation in  $\Omega^{\ell} := \ell^{-1}\Omega$ . If we take in particular  $\ell = |a|$ , we can assume |a| = 1, thus the curvature of the domain  $\Omega^{|a|}$  remains bounded. By Proposition 2.8

$$u(x) \le c_6' \quad \forall x \in B_{2r}(a) \cap \Omega$$
 (2.42)

where  $c'_6$  depends on N, q,  $\delta_1$ . Then we proceed as in [11], using Lemma 2.10 and internal Harnack inequality as quoted in [16, Corollary 10].

Since the solutions are Hölder continuous, the following statement holds as in [16, Theorem 4.2]:

**Lemma 2.12.** Let the assumptions on a and u of Lemma 2.11 be fulfilled. If  $b \in \partial \Omega \cap B_r(a)$  and  $0 < s \le 2^{-1}r$ , there exist two positive constants  $\delta$  and  $c_7$  depending on N, p, q and  $\Omega$  such that

$$u(x) \le c_7 \frac{|x-b|^{\delta}}{s^{\delta}} \max\{u(z) : z \in B_r(b) \cap \Omega\}$$
 (2.43)

for every  $x \in B_s(b) \cap \Omega$ .

As a consequence we derive the following Carleson type estimate.

**Lemma 2.13.** Assume  $a \in (\partial \Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \le |a|/8$ . Let  $u \in C(\Omega \cup ((\partial \Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial \Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exists a constant  $c_8$  depending only on N, p and q such that

$$u(x) \le c_8 u(a - \frac{r}{2}\mathbf{n}_a) \quad \forall x \in B_r(a) \cap \Omega.$$
 (2.44)

*Proof.* By Lemma 2.11 it is clear that for any integer h and  $x \in B_r(a) \cap \Omega$  such that  $d(x) \ge 2^{-h}r$ , there holds

$$u(x) \le c_6^h u(a - \frac{r}{2}\mathbf{n}_a). \tag{2.45}$$

Therefore u satisfies inequality (2.43) as any Hölder continuous function does. The proof that the constant is independent of r and u is more delicate. It is done in [1, Lemma 2.4] for linear equations, but it is based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case.

**Lemma 2.14.** Assume  $a \in (\partial \Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$  and  $0 < r \le |a|/8$ . Let  $u \in C(\Omega \cup ((\partial \Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$  be a positive solution of (1.1) vanishing on  $(\partial \Omega \setminus \{0\}) \cap B_{2\delta_1}$ . Then there exist  $\alpha \in (0, 1/2)$  and  $c_9 > 0$  depending on N, p and q such that

$$\frac{1}{c_9} \frac{t}{r} \le \frac{u(b - t\mathbf{n}_b)}{u(a - \frac{r}{2}\mathbf{n}_a)} \le c_9 \frac{t}{r}$$

$$\tag{2.46}$$

for any  $b \in B_r(a) \cap \partial \Omega$  and  $0 \le t < \frac{\alpha}{2}r$ .

*Proof.* It is similar to the one of [11, Lemma 3.15].

*Proof of Theorem 2.9.* Assume  $x \in B_{\frac{2\delta_1}{3}} \cap \Omega$  and set  $r = \frac{|x|}{8}$ .

Step 1: Tangential estimate: we suppose  $d(x) < \frac{\alpha}{2}r$ . Let  $a \in \partial \Omega \setminus \{0\}$  such that |a| = |x| and  $x \in B_r(a)$ . By Lemma 2.14,

$$\frac{8}{c_9} \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|} \le \frac{u(x)}{d(x)} \le 8c_9 \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|}.$$
 (2.47)

We can connect  $a - \frac{r}{2}\mathbf{n}_a$  with  $-2r\mathbf{n}_0$  by  $m_1$  (depending only on N) connected balls  $B_i = B_{\frac{r}{4}}(x_i)$  with  $x_i \in \Omega$  and  $d(x_i) \geq \frac{r}{2}$  for every  $1 \leq i \leq m_1$ . It follows from (2.44) that

$$c_6^{-m_1}u(-2r\mathbf{n}_0) \le u(a - \frac{r}{2}\mathbf{n}_a) \le c_6^{m_1}u(-2r\mathbf{n}_0),$$

which, together with (2.47) leads to

$$\frac{1}{c_{10}} \frac{u(-2r\mathbf{n}_0)}{|x|} \le \frac{u(x)}{d(x)} \le c_{10} \frac{u(-2r\mathbf{n}_0)}{|x|},\tag{2.48}$$

with  $c_{10} = 8c_9c_6^{m_1}$ .

Step 2: Internal estimate: we suppose  $d(x) \geq \frac{\alpha}{2}r$ . We can connect  $-2r\mathbf{n}_0$  with x by  $m_2$  (depending only on N) connected balls  $B_i' = B_{\frac{\alpha r}{4}}(x_i')$  with  $x_i' \in \Omega$  and  $d(x_i') \geq \frac{\alpha}{2}r$  for every  $1 \leq i \leq m_2$ . By Harnack and Carleson inequalities (2.40) and (2.44) and since  $\frac{\alpha}{4}|x| < d(x) \leq |x|$ , we get

$$\frac{\alpha}{4c_6'^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \le \frac{u(x)}{d(x)} \le \frac{4c_6'^{m_2}}{\alpha} \frac{u(-2r\mathbf{n}_0)}{|x|}.$$
 (2.49)

Step 3: End of proof. Suppose  $\frac{|x|}{2} \le s \le 2|x|$ , we can connect  $-2r\mathbf{n}_Q$  with  $-s\mathbf{n}_Q$  by  $m_3$  (depending only on N) connected balls  $B_i'' = B_{\frac{r}{2}}(x_i'')$  with  $x_i'' \in \Omega$  and  $d(x_i'') \ge r$  for every  $1 \le i \le m_3$ . This fact, jointly with (2.48) and (2.49), yields to

$$\frac{1}{c_{11}} \frac{u(-s\mathbf{n}_0)}{|x|} \le \frac{u(x)}{d(x)} \le c_{11} \frac{u(-s\mathbf{n}_0)}{|x|}$$
(2.50)

where  $c_{11} = c_{11}(N, q, \Omega)$ . Finally, if  $y \in B_{\frac{2r_0}{3}} \cap \Omega$  satisfies  $\frac{|x|}{2} \leq |y| \leq 2|x|$ , then by applying twice (2.50) we get (2.38) with  $c_5 = c_{11}^2$ .

The following inequality is a consequence of Theorem 2.9.

**Corollary 2.15.** Assume q > p-1 and  $0 \in \partial\Omega$ . Then there exists  $c_{12} > 0$  depending on p, q and  $\Omega$  such that for any positive solutions  $u_1, u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$  of (1.1) in  $\Omega$ , vanishing on  $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$ , there holds

$$\sup \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\} \le c_{12} \inf \left\{ \frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}} \right\}. \tag{2.51}$$

### 3 Boundary singularities

#### 3.1 Strongly singular solutions

In this section we consider the equation (1.1) in  $\mathbb{R}^N_+$ . We denote by  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  the spherical coordinates in  $\mathbb{R}^N$  and

$$S_+^{N-1} = \left\{ (\sin \phi \sigma', \cos \phi) : \sigma' \in S^{N-2}, \phi \in [0, \frac{\pi}{2}) \right\}.$$

If  $v(x) = r^{-\beta}\omega(\sigma)$  satisfies (1.1) in  $\mathbb{R}^N_+$  and vanishes on  $\partial \mathbb{R}^N_+ \setminus \{0\}$ , then  $\beta = \beta_q$  and  $\omega$  is a solution of

$$-div'\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S_+^{N-1}$$

$$\omega = 0 \quad \text{on } \partial S_\perp^{N-1}.$$
(3.1)

where  $\beta_q$  and  $\Lambda_{\beta_q}$  have been defined in (1.10). We denote by  $(\beta_*, \psi_*) \in \mathbb{R}_+^* \times C^2(\overline{S}_+^{N-1})$  the unique couple such  $\max \psi_* = 1$  with the property that the function  $(r, \sigma) \mapsto r^{-\beta_*} \psi_*(\sigma)$  is positive, p-harmonic in  $\mathbb{R}_+^N$  and vanishes on  $\partial \mathbb{R}_+^N \setminus \{0\}$ . Then  $\psi_* = \psi$  satisfies

$$-div'\left(\left(\beta_{*}^{2}\psi^{2}+|\nabla'\psi|^{2}\right)^{\frac{p-2}{2}}\nabla'\psi\right)-\beta_{*}\Lambda_{\beta_{*}}\left(\beta_{*}^{2}\psi^{2}+|\nabla'\psi|^{2}\right)^{\frac{p-2}{2}}\psi=0 \quad \text{in } S_{+}^{N-1}$$

$$\psi=0 \quad \text{on } \partial S_{+}^{N-1}.$$
(3.2)

Since the function  $\psi_*$  is unique it depends only on the azimuthal variable  $\theta_{N-1} = \cos^{-1}(\frac{x_N}{|x|})$  (see Appendix II). Our first result is the following

**Theorem 3.1.** If  $q \ge q_*$ , or equivalently  $\beta_q \le \beta_*$ , there exists no positive solution to problem (3.1).

*Proof.* Suppose such a solution  $\omega$  exists and put  $\theta = \beta_q/\beta_*$ , then  $0 < \theta \le 1$ . Set  $\eta = \psi^{\theta}$ , where  $\psi$  is a positive solution of (3.2), and define the operator  $\mathcal{T}$  by

$$\mathcal{T}(\eta) = -div' \left( \left( \beta_q^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \nabla' \eta \right) - \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{p-2}{2}} \eta + \left( \beta_q^2 \eta^2 + |\nabla' \eta|^2 \right)^{\frac{q}{2}}.$$
(3.3)

Since  $\nabla \eta = \theta \psi^{\theta-1} \nabla \psi$ ,

$$\left(\beta_q^2 \eta^2 + |\nabla' \eta|^2\right)^{\frac{p-2}{2}} = \theta^{p-2} \psi^{(\theta-1)(p-2)} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p-2}{2}},$$

$$\left(\beta_q^2 \eta^2 + |\nabla' \eta|^2\right)^{\frac{p-2}{2}} \nabla' \eta = \theta^{p-1} \psi^{(\theta-1)(p-1)} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p-2}{2}} \nabla' \psi,$$

therefore

$$\mathcal{T}(\eta) = -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \nabla' \psi \right)$$

$$- \theta^{p-1} (\theta - 1) (p - 1) \psi^{(\theta-1)(p-1)-1} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} |\nabla' \psi|^2$$

$$- \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{q}{2}}.$$

But  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_q} \theta^{p-1} \leq \beta_* \Lambda_{\beta_*} \theta^{p-1}$  since  $\beta_q \leq \beta_*$ . Using (3.2), we see that  $\mathcal{T}(\eta) \geq 0$ . Because Hopf Lemma is valid, there holds  $\partial_{\mathbf{n}} \psi < 0$  on  $\partial S_+^{N-1}$ . Since  $\omega$  is  $C^1$  in  $\overline{S_+^{N-1}}$  and  $\psi$  is defined up to an homothety, there exists a smallest function  $\psi$  such that  $\eta \geq \omega$ , and the graphs of  $\eta$  and  $\omega$  over  $\overline{S_+^{N-1}}$  are tangent, either at some  $\alpha \in S_+^{N-1}$ , or only at a point  $\alpha \in \partial S_+^{N-1}$ . We put  $w = \eta - \omega$ . Then

$$\mathcal{T}(\eta) = \mathcal{T}(\eta) - \mathcal{T}(\omega) = \Phi(1) - \Phi(0), \tag{3.4}$$

where  $\Phi(t) = \mathcal{T}(\omega_t)$  with  $\omega_t = \omega + tw$ .

We use local coordinates  $(\sigma_1, ..., \sigma_{N-1})$  on  $S^{N-1}$  near  $\alpha$ . We denote by  $g = (g_{ij})$  the metric tensor on  $S^{N-1}$  and by  $g^{jk}$  its contravariant components. Then, for any  $\varphi \in C^1(S^{N-1})$ ,

$$|\nabla \varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial \varphi}{\partial \sigma_j} \frac{\partial \varphi}{\partial \sigma_k} = \langle \nabla \varphi, \nabla \varphi \rangle_g.$$

If  $X=(X^1,...X^d)\in C^1(TS^{N-1})$  is a vector field, we lower indices by setting  $X^\ell=\sum_i g^{\ell i}X_i$  and define the divergence of X by

$$div'_{g}X = \frac{1}{\sqrt{|g|}} \sum_{\ell} \frac{\partial}{\partial \sigma_{\ell}} \left( \sqrt{|g|} X^{\ell} \right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial \sigma_{\ell}} \left( \sqrt{|g|} g^{\ell i} X_{i} \right).$$

We write  $\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t)$  where

$$\Phi_1(t) = -\beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \omega_t , \quad \Phi_2(t) = \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{q}{2}}$$

and

$$\Phi_3(t) = -\mathrm{div}' \left( \left( \beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2 \right)^{\frac{p-2}{2}} \nabla' \omega_t \right).$$

Then

$$\Phi_1(1) - \Phi_1(0) = -\sum_j a_j \frac{\partial w}{\partial \sigma_j} - bw \text{ and } \Phi_2(1) - \Phi_2(0) = \sum_j c_j \frac{\partial w}{\partial \sigma_j} + dw,$$

where

$$b = \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2} - 2} \left( (p - 1) \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right),$$

$$a_j = (p - 2) \beta_q \Lambda_{\beta_q} \left( \beta_q^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{p}{2} - 2} \omega_t \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k},$$

$$d = q \beta_q^2 \left( \beta^2 \omega_t^2 + |\nabla \omega_t|^2 \right)^{\frac{q}{2} - 1} \omega_t,$$

and

$$c_j = q \left(\beta_q^2 \omega_t^2 + |\nabla \omega_t|^2\right)^{\frac{q}{2} - 1} \sum_k g^{jk} \frac{\partial \omega_t}{\partial \sigma_k}.$$

**Furthermore** 

$$\Phi_3(1) - \Phi_3(0) = -(p-2)\operatorname{div}'\left(\left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left(\beta_q^2\omega_t w + \langle\nabla'\omega_t, \nabla'w\rangle_g\right)\nabla'\omega_t\right) - \operatorname{div}'\left(\left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-2}{2}}\nabla'w\right).$$

Therefore we can write  $\Phi(1) - \Phi(0)$  under the form

$$\Phi(1) - \Phi(0) = -\operatorname{div}'(A\nabla'w) + \langle B, \nabla'w \rangle_g + Cw := \mathcal{L}w$$
(3.5)

where

$$\langle AX, X \rangle_g = \left(\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2\right)^{\frac{p-4}{2}} \left(p-2\right) \langle \nabla' \omega_t, X \rangle_g^2 + |\nabla' \omega_t|^2 |X|^2\right)$$

$$\geq \left(\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2\right)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_t|^2 |X|^2.$$
(3.6)

and B and C can be computed from the previous expressions. It is important to notice that  $\beta_q^2 \omega_t^2 + |\nabla' \omega_t|^2$  is bounded between two positive constants  $m_1$  and  $m_2$  in  $\overline{S_+^{N-1}}$ . Thus the operator  $\mathcal L$  is uniformly elliptic with bounded coefficients. Since w is nonnegative and either at some point  $\alpha$ ,  $\nabla' w(\alpha) = 0$  and  $w(\alpha) > 0$ , or at some boundary point  $\alpha$  where  $w(\alpha) = 0$  and  $\partial_{\mathbf n} w(\alpha) < 0$ , it follows from the strong maximum principle or Hopf Lemma (see [7]) that w = 0, contradiction.

**Theorem 3.2.** Assume  $q < q_*$  or equivalently  $\beta_q > \beta_*$ . There exists a unique positive solution  $\omega_*$  to problem (3.1).

*Proof. Existence*. It will follow from [4]. Indeed problem (3.1) can be written under the form

$$\mathbf{A}(\omega) := -div' \, \mathbf{a}(\omega, \nabla'\omega) = \mathbf{B}(\omega, \nabla'\omega) \qquad \text{in } S_+^{N-1} \\ \omega = 0 \qquad \text{on } \partial S_+^{N-1}, \tag{3.7}$$

where

$$\mathbf{a}(r,\xi) = \left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{p-2}{2}} \xi, \mathbf{B}(r,\xi) = \beta_q \Lambda_{\beta_q} \left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{p-2}{2}} r - \left(\beta_q^2 r^2 + |\xi|^2\right)^{\frac{q}{2}}.$$
(3.8)

The operator  ${\bf A}$  is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [4, Theorem 2.1], and the term  ${\bf B}$  satisfies (1.9),(1.10) in the same article. Therefore the existence of a positive solution  $\omega \in W_0^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  is ensured whenever we can find a supersolution  $\overline{\omega} \in W^{1,p}(S_+^{N-1}) \cap L^\infty(S_+^{N-1})$  and a nontrivial subsolution  $\underline{\omega} \in W^{1,p}(S_+^{N-1})$  of (3.7) such that

$$0 \le \underline{\omega} \le \overline{\omega} \qquad \text{in } S_+^{N-1}. \tag{3.9}$$

First we note that  $\eta=\eta_0$  is a supersolution if the positive constant  $\eta_0$  is large enough. In order to find a subsolution, we set again  $\eta=\psi^\theta$  with  $\theta=\beta_q/\beta_*$  and  $\psi$  as in (3.2). Now  $\theta>1$ , thus  $\eta\in W^{1,p}_0(S^{N-1}_+)$ . As above we have

$$\mathcal{T}(\eta) = -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \nabla' \psi \right)$$

$$- \theta^{p-1} (\theta - 1) (p - 1) \psi^{(\theta-1)(p-1)-1} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} |\nabla' \psi|^2$$

$$- \beta_q \Lambda_{\beta_q} \theta^{p-2} \psi^{(\theta-1)(p-1)} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{q}{2}}.$$

Now  $\beta_q \Lambda_{\beta_q} \theta^{p-2} = \beta_* \Lambda_{\beta_q} \theta^{p-1} = \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} + \beta_* \Lambda_{\beta_*} \theta^{p-1}$  and  $\Lambda_{\beta_q} - \Lambda_{\beta_*} = (\beta_q - \beta_*)(p-1) = \beta_* (p-1)(\theta-1)$ , hence

$$\mathcal{T}(\eta) = -\theta^{p-1} \psi^{(\theta-1)(p-1)} \operatorname{div}' \left( \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \nabla' \psi \right)$$

$$- \theta^{p-1} (\theta - 1) (p - 1) \psi^{(\theta-1)(p-1)-1} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} |\nabla' \psi|^2$$

$$- \beta_* (\Lambda_{\beta_q} - \Lambda_{\beta_*}) \theta^{p-1} \psi^{(\theta-1)(p-1)} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \psi$$

$$- \beta_* \Lambda_{\beta_*} \theta^{p-1} \psi^{(\theta-1)(p-1)} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{p-2}{2}} \psi + \theta^q \psi^{(\theta-1)q} \left( \beta_*^2 \psi^2 + |\nabla' \psi|^2 \right)^{\frac{q}{2}}.$$

Using the equation satisfied by  $\psi$  yields to the relation

$$\mathcal{T}(\eta) = -\theta^{p-1}(\theta - 1)(p - 1)\psi^{(\theta - 1)(p - 1) - 1} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p-2}{2}} |\nabla' \psi|^2$$

$$- \beta_*^2 (p - 1)(\theta - 1)\theta^{p-1}\psi^{(\theta - 1)(p - 1) - 1} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p-2}{2}} \psi^2$$

$$+ \theta^q \psi^{(\theta - 1)q} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{q}{2}}$$

$$= -\theta^{p-1}(\theta - 1)(p - 1)\psi^{(\theta - 1)(p - 1) - 1} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p}{2}}$$

$$+ \theta^q \psi^{(\theta - 1)q} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{q}{2}}.$$

If we replace  $\eta := \eta_1 = \psi^{\theta}$  by  $\eta := \eta_m = (m\psi)^{\theta}$  in the above computation, the inequality  $\mathcal{T}\eta_m) \leq 0$  will be true provided

$$m^{\theta(q+1-p)}\psi^{(\theta-1)(q+1-p)+1} \le \theta^{p-1-q}(\theta-1)(p-1)\left(\beta_*^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-q}{2}},$$

which is satisfied if we choose m small enough so that  $(m\psi)^{\theta} \leq \eta_0$  and satisfying

$$m^{\theta(q+1-p)} \leq \beta_*^{(\theta-1)(q+1-p)+1} \theta^{p-1-q} (\theta-1) (p-1) \frac{\min_{x \in S_+^{N-1}} \left(\beta_*^2 \psi^2 + |\nabla' \psi|^2\right)^{\frac{p-q}{2}}}{\max_{x \in S_+^{N-1}} \psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore  $0 < \eta_m \le \eta_0$  and standard regularity implies that the solution  $\omega$  is  $C^1$  in  $\overline{S}_+^{N-1}$ . Actually  $\omega$  is  $C^{\infty}$  since the operator is not degenerate.

Uniqueness. We use the tangency method developed in the proof of Theorem 3.1. Assume  $\omega_1$  and  $\omega_2$  are two positive solutions of (3.2), then they are positive in  $S_+^{N-1}$  and  $\partial_{\mathbf{n}}\omega_i < 0$  on  $\partial S_+^{N-1}$ . Either the  $\omega_i$  are ordered and  $\omega_1 \leq \omega_2$ , or their graphs intersect. In any case we can define

$$\tau = \inf\{s > 1 : s\omega_1 \ge \omega_2\}.$$

We set  $\omega^* = \tau \omega_1$ . Then either the graphs of  $\omega_2$  and  $\omega^*$  are tangent at some interior point  $\alpha$ , or they are not tangent in  $S_+^{N-1}$ ,  $\partial_{\mathbf{n}}\omega^* \leq \partial_{\mathbf{n}}\omega_2 < 0$  on  $\partial S_+^{N-1}$  and there exists  $\alpha \in \partial S_+^{N-1}$  such that  $\partial_{\mathbf{n}}\omega^*(\alpha) = \partial_{\mathbf{n}}\omega_2(\alpha) < 0$ . Furthermore  $\mathcal{T}(\omega^*) \geq 0$ . If we set  $w = \omega^* - \omega_2$ , then, as in Theorem 3.1,

$$-\mathrm{div}'(\tilde{A}\nabla'w) + \langle \tilde{B}, \nabla'w \rangle_q + \tilde{C}w = \tilde{\mathcal{L}}w \ge 0$$

where

$$\langle \tilde{A}X, X \rangle_{g} = \left(\beta_{q}^{2} \omega_{t}^{2} + |\nabla' \omega_{t}|^{2}\right)^{\frac{p-4}{2}} \left(p-2\right) \langle \nabla' \omega_{t}, X \rangle_{g}^{2} + |\nabla' \omega_{t}|^{2} |X|^{2})$$

$$\geq \left(\beta_{q}^{2} \omega_{t}^{2} + |\nabla' \omega_{t}|^{2}\right)^{\frac{p-4}{2}} \min\{1, p-1\} |\nabla' \omega_{t}|^{2} |X|^{2},$$
(3.10)

in which  $\omega_t = \omega_2 + t(\omega^* - \omega_2)$  and  $t \in (0,1)$  is obtained by applying the mean value theorem and  $\tilde{B}$  and  $\tilde{C}$  are defined accordingly. Since  $\tilde{\mathcal{L}}$  is uniformly elliptic and has bounded coefficients, it follows from the strong maximum principle that w = 0. Thus  $\omega^* = \tau \omega_1 = \omega_2$  and  $\tau = 1$  from the equation. This ends the proof.

#### 3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

**Theorem 3.3.** Assume  $q^* \le q , <math>\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial \Omega$ , such that  $\Omega \cap B_{\delta} = B_{\delta}^+$  for some  $\delta > 0$ . If  $u \in C^1(\overline{\Omega} \setminus \{0\})$  is a nonnegative solution of (1.1) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$ , then it is identically 0.

*Proof.* Step 1: Assume  $\Omega \subset \mathbb{R}^N_+$ . For  $\epsilon > 0$ , we set  $\Omega'_{\epsilon} = \Omega \cap \overline{B^c_{\epsilon}}$  and  $H_{\epsilon} = \mathbb{R}^N_+ \cap \overline{B^c_{\epsilon}}$ . For  $k, n \in \mathbb{N}_*$ ,  $n \ge \operatorname{diam}(\Omega)$ , we denote by  $v_{k,n,\epsilon}$   $(n \in \mathbb{N}_*)$  the solution of the problem

$$-\Delta_p v + |\nabla v|^q = 0 \qquad \text{in } H_{\epsilon} \cap B_n v = k \chi_{\mathbb{R}^N_+ \cap \partial B_{\epsilon}} \qquad \text{on } \partial (H_{\epsilon} \cap B_n).$$
 (3.11)

If  $k>c_2\epsilon^{\frac{q-p}{q+1-p}}$  for a suitable  $c_2=c_2(p,q)>0$  (see Lemma 2.6), then  $v_{k,n,\epsilon}\geq u$  in  $\Omega'_\epsilon$ . Moreover there holds  $v_{k,n,\epsilon}\leq v_{k',n',\epsilon}$  for  $n\leq n'$  and  $k\leq k'$ . Furthermore the function

$$U_{\epsilon,n}(x) = c_2 \left( (|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n-\epsilon)^{\frac{q-p}{q+1-p}} \right)$$

is a supersolution in  $B_n \setminus B_{\epsilon}$ , and there holds  $v_{k,n,\epsilon} \leq U_{\epsilon,n}$ . By monotonicity and standard a priori estimate, we obtain that  $v_{k,n,\epsilon} \to v_{\epsilon}$  when  $n,k \to \infty$  and that the function  $v=v_{\epsilon}$  is solution of

$$-\Delta_{p}v + |\nabla v|^{q} = 0 \qquad \text{in } H_{\epsilon}$$

$$\lim_{|x| \to \epsilon} v(x) = \infty \qquad \qquad v = 0 \qquad \text{on } \partial \mathbb{R}^{N}_{+} \cap \overline{B^{c}_{\epsilon}}.$$
(3.12)

Furthermore

$$u(x) \le v_{\epsilon}(x) \le c_2(|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega_{\epsilon}'. \tag{3.13}$$

The function  $v_{\epsilon}$  may not be unique, however it is the minimal solution of the above problem since the  $v_{k,n,\epsilon}$  is unique, and monotonicity in n and k holds. Actually,  $v_{\epsilon} \leq v_{\epsilon'}$  if  $0 \leq \epsilon \leq \epsilon'$ . For  $\ell > 0$ , we recall that the transformation  $v \mapsto T_{\ell}[v]$  defined by (2.41) leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have  $T_{\ell}[v_{k,n,\epsilon}] = v_{\ell} \frac{p-q}{q+1-p} \frac{1}{k} \frac{1}{\ell} e^{-1} \frac{1}{n} \frac{1}{\ell} e^{-1}$ , which implies

$$T_{\ell}[v_{\epsilon}] = v_{\ell^{-1}\epsilon}.\tag{3.14}$$

Letting  $\epsilon \to 0$ , we derive from the monotonicity with respect to  $\epsilon$  and standard  $C^{1,\alpha}$  estimates, that the following identity holds:

$$T_{\ell}[v_0] = v_0 \qquad \forall \ell > 0. \tag{3.15}$$

The function  $v_0$  is a positive and separable solution of (1.1) in  $\mathbb{R}^N_+$  which vanishes on  $\partial\Omega\setminus\{0\}$ . It follows from Theorem 3.1 that  $v_0=0$ , and so is u.

Step 2: The general case. We assume that  $\Omega \cap B_{\delta} \subset \mathbb{R}^{N}_{+}$  and we denote by M the maximum of u on  $\partial B_{\delta} \cap \Omega$ . Then the function  $(u-M)_{+}$  is a subsolution of (1.1) in  $\Omega \cap B_{\delta}$  which vanishes on  $\partial \Omega \cap B_{\delta} \setminus \{0\}$ . By Step 1, it is dominated by  $v_{0}$ , which ends the proof.

*Remark.* The previous result is valid if u is a subsolution with the same regularity. If u is no longer assumed to be nonnegative, only  $u^+$  vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that  $\Omega$  is locally contained into a half space to the boundary of which 0 belongs.

*Remark.* If no geometric assumption is made on  $\partial\Omega$ , we can prove that  $u(x) = o(|x|^{-\beta_q})$  near 0. The next result shows that the removability holds if  $q > q_*$ .

**Theorem 3.4.** Assume  $q^* < q < p \le N$  and  $\Omega$  is a  $C^2$  bounded domain with  $0 \in \partial \Omega$ . If u is a nonnegative solution of (1.1) in  $\Omega$  which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$  and vanishes on  $\partial \Omega \setminus \{0\}$ , it is identically 0.

*Proof.* As it is proved in [12], for any smooth subdomain  $S \subset S^{N-1}$ , there exists a unique  $\beta_{*s} > 0$  and  $\psi_s^* > 0$ , unique up to an homothety, such that  $x \mapsto |x|^{-\beta_{*s}} \psi_s^*(|x|^{-1} x)$  is p harmonic in the cone  $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1} x \in S\}$  and  $\psi_s^*$  satisfies

$$-div'\left(\left(\beta_{*s}^{2}\psi_{s}^{*\,2} + |\nabla'\psi_{s}^{*}|^{2}\right)^{\frac{p-2}{2}}\nabla'\psi_{s}^{*}\right) - \beta_{*s}\Lambda_{\beta_{*s}}\left(\beta_{*s}^{2}\psi^{*\,2} + |\nabla'\psi_{s}^{*}|^{2}\right)^{\frac{p-2}{2}}\psi_{s}^{*} = 0 \quad \text{in } S$$

$$\psi_{s}^{*} = 0 \quad \text{on } \partial S,$$

$$(3.16)$$

Furthermore  $S\subset \tilde{S}\subset S^{N-1}$  implies  $\beta_{*\tilde{s}}\leq \beta_{*s}$ . Using the system of spherical coordinates defined in (6.5) in Appendix II, for  $\epsilon>0$  we denote by  $S:=S_{\epsilon}$  the spherical shell with vertex the north pole N and latitude angle  $\theta_{N-1}\in [0,\frac{\pi}{2}+\epsilon]$ . Because of uniqueness of  $\beta_{*s}$ ,  $\beta_{*s_{\epsilon}}\uparrow\beta_{*}$  as  $\epsilon\to 0$ . Therefore, if  $q>q_*$ , or equivalently  $\beta_q<\beta_*$ , there exists  $\delta,\epsilon>0$  such that  $\Omega\cap B_\delta\subset C_{S_\epsilon}\cap B_\delta$  and  $\beta_q<\beta_{*s_{\epsilon}}$ . Since Theorem 3.1 is valid if  $S_+^{N-1}$  is replaced by  $S_\epsilon$  and  $\beta_q<\beta_{*s_{\epsilon}}$  it follows that u=0 as in the proof of Theorem 3.3, Steps 1 and 2.

The next result, valid in the case p=N, is based upon the conformal invariance of the N-Laplacian. In this case the exponent  $\beta_*$  corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in  $S_+^{N-1}$  is  $x_N/|x|^2$ .

**Theorem 3.5.** Assume  $N - \frac{1}{2} \le q < N$ ,  $\Omega$  is a bounded domain and  $0 \in \partial \Omega$  is such that there exists a ball  $B \subset \Omega^c$  to the boundary of which 0 belongs. If u is a nonnegative solution of

$$-\Delta_N u + |\nabla u|^q = 0 \quad \text{in } \Omega, \tag{3.17}$$

which belongs to  $C(\overline{\Omega} \setminus \{0\}) \cap W_0^{1,N}(\Omega \setminus \overline{B}_{\epsilon}(0))$  for any  $\epsilon > 0$ , it is identically 0.

Proof. We assume that the inward normal unit vector to  $\partial\Omega$  at 0 is  $\mathbf{e}_N=(0,0,...,1)$  and that the ball  $B=B_{\frac{1}{2}}(a)$  of center  $a=-\frac{1}{2}\mathbf{e}_N$  and radius  $\frac{1}{2}$  touches  $\partial\Omega$  at 0 and is exterior to  $\Omega$  (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if  $\partial\Omega$  is  $C^2$ . We denote by  $\mathcal{I}_\omega$  the inversion of center  $\omega=-\mathbf{e}_N$  and power 1, i.e.  $\mathcal{I}_\omega(x)=\omega+\frac{x-\omega}{|x-\omega|^2}$ . Under this transformation, the complement of the ball  $B_{\frac{1}{2}}(a)$ , which contains  $\Omega$ , is transformed into the half space  $\mathbb{R}^N_-$  which contains the image  $\tilde{\Omega}$  of  $\Omega$ . Since u satisfies (3.17),  $\tilde{u}=u\circ\mathcal{I}_\omega$  satisfies

$$-\Delta_N \tilde{u} + |x - \omega|^{2(q-N)} |\nabla \tilde{u}|^q = 0 \quad \text{in } \tilde{\Omega}.$$
(3.18)

Furthermore since  $0 = \mathcal{I}_{\omega}(0)$  and  $\mathcal{I}_{\omega}$  is a diffeomorphism,  $\tilde{u} \in C(\overline{\tilde{\Omega}} \setminus \{0\}) \cap C^{1}(\tilde{\Omega})$  and it vanishes on  $\partial \tilde{\Omega} \setminus \{0\}$ . Since  $|x - \omega| \leq 1$  and q < N,  $\tilde{u}$  is a subsolution for (3.17) in  $\tilde{G}$ . By Theorem 3.4,  $\tilde{u} = 0$ .

#### 3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that  $\psi_*$  is unique positive solution of (1.11) such that  $\sup \psi_* = 1$ . Our first result is valid for any  $1 but it needs a geometric constraint on <math>\Omega$ .

**Theorem 3.6.** Let  $p-1 < q < q_* < p \le N$  and  $\Omega \subset \mathbb{R}^N_+$  be a bounded  $C^2$  domain such that  $0 \in \partial \Omega$ . Assume that there exists  $\delta > 0$  such that  $\Omega_\delta := \Omega \cap B_\delta = B_\delta^+$ . Then for any k > 0 there exists a unique positive solution  $u := u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies

$$\lim_{x \to 0} \frac{u_k(x)}{\Psi_*(x)} = k \tag{3.19}$$

in the  $C^1$ -topology of  $S^{N-1}_+$ , where

$$\Psi_*(x) = |x|^{-\beta_*} \, \psi_*(|x|^{-1}x).$$

The proof of this theorem is long and difficult and requires a certain number of intermediate results.

**Lemma 3.7.** Let the assumptions on p, q and  $\Omega$  of Theorem 3.6 be satisfied. There exists a unique positive p-harmonic function  $\Phi_*$  in  $\Omega$ , which is continuous in  $\overline{\Omega} \setminus \{0\}$ , vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies

$$\lim_{x \to 0} \frac{\Phi_*(x)}{\Psi_*(x)} = 1. \tag{3.20}$$

*Proof.* For  $0 < \epsilon < \delta$  let  $v_{\epsilon}$  be the unique nonnegative p-harmonic function in  $\Omega \backslash \overline{B_{\epsilon}^{+}}$  which is continuous in  $\overline{\Omega} \backslash B_{\epsilon}^{+}$ , vanishes on  $\partial \Omega \backslash B_{\epsilon}$  and achieves the value  $\Psi_{*}$  on  $\partial B_{\epsilon} \cap \Omega$ . Since  $\Omega \subset \mathbb{R}_{+}^{N}$ ,  $v_{\epsilon} \leq \Psi_{*}$  in  $\Omega \backslash B_{\epsilon}^{+}$ . Hence inequalities  $0 < \epsilon < \epsilon' \leq \delta$  imply  $v_{\epsilon} \leq v_{\epsilon'}$  in  $\Omega \backslash \overline{B_{\epsilon'}^{+}}$ . Because  $\Psi_{*} \leq \delta^{-\beta_{*}}$ , there holds

$$v_{\epsilon} + \delta^{-\beta_*} \ge \Psi_*, \tag{3.21}$$

in  $\Omega \setminus B_{\delta}^+$ . Since  $v_{\epsilon}$  and  $\Psi_*$  coincide on  $\partial B_{\epsilon}^+$  and vanish on  $\partial \mathbb{R}^N_+ \cap (B_{\delta}^+ \setminus B_{\epsilon}^+)$ , (3.21) holds also in  $B_{\delta}^+ \setminus B_{\epsilon}^+$ . Because  $v_{\epsilon} \geq 0$  there holds

$$(\Psi_* - \delta^{-\beta_*})_+ \le v_{\epsilon} \le \Psi_* \quad \text{in } \Omega \setminus B_{\epsilon}^+. \tag{3.22}$$

By a standard regularity result  $v_{\epsilon}$  converges to a function  $\Phi_*$  continuous in  $\overline{\Omega} \setminus \{0\}$ , p-harmonic in  $\Omega$  such that

$$(\Psi_* - \delta^{-\beta_*})_+ \le \Phi_* \le \Psi_*$$

in  $\Omega$ . Therefore (3.20) holds provided  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ . Let us define a function  $\tilde{\phi}_*$  by  $\tilde{\phi}_*(x) = |x|^{\beta_*} \Phi_*(x)$ , then  $\tilde{\phi}_*(r,\sigma) \leq \psi_*(\sigma)$  where r = |x| and  $\sigma = \frac{x}{|x|} \in S_+^{N-1}$ . By standard  $C^{1,\alpha}$  estimates,  $\tilde{\phi}_*(r,\cdot)$  is relatively compact in the  $C(\overline{S_+^{N-1}})$ -topology. Therefore the convergence of  $\frac{\Phi_*(x)}{\Psi_*(x)}$  to 1 when x to 0 holds not only when  $\frac{x}{|x|}$  remains in a compact subset of  $S_+^{N-1}$ , but uniformly on  $S_+^{N-1}$ , which implies (3.20). Uniqueness follows classically by (3.20) and the maximum principle.  $\square$ 

**Lemma 3.8.** Let the assumptions on p, q and  $\Omega$  of Theorem 3.6 be satisfied. If for some k > 0 there exists a solution  $u_k$  of (1.1) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies (3.19), then for any k > 0 there exists such a solution.

*Proof.* We notice that for any c < 1 (resp c > 1),  $cu_k$  is a subsolution (resp. supersolution) of (1.1) in  $\Omega$ . Let  $\Phi_*$  be as in Lemma 3.7. If c < 1, the function  $ck\Phi_*$  is a supersolution of (1.1) which vanishes on  $\partial\Omega\setminus\{0\}$ . Furthermore

$$\lim_{x \to 0} \frac{cu_k(x)}{\Psi_*(x)} = ck = \lim_{x \to 0} \frac{ck\Phi_*(x)}{\Psi_*(x)}.$$

Then there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $cu_k \leq u_{ck} \leq ck\Phi_*$ . If c>1, we set  $u^*:=T_{c^\theta}[u_k]$ , which means  $u^*(x)=c^{\beta_q\theta}u_k(c^\theta x)$  with  $\theta=(\beta_q-\beta_*)^{-1}$ . Then  $u^*$  is a solution of (1.1) in  $\Omega^{c^\theta}=\frac{1}{c^\theta}\Omega$ . In particular,  $u^*$  satisfies the equation in  $B^+_{\frac{\delta}{c^\theta}}(0)$ . Since  $c^\theta>1$ ,  $B^+_{\frac{\delta}{c^\theta}}(0)\subset B^+_{\delta}(0)$ . Put  $m=\max\{u^*:x\in\partial B^+_{\frac{\delta}{c^\theta}}(0)\}$ . The function  $(u^*-m)_+$ , extended by 0 outside  $B^+_{\frac{\delta}{c^\theta}}(0)$ , is a subsolution of (1.1) in  $\Omega$ . Furthermore it satisfies

$$\lim_{x \to 0} \frac{(u^* - m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of  $S^{N-1}_+$ . Therefore there exists a solution  $u_{ck}$  of (1.1) in  $\Omega$  which satisfies  $(u^*-m)_+ \leq u_{ck} \leq ck\Phi_*$ , and in particular it vanishes on  $\partial\Omega\setminus\{0\}$  and belongs to  $C^1(\overline{\Omega}\setminus\{0\})$ . By [13],  $u_{ck}$  is positive in  $\Omega$ . Thus  $u_{ck}$  belongs to  $C^{1,\alpha}(\overline{B^+_\delta}(0)\setminus\{0\})$  and satisfies

$$|x|^{\beta_*} |u_{ck}(x)| + |x|^{1+\beta_*} |\nabla u_{ck}(x)| + |x|^{1+\beta_* + \alpha} \sup_{\substack{|y| \le |x| \\ x \ne y}} \frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x - y|^{\alpha}} \le M$$

by (2.11). Therefore the set of functions  $\{r^{\beta_*+1}\nabla u_{ck}(r,.)\}_{r>0}$  is uniformly relatively compact in the topology of uniform convergence on  $\overline{S}_+^{N-1}$ . Since it converges to  $ck\nabla'\psi_*$  uniformly on compact subsets of  $S_+^{N-1}$  as  $r\to 0$ , this convergence holds in  $C(\overline{S_+^{N-1}})$ . This implies

$$\lim_{x \to 0} \frac{u_{ck}(x)}{\Psi_*(x)} = ck. \tag{3.23}$$

The next Lemma is the keystone of our construction. Its proof is very delicate and needs several intermediate steps.

**Lemma 3.9.** Under the assumptions of Theorem 3.6 there exists a real number  $R_0$  such that  $0 < R_0 \le \delta$  and a positive subsolution  $\tilde{u}$  of (1.1) in  $B_{R_0}^+$  which is Lipschitz continuous in  $\overline{B_{R_0}^+} \setminus \{0\}$ , vanishes on  $\overline{B_{R_0}^+} \cap \partial \mathbb{R}^N_+ \setminus \{0\}$ , is smaller than  $\Psi_*$  and satisfies

$$\lim_{x \to 0} \frac{\tilde{u}(x)}{\Psi_*(x)} = 1. \tag{3.24}$$

*Proof.* The construction of the function  $\tilde{u}$ . We look for a subsolution under the form  $\tilde{u} = \Psi_* - w$  for a suitable nonnegative function w.

Step 1: reduction of the problem. We use spherical coordinates for a  $C^1$  function  $u: x \mapsto u(x) = u(r,\sigma), r = |x|, \sigma = \frac{x}{|x|}$ . Then  $\nabla u = u_r \mathbf{e} + r^{-1} \nabla' u$  where  $\mathbf{e} = |x|^{-1} x, |\nabla u|^2 = u_r^2 + r^{-2} |\nabla' u|^2$  and  $|\nabla u|^q = \left(u_r^2 + r^{-2} |\nabla' u|^2\right)^{\frac{q}{2}}$ . The expression of the p-Laplacian in spherical coordinates is

$$\begin{split} -\Delta_p u &= -\left(\left(u_r^2 + r^{-2} \left| \nabla' u \right|^2\right)^{\frac{p-2}{2}} u_r\right)_r - \frac{N-1}{r} \left(u_r^2 + r^{-2} \left| \nabla' u \right|^2\right)^{\frac{p-2}{2}} u_r \\ &- \frac{1}{r^2} div' \left(\left(u_r^2 + r^{-2} \left| \nabla' u \right|^2\right)^{\frac{p-2}{2}} \nabla' u\right). \end{split}$$

Put  $v(t,\sigma) = r^{\beta_*} u(r,\sigma)$  with  $t = \ln r \in (-\infty, \ln \delta]$ , then v satisfies

Q[v] :=

$$-\left(\left((v_{t} - \beta_{*}v)^{2} + |\nabla'v|^{2}\right)^{\frac{p-2}{2}}(v_{t} - \beta_{*}v)\right)_{t} - div'\left(\left((v_{t} - \beta_{*}v)^{2} + |\nabla'v|^{2}\right)^{\frac{p-2}{2}}\nabla'v\right)$$

$$+ \Lambda_{\beta_{*}}\left((v_{t} - \beta_{*}v)^{2} + |\nabla'v|^{2}\right)^{\frac{p-2}{2}}(v_{t} - \beta_{*}v) + e^{\nu t}\left((v_{t} - \beta_{*}v)^{2} + |\nabla'v|^{2}\right)^{\frac{q}{2}} = 0$$

$$(3.25)$$

in  $(-\infty, \ln \delta) \times S_+^{N-1}$  where  $\nu = 1 - (q+1-p)(\beta_*+1) = 1 - \frac{\beta_*+1}{\beta_q+1} > 0$  and  $\Lambda_{\beta_*} = \beta_*(p-1) + p - N$ . Notice that  $\psi_*$  satisfies

$$- \operatorname{div'} \left( \left( \beta_*^2 \psi_*^2 + \left| \nabla' \psi_* \right|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* \right) - \beta_* \Lambda_{\beta_*} \left( \beta_*^2 \psi_*^2 + \left| \nabla' \psi_* \right|^2 \right)^{\frac{p-2}{2}} \psi_* = 0, \tag{3.26}$$

hence it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t,\sigma) = \psi_* - a(t)g(\psi_*)$$

where g is a continuous increasing function defined on  $\mathbb{R}_+$ , vanishing at 0 and smooth on  $\mathbb{R}_+^*$  and  $a(t)=e^{\gamma t}$  with  $\gamma>0$  to be chosen. Thus  $a'=\gamma a$ ,  $a''=\gamma^2 a$ ,  $V_t=-\gamma a g(\psi_*)$ ,  $V_t-\beta_* V=-\gamma a g(\psi_*)$ 

$$\begin{split} -\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*), & \nabla'V = (1 - ag'(\psi_*))\nabla'\psi_* \text{ and} \\ & (V_t - \beta_*V)^2 + |\nabla'V|^2 = (-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - ag'(\psi_*))^2 \left|\nabla'\psi_*\right|^2 \\ & = \left(\beta_*^2\psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*\right) + (1 - 2ag'(\psi_*))\left|\nabla'\psi_*\right|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\ & = \beta_*^2\psi_*^2 + \left|\nabla'\psi_*\right|^2 + 2a\left(\beta_*(\gamma - \beta_*)\psi_*g(\psi_*) - g'(\psi_*)\left|\nabla\psi_*\right|^2\right) + O(a^2 \|g(\psi_*)\|_{C^1}). \end{split}$$

Therefore

$$\left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \\
= \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ 1 + (p-2)a \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \\
+ O(a^2 \|g(\psi)\|_{C^1}),$$

and

$$e^{\nu t} \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{q}{2}}$$

$$= e^{\nu t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] + O(e^{\nu t} a^2 \|g(\psi_*)\|_{C^1}),$$

thus

$$\begin{split} \left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \left( V_t - \beta_* V \right) \\ &= -\beta_* \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma) \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} g(\psi_*) \\ &- a\beta_* (p-2) \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2)^{\frac{4-p}{2}}} \psi_* + O(a^2 \|g(\psi_*)\|_{C^1}). \end{split}$$

Finally,

$$-\left(\left((V_{t} - \beta_{*}V)^{2} + |\nabla'V|^{2}\right)^{\frac{p-2}{2}}(V_{t} - \beta_{*}V)\right)_{t}$$

$$= a\left[\left(\gamma^{2} - \beta_{*}\gamma\right)\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}g(\psi_{*})$$

$$+\beta_{*}(p-2)\frac{\beta_{*}(\gamma^{2} - \beta_{*}\gamma)\psi_{*}g(\psi_{*}) - \gamma g'(\psi_{*})|\nabla\psi_{*}|^{2}}{\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{4-p}{2}}}\psi_{*}\right] + O(a^{2}\|g(\psi_{*})\|_{C^{2}}).$$
(3.27)

Since

$$\left( (V_t - \beta_* V)^2 + |\nabla' V|^2 \right)^{\frac{p-2}{2}} \nabla' V = 
\left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} (1 - ag'(\psi_*)) \left[ 1 + a(p-2) \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \nabla' \psi_* 
+ O(a^2 ||g(\psi_*)||_{C^1}) 
= \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \nabla' \psi_* 
+ a \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_* (\gamma - \beta_*) \psi_* g(\psi_*) - g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} - g'(\psi_*) \right] \nabla' \psi_* 
+ O(a^2 ||g(\psi_*)||_{C^1}),$$

we get similarly

$$-div'\left(\left((V_{t} - \beta_{*}V)^{2} + |\nabla'V|^{2}\right)^{\frac{p-2}{2}} \nabla'V\right) = -div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}} \nabla'\psi_{*}\right)$$

$$-a div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}} \left[(p-2)\frac{\beta_{*}(\gamma - \beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*})|\nabla\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} - g'(\psi_{*})\right] \nabla'\psi_{*}\right)$$

$$+ O(a^{2} \|g(\psi_{*})\|_{C^{2}}).$$
(3.28)

Noting that

$$-div'\left(\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)\psi_* = \beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\psi_*,\tag{3.29}$$

we obtain

$$e^{-\gamma t}\mathcal{Q}[V]$$

$$= \left[ (\gamma^{2} - \beta_{*}\gamma) \left( \beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2} \right)^{\frac{p-2}{2}} g(\psi_{*}) + \beta_{*}(p-2) \frac{\beta_{*}(\gamma^{2} - \beta_{*}\gamma)\psi_{*}g(\psi_{*}) - \gamma g'(\psi_{*}) |\nabla\psi_{*}|^{2}}{(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2})^{\frac{4-p}{2}}} \psi_{*} \right]$$

$$- div' \left( \left( \beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2} \right)^{\frac{p-2}{2}} \left[ (p-2) \frac{\beta_{*}(\gamma - \beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*}) |\nabla\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} - g'(\psi_{*}) \right] \nabla'\psi_{*} \right)$$

$$- \Lambda_{\beta_{*}} \left( (\gamma - \beta_{*}) \left( \beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2} \right)^{\frac{p-2}{2}} g(\psi_{*}) + \beta_{*}(p-2) \frac{\beta_{*}(\gamma - \beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*}) |\nabla\psi_{*}|^{2}}{(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2})^{\frac{4-p}{2}}} \psi_{*} \right)$$

$$+ e^{(\nu - \gamma)t} \left( \beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2} \right)^{\frac{q}{2}} \left[ 1 + qa \frac{\beta_{*}(\gamma - \beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*}) |\nabla\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} \right] + O(a \|g(\psi_{*})\|_{C^{2}}).$$

$$(3.30)$$

In this expression we have in particular

$$-div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}\left[\left(p-2\right)\frac{\beta_{*}(\gamma-\beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*})|\nabla\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} - g'(\psi_{*})\right]\nabla'\psi_{*}\right)$$

$$= (p-1)div'\left[g'(\psi_{*})\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}\nabla\psi_{*}\right]$$

$$-\beta_{*}div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-4}{2}}\left[(p-2)\beta_{*}\psi_{*}g'(\psi_{*}) + (p-2)(\gamma-\beta_{*})g(\psi_{*})\right]\psi_{*}\right)$$

$$= (p-1)g''(\psi_{*})\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}|\nabla\psi_{*}|^{2}$$

$$+ (p-1)g'(\psi_{*})div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}\nabla\psi_{*}\right)$$

$$- (p-2)\beta_{*}div'\left[\frac{\left((\gamma-\beta_{*})g(\psi_{*})\psi_{*} + \beta_{*}g'(\psi_{*})\psi_{*}^{2}\right)}{\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{4-p}{2}}}\nabla'\psi_{*}\right].$$
(3.31)

Using the equation (3.26) satisfied by  $\psi_*$ , it infers that

$$-div'\left(\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}\left[\left(p-2\right)\frac{\beta_{*}(\gamma-\beta_{*})\psi_{*}g(\psi_{*}) - g'(\psi_{*})|\nabla\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} - g'(\psi_{*})\right]\nabla'\psi_{*}\right)$$

$$= (p-1)\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{p-2}{2}}\left(g''(\psi_{*})|\nabla'\psi_{*}|^{2} - \beta_{*}\Lambda_{\beta_{*}}g'(\psi_{*})\psi_{*}\right)$$

$$- (p-2)\beta_{*}div'\left[\frac{\left((\gamma-\beta_{*})g(\psi_{*})\psi_{*} + \beta_{*}g'(\psi_{*})\psi_{*}^{2}\right)}{\left(\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}\right)^{\frac{4-p}{2}}}\nabla'\psi_{*}\right].$$

$$(3.32)$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$e^{-\gamma t} \mathcal{Q}[V] = \left(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] + e^{(\nu - \gamma)t} R[V] + O(a \|g(\psi_*)\|_{C^2}), \tag{3.33}$$

where

$$R[V] = e^{\nu t} \left( \beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2 \right)^{\frac{q}{2}} \left[ 1 + q \frac{\beta_* (a' - \beta_* a) \psi_* g(\psi_*) - a g'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right], \tag{3.34}$$

and

$$Q_{1}[V] = (\gamma - \Lambda_{\beta_{*}})(\gamma - \beta_{*}) \left[ 1 + (p - 2) \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + |\nabla' \psi_{*}|^{2}} \right] - (p - 1) \beta_{*} \Lambda_{\beta_{*}} \frac{\psi_{*} g'(\psi_{*})}{g(\psi_{*})}$$

$$+ \left[ (p - 4) \beta_{*} \Lambda_{\beta_{*}} \psi_{*} - 2\Delta' \psi_{*} \right] \left( \gamma - \beta_{*} \left( 1 - \frac{\psi_{*} g'(\psi_{*})}{g(\psi_{*})} \right) \right) \frac{\beta_{*} \psi_{*}}{\beta_{*}^{2} \psi_{*}^{2} + |\nabla' \psi_{*}|^{2}}$$

$$- (p - 2) \left[ \frac{\psi_{*} g'(\psi_{*})}{g(\psi_{*})} \left( (\beta_{*} + 1) \gamma - \beta_{*} \Lambda_{\beta_{*}} + \beta_{*} \right) + \gamma - \beta_{*} + \beta_{*} \frac{\psi_{*}^{2} g''(\psi_{*})}{g(\psi_{*})} \right] \frac{|\nabla' \psi_{*}|^{2}}{\beta_{*}^{2} \psi_{*}^{2} + |\nabla' \psi_{*}|^{2}}$$

$$+ (p - 1) \frac{g''(\psi_{*})}{g(\psi_{*})} |\nabla' \psi_{*}|^{2}.$$

$$(3.35)$$

In this expression the difficult term to deal with is  $[(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*]$  since it has not a prescribed sign. However  $\Delta'\psi_* = O(\psi_*)$  by (6.19) in Appendix II.

Step 2: The perturbation method and the computation with  $g(\psi_*) = \psi_*$ . With such a choice of function g

$$Q_{1}[V] = (\gamma - \Lambda_{\beta_{*}})(\gamma - \beta_{*}) \left[ 1 + (p-2) \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + |\nabla' \psi_{*}|^{2}} \right] - (p-1)\beta_{*} \Lambda_{\beta_{*}} - (p-2) \left[ (\gamma - \Lambda_{\beta_{*}})\beta_{*} + 2\gamma \right] \frac{|\nabla' \psi_{*}|^{2}}{\beta_{*}^{2} \psi_{*}^{2} + |\nabla' \psi_{*}|^{2}} + \gamma O(\psi_{*}^{2}).$$
(3.36)

Equivalently

$$Q_{1}[V] = \left[1 + (p-2)\frac{\beta_{*}^{2}\psi_{*}^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}}\right] \left(\gamma^{2} - (\Lambda_{\beta_{*}} + \beta_{*})\gamma\right)$$
$$-\gamma \left[(p-2)(\beta_{*} + 2)\frac{|\nabla'\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} + O(\psi_{*}^{2})\right]$$

and finally

$$Q_1[V] = \left[1 + (p-2)\frac{\beta_*^2 \psi_*^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2}\right] \gamma \left[\gamma - (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)) + O(\psi_*^2)\right]. \quad (3.37)$$

Using the fact that  $\beta_* > \frac{N-1}{p-1}$  if  $1 and <math>1 < \beta_* < \frac{N-1}{p-1}$  if 2 (see Theorem 6.1 in Appendix II), we have

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) \ge \begin{cases} \Lambda_{\beta_*} + \beta_*(p-1) & \text{if } p \ge 2\\ N + 3(p-2) > N - 3 & \text{if } 1 (3.38)$$

When N=2, we have explicitly  $\beta_*=\frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$  (see [9, Th 3.3]). Therefore for all  $N\geq 2$  and p>1, there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) > 0. \tag{3.39}$$

We fix  $\epsilon_0 > 0$  such that, whenever  $\psi_* \leq \epsilon_0$ , there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) + O(\psi_*^2) > \frac{1}{2} (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)). \tag{3.40}$$

If we fix  $\gamma_0 > 0$  such that

$$\gamma_0 < \min \left\{ \frac{1}{2} \left( \Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2) \right), \nu, \beta_* \right\},$$
(3.41)

we obtain

$$Q_1[V] \le -\min\{1, p-1\}\gamma m^2 \qquad \forall 0 < \gamma \le \gamma_0, \tag{3.42}$$

whenever  $\psi_* \leq \epsilon_0$ , for some m depending only on p, q and N (through  $\psi_*$  and  $\nu$ ), which, in the same range of value of  $\psi_*$ , yields to

$$\left(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) \mathcal{Q}_1[V] \le -c_{17} \psi_* \qquad \forall \, 0 < \gamma \le \gamma_0, \tag{3.43}$$

for some  $c_{17}>0$  depending on N,p,q. This estimate is valid whatever is p>1, but only in a neighborhood of  $\psi_*=0$ . If we replace  $g(\psi_*)=\psi_*$  by  $g_k(\psi_*)=\psi_*e^{-k\psi_*}$  for 0< k<1, and denote by  $\mathcal{Q}_{1,k}[V]$  the corresponding expression of  $\mathcal{Q}_1[V]$  which becomes now  $\mathcal{Q}_{1,0}[V]$ . We define similarly  $\mathcal{Q}_k[V]$ , and  $\mathcal{Q}[V]$  becomes  $\mathcal{Q}_0[V]$ . Since  $g_k'(\psi_*)=e^{-k\psi_*}-kg_k(\psi_*)$  and  $g_k''=-2ke^{-k\psi_*}+k^2g_k(\psi_*)$ , we obtain

$$Q_{1,k}[V] = Q_{1,0}[V] + k(p-1)\beta_* \Lambda_{\beta_*} \psi_* + (p-1)\left(-\frac{2k}{\psi_*} + k^2\right) |\nabla' \psi_*|^2 + (2-p)\beta_* \left(-2k + k^2\right) \psi_* + O(\psi_*^2)$$
(3.44)

Notice that  $\nabla' \psi_*$  vanishes only at the North pole  $\mathbf{e}_N$ , thus there exists  $k_0 \in (0,1]$  such that

$$k(1-p)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(\frac{2k}{\psi_*} - k^2\right) \left|\nabla'\psi_*\right|^2 \ge \frac{1}{2}(2-p)_+\beta_*\left(-2k+k^2\right)\psi_* \qquad \forall k \le k_0$$

whenever  $\psi_* < \epsilon_0$  which yields to

$$\left(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2\right)^{\frac{p-2}{2}} g_k(\psi_*) \mathcal{Q}_{1,k}[V] \le -c_{18}k \qquad \forall k \le k_0 \tag{3.45}$$

for some  $c_{13} = c_{13}(N, p, q, \epsilon_0)$ . There exists  $c_{14} = c_{14}(N, p, q) > 0$  such that

$$\left(\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2\right)^{\frac{q}{2}} \left[ 1 + q e^{\gamma t} \frac{\beta_* (\gamma - \beta_*) \psi_* g_k(\psi_*) - g_k'(\psi_*) |\nabla \psi_*|^2}{\beta_*^2 \psi_*^2 + |\nabla' \psi_*|^2} \right] \le c_{14}$$
(3.46)

in  $S^{N-1}_+ \times (-\infty, \ln \delta]$ . Moreover

$$O(a \|g(\psi_*)\|_{C^2}) \le e^{\gamma t} \tilde{c}_k$$
 (3.47)

for some  $\tilde{c}_k = \tilde{c}_k(N,p,q) > 0$ . We derive from (3.45)-(3.47)

$$e^{-\gamma t} \mathcal{Q}_k[V] \le -c_{13}k + c_{14}e^{(\nu-\gamma)t} + e^{\gamma t}\tilde{c}_k \qquad \forall k \le k_0$$
 (3.48)

Thus there exists  $T_k \leq \ln \delta$  such that  $\mathcal{Q}_k[V] \leq 0$ , for all  $t \leq T_k$  and provided  $\psi_* \leq \epsilon_0$ . This local estimate will be used in the construction of the subsolution when  $p \geq 2$ .

Step 3: The case  $1 . Since the function <math>\psi^*$  depends only on the azimuthal angle  $\theta \in (0; \frac{\pi}{2}]$  we will write  $\psi_*(\sigma) = \psi_*(\theta)$  and  $\nabla' \psi_*(\sigma) = \psi_{*\theta}(\theta) \mathbf{n}$  where  $\mathbf{n}$  is the downward unit vector tangent to  $S^{N-1}$  in the hyperplane going through  $\sigma$  and the poles. From (6.8),

$$(p-4)\beta_* \Lambda_{\beta_*} \psi_* - 2\Delta' \psi_* = (p-2) \left( \beta_* \Lambda_{\beta_*} \psi_* + 2 \frac{\beta_*^2 \psi_* + \psi_{*\theta}}{\beta_*^2 \psi_*^2 + \psi_{*\theta}^2} \right), \tag{3.49}$$

since  $\psi_{*\theta}^{\,2} = \left| \nabla' \psi_* \right|^2$  and thus

$$((p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*) \frac{\beta_*\gamma\psi_*}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}$$

$$= (p-2)\gamma \left(\Lambda_{\beta_*} \frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} + 2\beta_* \frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\right).$$
(3.50)

From Theorem 6.1-Step 4 in Appendix II, we know that  $\beta_*^2 \psi_* + \psi_{*\theta\theta} \ge 0$ , thus the contribution of this term to  $\mathcal{Q}_1[V]$  is nonpositive. We replace this expression in  $\mathcal{Q}_1[V]$  with  $g(\psi_*) = \psi_*$  and obtain

$$\mathcal{Q}_{1}[V] = (\gamma - \Lambda_{\beta_{*}})(\gamma - \beta_{*}) \left( 1 + (p-2) \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} \right) - \Lambda_{\beta_{*}} \beta_{*}(p-1) 
+ (p-2)\gamma \Lambda_{\beta_{*}} \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} - (p-2) \left( (\beta_{*} + 2)\gamma - \Lambda_{\beta_{*}} \beta_{*} \right) \frac{\psi_{*\theta}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} 
+ 2\beta_{*}(p-2) \frac{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta} \psi_{*}}{(\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2})^{2}} \gamma$$

$$\leq \gamma \left( 1 + (p-2) \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} \right) (\gamma - \Lambda_{\beta_{*}} - \beta_{*}) - (p-2)\gamma \frac{(\beta_{*} + 2))\psi_{*\theta}^{2} - \Lambda_{\beta_{*}} \beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}}$$

$$\leq \gamma \left( 1 + (p-2) \frac{\beta_{*}^{2} \psi_{*}^{2}}{\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} \right) \left( \gamma - \left( \Lambda_{\beta_{*}} + \beta_{*} + (p-2) \frac{(\beta_{*} + 2)\psi_{*\theta}^{2} - \Lambda_{\beta_{*}} \beta_{*}^{2} \psi_{*}^{2}}{(p-1)\beta_{*}^{2} \psi_{*}^{2} + \psi_{*\theta}^{2}} \right) \right).$$
(3.51)

We can write

$$\Lambda_{\beta_*} + \beta_* + (p-2) \frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} 
= \frac{(\Lambda_{\beta_*} + (p-1)\beta_*)\beta_*^2\psi_*^2 + (\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2))\psi_{*\theta}^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2} 
\ge c_{15} (\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2))$$
(3.52)

for some positive constant  $c_{15}$ . This expression  $\Lambda_{\beta_*}+\beta_*(p-1)+2(p-2)$  is always positive: obviously if  $N\geq 3$  and by using the explicit expression of  $\beta_*$  if N=2. Thus there exists  $\gamma_0$  and  $c_{16}>0$  such that  $\mathcal{Q}_1[V]<-c_{16}$  for  $0<\gamma\leq\gamma_0$ . The perturbation method of Step 2, is valid in the whole range of values of  $\psi_*$  and we derive from (3.42)-(3.43) that (3.48) holds for all  $k\leq k_0$  and  $t\leq T_k$ . Therefore  $\mathcal{Q}_k[V]\leq 0$ .

Step 4: The case  $p \ge 2$ . For c > 0 to be fixed and  $\psi_* \ge \epsilon_0$ ,  $\gamma \in (0, \gamma_0]$ , we take  $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$ . Then

we derive from (3.35):

$$\mathcal{Q}_{1}[V] = (\gamma - \Lambda_{\beta_{*}})(\gamma - \beta_{*}) \frac{(p-1)\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} - (p-1)\beta_{*}\Lambda_{\beta_{*}} \left(1 - \frac{\gamma}{\beta_{*}}\right) 
- (p-1)\frac{\gamma(\beta_{*} - \gamma)}{\beta_{*}^{2}}\psi_{*}^{-1 - \frac{\gamma}{\beta_{*}}} |\nabla'\psi_{*}|^{2} - (p-2)(\beta_{*} - \gamma)(\gamma - \Lambda_{\beta_{*}}) \frac{|\nabla'\psi_{*}|^{2}}{\beta_{*}^{2}\psi_{*}^{2} + |\nabla'\psi_{*}|^{2}} 
= (1-p)\left[\gamma(\beta_{*} - \gamma) + \frac{\gamma(\beta_{*} - \gamma)}{\beta_{*}^{2}}\psi_{*}^{-1 - \frac{\gamma}{\beta_{*}}} |\nabla'\psi_{*}|^{2}\right].$$
(3.53)

For  $k \leq k_0$  we fix c such that  $c\epsilon_0^{1-\frac{\gamma}{\beta_*}} = \epsilon_0 e^{-k\epsilon_0} \iff c = \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0}$  and we define g by

$$g(\psi_*) = \min\left\{\psi_* e^{-k\psi_*}, \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}}\right\} = \begin{cases} \psi_* e^{-k\psi_*} & \text{if } 0 \le \psi_* \le \epsilon_0\\ \frac{\gamma}{\beta_*} e^{-k\epsilon_0} \psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0 \le \psi_* \le 1, \end{cases}$$
(3.54)

and we set  $V(t,\sigma)=\psi^*(\sigma)-a(t)g(\psi_*(\sigma))$  with  $(t,\sigma)\in(-\infty,T_k]\times S^{N-1}_+$  and define  $\tilde{u}(r,\sigma)=r^{-\beta_*}(\psi^*(\sigma)-a(\ln r)g(\psi_*(\sigma)))$  accordingly for  $(r,\sigma)\in(-\infty,e^{T_k}]\times S^{N-1}_+$ . Since  $\psi_*$  is a decreasing function the coincidence set  $\{\sigma\in S^{N-1}_+:\psi_*(\sigma)=\epsilon_0\}$  is a circular cone  $\Sigma_{\theta_0}$  with vertex 0, axis  $\mathbf{e}_N$  and angle  $\theta_0$ . We set  $R_0=e^{T_k}$ 

$$\Gamma_{1} = \left\{ x = (r, \theta) \in B_{R_{0}}^{+} : \theta_{0} < \theta < \frac{\pi}{2} \right\} = \left\{ (r, \sigma) \in [0, R_{0}) \times S_{+}^{N-1} : 0 < \psi_{*}(\sigma) < \epsilon_{0} \right\},$$

$$\Gamma_{2} = \left\{ x = (r, \theta) \in B_{R_{0}}^{+} : 0 < \theta < \theta_{0} \right\} = \left\{ (r, \sigma) \in [0, R_{0}) \times S_{+}^{N-1} : \epsilon_{0} < \psi_{*}(\sigma) < 1 \right\},$$

and define

$$\begin{split} \tilde{u}(r,\sigma) &= r^{-\beta_*} \left( \psi_*(\sigma) - r^{\gamma} g(\psi_*(\sigma)) \right) \\ &= \left\{ \begin{array}{ll} u_1(r,\sigma) &= r^{-\beta_*} (1 - r^{\gamma} e^{-k\psi_*(\sigma)}) \psi_*(\sigma) & \text{if } (r,\theta) \in \Gamma_1 \\ u_2(r,\sigma) &= r^{-\beta_*} \left( 1 - r^{\gamma} \epsilon_0^{\frac{\gamma}{\beta_*}} e^{-k\epsilon_0} (\psi_*(\sigma))^{1 - \frac{\gamma}{\beta_*}} \right) \psi_*(\sigma) & \text{if } (r,\theta) \in \Gamma_2. \end{array} \right. \end{split}$$

The function  $\tilde{u}$  is a subsolution separately on  $\Gamma_1$  and  $\Gamma_2$  and is Lipschitz continuous in  $\overline{\Omega} \setminus \{0\}$ . If we denote by  $g_1$  and  $g_2$  the restriction of g to  $\Gamma_1$  and  $\Gamma_2$  respectively, that is to  $\Omega_1$  and  $\Omega_2$ , then  $g_1'(\epsilon_0) > g_2'(\epsilon_0) > 0$ . Let  $\zeta \in C_c^1(B_{R_0}^+)$  which vanishes in neighborhoods of 0 and  $\partial B_{R_0}^+$ ,  $\zeta \geq 0$ , then

$$\int_{\Gamma_i} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega_i} |\nabla \tilde{u}|^q \zeta dx \le \int_{\Sigma_{\theta_0}} |\nabla u_i|^{p-2} \partial_{\mathbf{n}_i} u_i \zeta dS, \tag{3.55}$$

where  $\mathbf{n}_i$  is the normal unit vector on  $\Sigma_{\theta_0}$  outward from  $\Gamma_i$ . Actually,  $\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{n}$  thus

$$\nabla \tilde{u} = \tilde{u}_r \mathbf{e} + r^{-\beta_* - 1} (1 - r^{\gamma} g'(\psi_*)) \nabla' \psi_* = \tilde{u}_r \mathbf{e} + r^{-\beta_* - 1} (1 - r^{\gamma} g'(\psi_*)) \psi_{*\theta} \mathbf{n}.$$

and on  $\Sigma_{\theta_0}$ ,

$$\nabla \tilde{u} = \begin{cases} \tilde{u}_r \mathbf{e} - r^{-\beta_* - 1} (1 - r^{\gamma} g_1'(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_1 \\ \tilde{u}_r \mathbf{e} + r^{-\beta_* - 1} (1 - r^{\gamma} g_2'(\epsilon_0)) \psi_{*\theta} \mathbf{n} & \text{in } \Gamma_2 \end{cases}$$

Therefore

$$\begin{split} |\nabla u_1|^{p-2} \, \partial_{\mathbf{n}_1} u_1 \\ &= -r^{-\beta_* - 1} (1 - r^{\gamma} g_1'(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_* - 2} (1 - r^{\gamma} g_1'(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_1 \end{split}$$

and

$$\begin{split} |\nabla u_2|^{p-2} \, \partial_{\mathbf{n}_2} u_2 \\ &= r^{-\beta_* - 1} (1 - r^{\gamma} g_2'(\epsilon_0)) \left( \tilde{u}_r^2 + r^{-2\beta_* - 2} (1 - r^{\gamma} g_2'(\epsilon_0))^2 \psi_{*\theta}^2 \right)^{\frac{p-2}{2}} \psi_{*\theta} \quad \text{in } \Gamma_2 \end{split}$$

By adding the two inequalities (3.55)

$$\int_{\Omega} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \zeta dx + \int_{\Omega} |\nabla \tilde{u}|^{q} \zeta dx \le \int_{\Sigma_{\theta_{0}}} \left( |\nabla u_{1}|^{p-2} \partial_{\mathbf{n}_{1}} u_{1} + |\nabla u_{2}|^{p-2} \partial_{\mathbf{n}_{2}} u_{2} \right) \zeta dS. \quad (3.56)$$

By monotonicity of the function  $X \mapsto (\tilde{u}_r^2 + X^2)^{\frac{p}{2}}$  and since

$$r^{-\beta_*-1}(1-r^{\gamma}g_2'(\epsilon_0)) \ge r^{-\beta_*-1}(1-r^{\gamma}g_1'(\epsilon_0)) \ge 0,$$

we derive

$$r^{-\beta_*-1}(1-r^{\gamma}g_2'(\epsilon_0))\left(\tilde{u}_r^2+r^{-2\beta_*-2}(1-r^{\gamma}g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}$$

$$\geq r^{-\beta_*-1}(1-r^{\gamma}g_1'(\epsilon_0))\left(\tilde{u}_r^2+r^{-2\beta_*-2}(1-r^{\gamma}g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}$$

We derive that the right-hand side of (3.56) is nonpositive because  $\psi_{*\theta} \leq 0$ , and therefore  $\tilde{u}$  is a positive subsolution of (1.1) in  $B_{R_0}^+$  dominated by  $\Psi_*$  and satisfying (3.24).

Proof of Theorem 3.6. Let  $M = \max\{\Psi_*(x) : x \in \partial B_{R_0}^+\}$ , then  $M = R_0^{-\beta_*}$ . The function  $u^*$  defined by

$$u^*(x) = \begin{cases} (\tilde{u}(x) - M)_+ & \text{if } x \in B_{R_0}^+ \\ 0 & \text{if } x \in \Omega \setminus B_{R_0}^+, \end{cases}$$

is indeed a subsolution of (1.1) in whole  $\Omega$  where it satisfies  $u^* \leq \Psi_*$  and it vanishes on  $\partial \Omega \setminus \{0\}$ . Since  $\Phi_*$  is a positive p-harmonic function in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies (3.20), it is supersolution of (1.1) and therefore it dominates  $u^*$ . Therefore there exists a solution u of (1.1) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies  $u^* \leq u \leq \Phi_*$ . This implies that (3.19) holds with k=1 and we conclude with Lemma 3.8. This ends the proof of Lemma 3.9.

When p=N the statement of Theorem 3.6 holds without the flatness assumption on  $\partial\Omega$ . The proof of the next theorem is an easy adaptation to the one of Theorem 3.6, provided Lemma 3.7, Lemma 3.8 and Lemma 3.9 are modified accordingly.

**Theorem 3.10.** Assume  $N-1 < q < N-\frac{1}{2}$  and  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial \Omega$ . Then for any k > 0 there exists a unique positive solution  $u := u_k$  of (3.17) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies uniformly with respect to  $\sigma \in S^{N-1}_+$ 

$$\lim_{\begin{subarray}{l} x \to 0 \\ x/|x| \to \sigma \end{subarray}} |x| \, u_k(x) = k\psi_*(\sigma). \tag{3.57}$$

Since p=N, then  $\beta_*=1$  and  $\psi_*(\sigma)=\frac{x_N}{|x|}=\cos\theta_{N-1}$  with the identification of  $\sigma$  and  $\theta_{N-1}:=\theta$ . In a more intrinsic manner (3.57) can be written under the form

$$\lim_{\substack{x \to 0 \\ x \in \Omega}} |x|^2 \frac{u_k(x)}{d(x)} = k.$$
 (3.58)

We recall that if  $\omega \in \mathbb{R}^N$  and  $\mathcal{I}_{\omega}$  denotes the inversion of center  $\omega$  and power 1, i.e.  $\mathcal{I}_{\omega}(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$ , then  $\tilde{u} = u \circ \mathcal{I}_{\omega}$  satisfies (3.18).

**Lemma 3.11.** Assume  $\Omega$  be a bounded  $C^2$  domain such that  $0 \in \partial \Omega$ . Then there exists a unique N-harmonic function  $\Phi_*$  in  $\Omega$ , which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies

$$\lim_{\begin{subarray}{c} x \to 0 \\ x/|x| \to \sigma \end{subarray}} |x| \, \Phi_*(x) = \psi_*(\sigma), \tag{3.59}$$

uniformly with respect to  $\sigma \in S^{N-1}_+$ .

*Proof.* Uniqueness is standard. Let  $\omega = -\mathbf{e}_N \in \overline{\Omega}^c$ , with the notations of the proof of Theorem 3.5,  $\omega' = -\omega$ ,  $a = -\frac{1}{2}\mathbf{e}_N$  and a' = -a. We can assume that the balls  $B_{\frac{1}{2}}(a)$  and  $B_{\frac{1}{2}}(a')$  are tangent to  $\partial\Omega$  at 0 and respectively subset of  $\Omega^c$  and  $\Omega$ . The function  $x \mapsto \Psi(x) = -\frac{x_N}{|x|^2}$  which is N-harmonic in  $\mathbb{R}^N_-$  and vanishes on  $\partial\mathbb{R}^N_-\setminus\{0\}$  is transformed by the inversion  $\mathcal{I}_{\omega'}$  of center  $\omega'$  and power 1 into the function  $\Psi_{\omega'} = \Psi \circ \mathcal{I}_{\omega}$  which is positive and N-harmonic in  $B_{\frac{1}{2}}(a')$  and vanishes on  $\partial B_{\frac{1}{2}}(a')\setminus\{0\}$ . The function  $\hat{\Psi} = -\Psi$  which is N-harmonic in  $\mathbb{R}^N_+$  and vanishes on  $\partial\mathbb{R}^N_+\setminus\{0\}$  is transformed by the inversion  $\mathcal{I}_{\omega}$  of center  $\omega$  and power 1 into the function  $\Psi_{\omega} = \hat{\Psi} \circ \mathcal{I}_{\omega}$  which is positive and N-harmonic in  $B_{\frac{1}{2}}(a)$  and vanishes on  $\partial B_{\frac{1}{2}}(a)\setminus\{0\}$ . For  $\epsilon>0$  we denote by  $\Phi_{\epsilon}$  the solution of

$$\begin{aligned}
-\Delta_N \Phi_{\epsilon} &= 0 & \text{in } \Omega \cap B_{\epsilon}^c \\
\Phi_{\epsilon} &= 0 & \text{in } (B_{\frac{1}{2}}^c(a') \cap \partial B_{\epsilon}) \cup (\partial \Omega \cap B_{\epsilon}^c) \\
\Phi_{\epsilon} &= \Psi_{\omega'} & \text{in } B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}.
\end{aligned} (3.60)$$

If  $0 < \epsilon' < \epsilon$ ,  $\Phi_{\epsilon'} \ge \Psi_{\omega'}$  in  $B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}$ , thus  $\Phi_{\epsilon'} \ge \Phi_{\epsilon'}$  in  $\Omega \cap B_{\epsilon}^c$ . We also denote by  $\hat{U}_{\epsilon}$  the solution of

$$\begin{aligned}
& \Delta_N \hat{\Phi}_{\epsilon} = 0 & \text{in } \Omega \cap B_{\epsilon}^c \\
& \hat{\Phi}_{\epsilon} = 0 & \text{in } \partial \Omega \cap B_{\epsilon}^c \\
& \hat{\Phi}_{\epsilon} = \Psi_{\omega} & \text{in } \Omega \cap \partial B_{\epsilon}^c.
\end{aligned} \tag{3.61}$$

In the same way as above

$$0 < \epsilon' < \epsilon \Longrightarrow \hat{\Phi}_{\epsilon'} \le \hat{\Phi}_{\epsilon} \quad \text{in } \Omega \cap \partial B_{\epsilon}^c$$

Using the explicit form of  $\Psi$ ,  $\mathcal{I}_{\omega}: x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$  and  $\mathcal{I}_{\omega'}: x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$  we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_{\omega} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_{\epsilon}},$$

thus

$$\Phi_\epsilon \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap B^c_\epsilon.$$

Letting  $\epsilon \to 0$  we conclude that  $\Phi_{\epsilon}$  converges uniformly in  $\overline{\Omega} \setminus \{0\}$  to  $\Phi_{*}$  which vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies (3.59).

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications, so we omit it.

**Lemma 3.12.** Let the assumptions on q and  $\Omega$  of Theorem 3.10 be satisfied. If for some k > 0 there exists a solution  $u_k$  of (3.17) in  $\Omega$ , which belongs to  $C^1(\overline{\Omega} \setminus \{0\})$ , vanishes on  $\partial \Omega \setminus \{0\}$  and satisfies (3.57), then for any k > 0 there exists such a solution.

**Lemma 3.13.** Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution  $\tilde{u}$  of (3.17) in  $\Omega$  which vanishes on  $\partial\Omega\setminus\{0\}$ , is smaller than  $\Phi_*$  and satisfies

$$\lim_{\begin{subarray}{c} x \to 0 \\ x/|x| \to \sigma \end{subarray}} |x| \, \tilde{u}(x) = \sigma, \tag{3.62}$$

uniformly with respect to  $\sigma \in S_+^{N-1}$ .

*Proof.* Let  $\tau > 0$  to be fixed and let w be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \qquad \text{in } B_2^- \tag{3.63}$$

which vanishes on  $\partial B_2^- \setminus \{0\}$  and satisfies

$$\lim_{\begin{subarray}{c} x \to 0 \\ x/|x| \to \sigma \end{subarray}} |x|w(x) = \sigma \tag{3.64}$$

in the  $C^1$ -topology of  $S^{N-1}_-$ . Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function  $\Phi_*$  corresponding to this domain, obtained in Lemma 3.11. By  $\mathcal{I}_{\omega'}$ , the half-ball  $B^-_2$  is transform into the lunule  $G=B_{\frac{1}{2}}(a')\setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$  and  $\tilde{w}=w\circ\mathcal{I}_{\omega'}$  satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \quad \text{in } G.$$
(3.65)

Since  $|x - \omega'| \le 1$  in G,  $-\Delta_N \tilde{w} + |\nabla \tilde{w}|^q \le 0$  in G. We extend  $\tilde{w}$  by 0 in  $\Omega \setminus G$  and the resulting function  $\tilde{u}$  is a subsolution of (3.17) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ ), is smaller than the N-harmonic function  $\Phi_*$  obtained in Lemma 3.11, and satisfies (3.62).

# 4 Classification of boundary singularities

We assume that  $\Omega \subset \mathbb{R}^N$  is a  $C^2$  domain and  $0 \in \partial \Omega$ . Furthermore, in order to avoid extremely technical computations, we shall assume either that  $\partial \Omega$  is flat near 0 or p=N. We suppose that the tangent plane to  $\partial \Omega$  at 0 is  $\partial \mathbb{R}^N_+ = \{x = (x',0)\}$  and the normal inward unit vector at 0 is  $\mathbf{e}_N$ , therefore  $\mathbf{n} = -\mathbf{e}_N$  in the sequel. We denote by  $\omega_{s_+^{N-1}}$  the unique positive solution of (3.1) in  $S_+^{N-1}$  and by  $U_{s_+^{N-1}}$  the corresponding singular solution of (1.1) in  $\mathbb{R}^N_+$  defined by

$$U_{s_{+}^{N-1}}(x) = |x|^{-\beta_{q}} \,\omega_{s_{+}^{N-1}}(\frac{x}{|x|}). \tag{4.66}$$

We recall that  $\psi_*$  is the unique positive solution of (3.2) with maximum 1 and  $\Psi_*$  the corresponding p-harmonic function

$$\Psi_*(x) = |x|^{-\beta_*} \,\psi_*(\frac{x}{|x|}). \tag{4.67}$$

#### **4.1** The case 1

The first statement points out the link between weak and strong singularities.

**Proposition 4.1.** Under the assumptions of Theorem 3.6 there exists  $\lim_{k\to\infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$  which vanishes on  $\partial\Omega\setminus\{0\}$ , satisfies (1.1) in  $\Omega$  and

$$\lim_{x \to 0} \frac{u_{\infty}(x)}{U_{s_{\perp}^{N-1}}(x)} = 1. \tag{4.68}$$

Proof. Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping  $k\mapsto u_k$  is increasing and  $u_k\leq U_{s_+^{N-1}}$ , there exists  $\lim_{k\to\infty}u_k:=u_\infty\leq U_{s_+^{N-1}}$  and  $u_\infty\in C(\overline\Omega\setminus\{0\})\cap C^1(\Omega)$ . It vanishes on  $\partial B_\delta^+\setminus\{0\}$  and satisfies (1.1) in  $B_\delta^+$ . In order to take into account the domain  $B_\delta^+$  in the notations, we set  $u_k=u_{k,\delta}$ . Since the mapping  $\delta\mapsto u_{k,\delta}$  is also increasing and  $u_{k,\delta}\leq k\Psi_*$ , there also exists  $\lim_{\delta\to\infty}u_{k,\delta}:=u_{k,\infty}\leq k\Psi_*$  Then, for all  $\ell>0$ ,

$$T_{\ell}[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q},\ell^{-1}\delta}(x). \tag{4.69}$$

Letting  $k \to \infty$ , we obtain

$$T_{\ell}[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty,\ell^{-1}\delta}(x), \tag{4.70}$$

and letting  $\delta \to \infty$ , we obtain

$$T_{\ell}[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \tag{4.71}$$

This implies that

$$u_{\infty,\infty}(r,\sigma) = r^{-\beta_q}\omega'(\sigma),$$
 (4.72)

and  $\omega'$  is a positive solution of problem (3.1). Therefore  $\omega'=\omega_{s_+^{N-1}}$  by Theorem 3.2. If we let  $\ell\to 0$  in (4.69) and take  $|x|=1,\,x=\sigma$ , we derive

$$\lim_{\ell \to 0} \ell^{\beta_q} u_{\infty,\delta}(\ell,\sigma) = \lim_{\ell \to 0} u_{\infty,\ell^{-1}\delta}(1,\sigma) = u_{\infty,\infty}(1,\sigma) = \omega_{s_+^{N-1}}(\sigma). \tag{4.73}$$

This convergence holds in  $C^1(\overline{S_+^{N-1}})$  because of Lemma 2.5. This implies (4.68).

The main classification result is as follows.

**Theorem 4.2.** Assume  $1 , <math>p - 1 < q < q^*$  and  $\partial\Omega \cap B_{\delta} = \{x = (x', 0) : |x'| < \delta\}$ , for some  $\delta > 0$ . If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (1.1) in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ , then we have the following alternative:

(i) either there exists  $k \ge 0$  such that

$$\lim_{x \to 0} \frac{u(x)}{\Psi_*(x)} = k,\tag{4.74}$$

(ii) or

$$\lim_{x \to 0} \frac{u(x)}{U_{s_{\perp}^{N-1}}(x)} = 1. \tag{4.75}$$

Proof. Step 1. Assume

$$\liminf_{x \to 0} \frac{u(x)}{\Psi_*(x)} < \infty, \tag{4.76}$$

then we claim that (4.74) holds. We first note that if (4.76) holds, there also holds

$$\liminf_{x \to 0} \frac{u(x)}{u_1(x)} < \infty, \tag{4.77}$$

where  $u_1$  is the solution of (1.1) obtained in Theorem 3.6 with k = 1. If  $\{x_n\}$  is converging to 0 and such that for some k > 0

$$\liminf_{x \to 0} \frac{u(x)}{u_1(x)} = k = \lim_{n \to \infty} \frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.38) applied to both u and  $u_1$ ,

$$\frac{u(x_n)}{u_1(x_n)} = \frac{u(x_n)}{d(x_n)} \frac{d(x_n)}{u_1(x_n)} \ge c_5^{-2} \frac{u(x)}{u_1(x)} \quad \forall x \text{ s.t. } |x| = |x_n|.$$

This implies in particular

$$u(x) \le c_5^2(k + \epsilon_n)u_1(x)$$
  $\forall x \text{ s.t. } |x| = |x_n|$ 

where  $\{\epsilon_n\}$  is converging to  $0_+$ , and by the comparison principle

$$u(x) \le Ku_1(x)$$
  $\forall x \in \mathbb{R}^N_+ \text{ s.t. } |x_n| \le |x| \le \frac{\delta}{2}$ 

for some K > 0 and all  $n \in \mathbb{N}_*$ . Therefore

$$\limsup_{x \to 0} \frac{u(x)}{u_1(x)} < \infty. \tag{4.78}$$

We can assume that  $k \neq 0$ , otherwise (4.74) holds with k = 0 and actually u remains bounded near 0. As a consequence of the Hopf Lemma and  $C^1$  regularity, there exists K > 0 such that

$$u(x) \le K\Psi_*(x) \qquad \forall x \in B^+_{\frac{\delta}{2}}.$$
 (4.79)

Let  $m=\max\{u(x):|x|=\delta\}$ . For  $0<\tau<\delta$  we denote by  $k_{\tau}$  the minimum of the  $\kappa>0$  such that  $u(x)\leq\kappa\Psi_*(x)+m$  for  $\tau\leq|x|\leq\delta$ . Then  $u(x)\leq k_{\tau}\Psi_*(x)+m$ , and either the graphs of the mappings u(.) and  $k_{\tau}\Psi_*(.)+m$  are tangent at some  $x_{\tau}\in B_{\delta}^+\setminus\overline{B}_{\tau}^+$ , or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on  $|x|=\tau$ . Since

$$|\nabla \Psi_*(x)|^2 = |x|^{-2(\beta_*+1)} (\beta_*^2 \psi_*^2 + |\nabla \psi_*|^2),$$

it never vanishes. If we set  $w = u - (k_{\tau} \Psi_*(x) + m)$ , then

$$-\mathcal{L}w + |\nabla u|^q = 0 \tag{4.80}$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)$$

is uniformly elliptic in a neighborhood of  $x_{\tau}$  (see [6, Lemma 1.3]). Furthermore  $w \leq 0$  and  $w(x_{\tau}) = 0$  by the strong maximum principle  $\nabla u(x_{\tau})$  must vanish, which contradicts the fact that  $\nabla u(x_{\tau}) = \nabla w(x_{\tau})$  by the tangency condition, and  $\nabla w(x_{\tau}) \neq 0$ . Therefore  $|x_{\tau}| = \tau$  and  $x_{\tau} \notin \partial \mathbb{R}^{N}_{+}$ . If  $\tau' < \tau$ ,  $k_{\tau} \leq k_{\tau'}$ , and we set  $k = \lim_{\tau \to 0} k_{\tau}$ , which is finite because of (4.79). There exists  $\{\tau_{n}\}$  such that  $\sigma_{n} := \tau^{-1}x_{\tau_{n}} \to \sigma_{0}$ . Furthermore

$$r^{\beta_*}u(r,\sigma) \le k_{\tau}\psi_*(\sigma) + mr^{\beta_*}$$
 if  $\tau \le r \le \delta$  and  $\tau^{\beta_*}u(\tau,\sigma_{\tau}) = k_{\tau}\psi_*(\sigma_{\tau}) + m\tau^{\beta_*}$ . (4.81)

Put

$$u_{\tau}(x) = \tau^{\beta_*} u(\tau x) \tag{4.82}$$

Then

$$-\Delta_p u_\tau + \tau^{p-q-\beta_*(p+1-q)} \left| \nabla u_\tau \right|^q = 0 \qquad \text{in } B_{\frac{\delta}{2}}^+ \setminus \{0\}$$

and, by (4.79),

$$0 \le u_{\tau}(x) \le K |x|^{-\beta_*} \qquad \text{in } B^+_{\frac{\delta}{2\pi}} \setminus \{0\}.$$

By Lemma 2.5, the set of functions  $\{u_{\tau}(.)\}$  is relatively compact in the  $C^1_{loc}$  topology of  $\overline{\mathbb{R}^N_+}\setminus\{0\}$ . Therefore, as  $q< q^*$ , there exist a sequence  $\{\tau'_n\}\subset\{\tau_n\}$  converging to 0, and a positive p-harmonic function v in  $\mathbb{R}^N_+$ , continuous in  $\overline{\mathbb{R}^N_+}\setminus\{0\}$  and vanishing on  $\partial\mathbb{R}^N_+\setminus\{0\}$ , such that  $u_{\tau'_n}\to v$ , and v satisfies (4.79) in  $\overline{\mathbb{R}^N_+}\setminus\{0\}$ . By Theorem 5.1 in Appendix I, there exists  $k^*$  such that  $v=k^*\Psi_*$ . In particular,

$$\lim_{\tau_n' \to 0} u_{\tau_n'}(1, \sigma) = k^* \psi_*(\sigma)$$
(4.83)

in the  $C^1(\overline{S^{N-1}_+})$  topology. Combining (4.81), (4.82)and (4.83) we conclude that  $k^*=k$  and

$$\lim_{\tau_n' \to 0} \tau_n'^{\beta_*} u_{\tau_n'}(1, \sigma) = k \psi_*(\sigma) \tag{4.84}$$

Using Theorem 3.6, it is equivalent to

$$\lim_{\tau_n' \to 0} \frac{u(\tau_n', \sigma)}{u_k(\tau_n', \sigma)} = 1 \tag{4.85}$$

uniformly on  $S_+^{N-1}$ . For any  $\epsilon > 0$ , there exists  $n_{\epsilon} > 0$  such that  $n \geq n_{\epsilon}$  implies

$$u_{k-\epsilon}(\tau'_n, \sigma) \le u(\tau'_n, \sigma) \le u_{k+\epsilon}(\tau'_n, \sigma)$$

By comparison principle,

$$u_{k-\epsilon} \le u \le u_{k+\epsilon} + m \quad \text{in } B_{\delta}^+ \setminus B_{\tau'_n}^+,$$
 (4.86)

and finally

$$u_{k-\epsilon} \le u \le u_{k+\epsilon} + m \quad \text{in } B_{\delta}^+,$$
 (4.87)

Since  $\epsilon$  is arbitrary and using again Theorem 3.6, it implies

$$\lim_{r \to 0} \frac{u(r,\sigma)}{\Psi_*(r,\sigma)} = k,\tag{4.88}$$

locally uniformly on  $S^{N-1}$ . But since the convergence holds in  $C^1(\overline{S_+^{N-1}})$ , (4.74) follows.

Step 2. Assume

$$\lim_{x \to 0} \frac{u(x)}{\Psi_*(x)} = \infty. \tag{4.89}$$

For any  $0 < \epsilon < \delta$  and k > 0, there holds

$$u_k(x) \le u(x) \le v_{\epsilon}(x) \quad \text{in } B_{\delta}^+ \setminus B_{\epsilon}^+$$
 (4.90)

where  $v_{\epsilon}$  has been defined in (3.12) and  $u_k$  is given by Theorem 3.6. Letting  $\epsilon \to 0$ ,  $k \to \infty$ , and using Proposition 4.1, we derive

$$u_{\infty}(x) \le u(x) \le v_0(x) \qquad \text{in } B_{\delta}^+ \setminus \{0\}. \tag{4.91}$$

We have seen in Theorem 3.3 that  $v_0$  is a separable solution of (1.1) in  $\mathbb{R}^N_+$  which vanishes on  $\partial \mathbb{R}^N_+ \setminus \{0\}$ , therefore  $v_0(x) = U_{s^{N-1}_+}(x)$ . This implies

$$u_{\infty}(x) \le u(x) \le |x|^{-\beta_q} \omega_{s_+^{N-1}}(\frac{x}{|x|}) \quad \text{in } B_{\delta}^+ \setminus \{0\}.$$
 (4.92)

We conclude using Proposition 4.1.

#### **4.2** The case p = N

When p=N, the assumption that  $\partial\Omega$  is an hyperplane near 0 can be removed. The proof of the next results is based upon Theorem 3.10. The following result is the extension to the case p=N of Proposition 4.1.

**Proposition 4.3.** Under the assumptions of Theorem 3.10 there exists  $\lim_{k\to\infty} u_k = u_\infty$  which is the unique element of  $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  which satisfies (3.17) in  $\Omega$ , vanishes on  $\partial\Omega \setminus \{0\}$  and such that

$$\lim_{x \to 0} \frac{u_{\infty}(x)}{U_{s_{\perp}^{N-1}}(x)} = 1. \tag{4.93}$$

*Proof.* We denote by  $u_k^{\Omega}$  the unique positive solution of (3.17) satisfying (3.57) obtained in Theorem 3.6. Then

$$T_{\ell}[u_k^{\Omega}] = u_{\ell^{\beta_q - \beta_*}k}^{\Omega^{\ell}},\tag{4.94}$$

because of uniqueness. We denote by  $B:=B_{\frac{1}{2}}(a)$  and  $B':=B_{\frac{1}{2}}(a')$  the two balls tangent to  $\partial\Omega$  at 0 respectively interior and exterior to  $\Omega$  introduced in the proof of Lemma 3.11. Estimate (3.58) implies

$$u_k^{B^{\prime c}} \le u_k^{\Omega} \le u_k^B \tag{4.95}$$

the left-hand side inequality holding in  $\Omega$  and the right-hand side one in B. Therefore

$$T_{\ell}[u_k^{B'^c}] := u_{\ell}^{B'^c\ell} \leq T_{\ell}[u_k^{\Omega}] \leq T_{\ell}[u_k^B] := u_{\ell}^{B\ell} = u_{\ell}^{B\ell}$$

the domains of validity of these inequalities being modified accordingly. Using again (3.58) we obtain

$$T_{\ell'}[u_{k'}^{B'^c}] \le T_{\ell}[u_k^{B'^c}] \quad \text{in } B'^{c\ell'},$$
 (4.97)

for any  $0 < \ell' \le \ell$  and  $\ell'^{\beta_q - \beta_*} k' \le \ell^{\beta_q - \beta_*} k$ . In the same way

$$T_{\ell'}[u_{k'}^B] \ge T_{\ell}[u_k^B] \quad \text{in } B^{\ell},$$
 (4.98)

for any  $0<\ell'\le \ell$  and  $\ell'^{\beta_q-\beta_*}k'\ge \ell^{\beta_q-\beta_*}k$ . Since  $u_k^\Omega\ u_k^B,\ u_k^{B'^c}$  are increasing with respect to k, they converge respectively to  $u_\infty^\Omega\ u_\infty^B,\ u_\infty^{B'^c}$  and there holds for any  $\ell>0$ 

$$T_{\ell}[u_{\infty}^{B^{\prime c}}] \le T_{\ell}[u_{\infty}^{\Omega}] \le T_{\ell}[u_{\infty}^{B}],\tag{4.99}$$

from (4.96) and

(i) 
$$T_{\ell'}[u_{\infty}^{B'^c}] \le T_{\ell}[u_{\infty}^{B'^c}] \qquad \text{in } B'^{c\ell'}$$
(ii) 
$$T_{\ell'}[u_{\infty}^B] \ge T_{\ell}[u_{\infty}^B] \qquad \text{in } B^{\ell}$$
(4.100)

for any  $0 < \ell' \le \ell$ . Notice that , replacing  $\ell$  by  $\ell \ell'$  we can rewrite (4.99) as follows

$$T_{\ell'}[T_{\ell}[u_{\infty}^{B'^c}]] \le T_{\ell'}[T_{\ell}[u_{\infty}^{\Omega}]] \le T_{\ell'}[T_{\ell}[u_{\infty}^B]]. \tag{4.101}$$

Because of the monotonicity with respect to  $\ell$  the following limits exist

$$U^{B'^c} = \lim_{\ell \to 0} T_{\ell}[u_{\infty}^{B'^c}] \quad \text{and} \quad U^B = \lim_{\ell \to 0} T_{\ell}[u_{\infty}^B]. \tag{4.102}$$

By Lemma 2.5 applied with  $\phi(|x|)=|x|^{-\beta_q}$  and since there holds  $u_\infty^B(x)\leq c|x|^{-\beta_q}$  and  $u_\infty^{B'}(x)\leq c|x|^{-\beta_q}$  $c|x|^{-\beta_q}$ , we derive

(i) 
$$|\nabla T_{\ell}[u_{\infty}^B](x)| \le c_2|x|^{-\beta_q-1}$$
  $\forall x \in B^{\ell}$ 

$$\begin{array}{lll} (i) & |\nabla T_{\ell}[u_{\infty}^{B}](x)| \leq c_{2}|x|^{-\beta_{q}-1} & \forall x \in B^{\ell} \\ (ii) & |\nabla T_{\ell}[u_{\infty}^{B}](x) - \nabla T_{\ell}[u_{\infty}^{B}](y)| \leq c_{2}|x|^{-\beta_{q}-1-\alpha}|x-y|^{\alpha} & \forall x,y \in B^{\ell}, \ |x| \leq |y| \\ (iii) & T_{\ell}[u_{\infty}^{B}](x) \leq c_{2}|x|^{-\beta_{q}-1}(\text{dist}\ (x,\partial B^{\ell}))^{\alpha} & \forall x \in B^{\ell}, \end{array}$$
 (4.103)

$$(iii) \quad T_{\ell}[u_{\infty}^{B}](x) \le c_{2}|x|^{-\beta_{q}-1}(\operatorname{dist}(x,\partial B^{\ell}))^{\alpha}$$

and

(i) 
$$|\nabla T_{\ell}[u_{\infty}^{B^{\prime c}}](x)| \le c_2|x|^{-\beta_q - 1}$$
  $\forall x \in B^{\prime c\ell}$ 

(ii) 
$$|\nabla T_{\ell}[u_{\infty}^{B'c}](x) - \nabla T_{\ell}[u_{\infty}^{B'c}](y)| \le c_2|x|^{-\beta_q - 1 - \alpha}|x - y|^{\alpha} \quad \forall x, y \in B'^{c\ell}, |x| \le |y|^{-\beta_q - 1 - \alpha}|x - y|^{\alpha}$$

(i) 
$$|\nabla T_{\ell}[u_{\infty}^{B'^{c}}](x)| \leq c_{2}|x|^{-\beta_{q}-1}$$
  $\forall x \in B'^{c\ell}$   
(ii)  $|\nabla T_{\ell}[u_{\infty}^{B'^{c}}](x) - \nabla T_{\ell}[u_{\infty}^{B'^{c}}](y)| \leq c_{2}|x|^{-\beta_{q}-1-\alpha}|x-y|^{\alpha}$   $\forall x, y \in B'^{c\ell}, |x| \leq |y|$   
(iii)  $T_{\ell}[u_{\infty}^{B'^{c}}](x) \leq c_{2}|x|^{-\beta_{q}-1}(\text{dist }(x,\partial B'^{c\ell}))^{\alpha}$   $\forall x \in B'^{c\ell}.$ 
(4.104)

Thus the sets of functions  $\{T_\ell[u_\infty^B]\}$  and  $\{T_\ell[u_\infty^{B'}]\}$  are equicontinuous in the  $C^1$ -loc topology and by uniqueness, the limit in (4.102) below holds in this topology. Hence  $U^{B'^c}$  and  $U^{B^c}$  are positive solutions of (3.17) in  $\mathbb{R}^N_+$  which vanish on  $\partial \mathbb{R}^N_+ \setminus \{0\}$ . Furthermore  $U^{B'^c} \leq U^{B^c}$  Since for any  $\ell, \ell' > 0$ ,

 $T_{\ell'}[T_{\ell}[u_{\infty}^{B'^c}]] = T_{\ell\ell'}[u_{\infty}^{B'^c}]$ , it follows  $T_{\ell'}[U^{B'^c}] = U^{B'^c}$  and in the same way  $T_{\ell'}[U^B] = U^B$ . This means that  $U^B$  and  $U^{B'^c}$  are self-similar solutions of (3.17) in  $\mathbb{R}^N_+$  and they vanish on  $\partial \mathbb{R}^N_+ \setminus \{0\}$ . Hence

$$U^B = U^{B'^c} = U_{S_+^{N-1}}. (4.105)$$

Applying again Lemma 2.5 to  $u_{\infty}^{\Omega}$  with  $\phi(|x|)=|x|^{-\beta_q}$  we have

- $\begin{array}{ll} (i) & |\nabla T_\ell[u_\infty^\Omega](x)| \leq c_2 |x|^{-\beta_q-1} & \forall x \in \Omega^\ell \\ (ii) & |\nabla T_\ell[u_\infty^\Omega](x) \nabla T_\ell[u_k^\Omega](y)| \leq c_2 |x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in \Omega^\ell, \ |x| \leq |y| \\ (iii) & T_\ell[u_\infty^\Omega](x) \leq c_2 |x|^{-\beta_q-1} (\mathrm{dist}\,(x,\partial\Omega^\ell))^\alpha & \forall x \in \Omega^\ell. \end{array}$ (4.106)

This implies that the set of functions  $\{T_\ell[u_\infty^\Omega]\}_\ell$  is equicontinuous in the  $C^1$ -loc topology of  $\mathbb{R}_+^N$  and there exists a sequence  $\{\ell_n\}\to 0$  and a function U such that  $T_{\ell_n}[u_\infty^\Omega]\to U^\Omega$  in this topology of  $\mathbb{R}_+^N$ , and U is a positive solution of (3.17) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\mathbb{R}_+^N\setminus\{0\}$ . From (4.99) and (4.105) there holds  $U^{\Omega} = U_{S^{N-1}}$  and therefore

$$\lim_{\ell \to 0} T_{\ell}[u_{\infty}^{\Omega}] = U_{S_{+}^{N-1}}.$$
(4.107)

This implies (4.93) and

$$\lim_{r \to 0} r^{\beta_q} u_{\infty}^{\Omega}(r, \sigma) = \omega_{S_+^{N-1}}(\sigma) \tag{4.108}$$

uniformly on compact subsets of  $S_{+}^{N-1}$ .

Up to minor modifications the proof of the next classification theorem is similar to the one of Theorem 4.2.

**Theorem 4.4.** Assume  $N-1 < q < N-\frac{1}{2}$  If  $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$  is a positive solution of (3.17) in  $\Omega$  which vanishes on  $\partial \Omega \setminus \{0\}$ , then we have the following alternative:

- (i) either there exists k > 0 such that (4.74) holds,
- (ii) or (4.75) holds.

# Appendix I: Positive p-harmonic functions in a half space

In this section we prove the following rigidity result.

**Theorem 5.1.** Assume  $1 and <math>u \in C^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+} \setminus \{0\})$  is a positive p-harmonic function which vanishes on  $\partial \mathbb{R}^N_+ \setminus \{0\}$  and such that  $|x|^{\beta_*} u(x)$  is bounded. Then there exists  $k \ge 0$  such that

$$u(x) = k\Psi_*(x) \qquad \forall x \in \mathbb{R}^N_+.$$
 (5.1)

*Proof.* Since  $|x|^{\beta_*}u(x)$  is bounded,  $|x|^{\beta_*+1}\nabla u(x)$  is also bounded and there exists m>0 such that  $u(x) \leq m\Psi_*(x)$  in  $B_{\delta}^+$ . We denote by k the infimum of the c>0 such that  $u(x) \leq c\Psi_*(x)$ . Then

$$0 \le u(x) \le k\Psi_*(x) \qquad \forall x \in \mathbb{R}^N_+ \setminus \{0\}$$
 (5.2)

and we assume that k>0 otherwise u=0. Assume that the graphs over  $\mathbb{R}^N_+$  of the functions  $x\mapsto u(x)$  and  $x\mapsto k\Psi_*(x)$  are tangent at some point  $x_0\in\mathbb{R}^N_+$  or  $x_0\in\partial\mathbb{R}^N_+\setminus\{0\}$ . Since  $\nabla\Psi_*$  never vanishes in  $\overline{\mathbb{R}}^N_+\setminus\{0\}$  it follows from the strong maximum principle or Hopf Lemma that  $u=k\Psi_*$ . If the two graphs are not tangent in  $\overline{\mathbb{R}}^N_+\setminus\{0\}$ , either they are asymptotically tangent at 0, or at  $\infty$ .

(i) In the first case there exists two sequences  $\{k_n\}$  increasing to k and  $\{x_n\} \subset \mathbb{R}^N_+$  converging to zero such that  $\frac{u(x_n)}{\Psi_*(x_n)} = k_n$ . We set  $r_n = |x_n|$  and  $u_{r_n}(x) = r_n^{\beta_*} u(r_n x)$ . Then  $u_{r_n}$  is p-harmonic and positive and  $0 < u_{r_n}(x) \le k |x|^{-\beta_*} \psi_*(\frac{x}{|x|})$ ; therefore

$$|\nabla u_{r_n}(x)| \le C |x|^{-\beta_* - 1} \text{ and } |\nabla u_{r_n}(x) - \nabla u_{r_n}(x')| \le C |x|^{-\beta_* - 1 - \alpha} |x - x'|^{\alpha}$$
 (5.3)

for  $0<|x|\le |x'|$  and some constants C>0 and  $\alpha\in(0,1)$ . Up to a subsequence, we can assume that  $u_{r_n}$  converges to some U in the  $C^1_{loc}$  topology of  $\overline{\mathbb{R}}^N_+\setminus\{0\}$  and  $\frac{x_n}{r_n}\to\xi\in S^{N-1}_+$ . The function U is p-harmonic and positive in  $\mathbb{R}^N_+$  and satisfies  $0\le U\le k\Psi_*$  in  $\mathbb{R}^N_+$  and  $U(\xi)=k\Psi_*(\xi)$  if  $\xi\in S^{N-1}_+$  or  $U_{x_N}(\xi)=k\Psi_{*x_N}(\xi)$  if  $\xi\in\partial S^{N-1}_+$ . It follows from the strong maximum principle or Hopf Lemma that  $U=k\Psi_*$ . Therefore  $u_{r_n}\to k\Psi_*$  and in particular

$$\lim_{r_n \to 0} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1}. \tag{5.4}$$

For any  $\epsilon>0$ , there exists  $n_{\epsilon}\in\mathbb{N}_*$  such that for  $n\geq n_{\epsilon}$ ,  $(k-\epsilon)\Psi_*(x)\leq u(x)\leq (k+\epsilon)\Psi_*(x)$  if  $|x|=r_n$ . This implies  $(k-\epsilon)\Psi_*(x)\leq u(x)\leq (k+\epsilon)\Psi_*$  for  $|x|\geq r_n$  and therefore in  $\mathbb{R}^N$ . Since  $\epsilon$  is arbitrary, we deduce that  $u=k\Psi_*$ .

(ii) if the two graphs are tangent at infinity, there exist two sequences  $\{k_n\}$  increasing to k and  $\{x_n\}$  such that  $r_n = |x_n| \to \infty$  with  $u(x_n) = k_n \Psi_*(x_n)$  and

$$\lim_{r_n \to \infty} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S_+^{N-1}. \tag{5.5}$$

Therefore we look at the supremum of the c>0 such that  $u\geq c\Psi_*$ . If the set of such c is empty, it would mean that

$$\inf_{x \in \mathbb{R}^N_{\perp}} \frac{u(x)}{\Psi_*(x)} = 0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf Lemma imply  $u \equiv 0$ , contradicting (5.5), and this relation prevents also this infimum be achieved at infinity. We are left with the case where there exists a sequence  $\{z_n\} \subset \mathbb{R}^N_+$ , converging to 0, such that

$$\lim_{n \to \infty} \frac{u(z_n)}{\Psi_*(z_n)} = 0. \tag{5.6}$$

By boundary Harnack inequality [2, th 2.11], there exists c > 0 such that

$$c^{-1} \frac{u(z)}{\Psi_*(z)} \le \frac{u(z_n)}{\Psi_*(z_n)} \le c \frac{u(z)}{\Psi_*(z)} \quad \forall z \in \mathbb{R}_+^N \text{ s.t. } |z| = |z_n|$$
 (5.7)

Combining (5.6) and (5.7), we derive that

$$\lim_{n \to \infty} \sup_{|z| = |z_n|} \frac{u(z)}{\Psi_*(z)} = 0, \tag{5.8}$$

Denoting by  $\epsilon_n$  the supremum in the above relation, we obtain that  $u \leq \epsilon_n \Psi_*$  in  $\mathbb{R}^N_+ \setminus B_{\epsilon_n}$  and finally u=0, contradiction. Thus we are left with the case where there exists  $k' \in (0,k]$  which is the supremum of the c>0 such that  $u \geq c\Psi_*$ . In particular  $u \geq k'\Psi_*$ . Remembering that  $u \leq k\Psi_*$  we get k=k', which implies  $u=k\Psi_*$ .

Next we assume that k' < k. Clearly the graphs of u and  $k'\Psi_*$  cannot be tangent in  $\overline{\mathbb{R}}_+^N$ , because of strong maximum principle or Hopf Lemma. They cannot be tangent at infinity because of (5.5). Therefore there exist two sequences  $\{k'_n\}$  increasing to k' and  $\{x'_n\} \subset \mathbb{R}_+^N$  converging to 0 such that  $\frac{u(x'_n)}{\Psi_*(x'_n)} = k'_n$ . As in case (i) we obtain that

$$\lim_{r_n'\to 0} \frac{r_n'^{\beta_*} u(r_n', \sigma)}{\psi_*(\sigma)} = k' \quad \text{uniformly on } S_+^{N-1}, \tag{5.9}$$

where  $r'_n = |x'_n|$ , and finally derive that  $u = k'\Psi_*$ , a contradiction with (5.5). Therefore k = k', which ends the proof.

*Remark.* In the case p=N the result holds under the weaker assumption  $\lim_{|x|\to\infty}u(x)=0$ . This is due to the fact that this condition implies by regularity

$$\lim_{|x| \to \infty} \frac{u(x)}{\omega_{s_{\perp}^{N-1}}(\frac{x}{|x|})} = 0$$

and therefore

$$u(x) \le m\Psi_*(x) \quad \forall x \text{ s.t. } |x| \ge 1,$$

where  $m = \max_{|x|=1} \frac{u(x)}{\omega_{s_+^{N-1}}(\frac{x}{|x|})}$ . Using the inversion  $x \mapsto \frac{x}{|x|^2}$ , we obtain that the estimate  $u \le m\Psi_*$  holds  $\mathbb{R}^N$ , and we conclude by Theorem 5.1.

Remark. We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x| \to \infty} |x|^{-\tilde{\beta}} u(x) = 0, \tag{5.10}$$

were  $\tilde{\beta}$  is the (positive) exponent corresponding to the regular spherical p-harmonic function under the form

$$\tilde{\Psi} = |x|^{\tilde{\beta}} \, \tilde{\psi}(\frac{x}{|x|}),\tag{5.11}$$

see [14], [12]. Note that  $\tilde{\beta} = 1$  when p = N.

### 6 Appendix II: Estimates on $\beta_*$

When N=2 and 1 , it is proved in [9] that

$$\beta_* = \frac{3 - p + 2\sqrt{p^2 - 5p + 7}}{3(p - 1)}. (6.1)$$

Up to now no estimate is known when N>2 except in the cases p=2 where  $\beta_*=N-1$  and p=N where  $\beta_*=1$ , besides the classical one

$$\beta_* > \frac{N-p}{p-1},\tag{6.2}$$

valid when p < N. In this section we prove the following result

**Theorem 6.1.** Assume 1 . Then the following estimates hold:

$$1 \frac{N-1}{p-1},\tag{6.3}$$

$$2$$

*Remark.* It is worth noticing that when p=2 or p=N, there holds  $\beta_*=\frac{N-1}{p-1}$ .

*Proof of Theorem 6.1.* We consider the following set of spherical coordinates in  $\mathbb{R}^N_+$  with  $x=(x_1,...,x_N)$ 

$$x_{1} = r \sin \theta_{N-1} \sin \theta_{N-2} ... \sin \theta_{2} \sin \theta_{1}$$

$$x_{2} = r \sin \theta_{N-1} \sin \theta_{N-2} ... \sin \theta_{2} \cos \theta_{1}$$

$$\vdots$$

$$x_{N-1} = r \sin \theta_{N-1} \cos \theta_{N-2}$$

$$x_{N} = r \cos \theta_{N-1}$$

$$(6.5)$$

with  $\theta_1 \in [0, 2\pi]$  and  $\theta_k \in [0, \pi]$  for k = 2, ..., N - 2 and  $\theta_{N-1} \in [0, \frac{\pi}{2}]$ . Under this representation, a solution  $\omega$  of (3.2) verifies

$$-\frac{1}{\sin^{N-2}\theta_{N-1}} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_{*}^{2}\omega^{2} + \omega_{\theta_{N-1}}^{2} + \frac{1}{\sin^{2}\theta_{N-1}} |\nabla_{\theta'}\omega|^{2} \right)^{\frac{p-2}{2}} \omega_{\theta_{N-1}} \right]_{\theta_{N-1}}$$

$$-\frac{1}{\sin^{2}\theta_{N-1}} div'_{\theta'} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_{*}^{2}\omega^{2} + \omega_{\theta_{N-1}}^{2} + \frac{1}{\sin^{2}\theta_{N-1}} |\nabla_{\theta'}\omega|^{2} \right)^{\frac{p-2}{2}} \nabla_{\theta'}\omega \right]$$

$$= \beta_{*}\Lambda_{\beta_{*}} \left[ \sin^{N-2}\theta_{N-1} \left( \beta_{*}^{2}\omega^{2} + \omega_{\theta_{N-1}}^{2} + \frac{1}{\sin^{2}\theta_{N-1}} |\nabla_{\theta'}\omega|^{2} \right)^{\frac{p-2}{2}} \omega \right]$$
(6.6)

where  $\nabla_{\theta'}$  and  $div'_{\theta'}$  denotes respectively the spherical gradient the divergence in variables  $\theta'=(\theta_1,...,\theta_{N-2})$  parameterizing  $S^{N-2}$  and  $\Lambda_{\beta_*}$  is defined in Introduction. If  $\omega$  is the unique positive solution of (3.2)

(up to homothety), it depends only on  $\theta_{N-1}$  and is  $C^{\infty}$ . For simplicity we set  $\theta_{N-1} = \theta \in [0, \frac{\pi}{2}]$  and  $\omega = \omega(\theta)$  satisfies

Step 1: The eigenvalue identity. Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta} - (N-2)\cot\theta\,\omega_{\theta} - (p-2)\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_{\theta}^2}\omega_{\theta}^2 = \beta_*\Lambda_{\beta_*}\omega. \tag{6.8}$$

By multiplying (6.8) by  $\cos\theta\sin^{N-2}\theta$  and then integrating over  $(0,\frac{\pi}{2})$  we obtain

$$-\int_0^{\frac{\pi}{2}} (\omega_{\theta\theta} + (N-2)\cot\theta\,\omega_{\theta})\cos\theta\sin^{N-2}\theta d\theta = (N-1)\int_0^{\frac{\pi}{2}} \omega\cos\theta\sin^{N-2}\theta d\theta.$$

Noticing that

$$\beta_* \Lambda_{\beta_*} + 1 - N = (p-1) \left(\beta_* - \frac{N-1}{p-1}\right) (\beta_* + 1)$$

we derive

$$(2-p)\int_0^{\frac{\pi}{2}} \frac{\beta_*^2 \omega + \omega_{\theta\theta}}{\beta_*^2 \omega^2 + \omega_{\theta}^2} \omega_{\theta}^2 \omega \cos \theta \sin^{N-2} \theta d\theta$$

$$= (p-1)\left(\beta_* - \frac{N-1}{p-1}\right) (\beta_* + 1) \int_0^{\frac{\pi}{2}} \omega \cos \theta \sin^{N-2} \theta d\theta.$$
(6.9)

Step 2: Elliptic coordinates and reduction. Writing  $\omega(\theta) = \omega(0) + a\theta^2 + o(\theta^2)$ ,  $\omega_{\theta}(\theta) = 2a\theta + o(\theta)$  and  $\omega_{\theta\theta}(\theta) = 2a + o(1)$ , then  $-Na = \beta_* \Lambda_{\beta_*}$ . This implies that  $\omega$  is decreasing near 0. It is immediate that it cannot have a local minimum in  $(0, \frac{\pi}{2})$ , therefore it remains decreasing in the whole interval. We parameterize the ellipse

$$E_r = \{(x, y) : x > 0, y < 0, x^2 + \beta_*^{-2}y^2 = r^2\}$$

by setting

$$\omega = r \cos \phi$$
 and  $-\omega_{\theta} = \beta_* r \sin \phi$  with  $\phi = \phi(\theta)$  and  $r = r(\theta)$ .

The functions r and  $\phi$  are  $C^2$ . Hence  $r_{\theta} \cos \phi - r \sin \phi \phi_{\theta} = -\beta_* r \sin \phi$ , then  $r_{\theta} \cos \phi = (\phi_{\theta} - \beta_*) r \sin \phi$  and  $r_{\theta} = (\phi_{\theta} - \beta_*) r \tan \phi$ . Plugging this into (6.8), we derive

$$-\left((p-1)\frac{r_{\theta}}{r} + \phi_{\theta}\cot\phi + (N-2)\cot\theta\right) + \Lambda_{\beta_*}\cot\phi = 0, \tag{6.10}$$

and finally

$$(p-1)(\phi_{\theta} - \beta_*) \tan \phi + (\phi_{\theta} - \Lambda_{\beta_*}) \cot \phi = (2-N) \cot \theta.$$
(6.11)

Step 3: Estimates on  $\phi_{\theta}$ . We can write (6.11) under the equivalent form

$$(p-1)(\phi_{\theta} - \beta_*) \tan^2 \phi + \phi_{\theta} - \Lambda_{\beta_*} = (2-N) \frac{\cos \theta}{\cos \phi} \frac{\sin \phi}{\sin \theta}.$$
 (6.12)

Since

$$\lim_{\theta \to 0} \frac{\sin \phi}{\sin \theta} = \lim_{\theta \to 0} \frac{\cos \phi}{\cos \theta} \phi_{\theta} = \phi_{\theta}(0),$$

we derive  $\phi_{\theta}(0) - \Lambda_{\beta_*} = (2 - N)\phi_{\theta}(0)$  and thus  $\phi_{\theta}(0) = \frac{\Lambda_{\beta_*}}{N - 1}$ . Similarly, the expansion of  $\phi(\theta)$  near  $\theta = \frac{\pi}{2}$  yields to  $\phi_{\theta}(\frac{\pi}{2}) = \beta_*$ . Since p < N,  $\Lambda_{\beta_*}/(N - 1) < \beta_*$ . We claim now that

$$\phi_{\theta}(\theta) \le \beta_* \qquad \forall \theta \in (0, \frac{\pi}{2}).$$
(6.13)

If  $\Lambda_{\beta_*} \leq \beta_*$ , then

$$(2-N)\cot\theta = (p-1)(\phi_{\theta}-\beta_*)\tan\phi + (\phi_{\theta}-\Lambda_{\beta_*})\cot\phi \ge ((p-1)\tan\phi + \cot\phi)(\phi_{\theta}-\beta_*)$$

thus (6.13) holds.

Next we assume  $\beta_* < \Lambda_{\beta_*}$ . It means  $0 < (p-2)\beta_* - (N-p)$  and thus p > 2. We claim that

$$\beta_* \le \frac{N-2}{p-2}.\tag{6.14}$$

We proceed by contradiction and assume

$$\beta_* > \frac{N-2}{p-2}. (6.15)$$

Then

$$(p-2)\left(\beta_*^2 - \frac{N-p}{p-2}\beta_* - \frac{N-2}{p-2}\right) = (p-2)\left(\beta_* + 1\right)\left(\beta_* - \frac{N-2}{p-2}\right) > 0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*} - \beta_*) > N - 2.$$

Since

$$\lim_{\theta \to \frac{\pi}{2}} \cot \theta \tan \phi = \lim_{\theta \to \frac{\pi}{2}} \frac{\cos \theta}{\cos \phi} = \lim_{\theta \to \frac{\pi}{2}} \frac{\sin \theta}{\phi \theta \sin \phi} = \frac{1}{\beta_*}$$

and

$$(p-1)(\phi_{\theta}(\theta) - \beta_*) \tan^2 \phi = \Lambda_{\beta_*} - \phi_{\theta}(\theta) + (2-N) \frac{\cos \theta}{\cos \phi} \frac{\sin \phi}{\sin \theta}$$
$$= \frac{1}{\beta_*} (\beta_* (\Lambda_{\beta_*} - \beta_*) + 2 - N) + o(1),$$
 (6.16)

thus, if (6.15) holds there exists  $\epsilon > 0$  such that  $\phi_{\theta}(\theta) > \beta_*$  for any  $\theta \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$ . Since  $\phi_{\theta}(0) < \beta_*$ , there exists  $\bar{\theta} \in (0, \frac{\pi}{2})$  such that  $\phi_{\theta}(\bar{\theta}) = \beta_*$  and  $\phi_{\theta\theta}(\bar{\theta}) \geq 0$ . We compute  $\phi_{\theta\theta}$  and get

$$(p-1)\phi_{\theta}(\phi_{\theta}-\beta_{*})\sec^{2}\phi + ((p-1)\tan\phi + \cot\phi)\phi_{\theta\theta} - \phi_{\theta}(\phi_{\theta}-\Lambda_{\beta_{*}})\csc^{2}\phi = (N-2)\csc^{2}\theta$$

Hence, at  $\theta = \bar{\theta}$ 

$$\phi_{\theta\theta}(\bar{\theta})\left((p-1)\tan\phi(\bar{\theta}) + \cot\phi(\bar{\theta})\right) = \beta_*(\beta_* - \Lambda_{\beta_*})\csc^2\phi(\theta) + (N-2)\csc^2\bar{\theta}$$

From (6.11),

$$\cot \phi(\bar{\theta}) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \cot \bar{\theta}$$

Therefore

$$A(\bar{\theta}) := \phi_{\theta\theta}(\bar{\theta}) \left( (p-1) \tan \phi(\bar{\theta}) + \cot \phi(\bar{\theta}) \right)$$

$$= \left( 1 + \left( \frac{N-2}{\Lambda_{\beta_*} - \beta_*} \right)^2 \cot^2 \bar{\theta} \right) \beta_* (\beta_* - \Lambda_{\beta_*}) + (N-2)(1 + \cot^2 \bar{\theta})$$

$$= \beta_* (\beta_* - \Lambda_{\beta_*}) + N - 2 - \left( \frac{(N-2)^2}{\Lambda_{\beta_*} - \beta_*} + 2 - N \right) \cot^2 \bar{\theta}$$

$$= -(p-2)(\beta_* + 1) \left( \beta_* - \frac{N-2}{p-2} \right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_* (N-1) - \Lambda_{\beta_*}) \cot^2 \bar{\theta}$$

$$< 0,$$
(6.17)

using (6.15) and the fact that N > p. This is a contradiction, thus (6.14) holds.

Next, if  $\beta_* < \frac{N-2}{p-2}$ , it follows from (6.16) that there exists  $\epsilon > 0$  such that  $\phi_\theta < \beta_*$  in  $[\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$ . If (6.13) is not true, there exist  $0 < \theta_1 < \theta_2 < \frac{\pi}{2} - \epsilon$  such that  $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$ ,  $\phi_{\theta\theta}(\theta_1) \geq 0$ ,  $\phi_{\theta\theta}(\theta_2) \leq 0$ . Using the equation satisfied by  $\phi_{\theta\theta}$ , we obtain for i = 1, 2,

$$A(\theta_i) = (2 - p)(\beta_* + 1) \left(\beta_* - \frac{N - 2}{p - 2}\right) - \frac{N - 2}{\Lambda_{\beta_*} - \beta_*} (\beta_* (N - 1) - \Lambda_{\beta_*}) \cot^2 \theta_i.$$
 (6.18)

On one hand  $A(\theta_2) \leq 0 \leq A(\theta_1)$ , and on the other

$$A(\theta_2) - A(\theta_1) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) (\cot^2 \theta_1 - \cot^2 \theta_2) > 0,$$

since  $\cot$  is decreasing in  $(0, \frac{\pi}{2})$ ,  $\cot^2 \theta_1 > \cot^2 \theta_2$ , a contradiction. Therefore  $\phi_\theta \leq \beta_*$  in  $(0, \frac{\pi}{2})$ .

Finally, if  $\beta_* = \frac{N-2}{p-2}$  and the maximum of  $\phi_{\theta}$  on  $[0, \frac{\pi}{2})$  is larger than  $\beta_*$  and achieved at some  $\bar{\theta} < \frac{\pi}{2}$  the exists  $\theta_1 < \bar{\theta}$  such that  $\phi_{\theta}(\theta_1) = \beta_*$  and  $\phi_{\theta\theta}(\theta_1) \ge 0$ . In that case

$$0 \le A(\theta_1) = -\frac{N-2}{\Lambda_{\beta_1} - \beta_*} (\beta_*(N-1) - \Lambda_{\beta_*}) \cot^2 \theta_1 < 0$$

which is again a contradictions.

Step 4: End of the proof. Since  $r^2 = \beta_*^2 \omega^2 + \omega_\theta^2$ ,  $r_\theta = r(\phi_\theta - \beta_*) \tan \phi$ , we have

$$rr_{\theta} = (\beta_*^2 \omega + \omega_{\theta\theta}) \omega_{\theta} = r(\phi_{\theta} - \beta_*) \tan \phi.$$

Since  $\omega_{\theta} < 0$  on  $(0, \frac{\pi}{2})$ , it follows from Step 3 that  $\beta_*^2 \omega + \omega_{\theta\theta} \ge 0$  and thus

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2 \omega + \omega_{\theta\theta}}{\beta_*^2 \omega^2 + \omega_{\theta}^2} \omega_{\theta}^2 \omega \cos \theta \sin^{N-2} \theta d\theta > 0,$$

since the integrand cannot be identically 0. The conclusion follows from (6.9).

Remark. Since  $\omega_{\theta}(\frac{\pi}{2}) = -c^2 < 0$ , it follows  $\omega(\theta) = -\omega_{\theta}(\theta) \cot \theta + O(\frac{\pi}{2} - \theta)$  as  $\theta \to \frac{\pi}{2}$ , and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2 \omega + \omega_{\theta\theta}}{\beta_*^2 \omega^2 + \omega_{\theta}^2} \omega_{\theta}^2 = (\beta_*^2 \omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_*\Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1))$$
 as  $\theta \to \frac{\pi}{2}$ 

and since  $\Delta'\omega := \omega_{\theta\theta} + (N-2)\cot\theta\,\omega_{\theta}$ 

$$-\Delta'\omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1}\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}.$$

Because  $\omega$  is  $C^{\infty}$  we obtain finally

$$\left|\Delta'\omega\right| \le c\omega,\tag{6.19}$$

for some c > 0.

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