A NOTE ON THE STABILITY OF THE CHEEGER CONSTANT OF N-GONS

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ABSTRACT. The regular N-gon provides the minimal Cheeger constant in the class of all N-gons with fixed volume. This result is due to a work of Bucur and Fragalà in 2014. In this note, we address the stability of their result in terms of the L^1 distance between sets. Furthermore, we provide a stability inequality in terms of the Hausdorff distance between the boundaries of sets in the class of polygons having uniformly bounded diameter. Finally, we show that our results are sharp, both in the exponent of decay and in the notion of distance between sets.

Given a Borel set $\Omega \subset \mathbb{R}^n$ with finite measure, the *Cheeger constant* of Ω is defined by

$$h(\Omega) = \inf \left\{ \frac{P(E)}{|E|} \ \Big| \ E \text{ measureable, } E \subseteq \Omega \right\}.$$

One can show that the infimum above is always attained, and a set $E \subseteq \Omega$ such that $\frac{P(E)}{|E|} = h(\Omega)$ is called a *Cheeger set* of Ω.

The Cheeger constant is related to the first Dirichlet eigenvalue of the p-Laplacian of a set $\Omega \subset \mathbb{R}^n$:

$$\lambda_p(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^p \, \mathrm{d}x \ \Big| \ u \in W_0^{1,p}(\Omega), \ \|u\|_{L^p(\Omega)} = 1 \right\}. \tag{1}$$

Indeed, for every set Ω and every $p > 1$,

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p \quad \text{and} \quad \lim_{p \to 1} \lambda_p(\Omega) = h(\Omega). \tag{2}$$

See, for example, [KN08] for more details about the relation between the Cheeger constant and the first Dirchlet eigenvalue of the p-Laplacian or [Bu10] for more details about spectral problems. It is a well known fact that for every $p > 1$, the ball provides the minimal value for λ_p among all sets with fixed volume. The *Cheeger inequality* expresses this fact for the limit case $p = 1$:

$$|\Omega|^{\frac{1}{n}} h(\Omega) \geq h(B), \quad \text{whenever } \Omega \text{ is a Borel set,} \tag{3}$$

where B denotes the unit-area ball of $\mathbb{R}^n$. A celebrated conjecture due to Pólya and Szegő, states that if one considers the minimization problem for λ_p among the class of N-gons with fixed area then the solution is given by the regular N-gon. In [PS51] the authors show the validity of the conjecture for $p = 2$ in the case $N = 3$ and $N = 4$.

Recent progress has been made in [BF14], where the authors prove the validity of the conjecture for the limit case $p = 1$ (i.e. the Cheeger constant). In particular, they prove that for every fixed N, the regular N-gon minimizes the Cheeger constant among all N-gons with fixed volume:

$$\sqrt{|\Omega|} h(\Omega) \geq h(\Omega_0) \quad \text{for } \Omega \in P_N, \tag{4}$$

where here and in the sequel P_N is the set of all N-gons in $\mathbb{R}^2$ and Ω_0 denotes the unit-area regular N-gon. Our goal in this short note is to address the stability of (4) in the class of N-gons. Namely, our theorem is the following.

Theorem 1. *For each $N \geq 3$, there exists C and $\eta > 0$ depending only on N such that if $\Omega \in P_N$ with $|\Omega| = 1$ and $h(\Omega) - h(\Omega_0) \leq \eta$, then there exists a rigid motion ρ of $\mathbb{R}^2$ such that*

$$\mathrm{hd}(\partial \Omega, \partial \rho \Omega_0)^2 \leq C(h(\Omega) - h(\Omega_0)). \tag{5}$$

As a consequence, for every $N \geq 3$, there exists some constant C_1 depending only on N such that for any $\Omega \in P_N$ with $|\Omega| = 1$, there exists a rigid motion ρ of $\mathbb{R}^2$ such that

$$|\Omega \Delta \rho \Omega_0|^2 \leq C_1 \left(h(\Omega) - h(\Omega_0) \right). \tag{6}$$

Moreover, for every $M > 0$, there exists a constant C_2 depending on N and M such that for any $\Omega \in P_N$ with $\mathrm{diam}(\Omega) < M$ and $|\Omega| = 1$, the following holds:

$$\mathrm{hd}(\partial \Omega, \partial \rho \Omega_0)^2 \leq C_2 \left(h(\Omega) - h(\Omega_0) \right), \tag{7}$$

where ρ is a suitable rigid motion of $\mathbb{R}^2$.

Here $\mathrm{hd}(\cdot, \cdot)$ denotes the Hausdorff distance between sets. As we will show below, in general we cannot expect to have stability in the form of (5) when $h(\Omega)$ is far from being optimal, as the Cheeger constant does not detect the presence of small tentacles (see Figure 1), even in the class of N-gons.

Before moving on to the proof of Theorem 1, for sake of completeness we briefly retrieve results, definitions, and terminology that we use from [KL06] and [BF14]. For a convex subset $\Omega \subseteq \mathbb{R}^2$ there exists a unique Cheeger set C with the the following characterization: $C = \Omega^{(i)} \oplus B_R$, where $R = \frac{1}{h(\Omega)}$ and

$$\Omega^{(i)} = \{x \in \Omega \mid \mathrm{dist}(x, \partial \Omega) > R\}$$

is called the *inner parallel set of* Ω. Here, given two sets E and F, $E \oplus F = \{x + y \mid x \in E, y \in F\}$. An N-gon Ω is called *Cheeger regular* if the boundary of its Cheeger set touches all N sides of Ω. In [KL06], it is shown that if a set $\Omega \in P_N$ is convex and Cheeger regular, then

$$h(\Omega) = \frac{P(\Omega) + \sqrt{P(\Omega)^2 - 4\tau(\Omega)|\Omega|}}{2|\Omega|}, \tag{8}$$

where

$$\tau(\Omega) = \sum_{i=1}^{N} \left[\tan\left(\frac{\pi - \gamma_i}{2} \right) - \left(\frac{\pi - \gamma_i}{2} \right) \right]$$

and where $\{\gamma_i\}_{i=1}^{N}$ are the inner angles of the polygon. In the general case, the stability of the Cheeger inequality was obtained in [FMP10] as a consequence of the quantitative isoperimetric inequality for sets of finite perimeter ([FMP09], [FuMP06], [CL10]). As in the general case, Theorem 1 is a consequence of the stability of the polygonal isoperimetric inequality, proved in [IN14]. More precisely, we make use of the following proposition proved in [CM14] as a consequence of [IN14].

Proposition 2. *For every $N \geq 3$, there exists a positive constant C depending only on N with the following property: for every convex unit-area $\Omega \in P_N$, there exists a rigid motion ρ of $\mathbb{R}^2$ such that*

$$\mathrm{hd}(\partial \Omega, \partial \rho \Omega_0)^2 \leq C \left(P(\Omega)^2 - P(\Omega_0)^2 \right). \tag{9}$$

Finally, we recall the isoperimetric inequality for convex polygons, which states that, for every convex polygon Ω,

$$\frac{P(\Omega)^2}{4|\Omega|} \geq \tau(\Omega) + \pi, \tag{10}$$

with equality if and only if Ω is a circumscribed polygon (namely a polygon for which the largest inscribed circle touches all of his sides).

In their paper, Bucur and Fragalà remark that in the case of simple convex polygons, the proof of their theorem becomes straightforward by exploiting the characterization of the Cheeger set for a convex bounded Borel set Ω contained in [KL06]. We are able to reduce to this case with a compactness argument, then, by tracing through their argument and making use of Proposition 2, we can easily achieve the proof.

Proof of Theorem 1. Notice that $|\Omega \Delta \rho \Omega_0| \leq 2$ for all $\Omega \subset \mathbb{R}^2$ with $|\Omega| = 1$ and for every rigid motion ρ of $\mathbb{R}^2$, so inequality (6) follows immediately from (5) by choosing the constant to be sufficiently large. Furthermore, (7) is a consequence of (5). Indeed,

$$\mathrm{hd}(\partial E, \partial F) \leq \mathrm{diam}(E) + \mathrm{diam}(F),$$

for two generic closed sets E and F in $\mathbb{R}^2$ such that

$$\partial E \cap \partial F \neq \emptyset. \tag{11}$$

Up to a translation, (11) holds for Ω and Ω_0; this and the boundedness of the diameter imply that

$$\mathrm{hd}(\partial\Omega, \partial\Omega_0) \leq M + \mathrm{diam}(\Omega_0),$$

so by choosing the constant large enough depending on M, we obtain (7). Therefore let us focus on the proof of (5). We divide the proof into two steps.

Step 1: Qualitative stability. We prove that for every fixed $\varepsilon > 0$ there exists η_0 such that if Ω is a unit-area N-gon with $h(\Omega) - h(\Omega_0) < \eta_0$, then for some rigid motion ρ of $\mathbb{R}^2$, we have $\mathrm{hd}(\partial\Omega, \partial\rho\Omega_0) < \varepsilon$. Indeed, consider a sequence of unit-area N-gons Ω_k such that $h(\Omega_k) \to h(\Omega_0)$. Following the compactness argument in Proposition 9 in [BF14], we obtain a subsequence of Ω_k converging in L^1_{loc} to a limit Π with $|\Pi| \leq 1$. We let $\Pi_0 = \sqrt{|\Pi|^{-1}}\Pi$. Then, as shown in [BF14],

$$h(\Pi_0) = \sqrt{|\Pi|}h(\Pi) \leq \liminf_{k\to\infty} \sqrt{|\Pi|}h(\Omega_k) = h(\Omega_0).$$

On the other hand, by minimality,

$$h(\Omega_0) \leq h(\Pi_0).$$

Therefore, $\Pi_0 = \Omega_0$ up to a rigid motion, and so Π is a regular N-gon. Consider a ball B_R such that $\Pi \subset\subset B_R$ and suppose that there is some subsequence of Ω_k such that each Ω_k has a vertex not contained in B_R. Then we conclude that Π must have at most $N-1$ vertices, contradicting the fact that Π is a regular N-gon. Thus $\Omega_k \subset B_R$ for k large, and, in particular, $\Omega_k \to \Pi$ in L^1. Therefore $|\Pi| = 1$ and thus $\Pi = \rho\Omega_0$ for some rigid motion ρ. If a bounded sequence of N-gons converges in L^1 to an N-gon, then the boundaries converge in the Hausdorff distance, so

$$\mathrm{hd}(\partial\Omega_k, \partial\rho\Omega_0) \to 0$$

and we achieve the proof of Step 1.

Step 2: Quantitative stability. Fix $\varepsilon > 0$, let η_0 be the constant given from the Step 1 and let Ω be a unit-area N-gon such that $h(\Omega) - h(\Omega_0) < \eta$ for some $\eta < \eta_0$ to be fixed. Thanks to Step 1 and because Ω_0 is convex, up to choosing an ε small enough, Ω must be as well. Moreover, since Ω_0 is Cheeger regular, Ω will also be Cheeger regular up to further decreasing ε and with a suitable choice of η. Indeed, we note that a sufficient condition for an N-gon to be Cheeger regular is for its inner parallel set to also be an N-gon. The inner parallel set $\Omega^{(i)}$ have boundary given by

$$\partial\Omega^{(i)} = \left\{ x \in \Omega \ \middle| \ \mathrm{d}(x, \partial\Omega) = \frac{1}{h(\Omega)} \right\}.$$

Therefore, since $\mathrm{hd}(\partial\Omega, \partial\Omega_0) \leq \varepsilon$ and $h(\Omega) - h(\Omega_0) < \eta$ up to a rigid motion of $\mathbb{R}^2$:

$$\mathrm{hd}\left(\partial\Omega^{(i)}, \partial\Omega_0^{(i)}\right) \leq \omega(\varepsilon + \eta),$$

for a continuous function ω such that $\omega(0_+) = 0$. In particular, since the inner parallel set of Ω_0 is an N-gon, by choosing ε and $\eta < \eta_0(\varepsilon)$ small enough, the previous arguments and Step 1 prove that Ω is a convex, Cheeger regular N-gon. We are thus in the position to exploit (8), the characterization of the Cheeger constant for convex, Cheeger regular sets. We obtain the following:

$$\begin{aligned} 2(h(\Omega) - h(\Omega_0)) &= P(\Omega) + \sqrt{P(\Omega)^2 - 4\tau(\Omega)} - P(\Omega_0) + \sqrt{P(\Omega_0)^2 - 4\tau(\Omega_0)} \\ &= P(\Omega) - P(\Omega_0) + \sqrt{P(\Omega)^2 - 4\tau(\Omega)} - \sqrt{4\pi} \\ &\geq P(\Omega) - P(\Omega_0), \end{aligned} \tag{12}$$

where the final two lines both follow from (10). Moreover, we have

$$P(\Omega) - P(\Omega_0) \geq c(P(\Omega)^2 - P(\Omega_0)^2) \tag{13}$$

for ε small enough, because $\mathrm{hd}(\partial\Omega, \partial\Omega_0) \leq \varepsilon$ and thus $P(\Omega) < 2P(\Omega_0)$. Finally, Proposition 2 combined with (12) and (13) yields

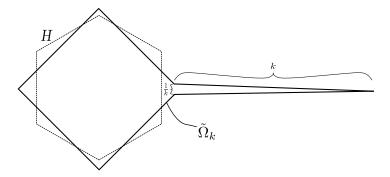$$h(\Omega) - h(\Omega_0) \geq c \ \mathrm{hd}(\partial\Omega, \partial\rho\Omega_0)^2$$

FIGURE 1. A sequence of hexagons with Cheeger constants uniformly bounded and diameters unbounded. In this case, the Cheeger deficit cannot control the Hausdorff distance from a unit-area regular hexagon.

for $c > 0$ where c depends only on N. ☐

Remark 3. We point out that (5) fails to hold when $h(\Omega) - h(\Omega_0)$ is large, as the Cheeger constant does not detect behavior of the set away from where the Cheeger set lies. Consider $N = 6$ for the sake of simplicity, and consider the following construction as in Figure 1. First obtain $\tilde{\Omega}_k$ by cutting away a triangle of base $\frac{1}{k}$ from a corner of a unit-area square S and replacing it with another triangle having the same base but with height k. Set $\Omega_k := \frac{\tilde{\Omega}_k}{|\tilde{\Omega}_k|}$ and note that their Cheeger constants are uniformly bounded. Indeed, if C_0 is the Cheeger set of S, then by construction, $C_0 \subset \tilde{\Omega}_k$ for all k, and thus $h(\tilde{\Omega}_k) \leq h(S)$, so

$$h(\Omega_k) = \sqrt{|\tilde{\Omega}_k|} h(\tilde{\Omega}_k) \leq \sqrt{|\tilde{\Omega}_k|} h(S) \leq \sqrt{\frac{3}{2}} h(S).$$

On the other hand, clearly

$$\inf\{\mathrm{hd}(\partial\Omega_k, \partial\rho\Omega_0) \mid \rho \text{ rigid motion of } \mathbb{R}^2\} \to \infty.$$
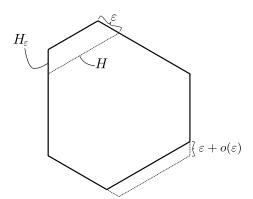

FIGURE 2. We choose a small perturbation of the regular hexagon H in an area-fixing way. Some easy calculations show that $h(H_\varepsilon) - h(H) \approx c_0 A(\varepsilon)^2 = c_1 |H_\varepsilon \Delta H|^2 \approx c_2 \mathrm{hd}(\partial H_\varepsilon, \partial H)^2$, where $A(\varepsilon)$ denotes the areas bounded by the dotted lines. Thus yields the sharpness of (6) and (7).

Remark 4. We also underline that the exponent 2 in (6) and (7) cannot be improved, due to the sharpness of the quantitative isoperimetric inequality for polygons. Indeed, suppose that, for example, inequality (6) were to hold as

$$f(|\Omega \Delta \rho \Omega_0|) \leq C_1 (h(\Omega) - h(\Omega_0)), \tag{14}$$

for some $f(x)$. Then by testing the inequality on a small area-fixing perturbations of the regular N-gon (as in figure 2) we immediately find that $f(|\Omega_\varepsilon \Delta \rho \Omega_0|) \leq c|\Omega_\varepsilon \Delta \rho \Omega_0|^2$.

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