On the Dimension of Finite Point Sets I. An Improved Incidence Bound for Proper 3D sets.

György Elekes

Abstract

We improve the well-known Szemerédi–Trotter incidence bound for proper 3–dimensional point sets (defined appropriately).

1 Introduction

1.1 The Szemerédi–Trotter incidence bound in the plane

The following estimate was conjectured by Erdős and proven by Szemerédi–Trotter for incidences of points and lines.

Proposition 1.1 (Szemerédi–Trotter Theorem) *The maximum number $I(n, m)$ of incidences between n points and m straight lines in the Euclidean plane satisfies*

$$I(n, m) = O(n^{2/3}m^{2/3} + n + m).$$

As a special case, given a set $\mathcal{P} \subset \mathbb{R}^2$ of n points, the number of k–rich lines (which contain at least k points of $\mathcal{P}$) is bounded by

$$C \cdot \max\Big\{\frac{n^2}{k^3}, \frac{n}{k}\Big\},$$

for an absolute constant $C > 0$ and any $2 \le k \le n$.

Proof: see [ST83] and, for a simple proof, [Szé97].
It is also true that these bounds are sharp, apart from the constant factors.

How much does the situation change if we consider point sets (and straight lines) in higher dimensional spaces? On the one hand, the foregoing bounds still apply, as shown by a projection to a generic plane. On the other hand, no better bound can be stated, since any planar point set which attains the order of magnitude in the Szemerédi–Trotter bounds, can be considered as a subset of $\mathbb{R}^3$ (or that of $\mathbb{R}^d$). However, one might have the feeling that the real question would be to consider *proper* 3–dimensional (or d–dimensional) sets.

The main goal of this paper is to improve the Szemerédi–Trotter bound(s) for proper 3–dimensional point sets (defined appropriately in the next section).

1.2 Proper d–dimensional point sets

Let H be a (finite) set of n planes in $\mathbb{R}^3$ (or, in general, of hyperplanes in $\mathbb{R}^d$). They cut the space into at most $\binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0} \sim n^3$ open convex cells (and into $\leq \sum_{i=0}^{d} \binom{n}{i} \sim n^d$ in $\mathbb{R}^d$), with equality iff H is in general position, i.e., if any three (in general, any d) share exactly one common point.

Definition 1.2 A set of N points is *proper d–dimensional up to a constant factor C,* (for short, "proper d–D") if it can be cut into singletons by at most $C\sqrt[d]{N}$ appropriate hyperplanes.

1.3 The main result

Theorem 1.3 *Assume that a set $\mathcal{P} \subset \mathbb{R}^3$ of N points is proper 3–dimensional, up to a constant factor C. Then*

(i) for any $k \leq C\sqrt[3]{N}$, the number of k–rich lines is

$$O\left(\frac{N^2}{k^4}\right);$$

(ii) more generally, for any $k \leq C\sqrt[3]{N}$, the number of incidences between $\mathcal{P}$ and the k–rich lines is $O(N^2/k^3)$;

(iii) the number of incidences between any M straight lines and the N points of $\mathcal{P}$ is

$$I = \begin{cases} O(M), & \text{if } N^2 < M; \\ O(N^{1/2}M^{3/4}), & \text{if } N^{2/3} < M \leq N^2; \\ O(N^{1/3}M), & \text{if } M \leq N^{2/3}. \end{cases}$$

In other words,

$$I = O\Big(\min\big\{M + N^{1/2}M^{3/4}\ ,\ N^{1/3}M\big\}\Big).$$

It is also true that these bounds give the best possible order of magnitude.

The forthcoming Section 2 provides examples which show that — apart from constant factors – our upper bounds cannot be improved. After some preparatory observations (including our Main Lemma 3.8) in Section 3, the proof of Theorem 1.3 comes in Section 4.

1.4 Micha Sharir's "joints".

Given a set of m straight lines $\mathcal{L} = \{L_1, L_2, \ldots, L_m\}$, a *joint* is a point where at least three non-coplanar L_i meet. In what follows we denote by $\mathcal{J}(\mathcal{L})$ the set of joints of $\mathcal{L}$.

It was conjectured by Micha Sharir [**?**] that if $|\mathcal{L}| = m$ then

$$|\mathcal{J}(\mathcal{L})| \leq C \cdot m^{3/2},$$

for an absolute constant C.

Here we show the validity of this conjecture if $\mathcal{J}(\mathcal{L})$ is proper three–dimensional — which is true for all known examples with many joints. Actually, we prove somewhat more (though the original problem still remains open).

Theorem 1.4 *If $|\mathcal{L}| = m$ and $\mathcal{J}_0$ is a proper three–dimensional subset (up to a constant factor $C \geq 1$) of the intersections — not necassarily of the joints! — then*

$$|\mathcal{J}_0| \leq C^{3/2} m^{3/2}.$$

Proof: Write $n := |\mathcal{J}_0|$ and consider $C\sqrt[3]{n}$ planes which cut $\mathcal{J}_0$ into singletons. Then each $L_i \in \mathcal{L}$ can pass through at most $C\sqrt[3]{n} + 1 \leq 2C\sqrt[3]{n}$ points of $\mathcal{J}_0$, yielding a total of $m \cdot 2C\sqrt[3]{n}$ incidences. Since each points in $\mathcal{J}_0$ is incident upon ≥ 2 lines, we have $2n \leq m \cdot 2C\sqrt[3]{n}$, whence $n \leq C^{3/2} m^{3/2}$. ∎

2 Lower bounds

Example 2.1 For the $N = n^3$ points of an $n \times n \times n$ cube lattice, and any $2 \leq k \leq n$,

(a) the number of k–rich lines is $\Omega(N^2/k^4)$;

(b) the number of incidences between the lattice points and the k–rich lines is $\Omega(N^2/k^3)$.

[In general, for any $d \geq 2$ and the $N = n^d$ points of an $n \times n \times \ldots \times n$ cube lattice in $\mathbb{R}^d$, we have at least $\Omega(N^2/k^{d+1})$ k–rich lines which, of course, produce at least $\Omega(N^2/k^d)$ incidences.]

Proof: It suffices to show part (a) since it immediately implies part (b).
Consider the $N = n^3$ points of $\{1, 2, \ldots, n\}^3$. First we construct $\Omega(n^3/k^3)$ straight lines which all go through the origin $(0, 0, 0)$, such that each of them contains approximately k points of the lattice. These lines will be defined in terms of their points (u, v, w) which is closest to the origin.

We let the coordinates of these points range through

$$u = \frac{n}{4k}, \ldots, \frac{n}{2k};$$
$$v = 1, \ldots, \frac{n}{2k};$$
$$w = 1, \ldots, v$$

such that $\gcd(v, w) = 1$.

For each such (u,v,w), the straight line which passes through it and the origin, will also pass through at least $2k$ and at most $4k$ points of the cube lattice. Moreover, the number of such points (u,v,w) is

$$\frac{n}{4k} \cdot \sum_{v=1}^{n/(2k)} \phi(v) = \frac{n}{4k} \cdot \Theta\left(\frac{n^2}{k^2}\right) = \Theta\left(\frac{n^3}{k^3}\right),$$

where ϕ — i.e., Euler's function — gives the number of $w \in \{1 \dots v\}$ which are coprime to v, and we used the well-known fact that $\sum_{i=1}^{m} \phi(i) = \Theta(m^2)$.

Now we shift these lines by each of the vectors $(a,b,c) \in \{1,2,\dots,n/2\}^3$. Then each new line will still pass through at most $4k$ and, this time, at least k lattice points. Of course, these $(n/2)^3 \cdot \Theta(n^3/k^3)$ lines are not all distinct. However, each occurs with multiplicity at most $4k$ whence

$$\text{number of } k\text{–rich lines } \geq \left(\frac{n}{2}\right)^3 \cdot \Theta\left(\frac{n^3}{k^3}\right) \cdot \frac{1}{4k} = \Theta\left(\frac{n^6}{k^4}\right) = \Theta\left(\frac{N^2}{k^4}\right). \blacksquare$$

Remark 2.2 A similar construction, with coordinates $u_1, u_2, \dots, u_d$ (in place of u,v,w), ranging through

$$\begin{aligned} u_1 &= \frac{n}{4k}, \dots, \frac{n}{2k}; \\ u_2, u_3, \dots, u_{d-1} &= 1, \dots, \frac{n}{2k}; \\ u_d &= 1, \dots, u_{d-1} \end{aligned}$$

such that $\gcd(u_{d-1}, u_d) = 1$, gives $\Theta(N^2/k^{d+1})$ lines for a d–dimensional $n \times n \times \dots \times n$ cube lattice with $N = n^d$ points.

Example 2.3 To show that the bounds in part (iii) of the Main Theorem 1.3 are best possible for all M and N, we again consider the $N = n^3$ points of and $n \times n \times n$ cube lattice.

(a) If $M > N^2/16$, we just draw M lines, each through at least one point of the lattice.

(b) For $M < N^{2/3} = n^2$, we pick any M of the n^2 lattice lines parallel to, say, the x–axis.

(c) If $N^2/16 > M \geq N^{2/3}$ then we define

$$2 \leq k \overset{\text{def}}{=} \frac{N^{1/2}}{M^{1/4}} \leq \frac{N^{1/2}}{(N^{2/3})^{1/4}} = N^{1/3} = n$$

and consider the k–rich lines of the lattice. According to Example 2.1.(b), the number of incidences between these lines and the lattice points is

$$\Omega\left(\frac{N^2}{k^3}\right) = \Omega\left(\frac{N^2}{N^{3/2}/M^{3/4}}\right) = \Omega(N^{1/2}M^{3/4}). \blacksquare$$

3 Arrangements of planes in $\mathbb{R}^3$.

3.1 Distances and neighborhoods.

Let H be a (finite) set of n planes in $\mathbb{R}^3$ (or, in general, of hyperplanes in $\mathbb{R}^d$), as in Section 1.2. If they are in general position then — as it was already mentioned there — they cut the space into at most $\sim n^3$ open convex cells (and into $\sim n^d$ in $\mathbb{R}^d$). The set of these cells, together with their vertices, edges, and faces, is called the *arrangement* defined by H. We shall denote it by $\mathcal{A}(H)$.

For two cells $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{A}(H)$, a natural notion of distance is

$$\operatorname{dist}(\mathcal{C}_i, \mathcal{C}_j) \stackrel{\text{def}}{=} \#\{h \in H \ ; \ h \text{ separates } \mathcal{C}_i \text{ and } \mathcal{C}_j\}.$$

A spectacular representation is the following: pick two points $P_i \in \mathcal{C}_i$, $P_j \in \mathcal{C}_j$ and connect them by a straight line segment. Then the foregoing distance equals the number of $h \in H$ which cut the segment $\overline{P_i P_j}$.

It is easy to see that "dist" is a metric, i.e. it satisfies the triangle inequality.

Our goal is to bound from above — in terms of $|H|$ — the number of pairs $(\mathcal{C}_i, \mathcal{C}_j)$ whose distance is at most a given $\varrho > 0$. This will be achieved in the Main Lemma 3.8.

To this end, we define the ϱ–neighborhood of a cell $\mathcal{C}_j$ by

$$B_\varrho(\mathcal{C}_j) = \{\mathcal{C}_i \in \mathcal{A}(H) \ ; \ \operatorname{dist}(\mathcal{C}_i, \mathcal{C}_j) \le \varrho\},$$

and we note that the number of (ordered) "ϱ–close pairs" mentioned above equals

$$\sum_{\mathcal{C}_j \in \mathcal{A}(H)} |B_\varrho(\mathcal{C}_j)|.$$

The next two subsections recall two well-known results, related to the foregoing ϱ–neighborhoods in some sense. Our main tool (Lemma 3.8.) comes after these.

3.2 Zones

For any (hyper)plane $h \in H$, the *zone* of h is the set of cells which "touch" h, i.e., which have a face on h. Also the *$\le \varrho$–zone* of h can be defined as the set of cells $\mathcal{C}_j$ for which there is another cell $\mathcal{C}_i$ in the zone of h for which $\operatorname{dist}(\mathcal{C}_i, \mathcal{C}_j) \le \varrho$. (In this sense the 0–zone coincides with the original zone of h.)

Theorem 3.1 (Matoušek) *The number of vertices (and, consequently, that of the cells, faces, edges) in the $\le \varrho$–zone of any $h \in H$ is $O(\varrho |H|^2)$ in $\mathbb{R}^3$ and $O(\varrho |H|^{d-1})$ in $\mathbb{R}^d$.*

Proof: see [Mat88] for the bound on the number of vertices. The rest is implied by the fact that — according to the "general position" assumption — each other object has a vertex furthest from h and each vertex is counted a bounded number of times (which, of course, depends on the dimension).

We also re-state this result in terms of ϱ–neighborhoods. To do so, we shall say that a (hyper)plane $h \in H$ and a ϱ–neighborhood $B_\varrho(\mathcal{C}_j)$ are *incident upon each other* if h contains at least one face of at least one cell $\mathcal{C}_i \in B_\varrho(\mathcal{C}_j)$. (It does not matter whether this face is located in the "interior" of the ϱ–neighborhood or on its boundary.)

The next result says that only $O(\varrho)$ (hyper)planes are incident upon an "average" neighborhood. More precisely, we have the following.

Corollary 3.2 *For each $\mathcal{C}_j \in \mathcal{A}(H)$, denote by n_j the number of $h \in H$ which are incident upon $B_\varrho(\mathcal{C}_j)$. Then*

$$\sum_{\mathcal{C}_j \in \mathcal{A}(H)} n_j = O(\varrho |H|^3)$$

in $\mathbb{R}^3$ and $O(\varrho |H|^d)$ in $\mathbb{R}^d$.

Proof: Note that h is incident upon $B_\varrho(\mathcal{C}_j)$ iff $\mathcal{C}_j$ is in the $\leq \varrho$–zone of h. The rest is just double–counting, using Theorem 3.1.

We also state yet another consequence which can be considered as the "younger brother" (i.e., 2–dimensional version) of the forthcoming Main Lemma 3.8.

Corollary 3.3 *In $\mathbb{R}^2$ we have*

$$\sum_{\mathcal{C}_j \in \mathcal{A}(H)} |B_\varrho(\mathcal{C}_j)| = O(\varrho^2 |H|^2).$$

Proof: Instead of summing the number of cells $\mathcal{C}_i$ in each $B_\varrho(\mathcal{C}_j)$, we double–count the triples $(\mathcal{C}_j, h, \mathcal{C}_i)$ such that $h \in H$ bounds $\mathcal{C}_i \in B_\varrho(\mathcal{C}_j)$ and separates it from $\mathcal{C}_j$.
On the one hand, the number of these triples cannot be smaller than the sum in question (each pair of cells is counted at least once).
On the other hand, for a fixed straight line $h \in H$ and a $\mathcal{C}_j$ in the $\leq \varrho$–zone of h, the number of the $\mathcal{C}_i$ to be counted is at most $2\varrho + 1$. (Any two such cells are at distance $\leq 2\varrho$ apart, along the line h.) Thus, using Theorem 3.1 for $d = 2$, we have

$$\sum_{\mathcal{C}_j \in \mathcal{A}(H)} |B_\varrho(\mathcal{C}_j)| \leq \#\text{ of triples } \leq |H| \cdot O(\varrho |H|) \cdot (2\varrho + 1) = O(\varrho^2 |H|^2). \blacksquare$$

As for the second moment $\sum |B_\varrho(\mathcal{C}_j)|^2$, it may not always be bounded by a quadratic function of $|H|$ (e.g., if the lines all surround a regular polygon then each of its $|H|$ triangular neighbours has $\geq |H|$ other cells in its $\varrho = 2$–neighborhood.).

Problem 3.4 *Let $\mathcal{A}(H)$ be a simple arrangement in $\mathbb{R}^2$. Is it true that it can be refined to an $\mathcal{A}(H^+)$ by adding $O(\sqrt{|H|})$ new straight lines such that $\sum_{\mathcal{C}_j \in \mathcal{A}(H^+)} |B_\varrho(\mathcal{C}_j)|^2 = O(\varrho^4 |H|^2)$?*

It may well be true that one can even force the stronger upper bound $|B_\varrho(\mathcal{C}_j)| = O(\varrho^2)$ for all $\mathcal{C}_j \in \mathcal{A}(H^+)$ — but it is "even more" unknown.

3.3 Levels.

During this subsection, we study arrangements located in a fixed Cartesian coordinate system and consider the positive half of the z–axis (or that of the x_d–axis in $\mathbb{R}^d$) as pointing "up". Thus we can say that a point is "below" or "above" a non-vertical (hyper)plane.

Also, while speaking about levels (to be defined immediately), we shall assume that none of the (hyper)planes are vertical.

The *level* of a cell $\mathcal{C}_j \in \mathcal{A}(H)$ is the number of $h \in H$ which lie below $\mathcal{C}_j$. This can also be visualized by picking a point $P \in \mathcal{C}_j$ and drawing a ray from P downward; the level of $\mathcal{C}_j$ is the number of $h \in H$ which cut this ray.

Theorem 3.5 (Clarkson) *The number of vertices, edges, faces, and cells of level $\leq \varrho$ is $O(\varrho^2|H|)$ in $\mathbb{R}^3$ and $O(\varrho^{\lceil d/2 \rceil}|H|^{\lfloor d/2 \rfloor})$ in $\mathbb{R}^d$.*

Proof: see [Cla88] and also Theorem 6.3.1 in [Mat02] for vertices; for the rest proceed as in the proof of Theorem 3.1.

From now on, we stop stating results for dimensions exceeding three. The reason for this is that the higher dimensional versions of the forthcoming bounds — though usually sharp — do not seem strong enough for extending our Main Lemma 3.8 to $d \geq 4$.

Corollary 3.6 *In $\mathbb{R}^3$, for any $\mathcal{C}_j \in \mathcal{A}(H)$ and $\varrho > 0$, we have*

$$|B_\varrho(\mathcal{C}_j)| = O(\varrho^2|H|).$$

Proof: First we pick a point $P \in \mathcal{C}_j$ and apply a projective transform π which maps P to the point at infinity of the z–axis. Consequently, since no $h \in H$ contains P, no plane h will be mapped into vertical position.

For any cell $\mathcal{C}_i \in B_\varrho(\mathcal{C}_j)$ and any point $P_i \in \mathcal{C}_i$, the segment $\overline{P_i P}$ intersects $\leq \varrho$ planes $h \in H$. Moreover, it is mapped to a vertical ray emanating from $\pi(P_i)$ which, of course, can point either downward or upward.

In the former case, the image $\pi(\mathcal{C}_i)$ is at level $\leq \varrho$ in $\mathcal{A}(\pi(H))$. According to Theorem 3.5, there are $O(\varrho^2|H|)$ such cells.

Otherwise, in the latter case, we reflect $\pi(H)$ and the arrangement about the x–y plane and apply the same Theorem to the reflected image.

To sum up, the number of cells in $B_\varrho(\mathcal{C}_j)$ is at most twice the bound in Theorem 3.5, which still makes $O(\varrho^2|H|)$. ∎

Corollary 3.7 *If $B_\varrho(\mathcal{C}_j)$ is incident upon n_j planes $h \in H$ then*

$$|B_\varrho(\mathcal{C}_j)| = O(\varrho^2 n_j).$$

Proof: Those planes which are not incident upon the ϱ–neighborhood cannot affect its size; we can just delete them and then apply Corollary 3.6. ∎

3.4 Graphs of short distances.

Given an arrangement $\mathcal{A}(H)$ and a $\varrho > 0$, we define a graph $G_{\leq\varrho}$ on the cells $\mathcal{C}_j \in \mathcal{A}(H)$ as vertices (one can visualize them as representative points $P_j \in \mathcal{C}_j$) and edge set $E_{\leq\varrho}$ by connecting two cells $\mathcal{C}_i$, $\mathcal{C}_j$ (or, equivalently, the points P_i and P_j) by an edge if $\mathrm{dist}(\mathcal{C}_i, \mathcal{C}_j) \leq \varrho$. Our prime tool bounds the number of edges of this graph in terms of ϱ and $|H|$.

Lemma 3.8 (Main Lemma) *In $\mathbb{R}^3$, we have*

$$|E_{\leq\varrho}| = O(\varrho^3|H|^3).$$

Proof: As in Corollaries 3.2 and 3.7, denote by n_j the number of planes $h \in H$ which are incident upon a cell $\mathcal{C}_j \in \mathcal{A}(H)$. Then

$$\begin{aligned} |E_{\leq\varrho}| &= \sum_{\mathcal{C}_j \in \mathcal{A}(H)} |B_\varrho(\mathcal{C}_j)| = \sum_{\mathcal{C}_j \in \mathcal{A}(H)} O(\varrho^2 n_j) = \\ &= \varrho^2 O\left(\sum_{\mathcal{C}_j \in \mathcal{A}(H)} n_j \right) = \varrho^2 O(\varrho |H|^3) = O(\varrho^3|H|^3). \ \blacksquare \end{aligned}$$

4 Proof of the Main Theorem 1.3.

We demonstrate parts (i)–(iii) one by one, following (and suitably adapting) an ingenious idea of J. Solymosi [**?**].

Proof of part (i): Assume that a set $\mathcal{P} \subset \mathbb{R}^3$ of N points can be cut into singletons by a set H of $n \leq C\sqrt[3]{N}$ planes. In other words, each cell contains at most one point $P \in \mathcal{P}$. Moreover, let $2 \leq k \leq n$ be arbitrary.

Define

$$\varrho = \frac{3n}{k}.$$

We shall make use of the graph $G_{\leq\varrho}$ of pairs of cells $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{A}(H)$, for which $\mathrm{dist}(\mathcal{C}_i, \mathcal{C}_j) \leq \varrho$.

First we consider a k–rich line l and assume that the points of $l \cap P$ are $P_1, P_2, \ldots, P_k$ in this linear order. This l intersects each of the n planes $h \in H$ at most once. Therefore, at most $k/3$ of the segments between consecutive pairs of points P_iP_{i+1} will intersect more than ϱ planes — otherwise there would be strictly more than $(k/3) \cdot (3n/k) = n$ intersections.

Hence there remain at least

$$(k-1) - \frac{k}{3} = \frac{3k-3-k}{3} \geq \frac{k}{6}$$

segments which are cut by $\leq \varrho$ planes $h \in H$. In terms of the graph $G_{\leq\varrho}$, each k–rich line contributes at least $k/6$ edges. (Moreover, the latter are all distinct since each cell contains at most one point and two points determine an unique line.)

Since the number of edges satisfies $|E_{\leq \varrho}| = O(\varrho^3 n^3)$ by the Main Lemma 3.8, we have

$$\text{number of } k\text{–rich lines } \leq \frac{O(\varrho^3 n^3)}{k/6} = \frac{O\left(\frac{n^3}{k^3} \cdot n^3\right)}{\frac{k}{6}} = O\left(\frac{n^6}{k^4}\right) = \\ = O\left(\frac{N^2}{k^4}\right). \blacksquare$$

Proof of (ii): As a generalization of what was said before, we assume that a straight line l passes through $k_l \geq k$ points of the proper 3–dimensional point set $\mathcal{P}$. Then, just as we have seen, at most $k/3$ (which is at most $k_l/3$) segments will be cut by more than $\varrho = 3n/k$ planes $h \in H$, giving way to at least

$$(k_l - 1) - \frac{k_l}{3} = \frac{3k_l - 3 - k_l}{3} \geq \frac{k_l}{6}$$

"close pairs" and thus at least this many edges of $G_{\leq \varrho}$. Turning this upside down, for each such line we have that the number of incidences generated by l is at most six times the number of edges of $G_{\leq \varrho}$ on l. Summing for all k–rich lines, the total number I of incidences satisfies

$$I \leq 6 \cdot |E_{\leq \varrho}| = 6 \cdot O\left(\frac{n^3}{k^3} \cdot n^3\right) = O\left(\frac{N^2}{k^3}\right). \blacksquare$$

Proof of (iii): Consider a set of N points, proper 3–dimensional up to a constant factor C. By definition, the set can be cut into singletons by a set H of some $n \leq C\sqrt[3]{N}$ planes.

First, for such sets and any M straight lines, the number of incidences is $O(M\sqrt[3]{N})$, since no line can pass through more than $C\sqrt[3]{N}+1 \leq (C+1)\sqrt[3]{N} = O(N^{1/3})$ cells of $\mathcal{A}(H)$, each of which contains at most one point of the given set.

Next, we show another bound which is better than the previous one for $M \geq N^{2/3}$.

Denote by I the number of incidences between our set of N points and M lines. (Thus an "average" line will be incident upon $\sim I/M$ points.)

Put $k = I/(2M)$ and discard all lines which pass through less than k points. Denote by M' and I' the number of preserved lines and incidences, respectively. In total, at most $Mk = I/2$ incidences could be discarded whence $I' \geq I/2$.

We distinguish two cases.

Case **I.** If $k = I/(2M) < 2$ then we have $I < 4M$.

Case **II.** Otherwise $k = I/(2M) \geq 2$, thus we can apply part (ii) of the Main Theorem, which yields

$$I/2 \leq I' = O\left(\frac{N^2}{k^3}\right) = O\left(\frac{N^2M^3}{I^3}\right),$$

whence $I^4 = O(N^2M^3)$ i.e. $I = O(N^{1/2}M^{3/4})$. Thus $I = O(M + N^{1/2}M^{3/4})$ anyway, since the right hand side is an upper bound in either case. ∎

Concluding remarks

The following questions remain open.

Problem 4.1 *Is it true for all $d \geq 2$ that if a set $\mathcal{P} \subset \mathbb{R}^d$ of N points is proper d–dimensional then*

$$\textit{number of } k\textit{–rich lines } = O\left(\frac{N^2}{k^{d+1}}\right)?$$

This order of magnitude, if true, is best possible (as a function of N and k), as shown by an $N = n \times n \times \ldots \times n$ cube lattice (see Remark 2.2). Perhaps a positive answer to the following question could help in solving the previous problem.

Problem 4.2 *Is it true for all $d \geq 2$ that the edge set $E_{\leq \varrho}$ of the graph $G_{\leq \varrho}$ of "short distances" defined in terms of an arrangement of n hyperplanes in $\mathbb{R}^d$ satisfies*

$$|E_{\leq \varrho}| = O(\varrho^d n^d)?$$

(For $d = 1$ the statement is obvious while the cases $d = 2$ and $d = 3$ are Corollary 3.3 and the Main Lemma 3.8, respectively.)

References

[Cla88] K Clarkson. Applications of random sampling in Computational Geometry II. *Proc. 4th Annu. ACM Sympos. Comput. Geom.*, pages 1–11, 1988.

[Mat88] Jiri Matoušek. Line arrangements and range search. *Inf. Process. Lett.*, 27:275–280, 1988.

[Mat02] Jiri Matoušek. *Lectures on Discrete Geometry.* Springer–Verlag, Berlin, Heidelberg, New York, 2002.

[ST83] Endre Szemerédi and W. T. Trotter Jr. Extremal problems in Discrete Geometry. *Combinatorica*, 3 (3–4):381–392, 1983.

[Szé97] László A Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combinatorics, Probability and Computing*, 6,No.3:353–358, 1997.