LONG TIME BEHAVIOR FOR A SEMILINEAR HYPERBOLIC EQUATION WITH ASYMPTOTICALLY VANISHING DAMPING TERM AND CONVEX POTENTIAL

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Abstract Recently, A. Cabot and P. Frankel studied the long time behavior of solutions to the following semilinear hyperbolic equation:

$$\text{(E)} \qquad \frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, the damping term, is a decreasing function, f is the gradient of a given convex function defined on an a real Hilbert space V, and $A : V \to V'$ is a linear and continuous operator assumed to be symmetric, monotone and semi-coercive. They proved that if the damping term $\gamma(t)$ behaves like $\frac{K}{t^\alpha}$ as $t \to +\infty$, for some $K > 0$ and $\alpha \in]0,1[$, then every <u>bounded</u> solution u to the equation (E) (i.e. $u \in L^\infty(0,+\infty;V)$) converges weakly in V as $t \to +\infty$ toward a solution to the stationary equation $Av + f(v) = 0$. They left open the question: Does convergence still hold without assuming the boundedness of the solution? In this paper, we give a positive answer to this question. Our approach relies on precise estimates on the decay rates for the energy function along trajectories of (E).

keywords: Dissipative hyperbolic equation, asymptotically small dissipation, asymptotic behavior, energy function, convex function.

AMS classification numbers: 34G10, 34G20, 35B40, 35L70.

1. Introduction and statement of results

Throughout this paper, we follow the same notations as in the paper [5]. Let H be a real Hilbert space with inner product and norm respectively denoted by $\langle .,.\rangle$ and $|.|$. Let V be a real Hilbert space such that $V \hookrightarrow H \hookrightarrow V'$ with continuous and dense injections, where V' is the dual space of V. Let $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a decreasing function which belongs to the space $W^{1,1}_{loc}(\mathbb{R}^+;\mathbb{R}^+)$. Let $A : V \to V'$ be a linear and continuous operator such that the associated

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bilinear form $a : V \times V \rightarrow \mathbb{R}$ $(u,v) \mapsto \langle Au, v \rangle_{V',V}$ is symmetric, positive and satisfies the following property:

(1.1) $$\exists \lambda \geq 0, \mu > 0 : \forall v \in V, \ a(v,v) + \lambda |v|^2 \geq \mu \|v\|_V^2 .$$

Let $f : V \rightarrow V'$ be a continuous function deriving from a convex potential i.e, there exists a C^1 convex function $F : V \rightarrow \mathbb{R}$ such that:

$$\forall u, v \in V, \ F'(u)(v) = \langle f(u), v \rangle_{V',V}.$$

It is clear that the function $\phi : V \rightarrow \mathbb{R}$ defined by:

$$\phi(v) = \frac{1}{2} a(v,v) + F(v)$$

is C^1, convex and satisfies the following property:

$$\forall u, v \in V, \phi'(u)(v) = \langle Au + f(u), v \rangle_{V',V}.$$

We assume moreover that the function ϕ is bounded from below and that the set

$$\arg \min \phi = \{v \in V : \phi(v) = \min \phi\}$$

is not empty. Notice that, since ϕ is convex, $\arg \min \phi$ coincides with the set $S = \{v \in V : Av + f(v) = 0\}$ of critical points of ϕ.

In this paper, our purpose is to investigate the asymptotic behavior of the semilinear hyperbolic equation:

(E) $$\frac{d^2 u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0.$$

This equation and its ODE version (called the heavy ball with friction) have been studied by many authors under various conditions on the damping and potential terms, see for instance, [1], [2], [3], [4], [5], [6], [7], [10], and references there in.

By a solution of (E) we mean a function $u : \mathbb{R}^+ \longrightarrow H$ which belongs to the class

$$W_{loc}^{1,1}(\mathbb{R}^+, V) \cap W_{loc}^{2,1}(\mathbb{R}^+, H)$$

and satisfies the equation (E) for almost every $t \geq 0$. A solution u to (E) is said to be bounded if it belongs moreover to the space $L^\infty(0, +\infty; H)$.

In [5], Cabot and Frankel proved the following interesting convergence result:

Theorem 1.1 (A. Cabot and P. Frankel). *Assume that there exist $\alpha \in]0, 1[$ and $K_1, K_2 > 0$ such that for every $t \geq 0$, $\frac{K_1}{(1+t)^\alpha} \leq \gamma(t) \leq \frac{K_2}{(1+t)^\alpha}$. Let u be a __bounded__ solution to (E). Then there exists $u_\infty \in \arg \min \phi$ such that $u(t)$ converges weakly in V to u_∞ as $t \rightarrow +\infty$.*

An open question left in the paper [5] was whether the condition $u \in L^\infty(0, +\infty; H)$ is really necessary in the previous theorem (see Remark 3.15 in [5]). In the present paper, we will show, without assuming the boundedness of the solution, that the weak convergence result still holds in the case $\alpha \in [0, \frac{1}{2}]$ and in the case $\alpha \in]\frac{1}{2}, 1[$ up to a supplementary assumption on the derivative of the damping term γ. Such assumption is satisfied, for instance, by functions of the form $\frac{K}{(1+t)^\alpha}$ where $K > 0$. Moreover, in each case, we will establish an estimate on the rate of the decay for the energy function on the trajectories of (E). More precisely, we will prove the following theorems:

Theorem 1.2. *Assume that there exist $\alpha \in [0, \frac{1}{2}]$ and $K > 0$ such that for every $t \geq 0$, $\gamma(t) \geq \frac{K}{(1+t)^\alpha}$. Then for every solution u to (E) there exists $u_\infty \in \arg\min\phi$ such that $u(t)$ converges weakly in V to u_∞ as $t \to +\infty$. Moreover,*

$$\frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min\phi = \circ(\frac{1}{t}) \text{ as } t \to +\infty.$$

Theorem 1.3. *Assume that there exist $\alpha \in [0, 1[$, $K > 0$ and $t_0 \geq 0$ such that $\gamma(t) \geq \frac{K}{(1+t)^\alpha}$ for every $t \geq 0$ and $\gamma'(t) \leq -\alpha\frac{\gamma(t)}{1+t}$ for almost every $t \geq t_0$. Let u be a solution to (E), then $u(t)$ converges weakly in V as $t \to +\infty$ toward some $u_\infty \in \arg\min\phi$. Moreover, for every $\bar{\alpha} < \alpha$,*

$$\frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min\phi = \circ(\frac{1}{t^{1+\bar{\alpha}}}) \text{ as } t \to +\infty.$$

2. Proof of Theorem 1.2 and Theorem 1.3

We will first prove some preliminary results under the following general hypothesis on the damping term γ :

$$(2.1) \qquad \exists K > 0 \text{ and } \alpha \in [0, 1[: \forall t \geq 0, \ \gamma(t) \geq \frac{K}{(1+t)^\alpha}.$$

These results will be useful in the proofs of Theorem 1.2 and Theorem 1.3.

Let u be a solution to the equation (E). Define the energy function

$$(2.2) \qquad \mathcal{E}(t) = \frac{1}{2}\left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min\phi, \ t \geq 0.$$

A simple computation yields

$$\frac{d\mathcal{E}}{dt}(t) = -\gamma(t)\left|\frac{du}{dt}(t)\right|^2, \ a.e. \ t \geq 0.$$

Thus the function $\mathcal{E}$ is decreasing and converges as $t \to +\infty$ to some real number $\mathcal{E}_\infty$ which will be identified later. Moreover

$$(2.3) \qquad \int_0^{+\infty} \gamma(t)\left|\frac{du}{dt}(t)\right|^2 dt < \infty$$

and

$$(2.4) \qquad \forall t \geq 0, \ \mathcal{E}(t) - \mathcal{E}_\infty = \int_t^{+\infty} \gamma(s) \left| \frac{du}{dt}(s) \right|^2 ds.$$

Let v be a fixed point in $\arg\min \phi$ and define the function $p(t) = \frac{1}{2} |u(t) - v|^2$, $t \geq 0$. Proceeding as in the proof of Proposition 3.5 in [5], one can easily prove that for almost every t in $\mathbb{R}^+$ we have

$$\ddot{p}(t) + \gamma(t)\dot{p}(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 - \mathcal{E}(t).$$

Multiplying the last inequality by $\lambda_r(t) = (1+t)^r$, $r \in \mathbb{R}$, and integrating by parts over the interval $[0, T]$, $T > 0$, we easily obtain after simplification

$$\int_0^T \lambda_r(t)\mathcal{E}(t)dt \ \leq \ \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt - \lambda_r(T)\dot{p}(T) + \left[\lambda_r' - (\gamma\lambda_r) \right](T)p(T)$$

$$(2.5) \qquad\qquad\qquad + \int_0^T \left[(\lambda_r\gamma)' - \lambda_r'' \right](t)p(t)dt + C_r$$

where $C_r = \dot{p}(0) + (\gamma(0) - r)p(0)$.

Since γ satisfies (2.1), $\lambda_r'(T) = \circ \left[(\gamma\lambda_r)(T) \right]$ as $T \to +\infty$. Thus, there exits $T_r \geq 0$ such

$$(2.6) \qquad \forall t \geq T_r, \ \lambda_r'(T) - (\gamma\lambda_r)(T) \leq -\frac{1}{2}(\gamma\lambda_r)(T).$$

On the other hand, thanks to Cauchy-Schwarz inequality, we have

$$(2.7) \qquad |\dot{p}(T)| \ \leq \ \left| \frac{du}{dt}(t) \right| |u(t) - v|$$

$$\leq \ 2\sqrt{\mathcal{E}(T)}\sqrt{p(T)}$$

Inserting estimates (2.6)-(2.7) into (2.5) and using hypothesis (2.1) and the following elementary inequality

$$\forall a > 0 \ \forall \ x, b \in \mathbb{R}, \ bx - ax^2 \leq \frac{b^2}{4a},$$

with $x = \sqrt{p(T)}$, we deduce that for every $T \geq T_r$ we have

$$\int_0^T \lambda_r(t)\mathcal{E}(t)dt \ \leq \ \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K}\lambda_{r+\alpha}(T)\mathcal{E}(T)$$

$$(2.8) \qquad\qquad\qquad + \int_0^T \left[(\lambda_r\gamma)'(t) - \lambda_r''(t) \right] p(t)dt + C_r.$$

Let us notice that if $r \leq 0$, $(\lambda_r\gamma)'(t) - \lambda_r''(t) \leq 0$ a.e. on $\mathbb{R}^+$ (since the function $\lambda_r\gamma$ is decreasing and the function λ_r is convex); then, in the case where $r \leq 0$, (2.8) becomes

$$(2.9) \qquad \forall T \geq T_r, \ \int_0^T \lambda_r(t)\mathcal{E}(t)dt \leq \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K}\lambda_{r+\alpha}(T)\mathcal{E}(T) + C_r.$$

Letting $r = -\alpha$ in the last inequality and using (2.3) and the fact that is $\mathcal{E}$ a decreasing function, we get

$$\forall T \geq T_{-\alpha}, \quad \int_0^T \lambda_{-\alpha}(t)\mathcal{E}(t)dt \leq \frac{3}{2K}\int_0^{+\infty} \gamma(t)\left|\frac{du}{dt}(t)\right|^2 dt + \frac{2}{K}\mathcal{E}(0) + C_{-\alpha},$$

which implies that

$$(2.10) \qquad \qquad \int_0^{+\infty} \lambda_{-\alpha}(t)\mathcal{E}(t)dt < \infty.$$

Recalling that $\alpha < 1$, we then deduce that the limit $\mathcal{E}_\infty$ of $\mathcal{E}(t)$ as $t \to +\infty$ is equal to zero. Let us now prove the following crucial lemma:

Lemma 2.1. *Let $r \in \mathbb{R}\setminus\{-1\}$. If $\int_0^{+\infty} \lambda_r(t)\mathcal{E}(t)dt < \infty$ then $\mathcal{E}(t) = \circ(1/t^{1+r})$ as $t \to +\infty$ and $\int_0^{+\infty} \lambda_{r+1-\alpha}(t)\left|\frac{du}{dt}(t)\right|^2 dt < \infty$.*

Proof. Since the energy function $\mathcal{E}$ is decreasing, we have

$$(2.11) \qquad \qquad \mathcal{E}(t)\int_{\frac{t}{2}}^t (1+s)^r ds \leq \int_{\frac{t}{2}}^{+\infty} \lambda_r(s)\mathcal{E}(s)ds.$$

A simple computation yields $\int_{\frac{t}{2}}^t (1+s)^r ds \simeq M_r t^{r+1}$ for t large enough where M_r is a nonnegative constant depending only on r. Inserting this last estimate into (2.11), we get $\lim_{t\to+\infty} t^{1+r}\mathcal{E}(t) = 0$. On the other hand, by using equality (2.4), the fact that $\mathcal{E}_\infty = 0$, and Fubini Theorem, we obtain

$$\int_0^{+\infty} \lambda_r(t)\mathcal{E}(t)dt = \frac{1}{1+r}\int_0^{+\infty} \gamma(s)\left[(1+s)^{r+1} - 1\right]\left|\frac{du}{dt}(s)\right|^2 ds,$$

which clearly implies that $\int_0^{+\infty} \lambda_{r+1-\alpha}(t)\left|\frac{du}{dt}(t)\right|^2 dt < \infty$ since $\int_0^{+\infty} \gamma(s)\left|\frac{du}{dt}(s)\right|^2 ds < \infty$ and $\gamma(s) \geq \frac{K}{(1+s)^\alpha}$. $\qquad\square$

Now we are in position to complete the proof of our first main theorem.

Proof of Theorem 1.2: In view of (2.10), Lemma 2.1 implies $\mathcal{E}(t) = \circ(t^{\alpha-1})$ as $t \to +\infty$ and $\int_0^{+\infty} \lambda_{1-2\alpha}(t)\left|\frac{du}{dt}(t)\right|^2 dt < \infty$. Hence by letting $r = 0$ in (2.9), we get, for T large enough,

$$\int_0^T \mathcal{E}(t)dt \leq \frac{3}{2}\int_0^T \left|\frac{du}{dt}(t)\right|^2 dt + \circ(T^{2\alpha-1}) + C_0.$$

Therefore, by letting $T \to +\infty$ and using the assumption $\alpha \leq \frac{1}{2}$, we get

$$\int_0^\infty \mathcal{E}(t)dt \leq \frac{3}{2}\int_0^\infty \lambda_{1-2\alpha}(t)\left|\frac{du}{dt}(t)\right|^2 dt + C_0,$$

Hence, by using once again Lemma 2.1, we deduce that $\mathcal{E}(t) = \circ(1/t)$ as $t \to +\infty$ and that $\int_0^{+\infty} \lambda_{1-\alpha}(t)\left|\frac{du}{dt}(t)\right|^2 dt < \infty$ which implies, since $\alpha \leq \frac{1}{2}$, that $\int_0^{+\infty}(1+t)^\alpha\left|\frac{du}{dt}(t)\right|^2 dt < \infty$. Therefore we deduce the weak convergence of $u(t)$ in V as $t \to +\infty$ from the following lemma which is implicitly proved in [5] (see the proofs of Theorem 3.7 and Theorem 3.13) by adapting a classical arguments originated by F. Alvarez [1] based on the famous Opial's lemma [9].

Lemma 2.2. *Assume (2.1). Let u be a solution to (E). If $\int_0^\infty (1+t)^\alpha \left| \frac{du}{dt}(t) \right|^2 dt < \infty$ then $u(t)$ converges weakly in V as $t \to +\infty$ to some $u_\infty \in \arg\min \phi$.*

Now we are going to prove our second main theorem. Hence, hereafter, we assume that the function γ satisfies (2.1) and the hypothesis on its derivative given in Theorem 1.3. First we will prove the following key lemma:

Lemma 2.3. *If $\nu < 2\alpha - 1$ and $\int_0^{+\infty} \lambda_\nu(t)\mathcal{E}(t)dt < +\infty$ then $\int_0^{+\infty} \lambda_{\nu+1-\alpha}(t)\mathcal{E}(t)dt < +\infty$.*

Proof of Lemma 2.3: Let $\nu < 2\alpha - 1$ such that $\int_0^{+\infty} \lambda_\nu(t)\mathcal{E}(t)dt < +\infty$. According to Lemma 2.1, we have:

$$(2.12) \qquad\qquad \mathcal{E}(t) = \circ(1/t^{1+\nu}) \text{ as } t \to +\infty$$

and

$$(2.13) \qquad\qquad \int_0^{+\infty} \lambda_{1+\nu-\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt < \infty.$$

Let $\rho = 1 + \nu - \alpha$. Using the hypothesis on the damping term γ and the fact that $\rho < \alpha$, we find that for almost every $t \geq t_0$ we have

$$
\begin{aligned}
\left[(\lambda_\rho \gamma)' - \lambda_\rho'' \right](t) &\leq (\rho - \alpha)\lambda_{\rho-1}(t)\gamma(t) - \rho(\rho-1)\lambda_{\rho-2}(t) \\
&\leq (\rho - \alpha)K\,\lambda_{\rho-\alpha-1}(t) - \rho(\rho-1)\lambda_{\rho-2}(t) \\
&\simeq (\rho - \alpha)K\,\lambda_{\rho-\alpha-1}(t) \text{ as } t \to +\infty.
\end{aligned}
$$

The last inequality implies that there exists $\tau_0 \geq \max(T_0, t_0)$ such that for almost every $t \geq \tau_0$ we have $\left[(\lambda_\rho \gamma)' - \lambda_\rho'' \right](t) \leq 0$. Inserting this last inequality into (2.8) with $r = \rho$, we obtain

$$(2.14) \qquad \int_0^T \lambda_\rho(t)\mathcal{E}(t)dt \leq \frac{3}{2}\int_0^T \lambda_\rho(t)\left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K}\lambda_{1+\nu}(T)\mathcal{E}(T) + A_\rho \text{ for a.e. } T \geq \tau_0,$$

where $A_\rho = C_\rho + \int_0^{\tau_0} \left[(\lambda_r \gamma)'(t) - \lambda_r''(t) \right]p(t)dt$. Hence, by using estimates (2.12)-(2.13) and by letting $T \to +\infty$ in (2.14), we deduce that $\int_0^{+\infty} \lambda_\rho(t)\mathcal{E}(t)dt < \infty$.

Now we are in position to prove our second main theorem.

Proof of Theorem 1.3: We will proceed as in the proof of Theorem 1.3 in [8]. Let $A = \{\nu \in \mathbb{R} : \int_0^{+\infty} \lambda_\nu(t)\mathcal{E}(t)dt < +\infty\}$. From (2.8), $-\alpha \in A$, thus A is a non empty interval of $\mathbb{R}$ which is on the forme $A =]-\infty, \alpha_0[$ or $A =]-\infty, \alpha_0]$ where $\alpha_0 = \sup A$. The previous lemma asserts that: if $\nu < \alpha_0$ and $\nu < 2\alpha - 1$ then $\nu + 1 - \alpha \leq \alpha_0$ which means that $\min(\alpha_0, 2\alpha - 1) \leq \alpha_0 + \alpha - 1$. Now since $\alpha - 1 < 0$, the last inequality reads as $2\alpha - 1 \leq \alpha_0 + \alpha - 1$, thus $\alpha \leq \alpha_0$. Therefore, by using the defintion of α_0 and Lemma 2.1 we infer that for all $\bar{\alpha} < \alpha$, $\mathcal{E}(t) = \circ(1/t^{1+\bar{\alpha}})$ as $t \to +\infty$ and $\int_0^{+\infty}(1+t)^{1+\bar{\alpha}-\alpha}\left| \frac{du}{dt}(t) \right|^2 dt < \infty$. Hence, by taking $\bar{\alpha}$ closed enough to α and using the fact that $\alpha < 1$, we deduce that $\int_0^{+\infty}(1+t)^\alpha \left| \frac{du}{dt}(t) \right|^2 dt < \infty$ which completes the proof thanks to Lemma 2.2.

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