The Well-posedness and Blow-up rate of Solution for the Generalized Zakharov equations with Magnetic field in $\mathbb{R}^d$

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Abstract

The present paper is devoted to the study of the well-posedness and the lower bound of blow-up rate to the Cauchy problem of the generalized Zakharov(GZ) equations with magnetic field in $\mathbb{R}^d$, $d \geq 1$. The work of well-posedness of the GZ system bases on the local well-posedness theory in [10]. At first, the existence, uniqueness and continuity of solution to the GZ system with magnetic field in $\mathbb{R}^d$ is proved. Next, we establish the lower bound of blow-up rate of blow-up solution in sobolev spaces to the GZ system, which is almost a critical index. Finally, we obtain the long time behavior of global solution, whose H^k-norm grows at k-exponentially in time.

Keywords: the generalized Zakharov equations, plasma, the Cauchy problem, Bourgain spaces, local well-posedness, the lower bound of blow-up rate, the long time behavior of global solution.

1 Introduction

In this paper, we consider the Cauchy problem of the generalized Zakharov system with magnetic field in $\mathbb{R}^d$ as follows

$$
\begin{cases}
i\partial_t E + \Delta E - nE + i(E \otimes B) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\frac{1}{c_0^2}\partial_{tt} n - \Delta(n + |E|^2) = 0, \\
\Delta B - i\eta \nabla \times \nabla \times (E \otimes \overline{E}) + A = 0, \\
E(0,x) = E_0(x), & x \in \mathbb{R}^d, \\
(n(0,x), \partial_t n(0,x)) = (n_0(x), n_1(x)),
\end{cases}
\tag{1.1}
$$

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where $c_0 > 0$ is a constant, $E(t,x)$ denotes a vector valued function from $\mathbb{R}^+ \times \mathbb{R}^d$ into $\mathbb{C}^d$. $n(t,x)$ is a function from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}$, $B(t,x)$ is a vector valued function from $\mathbb{R}^+ \times \mathbb{R}^d$ into $\mathbb{C}^d$, and A has the following two form:

(A_1) $\quad A = \beta B$, $\quad \beta$ is a nonpositive constant;

(A_2) $\quad A = -\gamma \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{B(t,y)}{|x-y|^2} dy.$

The system (1.1) describes the spontaneous generation of a magnetic field in a cold plasma (case A_1) or in a hot plasma (case A_2) [16]. E denotes the slowly varying complex amplitude of the high-frequency electric field, $n(t,x)$ represents the fluctuation of the electron density from its equilibrium, B is the self-generated magnetic field. $i^2 = -1$, constant $\eta > 0$, $\overline{E}$ denotes the complex conjugate of E, and $\otimes$ means the exterior product of vector-valued functions.

If we neglect the magnetic field B, system (1.1) becomes the classical Zakharov equation

$$\begin{cases} i\partial_t E + \Delta E - nE = 0, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \frac{1}{c_0^2}\partial_{tt}n - \Delta(n + |E|^2) = 0, \\ E(0,x) = E_0(x), x \in \mathbb{R}^d, \\ (n(0,x), \partial_t n(0,x)) = (n_0(x), n_1(x)), \end{cases} \quad (1.2)$$

which describes the propagation of Langmuir wave [27]. The Cauchy problem of Eq.(1.2) was established by several authors [1, 2, 5, 10, 15, 23, 25]. Such as, the local well-posedness was obtain in spaces $H^k \times H^l \times H^{l-1}$[10] for any dimensions d, C. Sulem and P.L. Sulem [25] proved the global existence of a weak solution in two and three dimensions for the small initial data. With the same assumptions, they also got the existence and uniqueness of the smooth solution $(E,n) \in \mathcal{C}([0,T[;H^m) \times \mathcal{C}([0,T[;H^{m-1}), m \geq 3$. Moreover, the solution was global in one dimension, and the global solution can be extended in two dimensions with small initial data [1]. Numerical simulations strongly suggest a finite blow-up time for some initial data, and global solution of the small initial data can be numerically verified by Papanicolaou, C. Sulem, P. L. Sulem, Wang, and Landman [17, 24]. By constructing a family of blow-up solutions of the following form

$$\begin{cases} E(t,x) = \frac{\omega}{T-t}e^{i(\theta + \frac{|x|^2 - 4\omega^2}{4(-T+t)})}P\left(\frac{x\omega}{T-t}\right), \\ n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{(T-t)}\right), \end{cases}$$

where $\omega > \omega_0, \theta \in \mathbb{R}$, and

$$P(x) = P(|x|), N(x) = N(|x|), \Delta P - P = NP,$$

$$\frac{1}{(c_0\omega)^2}(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta P^2,$$

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with $r = |x|, \Delta w = w_{rr} + \frac{1}{r}w_r$, L. Glangetas and F. Merle [11] proved the existence of self-similar blow-up solutions to the Hamiltonian case of Eq.(1.2) in two dimensions. i.e.

$$
\begin{cases}
i\partial_t E + \Delta E - nE = 0, \\
\partial_t n + divv = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\frac{1}{c_0^2}\partial_t v + \nabla(n + |E|^2) = 0, \\
E(0, x) = E_0(x), \quad x \in \mathbb{R}^2, \\
(n(0, x), \partial_t n(0, x)) = (n_0(x), n_1(x)),
\end{cases}
\tag{1.3}
$$

more results of Eq.(1.3) can be found in [12, 20, 21]. In fact, the existence and uniqueness of global solution is open problem in $d \geq 3$. It is interested to recall the situation in the case $c_0 = \infty$, that is the Zakharov equations reduce to the cube nonlinear Schrödinger equation [9, 13, 17, 22]

$$
i\partial_t u + \Delta u = -|u|^2 u.
$$

Returning to the generalized Zakharov system (1.1) with magnetic field. We consider the system (1.1) in the Hamiltonian case, i.e.

$$
\partial_t n(t, x) = -\Delta w(t, x) = -divV(t, x),
$$

then system (1.1) can be written in the form [19]

$$
\begin{cases}
i\partial_t E + \Delta E - nE + i(E \otimes B) = 0, \\
\partial_t n + divV = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\frac{1}{c_0^2}\partial_t V + \nabla(n + |E|^2) = 0, \\
\Delta B - i\eta\nabla \times \nabla \times (E \otimes \overline{E}) + A = 0, \\
E(0, x) = E_0(x), \quad x \in \mathbb{R}^d, \\
(n(0, x), V(0, x)) = (n_0(x), V_0(x)).
\end{cases}
\tag{1.4}
$$

In 1995, by the conservation laws of Eq.(1.4) in the case (A_1)

$$
H_1(E) := \|E\|_{L^2} = \|E_0\|_{L^2} = H_1(E_0), \tag{1.5}
$$

$$
H_2(E, n, V, B) := \|\nabla E\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{1}{2c_0^2}\|V\|_{L^2}^2 + \int_{\mathbb{R}^d} n|E|^2 dx
$$
$$
- \frac{i}{2}\int_{\mathbb{R}^d} B(E \otimes \overline{E})dx = H_2(E_0, n_0, V_0, B_0), \tag{1.6}
$$

C. Laurey [19] got the global existence of weak solution $(E, n, B) \in L^\infty(\mathbb{R}^+; H^1) \times L^\infty(\mathbb{R}^+; L^2) \times L^\infty(\mathbb{R}^+; L^2)$ to Eq.(1.4) in the case (A_1) with the small initial data $(E_0, n_0, B_0) \in H^1 \times L^2 \times L^2$. As the initial data $(E_0, n_0, B_0) \in H^{s+1} \times H^s \times H^{s+1}, s > \frac{d}{2}, d = 2, 3$, he established the local existence and uniqueness of a strong solution $(E, n, B) \in L^\infty([0, T[; H^{s+1}) \times L^\infty([0, T[; H^s) \times L^\infty([0, T[; H^{s+1})$ to system (1.1) in the case (A_1) and (A_2), for some $T > 0$.

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If $d = 2, s = 2$, in the case (A_1), the smooth solution was global with the small initial data. Recently, similar to [11], in two dimensions, Gan, Guo and Huang [8] constructed a family of blow-up solutions and proved the existence of self-similar blow-up solution to Eq.(1.4) in the case (A_1).

A natural problem of system (1.1) is to establish the global solution or construct the blow-up solution in dimensions $d \geq 3$. In this paper, at first, for the generalized Zakharov equation (1.1) with magnetic field in $\mathbb{R}^d$, similar to [10] the well-posedness is obtained in spaces

$$(E, B, n, \partial_t n) \in \mathcal{C}([0, T[; H^k) \times \mathcal{C}([0, T[; H^k) \times \mathcal{C}([0, T[; H^l) \times \mathcal{C}([0, T[; H^{l-1}),$$

if the initial data (E_0, B_0, n_0, n_1) belongs to $H^k \times H^k \times H^l \times H^{l-1}$. The difficult is how to deal with the nonlinearity in system $(1.1)_1$ and system $(1.1)_3$ with (A_1) or (A_2) is decisive. If the solution blows up in finite time T^*, then the lower bound for the blow-up rate of the blow-up solution to system (1.1) satisfies

$$\|E(t)\|_{H^l} + \|n(t)\|_{H^l} + \|\partial_t n\|_{H^{l-1}} > C \frac{1}{(T^* - t)^{\frac{3}{4}\vartheta_l - \epsilon}},$$

where $\vartheta_l = \frac{1}{4}(2l + 4 - d)$ and ϵ is a any positive constant. Moreover, for equation (1.2) in 3D, we have the following lower bound

$$\|E(t)\|_{H^l} + \|n(t)\|_{H^l} + \|\partial_t n\|_{H^{l-1}} > C \frac{1}{(T^* - t)^{\vartheta_l - \epsilon}},$$

which almost up to the bound of the following asymptotic self-similar blow-up solution to equation (1.2)

$$\begin{cases} E(t, x) = \frac{1}{T-t} P\left(\frac{|x|}{(T-t)^{1/2}}\right) + i \frac{1}{T-t} Q\left(\frac{|x|}{(T-t)^{1/2}}\right), \\ n(t, x) = \frac{1}{T-t} N\left(\frac{|x|}{(T-t)^{1/2}}\right), \end{cases}$$

where $P(x) = P(|x|), Q(x) = Q(|x|), N(x) = N(|x|)$, and (P, N) satisfies the ODEs

$$\begin{cases} \Delta P + \frac{1}{2} r Q_r + Q = NP, \\ \Delta Q + \frac{1}{2} r P_r + P = NQ, \\ \frac{1}{4} r^2 N_{rr} + \frac{7}{4} r N_r + 2N) = \Delta(P^2) + \Delta(Q^2). \end{cases}$$

The remainder of this paper is organized as follows. In Section 2, we first recall the definition of the weighted Bourgain spaces, some important lemmas, and present the proof's frame for the well-posedness of the Cauchy problem to system (1.1). In Section 3, the local well-posedness of the Cauchy problem to system (1.1) with the magnetic field B satisfying the case (A_1), (A_2) is established in $H^k \times H^l \times H^{l-1}$. Next, in Section 4, we derive the lower bound of blow-up rate of blow-up solution in Sobolev spaces to system (1.1) and Eq.(1.2), which is almost a critical index. In Section 5, we obtain the global solution to system (1.1) with the small initial data, and the H^k-norm of solution grows at k-exponentially in time.

2 The Preliminary

In this subsection, for the convenience of the readers, we recall the Bourgain method, in order to make this paper self-contained and to locate exactly the required nonlinear estimated, which are come from the paper[10], only make some little modification for our target. These important lemmas will be used repeatedly throughout this paper.

In order to deal with the second wave equation of system (1.1), without loss of generality, let $c_0 = 1$, we split n into its positive and negative frequency parts as

$$\varphi_\pm = n \pm i\Lambda^{-1}\partial_t n, \qquad (2.1)$$

where the operator $\Lambda = (1 - \Delta)^{\frac{1}{2}}$. Then one can easily check that

$$(i\partial_t \mp \Lambda)\varphi_\pm = \mp\Lambda^{-1}(\partial_{tt} - \Delta)n \mp \Lambda^{-1}n.$$

Therefore, system $(1.1)_2$ is equivalent to

$$i\partial_t\varphi_\pm = \pm\Lambda\varphi_\pm \mp \Lambda^{-1}(\Delta|E|^2) \mp \frac{1}{2}\Lambda^{-1}(\varphi_+ + \varphi_-). \qquad (2.2)$$

The generalized Zakharov system (1.1) then takes the following form

$$\begin{cases} i\partial_t E + \Delta E - \frac{1}{2}(\varphi_+ + \varphi_-)E + i(E \otimes B) = 0, \\ i\partial_t\varphi_\pm = \pm\Lambda\varphi_\pm \mp \Lambda^{-1}(\Delta|E|^2) \mp \frac{1}{2}\Lambda^{-1}(\varphi_+ + \varphi_-), \\ \Delta B - i\eta\nabla \times \nabla \times (E \otimes \overline{E}) + A = 0, \end{cases} \qquad (2.3)$$

with the initial data $(E_0, \varphi_{0\pm}, B_0) = (E_0, n_0 \pm i\Lambda^{-1}n_1, B_0)$.

Define the semigroup $S(t) = e^{it\Delta}$, $W^+(t) = e^{-it(1-\Delta)^{\frac{1}{2}}}$ and $W^-(t) = e^{-it(1-\Delta)^{\frac{1}{2}}}$, the solution $(E, \varphi_\pm)$ of the Cauchy problem to Eq.(2.3) is rewritten in a standard way as integral equation

$$E = S(t)E_0 - i\int_0^t S(t-\tau)[\frac{1}{2}(\varphi_+ + \varphi_-)E - i(E \otimes B)](\tau)d\tau \qquad (2.4)$$

and

$$\varphi_\pm = W^\pm(t)\varphi_{0\pm} + i\int_0^t W^\pm(t-\tau)[\mp\Lambda^{-1}(\Delta|E|^2) \mp \frac{1}{2}\Lambda^{-1}(\varphi_+ + \varphi_-)](\tau)d\tau. \qquad (2.5)$$

In order to use function space norms defined in terms of the space time Fourier transform of solutions $(E, \varphi_\pm)$ in the context on finite time interval $[-T, T]$, we introduce an even time cut-off function $\psi \in \mathcal{C}_0^\infty$ satisfying

$$\begin{cases} \psi(t) = 1, & if \quad |t| \leq 1, \\ 0 \leq \psi(t) \leq 1, & if \quad |t| \geq 0, \\ \psi(t) = 0, & if \quad |t| \geq 2. \end{cases}$$

Denote $\psi_T(t) = \psi(\frac{t}{T})$. Consider the cut-off equation

$$E = \psi_1(t)S(t)E_0 - i\psi_T(t) \int_0^t S(t-\tau)f_1(\tau)d\tau \qquad (2.6)$$

and

$$\varphi_\pm = \psi_1(t)W^\pm(t)\varphi_{0\pm} + i\psi_T(t) \int_0^t W^\pm(t-\tau) \left[f(\tau) \mp \frac{1}{2}\Lambda^{-1}\psi_{2T}(\varphi_+ + \varphi_-)(\tau) \right] d\tau, \qquad (2.7)$$

where $f_1 = \psi_{2T}^2[\frac{1}{2}(\varphi_+ + \varphi_-)E] - i\psi_{3T}^3[(E \otimes B)]$, and $f = \mp\Lambda^{-1}\psi_{2T}^2(\Delta|E|^2)$. One can easily check that (2.6), (2.7) is actually identical with (2.4), (2.5) respectively on $\text{Spp}\psi_T$, if $|t| \leq T \leq 1$. For convenience, let W denotes W^+ and W^- in this paper.

Introducing the space-time weighted Bourgain spaces with norms respectively given by

$$\|u\|_{X_S^{s,b}} := \| <\xi>^s < \tau + |\xi|^2 >^b \hat{u}(\tau, \xi)\|_{L_{\tau,\xi}^2},$$

$$\|v\|_{X_W^{s,b}} := \| <\xi>^s < \tau + |\xi| >^b \hat{v}(\tau, \xi)\|_{L_{\tau,\xi}^2},$$

where $< \cdot > = < 1 + |\cdot|^2 >^{1/2}$. In a similar way, define the Y_S^k and Y_W^k space with the norm

$$\|u\|_{Y_S^k} := \| <\xi>^k < \tau + |\xi|^2 >^{-1} \hat{u}(\tau, \xi)\|_{L_\xi^2(L_\tau^1)},$$

and

$$\|u\|_{Y_W^k} := \| <\xi>^k < \tau + |\xi| >^{-1} \hat{u}(\tau, \xi)\|_{L_\xi^2(L_\tau^1)}.$$

In order to solve the Cauchy problem of the generalized Zakharov system (1.1) in the form of the integral equation (2.6), (2.7) by the contraction mapping theorem in the space $X_S^{s,b}, X_W^{k,l}$, similar to the method in [10], we recall some important lemmas which make some little modification for our target.

Lemma 2.1 *Assume $s, b \in \mathbb{R}$. Then we have*

$$\|\psi_1 S(t)u_0\|_{X_S^{s,b}} \leq \|\psi_1\|_{H_t^b}\|u_0\|_{H_x^s},$$

and

$$\|\psi_1 W(t)u_0\|_{X_W^{s,b}} \leq \|\psi_1\|_{H_t^b}\|u_0\|_{H_x^s}.$$

Moreover, define $\psi(S *_R f) = \psi \int_0^t S(t-\tau)f(\tau)d\tau$, if $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$, and $T \leq 1$, then we have

$$\|\psi_T(S *_R f)\|_{X_S^{s,b}} \leq CT^{1-b+b'}\|f\|_{X_S^{s,b'}}, \qquad (2.8)$$

and

$$\|\psi_T(W *_R f)\|_{X_W^{s,b}} \leq CT^{1-b+b'}\|f\|_{X_W^{s,b'}}. \qquad (2.9)$$

Remark 2.1 *While we deal with the (2.6) and (2.7), the following inequality will be used in order to get the positive power of T,*

$$\|\psi_T u\|_{X_S^{s,b}} \leq CT^{-b+\frac{1}{q}} \|u\|_{X_S^{s,b}},$$

where $s \in \mathbb{R}$, $b \geq 0$, $2 \leq q$ and $1 < bq$. if $X_W^{s,b}$ take place of $X_S^{s,b}$, the above result is also right.

Lemma 2.2 *Let $b_0 > \frac{1}{2}$ and $0 \leq \gamma \leq 1$. Assume $a, a_1, a_2 \geq 0$ satisfy*

$$(1 - \gamma)a < b_0,$$

$$(1 - \gamma) \max(a, a_1, a_2) \leq b_0 \leq (1 - \gamma)(a + a_1 + a_2), \qquad (2.10)$$

$$m \geq \frac{d}{2} + 1 - (1 - \gamma)(a + a_1 + a_2)/b_0 \geq 0 \qquad (2.11)$$

with strict inequality in (2.11L) if equality holds in (2.10R) or if $a_1 = 0$. Let $v, v_1, v_2 \in L^2$ such that $\mathcal{F}^{-1}(\langle \tau + |\xi| \rangle^{\gamma a} \hat{v}), \mathcal{F}^{-1}(\langle \tau + |\xi_i| \rangle^{\gamma a_i} \hat{v}_i), i = 1, 2$ have support in $|t| \leq CT$. Then we deduce the following estimates

$$\int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \xi \rangle^m} \leq CT^{\gamma(a+a_1+a_2)} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2}, \qquad (2.12)$$

$$\int \frac{\hat{v}\hat{v}_1\hat{v}_2}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \xi_2 \rangle^m} \leq CT^{\gamma(a+a_1+a_2)} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2}, \qquad (2.13)$$

where $\sigma = \tau + |\xi|, \quad \sigma_i = \tau + |\xi_i|^2, i = 1, 2$.

By virtue of Lemma 2.2, we obtain Lemma 3.4 in [10] with the positive power θ_1 of T, the θ_1 can be up to $\gamma(b + b_1 + c_1)$, we also get Lemma 3.5, Lemma 3.6, Lemma 3.7 of [10], the positive power of T is $\theta_2 = \gamma(c + 2b_1)$, $\theta_3 = \gamma(b + 1 + b_1)$, $\theta_4 = \gamma(1 + 2b_1)$ respectively.

Similar to the proof of Proposition 3.1 in [10], taking advantage of Lemma 2.1 and Lemma 2.2 to the integral equation (2.6), (2.7) respectively, it follows that

$$\begin{aligned}
\|E\|_{X_S^{k,b_1}} &\leq \|\psi_1(t)S(t)E_0\|_{X_S^{k,b_1}} + \|\psi_T S *_R f_1\|_{X_S^{k,b_1}} \\
&\leq C\|E_0\|_{H^k} + CT^{1-b_1+b'} \|f_1\|_{X_S^{k,b'}},
\end{aligned} \qquad (2.14)$$

and

$$\begin{aligned}
\|\varphi_\pm\|_{X_W^{l,b}} &\leq \|\psi_1(t)W(t)\varphi_{0\pm}\|_{X_W^{l,b}} + \left\|\psi_T[W *_R (f + \Lambda^{-1}\psi_{2T}(\varphi_+ + \varphi_-))]\right\|_{X_W^{l,b}} \\
&\leq C\|\varphi_{0\pm}\|_{H^l} + CT^{1-b-c} \|f\|_{X_W^{l,-c}} + CT\|\Lambda^{-1}\psi_{2T}(\varphi_+ + \varphi_-)\|_{X_W^{l,b}},
\end{aligned}$$
$$(2.15)$$

where $-\frac{1}{2} < b' \leq 0 \leq b_1 \leq b' + 1$ and $0 \leq b \leq 1 - c$, $c \in [0, \frac{1}{2}[$.

Let $b' = -c_1$, if k, l, b satisfies the condition of Proposition 3.1 in [10], then the second term of (2.14R) can be dealt with as follows

$$
\begin{aligned}
T^{1-b_1+b'}\|f_1\|_{X_S^{k,b'}} &= T^{1-b_1-c_1}\|f_1\|_{X_S^{k,-c_1}} \\
&\leq CT^{1-b_1-c_1}\Big(T^{\gamma(b+b_1+c_1)}\|E\|_{X_S^{k,b_1}}\|\varphi_\pm\|_{X_W^{l,b}} \\
&\quad + \|\psi_{3T}^3(E\otimes B)\|_{X_S^{k,-c_1}}\Big).
\end{aligned}
\tag{2.16}
$$

Similarly, the second term of (2.15R) can be estimated by

$$
\|f\|_{X_W^{l,-c}} \leq CT^{\gamma(c+2b_1)}\|E\|_{X_S^{k,b_1}}^2.
\tag{2.17}
$$

The solution $(E, \varphi_\pm)$ is locally well-posedness in the space $X_S^{k,b_1} \times X_W^{l,b}$, if $\psi_{3T}(E\otimes B)$ satisfies

$$
\|\psi_{3T}^3(E\otimes B)\|_{X_S^{k,-c_1}} \leq C\|E\|_{X_S^{k,b_1}}^3.
\tag{2.18}
$$

If $b, b_1 > \frac{1}{2}$, using Sobolev embedding theorem for time, then the solution satisfies

$$
(E, \varphi_\pm, \partial_t\varphi_\pm) \in \mathcal{C}([0, T[; H^k) \times \mathcal{C}([0, T[; H^l) \times \mathcal{C}([0, T[; H^{l-1}).
\tag{2.19}
$$

While b, $b_1 \leq \frac{1}{2}$, in order to obtain the continuity of time (2.19), we need to prove

$$
\psi_{3T}^3(E\otimes B) \in Y_S^k.
\tag{2.20}
$$

In the next section, we devote to getting the estimates (2.18), (2.10) and producing additional power of T in the process.

3 The Nonlinear Estimates

If the magnetic field B satisfies the case (A_1), then it follows that

$$
B = \frac{i\eta}{(\Delta + \beta I)}\nabla \times \nabla \times (E\otimes\overline{E}).
\tag{3.1}
$$

Substituting (3.1) into (2.18L), in order to derive (2.18R), it is sufficient to show

$$
|\mathcal{Q}| \leq CT^{\vartheta_1}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},
\tag{3.2}
$$

where

$$
\mathcal{Q} = \int \frac{\langle\xi_1\rangle^k \hat{v}\hat{v}_1\hat{v}_2\hat{v}_3}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle^{c_1}\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\xi\rangle^k\langle z_2\rangle^k\langle\xi_3\rangle^k},
$$

$\xi = \xi_1 - \xi_2$, $z_2 = \xi_2 - \xi_3$, $\tau = \tau_1 - \tau_2$, $s_2 = \tau_2 - \tau_3$, $\widetilde{\sigma_2} = s_2 + |z_2|^2$, $\sigma_i = \tau_i + |\xi|^2$, $\hat{v} = \langle\xi\rangle^k\langle\sigma\rangle^{b_1}\hat{E}(\xi, \tau)$, $\hat{v}_2 = \langle z_2\rangle^k\langle\widetilde{\sigma_2}\rangle^{b_1}\hat{E}(z_2, s_2)$, $\hat{v}_3 = \langle\xi_3\rangle^k\langle\sigma_3\rangle^{b_1}\hat{E}(\xi_3, \tau_3)$ and $v_1 \in L^2$.

Similarly, in order to estimate (2.20), we only to establish

$$|\mathcal{R}| \leq CT^{\vartheta_2} \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2} \|v_3\|_{L^2}, \tag{3.3}$$

where

$$\mathcal{R} = \int \frac{\langle \xi_1 \rangle^k \hat{v} \hat{v}_1 \hat{v}_2 \hat{v}_3}{\langle \sigma \rangle^{b_1} \langle \sigma_1 \rangle \langle \widetilde{\sigma_2} \rangle^{b_1} \langle \sigma_3 \rangle^{b_1} \langle \xi \rangle^k \langle z_2 \rangle^k \langle \xi_3 \rangle^k},$$

with $v_1 \in L_x^2$, the other notation is the same as the above (3.2).

On the other hand, if

$$A = -\gamma \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{B(t,y)}{|x-y|^2} dy,$$

plugging it into system $(1.1)_3$, after taking the partial Fourier transformation with respect to the space variable, for $d > 2$,

$$|\xi|^2 \widehat{B}(\xi) - i\eta \xi \times \xi(\widehat{E \otimes \overline{E}})(\xi) + \gamma c_d |\xi|^{2-d} \widehat{B_t}(\xi) = 0, \tag{3.4}$$

where we have used the equality $\widehat{|\cdot|^{-2}} = c_d |\cdot|^{2-d}$, for $d > 2$. Consequently,

$$\partial_t \widehat{B} + \frac{1}{\gamma c_d} |\xi|^d \widehat{B} - \frac{i\eta}{\gamma c_d} |\xi|^{d-2} \xi \times \xi(\widehat{E \otimes \overline{E}})(\xi) = 0. \tag{3.5}$$

Therefore, we can solve $\widehat{B}$ as follows

$$\widehat{B} = e^{-\frac{1}{\gamma c_d} |\xi|^d t} \widehat{B}_0 + \int_0^t e^{-\frac{1}{\gamma c_d} |\xi|^d (t-\tau)} \frac{i\eta}{\gamma c_d} |\xi|^{d-2} \xi \times \xi(\widehat{E \otimes \overline{E}}) d\tau,$$

which is equivalent to

$$B = e^{-\frac{t}{\gamma c_d}(-\Delta)^{\frac{d}{2}}} B_0 + \int_0^t e^{-\frac{1}{\gamma c_d}(-\Delta)^{\frac{d}{2}}(t-\tau)} \frac{i\eta}{\gamma c_d} (-\Delta)^{\frac{d-2}{2}} (\nabla \times \nabla \times (E \otimes \overline{E})) d\tau. \tag{3.6}$$

By virtue of the fractional parabolic equation theory, we only to estimate (3.2), (3.3). In order to prove (3.2), (3.3), we first give the following lemma.

Lemma 3.1 *Let $b_0 > \frac{1}{2}$ and $\gamma \in [0,1]$. If $a, a_1, a_2, a_3 \geq 0$ and $0 < \eta, \eta_i \leq 1, i = 1, 2, 3$ satisfy*

$$(1 - \gamma)[\eta a + \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = 2b_0. \tag{3.7}$$

Let $v, v_1, v_2, v_3 \in L^2$ such that $\mathcal{F}^{-1}(\langle \tau + |\xi| \rangle^{\gamma a} \hat{v}), \mathcal{F}^{-1}(\langle \tau + |\xi_i| \rangle^{\gamma a_i} \hat{v}_i), i = 1, 2$ have support in $|t| \leq CT$. Then the following inequalities hold

$$\int \frac{\hat{v} \hat{v}_1 \hat{v}_2 \hat{v}_3}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \widetilde{\sigma_2} \rangle^{a_2} \langle \sigma_3 \rangle^{a_3} \langle \xi \rangle^m} \leq CT^\theta \|v\|_{L^2} \|v_1\|_{L^2} \|v_2\|_{L^2} \|v_3\|_{L^2}, \tag{3.8}$$

if $m \geq d - (1 - \gamma)[(1 - \eta_1)a_1 + (1 - \eta_2)a_2 + (1 - \eta_3)a_3]/b_0 \geq 0$.

$$\int \frac{\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\widetilde{\sigma_2}\rangle^{a_2}\langle\sigma_3\rangle^{a_3}\langle\xi_3\rangle^{m_3}} \leq CT^\theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \quad (3.9)$$

if $m_3 \geq d - (1 - \gamma)[(1 - \eta)a + (1 - \eta_1)a_1 + (1 - \eta_2)a_2]/b_0 \geq 0$.

$$\int \frac{\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\widetilde{\sigma_2}\rangle^{a_2}\langle\sigma_3\rangle^{a_3}\langle\xi\rangle^m\langle\xi_3\rangle^{m_3}} \leq CT^\theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},$$
$$(3.10)$$

if $m + m_3 \geq d - (1 - \gamma)[(1 - \eta_1)a_1 + (1 - \eta_2)a_2]/b_0 \geq 0$.

$$\int \frac{\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\widetilde{\sigma_2}\rangle^{a_2}\langle\sigma_3\rangle^{a_3}\langle z_2\rangle^{m_2}\langle\xi_3\rangle^{m_3}} \leq CT^\theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},$$
$$(3.11)$$

if $m_2 + m_3 \geq d - (1 - \gamma)[(1 - \eta)a + (1 - \eta_1)a_1]/b_0 \geq 0$.

$$\int \frac{\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3}{\langle\sigma\rangle^a\langle\sigma_1\rangle^{a_1}\langle\widetilde{\sigma_2}\rangle^{a_2}\langle\sigma_3\rangle^{a_3}\langle z_2\rangle^{m_2}\langle\xi\rangle^m} \leq CT^\theta\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},$$
$$(3.12)$$

if $m + m_2 \geq d - (1 - \gamma)[(1 - \eta_1)a_1 + (1 - \eta_3)a_3]/b_0 \geq 0$, *where* $\theta = \gamma(a + a_1 + a_2 + a_3)$.

Proof. We first prove (3.8), taking advantage of Hölder inequality in space time to (3.8L), we have

$$(3.8L) \leq \|\mathcal{F}^{-1}(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-a}|\hat{v}|)\|_{L_t^q(L_x^r)}\|\mathcal{F}^{-1}(\langle\widetilde{\sigma_2}\rangle^{-a_2}|\hat{v}_2|)\|_{L_t^{q_2}(L_x^{r_2})}$$
$$\times \prod_{i=1,3} \|\mathcal{F}^{-1}(\langle\sigma_i\rangle^{-a_i}|\hat{v}_i|)\|_{L_t^{q_i}(L_x^{r_i})} \qquad (3.13)$$
$$\leq C\|\mathcal{F}^{-1}(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-a}|\hat{v}|)\|_{L_t^q(L_x^r)} \prod_{i=1}^3 T^{\gamma a_i}\|v_i\|_{L^2},$$

where the coefficient satisfies

$$\frac{1}{q} + \sum_{i=1}^3 \frac{1}{q_i} = 1, \quad \frac{1}{r} + \sum_{i=1}^3 \frac{1}{r_i} = 1, \qquad (3.14)$$

the second inequality comes from Lemma 3.1 in [10] with the coefficient satisfies for $i = 1, 2, 3$

$$\frac{2}{q_i} = 1 - \eta_i(1 - \gamma)a_i/b_0, \quad \delta_i := (\frac{d}{2} - \frac{d}{r_i}) = (1 - \eta_i)(1 - \gamma)a_i/b_0, \quad (3.15)$$

Note that if $m \geq \delta(r) := \frac{d}{2} - \frac{d}{r}$, then $H^m \hookrightarrow L^r$, consequently

$$\|\mathcal{F}^{-1}(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-a}\hat{v})\|_{L_t^q(L_x^r)} \leq \|\mathcal{F}^{-1}(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-a}\hat{v})\|_{L_t^q(H_x^m)}$$
$$\leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-a}\hat{v})\|_{L_t^q(L_x^2)} \qquad (3.16)$$
$$\leq CT^{\gamma a}\|v\|_{L^2},$$

the last inequality comes from Lemma 2.4 in [10] with $\frac{2}{q} = 1 - (1-\gamma)\eta a/b_0$. Plugging (3.16) into (3.13), one can easily get (3.8). Moreover, we have

$$(1 - \gamma)[\eta a + \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = 2b_0$$

and

$$m \geq \delta(r) = d - (\delta_1 + \delta_2 + \delta_3) = d - (1-\gamma)[(1-\eta_1)a_1 + (1-\eta_2)a_2 + (1-\eta_3)a_3]/b_0 \geq 0.$$

Next, we will estimate (3.9) as follows

$$(3.9L) \leq \|\mathcal{F}^{-1}(\langle\sigma\rangle^{-a}|\hat{v}|)\|_{L_t^q(L_x^r)}\|\mathcal{F}^{-1}(\langle\sigma_1\rangle^{-a_1}|\hat{v}_1|)\|_{L_t^{q_1}(L_x^{r_1})}$$
$$\times \|\mathcal{F}^{-1}(\langle\widetilde{\sigma_2}\rangle^{-a_2}|\hat{v}_2|)\|_{L_t^{q_2}(L_x^{r_2})}\|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-a_3}\langle\xi_3\rangle^{-m_3}|\hat{v}_3|)\|_{L_t^{q_3}(L_x^{r_3})}$$
$$\leq C\|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-a_3}\langle\xi_3\rangle^{-m_3}|\hat{v}_3|)\|_{L_t^{q_3}(L_x^{r_3})}T^{\gamma a}\|v\|_{L^2}\prod_{i=1}^{2}T^{\gamma a_i}\|v_i\|_{L^2}$$
$$\leq CT^{\gamma(a+a_1+a_2+a_3)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},$$

$$(3.17)$$

where the constants satisfy (3.14), (3.15) and $m_3 \geq \delta_3 \geq 0$, which derives (3.7) and

$$m_3 \geq d - (1-\gamma)[(1-\eta)a + (1-\eta_1)a_1 + (1-\eta_2)a_2]/b_0 \geq 0.$$

We now show (3.10), applying Hölder inequality in space time to (3.10L) to yield

$$(3.10L) \leq \|\mathcal{F}^{-1}(\langle\xi\rangle^{-m}\langle\sigma\rangle^{-a}|\hat{v}|)\|_{L_t^q(L_x^r)}\|\mathcal{F}^{-1}(\langle\xi_3\rangle^{-m_3}\langle\sigma_3\rangle^{-a_3}$$
$$|\hat{v}_3|)\|_{L_t^q(L_x^r)} \times \prod_{i=1}^{2}\|\mathcal{F}^{-1}(\langle\sigma_i\rangle^{-a_i}|\hat{v}_i|)\|_{L_t^{q_i}(L_x^{r_i})} \quad (3.18)$$
$$\leq CT^{\gamma(a+a_1+a_2+a_3)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2},$$

where we used (3.16) and

$$\|\mathcal{F}^{-1}(\langle\xi_3\rangle^{-m_3}\langle\sigma_3\rangle^{-a_3}\hat{v}_3)\|_{L_t^{q_3}(L_x^{r_3})}$$
$$\leq \|\mathcal{F}^{-1}(\langle\xi_3\rangle^{-m_3}\langle\sigma_3\rangle^{-a_3}\hat{v}_3)\|_{L_t^{q_3}(H_x^{m_3})}$$
$$\leq \|\mathcal{F}^{-1}(\langle\sigma_3\rangle^{-a_3})\|_{L_t^{q_3}(L_x^2)} \quad (3.19)$$
$$\leq CT^{\gamma a_3}\|v\|_{L^2},$$

with $m_3 \geq \delta_3 := \frac{d}{2} - \frac{d}{r_3} \geq 0$, $\frac{2}{q_3} = 1 - \frac{\eta_3(1-\gamma)a_3}{b_0}$. Thus, we have

$$m + m_3 \geq \delta + \delta_3 = d - (\delta_1 + \delta_2) = d - \sum_{i=1}^{2}(1-\gamma)(1-\eta_i)a_i/b_0.$$

Similarly, one can easily check that (3.11) and (3.12). This completes the proof of Lemma 3.1. ∎

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Lemma 3.2 *Assume $b_0 > \frac{1}{2}, \gamma \in [0,1]$ and $\eta, \eta_i \in (0,1]$, $0 < b_1, c_1 < b_0$. Suppose the function $\mathcal{F}^{-1}(\langle\sigma\rangle^{-b_1}\hat{v})$ and $\mathcal{F}^{-1}(\langle\sigma_i\rangle^{-a_i}\hat{v}_i), i = 1,2,3$, have support in $t \leq CT$. If*

$$(1-\gamma)[(\eta + \eta_2 + \eta_3)b_1 + \eta_1 c_1] = 2b_0, \qquad and$$

$$
\begin{cases}
2k \geq d - (1-\gamma)[(1-\eta_1)c_1 + (1-\eta_2)b_1]/b_0 \geq 0, \\
2k \geq d - (1-\gamma)[(1-\eta)b_1 + (1-\eta_1)c_1]/b_0 \geq 0, \\
2k \geq d - (1-\gamma)[(1-\eta_1)c_1 + (1-\eta_3)b_1]/b_0 \geq 0.
\end{cases}
$$

Then the estimate (3.2) holds for all $T \leq T_0 < \infty$ with

$$|\mathcal{Q}| \leq CT^{\gamma(3b_1+c_1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}. \tag{3.20}$$

Proof. In order to derive (3.20), we divide the integration region into two subregions:

Case 1, if $|\xi_1| \leq 2|z_2|$, we estimate the contribution $\mathcal{Q}_1$ of that region to $\mathcal{Q}$ by

$$|\mathcal{Q}_1| \leq \int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle^{c_1}\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\xi\rangle^k\langle\xi_3\rangle^k}. \tag{3.21}$$

Thanks to (3.10) of Lemma 3.1 with $(a, a_1, a_2, a_3, m, m_3) = (b_1, c_1, b_1, b_1, k, k)$, it follows that

$$|\mathcal{Q}_1| \leq CT^{\gamma(3b_1+c_1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \tag{3.22}$$

and

$$2k \geq d - (1-\gamma)[(1-\eta_1)c_1 + (1-\eta_2)b_1]/b_0 \geq 0.$$

Case 2, if $|\xi_1| \geq 2|z_2|$, then $\frac{|\xi_1|}{2} \leq |\xi_1 - z_2| \leq \frac{3}{2}|\xi_1|$, we estimate the contribution $\mathcal{Q}_2$ of that region to $\mathcal{Q}$ by

$$
\begin{aligned}
|\mathcal{Q}_2| &\leq C \int \frac{\langle\xi_1 - z_2\rangle^k|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle^{c_1}\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\xi\rangle^k\langle z_2\rangle^k\langle\xi_3\rangle^k} \\
&\leq C \int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle^{c_1}\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle z_2\rangle^k\langle\xi_3\rangle^k} \\
&\quad + C \int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle^{c_1}\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle z_2\rangle^k\langle\xi\rangle^k} \\
&:= \mathcal{Q}_{21} + \mathcal{Q}_{22}.
\end{aligned}
\tag{3.23}
$$

By virtue of (3.11) of Lemma 3.1 with $(a, a_1, a_2, a_3, m, m_3) = (b_1, c_1, b_1, b_1, k, k)$, we deduce that

$$|\mathcal{Q}_{21}| \leq CT^{\gamma(3b_1+c_1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \tag{3.24}$$

and

$$2k \geq d - (1-\gamma)[(1-\eta)b_1 + (1-\eta_1)c_1]/b_0 \geq 0.$$

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Similarly, we can estimate $\mathcal{Q}_{22}$ as follows

$$|\mathcal{Q}_{22}| \leq CT^{\gamma(3b_1+c_1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \qquad (3.25)$$

and

$$2k \geq d - (1-\gamma)[(1-\eta_1)c_1 + (1-\eta_3)b_1]/b_0 \geq 0.$$

This completes the proof of Lemma 3.2. ∎

Lemma 3.3 *Let $b_0 > \frac{1}{2}$ and $\gamma \in [0,1]$. Given $\eta, \eta_i \in (0,1]$, and $0 < b_1, c_1 < b_0$. Assume that the function $\mathcal{F}^{-1}(\langle\sigma\rangle^{-b_1}\hat{v})$ and $\mathcal{F}^{-1}(\langle\sigma_i\rangle^{-a_i}\hat{v}_i), i = 1, 2, 3$, have support in $t \leq CT$. If*

$$(1-\gamma)[(\eta + \eta_2 + \eta_3)b_1 + \eta_1] = 2b_0, \qquad and$$

$$\begin{cases} 2k \geq d - (1-\gamma)[(1-\eta_1) + (1-\eta_2)b_1]/b_0 \geq 0, \\ k \geq d - (1-\gamma)[(2-\eta-\eta_2)b_1 + 1 - \eta_1]/b_0 \geq 0, \\ k \geq d - (1-\gamma)[1-\eta_1 + (2-\eta_2-\eta_3)b_1]/b_0 \geq 0. \end{cases}$$

Then the estimate (3.3) holds for all $T \leq T_0 < \infty$ with

$$|\mathcal{R}| \leq CT^{\gamma(3b_1+1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}. \qquad (3.26)$$

Proof. In order to estimate (3.26), we divide the integration region into two subregions:

Region $|\xi_1| \leq 2|z_2|$: We estimate the contribution $\mathcal{R}_1$ of that region to $\mathcal{R}$ by

$$|\mathcal{R}_1| \leq \int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\xi\rangle^k\langle\xi_3\rangle^k}. \qquad (3.27)$$

Thanks to (3.10) of Lemma 3.1 with $(a, a_1, a_2, a_3, m, m_3) = (b_1, 1, b_1, b_1, k, k)$, one can easily check that

$$|\mathcal{R}_1| \leq CT^{\gamma(3b_1+1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \qquad (3.28)$$

and

$$2k \geq d - (1-\gamma)[(1-\eta_1) + (1-\eta_2)b_1]/b_0 \geq 0.$$

Region $|\xi_1| \geq 2|z_2|$: Since $\frac{|\xi_1|}{2} \leq |\xi_1 - z_2| \leq \frac{3}{2}|\xi_1|$, we estimate the contribution $\mathcal{R}_2$ of that region to $\mathcal{R}$ by

$$\begin{aligned} |\mathcal{R}_2| \leq{}& C\int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\langle\xi_3\rangle^k} \\ &+ C\int \frac{|\hat{v}\hat{v}_1\hat{v}_2\hat{v}_3|}{\langle\sigma\rangle^{b_1}\langle\sigma_1\rangle\langle\widetilde{\sigma_2}\rangle^{b_1}\langle\sigma_3\rangle^{b_1}\langle\xi\rangle^k} \\ :={}& \mathcal{R}_{21} + \mathcal{R}_{22}. \end{aligned} \qquad (3.29)$$

By virtue of (3.8), (3.9) of Lemma 3.1 with $(a, a_1, a_2, a_3, m_3) = (b_1, 1, b_1, b_1, k)$, $(a, a_1, a_2, a_3, m) = (b_1, 1, b_1, b_1, k)$, we can end up with

$$|\mathcal{R}_{21}| + |\mathcal{R}_{22}| \leq CT^{\gamma(3b_1+1)}\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}\|v_3\|_{L^2}, \qquad (3.30)$$

with the efficient satisfies $k \geq d - (1-\gamma)[(2-\eta-\eta_2)b_1 + 1 - \eta_1]/b_0 \geq 0$, and

$$k \geq d - (1-\gamma)[1 - \eta_1 + (2-\eta_2-\eta_3)b_1]/b_0 \geq 0.$$

This completes the proof of Lemma 3.3. ∎

Analogous to Proposition 1.1 in [10], in view of Lemma 3.2, Lemma 3.3, we have the following well-posedness result.

Theorem 3.1 *Let the space dimension $d > 1$. Assume k, l satisfy*

$$k \in [l, l+1], \ l > \frac{d}{2} - 2, \ 2k - (l+1) > \frac{d}{2} - 2, \qquad if \quad d \geq 4, \qquad (3.31)$$

$$k \in [l, l+1], \qquad l \geq 0, \qquad 2k - (l+1) \geq 0, \qquad if \quad d = 2, 3. \qquad (3.32)$$

Then GZ system (1.1) in the case (A_1) (or in the case (A_2) if $d > 2$) with the initial data $(E_0, n_0, \partial_t n_0) \in H^k \times H^l \times H^{l-1}$ is locally well-posedness in $X_S^{k,b_1} \times X_W^{l,b} \times X_W^{l-1,b}$ with the $B_0 \in H^k$ for suitable b_1, b close to $\frac{1}{2}$. Moreover, the solutions satisfy

$$(E, n, \partial_t n) \in \mathcal{C}([0, T[; H^k \times H^l \times H^{l-1}). \qquad (3.33)$$

Proof. By virtue of Lemma 3.2 and Lemma 3.3. Substituting (2.18) into (2.16), then plugging (2.16), (2.17) into (2.14), (2.15) respectively, it follows from (2.14) and (2.15) that

$$\begin{aligned}
\|E\|_{X_S^{k,b_1}} \leq C\|E_0\|_{H^k} + CT^{1-b_1-c_1}\Big(T^{\gamma_1(b+b_1+c_1)}\|\psi_{2T}E\|_{X_S^{k,b_1}} \\
\times \|\psi_{2T}\varphi_\pm\|_{X_W^{l,b}} + T^{\gamma_2(3b_1+c_1)}\|\psi_{3T}E\|_{X_S^{k,b_1}}^3\Big),
\end{aligned} \qquad (3.34)$$

and

$$\begin{aligned}
\|\varphi_\pm\|_{X_W^{l,b}} \leq C\|\varphi_{0\pm}\|_{H^l} + CT^{1-b-c}\Big(T^{\gamma_3(2b_1+c)} \\
\times \|\psi_{2T}E\|_{X_S^{k,b_1}}^2 + T^{b+c}\|\psi_{2T}\varphi_\pm\|_{X_W^{l,b}}\Big),
\end{aligned} \qquad (3.35)$$

where k, l satisfy (3.31), (3.32) and the condition of Lemma 3.2. Consequently, we solve (3.34) and (3.35) by the contraction mapping argument for small enough time T in space $X_S^{k,b_1} \times X_W^{l,b}$. It remains only to be proved that under condition (3.31) and (3.32), we can choose $b_1, c_1, \eta_i \in (0, 1]$ and $\gamma \in [0, 1], i = 1, 2, 3$ satisfying the assumptions of Lemma 3.2 and Lemma 3.3 if needed. Since (3.31), (3.32) and the assumptions of Lemma 3.2 and Lemma 3.3 are consistency condition, which it is not difficult to check.

At this point we have obtained the existence, uniqueness and continuity of local solution in time for the cut-off equation (2.6) and (2.7). Similar to the method on page 415–416 in [10], one can easily check that the solutions are in fact not depend with the cut-off time. This completes the proof of Theorem 3.1. ∎

Remark 3.1 *Although the form of system (1.1) is more complex than it of Eq.(1.2), the result well-posedness of Theorem 3.1 to the GZ system (1.1) with magnetic field (A_1) (or (A_2), if $d > 2$) is the same as the Proposition 1.1 [10], which is proved by J. Ginibre, Y. Tsutsumi and G. Velo in 1996.*

Remark 3.2 *If the space dimension $d = 1$, then $E \otimes B = 0$, the system (1.1) with the case (A_1) becomes Eq.(1.2). Thus the GZ system (1.1) is locally well-posedness for $(E_0, n_0, \partial_t n_0) \in H^k \times H^l \times H^{l-1}$, if the indexes k and l satisfy*

$$-\frac{1}{2} \le k - l \le 1, \quad 0 \le l + \frac{1}{2} \le 2k.$$

Moreover, if the initial data $(E_0, n_0, \partial_t n_0) \in H^1 \times L^2 \times H^{-1}$, then there exists a global solution $(E, n, \partial_t n)$, which satisfies

$$(E, n, \partial_t n) \in \mathcal{C}(\mathbb{R}^+; H^1) \times \mathcal{C}(\mathbb{R}^+; L^2) \times \mathcal{C}(\mathbb{R}^+; H^{-1}).$$

Remark 3.3 *The initial data $B_0 \in H^k$ in Theorem 3.1 is necessary. In fact, if A satisfies the case (A_1), then we can derive from (3.6) that*

$$\|B\|_{X_S^{k,b_1}} \le \|B_0\|_{H^k} + C\|E\|_{X_S^{k,b_1}}^2.$$

Remark 3.4 *By the energy estimation, C. Laurey [19] proves the local existence of solution in the spaces*

$$(E, n, B) \in \mathcal{C}([0, T[; H^{s+1}) \times \mathcal{C}([0, T[; H^s) \times \mathcal{C}([0, T[; H^{s+1}),$$

to system (1.1) with the case (A_1), (A_2), for $d = 2, 3$. In fact, similarly, we also can prove for some $T > 0$ that the solutions satisfy

$$(E, n, B) \in \mathcal{C}([0, T[; B_{p,r}^{s+1}) \times \mathcal{C}([0, T[; B_{p,r}^s) \times \mathcal{C}([0, T[; B_{p,r}^{s+1}),$$

for $s > \frac{d}{2}$.

4   The lower bound for the blow-up rate of blow-up solutions

In 1994, L. Glangetas and F. Merle [11] proved the following form of self-similar blow-up solutions to equation (1.2) in $\mathbb{R}^2$, i.e.

$$\begin{cases} E(t, x) = \frac{\omega}{T-t} e^{i(\theta + \frac{|x|^2 - 4\omega^2}{4(-T+t)})} P\left(\frac{x\omega}{T-t}\right), \\ n(t, x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{(T-t)}\right). \end{cases} \tag{4.1}$$

where $\omega > \omega_0, \theta \in \mathbb{R}$, $P(x) = P(|x|), N(x) = N(|x|)$, and (P, N) satisfies the elliptic equation

$$\begin{cases} \Delta P - P = NP, \\ \frac{1}{(c_0\omega)^2}(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta P^2 \end{cases}$$

with $r = |x|, \Delta w = w_{rr} + \frac{1}{r}w_r$.

The situation for Zakharov equation in $\mathbb{R}^3$ is more complex. Until now, there are no known explicit blow-up solutions, M. Landman, etc. observed an asymptotic self-similar blow-up solution for Zakharov equation (1.2) in $\mathbb{R}^3$ of the form [18]

$$\begin{cases} E(t, x) = \frac{2}{3(T-t)}e^{i(T-t)^{-1/3}}P\left(\frac{|x|}{\sqrt{3}(T-t)^{2/3}}\right), \\ n(t, x) = \frac{1}{3(T-t)^{4/3}}N\left(\frac{|x|}{\sqrt{3}(T-t)^{2/3}}\right). \end{cases} \qquad (4.2)$$

where $P(x) = P(|x|), N(x) = N(|x|)$, and (P, N) satisfies the elliptic equation

$$\begin{cases} \Delta P - P = NP, \\ \frac{1}{(2}(2r^2 N_{rr} + 13rN_r + 14N) = \Delta P^2. \end{cases}$$

In this subsection, we consider the singular solution of the system (1.1) in the case (A_1) and (A_2) in finite time, we will establish the lower bound for the blow-up rate of the blow-up solution to system (1.1).

Theorem 4.1 *Let k, l satisfy (3.31) and (3.32). Assume that the initial data (E_0, B_0, n_0, n_1) belongs to $H^k \times H^k \times H^l \times H^{l-1}$. Then there exists a time $T > 0$ depending only on (E_0, B_0, n_0, n_1) and a unique solution $(E, n, \partial_t n)$ to system (1.1) with the initial data (E_0, n_0, n_1), which is guaranteed by Theorem 3.1. If the solution blows up in finite time T^* in the space $H^k \times H^l \times H^{l-1}$, then we have the lower bound for the blow-up rate of blow-up solution satisfies for any $\epsilon > 0$*

$$(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}}) > C\frac{1}{(T^* - t)^{\frac{3}{4}\vartheta_l - \epsilon}},$$

where $\vartheta_l = \frac{1}{4}(2l + 4 - d)$.

Proof. Let $b = b_1, c = c_1$ and $\gamma_1 = \gamma_3$, in view of (3.34) and (3.35), we have

$$\begin{aligned} \|E\|_{X_S^{k,b}} &\leq C\|E_0\|_{H^k} + CT^{1-b-c+\gamma_1(2b+c)}\|\psi_{2T}E\|_{X_S^{k,b}} \\ &\quad \times \|\psi_{2T}\varphi_\pm\|_{X_W^{l,b}} + CT^{1-b-c+\gamma_2(3b+c)}\|\psi_{3T}E\|_{X_S^{k,b}}^3 \\ &\leq C\|E_0\|_{H^k} + CT^{2-3b-c+\gamma_1(2b+c)}\|E\|_{X_S^{k,b}}\|\varphi_\pm\|_{X_W^{l,b}} \\ &\quad + CT^{\frac{5}{2}-4b-c+\gamma_2(3b+c)}\|E\|_{X_S^{k,b}}^3, \end{aligned} \qquad (4.3)$$

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and

$$\|\varphi_\pm\|_{X_W^{l,b}} \leq C\|\varphi_{0\pm}\|_{H^l} + CT^{1-b-c}\Big(T^{\gamma_1(2b+c)}\|\psi_{2T}E\|_{X_S^{k,b}}^2 + T^{b+c}\|\psi_{2T}\varphi_\pm\|_{X_W^{l,b}}\Big)$$

$$\leq C\|\varphi_{0\pm}\|_{H^l} + C\Big(T^{2-3b-c+\gamma_1(2b+c)}\|E\|_{X_S^{k,b}}^2 + T^{\frac{3}{2}-b}\|\varphi_\pm\|_{X_W^{l,b}}\Big),$$
(4.4)

the last inequality of (4.3), (4.4) comes from Remark 2.1 with $q = 2$, where k, l satisfy (3.31), (3.32) and the other constants satisfy

$$b_0 > \frac{1}{2}, \ 0 < b, c \leq b_0, \ \gamma_i \in [0, 1],$$
(4.5)

$$(1 - \gamma_1)\max\{b, c\} \leq b_0 \leq (1 - \gamma_1)(2b + c),$$
(4.6)

$$l \geq \frac{d}{2} + 1 - (1 - \gamma_1)(2b + c)/b_0,$$
(4.7)

and

$$\begin{cases} (1 - \gamma_2)[(\eta + \eta_2 + \eta_3)b_1 + \eta_1 c] = 2b_0, \quad \eta_i \in (0, 1], \\ 2k \geq d - (1 - \gamma_2)[(1 - \eta_1)c + (1 - \eta_2)b]/b_0 \geq 0, \\ 2k \geq d - (1 - \gamma_2)[(1 - \eta)b + (1 - \eta_1)c]/b_0 \geq 0, \\ 2k \geq d - (1 - \gamma_2)[(1 - \eta_1)c + (1 - \eta_3)b]/b_0 \geq 0, \end{cases}$$
(4.8)

In view of (4.7), in order to let γ_1 large enough, we choose

$$\gamma_1 = 1 - \frac{(\frac{d}{2} + 1 - l)b_0}{2b + c}.$$
(4.9)

Due to $c \in (0, \frac{1}{2}), b > \frac{1}{2}$. Substituting (4.9) into (4.6) to yield

$$\frac{d}{2} - 2 < l < \frac{d + 2}{2}.$$
(4.10)

By virtue of (4.9) and (4.10), let $b = \frac{1}{2} + \epsilon, b_0 = \frac{1}{2} + \epsilon_0, 0 < \epsilon \leq \epsilon_0$, it follows that

$$\begin{aligned} 2 - 3b - c + \gamma_1(2b + c) &= 2 - b + (l - 1 - \frac{d}{2})b_0 \\ &= \frac{4 - d}{4} + \frac{l}{2} - \epsilon - \left(\frac{d + 2}{2} - l\right)\epsilon_0 \\ &:= \vartheta_l^- \\ &< \frac{1}{4}(2l + 4 - d) := \vartheta_l. \end{aligned}$$
(4.11)

Let $\frac{5}{2} - 4b - c + \gamma_2(3b + c) = \frac{3}{2}\vartheta_l^-$, which can be guaranteed by (4.8).

Combining (4.3) with (4.4), in view of Hölder inequality to give by

$$\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}} \leq C(\|E_0\|_{X_S^{k,b}} + \|\varphi_{0\pm}\|_{X_W^{l,b}}) + CT^{\frac{3}{2}-b}\|\varphi_\pm\|_{X_W^{l,b}}$$
$$+ CT^{\vartheta_l^-}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}})^2 + CT^{\frac{3}{2}\vartheta_l^-}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}})^3$$
$$\leq C(\|E_0\|_{X_S^{k,b}} + \|\varphi_{0\pm}\|_{X_W^{l,b}}) + CT^{\frac{3}{2}\vartheta_l^-}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}})^3$$
$$+ CT^{\frac{3}{2}-b}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}}).$$
$$(4.12)$$

Note that $\frac{3}{2} - b > 0$, if T is small enough such that $CT^{\frac{3}{2}-b} \leq \frac{1}{2}$, then we have

$$\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}} \leq C(\|E_0\|_{X_S^{k,b}} + \|\varphi_{0\pm}\|_{X_W^{l,b}}) + CT^{\frac{3}{2}\vartheta_l^-}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}})^3.$$
$$(4.13)$$

Next, we will infer a lower bound on the blow-up rate of blow-up solution. Denote by T^* the supremum of the existence time $T > 0$ for which there exists a solution (E, n) of the Zakharov system (1.1) satisfying

$$(\|E\|_{X_S^{k,b}} + \|n\|_{X_W^{l,b}} + \|\partial_t n\|_{X_W^{l-1,b}}) < \infty.$$

Then for all time $t \in [0, T)[$, the solutions satisfy

$$\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}} < \infty,$$

which is guaranteed by the local well-posedness of Theorem 3.1. By the maximality of T^*, it follows that

$$\|E(t)\|_{L_{T^*}^\infty(H^k)} + \|n(t)\|_{L_{T^*}^\infty(H^l)} + \|\partial_t n(t)\|_{L_{T^*}^\infty(H^{l-1})} = \infty.$$

Otherwise, the Cauchy problem of system (1.1) at time T^* with the initial data $(E(T^*, \cdot), n(T^*, \cdot))$ would be well-defined and the local existence theory would extend the solution (E, n) beyond T^*. Thus, if $T^* < \infty$, the solution blows up and

$$\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}} \longrightarrow \infty \qquad t \to T^*.$$

Consider the solution (E, n) posed at some time $t \in [0, T^*[$. Assume for some M such that

$$C(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}}) + C(T - t)^{\frac{3}{2}\vartheta_l^-} M^3 \leq M.$$

Then $T < T^*$. Consequently, $\forall M > 0$

$$C(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}}) + C(T^* - t)^{\frac{3}{2}\vartheta_l^-} M^3 > M.$$

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Choosing $M = 2C(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}})$, we deduce that

$$C(T^* - t)^{\frac{3}{2}\vartheta_l^-} M^3 > M,$$

which is equivalent to

$$(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}}) > C\frac{1}{(T^* - t)^{\frac{3}{4}\vartheta_l^-}}.$$

This completes the proof of Theorem 4.1. ∎

Corollary 4.1 *Under the assumption of Theorem 4.1. If we neglect the magnetic B, then the classical Zakharov Eq.(1.2) is locally well-posedness. If the solution (E, n) blows up in finite time T^* in the space $H^k \times H^l \times H^{l-1}$, then we have the lower bound for the blow-up rate of blow-up solution satisfies for any $\epsilon > 0$*

$$(\|E\|_{H^k} + \|n\|_{H^l} + \|\partial_t n\|_{H^{l-1}}) > C\frac{1}{(T^* - t)^{\vartheta_l - \epsilon}},$$

where $\vartheta_l = \frac{1}{4}(2l + 4 - d)$.

Proof. As the process of (4.12), we have

$$\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}} \le C(\|E_0\|_{H^k} + \|\varphi_{0\pm}\|_{H^l}) + CT^{\vartheta_l^-}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}})^2$$
$$+ CT^{\frac{5}{2} - 2b - c}(\|E\|_{X_S^{k,b}} + \|\varphi_\pm\|_{X_W^{l,b}}),$$

(4.14)

where $\vartheta_l = \frac{1}{4}(2l + 4 - d)$. Similarly, one can easily get the lower bound for blow-up rate of blow-up solution to Eq.(1.2)

$$(\|E(t)\|_{H^k} + \|n(t)\|_{H^l} + \|\partial_t n(t)\|_{H^{l-1}}) > C\frac{1}{(T^* - t)^{\vartheta_l - \epsilon}}. \qquad (4.15)$$

This concludes the proof of Corollary 4.1. ∎

Remark 4.1 *If we consider the self-similar blow-up solution (E, n) of (4.1) to Eq.(1.2) which blows up in a finite time T^* in $\mathbb{R}^2$, then we obtain the blow-up rate of blow-up solution n satisfying*

$$\|n\|_{\dot{H}^l} = \left(\frac{\omega}{T^* - t}\right)^{l+1} \|N\|_{\dot{H}^l}.$$

In [21], F. Merle prove the optimal lower bound of the blow-up rate of the solution (E, n) in space $H^1 \times L^2$ in 2D is $C\frac{1}{(T^* - t)}$. However, for $d = 3$, the homogeneous norm of n the asymptotic self-similar blow-up solution (4.2) is

$$\|n\|_{\dot{H}^l} = \frac{C}{(T^* - t)^{\frac{1}{3}(2l+1)}} \|N\|_{\dot{H}^l}.$$

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Remark 4.2 *As $d = 3$, the result of Corollary 4.1 was obtained in [6]. If we consider the following asymptotic self-similar blow-up solution to Eq.(1.2) with $c_0 = 1$ in 3D*

$$\begin{cases} E(t,x) = \frac{1}{T-t} P\left(\frac{|x|}{(T-t)^{1/2}}\right) + i\frac{1}{T-t} Q\left(\frac{|x|}{(T-t)^{1/2}}\right), \\ n(t,x) = \frac{1}{T-t} N\left(\frac{|x|}{(T-t)^{1/2}}\right). \end{cases} \quad (4.16)$$

where $P(x) = P(|x|), Q(x) = Q(|x|), N(x) = N(|x|)$, and (P, N) satisfies the ODEs

$$\begin{cases} \Delta P + \frac{1}{2} r Q_r + Q = NP, \\ \Delta Q + \frac{1}{2} r P_r + P = NQ, \\ \frac{1}{4} r^2 N_{rr} + \frac{7}{4} r N_r + 2N) = \Delta(P^2) + \Delta(Q^2). \end{cases}$$

The homogeneous norm of the solution (4.16) satisfies

$$\|n\|_{\dot{H}^l} = \left(\frac{1}{T-t}\right)^{\frac{1}{4}(2l+1)} \|N\|_{\dot{H}^l},$$

and

$$\|E\|_{\dot{H}^k} = \left(\frac{1}{T-t}\right)^{\frac{1}{4}(2k+1)} (\|P\|_{\dot{H}^k} + \|Q\|_{\dot{H}^k}).$$

The lower bound of the solution in Corollary 4.1 is almost up to the optimal bound of the asymptotic blow-up rate $\frac{1}{4}(2l+1)$. Until now, we do not find explicit blow-up solution to Eq.(1.2) in $\mathbb{R}^3$, the blow-up rate of blow-up solution is open problem.

5 The global existence of solution

In this subsection, by the local well-posedness and conservation laws, in the space dimension $d = 2, 3, 4$, we shall establish the global solution of the GZ system with magnetic field in the case (A_1), the results are

Theorem 5.1 *Assume the initial data (E_0, n_0, n_1) belong to the Sobolev space $H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times H^{k-2}(\mathbb{R}^d), k \geq 1, d = 2, 3$. Let the initial data satisfy*

$$\begin{cases} (1 + \frac{\eta}{2}) K^4(2) \|E_0\|_{L^2}^2 < 1, & \text{if } d = 2, \\ \|E_0\|_{L^2}^2 H_2(0)(1 + \frac{\eta}{2})^2 < \frac{4}{27 K^8(3)}, \\ \text{and } \|\nabla E_0\|_{L^2}^2 \leq H_2(0), & \text{if } d = 3. \end{cases} \quad (5.1)$$

Then there exists a unique and global solution

$$(E, n, \partial_t n) \in \mathcal{C}(\mathbb{R}^+; H^k(\mathbb{R}^2) \times H^{k-1}(\mathbb{R}^2) \times H^{k-2}(\mathbb{R}^2))$$

and

$$(E, n, \partial_t n) \in \mathcal{C}(\mathbb{R}^+; H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3))$$

to system (1.1) in the case (A_1) with the initial data (E_0, n_0, n_1). Moreover, if $k = 1$, then the global solutions satisfy

$$\|E\|_{H^1} + \|n\|_{L^2} + \|V\|_{L^2} + \|\partial_t n\|_{H^{-1}} \leq C, \qquad (5.2)$$

uniformly bound for $t \in \mathbb{R}^+$. If $k \geq 2$ and $d = 2$, the global solutions grow at most k-exponential bounds and satisfy

$$\|E\|_{H^k} + \|n\|_{H^{k-1}} + \|\partial_t E\|_{H^{k-2}} + \|\partial_t n\|_{H^{k-2}} \leq C e^{e^{\cdots e^{ct}}}, \qquad (5.3)$$

where c and C are positive constants.

At first, in order to present the proof the Theorem 5.1, we recall the following two lemmas.

Lemma 5.1 *[26] Let the function $u \in H^1(\mathbb{R}^d), 2 \leq d < 4$. Then we have*

$$\|u\|_{L^4}^4 \leq K^4(d)\|u\|_{L^2}^{4-d}\|\nabla u\|_{L^2}^d,$$

where $K^4(d) = \frac{2}{\|\psi\|_{L^2}^2}$, the function ψ is the ground state solution of

$$\frac{d}{2}\Delta\psi + \frac{d-4}{2}\psi + \psi^3 = 0.$$

Lemma 5.2 *Given $f(t)$ be positive and continuous function on $\mathbb{R}^+$. Let $c_1, c_2 > 0$ and $k > 1$ such that*

$$f(t) \leq c_1 + c_2 f^k(t).$$

If the constants $c_1, c_2 > 0$ and $k > 1$ satisfy

$$c_1^{k-1} c_2 < \frac{(k-1)^{k-1}}{k^k}, \quad \text{and } f(0) \leq c_1,$$

then the function f is uniformly bounded on $\mathbb{R}^+$.

The proof of Lemma 5.2 is simple, which can be found in many books, we omit it here.

Proof of Theorem 5.1. The local well-posedness of solution to system (1.1) is guaranteed by Theorem 3.1. In view of the conservation law (1.5), (1.6) and Lemma 5.1, without loss of generality, let $c_0 = 1$, one can easily check that

$$\|\nabla E\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{2}\|V\|_{L^2}^2 \leq H_2(0) + (1 + \frac{\eta}{2})K^4(d)\|E_0\|_{L^2}^{4-d}\|\nabla E\|_{L^2}^d, \tag{5.4}$$

where we have used Young's inequality, Hölder's inequality and

$$\|n|E|^2\|_{L^1} \le \|n\|_{L^2}\|E\|_{L^4}^2$$
$$\le \frac{1}{4}\|n\|_{L^2}^2 + K^4(d)\|E_0\|_{L^2}^{4-d}\|\nabla E\|_{L^2}^d,$$

$$\|B(E \otimes \overline{E})\|_{L^1} \le \eta\|E \otimes \overline{E}\|_{L^2}^2$$
$$\le \eta\|E\|_{L^4}^4 \le \eta K^4(d)\|E_0\|_{L^2}^{4-d}\|\nabla E\|_{L^2}^d.$$

Consequently. we deduce from (5.4) that

$$\|\nabla E\|_{L^2}^2 \le H_2(0) + (1 + \frac{\eta}{2})K^4(2)\|E_0\|_{L^2}^2\|\nabla E\|_{L^2}^2, \qquad \text{if } d = 2, \qquad (5.5)$$

$$\|\nabla E\|_{L^2}^2 \le H_2(0) + (1 + \frac{\eta}{2})K^4(3)\|E_0\|_{L^2}\|\nabla E\|_{L^2}^3, \qquad \text{if } d = 3, \qquad (5.6)$$

By Lemma 5.2 to (5.5), (5.6) to yield (5.2) in the assumption (5.1) of Theorem 5.1.

If k=2 and $d = 2$, as the process of proof of Theorem 7.1 in [19], we have

$$\frac{d}{dt}(m(t)) \le Cm(t)(1 + \log m(t))$$

where $m(t) = \|\partial_t n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|\partial_t E\|_{L^2}^2 + 1$, i.e.

$$\frac{d}{dt}(1 + \log m(t)) \le C(1 + \log m(t)). \qquad (5.7)$$

By the Gronwall lemma to (5.7) is given by

$$\|\partial_t n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|\partial_t E\|_{L^2}^2 + 1 \le Ce^{e^{ct}}. \qquad (5.8)$$

Note that $\|\Delta E\|_{L^2} \le C(\|E_t\|_{L^2} + \|\nabla n\|_{L^2} + 1)$. Hence we obtain the result (5.3) as $k = 2$.

By mathematical induction, assume the result (5.3) of Theorem 5.1 is valid for the case m = k +1. We now consider the case $m = k + 2$, applying the second equation of system (1.1) by the operator ∂^k, taking the scalar product of $2\partial^k n$, integration by parts, we have

$$\frac{d}{dt}(\|\partial^k n_t\|_{L^2}^2 + \|\nabla \partial^k n\|_{L^2}^2) \le 2\|\partial^k n_t\|_{L^2}\|\Delta \partial^k |E|^2\|_{L^2}$$

$$\le 2\|\partial^k n_t\|_{L^2}\left(\|E\|_{L^\infty}\|\Delta \partial^k E\|_{L^2} + \sum_{\substack{1 \le i,j \le k+1}}^{i+j=k+2}\|\partial^i E \partial^j \overline{E}\|_{L^2}\right)$$

$$\le Ce^{e^{\cdot^{\cdot^{e^{ct}}}}}(\|\partial^k n_t\|_{L^2}^2 + \|\Delta \partial^k E\|_{L^2}^2 + 1),$$

$$(5.9)$$

the last inequality is guaranteed by

$$\|\partial^i E \partial^j \overline{E}\|_{L^2}^2 \leq \|\partial^{i-1} E\|_{L^2}^{1/2} \|\partial^{i+1} E\|_{L^2}^{1/2} \|\partial^{j-1} E\|_{L^2}^{1/2} \|\partial^{j+1} E\|_{L^2}^{1/2}$$
$$\leq C\|\Delta \partial^k E\|_{L^2}^{1/2}.$$

Differentiating the first equation of system (1.1) with respect to the time variable, then applying the operator ∂^k, Multiplying the resulting equation by $2\partial^k \overline{E}_t$, integration by parts, taking the imaginary part, it follows that

$$\frac{d}{dt}\|\partial^k E_t\|_{L^2}^2 = Im 2\int_{\mathbb{R}^2} \partial^k(nE)_t \partial^k \overline{E}_t dx - Im 2\int_{\mathbb{R}^2} i\partial^k(E\otimes B)_t \partial^k \overline{E}_t dx.$$
(5.10)

We first deal with the first term of right hand in (5.10) as follows

$$Im \int_{\mathbb{R}^2} \partial^k(nE)_t \partial^k \overline{E}_t dx = Im \int_{\mathbb{R}^2} \partial^k(n_t E)\partial^k \overline{E}_t dx$$
$$+ Im \int_{\mathbb{R}^2} \sum_{\substack{i+j=k \\ 0\leq j\leq k-1}} \partial^i n \partial^j E_t \partial^k \overline{E}_t dx,$$
(5.11)

By virtue of the induction and interpolation inequality, we have

$$\|\partial^k(n_t E)\partial^k \overline{E}_t\|_{L^1} \leq C\left(\|E\|_{L^\infty}\|\partial^k n_t\|_{L^2} + \sum_{\substack{i+j=k \\ 0\leq j\leq k-1}} \|\partial^i n_t \partial^j E\|_{L^2}\right)\|\partial^k E_t\|_{L^2}$$

$$\leq C\left(\|\partial^k n_t\|_{L^2} + \sum_{\substack{i+j=k \\ 0\leq j\leq k-1}} \|\partial^i n_t\|_{L^4}\|\partial^j E\|_{L^4}\right)\|\partial^k E_t\|_{L^2}$$

$$\leq C\|\partial^k n_t\|_{L^2}\|\partial^k E_t\|_{L^2} + C\sum_{\substack{i+j=k \\ 0\leq j\leq k-1}} \|\partial^i n_t\|_{L^2}^{1/2}\|\partial^{i+1} n_t\|_{L^2}^{1/2}$$

$$\times \|\partial^j E\|_{L^2}^{1/2}\|\partial^{j+1} E\|_{L^2}^{1/2}\|\partial^k E_t\|_{L^2}$$

$$\leq Ce^{e^{\cdot\cdot\cdot e^{ct}}}(\|\partial^k n_t\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + 1),$$
(5.12)

where $e^{e^{\cdot\cdot\cdot e^{ct}}}$ denotes $(k+1)$-exponent, we have used the inequality

$$\|E\|_{H^{k+1}} + \|n\|_{H^k} + \|\partial_t E\|_{H^{k-1}} + \|\partial_t n\|_{H^{k-1}} \leq Ce^{e^{\cdot\cdot\cdot e^{ct}}},$$

which is guaranteed by induction assumption. Similarly,

$$\sum_{\substack{i+j=k \\ 0\leq j\leq k-1}} \|\partial^i n \partial^j E_t \partial^k \overline{E}_t\|_{L^1} \leq C(\|\partial^{k+1} n\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + 1).$$
(5.13)

Substituting (5.12) and (5.13) into (5.11) to yield

$$\left| Im \int_{\mathbb{R}^2} \partial^k (nE)_t \partial^k \overline{E}_t dx \right| \leq C e^{e^{\cdots e^{ct}}} (\|\partial^{k+1} n\|_{L^2}^2 + \|\partial^k n_t\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + 1).$$

$$(5.14)$$

Next, we will investigate the second term of right hand in (5.10). Note that (iB) is real function, we have

$$Im \int_{\mathbb{R}^2} i\partial^k (E \otimes B)_t \partial^k \overline{E}_t dx = Im \int_{\mathbb{R}^2} i\partial^k (E \otimes B_t) \partial^k \overline{E}_t dx$$
$$+ Im \int_{\mathbb{R}^2} \sum_{\substack{0 \leq l \leq k-1}}^{l+j=k} (\partial^l E_t \otimes \partial^j (iB)) \partial^k \overline{E}_t dx, \tag{5.15}$$

Since

$$|Im \int_{\mathbb{R}^2} \sum_{\substack{0 \leq l \leq k-1}}^{l+j=k} (\partial^l E_t \otimes \partial^j (iB)) \partial^k \overline{E}_t dx|$$

$$\leq \sum_{\substack{0 \leq l \leq k-1}}^{l+j=k} \|\partial^l E_t\|_{L^4} \|\partial^j B\|_{L^4} \|\partial^k \overline{E}_t\|_{L^2}$$

$$\leq \sum_{\substack{0 \leq l \leq k-1}}^{l+j=k} \|\partial^l E_t\|_{L^2}^{1/2} \|\partial^{l+1} E_t\|_{L^2}^{1/2} \|\partial^j B\|_{L^2}^{1/2} \|\partial^{j+1} B\|_{L^2}^{1/2} \|\partial^k \overline{E}_t\|_{L^2}$$

$$\leq C e^{e^{\cdots e^{ct}}} (\|\partial^k E_t\|_{L^2}^{3/2} + \|\partial^k E_t\|_{L^2})$$

$$\leq C e^{e^{\cdots e^{ct}}} (\|\partial^k E_t\|_{L^2}^2 + 1)$$

$$(5.16)$$

and

$$|Im \int_{\mathbb{R}^2} i\partial^k (E \otimes B_t) \partial^k \overline{E}_t dx|$$

$$\leq \left(\|E\|_{L^\infty} \|\partial^k B_t\|_{L^2} + \sum_{\substack{0 \leq j \leq k-1}}^{l+j=k} \|\partial^l E\|_{L^4} \|\partial^j B_t\|_{L^4} \right) \|\partial^k \overline{E}_t\|_{L^2}$$

$$\leq \sum_{\substack{0 \leq l \leq k-1}}^{l+j=k} \|\partial^l E\|_{L^2}^{1/2} \|\partial^{l+1} E\|_{L^2}^{1/2} \|\partial^j B_t\|_{L^2}^{1/2} \|\partial^{j+1} B_t\|_{L^2}^{1/2} \|\partial^k \overline{E}_t\|_{L^2}$$

$$+ C e^{e^{\cdots e^{ct}}} \|\partial^k \overline{E}_t\|_{L^2}^2$$

$$\leq C e^{e^{\cdots e^{ct}}} (\|\partial^k E_t\|_{L^2}^{3/2} + \|\partial^k E_t\|_{L^2}^2)$$

$$\leq C e^{e^{\cdots e^{ct}}} (\|\partial^k E_t\|_{L^2}^2 + 1),$$

$$(5.17)$$

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where we have used the equality

$$B = \frac{i\eta}{\Delta + \beta I}\nabla \times \nabla \times (E \otimes \overline{E}).$$

Inserting (5.16) and (5.17) into (5.15), it follows that

$$\left| Im \int_{\mathbb{R}^2} i\partial^k (E \otimes B)_t \partial^k \overline{E}_t dx \right| \leq Ce^{e^{\cdot\cdot^{e^{ct}}}} (\|\partial^k E_t\|_{L^2}^2 + 1). \qquad (5.18)$$

Plugging (5.14) and (5.18) into (5.10) to deduce

$$\frac{d}{dt}\|\partial^k E_t\|_{L^2}^2 \leq Ce^{e^{\cdot\cdot^{e^{ct}}}} (\|\partial^{k+1} n\|_{L^2}^2 + \|\partial^k n_t\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + 1). \quad (5.19)$$

Adding (5.9) with (5.19), by the inequality

$$\|\Delta\partial^k E\|_{L^2} \leq C(\|\partial^k \nabla n\|_{L^2} + \|\partial^k E_t\|_{L^2} + 1), \qquad (5.20)$$

which is estimated by the first equation in system (1.1). Hence we deduce

$$\frac{d}{dt}(\|\partial^k n_t\|_{L^2}^2 + \|\partial^k \nabla n\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + 1)$$
$$\leq Ce^{e^{\cdot\cdot^{e^{ct}}}} (\|\partial^k n_t\|_{L^2}^2 + \|\partial^k E_t\|_{L^2}^2 + \|\partial^k \nabla n\|_{L^2}^2 + 1). \qquad (5.21)$$

By the Gronwall lemma to (5.21), using (5.20), we obtain

$$\|E\|_{H^{k+2}}^2 + \|n\|_{H^{k+1}}^2 + \|n_t\|_{H^k}^2 + \|E_t\|_{H^k}^2 \leq Ce^{e^{\cdot\cdot^{e^{ct}}}}, \qquad (5.22)$$

where the $e^{e^{\cdot\cdot^{e^{ct}}}}$ denotes the $(k+2)$-exponent. The proof of Theorem 5.1 is completed. ∎

Remark 5.1 *As $d = 4$, if the initial data $(E, n, V) \in H^1 \times L^2 \times L^2$, $\|\nabla E_0\|_{L^2}^2 \leq H_2(0)$ and $(1 + \eta/2)CH_2(0) < 1$, then system (1.1) in the case (A_1) has a global weak solution*

$$(E, n, V) \in \mathcal{C}(\mathbb{R}^+; H^1 \times L^2 \times H^{-1}),$$

where the constant C satisfies the inequality

$$\|E\|_{L^4} \leq C\|\nabla E\|_{L^2}.$$

The above proof is similar to the proof of the case $d = 2, 3$ in Theorem 5.1. In fact, we have

$$\|\nabla E\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 + \frac{1}{2}\|V\|_{L^2}^2 \leq H_2(0) + (1 + \frac{\eta}{2})\|E\|_{L^4}^4,$$
$$\leq H_2(0) + (1 + \frac{\eta}{2})C\|\nabla E\|_{L^2}^4.$$

Hence, it follows that

$$\|\nabla E\|_{L^2}^2 \leq H_2(0) + (1 + \frac{\eta}{2})C\|\nabla E\|_{L^2}^4. \qquad (5.23)$$

By virtue of Lemma 5.2 to (5.23) yields the above result.

Remark 5.2 *There exists a family of self-similar blow-up solution to the system (1.1) in 2D[8, 11, 12]. With assumption of small initial data in 2D, we prove the global solution $(E, n, \partial_t n) \in \mathcal{C}(\mathbb{R}^+; H^k \times H^{k-1} \times H^{k-2})$ to system (1.1), $k \geq 1$. In 3D, if the initial data is small enough, we obtain the unique and global solution $(E, n, \partial_t n) \in \mathcal{C}(\mathbb{R}^+; H^1 \times L^2 \times H^{-1})$. Moreover, for the 1D, system (1.1) becomes Eq.(1.2), the global well-posedness of solution $(E, n) \in L^2 \times H^{-1/2}$, which obtianed by J. Colliander et. in [7] is critial and optimal, because of the the ill-posedness of Eq.(1.2) in [14].*

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