ESTIMATES FOR THE FIRST EIGENVALUE OF JACOBI OPERATOR ON HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPHERES

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ABSTRACT. In this paper, we study the first eigenvalue of Jacobi operator on an n-dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$. We give an optimal upper bound for the first eigenvalue of Jacobi operator, which only depends on the mean curvature H and the dimension n. This bound is attained if and only if, $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ when $H \neq 0$ or $\varphi: M \to S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k=1,2,\cdots,n-1$ when H=0.

1. Introduction

Let $\varphi: M \to S^{n+1}(1)$ be an *n*-dimensional compact hypersurface in the unit sphere $S^{n+1}(1)$ of dimension n+1. We consider a variation of the hypersurface $\varphi: M \to S^{n+1}(1)$, for any $t \in (-\varepsilon, \varepsilon)$,

$$\varphi_t: M \to S^{n+1}(1)$$

is an immersion with $\varphi_0 = \varphi$. The area of φ_t is given by

$$A(t) = \int_{M} dA_{t}$$

and the volume of φ_t is defined by

$$V(t) = \frac{1}{n+1} \int_{M} \langle \varphi_t, N(t) \rangle dA_t,$$

where N(t) denotes the unit normal of φ_t . For any t, if V(t) = V(0), then the variation φ_t is called volume-preserving. If the variational vector $\frac{\partial \varphi_t}{\partial t}|_{t=0} = fN$ for a smooth function f, then the variation is called a normal variation, where N is the unit normal of φ . Let H denote the mean curvature of φ . The first variation formula of the area functional A(t) is given by

$$\frac{dA(t)}{dt}|_{t=0} = -\int_{M} nHfdA,$$

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where $f = \langle \frac{\partial \varphi_t}{\partial t} |_{t=0}, N \rangle$. Thus, we know that a compact hypersurface is minimal, that is, $H \equiv 0$ if and only if

$$\frac{dA(t)}{dt}|_{t=0} = 0.$$

Hence, compact minimal hypersurfaces are critical points of the area functional A(t). The second variation formula of A(t) is given by

$$\frac{d^2A(t)}{dt^2}|_{t=0} = -\int_M fJfdA$$

and

$$Jf = \Delta f + (S+n)f,$$

where S denotes the squared norm of the second fundamental form of φ and Δ stands for the Laplace-Beltrami operator. The J is called a Jacobi operator or a stability operator on the minimal hypersurface φ (cf. [2], [9]).

Let λ_1^J denote the first eigenvalue of the Jacobi operator J. Then

$$Ju = -\lambda_1^J u$$

and the λ_1^J is given by

$$\lambda_1^J = \inf_{f \neq 0} \frac{-\int_M f J f dA}{\int_M f^2 dA}.$$

For a compact minimal hypersurface in $S^{n+1}(1)$, Simons [10] proved

$$\lambda_1^J \le -n$$

and $\lambda_1^J=-n$ if and only if $\varphi:M\to S^{n+1}(1)$ is totally geodesic. Furthermore, Wu [11] proved that for an n-dimensional compact non-totally geodesic minimal hypersurface $\varphi:M\to S^{n+1}(1)$ in $S^{n+1}(1)$, then $\lambda_1^J\le -2n$ and $\lambda_1^J=-2n$ if and only if $\varphi:M\to S^{n+1}(1)$ is a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}})\times S^k(\sqrt{\frac{k}{n}})$, for $k=1,2,\cdots,n-1$. Thus, we know that the upper bound for the first eigenvalue λ_1^J due to Wu is optimal and it only depends on the dimension n, does not depends on the immersion.

On the other hand, if one considers the volume-preserving variation of φ , then we have

$$\int_{M} f dA = 0.$$

From the first variation formula:

$$\frac{dA(t)}{dt}|_{t=0} = -\int_{M} nHfdA,$$

we know that compact hypersurfaces with constant mean curvature are critical points of the area functional A(t) for the volume-preserving variation and the second variation formula of A(t) is given by

$$\frac{d^2A(t)}{dt^2}|_{t=0} = -\int_M fJfdA,$$

where the Jacobi operator J on compact hypersurfaces with constant mean curvature is the same as one of compact minimal hypersurfaces ([2], [4]).

Alias, Barros and Brasil [3] studied the first eigenvalue of the Jacobi operator J on compact hypersurfaces with constant mean curvature. They proved the following:

Theorem ABB. If $\varphi: M \to S^{n+1}(1)$ is an n-dimensional compact hypersurface with non-zero constant mean curvature H in the unit sphere $S^{n+1}(1)$, then either $\lambda_1^J = -n(1+H^2)$ and $\varphi: M \to S^{n+1}(1)$ is totally umbilical or

$$\lambda_1^J \le -2n(1+H^2) + \frac{n(n-2)|H|}{\sqrt{n(n-1)}} \max \sqrt{S-nH^2}$$

and the equality holds if and only if $\varphi: M \to S^{n+1}(1)$ is $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$, with $r^2 > \frac{1}{n}$ for $n \geq 2$.

According to this theorem, we know that, for n=2, the upper bund of the first eigenvalue λ_1^J of the Jacobi operator of non-totally umbilical compact hypersurfaces with constant mean curvature only depends on the mean curvature H and the dimension. But for $n\geq 3$, the upper bound of the first eigenvalue λ_1^J of the Jacobi operator on non-totally umbilical compact hypersurfaces with constant mean curvature includes the term $\max \sqrt{S-nH^2}$. Hence, the upper bound of the first eigenvalue λ_1^J does not only depend on the mean curvature H and the dimension n, but also depends on the immersion φ .

It is natural and important to propose the following:

Problem 1.1. To find an optimal upper bound for the first eigenvalue λ_1^J of the Jacobi operator on non-totally umbilical compact hypersurfaces with constant mean curvature, which only depends on the mean curvature H and the dimension n.

In this paper, we give an affirmative answer for the above problem 1.1.

Theorem 1.1. Let $\varphi: M \to S^{n+1}(1)$ be an n-dimensional non-totally umbilical compact hypersurface with constant mean curvature H in the unit sphere $S^{n+1}(1)$.

(1) If $2 \le n \le 4$ or $n \ge 5$ and $n^2H^2 < \frac{16(n-1)}{n(n-4)}$, then the first eigenvalue λ_1^J of the Jacobi operator J satisfies

$$\lambda_1^J \le -n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}$$

and the equality holds if and only if $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ with r > 0 satisfying

$$\begin{cases} 1 > r^2 > \frac{1}{n} & \text{for } 2 \le n \le 4, \\ \frac{n}{(n-2)^2} > r^2 > \frac{1}{n}, & \text{for } n \ge 5 \text{ and } n^2 H^2 < \frac{16(n-1)}{n(n-4)} \end{cases}$$

or $\varphi: M \to S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n-1$ with H = 0.

(2) If $n \geq 5$ and $n^2H^2 \geq \frac{16(n-1)}{n(n-4)}$, the first eigenvalue λ_1^J of the Jacobi operator J satisfies

$$\lambda_1^J \le -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

and the equality holds if and only if $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$.

Remark 1.1. Since the first eigenvalue of Jacobi operator J on totally umbilical hypersurfaces satisfies $\lambda_1^J = -n(1+H^2)$, according to our theorem, one knows that for $2 \le n \le 4$, there are no n-dimensional compact hypersurfaces in the unit sphere with constant mean curvature H so that the first eigenvalue λ_1^J of Jacobi operator J takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}, -n(1+H^2)\right).$$

For any $n \geq 2$, there are no n-dimensional compact hypersurfaces in the unit sphere with constant mean curvature H satisfying $n^2H^2 < \frac{16(n-1)}{n(n-4)}$ so that the first eigenvalue λ_1^J of Jacobi operator J takes a value in the internal

$$\left(-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2}{4(n-1)}, -n(1+H^2)\right).$$

One should compare the bound

$$-n(1+H^2) - \frac{n(\sqrt{4(n-1)+n^2H^2}-(n-2)|H|)^2}{4(n-1)}$$

with the pinching constant in the rigidity theorem of Cheng and Nakagawa [7] or Alencar and do Carmo [1].

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2. Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let $\varphi: M \to S^{n+1}(1)$ be an *n*-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a local orthonormal frame on M. Hence, we have

$$\omega_{n+1}=0$$

on M. From Cartan's lemma, we have

(2.1)
$$\omega_{in+1} = \sum_{j=1}^{n} h_{ij}\omega_{j}, \ h_{ij} = h_{ji}.$$

The mean curvature H and the second fundamental form II of $\varphi: M \to S^{n+1}(1)$ are defined, respectively, by

$$H = \frac{1}{n} \sum_{i=1}^{n} h_{ii}, II = \sum_{i,j=1}^{n} h_{ij} \omega_i \otimes \omega_j \mathbf{e}_{n+1}.$$

When the mean curvature H of $\varphi: M \to S^{n+1}(1)$ is identically zero, we recall that $\varphi: M \to S^{n+1}(1)$ is by definition a minimal hypersurface. From the structure equations of $\varphi: M \to S^{n+1}(1)$, Gauss equation is given by

$$(2.2) R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

From (2.2), we have

$$n(n-1)r = n(n-1) + n^2H^2 - S,$$

where n(n-1)r and S denote the scalar curvature and the squared norm of the second fundamental form of $\varphi: M \to S^{n+1}(1)$, respectively. Defining the covariant derivative of h_{ij} by

(2.3)
$$\sum_{k} h_{ijk}\omega_{k} = dh_{ij} + \sum_{k} h_{ik}\omega_{kj} + \sum_{k} h_{kj}\omega_{ki},$$

we obtain the Codazzi equations

$$(2.4) h_{ijk} = h_{ikj}.$$

By taking exterior differentiation of (2.3), and defining

(2.5)
$$\sum_{l} h_{ijkl}\omega_{l} = dh_{ijk} + \sum_{l} h_{ljk}\omega_{li} + \sum_{l} h_{ilk}\omega_{lj} + \sum_{l} h_{ijl}\omega_{lk},$$

we have the following Ricci identities:

$$(2.6) h_{ijkl} - h_{ijlk} = \sum_{m} h_{mj} R_{mikl} + \sum_{m} h_{im} R_{mjkl}.$$

For any C^2 -function f on M, we define its gradient and Hessian by

$$df = \sum_{i=1}^{n} f_i \omega_i,$$

$$\sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{j=1}^{n} f_j\omega_{ji}.$$

Thus, the Laplace-Beltrami operator Δ is given by

$$\Delta f = \sum_{i=1}^{n} f_{ii}.$$

Example 2.1. For totally umbilical sphere $S^n(r)$ of radius r > 0, the first eigenvalue $\lambda_1^J = -n(1+H^2)$ with $H = \frac{1}{r}$.

Example 2.2. For Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, k = 1, 2, ..., n, the first eigenvalue $\lambda_1^J = -2n$ with H = 0.

Example 2.3. For hypersurfaces $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ with 0 < r < 1, the principal curvatures are given by

$$k_1 = -\frac{\sqrt{1-r^2}}{r}, \quad k_2 = \dots = k_n = \frac{r}{\sqrt{1-r^2}}.$$

Hence, we know that

$$nH = \frac{nr^2 - 1}{r\sqrt{1 - r^2}}, \quad S = \frac{1 - 2r^2 + nr^4}{r^2(1 - r^2)}.$$

For $r^2 \geq \frac{1}{n}$, by a direct computation, we know that the first eigenvalue λ_1^J of the Jacobi operator J on $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$\lambda_1^J = -n(1+H^2) - \frac{n(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|)^2}{4(n-1)}.$$

For $n \geq 5$ and $\frac{1}{n} \leq r^2 < \frac{n}{(n-2)^2}$, we know the hypersurface $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ satisfies

$$n^2H^2 < \frac{16(n-1)}{n(n-4)}$$

and

$$\lambda_1^J = -n(1+H^2) - \frac{n(\sqrt{4(n-1) + n^2H^2} - (n-2)|H|)^2}{4(n-1)}.$$

The hypersurface $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$ satisfies

$$\lambda_1^J = -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

with $n^2H^2 = \frac{16(n-1)}{n(n-4)}$.

3. Proof of theorem 1.1.

In this section, we give a proof of the theorem 1.1.

Proof of theorem 1.1. When $H \equiv 0$, according to the result of Wu [11], we have $\lambda_1^J \leq -2n$ and $\lambda_1^J = -2n$ if and only if $\varphi: M \to S^{n+1}(1)$ is isometric to a Clifford torus $S^{n-k}(\sqrt{\frac{n-k}{n}}) \times S^k(\sqrt{\frac{k}{n}})$, for $k = 1, 2, \dots, n-1$.

From now we assume $H \neq 0$. By making use of the Codazzi equations, Ricci identities and a standard computation of Simons' type formula (cf. [7], [5, 6], [8] and [10]), we have

(3.1)
$$\frac{1}{2}\Delta S = \sum_{i,i,k=1}^{n} h_{ijk}^{2} + nS - n^{2}H^{2} + nHf_{3} - S^{2},$$

where $f_3 = \sum_{i=1}^n k_i^3$ and k_i , i = 1, 2, ..., n denote the principal curvatures. Putting $\mu_i = k_i - H$, we have

(3.2)
$$B := \sum_{i=1}^{n} \mu_i^2 = S - nH^2 \ge 0, \quad f_3 = B_3 + 3HB + nH^3,$$

where $B_3 = \sum_{i=1}^n \mu_i^3$. The following inequality is known (cf. [7] and [8]):

$$(3.3) |B_3| \le \frac{n-2}{\sqrt{n(n-1)}} B^{\frac{3}{2}},$$

and the equality holds if and only if at least n-1 of k_i , for $i=1,2,\ldots,n$, are equal with each other. Since H is constant, we can assume H>0. Thus, from (3.1), (3.2) and (3.3), we have

(3.4)
$$\frac{1}{2}\Delta B = \frac{1}{2}\Delta S \ge \sum_{i,i,k=1}^{n} h_{ijk}^2 + B(n+nH^2-B) - nH\frac{n-2}{\sqrt{n(n-1)}}B^{\frac{3}{2}}.$$

For any constant $\alpha > 0$ and $\varepsilon > 0$, we consider a function $f_{\varepsilon} = (B + \varepsilon)^{\alpha} > 0$. Hence, we have, from (3.4),

$$\Delta f_{\varepsilon} = \alpha(\alpha - 1)(B + \varepsilon)^{\alpha - 2} |\nabla B|^{2} + \alpha(B + \varepsilon)^{\alpha - 1} \Delta B$$

$$\geq \alpha(\alpha - 1)(B + \varepsilon)^{\alpha - 2} |\nabla B|^{2}$$

$$+ 2\alpha(B + \varepsilon)^{\alpha - 1} \left(\sum_{i,j,k=1}^{n} h_{ijk}^{2} + B(n + nH^{2} - B) - nH \frac{n - 2}{\sqrt{n(n - 1)}} B^{\frac{3}{2}} \right).$$

Since H is constant, we have

(3.6)
$$\nabla_k(nH) = \sum_{i=1}^n h_{iik} = 0, \quad h_{kkk}^2 \le (n-1) \sum_{i \ne k} h_{iik}^2$$
$$|\nabla B|^2 = \sum_{k=1}^n (2 \sum_{i=1}^n \mu_i h_{iik})^2 \le 4B \sum_{i=1}^n h_{iik}^2.$$

Thus, we obtain

$$|\nabla B|^2 \le 4B \sum_{i,k=1}^n h_{iik}^2$$

$$= 4B \left(\frac{n}{n+2} \sum_{k=1}^n h_{kkk}^2 + \frac{2}{n+2} \sum_{k=1}^n h_{kkk}^2 + \sum_{i \neq k} h_{iik}^2\right)$$

$$\le \frac{4n}{n+2} B \left(\sum_{k=1}^n h_{kkk}^2 + 3 \sum_{i \neq k} h_{iik}^2\right).$$

For any constant β , we have

$$\lambda_1^J \int_M f_\varepsilon^2 dA \le -\int_M f_\varepsilon J f_\varepsilon dA$$

$$= -\beta \int_{M} f_{\varepsilon} \Delta f_{\varepsilon} dA - \int_{M} \left((1 - \beta) f_{\varepsilon} \Delta f_{\varepsilon} + (S + n) f_{\varepsilon}^{2} \right) dA$$

$$= \beta \int_{M} |\nabla f_{\varepsilon}|^{2} dA - \int_{M} f_{\varepsilon} \left\{ (1 - \beta) \left(\alpha (\alpha - 1) (B + \varepsilon)^{\alpha - 2} |\nabla B|^{2} \right) \right\} + \alpha (B + \varepsilon)^{\alpha - 1} \Delta B + (B + nH^{2} + n) f_{\varepsilon} dA$$

$$= \alpha \int_{M} f_{\varepsilon} \left\{ 1 + 2\alpha \beta - \beta - \alpha \right\} (B + \varepsilon)^{\alpha - 2} |\nabla B|^{2} dA$$

$$- \int_{M} f_{\varepsilon}^{2} \left\{ \frac{\alpha (1 - \beta)}{B + \varepsilon} \Delta B + B + nH^{2} + n \right\} dA.$$

By taking α and β satisfying

(3.8)
$$\alpha > \frac{n-2}{4n}, \quad 1-\beta = \frac{2n\alpha}{4n\alpha + 2 - n},$$

we have

$$(n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha = 0.$$

Since

$$\sum_{i,j,k=1}^{n} h_{ijk}^2 = \sum_{k=1}^{n} h_{kkk}^2 + 3\sum_{i \neq k} h_{iik}^2 + \sum_{i \neq j \neq k \neq i}^{n} h_{ijk}^2,$$

from (3.7), we obtain

$$(3.9) \qquad (1 + 2\alpha\beta - \beta - \alpha)|\nabla B|^2 - 2(1 - \beta)(B + \varepsilon) \sum_{i,j,k=1}^n h_{ijk}^2$$

$$\leq \frac{2}{n+2} B \left\{ (n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha \right\} \left(\sum_{k=1}^n h_{kkk}^2 + 3\sum_{i \neq k} h_{iik}^2 \right) = 0.$$

Thus, we infer

$$\begin{split} &\lambda_1^J \int_M f_\varepsilon^2 dA \\ &\leq \alpha \int_M f_\varepsilon (B+\varepsilon)^{\alpha-2} \bigg\{ \Big(1+2\alpha\beta-\beta-\alpha\Big) |\nabla B|^2 - 2(1-\beta)(B+\varepsilon) \sum_{i,j,k=1}^n h_{ijk}^2 \bigg\} dA \\ &- \int_M f_\varepsilon^2 \bigg\{ \frac{2\alpha(1-\beta)B}{B+\varepsilon} \bigg((n+nH^2-B) - nH \frac{(n-2)}{\sqrt{n(n-1)}} B^{\frac{1}{2}} \bigg) + B + nH^2 + n \bigg\} dA \\ &\leq - \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} \bigg(\big\{ 1 - 2\alpha(1-\beta) \big\} B - \frac{2\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}} nH B^{\frac{1}{2}} + \varepsilon \bigg) dA \\ &- 2\alpha(1-\beta)(n+nH^2) \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} dA - (n+nH^2) \int_M f_\varepsilon^2 dA. \end{split}$$

For $1 - 2\alpha(1 - \beta) > 0$, we obtain

$$\begin{split} &\lambda_1^J \int_M f_\varepsilon^2 dA \\ &\leq \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} \bigg(\frac{\alpha^2 (1-\beta)^2 (n-2)^2}{(1-2\alpha(1-\beta))n(n-1)} (nH)^2 - \varepsilon \bigg) dA \\ &- 2\alpha (1-\beta)(n+nH^2) \int_M f_\varepsilon^2 \frac{B}{B+\varepsilon} dA - (n+nH^2) \int_M f_\varepsilon^2 dA. \end{split}$$

Since $\varphi: M \to S^{n+1}(1)$ is not totally umbilical, we have

$$\lim_{\varepsilon \to 0} \int_{M} f_{\varepsilon}^{2} dA = \int_{M} B^{2\alpha} dA > 0.$$

Letting $\varepsilon \to 0$, we derive

$$(3.10) \lambda_1^J \le -(1+2\alpha(1-\beta))n(1+H^2) + \frac{\alpha^2(1-\beta)^2}{(1-2\alpha(1-\beta))} \frac{(n-2)^2}{n(n-1)} n^2 H^2.$$

For n = 2, we have

$$\lambda_1^J \le -(1 + 2\alpha(1 - \beta))n(1 + H^2).$$

From (3.8), we have $\beta = \frac{1}{2}$ for any $0 < \alpha < 1$. Hence, we obtain

$$\lambda_1^J \le -2n(1+H^2).$$

For $2 < n \le 4$ or $n \ge 5$ and $n^2H^2 < \frac{16(n-1)}{n(n-4)}$, we have

(3.11)
$$\frac{1}{2} > \frac{1}{2} \left(1 - \sqrt{\frac{(n-2)^2 H^2}{4(n-1) + n^2 H^2}} \right) > \frac{1}{2} - \frac{1}{n} \ge \frac{1}{2} - \frac{1}{\sqrt{2n}}.$$

Observe from (3.8) that $1 - 2\alpha(1 - \beta) > 0$ if and only if

(3.12)
$$\frac{1}{2} - \frac{1}{\sqrt{2n}} < \alpha < \frac{1}{2} + \frac{1}{\sqrt{2n}}.$$

Defining

$$w(\alpha) = \alpha(1 - \beta) = \frac{2n\alpha^2}{4n\alpha + 2 - n},$$

 $w(\alpha)$ is an increasing function of α , for $\alpha > \frac{1}{2} - \frac{1}{n}$ and

$$w(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}, \quad w(\frac{1}{2} + \frac{1}{\sqrt{2n}}) = \frac{1}{2}.$$

According to (3.11) and (3.12), there exists a α satisfying

$$\frac{1}{2} - \frac{1}{n} < \alpha < \frac{1}{2} + \frac{1}{\sqrt{2n}}$$

such that

(3.13)
$$w(\alpha) = \frac{1}{2} \left(1 - \sqrt{\frac{(n-2)^2 H^2}{4(n-1) + n^2 H^2}} \right).$$

Therefore, we have, for this α ,

(3.14)
$$1 - 2\alpha(1 - \beta) = \sqrt{\frac{(n-2)^2 H^2}{4(n-1) + n^2 H^2}} > 0.$$

From (3.10), we obtain

(3.15)
$$\lambda_1^J \le -n(1+H^2) - 2\alpha(1-\beta)n \frac{4(n-1)(1-2\alpha(1-\beta))(1+H^2) - 2\alpha(1-\beta)(n-2)^2H^2}{4(n-1)(1-2\alpha(1-\beta))}.$$

From (3.14), we infer

$$\begin{aligned} &4(n-1)\left(1-2\alpha(1-\beta)\right)(1+H^2)-2\alpha(1-\beta)(n-2)^2H^2\\ &=\left\{4(n-1)(1+H^2)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}-\left(1-\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\right)(n-2)^2H^2\right\}\\ &=4(n-1)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}-(n-2)^2H^2+\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}n^2H^2\\ &=\sqrt{4(n-1)+n^2H^2}\sqrt{(n-2)^2H^2}-(n-2)^2H^2\\ &=\sqrt{(n-2)^2H^2}\bigg(\sqrt{4(n-1)+n^2H^2}-(\sqrt{(n-2)^2H^2}\bigg).\end{aligned}$$

From (3.14), (3.15) and the above equality, we obtain

$$\begin{split} \lambda_1^J &\leq -n(1+H^2) - \frac{n\bigg(1-\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\bigg)}{4(n-1)\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}} \times \\ &\times \sqrt{(n-2)^2H^2} \bigg(\sqrt{4(n-1)+n^2H^2} - (\sqrt{(n-2)^2H^2}\bigg) \\ &= -n(1+H^2) - \frac{n}{4(n-1)} (\sqrt{4(n-1)+n^2H^2} - (n-2)|H|)^2. \end{split}$$

If the equality holds, we know that $h_{ijk} = 0$, for any i, j, k = 1, 2, ..., n. Hence, we know that the second fundamental form is parallel and S is constant. Thus, we know that $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(r) \times S^{n-1}(\sqrt{1-r^2})$ since, from the (3.3), the n-1 of the principal curvatures are equal with each other. From the

examples in the section 2, we know that r satisfies

$$\begin{cases} r^2 > \frac{1}{n} & \text{for } 2 \le n \le 4, \\ \frac{1}{n} < r^2 < \frac{n}{(n-2)^2}, & \text{for } n \ge 5 \text{ and } n^2 H^2 < \frac{16(n-1)}{n(n-4)}. \end{cases}$$

If $n \ge 5$ and $n^2H^2 \ge \frac{16(n-1)}{n(n-4)}$, we take

$$\alpha(1-\beta) = \frac{1}{2} - \frac{1}{n},$$

that is,

$$\beta = 0$$
 and $\alpha = \frac{1}{2} - \frac{1}{n}$,

Thus, the inequality (3.10) becomes

$$\lambda_1^J \le -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

If the equality holds, we know

$$(1-2\alpha)\sqrt{B} = \frac{\alpha(n-2)}{\sqrt{n(n-1)}}nH.$$

Thus, we have

(3.16)
$$S = B + nH^2 = nH^2 + \frac{(n-2)^4}{16n(n-1)}n^2H^2.$$

because of

$$\alpha = \frac{1}{2} - \frac{1}{n}.$$

Since S is constant, the first eigenvalue λ_1^J of the Jacobi operator is given by

$$\lambda_1^J = -S - n = -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

Hence, we obtain

(3.17)
$$S = n - 2 + 2(n-1)H^2 - \frac{(n-2)^4}{8(n-1)}H^2.$$

From (3.16) and (3.17), we get

$$n-2 = (2-n)H^2 + \frac{(n-2)^4(n+2)}{16(n-1)}H^2,$$
$$1 = \frac{n(n-4)}{16(n-1)}n^2H^2,$$

that is,

$$n^2H^2 = \frac{16(n-1)}{n(n-4)}.$$

Since, from the (3.3), the n-1 of the principal curvatures are equal with each other, From the examples in the section 2, we know that $\varphi: M \to S^{n+1}(1)$ is isometric to $S^1(\frac{\sqrt{n}}{n-2}) \times S^{n-1}(\frac{\sqrt{(n-1)(n-4)}}{n-2})$. It completes the proof of theorem 1.1.

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