HARDY INEQUALITIES AND NON-EXPLOSION RESULTS FOR SEMIGROUPS

KRZYSZTOF BOGDAN, BARTŁOMIEJ DYDA, AND PANKI KIM

ABSTRACT. We prove non-explosion results for Schrödinger perturbations of symmetric transition densities and Hardy inequalities for their quadratic forms by using explicit supermedian functions of their semigroups.

1. Introduction

Hardy-type inequalities are important in harmonic analysis, potential theory, functional analysis, partial differential equations and probability. In PDEs they are used to obtain a priori estimates, existence and regularity results [24] and to study qualitative properties and asymptotic behaviour of solutions [28]. In functional and harmonic analysis they yield embedding theorems and interpolation theorems, e.g. Gagliardo-Nirenberg interpolation inequalities [21]. The connection of Hardy-type inequalities to the theory of superharmonic functions in analytic and probabilistic potential theory was studied, e.g., in [1], [14], [5], [11]. A general rule stemming from the work of P. Fitzsimmons [14] may be summarized as follows: if \mathcal{L} is the generator of a symmetric Dirichlet form \mathcal{E} and h is superharmonic, i.e. $h \ge 0$ and $\mathcal{L}h \le 0$, then $\mathcal{E}(u,u) \ge \int u^2(-\mathcal{L}h/h)$. The present paper gives an analogous connection in the setting of symmetric transition densities. When these are integrated against increasing weights in time and arbitrary weights in space, we obtain suitable (supermedian) functions h. The resulting analogues q of the Fitzsimmons' ratio $-\mathcal{L}h/h$ yield explicit Hardy inequalities which in many cases are optimal. The approach is very general and the resulting Hardy inequality is automatically valid on the whole of the ambient L^2 space.

We also prove non-explosion results for Schrödinger perturbations of the original transition densities by the ratio q, namely we verify that h is supermedian, in particular integrable with respect to the perturbation. For

Date: May 22, 2018.

¹⁹⁹¹ Mathematics Subject Classification. Primary: 31C25, 35B25; Secondary: 31C05, 35B44.

 $Key\ words\ and\ phrases.$ optimal Hardy equality, transition density, Schrödinger perturbation.

Krzysztof Bogdan and Bartłomiej Dyda were partially supported by NCN grant 2012/07/B/ST1/03356.

instance we recover the famous non-explosion result of Baras and Goldstein for $\Delta + (d/2 - 1)^2 |x|^{-2}$, cf. [2] and [25].

The results are illustrated by applications to transition densities with certain scaling properties.

The structure of the paper is as follows. In Theorem 1 of Section 2 we prove the non-explosion result for Schrödinger perturbations. In Theorem 2 of Section 3 we prove the Hardy inequality. In fact, under mild additional assumptions we have a Hardy equality with an explicit remainder term. Sections 4, 5 and 6 present applications. In Section 4 we recover the classical Hardy equalities for the quadratic forms of the Laplacian and fractional Laplacian. For completeness we also recover the best constants and the corresponding remainder terms, as given by Filippas and Tertikas [13], Frank, Lieb and Seiringer [15] and Frank and Seiringer [16]. In Section 5 we consider transition densities with weak global scaling in the setting of metric spaces. These include a class of transition densities on fractal sets (Theorem 10 and Corollary 11) and transition densities of many unimodal Lévy processes on \mathbb{R}^d (Corollary 12). We prove Hardy inequalities for their quadratic forms. In Section 6 we focus on transition densities with weak local scaling on \mathbb{R}^d . The corresponding Hardy inequality is stated in Theorem 13.

The calculations in Sections 4, 5 and 6, which produce explicit weights in Hardy inequalities, also give non-explosion results for specific Schrödinger perturbations of the corresponding transition densities by means of Theorem 1. These are stated in Corollary 6 and 8 and Remark 2, 5 and 6.

Currently our methods are confined to the (bilinear) L^2 setting. We refer to [16], [27] for other frameworks. Regarding further development, it is of interest to find relevant applications with less space homogeneity and scaling than required in the examples presented below, extend the class of considered time weights, prove explosion results for "supercritical" Schrödinger perturbations, and understand more completely when equality holds in our Hardy inequalities.

Below we use ":=" to indicate definitions, e.g. $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. For two nonnegative functions f and g we write $f \approx g$ if there is a positive number c, called constant, such that $c^{-1} g \leq f \leq c g$. Such comparisons are usually called sharp estimates. We write $c = c(a,b,\ldots,z)$ to claim that c may be so chosen to depend only on a,b,\ldots,z . For every function f, let $f_+ := f \vee 0$. For any open subset D of the d-dimensional Euclidean space \mathbb{R}^d , we denote by $C_c^{\infty}(D)$ the space of smooth functions with compact supports in D, and by $C_c(D)$ the space of continuous functions with compact supports in D. In statements and proofs, c_i denote constants whose exact values are unimportant. These are given anew in each statement and each proof.

Acknowledgement. We thank William Beckner for comments on the Hardy equality (29). We thank Tomasz Byczkowski, Tomasz Grzywny, Tomasz Jakubowski, Kamil Kaleta, Agnieszka Kałamajska and Dominika Pilarczyk

for comments, suggestions and encouragement. We also thank Rupert L. Frank and Georgios Psaradakis for remarks on the literature related to Section 4.

2. Non-explosion for Schrödinger perturbations

Let $(X, \mathcal{M}, m(dx))$ be a σ -finite measure space. Let $\mathcal{B}_{(0,\infty)}$ be the Borel σ -field on the halfline $(0,\infty)$. Let $p:(0,\infty)\times X\times X\to [0,\infty]$ be $\mathcal{B}_{(0,\infty)}\times \mathcal{M}\times \mathcal{M}$ -measurable and symmetric:

(1)
$$p_t(x,y) = p_t(y,x), \quad x,y \in X, \quad t > 0,$$

and let p satisfy the Chapman–Kolmogorov equations:

(2)
$$\int_{X} p_s(x,y) p_t(y,z) m(dy) = p_{s+t}(x,z), \qquad x,z \in X, \ s,t > 0,$$

and assume that for every $t > 0, x \in X$, $p_t(x, y)m(dy)$ is (σ -finite) integral kernel. Let $f : \mathbb{R} \to [0, \infty)$ be non-decreasing, and let f = 0 on $(-\infty, 0]$. We have $f' \geq 0$ a.e., and

(3)
$$f(a) + \int_a^b f'(s)ds \le f(b), \quad -\infty < a \le b < \infty.$$

Further, let μ be a nonnegative σ -finite measure on (X, \mathcal{M}) . We put

(4)
$$p_s\mu(x) = \int_X p_s(x,y)\,\mu(dy),$$

(5)
$$h(x) = \int_0^\infty f(s)p_s\mu(x) ds.$$

We denote, as usual, $p_th(x)=\int_X p_t(x,y)h(y)m(dy)$. By Fubini-Tonelli and Chapman-Kolmogorov, for t>0 and $x\in X$ we have

(6)
$$p_t h(x) = \int_t^\infty f(s-t) p_s \mu(x) \, ds$$
$$\leq \int_t^\infty f(s) p_s \mu(x) \, ds$$
$$\leq h(x).$$

In this sense, h is supermedian.

We define $q: X \to [0, \infty]$ as follows: q(x) = 0 if h(x) = 0 or ∞ , else

(8)
$$q(x) = \frac{1}{h(x)} \int_0^\infty f'(s) p_s \mu(x) \, ds.$$

For all $x \in X$ we thus have

(9)
$$q(x)h(x) \le \int_0^\infty f'(s)p_s\mu(x) ds.$$

We define the Schrödinger perturbation of p by q [7]:

(10)
$$\tilde{p} = \sum_{n=0}^{\infty} p^{(n)},$$

where $p_t^{(0)}(x, y) = p_t(x, y)$, and

(11)
$$p_t^{(n)}(x,y) = \int_0^t \int_X p_s(x,z) \, q(z) p_{t-s}^{(n-1)}(z,y) \, m(dz) \, ds, \quad n \ge 1.$$

It is well-known that \tilde{p} is a transition density [7].

Theorem 1. We have $\int_X \tilde{p}_t(x,y)h(y)m(dy) \leq h(x)$.

Proof. For $n = 0, 1, \ldots$ and $t > 0, x \in X$, we consider

$$p_t^{(n)}h(x) := \int_X p_t^{(n)}(x,y)h(y) \, m(dy),$$

and we claim that

(12)
$$\sum_{k=0}^{n} p_t^{(k)} h(x) \le h(x).$$

By (7) this holds for n = 0. By (11), Fubini-Tonelli, induction and (8),

$$\sum_{k=1}^{n+1} p_t^{(k)} h(x) = \int_X \int_0^t \int_X p_s(x, z) \, q(z) \sum_{k=0}^n p_{t-s}^{(k)}(z, y) h(y) \, m(dy) \, ds \, m(dz)$$

$$\leq \int_0^t \int_X p_s(x, z) \, q(z) h(z) \, m(dz) \, ds$$

$$= \int_0^t \int_X p_s(x, z) \int_0^\infty f'(u) \int_X p_u(z, w) \, \mu(dw) \, du \, m(dz) \, ds.$$

$$= \int_0^t \int_0^\infty f'(u) p_{s+u} \mu(x) \, du \, ds,$$

where in the last passage we used (2) and (4). By (3),

$$\sum_{k=1}^{n+1} p_t^{(k)} h(x) \le \int_0^\infty \int_0^{u \wedge t} f'(u - s) \, ds \, p_u \mu(x) \, du$$

$$\le \int_0^\infty [f(u) - f(u - u \wedge t)] \, p_u \mu(x) \, du$$

$$= \int_0^\infty [f(u) - f(u - t)] \, p_u \mu(x) \, du,$$

because f(s) = 0 if $s \le 0$. By this and (6) we obtain

$$\sum_{k=0}^{n+1} p_t^{(k)} h(x) \le \int_t^{\infty} f(u-t) p_u \mu(x) du + \int_0^{\infty} [f(u) - f(u-t)] p_u \mu(x) du = \int_0^{\infty} f(u) p_u \mu(x) du = h(x).$$

The claim (12) is proved. The theorem follows by letting $n \to \infty$.

Remark 1. Theorem 1 asserts that h is supermedian for \tilde{p} . This is much more than (7), but (7) may also be useful in applications [20, Lemma 5.2].

We shall see in Section 4 that the above construction gives integral finiteness (non-explosion) results for specific Schrödinger perturbations with rather singular q, cf. Corollaries 6 and 8. In the next section q will serve as an admissible weight in a Hardy inequality.

3. Hardy inequality

Throughout this section we let p, f, μ, h and q be as defined in Section 2. Additionally we shall assume that p is Markovian, namely $\int_X p_t(x,y)m(dy) \le 1$ for all $x \in X$. In short, p is a subprobability transition density. By Holmgren criterion [23, Theorem 3, p. 176], we then have $p_t u \in L^2(m)$ for each $u \in L^2(m)$, in fact $\int_X [p_t u(x)]^2 m(dx) \le \int_X u(x)^2 m(dx)$. Here $L^2(m)$ is the collection of all the real-valued square-integrable \mathcal{M} -measurable functions on X. As usual, we identify $u, v \in L^2(m)$ if u = v m-a.e. on X. The space (of equivalence classes) $L^2(m)$ is equipped with the scalar product $\langle u, v \rangle = \int_X u(x)v(x)m(dx)$. Since the semigroup of operators $(p_t, t > 0)$ is self-adjoint and weakly measurable, we have

$$\langle p_t u, u \rangle = \int_{[0,\infty)} e^{-\lambda t} d\langle P_{\lambda} u, u \rangle,$$

where P_{λ} is the spectral decomposition of the operators, see [19, Section 22.3]. For $u \in L^2(m)$ and t > 0 we let

$$\mathcal{E}^{(t)}(u,u) = \frac{1}{t} \langle u - p_t u, u \rangle.$$

By the spectral decomposition, $t \mapsto \mathcal{E}^{(t)}(u, u)$ is nonnegative and nonincreasing [17, Lemma 1.3.4], which allows to define the quadratic form of p,

(13)
$$\mathcal{E}(u,u) = \lim_{t \to 0} \mathcal{E}^{(t)}(u,u), \quad u \in L^2(m).$$

The domain of the form is defined by the condition $\mathcal{E}(u,u) < \infty$ [17]. The following is a Hardy-type inequality with a remainder.

Theorem 2. If $u \in L^2(m)$ and u = 0 on $\{x \in X : h(x) = 0 \text{ or } \infty\}$, then

(14)
$$\mathcal{E}(u,u) \ge \int_X u(x)^2 q(x) \, m(dx) + \liminf_{t \to 0} \int_X \int_X \frac{p_t(x,y)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right)^2 h(y) h(x) m(dy) m(dx).$$

If $f(t) = t_+^{\beta}$ with $\beta \ge 0$ in (5) or, more generally, if f is absolutely continuous and there are $\delta > 0$ and $c < \infty$ such that

(15)
$$[f(s) - f(s-t)]/t \le cf'(s)$$
 for all $s > 0$ and $0 < t < \delta$,

then for every $u \in L^2(m)$

(16)
$$\mathcal{E}(u,u) = \int u(x)^2 q(x) \, m(dx) + \lim_{t \to 0} \int_{Y} \int_{Y} \frac{p_t(x,y)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right)^2 h(y) h(x) m(dy) m(dx),$$

Proof. Let v = u/h, with the convention that v(x) = 0 if h(x) = 0 or ∞ . Let t > 0. We note that $|vh| \le |u|$, thus $vh \in L^2(m)$ and by (7), $vp_th \in L^2(m)$. We then have

(17)
$$\mathcal{E}^{(t)}(vh, vh) = \langle v \frac{h - p_t h}{t}, vh \rangle + \langle \frac{vp_t h - p_t(vh)}{t}, vh \rangle =: I_t + J_t.$$

By the definition of J_t and the symmetry (1) of p_t ,

$$J_{t} = \frac{1}{t} \int_{X} \int_{X} p_{t}(x, y) [v(x) - v(y)] h(y) m(dy) v(x) h(x) m(dx)$$
$$= \int_{X} \int_{X} \frac{p_{t}(x, y)}{2t} [v(x) - v(y)]^{2} h(x) h(y) m(dx) m(dy) \ge 0.$$

To deal with I_t , we let $x \in X$, assume that $h(x) < \infty$, and consider

$$(h - p_t h)(x) = \int_0^\infty f(s) p_s \mu(x) \, ds - \int_0^\infty f(s) p_{s+t} \mu(x) \, ds$$
$$= \int_0^\infty [f(s) - f(s-t)] p_s \mu(x) \, ds.$$

Thus,

$$I_{t} = \int_{X} v^{2}(x)h(x) \int_{0}^{\infty} \frac{1}{t} [f(s) - f(s-t)] p_{s}\mu(x) ds m(dx).$$

By (13) and Fatou's lemma.

(18)
$$\mathcal{E}(vh, vh) \ge \int_{X} \int_{0}^{\infty} f'(s) p_{s} \mu(x) \, ds \, v^{2}(x) h(x) \, m(dx)$$

$$+ \liminf_{t \to 0} \int_{X} \int_{X} \frac{p_{t}(x, y)}{2t} \left[v(x) - v(y) \right]^{2} h(y) h(x) m(dy) m(dx)$$

$$= \int_{X} v^{2}(x) h^{2}(x) q(x) m(dx)$$

$$+ \liminf_{t \to 0} \int_{Y} \int_{Y} \frac{p_{t}(x, y)}{2t} \left[v(x) - v(y) \right]^{2} h(y) h(x) m(dy) m(dx).$$

Since u = 0 on $\{x \in X : h(x) = 0 \text{ or } \infty\}$, we have u = vh, hence $\mathcal{E}^{(t)}(u, u) = \mathcal{E}^{(t)}(vh, vh)$ for all t > 0, and so $\mathcal{E}(u, u) = \mathcal{E}(vh, vh)$. From (18) we obtain (14).

If f is absolutely continuous on \mathbb{R} , then (3) becomes equality, and we return to (17) to analyse I_t and J_t more carefully. If $\int_X u(x)^2 q(x) m(dx) < \infty$, which is satisfied in particular when $\mathcal{E}(u,u) < \infty$, and if (15) holds, then we can apply Lebesgue dominated convergence theorem to I_t . In view

of (13) and (17), the limit of J_t then also exists, and we obtain (16). If $\int_X u(x)^2 q(x) \, m(dx) = \infty$, then (18) trivially becomes equality. Finally, (15) holds for $f(t) = t_+^{\beta}$ with $\beta \geq 0$.

Corollary 3. For every $u \in L^2(m)$ we have $\mathcal{E}(u,u) \geq \int_X u(x)^2 q(x) \, m(dx)$.

We are interested in non-zero quotients q. This calls for lower bounds of the numerator and upper bounds of the denominator. The following consequence of (14) applies when sharp estimates of p are known.

Corollary 4. Assume there are a $\mathcal{B}_{(0,\infty)} \times \mathcal{M} \times \mathcal{M}$ -measurable function \bar{p} and a constant $c \geq 1$ such that for every $(t, x, y) \in (0, \infty) \times X \times X$,

$$c^{-1}p_t(x,y) \le \bar{p}_t(x,y) \le cp_t(x,y).$$

Let

$$\bar{h}(x) = \int_0^\infty \int_X f(s)\bar{p}_s(x,y)\mu(dy) \, ds,$$

and let $\bar{q}(x) = 0$ if $\bar{h}(x) = 0$ or ∞ , else let

$$\bar{q}(x) = \frac{1}{\bar{h}(x)} \int_0^\infty f'(s) \bar{p}_s \mu(x) \, ds.$$

Then $c^{-1}h \leq \bar{h} \leq ch$, $c^{-2}q \leq \bar{q} \leq c^2q$, and for $u \in L^2(m)$ such that u = 0 on $X \cap \{\bar{h} = 0 \text{ or } \infty\}$, we have

(19)
$$\mathcal{E}(u,u) \ge c^{-2} \int u(x)^2 q(x) \, m(dx).$$

In the remainder of the paper we discuss applications of the results in Section 2 and Section 3 to transition densities with certain scaling properties.

4. Applications to (fractional) Laplacian

Let $0 < \alpha < 2$, $d \in \mathbb{N}$, $\mathcal{A}_{d,-\alpha} = 2^{\alpha} \Gamma((d+\alpha)/2) \pi^{-d/2}/|\Gamma(-\alpha/2)|$ and $\nu(x,y) = \mathcal{A}_{d,-\alpha}|y-x|^{-d-\alpha}$, where $x,y \in \mathbb{R}^d$. Let m(dx) = dx, the Lebesgue measure on \mathbb{R}^d . Throughout this section, g is the Gaussian kernel

(20)
$$g_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}, \quad t > 0, \quad x \in \mathbb{R}^d.$$

For $u \in L^2(\mathbb{R}^d, dx)$ we define

(21)
$$\mathcal{E}(u,u) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [u(x) - u(y)]^2 \nu(x,y) \, dy \, dx.$$

The important case $\beta=(d-\alpha)/(2\alpha)$ in the following Hardy equality for the Dirichlet form of the fractional Laplacian was given by Frank, Lieb and Seiringer in [15, Proposition 4.1] (see [3] for another proof; see also [18]). In fact, [15, formula (4.3)] also covers the case of $(d-\alpha)/(2\alpha) \leq \beta \leq (d/\alpha) - 1$ and smooth compactly supported functions u in the following Proposition. Our proof is different from that of [15, Proposition 4.1] because we do not use the Fourier transform.

Proposition 5. If $0 < \alpha < d \land 2$, $0 \le \beta \le (d/\alpha) - 1$ and $u \in L^2(\mathbb{R}^d)$, then

$$\mathcal{E}(u,u) = C \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^{\alpha}} \, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(x) h(y) \nu(x,y) \, dy \, dx \,,$$

$$\begin{array}{l} \textit{where } C = 2^{\alpha}\Gamma(\frac{d}{2} - \frac{\alpha\beta}{2})\Gamma(\frac{\alpha(\beta+1)}{2})\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2})^{-1}\Gamma(\frac{\alpha\beta}{2})^{-1}, \, h(x) = |x|^{\alpha(\beta+1)-d}. \\ \textit{We get a maximal } C = 2^{\alpha}\Gamma(\frac{d+\alpha}{4})^2\Gamma(\frac{d-\alpha}{4})^{-2} \; \textit{if } \beta = (d-\alpha)/(2\alpha). \end{array}$$

Proof. (21) is the Dirichlet form of the convolution semigroup of functions defined by subordination, that is we let $p_t(x,y) = p_t(y-x)$, where

(22)
$$p_t(x) = \int_0^\infty g_s(x)\eta_t(s) ds,$$

g is the Gaussian kernel defined in (20) and $\eta_t \geq 0$ is the density function of the distribution of the $\alpha/2$ -stable subordinator at time t, see, e.g., [4] and [17]. Thus, $\eta_t(s) = 0$ for $s \leq 0$, and

$$\int_0^\infty e^{-us} \eta_t(s) \, ds = e^{-tu^{\alpha/2}}, \quad u \ge 0.$$

Let $-1 < \beta < d/\alpha - 1$. The Laplace transform of $s \mapsto \int_0^\infty t^\beta \eta_t(s) dt$ is

$$\int_0^\infty \int_0^\infty t^{\beta} \eta_t(s) \, dt \, e^{-us} \, ds = \int_0^\infty t^{\beta} \int_0^\infty \eta_t(s) e^{-us} \, ds \, dt = \int_0^\infty t^{\beta} e^{-tu^{\alpha/2}} \, dt$$
$$= \Gamma(\beta + 1) u^{-\frac{\alpha(\beta + 1)}{2}}.$$

Since $\int_0^\infty e^{-us} s^{\gamma} ds = \Gamma(\gamma + 1) u^{-(\gamma + 1)}$,

(23)
$$\int_0^\infty t^\beta \eta_t(s) dt = \frac{\Gamma(\beta+1)}{\Gamma(\frac{\alpha(\beta+1)}{2})} s^{\frac{\alpha(\beta+1)}{2}-1}.$$

We consider $-\infty < \delta < d/2 - 1$ and calculate the following integral for the Gaussian kernel by substituting $s = |x|^2/(4t)$,

(24)
$$\int_0^\infty g_t(x)t^{\delta} dt = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} t^{\delta} dt$$
$$= (4\pi)^{-d/2} \left(\frac{|x|^2}{4}\right)^{\delta - d/2 + 1} \int_0^\infty s^{d/2 - \delta - 2} e^{-s} ds$$
$$= 4^{-\delta - 1} \pi^{-d/2} |x|^{2\delta - d + 2} \Gamma(d/2 - \delta - 1).$$

For $f(t):=t_+^\beta$ and σ -finite Borel measure $\mu\geq 0$ on \mathbb{R}^d we have

$$h(x) := \int_{0}^{\infty} \int_{\mathbb{R}^{d}} f(t) p_{t}(x - y) \mu(dy) dt$$

$$= \int_{0}^{\infty} \int_{\mathbb{R}^{d}} t^{\beta} \int_{0}^{\infty} g_{s}(x - y) \eta_{t}(s) ds \, \mu(dy) dt$$

$$= \int_{\mathbb{R}^{d}} \int_{0}^{\infty} \int_{0}^{\infty} t^{\beta} \eta_{t}(s) dt \, g_{s}(x - y) ds \, \mu(dy)$$

$$= \int_{\mathbb{R}^{d}} \int_{0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\frac{\alpha(\beta + 1)}{2})} s^{\frac{\alpha(\beta + 1)}{2} - 1} g_{s}(x - y) ds \, \mu(dy)$$

$$= \frac{\Gamma(\beta + 1)}{\Gamma(\frac{\alpha(\beta + 1)}{2})} \frac{\Gamma(\frac{d}{2} - \frac{\alpha(\beta + 1)}{2})}{4^{\frac{\alpha(\beta + 1)}{2}} \pi^{d/2}} \int_{\mathbb{R}^{d}} |x - y|^{\alpha(\beta + 1) - d} \, \mu(dy),$$

where in the last two equalities we assume $\alpha(\beta+1)/2-1 < d/2-1$ and use (23) and (24). If, furthermore, $\beta \geq 0$, then by the same calculation

$$\int_0^\infty \int_{\mathbb{R}^d} f'(t) p_t(x, y) \mu(dy) dt$$

$$= \beta \frac{\Gamma(\beta)}{\Gamma(\frac{\alpha\beta}{2})} 4^{-\frac{\alpha\beta}{2}} \pi^{-d/2} \Gamma(\frac{d}{2} - \frac{\alpha\beta}{2}) \int_{\mathbb{R}^d} |x - y|^{\alpha\beta - d} \mu(dy).$$

Here the expression is zero if $\beta = 0$. If $\mu = \delta_0$, then we get

(25)
$$h(x) = \frac{\Gamma(\beta+1)}{\Gamma(\frac{\alpha(\beta+1)}{2})} \frac{\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2})}{4^{\frac{\alpha(\beta+1)}{2}} \pi^{d/2}} |x|^{\alpha(\beta+1)-d}$$

and

(26)
$$q(x) = \frac{1}{h(x)} \int_0^\infty \int_{\mathbb{R}^d} f'(t) p_t(x, y) \mu(dy) dt$$
$$= \frac{4^{\alpha/2} \Gamma(\frac{d}{2} - \frac{\alpha\beta}{2}) \Gamma(\frac{\alpha(\beta+1)}{2})}{\Gamma(\frac{d}{2} - \frac{\alpha(\beta+1)}{2}) \Gamma(\frac{\alpha\beta}{2})} |x|^{-\alpha}.$$

By homogeneity, we may assume $h(x) = |x|^{\alpha(\beta+1)-d}$, without changing q. By the second statement of Theorem 2, it remains to show that

(27)
$$\lim_{t \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_t(x,y)}{2t} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(y)h(x)dydx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(y)h(x)\nu(x,y) dy dx.$$

Since $p_t(x,y)/t \leq \nu(x,y)$ [6] and $p_t(x,y)/t \to \nu(x,y)$ as $t \to 0$, (27) follows either by the dominated convergence theorem, if the right hand side of (27) is finite, or – in the opposite case – by Fatou's lemma. If $\alpha\beta = (d-\alpha)/2$,

then we obtain $h(x) = |x|^{-(d-\alpha)/2}$ and the maximal

$$q(x) = \frac{4^{\alpha/2} \Gamma(\frac{d+\alpha}{4})^2}{\Gamma(\frac{d-\alpha}{4})^2} |x|^{-\alpha}.$$

Finally, the statement of the proposition is trivial for $\beta = d/\alpha - 1$.

Corollary 6. If $0 \le r \le d - \alpha$, $x \in \mathbb{R}^d$ and t > 0, then

$$\int_{\mathbb{R}^d} p_t(y - x) |y|^{-r} dy \le |x|^{-r}.$$

If $0 < r < d - \alpha$, $x \in \mathbb{R}^d$, t > 0, $\beta = (d - \alpha - r)/\alpha$, q is given by (26) and \tilde{p} is given by (10), then

$$\int_{\mathbb{R}^d} \tilde{p}_t(y-x)|y|^{-r}dy \le |x|^{-r}.$$

Proof. By (7) and the proof of Proposition 5, we get the first estimate. The second estimate is stronger, because $\tilde{p} \geq p$, cf. (10), and it follows from Theorem 1, cf. the proof of Proposition 5. We do not formulate the second estimate for r = 0 and $d - \alpha$, because the extension suggested by (26) reduces to a special case of the first estimate.

For completeness we now give Hardy equalities for the Dirichlet form of the Laplacian in \mathbb{R}^d . Namely, (29) below is the optimal classical Hardy equality with remainder, and (28) is its slight extension, in the spirit of Proposition 5. For the equality (29), see for example [13, formula (2.3)], [16, Section 2.3] or [3]. Equality (28) may also be considered as a corollary of [16, Section 2.3].

Proposition 7. Suppose $d \geq 3$ and $0 \leq \gamma \leq d-2$. For $u \in W^{1,2}(\mathbb{R}^d)$,

(28)
$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx = \gamma (d - 2 - \gamma) \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx + \int_{\mathbb{R}^d} \left| h(x) \nabla \frac{u}{h}(x) \right|^2 dx,$$

where $h(x) = |x|^{\gamma+2-d}$. In particular,

$$(29) \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx = \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx + \int_{\mathbb{R}^d} \left| |x|^{\frac{2-d}{2}} \nabla \frac{u(x)}{|x|^{(2-d)/2}} \right|^2 dx.$$

Proof. The first inequality is trivial for $\gamma = d - 2$, so let $0 \le \gamma < d - 2$. We first prove that for $u \in L^2(\mathbb{R}^d, dx)$,

(30)
$$\mathcal{C}(u,u) = \gamma(d-2-\gamma) \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx + \lim_{t \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g_t(x,y)}{2t} \left(\frac{u}{h}(x) - \frac{u}{h}(y)\right)^2 h(y)h(x)dydx,$$

where g is the Gaussian kernel defined in (20), and C is the corresponding quadratic form. Even simpler than in the proof of Proposition 5, we let

$$f(t) = t^{\gamma/2}$$
 and $\mu = \delta_0$, obtaining

$$h(x) := \int_0^\infty f(s)g_s\mu(x) ds = \int_0^\infty \int_{\mathbb{R}^d} f(s)g_s(x-y)\mu(dy) ds$$
$$= \int_{\mathbb{R}^d} 4^{-\gamma/2-1}\pi^{-d/2}|x-y|^{\gamma-d+2}\Gamma(d/2-\gamma/2-1)\mu(dy)$$
$$= 4^{-\gamma/2-1}\pi^{-d/2}|x|^{\gamma-d+2}\Gamma(d/2-\gamma/2-1),$$

$$\int_0^\infty f'(s)g_s\mu(x) ds = \frac{\gamma}{2} 4^{-\gamma/2} |x|^{\gamma - d} \pi^{-d/2} \Gamma(d/2 - \gamma/2),$$

(31)
$$q(x) = \frac{\int_0^\infty f'(s)g_s\mu(x) \, ds}{h(x)} = \frac{\gamma(d-2-\gamma)}{|x|^2}.$$

By Theorem 2 we get (30). Since the quadratic form of the Gaussian semigroup is the classical Dirichlet integral, taking $\gamma = (d-2)/2$ and $q(x) = (d-2)^2/(4|x|^2)$ we recover the classical Hardy inequality:

(32)
$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \ge \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx, \qquad u \in L^2(\mathbb{R}^d, dx).$$

We, however, desire (28). It is cumbersome to directly prove the convergence of (30) to (28)¹. Here is an approach based on calculus. For $u \in C_c^{\infty}(\mathbb{R}^d \setminus \{0\})$ we have

$$\begin{split} \partial_j \left(|x|^{d-2-\gamma} u(x) \right) &= (d-2-\gamma) |x|^{d-4-\gamma} u(x) x_j + |x|^{d-2-\gamma} u_j(x), \\ \left| \nabla \left(|x|^{d-2-\gamma} u(x) \right) \right|^2 &= |x|^{2(d-4-\gamma)} \left[(d-2-\gamma)^2 u(x)^2 |x|^2 + |x|^4 |\nabla u(x)|^2 \right. \\ &+ (d-2-\gamma) \left\langle \nabla (u^2)(x), x \right\rangle |x|^2 \right], \end{split}$$

hence

$$\begin{split} \int_{\mathbb{R}^d} \left| \nabla \frac{u}{h}(x) \right|^2 h(x)^2 dx &= \int_{\mathbb{R}^d} \left| \nabla \left(|x|^{d-2-\gamma} u(x) \right) \right|^2 |x|^{2(\gamma+2-d)} dx \\ &= (d-2-\gamma)^2 \int_{\mathbb{R}^d} u(x)^2 |x|^{-2} dx + \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \\ &+ (d-2-\gamma) \int_{\mathbb{R}^d} \left\langle \nabla (u^2)(x), |x|^{-2} x \right\rangle dx. \end{split}$$

Since $\operatorname{div}(|x|^{-2}x) = (d-2)|x|^{-2}$, the divergence theorem yields (28). We then extend (28) to $u \in C_c^{\infty}(\mathbb{R}^d)$ as follows. Let $\psi \in C_c^{\infty}(\mathbb{R}^d)$ be such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $|x| \leq 1$, $\psi(x) = 0$ if $|x| \geq 2$. Let $u_n(x) = u(x)[1-\psi(nx)]$, $n \in \mathbb{N}$. We note the local integrability of $|x|^{-2}$ in \mathbb{R}^d with $d \geq 3$. We let $n \to \infty$ and have (28) hold for u by using the convergence in $L^2(|x|^{-2}dx)$, inequality $|\nabla u_n(x)| \leq |\nabla u(x)| + c_1|u(x)||x|^{-1}$, the identity $h(x)\nabla(u_n/h)(x) = \nabla u_n(x) - u_n(x)[\nabla h(x)]/h(x)$ for $x \neq 0$, the observation that $|\nabla h(x)|/h(x) \leq c|x|^{-2}$ and the dominated convergence theorem. We can now extend (28) to $u \in W^{1,2}(\mathbb{R}^d)$. Indeed, assume

¹But see a comment before [3, (1.6)] and our conclusion below.

that $C_c^{\infty}(\mathbb{R}^d) \ni v_n \to u$ and $\nabla v_n \to g$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$, so that $g = \nabla u$ in the sense of distributions. We have that $h(x)\nabla(v_n/h)(x) = \nabla v_n(x) - v_n(x)[\nabla h(x)]/h(x) \to g - u[\nabla h(x)]/h(x)$ in $L^2(\mathbb{R}^d)$. The latter limit is $h\nabla(u/h)$, as we understand it. We obtain the desired extension of (28). As a byproduct we actually see the convergence of the last term in (30). Taking $\gamma = (d-2)/2$ in (28) yields (29).

We note that (32) holds for all $u \in L^2(\mathbb{R}^d)$.

Corollary 8. If $0 \le r \le d-2$, $x \in \mathbb{R}^d$ and t > 0, then

$$\int_{\mathbb{R}^d} g_t(y - x) |y|^{-r} dy \le |x|^{-r}.$$

If 0 < r < d-2, $x \in \mathbb{R}^d$, t > 0, $\beta = (d-2-r)/2$, q is given by (31), and \tilde{g} is the Schrödinger perturbation of g by q as in (10), then

$$\int_{\mathbb{R}^d} \tilde{g}_t(y-x)|y|^{-r}dy \le |x|^{-r}.$$

The proof is similar to that of Corollary 6 and is left to the reader.

5. Applications to transition densities with global scaling

In this section we show how sharp estimates of transition densities satisfying certain scaling conditions imply Hardy inequalities. In particular we give Hardy inequalities for symmetric jump processes on metric measure space studied in [10], and for unimodal Lévy processes recently estimated in [6]. In what follows we assume that $\phi:[0,\infty)\to[0,\infty)$ is nondecreasing and left-continuous, $\phi(0)=0,\,\phi(x)>0$ if x>0 and $\phi(\infty^-):=\lim_{x\to\infty}\phi(x)=\infty$. We denote, as usual,

$$\phi^{-1}(u) = \inf\{s > 0 : \phi(s) > u\}, \qquad u \ge 0.$$

Here is a simple observation, which we give without proof.

Lemma 9. Let $r, t \geq 0$. We have $t \geq \phi(r)$ if and only if $\phi^{-1}(t) \geq r$.

We see that ϕ^{-1} is upper semicontinuous, hence right-continuous, $\phi^{-1}(\infty^{-}) = \infty$, $\phi(\phi^{-1}(u)) \leq u$ and $\phi^{-1}(\phi(s)) \geq s$ for $s, u \geq 0$. If ϕ is continuous, then $\phi(\phi^{-1}(u)) = u$, and if ϕ is strictly increasing, then $\phi^{-1}(\phi(s)) = s$ for $s, u \geq 0$. Both these conditions typically hold in our applications, and then ϕ^{-1} is the genuine inverse function.

We first recall, after [6, Section 3], [22, Section 2] and [29, (2.7) and (2.20)], the following weak scaling conditions. We say that a function φ : $[0,\infty) \to [0,\infty)$ satisfies the global weak lower scaling condition if there are numbers $\underline{\alpha} \in \mathbb{R}$ and $\underline{c} \in (0,1]$, such that

(33)
$$\varphi(\lambda\theta) \ge \underline{c} \lambda^{\underline{\alpha}} \varphi(\theta), \qquad \lambda \ge 1, \quad \theta > 0.$$

We then write $\varphi \in \text{WLSC}(\underline{\alpha}, \underline{c})$. Put differently, $\varphi(R)/\varphi(r) \geq \underline{c}(R/r)^{\underline{\alpha}}$, $0 < r \leq R$. The global weak upper scaling condition holds if there are

numbers $\overline{\alpha} \in \mathbb{R}$ and $\overline{c} \in [1, \infty)$ such that

(34)
$$\varphi(\lambda\theta) \leq \overline{c} \lambda^{\overline{\alpha}} \varphi(\theta), \qquad \lambda \geq 1, \quad \theta > 0,$$

or $\varphi(R)/\varphi(r) \leq \overline{c} (R/r)^{\overline{\alpha}}$, $0 < r \leq R$. In short, $\varphi \in \text{WUSC}(\overline{\alpha}, \overline{c})$. We note that φ has the lower scaling if and only if $\varphi(\theta)/\theta^{\underline{\alpha}}$ is almost increasing, i.e. comparable with a nondecreasing function on $[0, \infty)$, and φ has the upper scaling if and only if $\varphi(\theta)/\theta^{\overline{\alpha}}$ is almost decreasing, see [6, Lemma 11].

Let (F, ρ, m) be a metric measure space with metric ρ and Radon measure m with full support. We denote $B(x,r)=\{y\in F: \rho(x,y)< r\}$ and assume that there is a nondecreasing function $V:[0,\infty)\to[0,\infty)$ such that $V(0)=0,\,V(r)>0$ for r>0, and

(35)
$$c_1 V(r) \le m(B(x,r)) \le c_2 V(r) \quad \text{for all } x \in F \text{ and } r \ge 0.$$

We call V the volume function.

Theorem 10. Let p be a symmetric subprobability transition density on F, with Dirichlet form \mathcal{E} , and assume that

(36)
$$p_t(x,y) \approx \frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))}, \quad t > 0, \quad x, y \in F,$$

where $\phi, V : [0, \infty) \to (0, \infty)$ are non-decreasing, positive on $(0, \infty)$, $\phi(0) = V(0) = 0$, $\phi(\infty^-) = \infty$ and V satisfies (35). If $\underline{A} > \overline{\alpha} > 0$, $V \in \mathrm{WLSC}(\underline{A}, \underline{C})$ and $\phi \in \mathrm{WUSC}(\overline{\alpha}, \overline{c})$, then there is C > 0 such that

(37)
$$\mathcal{E}(u,u) \ge C \int_{F} \frac{u(x)^2}{\phi(\rho(x,y))} m(dx), \qquad y \in F, \quad u \in L^2(F,m).$$

Proof. Let $y \in F$ and $u \in L^2(F, m)$. The constants in the estimates below are independent of y and u. Let $0 < \beta < \underline{A}/\overline{\alpha} - 1$ and define

(38)
$$h(x) = \int_0^\infty t^\beta p_t(x, y) dt, \quad x \in F.$$

We shall prove that

(39)
$$\mathcal{E}(u,u) \approx \int_{F} \frac{u(x)^{2}}{\phi(\rho(x,y))} m(dx) + \liminf_{t \to 0} \int_{F} \int_{F} \frac{p_{t}(x,z)}{2t} \left(\frac{u(x)}{h(x)} - \frac{u(z)}{h(z)}\right)^{2} h(z)h(x)m(dz)m(dx).$$

To this end, we first verify

(40)
$$h(x) \approx \phi(\rho(x,y))^{\beta+1}/V(\rho(x,y)), \qquad \rho(x,y) > 0.$$

Indeed, letting $r = \rho(x, y) > 0$ we first note that Lemma 9 yields $t \ge \phi(r)$ equivalent to $tV(\phi^{-1}(t)) \ge V(r)\phi(r)$, from whence

$$h(x) \approx V(r)^{-1} \phi(r)^{-1} \int_0^{\phi(r)} t^{\beta+1} dt + \int_{\phi(r)}^{\infty} \frac{t^{\beta}}{V(\phi^{-1}(t))} dt$$
$$= (\beta + 2)^{-1} \phi(r)^{\beta+1} / V(r) + I.$$

To estimate I, we observe that the assumption $\phi \in \mathrm{WUSC}(\overline{\alpha}, \overline{c})$ implies $\phi^{-1} \in \mathrm{WLSC}(1/\overline{\alpha}, \overline{c}^{-1/\overline{\alpha}})$ [6, Remark 4]. If r > 0 and $t \ge \phi(r)$, then

$$\frac{V(\phi^{-1}(t))}{V(r)} \ge \frac{V(\phi^{-1}(t))}{V(\phi^{-1}(\phi(r)))} \ge \underline{C} \left(\frac{\phi^{-1}(t)}{\phi^{-1}(\phi(r))}\right)^{\underline{A}}$$
$$\ge \underline{C} \, \overline{c}^{-\underline{A}/\overline{\alpha}} \frac{t^{\underline{A}/\overline{\alpha}}}{\phi(r)^{\underline{A}/\overline{\alpha}}},$$

hence,

$$\frac{t^{\beta}}{V(\phi^{-1}(t))} \leq \frac{\overline{c}^{\underline{A}/\overline{\alpha}}}{\underline{C}} \frac{\phi(r)^{\underline{A}/\overline{\alpha}}}{V(r)} t^{\beta - \underline{A}/\overline{\alpha}}.$$

The claim (40) now follows because

$$(41) I \leq \frac{\overline{c}^{\underline{A}/\overline{\alpha}}}{\underline{C}} \frac{\phi(r)^{\underline{A}/\overline{\alpha}}}{V(r)} \int_{\phi(r)}^{\infty} t^{\beta - \underline{A}/\overline{\alpha}} dt = \frac{\overline{c}^{\underline{A}/\overline{\alpha}}}{\underline{C}(\underline{A}/\overline{\alpha} - 1 - \beta)} \frac{\phi(r)^{\beta + 1}}{V(r)}.$$

The function

$$k(x) := \int_0^\infty p_t(x, y)(t^\beta)' dt, \quad x \in F,$$

also satisfies

$$k(x) \approx \phi(\rho(x,y))^{\beta}/V(\rho(x,y)), \quad x \in F.$$

This follows by recalculating (40) for $\beta - 1$. We get

(42)
$$C_1\phi(\rho(x,y))^{-1} \le q(x) := \frac{k(x)}{h(x)} \le C_2\phi(\rho(x,y))^{-1},$$

by choosing any $\beta \in (0, \underline{A}/\overline{\alpha} - 1)$. The theorem follows from (16).

Remark 2. With the above notation, for each $0 < \beta < \underline{A}/\overline{\alpha} - 1$ there exists a constant c such that for all $x, y \in F$ we have

$$\int p_t(x,z)\phi(\rho(z,y))^{\beta+1}/V(\rho(z,y))m(dz) \le c\phi(\rho(x,y))^{\beta+1}/V(\rho(x,y))$$

and

$$\int \tilde{p}_t(x,z)\phi(\rho(z,y))^{\beta+1}/V(\rho(z,y))m(dz) \le c\phi(\rho(x,y))^{\beta+1}/V(\rho(x,y)),$$

where \tilde{p} is given by (10) with $q(x) = C_1 \phi(\rho(x,y))^{-1}$ on F and C_1 is the constant in the lower bound of the sharp estimate in (42). This is a non-explosion result for \tilde{p} , and it is proved in the same way as Corollary 6.

Remark 3. Interestingly, the Chapman-Kolmogorov equations and (36) imply that ϕ in Theorem 10 satisfies a lower scaling, too. We leave the proof of this fact to the interested reader because it is not used in the sequel. An analogous situation occurs in [6, Theorem26].

In [10] a wide class of transition densities are constructed on locally compact separable metric measure spaces (F, ρ, m) with metric ρ and Radon measure m of infinite mass and full support on F. Here are some of the assumptions of [10] (for details see [10, Theorem 1.2]). The functions $\phi, V : [0, \infty) \to (0, \infty)$ are increasing, $\phi(0) = V(0) = 0$, $\phi(1) = 1$, $\phi \in \text{WLSC}(\underline{\alpha}, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, \overline{c})$, $V \in \text{WLSC}(\underline{A}, \underline{C}) \cap \text{WUSC}(\overline{A}, \overline{C})$, and

$$\int_0^r \frac{s}{\phi(s)} \, ds \le c \frac{r^2}{\phi(r)} \quad \text{for every } r > 0.$$

A symmetric measurable function J(x, y) satisfying

(43)
$$J(x,y) \approx \frac{1}{V(\rho(x,y))\phi(\rho(x,y))}, \quad x,y \in F, x \neq y,$$

is considered in [10, Theorem 1.2] along with a symmetric pure-jump Markov process having J as jump kernel and symmetric p satisfying (36) as transition density. By Theorem 10, we obtain the following result.

Corollary 11. Under the assumptions of [10, Theorem 1.2], (37) and (39) hold if $\underline{A} > \overline{\alpha}$.

We now specialize to $F = \mathbb{R}^d$ equipped with the Lebesgue measure. Let ν be an infinite isotropic unimodal $L\acute{e}vy$ measure on \mathbb{R}^d i.e. $\nu(dx) = \nu(|x|)dx$, where $(0,\infty) \ni r \mapsto \nu(r)$ is nonincreasing, and

$$\nu(\mathbb{R}^d \setminus \{0\}) = \infty$$
 and $\int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1) \ \nu(dx) < \infty.$

Let

(44)
$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x \rangle) \, \nu(dx).$$

Because of rotational symmetry, ψ depends only on $|\xi|$, and we can write $\psi(r) = \psi(\xi)$ for $r = |\xi|$. This ψ is almost increasing [6], namely $\pi^2 \psi(r) \geq \psi^*(r) := \sup\{\psi(p) : 0 \leq p \leq r\}$. Let $0 < \underline{\alpha} \leq \overline{\alpha} < 2$. If $0 \not\equiv \psi \in \mathrm{WLSC}(\underline{\alpha},\underline{c}) \cap \mathrm{WUSC}(\overline{\alpha},\overline{c})$, then the following defines a convolution semigroup of functions,

(45)
$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} e^{-t\psi(\xi)} d\xi, \quad t > 0, x \in \mathbb{R}^d,$$

and the next two estimates hold [6, Theorem 21].

(46)
$$p_t(x) \approx \left[\psi^-(1/t)\right]^d \wedge \frac{t\psi(1/|x|)}{|x|^d}, \qquad t > 0, \ x \in \mathbb{R}^d,$$

(47)
$$\nu(|x|) \approx \frac{\psi(1/|x|)}{|x|^d}, \qquad x \in \mathbb{R}^d.$$

Here $\psi^-(u) = \inf\{s \geq 0 : \psi^*(s) \geq u\}$, the left-continuous inverse of ψ^* . The corresponding Dirichlet form is

$$\mathcal{E}(u,u) = (2\pi)^d \int_{\mathbb{R}^d} \hat{u}(\xi) \overline{\hat{v}(\xi)} \psi(\xi) \, d\xi = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \nu(y - x) \, dy \, dx$$
$$\approx \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 \frac{\psi(|x - y|^{-1})}{|y - x|^d} \, dy \, dx,$$

cf. [17, Example 1.4.1] and the special case discussed in the proof of Proposition 5 above.

Corollary 12. If $d > \overline{\alpha}$, then is c > 0 such that for all $u \in L^2(\mathbb{R}^d)$

(48)
$$\mathcal{E}(u,u) \ge c \int_{\mathbb{R}^d} u(x)^2 \psi(1/|x|) \, dx.$$

Proof. Let $0 < \beta < (d/\overline{\alpha}) - 1$, $h(x) = \int_0^\infty t^\beta p_t(x) \, dt$, and $k(x) = \int_0^\infty (t^\beta)' p_t(x) \, dt$. Considering $\rho(x,y) = |y-x|$, $\phi(r) = 1/\psi(1/r)$ and $V(r) = r^d$, by (37) we get (48) for all $u \in L^2(\mathbb{R}^d)$. To add some detail, we note that ϕ satisfies the same scalings as ψ^* and $\phi^{-1}(t) = 1/\psi^-(t^{-1})$. Thus (40) yields $h(x) \approx \psi(|x|^{-1})^{-\beta-1}|x|^{-d}$ and $k(x) \approx \psi(|x|^{-1})^{-\beta}|x|^{-d}$. In fact, we actually obtain Hardy equality. Indeed,

$$\lim_{t \to 0} \inf \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_t(x,y)}{2t} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(y)h(x)dydx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(y)h(x)\nu(|x-y|) dy dx ,$$

because if $t \to 0$, then $p_t(x,y)/t \le c\nu(|x-y|)$ by (46) and (47), and $p_t(x,y)/t \to \nu(|x-y|)$ (weak convergence of radially monotone functions implies convergence almost everywhere), and we can use the dominated convergence theorem or Fatou's lemma, as in the proof of Proposition 5. We thus have a strengthening of (48) for every $u \in L^2(\mathbb{R}^d)$:

(50)
$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d} u(x)^2 \frac{k(x)}{h(x)} dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{u(x)}{h(x)} - \frac{u(y)}{h(y)} \right]^2 h(y) h(x) \nu(|x-y|) dy dx.$$

For instance if we take $\psi(\xi) = |\xi| \sqrt{\log(1+|\xi|)}$, the Lévy-Kchintchine exponent of a subordinated Brownian motion [26], then we obtain

$$\int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 |\xi| \sqrt{\log(1+|\xi|)} d\xi \ge c \int_{\mathbb{R}^d} \frac{u(x)^2 \sqrt{\log(1+|x|^{-1})}}{|x|} dx, \quad u \in L^2(\mathbb{R}^d).$$

Remark 4. We note that [12, Theorem 1, the "thin" case (T)] gives (48) for continuous functions u of compact support in \mathbb{R}^d . Here we extend the result to all functions $u \in L^2(\mathbb{R}^d)$, as typical for our approach. We note in passing

that [12, Theorem 1, Theorem 5] offers a general framework for Hardy inequalities without the remainder terms and applications for quadratic forms on Euclidean spaces.

Here is an analogue of Remark 2.

Remark 5. Using the notation above, for every $0 < \beta < (d - \overline{\alpha})/\overline{\alpha}$, there exist constants c_1 , c_2 such that

$$\int p_t(y-x)\psi(|y|^{-1})^{-\beta-1}|y|^{-d}dy \le c_1\psi(|x|^{-1})^{-\beta-1}|x|^{-d}, \quad x \in \mathbb{R}^d,$$

and

$$\int \tilde{p}_t(x,z)dy \psi(|y|^{-1})^{-\beta-1}|y|^{-d}dy \le c_1\psi(|x|^{-1})^{-\beta-1}|x|^{-d}, \quad x \in \mathbb{R}^d,$$

where \tilde{p} is given by (10) with $q(x) = c_2 \psi(1/|x|)$ on \mathbb{R}^d . The result is proved as Remark 2. In particular we obtain non-explosion of Schrödinger perturbations of such unimodal transition densities with $q(x) = c_2 \psi(1/|x|)$. Naturally, the largest valid c_2 is of further interest.

6. Weak local scaling on Euclidean spaces

In this section we restrict ourselves to the Euclidean space and apply Theorem 2 to a large class of symmetric jump processes satisfying two-sided heat kernel estimates given in [9] and [6]. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function such that $\phi(0) = 0$, $\phi(1) = 1$, and

$$\underline{c}\left(\frac{R}{r}\right)^{\underline{\alpha}} \le \frac{\phi(R)}{\phi(r)} \le \overline{c}\left(\frac{R}{r}\right)^{\overline{\alpha}}$$
 for every $0 < r < R \le 1$.

Let J be a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d \cap \{x \neq y\}$ and let κ_1, κ_2 be positive constants such that

(51)
$$\frac{\kappa_1^{-1}}{|x-y|^d \phi(|x-y|)} \le J(x,y) \le \frac{\kappa_1}{|x-y|^d \phi(|x-y|)}, \quad |x-y| \le 1,$$

and

(52)
$$\sup_{x \in \mathbb{R}^d} \int_{\{y \in \mathbb{R}^d : |x-y| > 1\}} J(x,y) dy =: \kappa_2 < \infty.$$

We consider the quadratic form

$$\mathcal{E}(u,u) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [u(y) - u(x)]^2 J(x,y) dy dx, \qquad u \in L^2(\mathbb{R}^d, dx),$$

with the Lebesgue measure dx as the reference measure, for the symmetric pure-jump Markov processes on \mathbb{R}^d constructed in [8] from the jump kernel J(x,y).

Theorem 13. If $d \geq 3$, then

(53)
$$\mathcal{E}(u,u) \ge c \int_{\mathbb{R}^d} u(x)^2 \frac{dx}{\phi(|x|) \vee |x|^2}, \qquad u \in L^2(\mathbb{R}^d).$$

Proof. Let Q and $p_t(x, y)$ be the quadratic form and the transition density corresponding to the symmetric pure-jump Markov process in \mathbb{R}^d with the jump kernel $J(x, y)\mathbf{1}_{\{|x-y|<1\}}$ instead of J(x, y), cf. [9, Theorem 1.4]. Thus,

$$Q(u,u) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 J(x,y) \mathbf{1}_{\{|x-y| \le 1\}} dx dy, \qquad u \in C_c(\mathbb{R}^d).$$

We define h as $h(x) = \int_0^\infty p_t(x,0) t^\beta dt$, $x \in \mathbb{R}^d$ where $-1 < \beta < d/2 - 1$. We note that for every $T, M \ge 0$,

(54)
$$\int_{T}^{\infty} t^{\beta - \frac{d}{2}} e^{-\frac{Mr^2}{t}} dt = r^{2\beta - d + 2} \int_{0}^{\frac{r^2}{T}} u^{-2 - \beta + \frac{d}{2}} e^{-Mu} du.$$

We shall use [9, Theorem 1.4]. We however note that the term $\log \frac{|x-y|}{t}$ in the statement of [9, Theorem 1.4] should be replaced by $1 + \log_+ \frac{|x-y|}{t}$, to include the case $c^{-1}t \leq |x-y| \leq ct$ missed in the considerations in [9]. With this correction, our arguments are as follows. When $r = |x| \leq 1$, we have

(55)
$$c_0^{-1} \left(\frac{1}{\phi^{-1}(t)^d} \wedge \frac{t}{r^d \phi(r)} \right) \le p_t(x, 0) \le c_0 \left(\frac{1}{\phi^{-1}(t)^d} \wedge \frac{t}{r^d \phi(r)} \right), \quad t \in (0, 1]$$

and

$$(56) p_t(x,0) \le c_0 t^{-d/2} e^{-\overline{c}\left(\left(r\left(\log_+(\frac{r}{t})+1\right)\right) \wedge \frac{r^2}{t}\right)} \le c_0 t^{-d/2} e^{-\overline{c}\frac{r^2}{t}}, t > 1.$$

Thus, by (55), Lemma 9, (56), (41), (51) and (54),

$$c_{3} \frac{\phi(r)^{\beta+1}}{r^{d}} \leq c_{0}^{-1} \int_{0}^{\phi(r)} \frac{t^{\beta+1}}{r^{d}\phi(r)} dt \leq h(x)$$

$$\leq c_{0} \int_{0}^{\phi(r)} \frac{t^{\beta+1}}{r^{d}\phi(r)} dt + c_{0} \int_{\phi(r)}^{1} \frac{t^{\beta}}{(\phi^{-1}(t))^{d}} dt + c_{0} \int_{1}^{\infty} t^{\beta-\frac{d}{2}} e^{-\frac{\overline{c}r^{2}}{t}} dt$$

$$\leq c_{4} \frac{\phi(r)^{\beta+1}}{r^{d}} + \frac{c_{5}}{r^{d-2-2\beta}} \int_{0}^{\infty} u^{-2-\beta+d/2} e^{-\overline{c}u} du$$

$$\leq c_{6} \left(\phi(r)^{1+\beta} + r^{2+2\beta}\right) r^{-d}.$$

If r = |x| > 1, then by [9, Theorem 1.4], we have

(57)
$$c_0^{-1} e^{-\underline{c}r(\log_+(\frac{r}{t})+1)} \le p_t(x,0) \le c_0 e^{-\overline{c}r(\log_+(\frac{r}{t})+1)}, \quad t \in (0,1],$$

and for t > 1 we have

(58)
$$c_0^{-1} e^{-\underline{c}\left(\left(r(\log_+(\frac{r}{t})+1)\right)\wedge\frac{r^2}{t}\right)} \le p_t(x,0)/t^{-d/2} \le c_0 e^{-\overline{c}\left(\left(r(\log_+(\frac{r}{t})+1)\right)\wedge\frac{r^2}{t}\right)}.$$

In particular,

(59)
$$p_t(x,0) \ge c_0^{-1} t^{-d/2} e^{-\underline{c}\left(\left(r\left(\log_+(\frac{r}{t})+1\right)\right) \wedge \frac{r^2}{t}\right)} \ge c_7 t^{-d/2}, \quad t > r^2.$$

Then (57), (58), (59), (51) and (54) give

$$\frac{c_7}{r^{d-2-2\beta}} \int_0^1 u^{-2-\beta+d/2} du = c_7 \int_{r^2}^{\infty} t^{\beta-\frac{d}{2}} dt \le h(x)$$

$$\le c_8 \int_0^r t^{\beta} e^{-c_9 r} dt + c_8 \int_r^{\infty} t^{\beta-\frac{d}{2}} e^{-\frac{c_{10} r^2}{t}} dt$$

$$= c_8 (\beta+1)^{-1} r^{\beta+1} e^{-c_9 r} + \frac{c_8}{r^{d-2-2\beta}} \int_0^r u^{-2-\beta+d/2} e^{-c_{10} u} du$$

$$\le \frac{c_{11}}{r^{d-2-2\beta}}.$$

Thus,

$$h(x) \approx (\phi(|x|) \vee |x|^2)^{\beta+1} |x|^{-d}, \quad x \in \mathbb{R}^d.$$

In particular, if $0 < \beta < d/2 - 1$, and $k(x) = \int_0^\infty p_t(x,0)(t^\beta)' dt$, then

$$k(x) \approx (\phi(|x|) \vee |x|^2)^{\beta} |x|^{-d}, \quad x \in \mathbb{R}^d.$$

Therefore,

$$q(x) := \frac{k(x)}{h(x)} \approx \frac{1}{\phi(|x|) \vee |x|^2}.$$

Theorem 2 yields

(60)
$$\mathcal{E}(u,u) \ge \mathcal{Q}(u,u) \ge c_{12} \int_{\mathbb{R}^d} u(x)^2 \frac{dx}{\phi(|x|) \vee |x|^2}, \quad u \in L^2(\mathbb{R}^d).$$

Remark 6. As in Remark 5 we obtain non-explosion for Schrödinger perturbations by $q(x) = c/[\phi(|x|) \vee |x|^2]$.

Remark 7. The arguments and conclusions of Theorem 13 are valid for the unimodal Lévy processes, in particular for the subordinated Brownian motions, provided their Lévy-Khintchine exponent ψ satisfies the assumptions of local scaling conditions at infinity with exponents strictly between 0 and 2 < d made in [6, Theorem 21]:

$$\int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \psi(\xi) d\xi \ge c \int_{\mathbb{R}^d} u(x)^2 \left[\psi\left(\frac{1}{|x|}\right) \wedge \frac{1}{|x|^2} \right] dx, \qquad u \in L^2(\mathbb{R}^d).$$

References

- [1] A. Ancona. On strong barriers and an inequality of Hardy for domains in \mathbb{R}^n . J. London Math. Soc. (2), 34(2):274–290, 1986.
- [2] P. Baras and J. A. Goldstein. The heat equation with a singular potential. *Trans. Amer. Math. Soc.*, 284(1):121–139, 1984.
- [3] W. Beckner. Pitt's inequality and the fractional Laplacian: sharp error estimates. Forum Math., 24(1):177–209, 2012.
- [4] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondraček. Potential analysis of stable processes and its extensions, volume 1980 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009. Edited by Piotr Graczyk and Andrzej Stos.

- [5] K. Bogdan and B. Dyda. The best constant in a fractional Hardy inequality. Math. Nachr., 284(5-6):629-638, 2011.
- [6] K. Bogdan, T. Grzywny, and M. Ryznar. Density and tails of unimodal convolution semigroups. J. Funct. Anal., 266(6):3543-3571, 2014.
- [7] K. Bogdan, W. Hansen, and T. Jakubowski. Time-dependent Schrödinger perturbations of transition densities. Studia Math., 189(3):235–254, 2008.
- [8] Z.-Q. Chen, P. Kim, and T. Kumagai. On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. Acta Math. Sin. (Engl. Ser.), 25(7):1067–1086, 2009.
- [9] Z.-Q. Chen, P. Kim, and T. Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.*, 363(9):5021–5055, 2011.
- [10] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. Probab. Theory Related Fields, 140(1-2):277-317, 2008.
- [11] B. Dyda. Fractional calculus for power functions and eigenvalues of the fractional Laplacian. Fract. Calc. Appl. Anal., 15(4):536–555, 2012.
- [12] B. Dyda and A. V. Vähäkangas. A framework for fractional Hardy inequalities. Ann. Acad. Sci. Fenn., Math., 39(2):675–689, 2014.
- [13] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. J. Funct. Anal., 192(1):186–233, 2002.
- [14] P. J. Fitzsimmons. Hardy's inequality for Dirichlet forms. J. Math. Anal. Appl., 250(2):548–560, 2000.
- [15] R. L. Frank, E. H. Lieb, and R. Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators J. Amer. Math. Soc., 21(4):925–950, 2008.
- [16] R. L. Frank and R. Seiringer. Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal., 255(12):3407–3430, 2008.
- [17] M. Fukushima, Y. Oshima, and M. Takeda. Dirichlet forms and symmetric Markov processes, volume 19 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, extended edition, 2011.
- [18] I. W. Herbst. Spectral theory of the operator $(p^2 + m^2)^{1/2} Ze^2/r$. Comm. Math. Phys., 53(3):285–294, 1977.
- [19] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.
- [20] T. Jakubowski. Fundamental solution of the fractional diffusion equation with a singular drift. preprint, 2014.
- [21] A. Kałamajska and K. Pietruska-Pałuba. On a variant of the Gagliardo-Nirenberg inequality deduced from the Hardy inequality. Bull. Pol. Acad. Sci. Math., 59(2):133– 149, 2011.
- [22] P. Kim, R. Song, and Z. Vondraček. Global uniform boundary Harnack principle with explicit decay rate and its application. Stochastic Process. Appl., 124(1):235– 267, 2014.
- [23] P. D. Lax. Functional analysis. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2002.
- [24] V. Maz'ya. Sobolev spaces with applications to elliptic partial differential equations, volume 342 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, augmented edition, 2011.
- [25] D. Pilarczyk. Self-similar asymptotics of solutions to heat equation with inverse square potential. J. Evol. Equ., 13(1):69–87, 2013.
- [26] R. L. Schilling, R. Song, and Z. Vondraček. Bernstein functions, volume 37 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, second edition, 2012. Theory and applications.
- [27] I. Skrzypczak. Hardy-type inequalities derived from p-harmonic problems. Nonlinear Anal., 93:30–50, 2013.

- [28] J. L. Vazquez and E. Zuazua. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J. Funct. Anal., 173(1):103–153, 2000.
- [29] M. Zähle. Potential spaces and traces of Lévy processes on h-sets. Izv. Nats. Akad. Nauk Armenii Mat., 44(2):67–100, 2009.

Department of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: bogdan@pwr.edu.pl

Department of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego $27,\ 50\text{-}370$ Wrocław, Poland

E-mail address: Bartlomiej.Dyda@pwr.edu.pl

DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, BUILDING 27, 1 GWANAK-RO, GWANAK-GU SEOUL 151-747, REPUBLIC OF KOREA

E-mail address: pkim@snu.ac.kr