# BIFURCATION FROM INFINITY FOR AN ASYMPTOTICALLY LINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the asymptotically linear Schrödinger equation (1.1) and show that if  $\lambda_0$  is an isolated eigenvalue for the linearization at infinity, then under some additional conditions there exists a sequence  $(u_n, \lambda_n)$  of solutions such that  $||u_n|| \to \infty$  and  $\lambda_n \to \lambda_0$ . Our results extend those by Stuart [21]. We use degree theory if the multiplicity of  $\lambda_0$  is odd and Morse theory (or more specifically, Gromoll-Meyer theory) if it is not.

### 1. Introduction

In this paper we consider the Schrödinger equation

$$(1.1) -\Delta u + V(x)u = \lambda u + f(x, u), \quad x \in \mathbb{R}^N,$$

where  $\lambda$  is a real parameter,  $V \in L^{\infty}(\mathbb{R}^N)$ ,  $f(x,u)/u \to m(x)$  as  $|u| \to \infty$ ,  $m \in L^{\infty}(\mathbb{R}^N)$  and  $\lambda_0$  is an isolated eigenvalue of finite multiplicity for  $\mathcal{L} := -\Delta + V(x) - m(x)$ .  $\mathcal{L}$  will be considered as an operator in  $L^2(\mathbb{R}^N)$ . It is well known (see e.g. [18]) that  $\mathcal{L}$  is selfadjoint and its domain  $D(\mathcal{L})$  is the Sobolev space  $H^2(\mathbb{R}^N)$ . We shall show that if the distance from  $\lambda_0$  to the essential spectrum  $\sigma_e(\mathcal{L})$  of  $\mathcal{L}$  is larger than the Lipschitz constant of f - m (with respect to the u-variable), then there exists a sequence of solutions  $(u_n, \lambda_n) \subset H^2(\mathbb{R}^N) \times \mathbb{R}$  such that  $||u_n|| \to \infty$  and  $\lambda_n \to \lambda_0$ . See Theorems 1.3 and 1.4 for more precise statements. We shall say that these solutions bifurcate from infinity or that  $\lambda_0$  is an asymptotic bifurcation point. Our results extend those by Stuart [21] who has shown using degree theory that if f(x,u) = f(u) + h(x), then asymptotic bifurcation occurs if  $\lambda_0$  is of odd multiplicity and the bifurcating set contains a continuum.

Both here and in [21] (see also [20]) the result is first formulated in terms of an abstract operator equation. Let E be a Hilbert space,  $L:D(L)\to E$  a selfadjoint linear operator and let  $N:E\to E$  be a continuous nonlinear operator which is asymptotically linear in the sense of Hadamard (H-asymptotically linear for short, see Definition 2.1(i)). We show that if  $\lambda_0$  is an isolated eigenvalue of odd multiplicity for L and if the distance dist  $(\lambda_0, \sigma_e(L))$  from  $\lambda_0$  to the essential spectrum of L is larger than the asymptotic Lipschitz constant of N (introduced in Definition 2.1(ii)), then  $\lambda_0$  is an asymptotic bifurcation point for the equation

(1.2) 
$$Lu = \lambda u + N(u), \quad u \in D(L).$$

Here we have assumed for notational simplicity that the asymptotic derivative  $N'(\infty)$  of N is 0, see Theorem 1.1 for the full statement. This theorem slightly extends some results in [20, 21]

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where the distance condition on  $\lambda_0$  was somewhat stronger. If N is the gradient of a  $C^1$ -functional and  $\lambda_0$  is an isolated eigenvalue of finite (not necessarily odd) multiplicity, we show that under an additional hypothesis  $\lambda_0$  is an asymptotic bifurcation point for (1.2). The exact statement is given in Theorem 1.2. Existence of asymptotic bifurcation when the multiplicity of  $\lambda_0$  is even seems to be new and is the main abstract result of this paper. A related problem  $u = \lambda(Au + N(u))$  has been considered in [6, 23] under the assumptions that A is bounded linear, A + N is the gradient of a functional and a k-set contraction, and N is asymptotically linear in the stronger sense of Fréchet. It was then shown that each eigenvalue  $1/\lambda_0$  of A with  $|\lambda_0 k| < 1$  is an asymptotic bifurcation point. However, the arguments there seem to break down in our case.

The proofs in [20, 21] were effected by first making the inversion  $u \mapsto u/\|u\|^2$  (an idea that goes back to Rabinowitz [16] and Toland [22]). In this way the problem is transformed to that of looking for bifurcation from 0 instead of infinity. In the next step a finite-dimensional reduction is performed and finally it is shown that since  $\lambda_0$  has odd multiplicity, the Brouwer degree for the linearization of the reduced operator at u=0 changes as  $\lambda$  passes through  $\lambda_0$ . This forces bifurcation, and an additional argument which goes back to [15] and uses degree theory in an essential way, shows that there is a continuum bifurcating from  $(0, \lambda_0)$ . Since the degree does not change if the multiplicity of  $\lambda_0$  is even, in Theorem 1.2 we use Morse theory instead, and therefore we need the assumption that N is the gradient of a functional. Morse theory can only assert that there exists a sequence, and not necessarily a continuum, bifurcating from infinity. Let us also point out that in [20] a more general operator equation of the form  $F(\lambda, u) = 0$  has been considered  $(F(\lambda, \cdot))$  acts between two Banach spaces). Here we will only be concerned with (1.2), and this allows some simplifications of Stuart's arguments (in particular in the part involving the finite-dimensional reduction). Since we do not make inversion, we get a less restrictive bound for the distance from  $\lambda_0$  to the essential spectrum.

The fact that  $\operatorname{dist}(\lambda_0, \sigma_e(L))$  is larger than the Lipschitz constant of N at infinity is needed in order to perform a finite-dimensional reduction of Liapunov-Schmidt type. As we shall see, if the distance condition is satisfied, then one can find an orthogonal decomposition  $E = Z \oplus W$ , where  $\dim Z < \infty$ , such that writing  $u = z + w \in Z \oplus W$ , it is possible to use the contraction mapping principle in order to express w as a function of z and  $\lambda$ . Although one may think this is only a technical condition, it has been shown by Stuart [21, Section 5.2] that there exist examples where asymptotic bifurcation does not occur at eigenvalues of odd multiplicity (and in Section 5.3 there one finds an example where asymptotic bifurcation occurs when  $\lambda_0$  is not an eigenvalue). So the above condition, or some other, is needed.

The reason for requiring N to be H-asymptotically and not just asymptotically linear (in the sense of Fréchet) is that, in contrast to the situation when (1.1) is considered for x in a bounded domain, we cannot expect the Nemytskii operator N induced by f to be asymptotically linear. Indeed, it has been shown in [19] that if  $f(u)/u \to m$  as  $|u| \to \infty$ , then N is always H-asymptotically linear, and it is asymptotically linear if and only if f(u) = mu. In the proof of Theorem 1.3 we show that also the Nemytskii operator corresponding to f(x,u) is H-asymptotically linear if  $f(x,u)/u \to m(x)$  as  $|u| \to \infty$ . The related concept of H-differentiability in the context of elliptic equations in  $\mathbb{R}^N$  has been introduced in a series of papers by Evéquoz and Stuart, see e.g. [7].

Now we can state our main results. The symbols  $N'(\infty)$  and  $\operatorname{Lip}_{\infty}$  (denoting asymptotic H-derivative and asymptotic Lipschitz constant) which appear below are introduced in Definition 2.1.

**Theorem 1.1.** Let E be a Hilbert space and suppose that  $L: D(L) \to E$  is a selfadjoint linear operator. Suppose further that

- (i) N is H-asymptotically linear and  $N'(\infty): E \to E$  is selfadjoint,
- (ii)  $\lambda_0$  is an isolated eigenvalue of odd multiplicity for  $L-N'(\infty)$  and

$$\operatorname{Lip}_{\infty}(N-N'(\infty)) < \operatorname{dist}(\lambda_0, \sigma_e(L-N'(\infty))).$$

Then  $\lambda_0$  is an asymptotic bifurcation point for equation (1.2). Moreover, there exists a continuum bifurcating from infinity at  $\lambda_0$ .

By a continuum bifurcating from infinity at  $\lambda_0$  we mean a closed connected set  $\Gamma \subset E \times \mathbb{R}$  of solutions of (1.2) which contains a sequence  $(u_n, \lambda_n)$  such that  $||u_n|| \to \infty$ ,  $\lambda_n \to \lambda_0$ . This theorem should be compared with Theorem 4.2 and Corollary 4.3 in [21] (see also Theorem 6.3 in [20]) where the distance condition was somewhat stronger than in (ii) above. The main ingredient in the proof is a finite-dimensional reduction which roughly speaking goes as follows. Let W be an L-invariant subspace of E such that codim  $W < \infty$  and  $Z := W^{\perp} \subset D(L)$ . Let  $P : E \to W$  be the orthogonal projection and write W = Pu, Z = (I - P)u. Then (1.2) is equivalent to the system

$$Lw - \lambda w = PN(w+z),$$
  

$$Lz - \lambda z = (I - P)N(w+z).$$

Choosing an appropriate W,  $\delta > 0$  small enough and R > 0 large enough, one can solve uniquely for w in the first equation provided  $|\lambda - \lambda_0| \le \delta$  and  $||z|| \ge R$ . In this way we obtain  $w = w(\lambda, z)$  which inserted in the second equation gives a (finite-dimensional) problem on  $Z \setminus B_R(0)$ . See Proposition 3.4 for more details. Now the proof of Theorem 1.1 is completed by a well-known argument using Brouwer's degree.

If N is a potential operator, then the reduced problem has variational structure. More precisely, suppose  $N(u) = \nabla \psi(u)$  for some  $\psi \in C^1(E, \mathbb{R})$  and let  $\Phi_{\lambda}(u) := \frac{1}{2}\langle Lu - \lambda u, u \rangle - \psi(u)$ . Then the functional  $\varphi_{\lambda}$  given by  $\varphi_{\lambda}(z) = \Phi_{\lambda}(w(\lambda, z) + z)$  is of class  $C^1$  and  $z \in Z \setminus \overline{B}_R(0)$  is a critical point of  $\varphi_{\lambda}$  if and only if  $u = w(\lambda, z) + z$  is a solution of (1.2), see Proposition 3.6. Recall that a functional  $\varphi$  is said to satisfy the Palais-Smale condition ((PS) for short) if each sequence  $(z_n)$  such that  $\varphi(z_n)$  is bounded and  $\varphi'(z_n) \to 0$  contains a convergent subsequence.

**Theorem 1.2.** Let E be a Hilbert space and suppose that  $L: D(L) \to E$  is a selfadjoint linear operator. Suppose further that

- (i) N is a potential operator, i.e. there exists a functional  $\psi \in C^1(E, \mathbb{R})$  such that  $\nabla \psi(u) = N(u)$  for all  $u \in E$ ,
- (ii) N is H-asymptotically linear and  $N'(\infty): E \to E$  is selfadjoint,
- (iii)  $\lambda_0$  is an isolated eigenvalue of finite multiplicity for  $L-N'(\infty)$  and

$$\operatorname{Lip}_{\infty}(N-N'(\infty)) < \operatorname{dist}(\lambda_0, \sigma_e(L-N'(\infty))).$$

If  $\varphi_{\lambda_0}$  satisfies (PS), then  $\lambda_0$  is an asymptotic bifurcation point for equation (1.2).

Note that here we do not assume  $\lambda_0$  is of odd multiplicity. In Theorem 1.4 below we shall give sufficient conditions for f in order that such  $\lambda_0$  be an asymptotic bifurcation point for (1.1).

To formulate our results for equation (1.1) we introduce the following assumptions on f:

- $(f_1)$   $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$  satisfies the Carathéodory condition, i.e., it is continuous in s for almost all  $x \in \mathbb{R}^N$  and measurable in x for all  $s \in \mathbb{R}$ , and there exist  $\alpha \in L^2(\mathbb{R}^N)$ ,  $\beta \in \mathbb{R}^+$  such that  $|f(x,s)| \leq \alpha(x) + \beta|s|$  for all  $x \in \mathbb{R}^N$ ,  $s \in \mathbb{R}$ ;
- (f<sub>2</sub>) f is Lipschitz continuous in the second variable, with Lipschitz constant Lip(f) := inf{C :  $|f(x,s) f(x,t)| \le C|s-t|$  for all  $x \in \mathbb{R}^N$ ,  $s,t \in \mathbb{R}$ };
- $(f_3) \lim_{|s|\to\infty} f(x,s)/s = m(x), \text{ where } m \in L^{\infty}(\mathbb{R}^N);$
- $(f_4)$  g(x,s) := f(x,s) m(x)s is bounded by a constant independent of  $x \in \mathbb{R}^N$  and  $s \in \mathbb{R}$ ;
- ( $f_5$ ) Assume the limits  $g_{\pm}(x) := \lim_{s \to \pm \infty} g(x, s)$  exist and either  $\pm g_{\pm} \ge 0$  a.e. or  $\pm g_{\pm} \le 0$  a.e. In addition, there exists a set of positive measure on which none of  $g_{\pm}$  vanishes;
- (f<sub>6</sub>) Assume the limits  $h_{\pm}(x) := \lim_{s \to \pm \infty} g(x, s)s$  exist,  $h_{\pm} \in L^{\infty}(\mathbb{R}^N)$  and either  $g(x, s)s \geq 0$  or  $g(x, s)s \leq 0$  for all  $x \in \mathbb{R}^N$ ,  $s \in \mathbb{R}$ . In addition, there exists a set of positive measure on which none of  $h_{\pm}$  vanishes.

Note that if  $f(x,s) = \alpha(x) + f_0(s)$  and  $|f_0(s)| \leq \beta |s|$ , where  $\alpha \in L^2(\mathbb{R}^N)$ ,  $\beta > 0$  and  $f_0$  is continuous, then f satisfies  $(f_1)$ . As we have already mentioned, such functions f have been considered in [21].

**Theorem 1.3.** Suppose that  $V \in L^{\infty}(\mathbb{R}^N)$  and f satisfies  $(f_1)$ - $(f_3)$ . Let g(x,s) := f(x,s) - m(x)s. If  $\lambda_0$  is an isolated eigenvalue of odd multiplicity for  $-\Delta + V - m$  and  $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$ , then  $\lambda_0$  is an asymptotic bifurcation point for equation (1.1). Moreover, there exists a continuum bifurcating from infinity at  $\lambda_0$ .

This strengthens some of the results of [21, Theorem 5.2]. Using examples in [21, Theorems 5.4, 5.6] and the remarks following them we shall show in Remark 5.1 that the condition on Lip(g) above is sharp in the sense that if  $\text{Lip}(g) > \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$ , then there may be no bifurcation at a simple eigenvalue.

**Theorem 1.4.** Suppose that  $V \in L^{\infty}(\mathbb{R}^N)$  and f satisfies  $(f_1)$ - $(f_4)$  and either  $(f_5)$  or  $(f_6)$ . If  $\lambda_0$  is an isolated eigenvalue of finite multiplicity for  $-\Delta + V - m$  and  $\text{Lip}(g) < \text{dist}(\lambda_0, \sigma_e(-\Delta + V - m))$ , then  $\lambda_0$  is an asymptotic bifurcation point for equation (1.1).

To our knowledge there are no earlier results on asymptotic bifurcation for (1.1) if the multiplicity of  $\lambda_0$  is even.

The rest of the paper is organized as follows. Section 2 contains some preliminary material. In Section 3 a finite-dimensional reduction is performed. In Section 4 we prove Theorems 1.1 and 1.2, and Section 5 is concerned with the proofs of Theorems 1.3 and 1.4.

**Notation.**  $\langle \cdot, \cdot \rangle$  denotes the inner product in a (real) Hilbert space E and  $\| \cdot \|$  is the corresponding norm. If  $\Phi \in C^1(E, \mathbb{R})$ , then  $\Phi'(u) \in E^*$  is the Fréchet derivative of  $\Phi$  at u and  $\nabla \Phi(u)$  (the gradient of  $\Phi$  at u) is the corresponding element in E, i.e.,  $\langle \nabla \Phi(u), v \rangle = \Phi'(u)v$ . The graph norm

corresponding to a linear operator L will be denoted by  $\|\cdot\|_L$ . The symbol  $B_r(a)$  will stand for the open ball centered at a and having radius r, and we denote the  $L^p$ -norm of u by  $\|u\|_p$ .

# 2. Preliminaries

Let X, Y be (real) Banach spaces and let  $N: X \setminus B_R(0) \to Y$ .

**Definition 2.1.** (i) We say that N is asymptotically linear in the sense of Hadamard (H-asymptotically linear for short) if there is a bounded linear operator  $B: X \to Y$  such that

$$\lim_{n \to \infty} \frac{N(t_n u_n)}{t_n} = Bu$$

for all sequences  $(t_n) \subset \mathbb{R}$ ,  $(u_n) \subset X$  such that  $u_n \to u$  and  $||t_n u_n|| \to \infty$ . The operator B is called the asymptotic H-derivative and is denoted by  $N'(\infty)$ .

(ii) We say that N is Lipschitz continuous at infinity if

$$\operatorname{Lip}_{\infty}(N) := \lim_{R \to \infty} \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \ \|u\|, \|v\| \ge R \right\} < \infty.$$

Note that the limit is well defined because the supremum above decreases as R increases.

**Remark 2.2.** (i) The definition of *H*-asymptotic linearity given in [19] is in fact a little different but the one formulated above is somewhat more convenient and is equivalent to the original one as has been shown in [19, Theorem A.1].

(ii) Recall that N is asymptotically linear (in the sense of Fréchet) if there is a bounded linear operator B such that

(2.1) 
$$\lim_{\|u\| \to \infty} \frac{\|N(u) - Bu\|}{\|u\|} = 0.$$

It is clear that if N is asymptotically linear, then it is H-asymptotically linear and  $N'(\infty) = B$ . If, however, dim  $X < \infty$ , then H-asymptotic linearity is equivalent to asymptotic linearity and (2.1) above holds for  $B = N'(\infty)$ , see [19, Remark 2].

Recall that a linear operator  $L:D(L)\subset X\to Y$  is called a Fredholm operator if it is densely defined, closed, dim  $N(L)<\infty$  (where N(L) is the kernel of L), the range R(L) is closed and codim  $R(L)<\infty$ . The number

$$\operatorname{ind}(L) := \dim N(L) - \operatorname{codim} R(L)$$

is the index of L (cf. [17, Section 1.3]).

Suppose that E is a real Hilbert space and let  $L:D(L)\subset E\to E$  be a selfadjoint Fredholm operator. Then ind (L)=0,  $E=N(L)\oplus R(L)$  (orthogonal sum) and  $S:=L|_{R(L)\cap D(L)}$  is invertible with bounded inverse. Hence, in view of [9, Problem III.6.16],

$$||S^{-1}|| = r(S^{-1}) = \frac{1}{\operatorname{dist}(0, \sigma(S))} = \frac{1}{\operatorname{dist}(0, \sigma(L) \setminus \{0\})},$$

where  $r(S^{-1})$  denotes the spectral radius of  $S^{-1}$ . The first equality holds since  $S^{-1}$  is selfadjoint, see [9, (V.2.4)]. Recall that a selfadjoint operator is necessarily densely defined and closed.

It is clear that if W is a closed subspace of R(L), invariant with respect to L (i.e.  $L(W \cap D(L)) \subset W$ ), then  $L_W := L|_{W \cap D(L)}$  is also invertible and

$$||L_W^{-1}|| = \frac{1}{\text{dist}(0, \sigma(L_W))}.$$

**Remark 2.3.** Keeping the above notation observe that  $L_W^{-1}: W \to W \cap D(L)$  is bounded with respect to the graph norm  $\|\cdot\|_L$  in  $W \cap D(L)$  (recall that  $\|u\|_L := \|u\| + \|Lu\|$  for  $u \in D(L)$ ). In fact,

$$||L_W^{-1}w||_L = ||L_W^{-1}w|| + ||w|| \le \left(1 + \frac{1}{\operatorname{dist}(0, \sigma(L_W))}\right) ||w||, \ w \in W.$$

**Definition 2.4.** For a selfadjoint Fredholm operator  $L:D(L)\to E$ , let us put

(2.2) 
$$\gamma(L) := \inf\{\|(L|_{W \cap D(L)})^{-1}\| : W \in \mathcal{W}\},\$$

where W denotes the family of closed L-invariant linear subspaces of R(L) such that  $\operatorname{codim} W < \infty$  and  $W^{\perp} \subset D(L)$ .

**Definition 2.5.** By the essential spectrum  $\sigma_e(L)$  of a selfadjoint linear operator  $L: E \supset D(L) \to E$  we understand the set

$$\{\lambda \in \mathbb{C} : L - \lambda I \text{ is not a Fredholm operator}\}\$$

(see  $[17, \S 1.4]$ ).

It follows immediately from this definition that  $\sigma_e(L) \subset \sigma(L)$  and  $\sigma(L) \setminus \sigma_e(L)$  consists of isolated eigenvalues of finite multiplicity.

**Theorem 2.6.** Let  $L: E \supset D(L) \to E$  be a selfadjoint linear operator and let  $\lambda_0 \in \sigma(L) \setminus \sigma_e(L)$ . Then  $L - \lambda_0 I$  is a Fredholm operator and

$$\gamma(L - \lambda_0 I) = \frac{1}{\operatorname{dist}(\lambda_0, \sigma_e(L))}.$$

If  $\sigma_e(L) = \emptyset$  (this is the case e.g. if L is resolvent compact), then  $\gamma(L - \lambda_0 I) = 0$ .

*Proof.* Since  $\sigma_e(L) - \lambda_0 = \sigma_e(L - \lambda_0 I)$  and hence

$$\operatorname{dist}(\lambda_0, \sigma_e(L)) = \operatorname{dist}(0, \sigma_e(L - \lambda_0 I)),$$

we may assume without loss of generality that  $\lambda_0=0$  and we will show that

$$\gamma(L) = \frac{1}{\operatorname{dist}(0, \sigma_e(L))}.$$

If  $W \in \mathcal{W}$  and  $Z := W^{\perp}$ , then  $\dim Z < \infty$  and  $Z \subset D(L)$  is L-invariant. Hence  $\sigma(L) = \sigma(L|_{W \cap D(L)}) \cup \sigma(L|_{Z})$ . Obviously, any  $\lambda \in \sigma(L|_{Z})$  is an isolated eigenvalue of finite multiplicity; thus  $\sigma_e(L) \subset \sigma(L|_{W \cap D(L)})$ . This implies that

$$\|(L|_{W\cap D(L)})^{-1}\| = \frac{1}{\operatorname{dist}(0,\sigma(L|_{W\cap D(L)})} \ge \frac{1}{\operatorname{dist}(0,\sigma_e(L))} \text{ and therefore } \gamma(L) \ge \frac{1}{\operatorname{dist}(0,\sigma_e(L))}.$$

Take any  $0 < d < \text{dist}(0, \sigma_e(L))$  and let

$$D = [-d, d] \cap \sigma(L), \quad B := \sigma(L) \setminus D.$$

Clearly D is finite: if  $\lambda \in D$ , then  $\lambda \in \sigma(L) \setminus \sigma_e(L)$ , i.e.,  $\lambda$  is an isolated eigenvalue of finite multiplicity. Therefore B is closed and  $\sigma_e(L) \subset B$ . Obviously,  $\sigma(L) = D \cup B$ . Let Z be the subspace spanned by the eigenfunctions corresponding to the eigenvalues in D and let  $W = Z^{\perp}$ . Then  $Z \subset D(L)$ ,  $W \subset R(L)$ , Z, W are invariant with respect to L,  $L|_Z$  is bounded,  $D = \sigma(L|_Z)$  and  $B = \sigma(L|_{W \cap D(L)})$ . Clearly,  $W \in W$  since dim  $Z < \infty$ . Now

$$\|(L|_{W\cap D(L)})^{-1}\| = r((L|_{W\cap D(L)})^{-1}) = \frac{1}{\operatorname{dist}\left(0, \sigma(L|_{W\cap D(L)})\right)} = \frac{1}{\operatorname{dist}\left(0, B\right)} \le \frac{1}{d}.$$

This implies the assertion. Note that if  $\sigma_e(L) = \emptyset$ , we can choose any d > 0. Hence  $\gamma(L) = 0$ .  $\square$ 

**Remark 2.7.** Let L be a Fredholm operator of index 0 and let  $\mathcal{P}(L)$  denote the collection of all bounded operators K of finite rank and such that L + K is invertible. Clearly,  $\mathcal{P}(L) \neq \emptyset$ . Put

$$\widetilde{\gamma}(L) := \inf\{\|(L+K)^{-1}\| : K \in \mathcal{P}(L)\}.$$

Then  $\tilde{\gamma}(L)$  corresponds to the notion of essential conditioning number in [20, Section 5.1], see also [21, Section 3.1] where the definition above appears explicitly.

We claim that if L is a selfadjoint Fredholm operator, then  $\widetilde{\gamma}(L) = \gamma(L)$ . For  $K \in \mathcal{P}(L)$ ,  $\sigma_e(L) = \sigma_e(L+K) \subset \sigma(L+K)$ , hence

$$||(L+K)^{-1}|| \ge r((L+K)^{-1}) = \frac{1}{\operatorname{dist}(0, \sigma(L+K))} \ge \frac{1}{\operatorname{dist}(0, \sigma_e(L))}$$

So  $\widetilde{\gamma}(L) \geq \gamma(L)$  according to the definition of  $\widetilde{\gamma}$  and Theorem 2.6. On the other hand, take any  $W \in \mathcal{W}$  and let  $Z := W^{\perp}$ . As before, write  $u = z + w \in Z \oplus W$  and let  $Ku := \alpha z - Lz$ , where

$$\alpha := \inf\{\|Lw\| : w \in W \cap D(L), \|w\| = 1\} > 0.$$

Then K has finite rank and, for  $u \in D(L)$ ,  $Lu + Ku = Lw + \alpha z$ . Hence L + K is invertible and it is easy to see that

$$\inf\{\|Lu + Ku\| : u \in D(L), \|u\| = 1\} \ge \alpha.$$

So

$$\widetilde{\gamma}(L) \le \|(L+K)^{-1}\| \le \frac{1}{\alpha} = \|(L|_{W \cap D(L)})^{-1}\|$$

and  $\widetilde{\gamma}(L) \leq \gamma(L)$ . We have shown that  $\widetilde{\gamma}(L) = \gamma(L)$ . Therefore Theorem 2.6 may be considered as a refinement of [20, Theorem 5.5 and Corollary 5.6].

# 3. The problem and finite-dimensional reduction

Let E be a real Hilbert space and  $L: E \supset D(L) \to E$  a selfadjoint operator. We shall study the existence of solutions to the eigenvalue problem (1.2), i.e.,

$$Lu = \lambda u + N(u), \quad u \in D(L), \ \lambda \in \mathbb{R},$$

or, more precisely, the existence of asymptotic bifurcation of solutions to (1.2). Recall that  $\lambda_0 \in \mathbb{R}$  is an asymptotic bifurcation point for (1.2) if there exist sequences  $\lambda_n \to \lambda_0$  and  $(u_n) \subset D(L)$  such that  $||u_n|| \to \infty$  and  $Lu_n - N(u_n) = \lambda_n u_n$ .

By X we denote the domain D(L) furnished with the graph norm

$$||u||_L := ||u|| + ||Lu||, \ u \in D(L).$$

Then X is a Banach space, L is bounded as an operator from X to E and the inclusion  $i: X \hookrightarrow E$  is continuous.

If N is a potential operator, i.e. there exists  $\psi \in C^1(E,\mathbb{R})$  such that  $N = \nabla \psi$ , then along with (1.2) we can consider the existence of critical points of the functional  $\Phi_{\lambda} : X \to \mathbb{R}$ ,  $\lambda \in \mathbb{R}$ , given by

$$\Phi_{\lambda}(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u), \quad u \in X.$$

Since  $|\langle Lu,u\rangle| \leq ||Lu|| ||u|| \leq ||u||_L^2$ ,  $\Phi_{\lambda} \in C^1(X,\mathbb{R})$  and

(3.1) 
$$\Phi'_{\lambda}(u)v = \langle Lu - \lambda u, v \rangle - \langle N(u), v \rangle, \quad u, v \in X.$$

It is clear that if  $u \in X$  solves (1.2) for some  $\lambda \in \mathbb{R}$ , then  $\Phi'_{\lambda}(u)v = 0$  for all  $v \in X$ , i.e., u is a critical point of  $\Phi_{\lambda}$ . Conversely, if  $u \in X$  and  $\Phi'_{\lambda}(u) = 0$ , then u solves (1.2) since D(L) is dense in E. Note that if L is unbounded, then  $\Phi_{\lambda}$  is defined on D(L) and is not  $C^1$  with respect to the original norm  $\|\cdot\|$  of E on D(L).

In what follows we assume:

- **3.1.** N is H-asymptotically linear with  $N'(\infty) = 0$ ;
- **3.2.** N is Lipschitz continuous at infinity;
- **3.3.**  $\lambda_0 = 0 \in \sigma(L) \setminus \sigma_e(L)$  and  $\text{Lip}_{\infty}(N) < \text{dist}(0, \sigma_e(L))$ .

Observe that these assumptions cause no loss of generality in Theorems 1.1 and 1.2 since if  $N'(\infty) \neq 0$  is selfadjoint and  $\lambda_0 \neq 0$ , then we may replace L by  $L - N'(\infty) - \lambda_0 I$  and N by  $N - N'(\infty)$ .

As a first step towards showing that  $\lambda_0 = 0$  is an asymptotic bifurcation point for (1.2) we perform a kind of a Liapunov-Schmidt finite-dimensional reduction near infinity. Put

$$L_{\lambda}u := Lu - \lambda u$$
, where  $u \in D(L_{\lambda}) = D(L)$ ,  $\lambda \in \mathbb{R}$ 

and note that the norms  $\|\cdot\|_L$  and  $\|\cdot\|_{L_{\lambda}}$  are equivalent. Given  $W \in \mathcal{W}$ , let  $P: E \to W$  be the orthogonal projection and  $Z:=W^{\perp}$ . Observe that  $u=w+z\in D(L)$ , where  $w\in W$ ,  $z\in Z$ , solves (1.2) if and only if

$$(3.2) L_{\lambda}w = PN(w+z),$$

$$(3.3) L_{\lambda}z = (I - P)N(w + z).$$

**Proposition 3.4.** There are a subspace  $W \in \mathcal{W}$ , numbers  $\delta \in (0, \text{dist } (0, \sigma(L) \setminus \{0\}), R > 0 \text{ and a continuous map } w : [-\delta, \delta] \times (Z \setminus B_R(0)) \to W \cap D(L) \text{ such that } (3.2) \text{ holds for } w = w(\lambda, z) \text{ and:}$ 

(i) For any  $\lambda$  with  $|\lambda| \leq \delta$ ,  $z, z' \in Z \setminus B_R(0)$  and some constant c > 0,

$$(3.4) ||w(\lambda, z) - w(\lambda, z')|| \le ||w(\lambda, z) - w(\lambda, z')||_L \le c||z - z'||.$$

In particular,  $w(\cdot, \cdot)$  is continuous with respect to the graph norm.

- (ii)  $w(\lambda, \cdot)$  is H-asymptotically linear with  $w'(\lambda, \infty) = 0$ .
- (iii)  $z \in Z \setminus B_R(0)$  is a solution of (3.3) with  $w = w(\lambda, z)$  if and only if  $u = w(\lambda, z) + z$  is a solution of (1.2).

Note that the condition on  $\delta$  implies invertibility of  $L_{\lambda}$  for  $0 < |\lambda| \le \delta$ .

*Proof.* (i) According to Definition 2.4 of  $\gamma(L)$ , Theorem 2.6 and assumption 3.3, there is a closed subspace  $W \in \mathcal{W}$  for which

$$\operatorname{Lip}_{\infty}(N) \| (L|_{W \cap D(L)})^{-1} \| < 1.$$

Hence we can find  $\delta \in (0, \operatorname{dist}(0, \sigma(L) \setminus \{0\}))$  and R > 0 such that

$$k := \sup_{|\lambda| < \delta} \|(L_{\lambda}|_{W \cap D(L)})^{-1}\| \cdot \beta < 1,$$

where

(3.5) 
$$\beta := \sup \left\{ \frac{\|N(u) - N(v)\|}{\|u - v\|} : u \neq v, \ \|u\|, \|v\| \ge R \right\}.$$

Let  $Z:=W^{\perp}$  and let  $P:E\to W$  be the orthogonal projection. To facilitate the notation let us put

$$M_{\lambda}(w+z) := (L_{\lambda}|_{W \cap D(L)})^{-1} PN(w+z) \in W \cap D(L), \quad w \in W, \ z \in Z \text{ and } |\lambda| \le \delta.$$

Then (3.2) is equivalent to the fixed point equation

$$(3.6) w = M_{\lambda}(w+z).$$

Fix  $\lambda \in [-\delta, \delta]$  and  $z \in Z$ ,  $||z|| \ge R$ . If  $w, w' \in W$ , then  $||w + z||, ||w' + z|| \ge ||z|| \ge R$ , so taking into account that ||P|| = 1, we have

$$||M_{\lambda}(w+z) - M_{\lambda}(w'+z)|| \le k||w-w'||.$$

By the Banach contraction principle there is a unique  $w = w(\lambda, z) \in W \cap D(L)$ , continuously depending on  $\lambda$  and z, such that (3.6), and hence (3.2), holds. Moreover,

$$||w(\lambda, z) - w(\lambda, z')|| = ||M_{\lambda}(w(\lambda, z) + z) - M_{\lambda}(w(\lambda, z') + z')|| \le k||w(\lambda, z) - w(\lambda, z')| + z - z'|| \le k||w(\lambda, z) - w(\lambda, z')|| + k||z - z'||$$

for all  $|\lambda| \leq \delta$ ,  $z, z' \in Z \setminus B_R(0)$ . So  $||w(\lambda, z) - w(\lambda, z')|| \leq k(1 - k)^{-1}||z - z'||$ . Using this, (3.5) and arguing as above, we obtain

$$||L_{\lambda}w(\lambda,z) - L_{\lambda}w(\lambda,z')|| = ||PN(w(\lambda,z)+z) - PN(w(\lambda,z')+z')||$$
  
$$\leq \beta||w(\lambda,z) - w(\lambda,z')|| + \beta||z - z'|| \leq \frac{\beta}{1-k}||z - z'||.$$

Since  $\|\cdot\|_L$  and  $\|\cdot\|_{L_\lambda}$  are equivalent norms, the second inequality in (3.4) follows (the first one is obvious).

(ii) To show the *H*-asymptotic linearity of  $w(\lambda, \cdot)$  with  $w'(\lambda, \infty) = 0$ , let  $(z_n) \subset Z$  and  $(t_n) \subset \mathbb{R}$  be sequences such that  $z_n \to z$  and  $||t_n z_n|| \to \infty$ . Then, for sufficiently large n,  $||w(\lambda, t_n z_n) + t_n z_n|| \ge ||t_n z_n|| \ge R$  and

$$\|w(\lambda, t_n z_n)\| \leq \|M_{\lambda}(w(\lambda, t_n z_n) + t_n z_n) - M_{\lambda}(t_n z_n)\| + \|M_{\lambda}(t_n z_n)\| \leq k\|w(\lambda, t_n z_n)\| + \|M_{\lambda}(t_n z_n)\|.$$

Thus, in view of assumption 3.1,

(3.7) 
$$\frac{\|w(\lambda, t_n z_n)\|}{|t_n|} \le \frac{1}{1-k} \frac{\|M_{\lambda}(t_n z_n)\|}{|t_n|} \to 0.$$

(iii) is an immediate consequence of (i).

**Remark 3.5.** Suppose that  $z_n \to z$  in Z and take a sequence  $(t_n) \subset \mathbb{R}$  such that  $||t_n z_n|| \to \infty$ . Then, again in view of the H-asymptotic linearity of N and (3.7), we have

(3.8) 
$$\frac{N(w(\lambda, t_n z_n) + t_n z_n)}{t_n} = \frac{N\left(t_n\left(\frac{w(\lambda, t_n z_n)}{t_n} + z_n\right)\right)}{t_n} \to 0$$

for each fixed  $\lambda \in [-\delta, \delta]$ .

If  $N = \nabla \psi$ , then we let

(3.9) 
$$\varphi_{\lambda}(z) := \Phi_{\lambda}(w(\lambda, z) + z), \quad |\lambda| \le \delta, \ z \in Z \setminus \overline{B}_{R}(0).$$

**Proposition 3.6.** Let  $|\lambda| \leq \delta$ . Then  $\varphi_{\lambda} \in C^1(Z \setminus \overline{B}_R(0), \mathbb{R})$  and

(3.10) 
$$\nabla \varphi_{\lambda}(z) = L_{\lambda}z - (I - P)N(w(\lambda, z) + z).$$

Therefore  $z \in Z \setminus \overline{B}_R(0)$  is a critical point of  $\varphi_{\lambda}$  if and only if  $u = w(\lambda, z) + z$  solves (1.2). Moreover,  $\nabla \varphi_{\lambda}$  is asymptotically linear with  $(\nabla \varphi_{\lambda})'(\infty) = L_{\lambda}|_{Z}$ .

*Proof.* To show (3.10) we shall compute the derivative of  $\varphi_{\lambda}$  in the direction  $h \in \mathbb{Z}$ ,  $h \neq 0$ . For notational convenience we write w(z) for  $w(\lambda, z)$ . Let t > 0,

$$u := w(z) + z$$
 and  $\xi := w(z + th) - w(z) + th$ .

Then we have

$$\varphi_{\lambda}(z+th) - \varphi_{\lambda}(z) = \Phi_{\lambda}(u+\xi) - \Phi_{\lambda}(u) - \Phi_{\lambda}'(u)\xi + \Phi_{\lambda}'(u)\xi.$$

Clearly,  $\xi \neq 0$  as t > 0. In view of (3.1), (3.2) and since  $w(z + th) - w(z) \in W$ ,

$$\Phi'_{\lambda}(u)\xi = \langle L_{\lambda}u - N(u), \xi \rangle = \langle L_{\lambda}w(z) - PN(u), \xi \rangle + \langle L_{\lambda}z - (I - P)N(u), \xi \rangle$$
$$= \langle L_{\lambda}z - N(u), th \rangle = t\Phi'_{\lambda}(u)h.$$

Hence

(3.11) 
$$\frac{\varphi_{\lambda}(z+th) - \varphi_{\lambda}(z)}{t} = \Phi_{\lambda}'(u)h + \frac{\|\xi\|_{L}}{t} \cdot \frac{\Phi_{\lambda}(u+\xi) - \Phi_{\lambda}(u) - \Phi_{\lambda}'(u)\xi}{\|\xi\|_{L}}.$$

It follows from (3.4) that

$$\|\xi\|_L \leq td\|h\|$$

for some d > 0. This, together with the Fréchet differentiability of  $\Phi_{\lambda}$  on X (i.e., on D(L) with the graph norm) implies that the second term on the right-hand side of (3.11) tends to 0 as  $t \to 0$ . So

$$\lim_{t \to 0^+} \frac{\varphi_{\lambda}(z+th) - \varphi_{\lambda}(z)}{t} = \Phi_{\lambda}'(u)h = \langle L_{\lambda}z, h \rangle - \langle (I-P)N(w(z)+z), h \rangle.$$

Therefore  $\varphi_{\lambda}$  is continuously Gâteaux differentiable, hence continuously Fréchet differentiable as well, and the derivative is as claimed.

If  $z \in Z \setminus \overline{B}_R(0)$  is a critical point of  $\varphi_{\lambda}$ , then (3.3) with  $w = w(\lambda, z)$  is satisfied; this together with (3.2) shows that  $u = w(\lambda, z) + z$  solves (1.2).

Since dim  $Z < \infty$ , in order to prove the last part of the assertion it suffices to show that  $\nabla \varphi_{\lambda}$  is H-asymptotically linear (see Remark 2.2(ii)). If  $z_n \to z$  in Z,  $(t_n) \subset \mathbb{R}$  and  $||t_n z_n|| \to \infty$ , then, in view of (3.8),

$$\frac{\nabla \varphi_{\lambda}(t_n z_n)}{t_n} = L_{\lambda} z_n - \frac{(I - P)N(w(t_n z_n) + t_n z_n)}{t_n} \to L_{\lambda} z.$$

This concludes the proof.

Remark 3.7. (i) Using (3.4) and the fact that  $\beta$  in (3.5) is finite, it is easy to see that  $\nabla \varphi_{\lambda}$  is Lipschitz continuous on  $Z \setminus \overline{B}_R(0)$  and the Lipschitz constant may be chosen independently of  $\lambda \in [-\delta, \delta]$ .

(ii) In what follows we may (and will need to) assume that  $\varphi_{\lambda}$  is defined on Z and not only on  $Z \setminus \overline{B}_R(0)$ . Such an extension of  $\varphi_{\lambda}$  can be achieved e.g. as follows. Let  $\chi \in C^{\infty}(\mathbb{R}, [0, 1])$  be a cutoff function such that  $\chi(t) = 0$  for  $t \leq R + 1$  and  $\chi(t) = 1$  for  $t \geq R + 2$ . Set  $\widetilde{\varphi}_{\lambda}(z) := \chi(||z||)\varphi_{\lambda}(z)$ . Then  $\widetilde{\varphi}_{\lambda}$  is of class  $C^1$ , Lipschitz continuous and  $\widetilde{\varphi}_{\lambda}(z) = \varphi_{\lambda}(z)$  for ||z|| > R + 2. In particular,  $z \in Z \setminus \overline{B}_{\widetilde{R}}(0)$ , where  $\widetilde{R} := R + 2$ , is a critical point of  $\widetilde{\varphi}_{\lambda}$  if and only if  $u = w(\lambda, z) + z$  solves (1.2).

## 4. Proofs of Theorems 1.1 and 1.2

In the proof of Theorem 1.1 we shall need the following version of Whyburn's lemma which may be found in [1, Proposition 5]:

**Lemma 4.1.** Let Y be a compact space and  $A, B \subset Y$  closed sets. If there is no connected set  $\Gamma \subset Y \setminus (A \cup B)$  such that  $\overline{\Gamma} \cap A \neq \emptyset$  and  $\overline{\Gamma} \cap B \neq \emptyset$  ( $\overline{\Gamma}$  stands for the closure of  $\Gamma$  in Y), then A and B are separated, i.e. there are open sets  $U, V \subset Y$  such that  $A \subset U$ ,  $B \subset V$ ,  $U \cap V = \emptyset$  and  $Y = U \cup V$  (clearly, U, V are closed as well).

Proof of Theorem 1.1. By Proposition 3.4, it suffices to consider equation (3.3) with  $w = w(\lambda, z)$  which we re-write in the form

$$(4.1) F_{\lambda}(z) := L_{\lambda}z - (I - P)N(w(\lambda, z) + z) = 0.$$

As in assumptions 3.1–3.3, it causes no loss of generality to take  $\lambda_0 = 0$  and  $N'(\infty) = 0$ . Although  $F_{\lambda}$  in Proposition 3.4 has been defined for  $|\lambda| \leq \delta$  and  $||z|| \geq R$ , we may (and do) extend it continuously to  $[-\delta, \delta] \times Z$ . Since  $w'(\lambda, \infty) = 0$  (see (ii) of Proposition 3.4) and asymptotic linearity coincides with H-asymptotic linearity on Z (because dim  $Z < \infty$ ), we have, setting  $K_{\lambda}(z) := (I - P)N(w(\lambda, z) + z)$  and using Remark 3.5,

(4.2) 
$$\lim_{\|z\| \to \infty} \frac{\|K_{\lambda}(z)\|}{\|z\|} = 0.$$

Suppose there is no asymptotic bifurcation at  $\lambda_0 = 0$ . Taking smaller  $\delta$  and larger R if necessary,  $F_{\lambda}(z) \neq 0$  for any  $|\lambda| \leq \delta$  and  $||z|| \geq R$ . Therefore the Brouwer degree  $\deg(F_{\lambda}, B_R(0), 0)$  (see e.g. [2, Section 3.1]) is well defined and independent of  $\lambda \in [-\delta, \delta]$ . Since  $\delta < \operatorname{dist}(0, \sigma(L) \setminus \{0\})$ ,  $L_{\pm \delta}$  are invertible. It follows therefore from (4.2) that if  $R_0 \geq R$  is sufficiently large, then  $L_{\pm \delta}z - tK_{\pm \delta}(z) \neq 0$ 

0 for any  $||z|| \ge R_0$ ,  $t \in [0,1]$ . Hence by the excision property and the homotopy invariance of degree,

$$k = \deg(F_{\pm\delta}, B_R(0), 0) = \deg(F_{\pm\delta}, B_{R_0}(0), 0) = \deg(L_{\pm\delta}|_Z, B_{R_0}, 0)$$

for some  $k \in \mathbb{Z}$ . Let  $d_1, d_2$  be the number of negative eigenvalues (counted with their multiplicity) of respectively  $L_{\delta}|_{\mathbb{Z}}$  and  $L_{-\delta}|_{\mathbb{Z}}$ . Then  $k = (-1)^{d_1} = (-1)^{d_2}$  [2, Lemma 3.3]. However, since  $d_1 = d_2 + \dim N(L)$  and  $\dim N(L)$  is odd, this is impossible. So we have reached a contradiction to the assumption that there is no bifurcation.

It remains to prove that there exists a bifurcating continuum. Usually this is done by first making the inversion  $u \mapsto u/\|u\|^2$  and then showing there is a continuum bifurcating from 0 [16, 20, 22]. Here we give a slightly different argument avoiding inversion. Let

$$\Sigma := \{ (z, \lambda) \in (Z \setminus B_R(0)) \times [-\delta, \delta] : F_{\lambda}(z) = 0 \}.$$

Compactify Z by adding the point at infinity and let  $A := \overline{B}_R(0) \times [-\delta, \delta]$ ,  $B := \{(\infty, 0)\}$ ,  $Y := A \cup \Sigma \cup B$ . Then Y is compact, A and B are closed disjoint. We claim that if R is large enough, there is a connected set  $\Gamma \subset \Sigma$  such that  $\{(\infty, 0)\} \in \overline{\Gamma}$  (the closure taken in Y) and  $\overline{\Gamma} \cap A \neq \emptyset$ . Otherwise there exist U and V as in Lemma 4.1. Since U is compact and bounded, there exists a bounded open set  $\mathcal{O} \subset Z \times [-\delta, \delta]$  such that  $U \subset \mathcal{O}$  and  $\partial \mathcal{O} \cap \Sigma = \emptyset$ . Letting  $\mathcal{O}_{\lambda} := \{z : (z, \lambda) \in \mathcal{O}\}$  for  $\lambda \in [-\delta, \delta]$ , it follows from the excision property and the generalized version of the homotopy invariance property of degree [2, Theorem 4.1] that  $\deg(F_{\delta}, \mathcal{O}_{\delta}, 0) = \deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0)$ , a contradiction since by the same argument as above  $\deg(F_{\delta}, \mathcal{O}_{\delta}, 0) = (-1)^{k_1}$ ,  $\deg(F_{-\delta}, \mathcal{O}_{-\delta}, 0) = (-1)^{k_2}$  and  $k_1, k_2$  have different parity.

In the proof of Theorem 1.2 we shall use Gromoll-Meyer theory. Below we summarize some pertinent facts which are special cases of much more general results of [12] where functionals were considered in a Hilbert space E with filtration, i.e., with a sequence  $(E_n)$  of subspaces such that  $E_n \subset E_{n+1}$  for all n and  $\bigcup_{n=1}^{\infty} E_n$  is dense in E. In the terminology of [12], here we have the trivial filtration (i.e.,  $Z_n = Z$  for all n) which, together with the fact that dim  $Z < \infty$ , considerably simplifies the proofs. An alternative approach is via the Conley index theory, see e.g. [3, 4], in particular [3, Corollary 2.3] and [4, Theorem 2].

Let  $\varphi: Z \to \mathbb{R}$  be a function such that  $\nabla \varphi$  is locally Lipschitz continuous. Suppose also  $K = K(\varphi) := \{z \in Z : \nabla \varphi(z) = 0\}$  is bounded. A pair  $(\mathbb{W}, \mathbb{W}^-)$  of closed subsets of Z will be called *admissible* (for  $\varphi$  and K) if

- (i)  $K \subset \operatorname{int}(\mathbb{W})$  and  $\mathbb{W}^- \subset \partial \mathbb{W}$ ;
- (ii)  $\varphi|_{\mathbb{W}}$  is bounded;
- (iii) There exist a locally Lipschitz continuous vector field V defined in a neighbourhood N of  $\mathbb{W}$  and a continuous function  $\beta: N \to \mathbb{R}^+$  such that  $\|V(z)\| \le 1$ ,  $\langle V(z), \varphi(z) \rangle \ge \beta(z)$  for all  $z \in N$ , and  $\beta$  is bounded away from 0 on compact subsets of  $N \setminus K$  (we shall call V admissible for  $(\mathbb{W}, \mathbb{W}^-)$ );
- (iv)  $\mathbb{W}^-$  is a piecewise  $C^1$ -manifold of codimension 1, V is transversal to  $\mathbb{W}^-$ , the flow  $\eta$  of -V can leave  $\mathbb{W}$  only via  $\mathbb{W}^-$  and if  $z \in \mathbb{W}^-$ , then  $\eta(t,z) \notin \mathbb{W}$  for any t > 0.

Let  $H^*$  denote the Čech (or Alexander-Spanier) cohomology with coefficients in  $\mathbb{Z}_2$  and let the critical groups  $c^*(\varphi, K)$  of the pair  $(\varphi, K)$  be defined by

$$c^*(\varphi, K) := H^*(\mathbb{W}, \mathbb{W}^-).$$

Lemma 4.2. Suppose  $\varphi$  satisfies (PS).

- (i) For each R > 0 there exists a bounded admissible pair  $(\mathbb{W}, \mathbb{W}^-)$  for  $\varphi$  and K such that  $B_R(0) \subset \mathbb{W}$ .
- (ii) If  $(\mathbb{W}_1, \mathbb{W}_1^-)$  and  $(\mathbb{W}_2, \mathbb{W}_2^-)$  are two admissible pairs for  $\varphi$  and K, then  $H^*(\mathbb{W}_1, \mathbb{W}_1^-) \cong H^*(\mathbb{W}_2, \mathbb{W}_2^-)$  (i.e.,  $c^*(\varphi, K)$  is well defined).
- (iii) Suppose  $\{\varphi_{\lambda}\}_{{\lambda}\in[0,1]}$  is a family of functions satisfying (PS) and such that  $\nabla\varphi_{\lambda}$  is locally Lipschitz continuous,  ${\lambda}\mapsto\nabla\varphi_{\lambda}$  is continuous, uniformly on bounded subsets of Z, and  $K(\varphi_{\lambda})\subset B_R(0)$  for some R>0 and all  ${\lambda}\in[0,1]$ . Then  $c^*(\varphi_{\lambda},K(\varphi_{\lambda}))$  is independent of  ${\lambda}$ .

This lemma corresponds to Lemma 2.13 and Propositions 2.12, 2.14 in [12]. Note that condition (PS)\* there is in our setting (i.e. for trivial filtration) equivalent to (PS).

Outline of proof. (i) Choose R, a, b so that  $K \subset B_R(0), a < \varphi(z) < b$  for all  $z \in B_R(0)$  and let

$$(4.3) V(z) := \frac{\nabla \varphi(z)}{1 + \|\nabla \varphi(z)\|}.$$

Clearly, the flow  $\eta$  given by

$$\frac{d\eta}{dt} = -V(\eta), \quad \eta(0, z) = z$$

is defined on  $\mathbb{R} \times Z$ . Let

$$\mathbb{W} := \{ \eta(t, z) : t \ge 0, \ z \in B_R(0), \ \varphi(\eta(t, z)) \ge a \}, \quad \mathbb{W}^- := \mathbb{W} \cap \varphi^{-1}(a).$$

Then  $(\mathbb{W}, \mathbb{W}^-)$  is an admissible pair. The proof follows that of [12, Lemma 2.13] but is simpler - there is no need for using cutoff functions. Note that (here and below) the Palais-Smale condition rules out the possibility that  $\varphi(\eta(t,z)) > a$  and  $\|\eta(t,z)\| \to \infty$  as  $t \to \infty$ , hence  $t \mapsto \eta(t,z)$  either approaches K as  $t \to \infty$  or hits  $\mathbb{W}^- = \varphi^{-1}(a)$  in finite time.

(ii) Assume that  $\varphi$  is unbounded below and above (the other cases are simpler but somewhat different). Let  $(\mathbb{W}_0, \mathbb{W}_0^-)$  be an admissible pair and  $V_0$  a corresponding admissible vector field. As  $\varphi|_{\mathbb{W}_0}$  is bounded, we may choose a,b so that  $a<\varphi(z)< b$  for all  $z\in\mathbb{W}_0$ . Since  $(\mathbb{W}_1,\mathbb{W}_1^-):=(\varphi^{-1}([a,b]),\varphi^{-1}(a))$  is an admissible pair, it suffices to show that  $H^*(\mathbb{W}_0,\mathbb{W}_0^-)\cong H^*(\mathbb{W}_1,\mathbb{W}_1^-)$ . Put  $V(z):=\chi_0(z)V_0(z)+\chi_1(z)V_1(z)$ , where  $V_1$  is given by (4.3) and  $\{\chi_0,\chi_1\}$  is a Lipschitz continuous partition of unity such that  $\chi_0(z)=1$  on  $\mathbb{W}_0$  and  $\chi_1(z)=1$  in a neighbourhood of  $\partial\mathbb{W}_1$ . Denote the flow of -V by  $\eta$ . Let  $A:=\{\eta(t,z):t\geq 0,\ z\in\mathbb{W}_0^-\}\cap\mathbb{W}_1$  and  $\mathbb{W}=\mathbb{W}_0\cup A,\mathbb{W}^-:=\mathbb{W}\cap\mathbb{W}_1^-$ . Then  $(\mathbb{W},\mathbb{W}^-)$  is a an admissible pair and using  $\eta$  one obtains a strong deformation retraction of A onto  $\mathbb{W}^-$ . So  $H^*(A,\mathbb{W}^-)=0$  and by exactness of the cohomology sequence of the triple  $(\mathbb{W},A,\mathbb{W}^-)$  and the strong excision property we have  $H^*(\mathbb{W},\mathbb{W}^-)\cong H^*(\mathbb{W},A)\cong H^*(\mathbb{W}_0,\mathbb{W}_0^-)$ . We also have, by excision again,  $H^*(\mathbb{W},\mathbb{W}^-)\cong H^*(\mathbb{W}\cup\mathbb{W}_1^-,\mathbb{W}_1^-)$ . Finally, using the flow  $\eta$  once more, we obtain a deformation of  $(\mathbb{W}_1,\mathbb{W}_1^-)$  into  $(\mathbb{W}\cup\mathbb{W}_1^-,\mathbb{W}_1^-)$  which leaves  $\mathbb{W}\cup\mathbb{W}_1^-$  and  $\mathbb{W}_1^-$  invariant. Hence  $(\mathbb{W}\cup\mathbb{W}_1^-,\mathbb{W}_1^-)$  and  $(\mathbb{W}_1,\mathbb{W}_1^-)$  are homotopy equivalent and thus have the same

cohomology. Putting everything together gives  $H^*(\mathbb{W}_0, \mathbb{W}_0^-) \cong H^*(\mathbb{W}_1, \mathbb{W}_1^-)$ . More details of the proof may be found in [12, Propositions 2.12 and 2.7].

(iii) Let  $\lambda_0 \in [0,1]$ . It suffices to show that  $c^*(\varphi_{\lambda}, K(\varphi_{\lambda}))$  is constant for  $\lambda$  in a neighbourhood of  $\lambda_0$ . Denote the vector field for  $\varphi_{\lambda}$  given as in (4.3) by  $V_{\lambda}$  and choose an admissible pair  $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$  for  $\varphi_{\lambda_0}$  and  $K(\varphi_{\lambda_0})$  such that  $B_{R_1}(0) \subset \mathbb{W}_{\lambda_0}$ , where  $R_1 > R$ . By the construction in (i), we may assume  $V_{\lambda_0}$  is admissible for this pair. Let  $\widetilde{V}(z) := \chi_1(z)V_{\lambda}(z) + \chi_2(z)V_{\lambda_0}(z)$ , where  $\{\chi_1, \chi_2\}$  is a partition of unity subordinate to the sets  $B_{R_1}(0)$  and  $\mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$ . It is easy to see that if  $|\lambda - \lambda_0|$  is small enough, then  $(\mathbb{W}_{\lambda_0}, \mathbb{W}_{\lambda_0}^-)$  is an admissible pair for  $\varphi_{\lambda}$ ,  $K(\varphi_{\lambda})$  and  $\widetilde{V}$  is a corresponding admissible field. Note in particular that

$$\|\nabla \varphi_{\lambda}(z)\| \ge \|\nabla \varphi_{\lambda_0}(z)\| - \|\nabla \varphi_{\lambda}(z) - \nabla \varphi_{\lambda_0}(z)\| > 0$$

for 
$$z \in \mathbb{W}_{\lambda_0} \setminus \overline{B}_R(0)$$
, so indeed  $\widetilde{V}$  is admissible. Hence  $c^*(\varphi_{\lambda}, K(\varphi_{\lambda})) \cong c^*(\varphi_{\lambda_0}, K(\varphi_{\lambda_0}))$ .

Proof of Theorem 1.2. Let  $\varphi_{\lambda}$  be given by (3.9) and extend it to the whole space Z according to Remark 3.7. If  $\lambda_0 = 0$  is not an asymptotic bifurcation point for (1.2), then it follows from Proposition 3.6 that  $\nabla \varphi_{\lambda}(z) \neq 0$  for  $\lambda \in [-\delta, \delta]$  and ||z|| > R, possibly after choosing a smaller  $\delta$  and larger R. By assumption,  $\varphi_0$  satisfies (PS) and since  $L_{\lambda}$  has bounded inverse if  $0 < |\lambda| \leq \delta$ , we see using (4.2) that  $\nabla \varphi_{\lambda}$  is bounded away from 0 as  $||z|| \to \infty$ . Hence all  $\varphi_{\lambda}$ ,  $|\lambda| \leq \delta$ , satisfy (PS). By Lemma 4.2,  $c^*(\varphi_{\lambda}, K(\varphi_{\lambda}))$  is independent of  $\lambda \in [-\delta, \delta]$ . For  $\lambda = \delta$ , let  $Z = Z_{\delta}^+ \oplus Z_{\delta}^-$  and  $z = z^+ + z^- \in Z_{\delta}^+ \oplus Z_{\delta}^-$ , where  $Z_{\delta}^\pm$  are the maximal  $L_{\delta}$ -invariant subspaces of Z on which  $L_{\delta}$  is respectively positive and negative definite. Choose  $\varepsilon > 0$  such that  $\langle \pm L_{\delta}z^{\pm}, z^{\pm} \rangle \geq \varepsilon ||z^{\pm}||^2$  and let

$$\mathbb{W} := \{ z \in \mathbb{Z} : ||z^+|| \le R_0, ||z^-|| \le R_0 \}, \quad \mathbb{W}^- := \{ z \in \mathbb{W} : ||z^-|| = R_0 \}.$$

Recall  $K_{\lambda}(z) = (I - P)N(w(\lambda, z) + z)$ . Taking a sufficiently large  $R_0$ ,

$$\langle \nabla \varphi_{\delta}(z), z^{+} \rangle = \langle L_{\delta}z, z^{+} \rangle - \langle K_{\delta}(z), z^{+} \rangle \ge \varepsilon \|z^{+}\|^{2} - \frac{1}{4}\varepsilon \|z\| \|z^{+}\| > 0, \quad z \in \mathbb{W}, \ \|z^{+}\| = R_{0}.$$

Similarly,

$$\langle \nabla \varphi_{\delta}(z), z^{-} \rangle < 0, \quad z \in \mathbb{W}, \ z^{-} \in \mathbb{W}^{-}.$$

So the flow of  $-\nabla \varphi_{\delta}$  is transversal to  $\mathbb{W}^-$  and can leave  $\mathbb{W}$  only via  $\mathbb{W}^-$ . Hence  $(\mathbb{W}, \mathbb{W}^-)$  is an admissible pair for  $\varphi_{\delta}$  and  $K(\varphi_{\delta})$ , and  $V = \nabla \varphi_{\delta}$  is a corresponding admissible vector field. Note that this pair is also admissible for the quadratic functional  $\Psi_{\delta}(z) := \frac{1}{2} \langle L_{\delta} z, z \rangle$ . Since 0 is the only critical point of  $\Psi_{\delta}$ , it follows e.g. from [14, Corollary 8.3] that if m is the Morse index of  $\Psi_{\delta}$ , then

$$c^{q}(\varphi_{\delta}, K(\varphi_{\delta})) = c^{q}(\Psi_{\delta}, 0) = \delta_{q,m} \mathbb{Z}_{2}.$$

A similar argument shows that  $c^q(\varphi_{-\delta}, K(\varphi_{-\delta})) = \delta_{q,n}\mathbb{Z}_2$ , where n is the Morse index of  $\Psi_{-\delta}$ . As the Morse index changes (by dim N(L)) when  $\lambda$  passes through 0,  $m \neq n$  and  $c^*(\varphi_{\delta}, K(\varphi_{\delta})) \neq c^*(\varphi_{-\delta}, K(\varphi_{-\delta}))$ . This is the desired contradiction.

## 5. Proofs of Theorems 1.3 and 1.4

We assume throughout this section that  $V \in L^{\infty}(\mathbb{R}^N)$  and f satisfies  $(f_1)$ - $(f_3)$ . We consider equation (1.1) which we re-write in the form

$$(5.1) -\Delta u + V_0(x)u = \lambda u + g(x, u), \quad x \in \mathbb{R}^N,$$

where we have put  $V_0(x) := V(x) - m(x)$  and g(x, u) := f(x, u) - m(x)u. Let  $\lambda_0$  be an isolated eigenvalue of finite multiplicity for  $-\Delta + V_0$ . Replacing  $V_0(x)$  by  $V_0(x) - \lambda_0$  we may assume without loss of generality that  $\lambda_0 = 0$ .

Let  $E:=L^2(\mathbb{R}^N)$  and  $Lu:=-\Delta u+V_0(x)u$ . As we have pointed of in the introduction, L is a selfadjoint operator whose domain is the Sobolev space  $H^2(\mathbb{R}^N)$  and the graph norm of L is equivalent to the Sobolev norm. (A brief argument: using the Fourier transform one readily sees that  $-\Delta+1:H^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$  is an isomorphism; hence the conclusion follows because  $V\in L^\infty(\mathbb{R}^N)$ .)

We define the operator N (the Nemytskii operator) by setting

$$N(u) := g(\cdot, u(\cdot)), \quad u \in E.$$

It follows from  $(f_1)$  and Krasnoselskii's theorem [11, Theorems 2.1 and 2.3] that  $N: E \to E$  is well defined and continuous. Let

$$G(x,s) := \int_0^s g(x,\xi) d\xi, \quad x \in \mathbb{R}^N, \ s \in \mathbb{R}$$

and

$$\psi(u):=\int_{\mathbb{R}^N}G(x,u)\,dx,\quad u\in E.$$

Then  $\psi \in C^1(E,\mathbb{R})$  and

$$\nabla \psi(u) = N(u),$$

see [11, Lemma 5.1]. Furthermore, let

$$\Phi_{\lambda}(u) := \frac{1}{2} \langle Lu - \lambda u, u \rangle - \psi(u), \quad u \in X := H^{2}(\mathbb{R}^{N}).$$

Then  $\Phi_{\lambda} \in C^1(X, \mathbb{R})$  and  $\Phi'_{\lambda}(u) = 0$  if and only if u is a solution of (5.1).

Proof of Theorem 1.3. We verify the assumptions of Theorem 1.1. First we show that N is H-asymptotically linear and  $N'(\infty) = 0$ . Let  $u_n \to u$  and  $||t_n u_n|| \to \infty$  in E. Assume passing to a subsequence that  $u_n \to u$  a.e. Since

$$\frac{g(x, t_n u_n)^2}{t_n^2} \le \left(\frac{\alpha(x)}{t_n} + (\beta + ||m||_{\infty})|u_n|\right)^2$$

and  $g(x,s)/s \to 0$  as  $|s| \to \infty$ , it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \frac{\|N(u_n)\|^2}{t_n^2} = \int_{\mathbb{R}^N} \lim_{n \to \infty} \frac{g(x, t_n u_n)^2}{(t_n u_n)^2} u_n^2 dx = 0.$$

Hence (i) of Theorem 1.1 is satisfied. Since  $\operatorname{Lip}_{\infty}(N) = \operatorname{Lip}_{\infty}(g) \leq \operatorname{Lip}(g) < \operatorname{dist}(0, \sigma_e(L))$  (where the second inequality follows by assumption), also (ii) of this theorem holds. This completes the proof.

Remark 5.1. As we have mentioned in the introduction, the condition  $\operatorname{Lip}(g) < \operatorname{dist}(\lambda_0, \sigma_e(L))$  is sharp in the sense that there may be no asymptotic bifurcation if  $\operatorname{Lip}(g) > \operatorname{dist}(\lambda_0, \sigma_e(L))$  and other assumptions of Theorem 1.3 are satisfied. Let N=1 and suppose  $V_0 \in C^1(\mathbb{R}), \ V_0'(x) \leq 0$  for x large,  $\lim_{|x| \to \infty} V_0(x) = V_0(\infty)$  exists and  $\inf\{\langle Lu, u \rangle : ||u||_2 = 1\} < V_0(\infty)$ . Then  $\sigma_e(L) = [V_0(\infty), \infty)$  and  $\lambda_0 := \inf \sigma(L) < \inf \sigma_e(L)$  is a simple eigenvalue. Assume without loss of generality that  $\lambda_0 = 0$ . Assume also that g is independent of x, of class  $C^1$ ,  $g(0) = \lim_{|x| \to \infty} g(s)/s = 0$  and  $\xi := V_0(\infty) + g'(0) < 0$ . Given  $\varepsilon > 0$ , we may choose g so that  $\operatorname{Lip}(g) = -g'(0) \in (V_0(\infty), V_0(\infty) + \varepsilon)$ . So

$$\operatorname{Lip}(g) - \varepsilon < \operatorname{dist}(0, \sigma_e(L)) = V_0(\infty) < \operatorname{Lip}(g),$$

and according to [21, Theorem 5.4] and the remarks following it, there is no asymptotic bifurcation at any  $\lambda > \xi$ , in particular, not at  $\lambda_0 = 0$ . See also the explicit Example 1 after the proof of Theorem 5.4 in [21]. A similar conclusion holds for  $N \geq 2$ , see [21, Theorem 5.6].

In the proof of Theorem 1.4 we shall need an auxiliary result. Let  $\lambda_0 = 0$  and write w(z) = w(0, z). Then w(z) satisfies equation (3.2), i.e. we have

$$Lw(z) = PN(w(z) + z).$$

**Lemma 5.2.** Suppose  $(f_1)$ - $(f_4)$  are satisfied. Then  $||w(z)||_{\infty} \leq C$  for some constant C > 0 and all ||z|| > R.

Proof. Recall  $L := -\Delta + V_0$ , where  $L : D(L) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ . We also define  $\widetilde{L} := -\Delta + V_0$  when  $-\Delta + V_0$  is regarded as an operator in  $L^{\infty}(\mathbb{R}^N)$  (i.e.,  $\widetilde{L} : D(\widetilde{L}) \subset L^{\infty}(\mathbb{R}^N) \to L^{\infty}(\mathbb{R}^N)$ ). By [8, Theorem],  $\sigma(L) = \sigma(\widetilde{L})$  and isolated eigenvalues of L and  $\widetilde{L}$  have the same multiplicity. Since Z is spanned by eigenfunctions of  $-\Delta + V_0$  corresponding to isolated eigenvalues and such eigenfunctions decay exponentially [18, Theorem C.3.4],  $Z \subset L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ . It follows therefore from [9, Theorem III.6.17] that there is an L-invariant decomposition  $L^{\infty}(\mathbb{R}^N) = \widetilde{Z} \oplus \widetilde{W}$ , where  $\widetilde{Z} = Z$ . Moreover, by [9, (III.6.19)],

$$Q := I - P = -\frac{1}{2\pi i} \int_{\gamma} (L - \lambda I)^{-1} d\lambda,$$

where  $\gamma$  is a smooth simple closed curve (in  $\mathbb{C}$ ) which encloses all eigenvalues corresponding to Z and no other points in  $\sigma(L)$ . By [8, Proposition 2.1],  $(L - \lambda I)^{-1}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = (\widetilde{L} - \lambda I)^{-1}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$ . Hence  $Q|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)} = \widetilde{Q}|_{L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}$ , where  $\widetilde{Q}$  denotes the  $\widetilde{L}$ -invariant projection of  $L^\infty(\mathbb{R}^N)$  on Z, and the same equality holds for P and  $\widetilde{P} := I - \widetilde{Q}$ .  $\widetilde{P}$  is a projection on a subspace of finite codimension, hence it is continuous and therefore  $(f_1)$ ,  $(f_4)$  imply  $y = y(z) := PN(w(z) + z) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and  $||y||_\infty \leq C_1$  for some  $C_1$  independent of  $z \in Z \setminus \overline{B}_R(0)$ . Since  $L|_{\widetilde{W}}$  has bounded inverse,  $||\widetilde{w}||_\infty \leq C$ , where  $\widetilde{w} = \widetilde{w}(z) := \widetilde{L}^{-1}y$  (note that for  $w = w(z) = L^{-1}y$  we only have a z-dependent  $L^2$ -bound because N(w + z) is not uniformly bounded in  $L^2(\mathbb{R}^N)$ ).

We complete the proof by showing that  $w = \widetilde{w}$ . Let  $\mu_n \notin \sigma(L)$ ,  $\mu_n \to 0$ . By the resolvent equation [9, (I.5.5) and §III.6.1],

$$w = L^{-1}y = (L - \mu_n I)^{-1}y - \mu_n L^{-1}(L - \mu_n I)^{-1}y$$

and

$$\widetilde{w} = \widetilde{L}^{-1}y = (\widetilde{L} - \mu_n I)^{-1}y - \mu_n \widetilde{L}^{-1}(\widetilde{L} - \mu_n I)^{-1}y.$$

Let  $v_n := (L - \mu_n I)^{-1} y$ . Since  $y \in L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ , [8, Proposition 2.1] implies  $v_n = (\widetilde{L} - \mu_n I)^{-1} y$  as well. As the last term on each of the right-hand sides above tends to 0,  $(v_n)$  is a Cauchy sequence in  $L^2(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$  (with the norm  $\|\cdot\|_{L^2 \cap L^{\infty}} := \|\cdot\|_2 + \|\cdot\|_{\infty}$ ) which yields  $w = \widetilde{w}$ .

Proof of Theorem 1.4. We have already shown that assumptions (i)-(iii) of Theorem 1.2 are satisfied. Suppose first that  $(f_4)$  and  $(f_5)$  hold. We only need to verify that  $\varphi_0$  satisfies (PS). Recall from (3.9) that for ||z|| > R

$$\varphi_0(z) = \Phi_0(w(z) + z),$$

where we have put w(z) = w(0, z), and by Proposition 3.6, we have

(5.2) 
$$\langle \nabla \varphi_0(z), \zeta \rangle = \langle Lz, \zeta \rangle - \int_{\mathbb{R}^N} g(x, w(z) + z) \zeta \, dx \quad \text{for all } z, \zeta \in \mathbb{Z}, \ \|z\| > R.$$

Let  $z=z^++z^-+z^0\in Z^+\oplus Z^-\oplus Z^0$ , where  $Z^+,Z^-$  respectively denote the subspaces of Z corresponding to the positive and the negative part of the spectrum of  $L|_Z$  and  $Z^0:=N(L)\subset Z$ . Let  $(z_n)\subset Z$  be such that  $\nabla \varphi_0(z_n)\to 0$ . It suffices to consider  $z_n$  with  $||z_n||>R$ , and we shall show that  $(z_n)$  is bounded. Since Z is spanned by eigenfunctions of  $-\Delta+V_0$  and dim  $Z<\infty$ , it follows from [18, Theorem C.3.4] that there are constants  $\delta, C_0>0$  such that  $|z(x)|\leq C_0e^{-\delta|x|}$  for all  $x\in\mathbb{R}^N$  and all  $z\in Z$  with  $||z||\leq 1$ . In particular, such z are uniformly bounded in  $L^p(\mathbb{R}^N)$  for any  $p\in [1,\infty]$ . Using this,  $(f_4)$  and equivalence of norms in Z, we obtain

$$\left| \langle Lz_n^+, z \rangle \right| \le \left| \int_{\mathbb{R}^N} g(x, w(z_n) + z_n) z \, dx \right| + o(1) \|z\| \le c_1 \|z\| \le c_2 \quad \text{for all } z \in Z^+, \ \|z\| = 1.$$

Hence  $(z_n^+)$  is bounded and a similar argument shows that so is  $(z_n^-)$ . Suppose  $||z_n^0|| \to \infty$  and write  $z_n^0 = t_n \tilde{z}_n^0$ , where  $||\tilde{z}_n^0|| = 1$ . Passing to a subsequence,  $\tilde{z}_n^0 \to \tilde{z}^0 \in Z^0$ . Denote

$$v_n := w(z_n) + z_n^+ + z_n^-.$$

We shall obtain a contradiction with the assumption  $\nabla \varphi_0(z_n) \to 0$  by showing that

(5.3) 
$$\langle -\nabla \varphi_0(z_n), \widetilde{z}_n^0 \rangle = \int_{\mathbb{D}^N} g(x, v_n + t_n \widetilde{z}_n^0) \widetilde{z}_n^0 dx \not\to 0.$$

By Lemma 5.2, the sequence  $(w(z_n))$  is bounded in  $L^{\infty}(\mathbb{R}^N)$ , and since so are the sequences  $(z_n^{\pm})$ ,  $v_n(x) + t_n \tilde{z}_n^0(x) \to \pm \infty$  for all  $x \in A_{\pm} := \{x \in \mathbb{R}^N : \pm \tilde{z}^0(x) > 0\}$ .

Suppose  $\pm g_{\pm} \geq 0$ . Since g is bounded and  $\tilde{z}_n^0$  is uniformly bounded in  $L^1(\mathbb{R}^N)$ , we may use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n \to \infty} \int_{A_+} g(x, v_n + t_n \widetilde{z}_n^0) \widetilde{z}_n^0 dx = \int_{A_+} g_{\pm} \widetilde{z}^0 dx \ge 0.$$

By the unique continuation property [5, Proposition 3 and Remark 2],  $\tilde{z}^0(x) \neq 0$  a.e. Hence the measure of  $\mathbb{R}^N \setminus (A_+ \cup A_-)$  is 0 and thus

(5.4) 
$$\int_{A_{+}} g_{+} \tilde{z}^{0} dx + \int_{A_{-}} g_{-} \tilde{z}^{0} dx > 0.$$

This implies (5.3). If  $\pm g_{\pm} \leq 0$ , the same argument remains valid after making some obvious changes.

Suppose now that  $(f_4)$  and  $(f_6)$  are satisfied. Here we do not know whether (PS) holds for  $\varphi_0$ , however, we will construct an admissible pair directly by adapting an argument in [10], see in particular the proof of Theorem 4.5 there. Suppose  $g(x,s)s \geq 0$  in  $(f_6)$  and let

$$\mathbb{W} := \{ z \in \mathbb{Z} : ||z^{\pm}|| \le R_0, ||z^0|| \le R_1 \}, \quad \mathbb{W}^- := \{ z \in \mathbb{W} : ||z^-|| = R_0 \text{ or } ||z^0|| = R_1 \}$$

 $(R_0, R_1 \text{ to be determined})$ . Boundedness of g and equivalence of norms in Z yield

$$\left| \int_{\mathbb{R}^N} g(x, w(z) + z) z^+ \, dx \right| \le c_3 ||z^+||.$$

Since  $\langle \pm Lz, z^{\pm} \rangle \geq \varepsilon ||z^{\pm}||^2$  for some  $\varepsilon > 0$ ,  $\langle \nabla \varphi_0(z), z^+ \rangle \geq \varepsilon ||z^+||^2 - c_3 ||z^+|| > 0$  if  $||z^+|| = R_0$  and  $\langle \nabla \varphi_0(z), z^- \rangle < 0$  if  $||z^-|| = R_0$  provided  $R_0$  is large enough. We want to show that there exists a (large)  $R_1$  such that  $\langle \nabla \varphi_0(z), z^0 \rangle < 0$  for z with  $||z^-|| = R_0$  and  $||z^0|| = R_1$ . Assuming the contrary,  $\lim_{n \to \infty} \langle \nabla \varphi_0(z_n), z_n^0 \rangle \geq 0$  for a sequence  $(z_n)$  such that  $||z_n^0|| \to \infty$ . Below we use the same notation as in (5.3). We have

$$0 = \langle -\nabla \varphi_0(z_n), w(z_n) \rangle = \int_{\mathbb{R}^N} g(x, v_n + t_n \widetilde{z}_n^0) w(z_n) \, dx,$$

 $g(x,s) \to 0$  as  $|s| \to \infty$  (because  $h_{\pm} \in L^{\infty}(\mathbb{R}^N)$  by  $(f_6)$ ) and  $|g(x,v_n+t_n\tilde{z}_n^0)z_n^{\pm}| \le c_4 e^{-\delta|x|}$ . So according to the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n \widetilde{z}_n^0) v_n \, dx = 0.$$

Hence Fatou's lemma and  $(f_6)$  give

$$\liminf_{n\to\infty}\int_{A_{\pm}}g(x,v_n+t_n\widetilde{z}_n^0)t_n\widetilde{z}_n^0\,dx=\liminf_{n\to\infty}\int_{A_{\pm}}g(x,v_n+t_n\widetilde{z}_n^0)(v_n+t_n\widetilde{z}_n^0)\,dx\geq\int_{A_{\pm}}h_{\pm}\,dx\geq0.$$

Since by assumption at least one of the integrals on the right-hand side is positive (possibly infinite),

$$\liminf_{n \to \infty} \langle -\nabla \varphi_0(z_n), z_n^0 \rangle = \liminf_{n \to \infty} \int_{\mathbb{R}^N} g(x, v_n + t_n \widetilde{z}_n^0) t_n \widetilde{z}_n^0 dx > 0,$$

a contradiction. So  $R_1$  exists as required and  $(\mathbb{W}, \mathbb{W}^-)$  is an admissible pair. Now it is easy to see as in the proof of (iii) of Lemma 4.2 that this is also an admissible pair for  $\varphi_{\pm\delta}$  if  $\delta$  is small enough. As in the proof of Theorem 1.2 one shows that the critical groups for  $\varphi_{\delta}$  and  $\varphi_{-\delta}$  are different, and this forces bifurcation.

If  $g(x,s)s \leq 0$ , a similar argument shows that  $\langle \nabla \varphi_0(z), z^0 \rangle > 0$  for some  $R_1$ , hence the exit set for the flow is  $\mathbb{W}^- := \{z \in \mathbb{W} : ||z^-|| = R_0\}$ .

Remark 5.3. Note that (5.4) is a variant of the Landesman-Lazer condition introduced in [13] and Theorem 1.4 remains valid if one assumes (5.4) holds for all  $z \in N(L)$ . This is slightly less restrictive than  $(f_5)$ . The reason that we have chosen  $(f_5)$  is that it is a general condition on f, with no reference to eigenfunctions corresponding to  $\lambda_0$ .  $(f_6)$  is a kind of strong resonance condition because  $g(x,s) \to 0$  as  $|s| \to \infty$ . Note also that our arguments show that under the assumptions of Theorem 1.4 there is a uniform bound for solutions of (1.1) with  $\lambda = \lambda_0$ .

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