Intrinsic sound of anti-de Sitter manifolds

Toshiyuki Kobayashi

Abstract As is well-known for compact Riemann surfaces, eigenvalues of the Laplacian are distributed discretely and most of eigenvalues vary viewed as functions on the Teichmüller space. We discuss a new feature in the Lorentzian geometry, or more generally, in pseudo-Riemannian geometry. One of the distinguished features is that L^2-eigenvalues of the Laplacian may be distributed densely in $\mathbb{R}$ in pseudo-Riemannian geometry. For three-dimensional anti-de Sitter manifolds, we also explain another feature proved in joint with F. Kassel [Adv. Math. 2016] that there exist countably many L^2-eigenvalues of the Laplacian that are stable under any small deformation of anti-de Sitter structure. Partially supported by Grant-in-Aid for Scientific Research (A) (25247006), Japan Society for the Promotion of Science.
Keywords and phrases: Laplacian, locally symmetric space, Lorentzian manifold, spectral analysis, Clifford–Klein form, reductive group, discontinuous group *2010*

MSC: Primary 22E40, 22E46, 58J50: Secondary 11F72, 53C35

1 Introduction

Our "common sense" for music instruments says:

"shorter strings produce a higher pitch than longer strings",

"thinner strings produce a higher pitch than thicker strings".

Let us try to "hear the sound of pseudo-Riemannian locally symmetric spaces". Contrary to our "common sense" in the Riemannian world, we find a phenomenon that compact three-dimensional anti-de Sitter manifolds have "intrinsic sound"

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which is stable under any small deformation. This is formulated in the framework of spectral analysis of anti-de Sitter manifolds, or more generally, of pseudo-Riemannian locally symmetric spaces X_Γ. In this article, we give a flavor of this new topic by comparing it with the flat case and the Riemannian case.

To explain briefly the subject, let X be a pseudo-Riemannian manifold, and Γ a discrete isometry group acting properly discontinuously and freely on X. Then the quotient space $X_\Gamma := \Gamma\backslash X$ carries a pseudo-Riemannian manifold structure such that the covering map $X \to X_\Gamma$ is isometric. We are particularly interested in the case where X_Γ is a pseudo-Riemannian locally symmetric space, see Section 3.2.

Problems we have in mind are symbolized in the following diagram:

	existence problem	deformation *v.s.* rigidity
Geometry	Does cocompact Γ exist? (Section 4.1)	Higher Teichmüller theory *v.s.* rigidity theorem (Section 4.2)
Analysis	Does L^2-spectrum exist? (Problem A)	Whether L^2-eigenvalues vary or not (Problem B)

2 A program

In [5, 6, 12] we initiated the study of "spectral analysis on pseudo-Riemannian locally symmetric spaces" with focus on the following two problems:

Problem A *Construct eigenfunctions of the Laplacian Δ_{X_Γ} on X_Γ. Does there exist a nonzero L^2-eigenfunction?*

Problem B *Understand the behaviour of L^2-eigenvalues of the Laplacian Δ_{X_Γ} on X_Γ under small deformation of Γ inside G.*

Even when X_Γ is compact, the existence of countably many L^2-eigenvalues is already nontrivial because the Laplacian Δ_{X_Γ} is not elliptic in our setting. We shall discuss in Section 2.2 for further difficulties concerning Problems A and B when X_Γ is non-Riemannian.

We may extend these problems by considering *joint* eigenfunctions for "invariant differential operators" on X_Γ rather than the single operator Δ_{X_Γ}. Here by "invariant differential operators on X_Γ" we mean differential operators that are induced from G-invariant ones on $X = G/H$. In Section 7, we discuss Problems A and B in this general formulation based on the recent joint work [6, 7] with F. Kassel.

2.1 Known results

Spectral analysis on a pseudo-Riemannian locally symmetric space $X_\Gamma = \Gamma\backslash X = \Gamma\backslash G/H$ is already deep and difficult in the following special cases:

1) (noncommutative harmonic analysis on G/H) $\Gamma = \{e\}$.
In this case, the group G acts unitarily on the Hilbert space $L^2(X_\Gamma) = L^2(X)$ by translation $f(\cdot) \mapsto f(g^{-1}\cdot)$, and the irreducible decomposition of $L^2(X)$ (*Plancherel-type formula*) is essentially equivalent to the spectral analysis of G-invariant differential operators when X is a semisimple symmetric space. Noncommutative harmonic analysis on semisimple symmetric spaces X has been developed extensively by the work of Helgason, Flensted-Jensen, Matsuki–Oshima–Sekiguchi, Delorme, van den Ban–Schlichtkrull among others as a generalization of Harish-Chandra's earlier work on the regular representation $L^2(G)$ for group manifolds.
2) (automorphic forms) H is compact and Γ is arithmetic.
If H is a maximal compact subgroup of G, then $X_\Gamma = \Gamma\backslash G/H$ is a Riemannian locally symmetric space and the Laplacian Δ_{X_Γ} is an elliptic differential operator. Then there exist infinitely many L^2-eigenvalues of Δ_{X_Γ} if X_Γ is compact by the general theory for compact Riemannian manifolds (see Fact 1). If furthermore Γ is irreducible, then Weil's local rigidity theorem [18] states that nontrivial deformations exist only when X is the hyperbolic plane $SL(2,\mathbb{R})/SO(2)$, in which case compact quotients X_Γ have a classically-known deformation space modulo conjugation, *i.e.*, their Teichmüller space. Viewed as a function on the Teichmüller space, L^2-eigenvalues vary analytically [1, 20], see Fact 11.
Spectral analysis on X_Γ is closely related to the theory of automorphic forms in the Archimedean place if Γ is an arithmetic subgroup.
3) (abelian case) $G = \mathbb{R}^{p+q}$ with $H = \{0\}$ and $\Gamma = \mathbb{Z}^{p+q}$.
We equip $X = G/H$ with the standard flat pseudo-Riemannian structure of signature (p,q) (see Example 1). In this case, G is abelian, but $X = G/H$ is non-Riemannian. This is seemingly easy, however, spectral analysis on the $(p+q)$-torus $\mathbb{R}^{p+q}/\mathbb{Z}^{p+q}$ is much involved, as we shall observe a connection with Oppenheim's conjecture (see Section 5.2).

2.2 Difficulties in the new settings

If we try to attack a problem of spectral analysis on $\Gamma\backslash G/H$ in the more general case where H is noncompact and Γ is infinite, then new difficulties may arise from several points of view:

(1) Geometry. The G-invariant pseudo-Riemannian structure on $X = G/H$ is not Riemannian anymore, and discrete groups of isometries of X do not always act properly discontinuously on such X.

(2) Analysis. The Laplacian Δ_X on X_Γ is not an elliptic differential operator. Furthermore, it is not clear if Δ_X has a self-adjoint extension on $L^2(X_\Gamma)$.
(3) Representation theory. If Γ acts properly discontinuously on $X = G/H$ with H noncompact, then the volume of $\Gamma\backslash G$ is infinite, and the regular representation $L^2(\Gamma\backslash G)$ may have infinite multiplicities. In turn, the group G may not have a good control of functions on $\Gamma\backslash G$. Moreover $L^2(X_\Gamma)$ is not a subspace of $L^2(\Gamma\backslash G)$ because H is noncompact. All these observations suggest that an application of the representation theory of $L^2(\Gamma\backslash G)$ to spectral analysis on X_Γ is rather limited when H is noncompact.

Point (1) creates some underlying difficulty to Problem B: we need to consider locally symmetric spaces X_Γ for which proper discontinuity of the action of Γ on X is preserved under small deformations of Γ in G. This is nontrivial. This question was first studied by the author [9, 11]. See [4] for further study. An interesting aspect of the case of noncompact H is that there are more examples where nontrivial deformations of compact quotients exist than for compact H (*cf*. Weil's local rigidity theorem [18]). Perspectives from Point (1) will be discussed in Section 4.

Point (2) makes Problem A nontrivial. It is not clear if the following well-known properties in the *Riemannian* case holds in our setting in the *pseudo-Riemannian* case.

Fact 1 *Suppose M is a compact Riemannian manifold.*

(1) *The Laplacian Δ_M extends to a self-adjoint operator on $L^2(M)$.*
(2) *There exist infinitely many L^2-eigenvalues of Δ_M.*
(3) *An eigenfunction of Δ_M is infinitely differentiable.*
(4) *Each eigenspace of Δ_M is finite-dimensional.*
(5) *The set of L^2-eigenvalues is discrete in $\mathbb{R}$.*

Remark 1. We shall see that the third to fifth properties of Fact 1 may fail in the pseudo-Riemannian case, *e.g.*, Example 6 for (3) and (4), and $M = \mathbb{R}^{2,1}/\mathbb{Z}^3$ (Theorem 7) for (5).

In spite of these difficulties, we wish to reveal a mystery of spectral analysis of pseudo-Riemannian locally homogeneous spaces $X_\Gamma = \Gamma\backslash G/H$. We shall discuss self-adjoint extension of the Laplacian in the pseudo-Riemannian setting in Theorem 13, and the existence of countable many L^2-eigenvalues in Theorems 8, 12 and 13.

3 Pseudo-Riemannian manifold

3.1 Laplacian on pseudo-Riemannian manifolds

A *pseudo-Riemannian manifold M* is a smooth manifold endowed with a smooth, nondegenerate, symmetric bilinear tensor g of signature (p,q) for some p, $q \in \mathbb{N}$.

(M,g) is a Riemannian manifold if $q=0$, and is a Lorentzian manifold if $q=1$. The metric tensor g induces a Radon measure $d\mu$ on X, and the divergence div. Then the Laplacian

$$\Delta_M := \operatorname{div}\operatorname{grad},$$

is a differential operator of second order which is a symmetric operator on the Hilbert space $L^2(X,d\mu)$.

Example 1. Let (M,g) be the standard flat pseudo-Riemannian manifold:

$$\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2).$$

Then the Laplacian takes the form

$$\Delta_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}.$$

In general, Δ_M is an elliptic differential operator if (M,g) is Riemannian, and is a hyperbolic operator if (M,g) is Lorentzian.

3.2 Homogeneous pseudo-Riemannian manifolds

A typical example of pseudo-Riemannian manifolds X with "large" isometry groups is semisimple symmetric spaces, for which the infinitesimal classification was accomplished by M. Berger in 1950s. In this case, X is given as a homogeneous space G/H where G is a semisimple Lie group and H is an open subgroup of the fixed point group $G^\sigma = \{g \in G : \sigma g = g\}$ for some involutive automorphism σ of G. In particular, $G \supset H$ are a pair of reductive Lie groups.

More generally, we say G/H is a *reductive homogeneous space* if $G \supset H$ are a pair of real reductive algebraic groups. Then we have the following:

Proposition 1. *Any reductive homogeneous space $X = G/H$ carries a pseudo-Riemannian structure such that G acts on X by isometries.*

Proof. By a theorem of Mostow, we can take a Cartan involution θ of G such that $\theta H = H$. Then $K := G^\theta$ is a maximal compact subgroup of G, and $H \cap K$ is that of H. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Take an $\mathrm{Ad}(G)$-invariant nondegenerate symmetric bilinear form $\langle\,,\,\rangle$ on $\mathfrak{g}$ such that $\langle\,,\,\rangle|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite, $\langle\,,\,\rangle|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite, and $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal to each other. (If G is semisimple, then we may take $\langle\,,\,\rangle$ to be the Killing form of $\mathfrak{g}$.)

Since $\theta H = H$, the Lie algebra $\mathfrak{h}$ of H is decomposed into a direct sum $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p})$, and therefore the bilinear form $\langle\,,\,\rangle$ is non-degenerate when restricted to $\mathfrak{h}$. Then $\langle\,,\,\rangle$ induces an $\mathrm{Ad}(H)$-invariant nondegenerate symmetric bilinear form $\langle\,,\,\rangle_{\mathfrak{g}/\mathfrak{h}}$ on the quotient space $\mathfrak{g}/\mathfrak{h}$, with which we identify the tangent space $T_o(G/H)$

at the origin $o = eH \in G/H$. Since the bilinear form $\langle\, ,\, \rangle_{\mathfrak{g}/\mathfrak{h}}$ is $\mathrm{Ad}(H)$-invariant, the left translation of this form is well-defined and gives a pseudo-Riemannian structure g on G/H of signature $(\dim \mathfrak{p}/\mathfrak{h}\cap\mathfrak{p}, \dim \mathfrak{k}/\mathfrak{h}\cap\mathfrak{k})$. By the construction, the group G acts on the pseudo-Riemannian manifold $(G/H, g)$ by isometries. $\square$

3.3 Pseudo-Riemannian manifolds with constant curvature, Anti-de Sitter manifolds

Let $Q_{p,q}(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ be a quadratic form on $\mathbb{R}^{p+q}$ of signature (p,q), and we denote by $O(p,q)$ the indefinite orthogonal group preserving the form $Q_{p,q}$. We define two hypersurfaces $M_{\pm}^{p,q}$ in $\mathbb{R}^{p+q}$ by

$$M_{\pm}^{p,q} := \{x \in \mathbb{R}^{p+q} : Q_{p,q}(x) = \pm 1\}.$$

By switching p and q, we have an obvious diffeomorphism

$$M_{+}^{p,q} \simeq M_{-}^{q,p}.$$

The flat pseudo-Riemannian structure $\mathbb{R}^{p,q}$ (Example 1) induces a pseudo-Riemannian structure on the hypersurface $M_{+}^{p,q}$ of signature $(p-1,q)$ with constant curvature 1, and that on $M_{-}^{p,q}$ of signature $(p,q-1)$ with constant curvature -1.

The natural action of the group $O(p,q)$ on $\mathbb{R}^{p,q}$ induces an isometric and transitive action on the hypersurfaces $M_{\pm}^{p,q}$, and thus they are expressed as homogeneous spaces:

$$M_{+}^{p,q} \simeq O(p,q)/O(p-1,q), \quad M_{-}^{p,q} \simeq O(p,q)/O(p,q-1),$$

giving examples of pseudo-Riemannian homogeneous spaces as in Proposition 1.

The *anti-de Sitter space* $\mathrm{AdS}^n = M_{-}^{n-1,2}$ is a model space for n-dimensional Lorentzian manifolds of constant negative sectional curvature, or *anti-de Sitter n-manifolds*. This is a Lorentzian analogue of the real hyperbolic space H^n. For the convenience of the reader, we list model spaces of Riemannian and Lorentzian manifolds with constant positive, zero, and negative curvatures.

Riemannian manifolds with constant curvature:

$$\begin{array}{lll} S^n = M_{+}^{n+1,0} \simeq O(n+1)/O(n) & & : \text{standard sphere}, \\ \mathbb{R}^n & & : \text{Euclidean space}, \\ H^n = M_{-}^{n,1} \simeq O(1,n)/O(n) & & : \text{hyperbolic space}, \end{array}$$

Lorentzian manifolds with constant curvature:

$$
\begin{array}{rll}
\mathrm{dS}^n = M_+^{n,1} \simeq & O(n,1)/O(n-1,1) & : \text{de Sitter space,} \\
\mathbb{R}^{n-1,1} & & : \text{Minkowski space,} \\
\mathrm{AdS}^n = M_-^{n-1,2} \simeq & O(2,n-1)/O(1,n-1) & : \text{anti-de Sitter space,}
\end{array}
$$

4 Discontinuous groups for pseudo-Riemannian manifolds

4.1 Existence problem of compact Clifford–Klein forms

Let H be a closed subgroup of a Lie group G, and $X = G/H$, and Γ a discrete subgroup of G. If H is compact, then the double coset space $\Gamma\backslash G/H$ becomes a C^∞-manifold for any torsion-free discrete subgroup Γ of G. However, we have to be careful for noncompact H, because not all discrete subgroups acts properly discontinuously on G/H, and $\Gamma\backslash G/H$ may not be Hausdorff in the quotient topology. We illustrate this feature by two general results:

Fact 2(1) (Moore's ergodicity theorem [15]) *Let G be a simple Lie group, and Γ a lattice. Then Γ acts ergodically on G/H for any noncompact closed subgroup H. In particular, $\Gamma\backslash G/H$ is non-Hausdorff.*

(2) (Calabi–Markus phenomenon ([2, 8])) *Let G be a reductive Lie group, and Γ an infinite discrete subgroup. Then $\Gamma\backslash G/H$ is non-Hausdorff for any reductive subgroup H with* $\operatorname{rank}_{\mathbb{R}} G = \operatorname{rank}_{\mathbb{R}} H$.

In fact, determining which groups act properly discontinuously on reductive homogeneous spaces G/H is a delicate problem, which was first considered in full generality by the author; we refer to [13, Section 3.2] for a survey.

Suppose now a discrete subgroup Γ acts properly discontinuously and freely on $X = G/H$. Then the quotient space

$$X_\Gamma := \Gamma\backslash X \simeq \Gamma\backslash G/H$$

carries a C^∞-manifold structure such that the quotient map $p : X \to X_\Gamma$ is a covering, through which X_Γ inherits any G-invariant local geometric structure on X. We say Γ is a *discontinuous group for X* and X_Γ is a *Clifford–Klein form* of $X = G/H$.

Example 2.(1) If $X = G/H$ is a reductive homogeneous space, then any Clifford–Klein form X_Γ carries a pseudo-Riemannian structure by Proposition 1.

(2) If $X = G/H$ is a semisimple symmetric space, then any Clifford–Klein form $X_\Gamma = \Gamma\backslash G/H$ is a pseudo-Riemannian locally symmetric space, namely, the (local) geodesic symmetry at every $p \in X_\Gamma$ with respect to the Levi-Civita connection is locally isometric.

By *space forms*, we mean pseudo-Riemannian manifolds of constant sectional curvature. They are examples of pseudo-Riemannian locally symmetric spaces. For simplicity, we shall assume that they are geodesically complete.

Example 3. Clifford–Klein forms of $M_+^{p+1,q} = O(p+1,q)/O(p,q)$ (respectively, $M_-^{p,q+1} = O(p,q+1)/O(p,q)$) are pseudo-Riemannian space forms of signature (p,q) with positive (respectively, negative) curvature. Conversely, any (geodesically complete) pseudo-Riemannian space form of signature (p,q) is of this form as far as $p \neq 1$ for positive curvature or $q \neq 1$ for negative curvature.

A general question for reductive homogeneous spaces G/H is:

Question 1. Does compact Clifford–Klein forms of G/H exist?

or equivalently,

Question 2. Does there exist a discrete subgroup Γ of G acting cocompactly and properly discontinuously on G/H?

This question has an affirmative answer if H is compact by a theorem of Borel. In the general setting where H is noncompact, the question relates with a "global theory" of pseudo-Riemannian geometry: *how local pseudo-Riemannian homogeneous structure affects the global nature of manifolds?* A classic example is *space form problem* which asks the global properties (*e.g.* compactness, volume, fundamental groups, *etc.*) of a pseudo-Riemannian manifold of constant curvature (local property). The study of discontinuous groups for $M_+^{p+1,q}$ and $M_-^{p,q+1}$ shows the following results in pseudo-Riemannian space forms of signature (p,q):

Fact 3 *Space forms of positive curvature are*

(1) *always closed if* $q = 0$, *i.e., sphere geometry in the Riemannian case;*
(2) *never closed if* $p \geq q > 0$, *in particular, if* $q = 1$ *(de Sitter geometry in the Lorentzian case [2]).*

The phenomenon in the second statement is called the *Calabi–Markus phenomenon* (see Fact 2 (2) in the general setting).

Fact 4 *Compact space forms of negative curvature exist*

(1) *for all dimensions if* $q = 0$, *i.e., hyperbolic geometry in the Riemannian case;*
(2) *for odd dimensions if* $q = 1$, *i.e., anti-de Sitter geometry in the Lorentzian case;*
(3) *for* $(p,q) = (4m,3)$ $(m \in \mathbb{N})$ *or* $(8,7)$.

See [13, Section 4] for the survey of the space form problem in pseudo-Riemannian geometry and also of Question 1 for more general G/H.

A large and important class of Clifford–Klein forms X_Γ of a reductive homogeneous space $X = G/H$ is constructed as follows (see [8]).

Definition 1. A quotient $X_\Gamma = \Gamma \backslash X$ of X by a discrete subgroup Γ of G is called *standard* if Γ is contained in some reductive subgroup L of G acting properly on X.

If a subgroup L acts properly on G/H, then any discrete subgroup of Γ acts properly discontinuously on G/H. A handy criterion for the triple (G,H,L) of reductive groups such that L acts properly on G/H is proved in [8], as we shall recall below.

Let $G = K \exp \overline{\mathfrak{a}_+} K$ be a Cartan decomposition, where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ and $\overline{\mathfrak{a}_+}$ is the dominant Weyl chamber with respect to a fixed positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. This defines a map $\mu : G \to \overline{\mathfrak{a}_+}$ (*Cartan projection*) by

$$\mu(k_1 e^X k_2) = X \quad \text{for } k_1, k_2 \in K \text{ and } X \in \mathfrak{a}.$$

It is continuous, proper and surjective. If H is a reductive subgroup, then there exists $g \in G$ such that $\mu(gHg^{-1})$ is given by the intersection of $\overline{\mathfrak{a}_+}$ with a subspace of dimension $\operatorname{rank}_{\mathbb{R}} H$. By an abuse of notation, we use the same H instead of gHg^{-1}. With this convention, we have:

Properness Criterion 5 ([8]) *L acts properly on G/H if and only if $\mu(L) \cap \mu(H) = \{0\}$.*

By taking a lattice Γ of such L, we found a family of pseudo-Riemannian locally symmetric spaces X_Γ in [8, 13]. The list of symmetric spaces admitting standard Clifford–Klein forms of finite volume (or compact forms) include $M_-^{p,q+1} = O(p,q+1)/O(p,q)$ with (p,q) satisfying the conditions in Fact 4. Further, by applying Properness Criterion 5, Okuda [16] gave examples of pseudo-Riemannian locally symmetric spaces $\Gamma \backslash G/H$ of infinite volume where Γ is isomorphic to the fundamental group $\pi_1(\Sigma_g)$ of a compact Riemann surface Σ_g with $g \geq 2$.

For the construction of stable spectrum on X_Γ (see Theorem 10 and Theorem 12 (2) below), we introduced in [6, Section 1.6] the following concept:

Definition 2. A discrete subgroup Γ of G acts *strongly properly discontinuously* (or *sharply*) on $X = G/H$ if there exists $C, C' > 0$ such that for all $\gamma \in \Gamma$,

$$d(\mu(\gamma), \mu(H)) \geq C \|\mu(\gamma)\| - C'.$$

Here $d(\cdot,\cdot)$ is a distance in $\mathfrak{a}$ given by a Euclidean norm $\|\cdot\|$ which is invariant under the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. We say the positive number C is *the first sharpness constant* for Γ.

If a reductive subgroup L acts properly on a reductive homogeneous space G/H, then the action of a discrete subgroup Γ of L is strongly properly discontinuous ([6, Example 4.10]).

4.2 Deformation of Clifford–Klein forms

Let G be a Lie group and Γ a finitely generated group. We denote by $\operatorname{Hom}(\Gamma, G)$ the set of all homomorphisms of Γ to G topologized by pointwise convergence. By taking a finite set $\{\gamma_1, \cdots, \gamma_k\}$ of generators of Γ, we can identify $\operatorname{Hom}(\Gamma, G)$ as a subset of the direct product $G \times \cdots \times G$ by the inclusion:

$$\operatorname{Hom}(\Gamma, G) \hookrightarrow G \times \cdots \times G, \quad \varphi \mapsto (\varphi(\gamma_1), \cdots, \varphi(\gamma_k)). \tag{1}$$

If Γ is finitely presentable, then $\mathrm{Hom}(\Gamma,G)$ is realized as a real analytic variety via (1).

Suppose G acts continuously on a manifold X. We shall take $X = G/H$ with noncompact closed subgroup H later. Then not all discrete subgroups act properly discontinuously on X in this general setting. The main difference of the following definition of the author [9] in the general case from that of Weil [18] is a requirement of proper discontinuity.

$$\begin{aligned} R(\Gamma,G;X) := \{\varphi \in \mathrm{Hom}(\Gamma,G) : \varphi \text{ is injective,} \\ \text{and } \varphi(\Gamma) \text{ acts properly discontinuously and freely on } G/H\}. \end{aligned} \tag{2}$$

Suppose now $X = G/H$ for a closed subgroup H. Then the double coset space $\varphi(\Gamma)\backslash G/H$ forms a family of manifolds that are locally modelled on G/H with parameter $\varphi \in R(\Gamma,G;X)$. To be more precise on "parameter", we note that the conjugation by an element of G induces an automorphism of $\mathrm{Hom}(\Gamma,G)$ which leaves $R(\Gamma,G;X)$ invariant. Taking these unessential deformations into account, we define the *deformation space* (*generalized Teichmüller space*) as the quotient set

$$\mathscr{T}(\Gamma,G;X) := R(\Gamma,G;X)/G.$$

Example 4.(1) Let Γ be the surface group $\pi_1(\Sigma_g)$ of genus $g \geq 2$, $G = PSL(2,\mathbb{R})$, $X = H^2$ (two-dimensional hyperbolic space). Then $\mathscr{T}(\Gamma,G;X)$ is the classical Teichmüller space, which is of dimension $6g-6$.
(2) $G = \mathbb{R}^n$, $X = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$. Then $\mathscr{T}(\Gamma,G;X) \simeq GL(n,\mathbb{R})$ (see (4) below).
(3) $G = SO(2,2)$, $X = \mathrm{AdS}^3$, and $\Gamma = \pi_1(\Sigma_g)$. Then $\mathscr{T}(\Gamma,G;X)$ is of dimension $12g-12$ (see [6, Section 9.2] and references therein).

Remark 2. There is a natural isometry between $X_{\varphi(\Gamma)}$ and $X_{\varphi(g\Gamma g^{-1})}$. Hence, the set $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ of L^2-eigenvalues is independent of the conjugation of $\varphi \in R(\Gamma,G;X)$ by an element of G. By an abuse of notation we shall write $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ for $\varphi \in \mathscr{T}(\Gamma,G;X)$ when we deal with Problem B of Section 2.

5 Spectrum on $\mathbb{R}^{p,q}/\mathbb{Z}^{p+q}$ and Oppenheim conjecture

This section gives an elementary but inspiring observation of spectrum on flat pseudo-Riemannian manifolds.

5.1 *Spectrum of* $\mathbb{R}^{p,q}/\varphi(\mathbb{Z}^{p+q})$

Let $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$. Then the group homomorphism $\varphi : \Gamma \to G$ is uniquely determined by the image $\varphi(\mathbf{e}_j)$ $(1 \leq j \leq n)$ where $\mathbf{e}_1, \cdots, \mathbf{e}_n \in \mathbb{Z}^n$ are the standard basis, and thus we have a bijection

$$\mathrm{Hom}(\Gamma,G) \xleftarrow{\sim} M(n,\mathbb{R}), \quad \varphi_g \hookleftarrow g \tag{3}$$

by $\varphi_g(\mathbf{m}) := g\mathbf{m}$ for $\mathbf{m} \in \mathbb{Z}^n$, or equivalently, by $g = (\varphi_g(\mathbf{e}_1), \cdots, \varphi_g(\mathbf{e}_n))$.

Let $\sigma \in \mathrm{Aut}(G)$ be defined by $\sigma(\mathbf{x}) := -\mathbf{x}$. Then $H := G^\sigma = \{0\}$ and $X := G/H \simeq \mathbb{R}^n$ is a symmetric space. The discrete group Γ acts properly discontinuously on X via φ_g if and only if $g \in GL(n,\mathbb{R})$. Moreover, since G is abelian, G acts trivially on $\mathrm{Hom}(\Gamma,G)$ by conjugation, and therefore the deformation space $\mathscr{T}(\Gamma,G;X)$ identifies with $R(\Gamma,G;X)$. Hence we have a natural bijection between the two subsets of (3):

$$\mathscr{T}(\Gamma,G;X) \xleftarrow{\sim} GL(n,\mathbb{R}). \tag{4}$$

Fix p, $q \in \mathbb{N}$ such that $p+q=n$, and we endow $X \simeq \mathbb{R}^n$ with the standard flat indefinite metric $\mathbb{R}^{p,q}$ (see Example 1). Let us determine $\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) \simeq \mathrm{Spec}_d(\mathbb{R}^{p,q}/\varphi_g(\mathbb{Z}^n))$ for $g \in GL(n,\mathbb{R}) \simeq \mathscr{T}(\Gamma,G;X)$.

For this, we define a function on $X = \mathbb{R}^n$ by

$$f_{\mathbf{m}}(\mathbf{x}) := \exp(2\pi\sqrt{-1}\,{}^t\mathbf{m}g^{-1}\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^n)$$

for each $\mathbf{m} \in \mathbb{Z}^n$ where $\mathbf{x}$ and $\mathbf{m}$ are regarded as column vectors. Clearly, $f_{\mathbf{m}}$ is $\varphi_g(\Gamma)$-periodic and defines a real analytic function on $X_{\varphi_g(\Gamma)}$. Furthermore, $f_{\mathbf{m}}$ is an eigenfunction of the Laplacian $\Delta_{\mathbb{R}^{p,q}}$:

$$\Delta_{\mathbb{R}^{p,q}} f_{\mathbf{m}} = -4\pi^2 Q_{g^{-1}I_{p,q}{}^tg^{-1}}(\mathbf{m}) f_{\mathbf{m}},$$

where, for a symmetric matrix $S \in M(n,\mathbb{R})$, Q_S denotes the quadratic form on $\mathbb{R}^n$ given by

$$Q_S(\mathbf{y}) := {}^t\mathbf{y}S\mathbf{y} \qquad \text{for } \mathbf{y} \in \mathbb{R}^n.$$

Since $\{f_{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}^n\}$ spans a dense subspace of $L^2(X_{\varphi_g(\Gamma)})$, we have shown:

Proposition 2. *For any $g \in GL(n,\mathbb{R}) \simeq \mathscr{T}(\Gamma,G;X)$,*

$$\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) = \{-4\pi^2 Q_{g^{-1}I_{p,q}{}^tg^{-1}}(\mathbf{m}) : \mathbf{m} \in \mathbb{Z}^n\}.$$

Here are some observation in the $n = 1,2$ cases.

Example 5. Let $n=1$ and $(p,q)=(1,0)$. Then $\mathrm{Spec}_d(X_{\varphi_g(\Gamma)}) = \{-4\pi^2m^2/g^2 : m \in \mathbb{Z}\}$ for $g \in \mathbb{R}^\times \simeq GL(1,\mathbb{R})$ by Proposition 2. Thus the smaller the period $|g|$ is, the larger the absolute value of the eigenvalue $|-4\pi^2m^2/g^2|$ becomes for each fixed $m \in \mathbb{Z} \setminus \{0\}$. This is thought of as a mathematical model of a music instrument for which shorter strings produce a higher pitch than longer strings (see Introduction).

Example 6. Let $n=2$ and $(p,q)=(1,1)$. Take $g = I_2$, so that $\varphi_g(\Gamma) = \mathbb{Z}^2$ is the standard lattice. Then the L^2-eigenspace of the Laplacian $\Delta_{\mathbb{R}^{1,1}/\mathbb{Z}^2}$ for zero eigenvalue contains $W := \{\psi(x-y) : \psi \in L^2(\mathbb{R}/\mathbb{Z})\}$. Since W is infinite-dimensional and $W \not\subset C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$, the third and fourth statements of Fact 1 fail in this pseudo-Riemannian setting.

By the explicit description of $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathscr{T}(\Gamma, G; X)$ in Proposition 2, we can also tell the behaviour of $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ under deformation of Γ by φ. Obviously, any constant function on $X_{\varphi(\Gamma)}$ is an eigenfunction of the Laplacian $\Delta_{X_{\varphi(\Gamma)}} = \Delta_{\mathbb{R}^{p,q}}/\varphi(\mathbb{Z}^{p+q})$ with eigenvalue zero. We see that this is the unique stable L^2-eigenvalue in the flat compact manifold:

Corollary 1 (non-existence of stable eigenvalues). *Let $n = p+q$ with $p, q \in \mathbb{N}$. For any open subset V of $\mathscr{T}(\Gamma, G; X)$,*

$$\bigcap_{\varphi \in V} \mathrm{Spec}_d(X_{\varphi(\Gamma)}) = \{0\}.$$

5.2 Oppenheim's conjecture and stability of spectrum

In 1929, Oppenheim [17] raised a question about the distribution of an indefinite quadratic forms at integral points. The following theorem, referred to as Oppenheim's conjecture, was proved by Margulis (see [14] and references therein).

Fact 6 (Oppenheim's conjecture) *Suppose $n \geq 3$ and Q is a real nondegenerate indefinite quadratic form in n variables. Then either Q is proportional to a form with integer coefficients (and thus $Q(\mathbb{Z}^n)$ is discrete in $\mathbb{R}$), or $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

Combining this with Proposition 2, we get the following.

Theorem 7. *Let $p+q = n$, $p \geq 2$, $q \geq 1$, $G = \mathbb{R}^n$, $X = \mathbb{R}^{p,q}$ and $\Gamma = \mathbb{Z}^n$. We define an open dense subset U of $\mathscr{T}(\Gamma, G; X) \simeq GL(n, \mathbb{R})$ by*

$$U := \{g \in GL(n, \mathbb{R}) : g^{-1} I_{p,q} {}^t g^{-1} \text{ is not proportional to an element of } M(n, \mathbb{Z}).\}$$

Then the set $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ of L^2-eigenvalues of the Laplacian is dense in $\mathbb{R}$ if and only if $\varphi \in U$.

Thus the fifth statement of Fact 1 for compact Riemannian manifolds do fail in the pseudo-Riemannian case.

6 Main results—sound of anti-de Sitter manifolds

6.1 Intrinsic sound of anti-de Sitter manifolds

In general, it is not clear whether the Laplacian Δ_M admits infinitely many L^2-eigenvalues for compact pseudo-Riemannian manifolds. For anti-de Sitter 3-manifolds, we proved in [6, Theorem 1.1]:

Theorem 8. *For any compact anti-de Sitter 3-manifold M, there exist infinitely many L^2-eigenvalues of the Laplacian Δ_M.*

In the abelian case, it is easy to see that compactness of X_Γ is necessary for the existence of L^2-eigenvalues:

Proposition 3. *Let $G = \mathbb{R}^{p+q}$, $X = \mathbb{R}^{p,q}$, $\Gamma = \mathbb{Z}^k$, and $\varphi \in R(\Gamma, G; X)$. Then $\mathrm{Spec}_d(X_{\varphi(\Gamma)}) \neq \emptyset$ if and only if $X_{\varphi(\Gamma)}$ is compact, or equivalently, $k = p+q$.*

However, anti-de Sitter 3-manifolds M admit infinitely many L^2-eigenvalues even when M is of infinite-volume (see [6, Theorem 9.9]):

Theorem 9. *For any finitely generated discrete subgroup Γ of $G = SO(2,2)$ acting properly discontinuously and freely on $X = \mathrm{AdS}^3$,*

$$\mathrm{Spec}_d(X_\Gamma) \supset \{l(l-2) : l \in \mathbb{N}, l \geq 10C^{-3}\}$$

where $C \equiv C(\Gamma)$ is the first sharpness constant of Γ.

The above L^2-eigenvalues are stable in the following sense:

Theorem 10 (stable L^2-eigenvalues). *Suppose that $\Gamma \subset G = SO(2,2)$ and $M = \Gamma \backslash \mathrm{AdS}^3$ is a compact standard anti-de Sitter 3-manifold. Then there exists a neighbourhood $U \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion with the following two properties:*

$$U \subset R(\Gamma, G; \mathrm{AdS}^3), \tag{5}$$

$$\#\Big(\bigcap_{\varphi \in U} \mathrm{Spec}_d(X_\Gamma)\Big) = \infty. \tag{6}$$

The first geometric property (5) asserts that a small deformation of Γ keeps proper discontinuity, which was conjectured by Goldman [3] in the AdS^3 setting, and proved affirmatively in [11]. Theorem 10 was proved in [6, Corollary 9.10] in a stronger form (*e.g.*, without assuming "standard" condition).

Figuratively speaking, Theorem 10 says that compact anti-de Sitter manifolds have "intrinsic sound" which is stable under any small deformation of the anti-de Sitter structure. This is a new phenomenon which should be in sharp contrast to the abelian case (Corollary 1) and the Riemannian case below:

Fact 11 (see [20, Theorem 5.14]) *For a compact hyperbolic surface, no eigenvalue of the Laplacian above $\frac{1}{4}$ is constant on the Teichmüller space.*

We end this section by raising the following question in connection with the flat case (Theorem 7):

Question 3. Suppose M is a compact anti-de Sitter 3-manifold. Find a geometric condition on M such that $\mathrm{Spec}_d(M)$ is discrete.

7 Perspectives and sketch of proof

The results in the previous section for anti-de Sitter 3-manifolds can be extended to more general pseudo-Riemannian locally symmetric spaces of higher dimension:

Theorem 12 ([6, Theorem 1.5]). *Let X_Γ be a standard Clifford–Klein form of a semisimple symmetric space $X = G/H$ satisfying the rank condition*

$$\operatorname{rank} G/H = \operatorname{rank} K/H \cap K. \tag{7}$$

Then the following holds.

(1) *There exists an explicit infinite subset I of joint L^2-eigenvalues for all the differential operators on X_Γ that are induced from G-invariant differential operators on X.*
(2) *(stable spectrum) If Γ is contained in a simple Lie group L of real rank one acting properly on $X = G/H$, then there is a neighbourhood $V \subset \operatorname{Hom}(\Gamma, G)$ of the natural inclusion such that for any $\varphi \in V$, the action $\varphi(\Gamma)$ on X is properly discontinuous and the set of joint L^2-eigenvalues on $X_{\varphi(\Gamma)}$ contains the infinite set I.*

Remark 3. We do not require X_Γ to be of finite volume in Theorem 12.

Remark 4. It is plausible that for a general locally symmetric space $\Gamma\backslash G/H$ with G reductive, no nonzero L^2-eigenvalue is stable under nontrivial small deformation unless the rank condition (7) is satisfied. For instance, suppose $\Gamma = \pi_1(\Sigma_g)$ with $g \geq 2$ and $R(\Gamma, G; X) \neq \emptyset$. (Such semisimple symmetric space $X = G/H$ was recently classified in [16].) Then we expect the rank condition (7) is equivalent to the existence of an open subset U in $R(\Gamma, G; X)$ such that

$$\#\Big(\bigcap_{\varphi \in U} \operatorname{Spec}_d(X_{\varphi(\Gamma)})\Big) = \infty.$$

It should be noted that not all L^2-eigenvalues of compact anti-de Sitter manifolds are stable under small deformation of anti-de Sitter structure. In fact, we proved in [7] that there exist also countably many *negative* L^2-eigenvalues that are NOT stable under deformation, whereas the countably many stable L^2-eigenvalues that we constructed in Theorem 9 are all positive. More generally, we prove in [7] the following theorem that include both stable and unstable L^2-eigenvalues:

Theorem 13. *Let G be a reductive homogeneous space and L a reductive subgroup of G such that $H \cap L$ is compact. Assume that the complexification $X_\mathbb{C}$ is $L_\mathbb{C}$-spherical. Then for any torsion-free discrete subgroup Γ of L, we have:*

(1) *the Laplacian Δ_{X_Γ} extends to a self-adjoint operator on $L^2(X_\Gamma)$;*
(2) $\#\operatorname{Spec}_d(X_\Gamma) = \infty$ *if X_Γ is compact.*

By "$L_{\mathbb{C}}$-spherical" we mean that a Borel subgroup $L_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In this case, a reductive subgroup L acts transitively on X by [10, Lemma 5.1].

Here are some examples of the setting of Theorem 13, taken from [13, Corollary 3.3.7].

Table 1 1

	G	H	L
(i)	$SO(2n,2)$	$SO(2n,1)$	$U(n,1)$
(ii)	$SO(2n,2)$	$U(n,1)$	$SO(2n,1)$
(iii)	$SU(2n,2)$	$U(2n,1)$	$Sp(n,1)$
(iv)	$SU(2n,2)$	$Sp(n,1)$	$U(2n,1)$
(v)	$SO(4n,4)$	$SO(4n,3)$	$Sp(1)\times Sp(n,1)$
(vi)	$SO(8,8)$	$SO(8,7)$	$Spin(8,1)$
(vii)	$SO(8,\mathbb{C})$	$SO(7,\mathbb{C})$	$Spin(7,1)$
(viii)	$SO(4,4)$	$Spin(4,3)$	$SO(4,1)\times SO(3)$
(ix)	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(4,1)\times SO(2)$

Examples for Theorem 13 include Table 1 (ii) for all $n\in\mathbb{N}$, whereas we need $n\in 2\mathbb{N}$ in Theorem 12 for the rank condition (7).

The idea of the proof for Theorem 12 is to take an average of a (nonperiodic) eigenfunction on X with rapid decay at infinity over Γ-orbits as a generalization of Poincaré series. Geometric ingredients of the convergence (respectively, nonzeroness) of the generalized Poincaré series include "counting Γ-orbits" stated in Lemma 1 below (respectively, the Kazhdan–Margulis theorem, *cf*. [6, Proposition 8.14]). Let $B(o,R)$ be a "pseudo-ball" of radius $R>0$ centered at the origin $o=eH\in X=G/H$, and we set

$$N(x,R) := \#\{\gamma\in\Gamma : \gamma\cdot x\in B(o,R)\}.$$

Lemma 1 ([6, Corollary 4.7]).

(1) *If Γ acts properly discontinuously on X, then $N(x,R)<\infty$ for all $x\in X$ and $R>0$.*
(2) *If Γ acts strongly properly discontinuously on X, then there exists $A_x>0$ such that*

$$N(x,R)\le A_x\exp\left(\frac{R}{C}\right)\quad \textit{for all } R>0,$$

where C is the first sharpness constant of Γ.

The key idea of Theorem 13 is to bring branching laws to spectral analysis [10, 12], namely, we consider the restriction of irreducible representations of G that are realized in the space of functions on the homogeneous space $X=G/H$ and analyze the G-representations when restricted to the subgroup L. Details will be given in [7].

Acknowledgements This article is based on the talk that the author delivered at the eleventh International Workshop: Lie Theory and its Applications in Physics in Varna, Bulgaria, 15-21, June, 2015. The author is grateful to Professor Vladimir Dobrev for his warm hospitality.

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