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Stable rationality of higher dimensional conic bundles

Hamid Ahmadinezhad and Takuzo Okada

Abstract. We prove that a very general nonsingular conic bundle $X \to \mathbb{P}^{n-1}$ embedded in a projective vector bundle of rank 3 over $\mathbb{P}^{n-1}$ is not stably rational if the anti-canonical divisor of X is not ample and $n \geq 3$.

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Titre. Rationalité stable des fibrés en coniques de grande dimension

Résumé. Nous démontrons qu'un fibré en coniques non-singulier très général $X \to \mathbb{P}^{n-1}$ plongé dans le projectivisé d'un fibré vectoriel de rang 3 au dessus de $\mathbb{P}^{n-1}$ n'est pas stablement rationnel lorsque le diviseur anti-canonique de X n'est pas ample et $n \geq 3$.

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1.   Introduction

An important question in algebraic geometry is to determine whether an algebraic variety is rational; that is, birational to projective space. Two algebraic varieties are said to be birational if they become isomorphic after removing finitely many lower-dimensional subvarieties from both sides. The closest varieties to being rational are those that admit a fibration into a projective space with all fibres rational curves; so-called conic bundles.

In this article, we study stable (non-)rationality of conic bundles over a projective space of arbitrary dimension (greater than one). A non-rational variety X may become rational after being multiplied by a suitable projective space, i.e., $X \times \mathbb{P}^m$ is birational to $\mathbb{P}^{n+m}$, where $n = \dim X$, in which case we say X is stably rational.

Stable non-rationality of conic bundles in dimension 3 has been studied extensively in [1, 2] and [8], giving a satisfactory answer. In higher dimensions almost nothing is known except for a few examples of stably non-rational conic bundles over $\mathbb{P}^3$ given in [1] and [9].

Throughout this article, by a conic bundle we mean a Mori fibre space of relative dimension 1 (see Definition 2.5 for details). The following is our main result.

Theorem 1.1. *Let $n \geq 3$ and d be integers, and let $\mathcal{E}$ be a direct sum of three invertible sheaves on $\mathbb{P}^{n-1}$. Let X be a very general member of a complete linear system $|2D + dF|$ on $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$, where D is the tautological divisor and F is the pullback of the hyperplane on $\mathbb{P}^{n-1}$. Suppose that the natural projection $X \to \mathbb{P}^{n-1}$ is a conic bundle.*

(1) *If X is singular, then X is rational.*

(2) *If X is non-singular and $-K_X$ is not ample, then X is not stably rational.*

This result covers the following varieties as a special case.

Corollary 1.2. *Let X be a very general hypersurface of bi-degree $(d, 2)$ in $\mathbb{P}^{n-1} \times \mathbb{P}^2$. If $d \geq n \geq 3$, then X is not stably rational.*

This can be thought of as a higher dimensional generalisation of the main result of [2].

Corollary 1.3. *Let X be a double cover of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ branched along a very general divisor of bi-degree $(2d, 2)$. If $2d \geq n \geq 3$, then X is not stably rational.*

By a result of Sarkisov [16], a conic bundle is birational to a standard conic bundle which is by definition a nonsingular conic bundle flat over a smooth base. The following criterion for rationality in terms of the discriminant was conjectured by Shokurov [17] (see also [10, Conjecture I]). Remarkabe progress toward this conjecture has been made in [10] and [13].

Conjecture 1.4. ([17, Conjecture 10.3]) *Let $X \to S$ be a 3-dimensional standard conic bundle and $\Delta \subset S$ the discriminant divisor. If $|2K_S + \Delta| \neq \emptyset$, then X is not rational.*

Although the statement becomes weaker than Theorem 1.1, we can restate our main result in terms of the discriminant:

Corollary 1.5. *With notation and assumptions as in* Theorem 1.1*, assume in addition that X is nonsingular and let $\Delta \subset \mathbb{P}^{n-1}$ be the discriminant divisor of the conic bundle $X \to \mathbb{P}^{n-1}$.*

(1) *If $|3K_{\mathbb{P}^{n-1}} + \Delta| \ne \emptyset$, then X is not stably rational.*

(2) *If $n \ge 7$, $\pi\colon X \to \mathbb{P}^{n-1}$ is standard and $|2K_{\mathbb{P}^{n-1}} + \Delta| \ne \emptyset$, then X is not stably rational.*

This leads us to pose the following.

Conjecture 1.6. *Let $\pi\colon X \to S$ be an n-dimensional standard conic bundle with $n \ge 3$. If $|2K_S + \Delta| \ne \emptyset$, then X is not rational. If in addition X is very general in its moduli, then X is not stably rational.*

The argument of stable non-rationality. It is known that a stably rational smooth projective variety is universally CH_0-trivial; see [5, Lemme 1.5] and [18, theorem 1.1] and references therein. Let $\mathcal{X} \to \mathcal{B}$ be a flat family over a complex curve $\mathcal{B}$ with smooth general fibre. Then, by the specialisation theorem of Voisin [19, Theorem 2.1], the stable non-rationality of a very general fibre will follow if the special fibre X_0 is not universally CH_0-trivial and has at worst ordinary double point singularities. This was generalised by Colliot-Thélène and Pirutka [5, Théorème 1.14] to the case where

1. X_0 admits a universally CH_0-trivial resolution $\varphi\colon Y \to X_0$ such that Y is not universally CH_0-trivial,

2. in mixed characteristic, that is, when $\mathcal{B} = \operatorname{Spec} A$ with A being a DVR of possibly mixed characteristic.

Thus it is enough to verify the existence of such a resolution $\varphi\colon Y \to X_0$ over an algebraically closed field of characteristic $p > 0$. In view of [18, Lemma 2.2], the core of the proof of universal CH_0-nontriviality for Y in our case is done by showing that $H^0(Y, \Omega^i) \ne 0$ for some $i > 0$, following Kollár [11] and Totaro [18]. This is done in Section 3.

2.   Embedded conic bundles

2.A.   Weighted projective space bundles

In this subsection we work over a field k.

Definition 2.1. A *toric weighted projective space bundle over $\mathbb{P}^n$* is a projective simplicial toric variety with Cox ring
$$\operatorname{Cox}(P) = k[u_0, \ldots, u_n, x_0, \ldots, x_m],$$
which is $\mathbb{Z}^2$-graded as
$$\begin{pmatrix} 1 & \cdots & 1 & \lambda_0 & \cdots & \lambda_m \\ 0 & \cdots & 0 & a_0 & \cdots & a_m \end{pmatrix}$$
with the irrelevant ideal $I = (u_0, \ldots, u_n) \cap (x_0, \ldots, x_m)$, where $\lambda_0, \ldots, \lambda_m$ are integers and $n, m, a_0, \ldots, a_m$ are positive integers. In other words, P is the geometric quotient
$$P = (\mathbb{A}^{n+m+2} \setminus V(I))/\mathbb{G}_m^2,$$
where the action of $\mathbb{G}_m^2 = \mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^{n+m+2} = \operatorname{Spec} \operatorname{Cox}(P)$ is given by the above matrix.

The natural projection $\Pi \colon P \to \mathbb{P}^n$ by the coordinates $u_0,\ldots,u_n$ realizes P as a $\mathbb{P}(a_0,\ldots,a_m)$-bundle over $\mathbb{P}^n$. In this paper, we simply call P the $\mathbb{P}(a_0,\ldots,a_m)$-*bundle over* $\mathbb{P}^n$ *defined by*

$$\begin{pmatrix} u_0 & \cdots & u_n & x_0 & \cdots & x_m \\ 1 & \cdots & 1 & | & \lambda_0 & \cdots & \lambda_m \\ 0 & \cdots & 0 & | & a_0 & \cdots & a_m \end{pmatrix}.$$

In the following, let P be as in Definition 2.1. Let $\mathsf{p} \in P$ be a point and $\mathsf{q} \in \mathbb{A}^{n+m+2} \setminus V(I)$ a preimage of p via the morphism $\mathbb{A}^{n+m+2} \setminus V(I) \to P$. We can write $\mathsf{q} = (\alpha_0,\ldots,\alpha_n,\beta_0,\ldots,\beta_m)$, where $\alpha_i, \beta_j \in k$. In this case we express p as $(\alpha_0 : \cdots : \alpha_n; \beta_0 : \cdots : \beta_m)$.

Remark 2.2. We will frequently use the following coordinate change. Consider a point $\mathsf{p} = (\alpha_0 : \cdots : \alpha_n; \beta_0 : \cdots : \beta_m) \in P$ and suppose for example that $\alpha_0 \neq 0, \beta_j \neq 0$ and $a_j = 1$ for some j. Then for $l \neq j$ such that $\lambda_l / a_l \geq \lambda_j$, the replacement

$$x_l \mapsto \alpha_0^{\lambda_l - a_l \lambda_j} \beta_j^{a_l} x_l - \beta_l u_0^{\lambda_l - a_l \lambda_j} x_j^{a_l}$$

induces an automorphism of P. By considering the above coordinate change, we can transform p (via an automorphism of P) into a point for which the x_l-coordinate is zero for l with $\lambda_l / a_l \geq \lambda_j$.

We have the decomposition

$$\mathrm{Cox}(P) = \bigoplus_{(\alpha,\beta) \in \mathbb{Z}^2} \mathrm{Cox}(P)_{(\alpha,\beta)},$$

where $\mathrm{Cox}(P)_{(\alpha,\beta)}$ consists of the homogeneous elements of bi-degree (α,β). An element $f \in \mathrm{Cox}(P)_{(\alpha,\beta)}$ is called a (homogeneous) polynomial of bi-degree (α,β).

The Weil divisor class group $\mathrm{Cl}(P)$ is naturally isomorphic to $\mathbb{Z}^2$. Let F and D be the divisors on P corresponding to $(1,0)$ and $(0,1)$, respectively, which generate $\mathrm{Cl}(P)$. Note that F is the class of the pullback of a hyperplane on $\mathbb{P}^n$ via $\Pi \colon P \to \mathbb{P}^n$. We denote by $\mathcal{O}_P(\alpha,\beta)$ the rank 1 reflexive sheaf corresponding to the divisor class of type (α,β), that is, the divisor $\alpha F + \beta D$. More generally, for a subscheme $Z \subset P$, we set $\mathcal{O}_Z(\alpha,\beta) = \mathcal{O}_P(\alpha,\beta)|_Z$. We remark that there is an isomorphism

$$H^0(P, \mathcal{O}_P(\alpha,\beta)) \cong \mathrm{Cox}(P)_{(\alpha,\beta)}.$$

Definition 2.3. For integers k,l,m,n with $n \geq 3$, we define $P_n(k,l,m)$ (resp. $Q_n(k,l)$) to be the $\mathbb{P}^2$-bundle (resp. $\mathbb{P}^1$-bundle) over $\mathbb{P}^{n-1}$ defined by the matrix

$$\begin{pmatrix} u_0 & \cdots & u_{n-1} & x & y & z \\ 1 & \cdots & 1 & | & k & l & m \\ 0 & \cdots & 0 & | & 1 & 1 & 1 \end{pmatrix} \left(\text{resp.} \begin{pmatrix} u_0 & \cdots & u_{n-1} & x & y \\ 1 & \cdots & 1 & | & k & l \\ 0 & \cdots & 0 & | & 1 & 1 \end{pmatrix} \right).$$

Remark 2.4. Let P be as in Definition 2.1. When $a_0 = \cdots = a_m = 1$, P is a $\mathbb{P}^m$-bundle over $\mathbb{P}^n$. More precisely we have an isomorphism

$$P \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(-\lambda_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-\lambda_m)).$$

Here, for a vector bundle $\mathcal{E}$ over $\mathbb{P}^n$, $\mathbb{P}(\mathcal{E}) = \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ denotes the projective bundle of one-dimensional quotients of $\mathcal{E}$. Moreover, via the above isomorphism, the pullback of a hyperplane on $\mathbb{P}^{n-1}$ and the tautological divisor on $\mathbb{P}(\mathcal{E})$ are identified with the divisors on P corresponding to $(1,0)$ and $(0,1)$, respectively.

2.B. Embedded conic bundles

In the rest of this section we work over $\mathbb{C}$. By a *splitting vector bundle*, we mean a vector bundle which is a direct sum of invertible sheaves.

Definition 2.5. Let X be a normal projective $\mathbb{Q}$-factorial variety of dimension n. We say that a morphism $\pi\colon X \to \mathbb{P}^{n-1}$ is a *conic bundle* (over $\mathbb{P}^{n-1}$) if it is a Mori fibre space, that is, X has only terminal singularities, π has connected fibres, $-K_X$ is π-ample and $\rho(X) = 2$, where $\rho(X)$ denotes the rank of the Picard group.

An *embedded conic bundle* $\pi\colon X \to \mathbb{P}^{n-1}$ is a conic bundle such that X is embedded in a projective bundle $\mathbb{P}(\mathcal{E}) := \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$ as a member of $|dF + 2D|$ for some splitting vector bundle $\mathcal{E}$ of rank 3 on $\mathbb{P}^{n-1}$ and $d \in \mathbb{Z}$, and π coincides with the restriction of $\Pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-1}$ to X. Here F and D denote the pullback of a hyperplane on $\mathbb{P}^{n-1}$ and the tautological class D on $\mathbb{P}(\mathcal{E})$, respectively.

In the following let $\mathcal{E}$ be a splitting vector bundle of rank 3 on $\mathbb{P}^{n-1}$ and $X \in |dF + 2D|$ be a general member. We denote by $\pi\colon X \to \mathbb{P}^{n-1}$ the restriction of $\Pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-1}$ to X. Without loss of generality we may assume that

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-k) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-m)$$

for some $k \leq l \leq m$. Then, by Remark 2.4, we have $\mathbb{P}(\mathcal{E}) \cong P_n(k,l,m)$ and the linear system $|dF + 2D|$ on $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$ corresponds to $|\mathcal{O}_{P_n(k,l,m)}(d,2)|$. Here we do not assume that $\pi\colon X \to \mathbb{P}^{n-1}$ is a conic bundle. We study conditions on k,l,m and d that make $\pi\colon X \to \mathbb{P}^{n-1}$ a conic bundle.

Lemma 2.6. *Let k,l,m,d be integers such that $k \leq l \leq m$. Set $P = P_n(k,l,m)$ and let X be a general member of $|\mathcal{O}_P(d,2)|$.*

(1) *X is smooth if and only if $d \geq 2m$, $d = l + m$, or $d = k + m$.*

(2) *X is not smooth and has only terminal singularities if and only if $2m > d > l + m$.*

(3) *X is non-normal if and only if $k + m > d$.*

Proof. Suppose that $d \geq 2m$. Then $|\mathcal{O}_P(d,2)|$ is base point free and its general member X is smooth. In the following we assume that $2m > d \geq k + m$.

Suppose that $2m > d > l + m$. Then X is defined in P by

$$ax^2 + by^2 + fxy + gxz + hyz = 0,$$

where $a, b, f, g, h \in \mathbb{C}[u]$. We have $\deg h = d - (l + m) > 0$ and $\deg g = d - (k + m) > 0$. Then X is singular along $(x = y = g = h = 0) \neq \emptyset$. The singular locus is of codimension 3 in X. Since X is general, the hypersurfaces in $\mathbb{P}^{n-1}$ defined by $g = 0$ and $h = 0$ are both nonsingular and intersect transversally. It is then straightforward to check that the blowup $\sigma\colon X' \to X$ along the singular locus is a resolution and we have $K_{X'} = \sigma^* K_X + E$, where E is the exceptional divisor. Thus X has terminal singularities.

Suppose that $2m > d = l + m$. Then X is defined in P by

$$ax^2 + by^2 + fxy + gxz + yz = 0.$$

Replacing y and z suitably, we can eliminate the terms by^2, fxy and gzx, that is, X is defined by

$$ax^2 + yz = 0.$$

It is then clear that X is smooth, when a is general.

Suppose that $l + m > d > k + m$. Then X is defined in P by

$$ax^2 + by^2 + fxy + gxz = 0.$$

We have $\deg g = d - (k + m) > 0$. Then X is singular along $(x = y = g = 0) \neq \emptyset$, and the singularity is not terminal since the singular locus is of codimension 2 in X.

Suppose that $l + m > d = k + m$. Then X is defined in P by

$$ax^2 + by^2 + fxy + xz = 0.$$

Replacing z suitably, we may assume that X is defined by

$$by^2 + zx = 0.$$

It is easy to see that X is smooth.

Finally suppose that $k + m > d$. Then X is defined in P by

$$ax^2 + by^2 + fxy = 0,$$

where $a, b, f \in \mathbb{C}[u]$. In this case X is singular along the divisor $(x = y = 0) \subset X$. Thus X is not normal. The above arguments prove (1), (2) and (3). $\square$

Lemma 2.7. *In the same setting as in* Lemma 2.6, *suppose that either* $d = l + m$ *or* $d = k + m$. *Then the variety* X *is rational. Moreover we have* $\rho(X) \geq 3$ *unless* $k = l = m$.

Proof. Suppose that $d = l + m$, which implies $2m \geq d = l + m$. We claim that X is defined by an equation of the form $ax^2 + yz = 0$, where $a \in \mathbb{C}[u]$. This is already proved in Lemma 2.6, when $2m > d$. Suppose that $2m = d = l + m$. Then $l = m$ and X is defined by

$$ax^2 + y^2 + z^2 + fxy + gxz + \alpha yz = 0,$$

where $\alpha \in \mathbb{C}$ and $a, f, g \in \mathbb{C}[u]$. Replacing y and z, the above equation can be transformed into $ax^2 + yz = 0$ and the claim is proved.

We consider the projection $X \dashrightarrow Q := Q_n(k, l)$ Note that $Q \cong \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}(-l))$. Then the projection is birational, hence X is rational. The projection $X \dashrightarrow Q$ is defined outside $(x = y = 0) \subset X$. Let $\mathsf{p} \in (x = y = 0)$ be a point. Then z does not vanish at p and we have

$$y = \frac{yz}{z} = -\frac{ax^2}{z}.$$

From this we deduce that $X \dashrightarrow Q$ is everywhere defined. Now we assume that either $k \neq l$ or $l \neq m$. Then $\deg a = d - 2k = l + m - k > 0$. We see that $(y = a = 0) \subset X$ is a divisor and it is contracted by $X \to Q$ to a codimension 2 subset of Q. This shows $\rho(X) \geq 3$.

Next, suppose that $d = k + m$. Note that $l + m \geq d$. If in addition $l + m > d$, then, by the proof of Lemma 2.6, the defining equation of X can be written as $by^2 + xz = 0$. The statement follows from the same argument as above. If $l + m = d$, then $k = l$ and we have $d = l + m$. This case is already proved. $\square$

Lemma 2.8. *In the same setting as in* Lemma 2.6, $\pi \colon X \to \mathbb{P}^{n-1}$ *is a nonsingular conic bundle if and only if one of the following holds:*

(1) $d > 2m$,

(2) $d = 2m$ *and* $m > l$, *or*

(3) $d = 2m = 2l = 2k$.

Proof. This follows from Lemmas 2.6 and 2.7. $\square$

Proposition 2.9. *Let X be an embedded conic bundle over $\mathbb{P}^{n-1}$. If X is general (in the linear system) and singular, then X is rational.*

Proof. We may assume that $X \in |\mathcal{O}_P(d, 2)|$, where $P = P_n(k, l, m)$, for some $k \leq l \leq m$. By Lemma 2.6, we have $2m > d \geq k + m$. Then a general member X is defined by an equation of the form

$$ax^2 + by^2 + fxy + gxz + hyz = 0,$$

where $a, b, f, g, h \in \mathbb{C}[u]$. Here, note that, if for example $l + m > d$, then we know that the term hyz does not appear in the equation. The inequality $d \geq k + m$ implies that $g \neq 0$ since X is general. Let $P \dashrightarrow Q = Q_n(k, l)$ be the natural projection. Now we can write the defining equation as

$$z(gx + hy) + ax^2 + by^2 + fxy = 0,$$

which implies that the restriction $X \dashrightarrow Q$ is birational. Therefore X is rational. $\square$

The following can be considered as a "normal form" of conic bundles, which describes nonsingular embedded conic bundles (see Proposition 2.11).

Definition 2.10. Let (λ, μ, ν) be a triplet of integers λ, μ, ν. We say that $\pi \colon X \to \mathbb{P}^{n-1}$ (or X) is of type $[\lambda, \mu, \nu]$ if X belongs to $|\mathcal{O}_P(\lambda + \mu + \nu, 2)|$, where $P = P_n(\lambda, \mu, \nu)$, and π coincides with the restriction of $P \to \mathbb{P}^{n-1}$ to X.

Proposition 2.11. *Let $\pi \colon X \to \mathbb{P}^{n-1}$ be a nonsingular embedded conic bundle. Then X is either of type $[\lambda, \mu, \nu]$ for some λ, μ, ν such that $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$ or of type $[0, 0, 0]$.*

Proof. We may assume that X belongs to $|\mathcal{O}_{P_n(k,l,m)}(d, 2)|$ for some $k \leq l \leq m$ and d. Since the family X is non-singular, we have $d \geq 2m$ by Lemma 2.8 and X is defined in $P_n(k, l, m)$ by an equation of the form

$$ax^2 + by^2 + cz^2 + fxy + gxz + hyz = 0,$$

where $a, b, c, f, g, h \in \mathbb{C}[u]$. We set $\alpha = \deg a, \beta = \deg b, \gamma = \deg c, \lambda = \deg h, \mu = \deg g$ and $\nu = \deg f$. By comparing the weights, we have

$$\alpha + 2k = \beta + 2l = \gamma + 2m = \nu + k + l = \mu + k + m = \lambda + l + m.$$

Now we have

$$P_n(k, l, m) \cong P_n(k + (\nu - m), l + (\nu - m), m + (\nu - m)) \cong P_n(\lambda, \mu, \nu) =: P,$$

and the linear system $|\mathcal{O}_{P_n(k,l,m)}(d, 2)|$ is identified with $|\mathcal{O}_P(\lambda + \mu + \nu, 2)|$. Thus X is of type $[\lambda, \mu, \nu]$. By applying Lemma 2.8 for $k = \lambda, l = \mu, m = \nu$ and $d = \lambda + \mu + \nu$, we get the desired result. $\square$

Remark 2.12. In the language of [1, Definition 3.1], a conic bundle $\pi \colon X \to \mathbb{P}^{n-1}$ of type $[\lambda, \mu, \nu]$ with $\lambda \leq \mu \leq \nu \leq \lambda + \mu$ is a conic bundle of graded-free type over $\mathbb{P}^{n-1}$ corresponding to the triplet $(-\lambda + \mu + \nu, \lambda - \mu + \nu, \lambda + \mu - \nu)$.

3. Stable non-rationality

In this section we study stable (non-)rationality of nonsingular embedded conic bundles $\pi \colon X \to \mathbb{P}^{n-1}$. By Proposition 2.11, such a conic bundle is of type $[\lambda, \mu, \nu]$, where either $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$ or $\lambda = \mu = \nu = 0$. In case X is of type $[0, 0, 0]$, then $X \cong \mathbb{P}^{n-1} \times \mathbb{P}^1$ and it is obviously rational. We consider the remaining cases and thus we assume that

$$0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$$

throughout this section. In addition we assume $\nu \geq 3$ throughout.

We set $P = P_n(\lambda, \mu, \nu)$, $\delta = \lambda + \mu + \nu$, and consider special members $X \in |\mathcal{O}_P(\delta, 2)|$ defined in P by an equation of the form

$$ax^2 + by^2 + cz^2 + fxy = 0, \tag{1}$$

where a, b, c, f are general polynomials in variables $u_0, \ldots, u_{n-1}$. Recall that $\nu = \deg f$ and $\deg a = -\lambda + \mu + \nu$, $\deg b = \lambda - \mu + \nu$ and $\deg c = \lambda + \mu - \nu$.

Remark 3.1. By the assumptions on λ, μ, ν, we have $\deg a = -\lambda + \mu + \nu \geq 3$, $\deg b = \lambda - \mu + \nu \geq 1$, $\deg c = \lambda + \mu - \nu \geq 0$ and $\deg f = \nu \geq 3$.

Lemma 3.2. *If the ground field is an algebraically closed field of characteristic* 0, *then* X *is smooth.*

Proof. The variety X is a general member of the base point free sub linear system of $|\mathcal{O}_P(\delta, 2)|$ on the smooth variety P. Thus, by the Bertini theorem, a general X is smooth. $\qquad\square$

We use universal CH_0-triviality to test stable rationality of varieties.

Definition 3.3. Let V be a projective variety defined over a field k. We denote by $\mathrm{CH}_0(V)$ the *Chow group of* 0-*cycles* on V. We say that V is *universally* CH_0-*trivial* if for any field F containing k, the degree map $\mathrm{CH}_0(V_F) \to \mathbb{Z}$ is an isomorphism. A projective morphism $\varphi \colon W \to V$ defined over k is *universally* CH_0-*trivial* if for any field containing k, the push-forward map $\varphi_* \colon \mathrm{CH}_0(W_F) \to \mathrm{CH}_0(V_F)$ is an isomorphism.

In the rest of this section we work over an algebraically closed field $\Bbbk$ of characteristic 2. Let R be the $\mathbb{P}(1, 1, 2)$-bundle over $\mathbb{P}^{n-1}$ defined by

$$\begin{pmatrix} u_0 & u_1 & \cdots & u_{n-1} & x & y & \bar{z} \\ 1 & 1 & \cdots & 1 & \lambda & \mu & 2\nu \\ 0 & 0 & \cdots & 0 & 1 & 1 & 2 \end{pmatrix}$$

and let $Z \subset R$ be the hypersurface defined by

$$ax^2 + by^2 + c\bar{z} + fxy = 0.$$

We have a natural morphism $P \to R$ which is a (purely inseparable) double cover branched along $(\bar{z} = 0) \subset R$. The image of X under $P \to R$ is the hypersurface $Z \subset R$. Let $\tau \colon X \to Z$ be the induced morphism, which is a double cover branched along the divisor cut out on Z by $\bar{z} = 0$. We set $\mathcal{L} = \mathcal{O}_Z(\nu, 1)$. Then $\bar{z}$ is a global section of $\mathcal{L}^2$, and over the non-singular locus of Z, τ is the double cover obtained by taking the roots of $\bar{z} \in H^0(Z, \mathcal{L}^2)$ in the sense of [11, Construction 8].

In Sections 3.A and 3.B below we will analyse the singularities of X and Z, and finally we will show the existence of a universally CH_0-trivial resolution $\varphi \colon Y \to X$ such that $H^0(Y, \Omega_Y^{n-1}) \neq 0$ under some conditions on λ, μ, ν. The latter implies that Y is not universally CH_0-trivial by [18, Lemma 2.2].

3.A.   Singularities

Recall that the ground field $\Bbbk$ is an algebraically closed field of characteristic 2 and X is a hypersurface in $P = P_n(\lambda, \mu, \nu)$ defined by

$$ax^2 + by^2 + cz^2 + fxy = 0$$

for general $a, b, c, f \in \Bbbk[u_0, \ldots, u_{n-1}]$. Similarly Z is the hypersurface in R defined by

$$ax^2 + by^2 + c\bar{z} + fxy = 0.$$

We set

$$\Xi = (x = y = 0) \subset R, \ \Xi_Z = \Xi \cap Z = (x = y = c = 0),$$

and $R^\circ = R \setminus \Xi$, $Z^\circ = Z \setminus \Xi_Z$.

In order to analyze singularities of $Z^\circ \subset R^\circ$, we consider standard affine charts of R°. For $i = 0, \dots, n-1$ and a coordinate $w \in \{x, y\}$, we set $U_{u_i, w} = (u_i \neq 0) \cap (w \neq 0) \subset R^\circ$. We have

$$R^\circ = \bigcup_{i \in \{0, \dots, n-1\}, w \in \{x, y\}} U_{u_i, w}.$$

We remark that $U_{u_i, w}$ is an affine $(n+1)$-space and that the restriction of the sections

$$\{u_0, \dots, u_{n-1}, x, y, \bar{z}\} \setminus \{u_i, w\}$$

are affine coordinates of $U_{u_i, w}$. We only treat $U_{u_0, x}$ because the other open subsets can be understood by symmetry. We set

$$\tilde{u}_i = u_i / u_0, \quad \tilde{y} = y / x u_0^{\mu - \lambda}, \quad \tilde{z} = \bar{z} / x^2 u_0^{\nu - 2\lambda}.$$

Then $U_{u_0, w}$ is an affine $(n+1)$-space with affine coordinates $\tilde{u}_1, \dots, \tilde{u}_{n-1}, \tilde{y}, \tilde{z}$. By a slight abuse of notation, the affine coordinates $\tilde{u}_1, \dots, \tilde{u}_{n-1}, \tilde{y}, \tilde{z}$ are simply denoted by $u_1, \dots, u_{n-1}, y, \bar{z}$.

Lemma 3.4. Z° *is smooth.*

Proof. If $\deg c = 0$, then c is a non-zero constant and thus $\Xi_Z = \emptyset$. In this case $Z = Z^\circ$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^{n-1}$ and it is smooth.

In the following we assume that $\deg c > 0$ and set

$$U_x = (x \neq 0), \quad U_y = (y \neq 0) \subset R,$$

so that $R^\circ = U_x \cup U_y$. We will show that for any point $\mathsf{q} \in R^\circ$, the condition that Z° is singular at $\mathsf{q} \in Z$ imposes $n + 2$ independent conditions on a, b, c, f. Then the assertion will follow by a dimension count argument since $\dim R^\circ = n + 1$. We note that $\deg b = \lambda - \mu + \nu \geq 1$, $\deg c = \lambda + \mu - \nu \geq \lambda \geq 3$ and $\deg f = \lambda \geq 3$ by Remark 3.1.

Let $\mathsf{q} \in U_x$. Replacing coordinates, we may assume $\mathsf{q} = (1 : 0 : \dots : 0; 1 : 0 : 0)$. Then $U_{u_0, x} \subset Q$ is an affine space with coordinates $u_1, \dots, u_{n-1}, y, \bar{z}$ and $Z \cap U_{u_0, z}$ is defined by

$$\tilde{a} + \tilde{b} y^2 + \tilde{c} \bar{z} + \tilde{f} y = 0,$$

where we set $\tilde{h} = h(1, u_1, \dots, u_{n-1})$ for a polynomial $h(u_0, \dots, u_{n-1})$. Note that q corresponds to the origin. The variety Z° is singular at q if and only if $\tilde{a}, \tilde{c}, \tilde{f}$ vanish at q and the linear part of $\tilde{a}$ is zero. This imposes $n + 2$ independent conditions since $\deg a > 0$ and $\deg c, \deg f \geq 0$ (cf. Remark 3.1).

Suppose that $\mathsf{q} \in U_y$. Replacing coordinates, we may assume $\mathsf{q} = (1 : 0 : \dots : 0; 0 : 1 : 0)$. Then $U_{u_0, y} \subset Q$ is an affine space with coordinates $u_0, \dots, u_{n-1}, x, \bar{z}$ and $Z \cap U_{u_0, y}$ is defined by

$$\tilde{a} x^2 + \tilde{b} + \tilde{c} \bar{z} + \tilde{f} x = 0.$$

The variety Z° is singular at q if and only if $\tilde{b}, \tilde{c}, \tilde{f}$ vanish at q and the linear part of $\tilde{b}$ is zero. The latter imposes $n + 2$ independent conditions since $\deg b > 0$ and $\deg c, \deg f \geq 0$ (cf. Remark 3.1), and the proof is complete. $\square$

We set $X^\circ = \pi^{-1}(Z^\circ)$.

Lemma 3.5. X *is smooth along* $X \setminus X^\circ$.

Proof. Note that $X \setminus X^\circ = X \cap (x = y = 0)$. For a point $\mathsf{p} \in X \setminus X^\circ$, X is smooth at p if and only if the hypersurface $(c = 0) \subset \mathbb{P}^{n-1}$ is smooth at the image of p under $X \to \mathbb{P}^{n-1}$. Clearly the hypersurface $(c = 0) \subset \mathbb{P}^{n-1}$ is smooth since c is general, and the assertion follows. $\square$

3.B.   Analysis of critical points

We set $\mathcal{L}^\circ = \mathcal{L}|_{Z^\circ}$, where we recall $\mathcal{L} = \mathcal{O}_Z(\nu, 1)$. By Lemma 3.4, Z° is non-singular and by Kollár's result [12, V.5] there exists an invertible sheaf $\mathcal{Q}^\circ$ on Z° such that $\mathcal{M}^\circ := \tau^*\mathcal{Q}^\circ \subset (\Omega_{X^\circ}^{n-1})^{\vee\vee}$, where $\vee\vee$ denotes the double dual. Let $\mathcal{M}$ be the push-forward of the invertible sheaf $\mathcal{M}^\circ$ via the open immersion $X^\circ \hookrightarrow X$. By Lemma 3.5, $\mathcal{M}$ is an invertible sheaf on X.

Definition 3.6. Let V be a nonsingular variety of dimension n defined over an algebraically closed field $\Bbbk$ of characteristic 2, $\mathcal{N}$ an invertible sheaf on V and $s \in H^0(V, \mathcal{N}^2)$ a section. Let $\mathsf{p} \in V$ be a point, ξ a local generator of $\mathcal{N}$ at p and $s = f(x_1, \ldots, x_n)\xi^2$ a local description of s with respect to local coordinates $x_1, \ldots, x_n$ of V at p. We say that s has a *critical point* at p if the linear term of f is zero.

We say that s has an *admissible critical point* at p if for a suitable choice of coordinates $x_1, \ldots, x_n$,

$$f = \begin{cases} \alpha + x_1 x_2 + x_3 x_4 + \cdots + x_{n-1} x_n + g, & \text{if } n \text{ is even,} \\ \alpha + \beta x_1^2 + x_2 x_3 + \cdots + x_{n-1} x_n + g, & \text{if } n \text{ is odd,} \end{cases}$$

where $\alpha, \beta \in \Bbbk$, $g = g(x_1, \ldots, x_n) \in (x_1, \ldots, x_n)^3$ and, in case n is odd, the coefficient of x_1^3 in g is nonzero.

Lemma 3.7. *The section $\bar{z} \in H^0(Z, \mathcal{L}^2)$ has only admissible critical points on Z°.*

Proof. We choose and fix a general $c \in \Bbbk[u]$ so that the hypersurface $(c = 0) \subset \mathbb{P}^{n-1}$ is non-singular. Clearly $\bar{z}$ does not have a critical point on $(c = 0) \subset Z^\circ$. On $Z^\circ \cap (c \neq 0)$, the section c is invertible and thus the section $\bar{z}$ has an admissible critical point if and only if the section

$$s := c(ax^2 + by^2 + fxy) \ (= c^2\bar{z})$$

has an admissible critical point. It is then enough to show that the section s, viewed as a section on $Q = Q_n(\lambda, \mu)$, has only admissible critical points on $U_c = (c \neq 0) \subset Q$ for general a, b and f. We set $U_x = (x \neq 0) \subset Q$ and $\Pi_y = (x = 0) \cap (y \neq 0) \subset Q$ so that $Q = U_x \cup \Pi_y$.

We first show that s does not have a critical point on $\Pi_y \cap U_c$. Let $\mathsf{p} \in \Pi_y \cap U_c$ be a point. We may assume $\mathsf{p} = (1 : 0 : \cdots : 0; 0 : 1)$. We work on the open subset $U_{u_0, y} = (u_0 \neq 0) \cap (y \neq 0) \subset Q$ which is the affine space with coordinates $u_1, \ldots, u_{n-1}$ and x. For $e = e(u_0, \ldots, u_{n-1})$, we set $\tilde{e} = e(1, u_1, \ldots, u_{n-1})$. Moreover we denote by $\tilde{e}_i$ the degree i part of $\tilde{e}$. Then the restriction of s to $U_{u_0, y}$ is $\tilde{c}(\tilde{a}x^2 + \tilde{b} + \tilde{f}x)$ and the point p corresponds to the origin. Then s has a critical point at p if and only if

$$\tilde{c}_0(\tilde{b}_1 + \tilde{f}_0 x) + \tilde{c}_1 \tilde{b}_0 = 0.$$

Note that $\tilde{c}_0 \neq 0$. Since $\deg b \geq 1$, this imposes n independent conditions on a, b, f. Thus, for any point $\mathsf{p} \in \Pi_y$, n conditions are imposed in order for s to have a critical point at p. By counting dimensions we conclude that s does not have a critical point on $\Pi_y \cap U_c$ since $\dim \Pi_y = n - 1$.

Let $\mathsf{p} \in U_x \cap U_c$ be a point. We may assume $\mathsf{p} = (1 : 0 : \cdots : 0; 1 : 0)$. We work on the open subset $U_{u_0, x} = (u_0 \neq 0) \cap (x \neq 0) \subset R$ which is the affine space with coordinates $u_1, \ldots, u_{n-1}$ and y. We have $s|_{U_{u_0, y}} = \tilde{c}(\tilde{a} + \tilde{b}y^2 + \tilde{f}y)$. Let ℓ, q and h be the linear, quadratic and cubic parts of $s|_{U_{u_0, y}}$, respectively. We have

$$\ell = \tilde{c}_0(\tilde{a}_1 + \tilde{f}_0 y) + \tilde{c}_1 \tilde{a}_0.$$

Since $\deg a \geq 1$, n conditions are imposed in order for s to have a critical point at p. It remains to show the existence of a section $s = c(ax^2 + by^2 + fxy)$ which has an admissible critical point at p. Now suppose that s has a critical point at p, that is, $\ell = 0$. This implies that $\tilde{f}_0 = 0$ and $\tilde{a}_1 = \tilde{a}_0 \tilde{c}_1 / \tilde{c}_0$. Then, for the quadratic and cubic parts, we have

$$q = \tilde{c}_0(\tilde{a}_2 + \tilde{b}_0 y^2 + \tilde{f}_1 y) + \frac{\tilde{a}_0 \tilde{c}_1^2}{\tilde{c}_0} + \tilde{c}_2 \tilde{a}_0,$$

$$h = \tilde{c}_0(\tilde{a}_3 + \tilde{b}_1 y^2 + \tilde{f}_2 y) + \cdots.$$

Since $\deg a \geq 3$ and $\deg f \geq 3$, we can choose a, b, f so that

$$q = \begin{cases} yu_1 + u_2u_3 + u_4u_5 + \cdots + u_{n-2}u_{n-1}, & \text{if } n \text{ is even,} \\ yu_1 + u_2u_3 + u_4u_5 + \cdots + u_{n-3}u_{n-2} + u_{n-1}^2, & \text{if } n \text{ is odd.} \end{cases}$$

In case n is even, the section s has a nondegenerate critical point at p and we are done. Suppose that n is odd. Since $\deg a \geq 3$, then we can choose a, b, f so that q is as above and the coefficient of u_{n-1}^3 in h is non-zero. For this choice of a, b, c, f, the section s has an admissible critical point at p and the proof is completed by the dimension counting argument. $\qquad\square$

Proposition 3.8. *Let the notation and assumption as above. Assume in addition that $\nu \geq n$. Then there exists a universally CH_0-trivial resolution $\varphi \colon Y \to X$ of singularities such that $H^0(Y, \Omega_Y^{n-1}) \neq 0$. In particular Y is not universally CH_0-trivial.*

Proof. By [15, Proposition 4.1] or [6], if the singularities of X correspond to admissible critical points of the section $\bar{z}$, then there exists a universally CH_0-trivial resolution $\varphi \colon Y \to X$ such that $\varphi^* \mathcal{M} \hookrightarrow \Omega_Y^{n-1}$ (in fact, φ is just the composite of blowups at each (isolated) singular point). Thus, by Lemma 3.7, X admits such a resolution. The branch divisor $(\bar{z} = 0)$ is clearly reduced and, by [12, Lemma V.5.9], we have an isomorphism

$$\mathcal{M}^\circ \cong \tau^*(\omega_{Z^\circ} \otimes \mathcal{L}^{\circ 2}) \cong \mathcal{O}_{X^\circ}(\nu - n, 0),$$

so that $\mathcal{M} \cong \mathcal{O}_X(\nu - n, 0)$. By the assumption we have $\nu - n \geq 0$, which implies $H^0(X, \mathcal{M}) \neq 0$. Thus $H^0(Y, \Omega_Y^{n-1}) \neq 0$ and by [18, Lemma 2.2], Y is not universally CH_0-trivial. $\qquad\square$

3.C.   Proof of theorems and corollaries

Theorem 3.9. *Suppose that the ground field is $\mathbb{C}$ and let (λ, μ, ν) be a triplet of integers such that $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$. If in addition $\nu \geq n$, then a very general embedded conic bundle $\pi \colon X \to \mathbb{P}^{n-1}$ of type $[\lambda, \mu, \nu]$ is not stably rational.*

Proof. For a field (or more generally a ring) K, we denote by P_K the $\mathbb{P}^2$-bundle $P_n(\lambda, \mu, \nu)$ over $\mathbb{P}^{n-1}$ defined over K. Let $\Bbbk$ be an algebraically closed field of characteristic 2 and let $X \to \mathbb{P}^{n-1}$ be a very general hypersurface in $P_\Bbbk$ defined by an equation of the form (1). We take a mixed characteristic discrete valuation ring A whose residue field is $\Bbbk$, for example the Witt ring, and then we lift X to a hypersurface $\mathcal{X}$ of P_A defined by an equation of the form (1). We choose and fix an embedding of the quotient field of A into $\mathbb{C}$ and set $V = \mathcal{X} \times_A \mathbb{C}$. Then V is a very general hypersurface of $P_\mathbb{C}$ defined by an equation of the form (1). By Proposition 3.8, we can apply the specialization theorem [5, Théorème 1.14] and conclude that V is not universally CH_0-trivial. Note that V is nonsingular by Lemma 3.2. Note also that V is not a very general conic bundle of type $[\lambda, \mu, \nu]$. However a very general conic bundle of type $[\lambda, \mu, \nu]$ degenerates (over a complex curve) to V, hence the assertion follows from the specialization argument [19, Theorem 2.1] (or by [14, Theorem 4.2.10]). $\qquad\square$

Now we can prove the main theorem and corollaries in Section 1.

Proof of Theorem 1.1. The assertion (1) follows from Proposition 2.9.

Let $\pi \colon X \to \mathbb{P}^{n-1}$ be a non-singular embedded conic bundles over $\mathbb{P}^{n-1}$. By Proposition 2.11, we may assume that it is of type $[\lambda, \mu, \nu]$, where either $0 < \lambda \leq \mu \leq \nu \leq \lambda + \mu$ or $\lambda = \mu = \nu = 0$. By adjunction we have $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(n, 1)$. The complete linear system $|\mathcal{O}_P(n, 1)|$, where $P = P_n(\lambda, \mu, \nu)$, is base point free if and only if $n \geq \nu$. This shows that $\mathcal{O}_P(n, 1)$, and hence $\mathcal{O}_X(n, 1)$, is ample if $n < \nu$. Since $-K_X$ is not ample by assumption, we have $n \geq \nu$. Therefore (2) follows from Theorem 3.9. $\qquad\square$

Proof of Corollaries 1.2 and 1.3. Let X be a very general hypersurface of bi-degree $(d, 2)$ in $\mathbb{P}^{n-1} \times \mathbb{P}^2$. Then $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(n - d, 1)$. By assumption $d \geq n$ and this implies that $-K_X$ is not ample. Thus X is not stably rational by Theorem 1.1.

Let X be a double cover of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ branched along a very general divisor of bi-degree $(2d, 2)$. Then X is a very general member of $|\mathcal{O}_P(2d, 2)|$, where $P = P_n(0, 0, d)$, and hence it is of type $[d, d, 2d]$. By the assumption we have $2d \geq n$. Thus X is not stably rational by Theorem 3.9 $\qquad\square$

Proof of Corollary 1.5. Let $\pi \colon X \to \mathbb{P}^{n-1}$ be as in Corollary 1.5. Then we may assume that it is of type $[\lambda, \mu, \nu]$, where $0 < \lambda \leq \mu \leq \nu \leq \lambda + \nu$ or $\lambda = \mu = \nu = 0$. The discriminant divisor Δ is a hypersurface in $\mathbb{P}^{n-1}$ of degree $\lambda + \mu + \nu$. The condition $|3K_{\mathbb{P}^{n-1}} + \Delta| \neq \emptyset$ is equivalent to the condition $\lambda + \mu + \nu \geq 3n$. The latter implies $\nu \geq n$ since $\lambda \leq \mu \leq \nu$. Thus (1) follows from Theorem 3.9.

Now suppose in addition that $n \geq 7$ and $\pi \colon X \to \mathbb{P}^{n-1}$ is standard. Note that X is defined in $P_n(\lambda, \mu, \nu)$ by an equation of the form

$$ax^2 + by^2 + cz^2 + fxy + gxz + hyz = 0,$$

where $a, \ldots, h \in \mathbb{C}[u]$. If $\deg c = \lambda + \mu - \nu > 0$, then the system of equations $a = b = \cdots = h = 0$ has a non-trivial solution on $\mathbb{P}^{n-1}$ since $n \geq 7$. This implies that π cannot be flat, in particular, not standard. Thus $\nu = \lambda + \mu$ and in this case the condition $|2K_{\mathbb{P}^{n-1}} + \Delta| = |\mathcal{O}_{\mathbb{P}^{n-1}}(2(\nu - n))| \neq \emptyset$ is equivalent to $\nu \geq n$ which implies stable non-rationality of X again by Theorem 3.9. This proves (2). $\qquad\square$

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