L_p -EXPANDER COMPLEXES

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ABSTRACT. We discuss two combinatorical ways of generalizing the definition of expander graphs and Ramanujan graphs, to quotients of buildings of higher dimension. The two possible definitions are equivalent for affine buildings, giving the notion of an L_p -expander complex. We calculate explicit spectral gaps on many combinatorical operators, on any L_p -expander complex.

We associate with any complex a natural "zeta function", generalizing the Ihara-Hashimoto zeta function of a finite graph. We generalize a well known theorem of Hashimoto, showing that a complex is Ramanujan if and only if the zeta function satisfies the Riemann hypothesis.

1. Introduction

Expander graphs in general and Ramanujan graphs in particular, have been a topic of great interest in the last four decades. In recent years, a theory of high dimensional expanders has emerged (see [Lub14] and the references therein). Namely, high dimensional simplicial complexes which resemble the properties of expander graphs when specialized to dimension one. Several different notions of high dimensional expanders have been proposed (which in general are not equivalent), each with its own goal and motivation.

The goal of this paper is to propose yet another such a generalization, based on the representation theory of all dimensional Hecke Algebras (to be defined later) associated with buildings. One of the advantages of our approach is that it gives a generalization of expanders and Ramanujan graphs in a unified way. In addition, we deduce that quotients of buildings associated with semisimple groups over local fields with property (T) are indeed high dimensional expanders, as one expects, recalling the classical result of Margulis showing how to get expander graphs from groups with property (T) ([Mar73]).

In [Kam16] we presented an alternative definition of expander graphs. The goal of this work is to generalize it to higher dimensions.

Consider a regular, locally finite thick affine building B, with parameter q. The building B has a corresponding irreducible affine Coxeter group (W,S) of type \tilde{A}_n , where S is a finite set of generators of W. Denote by B_{ϕ} the set of chambers of B, i.e. the highest dimensional faces. Every panel (i.e. a codimension one face) σ is contained in q+1 chambers. The building B is a colored pure simplicial complex, which means that each panel σ has a natural color (or cotype) $t(\sigma) \in S$ and each chamber contains |S| = n+1 panels of different colors. The color or $\tau(\sigma) \subset S$ of a general face σ is the union of the colors of the panels containing it.

The W-metric approach to buildings allows us to define a distance between every two chambers $C_0, C_1 \in B_{\phi}$ by $d(C_0, C_1) \in W$. Each chamber C has $q_w = q^{l(w)}$ chambers C' with $d(C, C') = w \in W$.

Let Γ be a cocompact torsion free lattice in G. In this case the quotient space $X = \Gamma \backslash B$ is a finite colored simplicial complex. Identify its chambers by X_{ϕ} . Let $\pi: B \to X$ be the projection and define for $f \in \mathbb{C}^{X_{\phi}}$ the pullback $\pi^* f \in \mathbb{C}^{B_{\phi}}$. Let C_0 be a fixed chamber of B, and let $\rho_{C_0}: \mathbb{C}^{B_{\phi}} \to \mathbb{C}^{B_{\phi}}$ be the spherical average operator defined by $\rho_{C_0}(f)(C) = \frac{1}{q_{d(C_0,C)}} \sum_{C': d(C_0,C')=d(C_0,C)} f(C')$. Finally, define the non trivial space $L_2^0(X_{\phi}) = \left\{ f \in \mathbb{C}^{X_{\phi}}: \sum_{C \in X_{\phi}} f(C) = 0 \right\}$. We can now define:

Definition 1.1. For $2 \le p \le \infty$ call X an L_p -expander if for every $f \in L_2^0(X_\phi)$ and $C_0 \in B_\phi$, $\rho_{C_0}(\pi^*f) \in L_{p+\epsilon}(B_\phi)$ for every $\epsilon > 0$.

Call X Ramanujan if it is an L_2 -expander.

The definition is equivalent to the fact that every matrix coefficient of every subrepresentation of $L^2(\Gamma \backslash G)$ with Iwahori fixed vector is in $L_{p+\epsilon}(G)$ for every $\epsilon > 0$. Generalizing the p = 2 case, we say that a G-representation satisfying this property is p-tempered.

Let us say right away that the Ramanujan complexes constructed in [LSV05a, LSV05b] using Lafforgue work, are also Ramanujan in our sense (See [Fir16]). However, our definition is aperiori stronger than the one given in [LSV05a] as it requires the $L_{2+\epsilon}$ condition on functions on faces of all colors, in contrast with [LSV05a] where only functions on vertices are considered. This is reflected by the fact that [LSV05b] works only with the classical spherical Hecke algebra, while we consider the Iwahori-Hecke algebra of the all dimensional Hecke algebra (see below). This allows us in turn to analyze higher dimensional faces and not just vertices (compare [EGL15]). The definition is equivalent to strongly-Ramanujan in [Kan16] and flag-Ramanujan in [Fir16].

The definition of p-temperedness is intimately connected to property (T) of reductive groups. The results of Oh in [Oh02] express the quantitative property (T) of reductive groups in the following way:

Theorem 1.2. (Oh [Oh02]) Let k be a non-Archimedean local field with chark $\neq 2$. Let G be the group of k-rational points of a connected linear almost k-simple algebraic group with k-rank ≥ 2 . Then every irreducible infinite dimensional unitary representation of G is p_0 -tempered for some explicit p_0 depending only on the affine Coxeter group W. Explicitly, the bounds are:

ſ	W	\tilde{A}_n	\tilde{B}_n	\tilde{C}_n	\tilde{D}_n , n even	\tilde{D}_n , n odd	\tilde{E}_6	\tilde{E}_7	\tilde{E}_8	\tilde{F}_4	\tilde{G}_2
ſ	p_0	2n	2n	2n	2(n-1)	2n	16	18	29	11	6

As a corollary we get:

Corollary 1.3. Every finite quotient complex X of the building corresponding to G as above is a L_{p_0} -expander.

Let us now relate the above to the representation theory of Hecke algebras. For $w \in W$ we define the operator $h_w : \mathbb{C}^{B_\phi} \to \mathbb{C}^{B_\phi}$ by $h_w f(C) = \sum_{C': d(C,C')=w} f(C')$.

Definition 1.4. The Iwahori-Hecke algebra H_{ϕ} of B is the linear span of all the h_w , $w \in W$.

Those operators satisfy for $w \in W$, $s \in S$ the Iwahori-Hecke relations:

$$h_s^2 = q \cdot Id + (q-1)h_s$$

$$h_w h_s = h_{ws} if l(ws) = l(w) + 1$$

In terms of G, the Iwahori-Hecke algebra can be written as $H_{\phi} \cong C_c[G_{\phi}\backslash G/G_{\phi}]$, where G_{ϕ} (usually denoted I) is the Iwahori subgroup of G and $C_c[G_{\phi}\backslash G/G_{\phi}]$ has the algebra structure given by convolution. The algebra H_{ϕ} can also be defined as the set of all row and column finite (see definition 5.6) linear operators acting on $\mathbb{C}^{B_{\phi}}$ and commuting with the action of G on the space.

The Iwahori-Hecke algebra H_{ϕ} acts naturally on $\mathbb{C}^{X_{\phi}}$, i.e. functions on chambers of the complex. In fact, the standard inner product on $\mathbb{C}^{X_{\phi}} \cong L_2(X_{\phi})$ gives it the structure of a finite dimensional unitary representation of H_{ϕ} (as a *-algebra), and $L_2^0(X_{\phi})$ is a proper subrepresentation.

Definition 1.5. We say that a finite dimensional representation V of H_{ϕ} is p-tempered if for every $v \in V$, $u \in V^*$ and $\epsilon > 0$ we have $\sum_{w \in W} q^{l(w)(1-p-\epsilon)} \left| \langle u, h_w v \rangle \right|^{p+\epsilon} < \infty$.

An easy calculation shows that definition 1.1 is actually equivalent to the p-temperedness of the H_{ϕ} representation $L_2^0(X_{\phi})$.

There is an alternative possible definition of an L_p -expander. It is based on the following:

Definition 1.6. Assume V is a finite dimensional representations of H_{ϕ} . We say that V is weakly contained in $L_p(B_{\phi})$ if for each operator $h \in H_{\phi}$ and eigenvalue λ of h on V, λ belongs to the approximate point spectrum of h on $L_p(B_{\phi})$.

The following is the main theorem of this work. It says that the two possible definitions are equivalent in the case we consider (Compare [CHH88], for p = 2 only).

Theorem 1.7. A finite dimensional representation V of H_{ϕ} is p-tempered if and only if it is weakly contained in $L_p(B_{\phi})$.

Therefore X is an L_p -expander if and only if the H_{ϕ} -representation $L_2^0(X_{\phi})$ is weakly contained in $L_p(B_{\phi})$.

In theorem 1.7, the Iwahori-Hecke algebra H_{ϕ} can be extended to a larger algebra , which we call the *all dimensional Hecke (ADH) algebra H*. The simplest way to define it is:

Definition 1.8. Identify B with the set of all its faces. The ADH algebra H of B is the algebra of all row and columns finite linear operators acting on \mathbb{C}^{B_f} and commuting with the action of G.

The algebra H contains many interesting operators, such as boundary and coboundary operators, Laplacians and adjacency operators. It is described explicitly in section 4 (see also [APVM15]). Shortly, distances between general faces are parameterized by $d \in W_{I_1} \backslash W/W_{I_2}$, for $I_1, I_2 \subsetneq S$, where $W_{I_1}, W_{I_2} \subset W$ are the corresponding parabolic subgroups. The algebra H is spanned by operators h_d , for $d \in W_{I_1} \backslash W/W_{I_2}$.

Explicit bounds on the operators of H can be given by the following theorem. Notice that we have a length function $l:W\to\mathbb{N}$. Each operator $h_w\in H_\phi$ sums $q^{l(w)}$ different chambers of B_ϕ and therefore $q^{l(w)}$ is its trivial eigenvalue. Now:

Theorem 1.9. The norm of $h_w \in H_\phi$ is bounded on $L_p(B_\phi)$ by $D(w, l(w))q^{l(w)(p-1)/p}$, where $D(q, l) = |W_0| 2^{l(\tilde{w}_0)}q^{4\cdot l(\tilde{w}_0)} (l+1+l(\tilde{w}_0))^{l(\tilde{w}_0)}$, W_0 is the spherical Coxeter group corresponding to W and \tilde{w}_0 is the longest element of W_0 .

Therefore the same bound applies to the action of $h_w \in H_\phi$ on $L_2^0(X_\phi)$.

In conjugation with Oh's theorem 1.2 this theorem gives an explicit spectral gap of the operators $h_w \in H_{\phi}$, in any quotient of the building.

There exists a direct application of the last theorem . We measure distance between chambers in a quotient complex X by $gallery\ distance$, i.e the length of the shortest gallery connecting the two chambers.

Theorem 1.10. Let X be an L_p -expander of with N chambers and $C_0 \in X_{\phi}$. Let n be the dimension of X and \tilde{w}_0 is the longest element of the spherical Coxeter group W_0 . Then all but o(N) chambers $C \in X_{\phi}$ are of gallery distance $l(C_0, C)$ which satisfies

$$l(C_0, C) \le \frac{p}{2} \log_q N + (l(\tilde{w}_0) + 1) \log_q \log_q N + 1$$

and

$$l(C_0, C) \ge \log_q N - (n+1)\log_q \log N_q - 1$$

In addition, the diameter of X is at most $p \log_q N + 2(l(\tilde{w}_0) + 1) \log_q \log_q N + 1$.

Compare the graph case in [LP15] corollary 2, or [Sar15].

As a final result, recall that for a graph, the expander property is connected to the eigenvalues of both the vertex adjacency operator and of Hashimoto's non backtracking operator. For the high dimensional case, we can give a generalization of the non-backtracking operator. As a preliminary, one can extend the Iwahori-Hecke algebra H_{ϕ} to an extended Iwahori-Hecke algebra \hat{H}_{ϕ} , given by operators $h_w, w \in \hat{W}$, the extended Iwahori-Hecke algebra. The operators of \hat{H}_{ϕ} act naturally on function on "colored" chambers of B- $\mathbb{C}^{\hat{B}_{\phi}} = \mathbb{C}^{B_{\phi} \times \Omega}$. Within \hat{H}_{ϕ} we have n Bernstein-Luzstig operators $h_{\beta_1}, ..., h_{\beta_n}$, corresponding to the simple coweights $\beta_1, ..., \beta_n$ of the root system of \hat{W} . The one dimensional case agrees with Hashimoto's non-backtracking operator. Now:

Theorem 1.11. Let V be a finite dimensional representation of \hat{H}_{ϕ} . Then V is p-tempered if and only if for every i = 1, ..., n every eigenvalue λ of h_{β_i} on V satisfies $|\lambda| \leq q^{l(\beta_i)(p-1)/p}$.

The theorem encourages the following definition:

Definition 1.12. Consider the \hat{H}_{ϕ} -representation $L_2\left(\hat{B}_{\phi}\right)$. Let

$$\zeta_{\hat{B}_{\phi}}(u) = \frac{1}{\det(1 - h_{\beta_1} u^{l(\beta_1)}) \cdot \dots \cdot \det(1 - h_{\beta_n} u^{l(\beta_n)})}$$

Corollary 1.13. The complex X is an L_p -expander if and only if every pole λ of $\zeta_{\hat{B}_{\phi}}(u)$ satisfies $|\lambda| \leq q^{(p-1)/p}$ or $|\lambda| = q$.

The theorem is a generalization of the well known connection between the expander property and the graph Zeta function, i.e. a q+1 regular graph is an L_p -expander (in the notions of [Kam16]), if and only if every non trivial eigenvalue λ of Hashimoto's non backtracking operator, satisfies $|\lambda| \leq q^{(p-1)/p}$. See [Kam16], theorem 10.1.

Related Works. Expander graphs are classical and we will not discuss their history here. There are various works on how to extend the theory to high dimensions, and in particular on how to extend the definition of a Ramanujan graph to the definition of a Ramanujan complex, a quotient of an affine building of type \tilde{A}_n . All the different works are motivated (implicitly or explicitly) by the notion of a tempered representation of a reductive group.

The extension to "cubical complexes", i.e. quotient of buildings of type $W = \tilde{A}_1 \times ... \times \tilde{A}_1$ was considered in [JL99]. This case requires considering the adjacency operator for each summand separately.

Based on previous works on the geometry of A_n buildings ([Car99]), in [CSZ03], it was suggested to study the representation theory of the spherical Hecke algebra acting on functions on the vertices of the complex. The definition was slightly changed in [LSV05b], definition 1.1, so it was equivalent to the fact that every spherical non trivial subrepresentation of $L_2(\Gamma \setminus G)$ is tempered. An Alon-Boppana type theorem was proved in [Li04].

Following Laffourge's work Ramanujan complexes were constructed in [Li04, LSV05a, LSV05b, Sar07], satisfying the above definition.

The action of the Iwahori-Hecke algebra on functions on chambers of the building and its quotients is classical, and was considered in the seminal work of Borel ([Bor76]), from an algebraic group point of view. A combinatorical theory, applied to any locally finite regular building, appeared in [Par06]. Recently, the construction of the all dimensional Hecke algebra H appeared in [APVM15].

An approach to high-dimensional expanders is given in [Fir16], and is similar in spirit to definition 1.5. The approach there is slightly more general, dealing with arbitrary simplicial complexes, but focuses on the Ramanujan case only (i.e. p = 2), and does not contain the explicit results for affine buildings.

Philosophy and Context of the Work. Most of this work deals with general locally finite regular buildings (see section 2), generally without the assumption of the existence of an automorphism group G. We allow buildings with arbitrary parameter system $\vec{q} = (q_s)_{s \in S}$, not just a single parameter q (the introduction is stated with a single q for simplicity). In particular, the only if case of theorem 1.7 holds for any locally finite regular building, affine or not (see corollary 18.4), although some change is required in definition 1.1 to deal with the thin case (see definition 12.3 and lemma 12.8). The if case of theorem 1.7 holds for any affine building, and is actually based on theorem 1.9.

In the affine case, a well known theorem of Tits shows that in thick irreducible affine locally finite regular buildings of dimensions greater than 2 always comes from algebraic Lie group over a non-Archimedean local field. However, this is not the case in dimensions 1 and 2. Since our proofs are combinatorical, theorem 1.9 applies to any affine building, while theorem 1.11 actually applies even more generally to arbitrary affine Iwahori-Hecke algebras, with parameter systems $q_s > 1$.

Note, however, that some of the theorems we cite, most notably Oh's theorem 1.2, are only known for algebraic Lie groups over a non-Archimedean local field. We call this case shortly the algebraic-group case.

Structure of the Work. We divided this work into 4 parts. Very generally, in part I we present the alldimensional Hecke algebra H and some of its basic properties. Part II is devoted to the basic representation theory of H, and in particular to p-tempered representations and L_p -expanders. Part III is devoted to the spectrum of operators of H. While the first three parts do not assume in general that W is affine, part IV contains the specific results to the affine case, which are the main results of this work. Part I- The Hecke Algebra. In section 2 we present the W-metric approach to buildings. It is standard and used mainly to set notations for the rest of the work. In section 3 we discuss distances in buildings between two faces. This topic also appears in [AB08]. In section 4 we define the algebra H and describe it explicitly. The main result here is proposition 4.9 showing that H is indeed an algebra. Similar description of the algebra was given in [APVM15], although we follow a slightly different approach We also deduce that H commutes with spherical average operators, such as ρ_{C_0} in the introduction.

In section 5 and section 5 we relate H to a sufficiently transitive (i.e Weyl transitive complete) automorphism group G of the building. We also show that H can be defined, as in the introduction, as the algebra of row and column finite operators acting on \mathbb{C}^B and commuting with the natural G-action on this space (see proposition 5.7, [Kam16] proposition 2.3, and compare the approach in [Fir16]). We prove that the Iwahori-Hecke algebra H_{ϕ} is isomorphic to the Hecke algebra $H_{G_{\phi}}(G)$ of G with respect to the Iwahori subgroup G_{ϕ} (see 6.7. In the algebraic-group case this claim appeared back in [Bor76]). The extension to H is straightforward and is given in proposition 6.9.

Part III- Basic Representation Theory. In section 9 we discuss unitary representations of the algebras H and H_{ϕ} , which is rather standard *-algebras subject. In section 10 we prove that there is a strong bijection between isomorphism classes of irreducible representations of the two algebras (proposition 10.6, similar result also appears in [Fir16], proposition 4.30).

In section 11 we show that matrix coefficients allow us to consider every H-representation as a subrepresentation of the action of H on \mathbb{C}^{B_f} . This is analog to the standard matrix coefficients argument which enables us to see each group representation as a subrepresentation of the action of G on \mathbb{C}^G . Matrix coefficients lead us to study tempered representations in section 12. Our definition of p-tempered or "almost L_p ", definition 12.3, is a little different then the standard $L_{p+\epsilon}$, but is equivalent in the algebraic-group case. This is done to handle the amenable case, which happens if the building is a single apartment. See also the corresponding definition for a (compactly generated) group in section 14. The definition of temperedness allows us to give the definition of an expander complex (definition 13.5).

In section 14 we discuss the connection between the representations of G and of H. A basic bijection between the right equivalent classes in simple and well known (proposition 14.12). However, it seems to be unknown in general if a finite dimensional (respectively unitary) representation of H_{ϕ} induces to an admissible (resp. unitary) representation of G. In theorem 14.13 we cite two strong results of Borel ([Bor76]) and Barbasch and Moy ([BM96]) showing the answer is yes in the algebraic-group case. Oh's theorem 1.2 is discussed in section 15. As said above, we were unable to translate the proof of this theorem into the methodology of this work. Therefore the theorem is only cited under the algebraic-group assumption.

Part III- Spectrum of Operators. We present and discuss our definition of weak containment in section 17. Notice that our definition also covers non-unitary representations, which is not standard. A more complete treatment of weak containment, on the unitary case only, is given in [Fir16]. In section 18 we prove the "only if" part of theorem 1.7, which is rather abstract, works in a general settings and is analog to [CHH88], theorem 1.

We then move to prove two generalizations of the "Alon-Boppana theorem". In graphs there are two similar results connecting the spectrum of the adjacency operator A on a q+1 regular graph and on the q+1 regular tree. The first (sometimes called Serre's theorem) shows that as the injectivity radius of the graph grows the spectrum of A dense in $[-2\sqrt{q},2\sqrt{q}]$, and actually converges in distribution to the spectral distribution of A on the q+1 regular tree (see [McK81]). The density part of this theorem was generalized in [Li04], see also [Fir16], theorem 5.1. We present another version of this theorem in theorem 19.2. The classical Alon-Boppana theorem itself assume only that the graph is connected and is large enough, and concerns only the largest eigenvalue (sometimes is absolute value) of A. We prove a generalization of this theorem in theorem 20.2.

Part IV- The affine case. Before discussing the affine case we show how to extend the theory to color rotating automorphisms in section 21. Since it adds some confusion we did not start with it, but it is essential to the affine case since it allows working with the extended Iwahori-Hecke algebra. This algebra acts naturally

on "recolored" chambers of the building, a subject not treated usually in works about buildings and Hecke algebras.

In section 22 we discuss root systems and their connection to affine Coxeter groups. Most of the results are standard and presented without proof. Theorem 22.1 is a structure theorem of affine Coxeter group, which is of its own interest. Similar result appears in [GSS12].

In section 23 we discuss temperedness in the affine case. Using the polynomial growth of W in this case we give a couple of different equivalent conditions for temperedness in proposition 23.6. Then we use theorem 22.1 to prove theorem 1.11. We also explain the connection of the results to the generalized Poincare series of \hat{W} , a notion from [Gyo83] and [Hof03].

Sections 24-28 are devoted to theorem 1.9. Section 24 contains some consequences of this theorem. First, it derives the if part of theorem 1.7. Secondly, we discuss some versions of the Kunze-Stein theorem. In section 25 we prove theorem 1.10 using theorem 1.9.

We then turn to the proof of the theorem itself, which is based on [CHH88], theorem 2. Section 26 is devoted to the connection between the well known Bernstein presentation of the Iwahori-Hecke algebra and the building construction known as sectorial retraction. Both ideas are versions of the Iwasawa decomposition used in [CHH88]. The Bernstein presentation allows us to write every operator as a sum of "sectorial operators". We then show how to bound sectorial operators in section 28, thus proving theorem 24 using some bounds provided by the Bernstein presentation.

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Part 1. The Hecke Algebra

2. Buildings

This section discusses the definition and basic properties of buildings. We will follow the W-metric approach to buildings (as in Ronan's book [Ron09]).

Simplicial Complexes. A simplicial complex is (B, V), V some set, $B \subset P(V)$, such that if $\phi \neq \sigma_1 \subset \sigma_2 \in B$ then $\sigma_1 \in B$. The elements of B are called faces. If a face is a subset of another face we say that the first face is contained in the second. The dimension of a face is the number of elements it has minus 1. We always assume dimensions are finite.

A face is called maximal if it is not a proper subset of another face. We say that a simplicial complex is $pure\ of\ dimension\ n$ if all its maximal faces have the same dimension n. Maximal faces in a pure simplicial complex are called chambers. Faces of dimension n-1 are called panels. Faces of dimension 0 are called vertices.

A face that is contained in a finite number of chambers is called *spherical*. The complex B is called *vertex* spherical if every vertex (and hence every face) is contained in a finite number of faces (such a simplicial complex is sometimes called locally finite, but we reserve this term for a locally finite building. See below). Two chambers C_1, C_2 are adjacent if they contain a common panel. A pure simplicial complex is connected if the equivalence relation on chambers generated by adjacency has a single equivalence class.

We say that a pure simplicial complex of dimension n is colored if each panel is colored by a singleton $\{i\} \subset [n] = \{0, ..., n\}$, such that if two panels belong to the same chamber they have different colors. The color (sometimes called cotype in building theory) $t(\sigma)$ of a face σ is the union of the colors of the panels containing it. It is a subset of $[n] = \{0, ..., n\}$ and the color of a chamber is the empty set. We usually denote a color by $I \subset [n]$. We denote the faces of color $I \subset [n]$ by B_I . For example, the set of chambers is B_{ϕ} .

Two adjacent chambers C_1, C_2 in a simplicial complex are called *j-adjacent* for $j \in [n]$ if they share a panel of color $\{j\}$.

A color preserving isomorphism (or simply an isomorphism) between two colored complexes $f: \Sigma_1 \to \Sigma_2$ is a bijection from the faces of Σ_1 to the faces of Σ_2 that preserves colors and containment of faces. In particular a (color preserving) automorphism is an isomorphism from a colored complex to itself.

A color rotating isomorphism $f: \Sigma_1 \to \Sigma_2$ is a bijection that preserve containment and such that there exists a bijection $\tau: [n] \to [n]$ satisfying that the color of $f(\sigma)$ is $\tau(t(\sigma))$.

Coxeter Groups. A Coxeter group (W, S) is given by a group W and a finite set of generators $S = \{s_0, ..., s_n\}$, such that W is the group defined by the relations

$$W = \langle s_i, i \in [n] | s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1 \rangle$$

Whenever we write W in this work we will implicitly assume we also have a fixed set of generators S. We will always assume S has n+1 elements $s_0, ..., s_n$, so to each element $s_i \in S$ corresponds a color $i \in [n]$. We generally identify S with [n]. Therefore by abuse of notation we may relate to $I \subset [n]$ as $I \subset S$.

The parameters $m_{i,j}$, $i, j \in [n]$, $m_{i,i} = 1$ are called the *Coxeter numbers* of the group. For every $I \subset [n]$ we define a subgroup $W_I = \langle s_i : i \in I \rangle$. Such a subgroup is called *a parabolic subgroup* and it is also a Coxeter group.

A Coxeter group, or a parabolic subgroup W_I , is called *spherical* if it is finite, and in the parabolic subgroup case we also say that the color I is spherical. If every $I \subsetneq [n]$ is spherical, we say that the Coxeter group W is *vertex spherical*.

There exists a length function $l: W \to \mathbb{N}$. The length l(w) of $w \in W$ is the length of the shortest word in the generators s_i expressing w.

A Coxeter group is called *irreducible affine* if the following conditions hold: 1.W is an infinite subgroup of the isomorphism group of a euclidean vector space V, generated by affine reflections. 2. V has no nontrivial W invariant subspace. 3. W is discrete, i.e. the number of $w \in W$ fixing some point $p \in V$ is finite. If W is irreducible affine then W is vertex spherical.

A Coxeter group is called *affine* if it is a finite direct sum of irreducible affine Coxeter groups. For classifications of spherical and affine Coxeter groups, also called Weyl groups, see [Ron09].

Proposition 2.1. A Coxeter group (W,S) with |S| = n + 1 has a geometric realization as a connected colored simplicial complex of dimension n- the Coxeter complex W. The chambers of W correspond to elements $w \in W$. Two chambers w, w' are adjacent if $w = w's_i$ and in this case the color of the common panel is i. Faces of color $I \subseteq S$ correspond to cosets wW_I .

Proof. See [Ron09], p.10.
$$\Box$$

Notice that a face of type I of \mathbb{W} is spherical if and only if I is a spherical color. In particular, W is vertex spherical (as a Coxeter group) if and only if the Coxeter complex \mathbb{W} is vertex spherical (as a simplicial complex).

Buildings. Fix a Coxeter group (W, S). We may identify the free monoid on n+1 elements with $\{id\} \cup_{m\geq 1} [n]^m$. If |S| = n+1 there exists a unique projection $p: \{id\} \cup_{m\geq 1} [n]^m \to W$ sending $i \to s_i$.

Definition 2.2. Let B be a colored simplicial complex of dimension n. A gallery \mathbb{G} is a finite sequence of chambers $\mathbb{G} = (C_0, C_1, ..., C_m)$, such that C_i, C_{i+1} are adjacent and $C_i \neq C_{i+1}$

For every gallery \mathbb{G} we define the color $t(\mathbb{G}) = (\alpha_0, ..., \alpha_{m-1}) \in [n]^m$ when C_i, C_{i+1} are α_i -adjacent. The Coxeter color of the gallery \mathbb{G} is $t_W(\mathbb{G}) = p(t(\mathbb{G})) \in W$.

Definition 2.3. (See [Ron09] chapter 3) A building (B, W, d) is given by:

- 1. A connected colored simplicial complex B in which each panels belongs to at least 2 chambers.
- 2. A Coxeter group W with n+1 generators S.
- 3. A distance function $d: B_{\phi} \times B_{\phi} \to W$.

Such that for every gallery of minimal length $\mathbb G$ between C,C', the distance $d(C,C')\in W$ equals $t_W(\mathbb G)\in W$

Example 2.4. The Coxeter complex \mathbb{W} is a building, with the distance function $d(w, w') = w^{-1}w'$.

If every panel in B belongs to exactly 2 chambers we say that B is thin. If every panel in B belongs to at least 3 chambers we say that B is thick.

If every panel in B belongs to a finite number of chambers we say that B is locally finite. If B is locally finite and W is vertex spherical (as a Coxeter group) then B is also vertex spherical (as a Coxeter group).

A building is called *locally finite regular* if every chamber C has a constant number $q_i < \infty$ of adjacent chambers of type i where q_i does not depend on C. The numbers $\overrightarrow{q} = (q_i)_{i \in [n]}$ are called the *parameter system* of the building. We also write $q_i = q_{s_i}$ if s_i is the i-th element of S (using the identification of S and [n]).

Example 2.5. Let T be a tree (i.e. a graph without cycles) with no leaves. Color its vertices with 0 and 1 such that each edge contains a vertex of each color. The tree T is an affine building with Coxeter group $\tilde{A}_1 = D_{\infty} = \langle s_0, s_1 : s_0^2 = s_1^2 = 1 \rangle$. If each vertex is contained in a finite number of edges it is locally finite and vertex spherical. If each vertex of type i is contained in q_i edges (i.e. T is a biregular graph) then it is a locally finite regular building.

From now on we assume the building is locally finite regular with parameter system \overrightarrow{q} .

Example 2.6. The Coxeter complex \mathbb{W} is a thin building and every thin building is isomorphic to \mathbb{W} . The Coxeter complex is always a locally finite building, even if it is not a locally finite simplicial complex.

An apartment A in a building B is a colored subcomplex that is isomorphic to the Coxeter complex.

Lemma 2.7. Every two chambers belong to an apartment of the building.

We are mainly interested in buildings that are vertex spherical. However, some interesting examples are not vertex spherical:

Example 2.8. If W is an irreducible affine Coxeter group then every color $I \subsetneq [n]$ is spherical, the Coxeter group W is vertex spherical and the Coxeter complex has a structure of a locally finite simplicial complex. However, since a general affine Coxeter group is a finite direct sum of irreducible affine Coxeter groups, the Coxeter complexW is not a locally finite simplicial complex. This complex W can also be considered as a a locally finite polysimplicial complex in which each vertex is contained in a finite number of chambers. The two points of views are equivalent.

Consider for example $W = \tilde{A}_1 \times \tilde{A}_1$. It is the Coxeter group with 4 generators- s_0, s_1, s'_0, s'_1 and relations

$$s_0^2 = s_1^2 = s_0'^2 = s_1'^2 = (s_0 s_0')^2 = (s_0 s_1')^2 = (s_1 s_0')^2 = (s_1 s_1')^2 = 1$$

The corresponding Coxeter complex \mathbb{W} can be considered as a "cube complex" which is a product of two trees, with squares (of color ϕ), edges (of color s_0, s_1, s_0', s_1') and vertices (of colors $\{s_0, s_0'\}, \{s_0, s_1'\}, \{s_1, s_0'\}, \{s_1, s_1'\}\}$ as faces. It can also be considered as a simplicial complex with chambers of dimension 3, in which only the colors $\phi, s_0, s_1, s_0', s_1', \{s_0, s_0'\}, \{s_0, s_1'\}, \{s_1, s_0'\}, \{s_1, s_1'\}$ are spherical. As said, both views are equivalent and we use the simplicial one in this work. See [JL99] for an expander theory for cube complexes.

In terms of semisimple algebraic groups, an almost simple group over a non-Archimedean local field (e.g $SL_n(Q_p)$) has an irreducible affine Weyl group and acts as as automorphism group on a vertex spherical building. A product of two almost simple groups (e.g. $SL_n(Q_p) \times SL_m(Q_{p'})$) acts on a "polysimplicial" building- a non vertex spherical building which is the product of the two buildings.

3. Distances in Buildings

Definition 3.1. Let $\sigma_1 \in B_{I_1}, \sigma_2 \in B_{I_2}$ be two faces in B. Choose $\sigma_1 \subset C_1, \sigma_2 \subset C_2$ with minimal distance between them and define $\tilde{d}(\sigma_1, \sigma_2) = d(C_1, C_2) \in W$. The distance $d(\sigma_1, \sigma_2) \in W_{I_1} \setminus W/W_{I_2}$ is the projection of $\tilde{d}(\sigma_1, \sigma_2)$ from W to $W_{I_1} \setminus W/W_{I_2}$.

One should prove it is well defined. The picture is explained in details in [AB08], section 5.3.2 and we base our discussion on it. Let us start with Coxeter groups. The following lemma is well known:

Lemma 3.2. 1. Let $I \subset S$ be fixed. Each coset $d = wW_I \in W/W_I$ has a unique shortest element \tilde{d} and similarly in $W_I \backslash W$.

Write $W^I \subset W$ for the set of shortest elements in the cosets W/W_I . Similarly, write ${}^IW \subset W$ for the set of shortest elements in the cosets $W_I \setminus W$.

2. Every $w \in W$ can be written uniquely as $w = w^I w_I$, $w^I \in W^I$, $w_I \in W_I$ and in this case $l(w) = l(w^I) + l(w_I)$.

Proof. See [Hum92] 5.12. \Box

There exists a similar statement for double cosets of Coxeter groups which is a refinement of the lemma above. It is less standard and we are mainly interested in the first statement of the following lemma. Notice that if $I_1 \subset I_2$ then W_{I_2} is a parabolic subgroup of W_{I_1} . Therefore, $I_2(W_{I_1})$ and $(W_{I_1})^{I_2}$ are well defined.

Lemma 3.3. Let $I_1, I_2 \subsetneq S$ be fixed. Then each double coset $d = W_{I_1} w W_{I_2} \in W_{I_1} \backslash W / W_{I_2}$ has a unique shortest element $\tilde{d} \in W$ with $d = W_{I_1} \tilde{d} W_{I_2}$. Write $^{I_1} W^{I_2}$ for the set of such shortest elements.

Let $I_3 = I_1 \cap \tilde{d}I_2\tilde{d}^{-1} \subset I_1$, $I_4 = I_2 \cap \tilde{d}^{-1}I_1\tilde{d} \subset I_2$ (multiplication takes place in W, as $S \subset W$). We have a bijection \sim_d : $W_{I_3} \leftrightarrow W_{I_4}$, given by $w_3 \sim_d w_4$, $w_3 \in W_{I_3}$, $w_4 \in W_{I_4}$ if $w_3\tilde{d} = \tilde{d}w_4$. Every element $w \in W$ with $W_{I_1}wW_{I_2} = W_{I_1}\tilde{d}W_{I_2}$ can be decomposed in $|W_{I_3}| = |W_{I_4}|$ ways as $w = w_1w_3\tilde{d}w_4w_2$, with: $w_1 \in (W_{I_1})^{I_3}$, $w_3 \in W_{I_3}$, $\tilde{d} \in {}^{I_1}W^{I_2}$, $w_4 \in W_{I_4}$, $w_2 \in {}^{I_4}(W_{I_2})$ and in this case $l(w) = l(w_1) + l(w_3) + l(\tilde{d}) + l(w_4) + l(w_2)$. All the different decompositions are given by $w_3 \to w_3\hat{w}_3$, $w_4 \to \hat{w}_4w_4$ for $\hat{w}_3 \sim_d \hat{w}_4$.

Proof. See [AB08], section 2.3.2, proposition 2.23.

Let us return to buildings. The following proposition is a generalization of lemma 3.3 to buildings.

Proposition 3.4. 1. Let $p: W \to W_{I_1} \setminus W/W_{I_2}$ be the projection. Then $d(\sigma_1, \sigma_2) = p(d(C_1, C_2))$ does not depend on the chambers $\sigma_1 \subset C_1 \in B_{\phi}, \sigma_2 \subset C_2 \in B_{\phi}$.

- 2. The unique shortest representative \tilde{d} of $d = d(\sigma_1, \sigma_2)$ is the shortest distance between two chambers containing the faces.
- 3. Let $I_3 = I_1 \cap \tilde{d}I_2\tilde{d}^{-1}$, $I_4 = I_2 \cap \tilde{d}^{-1}I_1\tilde{d}$. There exists a face σ_3 of color I_3 containing σ_1 and a face of color σ_4 containing σ_2 such that:
 - 3.a. Every two chambers $\sigma_1 \subset C_1, \sigma_2 \subset C_2$ with $d(C_1, C_2) = \tilde{d}$ contain σ_3, σ_4 respectively.
- 3.b. There exists a bijection $F: C_{\sigma_3} \to C_{\sigma_4}$ between the faces containing σ_3 and the faces containing σ_4 such that $d(C, F(C)) = \tilde{d}$.

Proof. See [AB08], section 5.3.2. \Box

Notice that the distance we defined is not symmetric. However, for $C_1, C_2 \in B_{\phi}$ if $d(C_1, C_2) = w$ then $d(C_2, C_1) = w^{-1}$.

Definition 3.5. for $d = W_{I_1} w W_{I_2} \in W_{I_1} \setminus W / W_{I_2}$ we define $d^* = W_{I_2} w^{-1} W_{I_1} \in W_{I_2} \setminus W / W_{I_1}$.

Proposition 3.6. If $d(\sigma_1, \sigma_2) = W_{I_1} w W_{I_2} = d$ then $d(\sigma_2, \sigma_1) = W_{I_2} w^{-1} W_{I_1} = d^*$.

Proof. A minimal gallery from C_2 to C_1 is the reverse of a minimal gallery from C_1 to C_2 . Now use the definition of distance using minimal galleries.

Recall that we we assume the building is locally finite regular with parameter system $\overrightarrow{q} = (q_i)_{i \in [n]}$.

Definition 3.7. Assume that I_1 is spherical. The number of faces of distance $d \in W_{I_1} \backslash W / W_{I_2}$ from a face $\sigma \in B_{I_1}$ is denoted by q_d .

Our next goal is to prove that the definition of q_d does not depend on σ and to calculate q_d explicitly. It will be done in proposition 3.11.

Note that if I_1 is not spherical it is contained in an infinite number of chambers, so q_d is usually ∞ .

Definition 3.8. For a finite subset $A \subset W$, denote $q_A = \sum_{w \in A} q_w$. In particular $q_{W_I} = \sum_{w \in W_I} q_w$

Proposition 3.9. The number of chambers C' of distance $w \in W$ from a chamber C depends only on w. We denote it by q_w . If a minimal decomposition is $w = s_{\alpha_1} \cdot ... \cdot s_{\alpha_l}$ then $q_w = q_{\alpha_1} \cdot ... \cdot q_{\alpha_l}$.

Proof. If w = w's with l(w) = l(w') + l(s) and d(C, C') = w = w's then there exists a single chamber C'' such that d(C, C'') = w', d(C'', C') = s (it is standard-follows from 3.4 for example).

On the other hand, if d(C, C'') = w', d(C'', C') = s then d(C, C') = w = w's. Therefore the number of such C' is the number of pairs C'', C' such that d(C, C'') = w', d(C'', C') = s. So inductively we have $q_w = q_{w'}q_s$.

Proposition 3.10. Let $d = \tilde{d}W_{I_2} \in W/W_{I_2}$ (recall that \tilde{d} is the shortest element in the coset). Then $q_d = q_{\tilde{d}}$.

Proof. By 3.4, every face of distance d from C has a single chamber C_2 of distance \tilde{d} . On the other hand every chamber of distance \tilde{d} has a single face of color I_2 . The claim follows.

Proposition 3.11. Assume I_1 is spherical. Let $d = W_{I_1} \tilde{d} W_{I_2}$. Let $I_3 = I_1 \cap \tilde{d} I_2 \tilde{d}^{-1}$, $I_4 = I_2 \cap \tilde{d}^{-1} I_1 \tilde{d}$. Then $q_d = q_{W_{I_1}}/q_{W_{I_3}} \cdot q_{\tilde{d}} = q_{(W_{I_1})^{I_3}} q_{\tilde{d}}$.

Proof. Let $\sigma_1 \in B_{I_1}$ be a face of color I_1 . Look at the pairs (C_1, σ_2) where C_1 is a chamber, $\sigma_1 \in C_1$ and σ_2 is of distance $\tilde{d}W_{I_2}$ from C_1 . By the last claim the number of such pairs is $\left(\sum_{w \in W_{I_1}} q_w\right) q_{\tilde{d}} = q_{W_{I_1}} q_{\tilde{d}}$. From 3.4 every face σ_3 is counted $q_{W_{I_3}}$ times and the first equality follows.

Finally, the decomposition $W_{I_1} = (W_{I_1})^{I_3} W_{I_3}$, its uniqueness, and the fact that it agrees with lengths of elements give

$$q_{W_{I_1}} = \left(\sum_{w \in W_{I_1}} q_w\right) = \left(\sum_{w \in (W_{I_1})^{I_3}} q_w\right) \left(\sum_{w \in W_{I_3}} q_w\right) = q_{(W_{I_1})^{I_3}} q_{W_{I_3}}$$

Definition 3.12. Using the notations of 3.11, denote $n_d = q_{W_{I_3}}$.

4. The All Dimensional Hecke Algebra

We now define our algebra. We assume that all colors used below are spherical. It is useful to let our algebra work on all spherical faces simultaneously.

Definition 4.1. From now on identify $B_f = \bigcup_{I:I \ spherical} B_I$.

Definition 4.2. For $d \in W_{I_1} \backslash W / W_{I_2}$ we define the operator $h_d : \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ by

$$h_d(f)(\sigma_1) = \begin{cases} \sum_{\sigma_2: d(\sigma_1, \sigma_2) = d} f(\sigma_2) & \sigma_1 \text{ of color } I_1 \\ 0 & \sigma_1 \text{ not of color } I_1 \end{cases}$$

Remark 4.3. Notice that we assume here that the number of faces at distance d is finite. This is a result of the regularity of the building.

Definition 4.4. The all dimensional Hecke (ADH) algebra H is

$$H = span \{h_d : d \in W_{I_1} \backslash W / W_{I_2}, I_1, I_2 \text{ spherical}\}$$

The linear span of all the h_d , $d \in W_{I_1} \setminus W/W_{I_2}$ is denoted H_{I_1,I_2} . We also write $H_I = H_{I,I}$.

We identify H_I and H_{I_1,I_2} with their natural embedding in H.

We should prove that our Hecke algebra is indeed an algebra. Composition of operators can be used to define multiplication $H_{I_1,I_2} \times H_{I_2,I_3} \to Hom_{\mathbb{C}}(\mathbb{C}^{B_{I_3}},\mathbb{C}^{B_{I_1}})$. However, it is not so obvious why the result is in H_{I_1,I_3} and what it is. Let us start with a simple claim:

Proposition 4.5. The algebra H_{ϕ} it is isomorphic to the abstract Iwahori-Hecke algebra of W - the algebra generated by $h_s, s \in S$ with the Iwahori-Hecke relations:

$$h_w h_s = h_{ws}$$
 if $l(ws) = l(w) + 1$
 $h_s^2 = q_s \cdot Id + (q_s - 1)h_s$

Proof. First, H_{ϕ} satisfies the Iwahori-Hecke relations. The relation $h_s^2 = q_s \cdot Id + (q_s - 1)h_s$ is immediate. The relation $h_w h_s = h_{ws}$ for l(ws) = l(w) + 1 follows from the fact that if $d(C, C_1) = w$, $d(C_1, C_2) = s$ then $d(C, C_2) = ws$.

Let H'_{ϕ} be the Iwahori-Hecke algebra. It is well known that H'_{ϕ} is an algebra with basis h_w , $w \in W$ (see [Hum92], 7.1). Since H_{ϕ} satisfies the relations we have a homomorphism of algebra $H'_{\phi} \to H_{\phi}$. Since H_{ϕ} is spanned by the h_w , $w \in W$ this homomorphism is onto and it remains to prove it has a trivial kernel. It is therefore enough to prove that the h_w are linearly independent in H_{ϕ} . This is immediate since every operator $h = \sum_{W} \alpha_w h_w \in H_{\phi}$ with $\alpha_w \neq 0$ for some $w \in W$ acts non trivially on $\mathbb{C}^{B_{\phi}}$.

Assume we have colors $I_2 \subset I_1$. Notice that the larger I is, the face is smaller. That means that each face of color I_2 has exactly one subface of color I_1 , and each face of color I_1 is contained in a constant number of faces of color I_2 . We define:

Definition 4.6. The (unsigned colored) coboundary operator $\delta_{I_2,I_1} \in H_{I_2,I_1}$, $\delta_{I_2,I_1} : \mathbb{C}^{B_{I_1}} \to \mathbb{C}^{B_{I_2}}$ is the element $h_d \in H_{I_2,I_1}$ for $d = W_{I_2} \setminus id/W_{I_1} \in W_{I_2} \setminus W/W_{I_1}$.

The (unsigned colored) boundary operator $\partial_{I_1,I_2} \in H_{I_1,I_2}$, $\partial_{I_1,I_2} : \mathbb{C}^{B_{I_2}} \to \mathbb{C}^{B_{I_1}}$ is the element $h_d \in H_{I_1,I_2}$ for $d = W_{I_1} \setminus id/W_{I_2} \in W_{I_1} \setminus W/W_{I_2}$.

We denote $\delta_I = \delta_{\phi,I}$, $\partial_I = \partial_{I,\phi}$, $e_I = \delta_I \partial_I$.

The coboundary operator δ_{I_2,I_1} assigns to each face of color I_2 the value of its subface of color I_1 . The boundary operator ∂_{I_1,I_2} assigns to each face of color I_1 the sum of values of the faces of color I_2 containing it.

Remark 4.7. It is worth noting that (when all faces of dimension m are spherical), the usual signed boundary and coboundary operators of B between dimensions m, m+1, belong to our algebra. Since the complex B is colored, we have a natural ordering of the vertices of each simplex, and therefore every simplex has a natural orientation, given (for example) by ascending sequence of colors. The usual boundary and coboundary operators are therefore sums with ± 1 coefficients of δ_{I_2,I_1} , ∂_{I_1,I_2} , $|I_2|=m$, $|I_1|=m+1$, $I_2\subset I_1$. We will not use them at all in this work.

Lemma 4.8. Let $I_1, I_2, I \subset [n]$ be spherical.

- 1. Let $d = W_{I_1} \setminus \tilde{d}/W_{I_2} \in W_{I_1} \setminus W/W_{I_2}$. Then: $h_d = (1/n_d) \partial_{I_1} h_{\tilde{d}} \delta_{I_2}$. (recall- $\tilde{d} \in W$ the shortest element in the double coset. n_d is defined in 3.12).
 - 2. We have $\partial_I \delta_I = q_{W_I} 1_I$, $e_I = \delta_I \partial_I = \sum_{w \in W_I} h_w \in H_\phi$. Also $e_I^2 = \left(\sum_{w \in W_I} q_w\right) e_I = q_{W_I} e_I$.
 - 3. The algebra H_I can be embedded in H_{ϕ} by $h_d \to q_{W_I}^{-1} n_d^{-1} e_I h_{\tilde{d}} e_I$.

Proof. (1) follows from 3.4 and 3.11.

For (2), Let $f \in \mathbb{C}^{B_{\phi}}$. Then $e_I f(C) = \delta_I \partial_I f(C)$ is equal to the sum of f over all chambers sharing with C its face of color I. Every such chamber is of distance $w \in W_I$ from C, therefore $e_I = \sum_{w \in W_I} h_w$. Applying e_I twice counts each element $\sum_{w \in W_I} q_w = q_{W_I}$ times.

(3) follows from (1) and (2). \Box

Theorem 4.9. The ADH algebra H is indeed an algebra. It is spanned by h_d , $d \in W_{I_1} \backslash W/W_{I_2}$, for I_1, I_2 spherical. The relations defining it are the Iwahori-Hecke relations and the relations:

$$\begin{array}{lcl} h_d & = & (1/n_d)\,\partial_{I_1}h_{\tilde{d}}\delta_{I_2} \\ \partial_I\delta_I & = & q_{W_I}1_I \\ \delta_I\partial_I & = & \sum_{w\in W_I}h_w \\ \delta_{I_1}\partial_{I_2} & = & 0 & \text{for } I_1\neq I_2 \end{array}$$

The algebra H is generated by the coboundary and boundary operators $\delta_I = \delta_{\phi,I}$, $\partial_I = \partial_{I,\phi}$, I spherical, as well as the identity operator 1_{ϕ} of H_{ϕ} .

Proof. By lemma 4.8 we have $H_{I_1,I_2}=\operatorname{span}\left\{\partial_{I_1}\tilde{h}\delta_{I_2}:\tilde{h}\in H_\phi\right\}$. Therefore for $h_1=\partial_{I_1}\tilde{h}_1\delta_{I_2}\in H_{I_1,I_2},\ h_2=\partial_{I_2}\tilde{h}_2\delta_{I_3}\in H_{I_2,I_3}$ we have $h_1h_2=\partial_{I_1}\tilde{h}_1\delta_{I_2}\partial_{I_2}\tilde{h}_2\delta_{I_3}$. But $\tilde{h}_1\delta_{I_2}\partial_{I_2}\tilde{h}_2=\tilde{h}_1e_{I_2}\tilde{h}_2\in H_\phi$. Therefore $h_1h_2\in H_{I_1,I_3}$ and H is an algebra.

The algebra H_{ϕ} is generated by $h_s, s \in S$. Since $h_s = \delta_{\{s\}} \partial_{\{s\}} - 1_{\phi}$ we get the result for it. Then lemma 4.8 gives the result for the entire algebra.

The theorem actually shows that H can be defined abstractly, for any parameter system \overrightarrow{q} . The conditions on the parameter system here are that $q_s = q_{s'}$ when $m_{s,s'}$ is odd (otherwise the Iwahori-Hecke algebra is not well defined, see [Hum92], 7.1).

Definition 4.10. Denote by $1_I = h_d$, $d = W_I \setminus 1/W_I$ the identity operator of H_I .

The ADH algebra has an identity element $1 = \sum_{I \ spherical} 1_I$. It also has an adjunction, making it a *-algebra- the involution $d = W_{I_2} w W_{I_1} \to d^* = W_{I_1} w^{-1} W_{I_2}$ extends to $(\alpha h_d)^* = \bar{\alpha} h_{d^*}$.

Proposition 4.11. We have $(h_1h_2)^* = h_2^*h_1^*$ for every $h_1, h_2 \in H$.

Proof. Consider the action of H on $L_2(B_f)$, i.e. the L_2 norm on B_f defined by the inner product $\langle f, g \rangle = \sum_{\sigma \in B_f} \langle \bar{f}(\sigma), g(\sigma) \rangle$. Then for $d \in W_{I_1} \backslash W/W_{I_2}$,

$$\langle h_d f, g \rangle = \sum_{\sigma \in B_f} \sum_{\sigma': d(\sigma, \sigma') = d} \langle \bar{f}(\sigma'), g(\sigma) \rangle = \sum_{\sigma' \in B_f} \sum_{\sigma: d(\sigma', \sigma) = d^*} \langle \bar{f}(\sigma'), g(\sigma) \rangle = \langle f, h_{d^*} g \rangle$$

Therefore the *-operator on H agrees with the *-operator coming from the inner product on $L_2(B_f)$. Since the homomorphism $H \to \text{hom}(L_2(B_f), L_2(B_f))$ is an embedding of H, the result follows.

The fact that the ADH algebra is well defined and its algebra relations can help to understand the geometry of the building. In particular, one can show (Compare [APVM15] theorem 3.1):

Lemma 4.12. Let B be a locally finite regular building. Let $\sigma_0 \in B_{I_0}$, $\sigma_1 \in B_{I_1}$ be spherical faces of B of distance $d_1 \in W_{I_0} \backslash W/W_{I_1}$. Let d_2, \tilde{d} be distances $d_2 \in W_{I_0} \backslash W/W_{I_2}$, $\tilde{d} \in W_{I_1} \backslash W/W_{I_2}$. Let M be the number of faces $\sigma_2 \in B_{I_2}$ with $d(\sigma_0, \sigma_2) = d_2$, $d(\sigma_1, \sigma_2) = \tilde{d}$. Then M is a polynomial function on the parameter system (q_i) , $i \in S$ which depends only on d_0, d_1 and \tilde{d} . More precisely: Let α be the coefficient of h_{d_1} in the decomposition of $h_{d_2}h_{\tilde{d}^*}$ into a sum of basis elements. Then $M = \alpha$.

Proof. To show that $M=\alpha$ choose a function f_{σ_1} with $f_{\sigma_1}(\sigma_1)=1$, $f_{\sigma_1}(\sigma)=0$ for $\sigma\neq\sigma_1$. Then $M=h_{d_2}h_{\tilde{d}^*}f_{\sigma_1}(\sigma_0)=\alpha h_{d_1}f_{\sigma_1}(\sigma_0)=\alpha$.

Lemma 4.13. Let σ_0 be fixed, σ_1 be at distance $d_1 = d(\sigma_0, \sigma_1)$ and σ_2 be at distance $d_2 = d(\sigma_0, \sigma_2)$. Let \tilde{d} be some distance. Let M_1 be the number of σ'_1 with $d(\sigma_0, \sigma'_1) = d_1$, $d(\sigma'_1, \sigma_2) = \tilde{d}$. Let M_2 be the number of σ'_2 with $d(\sigma_0, \sigma'_2) = d_2$, $d(\sigma_1, \sigma'_2) = \tilde{d}$. Then $M_1q_{d_2} = M_2q_{d_1}$.

Proof. Consider the number M of pairs (σ'_1, σ'_2) with $d(\sigma_0, \sigma'_1) = d_1$, $d(\sigma_0, \sigma'_2) = d_2$, $d(\sigma'_1, \sigma'_2) = \tilde{d}$. By proposition 4.12 the number M_2 of σ'_2 corresponding to a single σ_1 does not depend on σ_1 . Therefore $M = M_2 q_{d_1}$ and by symmetry also $M = M_1 q_{d_2}$.

We can now show that the operators of H commute with spherical operators.

Definition 4.14. For any spherical face σ_0 we define a spherical average operator $\rho_{\sigma_0}: \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ by

$$\rho_{\sigma_0} f(\sigma) = \frac{1}{q_{d(\sigma_0,\sigma)}} \sum_{\sigma': d(\sigma_0,\sigma) = d(\sigma_0,\sigma')} f(\sigma') = \frac{1}{q_{d(\sigma_0,\sigma)}} h_{d(\sigma_0,\sigma)} f(\sigma_0)$$

Proposition 4.15. The ADH algebra commutes with spherical average operators. That is- for any $h \in H$ and any face σ_0 we have $\rho_{\sigma_0}h = h\rho_{\sigma_0}$.

Proof. Let $\tilde{d} \in W_I \backslash W/W_{I_2}$ be a distance. We need to prove that $(h_{\tilde{d}}\rho_{\sigma_0}f)(\sigma_1) = (\rho_{\sigma_0}h_{\tilde{d}}f)(\sigma_1)$ for every $\sigma_1 \in B_f$. It if enough to prove it for the functions f_{σ_2} , $\sigma_2 \in B_f$ fixed, defined by $f_{\sigma_2}(\sigma_2) = 1$, $f(\sigma) = 0$ for $\sigma \neq \sigma_2$.

Fix $\sigma_0, \sigma_1, \sigma_2, \tilde{d}$. By definition, $(h_{\tilde{d}}\rho_{\sigma_0}f_{\sigma_2})(\sigma_1)$ equals the number of σ'_2 with $d(\sigma_0, \sigma'_2) = d(\sigma_0, \sigma_2) = d_2$, $d(\sigma_1, \sigma'_2) = \tilde{d}$, divided by q_{d_2} . Similarly, $(\rho_{\sigma_0}h_{\tilde{d}}f_{\sigma_2})(\sigma_1)$ equals the number of σ'_1 with $d(\sigma_0, \sigma'_1) = d(\sigma_0, \sigma_1) = d_1$, $d(\sigma'_1, \sigma_2) = \tilde{d}$, divided by q_{d_1} . By lemma 4.13, both numbers are equal.

5. Building Automorphisms I

Recall that an automorphism γ of the building is an automorphism of the simplicial complex respecting colors of the faces.

Definition 5.1. Let G be a subgroup of the automorphism group of the building. The group G is called:

- Chamber transitive if for every 2 chambers $C_1, C_{,2}$ we have an automorphism $\gamma \in G$ such that $\gamma(C_1) = C_2$.
- Weyl transitive if for every 4 chambers C_1, C_2, C_3, C_4 such that $d(C_1, C_2) = d(C_3, C_4)$ we have an automorphism $\gamma \in G$ such that $\gamma(C_1) = C_2, \gamma(C_3) = C_4$.
- Strongly transitive, or has a BN- pair if for every 2 chambers $C_1, C_{,2}$ and two apartments containing them $C_1 \in A_1, C_2 \in A_2$ we have an automorphism $\gamma \in G$ such that $\gamma(C_1) = C_2, \gamma(A_1) = A_2$.

Notice that the strongly transitive notion actually depends on the choice of apartments of the building.

Lemma 5.2. An automorphism group that is strongly transitive is Weyl transitive. An automorphism group that is Weyl transitive is chamber transitive. A building that has a chamber transitive automorphism group is regular.

<i>Proof.</i> Follows from	n the definitions.	

Lemma 5.3. The distance between the faces σ_1, σ_2 is preserved by every (color preserving) automorphism $\gamma: B \to B$.

Proof. Such automorphisms preserve distances between chambers. Therefore the chambers $\sigma_1 \subset C_1, \sigma_2 \subset C_2$ with minimal distance between them go to two chambers $\gamma(\sigma_1) \subset \gamma(C_1), \gamma(\sigma_2) \subset \gamma(C_2)$ with minimal distance between them.

Remark 5.4. A color rotating automorphism γ defines a permutation $\omega: S \to S$. Then the distance is changed according to the extension $\omega: W \to W, \omega: W_I \to W_{\omega(I)}$ given by $\omega(s_i) = s_{\omega(i)}$.

Lemma 5.5. An automorphism group G is Weyl transitive if and only if for every 4 faces $\sigma_1, \sigma_{,2}, \sigma_3, \sigma_4$ such that $d(\sigma_1, \sigma_2) = d(\sigma_3, \sigma_4)$ we have an automorphism $\gamma \in G$ such that $\gamma(\sigma_1) = \sigma_2, \gamma(\sigma_3) = \sigma_4$.

Proof. Follows from the definition of a distance between faces. \Box

Definition 5.6. Let S be a discrete set. We say that a linear operator $h: \mathbb{C}^S \to \mathbb{C}^S$ is row and column finite if it can be written as $hf(x) = \sum_{y \in S} \alpha_{x,y} f(y)$, for some $\alpha: S \times S \to \mathbb{C}$, with $|\{y: \alpha_{x,y} \neq 0\}| < \infty$, $|\{y: \alpha_{y,x} \neq 0\}| < \infty$ for every $x \in S$.

If γ is an automorphism of the building we let γ act on \mathbb{C}^{B_f} , $\gamma: \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ by $\gamma f(\sigma) = f(\gamma^{-1}(\sigma))$.

Proposition 5.7. 1. The ADH Algebra commutes with every color preserving automorphism of the building. 2. Assume that G is Weyl transitive. If a row and column finite linear transform $h: \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ commutes with every automorphism $\gamma \in G$ then it belongs to the ADH algebra H. *Proof.* Claim (1) follows from the fact that automorphisms respect distances between faces.

As for (2), write $h: \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ as $hf(\sigma) = \sum_{y \in V_T} \alpha_{\sigma,\sigma'} f(\sigma')$, as in the definition of a row and column finite operator. Assume h commutes with every $\gamma \in \operatorname{Aut}(T)$. Let $\sigma_1, \sigma_{,2}, \sigma_3, \sigma_4$ be faces such that $d(\sigma_1, \sigma_2) = d(\sigma_3, \sigma_4)$. By the last lemma there exists $\gamma \in \operatorname{Aut}(T)$ such that $\gamma(\sigma_1) = \sigma_3, \gamma(\sigma_2) = \sigma_4$. Let f_{σ_2} be the characteristic function of $\{\sigma_2\}$ and write $h\gamma f_{\sigma_2} = \gamma h f_{\sigma_2}$. Then

$$\alpha_{\sigma_1,\sigma_2} = (h\gamma f_{\sigma_2})(\sigma_3) = (\gamma h f_{\sigma_2})(\sigma_3) = \alpha_{\sigma_3,\sigma_4}$$

Therefore $\alpha_{\sigma,\sigma'}$ depends only on $d(\sigma,\sigma')$ and we can write $\alpha_{\sigma,\sigma'} = \alpha_{d(\sigma,\sigma')}$. Therefore $hf(\sigma) = \sum_{y \in V_T} \alpha_{d(\sigma,\sigma')} f(\sigma') = \sum_{d} \alpha_d h_d f(\sigma')$ and $h \in H$.

Corollary 5.8. If G is Weyl transitive then the ADH algebra H is the algebra of row and column finite operators $h: \mathbb{C}^{B_f} \to \mathbb{C}^{B_f}$ commuting with every automorphism $\gamma \in G$.

6. Building Automorphisms II- Hecke Algebras of Groups

Every semisimple group G over a non-Archimedean local field acts on a certain affine building as a color rotating automorphism group which is Weyl transitive. There is usually some normal subgroup of G (containing the center) that acts trivially. By moving to a quotient of G by this normal subgroup we will generally identify G with its image in the automorphism group. We will further take the finite index subgroup of G which acts by color preserving automorphism. On the other hand if the dimension of an affine building is ≥ 3 a well known theorem of Tits says that its full automorphism group G is an algebraic group (see [Ron09]).

It will be useful to define our Hecke algebra in terms of the automorphism group directly. We start with some basic claims about general locally profinite groups (or totally disconnected locally compact groups). We follow [Cas74].

Let G' be a general locally profinite group, i.e G' is a Hausdorff topological group that has a basis of the identity composed of compact open subgroups. Let K be a compact open subgroup of G'. Fix a Haar measure μ of G'. We will also assume G' is unimodular.

Definition 6.1. The Hecke algebra with respect to K, $H_K(G') \subset H(G')$, is the set of compactly supported functions $f: G' \to \mathbb{C}$ with $f(k_1gk_2) = f(g)$ for every $k_1, k_2 \in K$, $g \in G'$. It is spanned by the characteristic functions 1_{KgK} , $KgK \in K \setminus G'/K$.

The Hecke Algebra H(G') of G' is $H(G') = \bigcup_K H_K(G')$ where K goes over the set of compact open subgroups.

We denote by $C(K\backslash G')$ the set of functions $f:G'\to\mathbb{C}$ such that f(kg)=f(g) for every $k\in K, g\in G'$ We denote $C_l^\infty(G')=\cup_K C(K\backslash G')$. Notice that $H(G')\subset C_l^\infty(G')$.

Proposition 6.2. We can define a convolution of $h \in H(G')$ on $f \in \mathbb{C}_l^{\infty}(G')$ by the integral $h * f(y) = \int_G h(x) f(x^{-1}y) dx$. When restricted to $H_K(G')$ the convolution defines an algebra structure on it, as $H_K(G') \subset H(G') \subset C_l^{\infty}(G')$ and $H_K(G') * H_K(G') = H_K(G')$. It also defines an action of $H_K(G')$ on $C(K \setminus G')$. The identity of $H_K(G')$ is $\mu^{-1}(K) \cdot 1_K$. This convolution defines an idempotented algebra structure on H(G').

One can define adjunction in $H_K(G')$. If G' is unimodular, the adjunction is given by $*: 1_{KgK} \to 1_{Kg^{-1}K}$ on $H_K(G')$.

Proof. See $\underline{[Cas74]}$, section 2.1 for everything except adjunction. Adjunction can be verified directly and is left to the reader.

Remark 6.3. As K is open and h is compactly supported the integral is actually a finite sum.

We now connect the general considerations above to an automorphism group G of the building and the Hecke algebras we defined in the previous sections. Let G be a subgroup of the color preserving automorphism group of the building and assume it is Weyl transitive. Let C_0 be a fixed chamber and $G_{\phi} \subset G$ its stabilizer (i.e an Iwahori subgroup, usually denoted I). Let G_I be the stabilizer of the face $\sigma_0 \subset C_0$ is of color I. Give B the discrete topology. We can topologize G by the compact open topology, i.e. a basis of open sets are sets containing, for $A \subset B_{\phi}$ finite, $U \subset B_{\phi}$ arbitrary, all automorphisms γ with $\gamma(A) \subset U$. As X and Y

are discrete, this topology is equivalent to the topology defined by pointwise convergence of a sequence of functions.

Definition 6.4. Call G complete if it is closed in the compact open topology defined on the entire automorphism group of B.

Equivalently, G is complete if and only for every sequence of automorphisms $\{\gamma_n\} \subset G$ that converges pointwise to $\gamma: B \to B$ we have $\gamma \in G$. A complete Weyl transitive automorphism group is also strongly transitive.

Lemma 6.5. If G is complete and I is spherical then G_I is an open compact subgroup.

Proof. We will prove for G_{ϕ} . The proof for G_{I} is similar. The fact that G_{ϕ} is open is by definition of the compact open topology.

The locally finite assumption $(q_s < \infty, s \in S)$ guarantee that the number of chambers at length at most l from C_0 is finite. Therefore every sequence $\{\gamma_n\} \subset G_\phi$ has a subsequence agreeing on all chambers of distance at most l from C_0 . Therefore $\{\gamma_n\}$ has a subsequence converging pointwise to an automorphism γ and by completeness $\gamma \in G$. Since $\gamma(C_0) = C_0$, $\gamma \in G_\phi$.

We will assume from now on that G is complete.

A basis for the identify of G is composed of the compact open subgroups K_m , $m \geq 0$, where $K_m \subset G_{\phi}$ is the subgroup containing all the automorphisms fixing all chambers C with $l(d(C_0, C)) \leq m$. Since the K_m , $m \in \mathbb{N}$ are compact the topology gives G the structure of a locally profinite group. We now prove that G is unimodular.

Proposition 6.6. Let G be a complete Weyl transitive automorphism group of a locally finite regular building. Then G is unimodular.

Proof. For K' a compact open subgroup and A a finite a sum of left K' cosets, let [A:K'] be the number of such cosets. Assume μ is left invariant. Let K be a compact open subgroup and assume $\mu(K) = 1$. The modular character does not depend on this normalization. We have the following formula for the modular character (see proof below):

$$\delta(g) = [KgK : K][Kg^{-1}K : K]^{-1}$$

Choosing $K = G_{\phi}$ we have $[KgK : K] = q_w$, where $w = d(C_0, g(C_0))$. Since $d(C_0, g^{-1}(C_0)) = w^{-1}$ we have $[Kg^{-1}K : K] = q_{w^{-1}} = q_w$ and $\delta(g) = 1$.

Proof of the formula:

$$\delta(q) = \mu(Kq) = \mu(q^{-1}Kq) = [K : K \cap q^{-1}Kq]^{-1}[q^{-1}Kq : K \cap q^{-1}Kq]$$

Let H_1, H_2 be subgroups of some big group. We have a set bijection $H_1/(H_1 \cap H_2) \cong H_1H_2/H_2$ which is a weak version of the second isomorphism theorem. Therefore:

$$\begin{array}{lcl} [g^{-1}Kg:K\cap g^{-1}Kg] & = & g^{-1}KgK:K] = [KgK:K] \\ [K:K\cap g^{-1}Kg] & = & gKg^{-1}:gKg^{-1}\cap K] = [Kg^{-1}K:K] \end{array}$$

The fact that G is chamber transitive allow us to identify the set of chambers of the building B_{ϕ} with cosets of G_{ϕ} in G, i.e $B_{\phi} \cong G_{\phi} \backslash G$. The faces of the building of color I correspond similarly to $B_I \cong G_I \backslash G$. We can therefore identify $\mathbb{C}^{B_I} \cong C(G_I \backslash G)$.

The fact that G is Weyl transitive allow us to identify distances between chambers with double cosets $G_{\phi} \backslash G/G_{\phi}$, and distances between faces of color I by $G_I \backslash G/G_I$. We can therefore identify $H_I \cong H_{G_I}(G)$ as a vector space.

Proposition 6.7. Let G be a complete Weyl transitive automorphism group of the building B. Let C_0 be a fixed chamber and let G_I be the stabilizer of the face σ_I^0 of color I of C_0 .

Then $H_I \cong H_{G_I}(G)$ as an algebra and its action on \mathbb{C}^{B_I} is the same as the action of $H_{G_I}(G)$ on $C(G_I \backslash G)$.

Proof. Let $d \in W_I \setminus W/W_I$ and then consider the operator $h_d \in H_I$. Let $G_d \subset G$ be the subset consisting of all automorphisms $\gamma \in G$ such that $d(\sigma_I^0, \gamma(\sigma_I^0)) = d$. We claim that the element $\tilde{h}_d = \frac{1}{|G_I|} 1_{G_d} \in H_{G_I}(G)$ defines a homomorphism $H_I \to H_{G_I}(G)$, $h_d \to \tilde{h}_d$. The following are immediate:

- defines a homomorphism $H_I \to H_{G_I}(G)$, $h_d \to \tilde{h}_d$. The following are immediate: a. The set G_d is a double coset- $G_d = G_I \gamma_d G_I$, where $\gamma_d \in G$ is some element of G_d , i.e sending σ_I^0 to a face of distance d.
- b. The set G_d is a sum of q_d right cosets in G/G_I , each right coset γG_I containing all automorphisms sending σ_I^0 to a specific face of distance d. Therefore $|G_d| = q_d |G_I|$
- c. For $\gamma_0 \in G$, the set $\gamma_0 G_d \subset G$ is the set of all automorphisms sending $\gamma_0(\sigma_I^0)$ to a face of distance d of it.

Let $f \in C(G_I \backslash G)$. Since $G_I \backslash G \cong B_I$, there corresponds a function $f_B \in \mathbb{C}^{B_I}$. Explicitly, $f(\gamma) = f_B(\gamma^{-1} \cdot \sigma_I^0)$. We have:

$$\tilde{h}_d * f(\gamma_0) = \int_G h(\gamma) f(\gamma^{-1} \gamma_0) d\gamma = \frac{1}{|G_I|} \int_{G_d} f(\gamma^{-1} \gamma_0) d\gamma = \frac{1}{|G_I|} \int_{\gamma_0^{-1} G_d} f(\gamma^{-1}) d\gamma$$

By b. and c. above and the correspondence between f and f_B , the last value is the sum of f_B over the q_d faces of distance d from $\gamma_0^{-1}(\sigma_I^0)$. Therefore the action of $\tilde{h}_d \in H_{G_I}(G)$ on $C(G_I \setminus G)$ and the action of $h_d \in H_I$ on $f_B \in \mathbb{C}^{B_I}$ agree, i.e. $(\tilde{h}_d * f)_B = h_d \cdot f_B$ and we are done.

We now extend the above to show similar results for H. In the general context as in the beginning of this section define:

Definition 6.8. For $K_1, K_2 \subset G'$ compact open let $H_{K_1,K_2}(G')$ be the set of compactly supported functions $f \in \mathbb{C}_c(G')$ with $f(k_1gk_2) = f(g)$ for every $k_1 \in K$, $k_2 \in K$.

Define convolution on $*: H_{K_1,K_2}(G') \times H_{K_3,K_4}(G') \to H_{K_1,K_4}$ by:

$$f_1 * f_2 = \begin{cases} 0 & K_2 \neq K_3 \\ \text{as in } H(G') & K_2 = K_3 \end{cases}$$

Define convolution on $*: H_{K_1,K_2}(G') \times C(K_3 \backslash G') \to C(K_1 \backslash G')$ by:

$$f_1 * f_2 = \begin{cases} 0 & K_2 \neq K_3 \\ \text{as in } H(G') & K_2 = K_3 \end{cases}$$

If K_1, K_2 are two different compact open subgroups the above can give an algebra structure on $H_{K_1,K_1} \oplus H_{K_2,K_1} \oplus H_{K_2,K_2}$. We can therefore state:

Proposition 6.9. Let G be a complete Weyl transitive automorphism group of the building B. Let C_0 be a fixed chamber and G_I the stabilizer of the face of color I contained in C_0 .

Then the ADH algebra H is isomorphic as algebra to $\bigoplus_{I,J \text{ finite types}} H_{G_I,G_J}(G)$. Its action on the building is given by the action of the algebra $\bigoplus_{I,J \text{ spherical}} H_{G_I,G_J}(G)$ on $\mathbb{C}^{B_f} \cong \bigoplus_{I \text{ spherical}} \mathbb{C}^{B_I} \cong \bigoplus_{I \text{ spherical}} C(G_I \setminus G)$.

Remark 6.10. If B has a complete Weyl transitive automorphism group G we can prove proposition 4.15 using proposition 5.7 as follows: we have a left action of H on $\mathbb{C}^{B_f} \cong \bigoplus_{I \text{ spherical}} C(G_I \setminus G)$. The same space has a commuting right action. Let G_{σ_0} be the stabilizer of σ_0 . Then the element $\frac{1}{|G_{\sigma_0}|} 1_{G_{\sigma_0}} \in H(G)$ defines a projection $\bigoplus_{I \text{ spherical}} C(G_I \setminus G)$ into $\bigoplus_{I \text{ spherical}} C(G_I \setminus G)$, commuting with the left H action. This projection is exactly ρ_{σ_0} .

We continue the discussion of this section and address the representation theory consequences in section 14.

7. Finite Quotients

Assume we have a discrete subgroup $\Gamma \subset Aut(B)$ (recall that unless otherwise stated automorphisms are color preserving). We may then construct the quotient complex $X \cong B/\Gamma$. Since Γ is color preserving X is a colored complex.

We will assume two further properties:

- 1. We assume that X is finite (i.e- the number of faces of X is finite). Such a subgroup Γ is called a cocompact lattice. Since X is finite $\mathbb{C}^{X_f} \cong \mathbb{C}[X_f]$.
 - 2. We assume that Γ is torsion free.

The discreteness of Γ means that its intersection with every compact group is finite and since we assume Γ is torsion free, the intersection is actually trivial. Therefore for every spherical face σ , $G_{\sigma} \cap \Gamma = \{id\}$ where G_{σ} is the stabilizer of σ . Therefore X looks locally like the building- each face of spherical color I is contained in q_{W_I} chambers. We remark that X is not necessarily a simplicial complex, as there might be double simplices (e.g two edges between the same vertices). This will not interfere with the analysis but one should remember this point.

Assume also we have a function $f \in \mathbb{C}^{X_f} \cong \mathbb{C}[X_f]$, i.e a function that assigns a value to every face of a spherical color on the quotient. Using the projection operator $\pi: B_f \to X_f$ we can define a function $\hat{f} \in \mathbb{C}^{B_f}$ by $\hat{f}(\sigma) = f(\pi(\sigma))$. Now for some element $h \in H$ we can act on \hat{f} and get a new function $h\hat{f} \in \mathbb{C}^{B_f}$. Since the ADH algebra commutes with automorphisms, \hat{f} is Γ invariant and therefore so is $h\hat{f}$. So we can project and receive a function $hf \in \mathbb{C}^{B_f}$ defined by $h\hat{f}(\sigma) = hf(\pi(\sigma))$. Therefore H acts on \mathbb{C}^X .

The algebra H can also be defined directly- the fact that X is colored means that we can define the coboundary and boundary operators $\delta_I : \mathbb{C}^{X_I} \to \mathbb{C}^{X_\phi}$, $\partial_I : \mathbb{C}^{X_\phi} \to \mathbb{C}^{X_I}$ for every spherical color I. Since Γ is torsion free there exist q_{W_I} chambers containing every face of color I. Therefore the two definitions are the same. Now we can define the rest of H using the generating elements.

Since X is finite the action of H on it is a finite dimensional representation. Moreover:

Proposition 7.1. Consider the inner product $\langle f, g \rangle = \sum_{\sigma \in X_f} \langle \bar{f}(\sigma), g(\sigma) \rangle$ on \mathbb{C}^{X_f} . The action of H on $L_2(X_f)$ is a unitary representation, i.e. $\langle hf, g \rangle = \langle f, h^*g \rangle$

Proof. 2. It is enough to prove that $\langle \partial_I f, g \rangle = \langle f, \delta_I g \rangle$ for every $f, g \in L_2(X_f)$ and I spherical. It is immediate, since both sides calculate $\sum_{\sigma \subset C, \sigma \in X_I, C \in X_{\sigma}} \bar{f}(\sigma) g(C)$ where σ is of color I.

Remark 7.2. It is sometimes useful to use the inner product $\langle f,g\rangle_w=\sum_{\sigma\in X_f}w(\sigma)\left\langle \bar{f}(\sigma),g(\sigma)\right\rangle$, where $w(\sigma)=\#\{C\in X_\phi:\sigma\subset C\}$. In this definition $\|\delta_{I_2,I_1}f\|_w=\|f\|_w$ if $f\in\mathbb{C}^{X_{I_1}}$. However, the representation of H on the L_2 -norm is not unitary as we defined it. To fix it, one should change the involution to $\delta_{I_1}^*=\partial_{I_1}/q_{W_{I_1}}$ (and extend to the other elements of H). We will not use this inner product in our work, but see [EK15] for example.

Remark 7.3. The arguments of this section apply more generally then stated; as a matter of fact we have an action of H on \mathbb{C}^{X_f} for every quotient $X \cong B/H$ for any subgroup $H \subset Aut(B)$.

Part 2. Representation Theory

In this part we will present some basic results about the representation theory of H and its connection to the building.

8. The Trivial Representation and the Steinberg Representation

Definition 8.1. Let V be a vector space over \mathbb{C} . A representation (π, V) of an algebra with unit H is an homomorphism of algebras $\pi: H \to Hom_{\mathbb{C}}(V, V)$ such that $\pi(1) = Id_V$.

We will usually omit π and write "let V be a representation of H" implicitly assuming π is given as well. We will also let elements of $h \in H$ act directly on the vector space V. We start this part by presenting two simple and important representations:

Proposition 8.2. The sets of functions that depend only on the color of each face form a representation space of H. Its dimension is the number of finite colors of faces (i.e. $2^{n+1} - 1$ in the vertex spherical case). This is called the trivial representation.

Proof. For $f \in \mathbb{C}^{B_f}$ depending only on the color of the face, define f(I) to be the value of f on a face of color I. If $d \in W_{I_1} \setminus W/W_{I_2}$ then

$$h_d f(\sigma) = \begin{cases} 0 & \sigma \text{ not of type } I_2 \\ q_d f(I_2) & \sigma \text{ of type } I_2 \end{cases}$$

It is therefore a representation, as required.

Lemma 8.3. For every non zero representation V of H there exists $0 \neq v \in V$ such that $1_{\phi}v = v$.

Proof. We know that $1_H = \sum_I 1_I$. Let $0 \neq v \in V$. Since $1_H v = v$ there exists a finite color I with $1_I v \neq 0$. Since $1_I = q_{W_I}^{-1} \partial_I \delta_I$ also $\delta_I v \neq 0$. Therefore $1_\phi \delta_I v = \delta_I v \neq 0$.

Proposition 8.4. There is only one irreducible representation V of H on which $\partial_{\{s\}}$ acts by 0 for every $s \in S$. It is one dimensional. This representation is called the Steinberg representation or the special representation.

Proof. Assume that V a representation on which $\partial_{\{s\}}$ acts by 0 for every $s \in S$. Let $0 \neq v \in V$ be an element such that $1_{\phi}v = v$. For $s \in S$ we have $h_sv = (\delta_{\{s\}}\partial_{\{s\}} - 1_{\phi})v = (0 - 1_{\phi})v = -v$. Inductively, $h_w, w \in W$ acts on v by $(-1)^{l(w)}$. For every other $d \in W_{I_1} \backslash W/W_{I_2}$, $I_1 \neq \phi$ or $I_2 \neq \phi$, h_d acts by zero since every every restriction operator is 0. Therefore V is one dimensional and uniquely determined if it exists. To determine existence, notice that the relations above define a representation, as required.

Remark 8.5. Both the trivial and the Steinberg representations are one dimensional representations of the Iwahori-Hecke algebra H_{ϕ} . By the Iwahori-Hecke relations in each such representation $h_s \to q_s$ or $h_s \to -1$. In addition, if $m_{s,s'}$ is odd (and therefore $q_s = q_{s'}$) we must have $h_s, h_{s'} \to q_s$ or $h_s, h_{s'} \to -1$. On the other hand each such correspondence defines a one dimensional representation. Therefore the number of such representations is 2^M , where M is the number of equivalence classes of S generated by the relation $m_{s,s'}$ is odd $\Longrightarrow s, s'$ equivalent.

For example in the bi-regular tree there are 4 such representations and those are exactly the one dimensional H_{ϕ} representations given in in [Kam16], proposition 11.6.

9. Unitary Representations

Recall we have an involution $*: H \to H$.

Definition 9.1. Let V_1, V_2 be complex vector spaces. A map $\phi: V_1 \times V_2 \to \mathbb{C}$ is called *sesquilinear* if it is additive and $\phi(av_1, bv_2) = \bar{a}b\phi(v_1, v_2)$. A sesquilinear map $\phi: V \times V \to \mathbb{C}$ is called *positive definite Hermitian* (or an inner product) if we have $\phi(v_1, v_2) = \overline{\phi(v_2, v_1)}$ and $\phi(x, x) \in \mathbb{R}_{>0}$ for every $x \neq 0$. The space V is called a *Hilbert space* if it has an inner product and it complete with respect to the topology it defines. A representation V of H is called *unitary* if V is an Hilbert space and the representation satisfies $\langle hv_1, v_2 \rangle = \langle v_1, h^*v_2 \rangle$ for every $v_1, v_2 \in V$ and $h \in H$.

A representation V of H is called *normed* if V is a Banach space and every element $h \in H$ acts as a bounded linear operator.

If V is finite dimensional or normed, the complex dual space V^* is the vector space of continuous linear functionals on V. We have an obvious map $\phi: V^* \times V \to \mathbb{C}$. We define a \mathbb{C} action of V^* by $\langle \alpha v^*, v \rangle = \langle v^*, \bar{\alpha}v \rangle$ so the form $\phi: V^* \times V \to \mathbb{C}$ is sesquilinear.

In practice, we deal in this work with representations that are either finite dimensional (unitary or not), or the action of H on $L_p(B_f)$.

Proposition 9.2. Every finite dimensional unitary representation of H decomposes into a direct sum of irreducible representations.

Proof. This result is standard in representation theory- assume $\{0\} \neq V' \subset V$ is a proper subrepresentation. Let $U = \{u \in V : \langle v, u \rangle = 0, \forall v \in V'\}$. Since it is an inner product we have $V = V' \oplus U$. Moreover if $u \in U$, $h \in H$ then for every $v \in v'$, $\langle v, hu \rangle = \langle h^*v, u \rangle = 0$. Therefore $hu \in U$ and U is also a subrepresentation. The claim follows by a simple inductive argument.

Remark 9.3. In general, a finite dimensional representation V of H does not decompose into a sum irreducible representations, but into a sum of indecomposable representations (i.e representations that cannot be written as a sum of two proper subrepresentations).

Proposition 9.4. 1. Given a representation (π, V) of H we can define a representation (π^*, V^*) , acting on the vector space V^* by $(\pi^*(h)v^*)(v) = v^*(\pi(h^*)v)$.

2. A unitary representation (π, V) is isomorphic to (π^*, V^*) .

Proof. Claim (1) is standard and left to the reader. Claim (2) follows from the fact that the inner product gives a vector space isomorphism between V and V^* . It is easy to see that this isomorphism is also an isomorphism of H representations.

Definition 9.5. The representation (π^*, V^*) of the last proposition is called the dual representation.

Lemma 9.6. If V is finite dimensional it is irreducible if and only if V^* is irreducible.

Proof. Since $(V^*)^* \cong V$ it is enough to prove that if V is irreducible so is V^* . Assume V^* is not irreducible. Then for some $0 \neq v_0^* \in V^*$ the space Hv_0^* is a proper subspace of V^* - a linear subspace of smaller dimension. Therefore there is some $0 \neq v_0 \in V$ such that $\langle hv_0^*, v_0 \rangle = 0$ for every $h \in H$. But in this case $\langle v_0^*, h^*v_0 \rangle = 0$ for every $h \in H$. Since $v_0^* \neq 0$, Hv_0 is a proper subrepresentation and V is not irreducible.

10. Induction and Restriction of Algebra Representations

The ADH algebra H is an algebra with a unit 1_H . Let $H' \subset H$ be a subalgebra with unit $1_{H'}$. We do not assume in general that $1_{H'}$ is equal to 1_H .

Definition 10.1. Let V be a representation of H. The subspace $V' = \{hv : v \in V, h \in H'\} = \{1_{H'}v : v \in V\}$ is a representation space of H'. We call this representation the *restriction* of V from H representation to H' representation and denote it $\operatorname{res}_{H'}^H V$.

Let V' be a representation of H'. The action of H on the space $H \otimes_{H'} V$ is called the *induction* of V from H' representation to H representation. It is denoted $\operatorname{ind}_{H'}^H V$.

We will be mainly concerned with the case in which H is the ADH algebra of the building and $H' = H_{\phi}$.

Lemma 10.2. Assume that $1_{H'}H1_{H'} = H'$. Let V be a representation of H'. Then $res_{H'}^H ind_{H'}^H V$ is isomorphic to V.

Proof. Define a homomorphism $T: V \to \operatorname{res}_{H'}^H \operatorname{ind}_{H'}^H V$, $v \to 1_{H'} \otimes v$. T is surely a H'-homomorphism. Define the map $S: \operatorname{res}_{H'}^H \operatorname{ind}_{H'}^H V \to V$, $1_{H'} h \otimes_{H'} v \to 1_{H'} h 1_{H'} v$ (using the fact that $1_{H'} h 1_{H'} \in H'$). We have $ST = id_V$. It remains to prove that its image is $\operatorname{res}_{H'}^H \operatorname{ind}_{H'}^H V$. For each $v \in V$ we have $1_{H'} v = v$. Therefore in $\operatorname{ind}_{H'}^H V$, for every $h \in H$, $1_{H'} (h \otimes v) = 1_{H'} h 1_{H'} \otimes v = 1 \otimes (1_{H'} h 1_{H'} v)$. Therefore every element of $\operatorname{res}_{H'}^H \operatorname{ind}_{H'}^H V$ can be written as $1_{H'} \otimes v$ for $v \in V$, as required.

Lemma 10.3. Let V be a representation of H_{ϕ} . Each element of $\operatorname{ind}_{H_{\phi}}^{H}V$ can be written as $\sum_{I} \partial_{I} \otimes_{H_{\phi}} v_{I}$, $v_{I} \in V$, $I \subset S$ is a (spherical) color and ∂_{I} is the boundary operator of definition 4.6.

Proof. Let $h \otimes v \in \operatorname{ind}_{H'}^H V$, $h \in H$, $v \in V$. Since h is a sum of basis elements, it is enough to assume $h = h_d, d \in W_{I_1} \backslash W / W_{I_2}$. If $I_2 \neq \phi$, $h_d \otimes_{H_{\phi}} v = h_d \otimes_{H_{\phi}} 1_{\phi} v = h_d 1_{\phi} \otimes_{H_{\phi}} v = 0$. Otherwise $I_2 = \phi$ and $h_d = \partial_{I_1} h_w$ for some $w \in W$. Then $h_d \otimes_{H_{\phi}} v = \partial_{I_1} h_w \otimes_{H_{\phi}} v = \partial_{I_1} \otimes_{H_{\phi}} h_w v$ and the claim proven. \square

Lemma 10.4. Let V be a representation of H. Then $ind_{H_{\phi}}^{H} res_{H_{\phi}}^{H} V$ is isomorphic to V.

Proof. Consider the linear transformation $T:V\to \operatorname{ind}_{H_\phi}^H\operatorname{res}_{H_\phi}^HV,\ Tv=\sum_I(1/q_{W_I})\cdot\partial_I\otimes_{H_\phi}1_\phi\delta_Iv,$ and the linear transformation $S:\operatorname{ind}_{H_\phi}^H\operatorname{res}_{H_\phi}^HV\to V,\ S\left(\sum_Ih\otimes_{H_\phi}1_\phi v\right)\to\sum_Ih1_\phi v.$ By lemma 10.3 T is onto. We have $1_H=\sum_I1_I=\sum(1/q_{W_I})\cdot\partial_I\delta_I.$ Therefore $ST=id_V$ and T is an isomorphism of linear spaces. Finally, for every $h\in H$ we have $ShTv=\sum_I(1/q_{W_I})\cdot h\partial_I1_\phi\delta_Iv=hv.$ Therefore hT=Th and T is an H-isomorphism.

Lemma 10.5. Let V be a representation of H_{ϕ} . If V is finite dimensional, so is $ind_{H_{\phi}}^{H}V$. If V is unitary, so is $ind_{H_{\phi}}^{H}V$.

Proof. The finite dimensional case follows from lemma 10.3. As a matter of fact we have $\dim \operatorname{ind}_{H_{\phi}}^H V \leq \dim V \cdot \#\{I : I \text{ spherical}\}$. For the unitary case, define norm on $\operatorname{ind}_{H_{\phi}}^H V$ by $\|\partial_I \otimes_{H_{\phi}} v\| = q_{W_I}^{-1} \|e_I v\|_V = q_{W_I}^{-1} \|\delta_I \partial_I v\|_V$.

Corollary 10.6. Induction and restriction induce an equivalence of categories between H_{ϕ} -representations and H-representations. This equivalence preserves irreducible representations, unitary representations and finite dimensional representations.

Remark 10.7. This proposition is not true for other H_I , $I \neq \phi$. In particular, the restriction of the Steinberg representation to H_I representation is 0 for every $I \neq \phi$.

Problem 10.8. Is it true in general that induction sends finite dimensional H_I representations to finite dimensional H (or H_{ϕ}) representations?

See also Problem 14.15, where a similar question is asked about admissibility of the induction of a finite dimensional H_{ϕ} representation to H(G) representation.

11. Matrix Coefficients

The following construction is well known in the representation theory of groups: let G be a group. The space \mathbb{C}^G has a $G \times G$ action given by $(g_1, g_2)f(x) = f(g_1^{-1}xg_2)$. Given any representation V of G, V^* the dual representation, $0 \neq v \in V$, $0 \neq v^* \in V^*$, $c_{v^*,v}(g) = \langle v^*, gv \rangle$ is called a *matrix coefficient* of the representation. Then $v^* \otimes v \to c_{v^*,v}$ is a homomorphism of representations of $G \times G$, non zero if V is irreducible. This allows us to consider every irreducible representation as a subrepresentation of \mathbb{C}^G .

Similarly, let H^* be the space of linear functionals $f: H \to \mathbb{C}$. This space has a natural $H \otimes H$ action $(h_1, h_2)\psi(x) = \psi(h_1^*xh_2)$ making it a representation of $H \times H$. We will focus on the right H action, i.e. the action given by $h\psi(x) = \psi(xh)$.

Definition 11.1. Let V be a representation of H. Let $0 \neq v^*$ be a non zero vector. The functional $c_{v^*,v} \in H^*$, $c_{v^*,v}(h) = \langle v^*, hv \rangle$ is called a *matrix coefficient* of the representation.

Lemma 11.2. Let V be a representation of H. Let $0 \neq v^* \in V^*$ be a non zero vector. Then the correspondence $V \to H^*$, $v \to c_{v^*,v}$ is a non zero homomorphism of representations of H.

Proof. The fact that the correspondence is a homomorphism is given by definition:

$$hc_{v^*,v}(h') = c_{v^*,v}(h'h) = \langle v^*, h'hv \rangle = c_{v^*,hv}(h')$$

It is non zero since $v^* \neq 0$, therefore there exists $v \in V$ with $\langle v^*, v \rangle \neq 0$ and then $c_{v^*,v}(1_H) = \langle v^*, v \rangle \neq 0$, so $c_{v^*,v} \neq 0$.

Let us shortly discuss representations of H_{ϕ} . In this case we have $H_{\phi}^* \cong \mathbb{C}^W$ as vector spaces (since the $h_w, w \in W$ are a basis for H_{ϕ}). There is a bijection between functions $f \in \mathbb{C}^{B_{\phi}}$ that are spherical around C_0 and functions $f^* \in H_{\phi}^*$ given by $f^*(h_w) = (h_w f)(C_0)$. By proposition 4.15 the set of $f \in \mathbb{C}^{B_{\phi}}$ spherical around C_0 is an H_{ϕ} representation and it is easy to see directly that $f \to f^*$ is an isomorphism of H_{ϕ} representations. Therefore, the chain of H_{ϕ} homomorphism:

$$V \to H_{\phi}^* \leftrightarrow \mathbb{C}^W \leftrightarrow \left\{ f \in \mathbb{C}^{B_{\phi}} : f \text{ spherical around } C_0 \right\}$$

defines an embedding (i.e. a "geometric realization") of V in $\mathbb{C}^{B_{\phi}}$.

To extend the result to H we will need the following definition:

Definition 11.3. A functional $\psi \in H^*$ is of color ϕ if it is zero on every base element h_d , $d \in W_{I_1} \setminus W/W_{I_2}$, $I_1 \neq \phi$. We denote the set of functionals of color ϕ by $H^*(\phi)$.

The vector space $H^*(\phi)$ can be naturally identified with $\mathbb{C}^{\oplus_{I_2}W/W_{I_2}}$ (i.e. the set of functions on $\oplus_{I_2}W/W_{I_2}$) Let C_0 be a chamber of B. proposition 4.15 can be stated as follows:

Lemma 11.4. 1. The set of $f \in \mathbb{C}^{B_f}$ spherical around C_0 is an H-representation.

2. The correspondence $f \to f^*$ of spherical functions around C_0 into $H^*(\phi)$, $f^*(h_d) = (h_d f)(C_0)$ is an isomorphism of H-representations. The inverse of this isomorphism is given by $f(\sigma) = q_{d(C_0,\sigma)}^{-1} f^*(h_{d(C_0,\sigma)})$.

Corollary 11.5. Each non zero representation V of H has a non zero homomorphism to a subrepresentation of the action of H on $\{f \in \mathbb{C}^{B_f} : f \text{ spherical around } C_0\}$. This homomorphism is given by a choice of vector $v^* \in V^*$ such that $1_{\phi}v^* = v^*$ and defining $f_{v^*,v}(\sigma) = q_{d(C_0,\sigma)}^{-1} \langle v^*, h_{d(C_0,\sigma)}v \rangle$. If V is irreducible it is an

Proof. Choose $0 \neq v^* \in V^*$ such that $1_{\phi}v^* = v^*$. Then $c_{v^*,v}(h) = \langle 1_{\phi}v^*, hv \rangle = \langle v^*, 1_{\phi}hv \rangle$, so $c_{v^*,v} \in H^*(\phi)$. By lemma 11.2 it is a non zero homomorphism of H representations. Thus, by lemma 11.4, $v \to f_v$ is an isomorphism of H representations.

Definition 11.6. We call each H-homomorphism of V into \mathbb{C}^{B_f} as in corollary 11.5 a geometric realization of the representation V.

12. p-Finite Representations and p-Tempered Representations

Definition 12.1. We say that a finite dimensional representation V of H_{ϕ} is p-finite if for every $v \in V$,

 $u \in V^*$ we have $\sum_{w \in W} q_w^{(1-p)} |\langle u, h_w v \rangle|^p < \infty$. We say that a finite dimensional representation V of H_{ϕ} is p-tempered if for every $v \in V$, $u \in V^*$ and $\epsilon > 0$ we have $\sum_{w \in W} q_w^{(1-p)} \left| \langle u, h_w v \rangle \right|^p (1-\epsilon)^{l(w)} < \infty$.

We want to explain the geometry of this definition and extend it to H-representations.

Definition 12.2. For distance $d = W_I \setminus \tilde{w}/W_J \in W_I \setminus W/W_J$, $\tilde{w} \in W$ the shortest element in the double coset, we define the distance length $l(d) = l(\tilde{w}) \in \mathbb{N}$.

For $f \in \mathbb{C}^{B_f}$ define $||f||_p = \left(\sum_{\sigma \in B_f} |f(\sigma)|^p\right)^{1/p}$, $||f||_{\infty} = \sup_{\sigma \in B_f} |f(\sigma)|$. Let $L_p(B_f) = \{f \in \mathbb{C}^{B_f} : f(\sigma) \mid f$ $||f||_p < \infty$.

Fix a chamber $C_0 \in B_f$.

Definition 12.3. For every face $\sigma \in B_f$, we define the distance length $l(\sigma) = l^{C_0}(\sigma) = l(d(C_0, \sigma))$. For a

function $f \in \mathbb{C}^{B_f}$ and $0 < \delta < 1$ define $f_{\delta} = f_{\delta}^{C_0} \in \mathbb{C}^{B_f}$ as $f_{\delta}(\sigma) = (1 - \delta)^{l(\sigma)} f(\sigma)$. A function $f \in \mathbb{C}^{B_f}$ is called *p-tempered* and we write $f \in T_p(B_f) = T_p^{C_0}(B_f)$ if $f_{\delta}^{C_0} \in L_p(B_f)$ for every $\delta > 0$. Define $T_p(B_I) = T_p^{C_0}(B_I)$ for I spherical as $T_p(B_I) = T_p(B) \cap \mathbb{C}^{B_I} \subset \mathbb{C}^{B_f}$.

Lemma 12.4. The definition does not depend on the choice of C_0 , that is $T_p^{C_0}(B_f) = T_p^{C_0'}(B_f)$ for every chamber C'_0 .

Proof. Assume we replace C_0 by C'_0 with $l(d(C_0, C'_0)) = L$. Then for every face σ we have $\left| l^{C_0}(\sigma) - l^{C'_0}(\sigma) \right| \le C_0$ L. Therefore $f_{\delta}^{C_0}(\sigma)(1-\delta)^L \leq f_{\delta}^{C'_0}(\sigma) \leq f_{\delta}^{C_0}(\sigma)(1-\delta)^{-L}$ for every $\delta > 0$. The claim follows.

The following proposition explains definition 12.1.

Proposition 12.5. A finite dimensional representation V of H_{ϕ} is p-finite (resp. p-tempered) if and only if every function $f \in \mathbb{C}^{B_{\phi}}$, in every geometric realization of V, is in $L_p(B_{\phi})$. (resp. $T_p(B_{\phi})$).

Proof. There are q_w chambers of distance w from C_0 . Therefore every function f in a geometric realization of V is p-finite if and only if for every $v \in V$, $u \in V^*$

$$\sum_{w \in W} q_w \left(\left| \left\langle u, h_w v \right\rangle \right| / q_w \right)^p = \sum_{w \in W} q_w^{(1-p)} \left| \left\langle u, h_w v \right\rangle \right|^p < \infty$$

The p-tempered case is very similar.

We can now define p-finite and p-tempered H-representations:

Definition 12.6. A finite dimensional representation V of H is called p-finite (resp. p-tempered) if every function $f \in \mathbb{C}^{B_f}$, in every geometric realization of V, is in $L_p(B_f)$ (resp. $T_p(B_f)$).

The following lemma is immediate and left to the reader. It will allow us to work with H_{ϕ} instead of H.

Lemma 12.7. A function $f \in \mathbb{C}^{B_f}$ satisfies $f \in L_p(B_f)$ (respectively $T_p(B_f)$) if and only if for every spherical color I (including ϕ) $\delta_I f \in T_p(B_\phi)$ (respectively $\delta_I f \in L_p(B_\phi)$). Therefore the equivalence of categories between H_{ϕ} -representations and H-representations also respects p-finiteness and p-temperedness.

The following claim relates our definition of temperedness to the definition stated in the introduction. Recall that a building is thick if $q_s > 1$ for all $s \in S$.

Lemma 12.8. If $f \in L_{p+\epsilon}(B_f)$ for every $\epsilon > 0$ then $f \in T_p(B_f)$.

If the building is thick and the function $f \in T_p(B_f)$ is spherical around C_0 , then $f \in L_{p+\epsilon}(B_f)$ for every $\epsilon > 0$.

Proof. Using the last lemma it is enough to prove this claim for $f \in \mathbb{C}^{B_{\phi}}$ (since $f \in L_{p+\epsilon}(B_f)$ if and only if every spherical color $I, \delta_I f \in L_{p+\epsilon}(B_{\phi})$.

Assume $f \in L_{p+\epsilon}(B_{\phi})$ for every $\epsilon > 0$. Notice that the number of chambers of distance l from C_0 is bounded by r_2^l , for some $r_2 > 0$, since for $\alpha_2 = \max_{s \in W} q_s$, $q_w \le \alpha_2^{l(w)}$ and $\#\{w : l(w) = m\} \le |S|^m$. Therefore $\sum_C r_2^{-l(C)}$ converges and if we define $g(C) = \max\{|f(C)|, r_2^{-p^{-1}l(C)}\}$ then $g \in L_{p+\epsilon}(B_{\phi})$ for every $\epsilon > 0$. For every $\delta > 0$ there exists some $\epsilon > 0$ such that $(1 - \delta) < r_2^{-\epsilon p^{-2}}$. Then

$$\left| f_{\delta}^{C_0}(C) \right|^p \leq g_{\delta}^{C_0}(C)^p = g(C)^p (1-\delta)^{l(C)p} < g(C)^p r_2^{-\epsilon p^{-1}l(C)} \leq g(C)^p g(C)^{\epsilon} \leq g(C)^{p+\epsilon p^{-1}l(C)} \leq g(C)^{p+\epsilon$$

Therefore $f_{\delta}^{C_0} \in L_p(B_{\phi})$ and $f \in T_p(B_{\phi})$. For the other direction, assume the building is thick, and the function $f \in T_p(B_f)$ is spherical around C_0 . We may therefore define $f_W \in \mathbb{R}_{\geq 0}^W$ by $f_W(w) = |(h_w f)(C_0)|$.

Since $f \in T_p(B_\phi)$ and f is spherical around C_0 , the series $\sum_{w \in W} q_w \left(f_W(w)/q_w \right)^p (1-\delta)^{p \cdot l(w)}$ converges for every $0 < \delta < 1$. Since $\alpha_1 = \min_{s \in S} q_s > 1$, $q_w > \alpha_1^{l(w)}$ and since the series converges, $(f_W(w)/q_w)^p \le \alpha_1^{-l(w)}$ for almost all $w \in W$. For every $\epsilon > 0$ choose $1 > \delta > 0$ such that $(1 - \delta)^p \ge \alpha_1^{-p\epsilon}$. Then $(1 - \delta)^{p \cdot l(w)\epsilon} \ge \alpha_1^{-p\epsilon}$. $\alpha_1^{-pl(w)\epsilon} \ge (f_W(w)/q_w)^{\epsilon}.$

Then

$$\sum_{w \in W} q_w \left(f_W(w) / q_w \right)^{p+\epsilon} \le \sum_{w \in W} q_w \left(f_W(w) / q_w \right)^p (1-\delta)^{p \cdot l(w)}$$

And therefore $f \in L_{p+\epsilon}$

If the building is thin or the function is not spherical then the lemma has simple counter examples.

Lemma 12.9. $T_p(B)$ is a representation of H. Moreover for every $h \in H$ there exists a number $M(h) \in R_{\geq 0}$ such that for every $f \in T_p(B)$, $0.5 > \delta > 0$, $\|(hf)_{\delta}\|_p \leq M(h) \|f_{\delta}\|_p$.

Proof. The second part clearly imply the first. It is enough to prove it for $h = \delta_I$ and $h = \partial_I$. Let L be the length of the longest element of W_I . Therefore for every $\sigma \subset C$ of color I we have $l(\sigma) \leq l(C) \leq l(\sigma) + L$. Then we have:

$$\begin{aligned} & \|(\delta_I f)_{\delta}\|_p^p & \leq & q_{W_I} \|f_{\delta}\|_p^p \\ & \|(\partial_I f)_{\delta}\|_p^p & \leq & (1-\delta)^{-pL} q_{W_I}^{p-1} \|f_{\delta}\|_p^p \leq 2^{pL} q_{W_I}^{p-1} \|f_{\delta}\|_p^p \end{aligned}$$

Lemma 12.10. Let V be a finite dimensional representation of H. Then V is p-finite (resp. p-tempered) if and only if V^* is p-finite (resp. p-tempered).

Proof. This is an immediate corollary of the definition.

Lemma 12.11. An irreducible finite dimensional representation is p-finite (resp. p-tempered) if a single function $f \neq 0$ in some geometric realization is in L(B) (resp. $T_p(B)$).

Proof. Fix $0 \neq v_0^* \in V^*, v_0 \in V$, $1_\phi v_0^* = v_0^*$. Assume that the geometric realization $f_{v_0^*, v_0} \in \mathbb{C}^B$ of v_0 corresponding to v_0^* is p-tempered (p-finite respectively). Consider changing v_0 to v_0' . Since V is irreducible there exists $h \in H$ with $hv_0 = v_0'$. Therefore $f_{v_0^*, v_0'} = f_{v_0^*, hv_0} = hf_{v_0^*, v_0}$. Since $T_p(B)$ ($L_p(B)$ respectively) is a representation of H, $f_{v_0^*, v_0'}$ is also p-tempered (p-finite respectively). To prove that it does not depend on v_0^* switch the roles of V, V^* and use the fact that V^* is also irreducible.

13. Expander Family of Complexes

Let $X = B/\Gamma$ be a finite quotient of B. We wish to understand the action of H on $\mathbb{C}[X_f] = L_2(X_f)$. Recall that this representation is unitary and finite dimensional (see proposition 7.1) and therefore decomposes into a finite direct sum of irreducible representations.

Recall that ρ_{C_0} is the spherical average around the chamber $C_0 \in B_{\phi}$ from definition 4.14.

Proposition 13.1. Let $f \in \mathbb{C}[X_f]$, $C_0 \in X_0$. Let $\tilde{f} \in \mathbb{C}^{B_f}$ be the lift of f from X to B. Let \tilde{C}_0 be a chamber covering C_0 .

- (1) The correspondence $h \to (hf)(C_0)$ is a matrix coefficient of the H-representation $\mathbb{C}[X_f]$.
- (2) A geometric realization around \tilde{C}_0 is given by $\rho_{\tilde{C}_0}\tilde{f}$.
- (3) For every irreducible representation V there exists $C_0 \in X_{\phi}$ such that the matrix coefficient defined on V is non zero.

Proof. (1) follows by definition, since $\mathbb{C}[X_f]$ is finite dimensional and $f \to f(C_0)$ is a functional on $f \in \mathbb{C}[X_f]$. For (2), notice that $h \to (hf)(C_0)$ is a functional in H_ϕ^* and therefore has a geometric realization around C_0 which equals exactly $\rho_{\tilde{C}_0}\tilde{f}$. For (3), note that in every nonzero subrepresentation V there exists a non zero $f \in \mathbb{C}[X_\phi] \cap V$ by lemma 8.3.

Proposition 13.2. The trivial representation appears exactly once in $L_2(X)$. It is the subrepresentation of H which contains the sets of functions that depend only on the color of each face.

Proof. The fact that the constant functions on every color span the trivial representation is immediate. To prove it is the only such representation, choose some function $f \in \mathbb{C}[X_{\phi}]$ which spans a representation isomorphic to the trivial representation. Let $C_0 \in X_{\phi}$ be the chamber on which f gets its maximal values. Since $h_s, s \in S$ acts by q_s , all the chambers adjacent to C_0 must have the same value. Therefore f is constant on the chambers and the representation spanned by f is the representation which contains the constant functions on every color of face.

Definition 13.3. The non trivial representation of H on $\mathbb{C}[X_f]$ is the action of H on $L_2^{00}(X_f) = \{f : X \to \mathbb{C} : \forall I \sum_{\sigma \in X_I} f(\sigma) = 0 \}$. This is the space perpendicular to the trivial representation.

Proposition 13.4. The number of times the Steinberg representation appears in $L_2^{00}(X_f)$ is the dimension of the subspace $\{f \in \mathbb{C}[X_{\phi}] : \partial_{\{s\}}f = 0 \text{ for all } s \in S\}.$

Proof. The claim follows immediately from proposition 8.4.

Definition 13.5. The complex X is an L_p -expander if the representation of H on $L_2^{00}(X)$ is p-tempered. The complex X is a Ramanujan complex if it is an L_2 -expander.

Corollary 13.6. The following are equivalent:

- 1. X is an L_p -expander.
- 2. For every $f \in L_2^{00}(X)$ and $C_0 \in B$, $\rho_{C_0}(\tilde{f}) \in L_{p+\epsilon}(B)$ for every $\epsilon > 0$.
- 3. The action of H_{ϕ} on $L_2^0(X_{\phi}) = \left\{ f : X_{\phi} \to \mathbb{C} : \forall I \sum_{C \in X_{\phi}} f(C) = 0 \right\}$ is p-tempered.

Proof. Follows from proposition 13.1 and lemma 12.7.

14. Representations of the Automorphism Group

In this section we continue the discussion of section 6, this time looking at the representation theory involved. As in section 6 let G' be a general locally profinite group and K is a compact open subgroup with Haar measure 1. Let $H_K = \mathbb{C}_c[K \setminus G'/K]$ be the corresponding Hecke algebra with respect to convolution.

We want to understand the connection between the representation theory of G' and H_K . We will denote a representation of G' by U and a representation of H_K by V. We base the general discussion mainly on [Cas74], part 2.

The case we will be interested in is when G' = G a Weyl transitive color preserving complete automorphism group, $K = G_{\phi}$ is a chamber stabilizer and $H_K = H_{\phi}$. We mainly follow [Bor76] for the results specific to this case.

Definition 14.1. A representation U of G' is called:

- Smooth if every $v \in U$ is fixed by an open subgroup of G'.
- Admissible if for every compact open subgroup $K' \subset G$ the subspace $U^{K'}$ of vectors fixed by K' is finite dimensional.
- Unitary if there exists a G' invariant inner product on U (so that the completion of U is a Hilbert space).

Proposition 14.2. Let U be a smooth representation of G'. Let U^K be the vectors of U fixed by K. Then U^K is a representation space of H_K . If U is irreducible as representation of G then U^K is an irreducible representation of H_K . If U is unitary so is U^K .

Proof. Define $e_K: U \to U^K$ by $e_K v = \int_K k v \, dk$. This integral is actually a finite sum since by smoothness there exists a finite index compact open subgroup $K' \subset K$ with k'v = v for every $k' \in K'$. Then the integral becomes $1/[K:K'] \cdot \sum_{k \in K/K'} k v$. It is easy to see that $e_K v \in U^K$, $e_K|_K = id$ and $e_K^2 = e_K$ (we use here the fact that |K| = 1).

Define an action of the algebra H(G') on U by $f \cdot v = \int_G f(g)gv \, dg$. It is standard to verify that it defines an algebra representation of H(G'). In the action above, the element $1_{KgK} \in H_K(G') \subset H(G')$ acts on U by $e_K g e_K$. Since $e_K U = U^K$ we see that U^K is a representation of $H_K(G')$.

by $e_K g e_K$. Since $e_K U = U^K$ we see that U^K is a representation of $H_K(G')$.

Suppose U is irreducible. Let $0 \neq u \in \subset U^K$. Since U is irreducible, for every $v' \in U^K$ there exist $g_1, ..., g_m \in G$ and $\alpha_1, ..., \alpha_m \in \mathbb{C}$ such that $v' = \sum \alpha_i g_i u$. Since $u, v' \in U^K$ we have $v' = \sum \alpha_i e_K g_i e_K u \in H_K u$. Therefore $H_K u = U^K$ and U^K is an irreducible H_K representation. See [Cas74] and in particular proposition 2.2.4(a) there for more details.

Finally if U is unitary one can use the same inner product on U^K which gives it a unitary H_K structure. This verification is immediate.

To discuss temperedness of G' representations, we define:

Definition 14.3. Let U be an admissible representation of G'. Its dual representation \hat{U} is the action of G on the algebraic dual \hat{U} of U by $(gv^*)(v) = v^*(gv)$.

The contragredient representation \tilde{U} of U are the smooth vectors of \hat{U} , i.e vectors $\tilde{v} \in \hat{U}$ that have a compact open subgroup K' with $k\tilde{v} = \tilde{v}$ for any $k \in K'$.

Lemma 14.4. Let U be an admissible representation of G'. Then $\hat{U}^K = \tilde{U}^K = (\hat{U^K})$ and $\tilde{\tilde{U}} \cong U$.

Proof. [Cas74] 2.1.10.

Definition 14.5. Let U be an admissible representation of G', \tilde{U} the contragredient representation and $v \in U$, $\tilde{v} \in \tilde{U}$. The function $c_{\tilde{v},v}: G' \to \mathbb{C}$, $c_{\tilde{v},v}(g) = \langle \tilde{v}, gv \rangle$ is called a *matrix coefficient* of the representation U.

Since we have a Haar measure on G' the space $L_p(G')$ is well defined. We also define temperedness similar to definition 12.3:

Definition 14.6. Assume that G' is generated by an open set A_0 with compact closure. Then define for $g \in G'$: $l_{A_0}(g) = \min\{n : g \in A_0^n\}.$

Define the space

$$T_p(G') = T_p^{A_0}(G') = \left\{ f : G' \to \mathbb{C} \text{ measurable} : \int |f(g)|^p (1 - \epsilon)^{l_{A_0}(g)} dg < \infty \text{ for every } \epsilon > 0 \right\}$$

Lemma 14.7. 1. The set $T_p^{A_0}(G')$ does not depend on the generating set A_0 . 2. We have $\cap_{\epsilon>0}L_{p+\epsilon}(G')\subset T_p(G')$.

Proof. Let A'_0 be another open generating set with compact closure. Since it is covered by $\cup A_0^n$ it is covered by a finite subset of which and therefore there exists C > 0 such that $l_{A_0}(g') \leq C$ for every $g' \in A'_0$. Therefore $l_{A_0'}(g) \leq C \cdot l_{A_0}(g)$ for every $g \in G'$. Then for every $\epsilon > 0$, $(1 - \epsilon)^{l_{A_0'}(g)} \geq (1 - \epsilon)^{Cl_{A_0}(g)}$ for every $g \in G'$. Therefore $T_p^{A_0}(G') \subset T_p^{A_0'}(G')$. By symmetry $T_p^{A_0'}(G') \subset T_p^{A_0}(G')$ and we have equality.

For (2), we claim that $\mu(A_0^n)$ grows at most exponentially-there exists r > 0, such that $\mu(A_0^n) \le r^{n-1}\mu(A_0)$. By compactness, cover A_0^2 by a finite number of translations $y_i A_0, y_i \in G', i = 1, ..., R$. Then

$$A_0^n \subset \bigcup_{i_i \in \{1, \dots, N\}} y_{i_1} \cdot \dots \cdot y_{i_{n-1}} A_0$$

and $\mu(A_0^n) \leq R^{n-1}\mu(A_0)$. The rest of the proof is as in lemma 12.8.

Definition 14.8. An admissible representation U of G' is p-finite (resp. p-tempered) if for every $v \in U$, $\tilde{v} \in \tilde{U}$ we have $c_{\tilde{v},v} \in L_p(G')$ (resp. $c_{\tilde{v},v} \in T_p(G')$).

Lemma 14.9. An irreducible representation U is p-finite (resp.p-tempered) if for some $0 \neq v \in U$, $\tilde{v} \in \tilde{U}$ we have $c_{\tilde{v},v} \in L_p(G')$ (resp. $c_{\tilde{v},v} \in T_p(G')$).

Proof. As in lemma
$$12.11$$
.

Proposition 14.10. Assume the building B is thick. Let G be a complete Weyl transitive automorphism group of B. Let U be an irreducible representation of G, with $U^{G_{\phi}} \neq \{0\}$. Then U is p-finite (resp. ptempered) if and only if $U^{G_{\phi}}$ is p-finite (resp. p-tempered) as a representation of H_{ϕ} .

Proof. By proposition 14.2, $U^{G_{\phi}}$ is irreducible. Using lemma 12.11, $U^{G_{\phi}}$ is p tempered as H_{ϕ} representation if and only if for some $0 \neq v \in U^{G_{\phi}}$, $0 \neq \tilde{v} \in (\tilde{U}^K)$ we have

(14.1)
$$\sum_{w \in W} q_w^{1-p} \left| \langle \tilde{v}, h_w v \rangle \right|^p (1-\epsilon)^{pl(w)} = \sum_{w \in W} q_w \left| \langle \tilde{v}, h_w / q_w v \rangle \right|^p (1-\epsilon)^{pl(w)} < \infty$$

for every $\epsilon > 0$.

Using the last lemma, the G-representation U is p-tempered if and only if $c_{\tilde{v},v} \in T_p(G')$ for every $\epsilon > 0$. Choose an a compact generating set $A_0 = G_{\phi} \cup (\cup_{s \in S} G_{\phi} g_s G_{\phi})$, where $g_s \in G$ is an element sending $C_0 \in B_{\phi}$ to $C' \in B_{\phi}$ with $d(C_0, C') = s$. One can easily see that unless $g \in G_{\phi}$, $l_{A_0}(g) = l(d(C_0, gC_0))$. Therefore for every $\epsilon > 0$:

$$\int_{G} |c_{\tilde{v},v}(g)|^{p} (1-\epsilon)^{l_{A_{0}}(g)} dg = \int_{G} |\langle \tilde{v}, gv \rangle|^{p} (1-\epsilon)^{l_{A_{0}}(g)} dg = \sum_{G_{\phi}gG_{\phi} \in G_{\phi} \backslash G/G_{\phi}} \mu(G_{\phi}gG_{\phi}) |\langle \tilde{v}, gv \rangle|^{p} (1-\epsilon)^{l_{A_{0}}(g)} = \sum_{Id \neq w \in W} q_{w} |\langle \tilde{v}, h_{w}/q_{w}v \rangle|^{p} (1-\epsilon)^{l(w)} + |\langle \tilde{v}, v \rangle|^{p} (1-\epsilon)$$

(we used the facts that if $d(C_0, gC_0) = w$, then $\mu(G_\phi gG_\phi) = q_w$ and $\langle \tilde{v}, gv \rangle = \langle \tilde{v}, e_K ge_K v \rangle = \langle \tilde{v}, h_w/q_w v \rangle$, since v, \tilde{v} are K-fixed). Since the two conditions 14.1,14.2 are equivalent, we are done.

Lemma 14.11. Every unitary and admissible representation U of G' decomposes into a countable direct sum of irreducible representations $U=\oplus U_i$. For each open compact subgroup $K\subset G'$ only a finite number of the $U_i \text{ have } U_i^K \neq 0.$

Proof. It is similar to 9.2. See [Cas74] proposition 2.1.14.

Let us now discuss how to induce H_K representations to G' representations.

Proposition 14.12. Let V be a representation of H_K .

- (1) The space $\mathbb{C}[G'/K] \otimes_{H_K} V$ is a representation of G' with the natural left action of G' on $\mathbb{C}[G'/K]$.
- (2) The space $\mathbb{C}[G'/K] \otimes_{H_K} V$ is generated as a G' module by its K fixed vectors and $(\mathbb{C}[G'/K] \otimes_{H_K} V)^K$ is naturally isomorphic to V as a H_K representation.
- (3) The functors $\mathbb{C}[G'/K] \otimes_{H_K}$ and $(\cdot)^K$ provide a natural bijection between equivalence classes of irreducible G-representations with K fixed vectors and equivalence classes of irreducible H_K -representations.

Proof. For (1),(2) see [Bor76]. For (3) see [BK93] 4.2.3.

Proposition 14.12 does not give a full description of the connection between G'-representations and H_{K^-} representations. Two important ingredient that are missing are the admissibility of $\mathbb{C}[G/K] \otimes_{H_K} V$ if V is finite dimensional and the unitarity of $\mathbb{C}[G/K] \otimes_{H_K} V$ if V is unitary. In the algebraic-group case we have an answer to those questions, due to Borel ([Bor76]) and Barbasch and Moy ([BM93]):

Theorem 14.13. Let G be the rational points of a connected semisimple algebraic group \mathbb{G} over a non- $Archimedean\ local\ field\ k$.

Let B be the locally finite regular affine building corresponding to G. Let G_{ϕ} be a chamber stabilizer (i.e an Iwahori subgroup). Then:

- (1) If V is a finite dimensional representation of H_{ϕ} , then $\mathbb{C}[G/G_{\phi}] \otimes_{H_{\phi}} V$ is admissible.
- (2) The functors $\mathbb{C}[G/G_{\phi}] \otimes_{H_{\phi}}$ and $(\cdot)^{G_{\phi}}$ are exacts functors between admissible G representations and finite dimensional H_{ϕ} representations.
- (3) Every admissible representation U of G is a direct sum of the representation U_1 generated by $U_1^{G_{\phi}}$ and a representation U_2 with $U_2^{G_{\phi}} = 0$. (4) If V is finite dimensional and unitary then $\mathbb{C}[G/G_{\phi}] \otimes_{H_{\phi}} V$ is unitary.

Proof. (1),(2) and (3) are the main results of [Bor76]. The unitarity question is harder and was open for a long time. It was solved using the classification of unitary finite dimensional H_{ϕ} -representations. See [BM93].

Remark 14.14. The representations considered in the theorem above are the well known unramified principal series representations of G- i.e. the representations induced from an unramified character of a maximal torus. See [Bor76].

Problem 14.15. Can a similar answer for unitarity and admissibility be given for arbitrary locally finite regular buildings, perhaps assuming some transitivity property of the automorphism group? In particular, can a similar theorem be stated for right-angles buildings?

Since theorem 14.13 is rather deep and does not always apply, we define:

Definition 14.16. A representation V of H is G-unitary if it is a restriction to H of a unitary representation of G.

From proposition 14.2 if a representation is G-unitary then it is unitary. The converse is true in the algebraic group case by theorem 14.13.

15. Oh's Theorem

The results of [Oh02], specifically theorem 7.4, state:

Theorem 15.1. Let k be a non-Archimedean local field with chark $\neq 2$. Let G be the group of k-rational points of a connected linear almost k-simple algebraic group with k-rank ≥ 2 . Then every non trivial infinite dimensional unitary representation of G is p_0 -tempered for some explicit p_0 depending only on the affine Weyl group W. Explicitly, the bounds are (for $n \geq 2$):

W	\tilde{A}_n	\tilde{B}_n	\tilde{C}_n	\tilde{D}_n , n even	\tilde{D}_n , n odd	\tilde{E}_6	\tilde{E}_7	\tilde{E}_8	\tilde{F}_4	\tilde{G}_2
p_0	2n	2n	2n	2(n-1)	2n	16	18	29	11	6

Remark 15.2. The bounds given by the theorem are not in general optimal (but are optimal in the \tilde{A}_n , $n \geq 2$ case). See the discussion in [Oh02], after theorem 7.4.

Remark 15.3. Recall from Example 23.3 that the trivial representation is not p-tempered for any $p < \infty$.

Recall that every G as above acts as an automorphism group on a building, which by taking a finite index subgroup we may assume is color preserving. As a corollary we can say:

Corollary 15.4. Let V be a non trivial G-unitary representation of H corresponding to G as above. Then V is p_0 tempered for some explicit p_0 .

Remark 15.5. The paper [Oh02] gives an emphasis to the Coxeter group involved and the bounds are found using the geometry of the root system. One may therefore expect that the same results would apply to general Affine Iwahori-Hecke algebras, with arbitrary parameter system \overrightarrow{q} (under the assumption that $q_s > 1$, $s \in S$). We are not aware of such a result.

16. Representations of the Automorphism Group - Quotients of Buildings

In this section we connect the action of H on quotients $X \cong B/\Gamma$ to the representation theory of G. This will allow us to use strong results from the representation theory of reductive groups in our combinatorical setting.

Recall we defined $C_l^{\infty}(G)$ as the set of functions $f: G \to \mathbb{C}$ such that there exists a compact open subgroup $K \subset G$ with f(kg) = f(g) for every $g \in G$, $k \in K$. Equivalently $f \in C_l^{\infty}(G)$ if for the left regular action of G on functions $f' \in \mathbb{C}^G$, $(g \cdot f')(x) = f'(g^{-1}x)$, f is stabilized by some compact open subgroup K.

Proposition 16.1. Let $\Gamma \subset G$ be a discrete cocompact subgroup of the automorphism group of the building. Let $C^{\infty}(G/\Gamma) = \{f \in C_l^{\infty}(G) | f(g\gamma) = f(g) \text{ for every } \gamma \in \Gamma, g \in G\}.$

1. $C^{\infty}(G/\Gamma)$ is a representation of G given by the left action

$$(q \cdot f)(x) = f(q^{-1} \cdot x)$$

- 2. This representation is (2.a) smooth, (2.b) admissible and (2.c) unitary.
- 3. Let $K \subset G$ be a compact open subgroup. Then $C^{\infty}(G/\Gamma)^K \cong \mathbb{C}[K\backslash G/\Gamma]$ as a finite dimensional H_K representation.

Proof. (1) It is enough to prove that if f is smooth so is $g \cdot f$ for every $C_l^{\infty}(G)$. This is immediate since if f is fixed by K, $g \cdot f$ is fixed by gKg^{-1} .

- (2.a.) By definition of a smooth function.
- (2.b. + 3) Let K be a compact open subgroup. By definition $C^{\infty}(G/\Gamma)^K$ is the set of functions $f: G \to \mathbb{C}$ such that $f(kx\gamma) = f(x)$, $\gamma \in \Gamma$, $x \in G$, $k \in K$, which can be identified by $\mathbb{C}^{K \setminus G/\Gamma}$. To prove that $C^{\infty}(G/\Gamma)^K \cong \mathbb{C}[K \setminus G/\Gamma]$ and it is finite dimensional it is enough therefore to prove that $K \setminus G/\Gamma$ is finite. By definition it is true for G_{ϕ} and for general compact open subgroup K, $G_{\phi} \cap K$ has finite index in G_{ϕ} . Therefore

$$|K \backslash G/\Gamma| \le |K \cap G_{\phi} \backslash G/\Gamma| \le [G_{\phi} : K \cap G_{\phi}] |G_{\phi} \backslash G/\Gamma|$$

and the last value is finite.

(2.c) We define an inner product: for every two functions $f_1, f_2 \in C^{\infty}(G/\Gamma)$ there exists a compact open subgroup $K \subset G_{\phi}$ such that f_1, f_2 can be written as a finite sum $f_1 = \sum_i \alpha_i 1_{Kg_i\Gamma}$, $f_2 = \sum_i \beta_i 1_{Kg_i\Gamma}$. Define $\langle f_1, f_2 \rangle = [G_{\phi} : K]^{-1} \sum_i \bar{\alpha}_i \beta_i$. It is easy to see that this is indeed an inner product and does not depend on the choice of K.

Corollary 16.2. Assume B has a complete Weyl transitive automorphism group G. Then the representation of H_{ϕ} on $\mathbb{C}[X_{\phi}] \cong \mathbb{C}[B_{\phi}/\Gamma]$ is G-unitary and is the restriction of the action of G on $\mathbb{C}^{\infty}(G/\Gamma)$.

The decomposition of $\mathbb{C}[X_{\phi}]$ into a sum of irreducible representation of H_{ϕ} is given by:

- 1. The decomposition of $\mathbb{C}^{\infty}(G/\Gamma)$ into a sum of irreducible representations of G.
- 2. Removing all representations without G_{ϕ} fixed vectors.
- 3. Restriction of the finite number of resulting representations to G_{ϕ} fixed vectors.

In addition, X is an L_p -expander if and only if every non trivial irreducible subrepresentation of G on $\mathbb{C}^{\infty}(G/\Gamma)$ with non zero G_{ϕ} fixed vectors is p-tempered.

Proof. The decomposition into irreducible representations follows from proposition 16.1 and proposition 14.2. The temperedness follows from proposition 14.10.

We can now state one of the main results of this work.

Theorem 16.3. Let k be a non-Archimedean local field with chark $\neq 2$. Let G be the group of k-rational points of a connected linear almost k-simple algebraic group with k-rank ≥ 2 .

Let B be the corresponding building on which G acts and Γ a cocompact torsion free lattice in G.

Then $X \cong B/\Gamma$ is an L_{p_0} -expander, where $p_0 = p_0(W)$ depends only on the affine Weyl group W and is given in the table in theorem 15.1.

Proof. Follows from proposition 16.2 and theorem 15.1.

Remark 16.4. Every affine building B of dimension ≥ 3 with irreducible Weyl group W corresponds to such a group G, so this theorem is quite general. A positive answer to the question in 15.5 would show that the theorem is true for dimension 2 as well.

Remark 16.5. By [Fir16], the complexes constructed in [LSV05a] are Ramanujan, i.e. satisfy definition 1.1.

Remark 16.6. The definition in [LSV05b] (and similarly the definition in [CSZ03]) only considers functions on vertices of the graphs and the eigenvalues of the spherical Hecke operators for buildings of type \tilde{A}_n . It is shown there that this property is equivalent to the 2-temperedness of any such function which is not part of the trivial representation (assuming for simplicity that Γ is color preserving). This is equivalent to the property that any subrepresentation of H on $\mathbb{C}[X_f]$ which is not trivial and non zero on vertices is 2-tempered (i.e. its restriction to some H_{I^-} I a color of vertex- is non trivial). In the context of the automorphism group, it is equivalent to considering only subrepresentations of $\mathbb{C}^{\infty}(G/\Gamma)$ with G_I fixed vectors, for some vertex color I. Therefore the definition in this paper is aperiori stronger than the definition in [LSV05b], although we do not know if there exists a complex X lying in the gap between the definitions. A necessary condition for the existence of X is the existence of a finite dimensional unitary representation V of V that is nullified on vertices (i.e. V is the existence of a finite dimensional unitary representation V does not exist for V does exist for

In any case if one wants to consider operators acting on all the colors of faces, the definition in this paper is more adequate.

Remark 16.7. Corollary 16.2 suggests a generalized way of defining an L_p -expander- the requirement that every non trivial irreducible subrepresentation of G on $\mathbb{C}^{\infty}(G/\Gamma)$ is p-tempered. However, this definition seems to be dependent on the group G considered and not only the complexes X and B. See also [Fir16].

Part 3. Spectrum of Operators

17. Spectrum and Weak Containment

Definition 17.1. Let V be a representation of the ADH algebra H and $h \in H$. The point spectrum $\Sigma_V^p(h)$ is the set of eigenvalues of h on V.

We say that V supports spectrum if it is either finite dimensional or normed with the elements of H acting as bounded operators. If V is finite dimensional, we define $\Sigma_V(h) = \Sigma_V^{ap}(h) = \Sigma_V^p(h)$. If V is normed, the spectrum $\Sigma_V(h)$ of h, is the set of $\lambda \in \mathbb{C}$ such that $h - \lambda$ does not have an inverse with bounded norm on V. The approximate point spectrum $\Sigma_V^{ap}(h)$ of h is the set of $\lambda \in \mathbb{C}$ such that there exists a series of vectors $v_n \in U$, $||v_n||_V = 1$, such that $||hv_n - \lambda v_n||_V \to 0$.

Denote the spectral radius $\lambda_V(h)$ of h by $\lambda_V(h) = \sup\{|\lambda| | \lambda \in \Sigma_V(h)\}$, and if V is normed the norm $||h||_V$ of h by $||h||_V = \sup\{||hv||_V : v \in V, ||v||_V = 1\}$.

The following is standard:

Lemma 17.2. We have Gelfand's formula $\lambda_V(h) = \limsup \|h^n\|_V^{1/n}$ and if V is unitary, then $\|h\|_V = \sqrt{\lambda_V(hh^*)}$.

Now we can now compare arbitrary representations that support spectrum:

Definition 17.3. Assume V_1, V_2 are H-representations that support spectrum. If for every $h \in H$, $\lambda_{V_1}(h) \le \lambda_{V_2}(h)$ we say that V_1 is weakly contained in V_2 .

Remark 17.4. The above definition is not standard as far as we know. Similar definitions exist for unitary representations of locally compact groups and C^* algebras (see [CHH88] and references therein). One of the equivalent definitions is that for every $h \in H$, $||h||_{V_1} \leq ||h||_{V_2}$. For unitary representations (but not for general normed representations) this is equivalent to our definition by lemma 17.2.

18. The Point Spectrum of
$$T_p(B_f)$$

We want to understand the eigenvalues of the action of H on $T_p(B_f)$. Recall definition 12.3 of f_δ for $f \in \mathbb{C}^{B_f}$. We will need the following lemma:

Lemma 18.1. Let $f \in T_p(B_f)$ and $h \in H$. Then $||h(f_\delta) - (hf)_\delta||_p / ||f_\delta||_p \to 0$ as $\delta \to 0$. Therefore for every $\lambda \in \mathbb{C}$, $||hf_\delta - \lambda f_\delta||_p / ||f_\delta||_p \to 0$ if and only if $||(hf)_\delta - \lambda f_\delta||_p / ||f_\delta||_p \to 0$.

Proof. Write $h = \sum_d \alpha_d h_d$. Let $L = \max_{d:\alpha(d) \neq 0} l(d)$ (l(d) is as in the temperedness definition), $|h| = \sum_d |\alpha_d| h_d$.

Let $F \in \mathbb{C}^{B_f}$, $F(\sigma) = (|h||f|)(\sigma)$. From 12.9 we have $||F_{\delta}||_p \leq M ||f_{\delta}||_p$ for some $M \in \mathbb{R}_{>0}$ and $\delta > 0$ small enough.

Let σ be some face. We wish to understand $|hf_{\delta}(\sigma) - (hf)_{\delta}(\sigma)|$. For $l(d(\sigma, \sigma')) \leq L$ we have $f_{\delta}(\sigma') = (1 - \delta)^{l(\sigma)} f(\sigma') + (1 - \delta)^{l(\sigma)} r_{\sigma}(\sigma') f(\sigma')$ for some

$$1 - (1 - \delta)^{-L} \le r_{\sigma}(\sigma') = 1 - (1 - \delta)^{l(\sigma') - l(\sigma)} \le 1 - (1 - \delta)^{L}$$

Notice that for δ small enough $|r_{\sigma}(\sigma')| \leq 4\delta L$.

Let $g_{\sigma} \in \mathbb{C}^{B_f}$ be defined by $g_{\sigma}(\sigma') = r_{\sigma}(\sigma')f(\sigma')$. We have $f_{\delta} = (1 - \delta)^{l(\sigma)}(f + g_{\sigma})$. For $l(d(\sigma, \sigma')) \leq L$ we have $|g_{\sigma}|(\sigma') = |r_{\sigma}(\sigma')| |f(\sigma')| \leq 4L\delta |f|(\sigma')$.

Now

$$|(hf_{\delta})(\sigma) - (hf)_{\delta}(\sigma)| = \left| (1 - \delta)^{l(\sigma)} ((hf)(\sigma) + (hg_{\sigma})(\sigma)) - (hf)_{\delta}(\sigma) \right| =$$

$$= \left| (1 - \delta)^{l(\sigma)} (hg_{\sigma})(\sigma) \right| \leq (1 - \delta)^{l(\sigma)} (|h| |g_{\sigma}|)(\sigma)$$

$$\leq (1 - \delta)^{l(\sigma)} 4L\delta(|h| |f|)(\sigma) \leq 4L\delta F_{\delta}(\sigma)$$

Taking the p-s power and summing over all $\sigma \in B_f$, we have:

$$\left\|hf_{\delta}(C)-(hf)_{\delta}\right\|_{p} \leq \delta 4L \left\|F_{\delta}\right\|_{p} \leq \delta 4LM \left\|f_{\delta}\right\|_{p}$$
 and as $\delta \to 0$, $\left\|hf_{\delta}(C)-(hf)_{\delta}\right\|_{p}/\left\|f_{\delta}\right\|_{p} \to 0$ as required. \Box

Corollary 18.2. The point spectrum of $h \in H$ on $T_p(B_f)$ is contained in the approximate point spectrum of h on $L_p(B_f)$.

Proof. Assume $f \in T_p(B_f)$ such that $hf = \lambda f$. Therefore we have $\|(hf)_{\delta} - \lambda f_{\delta}\|_p / \|f_{\delta}\|_p = 0$. By the last lemma $\|hf_{\delta} - \lambda f_{\delta}\|_p / \|f_{\delta}\|_p \to 0$ and λ is in the approximate point spectrum of h on $L_p(B_f)$.

Corollary 18.3. Let V be a representation of H, $0 \neq v \in V$, $h \in H$ and $hv = \lambda v$.

Assume that some non zero geometric embedding of v is p-tempered. Then λ belongs to the approximate point spectrum of h on $L_p(B_f)$.

Corollary 18.4. If a finite dimensional representation V is p-tempered then V is weakly contained in $L_p(B_f)$. More precisely, for every $h \in H$ the set of eigenvalues of h on V is contained in the approximate point spectrum of h on $L_p(B_f)$.

Remark 18.5. The same logic allows us to compare arbitrary admissible G representations with the left action of G on $L_p(G) \cap C_l^{\infty}(G)$, using the H(G) action on the two spaces. Notice that for every $K \subset G$ the action of H(G,K) on the K-fixed vectors of an admissible representation is finite dimensional and its action on the K-fixed vectors of $L_p(G) \cap C_l^{\infty}(G)$ is normed. In particular, the same proof shows that if V is admissible and p-tempered then it is weakly contained in $L_p(G) \cap C_l^{\infty}(G)$. It can be probably generalized to other locally profinite groups, since no essential property of the building was used, Thus generalizing theorem 1 on [CHH88] in the locally profinite case to $p \neq 2$.

19. Generalized Serre Theorem

The following proposition generalizes a well known theorem, usually attributed to Serre (but also appears in [McK81]), for graphs with large injectivity radius (or girth). It applies to any normal element of H. The proof is based on [Li04]. Compare also [Fir16], theorem 5.1.

Definition 19.1. Let X be a quotient of the building B. The *injectivity radius* of X is the length of the shortest distance $d \in W$ between two chambers $C_1 \neq C_2$ of the building that cover the same chamber in X.

Theorem 19.2. Let $h \in H$ normal operator and λ in the spectrum of h on $L_2(B_f)$. Then there exists an $\epsilon(N) = \epsilon_{B,h}(N)$ with $\epsilon(N) \to 0$ as $N \to \infty$, such that for every finite quotient X of the building B with injectivity radius greater than N, there exists λ' in the spectrum of h on \mathbb{C}^{X_f} , with $|\lambda - \lambda'| < \epsilon(N)$.

Proof. Let l=l(d) be the largest length of an element h_d appearing in h. We claim that for any $\epsilon>0$ there exists an $N\in\mathbb{N}$ and $0\neq f_N\in\mathbb{C}[B_f]$, such that f_N is supported on faces at distance N/2-l around C_0 and we have $\|hf_N-\lambda f_N\|_2/\|f_N\|_2<\epsilon$. Take an approximate eigenfunction $f\in L_2(B_f)$ with $\|hf-\lambda f\|/\|f\|<\epsilon/2$ and call its restriction to distance N/2-l, $f_N\in\mathbb{C}[B_f]$. Taking $N\to\infty$ we know that $\|hf_N-\lambda f_N\|/\|f_N\|\to\|hf-\lambda f\|/\|f\|<\epsilon/2$. So there exists a finite N with $\|hf_N-\lambda f_N\|/\|f_N\|<\epsilon$, as needed.

By the assumptions on f_N , if X_f has injectivity radius greater than N, both f_N and hf_N can be projected to $L_2(X_f)$, i.e. we have $f \in L_2(X_f)$ with $||hf - \lambda f||_X / ||f||_X < \epsilon$. By the normality of h, there exists an eigenvalue of $h - \lambda$ on $L_2(X_f)$, with absolute value smaller than ϵ , and the claim follows.

Problem 19.3. The original proof of [McK81] shows that as the injectivity radius grows, the spectrum of the adjacency operator converges to the spectral measure of the adjacency operator on the tree. We therefore ask if it holds here as well, i.e. the spectrum of every normal (or self-adjoint) h converges to the spectral measure of h on $L_2(B_f)$. Compare (in slightly different settings) [ABB+12], theorem 1.2.

20. Alon-Boppana Theorem

The following proposition generalizes directly the classical Alon-Boppana theorem. For simplicity we consider operators of H_{ϕ} only. Our treatment follows [Lub94] proposition 4.5.4.

Definition 20.1. An element $h \in H_{\phi}$ is called a random-walk operator if it is self adjoint and a non-negative sum of the basis operators h_w .

A random walk operator defines (after normalization) a random walk on B_{ϕ} . Since we have symmetry among all all chambers, Kesten's argument in [Kes59], lemma 2.11, gives that $\lambda_2(h) = ||h||_2 =$ $\limsup (\|h^n 1_C\|_2)^{1/n}$. We can now state:

Theorem 20.2. Let X be a quotient of the building B. Assume the largest distance (in gallery length) between two chambers in X is N. Let $h \in H_{\phi}$ be a random-walk operator.

Then there exists an $\epsilon(N) = \epsilon_{B,h}(N)$ with $\epsilon(N) \to 0$ as $N \to \infty$, such that the largest eigenvalue of h on $L_2^0(X_\phi)$ is at least $\lambda_2(h) - \epsilon(N)$.

Proof. Write $\|..\|_B$, $\|..\|_X$ for the L_2 norms of the two spaces. Choose two chambers C_0^X , $C_1^X \in X_\phi$ of distance N and let $C_0, C_1 \in B_\phi$ be two chambers which cover C_0^X, C_1^X and are of distance N. The fact that h is a non negative sum of h_w -s tells us that $h^n 1_{C_0}$ is positive in every coordinate. Since $h^n 1_{C_0^X}$ is the projection of $h^n 1_{C_0}$, the norm just grows, i.e. $\|h^n 1_{C_0}\|_B \le \|h^n 1_{C_0^X}\|_X$. Let l = l(w) the largest distance of an element $w \in W$ appearing in h. Then $h^n 1_{C_0^X}$ and $h^n 1_{C_1^X}$ have

disjoint supports for n < N/l. Therefore:

$$\left\|h^{n}(1_{C_{0}^{X}}-1_{C_{1}^{X}})\right\|_{X} = \left\|h^{n}1_{C_{0}^{X}}-h^{n}1_{C_{1}^{X}}\right\|_{X} = \left\|h^{n}1_{C_{0}^{X}}\right\|_{X} + \left\|h^{n}1_{C_{0}^{X}}\right\|_{X} \ge$$

$$\geq \|h^{n}1_{C_{0}}\|_{B} + \|h^{n}1_{C_{0}}\|_{B} \geq 2(\lambda_{2}(h) - \epsilon(n))^{n}$$

for some $\epsilon(n) \to 0$ as $n \to \infty$.

Notice that $1_{C_0^X} - 1_{C_1^X} \in L_2^0(X_\phi)$. Take the *n*-th root. We found that there exists a function $f \in L_2^0(X_\phi)$ with $(\|h^n f\|_X / \|f\|_X)^{1/n} \ge \lambda_2(h) - \delta(n)$. Since h is self adjoint the last inequality means it has an eigenvalue of absolute value $\ge \lambda_2(h) - \epsilon(n)$.

Remark 20.3. Notice that given $h \in H$ one can estimate $\epsilon(n)$ in this theorem, by analyzing the rate of convergence of $(\|h^n 1_C\|_2)^{1/n}$ to $\lambda_2(h)$.

Remark 20.4. To extend the result to all of random-like operators of H one should replace chambers by other faces. Then the same proof applies for $\tau(h) = \max_{I \text{ spherical }} \limsup \left(\|h^n 1_{\sigma}\|_2 \right)^{1/n}$ where σ is some face of color I, and by extension of Kesten's argument $\tau(h) = \lambda_2(h)$. The details are left to the reader.

Part 4. The Affine Case

21. Color Rotations

Before discussing the affine case we should extend our algebra a little by color rotations. It is useful since this way we can talk about quotients by type rotating automorphisms. It will also be easier to work with the affine Hecke algebra. Since the claims are simple and similar to previous ones, we skip the proofs.

Definition 21.1. The automorphism group of S is the group of bijections $\omega: S \to S$ preserving the Coxeter values $m_{i,j}$ - i.e. for every $s,t \in S$, $m_{s,t} = m_{\omega(s),\omega(t)}$. Denote by $\hat{\Omega}$ a subgroup of the automorphism group of S, such that $q_s = q_{\omega(s)}$ for every $s \in S$ and $\omega \in \hat{\Omega}$.

While the restriction $q_s = q_{\omega(s)}$ is not really necessary, it will be simpler to assume it. The action of $\omega \in \hat{\Omega}$ on S extends to a group automorphism $\omega:W\to W$. We can therefore define:

Definition 21.2. The group $\hat{W} = W \times \hat{\Omega}$ is called the $\hat{\Omega}$ -extended Coxeter group.

Our standard to semi-direct product is that multiplication in W is given by $\omega \cdot w = \omega(w) \cdot \omega$ and the relations in W, Ω .

Lemma 21.3. By defining $l(\omega) = 0$ for $\omega \in \hat{\Omega}$, we can extend the length function $l: W \to \mathbb{N}$ to $l: \hat{W} \to \mathbb{N}$.

Notice that every $\omega \in \hat{\Omega}$ acts on distances as well $\omega : W_{I_1} \setminus W/W_{I_2} \to W_{\omega(I_1)} \setminus W/W_{\omega(I_2)}, \ \omega(W_{I_1} \setminus w/W_{I_2}) =$ $W_{\omega(I_1)}\backslash\omega(w)/W_{\omega(I_2)}.$

Denote $\hat{B}_f = B_f \times \hat{\Omega}$ and $\hat{B}_I = B_I \times \hat{\Omega}$. For every face of the building, each $\omega \in \Omega$ can be associated to a possible recoloring of it. Therefore $(\sigma, \omega) \in B_f \times \hat{\Omega} = \hat{B}_f$ can be seen as a "recolored face" in the "recolored building". We define an action $h_{\omega}: \mathbb{C}^{\hat{B}_f} \to \mathbb{C}^{\hat{B}_f}$ by the "recoloring" $h_{\omega}f(\sigma,\omega') = f(\sigma,\omega'\omega)$. We also have an action of H on $\mathbb{C}^{\hat{B}_f}$ acting on every coloring separately, because $\mathbb{C}^{\hat{B}_f} \cong \bigoplus_{\omega \in \Omega} \mathbb{C}^{B \times \{\omega\}}$. Then it is easy to notice that $h_{\omega}h_d = h_{\omega(d)}h_{\omega}$ for every $d \in W_{I_1} \backslash W/W_{I_2}$.

Definition 21.4. The $\hat{\Omega}$ -extended Iwahori-Hecke algebra \hat{H}_{ϕ} is the algebra generated by the $h_{\omega}, \omega \in \Omega$ action on \mathbb{C}^{B_f} and by H_{ϕ} . The $\hat{\Omega}$ -color rotating all dimensional Hecke algebra \hat{H} is the algebra generated by $h_{\omega}, \omega \in \hat{\Omega}$ and by $h \in H$.

The following proposition states some basic properties of the algebras:

Proposition 21.5. We have $\hat{H}_{\phi} = H_{\phi} \times \hat{\Omega}$ and $\hat{H} = H \times \hat{\Omega}$, i.e. as sets we have a direct product and we have $h_{\omega}h_d = h_{\omega(d)}h_{\omega}$ for every $d \in W_{I_1} \backslash W/W_{I_2}$

The algebra \hat{H}_{ϕ} is generated by the Iwahori-Hecke relations as well as the relation $h_{\omega}h_{w}=h_{\omega(w)}h_{\omega}$ for $w \in W$, $\omega \in \Omega$. Define for $w' = w \cdot \omega \in \hat{W}$, $h_{w'} = h_w h_\omega$. The algebra \hat{H}_ϕ is spanned by h_w , $w \in \hat{W}$. For $w, w' \in \hat{W}$ with l(w) + l(w') = l(ww') we have $h_w h_{w'} = h_{ww'}$. The algebra \hat{H} is spanned as a vector space by $h_{\omega}h_d$, $\omega \in \Omega$, $d \in W_{I_1} \backslash W/W_{I_2}$.

Let us turn to some of the representation theory involved. First, similar to proposition 10.6 we have an equivalence of categories between H_{ϕ} -representations and H-representations. This equivalence preserves irreducible representations, unitary representations, finite dimensional representations, p-finite representations and p-tempered representations.

The induction and restriction operators of section 10 can be used to study the relations between representations of the ADH algebra H (or the Iwahori-Hecke algebra H_{ϕ}) and the Ω -color rotating ADH algebra H(or \hat{H}_{ϕ}) from section 21. This time H is the smaller algebra and \hat{H} contains it.

The main difference between this case and section 10 is that the unit is the same in both algebras, so the situation resembles induction and restriction between representations of a group and a subgroup. We will only state the proposition below. The proof is omitted.

Proposition 21.6. Let V be a representation of H and $V' = ind_H^{\hat{H}}V$ the induced representation of \hat{H} . Let U be a representation of \hat{H} and $U' = res_H^{\hat{H}}U$ the restricted representation of H. Then:

- 1. As a vector space $V' \cong V \otimes \mathbb{C}[\Omega]$ and therefore $\dim V' = |\Omega| \dim V$.
- 2. As vector spaces $U' \cong U$ and therefore they have the same dimension.
- 3. $res_H^{\hat{H}}V' = res_H^{\hat{H}}ind_H^{\hat{H}}V$ is isomorphic to a direct sum of $|\Omega|$ times the representation V.
- 4. V is unitary if and only if V' is unitary and U is unitary if and only if U' is unitary.
- 5. V is p-tempered if and only if V' is p-tempered and U is p-tempered if and only if U' is p-tempered. 6. If $V \cong V_1 \oplus V_2$ then $V' \cong ind_H^{\hat{H}}V_1 \oplus ind_H^{\hat{H}}V_2$. If $U \cong U_1 \oplus U_2$ then $U' \cong res_H^{\hat{H}}U_1 \oplus res_H^{\hat{H}}U_2$.
- 7. Matrix coefficients give geometric realization of U as a subrepresentation of \hat{H} on $\mathbb{C}^{\hat{B}_f} = \mathbb{C}^{B_f \times \Omega}$.
- 8. Let X = B or $X = B/\Gamma$ be a building or a quotient of a building. Then we have a unitary representation of H on $L_2(X_f) = L_2(X_f \times \Omega)$. The algebra H acts on the same space by restriction. If Γ is color preserving then $L_2(\hat{X}) \cong ind_H^{\hat{H}} L_2(X)$.

Notice that irreducibility is not necessarily preserved by the induction and restriction operations. In particular, we do not have an equivalence of categories between H representations and \hat{H} representations.

22. Affine Root Systems

Most of the following is very standard. We follow [Par06] for some details about reducible root systems that can be ignored when first reading this.

Let R be a possibly reduced, crystallographic irreducible root system in a euclidean space V_R of dimension n. In other words: (i) R is a finite set of elements $\alpha \in V$ which span V. (ii) For every $\alpha \in R$ we have $s_{\alpha}(R) = R$ where $s_{\alpha}: V_R \to V_R$ is the reflection defined by $s_{\alpha}(x) = x - 2(\langle \alpha, x \rangle / \langle \alpha, \alpha \rangle)\alpha$. (iii) We have $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$ for every $\alpha, \beta \in R$. (iv) The s_{α} -s do not stabilize any non trivial proper subspace of V_R .

The set of *simple roots* is denoted $\Delta = \{\alpha_i : i = 1, ..., n\}$. It is unique after the choice of positive roots. The corresponding *coroot system* is $R^{\vee} = \{\alpha^{\vee} | \alpha^{\vee} = 2\alpha / \langle \alpha, \alpha \rangle, \alpha \in R\}$ with a set of simple coroots $\{\alpha_i^{\vee} : i = 1, ..., n\}$. The set of *simple coweights* is $\hat{\Delta} = \{\beta_i : i = 1, ..., n\}$. It is the dual basis of Δ , i.e. we have $\langle \alpha_i, \beta_j \rangle = \delta_{i,j}$.

The coroot lattice is $Q = \{\sum_{i=1}^n z_i \alpha_i^{\vee} : z_i \in \mathbb{Z}\}$. The coweight lattice is $P = \{\lambda \in V : \langle \lambda, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in R\} = \{\sum_{i=1}^n z_i \hat{\alpha}_i : z_i \in \mathbb{Z}\}$. The coroot lattice Q is a sublattice of the coweight lattice P and the group $\hat{\Omega} = P/Q$ is finite and abelian. The set of dominant coweights is $P^+ = \{\lambda \in V : \langle \lambda, \alpha \rangle \in \mathbb{N} \ \forall \alpha \in R\} = \{\sum_{i=1}^n z_i \beta_i : z_i \in \mathbb{N}\}$. From this description it is immediate that every $\beta \in P$ can be written as $\beta_1 - \beta_2$, $\beta_1, \beta_2 \in P^+$.

The spherical Weyl group is $W_0 = \langle s_\alpha | s_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha, \alpha \in R \rangle$. It is generated by the reflections determined by R. The set of simple roots allows us to identify W_0 with the Coxeter group generated by $s_i = s_{\alpha_i}, i = 1, ..., n$.

The affine Weyl group is $W = Q \rtimes W_0$. W is the Coxeter group generated by $s_1, ..., s_n$ and another affine reflection s_0 , defined by $s_0(x) = x - (\langle \alpha_0^{\vee}, x \rangle - 1) \alpha_0$ where α_0 is the highest root.

The extended affine Weyl group is $\hat{W} = P \times W_0$. The (finite and abelian) group $\hat{\Omega} = P/Q$ is isomorphic to a subgroup of the automorphism group of S, and we have $\hat{W} = W \times \hat{\Omega}$. The results of section 21 apply to it. In general $\hat{\Omega}$ is not the full automorphism group of S.

Since we will work with vertices we will call vertices of color $\{0,...,n\}-i$, vertices of $type\ i$. A vertex type i is called good (as in the notations of [Par06], section 3.4) if there exists $\omega_i \in \hat{\Omega}$ with $\omega_i(0) = i$. The good vertices are equal to the more standard $special\ vertices$, except for root systems of type BC_n or C_n . For every $0 \neq \omega \in \hat{\Omega}$, we have $\omega(0) \neq 0$, and therefore there exists a bijection between $\hat{\Omega}$ and the good types. The coroot lattice Q corresponds to the vertices of type 0 in \mathbb{W} and the coweight lattice P corresponds to the vertices of good type in \mathbb{W} .

As explained in [Par06] section 3.8, we may assume that $q_s = q_{\omega(s)}$ for every $s \in S$ and $\omega \in \hat{\Omega}$. This is the reason we do not assume the root system is reduced, see also the bipartite graph example below. The results of section 21 apply to this case. We define the extended ADH algebra $\hat{H} = H_{\hat{\Omega}} = H \rtimes \hat{\Omega}$ and the extended Iwahori-Hecke algebra $\hat{H}_{\phi} = H_{\phi} \rtimes \hat{\Omega}$.

The Coxeter complex \mathbb{W} is isomorphic as a topological space to V_R . The different reflections cut V_R into chambers (sometimes called alcoves) and this defines a simplicial structure on V_R which is isomorphic to \mathbb{W} . The chambers correspond to the elements of W, and two chambers share a panel of type s if and only if they correspond to elements of the from w, ws.

The fundamental chamber is the set $\{v \in V_R : \langle \alpha_i, v \rangle > 0, i = 1, ..., n, \langle \alpha_0, v \rangle < 1\}$. The fundamental (or dominant) sector is the set $\{v \in V_R : \langle \alpha_i, v \rangle > 0, i = 1, ..., n\}$. The fundamental parallelotope is the set $\{\sum_{i=1}^n x_i \beta_i : 0 \le x_i \le 1\}$. We denote by A_0 the set of $w \in W$ corresponding to the chambers of the fundamental parallelotope and by \hat{A}_0 the set of $\hat{w} \in \hat{W}$ corresponding to such chambers. We have $\hat{A}_0 = A_0 \cdot \Omega$ (as sets, multiplication takes place in \hat{W}). It is standard that $|\hat{A}_0| = W_0$.

We now state a basic structure theorem for the extended Coxeter group \hat{W} . Surprisingly, we could not find a standard reference for this theorem in the literature (It does appear however in [GSS12], proof of theorem 8.2).

Theorem 22.1. Each element $w \in \hat{W}$ can be written uniquely as $w = w_0 \beta a$, with $w_0 \in W_0$, $\beta \in P^+$ and $a \in \hat{A}_0$. Moreover, this decomposition satisfies $l(w) = l(w_0) + l(\beta) + l(a)$.

The element $\beta \in P^+$ satisfies $\beta = \prod_{i=1}^n \beta_i^{m_i}$ for some unique $m_i \in \mathbb{N}$ and we have $l(\beta) = \sum m_i l(\beta_i)$.

As a corollary,
$$h_w = h_{w_0} \left(\prod_{i=1}^n h_{\beta_i}^{m_i} \right) h_a$$
 and $q_w = q_{w_0} \left(\prod_{i=1}^n q_{\beta_i}^{m_i} \right) q_a$.

Proof. It is enough to prove the first statement, since the decomposition of $\beta \in P^+$ is well known and the claims about q_w and h_w are a direct corollary.

Denote the chamber corresponding to $w \in \hat{W}$ by C_w . The correspondence $w \to C_w$ is $|\hat{\Omega}|$ to 1, and the fundamental chamber is $C_0 = C_{Id}$.

The decomposition $\hat{W} = W_{I_0}^{I_0} \hat{W} = W_0^{I_0} \hat{W}$ is well known and is a version of lemma 3.2 for the extended Coxeter group. It remains to prove that each $w \in {}^{I_0} \hat{W}$ can be written as $w = \beta a$, $\beta \in P^+$ and $a \in \hat{A}_0$.

The elements of I_0 \hat{W} are elements $w \in \hat{W}$ such that l(sw) > l(w) for any $s \in I_0$. Since the length of element in \hat{W} is the number of hyperplanes separating C_w from C_0 , C_w is on the same side on the s-hyperplane of C_0 . Therefore, C_w is in the fundamental sector. Choose now an internal point $v_w \in C_w$. Then $\langle \alpha_i, v_w \rangle > 0$ for every i = 1, ..., n. Let $\beta \in P^+$ be the unique element satisfying $\langle \alpha_i, \beta \rangle = \lfloor \langle \alpha_i, v_w \rangle \rfloor \geq 0$, and $a = \beta^{-1}w$. A point $v_a \in C_a$ satisfies $0 \leq \langle \alpha_i, v_w \rangle \leq 1$ for i = 1, ..., n, and therefore $a \in \hat{A}_0$. By this description it is also clear that β is the only element in P satisfying $\beta^{-1}w \in \hat{A}_0$. Finally, each hyperplane separating C_β and C_0 also separates each point $v \in V_R$ satisfying $\langle \alpha_i, v \rangle \geq \langle \alpha_i, \beta \rangle$ for i = 1, ..., n. Therefore Each such hyperplane also separates C_w from C_0 , and therefore $l(w) = l(\beta) + l(a)$

We have a direct nice corollary to the theorem. Define an abstract parameter system as a set of intermediates $\vec{u} = (u_s)_{s \in S}$, satisfying the parameter system condition, i.e $u_s = u_{s'}$ when $m_{s,s'}$ is odd, and also $u_s = u_{\omega(s)}$ for $\omega \in \hat{\Omega}$. We also define $u_{\omega} = 1$ for $\omega \in \hat{\Omega}$. Then there exists for every $w \in \hat{W}$ a well defined monomial u_w satisfying $u_{ww'} = u_w u_{w'}$ if l(ww') = l(w) + l(w'). In the single parameter case we simply have $u_w = w^{l(w)}$.

Definition 22.2. For a subset $A \subset \hat{W}$ we define the formal series $P_A(\vec{u}) = \sum_{w \in A} h_w u_w \in \hat{H}_{\phi}[[\vec{u}]]$. The formal series $P_{\hat{W}}(\vec{u}) = \sum_{w \in \hat{W}} h_w u_w$ is called the *generalized Poincare series* of the Iwahori-Hecke algebra.

Corollary 22.3. As a formal series, we have:

$$P_{\hat{W}}(\vec{u}) = P_{W_0}(\vec{u}) \left(\prod_{i=1}^{n} \frac{1}{(1 - h_{\beta_i} u_{\beta_i})} \right) P_{\hat{A}_0}(\vec{u})$$

Remark 22.4. This generalized Poincare series was first considered by Gyoja in [Gyo83] (see also [Hof03]), where is was proven that it is a rational function. The formal series $\sum_{w \in \hat{W}} u_w$ (or, in the single parameter case $\sum_{w \in \hat{W}} u^{l(w)}$) is called the *Poincare series* of the extended Coxeter group \hat{W} . Explicit formulas for it are classical. While it is usually defined for the regular Coxeter group W and not the extended version \hat{W} , it does not really matter as by $\hat{W} = W \times \hat{\Omega}$, $P_{\hat{W}}(u) = P_W(u)P_{\hat{\Omega}}(u) = P_{\hat{\Omega}}(u)P_W(u)$.

Example 22.5. Consider the root system of type A_1 . Let V_R be \mathbb{R}^1 with the standard inner product. We have $R = \{\pm e_1\}$, $R^{\vee} = \{\pm 2e_1\}$. The simple coroot is $\alpha_1^{\vee} = 2e_1$ and the simple coweight is $\beta_1 = e_1$. We have $Q = \{2ze_1 : z \in \mathbb{Z}\}$, $P = \{ze_1 : z \in \mathbb{Z}\}$ and $\hat{\Omega} = P/Q \cong \{Id, \omega\}$. The Coxeter group is $W = \langle s_0, s_1 : s_0^2 = s_1^2 = 1\rangle$ and the extended Coxeter group is $\hat{W} = W \rtimes \hat{\Omega} = \langle s_0, s_1, \omega : s_0^2 = s_1^2 = \omega^2 = 1, \omega s_0 = s_1\omega\rangle$. We have as elements of \hat{W} , $\beta_1 = s_0\omega$. We have $\hat{A}_0 = \hat{\Omega} = \{Id, \omega\}$. Each element of $w \in \hat{W}$ can be written uniquely as $w = s_1^{\delta_1} \beta_1^m \omega^{\delta_\omega}$ for δ_ω , $\delta_1 \in \{0, 1\}$ and $m \geq 0$.

There is a single abstract parameter u is the parameter system, and the generalized Poincare series of the Iwahori-Hecke algebra is

$$P_{\hat{W}}(u) = (1 + h_{s_1}u)\frac{1}{1 - h_{\beta_s}u}(1 + h_{\omega})$$

This case corresponds to the Iwahori-Hecke algebra of the regular graph, as described in [Kam16], section 7. As explained there (with slightly different notations), the operator h_{β_1} is Hashimoto's non backtracking operator, used to define the graph Zeta function.

Example 22.6. Consider the non-reduced root system of type BC_1 . Let again V_R be \mathbb{R}^1 with the standard inner product. We have $R = \{\pm e_1, \pm 2e_1\}$, $R^{\vee} = \{\pm e_1, \pm 2e_1\}$. The simple coroot is $\alpha_1^{\vee} = e_1$ and the simple coweight is $\beta_1 = \alpha_1^{\vee} = e_1$. we have $P = Q = \{ze_1 : z \in \mathbb{Z}\}$ and $\hat{\Omega} = \{1\}$. The Coxeter group is $W = \langle s_0, s_1 : s_0^2 = s_1^2 = 1 \rangle$ and the extended Coxeter group is $\hat{W} = W$. We have as elements of \hat{W} , $\beta_1 = s_0 s_1$. We have $\hat{A}_0 = \{Id, s_0\}$. Each element of $w \in \hat{W}$ can be written uniquely as $w = s_1^{\delta_1} \beta_1^m s_0^{\delta_0}$ for $\delta_0, \delta_1 \in \{0, 1\}$ and $m \geq 0$.

There are two abstract parameters u_0, u_1 is the parameter system, and the generalized Poincare series of the Iwahori-Hecke algebra is

$$P_{\hat{W}}(u_1, u_2) = (1 + h_{s_1} u_1) \frac{1}{1 - h_{s_0 s_1} u_1 u_2} (1 + h_{s_0} u_0)$$

This case corresponds to the Iwahori-Hecke algebra of the bipartite biregular graph, as described in [Kam16], section 11. The operator $h_{\beta_1} = h_{s_0 s_1}$ is once again Hashimoto's non backtracking operator in the bipartite case. See also the discussion in [Hof03].

Example 22.7. Let us describe the general A_n case, let $V_0 = R^{n+1}$ with the standard inner product and $V_R = \{v \in V_0, \sum v_i = 0\}$. The set of roots (or coroots, which are equal) is $R = R^{\vee} = \{e_i - e_j : 0 \le i \ne j \le n\}$ and the set of simple roots (and simple coroots) are $\alpha_i = \alpha_i^{\vee} = e_{i-1} - e_i$, i = 1, ..., n (they indeed span the subspace $V_R \subsetneq V$). The coroot lattice is $Q = \{(z_0, ..., z_n) \in \mathbb{Z}^{n+1} : \sum z_i = 0\}$.

The simple coweights are $\beta_i = e_0 + ... + e_i - \frac{i}{n+1}(1,...,1), i = 1,...,n$. The coweight lattice is

$$P = \left\{ (z_0, ..., z_n) - \frac{\sum z_i}{n+1} (1, ..., 1) : (z_0, ..., z_n) \in \mathbb{Z}^{n+1} \right\} = \left\{ \sum x_i \beta_i : x_i \in \mathbb{Z} \right\}$$

The dominant coweights are

$$P^{+} = \left\{ (z_0, ..., z_n) - \frac{\sum z_i}{n+1} (1, ..., 1) : (z_0, ..., z_n) \in \mathbb{Z}^{n+1}, z_i \ge z_{i+1} \right\} = \left\{ \sum x_i \beta_i : x_i \in \mathbb{N} \right\}$$

 $W_0 \cong S_{n+1}$ acts by permutations of the coordinates of V (or V_R). We have $W = Q \rtimes S_{n+1}$, $\hat{W} = P \rtimes S_{n+1}$ with the obvious action on V_R .

The Coxeter generators are the transpositions $s_i = id \times (i-1,i) \in Q \times S_{n+1} \subset W$ for i = 1,...,n and $s_0 = (-1,0,...,0,1) \times (0,n) \in Q \times S_{n+1} = W$ (the left multiplier is an element of \mathbb{Z}^{n+1} , the right multiplier is a transposition in S_{n+1}).

In this case every vertex type is good and special. The group $\hat{\Omega}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and its elements are $\omega_i: S \to S$, $\omega_i(s_j) = s_{i+j \bmod n}$. If $n \ge 2$, $\hat{\Omega}$ is a proper subgroup of index 2 of the full automorphism group Ω of S that is isomorphic to the dihedral group $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$ and also contains the elements $\tau_i(s_i) = s_{i-i \bmod n}$.

23. Temperedness in the Affine case

In this section we study different conditions for temperedness in affine Coxeter groups.

Definition 23.1. The exponential growth rate of W is $\limsup_{m\to\infty} \#\{w\in W: l(w)=m\}^{1/m}$.

If W if affine is has a slow growth rate:

Lemma 23.2. If W is affine irreducible of dimension n, the number of $w \in W$ with $l(w) \le m$ is bounded by $G(m) = |W_0|^2 (m+1)^n$. Therefore W has exponential growth rate 1.

Proof. Using theorem 22.1, all $w \in W$ with $w = w_0 \prod_{i=1}^n \beta_i^{m_i} a$, $l(w) \leq m$, satisfy $m_i \leq m$. There are at most $|W_0|^2 (m+1)^n$ such w.

A spherical Coxeter group has exponential growth rate 0 and an infinite Coxeter groups has growth rate 1 if and only if it is a direct product of an affine Coxeter group and a spherical Coxeter group. See [Ter13] for more about this.

Example 23.3. By Example 23.3, the trivial representation is generated by a function $f \in \mathbb{C}^{B_{\phi}}$ having a constant value 1 on every chamber. Such a function is of course spherical around every chamber C_0 . Since $f \in L_{\infty}(B_{\phi})$, the trivial representation is ∞ -tempered.

The trivial representation is p-tempered, $p < \infty$, if and only if the series $\sum q_w (1 - \delta)^{pl(w)}$ converges for every $\delta > 0$. Assume the building is thick, i.e. $q_s > 1$ for every $s \in S$, and W infinite. Then $q_w > (1 + \epsilon)^{l(w)}$ for every $w \in W$ for some fixed $\epsilon > 0$. Therefore the trivial representation is not p-tempered for any $p < \infty$.

If the building is thin we have $q_w = 1$ for any $w \in W$. The trivial representation in this case is p-tempered, $p \ge 1$ if and only if the exponential growth rate is ≤ 1 . In any case it is never p-finite.

Example 23.4. By the proof of proposition 8.4, the Steinberg representation is generated by a function $f \in \mathbb{C}^{B_{\phi}}$, spherical around a fixed chamber C_0 , with values

$$f(C) = (-1)^{l(d(C_0,C))} / q_{d(C_0,C)}$$

In this case $f \in L_p(B_\phi)$ if and only if

$$\sum_{C} |f(C)|^p = \sum_{w \in W} q_w (1/q_w)^p = \sum_{w} q_w^{1-p} < \infty$$

 $f \in T_p(B_\phi)$ if and only if for every $0 < \delta < 1$

$$\sum_{w} q_w^{1-p} (1-\delta)^{p \cdot l(w)} < \infty$$

Assume that W is affine. If the building is thin $q_w = 1$ for every $w \in W$ and therefore $f \notin L_p(B_\phi)$ for every $p < \infty$. However, using the previous lemma $f \in T_1(B_\phi)$. If the building is thick $\alpha_1^{l(w)} \leq q_w \leq \alpha_2^{l(w)}$ for some $\alpha_1, \alpha_2 > 1$. Using the previous lemma, $f \in L_p(B_\phi)$ for every p > 1, $f \in T_1(B_\phi)$ and $f \notin L_1(B_\phi)$. Therefore the Steinberg representation is always 1-tempered.

Using the growth rate we can give a nicer equivalent definitions of p-temperedness. First, we state an easy lemma. The proof is elementary and is omitted.

Lemma 23.5. Let $g: \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a series. Then the following conditions are equivalent:

- 1. For every $0 < \delta < 1$, $\sum_{l=0}^{\infty} g(l)(1-\delta)^{l} < \infty$.
- 2. $\limsup_{l} q(l)^{1/l} < 1$.
- 3. For every $\delta > 0$, for almost every l, $g(l) \leq (1 + \delta)^l$.

Moreover, the conditions hold for the absolute value of any polynomial and if the conditions hold for g_1 and g_2 , then they hold for $g_1 \cdot g_2$, $g_1^{\gamma}(0 < \gamma \in R)$.

We can now state the equivalent conditions. We state them for the Iwahori-Hecke algebra H_{ϕ} and as usual similar conditions can be stated for H itself.

Proposition 23.6. Assume that \hat{W} is affine and $f \in \mathbb{C}^{B_{\phi} \times \hat{\Omega}}$ is spherical around C_0 . We may assign to f a function $f_{\hat{W}} \in \mathbb{C}^{\hat{W}}$ defined by $f_{\hat{W}}(w) = (h_w f)(C_0)$ for $w \in \hat{W}$.

The following are equivalent:

- 1. f is p-tempered, i.e for every $0 < \delta < 1$, $\sum_{w} \left| f_{\hat{W}}(w) \right|^{p} q_{w}^{1-p} (1-\delta)^{l(w)} < \infty$.
- 2. $\limsup_{w} \left(q_w^{1-p} \left| f_{\hat{W}}(w) \right|^p \right)^{1/l(w)} \le 1.$
- 3. For every $\delta > 0$, for almost every $w \in \hat{W}$, $|f_{\hat{W}}(w)| < q_w^{(p-1)/p} (1+\delta)^{l(w)}$.
- 4. Assuming B is thick: $\sum_{w} |f_{\hat{W}}(w)| q_w^s$ converges for every s < (1-p)/p.
- 5. For every parameter system $\overrightarrow{u} = (u_i)_{i \in S} \in \mathbb{R}^S_{>0}$ satisfying $u_s < q_s^{(1-p)/p}$ for every $s \in S$, the series $\sum_w |f_{\hat{W}}(w)| u_w$ converges.

Proof. Let $g(l) = \sup_{w \in \hat{W}, l(w) = l} \left| f_{\hat{W}}(w) \right|^p q_w^{1-p}$. Since \hat{W} is affine there exists a polynomial P(l) such that $g(l) \leq \sum_{w:l(w) = l} \left| f_{\hat{W}}(l) \right|^p q_w^{1-p} \leq P(l)g(l)$. Now all the conditions in the proposition are equivalent to the fact that g satisfies the previous lemma. We leave the verification to the reader.

Corollary 23.7. A \hat{H}_{ϕ} representation V is p-tempered if and only if the conditions of the previous proposition hold for $f_{\hat{W}}(w) = c_{v^*,v}(w) = \langle v^*, h_w v \rangle$ for every $v^* \in V^*$, $v \in V$.

Corollary 23.8. A finite dimensional \hat{H}_{ϕ} representation V is p-tempered if and only if the generalized Poincare series $P_{\hat{W}}(\vec{u}) = \sum_{w \in \hat{W}} h_w^V u_w$ absolutely converges (as a series of matrices) for every parameter system $\vec{u} = (u_s)_{s \in S}$ satisfying $u_s < q_s^{(1-p)/p}$ for every $s \in S$.

Remark 23.9. In the case of representations of dimension 1 a very simple case of this corollary was used in by Borel in [Bor76] to identify the one dimensional square integrable representations of affine Iwahori-Hecke algebras.

Definition 23.10. Let V be a finite dimensional representation of H. Let $h \in H$. Define $\lambda_V(h)$ as the largest absolute value of an eigenvalue of h on V.

Proposition 23.11. Assume W is affine and irreducible. Let V be a representation of H_{ϕ} . The following are equivalent:

- 1. V is p-tempered.

- 2. $\lambda_V(h_{\alpha}) \leq q_{\alpha}^{(p-1)/p}$ for every $\alpha \in P$. 3. $\lambda_V(h_{\alpha}) \leq q_{\alpha}^{(p-1)/p}$ for every $\alpha \in P^+$. 4. $\lambda_V(h_{\beta_i}) \leq q_{\beta_i}^{(p-1)/p}$ for i = 1, ..., n.

Proof. The fact that $(2) \Rightarrow (3) \Rightarrow (4)$ is obvious. For $(1) \Rightarrow (2)$ notice that for every $\alpha \in P$, $l(\alpha^m) = m \cdot l(\alpha)$ and therefore $h_{\alpha^m} = h_{\alpha}^m$ and $q_{\alpha^m} = q_{\alpha}^m$. Assume V is p-tempered. By the limsup condition of proposition 23.6

$$\operatorname{limsup}_m \left(\left| q_{\alpha^m}^{1-p} \left\langle v^*, h_{\alpha^m} v \right\rangle \right|^p \right)^{1/l(\alpha^m)} = \operatorname{limsup}_m \left(q_{\alpha}^{1-p} \left| \left\langle v^*, h_{\alpha}^m v \right\rangle \right|^{p/m} \right)^{1/l(\alpha)} \leq 1$$

By lemma 23.5 we may change the exponent and get

$$\operatorname{limsup}_m \left| \left\langle v^*, h_\alpha^m v \right\rangle \right|^{1/m} \le q_\alpha^{(p-1)/p}$$

Choose for v an eigenvector for an eigenvalue λ of h with $\lambda_V(h_\alpha) = |\lambda|$, we get $\lambda_V(h_\alpha) \leq q_\alpha^{(p-1)/p}$ by the matrix equality stated above.

Assume (4) that holds. Recall the matrix equality $\limsup \|A^m\|^{1/m} = \lambda_{max}(A)$ where $A \in M_n(\mathbb{C})$, $\|\|$ is any matrix norm and $\lambda_{max}(A)$ is the largest absolute value of an eigenvalue of A. Applying this equality, since $\lambda_V(h_{\beta_i}) \leq q_{\beta_i}^{(p-1)/p}$ we know that for any $u^* \in V^*, u \in V$ we have

$$\limsup_{m_i} \left(\left(q_{\beta_i}^{m_i} \right)^{1-p} \left| \left\langle u^*, h_{\beta_i}^{m_i} u \right\rangle \right|^p \right)^{1/m_i} \le 1$$

Applying all the operators together, we deduce that for any $u^* \in V^*$, $u \in V$ we have:

$$\limsup_{m_i} \left(\left(\prod_{i=1}^n q_{\beta_i}^{m_i} \right)^{1-p} \left| \left\langle u^*, \prod_{i=1}^n h_{\beta_i}^{m_i} u \right\rangle \right|^p \right)^{1/\sum m_i} \le 1$$

Now using 22.1

$$\lim \sup_{w} \left(q_{w}^{1-p} \left| \left\langle v^{*}, h_{w} v \right\rangle \right|^{p} \right)^{1/l(w)} = \sup_{a \in \hat{A}_{0}, w_{0} \in W_{0}} \lim \sup_{m_{i}, i = 1, \dots, n} \left(\left(q_{w_{0}} \left(\prod_{i=1}^{n} q_{\beta_{i}}^{m_{i}} \right) q_{a} \right)^{1-p} \left| \left\langle v^{*}, h_{w_{0}} \prod_{i=1}^{n} h_{\beta_{i}}^{m_{i}} h_{a} v \right\rangle \right|^{p} \right)^{1/(l(w_{0}) + \sum m_{i} l(\beta_{i}) + l(a))} \leq \sup_{a \in \hat{A}_{0}, w_{0} \in W_{0}} \lim \sup_{m_{i}, i = 1, \dots, n} \left(\left(\prod_{i=1}^{n} q_{\beta_{i}}^{m_{i}} \right)^{1-p} \left| \left\langle h_{w_{0}}^{*} v^{*}, \prod_{i=1}^{n} h_{\beta_{i}}^{m_{i}} h_{a} v \right\rangle \right|^{p} \right)^{1/\sum m_{i}} \leq 1$$

and by proposition 23.6, V is p-tempered.

Definition 23.12. In the equal parameter case $(q_s = q)$ define

$$\zeta_{1,V}(u) = \frac{1}{\det(1 - h_{\beta_1} u^{l(\beta_1)}) \cdot \dots \cdot \det(1 - h_{\beta_n} u^{l(\beta_n)})}$$

In general define

$$\zeta_{2,V}(s) = \frac{1}{\det(1 - h_{\beta_1} q_{\beta_1}^s) \cdot \dots \cdot \det(1 - h_{\beta_n} q_{\beta_n}^s)}$$

Notice that the above definition is closely related to corollary 22.3. Proposition 23.6 is equivalent to:

Corollary 23.13. Assume W is affine and irreducible. Let V be a representation of H_{ϕ} . The following are eauivalent:

- 1. V is p-tempered.
- 2. The poles of $\zeta_{2,V}(s)$ are all for $Re(s) \geq (1-p)/p$.
- 3. Assume the equal parameter case, the poles of $\zeta_{u,V}(s)$ are all for $|u| \geq q^{(1-p)/p}$.

By Examples 22.5, 22.6 and the results of [Kam16], all the discussion above is a direct generalization of the zeta functions of graphs, and its connection to temperedness.

24. Bounds on Hecke Operators

In theorem 28.3 we will prove that:

Theorem. Let $p \geq 2$. Let $q_{max} = \max_{s \in S} \{q_s\}$ and let \tilde{w}_0 be the longest element of W_0 . The norm of the operator h_{β} , $\beta \in P^+ \subset \hat{W}$ is bounded on $L_p(\hat{B}_{\phi})$ by $|W_0| |2q_{max}|^{l(\tilde{w}_0)} (l(\beta) + 1)^{l(\tilde{w}_0)} q_{\beta}^{(p-1)/p}$.

Corollary 24.1. The norm of h_w , $w \in \hat{W}$ is bounded on $L_n(\hat{B}_\phi)$ by

$$||h_w||_p \le D(q_{max}, l(w))q_w^{(p-1)/p}$$

with $D(q_{max}, l) = |W_0| 2^{l(\tilde{w}_0)} q_{max}^{4 \cdot l(\tilde{w}_0)} (l + 1 + l(\tilde{w}_0))^{l(\tilde{w}_0)}$.

The same bound holds for any finite dimensional unitary p-tempered H_{ϕ} -representation V.

Proof. Recall the decomposition $\hat{W} = P \rtimes W_0$. In addition, each element of P is conjugate by an element of W_0 to an element of P^+ , which is of the same length (see lemma 26.5). Therefore we can write for every $w \in \hat{W}, w = w_0 \beta w_0'$ with $w_0, w_0' \in W_0$, and $l(\beta) \leq l(w) + l(\tilde{w}_0), q_\beta \leq q_w q_{max}^{l(\tilde{w}_0)}$. If $w_0 = s_{i_0} \cdot \ldots \cdot s_{i_l}, w_0' = s_{i_0'} \cdot \ldots \cdot s_{i_k'}$, then by the description of the Iwahori-Hecke algebra

$$h_w = h_{s_{i_0}}^{\epsilon_{i_0}} \cdot ... h_{s_{i_l}}^{\epsilon_{i_l}} h_\beta h_{s_{i'_0}}^{\epsilon'_{i_0}} \cdot ... h_{s_{i'_k}}^{\epsilon'_{i_k}}$$

Where $\epsilon_i, \epsilon_i' \in \{\pm 1\}$. Since $\|h_s\|_p \leq q_s \leq q_{max}$, $\|h_s^{-1}\|_p \leq 1 \leq q_{max}$, we have

$$\begin{aligned} \|h_w\|_p &\leq q_{max}^l \|h_\beta\|_p \, q_{max}^k \leq q_{max}^{l(\tilde{w}_0)} \, |W_0| \, 2^{l(\tilde{w}_0)} q_{max}^{l(\tilde{w}_0)} \, (l(\beta) + 1)^{l(\tilde{w}_0)} \, q_\beta^{(p-1)/p} q_{max}^{l(\tilde{w}_0)} \\ &\leq |W_0| \, 2^{l(\tilde{w}_0)} q_{max}^{4 \cdot l(\tilde{w}_0)} \, (l(w) + l(\tilde{w}_0) + 1)^{l(\tilde{w}_0)} \, q_w^{(p-1)/p} \end{aligned}$$

For the second claim, denote $F = D(q, l(w))q_w^{(p-1)/p}$. Notice that $h_w^* = h_{w^{-1}}$ and therefore the L_p -norm of $h_w h_w^*$ is bounded by $D(q, l(w))q_w^{(p-1)/p}D(q, l(w^{-1}) + 1)q_{w^{-1}}^{(p-1)/p} = F^2$. Therefore the spectrum of $h_w h_w^*$ on $L_p(\hat{B}_\phi)$ is bounded by F^2 and by corollary 18.4, F^2 is also a bound of the eigenvalues of any p-tempered finite dimensional H_{ϕ} -representation V. If \underline{V} is also unitary, $h_w h_w^*$ is self adjoint and its norm is bounded by its largest eigenvalue. Finally, $||h_w||_V = \sqrt{||h_w h_w^*||_V} \le F$.

Remark 24.2. The p > 2 of both theorem 28.3 and corollary 24.1 can be deduced (and actually slightly improved) from the case p=2 and the trivial case $p=\infty$, by the Riesz-Thorin interpolation theorem. However, even better L_p -bounds can be deduced by better analysis in theorem 28.3.

Corollary 24.3. The spectrum of h_{β} , $\beta \in Q^+ \subset \hat{W}$ on $L_p(\hat{B}_{\phi})$ is bounded in absolute value by $q_{\beta}^{(p-1)/p}$.

Proof. By Gelfand's formula,

$$\lambda_{L_{p}(\hat{B}_{\phi})}(h_{\beta}) = \limsup \|h_{\beta}^{n}\|_{L_{p}(\hat{B}_{\phi})}^{1/n} = \limsup \|h_{n\beta}\|_{L_{p}(\hat{B}_{\phi})}^{1/n} \leq \limsup \left(D\left(q, l(n\beta)\right) \cdot q_{\beta}^{n(p-1)/p}\right)^{1/n} = q_{\beta}^{(p-1)/p} \limsup D\left(q, n \cdot l(\beta)\right)^{1/n} = q_{\beta}^{(p-1)/p}$$

The last equality holds by the limit $n^{1/n} \to_{n \to \infty} 1$.

Corollary 24.4. If a finite dimensional H (respectively \hat{H}_{ϕ} , H_{ϕ}) representation is weakly contained in $L_p(B_f)$ (respectively $L_p(\hat{B}_{\phi})$, $L_p(B_{\phi})$), then it is p-tempered.

Proof. Follows by the last corollary and theorem 18.4.

Another results of the above is a tight version of the Kunze-Stein theorem. Define a twisted p-norm on \hat{H}_{ϕ} by $\|\sum \alpha_w h_w\|_p' = \sum_w D(q, l(w)) q_w^{1/p} |\alpha_w|$ (since this sum is finite it is always well defined). Let $\hat{H}_{\phi,p}$ be the completion of \hat{H}_{ϕ} with respects to this norm, i.e.

$$\hat{H}_{\phi,p} = \left\{ \sum \alpha_w h_w : \sum_w D(q, l(w))^{-1} q_w^{1/p} |\alpha_w| < \infty \right\}$$

where the sums can be infinite.

Corollary 24.5. There exists a bounded action of $\hat{H}_{\phi,(p-1)/p}$ on $L_p(\hat{B}_{\phi})$. The norm of $h \in \hat{H}_{\phi,(p-1)/p}$ on $L_p(\hat{B}_{\phi})$ is bounded by $||h||_p'$.

If terms of the group G, this can be stated as follows- we have an isomorphism $H_{G_{\phi}}(G) \cong H_{\phi}$. Let $L_{G_{\phi},p}(G)$ be the completion of $H_{G_{\phi}}(G) \subset H(G) \subset C_{c}(G)$ with respect to the usual p norm and let $L'_{G_{\phi},p}(G)$ be the completion of $H_{G_{\phi}}(G)$ with respect to the twisted p-norm as above. Notice that both $L_{G_{\phi},p}(G)$ and $L'_{G_{\phi},p}(G)$ are subspaces of $C(G_I \setminus G)$ and that $L'_{G_{\phi},p}(G) \subset L'_{G_{\phi},p'}$ if p' < p. Then the results above say that convolution is a bounded bilinear operator $L'_{G_{\phi},(p-1)/p}(G) \times L_{G_{\phi},p}(G) \to L_{G_{\phi},p}(G)$. This is a strong version of the Kunze-Stein theorem for Iwahori-fixed vectors.

25. APPLICATION: AVERAGE DISTANCE AND DIAMETER

We define a distance between chambers of the quotient X as follows, for $C, C' \in X_{\phi}$ let l(C, C') be the length of the shortest gallery between them. Equivalently, $l(C, C') = \min l(d(\tilde{C}, \tilde{C}'))$, where $\tilde{C}, \tilde{C}' \in B_{\phi}$ cover C, C'.

Theorem 25.1. Let X be an L_p -expander of irreducible affine Coxeter group W, with single parameter q, having N chambers and $C_0 \in X_{\phi}$. Let n be the dimension of X and \tilde{w}_0 is the longest element of the spherical Coxeter group W_0 . Then all but o(N) other chambers $C \in X_{\phi}$ are of gallery distance $l(C_0, C)$ which satisfies

$$l(C_0, C) \le \left\lceil \frac{p}{2} \log_q N + (l(\tilde{w}_0) + 1) \log_q \log_q N \right\rceil$$

and

$$l(C_0, C) \ge \lfloor \log_q N - (n+1) \log_q \log_q N \rfloor$$

In addition, the diameter of X for N large enough is at most $\lceil p \log_a N + 2(l(\tilde{w}_0) + 1) \log_a \log_a N \rceil$.

Proof. Let $w \in W$ and consider $q_w^{-1}h_w1_{C_0}$. Every chamber C for which $h_wq_w^{-1}1_{C_0}(C) \neq 0$ is at a distance at most l(w) from C_0 .

Let $\pi \in \mathbb{C}^{X_{\phi}}$ be the constant function $\pi(C) = 1/N$. We have $\|1_{C_0} - \pi\|_2^2 = (1 - 1/n)^2 + (n - 1)n^{-2} = 1 - 1/n < 1$.

Let $l = \lceil \frac{p}{2} \log_q N + K \log_q \log_q N \rceil$ and let $w \in W$ with l(w) = l. Since $h_w q_w^{-1} \pi = \pi$ and $h_w q_w^{-1} 1_{C_0} - \pi \in L_2^0(X_\phi)$, by corollary 24.1 we have

$$\|h_w q_w^{-1} 1_{C_0} - \pi\|_2 \le D(q, l) q^{l(p-1)/p} q^{-l} \le D(q, l) N^{-1/2} \log_q N^{-K}$$

Since by Cauchy-Schwartz $||f||_1 \le N^{1/2} ||f||_2$ for every $f \in \mathbb{C}^{X_\phi}$. Therefore $||h_w q_w^{-1} 1_{C_0} - \pi||_1 \le D(q, l) \log N^{-K}$. As a result, $h_w q_w^{-1} 1_{C_0}$ is 0 on at most

$$ND(q,l)\log N^{-K} = N \left| W_0 \right| 2^{l(\tilde{w}_0)} q^{4 \cdot l(\tilde{w}_0)} \left(\frac{p}{2} \log_q N + K \log_q \log_q N + 1 + l(\tilde{w}_0) \right)^{l(\tilde{w}_0)} \log_q N^{-K}$$

chambers. For $K > l(\tilde{w_0})$ this is o(N).

For the lower bound, there are at most $G(l) = |W_0|^2 (l+1)^n$ elements $w \in W$ with $l(w) \leq l$. Therefore there are at most $G(l)q^l$ chambers $C \in X_{\phi}$ of with $l(C_0, C) \leq l$. For $l \leq \lfloor \log_q N - (n+1) \log_q \log_q N - 1 \rfloor$, we have $G(l)q^l = o(N)$.

The claim about the diameter follows from the upper bound on the average distance. Let $C_0, C_1 \in X_\phi$. Let

$$l = \left\lceil \frac{p}{2} \log_q N + (l(\tilde{w}_0) + 1) \log_q \log_q N \right\rceil$$

For N large enough so that

$$|W_0| \, 2^{l(\tilde{w}_0)} q^{4 \cdot l(\tilde{w}_0)} \left(\frac{p}{2} \log_q N + K \log_q \log_q N + 1 + l(\tilde{w}_0) \right)^{l(\tilde{w}_0)} \log_q N^{-K} < 0.5$$

There are more than 0.5N elements $C_2 \in X_{\phi}$ with $l(C_0, C_2) \leq l$ and more than 0.5N elements $C_3 \in X_{\phi}$ with $l(C_1, C_3) \leq l$. By pigeonhole there is some $C_2 = C_3$, so $l(C_0, C_1) \leq 2l$, as required.

Remark 25.2. The proof actually shows that for every $w \in W$, $l(w) > \frac{p}{2} \log_q N + (l(w_0) + 1) \log_q \log_q N$, for almost every two chambers there is a gallery of type w connecting them.

26. The Bernstein Presentation

As an introduction to this section, let us recall the most important construction in affine Hecke algebras. Recall that any coweight $\beta \in P$ can be written as a difference of two dominant coweights- $\beta = \beta_1 - \beta_2$, with $\beta_1, \beta_2 \in P^+$.

Definition 26.1. For $\beta \in P$, $\beta = \beta_1 - \beta_2$, $\beta_1, \beta_2 \in P^+$, we will denote by $Y_{\beta} \in \hat{H}_{\phi}$ the element $Y_{\beta} = q_{\beta_1}^{-1/2} q_{\beta_2}^{1/2} h_{\beta_1} h_{\beta_2}^{-1}$.

It is immediate to verify that Y_{β} does not depend on the choice of $\beta_1, \beta_2 \in P^+$, and that $Y_{\beta}Y_{\beta'} = Y_{\beta+\beta'}$ for $\beta, \beta' \in P$.

The following theorem is called the *Bernstein-Luzstig presentation* of the Hecke algebra (see [Par06], theorem 6.6 and references therein):

Theorem 26.2. (Bernstein) The operators $Y_{\beta}h_{w_0}$, $w_0 \in W_0$, $\beta \in P$, are a basis for the extended Iwahori-Hecke algebra \hat{H}_{ϕ} . Multiplication in the algebra with respect to this basis is given by the Iwahori-Hecke relations for W_0 , $Y_{\beta}Y_{\beta'} = Y_{\beta+\beta'}$ for $\beta, \beta' \in P$ and the relations:

$$Y_{\beta}h_{s_{i}} = h_{s_{i}}Y_{s_{i}(\beta)} + (q_{s_{i}} - 1)\frac{Y_{\beta} - Y_{s_{i}(\beta)}}{1 - Y_{-\alpha_{i}}^{\vee}} \qquad (R, i) \neq (BC_{n}, n)$$

$$Y_{\beta}h_{s_{n}} = h_{s_{n}}Y_{s_{n}(\beta)} + q_{s_{n}}^{1/2} \left(q_{s_{n}}^{1/2} - q_{s_{n}}^{-1/2} + \left(q_{0}^{1/2} - q_{0}^{-1/2}\right)Y_{-(2\alpha_{n})^{\vee}}\right)\frac{Y_{\beta} - Y_{s_{n}(\beta)}}{1 - Y_{-2(2\alpha_{n})^{\vee}}} \qquad (R, i) = (BC_{n}, n)$$

The expression $\frac{Y_{\beta}-Y_{s_i(\beta)}}{1-Y_{-\alpha_i^{\vee}}}$ is actually a compact way of writing a finite sum of $|\langle \alpha_i, \beta \rangle|$ different $Y_{\beta'}$. Recall $s_i(\beta) = \beta - \langle \alpha, \beta \rangle \alpha^{\vee}$. Now:

$$\frac{Y_{\beta} - Y_{s_{i}(\beta)}}{1 - Y_{-\alpha_{i}^{\vee}}} = \begin{cases} 0 & \langle \alpha_{i}, \beta \rangle = 0 \\ Y_{\beta} + Y_{\beta - \alpha_{i}^{\vee}} + \dots + Y_{\beta - (\langle \alpha, \beta \rangle - 1)\alpha^{\vee}} & \langle \alpha_{i}, \beta \rangle > 0 \\ -Y_{\beta + \alpha^{\vee}} - Y_{\beta + 2\alpha^{\vee}} \dots - Y_{s_{i}(\beta)} & \langle \alpha_{i}, \beta \rangle < 0 \end{cases}$$

Similar relations holds for the $(R, i) = (BC_n, n)$, this time the sum contains $|\langle 2\alpha_n, \beta \rangle|$ elements. We will need for later the some estimates on the resulting presentation. Define (See [Mac03], 2.7):

Definition 26.3. The Bruhat order on W_0 is defined by: $w' \leq w$ if there exits a decomposition $w' = t_0 \cdot \cdot t_l$, $t_i \in S$ and $w = t_{i_0} \cdot ... \cdot t_{i_k}$ for some $0 \leq i_0 < ... < i_k \leq l$.

Define a partial order on P^+ by $\beta > \beta'$ if $\beta - \beta' \in Q^+$, where $Q^+ \subset Q$ is the set of coroots that are a non negative sum of the simple coroots. The *coweight order* on P is defined by: for $\beta \in P$ let β^+ be the unique element $\beta^+ \in P^+$ in the W_0 orbit of β . Let $w_0^\beta \in W_0$ be the shortest element sending β to β^+ . Then $\beta \leq \beta'$ if and only if $\beta^+ < \beta'^+$ or $\left(\beta^+ = \beta'^+ \text{ and } w_0^{\beta^+} \geq w_0^{\beta'^+}\right)$.

Remark 26.4. In our definition we let the dominant coweight be the largest in any W_0 -orbit, which is opposite to the standard where the anti-dominant coweight is the largest.

Lemma 26.5. Let β be dominant. The set of $\beta' \in P$ such that $\beta' \leq \beta$ is closed under $\beta' \to \beta' - j\alpha_i^{\vee}$ for all j between 0 and $\langle \alpha_i, \beta' \rangle$ (inclusive) (i.e. a saturated set as in [Mac03], 2.6).

All the coweight $\beta' \leq \beta$ satisfy for simple root $\alpha_i \in R$, $|\langle \alpha_i, \beta' \rangle| \leq l(\beta') \leq l(\beta)$ (and $2 |\langle \alpha_i, \beta' \rangle| \leq l(\beta')$ in the $(R, i) = (BC_n, n)$ case), and $q_{\beta'} \leq q_{\beta}$.

Proof. Assume first that we work in the non reduced case. The first statement is proved in [Mac03], 2.6, 2.7. The second statement follows from the well known formula in the non-reduced case $l(\beta') = \frac{1}{2} \sum_{\alpha \in B} |\langle \alpha, \beta' \rangle| =$

 $\sum_{\alpha \in R^+} |\langle \alpha, \beta' \rangle| \text{ ([Mac03], 2.4.1), and similarly } q_{\beta} = \prod_{\alpha \in R^+} q_{\alpha}^{|\langle \alpha, \beta' \rangle|}, \text{ from which it follows that for every } w_0 \in W_0 \ l(w_0(\beta')) = l(\beta') \text{ and that } |\langle \alpha, \beta' \rangle| \leq l(\beta').$

If β' is dominant, $|\langle \alpha, \beta' \rangle| = \langle \alpha, \beta' \rangle$, so $l(\beta') = \sum_{\alpha \in R^+} \langle \alpha, \beta' \rangle = \langle \rho, \beta' \rangle$. where $\rho = \sum_{\alpha \in R^+} \alpha$. Since ρ is in the dominant sector, $\langle \rho, \alpha^{\vee} \rangle > 0$ for any $\alpha \in R^+$. Therefore if $\beta' \leq \beta$ and both are dominant then $l(\beta') \leq l(\beta)$. The claim that $q_{\beta'} \leq q_{\beta}$ is proven similarly.

In the reduced case, we may consider the corresponding non-reduced Root system R' containing the non divisible roots of R. All the claims follow from the claims on R'.

Proposition 26.6. For $\beta, \beta' \in P$, $w_0, w_0' \in W_0$ there exist constants $\alpha_{w_0', w_0, \beta', \beta}$ (depending on the parameter system \overrightarrow{q}) such that:

$$Y_{\beta}h_{w_0} = \sum_{w'_0,\beta'} \alpha_{w'_0,w_0,\beta',\beta} h_{w'_0} Y_{\beta'}$$

The constants $\alpha_{w'_0,w_0,\beta',\beta}$ satisfy:

- (1) $\alpha_{w_0,w_0,\beta,w_0(\beta)} = 1.$
- (2) If $w'_0 \not\leq w_0$ in the Coxeter order of the Coxet1er group W_0 , then $\alpha_{w'_0, w_0, \beta', \beta} = 0$.
- (3) If β is dominant, and $\beta' \not\leq \beta$ in the coweight order, then $\alpha_{w'_0, w_0, \beta', \beta} = 0$.
- (4) For any $w'_0 \in W_0$, $\sum_{\beta'} |\alpha_{w'_0, w_0, \beta', \beta}| \le 2^{l(w_0)} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) l(w'_0)}$ for $q_{max} = \max\{q_s : s \in S\}$.

Proof. Everything is proved by induction on l(w) using the Bernstein relations. Write $w_0 = \hat{w}_0 s_i$, $l(w_0) = l(\hat{w}_0) + 1$ and assume the claim is true for \hat{w}_0 . Then in the non $(R, i) = (BC_n, n)$ -case:

$$\begin{split} Y_{\beta}h_{w_0} &= Y_{\beta}h_{\hat{w}_0}h_{s_i} = \sum_{w'_0,\beta'} \alpha_{w'_0,\hat{w}_0,\beta',\beta}h_{w'_0}Y_{\beta'}h_{s_i} \\ &= \sum_{w'_0,\beta'} \alpha_{w'_0,\hat{w}_0,\beta',\beta}h_{w'_0} \left(h_{s_i}Y_{s_i(\beta')} + (q_{s_i} - 1)\frac{Y_{\beta'} - Y_{s_i(\beta')}}{1 - Y_{-\alpha_i^{\vee}}}\right) \end{split}$$

This gives by induction 1,2 and 3.

We may bound the sum of coefficients for $h_{w'_0}$ on the right hand side using the induction hypothesis. For $l(w'_0s) = l(w'_0) + 1$:

$$\begin{split} \sum_{\beta'} \left| \alpha_{w'_0, w_0, \beta', \beta} \right| &\leq \sum_{\beta'} \left(\left| \alpha_{w'_0, \hat{w}_0, \beta', \beta} \right| (q_{max} - 1) \left| \left\langle \alpha_i, \beta' \right\rangle \right| + \left| \alpha_{w'_0 s_0, \hat{w}_0, \beta', \beta} \right| q_{max} \right) \\ &\leq 2^{l(w_0) - 1} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0) - 1} (q_{max} - 1) l(\beta) + 2^{l(w_0) - 1} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0) - 2} q_{max} \\ &\leq 2^{l(w_0)} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0)} \end{split}$$

For $l(w_0's) = l(w_0') - 1$:

$$\begin{split} \sum_{\beta'} \left| \alpha_{w'_0, w_0, \beta', \beta} \right| &\leq \sum_{\beta'} \left(\left| \alpha_{w'_0, \hat{w}_0, \beta', \beta} \right| \left((q_{max} - 1) + (q_{max} - 1) \left| \left\langle \alpha_i, \beta' \right\rangle \right| \right) + \left| \alpha_{w'_0 s_0, \hat{w}_0, \beta', \beta} \right| \right) \\ &\leq 2^{l(w_0) - 1} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0) - 1} \left(q_{max} - 1 \right) l(\beta) + 2^{l(w_0) - 1} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0)} \\ &\leq 2^{l(w_0)} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w'_0)} \end{split}$$

The case $(R, i) = (BC_n, n)$ is similar.

We will also need the following variation of the above:

Proposition 26.7. For $\beta, \beta' \in P$, $w_0, w'_0 \in W_0$ there exist constants $\alpha'_{w'_0, w_0, \beta', \beta}$ (depending on the parameter system \overrightarrow{q}) such that:

$$Y_{\beta}h_{w_0}^{-1} = \sum_{w_0' < w_0, \beta' < \beta} \alpha'_{w_0', w_0, \beta', \beta} h_{w_0'}^{-1} Y_{\beta'}$$

The constants $\alpha'_{w'_0,w_0,\beta',\beta}$ satisfy:

- (1) $\alpha'_{w_0,w_0,\beta,w_0(\beta)} = 1.$
- (2) If $w'_0 \not\leq w_0$ in the Coxeter order of the Coxet1er group W_0 , then $\alpha'_{w'_0,w_0,\beta',\beta} = 0$.
- (3) If β is dominant, and $\beta' \not\leq \beta$ in the coweight order, then $\alpha'_{w'_0, w_0, \beta', \beta} = 0$.
- (4) $\sum_{\beta'} \left| \alpha'_{w'_0, w_0, \beta', \beta} \right| \le 2^{l(w_0)} (l(\beta) + 1)^{l(w_0) l(w'_0)}.$

Proof. We have $h_s^{-1} = q_s^{-1} (h_s - (q_s - 1) Id)$. Therefore:

$$Y_{\beta}h_{s_{i}}^{-1} = h_{s_{i}}^{-1}Y_{s_{i}(\beta)} - q_{s}^{-1}(q_{s_{i}} - 1)(Y_{\beta} - Y_{s_{i}(\beta)}) + q_{s}^{-1}(q_{s_{i}} - 1)\frac{Y_{\beta} - Y_{s_{i}(\beta)}}{1 - Y_{-\alpha_{i}^{\vee}}}$$

$$= h_{s_{i}}^{-1}Y_{s_{i}(\beta)} + q_{s}^{-1}(q_{s_{i}} - 1)Y_{-\alpha_{i}^{\vee}}\frac{Y_{\beta} - Y_{s_{i}(\beta)}}{1 - Y_{-\alpha_{i}^{\vee}}}$$

$$= h_{s_{i}}^{-1}Y_{s_{i}(\beta)} + q_{s}^{-1}(q_{s_{i}} - 1)\frac{Y_{\beta} - Y_{s_{i}(\beta)}}{Y_{\alpha_{i}^{\vee}} - 1}$$

And similarly in the $(R, i) = (BC_n, n)$ case.

The rest of the proof is similar to proposition 26.6, using $h_s^{-2} = q_s^{-2} (q_s Id - (q_s - 1)h_s)$. The extra q_s^{-1}, q_s^{-2} factors allows us to give the slightly better bound.

27. Sectorial Retraction

The Bernstein presentation is a generalized version of the *Iwasawa decomposition*. The building analog of the Iwasawa decomposition is based on the notion of a *sector*, and *sectorial retraction*. The goal of this section is to explain the connection between the two.

Definition 27.1. The dominant sector in V_R is the set $\mathbb{S}_0^R = \{v \in V_R : \langle \alpha_i, v \rangle > 0, i = 1, ..., n\}$. A sector \mathbb{S} in V_R is an image of \mathbb{S}_0^R under the action of \hat{W} . A sector \mathbb{S} (respectively a dominant sector) in an apartment $\mathbb{A} \subset B$ is the preimage of any sector $\mathbb{S}' \subset \mathbb{W} \cong V_R$ (respectively \mathbb{S}_0^R) under an isomorphism of \mathbb{A} with the Coxeter complex \mathbb{W} . We identify a sector \mathbb{S} with the set of chambers it contains.

The following is very standard.

Lemma 27.2. Given an apartment \mathbb{A} of B and a chamber $C_0 \in \mathbb{A}$, there exists a retraction $\rho_{C_0}^{\mathbb{A}} : B_{\phi} \to \mathbb{A}$ such that $d(C_0, C) = d(C_0, \rho_{C_0}^{\mathbb{A}}(C))$.

In affine buildings we also have another type of retraction into an apartment \mathbb{A} , based on a sector \mathbb{S} of \mathbb{A} . Recall that given two apartments $\mathbb{A}, \mathbb{A}' \subset B$ with $\mathbb{A} \cap \mathbb{A}' \neq \phi$ we have a unique isomorphism (as colored simplicial complexes) $\phi_{\mathbb{A},\mathbb{A}'} : \mathbb{A}' \to \mathbb{A}$, that is the identity on $\mathbb{A}' \cap \mathbb{A}$.

Theorem 27.3. (See [AB08], theorem 11.63 and lemma 11.64) Given a sector \mathbb{S} of an apartment \mathbb{A} and a chamber $C \in B_{\phi}$ there exists a subsector $\mathbb{S}' \subset \mathbb{S}$ and an apartment \mathbb{A}' , such that $\mathbb{S}' \subset \mathbb{A}'$ and $C \in \mathbb{A}'$.

We may define $\rho_{\mathbb{S}}^{\mathbb{A}}: B_{\phi} \to \mathbb{A}_{\phi}$ by $\rho_{\mathbb{S}}(C) = \phi_{\mathbb{A},\mathbb{A}'}(C)$, where $\phi_{\mathbb{A},\mathbb{A}'}: \mathbb{A}' \to \mathbb{A}$ is the simplicial isomorphism. This definition does not depend on the choice of \mathbb{S}' or \mathbb{A}' and we furthermore have for any $C_0 \in \mathbb{S}'$, $\rho_{C_0}^{\mathbb{A}}(C) = \rho_{\mathbb{S}}^{\mathbb{A}}(C)$.

From now on we assume we fix a dominant sector \mathbb{S}_0 in the building. We can extend $\rho_{\mathbb{S}_0}^{\mathbb{A}} : B_{\phi} \to \mathbb{A}_{\phi}$ into its "extended" version $\hat{\rho}_{\mathbb{S}_0}^{\mathbb{A}} : \hat{B}_{\phi} \to \hat{\mathbb{A}}_{\phi}$. Now:

Definition 27.4. Define $\rho_{\mathbb{S}_0}: \hat{B}_{\phi} \to \hat{W}$ by $\hat{f}_{\phi} \circ \hat{\rho}_{\mathbb{S}_0}^{\mathbb{A}}$ where $\hat{f}_{\phi}: \mathbb{A}_{\phi} \to \hat{\mathbb{W}}_{\phi} \cong \hat{W}$ is an isomorphism of \mathbb{A} and the Coxeter complex \mathbb{W} .

We call $\rho_{\mathbb{S}_0}(C) \in \hat{W}$ the sectorial type of $C \in \hat{B}_{\phi}$.

In this section we consider the natural embedding $P \subset \hat{W}$, so we write addition in P multiplicatively. For example, recall that every $\beta \in P$ can be written as $\beta = \beta_1 \beta_2^{-1}$, with $\beta_1, \beta_2 \in P^+$.

Definition 27.5. For any element $w \in \hat{W}$, $w = \beta w_0$, with $w_0 \in W_0$, $\beta \in P$, $\beta = \beta_1 \beta_2^{-1}$, $\beta_1, \beta_2 \in P^+$, define $L(w) = L_{\mathbb{S}_0}(w) = l(w_0) + l(\beta_2) - l(\beta_1)$ and $Q_w = Q_{\mathbb{S}_0, w} = q_{w_0} \cdot q_{\beta_2} \cdot q_{\beta_1}^{-1}$.

One can verify that non of the definitions depends on the choice of $\beta_1, \beta_2 \in P^+$. We have for $\beta \in P^+$, $L(\beta) = -l(\beta)$ and $Q_\beta = q_\beta^{-1}$.

Lemma 27.6. If $C_1, ..., C_l \in B_{\phi}$, $\rho_{\mathbb{S}_0}(C_i) = w_i$, then there exists a chamber $C_0 \in \mathbb{S}_0$ such that $\rho_{C_0}^{\mathbb{A}}(C_i) = \rho_{\mathbb{S}_0}^{\mathbb{A}}(C_i)$ and $L(w_i) = l(d(C_0, C_i)) - l(\rho_{\mathbb{S}_0}(C_0))$.

Proof. We may assume that the sector \mathbb{S}' in theorem 27.3 is contained in all the sectors \mathbb{S}_i , i=1,...,l defined as the sector with dominant direction based on the 0-vertex of C_i . Let $C_0 \in \mathbb{S}'$ be such that $\rho_{\mathbb{S}_0}\left(C_0\right) = \beta_1 \in P^+$. There exists such C_0 since $\mathbb{S}' \subset \mathbb{S}_0$ and \mathbb{S}_0 is dominant. Since $C_0 \in \mathbb{S}_i$ we have $d(C_0, C_i) = \beta_2 w_{0,i}$, with $\beta_2^{-1} \in P^+$ (i.e. β_2 is anti-dominant) and $w_{0,i} \in W_0$. Therefore $w_i = \rho_{\mathbb{S}_0}(C) = \rho_{\mathbb{S}_0}(C_0) \cdot d(C_0, C) = \beta_1 \beta_2 w_{0,i}$, and

$$L(w) = l(\beta_2) - l(\beta_1) + l(w_{0,i}) = l(\beta_2 w_{0,i}) - l(\beta_1) = l(d(C_0, C_i)) - l(\rho_{S_0}(C_0))$$

Lemma 27.7. If $C \in B_{\phi}$, $\rho_{\mathbb{S}_0}(C) = w$, then:

I. If L(ws) = L(w) + 1, then C has q_s s-adjacent chambers C', with $\rho_{\mathbb{S}_0}(C') = ws$.

I. If L(ws) = L(w) - 1, then C has $q_s - 1$ s-adjacent chambers C', with $\rho_{\mathbb{S}_0}(C') = w$ and one s-adjacent chamber C', with $\rho_{\mathbb{S}_0}(C') = ws$.

Proof. Since $\rho_{\mathbb{S}_0}$ is a retraction, all those chambers are with $\rho_{\mathbb{S}_0}(C') = w$ or $\rho_{\mathbb{S}_0}(C') = ws$. Choose $C_0 \in \mathbb{S}$ such that lemma 27.6 holds for all the $q_s + 1$ chambers containing the s-panel of C. Recall that one of the chambers is closer then the q_s others to C_0 . If L(ws) = L(w) + 1 the closest chamber is C and the q_s others are with $\rho_{\mathbb{S}_0}(C') = ws$. If L(ws) = L(w) - 1, C is not the closest and the claim follows.

Definition 27.8. Let $w \in \hat{W}$ and $1_w^{\mathbb{S}_0} \in \mathbb{C}^{\hat{B}_{\phi}}$ be the function defined as $1_w^{\mathbb{S}_0}(C) = 1$ if $\rho_{\mathbb{S}_0}(C) = w$ and $1_w^{\mathbb{S}_0}(C) = 0$ otherwise.

Lemma 27.9. If $w, w' \in \hat{W}$ with $L(ww'^{-1}) = L(w) + l(w'^{-1})$, then $h_{w'}1_w^{\mathbb{S}_0} = 1_{ww'^{-1}}^{\mathbb{S}_0}$.

Proof. It is enough to prove for $w'=s\in S$. If L(ws)=L(w)+1 then by lemma 27.7, for every chamber $C\in \hat{B}_{\phi}$ with $\rho_{\mathbb{S}_0}(C)=w$, the q_s s-adjacent chambers C' to C, all satisfy $\rho_{\mathbb{S}_0}(C')=ws$. Moreover, each chamber C' with $\rho_{\mathbb{S}_0}(C')=ws$, has a unique chamber C with $\rho_{\mathbb{S}_0}(C)=w$. Therefore $h_s1^{\mathbb{S}_0}_w=1^{\mathbb{S}_0}_{ws}$.

Definition 27.10. For $\beta \in P$, $\beta = \beta_1 \beta_2^{-1}$, $\beta_1, \beta_2 \in P^+$, we denote by $X_\beta \in \hat{H}_\phi$ the element $X_\beta = h_{\beta_1} h_{\beta_2}^{-1}$.

Notice that $Y_{\beta} = Q(\beta)^{1/2} X_{\beta}$, where Y_{β} is as in the Bernstein presentation.

Lemma 27.11. For $w_0 \in W_0$, $\beta \in P$ we have $1_{\beta^{-1}w_0^{-1}}^{\mathbb{S}_0} = h_{w_0}X_{\beta}1_{Id}^{\mathbb{S}_0}$.

Proof. For $\beta \in P^+$, $\beta' \in P$, we have $L(\beta'\beta^{-1}) = L(\beta') + l(\beta^{-1})$. By lemma 27.9, we have $h_{\beta}1_{\beta'}^{\mathbb{S}_0} = 1_{\beta'\beta^{-1}}^{\mathbb{S}_0}$. If $\beta' = \beta$, $1_{Id}^{\mathbb{S}_0} = h_{\beta}1_{\beta}^{\mathbb{S}_0}$, or $h_{\beta}^{-1}1_{Id}^{\mathbb{S}_0} = 1_{\beta}^{\mathbb{S}_0}$. Therefore for $\beta \in P$, $\beta = \beta_1\beta_2^{-1}$, $\beta_1, \beta_2 \in P^+$,

$$X_{\beta}1_{Id}^{\mathbb{S}_0} = h_{\beta_1}h_{\beta_2}^{-1}1_{Id}^{\mathbb{S}_0} = 1_{\beta_1\beta_2^{-1}}^{\mathbb{S}_0} = 1_{\beta^{-1}}^{S_0}$$

Finally, $L(\beta w_0^{-1}) = L(\beta) + l(w_0)$ and therefore

$$h_{w_0} X_{\beta} 1_{Id}^{\mathbb{S}_0} = h_{w_0} 1_{\beta^{-1}}^{S_0} = 1_{\beta^{-1} w_0^{-1}}^{\mathbb{S}_0}$$

Corollary 27.12. The action of h_{β} , $\beta \in P^+$ on $span\{1_w^{\mathbb{S}_0}\}_{w \in \hat{W}}$ satisfies:

$$h_{\beta} 1_{\gamma^{-1} w_0^{-1}}^{\mathbb{S}_0} = q_{\beta}^{1/2} \sum_{w_0', \beta'} \alpha_{w_0', w_0, \beta', \beta} Q_{\beta'}^{1/2} 1_{\gamma^{-1} \beta'^{-1} w_0'^{-1}}^{\mathbb{S}_0}$$

where $\alpha_{w_0,w_0,\beta,\beta}$ are as in theorem 26.6.

Proof. We apply the Bernstein relations of proposition 26.6 and get

$$\begin{split} h_{\beta} \mathbf{1}_{\gamma^{-1}w_{0}^{-1}}^{\mathbb{S}_{0}} &= X_{\beta} h_{w_{0}} X_{\gamma} \mathbf{1}_{Id}^{\mathbb{S}_{0}} = Q_{\beta}^{-1/2} Y_{\beta} h_{w_{0}} \mathbf{1}_{\gamma^{-1}}^{\mathbb{S}_{0}} \\ &= q_{\beta}^{1/2} \sum_{w'_{0},\beta'} \alpha_{w'_{0},w_{0},\beta',\beta} h_{w'_{0}} Y_{\beta'} \mathbf{1}_{\gamma^{-1}}^{\mathbb{S}_{0}} \\ &= q_{\beta}^{1/2} \sum_{w'_{0},\beta'} \alpha_{w'_{0},w_{0},\beta',\beta} Q_{\beta'}^{1/2} \mathbf{1}_{\gamma^{-1}\beta'^{-1}w'_{0}^{-1}}^{\mathbb{S}_{0}} \end{split}$$

Definition 27.13. Let $\lambda_w^{\mathbb{S}_0} \in \mathbb{C}[\hat{B}_{\phi}]^*$ be the functionals $\lambda_w^{\mathbb{S}_0}(f) = \sum_{C: \rho_{\mathbb{S}_0}(C) = w} f(C)$.

Lemma 27.14. If $w, w' \in \hat{W}$ with L(ww') = L(w) + l(w'), then $\lambda_w^{\mathbb{S}_0} h_{w'} = \lambda_{ww'}^{\mathbb{S}_0}$.

Proof. By induction, it is enough to prove for $w' = s \in S$. Then it is a direct result of lemma 27.7, as in lemma 27.9.

Proposition 27.15. For $w_0 \in W_0$, $\beta \in P$ we have $\lambda_{\beta^{-1}\tilde{w}_0w_0^{-1}}^{\mathbb{S}_0} = \lambda_{\tilde{w}_0}^{\mathbb{S}_0}X_{\beta}h_{w_0}^{-1}$.

Proof. For $\beta \in P^+$, $\beta' \in P$ we have, $\beta' \tilde{w}_0 \beta = \beta' \beta^{-1} \tilde{w}_0$, so $L(\beta' \tilde{w}_0 \beta) = L(\beta' \beta^{-1} \tilde{w}_0) = l(\beta') + l(\beta^{-1} \tilde{w}_0)$. Therefore by lemma 27.14, $\lambda_{\beta' \tilde{w}_0}^{\mathbb{S}_0} h_{\beta} = \lambda_{\beta' \beta^{-1} \tilde{w}_0}^{\mathbb{S}_0}$.

If $\beta = \beta_1 \beta_2^{-1}$, $\beta \in P$, $\beta_1, \beta_2 \in P^+$ then

$$\lambda_{\tilde{w}_0}^{\mathbb{S}_0} X_{\beta} = \lambda_{\tilde{w}_0}^{\mathbb{S}_0} h_{\beta_1} h_{\beta_2}^{-1} = \lambda_{\beta_1^{-1} \beta_2 \tilde{w}_0}^{\mathbb{S}_0} = \lambda_{\beta^{-1} \tilde{w}_0}^{\mathbb{S}_0}$$

Similarly, since $L(\tilde{w}_0 w_0^{-1} w_0) = L(\tilde{w}_0 w_0^{-1}) + l(w_0)$,

$$\lambda_{\beta^{-1}\tilde{w}_0 w_0^{-1}}^{\mathbb{S}_0} h_{w_0} = \lambda_{\beta^{-1}\tilde{w}_0}^{\mathbb{S}_0}$$

Summarizing, we get $\lambda_{\beta^{-1}\tilde{w}_0w_0^{-1}}^{\mathbb{S}_0} = \lambda_{\tilde{w}_0}^{\mathbb{S}_0}X_{\beta}h_{w_0}^{-1}$.

Corollary 27.16. We have

$$\lambda_{\gamma^{-1}\tilde{w}_0w_0^{-1}}^{\mathbb{S}_0}h_{\beta} = q_{\beta}^{1/2} \sum \alpha'_{w_0',w_0,\beta',\beta}q_{w_0'}Q_{\beta'}^{1/2}\lambda_{\gamma^{-1}\beta'\tilde{w}_0w_0'^{-1}}^{\mathbb{S}_0}$$

Proof. Using the Bernstein relations of proposition 26.7 we have:

$$\begin{split} \lambda_{\gamma^{-1}\bar{w}_0w_0^{-1}}^{\mathbb{S}_0}h_{\beta} &= \lambda_{\bar{w}_0}^{\mathbb{S}_0}X_{\gamma}h_{w_0}^{-1}X_{\beta} = \lambda_{\gamma^{-1}\bar{w}_0}^{\mathbb{S}_0}Q_{\beta}^{-1/2}h_{w_0}^{-1}Y_{\beta} \\ &= \lambda_{\gamma^{-1}\bar{w}_0}^{\mathbb{S}_0}q_{\beta}^{1/2}\sum\alpha'_{w'_0,w_0,\beta',\beta}Y_{\beta'}h_{w'_0^{'-1}}^{-1} \\ &= \lambda_{\gamma^{-1}\bar{w}_0}^{\mathbb{S}_0}q_{\beta}^{1/2}\sum\alpha'_{w'_0,w_0,\beta',\beta}Q_{\beta'}^{1/2}X_{\beta'}h_{w'_0^{'-1}}^{-1} \\ &= q_{\beta}^{1/2}\sum\alpha'_{w'_0,w_0,\beta',\beta}Q_{\beta'}^{1/2}\lambda_{\gamma^{-1}\beta'^{-1}\tilde{w}_0w_0'^{-1}} \end{split}$$

The following theorem summarizes the discussion in this section. It relates the Bernstein presentation of the Iwahori-Hecke algebra and the sectorial geometry of the building:

Theorem 27.17. Let $w = \gamma^{-1} w_0^{-1}$, $w' = \gamma'^{-1} w_0'^{-1}$ be elements of \hat{W} and $\beta \in P^+$. Then:

- For every $C' \in \hat{B}_{\phi}$ with $\rho_{\mathbb{S}_0}(C') = w'$ there exist $N_{w',w} = q_{\beta}^{1/2} Q_{\gamma'\gamma^{-1}}^{1/2} \alpha_{w'_0,w_0,\gamma'\gamma^{-1},\beta}$ chambers $C \in \hat{B}_{\phi}$, $\rho_{\mathbb{S}_0}(C) = w$, with $d(C', C) = \beta$.
- For every $C \in \hat{B}_{\phi}$ with $\rho_{\mathbb{S}_0}(C) = w$ there exist $N'_{w',w} = q_{\beta}^{1/2} Q_{\gamma\gamma'^{-1}}^{1/2} \alpha'_{\tilde{w}_0 w_0, \tilde{w}_0 w'_0, \gamma\gamma'^{-1}, \beta}$ chambers $C' \in \hat{B}_{\phi}, \ \rho_{\mathbb{S}_0}(C') = w', \ with \ d(C', C) = \beta.$

Proof. With notations as above, and by definition of $1_{w'}^{\mathbb{S}_0}$ and h_{β} , $N_{w',w}$ is the coefficient of $1_{w'}^{\mathbb{S}_0}$ in the decomposition of $h_{\beta}1_w^{\mathbb{S}_0}$. By proposition 27.12, this number is $q_{\beta}^{1/2}Q_{\gamma'\gamma^{-1}}^{1/2}\alpha_{w'_0,w_0,\gamma'\gamma^{-1},\beta}$.

Similarly, $N'_{w',w}$ is the coefficient of $\lambda_w^{\mathbb{S}_0}$ in the decomposition of $1_{w'}^{\mathbb{S}_0}h_{\beta}$. Let $w_1 = w_0\tilde{w}_0$, $w'_1 = w'_0\tilde{w}_0$. Then $w = \gamma^{-1} \tilde{w}_0 w_1^{-1}, \ w' = \gamma'^{-1} \tilde{w}_0 w_1'^{-1}$. By proposition 27.12 this coefficient is $q_{\beta}^{1/2} Q_{\gamma \gamma'^{-1}}^{1/2} \alpha'_{w_1, w'_1, \gamma \gamma'^{-1}, \beta}$.

28. How to Bound Operators

The goal is this section is to use theorem 27.17 to bound the norm of h_{β} , $\beta \in P^+$. For all this section, β is fixed.

Lemma 28.1. Let $X = X_0 \cup X_1$ be a (possibly infinite) a biregular graph, such that every $x \in X_0$ is connected

to at most K_0 vertices in X_1 , and every $y \in X_1$ is connected to at most S_1 vertices in X_0 . Let $A_X : \mathbb{C}^{X_0} \to \mathbb{C}^{X_1}$ be the adjacency operator from X_0 to X_1 , i.e. $Af(y) = \sum_{x \sim y} f(x)$. Then as operator $A: L_p(X_0) \to L_p(X_1)$, we have $||A||_p \le K_0^{1/p} K_1^{(p-1)/p}$.

Proof. For X finite the result follows from the convexity of $s \to s^p$. Then it extends easily to X infinite. \square

Let $w, w' \in \hat{W}$. Create a graph bipartite graph $X_{w,w'}$. The vertices X_0 will be chambers $C \in B_{\phi}$ with $\rho_{\mathbb{S}_0}(C) = w = \gamma^{-1}w_0^{-1}$, and the vertices X_1 will be the chambers C' with $\rho_{\mathbb{S}_0}(C') = w' = \gamma'^{-1}w_0'^{-1}$. Connect two chambers C_0, C_1 if $d(C_1, C_0) = \beta$. Then by theorem 27.17, this graph is (K_0, K_1) -biregular, with $K_0 = Q_{\gamma\gamma'^{-1}}^{1/2} \alpha'_{\tilde{w}_0 w_0^{-1}, \tilde{w}_0 w_0'^{-1}, \gamma\gamma'^{-1}, \beta}$ and $K_1 = q_{\beta}^{1/2} Q_{\gamma'\gamma^{-1}}^{1/2} \alpha_{w'_0, w_0, \gamma'\gamma^{-1}, \beta}$. By the above lemma we have:

$$\|A_{w,w'}\|_p \le K_0^{1/p} K_1^{(p-1)/p} = q_\beta^{1/2} Q_{\gamma'\gamma^{-1}}^{(p-2)/2p} \alpha'^{1/p}_{\tilde{w}_0 w_0^{-1}, \tilde{w}_0 w_0'^{-1}, \gamma \gamma'^{-1}, \beta} \alpha_{w_0', w_0, \gamma \gamma'^{-1}, \beta}^{(p-1)/p}$$

Let us now extend this to a bound on h_{β} . First of all, one can consider the results above by fixing w_0, w'_0 and $\beta' = \gamma' \gamma^{-1}$, but letting γ change. For each different γ , we have a different graph $X_{w,w'}$, with the same bound on $\|A_{w,w'}\|_p$. All those graphs are disjoint. Therefore the same bound holds for the disjoint union $X_{w_0,w'_0,\beta'} = \sqcup X_{w,w'}$ of all those graphs, that is

Let $w_0 \in W_0$ and for $f \in L_p(\hat{B}_\phi)$ define $f^{w_0} \in L_p(B_\phi)$ by:

$$f^{w_0}(C) = \begin{cases} f(C) & \exists \gamma \in P : \, \rho_{\mathbb{S}_0}(C) = \gamma^{-1} w_0^{-1} \\ 0 & \text{else} \end{cases}$$

Surely $f = \sum_{w_0 \in W_0} f^{w_0}$ and $||f||_p^p = \sum ||f^{w_0}||_p^p$

Lemma 28.2. Using the above notations, we have:

$$(h_{\beta}f)^{w'_0} = \sum_{w'_0 \le w_0} \left(\sum_{\beta' \le \beta} A_{w_0, w'_0, \beta'} \right) f^{w_0}$$

Proof. Follows by definition of f^{w_0} and , the graph $X_{w_0,w'_0,\beta'}$ and the adjacency operator $A_{w_0,w'_0,\beta'}$. We can now prove:

Theorem 28.3. With the notations above, for $f \in L_p(\hat{B}_{\phi})$ and $\beta \in P^+$, we have:

$$\left\| (h_{\beta}f)^{w'_{0}} \right\|_{p} \leq q_{\beta}^{1/2} \sum_{w'_{0} \leq w_{0}} \left(\sum_{\beta' \leq \beta} Q_{\beta'}^{(p-2)/2p} \alpha'^{1/p}_{\tilde{w}_{0}w_{0}^{-1}, \tilde{w}_{0}w_{0}^{'-1}, \beta', \beta} \alpha_{w'_{0}, w_{0}, \beta'^{-1}, \beta}^{(p-1)/p} \right) \|f^{w_{0}}\|_{p}$$

As a corollary, $\|h_{\beta}\|_{p} \leq |W_{0}| |2q_{max}|^{l(\tilde{w}_{0})} (l(\beta) + 1)^{l(\tilde{w}_{0})} q_{\beta}^{(p-1)/p}$

Proof. The first inequality follows from lemma 28.2 and the bound 28.1. We now turn to simplify the expression $q_{\beta}^{1/2} \sum_{\beta' \leq \beta} Q_{\beta'}^{(p-2)/2p} \alpha_{w_0',w_0,\beta',\beta}^{(p-1)/p} \alpha_{\tilde{w}_0w_0^{-1},\tilde{w}_0w_0^{-1},\beta'^{-1},\beta}^{1/p}$. since $\beta' \leq \beta$ we have by lemma 26.5 $Q_{\beta'} \leq q_{\beta}$, so $q_{\beta}^{1/2} Q_{\beta'}^{(p-2)/2p} \leq q_{\beta}^{(p-1)/p}$. Moreover:

$$\begin{array}{lcl} \sum_{\beta'} \alpha'^{1/p}_{\tilde{w}_0 w_0^{-1}, \tilde{w}_0 w_0'^{-1}, \beta', \beta} \alpha^{(p-1)/p}_{w_0', w_0, \beta'^{-1}, \beta} & \leq & \left(\sum_{\beta'} \alpha'_{\tilde{w}_0 w_0^{-1}, \tilde{w}_0 w_0'^{-1}, \beta'^{-1}, \beta} \right)^{1/p} \left(\sum_{\beta'} \alpha_{w_0', w_0, \beta', \beta} \right)^{(p-1)/p} \\ & \leq & \left(2^{l(w_0)} (q_{max} \cdot (l(\beta) + 1))^{l(w_0) - l(w_0')} \right)^{(p-1)/p} \left(2^{l(w_0)} (l(\beta) + 1)^{l(w_0) - l(w_0')} \right)^{1/p} \\ & \leq & \left| 2q_{max} \right|^{l(\tilde{w}_0)} (l(\beta) + 1)^{l(\tilde{w}_0)} \end{array}$$

The first inequality follows from Hölder's inequality. The second from the bounds of propositions 26.6 and

Therefore we have $\left\| (h_{\beta}f)^{w'_0} \right\|_p \le q_{\beta}^{(p-1)/p} \left| 2q_{max} \right|^{l(\tilde{w}_0)} (l(\beta) + 1)^{l(\tilde{w}_0)} \sum_{w'_0 \le w_0} \left\| f^{w_0} \right\|_p$. Denote $\lambda = q_{\beta}^{(p-1)/p} \left| 2q_{max} \right|^{l(\tilde{w}_0)} (l(\beta) + 1)^{l(\tilde{w}_0)} \sum_{w'_0 \le w_0} \left\| f^{w_0} \right\|_p$. Using the convexity inequality $\left(\sum_{i=1}^{N} \alpha_i\right)^p \leq N^{p-1} \sum_{i=1}^{n} \alpha_i^p$, we have

$$||h_{\beta}f||_{p}^{p} = \sum_{w'_{0}} ||(h_{\beta}f)^{w'_{0}}||_{p}^{p}$$

$$\leq \sum_{w'_{0}} (\sum_{w_{0}} \lambda ||f^{w_{0}}||_{p})^{p}$$

$$\leq \lambda^{p} ||W_{0}||W_{0}|^{p-1} \sum_{w_{0}} ||f^{w_{0}}||_{p}^{p}$$

$$= \lambda^{p} ||W_{0}||^{p} ||f||_{p}^{p}$$

and $||h_{\beta}||_{p} \leq |W_{0}| \lambda$ as needed.

Remark 28.4. As said in the introduction, the proof presented here is based on the proof of [CHH88], theorem

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