UNIQUENESS OF GRIM HYPERPLANES FOR MEAN CURVATURE FLOWS

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ABSTRACT. In this paper we show that an immersed nontrivial translating soliton for mean curvature flow in $\mathbb{R}^{n+1}(n=2,3)$ is a grim hyperplane if and only if it is mean convex and has weighted total extrinsic curvature of at most quadratic growth. For an embedded translating soliton Σ with nonnegative scalar curvature, we prove that if the mean curvature of Σ does not change signs on each end, then Σ must have positive scalar curvature unless it is either a hyperplane or a grim hyperplane.

1. Introduction

A mean curvature flow (MCF) in \mathbb{R}^{n+1} is the negative gradient flow of the volume functional, which can be analyzed from the perspective of partial differential equations as shown by Huisken in [4]. MCF is smooth in a short time and singularities must happen over a longer time. According to the rate of blow-up of the second fundamental form A(t,p) of the hypersurface Σ_t , this finite time singularity T is called type-I, if there exists a constant C_0 such that

$$\sup_{p \in \Sigma_{t}} \left| A\left(t, p\right) \right|^{2} \leq \frac{C_{0}}{(T - t)}$$

for all t < T. Otherwise this finite time singularity is called type-II.

We will deal with translating solitons which are important in study of type-II singularities.

A complete connected isometrically immersed hypersurface (Σ, Φ) in \mathbb{R}^{n+1} is called a *translating soliton* if its mean curvature vector satisfies

$$\vec{H} = w^{\perp},$$

where $w \in \mathbb{R}^{n+1}$ is a unitary vector and w^{\perp} stands for the orthogonal projection of w onto the normal bundle of Φ . Let ν denote the unit normal along Φ , then it is equivalent to

$$H = -\langle \nu, w \rangle$$
.

In particular, considering $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined by $f(x) = -\langle x, w \rangle$, then $\overline{\nabla} f = -w$ and $H = \langle \overline{\nabla} f, \nu \rangle$, therefore by definition translating solitons are f-minimal hypersurfaces. Since MCF is invariant under isometries,

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without loss of generality we may suppose that w = (0, ..., 0, 1), then the function f is defined by $f(x) = -x_{n+1}$ and the L_f -stability operador of Σ is given by

$$(1) L_f = \Delta_f + |A|^2$$

There are some examples of translating solitons: vertical hyperplanes, grim hyperplanes, translating bowl solitons and translating catenoids. In this article we will give a characterization of grim hyperplanes in dimensions 2 and 3.

Recall that a *grim hyperplane* in \mathbb{R}^{n+1} is a hypersurface \mathcal{G} of \mathbb{R}^{n+1} which can be represented parametrically via the embedding $\Phi: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n+1}$ defined by

$$\Phi(t, y_1, \dots, y_{n-1}) = (t, y_1, \dots, y_{n-1}, -\ln(\cos t)).$$

The grim hyperplane \mathcal{G} satisfies the translating soliton equation with $w = (0, \ldots, 0, 1)$ i.e. it is f-minimal for $f(x_1, \ldots, x_{n+1}) = -x_{n+1}$. Also it has positive mean curvature. When n = 2 or 3, there exists a constant C > 0 such that

$$\int_{B_R} |A|^2 e^{-f} \le CR^2$$

for all R sufficiently large. The aim of this article is to prove that indeed the grim hyperplanes are the only ones with these properties when n = 2, 3.

Theorem 1. Let $\Phi: \Sigma^n \longrightarrow \mathbb{R}^{n+1}$ be a translating soliton, with n=2 or 3, which is not a hyperplane. Then Σ is a grim hyperplane if and only if $H = -\langle w, \nu \rangle \geq 0$ and there exists C > 0 such that

$$(3) \qquad \int_{B_R} |A|^2 e^{-f} \le CR^2,$$

for all R sufficiently large, where B_R is the geodesic ball of radius R and $f(x) = -\langle x, w \rangle$.

The expression (3) is not satisfied for $n \geq 4$ (see Proposition 1), thus Theorem 1 is sharp in this sense.

It has been known that if $H \geq 0$ on a translating soliton Σ , then either $H \equiv 0$ on Σ and Σ is a hyperplane, or H > 0 everywhere on Σ . Note that both hyperplane and grim hyperplane has vanishing scalar curvature. In [6], Martín-Savas-Halilaj-Smoczyk proved that flat hyperplane and grim hyperplane are the only translating soliton with vanishing scalar curvature. It would be interesting to ask if the following is true.

Problem: Let Σ be a translating soliton with nonnegative scalar curvature S. Is it true that either $S \equiv 0$ on Σ and Σ is a hyperplane or grim hyperplane, or S > 0 everywhere on Σ ?

This problem is related to a result proved by Huang-Wu in [3]. Let M be a closed embedded n-dimensional hypersurface in \mathbb{R}^{n+1} with nonnegative scalar curvature. Let M_t be a solution to the mean curvature flow with

initial hypersurface M. Then the scalar curvature of M_t is strictly positive for all t > 0.

For complete embedded translating solitons, we have

Theorem 2. Let (Σ^n, g) be a embedded translating soliton with nonnegative scalar curvature S. Assume H does not change signs on each end. Then either Σ is a hyperplane or a grim hypersurface; or Σ has positive scalar cuvature.

2. Total weighted extrinsic curvature

In this section we will give the asymptotic properties of the total weighted extrinsic curvatures of grim hyperplanes. We have

$$\partial_t = \sec(t)(\cos t, 0, \cdots, 0, \sin t).$$

We choose the unit normal ν to \mathcal{G} to be $\nu = (\sin t, 0, \dots, 0, -\cos t)$. A little computation shows that $\overline{\nabla}_{\partial_t} \nu = (\cos t) \partial_t$ and $\overline{\nabla}_{\partial_{y_i}} \nu = 0 \ (1 \le i \le n-1)$.

Then the principal curvatures are $\lambda_1 = \cos t$, $\lambda_2 = \ldots = \lambda_n = 0$, thus on the coordinates t, y_1, \ldots, y_{n-1} the mean curvature only depends on t and is given by $H(t) = \cos t$. Since $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have the norm of the second fundamental form is given by

$$|A|(t) = \cos t = H(t).$$

Now, consider the function $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined by $f(x) = -x_{n+1}$, then $\langle \overline{\nabla} f, \nu \rangle = \cos t = H$.

Proposition 1. The Grim Hyperplane \mathcal{G} in \mathbb{R}^{n+1} satisfies

$$\lim_{R \to +\infty} \frac{1}{R^{n-1}} \int_{B_R} |A|^2 e^{-f} = |B^{n-1}(1)| \pi,$$

where B_R is the geodesic ball with center at 0 and radius R and $B^{n-1}(1)$ is the open ball in \mathbb{R}^{n-1} of radius 1 and center at the origin.

Proof of Proposition 1. Observe that f and the metric on \mathcal{G} in the coordinates t, y_1, \ldots, y_{n-1} are given by

$$f(t) = \ln(\cos t)$$

and

$$g = \sec^2(t) dt^2 + dy_1^2 + \ldots + dy_{n-1}^2.$$

Thus

$$r = \int_0^t \sec(\xi) d\xi = -\ln\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - t\right)\right)\right),\,$$

we have $t = \frac{\pi}{2} - \eta(r)$, where $\eta(r) = 2\arctan(e^{-r})$. Then

$$g = dr^2 + dy_1^2 + \dots + dy_{n-1}^2.$$

Besides that |A| and f in the coordinates r, y_1, \dots, y_{n-1} are given by

$$|A|(r) = \sin(\eta(r))$$
,

and

$$f(r) = \ln\left(\sin\left(\eta(r)\right)\right).$$

Denoting by $\|.\|$ the standard norm of \mathbb{R}^{n-1} , we have

$$B_R = \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : r^2 + ||y||^2 \le R^2 \right\}$$
$$= \left\{ (r, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : -\sqrt{R^2 - ||y||^2} \le r \le \sqrt{R^2 - ||y||^2}, \qquad ||y|| \le R \right\}.$$

Since $-\eta'(r) = \sin(\eta(r))$ is an even function, then

$$\begin{split} \int_{B_R} |A|^2 \, e^{-f} &= \int_{\{\|y\| \le R\}} \left[\int_{-\sqrt{R^2 - \|y\|^2}}^{\sqrt{R^2 - \|y\|^2}} \sin\left(\eta\left(r\right)\right) dr \right] dy \\ &= \int_{\{\|y\| \le R\}} \left[\pi - 2\eta \left(\sqrt{R^2 - \|y\|^2}\right) \right] dy \\ &= \pi \int_{\{\|y\| \le R\}} 1 dy - 2 \int_{\{\|y\| \le R\}} \eta \left(\sqrt{R^2 - \|y\|^2}\right) dy \\ &= \pi \left| B^{n-1}(1) \right| R^{n-1} - 2 \int_0^R \left(\int_{\mathbb{S}_\rho^{n-2}} \eta \left(\sqrt{R^2 - \rho^2}\right) dA \right) d\rho \\ &= \pi \left| B^{n-1}(1) \right| R^{n-1} - 2 \mathrm{area} \left(\mathbb{S}^{n-2} \right) \int_0^R \eta \left(\sqrt{R^2 - \rho^2}\right) \rho^{n-2} d\rho. \end{split}$$

where we have used the co-area formula. Now, letting $\rho = R \sin \theta$ and using the fact area $(\mathbb{S}^{n-2}) = (n-1) |B^{n-1}(1)|$, we have

(5)
$$\frac{1}{R^{n-1}} \int_{B_n} |A|^2 e^{-f} = |B^{n-1}(1)| [\pi - 2(n-1) F_{n-1}(R)],$$

where

(6)
$$F_{n-1}(R) = \int_0^{\pi/2} \eta(R\cos\theta)\sin^{n-2}\theta\cos\theta d\theta.$$

Observe that

$$\lim_{R \longrightarrow +\infty} \eta \left(R \cos \theta \right) \sin^{n-2} \theta \cos \theta = 0 \quad \text{for all } \theta \in \left[0, \frac{\pi}{2} \right].$$

Fixing R > 0, we have $\left| \eta \left(R \cos \theta \right) \sin^{n-2} \theta \cos \theta \right| \leq \frac{\pi}{2} \sin^{n-2} \theta \cos \theta$ for all $\theta \in [0, \pi/2]$. Besides that

$$\int_0^{\pi/2} \sin^{n-2}\theta \cos\theta d\theta = 1/(n-1).$$

Then $\lim_{R \to +\infty} F_{n-1}(R) = 0$, and hence by (5), we get

$$\lim_{R\longrightarrow +\infty}\frac{1}{R^{n-1}}\int_{B_{R}}\left|A\right|^{2}e^{-f}=\left|B^{n-1}\left(1\right)\right|\pi.$$

3. Proofs of Theorem 1 and Theorem 2

We begin this section with the following lemma which is in a form more general than we need. The lemma may have its independent interest.

Lemma 1. Assume that on a complete weighted manifold $(M, \langle , \rangle, e^{-f} d_{\text{VOL}})$, the functions $u, v \in C^2(M)$, with u > 0 and $v \ge 0$ on M, satisfy

(7)
$$\Delta_f u + q(x) u \le 0$$
 and $\Delta_f v + q(x) v \ge 0$,

where $q(x) \in C^0(M)$. Suppose that there exists a positive function $\kappa > 0$ on \mathbb{R}^+ satisfying $\frac{t}{\kappa(t)}$ is nonincreasing and

$$\int^{+\infty} \frac{t}{\kappa(t)} dt = +\infty,$$

such that

(8)
$$\int_{B_R} v^2 e^{-f} \le \kappa(R)$$

for all R. Then there exists a constant C such that v = Cu.

Remark 1. Without loss of generality, we can assume $\kappa(t) \geq C(1+t^2)$. Some examples of $\kappa(t)$ are Ct^2 , $Ct^2 \log(1+t)$, $Ct^2 \log(1+t) \log\log(3+t)$, \cdots .

Proof of Lemma 1. Set $w = \frac{v}{u}$, then v = wu, thus by (7) we get

$$\Delta_f v = w \Delta_f u + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w$$

$$\leq -w(qu) + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w$$

$$= -qv + 2 \langle \nabla w, \nabla u \rangle + u \Delta_f w.$$

Then

(9)
$$\Delta_f w \ge -2 \langle \nabla w, \nabla (\ln u) \rangle.$$

On the other hand, let $\varphi \in C_o^2(M)$, then by (9), we have

$$\begin{split} \int_{M} \varphi^{2} |\nabla w|^{2} e^{-f} &= \int_{M} \left\langle \varphi^{2} \nabla w, \nabla w \right\rangle e^{-f} \\ &= \int_{M} \left\langle \nabla \left(\varphi^{2} w \right), \nabla w \right\rangle e^{-f} - 2 \int_{M} \varphi w \left\langle \nabla \varphi, \nabla w \right\rangle e^{-f} \\ &= - \int_{M} \varphi^{2} w \left(\Delta_{f} w \right) e^{-f} - 2 \int_{M} \varphi w \left\langle \nabla \varphi, \nabla w \right\rangle e^{-f} \\ &\leq 2 \int_{M} \varphi^{2} w \left\langle \nabla w, \nabla \left(\ln u \right) \right\rangle e^{-f} - 2 \int_{M} \varphi w \left\langle \nabla \varphi, \nabla w \right\rangle e^{-f} \\ &= 2 \int_{M} \left\langle \varphi \nabla w, w \left(\varphi \nabla \left(\ln u \right) - \nabla \varphi \right) \right\rangle \\ &\leq \frac{1}{2} \int_{M} \varphi^{2} |\nabla w|^{2} e^{-f} + 2 \int_{M} w^{2} |\varphi \nabla \left(\ln u \right) - \nabla \varphi |^{2} e^{-f}. \end{split}$$

Then

$$(10) \quad \int_{M} \varphi^{2} |\nabla w|^{2} e^{-f} \leq 4 \int_{M} w^{2} |\varphi \nabla (\ln u) - \nabla \varphi|^{2} e^{-f} \qquad \forall \ \varphi \in C_{o}^{2} \left(M\right).$$

If $\psi \in C_o^{\infty}(M)$, then $\varphi = \psi u \in C_o^2(M)$. Besides that, a little computation shows

$$\varphi \nabla (\ln u) - \nabla \varphi = - (\nabla \psi) u,$$

Thus, from (10), we have

$$\int_{M} \psi^{2} u^{2} |\nabla w|^{2} e^{-f} \leq 4 \int_{M} w^{2} |\nabla \psi|^{2} u^{2} e^{-f}
= 4 \int_{M} |\nabla \psi|^{2} v^{2} e^{-f} \quad \forall \psi \in C_{o}^{\infty}(M).$$

Define functions β , ξ on $[0, +\infty)$ as

$$\beta(t) := \int_0^t \frac{\tau}{\kappa(\tau)} d\tau,$$

and ξ is the inverse function of β . From the hypothesis we know β' is nonincreasing and ξ' is nondecreasing functions on $[0, +\infty)$. Now, we now choose a cutoff function

$$\psi_{R}(x) = \begin{cases} 1, & \text{on } B_{\xi(R)}; \\ 2 - \frac{\beta(r(x))}{R}, & \text{on } B_{\xi(2R)} \setminus B_{\xi(R)}; \\ 0, & \text{on } M \setminus B_{\xi(2R)}. \end{cases}$$

where r(x) = d(x, p), $p \in M$ is a fixed point and B_R is the geodesic ball with radius R and center p. We see that $|\nabla \psi_R| = \frac{\beta'(r)}{R} = \frac{r}{R\kappa(r)}$. Then, by (8), we get

$$\begin{split} \int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} &= \int_{B_{\xi(R)}} \psi_R^2 u^2 |\nabla w|^2 e^{-f} \\ &\leq \int_M \psi_R^2 u^2 |\nabla w|^2 e^{-f} \\ &\leq 4 \int_M v^2 |\nabla \psi_R|^2 e^{-f} \\ &= 4 \int_{B_{\xi(2R)} \backslash B_{\xi(R)}} v^2 |\nabla \psi_R|^2 e^{-f} \\ &= \frac{4}{R^2} \int_{\xi(R)}^{\xi(2R)} (\beta'(s))^2 \int_{\partial B_s} v^2 e^{-f} dA ds. \end{split}$$

Here we have used co-area formula. For convenience, we write $V(s)=\int_{B_s} v^2 e^{-f} dV$. Therefore

$$V(s) = \int_0^s \int_{\partial B_{\tau}} v^2 e^{-f} dA d\tau \le \kappa(s),$$

and

$$\begin{split} \int_{B_{\xi(R)}} u^2 |\nabla w|^2 e^{-f} & \leq \frac{4}{R^2} \int_{\xi(R)}^{\xi(2R)} (\beta'(s))^2 V'(s) ds \\ & = \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \left| \frac{\xi(2R)}{\xi(R)} - \int_{\xi(R)}^{\xi(2R)} 2V(s) (\beta'(s)) d\beta'(s) \right] \right. \\ & \leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \left| \frac{\xi(2R)}{\xi(R)} - 2 \int_{\xi(R)}^{\xi(2R)} s d\beta'(s) \right] \right. \\ & \leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \left| \frac{\xi(2R)}{\xi(R)} - 2s\beta'(s) \left| \frac{\xi(2R)}{\xi(R)} + 2 \int_{\xi(R)}^{\xi(2R)} \beta'(s) ds \right] \right. \\ & \leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \left| \frac{\xi(2R)}{\xi(R)} - 2s\beta'(s) \left| \frac{\xi(2R)}{\xi(R)} + \beta(s) \left| \frac{\xi(2R)}{\xi(R)} \right] \right. \\ & \leq \frac{4}{R^2} \left[V(s) (\beta'(s))^2 \left| \frac{\xi(2R)}{\xi(R)} - 2s\beta'(s) \left| \frac{\xi(2R)}{\xi(R)} + R \right. \right] \end{split}$$

Since

$$V(s)(\beta'(s))^2 = V(s)\beta'(s)\beta'(s) \le s\beta'(s),$$

and $\beta'(s) = \frac{s}{\kappa(s)}$, thus Remark 1 implies these terms are bounded, hence when $R \longrightarrow +\infty$, all the terms on the right hand side go to zero. So we get

$$\int_{M} u^2 |\nabla w|^2 e^{-f} = 0.$$

Then $\nabla w \equiv 0$, thus there is a constant C such that $w \equiv C$ and hence v = Cu.

Definition 1. A two-sided translating soliton Σ is said to be stable if

$$\int_{\Sigma} \left[|\nabla \varphi|^2 - |A|^2 \varphi^2 \right] e^{-f} d\sigma \ge 0 \quad \text{for all } \varphi \in C_o^{\infty}(\Sigma).$$

As a consequence of Lemma 1, we have the following:

Corollary 1. Let $\Phi: \Sigma^n \longrightarrow \mathbb{R}^{n+1}$ be a stable translating soliton and let $\omega \in C^2(\Sigma)$ be a positive solution of the stability equation

$$(12) \Delta_f \omega + |A|^2 \omega = 0.$$

Moreover, if $H \geq 0$ and there exists a constant C > 0 such that

(13)
$$\int_{B_R} H^2 e^{-f} \leq CR^2 \quad \text{for all } R \text{ large enough.}$$

Then there exists a constant \widetilde{C} such that $H = \widetilde{C}\omega$. In particular, if $H \not\equiv 0$, then $\widetilde{C} \in \mathbb{R} \setminus \{0\}$ and H > 0.

Now, we include here a result due to Li and Wang ([5]) which will be needed in the proof our main theorem.

Lemma 2. Suppose Σ is complete and there exists a nonnegative function $\varphi: \Sigma \longrightarrow \mathbb{R}$, not identically zero, such that $(\Delta_f + q)(\varphi) \leq 0$. Then $\Delta_f + q$ is stable.

Proof. Let Ω be a compact subdomain in Σ and let u be the first eigenfunction satisfying

(14)
$$\begin{cases} (\Delta_f + q)u = -\lambda_1(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We may assume that $u \geq 0$ on Ω . From regularity of u and Hopf Lemma, we have

- u > 0 in the interior of Ω .
- $\frac{\partial u}{\partial \nu} < 0$ on $\partial \Omega$, where ν is the outward unit normal of $\partial \Omega$.

Thus, integration by parts on u and φ and also the hypothesis, we have

$$\int_{\Omega} u \left(\Delta_{f} \varphi \right) e^{-f} - \int_{\Omega} \varphi \left(\Delta_{f} u \right) e^{-f} = \int_{\partial \Omega} u \frac{\partial \varphi}{\partial \nu} e^{-f} - \int_{\partial \Omega} \varphi \frac{\partial u}{\partial \nu} e^{-f} \\
= - \int_{\partial \Omega} \varphi \frac{\partial u}{\partial \nu} e^{-f} \ge 0.$$

From hypothesis and (14), we have

(16)
$$\begin{cases} \Delta_f \varphi + Q \varphi \leq 0, \\ \Delta_f u + Q u = -\lambda_1(\Omega) u. \end{cases}$$

Since u > 0, multiplying the first inequality of (16) by u and the second equation by $-\varphi$, and finally both by e^{-f} , we have

(17)
$$u\left(\Delta_{f}\varphi\right)e^{-f}-\varphi\left(\Delta_{f}u\right)e^{-f} \leq \lambda_{1}\left(\Omega\right)\left(\varphi u\right)e^{-f}$$

Since both u > 0 and $\varphi \ge 0$ are not identically zero, then combining (17) with (15), we have $\lambda_1(\Omega) \ge 0$ for all compact subdomains of Σ , then $\lambda_1(f,Q) \ge 0$, therefore $\Delta_f + q$ is stable.

We are now ready to give the proof of Theorem 1.

Proof of Theorem 1 Since $\Phi: \Sigma^n \longrightarrow \mathbb{R}^{n+1}$ is a translating soliton, then the mean curvature H satisfies $\Delta_f H + |A|^2 H = 0$ (see Proposition 3 in [1]). Since $H \geq 0$ and Σ is a non-planar translating soliton, then H is not identically zero, thus by Lemma 2, Σ is stable and hence the weighted version of a result by Fischer-Colbrie and Schoen [2] guarantees there exists a non-constant positive C^2 -function ω on Σ such that

(18)
$$\Delta_f \omega + |A|^2 \omega = 0.$$

As $\frac{H^2}{n} \leq |A|^2$ and |A| satisfies (3), then

$$\int_{B_R} H^2 e^{-f} \leq nCR^2.$$

Then, by Corollary 1 and the condition that $H \geq 0$ and not identically zero, there is a constant $C_1 > 0$ such that

$$(20) H = C_1 \omega.$$

In particular H > 0 everywhere on Σ . On the other hand, the Simons equation (see [1] or [6]) implies that

(21)
$$|A| \left\{ \Delta_f |A| + |A|^2 |A| \right\} = |\nabla A|^2 - |\nabla |A||^2 \ge 0.$$

Since |A| satisfies (3), then by Lemma 1, $\exists C_2 \geq 0$ such that

$$(22) |A| = C_2 \omega.$$

Besides that Σ^n is not a hyperplane, then |A| is not identically zero, thus $C_2 > 0$. Then by (20) and (22) we have $|A|^2H^{-2} = \text{constant} > 0$. In particular this function attains its local maximum on Σ . Theorem B in [6] says that Σ is a grim hyperplane if and only if the function $|A|^2H^{-2}$ attains a local maximum. Therefore Σ is a grim hyperplane.

We now prove Theorem 2.

Proof of Theorem 2. To prove Theorem 2, we will need a result of Huang-Wu[3]. Denote by M_+ a connected component of $\{p \in M, H \geq 0 \text{ at } p\}$ that contains a point of positive mean curvature. We say that the mean curvature H changes signs through Γ if Γ is a connected component of ∂M_+ and Γ intersects the boundary of a connected component of $M \setminus \partial M_+$. Theorem 2 of Huang-Wu[3] $S \geq 0$, says that if H changes sign along Γ then Γ is unbounded set. Since we have assumed that H does not changes signs at infitiy, H has a sign. Hence either

- (1) $H \equiv 0$, or
- $(2)H \ge 0$ but does not vanish at least one point.

In case (1), Σ must be a hyperplane.

In case (2), if there is point $p \in \Sigma$, such that S(p) = 0 then $|A|^2 = H^2 - S \le H^2$ and equality holds at p. Therefore the function $|A|^2H^2$ is well defined and attains its maximum at p. By Theorem B in [6] it must be a grim hyperplane.

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