BOUNDARY EXPANSIONS FOR MINIMAL GRAPHS IN THE HYPERBOLIC SPACE

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ABSTRACT. We study expansions near the boundary of solutions to the Dirichlet problem for minimal graphs in the hyperbolic space and characterize the remainders of the expansion by multiple integrals. With such a characterization, we establish optimal asymptotic expansions of solutions with boundary values of finite regularity and demonstrate a slight loss of regularity for nonlocal coefficients.

1. Introduction

Complete minimal hypersurfaces in the hyperbolic space \mathbb{H}^{n+1} demonstrate similar properties as those in the Euclidean space \mathbb{R}^{n+1} in the aspect of the interior regularity and different properties in the aspect of the boundary regularity. Anderson [3], [4] studied complete area-minimizing submanifolds and proved that, for any given closed embedded (n-1)-dimensional submanifold N at the infinity of \mathbb{H}^{n+1} , there exists a complete area minimizing integral n-current which is asymptotic to N at infinity. In the case $n \leq 6$, these currents are embedded smooth submanifolds; while in the case $n \geq 7$, as in the Euclidean case, there can be closed singular set of Hausdorff dimension at most n-7. Hardt and Lin [16] discussed the C^1 -boundary regularity of such hypersurfaces. Subsequently, Lin [22] studied the higher order boundary regularity. In a more general setting, Graham and Witten [14] studied n-dimensional minimal surfaces of any codimension in asymptotically hyperbolic manifolds and derived an expansion of the normalized area up to order n+1.

Assume Ω is a bounded domain in \mathbb{R}^n . Lin [22] studied the Dirichlet problem of the form

(1.1)
$$\Delta f - \frac{f_i f_j}{1 + |Df|^2} f_{ij} + \frac{n}{f} = 0 \quad \text{in } \Omega,$$

with the condition

(1.2)
$$f > 0 \quad \text{in } \Omega,$$

$$f = 0 \quad \text{on } \partial \Omega.$$

In this paper, we follow Lin by denoting solutions of (1.1) by f. We note that the equation (1.1) becomes singular on $\partial\Omega$ since f=0 there. If Ω is a C^2 -domain in \mathbb{R}^n with a nonnegative boundary mean curvature $H_{\partial\Omega} \geq 0$ with respect to the inward normal of $\partial\Omega$, then (1.1)-(1.2) admits a unique solution $f \in C(\bar{\Omega}) \cap C^{\infty}(\Omega)$. Moreover,

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the graph of f is a complete minimal hypersurface in the hyperbolic space \mathbb{H}^{n+1} with asymptotic boundary $\partial\Omega$. At each point of the boundary, the gradient of f blows up and hence the graph of f has a vertical tangent plane. Locally, the graph of f can be represented as the graph of a new function, say u, over the vertical tangent plane and u satisfies a quasilinear elliptic equation which also becomes singular on the boundary. By discussing the regularity of u up to the boundary, Lin [22] established several results on global regularity of the graph of f. Lin proved that, if $\partial\Omega$ is $C^{n,\alpha}$ for some $\alpha \in (0,1)$, then the graph of f is $C^{n,\alpha}$ up to the boundary. Tonegawa [29] discussed the higher regularity in this setting. He proved that, if $\partial\Omega$ is smooth, then the graph of f is smooth up to the boundary if the dimension n is even or if the dimension n is odd and the boundary $\partial\Omega$ satisfies an extra condition. See also [23].

In this paper, we discuss fine boundary regularity of the graph of f by expanding relevant functions in terms of the distance to the boundary. We will do this from two aspects. First, we adopt the setup by Lin [22] and study the expansion of u. Second, we study the expansion of f itself.

Locally near each boundary point, the graph of f can be represented by a function over its vertical tangent plane. Specifically, we fix a boundary point of Ω , say the origin, and assume that the vector $e_n = (0, \dots, 0, 1)$ is the interior normal vector to $\partial \Omega$ at the origin. Then, with $x = (x', x_n)$, the x'-hyperplane is the tangent plane of $\partial \Omega$ at the origin, and the boundary $\partial \Omega$ can be expressed in a neighborhood of the origin as a graph of a smooth function over $\mathbb{R}^{n-1} \times \{0\}$, say

$$x_n = \varphi(x').$$

We now denote points in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by (x', x_n, y_n) . The vertical hyperplane given by $x_n = 0$ is the tangent plane to the graph of f at the origin in \mathbb{R}^{n+1} , and we can represent the graph of f as a graph of a new function u defined in terms of $(x', 0, y_n)$ for small x' and y_n , with $y_n > 0$. In other words, we treat $\mathbb{R}^n = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$ as our new base space and write $u = u(y) = u(y', y_n)$, with y' = x'. Then, for some R > 0, u satisfies

(1.3)
$$\Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{n u_n}{y_n} = 0 \text{ in } B_R^+,$$

and

$$(1.4) u = \varphi on B'_R.$$

By a similar reasoning as in [14], we can establish formal expansions for solutions of (1.3)-(1.4) in the following form: for n even,

$$u = \varphi + c_2 y_n^2 + c_4 y_n^4 + \dots + c_n y_n^n + \sum_{i=n+1}^{\infty} c_i y_n^i,$$

and, for n odd,

$$u = \varphi + c_2 y_n^2 + c_4 y_n^4 + \dots + c_{n-1} y_n^{n-1} + \sum_{i=n+1}^{\infty} \sum_{j=0}^{N_i} c_{i,j} y_n^i (\log y_n)^j,$$

where c_i and $c_{i,j}$ are smooth functions of $y' \in B'_R$ and N_i is a nonnegative constant depending on i, with $N_{n+1} = 1$. A formal calculation can only determine finitely many terms in the formal expansion of u and demonstrates a parity of dimensions. In fact, the coefficients c_2, c_4, \dots, c_n , for n even, and $c_2, c_4, \dots, c_{n-1}, c_{n+1,1}$, for n odd, have explicit expressions in terms of φ . For example, we have, for any $n \geq 2$,

$$c_2 = rac{1}{2(n-1)} \left(\Delta_{y'} arphi - rac{arphi_{lpha} arphi_{eta}}{1 + |D_{y'} arphi|^2} arphi_{lpha eta}
ight),$$

or

(1.5)
$$c_2 = \frac{1}{2} \sqrt{1 + |D_{y'}\varphi|^2} H,$$

where H is the mean curvature of the graph $x_n = \varphi(y')$ with respect to the upward unit normal. Moreover, we have, for n = 3,

(1.6)
$$c_{4,1} = -\frac{1}{8}\sqrt{1 + |D_{y'}\varphi|^2} \{\Delta_{\Sigma}H + 2H(H^2 - K)\},$$

where H and K are the mean curvature and the Gauss curvature of the graph Σ given by $x_n = \varphi(y')$, respectively. Formal expansions have different forms depending on whether the dimension of the space is even or odd, and logarithmic terms appear when the dimension is odd. We note that $c_{4,1} = 0$ if and only if Σ is a Willmore surface. We point out that the coefficient $c_{4,1}$ given by (1.6) is related to the Willmore functional in the renormalized area expansion in [14].

Logarithmic terms also appear in other problems, such as the singular Yamabe problem in [6], [24] and [27], the complex Monge-Ampère equations in [8], [10] and [20], and the asymptotically hyperbolic Einstein metrics in [5], [7], [9] and [17]. In fact, Fefferman [10] first observed that logarithmic terms should appear in the expansion.

Our goal in this paper is to discuss the relation between u and its formal expansions for boundary values of finite regularity and derive sharp estimates of remainders for the asymptotic expansions. We will also investigate the regularity property of nonlocal coefficients in the expansions.

Let $k \ge n+1$ be an integer and set, for n even,

(1.7)
$$u_k = \varphi + c_2 y_n^2 + c_4 y_n^4 + \dots + c_n y_n^n + \sum_{i=n+1}^k c_i y_n^i,$$

and, for n odd,

(1.8)
$$u_k = \varphi + c_2 y_n^2 + c_4 y_n^4 + \dots + c_{n-1} y_n^{n-1} + \sum_{i=n+1}^k \sum_{j=0}^{\left[\frac{i-1}{n}\right]} c_{i,j} y_n^i (\log y_n)^j,$$

where c_i and $c_{i,j}$ are functions of $y' \in B'_R$. We point out that the highest order in u_k is given by y_n^k . According to the pattern in this expansion, if we intend to continue to expand u_k , the next term has an order of y_n^{k+1} , for n even, and $y_n^{k+1}(\log y_n)^{\left[\frac{k}{n}\right]}$, for n odd. In these expansions, c_{n+1} or $c_{n+1,0}$ is the coefficient of the first global term and has no explicit expressions in terms of φ .

In this paper, we study the regularity and growth of the remainder $u - u_k$. We will prove the following result.

Theorem 1.1. For some integers $\ell \geq k \geq n+1$ and some constant $\alpha \in (0,1)$, let $\varphi \in C^{\ell,\alpha}(B_R')$ be a given function and $u \in C(\bar{B}_R^+) \cap C^{\infty}(B_R^+)$ be a solution of (1.3)-(1.4). Then, there exist functions c_i , $c_{i,j} \in C^{\ell-i,\epsilon}(B_R')$, for $i=0,2,4,\cdots,n+1,\cdots,k$ and any $\epsilon \in (0,\alpha)$, such that, for u_k defined as in (1.7) or (1.8), for any $\tau = 0,1,\cdots,\ell-k$, any $m=0,1,\cdots,k$, any $m=0,1,\cdots,k$, any $m=0,1,\cdots,k$, and $m=0,1,\cdots,k$, and

$$D_{u'}^{\tau} \partial_{u_n}^m (u - u_k) \in C^{\epsilon}(\bar{B}_r^+),$$

and, for any $(y', y_n) \in B_{R/2}^+$,

$$(1.10) |D_{y'}^{\tau} \partial_{y_n}^m (u - u_k)(y', y_n)| \le C y_n^{k - m + \alpha},$$

for some positive constant C depending only on n, ℓ , α , R, the L^{∞} -norm of u in G_R and the $C^{\ell,\alpha}$ -norm of φ in B'_R .

We note that the estimate (1.10) is optimal and that there is a slight loss of regularity of $c_{i,j}$ and $u - u_k$, for $i, k \ge n + 1$. The result (1.9) demonstrates a pattern as similar as for finitely differentiable functions. Under the assumption of a fixed regularity of the boundary value, the remainder of u has a designated regularity; meanwhile, the more we expand u in terms of y_n , the better regularity in y_n the remainder has. The main difference here is that the expansion of u includes terms involving logarithmic factors.

We point out that there is actually no loss of regularity for coefficients of local terms. If $\varphi \in C^{\ell,\alpha}(B_R')$ for some $\ell \geq 2$ and $\alpha \in (0,1)$, then $c_i \in C^{\ell-i,\alpha}(B_R')$, for $0 \leq i \leq \min\{\ell,n\}$ and i even, and $c_{n+1,1} \in C^{\ell-n-1,\alpha}(B_R')$ if $\ell \geq n+1$. Moreover, if $\varphi \in C^{\ell,\alpha}(B_R')$ for some $2 \leq \ell \leq n$ and $\alpha \in (0,1)$, then $u \in C^{\ell,\alpha}(\bar{B}_r^+)$ for any r(0,R). (See [22].)

If $\varphi \in C^{\infty}(B'_R)$, then the estimate (1.10) holds for all $m \geq 0$, all $k \geq \max\{n+1, m\}$, all $\tau \geq 0$ and all $\alpha \in (0, 1)$. This implies in particular that u is polyhomogeneous. Refer to [6] or [27] for the definition of polyhomogeneity.

A similar result holds for solutions of (1.1). We will present it in Section 7. In fact, the most part of the paper is devoted to the study of a class of equations more general than (1.3). Theorem 5.3 should be considered as the main result in this paper and can be applied to solutions of (1.3) as well as (1.1).

We now compare our results with earlier results of similar nature. The polyhomogeneity was established for the singular Yamabe problem in [6] and [27], for the complex Monge-Ampère equations in [20], and for the asymptotically hyperbolic Einstein metrics in [7] and [9]. It is proved mostly in the smooth category or sufficiently smooth category.

Results in this paper are established based on PDE techniques, such as barrier functions and scalings, and an iteration of ODE. With this approach, we are able to track down easily the regularity of coefficients and the remainder of the expansion and present the estimate of the remainder under the assumption of the optimal regularity, as shown in Theorem 1.1.

We prove Theorem 1.1, or more generally Theorem 5.3, in two steps. We establish the regularity of solutions first along tangential directions and then along the normal direction. We follow Lin [22] closely for the proof of the tangential regularity, by the maximum principle and rescaling. As noted in [22], the tangential regularity is important in treating the underlying PDE by an ODE technique. The utilization of the ODEs in this paper is adapted from the recent work by Jian and Wang [18], [19]. The main focus there is the optimal regularity for a class of Monge-Ampère equations and for a class of fully nonlinear uniformly elliptic equations. As a result, their iteration of ODEs terminates before the logarithmic terms show up. In our case, analyzing the impact of logarithmic terms on certain combinations of derivatives constitutes an indispensable part of the study of the regularity of remainders.

We finish the introduction with a brief outline of the paper. In Section 2, we provide a calculation to determine all the local terms in the formal expansion. In Section 3, we estimate the difference of the solution and its expansion involving all the local terms. The proof is based on the maximum principle. In Section 4, we discuss a class of quasilinear elliptic equations with singularity and prove the tangential smoothness of solutions near boundary. In Section 5, we treat quasilinear elliptic equations as ordinary differential equations and prove the regularity along the normal direction. In Section 6, we discuss expansions of the minimal graphs by treating them as functions over their vertical tangent planes near each boundary point and prove Theorem 1.1. In Section 7, we discuss expansions of f.

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2. Formal Expansions

In this section, we derive expansions of the minimal surface operator. We denote by y = (y', t) points in \mathbb{R}^n , with $y_n = t$, and, set

(2.1)
$$Q(u) = \Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{n u_t}{t}.$$

In the following, we calculate the operator Q on polynomials of t. We set, for n even,

$$(2.2) u_* = \varphi + c_2 t^2 + c_4 t^4 + \dots + c_n t^n,$$

and, for n odd,

(2.3)
$$u_* = \varphi + c_2 t^2 + c_4 t^4 + \dots + c_{n-1} t^{n-1} + c_{n+1,1} t^{n+1} \log t,$$

where c_i and $c_{n+1,1}$ are functions of y'.

We first consider the case that n is even.

Lemma 2.1. Let n be even and $\ell \geq n+2$. Then for any $\varphi \in C^{\ell}(B'_r)$, there exists $c_i \in C^{\ell-i}(B'_r)$, for $i=2,4,\cdots,n$, such that, for u_* defined in (2.2),

$$|Q(u_*)| \le Ct^n,$$

where C is a positive constant depending only on n and the C^{n+2} -norm of φ .

Proof. For u_* defined in (2.2), a straightforward calculation yields

$$Q(u_*) = 2(1 - n)c_2 + \Delta_{y'}\varphi - \frac{\varphi_{\alpha}\varphi_{\beta}}{1 + |D_{y'}\varphi|^2}\varphi_{\alpha\beta} + \sum_{i=4 \text{ even}}^{n} \left[i(i - n - 1)c_i + F_i(\varphi, c_2, \dots, c_{i-2}) \right] t^{i-2} + O(t^n),$$

where F_i is a smooth function in $\varphi, c_2, \dots, c_{i-2}$ and their derivatives up to order 2. We take

(2.4)
$$c_2 = \frac{1}{2(n-1)} \left(\Delta_{y'} \varphi - \frac{\varphi_{\alpha} \varphi_{\beta}}{1 + |D_{y'} \varphi|^2} \varphi_{\alpha\beta} \right),$$

and then successively, for $i = 4, \dots, n$,

(2.5)
$$c_i = \frac{1}{i(n+1-i)} F_i(\varphi, c_2, \dots, c_{i-2}).$$

Then, we obtain the desired result.

Next, we consider the case that n is odd.

Lemma 2.2. Let n be odd and $\ell \geq n+3$. Then for any $\varphi \in C^{\ell}(B'_r)$, there exist $c_i \in C^{\ell-i}(B'_r)$, for $i=2,4,\cdots,n-1$, and $c_{n+1,1} \in C^{\ell-n-1}(B'_r)$ such that, for u_* defined in (2.3),

$$|Q(u_*)| \le Ct^{n+1} \log t^{-1},$$

where C is a positive constant depending only on n and the C^{n+3} -norm of φ .

Proof. For u_* defined in (2.3), a straightforward calculation yields

$$Q(u_*) = 2(1-n)c_2 + \Delta_{y'}\varphi - \frac{\varphi_{\alpha}\varphi_{\beta}}{1+|D_{y'}\varphi|^2}\varphi_{\alpha\beta}$$

$$+ \sum_{i=4, \text{ even}}^{n-1} \left[i(i-n-1)c_i + F_i(\varphi, c_2, \cdots, c_{i-2})\right]t^{i-2}$$

$$+ \left[(n+1)c_{n+1,1} + F_{n+1,1}(\varphi, c_2, \cdots, c_{n-1})\right]t^{n-1} + O(t^{n+1}|\log t|),$$

where F_i is a smooth function in $\varphi, c_2, \dots, c_{i-2}$ and their derivatives, for $i = 4, \dots, n-1$, and $F_{n+1,1}$ is a smooth function in $\varphi, c_2, \dots, c_{n-1}$ and their derivatives. First, we take c_2 as in (2.4) and c_i in (2.5), for $i = 4, \dots, n-1$. Next, we take

(2.6)
$$c_{n+1,1} = -\frac{1}{n+1} F_{n+1,1}(\varphi, c_2, \dots, c_{n-1}).$$

Then, we obtain the desired result.

The functions c_i and $c_{n+1,1}$ defined in (2.4), (2.5) and (2.6) are functions of $y' \in B'_r$. We will refer to the corresponding terms by *local terms*. We can relate these functions to geometric quantities. For example, by (2.4), we have (1.5). Next, we calculate $c_{4,1}$ for n=3.

Proposition 2.3. For n = 3, $c_{4,1}$ in (2.6) is given by (1.6).

Proof. For n=3, the operator Q is given by

$$Q(u) = \Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{3u_t}{t}.$$

Set

$$u_* = \varphi + c_2 t^2 + c_{4,1} t^4 \log t.$$

We need to calculate $F_{4,1}$ in the proof of Lemma 2.2. In fact, by a calculation as in the proof of Lemma 2.2, we have c_2 given by (2.4) for n=3 and

$$c_{4,1} = -\frac{1}{4} \left\{ \Delta_{y'} c_2 - \frac{\varphi_{\alpha} \varphi_{\beta}}{1 + |D_{y'} \varphi|^2} \partial_{\alpha \beta} c_2 - \frac{1}{1 + |D_{y'} \varphi|^2} \left[8c_2 D_{y'} \varphi \cdot D_{y'} c_2 + 2\varphi_{\alpha} \partial_{\beta} c_2 \varphi_{\alpha \beta} + 8c_2^3 \right] + \frac{\varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha \beta}}{(1 + |D_{y'} \varphi|^2)^2} \left[2D_{y'} \varphi \cdot D_{y'} c_2 + 4c_2^2 \right] \right\}.$$

We note that c_2 can be expressed by (1.5). Then, a straightforward calculation yields

$$c_{4,1} = -\frac{1}{8}\sqrt{1+|D_{y'}\varphi|^2} \left\{ \Delta_{y'}H - \frac{\varphi_{\alpha}\varphi_{\beta}}{1+|D_{y'}\varphi|^2} \partial_{\alpha\beta}H - \frac{1}{1+|D_{y'}\varphi|^2} \left(\Delta_{y'}\varphi - \frac{\varphi_{\alpha}\varphi_{\beta}\varphi_{\alpha\beta}}{1+|D_{y'}\varphi|^2} \right) D_{y'}\varphi \cdot D_{y'}H + 2H(H^2 - K) \right\}.$$

This implies the desired result.

We note that $c_{4,1} = 0$ if and only if Σ is a Willmore surface.

3. Estimates of Local Terms

In this section, we derive an estimate for an expansion involving all local terms by the maximum principle. We denote by y = (y', t) points in \mathbb{R}^n , with $y_n = t$, and set, for any r > 0,

$$G_r = \{ (y', t) : |y'| < r, \ 0 < t < r \}.$$

For some R > 0 and some $\varphi \in C^2(B_R')$, we consider

(3.1)
$$\Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{n u_t}{t} = 0 \text{ in } G_R,$$

and

$$(3.2) u = \varphi on B'_R.$$

First, we derive a decay estimate by a standard application of the maximum principle.

Lemma 3.1. Assume $\varphi \in C^2(B_R')$ and let $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (3.1)-(3.2). Then, for any $(y',t) \in G_{R/4}$,

$$(3.3) |u - \varphi| \le Ct^2,$$

where C is a positive constant depending only on n, $|u|_{L^{\infty}(G_R)}$ and $|\varphi|_{C^2(B'_R)}$.

Proof. For u given as in Lemma 3.1, set

$$\mathcal{L}v = \left(\delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}\right) v_{ij} - n \frac{v_t}{t}.$$

First, we note

$$\mathcal{L}(u-\varphi) = -\left(\Delta\varphi - \frac{u_i u_j}{1 + |Du|^2}\varphi_{ij}\right).$$

The right-hand side is bounded by a constant multiple of $|D^2\varphi|_{L^{\infty}(B'_R)}$. Set, for some positive constants a and b to be determined,

$$\overline{w}(y',t) = a|y'|^2 + bt^2$$
 in G_r .

In the following, we take r = R/4. On ∂G_r , we have $\overline{w}(y',0) = a|y'|^2$, $\overline{w}(y',r) = a|y'|^2 + br^2$ and $\overline{w}(y',t) = ar^2 + bt^2$ if |y'| = r. Since $u - \varphi = 0$ on t = 0 and $u - \varphi$ is bounded in \overline{G}_r , we can choose a and b large such that $u - \varphi \leq \overline{w}$ on ∂G_r . Next, we note

$$\mathcal{L}\overline{w} = \left(1 - \frac{u_t^2}{1 + |Du|^2}\right) 2b + \left(n - 1 - \frac{|D_{y'}u|^2}{1 + |Du|^2}\right) 2a - 2nb$$

$$\leq 2(1 - n)b + (n - 1)a.$$

By choosing b sufficiently large relative to a, we obtain $\mathcal{L}\overline{w} \leq \mathcal{L}(u-\varphi)$ in G_r . Therefore, $\mathcal{L}\overline{w} \leq \mathcal{L}(u-\varphi)$ in G_r and $\overline{w} \geq u-\varphi$ on ∂G_r . By the maximum principle, we get $u-\varphi \leq \overline{w}$ in G_r . By taking y'=0, we obtain $(u-\varphi)(0',t) \leq bt^2$ for any $t \in (0,r)$. For any fixed y'_0 , we consider, instead of \overline{w} ,

$$\overline{w}_{y_0'}(y',t) = a|y' - y_0'|^2 + bt^2.$$

By repeating the above argument, we conclude

$$(u - \varphi)(y', t) \le bt^2$$
 for any $(y', t) \in G_r$.

By considering $-\overline{w}$ or $-\overline{w}_{y'_0}$, we get a lower bound of $(u-\varphi)(y',t)$. Therefore, (3.3) holds.

Next, we prove an estimate for an expansion of solutions involving all the local terms by the maximum principle.

Theorem 3.2. Let $\ell = n+2$ for n even and $\ell = n+3$ for n odd. Assume $\varphi \in C^{\ell}(B'_R)$ and let $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (3.1)-(3.2). Then, there exists an $r \in (0,R)$, such that, for any $(y',t) \in G_r$,

$$|u - u_*| \le Ct^{n+1}.$$

where u_* is given by (2.2) and (2.3), the coefficients c_i and $c_{n+1,1}$ are functions on B'_R given as in Lemmas 2.1 and 2.2, and C is a positive constant depending only on n, the L^{∞} -norm of u in G_R and the C^{ℓ} -norm of φ in B'_R .

Proof. The proof consists of several steps.Step 1. Set

$$Q(u) = (1 + |Du|^2) \left(\Delta u - \frac{nu_t}{t}\right) - u_i u_j u_{ij}.$$

Here, Q differs from the operator in (2.1) by a positive factor. We will construct supersolutions and subsolutions of (2.1). Let u_* and ψ be C^2 -functions in Ω . A straightforward calculation yields

$$Q(u_* + \psi) = Q(u_*) + I_1(u_*, \psi) + I_2(u_*, \psi) + I_3(u_*, \psi),$$

where

$$I_{1}(u_{*},\psi) = (1 + |Du_{*}|^{2}) \left(\Delta \psi - \frac{n\psi_{t}}{t}\right) + 2Du_{*} \cdot D\psi \left(\Delta u_{*} - \frac{nu_{*t}}{t}\right) - u_{*i}u_{*j}\psi_{ij} - (u_{*i}\psi_{j} + u_{*j}\psi_{i})u_{*ij},$$

$$I_{2}(u_{*},\psi) = 2Du_{*} \cdot D\psi \left(\Delta \psi - \frac{n\psi_{t}}{t}\right) + |D\psi|^{2} \left(\Delta u_{*} - \frac{nu_{*t}}{t}\right) - (u_{*i}\psi_{j} + u_{*j}\psi_{i})\psi_{ij} - \psi_{i}\psi_{j}u_{*ij},$$

$$I_{3}(u_{*},\psi) = |D\psi|^{2} \left(\Delta \psi - \frac{n\psi_{t}}{t}\right) - \psi_{i}\psi_{j}\psi_{ij}.$$

Here, we arrange I_1, I_2 and I_3 according to the powers of ψ and their derivatives.

We set u_* by (2.2) and (2.3), and choose c_1, \dots, c_n and $c_1, \dots, c_{n-1}, c_{n+1,1}$ as in (2.4), (2.5) and (2.6). By Lemma 2.1 and Lemma 2.2, we have

$$(3.6) |Q(u_*)| \le Ct^n.$$

Step 2. We now construct supersolutions and prove an upper bound of u. For some positive constants A and q to be determined, we set

$$\overline{\psi}(d) = A\left((|y'|^2 + t)^{n+1} - (|y'|^2 + t)^q\right),$$

and

$$\overline{w} = u_* + \overline{\psi}.$$

In the following, we choose q such that

$$(3.7) n+1 < q < n+2.$$

Then, a straightforward calculation yields

$$(3.8) I_{1} = -A(1+|D_{y'}\varphi|^{2}) \left\{ q(q-n-1)(|y'|^{2}+t)^{q-2} + n(n+1)(|y'|^{2}+t)^{n-1} \frac{|y'|^{2}}{t} \right\} B(y',t),$$

$$I_{2} = A^{2} \left\{ (|y'|^{2}+t)^{2n-\frac{1}{2}} + (|y'|^{2}+t)^{2n-\frac{1}{2}} \frac{|y'|^{2}}{t} \right\} O(1),$$

$$I_{3} = -A^{3}(n+1)^{n} \left\{ (|y'|^{2}+t)^{3n-1} + n(|y'|^{2}+t)^{3n-1} \frac{|y'|^{2}}{t} \right\} B(y',t),$$

where B is a function of the form

$$B = 1 + O(|y'|^2) + O(t) + O\big((|y'|^2 + t)^{q-n-1}\big) + O\big((|y'|^2 + t)^{n+2-q}\big).$$

For an illustration, we calculate a key expression in I_1 . We note

$$\overline{\psi}_{tt} - \frac{n\overline{\psi}_t}{t} = -Aq(q - n - 1)(|y'|^2 + t)^{q-2} - An(n+1)(|y'|^2 + t)^{n-1} \frac{|y'|^2}{t} \left(1 - \frac{q}{n+1}(|y'|^2 + t)^{q-n-1}\right).$$

This provides the dominant terms in I_1 . Moreover, the power 1/2 in I_2 comes from $D|y'|^2 = 2y'$, which is controlled by $(|y'|^2 + t)^{1/2}$. This extra power 1/2 plays an important role. We also note that I_1 and I_3 are nonpositive. For powers of the first terms in I_1 , I_2 and I_3 , we note by (3.7)

$$2\left(2n - \frac{1}{2}\right) > (q - 2) + (3n - 1).$$

Similarly, for powers of the second terms in I_1, I_2 and I_3 , we have

$$2\left(2n - \frac{1}{2}\right) > (n-1) + (3n-1).$$

Then, by using the Cauchy inequality and choosing y' and t small, we get

$$I_1 + I_2 + I_3 \le -\frac{1}{2}Aq(q-n-1)(|y'|^2 + t)^{q-2}.$$

Hence, by (3.4) and (3.6),

$$Q(\overline{w}) \le O(t^n) - \frac{1}{2}Aq(q-n-1)(|y'|^2 + t)^{q-2}.$$

We obtain, for any $A \ge 1$ and any y' and t small,

$$Q(\overline{w}) \leq 0.$$

Next, we choose an appropriate domain so that $u \leq \overline{w}$ on its boundary. By Lemma 3.1, we have $u - u_* \leq Ct^2$ for any small y' and t. It suffices to prove $Ct^2 \leq \overline{\psi}$ on the boundary of an appropriate domain. First, for any small t_0 , take $A \geq C$ large so that $C \leq At_0^{n-1}$. Next, take $r_0 = \sqrt[n]{t_0}$. Then, we obtain

$$u \leq \overline{w}$$
 on $\partial (B'_{\sqrt[n]{t_0}} \times (0, t_0))$.

An application of the maximum principle yields

$$u \leq \overline{w}$$
 in $B'_{\sqrt[n]{t_0}} \times (0, t_0)$.

By taking y' = 0, we obtain

$$u(0',t) \le u_* + At^{n+1}$$
 for any $0 < t < t_0$.

Similarly as in the proof of Lemma 3.1, we have

(3.9)
$$u \le u_* + At^{n+1} \quad \text{in } B'_{n/t_0} \times (0, t_0).$$

Step 3. We now construct subsolutions and prove a lower bound of u. We take u_* as introduced in Step 1. Set

$$\underline{\psi}(d) = -A\left((|y'|^2 + t)^{n+1} - (|y'|^2 + t)^q\right),\,$$

and

$$\underline{w} = u_* + \psi.$$

In the expressions of I_1 , I_2 and I_3 in (3.8), there is a change of sign for I_1 and I_3 . Hence, we proceed similarly as in Step 2 and conclude, for any y' and t small,

$$Q(\underline{w}) \ge 0.$$

Correspondingly, we obtain

(3.10)
$$u \ge u_* - At^{n+1} \quad \text{in } B'_{\sqrt[n]{t_0}} \times (0, t_0).$$

We have the desired result by combining (3.9) and (3.10).

If we only assume $\varphi \in C^{k,\alpha}(B_R')$ for some $2 \le k \le n$ and $\alpha \in (0,1)$, we can prove

$$\left| u - \left(c_0 + c_2 t^2 + \dots + c_{\tilde{k}} t^{\tilde{k}} \right) \right| \le C t^{k+\alpha},$$

where \tilde{k} is the largest even integer not greater than k. The proof is more involved since the second derivatives of $c_{\tilde{k}}$ do not exist. In Theorem 4.3, we will prove a similar result for k=2 for a class of more general equations.

Next, we derive an estimate of the gradients near the boundary.

Lemma 3.3. Let $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (3.1)-(3.2). Then, for any $(y',t) \in G_{R/4}$,

$$(3.11) |D(u - \varphi)(y', t)| \le Ct,$$

where C is a positive constant depending only on $|u|_{L^{\infty}(G_R)}$ and $|\varphi|_{C^2(\bar{G}_R)}$.

This result was proved by Lin [22]. We indicate briefly the proof.

Proof. By the geometric invariance of the equation (3.1), we assume $\varphi(0) = 0$ and $D_{\eta'}\varphi(0) = 0$, and prove, for any $t_0 \in (0, R/4)$,

$$(3.12) |Du(0, t_0)| \le Ct_0.$$

By Lemma 3.1, we have

$$|u(y',t)| \le |\varphi(y')| + Ct^2 \le C(|y'|^2 + t^2).$$

Set $\delta = t_0$ and consider the transform $y = (y', t) \mapsto z = (z', s)$ defined by $y' = \delta z'$ and $t = \delta s$. Then, define

$$u^{\delta}(z) = \frac{1}{\delta}u(y).$$

The function u^{δ} satisfies

$$\Delta u^{\delta} - \frac{u_i^{\delta} u_j^{\delta}}{1 + |Du^{\delta}|^2} u_{ij}^{\delta} - \frac{n u_s^{\delta}}{s} = 0 \quad \text{in } G_2,$$

and, by (3.13),

$$|u^{\delta}| \le C\delta$$
 in G_2 .

By the interior estimates ([13]), we have

$$|Du^{\delta}| \le C\delta$$
 in $B_{1/2}((0',1))$.

By evaluating at (0', 1) and transforming back to u, we have (3.12).

To end this section, we discuss briefly how to proceed. Let $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (3.1)-(3.2). We set

$$v = u - \varphi$$
.

Firstly, we write the equation (3.1) for u as an equation for v and employ this equation to derive the tangential regularity of v. Secondly, we write this partial differential equation as an ordinary differential equation in t and derive the normal regularity of v. In the following two sections, we will formulate such tangential regularity and normal regularity for more general equations, which will be applied to u as well as f.

4. Tangential Smoothness

In the present and the next sections, we study a class of quasilinear elliptic equations with singularity and discuss the regularity of solutions near boundary. We study the regularity along tangential directions in this section and along the normal direction in the next section. Results in these two sections will be applied to minimal surface equations in Section 6 and Section 7.

We denote by y = (y', t) points in \mathbb{R}^n . Set, for any constant r > 0,

$$G_r = \{ (y', t) : |y'| < r, 0 < t < r \}.$$

For a fixed R > 0, we assume $v \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ satisfies

(4.1)
$$A_{ij}v_{ij} + P\frac{v_t}{t} + Q\frac{v}{t^2} + N = 0 \text{ in } G_R,$$

where A_{ij} , P, Q and N are functions of the form

$$A_{ij} = A_{ij} \left(y', t, Dv, \frac{v}{t} \right), \quad P = P \left(y', t, Dv, \frac{v}{t} \right), \quad Q = Q \left(y', t, Dv, \frac{v}{t} \right),$$

and

$$N = N\left(y', t, Dv, \frac{v}{t}, \frac{|D_{y'}v|^2}{t}\right).$$

Hence, A_{ij} , P, Q and N are functions of y', t and

$$(4.2) Dv, \frac{v}{t}, \frac{|D_{y'}v|^2}{t}.$$

We assume (4.1) is uniformly elliptic; namely, there exists a positive constant λ such that, for any $(y', t, p, s) \in G_R \times \mathbb{R}^n \times \mathbb{R}$ and any $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \le A_{ij}(y',t,p,s)\xi_i\xi_j \le \lambda|\xi|^2.$$

Concerning the solution v, we always assume, for some positive constant C_0 ,

$$(4.3) |v| \le C_0 t^2.$$

and

$$(4.4) |Dv| \le C_0 t.$$

Now, we derive an estimate of derivatives near boundary by a rescaling method, which reduces global estimates to local ones.

Lemma 4.1. Assume A_{ij} , P, Q and N are C^{α} in its arguments, for some $\alpha \in (0,1)$. Let $v \in C^1(\bar{G}_R) \cap C^{2,\alpha}(G_R)$ be a solution of (4.1) in G_R , for some R > 0, and satisfy (4.3) and (4.4). Then,

$$D^2v \in L^{\infty}(G_{R/4}),$$

and

(4.5)
$$\frac{v}{t}$$
, Dv , $\frac{v^2}{t^3}$, $\frac{vv_t}{t^2}$, $\frac{v_t^2}{t} \in C^{0,1}(\bar{G}_{R/4})$.

Moreover, the bounds on the L^{∞} -norm of D^2v and the $C^{0,1}$ -norms of the functions in (4.5) depend only on n, α , λ , C_0 in (4.3) and (4.4), and the C^{α} -norms of A_{ij} , P, Q and N.

Proof. To be consistent with the proof of the next result, we prove the first conclusion in Lemma 4.1 under a weaker assumption. Instead of (4.4), we assume

$$(4.6) |Dv|^2 \le C_0 t.$$

We claim

(4.7)
$$\frac{Dv}{t}, D^2v \in L^{\infty}(G_{R/4}).$$

Then, we have (4.5) easily, since derivatives of the functions in (4.5) are bounded by $\frac{v}{t^2}$, $\frac{Dv}{t}$, D^2v , which are bounded by (4.3) and (4.7). For a later reference, we list these relations as follows:

$$D\left(\frac{v}{t}\right) = \frac{Dv}{t} - \frac{v}{t^2}Dt,$$

$$D(Dv) = D^2v,$$

and

$$D\left(\frac{v^2}{t^3}\right) = 2\frac{v}{t^2} \cdot \frac{Dv}{t} - 3\left(\frac{v}{t^2}\right)^2 Dt,$$

$$D\left(\frac{vv_t}{t^2}\right) = \frac{Dv}{t} \cdot \frac{v_t}{t} + \frac{v}{t^2} Dv_t - 2\frac{v}{t^2} \cdot \frac{v_t}{t} Dt,$$

$$D\left(\frac{v_t^2}{t}\right) = 2\frac{v_t}{t} Dv_t - \left(\frac{v_t}{t}\right)^2 Dt.$$

The vector Dt is given by (0', 1).

To prove (4.7), we take any $(y'_0, t_0) \in G_{R/4}$ and set $\delta = t_0/2$. With $e_n = (0', 1)$, we now consider the transform $T : B_1(e_n) \to G_R$ given by

$$y' = y'_0 + \delta z', \quad t = \delta(s+1).$$

Note that T maps e_n to (y'_0, t_0) . Set

$$v^{\delta}(z',s) = \delta^{-2}v(y',t).$$

By (4.1), v^{δ} satisfies

(4.8)
$$A_{ij}v_{ij}^{\delta} + \frac{P}{1+s}v_s^{\delta} + \frac{Q}{(1+s)^2}v^{\delta} + N = 0 \quad \text{in } B_1(e_n).$$

We note that, in the equation (4.1), all coefficients A_{ij} , P, Q and the nonhomogeneous term N are C^{α} in quantities given by (4.2), which are bounded functions in G_R by (4.3) and (4.6). Hence, all coefficients and the nonhomogeneous term N in (4.8) are bounded in $B_{7/8}(e_n)$. Moreover, by (4.3), v^{δ} is bounded in $B_1(e_n)$. In fact,

$$|v^{\delta}(z)| = \delta^{-2}|v(y)| \le C_0 \delta^{-2} t^2 \le C_0 (1+s)^2.$$

We now fix a constant $\epsilon \in (0,1)$ sufficiently small. The standard $C^{1,\epsilon}$ -estimates yield $v^{\delta} \in C^{1,\epsilon}(B_{3/4}(e_n))$. (Refer to [13].) With

$$D_y v = \delta D_z v^{\delta}, \quad \frac{v}{t} = \frac{\delta v^{\delta}}{1+s}, \quad \frac{|D_y v|^2}{t} = \frac{\delta |D_z v^{\delta}|^2}{1+s},$$

we can check that, for the equation (4.8), A_{ij} , P, Q, $N \in C^{\epsilon}(B_{3/4}(e_n))$. The bound of the C^{ϵ} -norms of these functions is independent of δ . By the Schauder estimate, we obtain $v^{\delta} \in C^{2,\epsilon}(B_{1/2}(e_n))$ and a bound on the $C^{2,\epsilon}$ -norm of v^{δ} in $B_{1/2}(e_n)$, independent of δ . Note

$$\frac{D_y v}{t} = \frac{D_z v^{\delta}}{1+s}, \quad D_y^2 v = D_z^2 v^{\delta}.$$

By evaluating at (z', s) = (0, 1), we have

$$\frac{|Dv(y_0', t_0)|}{t_0} + |D^2v(y_0', t_0)| \le C.$$

This proves (4.7) since (y'_0, t_0) is an arbitrary point in $G_{R/4}$.

We now generalize the result for v in Lemma 4.1 to that for arbitrary tangential derivatives of v.

Theorem 4.2. Assume A_{ij} , P, Q and N are $C^{\ell,\alpha}$ in its arguments, for some $\ell \geq 0$ and $\alpha \in (0,1)$. Let $v \in C^1(\bar{G}_R) \cap C^{\ell+2,\alpha}(G_R)$ be a solution of (4.1) in G_R , for some R > 0, and satisfy (4.3) and (4.4). Assume

$$(4.9) (2A_{nn} + 2P + Q)(\cdot, 0) < 0 on B'_R.$$

Then, there exists a constant $r \in (0, R)$ such that, for $\tau = 0, 1, \dots, \ell$,

(4.10)
$$\frac{D_{y'}^{\tau}v}{t^2}, \frac{DD_{y'}^{\tau}v}{t}, D^2D_{y'}^{\tau}v \in L^{\infty}(G_r),$$

and

(4.11)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}(v^2)}{t^3}, \frac{D_{y'}^{\tau}(vv_t)}{t^2}, \frac{D_{y'}^{\tau}(v_t^2)}{t} \in C^{0,1}(\bar{G}_r).$$

Moreover, the cooresponding bounds in (4.10) and (4.11) depend only on n, ℓ , α , λ , C_0 in (4.3) and (4.4), and the $C^{\ell,\alpha}$ -norms of A_{ij} , P, Q and N.

Proof. We first note that (4.10) implies (4.11) easily as in the proof of Lemma 4.1. For $\tau=0$, (4.10) follows from (4.3) and (4.7) for r=R/4. We fix an integer $1 \le k \le \ell$ and assume (4.10) holds for $\tau=0,\cdots,k-1$. We now consider the case $\tau=k$.

By applying $D_{y'}^k$ to (4.1), we obtain

(4.12)
$$A_{ij}(D_{y'}^k v)_{ij} + P \frac{(D_{y'}^k v)_t}{t} + Q \frac{D_{y'}^k v}{t^2} + N_k = 0,$$

where N_k is given by

$$(4.13) N_k = \sum_{\substack{l+m=k\\m \le k-1}} a_{lm} \left(D_{y'}^l A_{ij} \cdot D_{y'}^m v_{ij} + D_{y'}^l P \cdot \frac{D_{y'}^m v_t}{t} + D_{y'}^l Q \cdot \frac{D_{y'}^m v}{t^2} \right) + D_{y'}^k N,$$

for some constant a_{lm} . Derivatives of A_{ij} , P, Q and N also result in derivatives of v. For example, the k-th derivative of the last argument in N yields

$$D_{y'}^{k}\left(\frac{|D_{y'}v|^{2}}{t}\right) = \sum_{l+m=k} a_{lm} \frac{D_{y'}^{l+1}vD_{y'}^{m+1}v}{t} = a_{0k} \frac{D_{y'}v}{t} \cdot D_{y'}^{k+1}v + \cdots$$

In conclusion, N_k is a polynomial of the expressions in (4.10), for $\tau \leq k-1$, except those in (4.2). Then, by the induction hypotheses, N_k is bounded in G_r . In the following, we set

$$\mathcal{L}w = A_{ij}w_{ij} + \frac{P}{t}w_t + \frac{Q}{t^2}w.$$

Consider, for some positive constants a and b to be determined,

$$\overline{w}(y',t) = a|y'|^2 + bt^2.$$

Note $D_{y'}^k v = 0$ on t = 0 and $|D_{y'}^k v| \leq Ct$ by the induction hypothesis. Then, for each fixed $r \in (0, R/4)$, we can choose a and b large such that $D_{y'}^k v \leq \overline{w}$ on ∂G_r . Next,

(4.14)
$$\mathcal{L}\overline{w} = (2A_{nn} + 2P + Q)b + 2a\sum_{\alpha=1}^{n-1} A_{\alpha\alpha}.$$

By (4.9), we have $(2A_{nn} + 2P + Q)(\cdot, 0) \leq -c_0$ in $B'_{3R/4}$, for some positive constant c_0 . Lemma 4.1 implies that the coefficients of \mathcal{L} are $C^{0,1}$. By taking r small and then b sufficiently large, we obtain $\mathcal{L}\overline{w} \leq \mathcal{L}(D^k_{y'}v)$ in G_r . The maximum principle implies $D^k_{y'}v \leq \overline{w}$ in G_r . By taking y' = 0, we obtain $D^k_{y'}v(0',t) \leq bt^2$ for any $t \in (0,r)$. Proceeding as in the proof of Lemma 3.1, we obtain

$$(4.15) |D_{y'}^k v| \le Ct^2 in G_r.$$

Next, we prove

$$(4.16) |DD_{y'}^k v| \le Ct, |D^2 D_{y'}^k v| \le C \text{in } G_{r/4}.$$

The proof is similar to that of Lemma 4.1. We take any $(y'_0, t_0) \in G_{r/4}$ and set $\delta = t_0/2$. With $e_n = (0', 1)$, we now consider the transform $T : B_1(e_n) \to G_r$ given by

$$y' = y_0' + \delta z', \quad t = \delta(s+1).$$

Note that T maps e_n to (y'_0, t_0) . Set, for each $\tau \leq k$,

$$w_{\tau}^{\delta}(z',s) = \delta^{-2} D_{y'}^{\tau} v(y',t).$$

By (4.12), w_k^{δ} satisfies

(4.17)
$$A_{ij}(w_k^{\delta})_{ij} + \frac{P}{1+s}(w_k^{\delta})_s + \frac{Q}{(1+s)^2}w_k^{\delta} + N_k = 0 \quad \text{in } B_1(e_n).$$

All coefficients A_{ij} , P, Q and the nonhomogeneous term N_k in (4.12) are bounded in $B_1(e_n)$. Moreover, by (4.15), w_k^{δ} is bounded in $B_1(e_n)$. We now fix a small constant $\epsilon \in (0,1)$. The standard $C^{1,\epsilon}$ -estimates yield $w_k^{\delta} \in C^{1,\epsilon}(B_{3/4}(e_n))$. We now express N_k in (4.13) in (z',s). For example, for $\tau \leq k-1$, the expressions in (4.10) are given by

$$\frac{w_{\tau}^{\delta}}{(1+s)^2}, \frac{D_z w_{\tau}^{\delta}}{1+s}, D_z^2 w_{\tau}^{\delta}.$$

Hence, $N_k \in C^{\epsilon}(B_{3/4}(e_n))$ and the bound of the C^{ϵ} -norms is independent of δ . By the Schauder estimate, we obtain $w_k^{\delta} \in C^{2,\epsilon}(B_{1/2}(e_n))$ and a bound on the $C^{2,\epsilon}$ -norm of w_k^{δ} in $B_{1/2}(e_n)$, independent of δ . By evaluating the first derivative and the second derivative of w_k^{δ} at (z',s)=(0,1) and rewriting for $D_{y'}^k v$, we have

$$\frac{|DD_{y'}^k v(y_0', t_0)|}{t_0} + |D^2 D_{y'}^k v(y_0', t_0)| \le C.$$

This proves (4.16). We conclude the proof of (4.10) for $\tau = k$.

There is a loss of regularity in Theorem 4.2. Under the assumptions $A_{ij}, P, Q, N \in C^{\ell,\alpha}$, we only proved $D_{y'}^{\ell}u \in C^{1,1}$. We now prove it is $C^{2,\alpha}$ under a slightly strengthened condition on coefficients.

Theorem 4.3. Assume A_{ij} , P, Q and N are $C^{\ell,\alpha}$ in its arguments, for some $\ell \geq 0$ and $\alpha \in (0,1)$. Let $v \in C^1(\bar{G}_R) \cap C^{\ell+2,\alpha}(G_R)$ be a solution of (4.1) in G_R , for some R > 0, and satisfy (4.3) and (4.4). Assume (4.9) and

$$(4.18) ((2+\alpha)(1+\alpha)A_{nn} + (2+\alpha)P + Q)(\cdot,0) < 0 on B'_R.$$

Then, there exists a constant $r \in (0, R)$ such that, for $\tau = 0, 1, \dots, \ell$,

(4.19)
$$\frac{D_{y'}^{\tau}v}{t^2}, \frac{DD_{y'}^{\tau}v}{t}, D^2D_{y'}^{\tau}v \in C^{\alpha}(\bar{G}_r),$$

and

(4.20)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}(v^2)}{t^3}, \frac{D_{y'}^{\tau}(vv_t)}{t^2}, \frac{D_{y'}^{\tau}(v_t^2)}{t} \in C^{1,\alpha}(\bar{G}_r).$$

Moreover, the cooresponding bounds in (4.19) and (4.20) depend only on n, ℓ , α , λ , C_0 in (4.3) and (4.4), and the $C^{\ell,\alpha}$ -norms of A_{ij} , P, Q and N.

Proof. Step 1. We first consider $\ell = 0$. We claim, for some $c_2 \in C^{\alpha}(B_R)$, some $r \in (0, R)$ and any $(y', t) \in G_r$,

$$(4.21) |v(y',t) - c_2(y')t^2| \le Ct^{2+\alpha}.$$

The expression of c_2 will be given in the proof below. We point out that since c_2 is only C^{α} , we cannot differentiate c_2 . For convenience, we set

$$\mathcal{L}(v) = A_{ij}v_{ij} + P\frac{v_t}{t} + Q\frac{v}{t^2}, \quad \mathcal{Q}(v) = \mathcal{L}(v) + N.$$

For some function c_2 in B'_R and some function ψ in G_R to be determined, we set

$$\overline{v} = c_2(0)t^2 + \psi.$$

A straightforward calculation yields

$$Q(\overline{v}) = \mathcal{L}(\psi) + (2A_{nn} + 2P + Q)c_2(0) + N,$$

where A_{ij} , P, Q and N are evaluated at y, t, $D\overline{v}$, \overline{v}/t and $|D_{y'}\overline{v}|^2/t$. In the following, we take

(4.22)
$$c_2(0) = -\left(\frac{N}{2A_{nn} + 2P + Q}\right)\Big|_0,$$

where the right-hand side is evaluated with all of its arguments replaced by zero. Hence, by the expression of \overline{v} and the C^{α} -regularity of A_{ij} , P, Q and N, we have

$$(4.23) Q(\overline{v}) \le \mathcal{L}(\psi) + C\left(|y'|^{\alpha} + t^{\alpha} + |D\psi|^{\alpha} + \left(\frac{\psi}{t}\right)^{\alpha} + \left(\frac{|D_{y'}\psi|^2}{t}\right)^{\alpha}\right).$$

Next, set, for some constants μ_1 and μ_2 to be determined,

$$\psi(y',t) = \mu_1 t^2 (|y'|^2 + t^2)^{\frac{\alpha}{2}} + \mu_2 t^{2+\alpha}.$$

By a straightforward calculation, we get

$$\mathcal{L}(\psi) = \mu_1 B_1 (|y'|^2 + t^2)^{\frac{\alpha}{2}} + \mu_2 B_2 t^{\alpha},$$

where

$$B_1 = 2A_{nn} + 2P + Q + \frac{\alpha}{|y'|^2 + t^2} (A_{ab}\delta_{ab}t^2 + 4A_{an}y_at + 5A_{nn}t^2 + Pt^2) + \alpha(\alpha - 2)\frac{t^2}{(|y'|^2 + t^2)^2} (A_{ab}y_ay_b + 2A_{an}y_at + A_{nn}t^2),$$

and

$$B_2 = (2 + \alpha)(1 + \alpha)A_{nn} + (2 + \alpha)P + Q.$$

By (4.9) and (4.18), we have $2A_{nn} + 2P + Q \le -2b_1$ and $B_2 \le -2b_2$ in B'_R , for some constants $b_1, b_2 > 0$. In the following, we always take r and ψ small. First, we have

$$B_2 < -b_2$$
 in G_r .

Next, we can find a constant M such that, for any $(y',t) \in G_r$ with $|y'| \ge Mt$,

$$B_1 < -b_1$$
.

Hence, for such (y', t), we have

$$\mathcal{L}(\psi) \le -\mu_1 b_1 (|y'|^2 + t^2)^{\frac{\alpha}{2}} - \mu_2 b_2 t^{\alpha}$$

If $|y'| \leq Mt$, then $B_1 \leq C$ and

$$\mathcal{L}(\psi) \le C\mu_1(|y'|^2 + t^2)^{\frac{\alpha}{2}} - \mu_2 b_2 t^{\alpha} \le -\left(\frac{b_2}{M^{\alpha}}\mu_2 - C\mu_1\right) (|y'|^2 + t^2)^{\frac{\alpha}{2}}$$
$$= -c\mu_1(|y'|^2 + t^2)^{\frac{\alpha}{2}},$$

by choosing μ_2 to be a constant multiple of μ_1 . Therefore, we obtain

(4.24)
$$\mathcal{L}(\psi) \le -c\mu_1(|y'|^2 + t^2)^{\frac{\alpha}{2}} \quad \text{in } G_r.$$

We point out that, if $B_1 < 0$ in G_r , we can simply take $\mu_2 = 0$ and there is no need to assume (4.18). By the explicit expression of ψ , we have

$$|D\psi|^{\alpha} + \left(\frac{\psi}{t}\right)^{\alpha} + \left(\frac{|D_{y'}\psi|^2}{t}\right)^{\alpha} \le C\left(\mu_1^{\alpha}t^{\alpha}(|y'|^2 + t^2)^{\frac{\alpha^2}{2}} + \mu_1^{2\alpha}t^{\alpha}(|y'|^2 + t^2)^{\alpha^2}\right).$$

In the following, we assume

$$\mu_1 r^{2\alpha} \le 1$$

Then, we have $\mu_1^{\alpha}(|y'|^2+t^2)^{\alpha^2}\leq 2$, and

$$(4.26) |D\psi|^{\alpha} + \left(\frac{\psi}{t}\right)^{\alpha} + \left(\frac{|D_{y'}\psi|^2}{t}\right)^{\alpha} \le C\left(\mu_1^{\frac{\alpha}{2}} + \mu_1^{\alpha}\right)(|y'|^2 + t^2)^{\frac{\alpha}{2}}.$$

By (4.23), (4.24) and (4.26), we obtain

$$Q(\overline{v}) \le -c\mu_1(|y'|^2 + t^2)^{\frac{\alpha}{2}} + C\left(\mu_1^{\frac{\alpha}{2}} + \mu_1^{\alpha}\right)(|y'|^2 + t^2)^{\frac{\alpha}{2}}.$$

By $\alpha \in (0,1)$, we can take μ_1 sufficiently large such that

$$Q(\overline{v}) \leq 0$$
 in G_r .

We now compare v and \overline{v} on ∂G_r . By (4.3), in order to have $v \leq \overline{v}$ on ∂G_r , it suffices to require

$$(4.27) C_0 + |c_2(0)| \le \mu_1 r^{\alpha}.$$

We can find r sufficiently small and μ_1 sufficiently large such that (4.25) and (4.27) hold. Therefore, we have $\mathcal{Q}(\overline{v}) \leq \mathcal{Q}(v)$ in G_r and $v \leq \overline{v}$ on ∂G_r . By the maximum principle, we get $v \leq \overline{v}$ in G_r and hence

$$v \le c_2(0)t^2 + \psi \quad \text{in } B_r.$$

Similarly, we have

$$v \ge c_2(0)t^2 - \psi \quad \text{in } B_r.$$

By taking y' = 0, we have (4.21) for y' = 0. We can prove (4.21) for any $(y', t) \in G_r$ by a similar method. Instead of (4.22), we have

(4.28)
$$c_2(y') = -\left(\frac{N}{2A_{nn} + 2P + Q}\right)(y', 0).$$

We note $c_2 \in C^{\alpha}(B'_R)$.

With (4.21), we will prove

(4.29)
$$\frac{v}{t^2}, \frac{Dv}{t}, D^2v \in C^{\alpha}(\bar{G}_r).$$

This is (4.19) for $\tau = 0$. To prove (4.29), we take any $(y'_0, t_0) \in G_r$ and set $\delta = t_0/2$. With $e_n = (0', 1)$, we now consider the transform $T : B_1(e_n) \to G_R$ given by

$$y' = y'_0 + \delta z', \quad t = \delta(s+1).$$

Note that T maps e_n to (y'_0, t_0) . Set

$$v^{\delta}(z',s) = \delta^{-2} (v(y',t) - c_2(y_0')t^2).$$

The rest of the proof is similar as that in the proof of Lemma 4.1. We omit the details and point out that (4.21) allows us to scale back the estimate of the Hölder semi-norms of the second derivatives.

Step 2. We prove for general ℓ by an induction. We fix an integer $1 \le k \le \ell$ and assume (4.19) holds for $\tau = 0, \dots, k-1$. We now consider the case $\tau = k$.

We first claim, for some $c_{k,2} \in C^{\alpha}(B'_R)$, some $r \in (0,R)$ and any $(y',t) \in G_r$,

$$(4.30) |D_{y'}^k v(y',t) - c_{k,2}(y')t^2| \le Ct^{2+\alpha}.$$

The proof is similar as the proof of (4.21). By the induction hypothesis, the coefficients and the nonhomogeneous term in (4.12) satisfy all the regularity assumptions. We omit the details.

With (4.30), we can prove (4.19) for $\tau = k$ by a similar scaling argument.

5. Regularity along the Normal Direction

In this section, we continue our study of the equation (4.1) and discuss the regularity along the normal direction. Our basic technique is to write the partial differential equation as an ordinary differential equation in the t-direction. Then, we iterate this ODE to derive the desired regularity.

As in Section 4, we denote by y = (y', t) points in \mathbb{R}^n and set, for any constant r > 0,

$$G_r = \{ (y', t) : |y'| < r, \ 0 < t < r \}.$$

We start with the equation (4.1) and assume we can write it in the form

(5.1)
$$v_{tt} + p \frac{v_t}{t} + q \frac{v}{t^2} + F = 0,$$

where p and q are constants and F is a function in y', t and

(5.2)
$$v_t, \frac{v}{t}, \frac{v^2}{t^3}, \frac{vv_t}{t^2}, \frac{v_t^2}{t}, D_{y'}v_t, D_{y'}^2v, \frac{D_{y'}v}{t}.$$

In the applications later on, F is smooth in all of its arguments except y'.

We assume results in Section 4 hold for solutions $v \in C^1(\bar{G}_R) \cap C^2(G_R)$ of (5.1) and proceed to discuss the regularity of v in t. We note that (5.1) is the equation discussed in Appendix B.

In the following, we denote by \prime the derivative with respect to t. This should not be confused with y', the first n-1 coordinates of the point.

Throughout this section, we assume that $t^{\underline{m}}$ and $t^{\overline{m}}$ are solutions of the linear homogeneous equation corresponding to (5.1); namely,

$$(5.3) p = 1 - (\underline{m} + \overline{m}), \quad q = \underline{m} \cdot \overline{m}.$$

We always assume that \underline{m} and \overline{m} are integers and satisfy

$$(5.4) \underline{m} \le 0, \quad \overline{m} \ge 2.$$

We first discuss the optimal regularity of solutions up to $C^{\overline{m}-1,\alpha}$. This method was adapted from [18].

Theorem 5.1. Assume that \underline{m} and \overline{m} are integers satisfying (5.3) and (5.4), with $\overline{m} \geq 3$, and that F is $C^{\ell-2,\alpha}$ in its arguments, for some $\ell \geq \overline{m} - 1$ and $\alpha \in (0,1)$. Let $v \in C^1(\overline{G}_R) \cap C^{\ell,\alpha}(G_R)$ be a solution of (5.1) in G_R , for some R > 0. Suppose that there exists a constant $r \in (0,R)$ such that, for any nonnegative integer $\tau = 0, 1, \dots, \ell-1$,

(5.5)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}v^2}{t^3}, \frac{D_{y'}^{\tau}(vv')}{t^2}, \frac{D_{y'}^{\tau}(v')^2}{t} \in C^{\alpha}(\bar{G}_r).$$

Then, for any $\tau = 0, 1, \dots, \ell - \overline{m} + 1$

$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}v^2}{t^3}, \frac{D_{y'}^{\tau}(vv')}{t^2}, \frac{D_{y'}^{\tau}(v')^2}{t} \in C^{\overline{m}-2,\alpha}(\bar{G}_r).$$

In particular, for any $\tau = 0, 1, \dots, \ell - \overline{m} + 1$,

$$D_{y'}^{\tau}v \in C^{\overline{m}-1,\alpha}(\bar{G}_r).$$

Proof. We fix an integer $k=1,\dots,\overline{m}-2$. Now we prove the following result: If, for any nonnegative integer $\tau=0,1,\dots,\ell-k$,

(5.6)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}v^2}{t^3}, \frac{D_{y'}^{\tau}(vv')}{t^2}, \frac{D_{y'}^{\tau}(v')^2}{t} \in C^{k-1,\alpha}(\bar{G}_r),$$

then, for any nonnegative integer $\tau = 0, 1, \dots, \ell - k - 1$,

(5.7)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}v^2}{t^3}, \frac{D_{y'}^{\tau}(vv')}{t^2}, \frac{D_{y'}^{\tau}(v')^2}{t} \in C^{k,\alpha}(\bar{G}_r).$$

We note that the maximal τ for (5.7) is one less than that for (5.6). Hence, we need only prove, for any $\tau = 0, 1, \dots, \ell - k - 1$,

$$(5.8) \quad D_{y'}^{\tau} \partial_t^k \left(\frac{v}{t}\right), \ D_{y'}^{\tau} \partial_t^k v', \ D_{y'}^{\tau} \partial_t^k \left(\frac{v^2}{t^3}\right), \ D_{y'}^{\tau} \partial_t^k \left(\frac{vv'}{t^2}\right), \ D_{y'}^{\tau} \partial_t^k \left(\frac{(v')^2}{t}\right) \in C^{\alpha}(\bar{G}_r).$$

Set $p_0 = p$, $q_0 = q$ and $v_0 = v$. We write (5.1) as

(5.9)
$$v_0'' + p_0 \frac{v_0'}{t} + q_0 \frac{v_0}{t^2} + F_0 = 0.$$

Set, for $l \geq 1$,

$$p_l = 2l + p_0, \quad q_l = l^2 + (p_0 - 1)l + q_0,$$

and, inductively,

$$v_l = v'_{l-1} - \frac{2v_{l-1}}{t}.$$

Then, v_l satisfies

$$v_l'' + p_l \frac{v_l'}{t} + q_l \frac{v_l}{t^2} + F_l = 0,$$

where $F_l = \partial_t^l F_0$. This is (B.3). We will take l = k - 1. Since F_0 is a function in y', t and quantities in (5.2), then the induction hypothesis (5.6) implies $\partial_{y'}^{\tau} F_0 \in C^{k-1,\alpha}(\bar{G}_r)$ and hence $\partial_{y'}^{\tau} F_{k-1} \in C^{\alpha}(\bar{G}_r)$, for $\tau = 0, 1, \dots, \ell - k - 1$. By Lemma B.1 with l = k - 1, we conclude, for any $\tau = 0, 1, \dots, \ell - k - 1$,

(5.10)
$$D_{y'}^{\tau}v_{k-1}'', \frac{D_{y'}^{\tau}v_{k-1}'}{t}, \frac{D_{y'}^{\tau}v_{k-1}}{t^2} \in C^{\alpha}(\bar{G}_r).$$

In the following, we will prove that (5.10) implies (5.8).

If k = 1, then (5.10) yields, for any $\tau = 0, 1, \dots, \ell - k - 1$,

$$D_{y'}^{\tau}v'', \frac{D_{y'}^{\tau}v'}{t}, \frac{D_{y'}^{\tau}v}{t^2} \in C^{\alpha}(\bar{G}_r).$$

A simple calculation implies

$$D_{y'}^{\tau} \partial_t \left(\frac{v}{t} \right), D_{y'}^{\tau} \partial_t (v'), D_{y'}^{\tau} \partial_t \left(\frac{v^2}{t^3} \right), D_{y'}^{\tau} \partial_t \left(\frac{vv'}{t^2} \right), D_{y'}^{\tau} \partial_t \left(\frac{(v')^2}{t} \right) \in C^{\alpha}(\bar{G}_r).$$

Hence, (5.8) holds for k = 1 and any $\tau = 0, 1, \dots, \ell - k - 1$.

If $k \geq 2$, we iterate (B.10). First, we set

$$w_k = \frac{v_{k-1}}{t^2}.$$

Then, (5.10) implies, for any $\tau = 0, 1, \dots, \ell - k - 1$,

(5.11)
$$D_{y'}^{\tau} w_k, t D_{y'}^{\tau} w_k', t^2 D_{y'}^{\tau} w_k'' \in C^{\alpha}(\bar{G}_r).$$

By taking l = k - 1 in (B.10), we have

$$v_{k-2}(y',t) = c_2(y')t^2 - t^2 \int_t^r w_k(y',s)ds,$$

where $c_2 = v_{k-2}(\cdot, r)/r^2$ and hence $c_2 \in C^{\ell-2,\alpha}(\bar{B}_r)$. We note that the maximal l is $\overline{m} - 3$. Substitute this in (B.10) for l = k - 2 and repeat successively until l = 1. We obtain

$$v(y',t) = c_2(y')t^2 + c_3(y')t^3 + \dots + c_k(y')t^k + t^2R(y',t),$$

where $c_i \in C^{\ell-i,\alpha}(\bar{B}_r)$, for $i = 2, \dots, k$, and

(5.12)
$$R(y',t) = (-1)^{k-1} \int_t^r \cdots \int_{s_{k-2}}^r w_k(y',s_{k-1}) ds_{k-1} ds_{k-2} \cdots ds_1.$$

In particular, $c_i \in C^{\ell-k,\alpha}(\bar{B}_r)$, for $i=2,\cdots,k$. The maximal k is $\overline{m}-2$. The right-hand side in (5.12) is a multiple integral of multiplicity k-1. To proceed, we write

$$v(y',t) = P(y',t) + t^2 R(y',t),$$

where

$$P(y',t) = c_2(y')t^2 + \dots + c_k(y')t^k.$$

Then,

(5.13)
$$\frac{v}{t} = \frac{P}{t} + tR, v' = P' + 2tR + t^2R',$$

and

$$\frac{v^2}{t^3} = \frac{P^2}{t^3} + 2\frac{P}{t^2} \cdot tR + tR^2,$$
(5.14)
$$\frac{vv'}{t^2} = \frac{PP'}{t^2} + \left(\frac{P'}{t} + \frac{2P}{t^2}\right) \cdot tR + 2tR^2 + \frac{P}{t^2} \cdot t^2R' + t^2RR',$$

$$\frac{(v')^2}{t} = \frac{(P')^2}{t} + \frac{4P'}{t} \cdot tR + \frac{2P'}{t} \cdot t^2R' + 4tR^2 + 4t^2RR' + t^3(R')^2.$$

Next, we note by (5.12)

$$\partial_t^{k-1} R(y', t) = w_k(y', t).$$

Then, (5.11) implies, for any $\tau = 0, 1, \dots, \ell - k - 1$,

$$D_{u'}^{\tau} \partial_t^{k-1} R, \, t D_{u'}^{\tau} \partial_t^{k} R, \, t^2 D_{u'}^{\tau} \partial_t^{k+1} R \in C^{\alpha}(\bar{G}_r).$$

With

$$\partial_t^k(tR) = t\partial_t^k R + k\partial_t^k R,$$

$$\partial_t^k(t^2R') = t^2\partial_t^{k+1} R + 2kt\partial_t^k R + k(k-1)\partial_t^{k-1} R,$$

and similar expressions of the k-th t-derivatives of tR^2 , t^2RR' and $t^3(R')^2$, we conclude

$$D_{y'}^{\tau} \partial_t^k \left(\frac{v}{t} \right), D_{y'}^{\tau} \partial_t^k (v'), D_{y'}^{\tau} \partial_t^k \left(\frac{v^2}{t^3} \right), D_{y'}^{\tau} \partial_t^k \left(\frac{vv'}{t^2} \right), D_{y'}^{\tau} \partial_t^k \left(\frac{(v')^2}{t} \right) \in C^{\alpha}(\bar{G}_r).$$

Therefore, we have (5.8) for any $\tau = 0, 1, \dots, \ell - k - 1$.

Next, we discuss the higher regularity of solutions. In the proof of Theorem 5.1, the maximal l allowed in v_l is $\overline{m} - 3$. In the following, we will calculate F_l and v_l inductively by increasing l and obtain an expression of v accordingly for each large l. We first consider the case $l = \overline{m} - 2$.

Lemma 5.2. Assume that \underline{m} and \overline{m} are integers satisfying (5.3) and (5.4) and that F is $C^{\ell-2,\alpha}$ in its arguments, for some $\ell \geq \overline{m}$ and $\alpha \in (0,1)$. Let $v \in C^1(\bar{G}_R) \cap C^{\ell,\alpha}(G_R)$ be a solution of (5.1) in G_R , for some R > 0. Suppose there exists a constant $r \in (0,R)$ such that (5.5) holds for any $\tau = 0, 1, \dots, \ell-1$. Then, for $\overline{m} = 2$ and any $(y', t) \in G_r$,

(5.15)
$$v(y',t) = c_{2,1}(y')t^2 \log t + c_{2,0}(y')t^2 + t^2 w_2(y',t),$$

and, for $\overline{m} \geq 3$ and any $(y',t) \in G_r$,

$$(5.16) v(y',t) = c_2(y')t^2 + \dots + c_{\overline{m}-1}(y')t^{\overline{m}-1} + c_{\overline{m},1}(y')t^{\overline{m}}\log t + c_{\overline{m},0}(y')t^{\overline{m}} + t^2 \int_0^t \dots \int_0^{s_{\overline{m}-3}} w_{\overline{m}}(y',s_{\overline{m}-2})ds_{\overline{m}-2}ds_{\overline{m}-3} \dots ds_1,$$

where c_i and $c_{\overline{m},j}$ are functions in B'_r with $c_i \in C^{\ell-i,\alpha}(B'_r)$ for $i=2,\cdots,\overline{m}-1$, $c_{\overline{m},1} \in C^{\ell-\overline{m},\alpha}(B'_r)$ and $c_{\overline{m},0} \in C^{\ell-\overline{m},\epsilon}(B'_r)$ for any $\epsilon \in (0,\alpha)$, and $w_{\overline{m}}$ is a function in G_r such that, for any $\tau = 0, 1, \cdots, \ell - \overline{m}$ and any $\epsilon \in (0,\alpha)$,

$$(5.17) D_{y'}^{\tau} w_{\overline{m}}, t D_{y'}^{\tau} w_{\overline{m}}', t^2 D_{y'}^{\tau} w_{\overline{m}}'' \in C^{\epsilon}(\bar{G}_r),$$

and

$$(5.18) |D_{y'}^{\tau} w_{\overline{m}}| + t |D_{y'}^{\tau} w_{\overline{m}}'| + t^2 |D_{y'}^{\tau} w_{\overline{m}}''| \le Ct^{\alpha} \quad in \ G_r.$$

Proof. We adopt notations in the proof of Theorem 5.1. Throughout the proof, we always denote by c_i and $c_{i,j}$ functions of y', which may change their values from line to line.

In the proof of Theorem 5.1, we set $v_0 = v$ and introduced v_l inductively by (B.2). We also calculated $v_1, \dots, v_{\overline{m}-3}$. We now calculate $v_{\overline{m}-2}$. Recall that F_0 is a function in y', t and quantities in (5.2). The derivatives up to order $\ell - \overline{m}$ in y' of these quantities are $C^{\overline{m}-2,\alpha}(\bar{G}_r)$ functions by Theorem 5.1. (We only need to apply Theorem 5.1 for $\overline{m} \geq 3$. If $\overline{m} = 2$, we simply employ (5.5).) Then, $D_{y'}^{\tau} F_0 \in C^{\overline{m}-2,\alpha}(\bar{G}_r)$, and hence $D_{y'}^{\tau} F_{\overline{m}-2} = D_{y'}^{\tau} \partial_t^{\overline{m}-2} F_0 \in C^{\alpha}(\bar{G}_r)$, for any $\tau = 0, \dots, \ell - \overline{m}$. By (B.9), we have

(5.19)
$$v_{\overline{m}-2}(y',t) = c_{2,1}(y')t^2 \log t + c_{2,0}(y')t^2 + t^2 w_{\overline{m}}(y',t),$$

where $c_{2,1} \in C^{\ell-\overline{m},\alpha}$ and by Lemma A.2, $c_{2,0} \in C^{\ell-\overline{m},\epsilon}$ and $w_{\overline{m}}$ satisfies (5.17) and (5.18), for any $\epsilon \in (0,\alpha)$.

If $\overline{m} = 2$, then (5.19) is (5.15). Next, we consider $\overline{m} \ge 3$. With (5.19), we have (B.11) for $l = \overline{m} - 2$ and hence, by (B.12),

$$v_{\overline{m}-3}(y',t) = c_2(y')t^2 + t^2 \int_0^t \frac{v_{\overline{m}-2}(y',s)}{s^2} ds$$
$$= c_2(y')t^2 + c_{3,1}(y')t^3 \log t + c_{3,0}(y')t^3 + t^2 \int_0^t w_{\overline{m}}(y',s)ds.$$

Note that $c_{3,0}$ is a linear combination of $c_{2,0}$ and $c_{2,1}$ in (5.19) and hence $c_{3,0} \in C^{\ell-\overline{m},\epsilon}$, for any $\epsilon \in (0,\alpha)$. By using (B.12) for $l = \overline{m} - 3, \dots, 1$ successively, we obtain (5.16). \square

In Lemma 5.2, the coefficients $c_2, \dots, c_{\overline{m}-1}$ and $c_{\overline{m},1}$ have explicit expressions in terms of p,q and F and hence their regularity can be determined by that of F. However, no such expression exists for $c_{\overline{m},0}$ and there is a slight loss of regularity for $c_{\overline{m},0}$. Under the assumption $c_{\overline{m},1} = 0$, Lin [22] and Tonegawa [29] claimed $u \in C^{\overline{m},\alpha}(\bar{B}_r)$, for the solution u of the equation (1.3) with $\overline{m} = n + 1$. Their proof actually yields $u \in C^{\overline{m},\epsilon}(\bar{B}_r)$, for any $\epsilon \in (0,r)$.

We now expand v to arbitrary orders and estimate remainders.

Theorem 5.3. Assume that \underline{m} and \overline{m} are integers satisfying (5.3) and (5.4) and that F is $C^{\ell-2,\alpha}$ in its arguments, for some integers $\ell \geq k \geq \overline{m}$ and some $\alpha \in (0,1)$. Let $v \in C^1(\overline{G}_R) \cap C^{\ell,\alpha}(G_R)$ be a solution of (5.1) in G_R , for some R > 0. Suppose there exists a constant $r \in (0,R)$ such that (5.5) holds for any $\tau = 0,1,\dots,\ell-1$. Then, for any $(y',t) \in G_r$,

(5.20)
$$v(y',t) = \sum_{i=2}^{\overline{m}-1} c_i(y')t^i + \sum_{i=\overline{m}}^k \sum_{j=0}^{N_i} c_{i,j}(y')t^i(\log t)^j + \int_0^t \cdots \int_0^{s_{k-1}} w_k(y',s_k)ds_k ds_{k-1} \cdots ds_1,$$

where c_i and $c_{i,j}$ are functions in B'_r with $c_i \in C^{\ell-i,\alpha}(B'_r)$ for $i=2,\cdots,\overline{m}-1$, $c_{\overline{m},1} \in C^{\ell-\overline{m},\alpha}(B'_r)$ and, for $(i,j)=(\overline{m},0)$ or $i>\overline{m}$, $c_{i,j} \in C^{\ell-i,\epsilon}(B'_r)$ for any $\epsilon \in (0,\alpha)$, N_i is a nonnegative integer depending on i, and w_k is a function in G_r such that, for any $\tau=0,1,\cdots,\ell-k$ and any $\epsilon \in (0,\alpha)$,

$$(5.21) D_{u'}^{\tau} w_k \in C^{\epsilon}(\bar{G}_r),$$

and

$$|D_{y'}^{\tau}w_k| \le Ct^{\alpha} \quad in \ G_r.$$

We point out that, for $\overline{m} = 2$, the first summation in the right-hand side of (5.20) does not appear.

Proof. We adopt notations in the proof of Theorem 5.1. Throughout the proof, we always denote by c_i and $c_{i,j}$ functions of y', which may change their values from line to line. We will prove, by an induction on k,

(5.23)
$$v(y',t) = \sum_{i=2}^{\overline{m}-1} c_i(y')t^i + \sum_{i=\overline{m}}^k \sum_{j=0}^{N_i} c_{i,j}(y')t^i(\log t)^j + t^2 \int_0^t \cdots \int_0^{s_{k-3}} w_k(y',s_{k-2})ds_{k-2}ds_{k-3} \cdots ds_1,$$

where w_k is a function in G_r such that, for any $\tau = 0, 1, \dots, \ell - k$ and any $\epsilon \in (0, \alpha)$,

(5.24)
$$D_{y'}^{\tau} w_k, t D_{y'}^{\tau} w_k', t^2 D_{y'}^{\tau} w_k'' \in C^{\epsilon}(\bar{G}_r),$$

and

$$(5.25) |D_{y'}^{\tau} w_k| + t |D_{y'}^{\tau} w_k'| + t^2 |D_{y'}^{\tau} w_k''| \le Ct^{\alpha} in G_r.$$

We note that (5.23) holds for $k = \overline{m}$ by Lemma 5.2. We now assume that (5.23) holds for $\overline{m}, \overline{m} + 1, \dots, k - 1$, for some $k \ge \overline{m} + 1$, and proceed to prove for k.

The proof of (5.23) for k consists of three steps. By the induction hypothesis, we write, for each $\tilde{k} = \overline{m}, \dots, k-1$,

$$(5.26) v = P_{\widetilde{k}} + t^2 R_{\widetilde{k}},$$

where

(5.27)
$$P_{\widetilde{k}}(y',t) = \sum_{i=2}^{\overline{m}-1} c_i(y')t^i + \sum_{i=\overline{m}}^{\widetilde{k}} \sum_{j=0}^{N_i} c_{i,j}(y')t^i(\log t)^j,$$

and

(5.28)
$$R_{\widetilde{k}}(y',t) = \int_0^t \dots \int_0^{s_{\widetilde{k}-3}} w_{\widetilde{k}}(y',s_{\widetilde{k}-2}) ds_{\widetilde{k}-2} ds_{\widetilde{k}-3} \dots ds_1,$$

for some $c_i \in C^{\ell-i,\alpha}(B'_r)$ for $i=2,\cdots,\overline{m}-1, c_{\overline{m},1} \in C^{\ell-\overline{m},\alpha}(B'_r)$ and, for $(i,j)=(\overline{m},0)$ or $\overline{m} < i \leq k-1, c_{i,j} \in C^{\ell-i,\epsilon}(B'_r)$ for any $\epsilon \in (0,\alpha)$, and some $w_{\widetilde{k}}$ satisfying (5.24) and (5.25), for $\tau=0,1,\cdots,\ell-\widetilde{k}$. In the following, we will take $\widetilde{k}=k-2,k-1$ and proceed to construct $c_{k,j}$ and w_k and discuss their regularity.

Step 1. We first calculate $F_{k-2} = \partial_t^{k-2} F_0$. If $F_{k-2} \in C^{\epsilon}(\bar{G}_r)$ for any $\epsilon \in (0, \alpha)$, we can skip this step and proceed to Step 2. However, P_{k-2} and P_{k-1} in (5.27) have logarithmic terms, and hence, F_{k-2} may have log t-terms coupled with factors of t with nonpositive powers, which blow up at t = 0. We claim

(5.29)
$$F_{k-2} = \sum_{i=\overline{m}-k+1}^{0} \sum_{j=0}^{N_i} b_{i,j}(y') t^i (\log t)^j + G,$$

where, for any $\tau = 0, 1, \dots, \ell - k$ and any $\epsilon \in (0, \alpha), D_{y'}^{\tau} b_{i,j} \in C^{\epsilon}(\bar{B}_r), D_{y'}^{\tau} G \in C^{\epsilon}(\bar{G}_r)$ and

$$|D_{y'}^{\tau}G| \le Ct^{\alpha}$$
 in G_r .

In fact, G is a linear combination of products of $t^i(\log t)^j$, $1 \le i \le I_k$ and $0 \le j \le J_i$ for some integers I_k and J_i , and the following forms: for $1 \le m \le k-3$,

(5.30)
$$t^{-m} \int_0^t \cdots \int_0^{s_{m-1}} \widetilde{w}(y', s_m) ds_m ds_{m-1} \cdots ds_1,$$

and, for any $\tau = 0, 1, \dots, \ell - k$ and any $\epsilon \in (0, \alpha)$,

$$(5.31) t^2 D_{u'}^{\tau} \widetilde{w}'', t D_{u'}^{\tau} \widetilde{w}', D_{u'}^{\tau} \widetilde{w} \in C^{\epsilon}(\bar{G}_r).$$

The proof below shows that $b_{i,j}$ has better regularity for i < 0. However, this fact is not important, since these $b_{i,j}$'s do not contribute to the calculation of $c_{k,j}$. We need only track the regularity of $b_{0,j}$.

Recall that F_0 is a function in y', t and quantities in (5.2). We need to calculate $D_{y'}^{\tau} \partial_t^{k-2}$ acting on these expressions, for $\tau = 0, 1, \dots, \ell - k$. For an illustration, we consider $\tau = 0$. Take an integer $l \leq k - 2$ and we first calculate ∂_t^l acting only on

(5.32)
$$v_t, \frac{v}{t}, \frac{v^2}{t^3}, \frac{vv_t}{t^2}, \frac{v_t^2}{t}, D_{y'}v_t, \frac{D_{y'}v}{t}.$$

With v in (5.26) for $\tilde{k} = k - 1$, the first five functions in (5.32) are expressed in terms of P_{k-1} and R_{k-1} and their derivatives by (5.13) and (5.14). We need to calculate ∂_t^l

acting on each term in the right-hand sides of (5.13) and (5.14). We illustrate this by v/t. By (5.27), we have

$$\partial_t^l \left(\frac{P_{k-1}}{t} \right) = \sum_{i=0}^{\overline{m}-2-l} \max\{\overline{m} - 2 - l, 0\} c_i t^i + \sum_{i=\overline{m}-1-l}^{k-2-l} \sum_{j=0}^{N_i} c_{i,j} t^i (\log t)^j.$$

We point out that negative powers of t appear only in association with $(\log t)^j$. Indeed, if $l \geq \overline{m} - 1$, $\log t$ -terms have factors of t with nonpositive powers. Moreover, all coefficients c_i and $c_{i,j}$ are at least $C^{\ell-k+1,\epsilon}(B'_r)$. Next, by (5.28), it is easy to check that we can write

$$\partial_t^l(tR_{k-1}) = t^{k-2-l}G,$$

where G is a linear combination of terms in (5.30) and (5.31). We point out that t above always has a nonnegative power since $l \leq k-2$ and this power is zero only if l=k-2, the maximal order of differentiation allowed. Therefore, v/t can be expressed by an expression similar as the right-hand side of (5.29). Similar expressions hold for the l-th derivatives in t of other functions in (5.32). We point out that, in the expressions of $D_{y'}v_t$ and $D_{y'}v/t$, all coefficients of c_i or $c_{i,j}$ are at least $C^{\ell-k,\epsilon}$ and the remainder is at least C^{ϵ} in y' since one derivative with respect to y' is taken. Next, we calculate ∂_t^l acting on $D_{y'}^2v$, the function in (5.2) missing from (5.32). To this end, we take v in (5.26) for $\tilde{k} = k-2$. Then, in $D_{y'}^2v$, all coefficients of t^i or $t^i(\log t)^j$ are $C^{\ell-k,\epsilon}$, and the remainder is $C^{\ell-k,\epsilon}$ in y' and has an order $t^{k-2+\alpha}$ in t. When we put all these expressions together and calculate F_{k-2} , we conclude (5.29).

We now make two important observations concerning G in (5.29). First, there is no negative power of t in G. Second, whenever $\log t$ appears, it is coupled with a factor of a positive power of t. To verify this, we consider a simple case: for some nonnegative integers l_1 and l_2 with $l_1 + l_2 \le k - 2$,

$$\partial_t^{l_1} \left(\frac{v}{t} \right) \cdot \partial_t^{l_2} \left(\frac{v}{t} \right).$$

In the first factor, the least power of t coupled with $\log t$ is $\overline{m} - 1 - l_1$; while in the second factor, terms of the forms in (5.30) have a factor t^{k-2-l_2} . For the product, the power of t is given by

$$(\overline{m} - 1 - l_1) + (k - 2 - l_2) = k + \overline{m} - 3 - (l_1 + l_2) \ge \overline{m} - 1.$$

We note that each expression in (5.30), with its y'-derivatives up to order $\ell - k$, is in C^{ϵ} , for any $\epsilon \in (0, \alpha)$, and has an order of t^{α} . Here, the C^{ϵ} -regularity of (5.30) follows from Corollary A.4. Then, each term in G, with its y'-derivatives up to order $\ell - k$, is also C^{ϵ} in \bar{G}_r and has an order of t^{α} .

Step 2. We calculate v_{k-2} . We first verify $t^{k-1}v_{k-2} \to 0$. By (B.2), we have

$$t^{k-1}v_{k-2} = t^{k-1}\left\{ (v_{k-3})' - \frac{v_{k-3}}{t} \right\} = \dots = t^{k-1} \sum_{j=0}^{k-2} C_j \frac{v_0^{(k-2-j)}}{t^j}$$
$$= \sum_{j=0}^{k-2} C_j v^{(k-2-j)} t^{k-1-j}.$$

This goes to 0 as $t \to 0$, by (5.26) for $\tilde{k} = k - 1$ and the regularity of R_{k-1} . Hence, (B.5) holds for l = k - 2. By taking l = k - 2 in (B.6), we have

$$v_{k-2}(y',t) = c_{\overline{m}-k+2}(y')t^{\overline{m}-k+2} + \frac{1}{\overline{m}-\underline{m}}t^{\underline{m}-k+2} \int_0^t s^{k-1-\underline{m}}F_{k-2}(y',s)ds + \frac{1}{\overline{m}-\underline{m}}t^{\overline{m}-k+2} \int_t^r s^{k-1-\overline{m}}F_{k-2}(y',s)ds.$$

We now substitute F_{k-2} by (5.29) and obtain

$$v_{k-2} = c_{\overline{m}+2-k} t^{\overline{m}+2-k}$$

$$+ t^{\underline{m}-k+2} \int_0^t \sum_{i=\overline{m}-k+1}^0 \sum_{j=0}^{N_i} c_{i,j} s^{i+k-1-\underline{m}} (\log s)^j ds + t^{\underline{m}-k+2} \int_0^t s^{k-1-\underline{m}} G ds$$

$$+ t^{\overline{m}-k+2} \int_t^r \sum_{i=\overline{m}-k+1}^0 \sum_{j=0}^{N_i} c_{i,j} s^{i+k-1-\overline{m}} (\log s)^j ds + t^{\overline{m}-k+2} \int_t^r s^{k-1-\overline{m}} G ds.$$

The integrals involving log t can be calculated directly. By $k-1 \ge \overline{m}$, we have

$$\int_{t}^{r} s^{k-1-\overline{m}} G ds = \int_{0}^{r} s^{k-1-\overline{m}} G ds - \int_{0}^{t} s^{k-1-\overline{m}} G ds.$$

The first integral in the right-hand side is regular and can be combined with $c_{\overline{m}+2-k}$. Hence, we obtain

$$(5.33) v_{k-2}(y',t) = c_{\overline{m}+2-k}(y')t^{\overline{m}+2-k} + \sum_{i=\overline{m}+3-k}^{2} \sum_{j=0}^{N_i} c_{i,j}(y')t^i(\log t)^j + t^2w_k(y',t),$$

where c_i and $c_{i,j}$ are at least $C^{\ell-k,\epsilon}$ and

$$w_k(y',t) = t^{\underline{m}-k} \int_0^t s^{k-1-\underline{m}} G(y',s) ds - t^{\overline{m}-k} \int_0^t s^{k-1-\overline{m}} G(y',s) ds.$$

By Lemma A.2, we have, for any $\tau = 0, 1, \dots, \ell - k$ and any $\epsilon \in (0, \alpha)$,

$$t^2 D_{v'}^{\tau} w_k'', t D_{v'}^{\tau} w_k', D_{v'}^{\tau} w_k \in C^{\epsilon}(\bar{G}_r),$$

and

$$t^2 \left| D_{y'}^{\tau} w_k'' \right| + t \left| D_{y'}^{\tau} w_k' \right| + \left| D_{y'}^{\tau} w_k \right| \le C t^{\alpha} \quad \text{in } G_r.$$

We note that $c_{2,j}$ in (5.33) is a linear combination of $b_{0,j}$ in (5.29).

Step 3. We now express $v=v_0$ by an iteration. Recall the iteration formula (B.10). We will take successively $l=k-2,k-3,\cdots,1$. We note that v_{k-2} in (5.33) has two parts. One part is a polynomial of t and $\log t$, and another part has an order of $t^{2+\alpha}$. In substituting v_{k-2} in (B.10) with l=k-2 to calculate the integral, the terms corresponding to the first part can be integrated directly and, for the second part, we can write the integral as

$$t^2 \int_t^r w_k(y', s) ds = c_2 t^2 + t^2 \int_0^t w_k(y', s) ds.$$

Hence, we have

$$\begin{aligned} v_{k-3}(y',t) &= c_2(y')t^2 + t^2 \int_t^r v_{k-2}(y',s)s^{-2}ds \\ &= c_{\overline{m}+3-k}(y')t^{\overline{m}+3-k} + c_2(y')t^2 \\ &+ \sum_{i=\overline{m}+4-k}^3 \sum_{j=0}^{N_i} c_{i,j}t^i(\log t)^j + t^2 \int_0^t w_k(y',s)ds. \end{aligned}$$

We note that $c_{3,j}$ in the expression of v_{k-3} above is a linear combination of $c_{2,j}$ in (5.33) and hence a linear combination of $b_{0,j}$ in (5.29). We continue this process. The first term will contribute t^1 for $v_{\overline{m}-1}$, which results in t^{-1} after divided by t^2 . Then, an integration will yield $\log t$. By continuing, we obtain

$$v(y',t) = \sum_{i=2}^{\overline{m}} c_i(y')t^i + c_{\overline{m},1}(y')t^{\overline{m}} \log t + \sum_{i=\overline{m}+1}^k \sum_{j=0}^{N_i} c_{i,j}(y')t^i (\log t)^j + t^2 \int_0^t \cdots \int_0^{s_{k-3}} w_k(y',s_{k-2})ds_{k-2}ds_{k-3} \cdots ds_1.$$

We point out that $c_{k,j}$ in the expression of v above is a linear combination of $b_{0,j}$ in (5.29). This ends the proof of (5.23) for k.

With (5.23) proved by induction, we write v in (5.23) as

$$v(y',t) = v_k(y,t) + R_k(y',t),$$

where

$$R_k(y',t) = t^2 \int_0^t \cdots \int_0^{s_{k-3}} w_k(y',s_{k-2}) ds_{k-2} ds_{k-3} \cdots ds_1,$$

for some function w_k satisfying (5.24) and (5.25). We take $\widehat{w}_k = \partial_t^k R_k$. Then,

$$R_k(y',t) = \int_0^t \cdots \int_0^{s_{k-1}} \widehat{w}_k(y',s_k) ds_k ds_{k-1} \cdots ds_1.$$

Moreover, \widehat{w}_k satisfies (5.21) and (5.22). This proves (5.20), with \widehat{w}_k here serving the role of w_k in (5.20).

Remark 5.4. Set

$$v_k(y',t) = \sum_{i=2}^{\overline{m}-1} c_i(y')t^i + \sum_{i=\overline{m}}^k \sum_{j=0}^{N_i} c_{i,j}(y')t^i(\log t)^j.$$

Then, (5.21) and (5.22) imply, for any $\tau = 0, 1, \dots, \ell - k$, any $m = 0, \dots, k$ and any $\epsilon \in (0, \alpha),$

$$D_{y'}^{\tau} \partial_t^m (v - v_k) \in C^{k-m,\epsilon}(\bar{G}_r),$$

and

$$|D_{y'}^{\tau} \partial_t^m (v - v_k)| \le C t^{k-m+\alpha}$$
 in G_r .

Next, we prove two corollaries. We first prove that, if the first logarithmic term does not appear, then there are no logarithmic terms in the expansion and the solutions are as regular as the nonhomogeneous terms allow.

Corollary 5.5. Assume that \underline{m} and \overline{m} are integers satisfying (5.3) and (5.4) and that F is $C^{\ell-2,\alpha}$ in its arguments, for some $\ell \geq \overline{m}$ and $\alpha \in (0,1)$. Let $v \in C^1(\bar{G}_R) \cap C^{\ell,\alpha}(G_R)$ be a solution of (5.1) in G_R , for some R>0. Suppose there exists a constant $r\in(0,R)$ such that (5.5) holds for any $\tau = 0, 1, \dots, \ell - 1$. If $c_{\overline{m},1} = 0$, then $c_{i,j} = 0$, for any $i = \overline{m}, \dots, \ell$ and $j = 1, \dots, N_i$, and $v \in C^{\ell, \epsilon}(\overline{G}_r)$, for any $\epsilon \in (0, \alpha)$.

Proof. First, we have $c_{\overline{m},1}=0$ by the assumption and $c_{\overline{m},j}=0$, for any j>1, by Lemma 5.2. Inductively for any $\overline{m} + 1 \le k \le \ell$, we assume $c_{\overline{m},j} = \cdots = c_{k-1,j} = 0$ for any $j \ge 1$. Now, we examine Steps 1-3 in the proof of Theorem 5.3 and prove $c_{k,j} = 0$ for any $j \ge 1$.

By the induction hypothesis, P_{k-1} in (5.27), with k = k - 1, is a polynomial, i.e.,

$$P_{k-1}(y',t) = \sum_{i=2}^{k-1} c_i(y')t^i.$$

Then, F_{k-2} does not have logarithmic terms. Hence, instead of (5.29), we have

$$F_{k-2} = b_0 + G$$
,

for some $b_0 \in C^{\epsilon}(B'_r)$, for any $\epsilon \in (0, \alpha)$. Then, instead of (5.33), we obtain

$$v_{k-2}(y',t) = c_{\overline{m}+2-k}(y')t^{\overline{m}+2-k} + c_2(y')t^2 + t^2w_k(y',t).$$

We now iterate as in the proof of Theorem 5.3. The term t^2 will contribute t^2, \dots, t^k . The first term will contribute a log t factor when the power of t becomes 1, which reduces to -1 after divided by t^2 . Hence,

$$v(y',t) = \sum_{i=2}^{k} c_i(y')t^i + c_{\overline{m},1}(y')t^{\overline{m}} \log t + t^2 \int_0^t \cdots \int_0^{s_{k-3}} w_k(y',s_{k-2})ds_{k-2}ds_{k-3} \cdots ds_1.$$

Therefore, $c_{k,j}=0$ for any $j\geq 1$. Note $c_{\overline{m},1}=0$ by assumption. Next, we prove $D_{y'}^{\ell-k}\partial_t^k v\in C^{\epsilon}(\bar{G}_r)$ for any $k=0,1,\cdots,\ell$ and any $\epsilon\in(0,\alpha)$. In fact, we can take $\epsilon = \alpha$ if $k \leq \overline{m} - 1$. For k = 0, this is implied by the assumption. For $1 \le k \le \overline{m} - 1$, this follows from the proof of Theorem 5.1. For $\overline{m} \le k \le \ell$, by Theorem 5.3, we have

$$v(y',t) = \sum_{i=2}^{k} c_i(y')t^i + \int_0^t \cdots \int_0^{s_{k-1}} w_k(y',s_k)ds_k ds_{k-1} \cdots ds_1,$$

where $c_i \in C^{\ell-k,\epsilon}(B'_r)$ for $i=2,\cdots,k$ and $D^{\tau}_{y'}w_k \in C^{\epsilon}(\bar{G}_r)$ for any $\tau=0,1,\cdots,\ell-k$ and any $\epsilon \in (0,\alpha)$. This implies the desired result.

In the next result, we estimate the largest power of the log-factors. For simplicity, we state in the smooth category.

Corollary 5.6. Let $v, \underline{m}, \overline{m}$ and r be as in Theorem 5.1. Assume F is smooth in all of its arguments and

(5.34)
$$F \text{ is linear in } \frac{v^2}{t^3}, \frac{vv_t}{t^2}, \frac{v_t^2}{t}.$$

Then, for any $i \geq \overline{m}$,

$$N_i \le \left\lceil \frac{i-1}{\overline{m}-1} \right\rceil.$$

Proof. We will prove the following statement: If $t^k(\log t)^j$ appears in v for some $k \geq \overline{m}$, then

$$(5.35) k > (\overline{m} - 1)j + 1.$$

We prove (5.35) by an induction on k. If $k = \overline{m}$, then the corresponding j is either 0 or 1, and hence (5.35) holds. Suppose (5.35) holds for any $k = \overline{m}, \dots, l-1$ for some $l \ge \overline{m} + 1$. We now consider k = l. The proof is by a computation based on (5.1) and (5.2).

Let $t^{l}(\log t)^{j}$ be one term in v. We substitute such a term in (5.1) and note

$$(t^{l}(\log t)^{j})'' + p\frac{(t^{l}(\log t)^{j})'}{t} + q\frac{t^{l}(\log t)^{j}}{t^{2}}$$

$$= (l - \overline{m})(l - \underline{m})t^{l-2}(\log t)^{j} + j(2l - 1 + p)t^{l-2}(\log t)^{j-1}$$

$$+ j(j-1)t^{l-2}(\log t)^{j-2}.$$

The term $t^{l-2}(\log t)^j$ has the highest power of $\log t$ and a nonzero coefficient since $l \ge \overline{m} + 1$. Next, we find the corresponding term in F. Set

$$\overline{v}(y',t) = \sum_{i=2}^{l-1} \sum_{j=0}^{N_i} c_{i,j}(y') t^i (\log t)^j.$$

By the induction hypothesis, each pair i and j in \overline{v} satisfy (5.35), with k replaced by i. We now substitute \overline{v} in (5.1) and identify $t^{l-2}(\log t)^j$. First, we note that all terms $t^i(\log t)^j$ in \overline{v}_t , $\frac{\overline{v}}{t}$ satisfy $i \geq (\overline{m}-1)j$ and that all terms $t^i(\log t)^j$ in $\frac{\overline{v}_t}{t}$, $\frac{\overline{v}}{t^2}$ satisfy $i \geq (\overline{m}-1)j-1$.

Hence, all terms $t^i(\log t)^j$ in $\frac{\overline{v}^2}{t^3}$, $\frac{\overline{v}v_t}{t^2}$, $\frac{\overline{v}_t^2}{t}$ with $i \leq l-2$ satisfy $i \geq (\overline{m}-1)j-1$. Therefore, these three terms are dominant. Recall that F is smooth in

$$\overline{v}_t, \, \frac{\overline{v}}{t}, \, \frac{\overline{v}^2}{t^3}, \, \frac{\overline{v}\overline{v}_t}{t^2}, \, \frac{\overline{v}_t^2}{t},$$

and is linear in the last three quantities by (5.34). If we expand F in terms of $t^i(\log t)^j$, then all terms $t^i(\log t)^j$ with $i \leq l-2$ satisfy $i \geq (\overline{m}-1)j-1$. With i=l-2, we obtain $l \geq (\overline{m}-1)j+1$. This is (5.35) for k=l.

6. Expansions for Vertical Graphs

In this section, we study the expansion of minimal graphs in the hyperbolic space by viewing them as graphs over their vertical tangent planes. We recall the set up in Section 3. We denote by y = (y', t) points in \mathbb{R}^n , with $y_n = t$, and set, for any r > 0,

$$G_r = \{ (y', t) : |y'| < r, 0 < t < r \}.$$

For some R > 0 and some $\varphi \in C^2(B'_R)$, we consider a solution $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ of (3.1) and (3.2); namely, u satisfies

$$\Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{n u_t}{t} = 0 \quad \text{in } G_R,$$

and

$$u = \varphi$$
 on B'_R .

In the following, we set

$$(6.1) v = u - \varphi.$$

Then,

(6.2)
$$\Delta v - \frac{u_i u_j}{1 + |Du|^2} v_{ij} - \frac{n v_t}{t} + \Delta \varphi - \frac{u_i u_j}{1 + |Du|^2} \varphi_{ij} = 0 \text{ in } G_R.$$

In putting (6.2) in the form of (4.1), we have

$$A_{ij} = \delta_{ij} - \frac{(v+\varphi)_i(v+\varphi)_j}{1+|D(v+\varphi)|^2},$$

$$P = -n, \quad Q = 0,$$

and

$$N = \Delta \varphi - \frac{(v+\varphi)_i(v+\varphi)_j}{1+|D(v+\varphi)|^2}\varphi_{ij}.$$

By (3.3) and (3.11), we have

$$(6.3) |v| \le C_0 t^2 in G_{R/2},$$

and

$$(6.4) |Dv| \le C_0 t in G_{R/2}.$$

These correspond to (4.3) and (4.4).

We now prove the tangential smoothness of v.

Theorem 6.1. Assume $\varphi \in C^{\ell,\alpha}(B_R')$ for some $\ell \geq 2$ and $\alpha \in (0,1)$. Let $v \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (6.2) and satisfy (6.3) and (6.4). Then, there exists a constant $r \in (0, R/2)$ such that, for $\tau = 0, 1, \dots, \ell - 2$,

(6.5)
$$\frac{D_{y'}^{\tau}v}{t^2}, \frac{DD_{y'}^{\tau}v}{t}, D^2D_{y'}^{\tau}v \in C^{\alpha}(G_r),$$

and

(6.6)
$$\frac{D_{y'}^{\tau}v}{t}, D_{y'}^{\tau}Dv, \frac{D_{y'}^{\tau}(v_t^2)}{t} \in C^{1,\alpha}(\bar{G}_r).$$

Moreover, the cooresponding bounds depend only on n, ℓ , α , C_0 in (6.3) and (6.4), and the $C^{\ell,\alpha}$ -norm of φ .

Proof. By $n \geq 2$, we have

$$2A_{nn} + 2P + Q \le 2 - 2n < 0,$$

and

$$(2+\alpha)(1+\alpha)A_{nn} + (2+\alpha)P + Q \le (2+\alpha)(1+\alpha-n) < 0.$$

Hence, the desired results follow from Theorem 4.3.

We point out that Lin [22] already proved the tangential smoothness of v. The present form is used in the expansions to be discussed next.

Based on (6.2), a straightforward calculation shows that v satisfies

(6.7)
$$v_{tt} - n \frac{v_t}{t} + F = 0,$$

where

(6.8)
$$F = F(v) = \left(\delta_{\alpha\beta} - \frac{u_{\alpha}u_{\beta}}{1 + |D_{y'}u|^2}\right) \left(v_{\alpha\beta} + \varphi_{\alpha\beta}\right) - \frac{2u_{\alpha}}{1 + |D_{y'}u|^2} v_t v_{\alpha t} + \frac{v_t^2}{1 + |D_{y'}u|^2} \left(\Delta_{y'}v + \Delta_{y'}\varphi\right) - \frac{nv_t}{1 + |D_{y'}u|^2} \frac{v_t^2}{t}.$$

We note that F is a smooth function in t and

$$v_t, \frac{v_t^2}{t}, D_{y'}v_t, D_{y'}^2v,$$

and that F depends on y' through derivatives of φ up to order 2. Moreover,

$$F$$
 is linear in $\frac{v_t^2}{t}$.

We now expand v to arbitrary orders and estimate remainders.

Theorem 6.2. Assume $\varphi \in C^{\ell,\alpha}(B'_R)$ for some $\ell \geq k \geq n+1$ and some $\alpha \in (0,1)$. Let $v \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (6.2) and satisfy (6.3) and (6.4). Then, there

exists a positive constant $r \in (0,R)$ such that, for any $(y',t) \in G_r$,

(6.9)
$$v(y',t) = \sum_{i=2}^{n} c_i(y')t^i + \sum_{i=n+1}^{k} \sum_{j=0}^{\left[\frac{i-1}{n}\right]} c_{i,j}(y')t^i(\log t)^j + \int_0^t \cdots \int_0^{s_{k-1}} w_k(y',s_k)ds_k ds_{k-1} \cdots ds_1,$$

where c_i and $c_{i,j}$ are $C^{\ell-i,\varepsilon}$ -functions in B'_r for any $\epsilon \in (0,\alpha)$, and w_k is a function in G_r such that, for any $\tau = 0, 1, \dots, \ell - k$ and any $\epsilon \in (0,\alpha)$,

$$(6.10) D_{u'}^{\tau} w_k \in C^{\epsilon}(\bar{G}_r),$$

and

$$(6.11) |D_{u'}^{\tau} w_k| \le C t^{\alpha} \quad in \ G_r,$$

where C is a positive constant depending only on n, ℓ , α , C_0 in (6.3) and (6.4), and the $C^{\ell,\alpha}$ -norm of φ .

Proof. We note that (6.7) is in the form (5.1), with p = -n and q = 0. The general solutions of the homogeneous linear equation corresponding to (6.7) are spanned by 1 and t^{n+1} . Hence, \underline{m} and \overline{m} in (5.3) are given by $\underline{m} = 0$ and $\overline{m} = n + 1$. Then, (5.4) is satisfied. The desired results follow from Theorem 5.3 and Corollary 5.6.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We recall $v = u - \varphi$. By taking k = n + 1 in (6.9) and comparing with the estimates in Theorem 3.2, we conclude $c_i = 0$ for odd $i \le n$ and $c_{n+1,1} = 0$ if n is even. Then, Theorem 1.1 follows from Theorem 6.2 and, in particular, (6.9), (6.10) and (6.11).

We now prove an easy consequence of Theorem 1.1.

Corollary 6.3. Let $\varphi \in C^{\ell,\alpha}(B_R')$ be a given function, for some $\ell \geq n+1$ and $\alpha \in (0,1)$, and $u \in C(\bar{G}_R) \cap C^{\infty}(G_R)$ be a solution of (1.3)-(1.4). Then,

- (1) for n even, u is $C^{\ell,\epsilon}$ in \bar{G}_r , for any $\epsilon \in (0,\alpha)$ and any $r \in (0,R)$;
- (2) for n odd, u is $C^{n,\epsilon}$ in \bar{G}_r , for any $\epsilon \in (0,1)$ and any $r \in (0,R)$. Moreover, if $c_{n+1,1}$ vanishes on B'_R , then u is $C^{\ell,\epsilon}$ in \bar{G}_r , for any $\epsilon \in (0,\alpha)$ and any $r \in (0,R)$. In particular, for n=3, if the graph $z=\varphi(y')$ is a Willmore surface, then u is $C^{\ell,\epsilon}$ in \bar{G}_r .

Corollary 6.3 follows from Corollary 5.5, since $c_{n+1,1} = 0$ if n is even. We point out that Corollary 6.3 was already proved by Lin [22] and Tonegawa [29]. Refer to the discussion after the proof of Lemma 5.2 concerning the loss of regularity.

7. Expansions of f

In this section, we discuss the expansion of solutions f of (1.1). To this end, we need to impose an extra condition that the boundary mean curvature is positive, i.e., $H_{\partial\Omega} > 0$. In this case, the solution f has an order \sqrt{d} near $\partial\Omega$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance of x to the boundary $\partial\Omega$. We first assume that Ω is a bounded smooth domain and hence d is a smooth function near $\partial\Omega$. We denote by (y', d) the principal coordinates near $\partial\Omega$. Then, a formal expansion of f is given by, for n even,

$$a_1\sqrt{d} + a_3(\sqrt{d})^3 + \dots + a_{n-1}(\sqrt{d})^{n-1} + \sum_{i=n}^{\infty} a_i(\sqrt{d})^i,$$

and, for n odd,

$$a_1\sqrt{d} + a_3(\sqrt{d})^3 + \dots + a_{n-2}(\sqrt{d})^{n-2} + \sum_{i=n}^{\infty} \sum_{j=0}^{N_i} a_{i,j}(\sqrt{d})^i (\log\sqrt{d})^j,$$

where a_i and $a_{i,j}$ are smooth functions of $y' \in \partial \Omega$, and N_i is a nonnegative constant depending on i, with $N_n = 1$. In the present case, the coefficients a_1, a_3, \dots, a_{n-1} , for n even, and $a_1, a_3, \dots, a_{n-2}, a_{n,1}$, for n odd, have explicit expressions on $\partial \Omega$. For example, for any $n \geq 2$,

$$a_1 = \sqrt{\frac{2}{H}},$$

and, for n=3,

$$a_{3,1} = \frac{1}{4\sqrt{2}H^{\frac{5}{2}}} \{ \Delta_{\partial\Omega} H + 2H(H^2 - K) \},\,$$

where H and K are the mean curvature and the Gauss curvature of $\partial\Omega$, respectively. We point out that the expansion is with respect to \sqrt{d} , instead of d. Hence, our regularity results will also be stated in terms of \sqrt{d} .

Let $k \geq n$ be an integer and set, for n even,

(7.1)
$$f_k = a_1 \sqrt{d} + a_3 (\sqrt{d})^3 + \dots + a_{n-1} (\sqrt{d})^{n-1} + \sum_{i=n}^k a_i (\sqrt{d})^i,$$

and, for n odd,

(7.2)
$$f_k = a_1 \sqrt{d} + a_3 (\sqrt{d})^3 + \dots + a_{n-2} (\sqrt{d})^{n-2} + \sum_{i=n}^k \sum_{j=0}^{\left[\frac{i-1}{n-1}\right]} a_{i,j} (\sqrt{d})^i (\log \sqrt{d})^j,$$

where a_i and $a_{i,j}$ are functions of $y' \in \partial \Omega$. Our main result characterizes the remainder $f - f_k$.

Theorem 7.1. For some integers ℓ , k with $\ell - 1 \ge k \ge n$ and some $\alpha \in (0,1)$, let Ω be a bounded $C^{\ell,\alpha}$ -domain in \mathbb{R}^n such that $\partial\Omega$ has a positive mean curvature, and (y',d) be the principal coordinates near $\partial\Omega$. Suppose that $f \in C(\bar{\Omega}) \cap C^{\infty}(\Omega)$ is a solution of (1.1)-(1.2). Then, there exist functions $a_i, a_{i,j} \in C^{\ell-i-1,\epsilon}(\partial\Omega)$, for $i = 1, 3, \dots, n, \dots, k$

and any $\epsilon \in (0, \alpha)$, and a positive constant d_0 such that, for f_k defined as in (7.1) or (7.2), for any $\tau = 0, 1, \dots, \ell - k - 1$, any $m = 0, 1, \dots, k$, and any $\epsilon \in (0, \alpha)$,

$$D_{y'}^{\tau} \partial_{\sqrt{d}}^{m} (f - f_k)$$
 is C^{ϵ} in $(y', \sqrt{d}) \in \partial \Omega \times [0, d_0]$,

and, for any $0 < d < d_0$,

$$(7.3) |D_{y'}^{\tau} \partial_{\sqrt{d}}^{m} (f - f_k)| \le C(\sqrt{d})^{k - m + \alpha},$$

where C is a positive constant depending only on n, ℓ , α and the $C^{\ell,\alpha}$ -norm of $\partial\Omega$.

Theorem 7.1 is formulated as a global result only for convenience. In fact, the corresponding local version holds; namely, if we assume (1.1)-(1.2) near a boundary point, then we can prove (7.3) near this point.

We now indicate briefly the proof of Theorem 7.1. We provide an outline only and skip all details. We fix a boundary point, say the origin, and assume, for some R > 0, $f \in C(\bar{\Omega} \cap \bar{B}_R) \cap C^{\infty}(\Omega \cap B_R)$ satisfies

(7.4)
$$\Delta f - \frac{f_i f_j}{1 + |Df|^2} f_{ij} + \frac{n}{f} = 0 \quad \text{in } \Omega \cap B_R,$$

with the condition

(7.5)
$$f > 0 \quad \text{in } \Omega \cap B_R, \\ f = 0 \quad \text{on } \partial \Omega \cap B_R.$$

It is convenient to write the equation (7.4) in principal coordinates. Consider principal coordinates y = (y', d) near the origin, with d the distance of the point y to the boundary. The expansions in (7.1) and (7.2) suggest that \sqrt{d} is a better variable than d. So we introduce a new variable $t = \sqrt{2d}$.

Step 1. Write the equation (7.4) in coordinates (y',t) in

$$G_R = \{(y', t); |y'| < R, 0 < t < R\}.$$

Step 2. Introduce a new function

$$g = f - \frac{1}{\sqrt{H}}t,$$

and derive estimates of the following forms:

$$(7.6) |g| \le Ct^2,$$

and

$$(7.7) |Dg| \le Ct.$$

These estimates can be derived for f from the original equation, by the methods used in Section 3.

Step 3. Write the equation for f from Step 1 as an equation for g of the form

(7.8)
$$A_{ij}g_{ij} + P\frac{g_t}{t} + Q\frac{g}{t^2} + N = 0 \text{ in } G_R,$$

where the coefficients A_{ij} , P, Q and the nonhomogeneous term N are functions of y', t and

$$Dg, \frac{g}{t}, \frac{|Dg|_G^2}{t}.$$

We employ Theorem 4.3 to get the tangential regularity. We point out that (4.9) and (4.18) are satisfied for $n \geq 3$. Extra work is needed for n = 2.

Step 4. Write the equation (7.8) as

(7.9)
$$g_{tt} - (n-2)\frac{g_t}{t} - n\frac{g}{t^2} + F = 0,$$

where F is a function in y', t and

$$g_t, \frac{g}{t}, \frac{g^2}{t^3}, \frac{gg_t}{t^2}, \frac{g_t^2}{t}, D_{y'}g_t, D_{y'}^2g, \frac{D_{y'}g}{t}.$$

In fact,

$$F$$
 is linear in $\frac{g^2}{t^3}, \frac{gg_t}{t^2}, \frac{g_t^2}{t}$.

Then, we employ Theorem 5.3 to get an expansion of g, similar as the expansion of v in Theorem 6.2.

APPENDIX A. CALCULUS LEMMAS

In this section, we list several calculus lemmas concerning Hölder continuous functions. We denote by y = (y', t) points in \mathbb{R}^n and set, for any constant r > 0,

$$G_r = B_r' \times (0, r).$$

The most basic result is given by the following lemma.

Lemma A.1. Let $\alpha \in (0,1)$ be a constant and $f \in C(\bar{G}_r) \cap C^1(G_r)$ satisfy f(y',0) = 0 and, for any $(y',t) \in G_r$,

$$|Df(y',t)| \le Mt^{\alpha-1}$$
.

Then, $f \in C^{0,\alpha}(\bar{G}_r)$.

Similarly, we have the Hölder continuity for functions in integral forms.

Lemma A.2. Let $a \geq -1$ and $\alpha \in (0,1]$ be constants. Suppose $f \in C^{0,\alpha}(\bar{G}_r)$ with f(y',0) = 0. Define, for any $(y',t) \in G_r$,

$$F(y',t) = \frac{1}{t^{a+1}} \int_0^t s^a f(y',s) ds.$$

Then, $F \in C^{0,\alpha}(\bar{G}_r)$ if a > -1, and $F \in C^{0,\epsilon}(\bar{G}_r)$ for any $\epsilon \in (0,\alpha)$ if a = -1.

We point out that there is a slight loss of regularity for a = -1. In fact, the loss occurs only in the y'-direction. Refer to Lemma 5.1 [18].

Lemma A.2 has following corollaries.

Corollary A.3. Let $\alpha \in (0,1]$ be a constant. Suppose $f \in C^{1,\alpha}(\bar{G}_r)$ with f(y',0) = 0 and $f_t(y',0) = 0$. Then, $\frac{f}{t} \in C^{0,\alpha}(\bar{G}_r)$.

Corollary A.4. Let $m \geq 1$ be an integer and $\alpha \in (0,1]$ be a constant. Suppose $f \in C^{0,\alpha}(\bar{G}_r)$ with f(y',0)=0. Then,

$$\frac{1}{t^m} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-1}} f(y', s_m) ds_m ds_{m-1} \cdots ds_1 \in C^{0, \alpha}(\bar{G}_r).$$

Similar to Lemma A.2, we have the following result.

Lemma A.5. Let $a \leq -2$ and $\alpha \in (0,1)$ be constants. Suppose $f \in C^{0,\alpha}(\bar{G}_r)$ with f(y',0) = 0. Define, for any $(y',t) \in G_r$,

$$F(y',t) = \frac{1}{t^{a+1}} \int_{t}^{r} s^{a} f(y',s) ds.$$

Then, $F \in C^{0,\alpha}(\bar{G}_r)$.

APPENDIX B. A CLASS OF ODES

We briefly discuss the regularity of solutions of a class of ordinary differential equations. We denote by y = (y', t) points in \mathbb{R}^n . For convenience, we also denote by t the derivative with respect to t. This should not be confused with y', the first n-1 coordinates of the point. We set, for any t > 0,

$$G_r = B'_r \times (0, r).$$

Let p and q be constants, and F be a function defined in G_r . We assume that v is a solution of

(B.1)
$$v'' + p\frac{v'}{t} + q\frac{v}{t^2} + F = 0 \text{ in } G_r.$$

Here and hereafter, we always assume that involved functions have sufficient degree of regularity so notations make sense. The function F is not required to be bounded near t = 0. Our main task is to discuss the global regularity of v up to t = 0.

In the following, we assume that the general solutions of the homogeneous linear equation corresponding to (B.1) are spanned by $t^{\underline{m}}$ and $t^{\overline{m}}$, for some integers \underline{m} and \overline{m} such that

$$\underline{m} \le 0, \quad \overline{m} \ge 2.$$

Then,

$$p = 1 - (\underline{m} + \overline{m}), \quad q = \underline{m} \cdot \overline{m}.$$

For convenience, we write $p_0 = p$, $q_0 = q$, $v_0 = v$, $F_0 = F$, $\underline{m}_0 = \underline{m}$ and $\overline{m}_0 = \overline{m}$. Set, for $l \ge 1$ inductively,

$$p_l = p_{l-1} + 2, \quad q_l = p_{l-1} + q_{l-1},$$

and

(B.2)
$$v_l = v'_{l-1} - \frac{2v_{l-1}}{t}.$$

A straightforward calculation yields

(B.3)
$$v_l'' + p_l \frac{v_l'}{t} + q_l \frac{v_l}{t^2} + F_l = 0 \text{ in } G_r,$$

where

$$F_l = \partial_t^l F$$
.

We also note

$$p_l = 2l + p$$
, $q_l = l^2 + (p-1)l + q$.

Then,

(B.4)
$$p_l = 1 + 2l - (\underline{m} + \overline{m}), \quad q_l = (\underline{m} - l)(\overline{m} - l),$$

and the general solutions of the homogeneous linear equation corresponding to (B.3) are spanned by $t^{\underline{m}-l}$ and $t^{\overline{m}-l}$. A standard calculation yields the following result: Let v_l be a solution of (B.3) satisfying

(B.5)
$$t^{l-\underline{m}}v_l \to 0 \quad as \ t \to 0.$$

Then,

$$v_{l}(y',t) = \left[v_{l}(y',r)r^{l-\overline{m}} - \frac{r^{\underline{m}-\overline{m}}}{\overline{m}-\underline{m}} \int_{0}^{r} s^{l+1-\underline{m}} F_{l}(y',s) ds\right] t^{\overline{m}-l}$$

$$+ \frac{1}{\overline{m}-\underline{m}} t^{\underline{m}-l} \int_{0}^{t} s^{l+1-\underline{m}} F_{l}(y',s) ds$$

$$+ \frac{1}{\overline{m}-\underline{m}} t^{\overline{m}-l} \int_{t}^{r} s^{l+1-\overline{m}} F_{l}(y',s) ds.$$

We note that the regularity of v_l in y' inherits from that of $v_l(\cdot, r)$ and F_l , as long as the integrals in (B.6) make sense. We now rewrite (B.6) so we can discuss the regularity of v_l in t.

We first assume that $\overline{m} \geq 3$ is an integer and consider the case $l = 0, \dots, \overline{m} - 3$. In the last two integrals in (B.6), we write

(B.7)
$$F_l(y',s) = F_l(y',0) + [F_l(y',s) - F_l(y',0)].$$

Then, a simple calculation yields

(B.8)
$$v_l(y',t) = c_2(y')t^2 + c_{\overline{m}-l}(y')t^{\overline{m}-l} + t^2w_l(y',s),$$

where

$$c_{2}(y') = \frac{1}{(l+2-\underline{m})(\overline{m}-l-2)} F_{l}(y',0),$$

$$c_{\overline{m}-l}(y') = v_{l}(y',r)r^{l-n} - \frac{r^{\underline{m}-\overline{m}}}{\overline{m}-\underline{m}} \int_{0}^{r} s^{l+1-\underline{m}} F_{l}(y',s) ds + \frac{r^{l+2-\overline{m}}}{(\overline{m}-\underline{m})(l+2-\overline{m})} F_{l}(y',0),$$

and

$$w_{l}(y',t) = \frac{1}{\overline{m} - \underline{m}} t^{\underline{m}-l} \int_{0}^{t} s^{l+1-\underline{m}} [F_{l}(y',t) - F_{l}(y',0)] ds + \frac{1}{\overline{m} - \underline{m}} t^{\overline{m}-l} \int_{t}^{r} s^{l+1-\overline{m}} [F_{l}(y',t) - F_{l}(y',0)] ds.$$

By using Lemma A.2 and Lemma A.5 to analyze the regularity of w_l and its t-derivatives, we obtain the following result.

Lemma B.1. Let $\alpha \in (0,1)$ and r > 0 be constants, and $F_l \in C^{\alpha}(\bar{G}_r)$, for $l = 0, 1, \dots, \overline{m} - 3$. Suppose that $v_l \in L^{\infty}(G_r)$ is a solution of (B.3). Then,

$$v_l'', \frac{v_l'}{t}, \frac{v_l}{t^2} \in C^{\alpha}(\bar{G}_r).$$

We now consider $l = \overline{m} - 2$. Similarly, we decompose $F_{\overline{m}-2}$ according to (B.7). When we calculate terms involving $F_{\overline{m}-2}(y',0)$, we note that a logarithmic term in t appears. Next, for the integral

$$\frac{1}{\overline{m} - \underline{m}} t^2 \int_t^r s^{-1} \big[F_{\overline{m} - 2}(y', s) - F_{\overline{m} - 2}(y', 0) \big] ds,$$

we write $\int_t^r = \int_0^r - \int_0^t$. Then, instead of (B.8), we have

(B.9)
$$v_{\overline{m}-2}(y',t) = c_2(y')t^2 + c_{2,1}(y')t^2 \log t + t^2 w_{\overline{m}-2}(y',t),$$

where

$$c_{2}(y') = v_{\overline{m}-2}(y',r)r^{-2} - \frac{r^{\underline{m}-\overline{m}}}{\overline{m}-\underline{m}} \int_{0}^{r} s^{\overline{m}-\underline{m}-1} F_{\overline{m}-2}(y',s) ds$$

$$+ \frac{1}{(\overline{m}-\underline{m})^{2}} F_{\overline{m}-2}(y',0) + \frac{\log r}{\overline{m}-\underline{m}} F_{\overline{m}-2}(y',0)$$

$$+ \frac{1}{\overline{m}-\underline{m}} \int_{0}^{r} s^{-1} \left[F_{\overline{m}-2}(y',s) - F_{\overline{m}-2}(y',0) \right] ds,$$

$$c_{2,1}(y') = -\frac{1}{\overline{m}-m} F_{\overline{m}-2}(y',0),$$

and

$$w_{\overline{m}-2}(y',t) = \frac{1}{\overline{m} - \underline{m}} t^{\underline{m} - \overline{m}} \int_0^t s^{\overline{m} - \underline{m} - 1} \left[F_{\overline{m}-2}(y',s) - F_{\overline{m}-2}(y',0) \right] ds$$
$$- \frac{1}{\overline{m} - \underline{m}} \int_0^t s^{-1} \left[F_{\overline{m}-2}(y',s) - F_{\overline{m}-2}(y',0) \right] ds.$$

To finish this appendix, we derive one more formula. By regarding (B.2) as a first order ODE of v_{l-1} , we obtain

(B.10)
$$\frac{v_{l-1}(y',t)}{t^2} = \frac{v_{l-1}(y',r)}{r^2} - \int_t^r \frac{v_l(y',s)}{s^2} ds.$$

If

(B.11)
$$\int_0^r \frac{v_l(y',s)}{s^2} ds < \infty,$$

we can also write it as

(B.12)
$$\frac{v_{l-1}(y',t)}{t^2} = \left[\frac{v_{l-1}(y',r)}{r^2} - \int_0^r \frac{v_l(y',s)}{s^2} ds \right] + \int_0^t \frac{v_l(y',s)}{s^2} ds.$$

The simple formula (B.10) or (B.12) plays an important role. In the application, we fix a positive integer k. By taking $l = k, k - 1, \dots, 1$ successively, we then obtain an expression of $v_0 = v$ in terms of v_k in the form of multiple integrals. Then, a low degree of regularity of v_k will yield a high degree of regularity of v_k . This is not surprising since v_k is essentially the k-th derivative of v_k . An extra term in (B.2) is introduced so that the equation for v_k has a similar form as that for v_0 .

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