Symmetry and nonexistence of positive solutions for fractional systems

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Abstract

This paper is devoted to study the nonexistence of positive solutions for the following fractional Hénon system

$$\begin{cases} (-\triangle)^{\alpha/2}u = |x|^a v^p, & x \in R^n, \\ (-\triangle)^{\alpha/2}v = |x|^b u^q, & x \in R^n, \\ u \geq 0, v \geq 0, \end{cases}$$

where $0 < \alpha < 2$, $1 \leq p, q < \infty$, $a, b \geq 0$, $n \geq 2$. Using a direct method of moving planes, we prove the non-existence of positive solution in the subcritical case.

Key words: The method of moving planes, Fractional Laplacian, Non-linear elliptic system.

1 Introduction

We study the following system involving the fractional Laplacian

$$\begin{cases} (-\triangle)^{\alpha/2}u = |x|^a v^p, \ x \in R^n \\ (-\triangle)^{\alpha/2}v = |x|^b u^q, \ x \in R^n \\ u \geq 0, v \geq 0, \ x \in R^n, \end{cases} \tag{1.1}$$

where $0 < \alpha < 2$, $1 \leq p, q < \infty$, $a, b \geq 0$, $n \geq 2$.

The fractional Laplacian in R^n is a nonlocal pseudo-differential operator taking the form

$$(-\triangle)^{\alpha/2}u(x) = C_{n,\alpha}PV \int_{R^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy = C_{n,\alpha}\lim_{\varepsilon \to 0}\int_{R^n \backslash B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy, \tag{1.2}$$

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where $C_{n,\alpha}$ is a normalization constant. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in R^n. In this space, it can also be equivalently defined in terms of the Fourier transform

$$\mathcal{F}[(-\triangle)^{\alpha/2}u](\xi) = |\xi|^\alpha \mathcal{F}u(\xi),$$

where $\mathcal{F}u$ is the Fourier transform of u. One can extend this operator to a wider space of distributions:

$$\mathcal{L}_\alpha = \{u : R^n \to R \mid \int_{R^n} \frac{|u(x)|}{1+|x|^{n+\alpha}}dx < \infty\}.$$

Then in this space, we defined $(-\triangle)^{\alpha/2}u$ as a distribution by

$$\langle (-\triangle)^{\alpha/2}u(x), \phi \rangle = \int_{R^n} u(x)(-\triangle)^{\alpha/2}\phi(x)dx, \ \forall \phi \in C_0^\infty(R^n).$$

In our paper, we use the direct method of moving planes introduced by [1] to derive the symmetry of positive solutions and then deduce the nonexistence of positive solutions under explain how this paper improved previous result by listing their conditions [8]. The following is our main theorems.

Theorem 1.1. *Let $(u,v) \in L_\alpha \cap L_{loc}^\infty$ be a pair of nonnegative solution to (1.1) and $1 < p < \frac{n+\alpha+a}{n-\alpha}$, $1 < q < \frac{n+\alpha+b}{n-\alpha}$. Then u and v are radially symmetric and decreasing about the origin.*

Theorem 1.2. *Assume $a + \alpha > 0$, $b + \alpha > 0$ and $1 \le p, q < \infty$, let $(u,v) \in L_\alpha \cap L_{loc}^\infty$ be a pair of nonnegative solution to (1.1), then u and v also satisfy*

$$\begin{cases} u(x) = C_0 \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}}dy, \\ v(x) = C_1 \int_{R^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}}dy, \end{cases} \tag{1.3}$$

and vice versa, where C_0 and C_1 are positive constants.

Using the above theorems, we can show the following nonexistence of positive solutions to (1.1) or (1.3).

Theorem 1.3. *Let $(u,v) \in L_\alpha \cap L_{loc}^\infty$ be a pair of non-negative solution to (1.1) or (1.3), $1 < p < \frac{n+\alpha+a}{n-\alpha}$ and $1 < q < \frac{n+\alpha+b}{n-\alpha}$. Then $(u,v) = (0,0)$.*

As a consequence of the proof in Theorem 1.3, we obtain the following nonexistence result immediately.

Theorem 1.4. *Let $(u, v) \in L_\alpha \cap L^\infty_{loc}$ be a pair of non-negative solution to (1.1) or (1.3). If*

$$\int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy < \infty \ and \ \int_{R^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} dy < \infty$$

with $\frac{n+b}{q+1} + \frac{n+a}{p+1} \neq n - \alpha$. Then $(u, v) = (0, 0)$.

Remark 1.1. *In [8], the authors applied the method of moving planes in integral forms to study the symmetry of positive solutions for the fractional system (1.1), then derived the nonexistence of positive solutions. However, due to technical restrictions to apply the integral method, the authors have to assume that $u \in L^\beta_{loc}(R^n)$ and $v \in L^\gamma_{loc}(R^n)$ where $\beta = \frac{n(q-1)}{(n-\alpha)q-b-n}$ and $\gamma = \frac{n(p-1)}{(n-\alpha)p-a-n}$. In this paper, we manage to derive the same nonexistence result without imposing extra integrability conditions, by using a direct application of the moving of the moving planes to the differential equations.*

In recent years, the fractional Hénon-type problem has received a lot of attention. In [15], the authors considered the integral equation

$$u(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}|y|^s} dy,$$

where $\frac{n-s}{n-\alpha} < p < \alpha^*(s) - 1$ with $\alpha^*(s) = \frac{2(n-s)}{n-\alpha}$. They proved the nonexistence of positive solutions for the equation by the method of moving planes in integral forms and established the equivalence between the above integral equation and the following partial differential equation

$$(-\triangle)^{\frac{\alpha}{2}} u(x) = |x|^{-s} u^p.$$

In [14], the authors studied the following weighted system of partial differential equations

$$\begin{cases} (-\triangle)^{\alpha/2} u = |x|^{-s} v^p, & in \quad R^n, \\ (-\triangle)^{\alpha/2} v = |x|^{-t} u^q, & in \quad R^n, \\ u \geq 0, v \geq 0. \end{cases}$$

They first established the equivalence between the differential system and an integral system

$$\begin{cases} u(x) = \int_{R^n} \frac{v^p(y)}{|x-y|^{n-\alpha}|y|^s} dy, \\ v(x) = \int_{R^n} \frac{u^q(y)}{|x-y|^{n-\alpha}|y|^t} dy. \end{cases}$$

Then, in the critical case $\frac{n-s}{p+1} + \frac{n-t}{q+1} = n - \alpha$, they showed that every pair of positive solutions $(u(x), v(x))$ is radially symmetric about the origin. While in the subcritical case, they proved the nonexistence of positive solutions.

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In [21], the authors considered the following fractional Laplacian equation

$$
\begin{cases}
(-\triangle)^{\alpha/2} u(x) = 0, \ in \ R^n \\
u(x) \geq 0, \ in \ R^n,
\end{cases}
$$

where $n \geq 2$ and α is any real number between 0 and 2. They proved that the only solution is constant. As an application, they obtained an equivalence between a semi-linear differential equation

$$
(-\triangle)^{\alpha/2} u(x) = u^p(x), \ x \in R^n,
$$

and the corresponding integral equation

$$
u(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy, \ x \in R^n.
$$

For more similar results, please see [3], [6], [9] and the references therein.

The paper is organized as follows. In Section 2, we use the method of moving planes to prove Theorem 1.1. In Section 3, we establish the equivalence between problem (1.1) and integral system (1.3). Finally, in Section 4 we complete the proof of Theorem 1.3 and 1.4.

2   The Symmetry of Positive Solutions

Without any decay conditions on u and v , we are not able to carry the method of moving planes on u and v directly. To circumvent this difficulty, we make a Kelvin transform. Let

$$
\begin{cases}
\overline{u}(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2}), \\
\overline{v}(x) = \frac{1}{|x|^{n-\alpha}} v(\frac{x}{|x|^2}).
\end{cases}
$$

Then

$$
\begin{aligned}
(-\triangle)^{\alpha/2}\overline{u}(x) &= \frac{1}{|x|^{n+\alpha}}(-\triangle)^{\alpha/2} u(\frac{x}{|x|^2}), \\
&= \frac{1}{|x|^{n+\alpha+a}} v^p(\frac{x}{|x|^2}), \\
&= \frac{1}{|x|^{n+\alpha+a-p(n-\alpha)}} \overline{v}^p(x).
\end{aligned}
$$

In a similar way, we have

$$
(-\triangle)^{\alpha/2}\overline{v}(x) = \frac{1}{|x|^{n+\alpha+b-q(n-\alpha)}} \overline{u}^q(x).
$$

Let $\gamma = n + \alpha + a - p(n - \alpha)$, $\beta = n + \alpha + b - q(n - \alpha)$, and (1.1) becomes

$$
\begin{cases}
(-\triangle)^{\alpha/2}\overline{u} = |x|^{-\gamma}\overline{v}^p, \\
(-\triangle)^{\alpha/2}\overline{v} = |x|^{-\beta}\overline{u}^q, \\
\overline{u} \geq 0, \overline{v} \geq 0.
\end{cases}
\tag{2.4}
$$

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We first give some basic notations before starting moving the planes. Then we start moving planes on system (2.4).

Let

$$T_\lambda = \{x \in R^n \mid x_1 = \lambda, \lambda \in R\}$$

be the moving plane,

$$\Sigma_\lambda = \{x \in R^n \mid x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of the point $x = (x_1, x_2, \cdots, x_n)$ about the plane T_λ.

Assume that $(\overline{u}, \overline{v})$ solves the fractional system (2.4). To compare the values of $\overline{u}(x)$ with $\overline{u}(x^\lambda)$ and $\overline{v}(x)$ with $\overline{v}(x^\lambda)$, we denote

$$\begin{cases} U_\lambda(x) = \overline{u}(x^\lambda) - \overline{u}(x), \\ V_\lambda(x) = \overline{v}(x^\lambda) - \overline{v}(x). \end{cases}$$

Then system (2.4) becomes

$$\begin{cases} (-\triangle)^{\alpha/2} U_\lambda(x) = \frac{1}{|x^\lambda|^\gamma} \overline{v}^p(x^\lambda) - \frac{1}{|x|^\gamma} \overline{v}^p(x), \\ (-\triangle)^{\alpha/2} V_\lambda(x) = \frac{1}{|x^\lambda|^\beta} \overline{u}^q(x^\lambda) - \frac{1}{|x|^\beta} \overline{u}^q(x). \end{cases} \tag{2.5}$$

Before starting moving planes, we need some lemmas in [1].

Lemma 2.1. (**Narrow region principle**[1]) *Let Ω be a bounded narrow region in Σ_λ, such that it is contained in $\{x | \lambda - l < x_1 < \lambda\}$ with small l. Suppose that $\varphi \in L_\alpha \bigcap C_{loc}^{1,1}(\Omega)$ and is lower semi-continuous on $\overline{\Omega}$. If $C(x)$ is bounded from below in Ω, then*

$$\begin{cases} (-\triangle)^{\alpha/2}\varphi(x) + C(x)\varphi(x) \geq 0 & in \quad \Omega, \\ \varphi(x) \geq 0 & in \quad \Sigma_\lambda \backslash \Omega, \\ \varphi(x^\lambda) = -\varphi(x) & in \quad \Sigma_\lambda, \end{cases} \tag{2.6}$$

then for sufficiently small δ, we have

$$\varphi(x) \geq 0 \quad in \quad \Omega. \tag{2.7}$$

Furthermore, if $\varphi = 0$ at some point in Ω, then

$$\varphi(x) = 0 \quad almost \ everywhere \ in \quad R_n. \tag{2.8}$$

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These conclusions hold for unbounded region Ω if we further assume that

$$\underline{\lim}_{|x|\to\infty}\varphi(x) \geq 0. \tag{2.9}$$

Lemma 2.2. (**Decay at infinity**[1]) *Let Ω be an unbounded region in Σ_λ. Assume $\varphi \in L_\alpha \cap C_{loc}^{1,1}(\Omega)$ is a solution of*

$$\begin{cases} (-\triangle)^{\alpha/2}\varphi(x) + C(x)\varphi(x) \geq 0 & in \quad \Omega, \\ \varphi(x) \geq 0 & in \quad \Sigma_\lambda\backslash\Omega, \\ \varphi(x^\lambda) = -\varphi(x) & in \quad \Sigma_\lambda, \end{cases} \tag{2.10}$$

with

$$\underline{\lim}_{|x|\to\infty}|x|^\alpha C(x) \geq 0, \tag{2.11}$$

then there exists a constant $R_0 > 0$ such that if

$$\varphi(x_0) = \min_\Omega \varphi(x) < 0, \tag{2.12}$$

then

$$|x_0| \leq R_0. \tag{2.13}$$

2.1 Proof of Theorem 1.1

Now,we start moving planes.

Step.1: we show that when λ sufficiently negative,

$$U_\lambda(x), V_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}. \tag{2.14}$$

Let $W_\lambda(x) = U_\lambda(x) + V_\lambda(x)$, it follows from(2.5)

$$\begin{aligned} (-\triangle)^{\alpha/2}W_\lambda(x) &= (-\triangle)^{\alpha/2}(U_\lambda(x) - V_\lambda(x)), \\ &= \frac{1}{|x^\lambda|^\gamma}\overline{v}^p(x^\lambda) - \frac{1}{|x|^\gamma}\overline{v}^p(x) + \frac{1}{|x^\lambda|^\beta}\overline{u}^q(x^\lambda) - \frac{1}{|x|^\beta}\overline{u}^q(x). \end{aligned}$$

We claim that $W_\lambda(x) \geq 0$. If not there exists $x_0 \in \Sigma_\lambda \setminus \{0^\lambda\}$, such that $W_\lambda(x_0) = \min_{\Sigma_\lambda\setminus\{0^\lambda\}} W_\lambda(x) < 0$. This is guaranteed by the fact that, for ϵ sufficiently small and λ sufficiently negative, it holds that

$$U_\lambda(x), V_\lambda(x) \geq C > 0, x \in B_\varepsilon(0^\lambda)\backslash\{0^\lambda\},$$

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we will prove it in the Appendices. Without loss of generality, we suppose that $U_\lambda(x_0) < 0$, then we have $V_\lambda(x_0) < 0$.

To see this, we suppose that $V_\lambda(x_0) \geq 0$. It follows from $U_\lambda(x_0) < 0$ that there exists x_1, such that $U_\lambda(x_1) = \min_{\sum_\lambda \setminus \{0^\lambda\}} U_\lambda(x) < 0$. Thus

$$
\begin{aligned}
(-\triangle)^{\alpha/2} U_\lambda(x_1) &= C_{n,\alpha} PV \int_{R^n} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y|^{n+\alpha}} dy, \\
&= C_{n,\alpha} PV \int_{\sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{R^n \setminus \sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y|^{n+\alpha}} dy, \\
&= C_{n,\alpha} PV \int_{\sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{\sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y^\lambda)}{|x_1 - y^\lambda|^{n+\alpha}} dy, \\
&= C_{n,\alpha} PV \int_{\sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{\sum_\lambda} \frac{U_\lambda(x_1) + U_\lambda(y)}{|x_1 - y^\lambda|^{n+\alpha}} dy, \\
&\leq C_{n,\alpha} \int_{\sum_\lambda} \frac{U_\lambda(x_1) - U_\lambda(y)}{|x_1 - y^\lambda|^{n+\alpha}} dy + \frac{U_\lambda(x_1) + U_\lambda(y)}{|x_1 - y^\lambda|^{n+\alpha}} dy, \\
&= C_{n,\alpha} \int_{\sum_\lambda} \frac{2 U_\lambda(x_1)}{|x_1 - y^\lambda|^{n+\alpha}} dy. \\
&< 0. \tag{2.15}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(-\triangle)^{\alpha/2} U_\lambda(x_1) &= \frac{1}{|x_1^\lambda|^\gamma} \overline{v}^p(x_1^\lambda) - \frac{1}{|x_1|^\gamma} \overline{v}^p(x_1), \\
&= \frac{1}{|x_1^\lambda|^\gamma} \overline{v}^p(x_1^\lambda) - \frac{1}{|x_1|^\gamma} \overline{v}^p(x_1^\lambda) + \frac{1}{|x_1|^\gamma} \overline{v}^p(x_1^\lambda) - \frac{1}{|x_1|^\gamma} \overline{v}^p(x_1), \\
&= \frac{1}{|x_1|^\gamma} (\overline{v}^p(x_1^\lambda) - \overline{v}^p(x_1)) + \overline{v}^p(x_1^\lambda)(\frac{1}{|x_1^\lambda|^\gamma} - \frac{1}{|x_1|^\gamma}), \\
&\geq \frac{1}{|x_1|^\gamma} (\overline{v}^p(x_1^\lambda) - \overline{v}^p(x_1)). \tag{2.16}
\end{aligned}
$$

By (2.15) and (2.16), we can deduce that

$$
V_\lambda(x_1) < 0.
$$

Then

$$
W_\lambda(x_1) < W_\lambda(x_0),
$$

which is a contradiction. This proves that $V_\lambda(x_0) < 0$.

Notice that

$$\begin{aligned}
(-\triangle)^{\alpha/2}W_\lambda(x_0) \;\geq\; & \frac{1}{|x_0|^\gamma}(\overline{v}^p(x_0^\lambda) - \overline{v}^p(x_0)) + \frac{1}{|x_0|^\beta}(\overline{u}^q(x_0^\lambda) - \overline{u}^q(x_0)), \\
\;=\; & \frac{1}{|x_0|^\gamma}p\xi^{p-1}(\overline{v}(x_0^\lambda) - \overline{v}(x_0)) + \frac{1}{|x_0|^\beta}q\eta^{q-1}(\overline{u}(x_0^\lambda) - \overline{u}(x_0)) \\
\;\geq\; & \frac{1}{|x_0|^\gamma}p\overline{v}^{p-1}(x_0)(\overline{v}(x_0^\lambda) - \overline{v}(x_0)) + \frac{1}{|x_0|^\beta}q\overline{u}^{q-1}(x_0)(\overline{u}(x_0^\lambda) - \overline{u}(x_0)) \\
\;\geq\; & \frac{1}{|x_0|^\gamma}p\overline{v}^{p-1}(x_0)\mathrm{V}_\lambda(x_0) + \frac{1}{|x_0|^\beta}q\overline{u}^{q-1}(x_0)\mathrm{U}_\lambda(x_0)
\end{aligned}$$

where $\xi \in (\overline{v}(x_0^\lambda), \overline{v}(x_0))$, $\eta \in (\overline{u}(x_0^\lambda), \overline{u}(x_0))$. Let $C_1 = \frac{1}{|x_0|^\gamma}p\overline{v}^{p-1}(x_0)$, $C_2 = \frac{1}{|x_0|^\beta}q\overline{u}^{q-1}(x_0)$, $C = \max\{C_1, C_2\}$, then

$$(-\triangle)^{\alpha/2}W_\lambda(x_0) - CW_\lambda(x_0) \geq 0. \tag{2.17}$$

Similar to(2.15), we have

$$(-\triangle)^{\alpha/2}W_\lambda(x_0) - CW_\lambda(x_0) \leq W_\lambda(x_0)\left(\int_{\sum_\lambda} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}}dy - C(x_0)\right) < 0. \tag{2.18}$$

Indeed,

$$\begin{aligned}
\int_{\sum_\lambda} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}}dy \;\geq\; & \int_{\{x_1 \geq 0\}} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}}dy \\
\;=\; & \frac{1}{2}\int_{R^n} \frac{1}{(|x_0| + |y^\lambda|)^{n+\alpha}}dy \\
\;=\; & \frac{1}{2}\int_0^\infty \int_{B_r^0} \frac{1}{(|x_0| + |r|)^{n+\alpha}}d\sigma dr \\
\;=\; & \frac{1}{2}\int_0^\infty \frac{w_{n-1}r^{n-1}}{(|x_0| + |r|)^{n+\alpha}}dr \\
\;=\; & \frac{w_{n-1}}{2|x_0|^\alpha}\int_0^\infty \frac{t^{n-1}}{(1+t)^{n+\alpha}}dr, r = t|x_2| \\
\;\sim\; & \frac{C_3}{|x_0|^\alpha}, \tag{2.19}
\end{aligned}$$

and

$$\begin{aligned}
C_1 \;=\; & \frac{1}{|x_0|^\gamma}p\overline{v}^{p-1}(x_0) \\
\;\sim\; & p\frac{1}{|x_0|^\gamma}\left(\frac{1}{|x_0|^{n-\alpha}}\right)^{p-1} \\
\;\sim\; & \frac{1}{|x_0|^{\gamma+(p-1)(n-\alpha)}},
\end{aligned}$$

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$$C_2 = \frac{1}{|x_0|^\beta} q\overline{u}^{q-1}(x_0)$$

$$\sim q\frac{1}{|x_0|^\beta}(\frac{1}{|x_0|^{n-\alpha}})^{q-1}$$

$$\sim \frac{1}{|x_0|^{\beta+(q-1)(n-\alpha)}}.$$

Note that $\gamma + (p-1)(n-\alpha) = 2\alpha + a > \alpha$ and $\beta + (q-1)(n-\alpha) = 2\alpha + b > \alpha$, then

$$\int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} dy - C(x_0) > 0.$$

(2.18) is a contraction with (2.17). This shows that $W_\lambda(x) \geq 0$. In a similar process as in (2.15) and (2.16) we can prove

$$U_\lambda(x) \geq 0, \ V_\lambda(x) \geq 0.$$

Step.2: Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (2.14) holds to its limiting position.

Let

$$\lambda_0 = \sup\{\lambda \leq 0 | U_\mu \geq 0, V_\mu \geq 0, \forall x \in \Sigma_\mu \setminus \{0^\lambda\}, \mu \leq \lambda\}$$

In this part, we show that

$$\lambda_0 = 0$$

Suppose that

$$\lambda_0 < 0$$

we show that the plane T_λ can be moved further right. To be more rigorous, there exists some $\delta > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \delta)$, we have

$$W_\lambda(x) \geq 0, \ \ x \in \Sigma_\lambda \setminus \{0^\lambda\}. \tag{2.20}$$

This is a contradiction with the definition of λ_0. In fact, when $\lambda_0 < 0$, we have

$$W_{\lambda_0}(x) > 0, \ \ x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}. \tag{2.21}$$

If not, there exists some $\hat{x}$ such that

$$W_{\lambda_0}(\hat{x}) = \min_{x \in \Sigma_{\lambda_0}\setminus\{0^\lambda\}} W_{\lambda_0}(x) = 0, i.e. U_{\lambda_0}(\hat{x}) = 0, V_{\lambda_0}(\hat{x}) = 0.$$

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It follows that

$$
\begin{aligned}
(-\triangle)^{\alpha/2} U_{\lambda_0}(\hat{x}) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-U_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-U_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y^\lambda)}{|\hat{x}-y^\lambda|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\hat{x}-y|^{n+\alpha}} dy + C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{U_{\lambda_0}(y)}{|\hat{x}-y^\lambda|^{n+\alpha}} dy \\
&= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|\hat{x}-y^\lambda|^{n+\alpha}} - \frac{1}{|\hat{x}-y|^{n+\alpha}} \right) U_{\lambda_0}(y) dy \\
&\leq 0. \tag{2.22}
\end{aligned}
$$

On the other hand

$$
(-\triangle)^{\alpha/2} U_{\lambda_0}(\hat{x}) = \frac{1}{|\hat{x}^\lambda|^\gamma} \overline{v}^p(\hat{x}^\lambda) - \frac{1}{|\hat{x}|^\gamma} \overline{v}^p(\hat{x}) = \frac{1}{|\hat{x}^\lambda|^\gamma} \overline{v}^p(\hat{x}) - \frac{1}{|\hat{x}|^\gamma} \overline{v}^p(\hat{x}) > 0 \tag{2.23}
$$

A contradiction with (2.22), so does V_{λ_0}. This proves (2.21). We claim that for $\lambda_0 < 0$ and $\varepsilon > 0$ sufficiently small,

$$
U_{\lambda_0}(x), V_{\lambda_0}(x) \geq C > 0, x \in B_\varepsilon(0^{\lambda_0}) \setminus \{0^{\lambda_0}\},
$$

the proof will be given in the appendix. It follows from (2.21) that there exists a constant $C_0 > 0$, such that

$$
W_{\lambda_0}(x) \geq C_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0-\delta} \cap B_R(0)} \setminus \{0^{\lambda_0}\}.
$$

Then

$$
W_\lambda(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_0-\delta} \cap B_R(0) \setminus \{0^\lambda\}. \tag{2.24}
$$

From decay at infinity, we have

$$
W_\lambda(x) \geq 0, \quad \forall x \in B_R^c. \tag{2.25}
$$

From narrow region principle,

$$
W_\lambda(x) \geq 0, \forall x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}) \setminus \{0^\lambda\}. \tag{2.26}
$$

To see this, in Lemma 2.1, we let, for any sufficiently small $\eta > 0$, $H = \Sigma_\lambda \setminus B_\eta(0^\lambda)$ and the narrow region $\Omega = (\Sigma_\lambda^- \setminus \Sigma_{\lambda_0-\delta}) \setminus B_\eta(0^\lambda)$, while the lower bound of $C(x)$ can be seen from (2.19).

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Combining (2.24) (2.26) and (2.25), we conclude that

$$W_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \backslash \{0^\lambda\}.$$

This contradicts the definition of λ_0. Therefore, we must have $\lambda_0 = 0$ and $W_{\lambda_0} \geq 0, \forall x \in \Sigma_\lambda$. Similarly, one can move the plane T_λ from the $+\infty$ to the left and show that $W_{\lambda_0} \leq 0, \forall x \in \Sigma_\lambda$. Now we have shown that

$$\lambda_0 = 0, W_{\lambda_0} \equiv 0, \forall x \in \Sigma_\lambda. \tag{2.27}$$

2.2 Proof of theorem 1.1

Proof. So far, we have proved that $\overline{u}, \overline{v}$ is symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that $\overline{u}, \overline{v}$ is radially symmetric about 0. Let $x_1 \ x_2$ be any points centered at 0, i.e.,

$$0 = \frac{x_1 + x_2}{2}, |x_1| = |x_2|.$$

Then,

$$\overline{u}(x_1) = \overline{u}(x_2), \overline{v}(x_1) = \overline{v}(x_2).$$

Let

$$y_1 = \frac{x_1}{|x_1|^2}, y_2 = \frac{x_2}{|x_2|^2},$$

then

$$\frac{y_1 + y_2}{2} = 0.$$

Hence,

$$u(y_1) = u(\frac{x_1}{|x_1|^2}) = |x_1|^{n-\alpha}\overline{u}(x_1) = |x_2|^{n-\alpha}\overline{u}(x_2) = u(y_2),$$

$$v(y_1) = u(\frac{x_1}{|x_1|^2}) = |x_2|^{n-\alpha}\overline{v}(x_1) = |x_2|^{n-\alpha}\overline{v}(x_2) = v(y_2).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 The equivalence between problem (1.1) and the integral form (1.3)

In this section, we prove the equivalence between problem (1.1) and (1.3).

Proof of Theorem 1.2 Let (u, v) be a pair of positive solution to (1.1), we first show that

$$
\begin{cases}
u(x) = c_1 + \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy, \\
v(x) = c_2 + \int_{R^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} dy.
\end{cases}
\tag{3.28}
$$

Let

$$
\begin{cases}
u_R(x) = \int_{B_R(0)} G_R(x, y) |y|^a v^p(y) dy, \\
v_R(x) = \int_{B_R(0)} G_R(x, y) |y|^b u^q(y) dy,
\end{cases}
\tag{3.29}
$$

where $G_R(x, y)$ is the Green function of fractional Laplacian on $B_R(0)$.

It is easy to see that

$$
\begin{cases}
(-\triangle)^{\alpha/2} u_R(x) = |x|^a v^p(x), \ in \ B_R(0), \\
(-\triangle)^{\alpha/2} v_R(x) = |x|^b u^q(x), \ in \ B_R(0), \\
u_R(x) = v_R(x) = 0, \quad on \ B_R^c(0).
\end{cases}
\tag{3.30}
$$

Let $\varphi_R(x) = u(x) - u_R(x)$ and $\psi_R(x) = v(x) - v_R(x)$. From (1.1) and (3.30), we have

$$
\begin{cases}
(-\triangle)^{\alpha/2} \varphi_R(x) = 0, \ in \ B_R(0), \\
(-\triangle)^{\alpha/2} \phi_R(x) = 0, \ in \ B_R(0), \\
\varphi_R(x), \phi_R(x) \geq 0, \quad on \ B_R^c(0).
\end{cases}
$$

By the Maximum Principle, we derive

$$
\varphi_R(x), \phi_R(x) \geq 0, \ x \in R^n.
\tag{3.31}
$$

Therefore, when $R \to \infty$,

$$
\begin{cases}
u_R(x) \to \tilde{u}(x) = \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy, \\
v_R(x) \to \tilde{v}(x) = \int_{R^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} dy,
\end{cases}
\tag{3.32}
$$

Moreover,

$$
\begin{cases}
(-\triangle)^{\alpha/2} \tilde{u}(x) = |y|^a v^p(y), \ x \in R^n, \\
(-\triangle)^{\alpha/2} \tilde{v}(x) = |y|^b u^q(y), \ x \in R^n.
\end{cases}
\tag{3.33}
$$

Now let $\Phi(x) = u(x) - \tilde{u}(x)$ and $\Psi(x) = v(x) - \tilde{v}(x)$. From (1.1) and (3.33), we have

$$
\begin{cases}
(-\triangle)^{\alpha/2} \Phi(x) = 0, \ x \in R^n, \\
(-\triangle)^{\alpha/2} \Psi(x) = 0, \ x \in R^n, \\
\Phi(x), \Psi(x) \geq 0, \ x \in R^n.
\end{cases}
$$

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From Proposition 2 in [21], we have

$$\Phi(x) = c_1, \Psi(x) = c_2.$$

Thus we proved (3.28).

Next, we will show that $c_1 = c_2 = 0$. Without lose of generality, we may assume that $c_2 > 0$, then from (3.28), we have

$$u(x) = c_1 + \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy \geq c_1 + \int_{R^n} \frac{c_2 |y|^a}{|x-y|^{n-\alpha}} dy = \infty.$$

But it is impossible, hence $c_1 = c_2 = 0$. Therefore

$$\begin{cases} u(x) = \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy, \\ v(x) = \int_{R^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} dy. \end{cases}$$

We complete our proof.

4 The nonexistence of positive solutions

In this section, we prove Theorem 1.3 and Theorem 1.4. First we need some Lemmas.

Lemma 4.1. *Assume (u, v) is a pair of positive radial solution for (1.1) then for $r = |x| > 0$ it holds*

$$u(r) \leq C r^{-\frac{(b+\alpha)p+a+\alpha}{pq-1}}, \tag{4.34}$$

$$v(r) \leq C r^{-\frac{(a+\alpha)q+b+\alpha}{pq-1}}. \tag{4.35}$$

Proof. From (1.3) and the decreasing property of radial solution, we have

$$\begin{aligned} u(r) &= \int_{R^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy \\ &\geq \int_{B_r(0)} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy \\ &\geq \int_{B_r(0)} \frac{|y|^a v^p(r)}{|x-y|^{n-\alpha}} dy \\ &\geq |x|^n v^p(r) \int_{B_1(0)} \frac{|z|^a |x|^a}{|x-z|x||^{n-\alpha}} dy \\ &= C v^p(r) |x|^{a+\alpha} \\ &= C v^p(r) r^{a+\alpha}. \tag{4.36} \end{aligned}$$

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Similarly, we have

$$v(r) \geq Cu^q(r)r^{b+\alpha}. \qquad (4.37)$$

Combining (4.36) and (4.37), it gives

$$
\begin{aligned}
u(r) &\geq C[u^q(r)r^{b+\alpha}]^p r^{a+\alpha} \\
&= Cu^{pq}r^{(b+\alpha)p+a+\alpha},
\end{aligned}
$$

and

$$
\begin{aligned}
v(r) &\geq C[v^p(r)r^{a+\alpha}]^q r^{b+\alpha} \\
&= Cv^{pq}r^{(a+\alpha)q+b+\alpha}.
\end{aligned}
$$

(4.34) and (4.35) follow immediately from the above inequalities. $\qquad \square$

Lemma 4.2. *Assume (u,v) is a pair of positive radial solution for (1.1), then it holds*

$$\int_{R^n} |x|^b u^{q+1} dx < \infty, \quad and \quad \int_{R^n} |x|^b v^{q+1}(x \cdot \nabla v(x))dx < \infty, \qquad (4.38)$$

$$\int_{R^n} |x|^a v^{p+1} dx < \infty, \quad and \quad \int_{R^n} |x|^a u^{p+1}(x \cdot \nabla u(x))dx < \infty. \qquad (4.39)$$

Proof. For any $R > 0$, we need to show

$$\int_{B_R(0)} |x|^b v^{q+1} dx < \infty, \ as \ R \to \infty,$$

$$\int_{B_R(0)} |x|^a u^{p+1} dx < \infty, \ as \ R \to \infty.$$

Here we only show $\int_{B_R(0)} |x|^b v^{q+1} dx < \infty, R \to \infty$, the other can be proved in a similar way. The integral will converge only when

$$|x|^b u^{q+1} \sim o(\frac{1}{|x|^n}) \ for \ |x| \ large.$$

Since

$$|x|^b u^{q+1} \leq |x|^b (|x|^{-\frac{(b+\alpha)p+a+\alpha}{pq-1}})^{q+1},$$

it is sufficient to show that

$$\frac{(b+\alpha)p+a+\alpha}{pq-1}(q+1) - b > n. \qquad (4.40)$$

Or

$$p(n-\alpha)(q - \frac{b+\alpha}{n-\alpha}) \leq (n+b) + (a+\alpha)(q+1). \qquad (4.41)$$

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Case i: if $q \leq \frac{b+\alpha}{n-\alpha}$, (4.41) is automatically true.

Case ii: if $q \geq \frac{b+\alpha}{n-\alpha}$, then (4.41) becomes

$$\frac{p(n-\alpha)}{a+\alpha} < \frac{q + \frac{n+b}{a+\alpha} + 1}{q - \frac{b+\alpha}{n-\alpha}} = 1 + \frac{\frac{n+b}{a+\alpha} + \frac{b+\alpha}{n-\alpha} + 1}{q - \frac{b+\alpha}{n-\alpha}}. \tag{4.42}$$

Notice that $1 < p < \frac{n+\alpha+a}{n-\alpha}$ and $1 < q < \frac{n+\alpha+b}{n-\alpha}$,

$$LHS \quad of \quad (4.42) \leq \frac{n+\alpha+a}{n-\alpha} \cdot \frac{n-\alpha}{a+\alpha} = 1 + \frac{n}{a+\alpha}.$$

And

$$
\begin{aligned}
RHS \quad of \quad (4.42) \quad & \geq \quad 1 + \frac{1 + \frac{n+b}{a+\alpha} + \frac{b+\alpha}{n-\alpha}}{\frac{n+b+\alpha}{n-\alpha} - \frac{b+\alpha}{n-\alpha}} \\
& = \quad 1 + \frac{1 + \frac{n+b}{a+\alpha} + \frac{b+\alpha}{n-\alpha}}{\frac{n}{n-\alpha}} \\
& = \quad 1 + \frac{n+b}{a+\alpha} \cdot \frac{n-\alpha}{n} + \frac{n-\alpha}{n} + \frac{b+\alpha}{n} \\
& = \quad 1 + \frac{n+b}{n} \cdot \frac{n+a}{a+\alpha}.
\end{aligned}
$$

(4.42) is true as long as

$$\frac{n}{a+\alpha} < \frac{n+b}{n} \cdot \frac{n+a}{a+\alpha}, \tag{4.43}$$

since $a, b > 0$ and $a + \alpha > 0$. This completes the proof. $\qquad\square$

Proof of Theorem 1.3 Let $(u,v) \in L_\alpha \cap L_{loc}^\infty$ be a pair of positive solution to (1.1) or (1.3). By (1.3), we have

$$
\begin{cases}
u(kx) = \int_{R^n} \frac{|y|^a v^p(y)}{|kx-y|^{n-\alpha}} dy, \\
v(kx) = \int_{R^n} \frac{|y|^b u^q(y)}{|kx-y|^{n-\alpha}} dy.
\end{cases}
\tag{4.44}
$$

We differentiate the first equation of (4.44) with respect to k,

$$x \cdot \nabla u(kx) = (\alpha - n) \int_{R^n} \frac{x \cdot (kx-y)|y|^a v^p(y)}{|kx-y|^{n-\alpha+2}} dy, \; x \neq 0.$$

Let $k = 1$, then

$$x \cdot \nabla u(x) = (\alpha - n) \int_{R^n} \frac{x \cdot (x-y)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}} dy, \; x \neq 0. \tag{4.45}$$

Multiply both sides of (4.45) by $|x|^b u^q(x)$ and integrate on R^n, we have

$$\int_{R^n} |x|^b u^q(x)(x \cdot \nabla u(x))dx = (\alpha - n) \int_{R^n} \int_{R^n} \frac{x \cdot (x-y)|x|^b u^q(x)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}} dy dx.$$

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On the other hand, from the integration by parts formula, it follows

$$\int_{B_R(0)} |x|^b u^q(x)(x \cdot \nabla u(x))dx$$

$$= \frac{1}{q+1} \int_{B_R(0)} |x|^b (x \cdot \nabla u^{q+1}(x))dx$$

$$= -\frac{n+b}{q+1} \int_{B_R(0)} |x|^b u^{q+1}(x)dx + \frac{1}{p+1} \int_{\partial B_R(0)} R^{b+1} u^{q+1} d\sigma.$$

From Lemma 4.1, we have

$$\frac{1}{p+1} \int_{\partial B_R(0)} R^{b+1} u^{q+1} d\sigma \to 0, \ R \to \infty.$$

Hence when $R \to \infty$,

$$\int_{R^n} |x|^b u^q(x)(x \cdot \nabla u(x))dx = -\frac{n+b}{q+1} \int_{R^n} |x|^b u^{q+1}(x)dx.$$

Therefore,

$$-\frac{n+b}{q+1} \int_{R^n} |x|^b u^{q+1}(x)dx = (\alpha - n) \int_{R^n} \int_{R^n} \frac{x \cdot (x-y)|x|^b u^q(x)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}} dydx. \qquad (4.46)$$

Using the similar argument on the second equation of (4.44), we also have

$$\int_{R^n} |x|^a v^p(x)(x \cdot \nabla v(x))dx = (\alpha - n) \int_{R^n} \int_{R^n} \frac{x \cdot (x-y)|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha+2}} dydx,$$

and

$$\int_{R^n} |x|^a v^p(x)(x \cdot \nabla v(x))dx = -\frac{n+a}{p+1} \int_{R^n} |x|^a v^{p+1}(x)dx.$$

Consequently,

$$-\frac{n+a}{p+1} \int_{R^n} |x|^a v^{p+1}(x)dx = (\alpha - n) \int_{R^n} \int_{R^n} \frac{x \cdot (x-y)|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha+2}} dydx. \qquad (4.47)$$

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Adding (4.46) and (4.47) together, and using the fact $|x-y|^2 = x \cdot (x-y) + y \cdot (y-x)$, we have

$$
\begin{aligned}
& -\frac{n+a}{p+1}\int_{R^n}|x|^a v^{p+1}(x)dx - \frac{n+b}{q+1}\int_{R^n}|x|^b u^{q+1}(x)dx \\
= \ & (\alpha-n)\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^b u^q(x)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}}dydx \\
& +(\alpha-n)\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha+2}}dydx \\
= \ & \frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^b u^q(x)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{y \cdot (y-x)|y|^b u^q(y)|x|^a v^p(x)}{|x-y|^{n-\alpha+2}}dxdy \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha+2}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{y \cdot (y-x)|y|^a v^p(y)|x|^b u^q(x)}{|x-y|^{n-\alpha+2}}dxdy \\
= \ & \frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^b u^q(x)|y|^a v^p(y)}{|x-y|^{n-\alpha+2}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{y \cdot (y-x)|y|^b u^q(y)|x|^a v^p(x)}{|x-y|^{n-\alpha+2}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{x \cdot (x-y)|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha+2}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{y \cdot (y-x)|y|^a v^p(y)|x|^b u^q(x)}{|x-y|^{n-\alpha+2}}dydx \\
= \ & \frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha}}dydx \\
& +\frac{\alpha-n}{2}\int_{R^n}\int_{R^n}\frac{|y|^a v^p(y)|x|^b u^q(x)}{|x-y|^{n-\alpha}}dydx \\
= \ & \frac{\alpha-n}{2}\int_{R^n}|x|^a v^{p+1}(x)dx + \frac{\alpha-n}{2}\int_{R^n}|x|^b u^{q+1}(x)dx, \quad\quad (4.48)
\end{aligned}
$$

since

$$
\begin{aligned}
\int_{R^n}|x|^a v^{p+1}(x)dx \ &= \ \int_{R^n}\frac{|x|^a v^p(x)|y|^b u^q(y)}{|x-y|^{n-\alpha}}dydx \\
&= \ \int_{R^n}\frac{|y|^a v^p(y)|x|^b u^q(x)}{|x-y|^{n-\alpha}}dxdy \\
&= \ \int_{R^n}\frac{|y|^a v^p(y)|x|^b u^q(x)}{|x-y|^{n-\alpha}}dydx \\
&= \ \int_{R^n}|x|^b u^{q+1}(x)dx.
\end{aligned}
$$

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Then (4.48) becomes

$$(\frac{\alpha - n}{2} + \frac{n + b}{q + 1}) \int_{R^n} |x|^b u^{q+1}(x)dx + (\frac{\alpha - n}{2} + \frac{n + a}{p + 1}) \int_{R^n} |x|^a v^{p+1}(x)dx = 0.$$

That is

$$(\alpha - n + \frac{n + b}{q + 1} + \frac{n + a}{p + 1}) \int_{R^n} |x|^b u^{q+1}(x)dx = (\alpha - n + \frac{n + b}{q + 1} + \frac{n + a}{p + 1}) \int_{R^n} |x|^a v^{p+1}(x)dx = 0.$$

Because $1 < p < \frac{n+\alpha+a}{n-\alpha}$ and $1 < q < \frac{n+\alpha+b}{n-\alpha}$, thus $\frac{n+b}{q+1} + \frac{n+a}{p+1} \neq n - \alpha$ and problem (1.1) admits no positive solutions.

This completes our proof.

Proof of Theorem 1.4 Theorem 1.4 is a a direct consequence of Theorem 1.3.

5 Appendices

Lemma 5.1. *For λ negative large, there exists a constant $C > 0$, such that*

$$U_\lambda(x), V_\lambda(x) \geq C > 0, x \in B_\varepsilon(0^\lambda) \backslash \{0^\lambda\}. \tag{5.49}$$

Proof. For $x \in \Sigma_\lambda$, as $\lambda \to -\infty$, it is easy to see that

$$\overline{u}(x) \to 0. \tag{5.50}$$

To prove (5.49), it is sufficient to show

$$\overline{u}_\lambda(x) \geq C > 0, x \in B_\varepsilon(0^\lambda) \backslash \{0^\lambda\}.$$

Or equivalently,

$$\overline{u}(x) \geq C > 0, x \in B_\varepsilon(0) \backslash \{0\}.$$

Let η be a smooth cut-off function such that $\eta \in [0, 1]$ in R^n, $supp\ \eta \subset B_2$ and $\eta \equiv 1$ in B_1. Let

$$(-\triangle)^{\alpha/2}\phi(x) = \eta(x)|x|^a v^p(x).$$

Then,

$$\phi(x) = C_{n,-\alpha} \int_{R^n} \frac{\eta(y)|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy = C_{n,-\alpha} \int_{B_2(0)} \frac{\eta(y)|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy.$$

It is trivial for $|x|$ sufficiently large,

$$\phi(x) \sim \frac{1}{|x|^{n-\alpha}}. \tag{5.51}$$

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Since

$$\begin{cases} (-\triangle)^{\alpha/2}(u-\phi) \geq 0, & x \in B_R, \\ (u-\phi)(x) \geq 0, & x \in B_R^c, \end{cases} \tag{5.52}$$

by the maximum principle, we have

$$(u-\phi)(x) \geq 0, x \in B_R,$$

thus

$$(u-\phi)(x) \geq 0, x \in R^n.$$

For $|x|$ sufficiently large, from (5.51), one can see that for some constant $C > 0$,

$$u(x) \geq \frac{C}{|x|^{n-\alpha}}. \tag{5.53}$$

Hence for $|x|$ small

$$u(\frac{x}{|x|^2}) \geq C|x|^{n-\alpha},$$

and

$$\overline{u}(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2}) \geq C.$$

Together with (5.50), it yields that

$$U_\lambda(x) \geq \frac{C}{2} > 0, x \in B_\varepsilon(0^\lambda) \setminus \{0^\lambda\}. \tag{5.54}$$

Through an identical argument, one can show that (5.6) holds for $V_\lambda(x)$ as well. $\qquad \square$

Lemma 5.2. *For $\lambda_0 < 0$, if either of U_{λ_0} V_{λ_0} is not identically 0, then there exist some constant C and $\varepsilon > 0$ small such that*

$$U_{\lambda_0}(x), V_{\lambda_0}(x) \geq C > 0, x \in B_\varepsilon(0^{\lambda_0}) \setminus \{0^{\lambda_0}\}.$$

Proof. From Lemma 2.2 in [7], we have the integral equation

$$\begin{aligned} U_{\lambda_0}(x) &= \overline{u}_{\lambda_0}(x) - \overline{u}(x) \\ &= C_{n,\alpha} \int_{\Sigma_{\lambda_0}} (|y^{\lambda_0}|^{-\gamma}\overline{v}^p(y^{\lambda_0}) - |y|^{-\gamma}\overline{v}^p(y))(\frac{1}{|x-y|^{n+\alpha}} - \frac{1}{|x-y^{\lambda_0}|^{n+\alpha}})dy \\ &\geq C_{n,\alpha} \int_{\Sigma_{\lambda_0}} \frac{\overline{v}_{\lambda_0}^p(y) - \overline{v}^p(y)}{|y|^\gamma} \cdot (\frac{1}{|x-y|^{n+\alpha}} - \frac{1}{|x-y^{\lambda_0}|^{n+\alpha}})dy \end{aligned}$$

Since

$$V_{\lambda_0}(x) \not\equiv 0, x \in \Sigma_{\lambda_0},$$

there exists some x_0 such that $V_{\lambda_0}(x_0) > 0$. Thus, for some $\delta > 0$ small, it holds that

$$\overline{v}^p_{\lambda_0}(y) - \overline{v}^p(y) \geq C > 0, y \in B_\delta(x_0).$$

Therefore,

$$U_{\lambda_0}(x) \geq \int_{B_\delta(x_0)} Cdy \geq C > 0. \tag{5.55}$$

In a same way, one can show that $V_{\lambda_0}(x)$ also satisfies (5.55). $\qquad\square$

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