Sharp Well-Posedness Results for the Schrödinger-Benjamin-Ono System

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December 16, 2014

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Abstract

This work is concerned with the Cauchy problem for a coupled Schrödinger-Benjamin-Ono system

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv, & t \in [-T,T], \ x \in \mathbb{R}, \\ \partial_t v + \nu \mathcal{H}\partial_x^2 v = \beta \partial_x(|u|^2), & \\ u(0,x) = \phi, \ v(0,x) = \psi, & (\phi, \psi) \in H^s(\mathbb{R}) \times H^{s'}(\mathbb{R}). \end{cases}$$

In the *non-resonant* case ($|\nu| \neq 1$), we prove local well-posedness for a large class of initial data. This improves the results obtained by Bekiranov, Ogawa and Ponce (1998). Moreover, we prove C^2-*ill-posedness* at *low-regularity*, and also when the difference of regularity between the initial data is large enough. As far as we know, this last ill-posedness result is the first of this kind for a nonlinear dispersive system. Finally, we also prove that the local well-posedness result obtained by Pecher (2006) in the *resonant* case ($|\nu| = 1$) is sharp except for the end-point.

1 Introduction

In [9], Funakoshi and Oikawa deduced the following Schrödinger-Benjamin-Ono system,

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv, & t \in [-T,T], \ x \in \mathbb{R}, \\ \partial_t v + \nu \mathcal{H}\partial_x^2 v = \beta \partial_x(|u|^2), & \\ u(0,x) = \phi, \ v(0,x) = \psi, & (\phi, \psi) \in H^s(\mathbb{R}) \times H^{s'}(\mathbb{R}), \end{cases} \tag{1.1}$$

where $\mathcal{H}$ denotes the Hilbert transform, $u = u(t,x)$ is a complex-valued function, $v = v(t,x)$ is a real-valued function and α, β, ν are real constants such that $\alpha\beta \neq 0$.

The Schrödinger-Benjamin-Ono system (1.1) describes the motion of two fluids with different densities under capillary-gravity waves in a deep water flow. The short surface wave is usually described by a Schrödinger type equation and the long internal wave is described by some sort of wave equation accompanied by a dispersive term (which is a

Benjamin-Ono type equation in this case).

The natural function spaces to study the local well-posedness (L.W.P.) of this system are the Sobolev $H^s \times H^{s'}$-type spaces. Indeed, for a smooth solution (u, v), the following quantities are conserved for every $t \in [-T, T]$

$$\begin{cases} \|u(t)\|_2^2, \\ Im \int u(t, x) \partial_x \overline{u}(t, x) dx + \frac{\alpha}{2\beta} \|v(t)\|_2^2, \\ \|\partial_x u(t)\|_2^2 + \alpha \int v(t, x) |u(t, x)|^2 dx - \frac{\alpha\nu}{2\beta} \|D_x^{1/2} v(t)\|_2^2, \end{cases}$$

where $D_x = \mathcal{H}\partial_x$.

For $|\nu| \neq 1$, the *non-resonant* case, Bekiranov, Ogawa and Ponce proved in [4] the L.W.P. of the system (1.1), for (s, s') in the half-line

$$\ell := \left\{ (s, s') \in \mathbb{R}^2 \ : \ s' = s - 1/2, \ s \geq 0 \right\}.$$

In [1], Angulo, Matheus and Pilod obtained global well-posedness (G.W.P.), also for $(s, s') \in \ell$, by using an idea of Colliander, Holmer and Tzirakis [6].

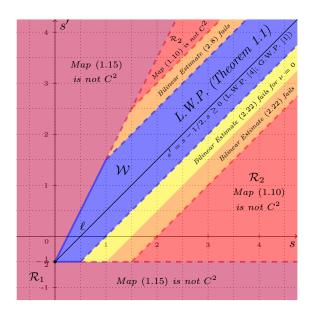In Theorem 1.1 of the present paper, we prove the L.W.P. of (1.1) for (s, s') in the

Figure 1: Problem (1.1) for $|\nu| \neq 1$ and $(\phi, \psi) \in H^s \times H^{s'}$.

region

$$\mathcal{W} := \{(s, s') \in \mathbb{R}^2 : -1/2 < s' - (s - 1/2) < 1, \ -1/2 \leq s' \leq 2s - 1/2\}.$$

Moreover, we establish C^2-*ill-posedness* of (1.1) for (s, s') in the regions

$$\mathcal{R}_1 := \{(s, s') \in \mathbb{R}^2 : s' < -1/2 \ \text{ or } \ 2s - 1/2 < s'\}$$

and

$$\mathcal{R}_2 := \{(s, s') \in \mathbb{R}^2 : |s' - (s - 1/2)| > 3/2\}.$$

Actually, the ill-posedness result holds in a slightly stronger sense in the region $\mathcal{R}_1$ (see Theorem 1.2 for the precise statement). Furthermore, Theorem 4.2 states that the bilinear estimates used to prove Theorem 1.1 fails in a part of the remaining region. For the case $\nu = 0$, it fails in the entire remaining region. All these results are summarized in Figure 1. In particular, we observe that our results are sharp[1] at *low-regularity*.

For $|\nu| = 1$, the *resonant* case, Pecher showed in [13] the L.W.P. of the system (1.1) for $(s, s') \in \ell$, except for the end-point $(0, -1/2)$. In the present paper, we prove in Theorem 1.3 the C^2-*ill-posedness* of (1.1) for $(s, s') \notin \ell$. Furthermore, we prove in Theorem 4.3 that the key bilinear estimate of Pecher's proof fails at the end-point.

Bekiranov, Ogawa and Ponce also obtained L.W.P. for other nonlinear dispersive systems such as the Schrödinger-Korteweg-de Vries system (in [3]) and the Benney system (in [4]), in both cases, for initial data in $H^s \times H^{s'}$ with $(s, s') \in \ell$. For the last system, due to scaling properties, the L.W.P. was only investigated in $H^s(\mathbb{R}) \times H^{s - \frac{1}{2}}(\mathbb{R})$ (see Remark 2 in [4]). In the case of the system (1.1), one can scale a solution (u, v) as $u_\lambda(t, x) = \lambda^{\frac{3}{2}} u(\lambda^2 t, \lambda x)$, $v_\lambda(t, x) = \lambda^2 v(\lambda^2 t, \lambda x)$. Then (u_λ, v_λ) solves (1.1) with initial data $\phi_\lambda(x) = \lambda^{\frac{3}{2}} \phi(\lambda x)$ and $\psi_\lambda(x) = \lambda^2 \psi(\lambda x)$ satisfying $\|\phi_\lambda\|_{\dot{H}^s} = \lambda^{1+s} \|\phi\|_{\dot{H}^s}$ and $\|\psi_\lambda\|_{\dot{H}^{s'}} = \lambda^{\frac{3}{2}+s'} \|\psi\|_{\dot{H}^{s'}}$. Thus $s' = s - 1/2$ keeps each norm equivalent under scaling. However, Theorem 1.1 shows that the regime $s' = s - 1/2$ is not necessary for the L.W.P. of the system (1.1). Also, note that Theorem 1.2 establishes C^2-*ill-posedness* for (s, s') in a neighborhood of $(-1, -3/2)$, which is a point of critical regularity, in the sense that the scaling transformation leaves the $\dot{H}^s \times \dot{H}^{s'}$-norm invariant at this regularity.

In [11], Ginibre, Tsutsumi and Velo proved the L.W.P. of the Benney system and of the 1D Zakharov system, for the region $\{(s, s') \in \mathbb{R}^2 : -1/2 < s - s' \le 1, \ 0 \le s' + 1/2 \le 2s\}$. In [7], Corcho and Linares proved the L.W.P. of the Schrödinger-Korteweg-de Vries system, for a region containing the half-line ℓ. Ill-posedness was not investigated in all these works ([3], [11], [4], [7]).

In [15], Wu improved the L.W.P. of the Schrödinger-Korteweg-de Vries system obtained in [7] to a larger region. Furthermore, he also proved C^2-*ill-posedness* results. In particular, he showed that his L.W.P. result is sharp[1] at *low-regularity*.

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[1] Sharp in the sense that one can not improves the result by performing a Picard iteration, since this method provides an analytic flow map data-solution (and hence C^∞ Fréchet differentiable).

To state our results, we introduce the integral equations associated to the system (1.1),

$$u(t) = e^{it\partial_x^2}\phi - i\alpha \int_0^t e^{i(t-t')\partial_x^2}\left(u(t') \cdot v(t')\right) dt', \tag{1.2}$$

$$v(t) = e^{-\nu t \mathcal{H}\partial_x^2}\psi + \beta \int_0^t e^{-\nu(t-t')\mathcal{H}\partial_x^2}(\partial_x |u(t')|^2) dt', \tag{1.3}$$

where $e^{it\partial_x^2}$ and $e^{-\nu t\mathcal{H}\partial_x^2}$ denote the unitary operators for the linear Schrödinger and Benjamin-Ono equations respectively. We need also to introduce the Bourgain spaces for constructing the local solutions. For $s, b, s', b', \nu \in \mathbb{R}$, we let $X^{s,b}$ and $Y_\nu^{s',b'}$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ under the norms

$$\|f\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b \widehat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} = \|e^{-it\partial_x^2}f\|_{H_t^b(\mathbb{R};H_x^s)}, \tag{1.4}$$

$$\|g\|_{Y_\nu^{s',b'}} := \|\langle\xi\rangle^{s'}\langle\tau+\nu|\xi|\xi\rangle^{b'} \widehat{g}(\tau,\xi)\|_{L^2_{\tau,\xi}} = \|e^{-\nu t\mathcal{H}\partial_x^2}g\|_{H_t^{b'}(\mathbb{R};H_x^{s'})}, \tag{1.5}$$

where $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$ and $\widehat{f}$ is the Fourier transform of f in both x and t variables

$$\widehat{f}(\tau,\xi) := \iint e^{-2\pi i(t\tau+x\xi)} f(x,t) dt dx.$$

Hereafter, we will simply denote $Y^{s',b'}$ instead of $Y_\nu^{s',b'}$.

Let $b, b' > 1/2$, the Sobolev lemma implies that

$$X^{s,b} \hookrightarrow C^0(\mathbb{R}; H^s(\mathbb{R})), \tag{1.6}$$

$$Y^{s',b'} \hookrightarrow C^0(\mathbb{R}; H^{s'}(\mathbb{R})). \tag{1.7}$$

Thus, for an interval I, $M_I := \{f \in X^{s,b} : f(t) = 0, \forall t \in I\}$ is a closed subspace of $X^{s,b}$. We define $X_I^{s,b}$ to be the quotient space $X^{s,b}/M_I$, which is a Banach space with the norm

$$\|f\|_{X_I^{s,b}} := \inf\{\|\tilde{f}\|_{X^{s,b}} : \tilde{f}(t) = f(t), \forall t \in I\}.$$

We write $X_T^{s,b}$ for $X_I^{s,b}$, when $I = [-T, T]$. We define $Y_T^{s',b'}$ similarly.

Now we are ready to enunciate our results. The first theorem states the L.W.P. of the system (1.1), in the *non-resonant* case, for $(s, s') \in \mathcal{W}$ (see Figure 1).

Theorem 1.1. *Let $|\nu| \neq 1$ and $s, s' \in \mathbb{R}$ satisfying*

$$-1/2 \leq s' \leq 2s - 1/2, \tag{1.8}$$

$$s - 1 < s' < s + 1/2. \tag{1.9}$$

The Cauchy problem (1.1) *is locally well-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, in the following sense: For every $R > 0$, there exist $T = T(R) > 0$ and $b, b' > 1/2$ such that if $\|\phi\|_{H^s} + \|\psi\|_{H^{s'}} < R$,*

there exists a unique solution $(u, v) \in X_T^{s,b} \times Y_T^{s',b'}$ satisfying (1.2)-(1.3) for all $t \in [-T, T]$. Moreover, this solution satisfies

$$(u, v) \in C^0([-T, T]; H^s(\mathbb{R})) \times C^0([-T, T]; H^{s'}(\mathbb{R})),$$

and the associated flow map data-solution,

$$S : B_R \to C^0([-T, T]; H^s(\mathbb{R})) \times C^0([-T, T]; H^{s'}(\mathbb{R})), \quad (\phi, \psi) \mapsto (u, v), \qquad (1.10)$$

is Lipschitz continuous, where B_R is the open ball in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, centered at the origin with radius R.

Next, we give the main ingredients in the proof of Theorem 1.1. Following the procedure employed in [4], we use the Banach Fixed Point theorem and the Fourier restriction norm method introduced by Bourgain in [5]. So the difficulty is to extend the following bilinear estimates found in [4]

$$\|\partial_x(u_1 \overline{u_2})\|_{Y^{s-\frac{1}{2},a}} \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} , \qquad b > 1/2, \ a \leq 0, \ s \geq 0, \qquad (1.11)$$

$$\|uv\|_{X^{s,a}} \leq C \|u\|_{X^{s,b}} \|v\|_{Y^{s-\frac{1}{2},b}} , \qquad 3/4 > b > 1/2, \ a < -1/4, \ s \geq 0, \qquad (1.12)$$

to new ones. Proceeding as in [11], we decouple the modulation regularities of the spaces X and Y in order to gain spatial regularity (i.e., we replace $(s-1/2, b)$ by (s', b') in Y). Then, by choosing $1/2 < b < c < 3/4$ and $1/2 < b' < c' < 3/4$ depending on (s, s'), we prove the following estimates

$$\|\partial_x(u_1 \overline{u_2})\|_{Y^{s',c'-1}} \leq C \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} , \qquad (1.13)$$

$$\|uv\|_{X^{s,c-1}} \leq C \|u\|_{X^{s,b}} \|v\|_{Y^{s',b'}} , \qquad (1.14)$$

for (s, s') in larger regions (c.f. Theorems 2.2 and 2.3). Hence, the system (1.1) is L.W.P. for $(s, s') \in \mathcal{W}$, where both estimates (1.13) and (1.14) hold. The estimate (1.11) offers minor difficulty in [4], since the regime $s' = s - 1/2$, $s \geq 0$ allows good cancellations in the frequency interactions. However, those cancellations do not occur anymore for (s, s') in the larger region where the estimate (1.13) holds. Thus, we need to perform a new decomposition of the Euclidean space (c.f. (2.11)-(2.14)) in order to obtain (1.13). On the other hand, there are no good cancellations for the estimate (1.12), even in the regime $s' = s - 1/2$, $s \geq 0$. However, we are able to prove the estimate (1.14) for (s, s') in a larger region by performing the decomposition (2.25)-(2.29), which is slightly different from the one used in [4] to obtain the estimate (1.12).

In the next theorem, we state an ill-posedness result for the *non-resonant* case.

Theorem 1.2. *Let $|\nu| \neq 1$ and $s, s' \in \mathbb{R}$. Suppose that the Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, in the sense of Theorem 1.1.*

(i) If (1.8) *is not verified, the associated flow map data-solution,*

$$S^t : B_R \to H^s(\mathbb{R}) \times H^{s'}(\mathbb{R}), \quad (\phi, \psi) \mapsto (u(t), v(t)), \tag{1.15}$$

is not C^2 at zero[1], for $t \in [-T, 0) \cup (0, T]$. Neither is, a fortiori, the flow map (1.10).

(ii) If $|s' - (s - 1/2)| > 3/2$, the associated map data-solution (1.10) *is not C^2 at zero[1].*

The first C^2-*ill-posedness* result of this kind was proved by Tzvetkov in [14] for the KdV equation. We essentially follow his argument to prove Theorem 1.2 (i). There is an additional technical difficulty to prove (ii). To overcome this difficulty, we allow the variable t to move. Therefore, (ii) presents a conclusion for the flow map (1.10) instead of the flow map (1.15). We emphasize that this approach has already been used in previous works (e.g., [2] and [8]).

Remark. As far as we know, Theorem 1.2 (ii) is the first result concerning the ill-posedness of a nonlinear dispersive system when the difference of regularity between the initial data is large enough (see region $\mathcal{R}_2$ in Figure 1). Such result seems natural, due to the coupling of the system via the nonlinearities. We believe that the same approach used to prove Theorem 1.2 (ii) can provide similar results for other nonlinear dispersive systems such as the Zakharov system and the Schrödinger-Korteweg-de Vries system. We plan to address this issue in a forthcoming paper.

Finally, we state an ill-posedness result for the *resonant* case.

Theorem 1.3. *Let $|\nu| = 1$ and $(s, s') \notin \ell$, i.e., $s' \neq s - 1/2$ or $s < 0$. If the Cauchy problem* (1.1) *is locally well-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, the flow map data-solution* (1.15) *is not C^2 at zero[1], for $t \in [-T, 0) \cup (0, T]$ and, a fortiori, neither is the flow map* (1.10).

Throughout the whole text, we use the following notations:

- For any $x \in \mathbb{R}$, we define $\operatorname{sgn}(x) := x|x|^{-1}$ if $x \neq 0$ and $\operatorname{sgn}(x) := 1$ if $x = 0$.

- Let $\mathbf{1}_\Omega$ denotes the characteristic function of an arbitrary set Ω, i.e., $\mathbf{1}_\Omega(x) = 1$ if $x \in \Omega$ and $\mathbf{1}_\Omega(x) = 0$ if $x \notin \Omega$.

- Fix η a smooth function supported on the interval $[-2, 2]$ such that $\eta(x) \equiv 1$ for all $|x| \leq 1$ and, for each $T > 0$, $\eta_T(t) := \eta(t/T)$.

- For positive quantities X and Y, the notation $X \lesssim Y$ means that there exist a constant $C > 0$ such that $X \leq CY$, depending only on the parameters α, β and ν related to (1.1), on the indices s, s', b, c, b' and c' related to the Bourgain spaces in the bilinear estimates (2.8) and (2.22), and on certain norms of the fixed cut-off function η. We denote $X \gtrsim Y$ when $Y \lesssim X$, and denote $X \sim Y$ when $X \lesssim Y \lesssim X$.

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[1] Actually, we prove that these maps are not two times Fréchet differentiable at zero.

This paper is organized as follows. In Section 2, we establish the new bilinear estimates that we use to prove Theorem 1.1 in Section 3. In Section 4, we prove Theorems 1.2 , 1.3, 4.2 and 4.3.

2   Bilinear Estimates

In this section, we improve the bilinear estimates presented in [4]. First, we state some calculus inequalities which will be useful in the proofs of Theorems 2.2 and 2.3.

Lemma 2.1. *Let $\alpha > 1/2$ and $1/2 < \beta, \gamma \le 1$. Then, for all $p \ne 0$ and $q, r \in \mathbb{R}$, the following estimates hold:*

(i) $\displaystyle \int \frac{dx}{\langle x - q \rangle^{2\beta} \langle x - r \rangle^{2\gamma}} \lesssim \frac{1}{\langle q - r \rangle^{2\min\{\beta,\gamma\}}}$,   (2.1)

(ii) $\displaystyle \int \frac{dx}{\langle x - q \rangle^{2\beta} \langle x - r \rangle^{2(1-\gamma)}} \lesssim \frac{1}{\langle q - r \rangle^{2(1-\gamma)}}$,   (2.2)

(iii) $\displaystyle \int \frac{dx}{\langle px^2 + qx + r \rangle^{\alpha}} \lesssim \frac{1}{|p|}$.   (2.3)

The estimates (2.1) and (2.2) are particular cases of the estimates established in Lemma 4.2 of [11]. The estimate (2.3) follows from elementary computations (for the ideas, see (2.14) of [3] and note that $\langle \cdot \rangle \sim 1 + | \cdot |$).

Theorem 2.2. *Assume that $|\nu| \ne 1$. Let $s, s' \in \mathbb{R}$ be such that $s \ge 0$,*

$$s' \le 2s - 1/2,$$ (2.4)
$$s' < s + 1/2.$$ (2.5)

Then, for all $b, c' \in \mathbb{R}$ such that

$$\max\{1/2 , \ (s' - s)/2 + 1/2\} < b,$$ (2.6)

$$c' < \min\{3/4 - (s' - s)/2 , \ 3/4\},$$ (2.7)

the following estimate holds:

$$\|\partial_x(u_1 \overline{u_2})\|_{Y^{s',c'-1}} \lesssim \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}} , \quad \forall u_1, u_2 \in X^{s,b}.$$ (2.8)

Proof. It is sufficient to show (2.8) for $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$. Thus, letting

$$f(\tau, \xi) := \langle \xi \rangle^s \langle \tau + \xi^2 \rangle^b \, \widehat{u_1}(\tau, \xi), \quad g(\tau, \xi) := \langle \xi \rangle^s \langle \tau - \xi^2 \rangle^b \, \overline{\widehat{u_2}}(-\tau, -\xi),$$

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and denoting $\tau_2 := \tau - \tau_1$ and $\xi_2 := \xi - \xi_1$, the estimate (2.8) is equivalent to

$$\left\| \frac{i\xi\langle\xi\rangle^{s'}}{\langle\tau + \nu|\xi|\xi\rangle^{1-c'}} \iint \frac{f(\tau_2,\xi_2)g(\tau_1,\xi_1)d\tau_1 d\xi_1}{\langle\xi_2\rangle^s \langle\tau_2 + \xi_2^2\rangle^b \langle\xi_1\rangle^s \langle\tau_1 - \xi_1^2\rangle^b} \right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2}\|g\|_{L^2}, \ \forall f,g \in \mathcal{S}(\mathbb{R}^2).$$

For convenience, we rewrite this estimate as

$$\left\| \iint \Phi(\tau,\xi,\tau_1,\xi_1)f(\tau_2,\xi_2)g(\tau_1,\xi_1)d\tau_1 d\xi_1 \right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2}\|g\|_{L^2}, \ \forall f,g \in \mathcal{S}(\mathbb{R}^2), \quad (2.9)$$

where

$$\Phi(\tau,\xi,\tau_1,\xi_1) := \frac{i\xi\langle\xi\rangle^{s'}\langle\sigma\rangle^{c'-1}}{\langle\xi_2\rangle^s \langle\sigma_2\rangle^b \langle\xi_1\rangle^s \langle\sigma_1\rangle^b} \ ,$$

with the additional notation $\sigma := \tau + \nu|\xi|\xi$, $\sigma_1 := \tau_1 - \xi_1^2$ and $\sigma_2 := \tau_2 + \xi_2^2$. With this notation, the algebraic relation associated to (2.9) is given by

$$\sigma - \sigma_1 - \sigma_2 = 2\xi\xi_1 - (1 - \nu\,\mathrm{sgn}(\xi))\xi^2 = (1 + \nu\,\mathrm{sgn}(\xi))\xi^2 - 2\xi\xi_2. \quad (2.10)$$

We split $\mathbb{R}^4$ into the following regions

$$\begin{aligned}
\mathcal{A} &= \left\{ (\tau,\xi,\tau_1,\xi_1) \in \mathbb{R}^4 \ : \ |(1 - \nu\,\mathrm{sgn}(\xi))\xi - 2\xi_1| < c_\nu|\xi| \right\}, & (2.11) \\
\mathcal{B} &= \left\{ (\tau,\xi,\tau_1,\xi_1) \in \mathcal{A}^c \ : \ |\sigma| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \right\}, & (2.12) \\
\mathcal{B}_1 &= \left\{ (\tau,\xi,\tau_1,\xi_1) \in \mathcal{A}^c \ : \ |\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \right\}, & (2.13) \\
\mathcal{B}_2 &= \left\{ (\tau,\xi,\tau_1,\xi_1) \in \mathcal{A}^c \ : \ |\sigma_2| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \right\}, & (2.14)
\end{aligned}$$

where $c_\nu := \frac{|1-|\nu||}{2} > 0$.

We use the Cauchy-Schwarz inequality and the Fubini theorem to estimate the left-hand side of (2.9) restricted to each one of these sets (also perform a change of variables in the region $\mathcal{B}_2$). Thus (2.9) is a consequence of the following estimates

$$\|\mathbf{1}_{\mathcal{A}\cup\mathcal{B}}\Phi\|_{L^\infty_{\tau,\xi}(L^2_{\tau_1,\xi_1})} \lesssim 1, \quad (2.15)$$

$$\|\mathbf{1}_{\mathcal{B}_1}\Phi\|_{L^\infty_{\tau_1,\xi_1}(L^2_{\tau,\xi})} \lesssim 1, \quad (2.16)$$

$$\|\mathbf{1}_{\widetilde{\mathcal{B}}_2}\widetilde{\Phi}\|_{L^\infty_{\tau_2,\xi_2}(L^2_{\tau,\xi})} \lesssim 1, \quad (2.17)$$

where

$$\begin{aligned}
\widetilde{\Phi}(\tau,\xi,\tau_2,\xi_2) &:= \Phi(\tau,\xi,\tau - \tau_2,\xi - \xi_2), \ \forall(\tau,\xi,\tau_2,\xi_2) \in \mathbb{R}^4, \\
\widetilde{\mathcal{B}}_2 &:= \left\{ (\tau,\xi,\tau_2,\xi_2) \in \mathbb{R}^4 \ : \ (\tau,\xi,\tau - \tau_2,\xi - \xi_2) \in \mathcal{B}_2 \right\}.
\end{aligned}$$

Proof of the estimate (2.15): In the region $\mathcal{A}$, $|\xi| \sim |\xi_1| \sim |\xi_2|$. In fact, rewriting

$$2|\xi_1| = |(1 - \nu\,\mathrm{sgn}(\xi))\xi - 2\xi_1 - (1 - \nu\,\mathrm{sgn}(\xi))\xi|,$$

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we conclude that $c_\nu|\xi| \leq 2|\xi_1| \leq (c_\nu + 1 + |\nu|)|\xi|$. Similarly, we have

$$2|\xi_2| = |(1 - \nu\,\mathrm{sgn}(\xi))\xi - 2\xi_1 + (1 + \nu\,\mathrm{sgn}(\xi))\xi|,$$

thus $c_\nu|\xi| \leq 2|\xi_2| \leq (c_\nu + 1 + |\nu|)|\xi|$. Hence, we get from $c' < 3/4$ and (2.4) that

$$|\Phi(\tau, \xi, \tau_1, \xi_1)| \lesssim \frac{\langle\xi\rangle^{s'-2s+\frac{1}{2}}|\xi|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b} \lesssim \frac{|\xi|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}, \ \forall(\tau, \xi, \tau_1, \xi_1) \in \mathcal{A}.$$

The same estimate holds in $\mathcal{B}$. In fact, in the region $\mathcal{A}^c$, we have from (2.10) that

$$c_\nu|\xi|^2 \leq |(1 - \nu\,\mathrm{sgn}(\xi))\xi^2 - 2\xi\xi_1| = |\sigma - \sigma_1 - \sigma_2|. \tag{2.18}$$

In particular, $|\xi|^2 \lesssim |\sigma|$ in the region $\mathcal{B}$. Note also that $\langle\xi_1 + \xi_2\rangle \lesssim \langle\xi_1\rangle\langle\xi_2\rangle$. Thus, we deduce from (2.7) that

$$|\Phi(\tau, \xi, \tau_1, \xi_1)| \lesssim \frac{\langle\xi\rangle^{s'-s+\frac{1}{2}}}{\langle\sigma\rangle^{1-c'}} \cdot \frac{|\xi|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b} \lesssim \frac{|\xi|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}, \ \forall(\tau, \xi, \tau_1, \xi_1) \in \mathcal{B}.$$

Now, observe from (2.1) and $b > 1/2$, that

$$\sup_{\tau,\xi}\left\|\frac{|\xi|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\right\|_{L^2_{\tau_1,\xi_1}} \lesssim \sup_{\tau,\xi}\left[\int \frac{|\xi|}{\langle 2\xi\xi_1 - \tau - \xi^2\rangle^{2b}}d\xi_1\right]^{\frac{1}{2}} \lesssim 1.$$

This concludes the proof of (2.15).

Proof of the estimate (2.16): By (2.18), $|\xi|^2 \lesssim |\sigma_1|$ in the region $\mathcal{B}_1$. Thus (2.6) implies that

$$|\Phi(\tau, \xi, \tau_1, \xi_1)| \lesssim \frac{\langle\xi\rangle^{s'-s+1}}{\langle\sigma_1\rangle^b} \cdot \frac{1}{\langle\sigma_2\rangle^b\langle\sigma\rangle^{1-c'}} \lesssim \frac{1}{\langle\sigma_2\rangle^b\langle\sigma\rangle^{1-c'}}, \ \forall(\tau, \xi, \tau_1, \xi_1) \in \mathcal{B}_1.$$

From (2.2), (2.10), (2.3) and $c' < 3/4$, we deduce that

$$\sup_{\tau_1,\xi_1}\left\|\frac{1}{\langle\sigma_2\rangle^b\langle\sigma\rangle^{1-c'}}\right\|_{L^2_{\tau,\xi}} \lesssim \sup_{\tau_1,\xi_1}\left[\int \frac{d\xi}{\langle(1 - \nu\,\mathrm{sgn}(\xi))\xi^2 - 2\xi\xi_1 - \sigma_1\rangle^{2(1-c')}}\right]^{\frac{1}{2}} \lesssim 1,$$

which yields (2.16).

Proof of the estimate (2.17): Denoting $\tau_1 := \tau - \tau_2$, $\xi_1 := \xi - \xi_2$ and σ, σ_1, σ_2 as before, we have $|\xi|^2 \lesssim |\sigma_2|$ in the region $\widetilde{\mathcal{B}}_2$. Then, we deduce from (2.6) that

$$|\widetilde{\Phi}(\tau, \xi, \tau_2, \xi_2)| \lesssim \frac{\langle\xi\rangle^{s'-s+1}}{\langle\sigma_2\rangle^b} \cdot \frac{1}{\langle\sigma_1\rangle^b\langle\sigma\rangle^{1-c'}} \lesssim \frac{1}{\langle\sigma_1\rangle^b\langle\sigma\rangle^{1-c'}}, \ \forall(\tau, \xi, \tau_2, \xi_2) \in \widetilde{\mathcal{B}}_2,$$

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and from (2.2), (2.10), (2.3) and $c' < 3/4$ that

$$\sup_{\tau_2,\xi_2}\left\|\frac{1}{\langle\sigma_1\rangle^b\langle\sigma\rangle^{1-c'}}\right\|_{L^2_{\tau,\xi}} \lesssim \sup_{\tau_2,\xi_2}\left[\int\frac{1}{\langle(1+\nu\,\mathrm{sgn}(\xi))\xi^2 - 2\xi\xi_2 + \sigma_2\rangle^{2(1-c')}}d\xi\right]^{\frac{1}{2}} \lesssim 1,$$

which concludes (2.17). This finishes the proof of (2.8). $\qquad\qquad\square$

Theorem 2.3. *Assume that $|\nu| \neq 1$. Let $s, s' \in \mathbb{R}$ be such that $s \geq 0$,*

$$-1/2 \ \leq \ s', \tag{2.19}$$
$$s - 1 \ < \ s'. \tag{2.20}$$

Then, for all $b, b', c \in \mathbb{R}$ such that $1/2 < b, b'$ and

$$1/2 < c < \min\left\{3/4 \ , \ (s' - s)/2 + 1\right\}, \tag{2.21}$$

the following estimate holds:

$$\|uv\|_{X^{s,c-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{Y^{s',b'}} \ , \quad \forall u \in X^{s,b}, \ \forall v \in Y^{s',b'}. \tag{2.22}$$

Proof. It is sufficient to show (2.8) for $u, v \in \mathcal{S}(\mathbb{R}^2)$. Thus letting

$$f(\tau,\xi) := \langle\xi\rangle^s\langle\tau + \xi^2\rangle^b\hat{u}(\tau,\xi), \quad g(\tau,\xi) := \langle\xi\rangle^{s'}\langle\tau + \nu|\xi|\xi\rangle^{b'}\hat{v}(\tau,\xi),$$

and denoting $\tau_2 := \tau - \tau_1$ and $\xi_2 := \xi - \xi_1$, the estimate (2.22) is equivalent to

$$\left\|\frac{\langle\xi\rangle^s}{\langle\tau + \xi^2\rangle^{1-c}}\iint\frac{f(\tau_2,\xi_2)g(\tau_1,\xi_1)d\tau_1 d\xi_1}{\langle\xi_2\rangle^s\langle\tau_2 + \xi_2^2\rangle^b\langle\xi_1\rangle^{s'}\langle\tau_1 + \nu|\xi_1|\xi_1\rangle^{b'}}\right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2}\|g\|_{L^2}, \ \forall f, g \in \mathcal{S}(\mathbb{R}^2).$$

For convenience, we rewrite this estimate as

$$\left\|\iint \Psi(\tau,\xi,\tau_1,\xi_1)f(\tau_2,\xi_2)g(\tau_1,\xi_1)d\tau_1 d\xi_1\right\|_{L^2_{\tau,\xi}} \lesssim \|f\|_{L^2}\|g\|_{L^2}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^2). \tag{2.23}$$

where

$$\Psi(\tau,\xi,\tau_1,\xi_1) := \frac{\langle\xi\rangle^s\langle\sigma\rangle^{c-1}}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^b\langle\xi_1\rangle^{s'}\langle\sigma_1\rangle^{b'}} \ ,$$

with the additional notation $\sigma := \tau + \xi^2$, $\sigma_1 := \tau_1 + \nu|\xi_1|\xi_1$ and $\sigma_2 := \tau_2 + \xi_2^2$. With this notation, the algebraic relation associated to (2.23) is given by

$$\sigma - \sigma_1 - \sigma_2 = 2\xi\xi_1 - (1 + \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 = (1 - \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 + 2\xi_1\xi_2 \ . \tag{2.24}$$

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We split $\mathbb{R}^4$ into the following regions

$$
\begin{aligned}
\mathcal{A} &= \left\{ (\tau, \xi, \tau_1, \xi_1) \in \mathbb{R}^4 \ : \ |\xi_1| \leq 1 \right\}, & (2.25) \\
\mathcal{B} &= \left\{ (\tau, \xi, \tau_1, \xi_1) \in \mathcal{A}^c \ : |(1 + \nu \operatorname{sgn}(\xi_1))\xi_1 - 2\xi| < c_\nu |\xi_1| \right\}, & (2.26) \\
\mathcal{C} &= \left\{ (\tau, \xi, \tau_1, \xi_1) \in \mathcal{A}^c \cap \mathcal{B}^c \ : |\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \right\}, & (2.27) \\
\mathcal{C}_1 &= \left\{ (\tau, \xi, \tau_1, \xi_1) \in \mathcal{A}^c \cap \mathcal{B}^c \ : |\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \right\}, & (2.28) \\
\mathcal{C}_2 &= \left\{ (\tau, \xi, \tau_1, \xi_1) \in \mathcal{A}^c \cap \mathcal{B}^c \ : |\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \right\}, & (2.29)
\end{aligned}
$$

where $c_\nu := \frac{|1 - |\nu||}{2} > 0$.

Arguing similarly to the proof of Theorem 2.2, it is enough to show

$$
\begin{aligned}
\|\mathbf{1}_{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}} \Psi\|_{L^\infty_{\tau,\xi}(L^2_{\tau_1,\xi_1})} &\lesssim 1, & (2.30) \\
\|\mathbf{1}_{\mathcal{C}_1} \Psi\|_{L^\infty_{\tau_1,\xi_1}(L^2_{\tau,\xi})} &\lesssim 1, & (2.31) \\
\|\mathbf{1}_{\widetilde{\mathcal{C}}_2} \widetilde{\Psi}\|_{L^\infty_{\tau_2,\xi_2}(L^2_{\tau_1,\xi_1})} &\lesssim 1, & (2.32)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\Psi}(\tau_2, \xi_2, \tau_1, \xi_1) &:= \Psi(\tau_1 + \tau_2, \xi_1 + \xi_2, \tau_1, \xi_1), \ \forall (\tau_2, \xi_2, \tau_1, \xi_1) \in \mathbb{R}^4, \\
\widetilde{\mathcal{C}}_2 &:= \left\{ (\tau_2, \xi_2, \tau_1, \xi_1) \in \mathbb{R}^4 \ : (\tau_1 + \tau_2, \xi_1 + \xi_2, \tau_1, \xi_1) \in \mathcal{C}_2 \right\}.
\end{aligned}
$$

Proof of the estimate (2.30): In the region $\mathcal{A}$, we get that

$$
|\Psi(\tau, \xi, \tau_1, \xi_1)| \lesssim \frac{\langle \xi_1 \rangle^{s-s'}}{\langle \sigma_2 \rangle^b \langle \sigma_1 \rangle^{b'}} \lesssim \frac{1}{\langle \sigma_2 \rangle^b \langle \sigma_1 \rangle^{b'}}, \ \forall (\tau, \xi, \tau_1, \xi_1) \in \mathcal{A},
$$

since $c < 3/4$ and $|\xi_1| \leq 1$. Therefore, we deduce from (2.1), (2.24) and (2.3) that

$$
\|\mathbf{1}_{\mathcal{A}} \Psi\|_{L^\infty_{\tau,\xi}(L^2_{\tau_1,\xi_1})} \lesssim \sup_{\tau,\xi} \left[\int \frac{d\xi_1}{\langle (1 + \nu \operatorname{sgn}(\xi_1))\xi_1^2 - 2\xi\xi_1 + \sigma \rangle^{2\min\{b,b'\}}} \right]^{\frac{1}{2}} \lesssim 1. \quad (2.33)
$$

In the region $\mathcal{B}$, $|\xi| \lesssim |\xi_2|$. Indeed, the identities

$$
\begin{aligned}
2|\xi| &= |(1 + \nu \operatorname{sgn}(\xi_1))\xi_1 - 2\xi - (1 + \nu \operatorname{sgn}(\xi_1))\xi_1| \\
2|\xi_2| &= |(1 + \nu \operatorname{sgn}(\xi_1))\xi_1 - 2\xi + (1 - \nu \operatorname{sgn}(\xi_1))\xi_1|
\end{aligned}
$$

imply that $2|\xi| \leq (c_\nu + 1 + |\nu|)|\xi_1|$ and $c_\nu |\xi_1| \leq 2|\xi_2|$. Therefore,

$$
|\Psi(\tau, \xi, \tau_1, \xi_1)| \lesssim \frac{1}{\langle \xi_1 \rangle^{s'} \langle \sigma_2 \rangle^b \langle \sigma_1 \rangle^{b'}} \lesssim \frac{|2(1 + \nu \operatorname{sgn}(\xi_1))\xi_1 - 2\xi|^{\frac{1}{2}}}{\langle \xi_1 \rangle^{s'+\frac{1}{2}} \langle \sigma_2 \rangle^b \langle \sigma_1 \rangle^{b'}}, \ \forall (\tau, \xi, \tau_1, \xi_1) \in \mathcal{B},
$$

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since $\langle\xi_1\rangle \leq \sqrt{2}|\xi_1| \leq \frac{\sqrt{2}}{c_\nu}|2(1+\nu\,\mathrm{sgn}(\xi_1))\xi_1 - 2\xi|$. Hence, performing the change of variable $\eta := (1 + \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 - 2\xi\xi_1 + \sigma$, we conclude from (2.19), (2.1), (2.24) and $b', b > 1/2$,

$$\|\mathbf{1}_\mathcal{B}\Psi\|_{L^\infty_{\tau,\xi}(L^2_{\tau_1,\xi_1})} \lesssim \sup_{\tau,\xi}\left[\int \frac{d\eta}{\langle\eta\rangle^{2\min\{b,b'\}}}\right]^{\frac{1}{2}} \lesssim 1. \tag{2.34}$$

In the region $\mathcal{A}^c \cap \mathcal{B}^c$, we have

$$c_\nu|\xi_1|^2 \leq |(1 - \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 - 2\xi\xi_1| = |\sigma - \sigma_1 - \sigma_2|. \tag{2.35}$$

In particular, $|\xi_1|^2 \lesssim |\sigma|$ in the region $\mathcal{C}$. Thus, (2.21) implies

$$|\Psi(\tau,\xi,\tau_1,\xi_1)| \lesssim \frac{\langle\xi_1\rangle^{s-s'}}{\langle\sigma\rangle^{1-c}} \cdot \frac{1}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^{b'}} \lesssim \frac{1}{\langle\sigma_2\rangle^b\langle\sigma_1\rangle^{b'}}, \ \forall(\tau,\xi,\tau_1,\xi_1) \in \mathcal{C}.$$

Hence, we deduce from (2.1), (2.24) and (2.3), that

$$\|\mathbf{1}_\mathcal{C}\Psi\|_{L^\infty_{\tau,\xi}(L^2_{\tau_1,\xi_1})} \lesssim \sup_{\tau,\xi}\left[\int \frac{d\xi_1}{\langle(1+\nu\,\mathrm{sgn}(\xi_1))\xi_1^2 - 2\xi\xi_1 + \sigma\rangle^{2\min\{b,b'\}}}\right]^{\frac{1}{2}} \lesssim 1. \tag{2.36}$$

Therefore, we conclude the proof of (2.30) gathering (2.33), (2.34) and (2.36).

Proof of the estimate (2.31): By (2.35), $1 \leq |\xi_1|^2 \lesssim |\sigma_1|$ in the region $\mathcal{C}_1$, thus we have

$$|\Psi(\tau,\xi,\tau_1,\xi_1)| \lesssim \frac{\langle\xi_1\rangle^{s-s'-\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{b'}{2}}\langle\sigma_1\rangle^{\frac{b'}{2}}} \cdot \frac{|2\xi_1|^{\frac{1}{2}}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^{1-c}} \lesssim \frac{|2\xi_1|^{\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{1}{4}}\langle\sigma_2\rangle^b\langle\sigma\rangle^{1-c}}, \ \ \forall(\tau,\xi,\tau_1,\xi_1) \in \mathcal{C}_1,$$

since (2.20) implies $s - s' - 1/2 < 1/2 < b'$. Hence, using (2.2), (2.24) and performing the change of variable $\eta := 2\xi_1\xi - (1 + \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 + \sigma_1 = \sigma - \sigma_2$, we get that

$$\|\mathbf{1}_{\mathcal{C}_1}\Psi\|_{L^\infty_{\tau_1,\xi_1}(L^2_{\tau,\xi})} \lesssim \sup_{\tau_1,\xi_1}\langle\sigma_1\rangle^{-\frac{1}{4}}\left[\int_{|\eta|\leq 2|\sigma_1|} \frac{d\eta}{\langle\eta\rangle^{2(1-c)}}\right]^{\frac{1}{2}} \lesssim \sup_{\tau_1,\xi_1}\langle\sigma_1\rangle^{-\frac{1}{4}-\frac{1}{2}+c} \lesssim 1,$$

since $1/2 < c < 3/4$. This concludes the proof of (2.31).

Proof of the estimate (2.32): Denoting $\tau := \tau_1 + \tau_2$, $\xi := \xi_1 + \xi_2$ and σ, σ_1, σ_2 as before, we have $|\xi_1|^2 \lesssim |\sigma_2|$ in the region $\widetilde{\mathcal{C}}_2$. Also, $s - s' < 1 < 2b$ by (2.20). Therefore,

$$|\widetilde{\Psi}(\tau_2,\xi_2,\tau_1,\xi_1)| \lesssim \frac{\langle\xi_1\rangle^{s-s'}}{\langle\sigma_2\rangle^b} \cdot \frac{1}{\langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{1-c}} \lesssim \frac{1}{\langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{1-c}}, \ \forall(\tau_2,\xi_2,\tau_1,\xi_1) \in \widetilde{\mathcal{C}}_2.$$

Hence, from (2.2), (2.24), (2.3) and $c < 3/4$, we conclude that

$$\|\mathbf{1}_{\widetilde{\mathcal{C}}_2}\widetilde{\Psi}\|_{L^\infty_{\tau_2,\xi_2}(L^2_{\tau_1,\xi_1})} \lesssim \sup_{\tau_2,\xi_2}\left[\int \frac{1}{\langle(1 - \nu\,\mathrm{sgn}(\xi_1))\xi_1^2 + 2\xi_1\xi_2 + \sigma_2\rangle^{2(1-c)}}d\xi_1\right]^{\frac{1}{2}} \lesssim 1.$$

This finishes the proof of (2.22) . $\qquad\qquad\square$

3   Local Well-Posedness

Using the new bilinear estimates of the previous section, Theorem 1.1 can be proven, with minor adjustments, in the same way that Bekiranov, Ogawa and Ponce proved L.W.P. of the system (1.1) for the case $s \geq 0$, $s' = s - 1/2$. In this section, we detail the proof for the convenience of the reader. First, we need to state the linear estimates for the Fourier restriction norm method (see, e.g., [10], [4], [5]).

Lemma 3.1. *Let $T \in (0, 1)$, $s \in \mathbb{R}$ and $1/2 < b \leq c \leq 1$, then the following estimates hold:*

$$(i) \qquad \|f\|_{C_t^0(\mathbb{R}; H_x^s)} \lesssim \|f\|_{X^{s,b}} \ , \tag{3.1}$$

$$(ii) \qquad \|\eta(t) e^{it\partial_x^2} \phi\|_{X^{s,b}} \lesssim \|\phi\|_{H^s} \ , \tag{3.2}$$

$$(iii) \qquad \left\| \eta_T(t) \int_0^t e^{i(t-t')\partial_x^2} f(t') dt' \right\|_{X^{s,b}} \lesssim T^{c-b} \|f\|_{X^{s,c-1}} \ . \tag{3.3}$$

Similar estimates hold for $e^{-\nu t \mathcal{H} \partial_x^2}$ and $Y^{s,b}$ replacing $e^{it\partial_x^2}$ and $X^{s,b}$, respectively.

Proof of Theorem 1.1. Let $s, s' \in \mathbb{R}$ satisfy (1.8) and (1.9). Then $-\frac{1}{2} < \frac{s'-s}{2} < \frac{1}{4}$ and we can fix $b, c, b', c' \in \mathbb{R}$ such that

$$\max\left\{ 1/2 \ , \ (s'-s)/2 + 1/2 \right\} < b < c < \min\left\{ 3/4 \ , \ (s'-s)/2 + 1 \right\}$$

and

$$1/2 < b' < c' < \min\left\{ 3/4 - (s'-s)/2 \ , \ 3/4 \right\}.$$

Thus the hypotheses of Theorems 2.2 and 2.3 are verified. Fix $R > 0$, $(\phi, \psi) \in B_R$ and a constant $C > 0$ greater than all the implicit constants in the estimates (2.8), (2.22), (3.2) and (3.3), and also greater than $|\alpha| + |\beta|$. Let

$$\mathfrak{B} := \left\{ (u, v) \in X^{s,b} \times Y^{s',b'} \ : \ \|(u, v)\|_{\mathfrak{B}} := \|u\|_{X^{s,b}} + \|v\|_{Y^{s',b'}} \leq 2CR \right\},$$

which is a complete metric space. For each $T \in (0, 1)$ such that

$$T^{\min\{c-b \ , \ c'-b'\}} < (8C^4 R)^{-1},$$

we consider the map $\Xi = \Xi[\phi, \psi, T] : \mathfrak{B} \to X^{s,b} \times Y^{s',b'}$, $(u,v) \mapsto (\Xi_1(u,v)\ ,\ \Xi_2(u,v))$ defined by

$$\Xi_1(u,v) := \eta(t)e^{it\partial_x^2}\phi - i\alpha\eta_T(t)\int_0^t e^{i(t-t')\partial_x^2}[u(t')v(t')]dt',$$

$$\Xi_2(u,v) := \eta(t)e^{-\nu t\mathcal{H}\partial_x^2}\psi + \beta\eta_T(t)\int_0^t e^{-\nu(t-t')\mathcal{H}\partial_x^2}\partial_x|u(t')|^2 dt',$$

From the estimates (2.8), (2.22), (3.2) and (3.3), we conclude that

$$\|\Xi(u,v)\|_{X^{s,b}\times Y^{s',b'}} \leq CR + (2C^4 RT^{\min\{c-b\,,\,c'-b'\}})(2CR), \quad \forall (u,v) \in \mathfrak{B},$$

which means that Ξ maps $\mathfrak{B}$ on itself, moreover

$$\|\Xi(u,v) - \Xi(\tilde{u},\tilde{v})\|_{\mathfrak{B}} \leq 8C^4 RT^{\min\{c-b\,,\,c'-b'\}}\|(u,v) - (\tilde{u},\tilde{v})\|_{\mathfrak{B}}, \quad \forall (u,v), (\tilde{u},\tilde{v}) \in \mathfrak{B}.$$

Hence, $\Xi : \mathfrak{B} \to \mathfrak{B}$ is a contraction and has a unique fixed point. This establishes the existence of solution (u,v) satisfying (1.2) and (1.3) for every $t \in [-T, T]$, and from (1.6) and (1.7) we have

$$(u,v) \in C^0([-T,T]; H^s(\mathbb{R})) \times C^0([-T,T]; H^{s'}(\mathbb{R})).$$

Thus, the flow map data-solution S in (1.10) is defined at $(\phi, \psi) \in B_R$ to be the fixed point of $\Xi[\phi, \psi, T]$. From (3.2), (3.3), (2.8) and (2.22), we get that

$$\|S(\phi, \psi) - S(\tilde{\phi}, \tilde{\psi})\|_{\mathfrak{B}} \leq \lambda\|(\phi, \psi) - (\tilde{\phi}, \tilde{\psi})\|_{H^s \times H^{s'}}, \quad \forall (\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in B_R,$$

where $\lambda = C(1 - 8C^4 RT^{\min\{c-b\,,\,c'-b'\}})^{-1}$. Hence, from (3.1), we conclude that the flow (1.10) is Lipschitz.

Finally, we will prove the uniqueness of the solution in the class $X_T^{s,b} \times Y_T^{s',b'}$. Suppose that $(u_1, v_1), (u_2, v_2) \in X^{s,b} \times Y^{s',b'}$ satisfy (1.2) and (1.3) for every $t \in [-T, T]$.

Let $T^* \leq T$, such that

$$2C^3(\|(u_1,v_1)\|_{X^{s,b}\times Y^{s',b'}} + \|(u_2,v_2)\|_{X^{s,b}\times Y^{s',b'}})T^{*\min\{c-b\,,\,c'-b'\}} \leq \frac{1}{2}. \tag{3.4}$$

For any $\epsilon > 0$ there exists $(\tilde{u}, \tilde{v}) \in X^{s,b} \times Y^{s',b'}$ such that $\tilde{u}(t) = u_1(t) - u_2(t)$ and $\tilde{v}(t) = v_1(t) - v_2(t)$ for every $t \in [-T^*, T^*]$ and

$$\|(\tilde{u}, \tilde{v})\|_{X^{s,b}\times Y^{s',b'}} \leq \|(u_1,v_1) - (u_2,v_2)\|_{X_{T^*}^{s,b}\times Y_{T^*}^{s',b'}} + \epsilon. \tag{3.5}$$

Therefore, for every $t \in [-T^*, T^*]$,

$$u_1(t) - u_2(t) = -i\alpha\eta_{T^*}(t)\int_0^t e^{i(t-t')\partial_x^2}[\tilde{u}(t')v_1(t') + u_2(t')\tilde{v}(t')]dt',$$

$$v_1(t) - v_2(t) = \beta\eta_{T^*}(t)\int_0^t e^{-\nu(t-t')\mathcal{H}\partial_x^2}\partial_x[\tilde{u}(t')\overline{u_1(t')} + u_2(t')\overline{\tilde{u}(t')}]dt'.$$

Thus from (3.3), (2.8) and (2.22) yields

$$
\begin{aligned}
\|u_1 - u_2\|_{X_{T^*}^{s,b}} &\leq \left\| -i\alpha\eta_{T^*}(t) \int_0^t e^{i(t-t')\partial_x^2}[\tilde{u}(t')v_1(t') + u_2(t')\tilde{v}(t')]dt' \right\|_{X^{s,b}} \\
&\leq C^3 T^{*c-b} \left(\|\tilde{u}\|_{X^{s,b}} \|v_1\|_{Y^{s',b'}} + \|u_2\|_{X^{s,b}} \|\tilde{v}\|_{Y^{s',b'}} \right)
\end{aligned}
\tag{3.6}
$$

and

$$
\|v_1 - v_2\|_{Y_{T^*}^{s',b'}} \leq C^3 T^{*c'-b'} \left(\|\tilde{u}\|_{X^{s,b}} \|u_1\|_{X^{s,b}} + \|u_2\|_{X^{s,b}} \|\tilde{u}\|_{X^{s,b}} \right).
\tag{3.7}
$$

Combining (3.4), (3.6) and (3.7) we have

$$
\|(u_1, v_1) - (u_2, v_2)\|_{X_{T^*}^{s,b} \times Y_{T^*}^{s',b'}} \leq \frac{1}{2} \|(\tilde{u}, \tilde{v})\|_{X^{s,b} \times Y^{s',b'}}.
\tag{3.8}
$$

From (3.5) and (3.8), we conclude that $\|(u_1, v_1) - (u_2, v_2)\|_{X_{T^*}^{s,b} \times Y_{T^*}^{s',b'}} \leq \epsilon$. Hence, since ϵ is arbitrary, $(u_1, v_1) = (u_2, v_2)$ on $[-T^*, T^*]$. Using translations in time, one can repeat this argument a finite number of times to conclude that $(u_1, v_1) = (u_2, v_2)$ on $[-T, T]$. $\square$

4   Ill-Posedness Results

Suppose that there exists $T > 0$ such that the Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R}) \times H^{s'}(\mathbb{R})$, in the time interval $[-T, T]$. Suppose also that there exists $t \in [-T, 0) \cup (0, T]$ such that the associated flow map data-solution (1.15) is two times Fréchet differentiable at zero. Then the second Fréchet derivative of S^t at zero belongs to $\mathcal{B}$, the normed space of bounded bilinear applications from $(H^s \times H^{s'})^2$ to $H^s \times H^{s'}$. In particular, we have the following estimate for the second Gâteaux derivative of S^t at zero,

$$
\left\| \frac{\partial^2 S^t}{\partial(\phi, \psi)^2}(0, 0) \right\|_{H^s \times H^{s'}} \leq \|D^2 S^t(0, 0)\|_{\mathcal{B}} \cdot \|(\phi, \psi)\|_{H^s \times H^{s'}}^2, \quad \forall \phi, \psi \in \mathcal{S}(\mathbb{R}).
\tag{4.1}
$$

We will denote $(u_{\phi,\psi}(t), v_{\phi,\psi}(t)) := S^t(\phi, \psi)$. This means that $(u_{\phi,\psi}(t), v_{\phi,\psi}(t))$ is a solution of the associated integral equations

$$
u_{\phi,\psi}(t) = e^{it\partial_x^2}\phi - i\alpha \int_0^t e^{i(t-t')\partial_x^2} \left(u_{\phi,\psi}(t') \cdot v_{\phi,\psi}(t') \right) dt',
\tag{4.2}
$$

$$
v_{\phi,\psi}(t) = e^{-\nu t \mathcal{H}\partial_x^2}\psi + \beta \int_0^t e^{-\nu(t-t')\mathcal{H}\partial_x^2}(\partial_x |u_{\phi,\psi}(t')|^2)dt'.
\tag{4.3}
$$

Since $(u_{0,0}(t), v_{0,0}(t)) = S^t(0, 0) = (0, 0)$, we have

$$
\frac{\partial S^t}{\partial(\phi, \psi)}(0, 0) = \left(\frac{\partial u_{0,0}}{\partial(\phi, \psi)}(t), \frac{\partial v_{0,0}}{\partial(\phi, \psi)}(t) \right) = \left(e^{it\partial_x^2}\phi, e^{-\nu t \mathcal{H}\partial_x^2}\psi \right), \quad \forall \phi, \psi \in \mathcal{S}(\mathbb{R}).
$$

Thus, using (4.2) to compute the second Gâteaux derivative of u at zero, in direction $(\phi, \psi) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$, yields

$$\frac{\partial^2 u_{0,0}}{\partial(\phi,\psi)^2}(t) = -2i\alpha \int_0^t e^{i(t-t')\partial_x^2} \left(e^{it'\partial_x^2}\phi \cdot e^{-\nu t' \mathcal{H}\partial_x^2}\psi \right) dt'.$$

Therefore, denoting $\xi_2 := \xi - \xi_1$, we have

$$
\begin{aligned}
\left\| \frac{\partial^2 u_{0,0}}{\partial(\phi,\psi)^2}(t) \right\|_{H^s} &= \left\| 2\alpha \langle\xi\rangle^s \int_0^t e^{-i(t-t')\xi^2} \left((e^{it'\partial_x^2}\phi)^\frown * (e^{-\nu t' \mathcal{H}\partial_x^2}\psi)^\frown \right)(\xi) dt' \right\|_{L_\xi^2} \\
&= \left\| 2\alpha \langle\xi\rangle^s \int_0^t e^{it'\xi^2} \int e^{-it'(\xi_2^2 + \nu|\xi_1|\xi_1)} \widehat{\phi}(\xi_2)\widehat{\psi}(\xi_1) d\xi_1 dt' \right\|_{L_\xi^2} \\
&= \left\| \int_0^t \int \Theta(t',\xi,\xi_1) f(\xi_2) g(\xi_1) d\xi_1 dt \right\|_{L_\xi^2},
\end{aligned}
$$

where $f(\xi_2) := \langle\xi_2\rangle^s \widehat{\phi}(\xi_2)$, $g(\xi_1) := \langle\xi_1\rangle^{s'} \widehat{\psi}(\xi_1)$ and

$$\Theta(t',\xi,\xi_1) := \frac{2|\alpha|\langle\xi\rangle^s}{\langle\xi_2\rangle^s \langle\xi_1\rangle^{s'}} \cdot e^{it'(\xi^2 - \xi_2^2 - \nu|\xi_1|\xi_1)}. \tag{4.4}$$

Hence, the assumption that the flow map (1.15) is C^2 at zero implies

$$\left\| \int_0^t \int \Theta(t',\xi,\xi_1) f(\xi_2) g(\xi_1) d\xi_1 dt' \right\|_{L_\xi^2} \leq \|D^2 S^t(0,0)\|_{\mathcal{B}} (\|f\|_{L^2} + \|g\|_{L^2})^2, \forall f,g \in \mathcal{S}(\mathbb{R}). \tag{4.5}$$

Similarly, differentiating the equation (4.3) twice, in direction $(\phi, 0) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$, yields

$$\frac{\partial^2 v_{0,0}}{\partial(\phi,0)^2}(t) = 2\beta \int_0^t e^{-\nu(t-t')\mathcal{H}\partial_x^2}\partial_x \left(e^{it'\partial_x^2}\phi \cdot \overline{e^{it'\partial_x^2}\phi} \right) dt'.$$

Thus, that assumption for the flow map (1.15) also implies

$$\left\| \int_0^t \int \Upsilon(t',\xi,\xi_1) f(\xi_2) \overline{f(-\xi_1)} d\xi_1 dt' \right\|_{L_\xi^2} \leq \|D^2 S^t(0,0)\|_{\mathcal{B}} \cdot \|f\|_{L^2}^2, \quad \forall f \in \mathcal{S}(\mathbb{R}), \tag{4.6}$$

where,

$$\Upsilon(t',\xi,\xi_1) := \frac{2|\beta||i\xi|\langle\xi\rangle^{s'}}{\langle\xi_2\rangle^s \langle\xi_1\rangle^s} \cdot e^{it'(\nu|\xi|\xi - \xi_2^2 + \xi_1^2)}. \tag{4.7}$$

Next, we will state an elementary result that will be useful in the proofs of Theorems 1.2, 1.3, 4.2 and 4.3.

Lemma 4.1. *Let $A, B, R \subset \mathbb{R}^n$. If $R - B \subset A$ then*

$$\|\mathbf{1}_R\|_{L^2(\mathbb{R}^n)} \|\mathbf{1}_B\|_{L^1(\mathbb{R}^n)} \leq \|\mathbf{1}_A * \mathbf{1}_B\|_{L^2(\mathbb{R}^n)}. \tag{4.8}$$

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Proof. If $R - B \subset A$, then

$$\mathbf{1}_A * \mathbf{1}_B(x) = \int_A \mathbf{1}_B(x - y)dy = \int_A \mathbf{1}_{x-B}(y)dy \geq \mathbf{1}_R(x)\|\mathbf{1}_B\|_{L^1(\mathbb{R}^n)}, \quad \forall x \in \mathbb{R}^n,$$

taking the L^2-norm, (4.8) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Theorem 1.2. *(i)* It is enough to show that (4.5) or (4.6) fails.
Case $s' < -1/2$: In this case, (4.5) fails. Indeed, for each $N \in \mathbb{N}$, define

$$\begin{aligned} A_N &:= \{\xi_1 \in \mathbb{R} : |(1 + |\nu|)\xi_1 + \mathrm{sgn}(\nu)N| < (1 + |\nu|)(4\langle t\rangle N)^{-1}\}, \\ B_N &:= \{\xi_2 \in \mathbb{R} : |2\xi_2 - \mathrm{sgn}(\nu)N| < (4\langle t\rangle N)^{-1}\}. \end{aligned}$$

For N sufficiently large (precisely $N > 1 + |\nu|$), we have that

$$\langle\xi_1\rangle \sim \langle\xi_2\rangle \sim \langle\xi_1 + \xi_2\rangle \sim N, \qquad \forall\xi_1 \in A_N, \ \forall\xi_2 \in B_N, \tag{4.9}$$

since $1 + |\nu| \neq 2$. Moreover, $\mathrm{sgn}(\xi_1) = -\mathrm{sgn}(\nu)$ for all $\xi_1 \in A_N$. Thus, we also have

$$|(\xi_1 + \xi_2)^2 - \nu|\xi_1|\xi_1 - \xi_2^2| = |\xi_1| \cdot |(1 + |\nu|)\xi_1 + 2\xi_2| < \frac{2N}{1 + |\nu|} \cdot \frac{2 + |\nu|}{4\langle t\rangle N} \leq \frac{1}{|t|}. \tag{4.10}$$

Observe that $\cos(x) \geq 1/2$ for $|x| \leq 1$. Hence, we deduce from (4.4), (4.9) and (4.10) that

$$Re\left(\Theta(t', \xi_1 + \xi_2, \xi_1)\right) \gtrsim \frac{1}{N^{s'}}, \qquad \forall|t'| \leq |t|, \forall\xi_1 \in A_N, \forall\xi_2 \in B_N. \tag{4.11}$$

Now, taking $f_N, g_N \in \mathcal{S}(\mathbb{R})$ such that $\mathbf{1}_{A_N} \leq g_N$, $\mathbf{1}_{B_N} \leq f_N$, $\|g_N\|_{L^2} \leq 2\|\mathbf{1}_{A_N}\|_{L^2}$ and $\|f_N\|_{L^2} \leq 2\|\mathbf{1}_{B_N}\|_{L^2}$, and using (4.11), we get that

$$\left|\int_0^t \int \Theta(t', \xi, \xi_1)f_N(\xi - \xi_1)g_N(\xi_1)d\xi_1 dt'\right| \geq \left|\int_0^t \int Re(\Theta(t', \xi, \xi_1))\mathbf{1}_{B_N}(\xi - \xi_1)\mathbf{1}_{A_N}(\xi_1)d\xi_1 dt'\right|$$

$$\gtrsim |t| \cdot \frac{\mathbf{1}_{A_N} * \mathbf{1}_{B_N}(\xi)}{N^{s'}}, \qquad\qquad \forall\xi \in \mathbb{R}. \quad (4.12)$$

Combining (4.12) with (4.5), yields

$$|t| \cdot \frac{\|\mathbf{1}_{A_N} * \mathbf{1}_{B_N}\|_{L^2}}{N^{s'}} \lesssim \|D^2 S^t(0,0)\|_{\mathcal{B}} \cdot (\|\mathbf{1}_{A_N}\|_{L^2} + \|\mathbf{1}_{B_N}\|_{L^2})^2. \tag{4.13}$$

On the other hand, defining

$$R_N := \{\xi \in \mathbb{R} : |\xi + b_\nu \, \mathrm{sgn}(\nu)N| < (8\langle t\rangle N)^{-1}\},$$

where $b_\nu = \frac{1}{1+|\nu|} - \frac{1}{2} \neq 0$, we have $R_N - B_N \subset A_N$. Hence, from (4.8) and (4.13), we conclude that

$$|t| \cdot \frac{N^{-\frac{1}{2}}N^{-1}}{N^{s'}} \lesssim \frac{\|D^2 S^t(0,0)\|_{\mathcal{B}}}{N}, \tag{4.14}$$

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which is false in the case $s' < -1/2$, since N can be chosen arbitrarily large.

Case $s' > 2s - 1/2$: For this case, we will show that (4.6) fails, using the same ideas used in the previous case. For $N \in \mathbb{N}$ sufficiently large (precisely $N > |1 - |\nu||^{-1}$), define

$$
\begin{aligned}
A_N &:= \{\xi_1 \in \mathbb{R} \ : \ |a_\nu \xi_1 + \mathrm{sgn}(\nu)(1 + |\nu|)N| < (c_t N)^{-1}\}, \\
B_N &:= \{\xi_2 \in \mathbb{R} \ : \ |a_\nu \xi_2 + \mathrm{sgn}(\nu)(1 - |\nu|)N| < (2c_t N)^{-1}\}, \\
R_N &:= \{\xi \in \mathbb{R} \ : \ |a_\nu \xi + 2\,\mathrm{sgn}(\nu)N| < (2c_t N)^{-1}\},
\end{aligned}
$$

where $a_\nu := |1 - |\nu|| \cdot |1 + |\nu|| \neq 0$ and $c_t := 1 + 8|t|(1 - |\nu|)^{-2}$. Then $R_N - B_N \subset A_N$. And if $\xi_1 \in A_N$ and $\xi_2 \in B_N$, then $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle \sim \langle \xi_1 + \xi_2 \rangle \sim N$, $\mathrm{sgn}(\xi_1 + \xi_2) = -\mathrm{sgn}(\nu)$ and

$$
\begin{aligned}
\left| \nu(\xi_1 + \xi_2)|\xi_1 + \xi_2| - \xi_2^2 + \xi_1^2 \right| &= |\xi_1 + \xi_2| \cdot |(1 - |\nu|)\xi_1 - (1 + |\nu|)\xi_2| \\
&< \frac{4N}{a_\nu} \cdot \frac{2}{c_t|1 - |\nu||N} \leq \frac{1}{|t|} \ . \tag{4.15}
\end{aligned}
$$

Following the arguments used in (4.9)-(4.11), we get from (4.7) and (4.15) that

$$
Re\left(\Upsilon(t', \xi_1 + \xi_2, \xi_1) \right) \gtrsim \frac{N^{s'+1}}{N^{2s}} \ , \qquad \forall |t'| \leq |t|, \forall \xi_1 \in A_N, \forall \xi_2 \in B_N. \tag{4.16}
$$

Now, taking $f_N \in \mathcal{S}(\mathbb{R})$ such that $\mathbf{1}_{-A_N \cup B_N} \leq f_N$ and $\|f_N\|_{L^2} \leq 2\|\mathbf{1}_{-A_N \cup B_N}\|_{L^2} \lesssim N^{-\frac{1}{2}}$, yields

$$
f_N(\xi - \xi_1)\overline{f_N(-\xi_1)} \geq \mathbf{1}_{-A_N \cup B_N}(\xi - \xi_1)\mathbf{1}_{-A_N \cup B_N}(-\xi_1) \geq \mathbf{1}_{B_N}(\xi - \xi_1)\mathbf{1}_{A_N}(\xi_1),
$$

for all $\xi, \xi_1 \in \mathbb{R}$. Thus, similarly to (4.12), we deduce from (4.16) that

$$
\left| \int_0^t \int \Upsilon(t', \xi, \xi_1) f_N(\xi - \xi_1)\overline{f_N(-\xi_1)} d\xi_1 dt' \right| \gtrsim |t| \cdot \frac{\mathbf{1}_{A_N} * \mathbf{1}_{B_N}(\xi) \cdot N^{s'+1}}{N^{2s}}, \qquad \forall \xi \in \mathbb{R}. \tag{4.17}
$$

Combining (4.17), (4.6) and (4.8), we conclude

$$
|t| \cdot \frac{N^{s'+1} \cdot N^{-\frac{1}{2}} \cdot N^{-1}}{N^{2s}} \lesssim \frac{\|D^2 S^t(0,0)\|_\mathcal{B}}{N}, \tag{4.18}
$$

which is false in the case $2s - 1/2 < s'$, since N can be chosen arbitrarily large.

(ii) If the map (1.10) is C^2 at zero then (4.5) and (4.6) hold for every $t \in [-T, T]$ and

$$
\sup_{t \in [-T, T]} \|D^2 S^t(0, 0)\|_\mathcal{B} < \infty. \tag{4.19}
$$

Thus, it is enough to show that (4.19) fails for $|s' - (s - 1/2)| > 3/2$, i.e., for $s' < s - 2$ or $s + 1 < s'$. Indeed, for each $N \in \mathbb{N}$, defining

$$
\begin{aligned}
A_N &:= \{\xi_1 \in \mathbb{R} \ : \ |\xi_1 - N| < 1/2\}, \\
B_N &:= \{\xi_2 \in \mathbb{R} \ : \ |\xi_2| < 1/4\}, \\
R_N &:= \{\xi \in \mathbb{R} \ : \ |\xi - N| < 1/4\},
\end{aligned}
$$

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we have $R_N - B_N \subset A_N$. Also, if $\xi_1 \in A_N$ and $\xi_2 \in B_N$ then

$$\langle \xi_1 \rangle \sim N, \quad \langle \xi_2 \rangle \sim 1, \quad \langle \xi_1 + \xi_2 \rangle \sim N,$$

and

$$\left| (\xi_1 + \xi_2)^2 - \nu |\xi_1| \xi_1 - \xi_2^2 \right| = |\xi_1| \cdot |(1 - \nu \operatorname{sgn}(\xi_1))\xi_1 + 2\xi_2| < 6(1 + |\nu|)N^2.$$

In addition, for $N > (6(1+|\nu|)T)^{-\frac{1}{2}}$, we define $t_N := (6(1+|\nu|)N^2)^{-1} \in (0, T]$. Therefore, following the arguments used in (4.9)-(4.14), we get that

$$N^{s-2-s'} \lesssim \frac{N^s \cdot t_N}{N^{s'}} \lesssim \|D^2 S^{t_N}(0,0)\|_{\mathcal{B}},$$

contradicting (4.19) when $s' < s - 2$ (since N can be chosen arbitrarily large). Moreover, for $\xi_1 \in A_N$ and $\xi_2 \in B_N$, we have

$$\left| \nu(\xi_1 + \xi_2)|\xi_1 + \xi_2| - \xi_2^2 + \xi_1^2 \right| < 6(1 + |\nu|)N^2.$$

Now following (4.15)-(4.18) we conclude that

$$N^{s'-s-1} \lesssim \frac{N^{s'+1} \cdot t_N}{N^s} \lesssim \|D^2 S^{t_N}(0,0)\|_{\mathcal{B}},$$

contradicting (4.19) when $s + 1 < s'$. This finishes the proof of the theorem. $\qquad \square$

Proof of Theorem 1.3. The proof use the same arguments used in the proof of Theorem 1.2(i). Suppose that we have some $t \in [-T, 0) \cup (0, T]$ such that the flow map (1.15) is C^2 at zero. For $N \in \mathbb{N}$, defining

$$\begin{aligned}
A_N &:= \{\xi_1 \in \mathbb{R} : |\xi_1 - \operatorname{sgn}(\nu)N| < (2\langle t \rangle N)^{-1}\}, \\
B_N &:= \{\xi_2 \in \mathbb{R} : |\xi_2| < (4\langle t \rangle N)^{-1}\}, \\
R_N &:= \{\xi \in \mathbb{R} : |\xi - \operatorname{sgn}(\nu)N| < (4\langle t \rangle N)^{-1}\},
\end{aligned}$$

we have $R_N - B_N \subset A_N$. Also, if $\xi_1 \in A_N$, $\xi_2 \in B_N$ then $\langle \xi_1 \rangle \sim N$, $\langle \xi_2 \rangle \sim 1$, $\langle \xi_1 + \xi_2 \rangle \sim N$, $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(\nu)$ and

$$\left| (\xi_1 + \xi_2)^2 - \nu |\xi_1| \xi_1 - \xi_2^2 \right| = |2\xi_1 \xi_2| < 4N \cdot (4\langle t \rangle N)^{-1} \le |t|^{-1}.$$

Following the arguments used in (4.9)-(4.14), we deduce from (4.5) that

$$|t| \cdot N^{s-\frac{1}{2}-s'} \lesssim \|D^2 S^t(0,0)\|_{\mathcal{B}}, \quad \forall N \in \mathbb{N}.$$

Hence $s' \ge s - 1/2$. On the other hand, defining

$$\begin{aligned}
A_N &:= \{\xi_1 \in \mathbb{R} : |\xi_1 + \operatorname{sgn}(\nu)N| < (2\langle t \rangle N)^{-1}\}, \\
B_N &:= \{\xi_2 \in \mathbb{R} : |\xi_2| < (4\langle t \rangle N)^{-1}\}, \\
R_N &:= \{\xi \in \mathbb{R} : |\xi + \operatorname{sgn}(\nu)N| < (4\langle t \rangle N)^{-1}\},
\end{aligned}$$

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we have $R_N - B_N \subset A_N$. Also, if $\xi_1 \in A_N$, $\xi_2 \in B_N$ then $\langle \xi_1 \rangle \sim N$, $\langle \xi_2 \rangle \sim 1$, $\langle \xi_1 + \xi_2 \rangle \sim N$, $\operatorname{sgn}(\xi_1 + \xi_2) = -\operatorname{sgn}(\nu)$ and

$$\left| \nu(\xi_1 + \xi_2)|\xi_1 + \xi_2| - \xi_2^2 + \xi_1^2 \right| = |2(\xi_1 + \xi_2)\xi_2| < 4N \cdot (4\langle t \rangle N)^{-1} \le |t|^{-1}.$$

Now following the arguments used in (4.15)-(4.18), we get from (4.6) that

$$|t| \cdot N^{s'-s+\frac{1}{2}} \lesssim \|D^2 S^t(0,0)\|_{\mathcal{B}}, \quad \forall N \in \mathbb{N}.$$

Hence $s' \le s - 1/2 \le s'$.

Finally, we will conclude that $s \ge 0$. Defining,

$$
\begin{aligned}
A_N &:= \{\xi_1 \in \mathbb{R} \ : \ |\xi_1 + \operatorname{sgn}(\nu)N| < (8\langle t \rangle N)^{-1}\}, \\
B_N &:= \{\xi_2 \in \mathbb{R} \ : \ |\xi_2 - \operatorname{sgn}(\nu)N| < (16\langle t \rangle N)^{-1}\}, \\
R_N &:= \{\xi \in \mathbb{R} \ : \ |\xi| < (16\langle t \rangle N)^{-1}\},
\end{aligned}
$$

we have $R_N - B_N \subset A_N$. Also, if $\xi_1 \in A_N$, $\xi_2 \in B_N$ then $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle \sim N$, $\langle \xi_1 + \xi_2 \rangle \sim 1$, $\operatorname{sgn}(\xi_1) = -\operatorname{sgn}(\nu)$ and

$$\left| (\xi_1 + \xi_2)^2 - \nu|\xi_1|\xi_1 - \xi_2^2 \right| = |2\xi_1(\xi_1 + \xi_2)| < 4N \cdot (4\langle t \rangle N)^{-1} \le |t|^{-1}.$$

Thus, similarly to (4.14), we conclude from (4.5) that $|t| \cdot N^{-\frac{1}{2}-s'-s} \lesssim \|D^2 S^t(0,0)\|_{\mathcal{B}}$, for every $N \in \mathbb{N}$. Hence $-2s = -1/2 - s' - s \le 0$, and this finishes the proof. $\qquad \square$

We finish this section giving some results about the remaining regions. For the *non-resonant* case, Theorem 4.2 states that, in a part of the remaining region, the L.W.P. of (1.1) can not be obtained by using the method of proof employed in this paper. Note that, in the case where $\nu = 0$, the method fails in the whole remaining region. In the *resonant* case, Theorem 4.3 ensures that the method used in [13] can not provide L.W.P. for (1.1) at the end-point.

Our proofs of Theorems 4.2 and 4.3 follow the arguments used by Kenig, Ponce and Vega in [12] to prove that their $X^{s,b}$ bilinear estimate for KdV equation fails for $s < -3/4$. But in our setting, Lemma 4.1 allows to give slightly more direct proofs.

Theorem 4.2. *Let $|\nu| \ne 1$ and $s, s', c, c' \in \mathbb{R}$. For every $c', c > 1/2$,*

(i) the bilinear estimate (2.8) fails for $s + 1/2 \le s'$;

(ii) the bilinear estimate (2.22) fails for $s' \le s - 3/2$;

(iii) the bilinear estimate (2.22) fails for $s' \le s - 1$, when $\nu = 0$.

Proof. *(i)* Recalling the notations of the proof of Theorem 2.2, we just have to show that (2.9) fails when $s + 1/2 \le s'$. For $N \in \mathbb{N}$, defining

$$
\begin{aligned}
A_N &:= \{(\tau_1, \xi_1) \in \mathbb{R}^2 \ : \ |\xi_1 - N| < N^{-1}, \ |\sigma_1| < 6\}, \\
B_N &:= \{(\tau_2, \xi_2) \in \mathbb{R}^2 \ : \ |\xi_2| < (2N)^{-1}, \ |\sigma_2| < 1\}, \\
R_N &:= \{(\tau, \xi) \in \mathbb{R}^2 \ : \ |\xi - N| < (2N)^{-1}, \ |\tau + \xi^2 - 2\xi N| < 1\},
\end{aligned}
$$

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we have $R_N - B_N \subset A_N$, since $\sigma_1 + \sigma_2 = \tau + \xi^2 - 2\xi\xi_1$. Moreover, for all $(\tau_1, \xi_1) \in A_N$ and $(\tau_2, \xi_2) \in B_N$,

$$\langle \xi_1 \rangle \sim N, \quad \langle \xi_2 \rangle \sim 1, \quad \langle \xi \rangle \sim N, \quad \langle \sigma \rangle \lesssim N^2.$$

Therefore, for all $(\tau, \xi, \tau_1, \xi_1) \in \mathbb{R}^4$,

$$\frac{N^{s'+1} \cdot \mathbf{1}_{B_N}(\tau_2, \xi_2)\mathbf{1}_{A_N}(\tau_1, \xi_1)}{N^{2(1-c')} \cdot N^s} \lesssim |\Phi(\tau, \xi, \tau_1, \xi_1)\mathbf{1}_{B_N}(\tau_2, \xi_2)\mathbf{1}_{A_N}(\tau_1, \xi_1)|. \tag{4.20}$$

Now, taking $f_N, g_N \in \mathcal{S}(\mathbb{R}^2)$ such that $\mathbf{1}_{A_N} \leq g_N$, $\mathbf{1}_{B_N} \leq f_N$, $\|g_N\|_{L^2} \lesssim \|\mathbf{1}_{A_N}\|_{L^2}$, $\|f_N\|_{L^2} \lesssim \|\mathbf{1}_{B_N}\|_{L^2}$ and combining (2.9), (4.20) and (4.8), yields the estimate

$$\frac{N^{s'+1} \cdot N^{-\frac{1}{2}} \cdot N^{-1}}{N^{2(1-c')} \cdot N^s} \lesssim N^{-\frac{1}{2}} \cdot N^{-\frac{1}{2}},$$

which is false for N sufficiently large whenever $s + 1/2 \leq s'$ and $c' > 1/2$.

(ii) Recalling the notations of the proof of Theorem 2.3, we just have to show that (2.23) fails when $s' \leq s - 3/2$. For $N \in \mathbb{N}$, defining

$$\begin{aligned}
A_N &:= \{(\tau_1, \xi_1) \in \mathbb{R}^2 \ : \ |\xi_1 - \mathrm{sgn}(\nu)N| < N^{-1}, \ |\sigma_1| < 7(1 + |\nu|)\}, \\
B_N &:= \{(\tau_2, \xi_2) \in \mathbb{R}^2 \ : \ |\xi_2| < (2N)^{-1}, \ |\sigma_2| < 1\}, \\
R_N &:= \{(\tau, \xi) \in \mathbb{R}^2 \ : \ |\xi - \mathrm{sgn}(\nu)N| < (2N)^{-1}, \ |\tau + \xi^2 + a_\nu \mathrm{sgn}(\nu)N\xi| < 1\},
\end{aligned}$$

where $a_\nu := |\nu| - 1$, we have $R_N - B_N \subset A_N$, since

$$\sigma_1 + \sigma_2 = [\tau + \xi^2 + a_\nu \mathrm{sgn}(\nu)N\xi] + [(1 + |\nu|)\xi_1(\xi_1 - \xi)] + [a_\nu(\xi_1 - \mathrm{sgn}(\nu)N)\xi].$$

Arguing as in the previous case, we get from (2.23) the following estimate

$$\frac{N^s \cdot N^{-\frac{1}{2}} \cdot N^{-1}}{N^{2(1-c)} \cdot N^{s'}} \lesssim N^{-\frac{1}{2}} \cdot N^{-\frac{1}{2}},$$

which is false for N sufficiently large when $s' \leq s - 3/2$ and $c > 1/2$.

(iii) Recalling the notations of the proof of Theorem 2.3 and defining for each $N \in \mathbb{N}$,

$$\begin{aligned}
A_N &:= \{(\tau_1, \xi_1) \in \mathbb{R}^2 \ : \ |\xi_1 - N| < 1, \ |\sigma_1| < 3\}, \\
B_N &:= \{(\tau_2, \xi_2) \in \mathbb{R}^2 \ : \ |\xi_2| < 1/2, \ |\sigma_2| < 1\}, \\
R_N &:= \{(\tau, \xi) \in \mathbb{R}^2 \ : \ |\xi - N| < 1/2, \ |\tau| < 1\},
\end{aligned}$$

we have $R_N - B_N \subset A_N$, since $\sigma_1 + \sigma_2 = \tau + \xi_2^2$ in the particular case $\nu = 0$. Arguing as in the case *(i)*, we get from (2.23) the following estimate

$$\frac{N^s}{N^{2(1-c)} \cdot N^{s'}} \lesssim 1,$$

which is false for N sufficiently large when $s' \leq s - 1$ and $c > 1/2$. $\qquad\square$

Theorem 4.3. *Let* $|\nu| = 1$, $(s, s') = (0 - 1/2)$ *and* $c, b, b' \in \mathbb{R}$. *The estimate* (2.22) *fails for every* $c > 1/2$.

Proof. Recalling the notations of the proof of Theorem 2.3, we just have to show that (2.23) fails. For $N \in \mathbb{N}$, defining

$$
\begin{aligned}
A_N &:= \{(\tau_1, \xi_1) \in \mathbb{R}^2 \; : \; |\xi_1 + \mathrm{sgn}(\nu)N| < 1/2, \; |\sigma_1| < 1\}, \\
B_N &:= \{(\tau_2, \xi_2) \in \mathbb{R}^2 \; : \; |\xi_2 - \mathrm{sgn}(\nu)N| < 1/4, \; |\sigma_2| < 1/3\}, \\
R_N &:= \{(\tau, \xi) \in \mathbb{R}^2 \; : \; |\xi| < 1/4, \; |\sigma + 2\,\mathrm{sgn}(\nu)N\xi| < 1/3\},
\end{aligned}
$$

we have $R_N - B_N \subset A_N$. Moreover, for all $(\tau_1, \xi_1) \in A_N$ and $(\tau_2, \xi_2) \in B_N$,

$$
\langle \xi_1 \rangle \sim N, \quad \langle \xi_2 \rangle \sim N, \quad \langle \xi \rangle \sim 1, \quad \langle \sigma \rangle \lesssim N.
$$

Therefore, for all $(\tau, \xi, \tau_1, \xi_1) \in \mathbb{R}^4$,

$$
\frac{N^{c-1} \mathbf{1}_{B_N}(\tau_2, \xi_2) \mathbf{1}_{A_N}(\tau_1, \xi_1)}{N^{-\frac{1}{2}}} \lesssim |\Psi(\tau, \xi, \tau_1, \xi_1) \mathbf{1}_{B_N}(\tau_2, \xi_2) \mathbf{1}_{A_N}(\tau_1, \xi_1)|. \tag{4.21}
$$

Now, taking $f_N, g_N \in \mathcal{S}(\mathbb{R}^2)$ such that $\mathbf{1}_{A_N} \leq g_N$, $\mathbf{1}_{B_N} \leq f_N$, $\|g_N\|_{L^2} \lesssim \|\mathbf{1}_{A_N}\|_{L^2}$, $\|f_N\|_{L^2} \lesssim \|\mathbf{1}_{B_N}\|_{L^2}$ and combining (2.23), (4.21) and (4.8), follows the estimate

$$
N^{c-\frac{1}{2}} \lesssim 1,
$$

which is false for N sufficiently large whenever $c > 1/2$. $\qquad\square$

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