# REPRESENTABLE CHOW CLASSES OF A PRODUCT OF PROJECTIVE SPACES

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ABSTRACT. Inside a product of projective spaces, we try to understand which Chow classes come from irreducible subvarieties. The answer is closely related to the theory of integer polymatroids. The support of a representable class can be (partially) characterized as some integer point inside a particular polymatroid. If the class is multiplicity-free, we obtain a complete characterization in terms of representable polymatroids. We also generalize some of the results to the case of products of Grassmannians.

#### 1. Introduction.

In [9], June Huh classified, up to a multiple, all Chow classes of  $\mathbb{P}^m \times \mathbb{P}^m$  that are representable by irreducible subvarities. He considers a Chow class as a *sequence* of integers, and part of his result is that sequences coming from representable classes have no internal zeroes. As a first step towards generalizing this result to an arbitrary number of projective spaces, we focus on the general version of the *no internal zeros* condition.

In general, the Chow ring of  $(\mathbb{P}^m)^n$  ha an easy presentation as a quotient of a polynomial ring in n variables. The Chow class of a subvariety, in this particular presentation, is also known as its multidegree. We can consider the support of this polynomial as points in  $\mathbb{R}^n$ , actually lying on a dilation of the standard n-1 simplex. When n=2 we get points in a line, i.e. a sequence. In this context, the natural generalization of no internal zeros is the fact that the support is the set of integer points of a polytope. Our main result is the following:

**Theorem 1.1.** Let  $Y \subset (\mathbb{P}^m)^n$  be an irreducible subvariety of dimension d. The support  $\mathrm{MSupp}_Y$  of the multidegree of Y forms an integer polytope  $P_X(Y)$  in  $\mathbb{R}^n$  defined by

$$\sum_{i \in [n]} t_i = nm - d$$

$$\sum_{i \in I} t_i \ge |I|m - \dim(\operatorname{pr}_I(Y)) \quad \forall I \subset [n]$$

which is an integer polymatroid. Moreover, in characteristic zero, a multiplicity-free Chow class of  $(\mathbb{P}^m)^n$  is representable if and only if the corresponding polymatroid is linearly representable.

For the proof we make the intersection theoretic analysis more general than what is necessary, hoping that some of the results can be applied to other situations. For instance, a very similar theorem will hold for products of grassmannians, and we give a brief application to the Flag variety embedded using the Plücker coordinates.

It is also worth mentioning that multiplicity-free subvarieties of products of projective spaces arise naturally in other mathematical contexts, both within the realm of algebraic geometry and

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beyond. For example, it comes up in the study of computer vision ([1]), Mustafin degeneration ([13], [4]) and algebraic statistics ([12]).

The paper is organized as follows. In Section 2 we set up the necessary background to state our main result on the multidegree support (Theorem 2.4). In Section 3 we carry out the intersection theoretic analysis in a general context. Then in Section 4 we apply the previous results to cases of products of projective spaces and products of Grassmannians and prove Theorem 3.1 and 4.1. We follow this with a quick application to the complete flag variety inside a product of Grassmanian case in Section 5. In Sections 6 and 7, we translate our results to the language of polymatroids, which among other things explain why our necessary conditions are not sufficient. In Section 8 we describe all possible supports for the multidegree and finally in Section 9 we point out some further directions.

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Convention. Throughout the paper, we work over an algebraically closed field K of characteristic zero.

#### 2. Background.

2.1. **Product of projective spaces.** For most of the paper we are interested in subvarieties of a product of projective spaces. In this section we set up some notations that will be used hereafter. Let  $[n] = \{1, 2, \dots, n\}$  and consider the variety

$$X = \underbrace{\mathbb{P}^m \times ... \times \mathbb{P}^m}_{n \text{ factors}} = \prod_{i \in [n]} \mathbb{P}^m$$

(i.e. we always label factors of X by integers between 1 and n). X comes naturally equipped with several projections, namely, for each  $\emptyset \neq I \subset [n]$  we denote

$$\operatorname{pr}_I:X\longrightarrow\prod_{i\in I}\mathbb{P}^m$$

to be the projection to the product of factors labeled by integers in I.

2.1.1. Multidegree. Recall that the Chow ring of  $\mathbb{P}^m$ , denoted  $A^*(\mathbb{P}^m)$ , is generated by the hyperplane class. More precisely it can be described as

$$A^*(\mathbb{P}^m) \cong \mathbb{Z}[H]/\langle H^{m+1}\rangle,$$

where H is the hyperplane class, and in general  $H^k$  is the class of codimension k linear subspaces. In particular,  $H^m$  is the class of a point and we can define the deg operator

$$\deg: A^*(\mathbb{P}^m) \longrightarrow \mathbb{Z}$$

by sending an element  $\sum b_i H^i$  to  $b_m$ .

Given any irreducible subvariety  $Y \subset \mathbb{P}^m$  of dimension d, its Chow class [Y] is equal to  $cH^{m-d}$ , where c is given by  $\deg([Y] \cdot H^d)$ , the number of points in the intersection of Y with a generic linear subspace of codimension d. The coefficient c is called the degree of Y.

Similarly, the Chow ring of a product of projective spaces,  $X = \prod_{i \in [n]} \mathbb{P}^m$ , is generated by pulling back the hyperplane classes of each factor. More precisely:

$$\mathbb{Z}[H_1, H_2, \cdots, H_n]/\langle H_1^{m+1}, H_2^{m+1}, \cdots, H_n^{m+1}\rangle$$

As above, the element  $H_1^{k_1}H_2^{k_2}\cdots H_n^{k_n}$  can be thought as the class of a product of generic linear subspaces of codimension  $k_i$  in the *i*-th factor of X, for all  $1 \le i \le n$ . We also have the deg map, giving the coefficient of the monomial  $H_1^m \cdots H_n^m$ . Any irreducible subvariety  $Y \subset X$  of dimension d can be written as

(2) 
$$[Y] = \sum_{\substack{a_1 + \dots + a_n = nm - d \\ a_i \in \mathbb{Z}^+}} c_{a_1, \dots, a_n}(Y) H_1^{a_1} \dots H_n^{a_n}$$

In what follows, to abbreviate the notation we use boldface to indicate vectors. So **a** is the vector  $(a_1, \dots, a_n)$ , we use  $\mathbf{1} = (1, \dots, 1)$ , and then  $\mathbf{H}^{m\mathbf{1}-\mathbf{a}}$  refers to  $H_1^{m-a_1} \cdots H_n^{m-a_n}$ . The coefficients c are the *multidegree* and they are defined by

$$c_{\mathbf{a}}(Y) = \deg([Y] \cdot \mathbf{H}^{m\mathbf{1} - \mathbf{a}})$$

Knowing the behavior of the irreducible varieties determine all varieties, since unions correspond to sums in the Chow ring. We want to understand the multidegrees of irreducible subvarieties, so we develop a bit more of notation.

The index set appearing in Equation (2) is the set of integer points in the dilated simplex  $(nm-d)\Delta_{n-1}$ . We define  $S_n^d := (nm-d)\Delta_{n-1} \cap \mathbb{Z}^n$ . The multidegree can be considered as an integer valued function:

$$m_Y: S_n^d \longrightarrow \mathbb{Z}^{\geq 0}$$
  
 $\mathbf{a} \longrightarrow c_{\mathbf{a}}(Y)$ 

One of our main objects of study is the support of the function  $m_Y$ .

**Definition 2.1.** The *multidegree support* of an irreducible subvariety  $Y \subset X = \prod_{i \in [n]} \mathbb{P}^m$  of dimension d, is given by the finite set

$$MSupp_Y = \{ \mathbf{a} \in S_n^d : m_Y(\mathbf{a}) > 0 \}$$

2.1.2. Understanding multidegrees. The driving question is:

Question 2.2. What are all possible functions  $f: S_n^d \longrightarrow \mathbb{Z}$  that can be constructed as  $m_Y$  for some subvariety  $Y \subset X$ ?

When n=1, just one copy of  $\mathbb{P}^m$ , the situation is fairly simple. Any positive integer can be a degree. When n=2 the situation is related to very interesting combinatorics. In this case the simplex  $(2m-d)\Delta_1$  is a line, and the set  $S_2^d$  is a string of 2m-d+1 points:

$$\{(2m-d,0),(2m-d-1,1),\cdots,(0,2m-d)\}$$

The multidegree can be expressed in the sequence of nonnegative integers

(3) 
$$(c_{(2m-d,0)}(Y), c_{(2m-d-1,1)}(Y), \cdots, c_{(0,2m-d)}(Y))$$

which we refer to as the multidegree sequence. June Huh proved the following remarkable theorem classifying multidegree sequences up to a positive integer multiple.

**Theorem 2.3.** A positive integer multiple of sequence of integers  $(c_0, \dots, c_d)$  is the multidegree sequence of an irreducible subvariety  $Y \subset \mathbb{P}^m \times \mathbb{P}^m$  of dimension d if and only it is a nonzero log-concave sequence of nonnegative integers with no internal zeros.

As a first step to generalize this result to m > 2, we focus on the last part of the statement. No internal zeros can be traslated as the support being convex. We shall prove the following theorem.

**Theorem 2.4.** Let  $Y \subset X$  be an irreducible subvariety of dimension d. The set  $MSupp_Y$  is the set of integer points in the polytope  $P_X(Y)$  in  $\mathbb{R}^n$  defined by

$$\sum_{i \in [n]} t_i = nm - d$$

$$\sum_{i \in I} t_i \ge |I|m - \dim(\operatorname{pr}_I(Y)) \quad \forall I \subset [n]$$

**Example 2.1.** Let's consider the variety  $Y \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$  cut out by the multihomogeneous prime ideal

$$J = \langle y_1 - 11y_3, y_0z_2 - 42y_3z_3, x_1z_2 + 39x_3z_3, y_2z_1 + 36y_3z_3, x_1z_0 - 28x_3z_0 - 19x_0z_3 + 46x_2z_3, x_3y_0 - 30x_1y_3 \rangle$$

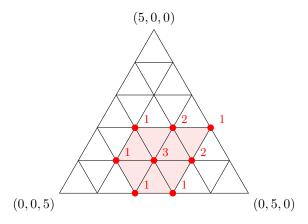
In the Chow ring it can be represented by

$$H_1^2H_2^3 + 2H_1^2H_2^2H_3 + H_1^2H_2H_3^2 + 2H_1H_2^3H_3 + 3H_1H_2^2H_3^2 + H_1H_2H_3^3 + H_1^3H_2^2 + H_2^2H_3^3 + H_1^3H_2^2 + H_2^2H_3^3 + H_1^3H_2^2 + H_2^3H_3^2 + H_3^3H_3^2 + H_$$

Which means that the multidegree is supported in the set

$$((2,3,0),(2,2,1),(2,1,2),(1,3,1),(1,2,2),(1,1,3),(0,3,2),(0,2,3))$$

This set can be represented graphically as



This is the set given by the inequalities:

$$t_1 + t_2 + t_3 = 5,$$
  $t_1 + t_2 \ge 2,$   $t_2 + t_3 \ge 3,$   $t_1 + t_2 \ge 2$   
 $t_1 + t_3 \ge 2,$   $t_1 \ge 0,$   $t_2 \ge 1,$   $t_3 \ge 0$ 

# 3. Intersection Theoretic Analysis.

An equivalent way of phrasing Theorem 2.4 is

**Theorem 3.1.** Let  $X = \prod_{i \in [n]} \mathbb{P}^m$ , and  $H_i$  the pullback of the hyerplane section of  $\mathbb{P}^m$  through the projection  $\operatorname{pr}_i$ . Then for Y an irreducible subvariety of dimension d we have that:

$$\deg\left([Y] \cdot H_1^{a_1} \cdot H_2^{a_2} \cdots H_n^{a_n}\right) > 0 \Longleftrightarrow \sum_{i \in [n]} a_i = d$$

$$\sum_{i \in I} a_i \le \dim\left(\operatorname{pr}_I(Y)\right) \quad \forall I \subset [n]$$

In this section we will use a more general setup and specify extra assumptions whenever necessary.

**Set-Up 3.2.** Let X be an irreducible complete variety of dimension m and  $D_1, \dots, D_n$  be distinct nef divisor classes on X. Suppose Y is a d-dimensional irreducible subvariety of X. We associate the following two subsets of  $\mathbb{Z}_{>0}^n$  to Y:

$$M_{D_1,\dots,D_n}(Y) := \{ \boldsymbol{a} \in \mathbb{Z}^n : \sum_{i \in [n]} a_i = d, \quad \deg\left([Y] \cdot \boldsymbol{D}^{\boldsymbol{a}}\right) > 0 \}$$
$$Q_{D_1,\dots,D_n}(Y) := \{ \boldsymbol{a} \in \mathbb{Z}^n : \sum_{i \in [n]} a_i = d, \quad \sum_{i \in I} a_i \le u_I, \forall I \subset [n] \}$$

where 
$$u_I = \max\{\sum_{i \in I} a_i : \boldsymbol{a} \in M_{D_1, \dots, D_n}(Y)\}$$

By definition,  $M \subset Q$ , and we would like to know under which conditions we have equality. Note that Q is the set of integer points of a polytope. A priori is not even clear if M is convex.

**Proposition 3.3.** In the case where  $X = \prod_{i \in [n]} \mathbb{P}^m$  and the  $D_i$ 's are pullbacks of hyperplane sections we have  $u_I = \dim(\operatorname{pr}_I(Y))$ .

Proof. On one hand, if  $\deg([Y] \cdot \prod_{i \in I} H_i^{a_i}) > 0$ , take general representatives of  $H_i$ . Any point in the set-theoretic intersection of the hyperplanes with Y will map, through  $\operatorname{pr}_I$ , into the intersection of  $\operatorname{pr}_I(Y)$  with the corresponding hyperplanes in  $\prod_{i \in I} \mathbb{P}^m$ . On the other hand, a point in the set-theoretic intersection of the irreducible variety  $\operatorname{pr}_I(Y)$  with some hyperplanes in  $\prod_{i \in I} \mathbb{P}^m$  has preimage (under  $\operatorname{pr}_I$ ) inside the intersection of Y with the preimages of the hyperplanes.

Following the above definitions it is easy to see that for any nonempty subset  $\{i_1, \dots, i_s\} \subset [n]$  we have

$$M_{D_{i_1}, \dots, D_{i_s}}(Y) \subset M_{D_1, \dots, D_n}(Y), \quad Q_{D_{i_1}, \dots, D_{i_s}}(Y) \subset Q_{D_1, \dots, D_n}(Y)$$

We identify elements of  $M_{D_{i_1},\dots,D_{i_s}}(Y)$  as elements in  $M_{D_1,\dots,D_n}(Y)$  with zero entries outside of  $\{i_1,\dots,i_s\}$ . We can make the following decompositions:

$$Q_{D_{i_1},\dots,D_{i_s}}(Y) = Q_0 \cup Q_1$$
 
$$Q_0 = \{ \mathbf{a} \in M_{D_{i_1},\dots,D_{i_s}}(Y) : a_1 = 0 \}$$
 
$$Q_1 = \{ \mathbf{a} \in M_{D_{i_1},\dots,D_{i_s}}(Y) : a_1 > 0 \}$$

And also for M:

$$M_{D_{i_1},\dots,D_{i_s}}(Y) = M_0 \cup M_1$$

$$M_0 = \{ \mathbf{a} \in M_{D_{i_1},\dots,D_{i_s}}(Y) : a_1 = 0 \}$$

$$M_1 = \{ \mathbf{a} \in M_{D_{i_1},\dots,D_{i_s}}(Y) : a_1 > 0 \}$$

We have M = Q if and only if

$$(4) Q_0 = M_0$$

$$(5) Q_1 = M_1$$

Now we try to impose numerical conditions on the  $u_I$ 's to make sure both equalities happen.

3.1. When do we have  $Q_0 = M_0$ ? Recall that  $M_0 \subset Q_0$ . We can assume that  $u_{\{2,\dots,n\}} = d$ , since otherwise  $Q_0$  is empty.

Note that  $M_0$  can be naturally identified with  $M_{D_2...D_n}(Y)$ , so by induction (in the number of divisors) let's suppose that it is equal to  $Q_{D_2...D_n}(Y)$ . We need to compare the inequalities for all J that do not contain 1.

$$\sum_{j \in J} t_j \le u_J = \max\{\sum_{j \in J} a_j : \mathbf{a} \in M_{D_1, \dots D_n}(Y)\}$$
$$\sum_{j \in J} t_j \le \hat{u}_J = \max\{\sum_{j \in J} a_j : \mathbf{a} \in M_{D_2, \dots D_n}(Y)\}$$

Clearly  $\hat{u}_J \leq u_J$ , and if they are equal then we would have  $Q_0 = M_0$ . We would like to have equality, i.e. the maximum is already attained in any subset that is not empty. Thus we pose the following condition:

Condition A: Given  $D_1, ..., D_n$ , for any fixed nonempty subset  $I_1 \subset [n]$  and any  $I_2 \supset I_1$  such that  $M_{\{D_i|i\in I_2\}}(Y) \neq \emptyset$ , we have

$$\max\{\sum_{j\in J} a_j : \mathbf{a} \in M_{\{D_i|i\in I_1\}}(Y) = \max\{\sum_{j\in J} a_j : \mathbf{a} \in M_{\{D_i|i\in I_2\}}(Y)\}$$

for all  $J \subset I_1$ .

**Example 3.1.** In the case where  $X = \prod_{i \in [n]} \mathbb{P}^m$ , Y is an irreducible subvariety, and  $D_i$  are the pullbacks of hyperplane sections, Condition **A** holds, as  $u_J = \dim \operatorname{pr}_J(Y)$  and the projection  $\operatorname{pr}_J$  factors through  $\operatorname{pr}_J$ .

3.2. When do we have  $Q_1 = M_1$ ? In this case, note that we can replace Y by Y' which represents the intersection product  $[Y] \cdot D_1^{-1}$ . Clearly

$$(a_1, \cdots, a_n) \in M_1 \iff (a_1 - 1, a_2, \cdots, a_n) \in M_{D_1, \cdots, D_m}(Y')$$

Since dim  $Y' = \dim Y - 1$ , we can assume by induction that  $M_{D_1, \dots, D_m}(Y') = Q_{D_1, \dots, D_m}(Y')$ . We assume  $u_1 \ge 1$  since otherwise  $M_1 = \emptyset$ .

<sup>&</sup>lt;sup>1</sup>We may assume Y' is irreducible, as we can always take an irreducible component of Y' if necessary.

By definition we have  $u_J \ge u_J' = \max\{\sum_{i \in J} a_i : \mathbf{a} \in M_{D_1, \dots D_n}(Y')\}$ , and  $u_J \ge u_J' + 1$  if  $1 \in J$ .

Now we compare with  $Q_1$  and  $M_1$  by first passing to Y'. Given  $\mathbf{a} = (a_1, \dots, a_n) \in Q_1$ , this means

$$\sum_{j \in J} a_j \le u_J \quad \forall J.$$

If, additionally, we had

(6) 
$$\left(\sum_{j\in J} a_j\right) - 1 \le u_J' \quad 1 \in J$$

(7) 
$$\sum_{j \in J} a_j \le u_J' \quad 1 \notin J$$

Then, by induction hypothesis,  $(a_1 - 1, a_2, \dots, a_n) \in Q_{D_1, \dots, D_n}(Y') = M_{D_1, \dots, D_n}(Y')$ , which implies  $\mathbf{a} \in M_{D_1, \dots, D_n}(Y)$ , more specifically, in  $M_1$ . We pose the following condition:

**Condition B:** For any j, denote Y' to be the intersection of a general element in  $|D_j|$  with Y. If  $M_{D_1,\dots,D_n}(Y') \neq \emptyset$ , then  $u'_J \geq u_J - 1$  for all nonempty subset  $j \in J$ , and  $u'_J = u_J$  whenever  $u_J < u_{J \cup \{j\}}$ .

**Proposition 3.4.** When Condition B holds,  $Q_1 = M_1$ .

*Proof.* Without lost of generality, let's assume j=1. For any  $\mathbf{a}=(a_1,\cdots,a_n)$ , let  $\mathbf{a}':=(a_1-1,\cdots,a_n)$ . We will check equations (5) and (6). If  $1 \in J$ , then  $\sum_{j\in J} a_j \leq u_J$  implies Equation (5)

$$\left(\sum_{j\in J} a_j\right) - 1 \le u_J - 1 \le u'_J.$$

If  $u_J$  is attained by some **a** with  $a_1 > 0$ , then  $u'_J = u_J$  and Equation (6) follows. It is left to check Equation (6) in the case where  $1 \notin J$  and  $u_J$  not attained by any point with  $a_1 > 0$ . In this case we have  $u_J \leq u_{J \cup \{1\}}$ . We are assuming that if  $u_J < u_{J \cup \{1\}}$ , then  $u_J = u'_J$  in which case Equation (6) follows. So the only case left to check is where  $1 \notin J$  and  $u_J = u_{J \cup \{1\}}$ .

For  $\mathbf{a} \in Q_1$  we have

$$a_1 + \sum_{i \in J} a_i \le u_{J \cup \{1\}},$$

so we have

$$\sum_{i \in J} a_i \le u_{J \cup \{1\}} - a_1 = u_J - a_1 \le u'_J,$$

which is what we wanted to show.

#### 4. Multiprojective space.

We now focus on concrete varieties and divisors. In order to prove Theorem 3.1, following the previous section, we need to check that both conditions  $\bf A$  and  $\bf B$  are met.

Proof of Theorem 3.1. Let  $X = \prod_{i \in [n]} \mathbb{P}^m$ , Y an irreducible subvariety of dimension d, and  $D_i$  the pullbacks of hyperplane sections. In Example 3.1 we already checked that condition **A** holds. We now prove that condition **B** holds too (we assume without loss of generality that j = 1). We may

assume dim  $Y \geq 2$ , otherwise the result is trivial.

Let Y' be the intersection of Y with a general element in  $|H_1|$ . By Bertini's theorem, we may assume it is an irreducible subvariety of dimension d-1. For all  $J \subset [n]$  we have the commutative diagram:

(8) 
$$Y' \longrightarrow \operatorname{pr}_{J}(Y')$$

$$\downarrow \qquad \qquad \downarrow$$

$$Y \longrightarrow \operatorname{pr}_{J}(Y)$$

We claim that for a general element in  $|H_1|$ , that the dimension of generic fibers in both horizontal morphisms is the same. This will imply that  $\dim Y' - \dim (\operatorname{pr}_J(Y')) = \dim Y - \dim (\operatorname{pr}_J(Y))$ , or equivalently  $\dim (\operatorname{pr}_J(Y)) = \dim (\operatorname{pr}_J(Y')) + 1$ . This will settle the first part of Condition  $\mathbf{B}$ , since  $u_J = \dim \operatorname{pr}_J(Y)$  in this case.

We proceed to prove the claim. Let U be a dense open subset of Y such that for any  $x \in U$ , the fiber dimension along  $p_J$  is generic, i.e. there exist s such that for all  $x \in U$  we have  $\dim \operatorname{pr}_J^{-1}(\operatorname{pr}_J(x)) = s$ . Since  $1 \in J$ , U dominates  $\operatorname{pr}_1(Y)$  so it contains a smooth dense open subvariety of  $\operatorname{pr}_1(Y)$ , also note that we can assume  $\operatorname{pr}_1(Y)$  is reduced since we are just interested in intersection theoretic and dimension issues. Applying Bertini's Theorem on  $\mathbb{P}^m$  to  $\operatorname{pr}_1(Y)$  and pulling back to X through  $\operatorname{pr}_1$ , we get that for a general point  $p \in \operatorname{pr}_J(U')$  we have  $\dim \operatorname{pr}_J^{-1}(p) = s$ , where U' is the intersection of U with general element of  $|H_1|$ . In particular, for a general (closed) point  $y \in \operatorname{pr}_J(Y')$ , we have that  $\operatorname{pr}_J^{-1}(y)$  contains some point in U' and hence  $\dim \operatorname{pr}_J^{-1}(y) = s$ , which is what we wanted to show.

Now suppose  $\dim \operatorname{pr}_J(Y) < \dim \operatorname{pr}_{J \cup 1}(Y)$ . Focusing on dimension, we may take  $\operatorname{pr}_{J \cup 1}(Y)$  as an irreducible subvariety of  $\prod_{j \in J \cup 1} \mathbb{P}^m$ . Denote  $\pi : \operatorname{pr}_{J \cup 1}(Y) \to \operatorname{pr}_J(Y)$  the projection. For a general point  $y \in \operatorname{pr}_J(Y)$ ,  $\operatorname{pr}_1(y)$  is closed and has a component of dimension at least 1, since the generic fiber dimension of  $\pi$  is positive and  $\operatorname{pr}_1(y)$  is homeomorphic to  $\pi^{-1}(y)$ . Thus  $\operatorname{pr}_1(y)$  intersects with a given hyperplane. By pulling back to  $\operatorname{pr}_{J \cup 1}(Y)$  along  $\operatorname{pr}_1$ , we conclude that the intersection Y'' of  $\operatorname{pr}_{J \cup 1}(Y)$  with a hyperplane in  $|H_1|$  maps dominantly and hence onto  $\operatorname{pr}_J(Y)$  via  $\pi$ . Since Y' maps onto Y'' via  $\operatorname{pr}_{J \cup 1}$ , we get  $\operatorname{dim} \operatorname{pr}_J(Y') = \operatorname{dim} \pi(Y'') = \operatorname{dim} \operatorname{pr}_J(Y)$ . This finishes the verification of condition  $\mathbf{B}$ .

We present another situation, more general, in which the results apply.

**Theorem 4.1.** Fix a finite dimensional vector space V. Let  $X = \prod_{i \in [n]} \operatorname{Gr}(k_i, n_i)$ , and  $D_i$  the pullback of the schubert divisor  $\operatorname{Gr}(k_i, n_i)$  through the projection  $\operatorname{pr}_i$ . Then for Y an irreducible subvariety of dimension d we have that:

$$\deg\left([Y]\cdot D_1^{a_1}\cdot D_2^{a_2}\cdots D_n^{a_n}\right) > 0 \Longleftrightarrow \sum_{i\in[n]} a_i = d$$

$$\sum_{i\in I} a_i \le \dim\left(\operatorname{pr}_I(Y)\right) \quad \forall I\subset[n]$$

*Proof.* For Condition A, we claim that  $u_J = \dim p_J(Y)$  holds in this case as well.

First recall the well-known fact that Künneth formula holds Chow rings of Grassmannians [15, Section 3]. In particular,

$$[Y] = \sum a_{\lambda_1, \dots, \lambda_m} \left( \bigotimes_{j=1}^m \sigma_{\lambda_j} \right),$$

where  $a_{\lambda_1,...,\lambda_m} > 0$ ,  $\sigma_{\lambda_j}$  is a Schubert cycle on  $Gr(k_j, n_j)$  and the codimensions of  $\sigma_{\lambda_j}$  (j = 1, ..., m) add up to  $\operatorname{codim}_X(Y)$ .

Note that  $u_J \leq \dim \operatorname{pr}_J(Y)$  is straightforward: if  $[Y] \cdot \prod D_j^{d_j} > 0$ , then  $[\operatorname{pr}_J(Y) \times \prod_{j \notin J} \operatorname{Gr}(k_j, n_j)] \cdot \prod_{j \in J} D_j^{d_j}$  is non-trivial, as  $\operatorname{pr}_J(Y) \times \prod_{j \notin J} \operatorname{Gr}(k_j, n_j)$  contains Y and intersection product is functorial (cf. miss, section 6.5).

This further implies  $[\operatorname{pr}_J(Y)] \cdot \prod_{j \in J} D_j^{\prime d_j}$  is non-trivial, as

$$[\operatorname{pr}_J(Y) \times \prod_{j \not\in J} \operatorname{Gr}(k_j, n_j)] \cdot \prod_{j \in J} D_j^{d_j} = ([\operatorname{pr}_J(Y)] \cdot \prod_{j \in J} {D_j'}^{d_j}) \times \prod_{j \not\in J} \operatorname{Gr}(k_j, n_j)$$

(cf. ibid, Example 2.3.1). Hence,  $[\operatorname{pr}_J(Y)] \cdot \prod_{j \in J} D_j'^{d_j} \neq 0$  and  $\dim \operatorname{pr}_J(Y) \geq \sum d_j$ . Since this inequality holds for any  $(d_j) \in M_{D_1,\dots,D_m}(Y)$ ,  $u_J \leq \dim \operatorname{pr}_J(Y)$ .

Suppose  $\operatorname{pr}_J(Y)$  is a codimension r subvariety of  $\prod_{i\in J}\operatorname{Gr}(k_i,n_i)$ , so that

$$[\operatorname{pr}_{J}(Y)] = \sum a_{\{\lambda_{j}|j\in J\}}(\bigotimes_{j\in J} \sigma_{\lambda_{j}}),$$

where  $a_{\{\lambda_j|j\in J\}} > 0$ ,  $\sigma_{\lambda_j}$  is a Schubert cycle on  $Gr(k_j, n_j)$  and the codimensions of  $\sigma_{\lambda_j}$   $(j \in J)$  add up to r. Over  $Gr(k_j, n_j)$ , the divisor class  $\sigma_1$  intersects positive-dimensional Schubert cycle positively. Hence, there exists integers  $d_j$  for  $j \in J$  such that  $\sum d_j = r$  and

$$\left(\bigotimes_{j\in J}\sigma_{\lambda_j}\right)\cdot\prod_{j\in J}D_j^{d_j}=\prod_{j\in J}(\sigma_{\lambda_j}\cdot\sigma_1^{d_j})>0,$$

where  $D_j$  is the pull-back of the divisor class over  $\operatorname{Gr}(k_j,n_j)$ . Pulling back to X, one can conclude that  $[Y] \cdot \prod_{j \in J} D_j^{d_j}$  is a non-trivial effective cycle, i.e. a positive linear combination of products of Schubert cycles on each  $\operatorname{Gr}(k_j,n_j)$ . Using again the fact that over each  $\operatorname{Gr}(k_j,n_j)$  the divisor class  $\sigma_1$  intersects positive-dimensional Schubert cycle positively, one can find  $e_j$   $(j \in [m])$  such that  $(e_j) \in M_{D_1,\ldots,D_m}(Y)$  and  $e_j \geq d_j$  for  $j \in J$ . This implies that  $u_J \geq \dim \operatorname{pr}_J(Y)$ . Thus, Condition A follows.

Next, we verify Condition **B** in case where dim  $Y = d \ge 2$ . The argument is more or less the same as for the case of product of projective spaces. Again, by Bertini's Theorem, we may assume that the intersection Y' of Y with a general element of  $|D_1|$  is of pure dimension d-1. We still have the commutative diagram (8). Moreover, the claim still holds, as a consequence of applying Bertini's Theorem over  $Gr(k_1, n_1)$ , which verifies the first part of Condition **B**. Finally, it only remains to check dim  $\operatorname{pr}_J(Y') = \dim \operatorname{pr}_J(Y)$ , when dim  $\operatorname{pr}_J(Y) < \dim \operatorname{pr}_{J \cup \{1\}}(Y)$ . The key point is that for a general cycle of  $Gr(k_1, n_1)$  of dimension  $s \ge 1$ , it intersects a divisor in  $Gr(k_1, n_1)$ , which is a direct consequence of Schubert calculus.

#### 5. Example: Flag varieties.

In this section we apply the previous results in a concrete combinatorial example. Consider the complete flag variety Fl(V) with V a vector space of dimension n+1. This variety parametrizes complete flags, i.e. sequences

$$V_{\bullet} := V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} = V$$

where each  $V_i$  is a linear subspace of V of dimension i respectively. One can embed this variety in a product of grassmannians:

$$Fl_n(V) \subset Gr(1,V) \times Gr(2,V) \times \cdots \times Gr(n,V)$$

as the subvariety cut out by incidence relations. For more information see [6, Part III] or the first lecture of [3].

The Picard group of each grassmannian Gr(k, V) is isomorphic to  $\mathbb{Z}$  with generator

$$\sigma_k := \{ W \in \operatorname{Gr}(k, V) : \dim W \cap U \ge 1 \},$$

where U is a fixed codimension k subspace. Pulling back and restricting to the flag variety, we obtain the *Schubert* divisors  $X_{w_0s_k}$ , the Schubert varieties of codimension 1. Our main theorem in this case becomes:

**Theorem 5.1.** Let  $z_k$  be the cohomological classes of the Schubert divisors  $X_{w_0s_k}$  in the flag varierty Fl(V), then we have

$$\deg (z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}) > 0 \Longleftrightarrow \sum_{i \in [n-1]} a_i = d$$

$$\sum_{i \in I} a_i \le \dim (Fl_I(V)) \quad \forall I \subset [n]$$

where  $Fl_I(V)$  is the partial flag variety, where the flags include subspaces only in dimensions given by  $I \subset [n]$ 

The cohomology ring of Fl(V) is well understood. It has a presentation given by

$$H^*(Fl(V)) = k[x_1, \dots, x_{n+1}]/\langle e_r(x_1, \dots, x_{n+1}) : 1 \le k \le n+1 \rangle$$

where each variable  $x_i$  has degree 2 and  $e_r$  is the r- elementary symmetric polynomials. In this presentation we can write the Schubert classes  $z_k$  using the so called Schubert polynomials:

$$z_k = x_1 + x_2 + \dots + x_k$$

And then it is a question of when do  $\frac{n(n-1)}{2}$  of these polynomials have a nonzero product.

**Example 5.1.** Consider the case n=4. We have three classes  $z_1, z_2, z_3$ , and we can compute the degree of  $z_1^{a_1} z_2^{a_2} z_3^{a_3}$  for each  $(a_1, a_2, a_3)$  with  $a_1 + a_2 + a_3 = 6$  and represent it as in Figure 5.1.

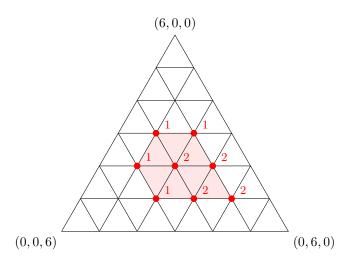


FIGURE 1. The multidegree of  $Fl(K^4)$  inside a product of grassmannians.

#### 6. Polymatroids.

**Definition 6.1.** A polymatroid is a pair (E, r), where E is a finite set and r is a function  $f: 2^E \to \mathbb{Z}$ , called a polymatroid rank satisfying the following:

- (R1)  $f(\emptyset) = 0$ .
- (R2) For all  $I \subset J \subset E$ , then  $f(I) \leq f(J)$ .
- (R3) For all  $I, J \subset E$ , then  $f(I \cap J) + f(I \cup J) \leq f(I) + f(J)$ . This condition is called submodularity.

We can define a polytope  $P_f$  as follows:

$$P_f := \left\{ \mathbf{t} \in \mathbb{R}^E : \sum_{i \in E} t_i = f([n]), \quad \sum_{i \in I} t_i \le f(I) \quad \forall I \subset [n] \right\}.$$

Any such polytope is called a polymatroid polytope.

The polytope given by Theorem 3.1 depends on the set function

$$\begin{array}{ccc} f_Y: 2^{[n]} & & \mathbb{Z} \\ & I & \longrightarrow & \dim(\mathrm{pr}_I(Y)) \end{array}$$

This function determines a polymatroid on the set [n].

**Proposition 6.2.** Suppose  $X = X_1 \times ... \times X_n$  is a product of irreducible complete varieties and  $D_j$  are pull-backs of nef divisors on  $X_j$ . Then, for any d-dimensional irreducible subvariety Y of X,  $f_Y$  is a polymatroid rank function. The polytopes  $P_X(Y)$  arising in Theorem 2.4 are polymatroid polytopes.

*Proof.* The function  $f_Y$  satisfies properties (R1) and (R2). To prove the claim we must prove it satisfies (R3), submodularity.

There is a natural morphism  $\operatorname{pr}_{I\cup J}(Y) \longrightarrow \operatorname{pr}_{I}(Y) \times_{\operatorname{pr}_{I\cap J}(Y)} \operatorname{pr}_{J}(Y)$ , where the fibered product is taken in the category of varieties. One can see it is injective at the level of closed points.

Moreover,  $\operatorname{pr}_I(Y) \longrightarrow \operatorname{pr}_{I\cap J}(Y)$  and  $\operatorname{pr}_I(Y) \longrightarrow \operatorname{pr}_{I\cap J}(Y)$  are surjective morphisms of varieties. The generic fiber dimension of  $\operatorname{pr}_J(Y) \to \operatorname{pr}_{I\cap J}(Y)$  is  $f_Y(J) - f_Y(I\cap J)$  and hence the generic fiber dimension of the projection  $\operatorname{pr}_I(Y) \times_{\operatorname{pr}_{I\cap J}(Y)} \operatorname{pr}_J(Y) \to \operatorname{pr}_I(Y)$  is  $f_Y(J) - f_Y(I\cap J)$ . In particular,  $\operatorname{dim} \operatorname{pr}_I(Y) \times_{\operatorname{pr}_{I\cap J}(Y)} \operatorname{pr}_J(Y) = f_Y(I) + f_Y(J) - f_Y(I\cup J)$ .

Thus, 
$$f_Y(I \cup J) \leq f_Y(I) + f_Y(J) - f_Y(I \cap J)$$
. Hence  $f_Y$  is submodular.

**Definition 6.3.** We call a polymatroid ([n], r) Chow if there exists a subvariety  $Y \subset X := \prod_{i \in [n]} \mathbb{P}^m$ , such that  $r = f_Y$ .

With these definitions and developments, Question 2.2 translates into the questions of which polymatroids are Chow. We will give an answer, but first we have to review some facts about matroid theory.

## 7. Matroids.

A matroid is a particular type of polymatroid.

**Definition 7.1.** A matroid is a polymatroid such that the rank function r satisfies one extra property:

(R4) If 
$$I \subset E$$
, then  $0 \le r(I) \le |I|$ .

Naturally, the function r is called the rank function of the matroid.

**Example 7.1.** In the general setting, the function  $f_Y$  does not satisfy (R4), but in the case  $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$  it does. For any subvariety Y we have the matroid  $([n], f_Y)$ .

For us there are two important sources of matroids.

**Example 7.2.** Let V be a vector space over k. Any finite set of vectors  $E = \{v_1, \dots, v_n\} \subset V$ , together with the function  $r(I) = \dim(\operatorname{span}(v_i : i \in I))$  makes (E, r) into a matroid. Matroids arising in this way are called *linearly representable* matroids.

Ardila and Boocher [2] proved that all linearly representable matroids are Chow in  $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ .

**Example 7.3.** Let  $k \to L$  a field extension. A finite set of elements  $E = \{x_1, \dots, x_n\} \subset L$  together with the function  $r(I) = \text{tr.deg}(k(x_{i_1}, \dots, x_{i_s}) : k)$ , where  $I = \{i_1, \dots, i_s\}$ , makes (E, r) a matroid. Matroids arising in this way are called *algebraic* matroids.

The theory of matroids and polymatroids are closely related. Indeed we have the following proposition.

**Proposition 7.2.** For any polymatroid (E, r) there exists a matroid (E', r') and a map, called an embedding,  $\sigma: E \longrightarrow 2^{E'}$  such that

$$r(I) = r' \left( \bigcup_{S \in \sigma(I)} S \right).$$

This map can be taken to be one to one and such that  $\bigcup_{e \in E} \sigma(e) = E'$ .

We will see this explicitly in the case needed in the next proposition.

**Theorem 7.3.** Let  $Y \subset (\mathbb{P}^m)^n$  be a subvariety. The polymatroid  $([n], f_Y)$  can be embedded in an algebraic matroid.

*Proof.* Let's consider a dense open affine of X isomorphic to  $(\mathbb{A}^m)^n$ . In this set, the variety Y is defined by an ideal  $\mathfrak{a}$  in  $k[x_{ij}:1\leq i\leq n,1\leq j\leq m]$ . Consider the field extension  $k\longrightarrow k(Y)$ , and the set  $E=\{x_{ij}\}_{i,j}$ , the set of the images of the generators in k(Y).

The projection of Y onto the subring  $k[x_{ij}:i\in I]$  is defined by the ideal  $\mathfrak{a}_I:=\mathfrak{a}\cap k[x_{ij}:i\in I]\subset k[x_{ij}]$ , in other words by the ideal obtained by *eliminating* the variables  $x_{ij}$  for  $i\notin I$ . By definition, we have

$$f_Y(I) = \dim \operatorname{pr}_I(Y) = \operatorname{tr.deg} (k(x_{ij} : i \in I)/\mathfrak{a}_I : k).$$

By definition,  $\mathfrak{a}_I$  is the subset of  $\mathfrak{a}$  that express all possible relations among the variables  $x_{ij}$  for  $i \in I$ . So there is no harm in computing the right hand side as a quotient over  $\mathfrak{a}$ . This means that  $\dim \operatorname{pr}_I(Y) = \operatorname{tr.deg}(k(x_{ij} : i \in I)/\mathfrak{a} : k)$  and that's precisely the rank, as an algebraic matroid, of I in k(V).

The embedding is given by:

$$\sigma: [n] \longrightarrow E$$
 
$$\sigma(i) = \{x_{ij} : 1 \le j \le m\}.$$

Furthermore, the following theorem (See [10]) ensures that, over the complex numbers, the notions of algebraic and linear matroids coincide.

**Theorem 7.4.** Over any field of characteristic zero, in particular over  $\mathbb{C}$ , any algebraic matroid is linear.

*Remark* 7.5. Here we are crucially using the complex numbers, more concretely being algebraically closed. Previous theorem fails in finite fields.

There are matroids that are not algebraic over any field. The smallest example is the *Vamos* matroid, with 8 elements and rank 4. See [14, Example 2.1.22]. Hence we have the following proposition.

**Proposition 7.6.** There are polymatroids that are not Chow.

On a positive direction we will describe all Chow polymatroids.

#### 8. Chow polymatroids.

We can expand the definition of linear representability of matroids to polymatroids.

**Definition 8.1.** A polymatroid rank function f is representable (over  $\mathbb{C}$ ) if there exist a complex vector space V together with subspaces  $V_1, \dots, V_n$ , such that for  $I \subset [n]$ :

$$f(I) = \dim\left(\sum_{i \in I} V_i\right).$$

In other words, a polymatroid is linear if it can be embedded in a linear matroid.

It turns out that all linear polymatroids are Chow polymatroids. This follows from Li's construction in [11]. We review the approach:

Given a vector space V and subspaces  $V_1, \dots, V_n$  such that  $\bigcap_{i \in [n]} V_i = \emptyset$ , in [11] the author consider the closure of the rational map

$$\mathbb{P}(V) \dashrightarrow \prod_{i \in [n]} \mathbb{P}(V/V_i).$$

In our context, it is better for notation if we choose complements, say by endowing V with an inner product and taking orthogonal complements,  $W_i$  to each  $V_i$ , and view them as subsets of  $\mathbb{P}(V)$ . The condition  $\bigcap_{i \in [n]} V_i = \emptyset$  is equivalent to  $V = \text{span}\{W_i : i \in [n]\}$ . So we instead consider the image of the composition

$$\mathbb{P}(V) \longrightarrow \prod_{i \in [n]} \mathbb{P}(W_i) \hookrightarrow \prod_{i \in [n]} \mathbb{P}(V) = X.$$

For us  $V = \mathbb{C}^m$ . We call this subvariety  $X(V; W_1, \dots, W_n)$ . From [11] we have:

**Theorem 8.2.** The subvariety  $Y = X(V; W_1, \dots, W_n)$  is irreducible. Its Chow class is multiplicity free. Its associated submodular function is

$$f_Y(I) = \dim\left(\sum_{i \in I} W_i\right) - 1.$$

Remark 8.3. To get rid of that -1 we can simply add an extra coordinate to V and each  $V_i$  as we'll see in an example below.

The construction of Ardila-Boocher in [2] starts with a linear subspace  $L \subset \mathbb{A}^n$ , given by the nullspace of a full rank  $m \times n$  matrix M, then it takes the closure of the composition:

$$L \hookrightarrow \mathbb{A}^n \to (\mathbb{A}^1) \times \cdots \times (\mathbb{A}^1) \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^1.$$

Considering the column vectors  $v_1, \dots, v_n$  of M, the corresponding subvariety  $\tilde{L}$  has an associated submodular function

$$f_{\tilde{L}}(I) = \dim \operatorname{span}\{v_i : i \in I\}.$$

In other words it is the rank function of the linear matroid given by the columns of M.

To recover this same submodular function in the context of this section we can do as follows: We are given  $V \cong K^m$  and n vectors  $v_1, \dots, v_n \in K^n$ , which span the whole space. We cannot take the lines they generated as our  $W_i$  because by projectivizing the dimension will drop, and in [2] the idea is to complete a line rather than to make it a point. What we can do is we can add an extra coordinate  $V \oplus K\{\mathbf{e}_0\}$  and consider

$$W_i := \operatorname{span}\{v_i, \mathbf{e}_0\}$$

Then, for  $Y = X(W_1, \dots, W_n)$ , we have

$$f_Y(I) = \dim \operatorname{span}\{W_i : i \in I\} - 1 = \dim \operatorname{span}\{v_i : i \in I\}.$$

So we recover the rank function of the matroid.

**Theorem 8.4.** A polymatroid is Chow if and only if it is linear.

In other words, all possible supports for representable Chow classes are given by Li's construction.

*Proof.* By Theorem 7.3 we can embed the polymatroid ( $[n], f_Y$ ) into an algebraic matroid. By Theorem 7.4, this matroid is linear. Hence ( $[n], f_Y$ ) is a linear polymatroid.

Conversely, Li's construction shows that all linear polymatroids are Chow.  $\Box$ 

In the multiplicity free case, the support is enough to determine the whole Chow class, since all the values are 1.

**Theorem 8.5.** Any multiplicity-free representable Chow class of a product of projective spaces is representable by some  $X(V; W_1, \dots, W_k)$ .

# 9. Further Questions.

9.1. **Small number of copies.** Our Question 2.2 is relating representability of Chow classes to the question of linear representability of polymatroids. In general it seems that a complete classification of representable matroids or polymatroids is intractable. However there are some well-known results over small sets.

**Proposition 9.1** (Ingleton's Inequality (cf. [10], or p.177 in [14])). The rank function of a subspace arrangement  $V_1$ ,  $V_2$ ,  $V_3$ ,  $V_4$  (over any field) satisfies the following inequality:

(9) 
$$[1,2] + [3] + [4] + [1,3,4] + [2,3,4] \le [1,3] + [1,4] + [2,3] + [2,4] + [3,4].$$
Here,  $[I] := \dim \sum_{i \in I} V_i$ .

**Proposition 9.2** (cf. Theorem 3,5 in [7], [5]). *Identify the set of all representable polymatroids* over a finite set E with integer vectors inside  $\mathbb{R}^{2^E}$ .

- (1) When  $|E| \leq 3$ , the convex hull  $\Gamma_E$  of all representable polymatroids are cut out by the basic inequalities in the definition of a polymatroid.
- (2) When |E| = 4,  $\Gamma_E$  is cut out by those basic inequalities together with Ingleton's inequality.
- (3) When |E| = 5,  $\Gamma_E$  is cut out by the basic inequalities, Ingleton's inequality, together with another 24 inequalities.<sup>2</sup>

In particular, for the case of three copies of projective spaces we see that the set of submodular functions coincide with the set of representable classes. So perhaps in this case a complete classification is possible.

9.2. **Generalizing log concavity.** Log concavity can be understood in terms of syzygies of a one dimensional toric ideal.

The left hand side generalizes naturally in higher dimensions. One can could consider all linear relations between the points. For example we could have an inequality as in the Figure below.

A conjecture of Neil White(see [16] and section 5 of [8]) predicts that all this relations between points in polymatroid are generated by quadrics. This seems like the natural place to start looking for a generalization of the log concavity statement.

 $<sup>^{2}</sup>$ For every inequality, all possible choices of elements in E are considered.

Linear relation between points:

Log concave inequality:

$$2p_2 = p_1 + p_3 a_k^2 \ge a_{k-1} a_{k+1}$$



Linear relation between points:

Inequality:

 $a_2a_3 \ge a_1a_4$ 

$$p_2 + p_3 = p_1 + p_4$$



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