DIRECT IMAGE OF PARABOLIC LINE BUNDLES

ROBERT AUFFARTH AND INDRANIL BISWAS

ABSTRACT. Given a vector bundle E, on an irreducible projective variety X, we give a necessary and sufficient criterion for E to be a direct image of a line bundle under a surjective étale morphism. The criterion in question is the existence of a Cartan subalgebra bundle of the endomorphism bundle $\operatorname{End}(E)$. As a corollary, a criterion is obtained for E to be the direct image of the structure sheaf under an étale morphism. The direct image of a parabolic line bundle under any ramified covering map has a natural parabolic structure. Given a parabolic vector bundle, we give a similar criterion for it to be the direct image of a parabolic line bundle under a ramified covering map.

1. Introduction

This work was inspired by [DP], and also by [Bo1], [Bo2]. In [DP], the authors address the following question: Given a vector bundle E on a smooth projective curve X over a field of characteristic zero, is there a branched covering of X and a line bundle L on X such that $E \otimes L$ is the direct image of the structure sheaf under the covering map? They answer this question affirmatively.

Let X and Y be smooth projective curves defined over an algebraically closed field k, where X is irreducible but Y need not be, and let $f: Y \longrightarrow X$ be a finite separable morphism. Then for any parabolic line bundle L_* on Y, the direct image f_*L_* has a natural parabolic structure.

Here we address the following questions:

Given a parabolic vector bundle E_* on X, when there is a pair (Y, f) as above such that

- (1) E_* is isomorphic to the parabolic vector bundle $f_*\mathcal{O}_Y$ (the parabolic structure on \mathcal{O}_Y is the trivial one), and more generally,
- (2) E_* is isomorphic to the parabolic vector bundle f_*L_* , where L_* is a parabolic line bundle on Y?

Under the assumption that the characteristic of k is zero, the first question is answered in Corollary 3.2 and the second question is answered in Theorem 3.1. When the characteristic of k is positive, these results remain valid if the parabolic structure on E_* is tame; see Section 3.2.

To understand the criteria in Corollary 3.2 and Theorem 3.1 the key step is to consider the étale case. More precisely, consider the following questions:

(1) Given a vector bundle E on X, when there is an étale covering $f: Y \longrightarrow X$, and a line bundle L on Y, such that $f_*L = E$.

²⁰¹⁰ Mathematics Subject Classification. 14E20, 14J60, 14L15.

Key words and phrases. Direct image, Cartan subalgebra, parabolic bundle, equivariant bundle.

(2) With X and E as above, when there is an étale covering $f: Y \longrightarrow X$ such that $f_*\mathcal{O}_Y = E$.

We prove the following (see Theorem 2.4 and Corollary 2.6):

Theorem 1.1. Let X be an irreducible projective variety defined over an algebraically closed field k. Given a vector bundle E on X of rank d, the following two are equivalent:

(1) There is an étale covering of degree d

$$f: Y \longrightarrow X$$

and a line bundle L on Y, such that $f_*L = E$.

(2) There is a subbundle $A \subset \operatorname{End}(E)$ of rank d such that for each closed point $x \in X$, the subspace $A_x \subset \operatorname{End}(E)_x = \operatorname{End}(E_x)$ is a Cartan subalgebra.

If there is a subbundle $A \subset \operatorname{End}(E)$ as in the second statement, then (Y, f) in the first statement can be so chosen that $A = f_*\mathcal{O}_Y$. In that case, the number of connected components of the scheme Y coincides with $\dim H^0(X, A)$.

Corollary 1.2. Given X and E as in Theorem 1.1, the following two are equivalent:

(1) There is an étale covering

$$f: Y \longrightarrow X$$

such that $f_*\mathcal{O}_Y = E$.

(2) There is a fiberwise injective homomorphism $\alpha: E \longrightarrow \operatorname{End}(E)$ such that for each closed point $x \in X$, the subspace $\alpha(E_x) \subset \operatorname{End}(E)_x = \operatorname{End}(E_x)$ is a Cartan subalgebra.

When these hold, the number of connected components of the scheme Y coincides with $\dim H^0(X, E)$.

In Section 4 we describe, in terms of the above criterion, when an étale cover factors through another given étale covering.

- 2. Direct image of line bundles by étale coverings
- 2.1. Construction of homomorphism of vector bundles from direct image. Let k be an algebraically closed field. Let X be an irreducible projective variety defined over k. Take any pair (Y, f), where Y is a projective scheme and

$$f: Y \longrightarrow X$$

is an étale covering of degree d. We do not assume that Y is connected. Take any line bundle L over Y. The direct image

$$E := f_*L \longrightarrow X \tag{2.1}$$

is a vector bundle of rank d. The direct image $f_*\mathcal{O}_Y$ of the structure sheaf will be denoted by W.

We have the natural homomorphism $E \otimes W \longrightarrow E$ which on any open set $U \subseteq X$ is induced by the bilinear map

$$\mathcal{O}_Y(f^{-1}U) \times L(f^{-1}U) \longrightarrow L(f^{-1}U), \quad (y, l) \longmapsto y \cdot l,$$

from the universal property of tensor-product. This homomorphism defines a homomorphism

$$\varphi: W \longrightarrow \operatorname{End}(E) = E \otimes E^*,$$
 (2.2)

where E is the vector bundle in (2.1).

The fibers of $\operatorname{End}(E)$ are reductive Lie algebras isomorphic to $\operatorname{Lie}(\operatorname{GL}(d,k)) = \operatorname{M}(d,k)$ (the $d \times d$ matrices with entries in k). We recall that a Lie subalgebra A of $\operatorname{M}(d,k)$ is a Cartan subalgebra if

- $\dim A = d$, and
- there is an element $T \in GL(d, k)$ such that the conjugation of M(d, k) by T takes A into the space of diagonal matrices.

Lemma 2.1. The homomorphism φ in (2.2) satisfies the condition that for every closed point $x \in X$, the image $\varphi(x)(W_x)$ is a Cartan subalgebra of $\operatorname{End}(E)_x = \operatorname{End}(E_x)$.

Proof. Fix an ordering of the elements of the set $f^{-1}(x)$ of cardinality d. Let $\{y_1, \dots, y_d\}$ be this ordered set $f^{-1}(x)$. Fix a nonzero element $v_i \in L_{y_i}$ for each $1 \leq i \leq d$. Since

$$E_x = \bigoplus_{i=1}^d L_{y_i},$$

the collection $\{v_1, \dots, v_d\}$ defines an ordered basis of the vector space E_x . Similarly, we have

$$W_x = \bigoplus_{i=1}^d k_i,$$

where k_i is a copy of k. For any $c \in k_i \subset W_x$, the endomorphism $\varphi(c) \in \operatorname{End}(E_x)$ sends the basis element

- v_i to $c \cdot v_i$, and
- v_i , $i \neq i$, to 0.

Therefore, the image $\varphi(W_x)$ is the space of all diagonal matrices with respect to the above basis $\{v_1, \dots, v_d\}$.

Since $H^0(Y, \mathcal{O}_Y) = H^0(X, f_*\mathcal{O}_Y)$, and the dimension of $H^0(Y, \mathcal{O}_Y)$ coincides with the number of connected components of the scheme Y, Lemma 2.1 has the following corollary:

Corollary 2.2. The number of connected components of the scheme Y coincides with the dimension of $H^0(X, W)$.

2.2. Criterion for direct images under étale maps. Let F and V be vector bundles on X of same rank d. Let

$$\varphi: V \longrightarrow \operatorname{End}(F) = F \otimes F^*$$
 (2.3)

be an \mathcal{O}_X -linear homomorphism.

Proposition 2.3. Assume that φ in (2.3) satisfies the condition that for every closed point $x \in X$, the image $\varphi(V_x)$ is a Cartan subalgebra of the Lie algebra $\operatorname{End}(F)_x = \operatorname{End}(F_x)$. Then there is an étale covering

$$f: Y \longrightarrow X$$

and a line bundle L over Y, such that

$$f_*L = F$$
 and $f_*\mathcal{O}_Y = V$.

Furthermore, the homomorphism φ coincides with the homomorphism in (2.2) corresponding to the above triple (Y, f, L).

Proof. For any closed point $x \in X$, consider the Cartan subalgebra $\varphi(V_x) \subset \operatorname{End}(F_x)$. It produces an *unordered* set of d lines in F_x

$$\{l_t^x\}_{t \in B_x}, \quad \#B_x = d$$
 (2.4)

such that

- the d lines $\{l_t^x\}_{t\in B_x}$ together generate the fiber F_x , and
- for each $t \in B_x$, there is a unique functional $\mu_t^x \in V_x^*$ with the property that for all $v \in V_x$, we have

$$\varphi(v)(w) = \begin{cases} \mu_t^x(v) \cdot w & \text{if } w \in l_t^x \\ 0 & \text{if } w \in l_s^x, s \neq t. \end{cases}$$

In (2.4) we use the notation $\{l_t^x\}_{t\in B_x}$ instead of $\{l_1^x, \dots, l_d^x\}$ in order to emphasize that in general these d lines do not have any ordering which can be chosen uniformly over X. We note that the d elements $\{\mu_t^x\}_{t\in B_x}$ of V_x^* are distinct. In fact, $\{\mu_t^x\}_{t\in B_x}$ is a basis of the dual vector space V_x^* .

The quasiprojective variety defined by the total space of the dual vector bundle V^* will be denoted by \mathbb{V}^* . The locus in \mathbb{V}^* of the above collection $\{\mu_t^x\}_{t\in B_x, x\in X}$ defines a reduced subscheme $Y\subset\mathbb{V}^*$. Let

$$f: Y \longrightarrow X$$
 (2.5)

be the restriction to Y of the natural projection $\mathbb{V}^* \longrightarrow X$. This map f defines an étale covering of X of degree d because the d lines $\{l_t^x\}_{t\in B_x}$ in (2.4) can be uniformly ordered over suitable étale open subsets of X.

Consider the pulled back vector bundle $f^*F \longrightarrow Y$. It has a line subbundle L whose fiber over any closed point $\mu_t^x \in Y$ is the line l_t^x contained in F_x .

The projection formula gives that $f_*f^*F = F \otimes f_*\mathcal{O}_Y$. This and the trace homomorphism $f_*\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ together give a homomorphism $f_*f^*F \longrightarrow F$. This homomorphism and the inclusion map $L \hookrightarrow f^*F$ together produce a homomorphism

$$\eta: f_*L \longrightarrow F.$$
(2.6)

This η is an isomorphism because the d lines $\{l_t^x\}_{t\in B_x}$ in (2.4) generate F_x .

Now, as constructed in (2.2), we have a fiberwise injective homomorphism

$$\widehat{\varphi}: f_*\mathcal{O}_Y \longrightarrow \operatorname{End}(f_*L) = \operatorname{End}(F).$$

It is straight-forward to check that the image of $\widehat{\varphi}$ coincides with the image of φ . Hence the composition $(\widehat{\varphi}^{-1}|_{\varphi(V)}) \circ \varphi$ is an isomorphism from V to $f_*\mathcal{O}_Y$.

In terms of the above isomorphisms $V \xrightarrow{\sim} f_* \mathcal{O}_Y$ and η in (2.6), the homomorphism φ in the statement of the proposition coincides with the homomorphism in (2.2).

Note that the condition in Proposition 2.3 that $\varphi(V_x)$ is a Cartan subalgebra of $\operatorname{End}(F_x)$ for every $x \in X$ implies that the homomorphism φ is fiberwise injective.

Combining Lemma 2.1, Corollary 2.2 and Proposition 2.3 we have the following:

Theorem 2.4. Given a vector bundle E on X of rank d, the following two are equivalent:

(1) There is an étale covering

$$f: Y \longrightarrow X$$

where Y is a projective scheme, and a line bundle L on Y, such that $f_*L = E$.

(2) There is a subbundle $A \subset \operatorname{End}(E)$ of rank d such that for each closed point $x \in X$, the subspace $A_x \subset \operatorname{End}(E)_x = \operatorname{End}(E_x)$ is a Cartan subalgebra.

If there is a subbundle $A \subset \operatorname{End}(E)$ as in the second statement, then (Y, f) in the first statement can be so chosen that $A = f_*\mathcal{O}_Y$. In that case, the number of connected components of Y coincides with dim $H^0(X, A)$.

Corollary 2.5. A vector bundle F on X of rank d splits into a direct sum of d line bundles if and only if there is a trivial subbundle of rank d

$$\iota : \mathcal{O}_X^{\oplus d} \hookrightarrow \operatorname{End}(F)$$

such that for each closed point $x \in X$, the subspace

$$\operatorname{image}(\iota(x)) \subset \operatorname{End}(F)_x = \operatorname{End}(F_x)$$

is a Cartan subalgebra.

Proof. If $F = \bigoplus_{i=1}^d L_i$, then the homomorphism

$$\mathcal{O}_X^{\oplus d} = \bigoplus_{i=1}^d \operatorname{End}(L_i) \hookrightarrow \operatorname{End}(F)$$

satisfies the condition in the statement of the corollary.

To prove the converse, assume that there is a trivial subbundle of rank d

$$\iota: \mathcal{O}_{X}^{\oplus d} \hookrightarrow \operatorname{End}(F)$$

satisfying the condition in the corollary. Now the unordered set in (2.4) becomes uniformed ordered over entire X. Therefore, the covering Y in (2.5) becomes a disjoint union of d copies of X. Consequently, the vector bundle f_*L in (2.6) is a direct sum of line bundles. Since η in (2.6) is an isomorphism, the vector bundle F splits into a direct sum of d line bundles.

Setting $L = \mathcal{O}_Y$ in Theorem 2.4 we have the following:

Corollary 2.6. Given a vector bundle E on X, the following two are equivalent:

(1) There is an étale covering

$$f: Y \longrightarrow X$$

such that $f_*\mathcal{O}_Y = E$.

(2) There is a fiberwise injective homomorphism $\alpha: E \longrightarrow \operatorname{End}(E)$ such that for each closed point $x \in X$, the subspace $\alpha(E_x) \subset \operatorname{End}(E)_x = \operatorname{End}(E_x)$ is a Cartan subalgebra.

When these statements hold, the number of connected components of the scheme Y coincides with dim $H^0(X, E)$.

Remark 2.7. Let E be a vector bundle on X along with two subbundles A, $B \subseteq \operatorname{End}(E)$ such that for each closed point $x \in X$, both A_x and B_x are Cartan subalgebras of $\operatorname{End}(E_x)$. Let $f: Y \longrightarrow X$ (respectively, $g: Z \longrightarrow X$) be the étale covering and $L \longrightarrow Y$ (respectively, $L' \longrightarrow Z$) be the line bundle associated to A (respectively, B). We note that if T is an automorphism of E such that $B = T^{-1}AT$, then there exists an isomorphism of E-schemes E such E such E such that E is an automorphism of E such that E such that E such that E is an automorphism of E such that E is a such

2.3. An example. Let E be a vector bundle on X such that there are two Cartan subalgebra bundles A and B of End(E). It is natural to ask whether there is an automorphism T of E such that $B = T^{-1}AT$. We give an example where there is no such T.

Let X be a smooth projective elliptic curve defined over \mathbb{C} . It has exactly three non-trivial line bundles of order two. Let L and M be two distinct nontrivial line bundles on X of order two. Therefore, the third nontrivial line bundle of order two is $L \otimes M$.

A theorem of Atiyah says that there are stable vector bundles on X of rank two and degree one, and any two of them differ by tensoring with a holomorphic line bundle of degree zero [At, p. 433,Theorem 6], [At, p. 434,Theorem 7] (see [BB, p. 70, Theorem 4.6], [BB, p. 70–71, Theorem 4.7] for an exposition). Take a stable vector bundle E on X of rank two and degree one. If N is a holomorphic line bundle on X, then

$$\operatorname{End}(E) = E \otimes E^* = (E \otimes N) \otimes (E \otimes N)^* = \operatorname{End}(E \otimes N).$$

Therefore, the endomorphism bundle $\operatorname{End}(E)$ does not depend on the choice of the stable vector bundle E of rank two and degree one. Since E is stable, the vector bundle $\operatorname{End}(E)$ is polystable. From the classification of vector bundles on X we know that any polystable vector bundle on X of degree zero is a direct sum of holomorphic line bundles [At, p. 433, Theorem 6] (see also [BB, p. 70, Theorem 4.6]). It is known that

$$\operatorname{End}(E) = \mathcal{O}_X \oplus L \oplus M \oplus (L \otimes M). \tag{2.7}$$

This can also be seen as follows. Let $f: Y \longrightarrow X$ be the unramified double cover corresponding to L. Then there is a holomorphic line bundle ξ on Y of degree one such that $E = f_*\xi$. Since $f_*\mathcal{O}_Y = L \oplus \mathcal{O}_X$, this implies that L is a subbundle of $\operatorname{End}(E)$. Hence L is a direct summand of $\operatorname{End}(E)$ because $\operatorname{End}(E)$ is polystable of degree zero. Similarly, M and $M \otimes L$ are also direct direct summands of $\operatorname{End}(E)$. Therefore, it follows that $\operatorname{End}(E)$ decomposes as in (2.7).

Since $E = f_*\xi$, we know that $f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L$ is a Cartan subalgebra bundle of End(E). Similarly, $\mathcal{O}_X \oplus M$ and $\mathcal{O}_X \oplus (M \otimes L)$ are also Cartan subalgebra bundles of End(E).

On the other hand $H^0(X, \operatorname{End}(E)) = \mathbb{C}$ because E is stable; note that this also follows from (2.7). Hence the automorphisms of E act trivially on $\operatorname{End}(E)$. Therefore, the above Cartan subalgebra bundles $\mathcal{O}_X \oplus L$, $\mathcal{O}_X \oplus M$ and $\mathcal{O}_X \oplus (M \otimes L)$ are not related by automorphism of E.

3. Ramified coverings of curves

Throughout this section we assume that X is an irreducible smooth projective curve defined over an algebraically closed field k.

A quasiparabolic structure on a vector bundle E over X consists of the following:

- a finite set of reduced distinct closed points $S = \{x_1, \dots, x_n\} \subset X$, and
- for each $x_i \in S$, a filtration of subspaces

$$0 \subsetneq F_1^i \subsetneq \cdots \subsetneq F_{\ell_i}^i = E_{x_i}$$

of the fiber E_{x_i} .

The subset S is called the *parabolic divisor*. A *quasiparabolic bundle* is a vector bundle equipped with a quasiparabolic structure. A *parabolic vector bundle* is a quasiparabolic bundle $(E, S, \{F_j^i\})$ as above together with real numbers λ_j^i , $1 \leq i \leq n$, $1 \leq j \leq \ell_i$, such that

$$1 > \lambda_1^i > \lambda_2^i > \dots > \lambda_{\ell_i}^i \ge 0.$$

These numbers λ_j^i are called *parabolic weights*. For notational convenience, a parabolic vector bundle $(E, S, \{F_j^i\}, \{\lambda_j^i\})$ is abbreviated as E_* . See [MS] for more on parabolic vector bundles.

We will consider only rational parabolic weights. Henceforth, we will assume that all the parabolic weights are rational numbers.

Let Y be a smooth projective curve, which need not be irreducible, and let

$$f: Y \longrightarrow X$$

be a finite separable morphism. Let L_* be a parabolic line bundle on Y with L being the underlying line bundle. The direct image $E := f_*L$ has a natural parabolic structure which will be described below.

Let $R \subset Y$ be the set of points where f is ramified. Let $P \subset Y$ be the parabolic divisor for L_* . The parabolic divisor for the parabolic structure on E is the image $f(R \cup P)$. Take a point $x \in f(R \cup P) \setminus f(R)$ in the complement of f(R). Then

$$(f_*L)_x = \bigoplus_{y \in f^{-1}(x)} L_y.$$

The quasiparabolic filtration of $(f_*L)_x$ is constructed using this decomposition. The parabolic weight of the line $L_y \subset (f_*L)_x$ is the parabolic weight of L_* at the point y. If y is not a parabolic point of L_* , then the parabolic weight of the line $L_y \subset (f_*L)_x$ is taken to be zero. Combining these we get a parabolic structure on E_x .

Now take any $x \in f(R)$. Let $\{y_1, \dots, y_m\}$ be the reduced inverse image $f^{-1}(x)_{\text{red}}$. The multiplicity of f at y_i will be denoted by b_i ; so $f^{-1}(x) = \sum_{i=1}^m b_i y_i$. For every $1 \leq i \leq m$, let $V_i \subset E_x$ be the image in the fiber E_x for the natural homomorphism

$$f_*(L \otimes \mathcal{O}_Y(-\sum_{j=1, j \neq i}^m b_j y_j)) \longrightarrow f_*L = E.$$

We have dim $V_i = b_i$, and

$$E_x = \bigoplus_{i=1}^m V_i. \tag{3.1}$$

We will construct a weighted filtration on each V_i ; these combined together will give the weighted filtration of E_x using (3.1). For each $0 \le \ell \le b_i$, let $F_\ell^i \subset E_x$ be the image for

the natural homomorphism

$$f_*(L \otimes \mathcal{O}_Y(-\ell y_i - \sum_{j=1, j \neq i}^m b_j y_j)) \longrightarrow f_*L.$$

Note that $F_{b_i}^i = 0$ and $F_0^i = V_i$; in particular, $F_\ell^i \subset V_i$ for all ℓ . It is easy to see that dim $F_\ell^i = b_i - \ell$, so $\{F_\ell^i\}_{\ell=0}^{b_i}$ is a complete flag of subspaces of V_i . The weight of the subspace $F_\ell^i \subset V_i$, $0 \leq \ell < b_i$, is $(\ell + \lambda_{y_i})/b_i$, where λ_{y_i} is the parabolic weight of L_* at y_i ; if y_i is not a parabolic point of L_* , then λ_{y_i} is taken to be zero. Note that $0 \leq (\ell + \lambda_{y_i})/b_i < 1$.

Now the parabolic structure on E over x is given by these weighted filtrations using (3.1). More precisely, for any $0 \le c < 1$, if $S_x^i(c) \subset V_i$, $1 \le i \le m$, is the subspace of V_i of weight c, then the subspace of E_x of parabolic weight c is the direct sum $\bigoplus_{i=1}^m S_x^i(c)$.

The direct image f_*L equipped with the above parabolic structure will be denoted by f_*L_* .

Since $H^i(Y, L) = H^i(X, f_*L)$, using the Riemann–Roch theorem for L and f_*L , we have

$$\operatorname{degree}(f_*L) = \operatorname{degree}(L) - \operatorname{genus}(Y) + 1 + \operatorname{degree}(f)(\operatorname{genus}(X) - 1),$$

where genus(Y) = dim $H^1(Y, \mathcal{O}_Y)$ (recall that Y need not be connected). On the other hand,

$$2({\rm genus}(Y)-1) \, = \, {\rm degree}(K_Y) \, = \, {\rm degree}(K_X) + \sum_{y \in R} (b_y-1) \, = \, 2({\rm genus}(X)-1) + \sum_{y \in R} (b_y-1) \, ,$$

where b_y is the multiplicity of f at y while K_X and K_Y are the canonical line bundles of X and Y respectively. From these it follows that

$$\operatorname{par-deg}(f_*L_*) = \operatorname{par-deg}(L_*).$$

Generalizing the constructions of direct sum, tensor product and dual of vector bundles, there are direct sum, tensor product and dual of parabolic vector bundles [Yo], [MY], [Bi2]. It should be mentioned that for two parabolic vector bundles E_* and F_* , while the underlying vector bundle for the parabolic direct sum $E_* \bigoplus F_*$ is the direct sum of the vector bundles underlying E_* and F_* , the underlying vector bundle for the parabolic tensor product $E_* \otimes F_*$ is not necessarily the tensor product of the vector bundles underlying E_* and F_* . Similarly, the underlying vector bundle for the parabolic dual E_*^* is different from the dual of the vector bundles underlying E_* , unless the parabolic structure on E_* is trivial (meaning there are no nonzero parabolic weights).

The endomorphism bundle for a parabolic vector bundle E_* is defined to be

$$\operatorname{End}(E_*) := E_* \otimes E_*^*; \tag{3.2}$$

it should be emphasized that both the tensor product and dual in (3.2) are in the parabolic category.

3.1. When the characteristic is zero. In this subsection we assume that the characteristic of the base field k is zero.

Let E_* be a parabolic vector bundle on X. Let $S \subset X$ be the parabolic divisor for E_* . The vector bundle underlying E_* will be denoted by E_0 . Consider the endomorphism (parabolic) bundle $\operatorname{End}(E_*)$ defined in (3.2). The vector bundle underlying it will be

denoted by $\operatorname{End}(E_*)_0$. The two vector bundles $\operatorname{End}(E_*)_0$ and $\operatorname{End}(E_0)$ are identified over the complement $X \setminus S$. This isomorphism extends to a homomorphism

$$\beta : \operatorname{End}(E_*)_0 \longrightarrow \operatorname{End}(E_0)$$

over entire X. For any point $x \in S$, the subspace $\beta(x)((\operatorname{End}(E_*)_0)_x) \subset \operatorname{End}(E_0)_x$ coincides with the space of endomorphisms of the fiber $(E_0)_x$ that preserve the quasiparabolic filtration of $(E_0)_x$.

A parabolic vector bundle on X can be expressed as the invariant direct image of an equivariant vector bundle over a (ramified) Galois cover of X [Bi1], [Bo1], [Bo2]; recall the assumption that all the parabolic weights are rational numbers. Let \widetilde{X} be an irreducible smooth projective curve,

$$\gamma: \widetilde{X} \longrightarrow X \tag{3.3}$$

a Galois covering which may be ramified, and \mathcal{E} a Γ -linearized vector bundle on \widetilde{X} , where $\Gamma := \operatorname{Gal}(\gamma)$, such that E_* corresponds to \mathcal{E} . The vector bundle E underlying E_* is the invariant direct image $(\gamma_*\mathcal{E})^{\Gamma}$; note that the action of Γ on \mathcal{E} produces an action of Γ on $\gamma_*\mathcal{E}$. In particular, we have $\operatorname{rank}(E) = \operatorname{rank}(\mathcal{E})$. Consider the finite subset D' of \widetilde{X} consisting of all points $y \in \widetilde{X}$ satisfying the following two conditions:

- y has nontrivial isotropy for the action of Γ on \widetilde{X} , and
- the action of isotropy subgroup Γ_y for y on the fiber \mathcal{E}_y is nontrivial.

The image $\gamma(D') \subset X$ is the subset of S consisting of all points over which E_* has nontrivial parabolic weight. The above isotropy subgroup Γ_y for y is cyclic; let m_y be the order of Γ_y . Fix a generator ν of Γ_y . A rational number $0 \le c < 1$ is a parabolic weight for E_* at $\gamma(y)$ if and only if $\exp(2\pi\sqrt{-1}c)$ is an eigenvalue for the action of ν on \mathcal{E}_y . In particular, $c \cdot m_y$ is an integer.

The subbundles of E_0 with the parabolic structure induced by E_* correspond to subbundles of \mathcal{E} preserved by the action of Γ . The parabolic vector bundles E_*^* and $\operatorname{End}(E_*)$ correspond to \mathcal{E}^* and $\operatorname{End}(\mathcal{E})$ respectively; note that the Γ -linearization of \mathcal{E} induces Γ -linearizations on both \mathcal{E}^* and $\operatorname{End}(\mathcal{E})$.

A subbundle $\mathcal{A} \subset \operatorname{End}(E_*)_0$ will be called a *Cartan subalgebra bundle* if the following two conditions hold:

- (1) For each closed point $x \in X \setminus S$, the fiber $\mathcal{A}_x \subset (\operatorname{End}(E_*)_0)_x = \operatorname{End}((E_0)_x)$ is a Cartan subalgebra of the Lie algebra $\operatorname{End}((E_0)_x)$.
- (2) For each point $x \in S$, the fiber of the Γ -linearized subbundle of $\operatorname{End}(\mathcal{E})$ corresponding to \mathcal{A} over a point $y \in \gamma^{-1}(x)$ is a Cartan subalgebra of the Lie algebra $\operatorname{End}(\mathcal{E}_y)$. (If this condition holds for one point of $\gamma^{-1}(x)$ then it holds for all points of $\gamma^{-1}(x)$; this is because of the action of Γ .)

The above definition of a Cartan subalgebra bundle of $\operatorname{End}(E_*)_0$ does not depend on the choice of the covering γ . To see this, if

$$\gamma'\,:\, \widetilde{X}'\,\longrightarrow\, X$$

is another such covering, then consider the normalization \mathcal{X} of the fiber product $\widetilde{X} \times_X \widetilde{X}'$. This covering \mathcal{X} of X also satisfies the conditions. If \mathcal{E}' is the equivariant vector bundle on \widetilde{X}' corresponding to E_* , then the pullbacks of \mathcal{E} and \mathcal{E}' to \mathcal{X} are equivariantly isomorphic to the equivariant bundle on \mathcal{X} corresponding to E_* . Hence a fiberwise decomposition of

 \mathcal{E} produces a fiberwise decomposition of its pullback to \mathcal{X} , which in turn descends to a fiberwise decomposition of \mathcal{E}' .

Note that $\mathcal{A} \subset \operatorname{End}(E_*)_0$ is a Cartan subalgebra bundle if and only if the Γ -linearized subbundle $\widetilde{\mathcal{A}} \subset \operatorname{End}(\mathcal{E})$ corresponding to \mathcal{A} has the property that for every $y \in \widetilde{X}$, the subspace $\widetilde{\mathcal{A}}_y \subset \operatorname{End}(\mathcal{E})_y$ is a Cartan subalgebra of the Lie algebra $\operatorname{End}(\mathcal{E})_y$.

Theorem 3.1. Given a parabolic vector bundle E_* on X, the following two are equivalent:

(1) There is a finite surjective map

$$f: Y \longrightarrow X$$
,

where Y is a smooth projective curve not necessarily connected, and a parabolic line bundle L_* on Y, such that

- E_* has a nontrivial parabolic weight at $x \in X$ if and only if there is a point $y \in f^{-1}(x)$ satisfying the condition that either y is a parabolic point for L_* or f is ramified at y (or both), and
- the parabolic vector bundle f_*L_* is isomorphic to E_* .
- (2) There is a Cartan subalgebra bundle \mathcal{A} of End $(E_*)_0$.

When there is a Cartan subalgebra bundle $\mathcal{A} \subset \operatorname{End}(E_*)_0$, the pair (Y, f) in the first statement can be so chosen that the subbundle \mathcal{A} equipped with the parabolic structure induced by $\operatorname{End}(E_*)$ is isomorphic to $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure (the parabolic structure on \mathcal{O}_Y is the trivial one, meaning it has no nonzero parabolic weight). In that case, the number of connected components of Y coincides with $\dim H^0(X, \mathcal{A})$.

Proof. Fix a Galois covering (\widetilde{X}, γ) as in (3.3) with $\Gamma = \operatorname{Gal}(\gamma)$ such that there is a Γ -linearized vector bundle \mathcal{E} on \widetilde{X} that corresponds to the parabolic vector bundle E_* .

Assume that the first statement in the theorem holds. Take (Y, f, L_*) as in the first statement. Let \widetilde{Y} denote the normalization of the fiber product $Y \times_X \widetilde{X}$. Let

$$p_1: \widetilde{Y} \longrightarrow Y \text{ and } p_2: \widetilde{Y} \longrightarrow \widetilde{X}$$

be the natural projections. We note that the normalization of a fiber product has the following property: Consider two ramified coverings $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$ defined by $z \longmapsto z^a$ and $z \longmapsto z^b$ respectively; then the projection to the second factor of the normalization of the fiber product $\mathbb{A}^1 \times_{\mathbb{A}^1} \mathbb{A}^1$ is an étale covering of \mathbb{A}^1 if b is a multiple of a. From this it follows that the above projection p_2 is an étale covering.

The action of Γ on \widetilde{X} produces an action of Γ on $Y \times_X \widetilde{X}$, hence \widetilde{Y} is equipped with an action of Γ ; the map p_2 intertwines the actions of Γ on \widetilde{Y} and \widetilde{X} . The above morphism p_1 is evidently Γ invariant, and hence it produces a morphism

$$\widetilde{Y}/\Gamma \longrightarrow Y;$$

it is easy to see that this morphism is an isomorphism. There is a Γ -linearized line bundle

$$\widetilde{L} \longrightarrow \widetilde{Y}$$

that corresponds to the parabolic line bundle L_* on Y.

Since p_2 is Γ -equivariant, the action of Γ on \widetilde{L} produces an action of Γ on the direct image $p_{2*}\widetilde{L}$. The parabolic vector bundle f_*L_* corresponds to this Γ -linearized vector bundle $p_{2*}\widetilde{L}$.

Since p_2 is étale, by Lemma 2.1, there is a homomorphism from the direct image

$$\xi: p_{2*}\mathcal{O}_{\widetilde{Y}} \longrightarrow \operatorname{End}(p_{2*}\widetilde{L})$$

whose fiberwise images are Cartan subalgebras. The action of Γ on \widetilde{Y} produces a Γ linearization on $\mathcal{O}_{\widetilde{Y}}$. Since p_2 is Γ -equivariant, the action of Γ on $\mathcal{O}_{\widetilde{Y}}$ produces an action
of Γ on $p_{2*}\mathcal{O}_{\widetilde{Y}}$. The above homomorphism ξ is Γ -equivariant for the action of Γ on $\operatorname{End}(p_{2*}\widetilde{L})$ induced by the action of Γ on $p_{2*}\widetilde{L}$ and the action of Γ on $p_{2*}\mathcal{O}_{\widetilde{Y}}$. Therefore,
the second statement in the theorem holds.

Now assume that the second statement in the theorem holds. Let \mathcal{A} be a Cartan subalgebra bundle of $\operatorname{End}(E_*)_0$. The Γ -linearized vector bundle $\operatorname{End}(\mathcal{E})$ corresponds to the parabolic vector bundle $\operatorname{End}(E_*)$, because E_* corresponds to \mathcal{E} . Let \mathcal{B} be the subbundle of $\operatorname{End}(\mathcal{E})$ preserved by the action of Γ such that \mathcal{B} corresponds to the subbundle \mathcal{A} of $\operatorname{End}(E_*)_0$. So for any closed point $x \in \widetilde{X}$, the subspace $\mathcal{B}_x \subset \operatorname{End}(\mathcal{E}_x)$ is a Cartan subalgebra.

By Proposition 2.3, there is an étale covering

$$\phi: \widetilde{Y} \longrightarrow \widetilde{X}$$

and a line bundle \mathcal{L} on \widetilde{Y} such that

$$\phi_* \mathcal{L} = \mathcal{E} \,. \tag{3.4}$$

Since \mathcal{B} is preserved by the action of Γ on $\operatorname{End}(\mathcal{E})$ induced by the action of Γ on \mathcal{E} , from the construction in Proposition 2.3 it follows that

- Γ acts on \widetilde{Y} ,
- the map ϕ intertwines the actions of Γ on \widetilde{Y} and \widetilde{X} ,
- ullet the line bundle ${\mathcal L}$ is Γ -linearized, and
- the isomorphism in (3.4) is Γ -equivariant.

Since ϕ is Γ -equivariant, the composition

$$\gamma \circ \phi : \widetilde{Y} \longrightarrow X$$

factors through a map

$$f: Y := \widetilde{Y}/\Gamma \longrightarrow X.$$
 (3.5)

Let $q:\widetilde{Y}\longrightarrow \widetilde{Y}/\Gamma=Y$ be the quotient map. The parabolic line bundle on Y corresponding to the Γ -linearized line bundle \mathcal{L} on \widetilde{Y} will be denoted by L_* .

For any vector bundle $W \longrightarrow \widetilde{Y}$, there is a canonical isomorphism

$$(f \circ q)_* W = f_* q_* W \xrightarrow{\sim} \gamma_* \phi_* W = (\gamma \circ \phi)_* W. \tag{3.6}$$

The action of Γ on \mathcal{L} produces actions of Γ on both $(f \circ q)_*\mathcal{L}$ and $(\gamma \circ \phi)_*\mathcal{L}$. The isomorphism in (3.6) is Γ -equivariant for $W = \mathcal{L}$.

Since the isomorphism in (3.6) for \mathcal{L} is Γ -equivariant, it can be deduced that for the above parabolic line bundle L_* on Y, the parabolic vector bundle f_*L_* on X, where f is defined in (3.5), corresponds to the Γ -linearized vector bundle $\phi_*\mathcal{L}$ on \widetilde{X} . Also, as the isomorphism in (3.4) for \mathcal{L} is Γ -equivariant, and the parabolic vector bundle E_* corresponds to the Γ -linearized vector bundle \mathcal{E} , from the above observation on (3.6) it also follows that the two parabolic vector bundle f_*L_* and F_* are isomorphic.

In view of the above proof, from Theorem 2.4 we conclude the following: If there is a Cartan subalgebra bundle $\mathcal{A} \subset \operatorname{End}(E_*)_0$, then the pair (Y, f) in the first statement of the theorem can be so chosen that the subbundle \mathcal{A} equipped with the parabolic structure induced by $\operatorname{End}(E_*)$ is isomorphic to $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure (the parabolic structure on \mathcal{O}_Y is the trivial one). In that case, the number of connected components of Y coincides with dim $H^0(X, \mathcal{A})$.

In Theorem 3.1, setting $L = \mathcal{O}_Y$ equipped with the trivial parabolic structure, we have the following:

Corollary 3.2. Given a parabolic vector bundle E_* on X, the following two are equivalent:

(1) There is a finite surjective map

$$f: Y \longrightarrow X$$

where Y is a smooth projective curve not necessarily connected, such that $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure is isomorphic to E_* (the parabolic structure on \mathcal{O}_Y is the trivial one).

- (2) There is a homomorphism of parabolic bundles $\alpha: E_* \longrightarrow \operatorname{End}(E_*)$ such that
 - α is an isomorphism of E_* with the image $\alpha(E_*)$ equipped with the parabolic structure induced by the parabolic structure of $\operatorname{End}(E_*)$, and
 - $\alpha(E_0)$ is a Cartan subalgebra bundle of $\operatorname{End}(E_*)_0$, where E_0 is the vector bundle underlying E_* .

When these hold, the number of connected components of Y coincides with dim $H^0(X, E_0)$.

3.2. The case of positive characteristic. Assume that the base field k is of positive characteristic. Let p denote the characteristic of k.

The correspondence between parabolic vector bundles and equivariant vector bundles used extensively in Section 3.1 remains valid under the following tameness condition (see [Bo1], [Bo2]):

If a/b is a parabolic weight, where a and b are nonzero coprime integers, then we assume that b is not a multiple of p.

Once we impose the above condition on E_* , the proof of Theorem 3.1 goes through without any change. Similarly, Corollary 3.2 remains valid after the above tameness condition on E_* is imposed.

Remark 3.3. Theorem 2.4 and Corollary 2.6 remain valid when X is a root-stack; see [Ca], [Bo1], [Bo2] for root-stacks. When the parabolic divisor is a simple normal crossing divisor, quasiparabolic filtrations satisfy certain conditions and the parabolic weights are tame (in positive characteristic case), there is an equivalence between parabolic vector bundles over a smooth projective variety Y and vector bundles over smooth root-stack whose underlying coarse moduli space is Y; see [Bo1], [Bo2]. Therefore, under the above assumptions on parabolic structure, Theorem 3.1 and Corollary 3.2 extend to higher dimensions.

4. Factoring of covering maps

In this section we assume that the characteristic of k is zero.

Let

$$Y \xrightarrow{f} X$$

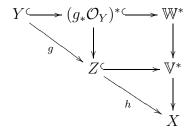
$$Z$$

$$(4.1)$$

be a commutative diagram of étale coverings, and define $V := h_* \mathcal{O}_Z$ and $W := f_* \mathcal{O}_Y$. Notice that the homomorphism $\mathcal{O}_Z \longrightarrow g_* \mathcal{O}_Y$ induces a homomorphism

$$V \longrightarrow W$$
 (4.2)

which is fiberwise injective, and so V is a subbundle of W. Notice moreover that V gives way to a Cartan subalgebra of $\operatorname{End}(V)$ and W produces a Cartan subalgebra of $\operatorname{End}(W)$. Let \mathbb{V}^* and \mathbb{W}^* be the total spaces of V^* and W^* , respectively. We have the following commutative diagram



where the map $\mathbb{W}^* \longrightarrow \mathbb{V}^*$ is the dual of the homomorphism in (4.2), and $(g_*\mathcal{O}_Y)^*$ is the restriction of this fiber bundle \mathbb{W}^* to $Z \subset \mathbb{V}^*$. Over $x \in X$, in \mathbb{W}^* we have the functionals $\mathcal{G}_x := \{\mu^x_{t,W}\}_{t \in B_{x,W}}$ that define the preimage of x in Y and in \mathbb{V}^* we have the linear functionals $\mathcal{H}_x := \{\mu^x_{t,V}\}_{t \in B_{x,V}}$ that define the preimage of x in Z. We note that under the map $\mathbb{W}^* \longrightarrow \mathbb{V}^*$, \mathcal{G}_x is taken to \mathcal{H}_x . This implies that for every $v \in V$, there exists $t' \in B_{x,W}$ such that for every $v' \in \ell^x_{t,V}$,

$$v(v') = \mu_{t',W}^x(v) \cdot v'.$$

Now notice that V is a direct summand of W, and therefore we have homomorphisms $i_V: V \longrightarrow W$ and $p_V: W \longrightarrow V$ such that $p_V i_V = \mathrm{id}_V$. These induce a homomorphism $\psi: \mathrm{End}(W) \longrightarrow \mathrm{End}(V)$, $f \longmapsto p_V \circ f \circ i_V$. Now, the previous condition just means that if V_x is embedded in $\mathrm{End}(V_x)$ as a Cartan subalgebra, W_x is embedded in $\mathrm{End}(W_x)$ as a Cartan subalgebra, then the following diagram commutes:

$$V_x \xrightarrow{i_{V,x}} W_x$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{End}(V_x) \xrightarrow{\psi_x} \operatorname{End}(W_x)$$

Indeed, we see that necessarily there is a subset of $\{\ell_t^x\}_{t\in B_{W,x}}$ that generates V_x , and so the commutativity of the previous diagram just means that $\mu_{t,W}^x$ is taken to a $\mu_{t,V}^x$.

By retracing our steps, we have proved the following proposition:

Proposition 4.1. Let $f: Y \longrightarrow X$ be an étale covering. Then there is a bijection between intermediate étale coverings as in (4.1) and direct summands V of $f_*\mathcal{O}_Y$ such that

 V_x has an embedding as a Cartan subalgebra of $\operatorname{End}(V_x)$ for every $x \in X$ and such that the induced diagram $V_x \hookrightarrow (f_* \mathcal{O}_Y)_x$

 $V_x \xrightarrow{} (f_* \mathcal{O}_Y)_x$ $\downarrow \qquad \qquad \downarrow$ $\operatorname{End}(V_x) \longleftarrow \operatorname{End}((f_* \mathcal{O}_Y)_x)$

commutes.

ACKNOWLEDGEMENTS

We thank the referee for helpful comments to improve the exposition. The first author is partially supported by Fondecyt Grant 3150171 and CONICYT PIA ACT1415. The second author wishes to thank the Universidad de Chile for hospitality while the work was carried out. He is partially supported by a J. C. Bose Fellowship.

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DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE CHILE, SANTIAGO, CHILE

E-mail address: rfauffar@uchile.cl

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in