ALGEBRAIC ISOMONODROMIC DEFORMATIONS AND THE MAPPING CLASS GROUP

GAËL COUSIN AND VIKTORIA HEU

ABSTRACT. The germ of the universal isomonodromic deformation of a logarithmic connection on a stable n-pointed genus-g curve always exists in the analytic category. The first part of this article investigates under which conditions it is the analytic germification of an algebraic isomonodromic deformation. Up to some minor technical conditions, this turns out to be the case if and only if the monodromy of the connection has finite orbit under the action of the mapping class group. The second part of this work studies the dynamics of this action in the particular case of reducible rank 2 representations and genus $g > 0$, allowing to classify all finite orbits. Both of these results extend recent ones concerning the genus 0 case.

CONTENTS

1. INTRODUCTION

The mapping class group. Let g and n be nonnegative integers. Let Σ_g be a compact oriented real surface of genus g, let $y^n = (y_1, \ldots, y_n)$ be a sequence of n distinct points in Σ_g. We shall denote by $Y^n := \{y_1, \ldots, y_n\}$ the corresponding (unordered) set of points. The (pure) mapping class group of (Σ_g, y^n) is defined to be the set of orientation preserving homeomorphisms h of Σ_g such that $h(y_i) = y_i$ for all $i \in [\![1, n]\!] := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$, quotiented by isotopies:

$$\Gamma_{g,n} := \mathrm{Homeo}_+(\Sigma_g, y^n)/\{\text{isotopies relative to } Y^n\}.$$

We can also consider homeomorphisms of Σ_g that preserve the set Y^n, but do not necessarily preserve the labelling of the punctures. This leads to the full mapping class group

$$\hat{\Gamma}_{g,n} := \mathrm{Homeo}_+(\Sigma_g, Y^n)/\{\text{isotopies relative to } Y^n\}.$$

Note that we have an exact sequence of groups

$$1 \longrightarrow \Gamma_{g,n} \longrightarrow \hat{\Gamma}_{g,n} \longrightarrow \mathfrak{S}_n \longrightarrow 1,$$

where $\mathfrak{S}_n$ denotes the symmetric group of degree n. In particular, $\Gamma_{g,n}$ is a subgroup of $\hat{\Gamma}_{g,n}$ of finite index $n!$. Let now $y_0 \in \Sigma_g \setminus Y^n$ be a point. We denote the fundamental group of $\Sigma_g \setminus Y^n$ with respect to the base point y_0 by

$$(1) \qquad \Lambda_{g,n} := \pi_1(\Sigma_g \setminus Y^n, y_0) \,.$$

The composition $\alpha . \alpha'$ of two paths $\alpha, \alpha' \in \Lambda_{g,n}$ shall denote the usual concatenation "first α, then α'". For any group G, we may consider the space $\mathrm{Hom}(\Lambda_{g,n}, G)$ of representations as well as the space of representations modulo conjugation, which we shall denote

$$(2) \qquad \chi_{g,n}(G) := \mathrm{Hom}(\Lambda_{g,n}, G)/G \,.$$

The mapping class group acts on $\chi_{g,n}(G)$. Define the groups of orientation preserving homeomorphisms h of Σ_g such that $h(y_0) = y_0$ and $h(y^n) = y^n$, respectively $h(Y^n) = Y^n$, modulo isotopy:

$$\Gamma_{g,n+1} \quad := \quad \mathrm{Homeo}_+(\Sigma_g, y^n, y_0)/\{\text{isotopies relative to } Y^n \cup \{y_0\}\} \,,$$

$$\hat{\Gamma}_{g,n}^\bullet \quad := \quad \mathrm{Homeo}_+(\Sigma_g, Y^n, y_0)/\{\text{isotopies relative to } Y^n \cup \{y_0\}\} \,.$$

Now $\hat{\Gamma}_{g,n}^\bullet$ naturally acts on the fundamental group $\Lambda_{g,n}$: for $h \in \hat{\Gamma}_{g,n}^\bullet$ and $\alpha \in \Lambda_{g,n}$, we set

$$\mathfrak{a}(h)(\alpha) := h_* \alpha \,.$$

Via the forgetful maps $\Gamma_{g,n+1} \to \Gamma_{g,n}$ and $\hat{\Gamma}_{g,n}^\bullet \to \hat{\Gamma}_{g,n}$ we obtain a commutative diagram

$$
\begin{array}{ccccc}
\Gamma_{g,n+1} & \hookrightarrow & \hat{\Gamma}_{g,n}^\bullet & \xrightarrow{\;\mathfrak{a}\;} & \mathrm{Aut}(\Lambda_{g,n}) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_{g,n} & \hookrightarrow & \hat{\Gamma}_{g,n} & \longrightarrow & \mathrm{Out}(\Lambda_{g,n}) := \!\!=\!\! = \mathrm{Aut}(\Lambda_{g,n})/\mathrm{Inn}(\Lambda_{g,n}) \,.
\end{array}
$$

Indeed, any element $h \in \mathrm{Homeo}_+(\Sigma_g, y^n)$ may be lifted to an element $h_0 \in \mathrm{Homeo}_+(\Sigma_g, y^n, y_0)$. Let $h_1 \in \mathrm{Homeo}_+(\Sigma_g, y^n, y_0)$ be another representative. Then they are the extremities of an isotopy $(h_t)_{t \in [0,1]}$ relative to Y^n. We have a loop $\gamma \in \Lambda_{g,n}$ defined by $\gamma(t) = h_t(y_0)$. Then for any $\alpha \in \Lambda_{g,n}$, we have $\mathfrak{a}(h_1)(\alpha) = \gamma^{-1} . \mathfrak{a}(h_0)(\alpha) . \gamma$.

In particular, for any group G, the mapping class group $\hat{\Gamma}_{g,n}$ acts on the space $\chi_{g,n}(G)$, and this action lifts to an action of $\hat{\Gamma}_{g,n}^\bullet$ on the space $\mathrm{Hom}(\Lambda_{g,n}, G)$. More precisely, for all $\rho \in \mathrm{Hom}(\Lambda_{g,n}, G), h \in \hat{\Gamma}_{g,n}^\bullet$ and $\alpha \in \Lambda_{g,n}$, we define

$$(3) \qquad ([h] \cdot \rho)(\alpha) := \rho(\mathfrak{a}(h^{-1})(\alpha)) \,.$$

Application to isomonodromic deformations and a dynamical study. In this paper, we establish two results about finite orbits of the mapping class group action on $\chi_{g,n}(G)$ for $G = \mathrm{GL}_r\mathbb{C}$. These results and their respective proofs can be read independently. In Theorem A, which will be stated in Section 1.A and proven in Part A, we relate such finite orbits to the existence of an algebraic universal isomonodromic deformation of a logarithmic connection over a curve, whose monodromy belongs to that orbit. This motivates Theorem B, which will be stated in Section 1.B and proven in Part B, classifying conjugacy classes of reducible rank 2 representations with finite orbit. To that end, we introduce a specific presentation of $\Lambda_{g,n}$ and explicit *formulae* for the mapping class group action.

Remark 1.0.1. Recall that a representation $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_r\mathbb{C})$ is called *irreducible* if the only subvector spaces $V \subset \mathbb{C}^r$ stable under $\mathrm{Im}(\rho)$ are $\{0\}$ and $\mathbb{C}^r$. A *semisimple* representation is a direct sum of irreducible representations.

1.A. Algebraization of universal isomonodromic deformations. We need to introduce some additional vocabulary before stating our main result, which can be seen as a criterion under which a GAGA-type theorem holds for isomonodromic deformations. In order to avoid having to introduce each definition twice, we adopt the ϖ-notation described in Table 1.

ϖ	ϖ − manifold	ϖ − open
analytic	connected Hausdorff complex analytic manifold	open with respect to Euclidean topology
algebraic	smooth irreducible quasi-projective variety over $\mathbb{C}$	open with respect to Zariski topology

<center>TABLE 1.</center>

Logarithmic connections. Let X be a ϖ-manifold and let D be a (possibly empty) reduced divisor on X. Denote by $D_1, \ldots, D_n$ the irreducible components of D. A *logarithmic ϖ-connection* of rank r over X with polar divisor D is a pair (E, ∇), where $E \to X$ is a ϖ-vector bundle of rank r over X, whose sheaf of sections we shall denote by $\mathcal{E}$, and ∇ is a $\mathbb{C}$-linear morphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(\log D),$$

which satisfies the Leibniz rule

$$\nabla(f \cdot e) = f \cdot \nabla(e) + e \otimes \mathrm{d}f$$

for any $f \in \mathcal{O}_X(\Delta), e \in \mathcal{E}(\Delta)$, where $\Delta \subset X$ is any ϖ-open subset. We require D to be minimal in the sense that for any $i \in [\![1, n]\!]$, ∇ does not factor through

$$\mathcal{E} \otimes \Omega^1(\log(D - D_i)) \hookrightarrow \mathcal{E} \otimes \Omega^1_X(\log D).$$

Such a logarithmic connection (E, ∇) is called *flat* if its curvature ∇^2 is zero.

We are particularly interested in the case where X is a smooth projective curve (a compact Riemann surface). Since then X is of complex dimension one, any logarithmic connection over X is automatically flat. Moreover, since then X is projective, any analytic logarithmic connection over X is isomorphic to the analytification of a unique algebraic logarithmic connection over X by one of Serre's GAGA theorems [Ser56, Prop. 18].

Monodromy. The notion of the monodromy representation of a flat connection varies slightly in the literature. For introductory and technical purposes, let us give the definition we are going to use. This definition can only be formulated in the analytic category; in the algebraic case the monodromy representation is defined *via* analytification. Let X and D be as above (X has arbitrary dimension). Denote $X^0 := X \setminus D$. Let (E, ∇) be an analytic logarithmic connection over X with polar divisor D. Assume moreover that this analytic connection is flat, which is equivalent to it being integrable, *i.e.* $\mathcal{S} := \ker(\nabla|_{X^0})$ is a locally constant sheaf of rank r over X^0. Let Σ and $Y \subset \Sigma$ be topological spaces such that there is a homeomorphism

$$\Phi : (\Sigma, Y) \xrightarrow{\sim} (X, D).$$

Fix such a homeomorphism and fix a point $\tilde{y}_0 \in \Sigma \setminus Y$. Denote $\tilde{x}_0 := \Phi(\tilde{y}_0)$. For any path $\gamma : [0, 1] \to \Sigma \setminus Y$, the pull back $(\Phi \circ \gamma)^* \mathcal{S}$ is locally constant and thus isomorphic to a constant sheaf. Hence γ defines an isomorphism $\gamma(\mathcal{S}) : \mathcal{S}_{\gamma(1)} \to \mathcal{S}_{\gamma(0)}$. This isomorphism is invariant by homotopy relative to $\{\gamma(0), \gamma(1)\}$ and satisfies $\gamma_1 . \gamma_2(\mathcal{S}) = \gamma_1(\mathcal{S}) \circ \gamma_2(\mathcal{S})$ for any pair of paths (γ_1, γ_2). We obtain a representation $\pi_1(\Sigma \setminus Y, \tilde{y}_0) \to \mathrm{GL}(\mathcal{S}_{\tilde{x}_0})$. *Via* an isomorphism $\mathcal{S}_{\tilde{x}_0} \to \mathbb{C}^r$, one deduces a (non canonical) representation $\rho_\nabla \in \mathrm{Hom}(\pi_1(\Sigma, \tilde{y}_0), \mathrm{GL}_r\mathbb{C})$ and a canonical conjugacy class of representation

$$[\rho_\nabla] \in \mathrm{Hom}(\pi_1(\Sigma \setminus Y, \tilde{y}_0), \mathrm{GL}_r\mathbb{C})/\mathrm{GL}_r\mathbb{C}.$$

We refer to ρ_∇ as the *monodromy representation* and to $[\rho_\nabla]$ as the *monodromy* of (E, ∇) with respect to Φ. Conversely, given Φ, given a conjugacy class of representation $[\rho] \in \mathrm{Hom}(\pi_1(\Sigma \setminus Y, \tilde{y}_0), \mathrm{GL}_r\mathbb{C})/\mathrm{GL}_r\mathbb{C}$ and a compatible choice of *mild transversal models* (see Section 3.2), there

is a flat logarithmic analytic connection (E, ∇) over X, unique up to isomorphism, inducing these transversal models and such that $[\rho_\nabla] = [\rho]$ (see [Cou17, Th. 3.3], adapted from [Del70, Prop. 5.4]). In our work, the use of the marking Φ is essential, as we wish to compare the monodromies of connections over various homeomorphic curves.

Isomonodromy. Let C be a smooth projective curve of genus g, let D_C be a reduced divisor of degree n on C. Let (E_0, ∇_0) be a logarithmic connection over C with polar divisor D_C.

A ω-*isomonodromic deformation of* (C, E_0, ∇_0) consists in the following data:

- a ω-family $(\kappa : X \to T, D)$ of n-pointed smooth curves of genus g (see Section 2.2);
- a flat logarithmic ω-connection (E, ∇) over X with polar divisor D;
- a point t_0 in T ; we denote $X_{t_0} := \kappa^{-1}(\{t_0\})$; and
- an isomorphism of pointed curves with logarithmic connections

$$(\psi, \Psi) : ((C, D_C), (E_0, \nabla_0)) \xrightarrow{\sim} ((X_{t_0}, D|_{X_{t_0}}), (E, \nabla)|_{X_{t_0}}) .$$

Why are such deformations called isomonodromic? Again we have to work in the analytic category. Up to shrinking T to a sufficiently small polydisc Δ containing t_0, the family $\kappa : (X, D) \to \Delta$ is topologically trivial. Hence there is a homeomorphism

$$\Phi : (\Sigma_g, Y^n) \times \Delta \xrightarrow{\sim} (X, D)$$

commuting with the natural projections to Δ. Now for any $t \in \Delta$, the morphism

$$\pi_1(\Sigma_g \setminus Y^n, y_0) \longrightarrow \pi_1((\Sigma_g \setminus Y^n) \times \Delta, (y_0, t)) ,$$

induced by the inclusion of the fiber at t, is an isomorphism. On the other hand, $(E, \nabla)|_{X_t}$ is a logarithmic connection over X_t with polar divisor $D|_{X_t}$. By flatness of ∇, its monodromy representation with respect to $\Phi|_t$ and the base point y_0 can be identified with the monodromy representation of (E, ∇) with respect to Φ and the base point (y_0, t). For $t = t_0$, this means we can identify the monodromy representation of (E, ∇) over X with respect to Φ with the monodromy representation of (E_0, ∇_0) over C with respect to $\psi^{-1} \circ \Phi|_{t_0}$. In that sense, we may say that with respect to some continuous "base point section" $t \mapsto (y_0, t)$, the monodromy representation along a germ of isomonodromic deformation is constant and given by the monodromy representation of (E_0, ∇_0). More generally, one can say that an isomonodromic deformation induces a topologically locally trivial family of monodromy representations, leading to a phenomenon of monodromy of the monodromy representation. The latter will become tangible in Section 4.

Statement of Theorem A. Following [Heu10, Th. 3.4](see also [Mal83, Kri02]), any triple (C, E_0, ∇_0) as before admits a universal analytic isomonodromic deformation, which is unique up to unique isomorphism, and whose parameter space T is the Teichmüller space $\mathcal{T}_{g,n}$. This universal analytic isomonodromic deformation satisfies a universal property with respect to germs of analytic isomonodromic deformations of (C, E_0, ∇_0). A universal algebraic isomonodromic deformation of (C, E_0, ∇_0), if it exists, would be an algebraic isomonodromic deformation whose analytic germification is isomorphic to the germification of the universal analytic isomonodromic deformation of (C, E_0, ∇_0). In Section 2.4, we give an alternative definition and state a universal property of universal algebraic isomonodromic deformations. Our main result is the following.

Theorem A. *Let C be a smooth complex projective curve of genus g. Let D_C be a set of n distinct points in C and let $\Phi : (\Sigma_g, Y^n) \to (C, D_C)$ be an orientation preserving homeomorphism. Let (E_0, ∇_0) be an algebraic logarithmic connection of rank r over C with polar divisor D_C and monodromy $[\rho] \in \chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ with respect to Φ. Assume that $2g - 2 + n > 0$ and that ∇_0 is mild. If $r > 2$, then assume further that ρ is semisimple. The following are equivalent:*

(1) There is a universal algebraic isomonodromic deformation of (C, E_0, ∇_0).

(2) The orbit $\Gamma_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ is finite.

Remark 1.A.1. Note that the orbit $\Gamma_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ does not depend on the choice of Φ.

We prove this theorem by adapting the proof for the special case of genus $g = 0$, which has been established in [Cou17]. The main ingredients of the proof of Theorem A are: the logarithmic Riemann-Hilbert correspondence (see Section 3.2); the introduction of a base point section for a family of punctured curves and the splitting of the fundamental group of the total space of the family (see Section 3.1), together with its relation to the mapping class group (see Section 4.2). Both implications to be proven appear as special cases of stronger results: Theorem A1 and Theorem A2, respectively. We give their statements and proofs in Section 5.2.

The statement of Theorem A is natural in the following sense. As we recall in Section 2.1, the (algebraic) moduli space $\mathcal{M}_{g,n}$ of stable smooth n-pointed genus-g curves is the quotient of the (analytic) Teichmüller space $\mathcal{T}_{g,n}$ by the natural action of $\Gamma_{g,n}$. Intuitively, a universal algebraic isomonodromic deformation should be the quotient of the universal analytic isomonodromic deformation with respect to a sufficiently large subgroup of $\Gamma_{g,n}$ that fixes $[\rho]$.

1.B. **Dynamical study of finite orbits in the reducible rank** 2 **case.** Since the pure mapping class group is a finite index subgroup of the full mapping class group, for any representation $\rho \in \mathrm{Hom}(\Lambda_{g,n}, G)$, the conjugacy class $[\rho] \in \chi_{g,n}(G)$ has finite orbit under $\Gamma_{g,n}$ if and only if it has finite orbit under $\hat{\Gamma}_{g,n}$. Note that the size of $\hat{\Gamma}_{g,n} \cdot [\rho]$ equals the size of the set of conjugacy classes of m-tuples

$$\left\{ (\rho'(s_1), \ldots, \rho'(s_m)) \;\middle|\; \rho' \in \mathrm{Hom}(\Lambda_{g,n}, G) \text{ and } [\rho'] \in \hat{\Gamma}_{g,n} \cdot [\rho] \right\}/G \,,$$

where $\{s_1, \ldots, s_m\}$ is a set of generators of $\Lambda_{g,n}$. We introduce a specific presentation

$$\Lambda_{g,n} = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle$$

and a subgroup

$$\hat{\Gamma}_{g,n}^{\circ} = \langle \tau_1, \ldots, \tau_{3g+n-2}, \sigma_1, \ldots, \sigma_{n-1} \rangle$$

of $\hat{\Gamma}_{g,n}^{\bullet}$ which, as such, acts on $\mathrm{Hom}(\Lambda_{g,n}, G)$, and which is sufficiently large in the sense that the $\hat{\Gamma}_{g,n}^{\circ}$-orbit of $[\rho] \in \chi_{g,n}(G)$ equals its $\hat{\Gamma}_{g,n}$-orbit. Moreover, the action of $\hat{\Gamma}_{g,n}^{\circ}$ on $\Lambda_{g,n}$ can be explicitly described (see Section 6). Table 2 summarizes the explicit action of the generators $\tau_1, \ldots, \tau_{3g+n-2}, \sigma_1, \ldots, \sigma_{n-1}$ of $\hat{\Gamma}_{g,n}^{\circ}$ on the generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n$ of $\Lambda_{g,n}$. Here we only indicate the action on those of our generators of $\Lambda_{g,n}$ that are not fixed by the action of the generator of $\hat{\Gamma}_{g,n}^{\circ}$ under consideration.

τ_{2k}	$k \in [\![1, g]\!]$	α_k	$\mapsto$	$\alpha_k \beta_k$
τ_{2k-1}	$k \in [\![1, g]\!]$	β_k	$\mapsto$	$\beta_k \alpha_k$
τ_{2g+k}	$k \in [\![1, g-1]\!]$	α_{k+1}	$\mapsto$	$\Theta_k^{-1} \alpha_{k+1}$
		α_k	$\mapsto$	$\alpha_k \Theta_k$
		β_k	$\mapsto$	$\Theta_k^{-1} \beta_k \Theta_k$
	where	Θ_k	$:=$	$\alpha_{k+1} \beta_{k+1}^{-1} \alpha_{k+1}^{-1} \beta_k$
τ_{3g-1+k}	$k \in [\![1, n-1]\!]$	α_g	$\mapsto$	$\alpha_g \Xi_k$
		β_g	$\mapsto$	$\Xi_k^{-1} \beta_g \Xi_k$
	$i \in [\![1, k]\!]$	γ_i	$\mapsto$	$\Xi_k^{-1} \gamma_i \Xi_k$
	where	Ξ_k	$=$	$(\gamma_1 \ldots \gamma_k)^{-1} \beta_g$
σ_k	$k \in [\![1, n-1]\!]$	γ_k	$\mapsto$	$\gamma_k \gamma_{k+1} \gamma_k^{-1}$
		γ_{k+1}	$\mapsto$	γ_k

TABLE 2. Action of $\hat{\Gamma}_{g,n}^{\circ}$ on $\Lambda_{g,n}$.

We then apply this explicit description of the mapping class group action to the specific study of finite $\Gamma_{g,n}$-orbits on $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ that correspond to reducible representations. For $g = 0$, this study has been completely carried out in [CM16]. In this special case, the study can be reduced

to linear dynamics. More precisely, any reducible representation $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ is conjugated to the tensor product of a character $\rho_{\mathbb{C}^*} \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ and an affine representation $\rho_{\mathrm{Aff}} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$:

$$[\rho] = [\rho_{\mathbb{C}^*} \otimes \rho_{\mathrm{Aff}}].$$

Moreover, $[\rho]$ has finite orbit under $\Gamma_{g,n}$ in $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ if and only if $[\rho_{\mathbb{C}^*}]$ and $[\rho_{\mathrm{Aff}}]$ have finite orbit under $\Gamma_{g,n}$ in $\chi_{g,n}(\mathbb{C}^*)$ and $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ respectively. On the other hand, for $g = 0$, the pure mapping class group acts trivially on $\chi_{g,n}(\mathbb{C}^*)$ and on the linear part of ρ_{Aff}. Hence in the special case $g = 0$, the study of finite orbits reduces to the study of a certain linear action on the translation part of ρ_{Aff}.

For $g > 0$, the study of finite orbits of conjugacy classes of reducible $\mathrm{GL}_2\mathbb{C}$-representations also reduces to the case of scalar and affine representations, but the linear part of ρ_{Aff} is no longer invariant and there is no effective means to reduce the study to linear dynamics. However, Table 2 allows to study the orbits explicitly. In the case $g = 1$ and $n > 0$, we find a particular type of representations whose conjugacy classes have finite orbit under $\Gamma_{g,n}$, namely the representations $\rho_{\mu,\mathbf{c}} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ defined by

$$\rho_{\mu,\mathbf{c}}(\alpha_1) := \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_{\mu,\mathbf{c}}(\beta_1) := \begin{pmatrix} 1 & -\frac{1}{\mu-1} \\ 0 & 1 \end{pmatrix} \quad \rho_{\mu,\mathbf{c}}(\gamma_i) := \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \quad \forall i \in [\![1, n]\!]$$

where $\mu \in \mathbb{C}^* \setminus \{1\}$ is a root of unity and $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$ with $\sum_{i=1}^n c_i = 1$. Note that the condition $\sum_{i=1}^n c_i = 1$ is necessary for $\rho_{\mu,\mathbf{c}}$ to be well defined. The complete classification, for every $g > 0$ and $n \geq 0$, of reducible rank-2 representations with finite $\Gamma_{g,n}$-orbit is the following.

Theorem B. *Let $g > 0, n \geq 0$. Let $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ be a reducible representation. Consider its conjugacy class $[\rho] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$. Then the orbit $\Gamma_{g,n} \cdot [\rho]$ is finite if and only if one of the following conditions is satisfied.*

(1) The representation ρ is a direct sum of scalar representations with finite images.

(2) We have $g = 1$, $n > 0$, there are a root of unity $\mu \in \mathbb{C}^* \setminus \{1\}$, $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$ with $\sum_{i=1}^n c_i = 1$ and a scalar representation λ with finite image such that

$$[\rho] \in \Gamma_{g,n} \cdot [\lambda \otimes \rho_{\mu,\mathbf{c}}].$$

Moreover, if the orbit $\Gamma_{g,n} \cdot [\rho]$ is finite, we can give an estimate for its cardinality, which for $\rho = \lambda_1 \oplus \lambda_2$ and $\rho = \lambda \otimes \rho_{\mu,\mathbf{c}}$ in the cases (1) and (2) respectively is

(1) $\frac{1}{2} \cdot \max \left\{ \mathrm{card}(\mathrm{Im}(\lambda_i))^{2g-1} \mid i \in \{1, 2\} \right\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\mathrm{Im}(\rho))^{2g}$ and

(2) $\max \left\{ N_2 , \phi(N)(2N - \phi(N))N^{n'-1} \right\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq (N^2 - 1)N^{n'-1}N_2^2,$

where ϕ denotes the Euler totient function, $n' := \mathrm{card}\{i \in [\![1, n]\!] \mid \rho(\gamma_i) \notin \mathbb{C}^* I_2\}$, $N := \mathrm{order}(\mu)$ and $N_2 := \mathrm{card}(\mathrm{Im}(\lambda))$.

The heart of the proof of Theorem B is the complete classification of finite $\hat{\Gamma}_{g,n}$-orbits in $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ under the full mapping class group (see the beginning of Section 7 for details on how we proceed). In Section 8.1, we deduce an explicit description of the finite $\Gamma_{g,n}$-orbits for scalar and affine representations. The decomposition of reducible representations into a tensor product of such representations then yields the result (see Sections 8.2 and 8.3).

Part A. Algebraization

2. Universal isomonodromic deformations

In this section, we will recall some well known results about moduli spaces and universal families of curves. For a more detailed exposition, see for example [ACG11, Chap. 15] and [Hub06, Chap. 6]. Then we turn to the existence of analytic and algebraic universal isomonodromic deformations of connections over curves, and their respective universal properties. The main

purpose of this section is the precise setup of our notation and definitions. The reader might want to skip this section at first and come back to it when needed (references will be given).

2.1. **Moduli spaces of curves.** We define *a curve of genus g* to be a *smooth projective complex* curve C with $H^1(C, \mathbb{Z}) = \mathbb{Z}^{2g}$. From now on, we will assume

(4) $$2g - 2 + n > 0\,.$$

As a set, the Teichmüller space $\mathcal{T}_{g,n}$ of n-pointed genus-g curves is the set of isomorphism classes $[C, D_C, \varphi]$ of triples (C, D_C, φ), where C is a genus g curve, $D_C = \{x_1, \ldots, x_n\}$ is a set of n distinct points in C and φ is a Teichmüller structure, *i.e.* an orientation-preserving homeomorphism $\varphi : (\Sigma_g, Y^n) \to (C, D_C)$. Two n-pointed genus-g curves with Teichmüller structure (C, D_C, φ) and $(C', D'_{C'}, \varphi')$ are said to be isomorphic if there exists an isomorphism of pointed curves $\psi : (C', D'_{C'}) \to (C, D_C)$ such that $[\varphi'] = [\psi \circ \varphi]$, where $[\varphi]$ denotes the isotopy class of φ. We have a natural action of $\Gamma_{g,n}$ on $\mathcal{T}_{g,n}$ given by

$$[h] \cdot [C, D_C, \varphi] := [C, D_C, \varphi \circ h^{-1}]\,; \qquad [h] \in \Gamma_{g,n}\,, [C, D_C, \varphi] \in \mathcal{T}_{g,n}\,.$$

The kernel of this action is finite. More precisely, we have (see [ACG11, Prop 4.11 p. 189]):

Lemma 2.1.1. *If the natural morphism $\Gamma_{g,n} \to \mathrm{Aut}(\mathcal{T}_{g,n})$ has nontrivial kernel $K_{g,n}$, then $K_{g,n} \simeq \mathbb{Z}/2\mathbb{Z}$ and one of the following holds.*

- *$(g, n) = (2, 0)$ and the non-trivial element of $K_{g,n}$ is the hyperelliptic involution of Σ_2.*
- *$(g, n) = (1, 1)$ and the non-trivial element of $K_{g,n}$ is the order 2 symmetry about the puncture, given, for $(\Sigma_1, y_1) = (\mathbb{C}/\mathbb{Z}^2, 0)$, by $z \mapsto -z$.*

As a set, the moduli space $\mathcal{M}_{g,n}$ of curves of genus g with n (labeled) punctures is the set of isomorphism classes $[C, \mathbf{x}]$ of pairs $(C, \mathbf{x})$, where C is a genus g curve and $\mathbf{x} = (x_1, \ldots, x_n)$ is a tuple of n distinct points in C. The isomorphisms are isomorphisms of pointed curves that *respect the labellings* of the n-tuples. Notice that a Teichmüller structure (C, D_C, φ) defines such a pair $(C, \mathbf{x})$, by setting $\mathbf{x} := (\varphi(y_i))_{i \in [\![1,n]\!]}$. In this way, we obtain a forgetful map

(5) $$\pi_{g,n} : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$$

whose fibers are globally fixed by the action of $\Gamma_{g,n}/K_{g,n}$ on $\mathcal{T}_{g,n}$. Denote by $\mathcal{R}_{g,n} \subset \mathcal{T}_{g,n}$ the set consisting in points with non-trivial stabilizer for the action of $\Gamma_{g,n}/K_{g,n}$. The subset $\mathcal{B}_{g,n} := \pi_{g,n}(\mathcal{R}_{g,n})$ of $\mathcal{M}_{g,n}$ characterizes pointed curves with automorphism groups not isomorphic to $K_{g,n}$. We say that these curves have *exceptional automorphisms*.

Recall that $\mathcal{T}_{g,n}$ has a natural structure of a complex analytic manifold, and $\mathcal{M}_{g,n}$ has a natural structure of a complex quasi-projective variety (see [ACG11, chap. XIV]). The set $\mathcal{B}_{g,n}$ of curves with exceptional automorphisms is a Zariski closed subset of $\mathcal{M}_{g,n}$ (see [ACG11, Rem. 5.13 p. 202 and Th. 6.5 p. 207]) which is a proper subset (see [Bai62, Mon62, Poo00, Cor08]). Moreover, the map $\pi_{g,n}|_{\mathcal{T}_{g,n} \setminus \mathcal{R}_{g,n}} : \mathcal{T}_{g,n} \setminus \mathcal{R}_{g,n} \to \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$ is a non branched analytic cover, with Galois group $\Gamma_{g,n}/K_{g,n}$. We thereby obtain a tautological morphism

(6) $$\mathrm{taut}_{g,n} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star) \twoheadrightarrow \Gamma_{g,n}/K_{g,n}\,;$$

such that any lift $\tilde{\gamma}$ in $\mathcal{T}_{g,n}$ for a loop γ of $\pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star)$ satisfies $\tilde{\gamma}(1) = \mathrm{taut}_{g,n}(\gamma) \cdot \tilde{\gamma}(0)$.

2.2. **Families of pointed curves.** Let C be a curve of genus g and D_C a reduced divisor of degree n on C. Recall that we always assume (4); *i.e.* the pointed curve (C, D_C) is stable.

A *ω-family of n-pointed genus-g curves with central fiber (C, D_C)* is a datum

$$\mathcal{F}_{(C, D_C)} = (\kappa : X \to T, D, t_0, \psi)\,,$$

where

- $\kappa : X \to T$ is a proper surjective smooth morphism of ω-manifolds;
- $D = \sum_{i=1}^{n} D_i$ is a reduced divisor on X such that
- there are pairwise disjoint sections $\sigma_1, \ldots, \sigma_n$ of κ with $\sigma_i(T) = D_i$;

- $t_0 \in T$ is a point and X_{t_0} denotes $X_{t_0} := \kappa^{-1}(\{t_0\})$;
- $\psi : (C, D_C) \xrightarrow{\sim} (X_{t_0}, D|_{X_{t_0}})$ is an isomorphism of ω-manifolds.

We shall always denote by $X_t := \kappa^{-1}(\{t\})$ the fiber at a parameter $t \in T$. When there is a smooth connected ω-neighborhood Δ of t_0 such that $\mathcal{F}_{(C,D_C)}|_\Delta$ satisfies a certain property, we may say that $\mathcal{F}_{(C,D_C)}$ satisfies this property "up to shrinking".

A morphism $\mathfrak{f} : \mathcal{F}'_{(C,D_C)} \to \mathcal{F}_{(C,D_C)}$ is a pair $\mathfrak{f} = (\mathfrak{f}^a, \mathfrak{f}^b)$, where $\mathfrak{f}^a : X' \to X$ and $\mathfrak{f}^b : T' \to T$ are morphisms of ω-varieties such that the following diagram commutes (and in particular $\mathfrak{f}^b(t'_0) = t_0$).

$$
\begin{array}{ccc}
(C, D_C) & \xlongequal{\mathrm{id}} & (C, D_C) \\
\psi' \downarrow & & \downarrow \psi \\
(X', D') & \xrightarrow{\mathfrak{f}^a} & (X, D) \\
\kappa' \downarrow & & \downarrow \kappa \\
T' & \xrightarrow{\mathfrak{f}^b} & T
\end{array}
$$

Remark 2.2.1. Note that this definition implies that (X', D') is isomorphic to the pullback $\mathfrak{f}^{b*}(X, D)$ (the fibered product with respect to $\mathfrak{f}^b$ and κ).

Suppose now that we have a Teichmüller structure for (C, D_C), given by an orientation preserving homeomorphism $\varphi : (\Sigma_g, Y^n) \to (C, D_C)$.

A ω-family of n-pointed genus-g curves with Teichmüller structure with central fiber (C, D_C, φ) is a datum $\mathcal{F}^+_{(C,D_C,\varphi)} = (\mathcal{F}_{(C,D_C)}, \Phi)$, where $\mathcal{F}_{(C,D_C)}$ is as above and $\Phi : (\Sigma_g, Y^n) \times T \xrightarrow{\sim} (X, D)$ is a homeomorphism such that the following diagram commutes, where pr denotes the projection to the second factor.

$$
\begin{array}{ccc}
(\Sigma_g, Y^n) \times \{t_0\} & \xrightarrow{\varphi} & (C, D_C) \\
\downarrow & & \downarrow \psi \\
(\Sigma_g, Y^n) \times T & \xrightarrow{\Phi} & (X, D) \\
& \searrow{\mathrm{pr}} & \downarrow \kappa \\
& & T
\end{array}
$$

In particular, if we denote

$$\Phi_t := \Phi|_{(\Sigma_g, Y^n) \times \{t\}},$$

then $\Phi_{t_0} = \psi \circ \varphi$. Notice that by definition, a ω-family with Teichmüller structure is topologically trivial. For a given φ as above, up to shrinking, any analytic family $\mathcal{F}_{(C,D_C)}$ lifts to a family $\mathcal{F}^+_{(C,D_C,\varphi)}$ with Teichmüller structure.

Let $\mathcal{F}^+_{(C,D_C,\varphi)} = (\mathcal{F}_{(C,D_C)}, \Phi)$ and $\mathcal{F}'^+_{(C,D_C,\varphi')} = (\mathcal{F}'_{(C,D_C)}, \Phi')$ be two ω-families with Teichmüller structures. A morphism $\mathcal{F}'^+_{(C,D_C,\varphi')} \to \mathcal{F}^+_{(C,D_C,\varphi)}$ is a datum $\mathfrak{f}^+ = (\mathfrak{f}, \mathfrak{f}^{\mathrm{top}})$, where $\mathfrak{f} = (\mathfrak{f}^a, \mathfrak{f}^b)$ is as before and $\mathfrak{f}^{\mathrm{top}}$ is a continous map, such that the following diagram commutes.

$$
\begin{array}{ccc}
(\Sigma_g, Y^n) \times T' & \xrightarrow{\mathfrak{f}^{\mathrm{top}}} & (\Sigma_g, Y^n) \times T \\
\Phi' \downarrow & & \uparrow \Phi^{-1} \\
(X', D') & \xrightarrow{\mathfrak{f}^a} & (X, D)
\end{array}
$$

Notice that the central fiber forces the map $\mathfrak{f}^{\mathrm{top}}$ to be $(\varphi^{-1} \circ \varphi') \times \mathfrak{f}^b$, up to a fiberwise isotopy.

To a ω-family $\mathcal{F}_{(C,D_C)}$ [resp. ω-family with Teichmüller structure $\mathcal{F}^+_{(C,D_C,\varphi)}$] as before, one can associate a so-called ω-family $\mathcal{F}$ [resp. ω-family with Teichmüller structure $\mathcal{F}^+$] *with non specified central fiber*, by forgetting (C, D_C) [resp. (C, D_C, φ)] and the marking t_0, ψ. A

morphism of ω-families with non specified central fiber is a datum $\mathfrak{f}$ [respectively $\mathfrak{f}^+$] as above for a convenient choice of a marked central fiber.

2.3. Universal families of pointed curves.

Let $\mathcal{F}^+_{(C,D_C,\varphi)} = (\mathcal{F}_{(C,D_C)}, \Phi)$ be a ω-family with Teichmüller structure. Then the classifying map

$$\mathrm{class}^+(\mathcal{F}^+) : \begin{cases} T & \to & \mathcal{T}_{g,n} \\ t & \mapsto & [X_t, D|_{X_t}, \Phi_t] \end{cases}$$

is holomorphic with respect to the natural analytic manifold structure of $\mathcal{T}_{g,n}$. The Teichmüller space $\mathcal{T}_{g,n}$ carries a universal family (see for example [Hub06, Chap. 6]), which is an analytic family with Teichmüller structure $\mathcal{F}^+_{g,n} = (\mathcal{F}_{g,n}, \Phi_{g,n})$ and non specified central fiber, satisfying

$$\mathrm{class}^+(\mathcal{F}^+_{g,n}) = \mathrm{id}_{\mathcal{T}_{g,n}} .$$

The universal Teichmüller curve enjoys the following universal property: If $\mathcal{F}^+ = (\mathcal{F}, \Phi)$ is an analytic family with Teichmüller structure and non specified central fiber, then there is a unique isomorphism

$$\mathfrak{f}^+ : \mathcal{F}^+ \overset{\sim}{\longrightarrow} \mathrm{class}^+(\mathcal{F}^+)^*(\mathcal{F}_{g,n})$$

with $\mathfrak{f}^b = \mathrm{id}_T$.

Let $\mathcal{F}_{(C,D_C)}$ be a ω-family. Assume we have a *labelling* $\mathbf{x}$ of D_C, *i.e.* $\mathbf{x} = (x_i)_{i \in [\![1,n]\!]} \in C^n$ and $D_C = \sum_{i=1}^n x_i$. Then there is a well defined labelling $\mathbf{D} = (D_i)_{i \in [\![1,n]\!]}$ of D defined by $D = \sum_{i=1}^n D_i$ and $\psi(x_i) \in D_i$ for all $i \in [\![1,n]\!]$. We then have a well defined classifying map

$$\mathrm{class}(\mathcal{F}) : \begin{cases} T & \to & \mathcal{M}_{g,n} \\ t & \mapsto & [X_t, \mathbf{D}|_{X_t}] \end{cases} ,$$

which is a morphism of ω-varieties with respect to the natural structure of ω-variety on $\mathcal{M}_{g,n}$. We say that the fiber $(X_t, D|_{X_t})$ of $\mathcal{F}$ at $t \in T$ has *exceptional automorphisms* if $\mathrm{class}(\mathcal{F})(t) \in \mathcal{B}_{g,n}$. This notion does not depend on the choice of a labelling.

Although there is no universal family of curves over $\mathcal{M}_{g,n}$ in the strict sense, we can consider algebraic Kuranishi families. Let $\mathcal{F}$ be a ω-family and let $t \in T$ be a parameter. Denote $\mathcal{F}|_{\Delta^{\mathrm{an}}}$ the analytic germification of $\mathcal{F}$ at t, which can be endowed with a Teichmüller structure $\Phi_{\Delta^{\mathrm{an}}}$. We say that $\mathcal{F}$ is *Kuranishi at t* if $\mathrm{class}^+(\mathcal{F}|_{\Delta^{\mathrm{an}}}, \Phi_{\Delta^{\mathrm{an}}})$ is an isomorphism. The notion of being Kuranishi at t does not depend on the choice of $\Phi_{\Delta^{\mathrm{an}}}$. We say that $\mathcal{F}$ is *Kuranishi* if it is Kuranishi at each $t \in T$. Notice that if $\mathcal{F}^{\mathrm{Kur}}$ is an algebraic Kuranishi family, then for any labelling, the classifying map $\mathrm{class}(\mathcal{F}^{\mathrm{Kur}})$ is dominant and has finite fibers.

For any stable n-pointed genus-g curve (C, D_C), there exists an algebraic Kuranishi family $\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ with central fiber (C, D_C). Moreover, we have (see [ACG11, Rem. 6.9, p. 208]):

Proposition 2.3.1 (Universal property of Kuranishi families). *Let (C, D_C) and $\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ be as above. Let $\mathcal{F}'_{(C,D_C)}$ be an algebraic family with central fiber (C, D_C). Then there are*

- *an étale base change $p : (T'', t_0'') \to (T', t_0')$; denote $\mathcal{F}''_{(C,D_C)} := p^*\mathcal{F}'_{(C,D_C)}$;*
- *a morphism $q : (T'', t_0'') \to (T, t_0)$ and*
- *an isomorphism $\mathfrak{f} : \mathcal{F}''_{(C,D_C)} \overset{\sim}{\longrightarrow} q^*\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ with $\mathfrak{f}^b = id_{\Delta''}$.*

2.4. Universal isomonodromic deformations.

Let again (C, D_C) be a stable n-pointed genus-g curve. Let (E_0, ∇_0) be logarithmic ω-connection over C with polar divisor D_C.

Isomonodromic deformations. A ω-*isomonodromic deformation* of (C, E_0, ∇_0) is a tuple $\mathcal{I}_{(C,E_0,\nabla_0)} = (\mathcal{F}_{(C,D_C)}, E, \nabla, \Psi)$, where

- $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ is a ω-family with central fiber (C, D_C),
- (E, ∇) is a flat logarithmic ω-connection over X with polar divisor D and
- $(\psi, \Psi) : (E_0 \to C, \nabla_0) \to (E \to X, \nabla)|_{X_{t_0}}$ is an isomorphism of ω-logarithmic connections, *i.e.* $\Psi : E_0 \to E|_{X_{t_0}}$ is a ω-vector bundle isomorphism over $\psi : C \to X_{t_0}$ satisfying $\Psi^* \left(\nabla|_{X_{t_0}} \right) = \nabla^0$.

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Let $\mathcal{I}_{(C,E_0,\nabla_0)}$ and $\mathcal{I}'_{(C,E_0,\nabla_0)}$ be two ω-isomonodromic deformations of (C, E_0, ∇_0). A morphism $\mathfrak{f} : \mathcal{I}_{(C,E_0,\nabla_0)} \to \mathcal{I}'_{(C,E_0,\nabla_0)}$ is a datum $\mathfrak{f} = (\mathfrak{f}^a, \mathfrak{f}^b, \mathfrak{f}^{\mathrm{vb}})$, where $(\mathfrak{f}^a, \mathfrak{f}^b)$ is a morphism $\mathcal{F}'_{(C,D_C)} \to \mathcal{F}_{(C,D_C)}$ as in Section 2.2, and $\mathfrak{f}^{\mathrm{vb}}$ is a morphism of ω-vector bundles over $\mathfrak{f}^a$ with $\nabla = \mathfrak{f}^{vb*}\nabla'$.

An algebraic isomonodromic deformation $\mathcal{I}_{(C,E_0,\nabla_0)}$ of (C, E_0, ∇_0) as above is called *regular* if moreover (E, ∇) is regular (with respect to a suitable meromorphic structure at infinity). The definition of regularity can be found in [Del70, Th. 4.1]. Putting this regularity condition on $\mathcal{I}_{(C,E_0,\nabla_0)}$ may be seen as a way of standardizing algebraic isomonodromic deformations, as illustrated by the following statement.

Lemma 2.4.1. *If (E_0, ∇_0) is mild and $\mathcal{I}_{(C,E_0,\nabla_0)}$ is an algebraic isomonodromic deformation of (C, E_0, ∇_0), then the analytification of $\mathcal{I}_{(C,E_0,\nabla_0)}$ is isomorphic to the analytification of a regular algebraic isomonodromic deformation $\mathcal{I}'_{(C,E_0,\nabla_0)}$ of (C, E_0, ∇_0).*

This Lemma will be proven in Section 3.2, where we will also recall the notion of mildness, which is a minor technical condition.

Analytic universal isomonodromic deformations. Let $\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D_C)$ be an orientation preserving homeomorphism. Consider the universal Teichmüller family $\mathcal{F}^+_{g,n} = (\mathcal{F}_{g,n}, \Phi_{g,n})$. We shall denote

$$\mathcal{F}_{g,n} = (\kappa_{g,n} : \mathcal{X} \to \mathcal{T}_{g,n}, \mathcal{D}); \quad \Phi_{g,n} : (\Sigma_g, Y^n) \times \mathcal{T}_{g,n} \xrightarrow{\sim} (\mathcal{X}, \mathcal{D}); \quad t_0 := [C, D_C, \varphi] \in \mathcal{T}_{g,n}.$$

By the definition of $\mathcal{F}^+_{g,n}$, we then have an isomorphism $\psi : (C, D_C) \xrightarrow{\sim} (\mathcal{X}_{t_0}, \mathcal{D}|_{\mathcal{X}_{t_0}})$. In particular, $\mathcal{F}^{\mathrm{Teich}}_{(C,D_C)} := (\mathcal{F}_{g,n}, t_0, \psi)$ is an analytic family with central fiber (C, D_C), which moreover is topologically trivial and has simply connected parameter space. The inclusion $\Phi^{-1}_{g,n} \circ \psi \circ \varphi$ of the topological fiber at t_0 then defines an isomorphism

$$(7) \qquad \Lambda_{g,n} = \pi_1(\Sigma_g \setminus Y^n, y_0) \xrightarrow{\sim} \pi_1((\Sigma_g \setminus Y^n) \times \mathcal{T}_{g,n}, (y_0, t_0)).$$

Now let $[\rho_{\nabla_0}]$ be the monodromy of (E_0, ∇_0) with respect to φ. The representation ρ_{∇_0} can then be trivially extended to a representation ρ of $\pi_1((\Sigma_g \setminus Y^n) \times \mathcal{T}_{g,n}, (y_0, t_0))$. It turns out that the conjugacy class of this "extended representation" is the monodromy representation, with respect to $\Phi_{g,n}$, of a certain flat logarithmic connection (E, ∇) over $\mathcal{X}$ with polar divisor $\mathcal{D}$ such that the pullback $\psi^*(E, \nabla)$ restricted to $\mathcal{X}_{t_0}$ is canonically isomorphic to (E_0, ∇_0). We obtain the *universal analytic isomonodromic deformation*

$$\mathcal{I}^{\mathrm{univ, an}}_{(C,E_0,\nabla_0)} := (\mathcal{F}^{\mathrm{Teich}}_{(C,D_C)}, E, \nabla, \Psi^{\mathrm{can}}).$$

Its construction has been carried out in [Heu10], using Malgrange's Lemma (see [Mal83]) and the fact that $\mathcal{T}_{g,n}$ is contractible by Fricke's Theorem. It satisfies the following universal property: if $\mathcal{I}'_{(C,E_0,\nabla_0)} = (\mathcal{F}'_{(C,D_C)}, E', \nabla', \Psi')$ is an analytic isomonodromic deformation of (C, E_0, ∇_0), and if Δ' is a sufficiently small neighborhood of its central parameter t'_0, then there is a morphism $q : (\Delta', t'_0) \to (\mathcal{T}_{g,n}, t_0)$ and a canonical isomorphism

$$\mathcal{I}'_{(C,E_0,\nabla_0)}|_{\Delta'} \simeq q^* \mathcal{I}^{\mathrm{univ, an}}_{(C,E_0,\nabla_0)}.$$

The construction of this analytic universal isomonodromic deformation and the proof of its universal property rely on the fact that up to shrinking the parameter space, analytic families of curves are topologically trivial and have simply connected parameter space. This is of course no longer the case in the algebraic category, the "extension of the monodromy representation" of (C, E_0, ∇_0) being the main challenge.

Algebraic universal isomonodromic deformations. An *algebraic universal isomonodromic deformation* of (C, E_0, ∇_0) is an algebraic isomonodromic deformation $\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)} = (\mathcal{F}_{(C,D_C)}, E, \nabla, \Psi)$, where $\mathcal{F}_{(C,D_C)} = \mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ is an algebraic Kuranishi family with central fiber (C, D_C). Note that an algebraic universal isomonodromic deformation of (C, E_0, ∇_0) does not

need to exist; its existence is precisely the subject of Theorem A. When it does exist, it satisfies the following universal property, which will be proven in Section 3.3.

Proposition 2.4.2 (Universal property of universal algebraic isomonodromic deformations). *Let (C, E_0, ∇_0) and $\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)}$ be as above. Let $\mathcal{I}'_{(C,E_0,\nabla_0)}$ be another algebraic isomonodromic deformation of (C, E_0, ∇_0). Assume that*

- *(E_0, ∇_0) is mild;*
- *the monodromy representation of (E_0, ∇_0) is irreducible;*
- *$\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)}$ and $\mathcal{I}'_{(C,E_0,\nabla_0)}$ are both regular.*

Then there are

- *an étale base change $p : (T'', t_0'') \to (T', t_0')$;*
  denote $(\mathcal{F}''_{(C,D_C)}, E'', \nabla'', \Psi'') := p^* \mathcal{I}'_{(C,E_0,\nabla_0)}$;
- *a flat algebraic connection (L, ξ) of rank 1 over T'' with empty polar divisor;*
- *a morphism $q : (T'', t_0'') \to (T, t_0)$ and*
- *an isomorphism $\mathfrak{f} : (\mathcal{F}''_{(C,D_C)}, (E'', \nabla'') \otimes \kappa''^*(L, \xi), \Psi'') \xrightarrow{\sim} q^* \mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)}$ with $\mathfrak{f}^b = id_{\Delta''}$.*

Remark 2.4.3. It is not possible without further assumptions to prove a similar statement for initial connections (E_0, ∇_0) with merely semisimple monodromy representations.

3. Fundamental groups and the Riemann-Hilbert correspondence

In this section, we shall see that up to an étale base change, any algebraic family of pointed curves can be endowed with a section avoiding the punctures. The existence of such a base point section allows us to decompose the fundamental group of the total space of the family of curves into an semi-direct product of the fundamental groups of the central fiber and the parameter space. Together with the logarithmic Riemann-Hilbert correspondence, this will be used to prove the universal property of universal algebraic isomonodromic deformations.

3.1. Splitting of the fundamental group.

Lemma 3.1.1 (Existence of a base point section). *Let $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be an algebraic family of pointed curves with central fiber (C, D_C) as in Section 2.2. Let x_0 be a point in $C \setminus D_C$. Then there are*

- *a Zariski open neighborhood Δ of t_0 in T and*
- *a finite étale cover $p : (\Delta', t_0') \to (\Delta, t_0)$*

such that for $\mathcal{F}'_{(C,D_C)} = (\kappa' : X' \to \Delta', D', t_0', \psi')$, defined by $\mathcal{F}'_{(C,D_C)} := p^* \mathcal{F}_{(C,D_C)}$, there exists a section σ of κ' with values in $X' \setminus D'$ such that $\sigma(t_0') = \psi'(x_0)$.

Proof. Since X is embedded in some projective space $\mathbb{P}^N$, by Bertini's Theorem, there exists a hyperplane H of $\mathbb{P}^N$ which intersects X_{t_0} transversely, is disjoint from $D|_{X_{t_0}}$ and satisfies $\psi(x_0) \in H$. Since H is ample, we have $\deg(X_t \cap H) > 0$ for each $t \in T$. In particular, $H \cap X_t \neq \varnothing$ for each parameter $t \in T$. By irreducibility of T, there exists an irreducible component T' of $X \cap H$ such that $\kappa(T') = T$ and $\psi(x_0) \in T'$. Now $\kappa|_{T'} : T' \to T$ is a connected finite ramified covering. Denote by $Z_1 \subset T$ its branching locus. Further, denote by Z_2 the adherence of $\kappa(T' \cap D)$. By construction, $Z := Z_1 \cup Z_2$ is a Zariski closed proper subset of T not containing t_0. Denote $\Delta := T \setminus Z$ and

$$\Delta' := \kappa^{-1}(\Delta) \cap T'.$$

We now have $t_0' := \psi(x_0) \in \Delta'$ and

$$p := \kappa|_{\Delta'} : (\Delta', t_0') \to (\Delta, t_0)$$

is a connected finite étale cover. Consider the algebraic family $\mathcal{F}'_{(C,D_C)} := p^* \mathcal{F}_{(C,D_C)}$. By definition of the pullback, its total space X' is given by a fibered product

$$X' = \{(x,t') \in X|_{\kappa^{-1}(\Delta)} \times \Delta' \mid \kappa(x) = p(t')\}$$

and we have $\kappa' : X' \to \Delta'$; $(x,t') \mapsto t'$. On the other hand, Δ' is a subset of $X|_{\kappa^{-1}(\Delta)}$ by construction and we can define a section σ of κ' by

$$\sigma : \Delta' \to X' ; \ t' \mapsto (t',t') .$$

Since moreover $\Delta' \cap D = \varnothing$ by the choice of Z_2, we have $\sigma(\Delta') \cap D' = \varnothing$. We conclude by noticing $\sigma(t'_0) = (\psi(x_0), t'_0) = \psi'(x_0)$. $\qquad\square$

To fix notations, let us recall the definition of (inner) semi-direct products.

Let G be a group and A a subgroup. Assume we have a group $\widetilde{B}$ fitting into a split short exact sequence of groups, as follows.

$$\{1\} \longrightarrow A \longrightarrow G \overset{\sigma}{\underset{}{\longrightarrow}} \widetilde{B} \longrightarrow \{1\}$$

Assume further that the map $A \to G$ in that sequence is defined by the inclusion map. Then A is a normal subgroup of G; for $B := \sigma(\widetilde{B})$ we have a natural morphism $\eta \in \mathrm{Hom}(B, \mathrm{Aut}(A))$ defined by $\eta(b)(a) = b \cdot a \cdot b^{-1}$ for all $a \in A$, $b \in B$; we have a group $A \rtimes_\eta B$ defined as the set $A \times B$ endowed with the group law

$$(a,b) \cdot (a',b') = (a \cdot \eta(b)(a'), b \cdot b') ,$$

and the natural morphism $A \rtimes_\eta B \to G$ defined by $(a,b) \mapsto a \cdot b$ is bijective, allowing us to identify $G = A \rtimes_\eta B$.

Lemma 3.1.2 (Splitting). *Let $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be an algebraic family as in Section 2.2. Let $\sigma : T \to X$ be a section of κ such that $\sigma(T) \subset X^0 = X \backslash D$. Denote $C^0 := C \backslash D_C$ and $x_0 := \psi^{-1}(\sigma(t_0))$. Then*

(8) $$\pi_1(X^0, \sigma(t_0)) = \psi_* \pi_1(C^0, x_0) \rtimes_\eta \sigma_* \pi_1(T, t_0) ,$$

where for all $\gamma \in \pi_1(C^0, x_0)$ and $\beta \in \pi_1(T, t_0)$ we have

$$\eta(\sigma_* \beta)(\psi_* \gamma) \ = \ \sigma_* \beta \cdot \psi_* \gamma \cdot \sigma_* \beta^{-1} .$$

Proof. Since σ takes values in X^0, we have a morphism of fundamental groups $\sigma_* : \pi_1(T, t_0) \to \pi_1(X^0, \sigma(t_0))$. From the embedding of the central fiber, we get the morphism $\psi_* : \pi_1(C^0, x_0) \to \pi_1(X^0, \sigma(t_0))$. Consider now the family of n-punctured curves given by $\kappa : X^0 \to T$. This family is a topologically locally trivial fibration and the fiber over t_0 identifies, *via* ψ, with C^0. Hence we have a long homotopy exact sequence

$$\cdots \longrightarrow \pi_2(X^0, \sigma(t_0)) \overset{\kappa^*}{\longrightarrow} \pi_2(T, t_0) \longrightarrow \pi_1(C^0, \sigma(t_0)) \overset{\psi^*}{\longrightarrow} \pi_1(X^0, \sigma(t_0)) \overset{\kappa^*}{\longrightarrow} \pi_1(T, t_0) \longrightarrow \{1\}.$$

The maps $\sigma_* : \pi_j(T, t_0) \to \pi_j(X^0, \tau(t_0))$ are sections for the corresponding κ_* and we may derive the following split short exact sequence:

$$\{1\} \longrightarrow \pi_1(C^0, x_0) \overset{\psi_*}{\longrightarrow} \pi_1(X^0, \tau(t_0)) \overset{\sigma_*}{\underset{\kappa_*}{\longrightarrow}} \pi_1(T, t_0) \longrightarrow \{1\} . \qquad\square$$

Given a decomposition (8), the monodromy representation of the flat connection underlying an isomonodromic deformation can be seen as an extension of the monodromy representation of the initial connection. When does such an extension exist, and is it somehow unique? Again we need a little group theory.

Lemma 3.1.3 (Extension of representations). *Let $G = A \rtimes_\eta B$ be as before and let $\rho_A \in \mathrm{Hom}(A, \mathrm{GL}_r \mathbb{C})$ be a representation.*

- *There exists a representation $\rho \in \mathrm{Hom}(G, \mathrm{GL}_r\mathbb{C})$ such that $\rho|_A = \rho_A$ if and only if there exists a representation $\rho_B \in \mathrm{Hom}(B, \mathrm{GL}_r\mathbb{C})$ such that for all $(a, b) \in A \times B$ we have*

$$\rho_A(b \cdot a \cdot b^{-1}) = \rho_B(b) \cdot \rho_A(a) \cdot \rho_B(b^{-1}) \, .$$

- *Let $\rho, \rho' \in \mathrm{Hom}(G, \mathrm{GL}_r\mathbb{C})$ be representations such that $\rho|_A = \rho'|_A = \rho_A$. Assume that ρ_A is irreducible. Then there is $\lambda \in \mathrm{Hom}(B, \mathbb{C}^*)$ such that*

$$\rho = \lambda \otimes \rho' \, .$$

The proof of this Lemma is elementary and will be left to the reader. A similar statement can be found in [Cou17, Lem. 1].

3.2. **Logarithmic Riemann-Hilbert correspondence.** Let us briefly recall some notions and results from [Cou17], allowing to construct isomonodromic deformations from extensions of monodromy representations.

Denote by $\mathbb{D}$ the unit disc around 0 in the complex line and denote by $\mathbb{V}$ the trivial vector bundle of rank r over $\mathbb{D}$. Its sheaf of holomorphic sections shall be denoted by $\mathcal{V} = \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{D}}$. A (logarithmic) *transversal model* is an analytic logarithmic connection $(\mathbb{V}, \xi)$ over $\mathbb{D}$ with polar locus $\{0\}$. It is called a *mild transversal model* if any automorphism of the locally constant sheaf $\ker(\xi|_{\mathbb{D}\setminus\{0\}})$ is obtained by the restriction to $\mathbb{D} \setminus \{0\}$ of an automorphism of the sheaf $\mathcal{V}$. The isomorphism class of a transversal model is called a *transversal type*. Accordingly, a *mild transversal type* is the transversal type of a mild transversal model.

Let X be a ω-manifold, and let $D \subset X$ be a normal crossing hypersurface. Denote $(D_i)_{i \in I}$ the irreducible components of D. Let

$$\rho \in \mathrm{Hom}(\pi_1(X \setminus D), \mathrm{GL}_r\mathbb{C})$$

be a representation and $\mathcal{L}$ be a locally constant sheaf over $X \setminus D$ with monodromy ρ. For each $i \in I$, choose a holomorphic embedding $f_i : \mathbb{D} \hookrightarrow X$ such that $f_i(\mathbb{D})$ intersects D_i transversely exactly once, at $f_i(0)$, a smooth point of D. We say that a transversal model $(\mathbb{V}, \xi_i)$ is *compatible* with ρ at D_i if its monodromy is isomorphic to the one of $f_i^* \mathcal{L}$. This is a well defined notion, independant of the choice of f_i. By isomorphism invariance, this adapts to a notion of *compatible transversal type*. Compatible mild transversal models always exist, *e.g.* one can choose non-resonant models.

Assume we have a flat ω-logarithmic connection ∇ over X, with polar locus in D. By [Cou17, Prop. 3.2.1], the transversal type defined by $f_i^* \nabla$ is independant of the choice of f_i, it depends only of D_i and ∇. It is called the *transversal type of ∇ at D_i*. The connection ∇ is said to be *mild* if for every component D_i, the transversal type of ∇ at D_i is mild.

Theorem 3.2.1 (Logarithmic Riemann-Hilbert)**.** *Let X be a ω-manifold, D a normal crossing divisor on X and $\rho : \pi_1(X \setminus D) \to \mathrm{GL}_r\mathbb{C}$ a representation. For each $i \in I$, let $(\mathbb{V}, \xi_i)$ be a mild transversal model compatible with ρ. Then up to isomorphism there is a unique flat ω-logarithmic connection (E, ∇) over X with polar locus D such that*

- *the monodromy of (E, ∇) is given by $[\rho]$ and*
- *for each $i \in I$, the transversal type of ∇ at D_i is given by $(\mathbb{V}, \xi_i)$;*
- *if "ω = algebraic", then (E, ∇) is regular.*

Proof. The proof of this theorem in the analytic category can be found in [Cou17, Section 3.2]. We only need to check that it also holds in the algebraic category. So assume now X is a smooth irreducible quasiprojective variety. By definition, there exists a smooth irreducible projective variety $\widehat{X}$ containing X as a Zariski open subset. Denote by $\widehat{D}_j, j \in J$, the irreducible components of $\widehat{X} \setminus X$ and by $\widehat{D}_i$ the Zariski closure of D_i in $\widehat{X}$ for each $i \in I$. By Hironaka's desingularization, we may suppose that $\widehat{D} := \sum_{i \in I \cup J} \widehat{D}$ is a normal crossing divisor. Moreover, since $X \setminus D = \widehat{X} \setminus \widehat{D}$, ρ defines $\widehat{\rho} = \rho \in \mathrm{Hom}(\pi_1(\widehat{X} \setminus \widehat{D}), \mathrm{GL}_r\mathbb{C})$. For each $j \in J$, choose an arbitrary mild transversal model $(\mathbb{V}, \xi_i)$ on $(\mathbb{D}, 0)$ compatible with $\widehat{\rho}$. Then there exists an analytic logarithmic connection $(\widehat{E}^{\mathrm{an}}, \widehat{\nabla}^{\mathrm{an}})$ over $\widehat{X}$ with polar divisor $\widehat{D}$ by the analytic statement. Since

$\widehat{X}$ is projective, this connection is however analytically isomorphic to an algebraic logarithmic connection $(\widehat{E}, \widehat{\nabla})$ on $\widehat{X}$ by GAGA [Ser56, Prop. 18]. Since any logarithmic connection on $\widehat{X}$ restricts to a regular connection on X (see [Del70, Thm 4.1]), $(E, \nabla) := (\widehat{E}, \widehat{\nabla})|_X$ has the desired properties. It remains to show uniqueness up to algebraic isomorphism. By the analytic statement, we already know that the (analytic) isomorphism class of $(E^{\mathrm{an}}, \nabla^{\mathrm{an}})$ is unique. Yet any analytic isomorphism between regular algebraic logarithmic connections ∇_1, ∇_2 over X is algebraic, for the isomorphism can be seen as a horizontal section of $\nabla_1 \otimes \nabla_2^{\vee}$, which is regular by [Del70, Prop. 4.6]. $\qquad\square$

3.3. **Proof of the universal property.** Lemma 2.4.1, stated in Section 2.4, implying that under suitable generic conditions, algebraic universal isomonodromic deformations, if they exist, may be chosen to be regular is now an immediate consequence of the logarithmic Riemann-correspondence. Moreover, we are now able to prove their universal property, also stated in Section 2.4.

Proof of Lemma 2.4.1. Let $\mathcal{I}_{(C,E_0,\nabla_0)} = (\mathcal{F}_{(C,D_C)}, E, \nabla, \Psi)$ with $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be an algebraic isomonodromic deformation of (C, E_0, ∇_0). Let $\rho \in \mathrm{Hom}(\pi_1(X \setminus D, x_0), \mathrm{GL}_r\mathbb{C})$ be a representative of the monodromy $[\rho]$ of (E, ∇). For $i \in [\![1, n]\!]$, let D_i be the component of D passing through $\psi(x_i)$. Since by assumption (C, E_0, ∇_0) is mild, Theorem 3.2.1 yields a regular algebraic connection (E', ∇') over X with polar divisor D, monodromy $[\rho]$ and the same transversal types as ∇ at the components $(D_i)_{i \in [\![1,n]\!]}$. Moreover, also by Theorem 3.2.1, there is an isomorphism $\widetilde{\Psi} : (E, \nabla)|_{X_{t_0}} \xrightarrow{\sim} (E', \nabla')|_{X_{t_0}}$. Then $\mathcal{I}'_{(C,E_0,\nabla_0)} := (\mathcal{F}_{(C,D_C)}, E', \nabla', \widetilde{\Psi} \circ \Psi)$ is a regular algebraic isomonodromic deformation of (C, E_0, ∇_0) and there is an analytic isomorphism $(E^{\mathrm{an}}, \nabla^{\mathrm{an}}) \simeq (E'^{\mathrm{an}} \nabla'^{\mathrm{an}})$. In particular, the analytification of $\mathcal{I}_{(C,E_0,\nabla_0)}$ is isomorphic to the analytification of $\mathcal{I}'_{(C,E_0,\nabla_0)}$. $\qquad\square$

Proof of Proposition 2.4.2. Let $\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)} = (\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}, E, \nabla, \Psi)$ be a regular algebraic universal isomonodromic deformation of (C, E_0, ∇_0) with parameter space (T, t_0) and let $\mathcal{I}'_{(C,E_0,\nabla_0)} = (\mathcal{F}'_{(C,D_C)}, E', \nabla', \Psi')$ be a regular algebraic isomonodromic deformation of (C, E_0, ∇_0) with parameter space (T', t'_0). By Lemma 3.1.1, there is an étale base change $\tilde{p} : (\widetilde{T}, \tilde{t}_0) \to (T, t_0)$, such that for $\widetilde{\mathcal{F}}^{\mathrm{Kur}}_{(C,D_C)} := \tilde{p}^* \mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$, there is a section $\sigma : \widetilde{T} \to \widetilde{X}$ avoiding the marked points. Since $\widetilde{\mathcal{F}}^{\mathrm{Kur}}_{(C,D_C)}$ is still Kuranishi, by the universal property of Kuranishi families, we have an étale base change $p : (T'', t''_0) \to (T', t'_0)$, a morphism $\tilde{q} : (T'', t''_0) \to (\widetilde{T}, \tilde{t}_0)$ and an isomorphism

$$\tilde{\mathfrak{f}} : \mathcal{F}''_{(C,D_C)} := p^* \mathcal{F}'_{(C,D_C)} \xrightarrow{\sim} \tilde{q}^* \widetilde{\mathcal{F}}^{\mathrm{Kur}}_{(C,D_C)}.$$

In particular, σ lifts to a section $\sigma'' := \tilde{f}^* \tilde{q}^* \sigma : T'' \to X''$ avoiding the marked points of $\mathcal{F}''_{(C,D_C)}$. Denote by

$$\rho'', \tilde{\rho} \in \mathrm{Hom}(\pi_1(X'' \setminus D'', \sigma''(t''_0)), \mathrm{GL}_r\mathbb{C})$$

representatives of the conjugacy classes of the monodromy representations of $(E''; \nabla'')$ and $\tilde{f}^* \tilde{q}^* \tilde{p}^*(E, \nabla)$ respectively (with respect to the identity). By the Splitting Lemma 3.1.2, we have

$$\pi_1(X'' \setminus D'', \sigma''(t''_0)) = \psi''_* \pi_1(C \setminus D_C, x_0) \rtimes_\eta \sigma''_* \pi_1(T'', t''_0).$$

Moreover, if ρ_{∇_0} denotes a representative of the monodromy representation of (E_0, ∇_0) (with respect to the identity), then ρ'' and $\tilde{\rho}$ could be chosen so that

$$\rho''|_{\psi''_* \pi_1(C \setminus D_C, x_0)} = \tilde{\rho}|_{\psi''_* \pi_1(C \setminus D_C, x_0)} = \psi''_* \rho_{\nabla_0}.$$

Since ρ_{∇_0} is irreducible, by Lemma 3.1.3 there is a representation $\lambda \in \mathrm{Hom}(\pi_1(T'', t''_0), \mathbb{C}^*)$ such that $\lambda \otimes (\sigma'')^* \rho'' = (\sigma'')^* \tilde{\rho}$. By the Riemann-Hilbert correspondence, there is a regular flat algebraic connection (L, ξ) of rank 1 over T'', without poles, whose monodromy representation is λ^{-1}. The monodromy representation of its lift $\kappa''^*(L, \xi)$ is the trivial extension of $\sigma''^* \lambda^{-1}$ to a representation $\psi''_* \pi_1(C \setminus D_C, x_0) \rtimes_\eta \sigma''_* \pi_1(T'', t''_0) \to \mathbb{C}^*$. Now the monodromy representations

of $(E''; \nabla'') \otimes \kappa''^*(L, \xi)$ and $\tilde{f}^* q^* p^*(E, \nabla)$ coincide. Both connections are regular, have same monodromy representations and same transversal models, given by (E_0, ∇_0). Hence they are isomorphic by the logarithmic Riemann-Hilbert correspondence. $\square$

4. The monodromy of the monodromy

In this section, we introduce the so-called group of mapping classes of a ω-family, which is the image of a canonical morphism from the fundamental group of the parameter space of the family to the fundamental group of the central fiber. For an isomonodromic deformation, the action on the monodromy representation of the initial connection by the group of mapping classes of the underlying family of curves corresponds to the monodromy of the monodromy representation. Under suitable conditions, this group can be canonically translated into a subgroup of $\Gamma_{g,n}$.

4.1. **Mapping classes of the central fiber.** As usual, let (C, D_C) be a stable n-pointed genus-g curve. Let $\mathcal{F}_{(C, D_C)}$ be a ω-family with parameter space (T, t_0). Let $\beta : [0, 1] \to T$ be a closed path with end point t_0, *i.e.* a continous map such that $\beta(0) = \beta(1) = t_0$. By [Hus94, Cor. 10.3], the pullback bundle $\beta^*(X, D) \to [0, 1]$ possesses a topological trivialization $\Phi : (C, D_C) \times [0, 1] \xrightarrow{\sim} \beta^*(X, D)$. For $s \in [0, 1]$, we denote

$$\Phi_s := \Phi|_{(C, D_C) \times \{s\}}$$

and deduce a homeomorphism from the central fiber seen over $\{1\}$ to the central fiber seen over $\{0\}$ given by

$$\psi^{-1} \circ \Phi_0 \circ \Phi_1^{-1} \circ \psi : (C, D_C) \xrightarrow{\sim} (C, D_C).$$

Its isotopy class shall be called the *mapping class associated to β and $\mathcal{F}_{(C, D_C)}$* and denoted

$$\mathrm{map}_{\mathcal{F}_{(C, D_C)}}(\beta).$$

Lemma 4.1.1. *The mapping class $\mathrm{map}_{\mathcal{F}_{(C, D_C)}}(\beta)$ is well defined, i.e. it does not depend on the choice of a trivialization Φ. Moreover, $\mathrm{map}_{\mathcal{F}_{(C, D_C)}}(\beta)$ only depends on the homotopy class of β.*

Proof. For fixed β, take two trivializations : $\Phi, \widetilde{\Phi} : (C, D_C) \times [0, 1] \xrightarrow{\sim} \beta^*(X, D)$. The family $\widetilde{\Phi}_0 \circ \widetilde{\Phi}_s^{-1} \circ \Phi_s \circ \Phi_1^{-1}$ gives an isotopy from $\Phi_0 \circ \Phi_1^{-1}$ to $\widetilde{\Phi}_0 \circ \widetilde{\Phi}_1^{-1}$.
Consider now two paths β_1 and β_2 that are homotopic relative to their endpoints. By definition, there exists a continuous map $\theta : \overline{\mathbb{D}} \to T$, where $\overline{\mathbb{D}}$ denotes the closed unit disc, such that $\beta_2(s) = \theta(e^{i\pi(1+s)})$ and $\beta_1(s) = \theta(e^{i\pi(1-s)})$. Since $\overline{\mathbb{D}}$ is contractible, by [Hus94, Cor. 10.3], there is a trivialization Φ of $\theta^*(X, D)$. It induces trivializations Φ^i of $\beta_i^* X$ for $i = 1, 2$. Since they are both induced by Φ, we have $\Phi_0^1 = \Phi_0^2 = \Phi_{-1}$ and $\Phi_1^1 = \Phi_1^2 = \Phi_1$. $\square$

Proposition 4.1.2. *Let $\mathcal{F}_{(C, D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be a ω-family as in Section 2.2. Assume that none of the fibers $(X_t, D|_{X_t})$ has exceptional automorphisms. Let $\mathbf{x}$ be a labelling of D_C and denote $\mathrm{cl} : T \to \mathcal{M}_{g,n} \backslash \mathcal{B}_{g,n}$ the corestriction of the induced classifying map $\mathrm{class}(\mathcal{F})$ (see Section 2.3). Then there exists an orientation preserving homeomorphism $\varphi : (\Sigma_g, y^n) \xrightarrow{\sim} (C, \mathbf{x})$ such that for all $\beta \in \pi_1(T, t_0)$, the following equation holds in $\Gamma_{g,n}/K_{g,n}$:*

$$\varphi^{-1} \circ \mathrm{map}_{\mathcal{F}_{(C, D_C)}}(\beta) \circ \varphi = \mathrm{taut}_{g,n}(\mathrm{cl}_* \beta),$$

where $\mathrm{taut}_{g,n}$ is the tautological morphism $\mathrm{taut}_{g,n} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star) \to \Gamma_{g,n}/K_{g,n}$ (see (6)) and $\star := [C, \mathbf{x}]$.

Proof. Fix an orientation preserving homeomorphism $\widetilde{\varphi} : (\Sigma_g, y^n) \xrightarrow{\sim} (C, \mathbf{x})$ and denote

$$\hat{\star} := [C, D_C, \widetilde{\varphi}] \in \mathcal{T}_{g,n}.$$

As usual, denote $\mathcal{F}_{g,n}^+ = (\mathcal{F}_{g,n}, \Phi_{g,n})$ the universal Teichmüller curve $\mathcal{F}_{g,n} = (\kappa_{g,n} : \mathcal{X} \to \mathcal{T}_{g,n}, \mathcal{D})$ endowed with the Teichmüller structure $\Phi_{g,n} : (\Sigma_g, Y^n) \times \mathcal{T}_{g,n} \xrightarrow{\sim} (\mathcal{X}, \mathcal{D})$. For any point $t \in \mathcal{T}_{g,n}$, we shall denote

$$\Phi_t^{g,n} := \Phi_{g,n}|_{(\Sigma_g, Y^n) \times \{t\}} : (\Sigma_g, Y^n) \times \{t\} \xrightarrow{\sim} (\mathcal{X}_t, \mathcal{D}|_{\mathcal{X}_t}).$$

Let $p : (\widetilde{T}, \tilde{t}_0) \to (T, t_0)$ be a universal cover and consider the pulled-back family
$$\widetilde{\mathcal{F}} = (\widetilde{\kappa} : \widetilde{X} \to \widetilde{T}, \widetilde{D}) := p^*(\kappa : X \to T, D).$$
Now for any contractible analytic submanifold $\widetilde{\Delta} \subset \widetilde{T}$ containing $\tilde{t}_0$, there is a trivialization
$$\widetilde{\Phi} : (C, D_C) \times \widetilde{\Delta} \xrightarrow{\sim} (\widetilde{X}, \widetilde{D})|_{\widetilde{\Delta}}$$
of $\widetilde{\mathcal{F}}|_{\widetilde{\Delta}}$, unique up to isotopy, such that $\widetilde{\Phi}_{\tilde{t}_0} = \psi$ with respect to the pullback identification $(\widetilde{X}_{\tilde{t}_0}, \widetilde{D}|_{\widetilde{X}_{\tilde{t}_0}}) = (X_{t_0}, D|_{X_{t_0}})$. Setting $\widetilde{\mathcal{F}}^+ := (\widetilde{\mathcal{F}}|_{\widetilde{\Delta}}, \widetilde{\Phi} \circ (\widetilde{\varphi} \times \mathrm{id}))$ defines an analytic family with Teichmüller structure. By the universal property of the Teichmüller curve, we have an isomorphism $\mathfrak{f} = (\mathrm{id}_{\widetilde{\Delta}}, \mathfrak{f}^a, \mathfrak{f}^{\mathrm{top}})$ fitting into the following commutative diagram.

$$
\begin{array}{ccc}
(\Sigma_g, Y^n) \times \widetilde{\Delta} & \xrightarrow[\sim]{\mathfrak{f}^{\mathrm{top}}} & (\Sigma_g, Y^n) \times \widetilde{\Delta} \\
{\scriptstyle \widetilde{\Phi}\circ(\widetilde{\varphi}\times\mathrm{id})} \downarrow & & \downarrow {\scriptstyle \mathrm{class}^+(\widetilde{\mathcal{F}}^+)^*\Phi_{g,n}} \\
(\widetilde{X}, \widetilde{D})|_{\widetilde{\Delta}} & \xrightarrow[\sim]{\mathfrak{f}^a} & \mathrm{class}^+(\widetilde{\mathcal{F}}^+)^*(\mathcal{X}, \mathcal{D}) \\
{\scriptstyle \widetilde{\kappa}} \downarrow & & \downarrow {\scriptstyle \mathrm{class}^+(\widetilde{\mathcal{F}}^+)^*\kappa_{g,n}} \\
\widetilde{\Delta} & =\!=\!=\!=\!= & \widetilde{\Delta}
\end{array}
$$

By commutativity of the diagram, we have $\mathfrak{f}^{\mathrm{top}} = \mathfrak{f}^{\mathrm{top}}_{\tilde{t}_0} \times \mathrm{id}_{\widetilde{\Delta}}$ up to fiberwise isotopy. Define

$$(9) \qquad \qquad \varphi := \widetilde{\varphi} \circ \left(\mathfrak{f}^{\mathrm{top}}_{\tilde{t}_0}\right)^{-1}.$$

Now let $[\beta] \in \pi_1(T, t_0) \setminus \{1\}$ and consider $\tilde{\beta} : [0, 1] \to \widetilde{T}$, the lift of β with starting point $\tilde{t}_0$. If the representative β of the homotopy class $[\beta]$ is well chosen, then $\tilde{\beta}$ is a $\mathcal{C}^\infty$-embedding. By existence of tubular neighborhoods, there is a contractible neighborhood $\widetilde{\Delta}$ of $\tilde{t}_0$ as above, containing $\tilde{\beta}$. We claim that, up to isotopy,

$$(10) \qquad \qquad \mathrm{map}_{\mathcal{F}_{(C, D_C)}}(\beta) = \widetilde{\Phi}^{-1}_{\tilde{\beta}(1)} \circ \widetilde{\Phi}_{\tilde{t}_0}.$$

Indeed, we have $\beta^*(X, D) = (p \circ \tilde{\beta})^*(X, D) = \tilde{\beta}^* p^*(X, D) = \tilde{\beta}^*(\widetilde{X}, \widetilde{D})$. Since moreover $\tilde{\beta}$ is an embedding, we have $\beta^*(X, D) = (\widetilde{X}, \widetilde{D})|_{\tilde{\beta}([0,1])}$. The claim then follows from the fact that $\psi^{-1} \circ \widetilde{\Phi}_{\tilde{t}_0}$ is the identity and from the definition of the mapping class.

Denote $\widehat{\beta} := \mathrm{class}^+(\widetilde{\mathcal{F}}^+)_* \tilde{\beta}$, which is a path in $\mathcal{T}_{g,n}$ with starting point $\hat{\star}$. By our definitions, the black part of the following diagram is commutative.

$$
\begin{array}{ccccccc}
(\Sigma_g, Y^n) \times \{\tilde{t}_0\} & \xrightarrow[\sim]{\widetilde{\Phi}\circ(\widetilde{\varphi}\times\{\tilde{t}_0\})} & (\widetilde{X}_{\tilde{t}_0}, \widetilde{D}|_{\widetilde{X}_{\tilde{t}_0}}) & =\!=\!= & (\widetilde{X}_{\tilde{\beta}(1)}, \widetilde{D}|_{\widetilde{X}_{\tilde{\beta}(1)}}) & \xleftarrow[\sim]{\widetilde{\Phi}\circ(\widetilde{\varphi}\times\{\tilde{\beta}(1)\})} & (\Sigma_g, Y^n) \times \{\tilde{\beta}(1)\} \\
{\scriptstyle \mathfrak{f}^{\mathrm{top}}_{\tilde{t}_0}} \downarrow \wr & & {\scriptstyle \mathfrak{f}^a_{\tilde{t}_0}} \downarrow \wr & & \wr \downarrow {\scriptstyle \mathfrak{f}^a_{\tilde{\beta}(1)}} & & \wr \downarrow {\scriptstyle \mathfrak{f}^{\mathrm{top}}_{\tilde{\beta}(1)}} \\
(\Sigma_g, Y^n) \times \{\hat{\star}\} & \xrightarrow[\Phi^{g,n}_{\hat{\star}}]{\sim} & (\mathcal{X}_{\hat{\star}}, \mathcal{D}|_{\mathcal{X}_{\hat{\star}}}) & \xrightarrow[\widehat{\psi}]{\sim} & (\mathcal{X}_{\widehat{\beta}(1)}, \mathcal{D}|_{\mathcal{X}_{\widehat{\beta}(1)}}) & \xleftarrow[\Phi^{g,n}_{\widehat{\beta}(1)}]{\sim} & (\Sigma_g, Y^n) \times \{\widehat{\beta}(1)\}
\end{array}
$$

We define

$$(11) \qquad \qquad \widehat{\psi} = \mathfrak{f}^a_{\widehat{\beta}(1)} \circ (\mathfrak{f}^a_{t_0})^{-1},$$

so that adding the gray arrow maintains this commutativity. Since $\mathfrak{f}^{\mathrm{top}}_{\widehat{\beta}(1)}$ is isotopic to $\mathfrak{f}^{\mathrm{top}}_{\tilde{t}_0}$, the following equations hold up to isotopy:

$$(12) \qquad \qquad \begin{cases} \Phi^{g,n}_{\hat{\star}} & = \ \mathfrak{f}^a_{\tilde{t}_0} \circ \widetilde{\Phi}_{\tilde{t}_0} \circ \varphi \\ \Phi^{g,n}_{\widehat{\beta}(1)} & = \ \mathfrak{f}^a_{\widetilde{\beta}(1)} \circ \widetilde{\Phi}_{\widetilde{\beta}(1)} \circ \varphi. \end{cases}$$

On the other hand, $\mathrm{cl}_*\beta$ is a closed path in $\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$ with end point $\star$. By construction, it lifts, with respect to the forgetful map $\pi_{g,n}$, to $\widehat{\beta}$, with $\widehat{\beta}(0) = \widehat{\star}$. By definition of the tautological morphism $\mathrm{taut}_{g,n}$, we thus have, for $[h] := \mathrm{taut}_{g,n}(\mathrm{cl}_*\beta) \in \Gamma_{g,n}/K_{g,n}$:

$$[h] \cdot \left[\mathcal{X}_{\widehat{\star}}, \mathcal{D}|_{\mathcal{X}_{\widehat{\star}}}, \Phi_{\widehat{\star}}^{g,n} \right] = \left[\mathcal{X}_{\widehat{\beta}(1)}, \mathcal{D}|_{\mathcal{X}_{\widehat{\beta}(1)}}, \Phi_{\widehat{\beta}(1)}^{g,n} \right].$$

Note that $\widehat{\psi}$ induces an isomorphism of pointed curves with Teichmüller structure. By the definition of the action of the mapping class group on $\mathcal{T}_{g,n}$, we now have

$$[h] \cdot \left[\mathcal{X}_{\widehat{\star}}, \mathcal{D}|_{\mathcal{X}_{\widehat{\star}}}, \Phi_{\widehat{\star}}^{g,n} \right] = [h] \cdot \left[\mathcal{X}_{\widehat{\beta}(1)}, \mathcal{D}|_{\mathcal{X}_{\widehat{\beta}(1)}}, \widehat{\psi} \circ \Phi_{\widehat{\star}}^{g,n} \right] = \left[\mathcal{X}_{\widehat{\beta}(1)}, \mathcal{D}|_{\mathcal{X}_{\widehat{\beta}(1)}}, \widehat{\psi} \circ \Phi_{\widehat{\star}}^{g,n} \circ h^{-1} \right].$$

Hence there is an element $[k] \in K_{g,n}$ such that, up to isotopy,

$$\widehat{\psi} \circ \Phi_{\widehat{\star}}^{g,n} = \Phi_{\widehat{\beta}(1)}^{g,n} \circ h \circ k.$$

Combined with (11) and (12), this implies, up to isotopy,

$$\widetilde{\Phi}_{\widehat{\beta}(1)}^{-1} \circ \widetilde{\Phi}_{\tilde{t}_0} = \varphi \circ h \circ k \circ \varphi^{-1},$$

which by (10) and the definitions of h and k yields the desired result. $\qquad\square$

4.2. Splitting and the mapping class group.
Let $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be a ω-family of n-pointed genus-g curves as in Section 2.2. Assume there is a section $\sigma : T \to X^0 := X \setminus D$ of κ. Then we can define a ω-family of $n+1$-pointed genus-g curves $\mathcal{F}_{(C,D_C^\bullet)}^\bullet := (\kappa : X \to T, D^\bullet, t_0, \psi)$ by setting $D^\bullet := D + \sigma(T)$ and $D_C^\bullet := D_C + x_0$, where $x_0 := \psi^{-1}(\sigma(t_0))$. To a labelling $\mathbf{x} = (x_1, \ldots, x_n)$ of D_C we can associate a labelling $\mathbf{x}^\bullet := (x_1, \ldots, x_n, x_0)$ of $D_C^\bullet$. Note that if a fiber of $\mathcal{F}^\bullet$ has exceptional automorphisms, then the corresponding fiber of $\mathcal{F}$ also has exceptional automorphisms.

Proposition 4.2.1. *Let (C, D_C) be a stable n-pointed genus-g curve. Let $\mathcal{F}_{(C,D_C^\bullet)}^\bullet = (\kappa : X \to T, D^\bullet, t_0, \psi)$ be a ω-family of $n+1$-pointed genus-g curves as above. Assume that none of the fibers of $\mathcal{F}^\bullet$ has exceptional automorphisms. Let $\mathbf{x}^\bullet$ be a labelling of $D_C^\bullet$ as above. Denote $\mathrm{cl}^\bullet : T \to \mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}$ the corestriction of the induced classifying map $\mathrm{class}(\mathcal{F}^\bullet)$.*

For a suitable homeomorphism $\varphi : (\Sigma_g, y^n, y_0) \xrightarrow{\sim} (C, \mathbf{x}, x_0)$, we have

$$(13) \qquad \pi_1(X^0, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_\eta \sigma_* \pi_1(T, t_0),$$

where for all $\alpha \in \Lambda_{g,n}$ and $\beta \in \pi_1(T, t_0)$, we have

$$\begin{aligned} \eta(\sigma_*\beta)((\psi \circ \varphi)_*\alpha) &= \sigma_*\beta \cdot (\psi \circ \varphi)_*\alpha \cdot \sigma_*\beta^{-1} \\ &= (\psi \circ \varphi)_* \left(\mathrm{taut}_{g,n+1}(\mathrm{cl}^\bullet_* \beta) \right)(\alpha), \end{aligned}$$

where, as we recall from the introduction, $\mathfrak{a}(h)(\alpha) = h_*\alpha$ for all $h \in \Gamma_{g,n+1}$ and $\alpha \in \Lambda_{g,n}$.

Proof. Since $2g - 2 + n > 0$ by assumption, we have $K_{g,n+1} = \{1\}$ according to Lemma 2.1.1. Then by Proposition 4.1.2, for a convenient choice of φ, the following equation holds in $\Gamma_{g,n+1}$ for every $\beta \in \pi_1(T, t_0)$:

$$(14) \qquad \varphi^{-1} \circ \mathrm{map}_{\mathcal{F}_{(C,D_C^\bullet)}^\bullet}(\beta) \circ \varphi = \mathrm{taut}_{g,n+1}(\mathrm{cl}^\bullet_* \beta).$$

Denote $C^0 := C \setminus D_C$. We claim that for any $\gamma \in \pi_1(C^0, x_0)$ and any $\beta \in \pi_1(T, t_0)$, the following equation holds in $\pi_1(X^0, \sigma(t_0))$:

$$(15) \qquad \psi_* \mathrm{map}_{\mathcal{F}_{(C,D_C^\bullet)}^\bullet}(\beta)_* \gamma = \sigma_*\beta \cdot \psi_*\gamma \cdot \sigma_*\beta^{-1}.$$

Indeed, let $\gamma : [0,1] \to C^0$ be a closed path with end point x_0. For any $s_0 \in [0,1]$, we have a closed path $\gamma_{s_0} := \gamma \times \{s_0\}$ in the product space $C^0 \times [0,1]$. We also have a path $\theta : [0,1] \to C^0 \times [0,1]; s \mapsto (x_0, s)$. The path $\theta \cdot \gamma_1 \cdot \theta^{-1}$ is closed and homotopic to γ_0. Now

<center>17</center>

let $\beta \in \pi_1(T, t_0)$ and let $\Phi : (C^0, x_0) \times [0,1] \overset{\sim}{\to} \beta^*(X^0, \sigma(T))$ be a trivialization commuting with the natural projections to $[0,1]$. Define the homeomorphism

$$\widetilde{\Phi} := \Phi \circ ((\Phi_1^{-1} \circ \psi) \times \mathrm{id}_{[0,1]}) : (C^0, x_0) \times [0,1] \overset{\sim}{\to} \beta^*(X^0, \sigma(T)),$$

which is another trivialization, satisfying $\widetilde{\Phi}_1 = \psi$ and $\widetilde{\Phi}_0 = \psi_* \mathrm{map}_{\mathcal{F}^\bullet_{(C,D_{\mathbb{C}}^\bullet)}}(\beta)$. Since $\widetilde{\Phi}$ is continuous, the closed paths $\widetilde{\Phi}_* \gamma_0$ and $\widetilde{\Phi}_* \theta \cdot \widetilde{\Phi}_* \gamma_1 \cdot \widetilde{\Phi}_* \theta^{-1}$ are homotopic in $\beta^*(X^0, \sigma(T))$. Considering the natural projection $\kappa : \beta^*(X^0, \sigma(T)) \to (X^0, \sigma(T))$, we have $\kappa_* \widetilde{\Phi}_* \gamma_0 = \widetilde{\Phi}_{0*} \gamma$ and $\kappa_* \widetilde{\Phi}_* \gamma_1 = \widetilde{\Phi}_{1*} \gamma$. Since moreover $\kappa_* \widetilde{\Phi}_* \theta = \sigma_* \beta$, we have (15).

Since φ is a homeomorphism, the induced map $\varphi_* : \Lambda_{g,n} \to \pi_1(C^0, x_0)$ is an isomorphism. The statement then follows from (14), (15) and the Splitting Lemma 3.1.2 . $\qquad\square$

5. Necessary and sufficient conditions for algebraizability

We shall see in Section 5.2 that Theorem A is a corollary of the juxtaposition of Theorem A1, showing that our algebraizability criterion for germs of universal isomonodromic deformations is necessary, and Theorem A2, showing that it is also sufficient. We have already established the main ingredients for the proofs of both theorems. For Theorem A2, we moreover need a representation-theoretical result developed in Section 5.1.

5.1. Extensions of representations.
We shall now consider the problem of extending a representation of the fundamental group of a fiber of a family of pointed curves to a representation for the whole family in light of Lemma 3.1.3 and Proposition 4.2.1. We begin with the elementary case of non semisimple rank 2 representations.

Let A, B be groups. Consider a representation $\rho \in \mathrm{Hom}(A, \mathrm{Upp})$, where Upp is the group of invertible upper triangular matrices of rank 2. To such a representation, we may associate two other ones : the *scalar part* $\rho_{\mathbb{C}^*} : \alpha \mapsto \rho(\alpha)_{2,2}$ and the *affine part* $\rho_{\mathrm{Aff}} := \rho_{\mathbb{C}^*}^{-1} \otimes \rho$. The latter takes values in

$$\mathrm{Aff}(\mathbb{C}) := \{(a_{i,j}) \in \mathrm{Upp} \mid a_{2,2} = 1\}$$

which is isomorphic to the affine group of the complex line.

Lemma 5.1.1. *Let $\rho = \rho_{\mathbb{C}^*} \otimes \rho_{\mathrm{Aff}}$ and $\rho' = \rho'_{\mathbb{C}^*} \otimes \rho'_{\mathrm{Aff}}$ as above, and assume that they are not semisimple. We have $[\rho] = [\rho'] \in \mathrm{Hom}(A, \mathrm{GL}_2\mathbb{C})/\mathrm{GL}_2\mathbb{C}$ if and only if $\rho_{\mathbb{C}^*} = \rho'_{\mathbb{C}^*}$ and $[\rho_{\mathrm{Aff}}] = [\rho'_{\mathrm{Aff}}] \in \mathrm{Hom}(A, \mathrm{Aff}(\mathbb{C}))/\mathrm{Aff}(\mathbb{C})$.*

Proof. The "if"-part is trivial. Assume $[\rho] = [\rho']$. Since they take values in Upp, both representations ρ and ρ' leave the line $\mathrm{span}(e_1)$ of $\mathbb{C}^2$ invariant. By non semisimplicity, for each of the representations, there is no other globally invariant line. Let $M = (m_{i,j}) \in \mathrm{GL}_2\mathbb{C}$ conjugate both representations. Then M must leave $\mathrm{span}(e_1)$ invariant, *i.e.* $M \in \mathrm{Upp}$. As the scalars are central in $\mathrm{GL}_2\mathbb{C}$, the element $M/m_{2,2} \in \mathrm{Aff}(\mathbb{C})$ conjugates both representations. In particular $\rho_{\mathbb{C}^*} = \rho'_{\mathbb{C}^*}$ and $M/m_{2,2}$ conjugates ρ_{Aff} and ρ'_{Aff}. $\qquad\square$

Lemma 5.1.2. *Let $\rho_A \in \mathrm{Hom}(A, \mathrm{GL}_2\mathbb{C})$ be non semisimple. Let $\theta \in \mathrm{Hom}(B, \mathrm{Aut}(A))$ such that for all $h \in \mathrm{Im}(\theta)$, we have $[\rho_A] = h \cdot [\rho_A] := [\rho_A \circ h^{-1}]$. Then there exists a representation $\rho_B \in \mathrm{Hom}(B, \mathrm{GL}_2\mathbb{C})$ such that*

$$\rho_A(\theta(\beta)^{-1}(\alpha)) = \rho_B(\beta)^{-1} \rho_A(\alpha) \rho_B(\beta) \quad \forall \alpha \in A, \beta \in B.$$

Proof. We may assume that ρ_A takes values in Upp. By assumption, for each $\beta \in B$, there exists a matrix $M_\beta \in \mathrm{GL}_2\mathbb{C}$ such that $\rho_A(\theta(\beta)^{-1}(\bullet)) = M_\beta^{-1} \rho_A(\bullet) M_\beta$. By Lemma 5.1.1, we may assume $M_\beta \in \mathrm{Aff}(\mathbb{C})$. If $\mathrm{Im}(\rho_A) \subset \mathrm{Upp}$ is non abelian, then it has trivial centralizer and the matrices $M_\beta \in \mathrm{Aff}(\mathbb{C})$ are uniquely defined. Otherwise, we have $\mathrm{Im}(\rho_A) \subset \{\lambda \left(\begin{smallmatrix} 1 & \tau \\ 0 & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{C}^*, \tau \in \mathbb{C}\}$ and the matrices M_β are uniquely defined if we impose $M_\beta \in \left\{\left(\begin{smallmatrix} \mu & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid \mu \in \mathbb{C}^*\right\}$. It is now straightforward to check that given these choices, the well-defined map $\beta \mapsto M_\beta$ is a morphism of groups. $\qquad\square$

For a similar result for semisimple representations ρ_A (of arbitrary rank), the group B, which in our case will be the fundamental group of a parameter space, might have to be enlarged, in order to take into account the non-unicity of the matrices M_β due to possible permutations of irreducible components.

Proposition 5.1.3. *Let $\rho_A \in \mathrm{Hom}(A, \mathrm{GL}_r\mathbb{C})$ be semisimple. Let (T, t_0) be a smooth connected quasi-projective variety, and let $\theta \in \mathrm{Hom}(\pi_1(T, t_0), \mathrm{Aut}(A))$ such that $H := \mathrm{Im}(\theta)$ stabilizes $[\rho_A]$. Then there is an étale base change $p : (T', t_0') \to (T, t_0)$ and a representation $\rho_B \in \mathrm{Hom}(\pi_1(T', t_0'), \mathrm{GL}_r\mathbb{C})$ such that*

$$\rho_A(\theta(p_*\beta)^{-1}(\alpha)) = \rho_B(\beta)^{-1} \cdot \rho_A(\alpha) \cdot \rho_B(\beta) \quad \forall \alpha \in A \,, \beta \in \pi_1(T', t_0') \,.$$

Proof. Let $\rho_A = \bigoplus_{i \in I} \rho_A^i$ be a decomposition such that each ρ_A^i is irreducible. The subgroup

$$\bigcap_{i \in I} \mathrm{Stab}_{\mathrm{Aut}(A)}[\rho_A^i] \subset \mathrm{Stab}_{\mathrm{Aut}(A)}[\rho_A] \,,$$

stabilizing the conjugacy class $[\rho_A^i]$ for each $i \in I$, is of finite index (see for example [Cou17, Lemma 3]). Hence the subgroup $\widetilde{H} := H \cap_{i \in I} \mathrm{Stab}_{\mathrm{Aut}(A)}[\rho_A^i]$ is of finite index in H. Consider now the finite connected unramified covering $\tilde{p} : (\widetilde{T}, \tilde{t}_0) \to (T, t_0)$ characterized by $\tilde{p}_*\pi_1(\widetilde{T}, \tilde{t}_0) = \theta^{-1}(\widetilde{H})$. Note that $\tilde{p}$ induces a structure of smooth quasi-projective variety on $\widetilde{T}$. Since $\widetilde{H}$ stabilizes $[\rho_A^i]$, for every $h \in \widetilde{H}$ and every $i \in I$, there is a matrix $M_h^i \in \mathrm{GL}_{r_i}\mathbb{C}$ such that

$$(16) \qquad\qquad (M_h^i)^{-1} \cdot \rho_A^i \cdot M_h^i = [h] \cdot \rho_A^i \,.$$

Given i and h, the choice of M_h^i is unique up to an element of the centralizer of ρ_A^i. Since ρ_A^i is irreducible, this centralizer is given by the set of scalar matrices. Denote by $\overline{M_h^i} \in \mathrm{PGL}_{r_i}\mathbb{C}$ the projectivization of $M_h^i \in \mathrm{GL}_{r_i}\mathbb{C}$. Then $\overline{\rho_B}^i : \beta \mapsto \overline{M_{\theta_* \tilde{p}_* \beta}^i}$ is a well defined element of $\mathrm{Hom}(\pi_1(\widetilde{T}, \tilde{t}_0), \mathrm{PGL}_{r_i}\mathbb{C})$. According to the Lifting Theorem [Cou15, Th. 3.1], there exists a Zariski closed subset $\widetilde{Z}$ of $\widetilde{T}$ not containing $\tilde{t}_0$, a finite morphism of smooth quasi-projective varieties

$$p' : (\widetilde{T}', t_0') \to (\widetilde{T} \setminus \widetilde{Z}, \tilde{t}_0) \,,$$

étale in a neighborhood T' of t_0', and a representation $\rho_B^i \in \mathrm{Hom}(\pi_1(\widetilde{T}', t_0'), \mathrm{GL}_{r_i}\mathbb{C})$ whose projectivization is $p'^* \overline{\rho_B}^i$. For a convenient choice of p', this property is satisfied for all $i \in I$ at once. We obtain a representation $\rho_B := \bigoplus_{i \in I} \rho_B^i$ in $\mathrm{Hom}(\pi_1(T', t_0'), \mathrm{GL}_{r_i}\mathbb{C})$ satisfying the required properties with respect to $p := \tilde{p} \circ p'|_{T'}$. $\qquad\square$

5.2. Finiteness and algebraization.

Theorem A1. *Let (C, D_C) be a stable n-pointed genus g-curve as in Section 2.2. Let $\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D_C)$ be an orientation preserving homeomorphism. Let (E_0, ∇_0) be an algebraic logarithmic connection over C with polar divisor D_C and denote by $[\rho_{\nabla_0}] \in \chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ its monodromy with respect to φ. Let $\mathcal{I}_{(C,E_0,\nabla_0)} = (\mathcal{F}_{(C,D_C)}, E, \nabla, \Psi)$ be an algebraic isomonodromic deformation of (C, E_0, ∇_0) with parameter space T as in Section 2.4. Assume that*

- *the classifying map $\mathrm{class}(\mathcal{F}) : T \to \mathcal{M}_{g,n}$ is dominant (see Section 2.3).*

Then

- *the $\Gamma_{g,n}$-orbit of $[\rho_{\nabla_0}]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ is finite.*

Proof. The orbit $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$ does not depend on the choice of φ. Moreover, it is canonically identified, for any $t_1 \in T$, with the orbit $\Gamma_{g,n} \cdot [\rho_{t_1}]$ of the monodromy of the connection (E, ∇) restricted to the fiber over t_1 of the family $\mathcal{F}$. Since $\mathrm{class}(\mathcal{F})$ is dominant we may assume, without loss of generality, that $\star := \mathrm{class}(\mathcal{F})(t_0) \in \mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}$. Moreover, up to restricting $\mathcal{I}_{(C,E_0,\nabla_0)}$ to a Zariski open neighborhood Δ of t_0 in T, we may assume that $\mathrm{class}(\mathcal{F})(T) \cap \mathcal{B}_{g,n} = \varnothing$. Notice that this property, as well as the assumption of $\mathrm{class}(\mathcal{F})$ being dominant is not altered by finite covers and further excision of strict subvarieties not containing t_0. According to Lemma 3.1.1, up to such a manipulation, we may assume that $\mathcal{F}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ admits

a section $\sigma : T \to X$ of κ with values in $X^0 := X \setminus D$ such that $\sigma(t_0) = \psi \circ \varphi(y_0)$. Denote by ρ a representative of the monodromy representation of (E, ∇) with respect to the identity such that the restriction of ρ to the subgroup $(\psi \circ \varphi)_* \Lambda_{g,n}$ of $\pi_1(X^0, \sigma(t_0))$, given by the inclusion of the central fiber, is identical to $(\psi \circ \varphi)_* \rho_{\nabla_0}$. Such a representative exists, as implies for example Theorem 3.2.1. Since the $\Gamma_{g,n}$-orbit of $[\rho_{\nabla_0}]$ does not depend on the choice of φ, we may assume that φ is convenient in the sense of Proposition 4.2.1. We then have a semi-direct product decomposition

$$\pi_1(X^0, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_\eta \sigma_* \pi_1(T, t_0),$$

where we have two different expressions for its structure morphism η, proving that

$$H := \mathrm{taut}_{g,n+1}(\mathrm{cl}^\bullet_* \pi_1(T, t_0)) \subset \Gamma_{g,n+1}$$

acts on $\rho_{\nabla_0} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_r\mathbb{C})$ by conjugation. More precisely, for all $\alpha \in \Lambda_{g,n}$ and $[h] = \mathrm{taut}_{g,n+1}(\mathrm{cl}^\bullet_* \beta) \in H$, we have

$$\rho_{\nabla_0}(\mathfrak{a}(h)(\alpha)) = \rho(\sigma_* \beta) \cdot \rho_{\nabla_0}(\alpha) \cdot \rho(\sigma_* \beta^{-1})$$

and in particular $[h^{-1}] \cdot [\rho_{\nabla_0}] = [\rho_{\nabla_0}]$. In other words, H is a subgroup of the stabilizer of $[\rho_{\nabla_0}]$ in $\Gamma_{g,n+1}$. By definition of the mapping class group action, we then have

$$\pi(H) \subset \mathrm{Stab}_{\Gamma_{g,n}}[\rho_{\nabla_0}],$$

where $\pi : \Gamma_{g,n+1} \to \Gamma_{g,n}$ is the projection forgetting the marking y_0. Since the size of the orbit $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$ equals the index of $\mathrm{Stab}_{\Gamma_{g,n}}[\rho_{\nabla_0}]$ in $\Gamma_{g,n}$, it now suffices to prove that $\pi(H)$ has finite index in $\Gamma_{g,n}$. Denote by $q : \Gamma_{g,n} \to \Gamma_{g,n}/K_{g,n}$ the quotient by the normal subgroup $K_{g,n}$, which, by Lemma 2.1.1, has order at most 2. Hence for the indices, we have

$$[\Gamma_{g,n} : \pi(H)] \leq 2 \cdot [\Gamma_{g,n}/K_{g,n} : q(\pi(H))].$$

We have a commutative diagram

$$
\begin{array}{ccc}
\Gamma_{g,n+1} & \xleftarrow{\;\mathrm{taut}_{g,n+1} \circ \mathrm{cl}^\bullet_*\;} & \pi_1(T, t_0) \\
{\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle \mathrm{taut}_{g,n} \circ \mathrm{cl}_*} \\
\Gamma_{g,n} & \xrightarrow{\qquad q \qquad} & \Gamma_{g,n}/K_{g,n}\,,
\end{array}
$$

where $\mathrm{cl} : T \to \mathcal{M}_{g,n} \backslash \mathcal{B}_{g,n}$ denotes the corestriction of $\mathrm{class}(\mathcal{F})$. On the other hand, by the dominance assumption and [Deb01, Lemma 4.19], the subgroup $\mathrm{cl}_* \pi_1(T, t_0)$ of $\pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star)$ is of finite index. In particular, since the tautological morphism $\mathrm{taut}_{g,n} : \pi_1(\mathcal{M}_{g,n} \setminus \mathcal{B}_{g,n}, \star) \twoheadrightarrow \Gamma_{g,n}/K_{g,n}$ is onto, the subgroup $q(\pi(H)) = \mathrm{taut}_{g,n}(\mathrm{cl}_* \pi_1(T, t_0))$ of $\Gamma_{g,n}/K_{g,n}$ has finite index. $\qquad\square$

Theorem A2. *Let $\mathcal{F}_{(C, D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be an algebraic family of stable n-pointed genus-g curves with central fiber (C, D_C) as in Section 2.2. Let (E_0, ∇_0) be an algebraic logarithmic connection over C with polar divisor D_C and denote by $[\rho_{\nabla_0}] \in \chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ its monodromy with respect to an orientation preserving homeomorphism $\varphi : (\Sigma_g, Y^n) \xrightarrow{\sim} (C, D_C)$. Assume that*

- *(E_0, ∇_0) is mild,*
- *$r = 2$ or ρ_{∇_0} is semisimple, and*
- *the $\Gamma_{g,n}$-orbit of $[\rho_{\nabla_0}]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ is finite.*

Then there are

- *an étale base change $p : (T', t_0') \to (T, t_0)$ and*
- *a flat algebraic logarithmic connection (E, ∇) over $X' := p^* X$ with polar divisor $p^* D$,*

such that $\psi^*(E, \nabla)|_{X'_{t_0'}}$ is isomorphic to (E_0, ∇_0).

Proof. Since (C, D_C) is stable by assumption, it only admits a finite number of automorphisms. Let $x_0 \in C \setminus D_C$ be a point fixed by no automorphism other than the identity. Up to isotopy, we may assume $\varphi(y_0) = x_0$. Let $\mathbf{x}^\bullet$ be the labelling of $D_C^\bullet = D_C + x_0$ induced by φ. By construction, we have $\star := [C, \mathbf{x}^\bullet] \in \mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}$. Up to an étale base change, we may

assume, by Lemma 3.1.1, that there is a section $\sigma : T \to X$ of κ with values in $X^0 := X \setminus D$ and such that $\sigma(t_0) = \psi(x_0)$. With the notation of Section 4.2, we may consider the family of $n+1$-pointed genus-g curves $\mathcal{F}^\bullet_{(C,D^\bullet_C)} = (\kappa : X \to T, D + \sigma(T), t_0, \psi)$. Altering neither $D^\bullet_C$, nor the labelling, nor the orbit $\Gamma_{g,n} \cdot [\rho_{\nabla_0}]$, we may assume that φ is conveniently chosen in the sense of Proposition 4.2.1. We then have a semi-direct product decomposition $\pi_1(X \setminus D, \sigma(t_0)) = (\psi \circ \varphi)_* \Lambda_{g,n} \rtimes_\eta \sigma_* \pi_1(T, t_0)$, where

$$\eta(\sigma_*\beta)((\psi \circ \varphi)_*\alpha) = \sigma_*\beta \cdot (\psi \circ \varphi)_*\alpha \cdot \sigma_*\beta^{-1} = (\psi \circ \varphi)_*\mathfrak{a}(\theta_*\beta)(\alpha)$$

and $\theta := \mathrm{taut}_{g,n+1} \circ \mathrm{cl}^\bullet_* : \pi_1(T, t_0) \to \Gamma_{g,n+1}$.

Since the $\Gamma_{g,n+1}$-orbit of $[\rho_{\nabla_0}]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ is finite, the stabilizer

$$H := \mathrm{Stab}_{\Gamma_{g,n+1}}[\rho_{\nabla_0}]$$

of the conjugacy class of ρ_{∇_0} under the action of $\Gamma_{g,n+1}$ has finite index in $\Gamma_{g,n+1}$. Since the tautological morphism is onto, the subgroup $\mathrm{taut}^{-1}_{g,n+1}(H)$ of $\pi_1(\mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}, \star)$ then has also finite index. In particular, there is a finite connected étale cover $q : (U, u_0) \to (\mathcal{M}_{g,n+1} \setminus \mathcal{B}_{g,n+1}, \star)$ such that $\pi_1(U, u_0) = \mathrm{taut}^{-1}_{g,n+1}(H)$. Now consider the fibered product

$$
\begin{array}{ccc}
(T', t'_0) & \xrightarrow{\ p\ } & (T, t_0) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{class}(\mathcal{F}^\bullet)} \\
(U, u_0) & \xrightarrow{\ q\ } & (\mathcal{M}_{g,n+1}, \star).
\end{array}
$$

We denote the pullback family of curves by $\mathcal{F}'_{(C,D^\bullet_C)} = (\kappa' : X' \to T', D' + \sigma'(T'), t'_0, \psi') := p^*\mathcal{F}^\bullet_{(C,D^\bullet_C)}$. We further denote $\mathrm{cl}' = \mathrm{cl}^\bullet \circ p$, which is the corestriction of $\mathrm{class}(\mathcal{F}')$. By construction, the morphism $\theta' := \theta \circ p = \mathrm{taut}_{g,n+1} \circ \mathrm{cl}'_* : \pi_1(T', t'_0) \to \Gamma_{g,n+1}$ takes values in H.

Again up to an étale base change of (T', t'_0), by Proposition 5.1.3 and Lemma 5.1.2, there is a representation $\rho_B \in \mathrm{Hom}(\pi_1(T', t'_0), \mathrm{GL}_r\mathbb{C})$ such that for all $\beta \in \pi_1(T', t'_0)$, $\alpha \in \Lambda_{g,n}$, we have

$$\left([\theta'_*\beta]^{-1} \cdot \rho_{\nabla_0}\right)(\alpha) = \rho_B(\beta) \cdot \rho_{\nabla_0}(\alpha) \cdot \rho_B(\beta^{-1}).$$

Since by definition $\left([\theta'_*\beta]^{-1} \cdot \rho_{\nabla_0}\right)(\alpha) = \rho_{\nabla_0}(\mathfrak{a}(\theta'_*\beta)(\alpha))$, we obtain a well defined representation

$$\rho : \begin{cases} \pi_1(X' \setminus D', \sigma'(t'_0)) & \to & \mathrm{GL}_r\mathbb{C} \\ (\psi' \circ \varphi)_*\alpha \cdot \sigma'_*\beta & \mapsto & \rho_{\nabla_0}(\alpha) \cdot \rho_B(\beta) \end{cases}$$

(see Lemma 3.1.3) with respect to the semi-direct product decomposition $\pi_1(X' \setminus D', \sigma'(t'_0)) = (\psi' \circ \varphi)_*\Lambda_{g,n} \rtimes_\eta \sigma'_*\pi_1(T', t'_0)$. By construction, ρ extends ρ_{∇_0}. We conclude by the logarithmic Riemann-Hilbert correspondence (see Theorem 3.2.1). $\qquad\square$

Proof of Theorem A. Let us first prove the implication $(1) \Rightarrow (2)$. Let $\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)} = (\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}, E, \nabla, \Psi)$ be an algebraic universal isomonodromic deformation of (C, E_0, ∇_0) as in Section 2.4. Then by definition, the family $\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ is Kuranishi. In particular, the classifying map $\mathrm{class}(\mathcal{F}^{\mathrm{Kur}}) : T \to \mathcal{M}_{g,n}$ is dominant. Then by Theorem A1, the $\Gamma_{g,n}$-orbit of $[\rho_{\nabla_0}]$ in $\chi_{g,n}(\mathrm{GL}_r\mathbb{C})$ is finite.

Let us now prove the implication $(2) \Rightarrow (1)$. Let $\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)} = (\kappa : X \to T, D, t_0, \psi)$ be any algebraic Kuranishi family with central fiber (C, D_C) as in Section 2.2. Note that such a family exists since (C, D_C) is stable, and that it remains Kuranishi after pullback *via* an étale base change. Up to such a manipulation, according to Theorem A2, the family $\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}$ can be endowed with a flat algebraic logarithmic connection (E, ∇) over X with polar divisor D such that there is an isomorphism $\Psi : (E_0, \nabla_0) \to (E, \nabla)|_{X_{t_0}}$ commuting with ψ via the natural projections to (C, D_C) and $(X_{t_0}, D|_{X_{t_0}})$ respectively. Now $\mathcal{I}^{\mathrm{univ,alg}}_{(C,E_0,\nabla_0)} := (\mathcal{F}^{\mathrm{Kur}}_{(C,D_C)}, E, \nabla, \Psi)$ defines an algebraic universal isomonodromic deformation of (C, E_0, ∇_0) (see Section 2.4). $\qquad\square$

Part B. **Dynamics**

<h2 style="text-align:center">6. Effective description of the mapping class group action</h2>

In this section we describe the action of $\hat{\Gamma}_{g,n}$ on $\Lambda_{g,n}$ in terms of specified generators for both groups.

6.1. Presentation of the fundamental group.
To give an effective description of $\Lambda_{g,n}$ and how $\hat{\Gamma}_{g,n}$ acts, we will assume that Σ_g is the subsurface of genus g of $\mathbb{R}^3$ depicted in Figure 1. On this surface we also depicted, in gray, an embedded closed disk $\bar{\Delta} \subset \Sigma_g$, we will denote Δ its interior. We fix n and we consider a subset $Y^n = \{y_1, \ldots, y_n\} \subset \Delta$ of cardinality n, as well as a point $y_0 \in \bar{\Delta} \setminus \Delta$. We have

$$\pi_1(\Sigma_g \setminus \Delta, y_0) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \delta \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = \delta^{-1} \right\rangle,$$

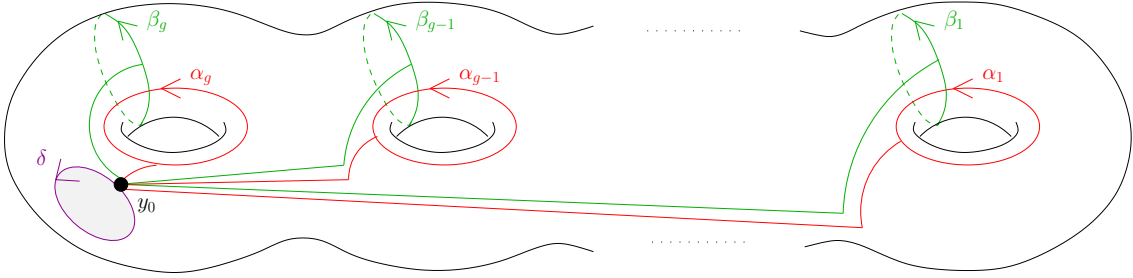where the mentioned generators correspond to the loops in Figure 1.

FIGURE 1. Preferred elements of the fundamental group, I

The loops in Figure 2 correspond to the following presentation.

$$\pi_1(\bar{\Delta} \setminus Y^n, y_0) = \left\langle \gamma_1, \ldots, \gamma_n, \delta \mid \gamma_1 \cdots \gamma_n = \delta \right\rangle.$$
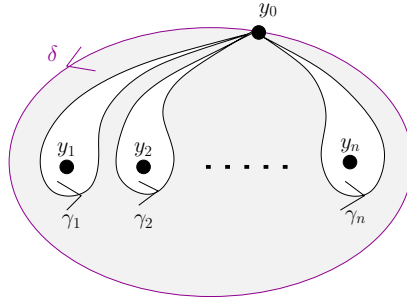

FIGURE 2. Preferred elements of the fundamental group, II

By the Van Kampen theorem, we have

$$
\begin{aligned}
\Lambda_{g,n} &= \pi_1(\Sigma \setminus \Delta, y_0) *_\delta \pi_1(\bar{\Delta} \setminus Y^n, y_0) \\
&= \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = ([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g])^{-1} \right\rangle.
\end{aligned}
$$

In the sequel, writing "the generators" of $\Lambda_{g,n}$, we will refer to the above

$$(\alpha_i)_{i \in [\![1,g]\!]}, \ (\beta_i)_{i \in [\![1,g]\!]}, \ (\gamma_j)_{j \in [\![1,n]\!]}.$$

6.2. **Mapping class group generators.** We define Γ_g^1 to be the mapping class group of orientation preserving homeomorphisms of $\Sigma \setminus \Delta$ that restrict to the identity on $\partial \Delta$. Continuating such homeomorphisms by the identity on Δ, we get a morphism

$$\varphi_g : \Gamma_g^1 \to \hat{\Gamma}_{g,n}^{\bullet} \, .$$

After Lickorish [Lic64] (see also [FM12, Th. 4.13]), the group Γ_g^1 is generated by the (right) Dehn-twists along the loops $\tau_1, \ldots, \tau_{3g-1}$ represented in Figure 3.
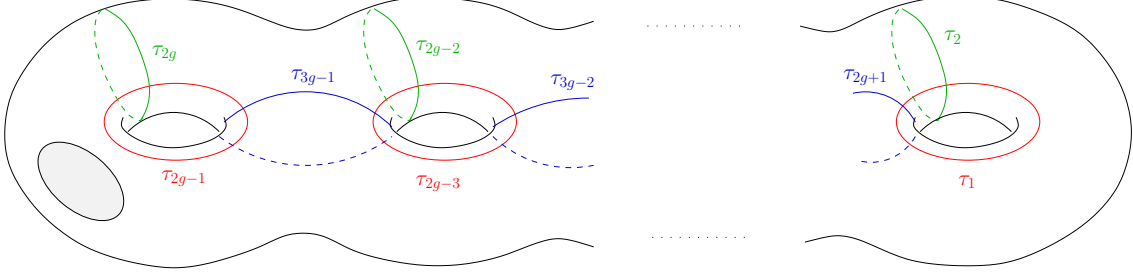
FIGURE 3. Dehn-twists

A right Dehn twist acts on paths which cross the corresponding Dehn curve as depicted in Figure 4. This action can be summarized as "a path crossing the Dehn curve has to turn right". A left Dehn twist is the inverse of a right Dehn twist.
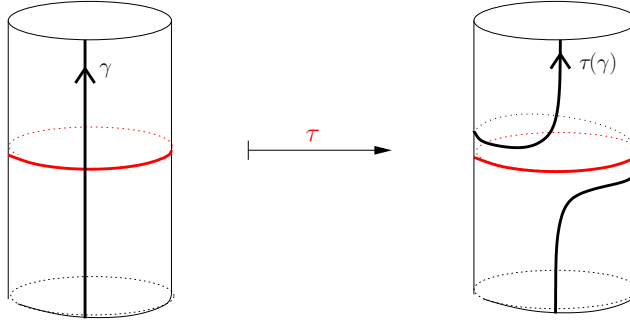

FIGURE 4. Dehn-twist action

One can now easily check the following.

Lemma 6.2.1 (Dehn-twists). *The action of the Dehn twists above on the fundamental group $\pi_1(\Sigma_g \setminus \Delta, y_0)$ is given in Table 3, where we only indicate the non-trivial actions on the generators. Here for τ_{2k-1} we give the formula for the left Dehn twist. The other generators all correspond to right Dehn twists. Moreover, for $k \in [\![1, g-1]\!]$, the element Θ_k described in Table 3 is fixed by τ_{2g+k}.*

τ_{2k}	$k \in [\![1,g]\!]$	α_k	$\mapsto$	$\alpha_k \beta_k$
τ_{2k-1}	$k \in [\![1,g]\!]$	β_k	$\mapsto$	$\beta_k \alpha_k$
τ_{2g+k}	$k \in [\![1,g-1]\!]$	α_{k+1}	$\mapsto$	$\Theta_k^{-1} \alpha_{k+1}$
		α_k	$\mapsto$	$\alpha_k \Theta_k$
		β_k	$\mapsto$	$\Theta_k^{-1} \beta_k \Theta_k$
	where	Θ_k	$:=$	$\alpha_{k+1} \beta_{k+1}^{-1} \alpha_{k+1}^{-1} \beta_k$

TABLE 3.

On the other hand, one can define the mapping class group of orientation preserving homeomorphisms of $\bar{\Delta}$ that preserve the set Y^n and restrict to the identity on $\partial\Delta$. It is classically called the braid group on n strands and denoted B_n. Continuating such homeomorphisms by the identity on the complement of Δ in Σ_g, we get a morphism

$$\varphi_0 : B_n \to \hat{\Gamma}_{g,n}^{\bullet}.$$

After Artin [Art25], the group B_n is generated by half-twists $\sigma_1, \ldots, \sigma_{n-1}$, whose action is depicted in Figure 5.
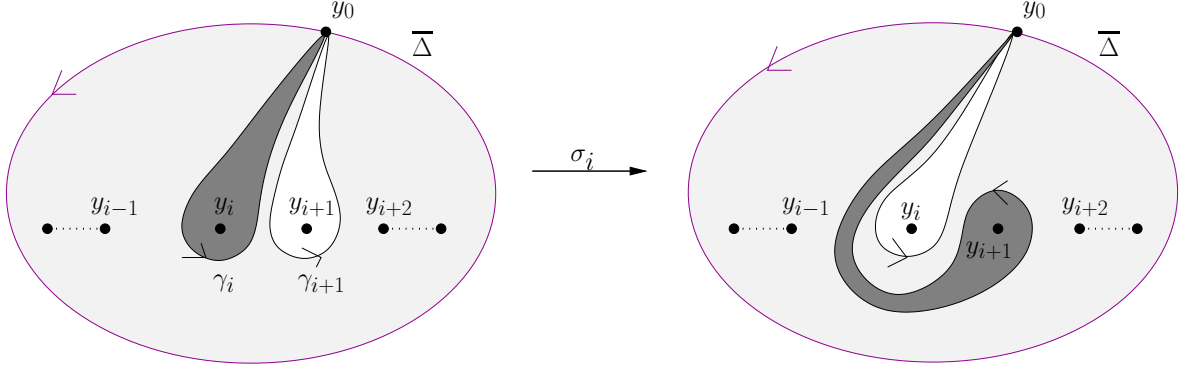

FIGURE 5. half-twists

Lemma 6.2.2 (half-twists). *The action of $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ on the fundamental group $\pi_1(\bar{\Delta} \setminus Y^n, y_0)$ is described in Table 4, where we only indicate the non-trivial actions on the generators. Moreover, Table 4 indicates the action of $\sigma_{cycl} := \sigma_{n-1} \circ \cdots \circ \sigma_1 \in B_n$ and some of its powers.*

σ_k	$k \in [\![1, n-1]\!]$	γ_k	$\mapsto$	$\gamma_k \gamma_{k+1} \gamma_k^{-1}$
		γ_{k+1}	$\mapsto$	γ_k
σ_{cycl}		γ_1	$\mapsto$	$\delta \gamma_n \delta^{-1}$
	$i \in [\![2, n]\!]$	γ_i	$\mapsto$	γ_{i-1}
σ_{cycl}^k	$k \in [\![1, n]\!]$			
	$i \in [\![1, k]\!]$	γ_i	$\mapsto$	$\delta \gamma_{n+i-k} \delta^{-1}$
	$j \in [\![k+1, n]\!]$	γ_j	$\mapsto$	γ_{j-k}

TABLE 4.

Remark 6.2.3. Note that σ_{cycl} is almost a cyclic permutation of the generators of $\pi_1(\bar{\Delta} \setminus Y^n, y_0)$. More precisely, it acts as such on the representations ρ that satisfy $\rho(\delta) = \mathrm{id}$, *e.g.* representations with abelian image.

By construction, the subgroups $\varphi_0(B_n)$ and $\varphi_g(\Gamma_g^1)$ of $\hat{\Gamma}_{g,n}^{\bullet}$ commute, and we have a morphism

$$B_n \times \Gamma_g^1 \xrightarrow{\varphi_0 \times \varphi_g} \hat{\Gamma}_{g,n}^{\bullet}.$$

Composing with the canonical map $\pi : \hat{\Gamma}_{g,n}^{\bullet} \to \hat{\Gamma}_{g,n}$ (forgetting that y_0 is fixed) yields a morphism $B_n \times \Gamma_g^1 \to \hat{\Gamma}_{g,n}$, which is not surjective. In order to generate the whole mapping class group $\hat{\Gamma}_{g,n}$, it suffices to add $\min(0, n-1)$ Dehn twists, namely the ones corresponding to the loops $\tau_{3g}, \ldots, \tau_{3g+n-2}$ of Figure 6 (see [FM12, Sec. 4.4.4]). We call them *mixing twists*.

Lemma 6.2.4 (Mixing twists). *The action of the (right) mixing twists $\tau_{3g}, \ldots, \tau_{3g+n-2}$ on the fundamental group $\Lambda_{g,n}$ is described in Table 5, where we only indicate the non-trivial actions on the generators. Moreover, for $k \in [\![1, n-1]\!]$, the element Ξ_k described there is fixed by τ_{3g-1+k}.*

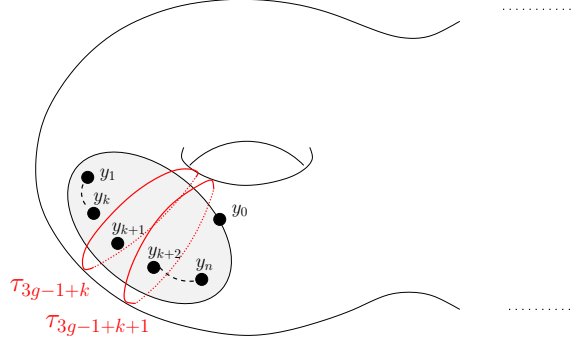FIGURE 6. Mixing twists

τ_{3g-1+k}	$k \in [\![1, n-1]\!]$	α_g	$\mapsto$	$\alpha_g \Xi_k$
		β_g	$\mapsto$	$\Xi_k^{-1} \beta_g \Xi_k$
	$i \in [\![1, k]\!]$	γ_i	$\mapsto$	$\Xi_k^{-1} \gamma_i \Xi_k$
	where	Ξ_k	$=$	$(\gamma_1 \dots \gamma_k)^{-1} \beta_g$

TABLE 5.

The twists, mixing twists and braids we introduced all fix y_0. We denote by $\hat{\Gamma}_{g,n}^\circ$ the subgroup of $\hat{\Gamma}_{g,n}^\bullet$ they generate. If $g = 0$, then we have $\hat{\Gamma}_{g,n}^\circ = B_n$. We are interested in the case $g > 0$, where we have

$$\hat{\Gamma}_{g,n}^\circ := \langle \tau_i, \sigma_j \mid i \in [\![1, 3g-1+\min(0, n-1)]\!], \, j \in [\![1, n-1]\!] \rangle .$$

As mentioned, the image of $\hat{\Gamma}_{g,n}^\circ$ under $\pi : \hat{\Gamma}_{g,n}^\bullet \to \hat{\Gamma}_{g,n}$ is $\hat{\Gamma}_{g,n}$.

Remark 6.2.5. We did not call $\delta = \gamma_1 \cdots \gamma_n = ([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g])^{-1}$ a generator of the fundamental group. It will nevertheless be useful to notice that among our preferred generators of $\hat{\Gamma}_{g,n}^\circ$, only the mixing twists act non trivially on δ. More precisely, for $k \in [\![1, n-1]\!]$ we have

$$\tau_{3g-1+k}(\delta) = [\Xi_k^{-1}, \beta_g] \delta .$$

7. AFFINE REPRESENTATIONS WITH FINITE ORBIT

We have now established an explicit description of the full mapping class group action on $\Lambda_{g,n}$, which is resumed in Table 2. This description at hand, we will now classify affine representations $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ with finite orbit $\hat{\Gamma}_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ for $g > 0$:

- We establish that for those representations $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ such that the group $\mathrm{Im}(\rho)$ is abelian, the orbit $\hat{\Gamma}_{g,n} \cdot [\rho]$ is finite if and only if $\mathrm{Im}(\rho)$ is finite (see Proposition 7.1.2).

- We then consider representations $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ such that the group $\mathrm{Im}(\rho)$ is not abelian. We classify all finite orbits in this case in three steps.
  - We give a necessary condition for the finiteness of $\hat{\Gamma}_{g,n} \cdot [\rho]$ in Lemma 7.2.2.
  - We prove that in the genus one case, this necessary condition is also sufficient (see Proposition 7.3.1).
  - We prove that in the higher genus case, this necessary condition can be enforced (see Lemma 7.4.1), and this enforced necessary condition cannot hold for every conjugacy class $[\rho'] \in \hat{\Gamma}_{g,n} \cdot [\rho]$. We conclude that in the higher genus case, there are no conjugacy classes of non abelian $\mathrm{Aff}(\mathbb{C})$-representations with finite orbit under $\hat{\Gamma}_{g,n}$ (see Proposition 7.4.2).

The group $\mathrm{Aff}(\mathbb{C}) = \{(a_{ij}) \in \mathrm{GL}_2(\mathbb{C}) \mid a_{21} = 0, a_{22} = 1\}$ identifies with the group $\{z \mapsto az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$ of affine transformations of $\mathbb{C}$. For shortness, its elements will be denoted as polynomials $az + b$. Our explicit calculations are easier to check with the following formulas in mind.

$$
\begin{array}{rcl}
(\lambda z) \circ (az + b) \circ (\lambda z)^{-1} &=& az + \lambda b \\
(z + c) \circ (az + b) \circ (z + c)^{-1} &=& az + b - c(a - 1) \\
[\lambda z + c, az + b] &=& z - c(a - 1) + (\lambda - 1)b
\end{array}
$$

Also, recall that by definition, for all $\tau \in \hat{\Gamma}_{g,n}^{\circ}$, $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ and $\alpha \in \Lambda_{g,n}$, we have

$$
(\tau \cdot \rho)(\alpha) = \rho(\tau_*^{-1}\alpha).
$$

7.1. Abelian case.

Lemma 7.1.1 (Finding a non trivial subgroup). *Let $g > 0$, $n \in \mathbb{N}$. Let G be a group with identity element* id *and let $\rho : \Lambda_{g,n} \to G$ be a representation. Assume that for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, we have*

$$
\rho'(\alpha_g) = \mathrm{id}.
$$

Then ρ is the trivial representation, i.e. $\mathrm{Im}(\rho) = \{\mathrm{id}\}$.

Proof. Note that our assumption on ρ is $\hat{\Gamma}_{g,n}^{\circ}$-invariant, so that anything we prove for ρ also holds for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$. We denote

$$
R' := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_1), \rho'(\beta_1) \rangle, \quad S' := \langle \rho'(\gamma_1), \dots, \rho'(\gamma_n) \rangle.
$$

- First step: *for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group R' is trivial.*
  For $k \in [\![1, g]\!]$, define the following property, which we shall denote $H(k)$:

  > *For any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group $R_k' := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_k), \rho'(\beta_k) \rangle$ is trivial.*

  Let us first prove that our assumption implies $H(g)$. Consider $\tau := \tau_{2g}^{-1}$ and $\rho' = \tau \cdot \rho$. Then $\rho'(\alpha_g) = \rho(\alpha_g \beta_g) = \rho(\beta_g)$. We have $\rho'(\alpha_g) = \rho(\alpha_g) = \mathrm{id}$, hence $\rho(\beta_g) = \rho(\alpha_g) = \mathrm{id}$. By $\hat{\Gamma}_{g,n}^{\circ}$-invariance, we have $H(g)$.
  Let now ρ be a representation satisfying $H(k)$. In particular, we have

  $$
  \rho(\alpha_i) = \rho(\beta_i) = \mathrm{id} \quad \forall i \in [\![k, g]\!].
  $$

  For $\rho' = \tau \cdot \rho$, with $\tau = \tau_{2g+k-1}^{-1}$ we have $\rho'(\alpha_k) = \rho(\beta_{k-1}^{-1}\alpha_k\beta_k) = \rho(\beta_{k-1})^{-1}$.
  For $\rho' = \tau \cdot \rho$, with $\tau = (\tau_{2k-3} \circ \tau_{2g+k-1})^{-1}$, we have $\rho'(\alpha_k) = \rho(\beta_{k-1}\alpha_{k-1})^{-1}$.
  Hence ρ satisfying $H(k)$ implies

  $$
  \rho(\alpha_i) = \rho(\beta_i) = \mathrm{id} \quad \forall i \in [\![k-1, g]\!].
  $$

  As $H(k)$ is $\hat{\Gamma}_{g,n}^{\circ}$-invariant, this proves $H(k-1)$. We conclude by noticing $R' = R_1'$.

- Second step: *for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group S' is trivial.*
  If $n = 0$ or $n = 1$, there is nothing to prove. Assume $n > 1$. We have already proven that R' is trivial for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$. In particular $\rho'(\delta) = \mathrm{id}$. Considering, for $i \in [\![1, n]\!]$, the action of $\tau = (\sigma_{cycl}^{n-i} \circ \tau_{3g+n-2})^{-1}$ on α_g then shows that for $\rho' = \tau \cdot \rho$ we have $\mathrm{id} = \rho'(\alpha_g) = \rho(\gamma_i)$ (see Table 4 page 24). Hence

  $$
  \langle \rho(\gamma_1), \dots, \rho(\gamma_n) \rangle = \{\mathrm{id}\}.
  $$

  Since the assertion is $\hat{\Gamma}_{g,n}^{\circ}$-invariant, we have proven that S' is trivial for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$.

We conclude that $\mathrm{Im}(\rho) = \mathrm{Im}(\rho') = \langle S', R' \rangle = \{\mathrm{id}\}$. $\qquad \square$

Proposition 7.1.2 (Abelian case). *Let $g > 0$. Let $\rho : \Lambda_{g,n} \to \mathrm{Aff}(\mathbb{C})$ be a representation such that the group $\mathrm{Im}(\rho)$ is abelian. Then the orbit of the conjugacy class $[\rho]$ under the action of $\hat{\Gamma}_{g,n}$ is finite if and only if $\mathrm{Im}(\rho)$ is finite.*

Proof. If $\mathrm{Im}(\rho)$ is finite, then the orbit $\hat{\Gamma}_{g,n}^{\circ} \cdot \rho$ is finite. *A fortiori*, the orbit $\hat{\Gamma}_{g,n}^{\circ} \cdot [\rho]$ is finite. Assume now that ρ is abelian and the orbit of $[\rho]$ is finite. Since $\mathrm{Im}(\rho)$ is an abelian subgroup of $\mathrm{Aff}(\mathbb{C})$ it is, up to conjugation, either a non trivial subgroup of the translation group

$$\{z \mapsto z + c \mid c \in \mathbb{C}\} \subset \mathrm{Aff}(\mathbb{C}),$$

or it is a subgroup of the linear group

$$\{z \mapsto \lambda z \mid \lambda \in \mathbb{C}^*\} \subset \mathrm{Aff}(\mathbb{C}).$$

- $\mathrm{Im}(\rho)$ *cannot be a non trivial translation group.*
  Indeed, if it would be the case, by Lemma 7.1.1, we might assume $\rho(\alpha_g) \neq \mathrm{id}$. Up to conjugation, we would then have

$$\rho \left(\begin{array}{c} \alpha_g \\ \beta_g \end{array} \right) = \left(\begin{array}{c} z + 1 \\ z + c \end{array} \right)$$

for a certain $c \in \mathbb{C}$. Considering the action of τ^{-m} with $\tau := \tau_{2g-1}$:

$$\tau^{-m} \cdot \rho \left(\begin{array}{c} \alpha_g \\ \beta_g \end{array} \right) = \rho \left(\begin{array}{c} \alpha_g \\ \beta_g \alpha_g^m \end{array} \right) = \left(\begin{array}{c} z + 1 \\ z + c + m \end{array} \right),$$

we would deduce that, for $m \neq m'$, the conjugacy classes of $\tau^m \cdot \rho$ and $\tau^{m'} \cdot \rho$ are distinct. Hence $\hat{\Gamma}_{g,n}^{\circ} \cdot [\rho]$ would be infinite, yielding a contradiction.

- *If* $\mathrm{Im}(\rho)$ *is a subgroup of the linear group, then it is finite.*
  Note that two distinct linear representations are not conjugated. For any $i \in [\![1, g]\!]$, finiteness of the orbit under $\langle \tau_{2i} \rangle$ yields that $\rho(\beta_i)$ is torsion. Similarly, considering $\langle \tau_{2i-1} \rangle$ yields that $\rho(\alpha_i)$ is torsion for all $i \in [\![1, g]\!]$. For $j \in [\![1, n-1]\!]$, finiteness of the orbit under $\langle \tau_{3g-1+j} \rangle$ yields that $\rho(\gamma_1 \dots \gamma_j)$ is torsion. Consequently, γ_j is torsion for all $j \in [\![1, n-1]\!]$. Hence

$$\mathrm{Im}(\rho) = \langle \rho(\alpha_i), \rho(\beta_i), \rho(\gamma_j) \mid i \in [\![1, g]\!], \ j \in [\![1, n-1]\!] \rangle$$

is an abelian group generated by finitely many torsion elements, whence the conclusion.

$\square$

7.2. **Preparation lemmata.**

Lemma 7.2.1 (Finding a non abelian subgroup). *Let* $g > 0$. *Let* $\rho : \Lambda_{g,n} \to \mathrm{Aff}(\mathbb{C})$ *be a representation. Assume that for any* $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, *the subgroup*

$$\langle \rho'(\alpha_g), \rho'(\beta_g) \rangle$$

of $\mathrm{Im}(\rho)$ *is abelian. Then* ρ *is an abelian representation,* i.e. $\mathrm{Im}(\rho)$ *is abelian.*

Proof. Denote

$$R_k' := \langle \rho'(\alpha_g), \rho'(\beta_g), \dots, \rho'(\alpha_k), \rho'(\beta_k) \rangle, \quad S' := \langle \rho'(\gamma_1), \dots, \rho'(\gamma_n) \rangle.$$

- First step: *For any* $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, *the group* R_g' *is contained in the center of* R_1'.
  For $k \in [\![1, g]\!]$, define the following property.

$H(k) :$ $\boxed{\text{For any } \rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho, \text{ the group } R_g' \text{ is a subgroup of the center of } R_k'.}$

By assumption, we have $H(g)$. Assume now $H(k)$ is proven. In particular, $R_g := \langle \rho(\alpha_g), \rho(\beta_g) \rangle$ is a subgroup of the center of $R_k := \langle \rho(\alpha_g), \rho(\beta_g), \dots, \rho(\alpha_k), \rho(\beta_k) \rangle$. Note that $H(k)$ is $\hat{\Gamma}_{g,n}^{\circ}$-invariant. Hence in order to prove $H(k-1)$, is suffices to prove that R_g is also a subgroup of the center of R_{k-1}.
For $\rho' = \tau \cdot \rho$, with $\tau = \tau_{2g+k-1}^{-1}$, only one of the generators of R_k is modified, namely

$$\rho'(\alpha_k) = \rho(\beta_{k-1}^{-1} \alpha_k \beta_k) = \rho(\beta_{k-1})^{-1} \rho(\alpha_k \beta_k).$$

27

In particular, we have $\rho'(\beta_g) = \rho(\beta_g)$. Then $H(k)$ implies that $\rho(\beta_g)$ belongs to the center of $\langle R_k, R'_k \rangle = \langle R_k, \rho(\beta_{k-1}) \rangle$. For $\rho'' = \tau' \cdot \rho$, with $\tau' = \tau \circ \tau_{2k-3}^{-1}$, we have

$$\rho''(\alpha_k) = \rho(\beta_{k-1}\alpha_{k-1})^{-1}\rho(\alpha_k\beta_k).$$

Then $H(k)$ implies that $\rho''(\beta_g) = \rho(\beta_g)$ belongs to the center of $\langle R_k, R''_k \rangle = \langle R_k, \rho(\beta_{k-1}\alpha_{k-1}) \rangle$. We have now proven that for any representation ρ such that $H(k)$ holds, $\rho(\beta_g)$ is an element of the center of

$$R_{k-1} = \langle R_k, R'_k, R''_k \rangle.$$

This assertion applied to $\tau_{2g-1}^{-1} \cdot \rho$ shows that $\rho(\beta_g\alpha_g)$ is an element of the center of R_{k-1}. Hence $R_g = \langle \rho(\beta_g), \rho(\beta_g\alpha_g) \rangle$ is a subgroup of the center of R_{k-1}.

- Second step: *For any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group $R' := R'_1$ is abelian.*
  If R' is trivial, then in particular it is abelian. If R' is non trivial, then by the first step in the proof of Lemma 7.1.1, we can find $\tau' \in \hat{\Gamma}_{g,n}^{\circ}$ such that for the induced representation $\rho'' = \tau' \cdot \rho$ we have $R'' = R'$ and R''_g is non trivial. Hence by the first step of the current lemma, R' has a non trivial center. Yet any subgroup of $\text{Aff}(\mathbb{C})$ with non trivial center is abelian.

- Third step: *For any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group R'_g is a subgroup of the center of $\text{Im}(\rho')$.*
  We have now proven that under our assumption, R' is abelian for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$. In particular, $\rho(\delta) = \text{id}$. Recall, from Remark 6.2.5, the action of the mixing twist τ_{3g+n-2} on δ. It is given by $\delta \mapsto [\beta_g^{-1}\delta\gamma_n^{-1}, \beta_g]\delta$.
  Hence, for $\rho' = \tau \cdot \rho$ with $\tau = (\sigma_{cycl}^{n-i} \circ \tau_{3g+n-2})^{-1}$, we have

$$\rho'(\delta) = [\rho(\beta_g^{-1}\gamma_i^{-1}), \rho(\beta_g)] = [\rho(\beta_g)^{-1}, \rho(\gamma_i)^{-1}]$$

(see Table 4 page 24). Consequently, $\rho(\beta_g)$ centralizes $S := \langle \rho(\gamma_i) \mid i \in [\![1,n]\!] \rangle$. Yet we could have applied the same argument to $\rho'' = \tau' \cdot \rho$, where $\tau' = \tau_{2g-1}^{-1}$ is the inverse of the Dehn-twist $\beta_g \mapsto \beta_g\alpha_g$, and we would have obtained that $\rho(\beta_g\alpha_g)$ centralizes S. It follows that R_g centralizes S. By $\hat{\Gamma}_{g,n}^{\circ}$-invariance of the statement, we deduce that for any $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$, the group R'_g centralizes $\text{Im}(\rho') = \langle R', S' \rangle$.

- Fourth step: *$\text{Im}(\rho)$ is abelian.*
  If ρ is the trivial representation, there is nothing to prove. Otherwise, by Lemma 7.1.1, there is a representation $\rho' \in \hat{\Gamma}_{g,n}^{\circ} \cdot \rho$ in the orbit of ρ such that $R'_g = \langle \rho'(\alpha_g), \rho'(\beta_g) \rangle$ is not the trivial group. On the other hand, we have proven that R'_g is a subgroup of the center of $\text{Im}(\rho')$. Hence $\text{Im}(\rho) = \text{Im}(\rho')$ is abelian.

$\square$

Lemma 7.2.2 (Prepared form). *Let $g > 0$. Let $\rho : \Lambda_{g,n} \to \text{Aff}(\mathbb{C})$ be a representation. Assume that $\text{Im}(\rho)$ is non abelian and $\hat{\Gamma}_{g,n} \cdot [\rho]$ is finite. Then up to the action of a certain element of the mapping class group and up to conjugation, ρ is of the following "prepared form"*

(17)
$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_i \\ \beta_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{m_g}z \\ z+1 \\ \mu^{m_i}z + a_i \\ z + b_i \\ z + c_j \end{pmatrix},$$

for $i \in [\![1, g-1]\!]$ and $j \in [\![1, n]\!]$, where $\mu \in \mathbb{C}^* \setminus \{1\}$ is a root of unity, $m_g, m_i \in \mathbb{Z}$, $a_i, b_i, c_j \in \mathbb{C}$ and $\mu^{m_g} \neq 1$.

Proof. According to Lemma 7.2.1, up to the action of an element of the mapping class group, we may assume $\rho([\alpha_g, \beta_g]) \neq \text{id}$. Since $\hat{\Gamma}_{g,n} \cdot [\rho]$ is finite, the linear part ρ_{lin} of ρ also has finite

orbit. After Proposition 7.1.2, ρ_{lin} takes values in a finite cyclic group $\langle \mu \rangle \subset \mathbb{C}^*$. Hence for each $i \in [\![1, g]\!]$, we have

$$\rho(\alpha_i) = \mu^{m_i} z + a_i, \quad \rho(\beta_i) = \mu^{n_i} z + b_i$$

for integers $m_i, n_i \in \mathbb{Z}$ and complex numbers $a_i, b_i \in \mathbb{C}$. Consider the actions of τ_{2i}^{-1} and τ_{2i-1}^{-1} on (m_i, n_i) (the other exponents are not altered) :

$$
\begin{array}{c|ccc}
\tau_{2i-1}^{-1} & \begin{pmatrix} m_i \\ n_i \end{pmatrix} & \mapsto & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix} \\
\hline
\tau_{2i}^{-1} & \begin{pmatrix} m_i \\ n_i \end{pmatrix} & \mapsto & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_i \\ n_i \end{pmatrix}
\end{array}
$$

These actions generate the action of $\mathrm{SL}_2\mathbb{Z}$ on $(m_i, n_i) \in \mathbb{Z}^2$. If $(m_i, n_i) \neq (0,0)$, then $\widetilde{m}_i := \gcd(m_i, n_i)$ is a well defined positive integer. Let p_i and q_i be integers such that $p_i m_i + q_i n_i = \widetilde{m}_i$. The matrix

$$\begin{pmatrix} p_i & q_i \\ -\frac{n_i}{\widetilde{m}_i} & \frac{m_i}{\widetilde{m}_i} \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}$$

then sends (m_i, n_i) to $(\widetilde{m}_i, 0)$. Hence, up to the action of a word in the twists $(\tau_{2i})_{i \in [\![1,g]\!]}$, $(\tau_{2i-1})_{i \in [\![1,g]\!]}$, we may assume $n_i = 0$ for each $i \in [\![1, g]\!]$. The property $\rho([\alpha_g, \beta_g]) \neq \mathrm{id}$ is not altered by such a word, hence $\mu^{m_g} \neq 1$. Up to conjugation by an element of $\mathrm{Aff}(\mathbb{C})$, we may moreover assume

(18)
$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \end{pmatrix}.$$

For $j \in [\![1, n]\!]$, let $c_j, d_j \in \mathbb{C}$ be defined by $\rho(\gamma_j) = d_j z + c_j$. For $k \in \mathbb{Z}$, consider the action of τ_2^{-k} :

$$\tau_2^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z + k\mu^{m_g} \\ z + 1 \\ d_j z + c_j \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ d_j z + c_j - k\frac{(d_j-1)\mu^{m_g}}{\mu^{m_g}-1} \end{pmatrix}.$$

For these sequences of normalized triples to be finite, we must have $d_j = 1$ for each $j \in [\![1, n]\!]$. $\quad \square$

7.3. Non abelian case in genus one.

Proposition 7.3.1 (Non abelian representations for $g = 1$). *Assume $g = 1$. Let $\rho : \Lambda_{g,n} \to \mathrm{Aff}(\mathbb{C})$ be a representation with non abelian image (in particular $n \geq 1$). Then the orbit $\hat{\Gamma}_{g,n} \cdot [\rho]$ is finite if and only if there is a root of unity $\mu \neq 1$ and $\mathbf{c} := (c_1, \ldots, c_n) \in \mathbb{C}^n$ with $\sum_{i=1}^n c_i = 1$ such that $[\rho] \in \hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$, where $\rho_{\mu,\mathbf{c}}$ is the representation given by*

$$\rho_{\mu,\mathbf{c}}(\alpha_1) = \mu z; \quad \rho_{\mu,\mathbf{c}}(\beta_1) = z - \frac{1}{\mu - 1}; \quad \rho_{\mu,\mathbf{c}}(\gamma_i) = z + c_i \quad \forall i \in [\![1, n]\!].$$

Proof. Recall that for $g = 1$, the fundamental group $\Lambda_{g,n}$ has the following presentation

$$\Lambda_{g,n} = \langle \alpha_1, \beta_1, \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = [\alpha_1, \beta_1] \rangle,$$

and the mapping class group $\hat{\Gamma}_{g,n}$ is generated by the elements of Table 6. Assume $[\rho]$ has finite orbit, then by Lemma 7.2.2, we have $\hat{\Gamma}_{g,n} \cdot [\rho] = \hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$ for a convenient choice of $\mathbf{c} \in \mathbb{C}^n$ and a root of unity $\mu \neq 1$. Let us now prove that $[\rho_{\mu,\mathbf{c}}]$ has finite orbit. Denote

$$N := \mathrm{order}(\mu); \qquad D_{\mathbf{c}} := \mu^{\mathbb{Z}} c_1 + \ldots + \mu^{\mathbb{Z}} c_n.$$

τ_1		β_1	$\mapsto$	$\beta_1 \alpha_1$
τ_2		α_1	$\mapsto$	$\alpha_1 \beta_1$
$\widetilde{\tau}_{2+k} := \tau_2^{-1} \circ \tau_{2+k}$	$k \in [\![1, n-1]\!]$	α_1	$\mapsto$	$\alpha_1 \beta_1^{-1} \Xi_k$
		β_1	$\mapsto$	$\Xi_k^{-1} \beta_1 \Xi_k$
	$i \in [\![1, k]\!]$	γ_i	$\mapsto$	$\Xi_k^{-1} \gamma_i \Xi_k$
	where	Ξ_k	$=$	$(\gamma_1 \dots \gamma_k)^{-1} \beta_1$
σ_i	$i \in [\![1, n-1]\!]$	γ_i	$\mapsto$	$\gamma_i \gamma_{i+1} \gamma_i^{-1}$
		γ_{i+1}	$\mapsto$	γ_i

$$\textsc{Table 6.}$$

Denote the following sets of tuples of affine transformations

$$S^1_{\mu,d} := \left\{ \left(\begin{array}{c} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) \middle| \begin{array}{l} k_1, k_2 \in \mathbb{Z}, \quad k_1 \notin N\mathbb{Z}, \\ \gcd(k_1, k_2, N) = 1 \end{array} \right\}$$

$$S^2_{\mu,d} := \left\{ \left(\begin{array}{c} \mu^{k_1} z + \frac{d}{\mu^{k_2}-1} \\ \mu^{k_2} z \end{array} \right) \middle| \begin{array}{l} k_1, k_2 \in \mathbb{Z}, \quad k_2 \notin N\mathbb{Z}, \\ \gcd(k_1, k_2, N) = 1 \end{array} \right\}$$

$$R_{\mu,\mathbf{c},d} := \left\{ \left(\begin{array}{c} z + \tilde{c}_1 \\ \vdots \\ z + \tilde{c}_n \end{array} \right) \middle| \begin{array}{l} (\hat{c}_1, \dots, \hat{c}_n) \in \mathfrak{S}_n \cdot (c_1, \dots, c_n), \\ \tilde{c}_i \in \mu^{\mathbb{Z}} \hat{c}_i \quad \forall i \in [\![1, n]\!], \\ d = \sum_{i=1}^{n} \tilde{c}_i \end{array} \right\} .$$

Moreover, we set $S_{\mu,d} := S^1_{\mu,d} \cup S^2_{\mu,d}$. Then by definition, we have

$$\rho_{\mu,\mathbf{c}} \left(\begin{array}{c} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{array} \right) \in O_{\mu,\mathbf{c}} := \bigcup_{d \in D_{\mathbf{c}}} \left\{ \left(\begin{array}{c} \varphi_\alpha \\ \varphi_\beta \\ \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) \middle| \left(\begin{array}{c} \varphi_\alpha \\ \varphi_\beta \end{array} \right) \in S_{\mu,d}, \quad \left(\begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) \in R_{\mu,\mathbf{c},d} \right\} .$$

Note that $O_{\mu,\mathbf{c}}$ is a finite set, and we will prove that each conjugacy class in the orbit of $\rho_{\mu,\mathbf{c}}$ under the action of the mapping class group has a representative in $O_{\mu,\mathbf{c}}$. We shall denote $[O_{\mu,\mathbf{c}}]$ the image of $O_{\mu,\mathbf{c}}$ in $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$.

- *The set $[O_{\mu,\mathbf{c}}]$ is stable under the inverses of τ_1 and τ_2.*
  In order to prove this first assertion, it is enough to prove that the sets $S^1_{\mu,d}$ and $S^2_{\mu,d}$ are stable under the action of τ_1^{-1} and τ_2^{-1} modulo conjugation by translations. Let

$$\rho \left(\begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) = \left(\begin{array}{c} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) \in S^1_{\mu,d}.$$

Then

$$\tau_1^{-1} \cdot \rho \left(\begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) = \left(\begin{array}{c} \mu^{k_1} z \\ \mu^{k_1+k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) \in S^1_{\mu,d}$$

and

$$\tau_2^{-1} \cdot \rho \left(\begin{array}{c} \alpha_1 \\ \beta_1 \end{array} \right) = \left(\begin{array}{c} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1}-1} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) .$$

To see that, up to conjugation by a translation, the latter image also belongs to $S_{\mu,d}$, we need to distinguish two cases. Firstly, if $k_1 + k_2 \in N\mathbb{Z}$, then $k_2 \notin N\mathbb{Z}$ and we obtain

$$\left(\begin{array}{c} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1}-1} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) = \left(\begin{array}{c} z + \frac{d}{\mu^{k_2}-1} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{array} \right) \approx \left(\begin{array}{c} z + \frac{d}{\mu^{k_2}-1} \\ \mu^{k_2} z \end{array} \right) \in S^2_{\mu,d}$$

Secondly, if $k_1 + k_2 \notin N\mathbb{Z}$, then we obtain

$$\begin{pmatrix} \mu^{k_1+k_2} z - \frac{\mu^{k_1} d}{\mu^{k_1}-1} \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \end{pmatrix} \approx \begin{pmatrix} \mu^{k_1+k_2} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1+k_2}-1} \end{pmatrix} \in S^1_{\mu,d}.$$

In a similar way, one can show that up to conjugation by translations, we have $\tau_1^{-1} \cdot S^2_{\mu,d} \subset S_{\mu,d}$ and $\tau_2^{-1} \cdot S^2_{\mu} \subset S^2_{\mu}$.

- *The set $[O_{\mu,\mathbf{c}}]$ is stable under the inverses of $\sigma_1, \dots, \sigma_{n-1}$.*
  Indeed, for every $\rho \in O_{\mu,\mathbf{c}}$, the group $\langle \rho(\gamma_1), \dots, \rho(\gamma_n) \rangle$ is a translation group. In particular, it is abelian. Hence the elements σ_i act as permutations. But permutations stabilize the set $R_{\mu,\mathbf{c},d}$.

- *The set $[O_{\mu,\mathbf{c}}]$ is stable under the inverse of the modified mixing twist $\widetilde{\tau}_{2+k}$ for every $k \in [\![1, n-1]\!]$.*
  Note that for $k \in [\![1, n-1]\!]$, up to a common conjugation by $\rho(\Xi_k)$, the representation $\rho' := \widetilde{\tau}_{2+k}^{-1} \cdot \rho$ may be described as follows, where $\Xi_k = (\gamma_1 \dots \gamma_k)^{-1}\beta_1$.

$$\begin{cases} \rho'(\alpha_1) = \rho(\Xi_k \alpha_1 \beta_1^{-1}) \\ \rho'(\beta_1) = \rho(\beta_1) \\ \rho'(\gamma_i) = \rho(\gamma_i) & i \in [\![1, k]\!]; \\ \rho'(\gamma_j) = \rho(\Xi_k \gamma_j \Xi_k^{-1}) & j \in [\![k+1, n]\!]. \end{cases}$$

In the following calculations, i represents an index less or equal to k (if such an index exists) and j represents an index greater than k.
Assume first that $\rho(\alpha_1, \beta_1) \in S^1_{\mu,d}$. Then

$$\rho \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \\ z + \tilde{c}_i \\ z + \tilde{c}_j \end{pmatrix} \quad \text{and} \quad \rho(\Xi_k) = \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} - \sum_{i=1}^{k} \tilde{c}_i.$$

Hence

$$\rho' \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_i \\ \gamma_j \end{pmatrix} = \begin{pmatrix} \mu^{k_1} z + d - \sum_{h=1}^{k} \tilde{c}_h \\ \mu^{k_2} z - \frac{d}{\mu^{k_1}-1} \\ z + \tilde{c}_i \\ z + \mu^{k_2} \tilde{c}_j \end{pmatrix} \approx \begin{pmatrix} \mu^{k_1} z \\ \mu^{k_2} z - \frac{d'}{\mu^{k_1}-1} \\ z + \tilde{c}_i \\ z + \mu^{k_2} \tilde{c}_j \end{pmatrix},$$

where

$$d' = \mu^{k_2} d - (\mu^{k_2} - 1) \sum_{i=1}^{k} \tilde{c}_i = \sum_{i=1}^{k} \tilde{c}_i + \sum_{j=k+1}^{n} \mu^{k_2} \tilde{c}_j,$$

since $d = \sum_{i=1}^{k} \tilde{c}_i + \sum_{j=k+1}^{n} \tilde{c}_j$. In other words, up to conjugation by a translation, we have $\rho' \in O_{\mu,\mathbf{c}}$. By an almost identical argumentation, we show that if $\rho \in O_{\mu,\mathbf{c}}$ with $\rho(\alpha_1, \beta_1) \in S^2_{\mu,d}$, then $\widetilde{\tau}_{2+k}^{-1} \cdot \rho$ is also in $O_{\mu,\mathbf{c}}$ modulo conjugation.

Since every element of $\hat{\Gamma}^{\circ}_{g,n}$ induces a bijection of $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ and we have proven that $[O_{\mu,\mathbf{c}}]$ is stable under τ_i^{-1} for every $i \in [\![1, n+1]\!]$ and σ_j^{-1} for every $j \in [\![1, n-1]\!]$, these generators of $\hat{\Gamma}^{\circ}_{g,n}$ induce bijections of

$$[O_{\mu,\mathbf{c}}] \subset \chi_{g,n}(\mathrm{Aff}(\mathbb{C})).$$

Hence $[O_{\mu,\mathbf{c}}]$ is also stable under τ_i for every $i \in [\![1, n+1]\!]$ and σ_j for every $j \in [\![1, n-1]\!]$. We conclude that the orbit $\hat{\Gamma}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}] = \hat{\Gamma}^{\circ}_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$ is contained in the finite set $[O_{\mu,\mathbf{c}}]$. $\qquad\square$

7.4. **Non abelian case in higher genus.** We are now considering the case $g > 1$, and arbitrary $n \geq 0$. Recall that $\Lambda_{g,n}$ then contains the group

$$G := \langle \alpha_{g-1}, \beta_{g-1}, \alpha_g, \beta_g \rangle \subset \Lambda_{g,n}$$

and $\hat{\Gamma}^\circ_{g,n}$ contains a subgroup

$$H := \langle \tau_{2g-3}, \tau_{2g-2}, \tau_{2g-1}, \tau_{2g}, \tau_{3g-1} \rangle \subset \hat{\Gamma}^\circ_{g,n}$$

acting on G as summarized by Table 7.

τ_{2k}	$k \in [\![g-1, g]\!]$	α_k	$\mapsto$	$\alpha_k \beta_k$
τ_{2k-1}	$k \in [\![g-1, g]\!]$	β_k	$\mapsto$	$\beta_k \alpha_k$
τ_{3g-1}		α_g	$\mapsto$	$\Theta^{-1} \alpha_g$
		α_{g-1}	$\mapsto$	$\alpha_{g-1} \Theta$
		β_{g-1}	$\mapsto$	$\Theta^{-1} \beta_{g-1} \Theta$
	where	Θ	$:=$	$\alpha_g \beta_g^{-1} \alpha_g^{-1} \beta_{g-1}$

TABLE 7.

Lemma 7.4.1 (Elimination criterion). *Let $g \geq 2$. Let $\rho : \Lambda_{g,n} \to \mathrm{Aff}(\mathbb{C})$ be a representation of the following "weak prepared form"*

$$(19) \qquad \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a \\ z + b \end{pmatrix},$$

where μ is a root of unity, $a, b \in \mathbb{C}$, $m_g, m_{g-1} \in \mathbb{Z}$ and $\mu^{m_g} \neq 1$. If $\hat{\Gamma}_{g,n} \cdot [\rho]$ is finite, then the conditions of Table 8 are fulfilled.

$\mu^{m_{g-1}}$	$=$	$\frac{1}{\mu^{m_g}}$
a	$=$	0
b	$=$	0

TABLE 8.

Proof. Note that if two representations ρ, ρ' of the form (19) are conjugated, then they their restrictions to G are equal. Assume

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a \\ z + b \end{pmatrix}$$

Now consider the action of τ_{2g-2}^{-k} for $k \in \mathbb{Z}$:

$$\tau_{2g-2}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a + k \cdot \mu^{m_{g-1}} b \\ z + b \end{pmatrix}$$

Since the suborbit $(\tau_{2g-2}^{-k} \cdot [\rho])_k$ is supposed to take finitely many values, we have $\boxed{b = 0}$.

Now consider the action of τ_{3g-1}^{-k}. We have

$$\tau_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z + k\mu^{m_g} \\ z + 1 \\ \mu^{m_{g-1}} z + a - k\mu^{m_g + m_{g-1}} \\ z \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \mu^{m_{g-1}} z + a - k \cdot \frac{\mu^{2m_g + m_{g-1}} - \mu^{m_g}}{\mu^{m_g} - 1} \\ z \end{pmatrix}.$$

As the corresponding suborbit is supposed to be finite, we have $\boxed{\mu^{m_g-1} = \mu^{-m_g}}$.

In order to conclude, consider $\tilde\tau_{3g-1} = \tau^{-1} \circ \tau_{3g-1} \circ \tau$, where $\tau := \tau_{2g-3} \circ \tau_{2g} \circ \tau_{2g-1}^{-1} \circ \tau_{2g}$. We have

$$\tilde\tau_{3g-1_*}^k : \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \mapsto \begin{pmatrix} \widetilde\Theta^{-k} \alpha_g \widetilde\Theta^k \\ \beta_g \widetilde\Theta^k \\ \alpha_{g-1} \widetilde\Theta^k \\ \widetilde\Theta^{-k} \beta_{g-1} \alpha_{g-1}^{-1} \widetilde\Theta^k \alpha_{g-1} \widetilde\Theta^k \end{pmatrix},$$

where $\widetilde\Theta := \tau_*^{-1} \Theta = \alpha_g^{-1} \beta_{g-1} \alpha_{g-1}^{-1}$. We have

$$\rho\left(\widetilde\Theta^k\right) = z - k \cdot a.$$

Hence, modulo conjugation by $\rho\left(\widetilde\Theta^k\right)$, we have

$$\tilde\tau_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \approx \rho \begin{pmatrix} \alpha_g \\ \widetilde\Theta^k \beta_g \\ \widetilde\Theta^k \alpha_{g-1} \\ \beta_{g-1} \alpha_{g-1}^{-1} \widetilde\Theta^k \alpha_{g-1} \end{pmatrix} = \begin{pmatrix} \mu^{m_g} z \\ z + 1 - k \cdot a \\ \frac{1}{\mu^{m_g}} z + (1 - k) \cdot a \\ z - k \cdot a \mu^{m_g} \end{pmatrix}.$$

Provided $1 - k \cdot a \neq 0$ (which is the case for an infinite number of $k \in \mathbb{Z}$ anyway), we obtain

$$\tilde\tau_{3g-1}^{-k} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} \approx \begin{pmatrix} \mu^{m_g} z \\ z + 1 \\ \frac{1}{\mu^{m_g}} z + \frac{(1-k) \cdot a}{1 - k \cdot a} \\ z - \frac{k \cdot a}{1 - k \cdot a} \mu^{m_g} \end{pmatrix}.$$

Again by finiteness, we have $\boxed{a = 0}$. $\qquad\qquad\square$

Proposition 7.4.2 (Non abelian representations for $g > 1$). *Assume $g \geq 2$ and $n \geq 0$. Let $\rho : \Lambda_{g,n} \to \mathrm{Aff}(\mathbb{C})$ be a representation with non abelian image. Then the orbit $\hat\Gamma_{g,n} \cdot [\rho]$ is infinite.*

Proof. Let $g \geq 2$ and let ρ be a representation with finite orbit modulo conjugation. Let us assume for a contradiction that $\rho(\Lambda_{g,n})$ is non-abelian. We may then assume that ρ is of "prepared form" as in Lemma 7.2.2. In particular, we may assume that ρ is of "weak prepared form" and hence, by Lemma 7.4.1, ρ satisfies the conditions of Table 8. In other words, we may assume that ρ is of the following form.

$$\rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z \\ z + 1 \\ \frac{1}{\mu} z \\ z \end{pmatrix},$$

where $\mu \neq 1$ is a root of unity. We have

$$\tau_{3g-1}^{-1} \cdot \rho \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z + \mu \\ z + 1 \\ \frac{1}{\mu} z - 1 \\ z \end{pmatrix} ; \qquad \tau_{2g} \cdot (\tau_{3g-1}^{-1} \cdot \rho) \begin{pmatrix} \alpha_g \\ \beta_g \\ \alpha_{g-1} \\ \beta_{g-1} \end{pmatrix} = \begin{pmatrix} \mu z \\ z + 1 \\ \frac{1}{\mu} z - 1 \\ z \end{pmatrix}.$$

Now $\tau_{2g} \circ \tau_{3g-1}^{-1} \cdot \rho$ is also of weak prepared form, but is not compatible with the elimination criterion of Table 8, whence the contradiction. $\qquad\qquad\square$

8. Reducible rank 2 representations with finite orbit

Theorem B concerns representations $\rho : \Lambda_{g,n} \to \mathrm{GL}_2\mathbb{C}$ that are reducible, *i.e.* that globally fix a line in $\mathbb{C}^2$. A particular case of reducible rank 2 representations are those that are totally reducible, *i.e.* that globally fix two distinct lines in $\mathbb{C}^2$. In Theorem B1, we will prove the statement in the totally reducible case, and in Theorem B2, we will prove it in the reducible but not totally reducible case. The juxtaposition of these two results yields Theorem B. First, we are going to estimate the size of finite orbits of conjugacy classes of affine representation under the pure mapping class group and prove the reduction to the affine case.

8.1. The size of some finite orbits.
Note that since $\mathbb{C}^*$ is abelian, we have a natural identification between scalar representations and their conjugacy classes: $\chi_{g,n}(\mathbb{C}^*) = \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$. In particular, the pure mapping class group $\Gamma_{g,n}$ acts on $\mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$.

Proposition 8.1.1. *Let $g > 0, n \geq 0$. Let $\lambda \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ be a scalar representation with finite image. Then*

$$\mathrm{card}(\mathrm{Im}(\lambda))^{2g-1} \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda) \leq \mathrm{card}(\mathrm{Im}(\lambda))^{2g} \tag{20}$$

Proof. Since $\mathrm{Im}(\lambda)$ is finite, there is a root of unity $\mu \in \mathbb{C}^*$ such that $\mathrm{Im}(\lambda) = \mu^{\mathbb{Z}}$. For each $j \in [\![1, n]\!]$, choose an integer $m_j \in \mathbb{Z}$ such that $\lambda(\gamma_j) = \mu^{m_j}$. Denote $N := \mathrm{order}(\mu)$ and

$$O_\lambda := \left\{ (\mu^{k_g}, \mu^{\ell_g}, \dots, \mu^{k_1}, \mu^{\ell_1}) \,\middle|\, \begin{array}{l} \mathbf{k} := (k_g, \dots, k_1) \in \mathbb{Z}^g, \quad \ell := (\ell_g, \dots, \ell_1) \in \mathbb{Z}^g, \\ \gcd(k_g, \dots, k_1, \ell_g, \dots, \ell_1, m_1, \dots, m_n, N) = 1 \end{array} \right\}.$$

Note that to any element $(\mu^{k_g}, \mu^{\ell_g}, \dots, \mu^{k_1}, \mu^{\ell_1}) \in O_\lambda$ we can associate a well defined representation $\lambda' \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ by setting $\lambda'(\alpha_i) = \mu^{k_i}$; $\lambda'(\beta_i) = \mu^{\ell_i}$ for all $i \in [\![1, g]\!]$ and $\lambda'(\gamma_j) = \mu^{m_j}$ for all $j \in [\![1, n]\!]$. In that sense, we can see O_λ as a subset of $\mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$.

We claim that $\Gamma_{g,n} \cdot \lambda = O_\lambda$. Notice that this claim implies (20). Indeed, the second inequality is obvious, and the first one follows from the fact that if we set for example $k_g = 1$, then we can choose all other exponents freely.

Let us now prove the claim. We clearly have $\lambda \in O_\lambda$. Each pure element τ of $\hat{\Gamma}^\circ_{g,n}$ transforms the generators γ_i into conjugates $\zeta_i^{-1}\gamma_i\zeta_i$. Since $\mathbb{C}^*$ is abelian, this implies that for any representation λ' corresponding to an element of O_λ, we have $(\tau \cdot \lambda')(\gamma_i) = \lambda'(\gamma_i) = \mu^{m_i}$. Consequently, $\Gamma_{g,n} \cdot O_\lambda = O_\lambda$ and in particular $\Gamma_{g,n} \cdot \lambda \subset O_\lambda$.

The orbits of $\hat{\Gamma}_{g,n}$ on $\chi_{g,n}(\mathbb{C}^*) = \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ are the ones of $\hat{\Gamma}^\circ_{g,n}$. Note that the subgroup $H := \langle \tau_i \mid i \in [\![1, 3g - 1 + \min(0, n-1)]\!] \rangle \subset \hat{\Gamma}^\circ_{g,n}$ is generated by pure elements. Translating Table 2 into an action of $\hat{\Gamma}^\circ_{g,n}$ on the powers of μ corresponding to the generators of $\Lambda_{g,n}$ then yields the following.

(a) For a given $(\tilde{k}_g, \dots, \tilde{k}_1) \in \{1, \dots, N\}^g$ such that $\gcd(\tilde{k}_g, \dots, \tilde{k}_1, m_1, \dots, m_n, N) = 1$, the subgroup $\langle \tau_{2i}, \tau_{2i-1} \mid i \in [\![1, g]\!] \rangle \subset H$ acts transitively on those elements of O_λ satisfying $\gcd(k_i, \ell_i) = \tilde{k}_i$ for all $i \in [\![1, g]\!]$ (see also the proof of Lemma 7.2.2).

(b) For all $\widetilde{\mathbf{k}} := (\tilde{k}_g, \dots, \tilde{k}_1) \in \{1, \dots, N\}^g$ such that $\gcd(\tilde{k}_g, \dots, \tilde{k}_1, m_1, \dots, m_n, N) = 1$, there is an element of the subgroup $\langle \tau_{2i}, \tau_{2i-1}, \tau_{2g+i'} \mid i \in [\![1, g]\!], i' \in [\![1, g-1]\!] \rangle \subset H$, which sends the element of O_λ given by $\mathbf{k} = (\gcd(\tilde{k}_g, \dots, \tilde{k}_1), 0, \dots, 0)$ and $\ell = \vec{0}$ to the element of O_λ given by $\mathbf{k} = \widetilde{\mathbf{k}}$ and $\ell = \vec{0}$.

(c) The subgroup $\langle \tau_{3g-1+j} \mid j \in [\![1, \min(0, n-1)]\!] \rangle \subset H$ acts transitively on those elements of O_λ satisfying $\ell = \vec{0}$ and $k_i = 0$ for all $i \in [\![1, g-1]\!]$.

Consequently, the pure subgroup H acts transitively on O_λ. This implies $\Gamma_{g,n} \cdot \lambda = O_\lambda$. $\quad\square$

Recall that we denote

$$\mathrm{Aff}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C}, \ a \neq 0 \right\}. \tag{21}$$

Proposition 8.1.2. *Let $g = 1$ and let $n > 0$. Let $\mu \in \mathbb{C}^*$ be a root of unity of order $N > 1$ and let $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$ with $\sum_{i=1}^n c_i = 1$. Consider the representation $\rho_{\mu, \mathbf{c}} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ defined by*

$$\rho_{\mu, \mathbf{c}}(\alpha_1) := \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}; \quad \rho_{\mu, \mathbf{c}}(\beta_1) := \begin{pmatrix} 1 & -\frac{1}{\mu - 1} \\ 0 & 1 \end{pmatrix}; \quad \rho_{\mu, \mathbf{c}}(\gamma_i) := \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \quad \forall i \in [\![1, n]\!].$$

Its orbit $\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}]$ is finite in $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$. More precisely, we have

(22) $$\phi(N)(2N - \phi(N)) \cdot N^{n'-1} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}]) \leq (N^2 - 1)N^{n'-1},$$

where $n' := \mathrm{card}\{i \in [\![1, n]\!] \mid c_i \neq 0\}$ and ϕ denotes the Euler totient function.

Remark 8.1.3. Observe that the estimate (22) yields an equality if N is a prime number.

Proof. For convenience we shall represent the elements of $\mathrm{Aff}(\mathbb{C})$ by degree one polynomials $az + b$, as in page 25. Denote

$$D_{\mathbf{c}} := \mu^{\mathbb{Z}} c_1 + \dots + \mu^{\mathbb{Z}} c_n; \qquad S_{\mu, d} := S_{\mu, d}^1 \cup S_{\mu, d}^2; \qquad R_{\mu, \mathbf{c}, d}; \qquad O_{\mu, \mathbf{c}}$$

as in the proof of Proposition 7.3.1. Moreover, denote

$$R_{\mu, \mathbf{c}, d}^{pure} := \left\{ \begin{pmatrix} z + \tilde{c}_1 \\ \vdots \\ z + \tilde{c}_n \end{pmatrix} \;\middle|\; \begin{array}{l} \tilde{c}_i \in \mu^{\mathbb{Z}} c_i \quad \forall i \in [\![1, n]\!] \\ d = \sum_{i=1}^n \tilde{c}_i \end{array} \right\}$$

$$O_{\mu, \mathbf{c}}^{pure} := \bigcup_{d \in D_{\mathbf{c}}} \left\{ (\varphi_\alpha, \varphi_\beta, \varphi_1, \dots, \varphi_n) \;\middle|\; \begin{pmatrix} \varphi_\alpha \\ \varphi_\beta \end{pmatrix} \in S_{\mu, d}, \; \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \in R_{\mu, \mathbf{c}, d}^{pure} \right\}.$$

We shall denote by $[O_{\mu, \mathbf{c}}]$ and $[O_{\mu, \mathbf{c}}^{pure}]$ the respective images of $O_{\mu, \mathbf{c}}$ and $O_{\mu, \mathbf{c}}^{pure}$ in $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$. By a slight refinement of the proof of Proposition 7.3.1, we have

(23) $$\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}] \subset [O_{\mu, \mathbf{c}}^{pure}].$$

Indeed, recall that the pure subgroup $\Gamma_{g,n}$ of $\hat{\Gamma}_{g,n}$ is the subgroup that respects the labellings of the punctures. Each pure element τ of $\hat{\Gamma}_{g,n}^\circ$ transforms the generators γ_i into conjugates $\zeta_i^{-1} \gamma_i \zeta_i$. As we have $\rho(\zeta_i) = \mu^m z + d$ for suitable $m \in \mathbb{Z}$, $d \in \mathbb{C}$, we deduce $(\tau \cdot \rho)(\gamma_i) = \mu^m c_i$. This proves the inclusion (23).

Moreover, using Table 6 page 30, we can check successively:

(a) as observed in the proof of Lemma 7.2.2, any element $\rho = [*_1, *_2, z + \tilde{c}_1, \dots z + \tilde{c}_n]$ of $[O_{\mu, c}]$ can be transformed into an element $\rho' = [z + d/(\mu - 1), \mu z, z + \tilde{c}_1, \dots, z + \tilde{c}_n]$ by an element of $\langle \tau_1, \tau_2 \rangle$, where $d = \sum_{i=1}^n \tilde{c}_i$;

(b) for any $j \in [\![1, n]\!]$, by the action of an element of $B_n = \langle \sigma_i \mid i \in [\![1, n-1]\!] \rangle$, the element ρ' can be transformed into $[z + d/(\mu - 1), \mu z, z + c'_1, \dots, z + c'_n]$, where $c'_1 = \tilde{c}_j$, $c'_j = \tilde{c}_1$ and $c'_i = \tilde{c}_i$ for $i \neq 1, j$;

(c) for any $m_j \in \mathbb{Z}$, using a power of $\tilde{\tau}_3$, we transform this latter element into $[z + d'/(\mu - 1), \mu z, z + \mu^{m_j} c'_1, \dots, z + c'_n]$, where $d' = d + (\mu^{m_j} - 1)\tilde{c}_j$.

(d) reusing an element of B_n, one gets ρ' altered only by replacing $\tilde{c}_j$ by $\mu^{m_j} \tilde{c}_j$ and d by d'.

This allows to infer that any element $\rho = [*_1, *_2, z + \tilde{c}_1, \dots z + \tilde{c}_n]$ of $[O_{\mu, c}]$ can be transformed into $[z + 1/(\mu - 1), \mu z, z + c_1, \dots, z + c_n]$ by a suitable element of $\hat{\Gamma}_{g,n}$. Reusing (a), we deduce $\hat{\Gamma}_{g,n} \cdot [\rho_{\mu, \mathbf{c}}] = [O_{\mu, \mathbf{c}}]$. The conjunction of this equality and the inclusion (23) yields

$$\Gamma_{g,n} \cdot [\rho_{\mu, \mathbf{c}}] = [O_{\mu, \mathbf{c}}^{pure}].$$

Denote by $[S_{\mu, d}]_t$ and $[O_{\mu, \mathbf{c}}^{pure}]_t$ the set of equivalence classes of $S_{\mu, d}$ and $O_{\mu, \mathbf{c}}^{pure}$ respectively modulo conjugation by translations. For each $d \in D_{\mathbf{c}}$, the cardinality of $[S_{\mu, d}]_t$ equals the cardinality of

$$K_N := \left\{ (k_1, k_2) \in [\![1, N]\!]^2 \;\middle|\; \gcd(k_1, k_2, N) = 1 \right\}.$$

Indeed, for $\{i, j\} = \{1, 2\}$, the elements of $S^i_{\mu,d}$ that are not conjugated by a translation to an element of $S^j_{\mu,d}$ are precisely those corresponding to $k_j = 0$ and $\gcd(k_i, N) = 1$. We can estimate

$$\phi(N)(2N - \phi(N)) \leq \mathrm{card}([S_{\mu,d}]_t) \leq N^2 - 1\,.$$

These inequalities are readily derived from the inclusions

$$\big\{(k_1, k_2) \in [\![1, N]\!]^2 \mid \gcd(k_1, N) = 1 \text{ or } \gcd(k_2, N) = 1\big\} \subset K_N \subset [\![1, N]\!]^2 \setminus \{(0, 0)\}\,.$$

On the other hand, conjugations by translations act trivially on $R^{pure}_{\mu,\mathbf{c},d}$. By definition of n', we have

$$\mathrm{card}\left(\bigcup_{d \in D_{\mathbf{c}}} R^{pure}_{\mu,\mathbf{c},d}\right) = N^{n'}\,.$$

We deduce

$$\phi(N)(2N - \phi(N))N^{n'} \leq \mathrm{card}[O^{pure}_{\mu,\mathbf{c}}]_t \leq (N^2 - 1)N^{n'}\,.$$

The condition $\sum_{i=1}^n c_i = 1$ ensures $n' > 0$. In particular, there is an index $i_0 \in [\![1, n]\!]$ such that $c_{i_0} \neq 0$. Up to conjugation by powers of the linear transformation μz, we can normalize $\tilde{c}_{i_0} = c_{i_0}$ for each element in $[O^{pure}_{\mu,\mathbf{c}}]_t$, which yields $\mathrm{card}[O^{pure}_{\mu,\mathbf{c}}] = \frac{1}{N}\,\mathrm{card}[O^{pure}_{\mu,\mathbf{c}}]_t$. $\qquad\square$

8.2. **Reduction to the affine case.** Consider the natural inclusion

$$\iota : (\mathbb{C}^*)^2 \to \mathrm{GL}_2\mathbb{C}\,; \quad (a_1, a_2) \mapsto \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)\,.$$

Lemma 8.2.1. *Let $g \geq 0, n \geq 0$. Let $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ be a totally reducible representation. Then there are scalar representations $\lambda_1, \lambda_2 \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ such that $[\rho] = [\iota_*(\lambda_1, \lambda_2)] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$. Moreover, we have*

$$(24) \quad \frac{1}{2}\max\{\mathrm{card}(\Gamma_{g,n} \cdot \lambda_i) \mid i \in \{1, 2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda_1) \cdot \mathrm{card}(\Gamma_{g,n} \cdot \lambda_2)\,.$$

In particular, the following are equivalent:

- $\Gamma_{g,n} \cdot [\rho]$ *is a finite subset of* $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$.
- $\Gamma_{g,n} \cdot \lambda_i$ *is a finite subset of* $\mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ *for* $i = 1, 2$.

Proof. The image of the map ι_* from $\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$ to $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ is obviously the set of conjugacy classes of totally reducible representations.

Note that by definition, the action of $\Gamma_{g,n}$ on $\iota_*\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$, induced by the action on $\mathrm{Hom}(\Lambda_{g,n}, (\mathbb{C}^*)^2)$, coincides with the action of $\Gamma_{g,n}$ on $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$. Moreover, we have

$$(25) \quad \frac{1}{2} \cdot \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2)) \leq \mathrm{card}(\Gamma_{g,n} \cdot [\iota_*(\lambda_1, \lambda_2)]) \leq \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2))\,.$$

Indeed, the second inequality is obvious, and the first one follows from the fact that if $[\iota_*(\lambda_1, \lambda_2)] = [\iota_*(\lambda_1', \lambda_2')]$ then either $(\lambda_1, \lambda_2) = (\lambda_1', \lambda_2')$ or $(\lambda_1, \lambda_2) = (\lambda_2', \lambda_1')$.

On the other hand, we have

$$(26) \quad \max\{\mathrm{card}(\Gamma_{g,n} \cdot \lambda_i) \mid i \in \{1, 2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot (\lambda_1, \lambda_2)) \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda_1) \cdot \mathrm{card}(\Gamma_{g,n} \cdot \lambda_2)\,.$$

We conclude by noticing that (25) and (26) imply (24). $\qquad\square$

Remark 8.2.2. The equality $[\rho] = [\iota_*(\lambda_1, \lambda_2)] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ in the above Lemma is commonly written as $\rho = \lambda_1 \oplus \lambda_2$. We adopted this notation in the statement of Theorem B, and we will use it in its proof.

Consider the natural inclusion

$$\upsilon : \mathrm{Aff}(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{C},\ a \neq 0\right\} \hookrightarrow \mathrm{GL}_2\mathbb{C}\,.$$

Lemma 8.2.3. *Let $g > 0, n \geq 0$ and let $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ be a reducible but not totally reducible representation. Then there is a unique $\lambda \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ and a unique conjugacy class $[\rho_{\mathrm{Aff}}] \in \chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ such that $[\rho] = [\lambda \otimes \iota_* \rho_{\mathrm{Aff}}] \in \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$. Moreover, we have*

$$(27) \quad \max\{\mathrm{card}(\Gamma_{g,n} \cdot \lambda), \mathrm{card}(\Gamma_{g,n} \cdot [\rho_{\mathrm{Aff}}])\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\Gamma_{g,n} \cdot \lambda) \cdot \mathrm{card}(\Gamma_{g,n} \cdot [\rho_{\mathrm{Aff}}]).$$

In particular, the following are equivalent.

- $\Gamma_{g,n} \cdot [\rho]$ *is a finite subset of* $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$.
- $\Gamma_{g,n} \cdot \lambda$ *is a finite subset of* $\mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ *and* $\Gamma_{g,n} \cdot [\rho_{\mathrm{Aff}}]$ *is a finite subset of* $\chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$.

Proof. The unique decomposition statement has been proven in Lemma 5.1.1 in Part A of the present paper. This Lemma also yields (27). □

8.3. Proof of Theorem B.

Theorem B1. *Let $g \geq 0, n \geq 0$. Let $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ be totally reducible, i.e. $\rho = \lambda_1 \oplus \lambda_2$ is a direct sum of scalar representations. The following are equivalent:*

- *the orbit $\Gamma_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ is finite.*
- *the subgroup $\mathrm{Im}(\rho)$ of $\mathrm{GL}_2\mathbb{C}$ has finite order.*

Moreover, if the orbit $\Gamma_{g,n} \cdot [\rho]$ is finite, then its size can be estimated as follows:

$$(28) \quad \frac{1}{2}\max\{\mathrm{card}(\mathrm{Im}(\lambda_i))^{2g-1} \mid i \in \{1,2\}\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq \mathrm{card}(\mathrm{Im}(\rho))^{2g}.$$

Proof. From Lemma 8.2.1, finiteness of the orbit $\Gamma_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ is tantamount to the finiteness of the orbits $\Gamma_{g,n} \cdot \lambda_i \subset \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ for $i = 1, 2$. Since $[\hat{\Gamma}_{g,n} : \Gamma_{g,n}] = n!$ is finite, finiteness of $\Gamma_{g,n} \cdot \lambda_i$ is equivalent to the finiteness of $\hat{\Gamma}_{g,n} \cdot \lambda_i$. Proposition 7.1.2 establishes that $\hat{\Gamma}_{g,n} \cdot \lambda_i$ is finite if and only if $\mathrm{Im}(\lambda_i)$ is finite. This proves the equivalence in the statement.

The left inequality in (28) follows from Lemma 8.2.1 and Proposition 8.1.1.

Each pure element τ of $\hat{\Gamma}_{g,n}^{\circ}$ transforms the generators γ_i into conjugates. By abelianity, for $\rho' = \tau \cdot \rho$ and any $i \in [\![1, n]\!]$, we get $\rho'(\gamma_i) = \rho(\gamma_i)$. We deduce the right inequality in (28). □

Theorem B2. *Let $g > 0, n \geq 0$ and let $\rho \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{GL}_2\mathbb{C})$ be a reducible but not totally reducible representation. The following are equivalent:*

- *the orbit $\Gamma_{g,n} \cdot [\rho]$ in $\chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ is finite.*
- $g = 1, n > 0$, *there are a scalar representation $\lambda \in \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ and an affine representation $\rho_{\mu, \mathbf{c}} \in \mathrm{Hom}(\Lambda_{g,n}, \mathrm{Aff}(\mathbb{C}))$ as in Proposition 8.1.2, such that*

$$[\rho] \in \Gamma_{g,n} \cdot [\lambda \otimes \rho_{\mu, \mathbf{c}}].$$

Moreover, if the orbit $\Gamma_{g,n} \cdot [\rho]$ is finite, then its size can be estimated as follows:

$$(29) \quad \max\left\{N_2, \phi(N)(2N - \phi(N))N^{n'-1}\right\} \leq \mathrm{card}(\Gamma_{g,n} \cdot [\rho]) \leq (N^2 - 1)N^{n'-1}N_2^2,$$

where $n' := \mathrm{card}\{i \in [\![1, n]\!] \mid \rho(\gamma_i) \notin \mathbb{C}^* I_2\}$, $N := \mathrm{order}(\mu)$, $N_2 = \mathrm{card}(\mathrm{Im}(\lambda))$ and ϕ is the Euler totient function.

Proof. From Lemma 8.2.3, we know that $[\rho]$ admits a unique decomposition $[\rho] = [\lambda \otimes \rho_{\mathrm{Aff}}]$, where λ is a scalar representation and ρ_{Aff} is an affine representation. Moreover, since ρ is not totally reducible, the affine representation ρ_{Aff} has non abelian image. Still by Lemma 8.2.3, the orbit $\Gamma_{g,n} \cdot [\rho] \subset \chi_{g,n}(\mathrm{GL}_2\mathbb{C})$ is finite if and only if the orbits $\Gamma_{g,n} \cdot \lambda \subset \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ and $\Gamma_{g,n} \cdot [\rho_{\mathrm{Aff}}] \subset \chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ are finite. From Proposition 7.1.2, the orbit $\Gamma_{g,n} \cdot \lambda \subset \mathrm{Hom}(\Lambda_{g,n}, \mathbb{C}^*)$ is finite if and only if λ has finite image. Since $[\hat{\Gamma}_{g,n} : \Gamma_{g,n}] = n!$ is finite, finiteness of the orbit $\Gamma_{g,n} \cdot [\rho_{\mathrm{Aff}}] \subset \chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$ is equivalent to the finiteness of the orbit $\hat{\Gamma}_{g,n} \cdot [\rho_{\mathrm{Aff}}] \subset \chi_{g,n}(\mathrm{Aff}(\mathbb{C}))$. Since ρ_{Aff} has non abelian image, by the Propositions 7.4.2 and 7.3.1, the finiteness of the latter orbit is equivalent to $g = 1, n > 0$ and $[\rho_{\mathrm{Aff}}] \in \hat{\Gamma}_{g,n} \cdot [\rho_{\mu, \mathbf{c}'}]$ for a convenient choice of a non trivial root of unity μ and $\mathbf{c}' = (c'_1, \ldots, c'_n) \in \mathbb{C}^n$ with $\sum_{i=1}^{n} c'_i = 1$. Composing with a suitable

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element of B_n shows this is also equivalent to $[\rho_{\mathrm{Aff}}] \in \Gamma_{g,n} \cdot [\rho_{\mu,\mathbf{c}}]$, for some $\mathbf{c} \in \mathfrak{S}_n \cdot \mathbf{c}'$. This proves the equivalence in the statement.

The estimate (29) follows from Lemma 8.2.3, Proposition 8.1.1 and Proposition 8.1.2, taken into account that

$$\mathrm{card}\{i \in [\![1, n]\!] \mid \rho(\gamma_i) \notin \mathbb{C}^* I_2\} = \mathrm{card}\{i \in [\![1, n]\!] \mid \rho_{\mathrm{Aff}}(\gamma_i) \neq \mathrm{id}\} = \mathrm{card}\{i \in [\![1, n]\!] \mid c_i \neq 0\}.$$

$\square$

References

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011, With a contribution by Joseph Daniel Harris. MR 2807457

[Art25] Emil Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), no. 1, 47–72. MR 3069440

[Bai62] Walter L. Baily, Jr., *On the automorphism group of a generic curve of genus > 2*, J. Math. Kyoto Univ. **1** (1961/1962), no. 101–108; correction, 325. MR 0142552

[CM16] Gaël Cousin and Delphine Moussard, *Finite braid group orbits in Aff(C)-character varieties of the punctured sphere*, To appear in International Math. Research Notices (2016), 1–55.

[Cor08] Maurizio Cornalba, *Erratum: "On the locus of curves with automorphisms" [Ann. Mat. Pura Appl. (4) **149** (1987), 135–151; mr0932781]*, Ann. Mat. Pura Appl. (4) **187** (2008), no. 1, 185–186. MR 2346015

[Cou15] Gaël Cousin, *Projective representations of fundamental groups of quasiprojective varieties: a realization and a lifting result*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 2, 155–159. MR 3300949

[Cou17] Gaël Cousin, *Algebraic isomonodromic deformations of logarithmic connections on the Riemann sphere and finite braid group orbits on character varieties*, Math. Ann. **367** (2017), no. 3-4, 965–1005. MR 3623216

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR 1841091

[Del70] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970. MR 0417174 (54 #5232)

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125

[Heu10] Viktoria Heu, *Universal isomonodromic deformations of meromorphic rank 2 connections on curves*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 2, 515–549. MR 2667785

[Hub06] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006, Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle. MR 2245223

[Hus94] Dale Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994. MR 1249482

[Kri02] I. Krichever, *Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations*, Mosc. Math. J. **2** (2002), no. 4, 717–752, 806. MR 1986088

[Lic64] W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge Philos. Soc. **60** (1964), 769–778. MR 0171269

[Mal83] B. Malgrange, *Sur les déformations isomonodromiques. I et II.*, Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 401–438. MR MR728432 (85m:58094b)

[Mon62] Paul Monsky, *The automorphism groups of algebraic curves*, ProQuest LLC, Ann Arbor, MI, 1962, Thesis (Ph.D.)–The University of Chicago. MR 2611511

[Poo00] Bjorn Poonen, *Varieties without extra automorphisms. I. Curves*, Math. Res. Lett. **7** (2000), no. 1, 67–76. MR 1748288

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42. MR 0082175

LAREMA, 2 Boulevard Lavoisier, 49045 Angers, France
E-mail address: cousin@math.univ-angers.fr

IRMA, 7 rue René Descartes, 67084 Strasbourg, France
E-mail address: heu@math.unistra.fr