BETTI NUMBERS AND PSEUDOEFFECTIVE CONES IN 2-FANO VARIETIES

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ABSTRACT. The 2-Fano varieties, defined by De Jong and Starr, satisfy some higher dimensional analogous properties of Fano varieties. We propose a definition of (weak) k-Fano variety and conjecture the polyhedrality of the cone of pseudoeffective k-cycles for those varieties in analogy with the case k=1. Then, we calculate some Betti numbers of a large class of k-Fano varieties to prove some special case of the conjecture. In particular, the conjecture is true for all 2-Fano varieties of index $\geq n-2$, and also we complete the classification of weak 2-Fano varieties answering Questions 39 and 41 in [AC13].

1. Introduction

The study of cones of curves or divisors on smooth complex projective varieties X is a classical subject in Algebraic Geometry and is still an active research topic. However, little is known when we pass to higher dimensions. For example it is a classical result that the cone of nef divisors is contained in the cone of pseudoeffective divisors, but in general $\operatorname{Nef}_k(X) \subseteq \overline{\operatorname{Eff}}_k(X)$ is not true. These phenomena can appear only if $\dim X \geq 4$ and very few examples are known. In particular [DELV11] gives two examples of such varieties. Furthermore [Ott15] proves that if X is the variety of lines of a very general cubic fourfold in \mathbb{P}^5 , then the cone of pseudoeffective 2-cycles on X is strictly contained in the cone of nef 2-cycles.

The central subject of this paper will be the k-Fano varieties.

Definition 1.1. A smooth Fano variety X is k-Fano if the s^{th} Chern character $ch_s(X)$ is positive (see Definition 2.3) for $1 \le s \le k$, and weak k-Fano for k > 1 if X is (k-1)-Fano and $ch_k(X)$ is nef.

There is a large interest in studying varieties with positive Chern characters. For example varieties with positive $ch_1(X)$ are Fano, hence uniruled, that is there is a rational curve through a general point. Fano varieties with positive second Chern character were introduced by J. de Jong and J. Starr in [dJS06, dJS07]. They proved a (higher dimensional) analogue of this result: weak 2-Fano varieties of pseudo-index at least 3 have a rational surface through a general point. Furthermore if X is weak 3-Fano then there is a rational threefold through a general point of X (under some hypothesis on the polarized minimal family of rational curves through a general point of X, [AC12, Theorem 1.5(3)]).

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Another problem concerns how the geometry of the cones of pseudoeffective k-cycles depends on the positivity of the Chern characters $ch_s(X)$. Mori's Cone Theorem resolves this problem for k = 1: the positivity of $ch_1(X)$ implies the polyhedrality of the cone of pseudoeffective 1-cycles and the extremal rays are spanned by classes of rational curves. By Kleiman's Theorem, a variety with positive $ch_1(X)$ is just a Fano variety, that is with $c_1(X)$ ample, but this is not enough, in general, for the polyhedrality of cones of pseudoeffective k-cycles for k > 1: Tschinkel showed a Fano variety where $\text{Eff}_2(X)$ has infinitely many extremal rays. Therefore more positivity is needed in order to obtain polyhedrality of cones of pseudoeffective k-cycles for k > 1.

In this paper we investigate a possible way of generalizing Mori's result:

Conjecture 1.2. If X is k-Fano, then $\overline{\mathrm{Eff}}_k(X)$ is a polyhedral cone.

The computing of the fourth Betti number is enough to show the polyhedrality of some of the cones of 2-cycles for a large class of varieties: complete intersections in weighted projective spaces, rational homogeneous varieties and most complete intersections in them, etc. This allows us to test the conjecture for many 2-Fano varieties, and in particular we prove that it holds for del Pezzo and Mukai varieties. Using the classification of Araujo-Castravet, we also prove the following.

Theorem 1.3. Let X be a n-dimensional 2-Fano variety with $i_X \geq n-2$. Then $\overline{\mathrm{Eff}}_2(X)$ is polyhedral. Also, $\overline{\mathrm{Eff}}_3(X)$ is polyhedral with the possible exception of the complete intersection of type (2,2) in \mathbb{P}^8 .

In particular, Conjecture 1.2 is true for any n-dimensional k-Fano variety with $i_X \ge n-2$ and k=2,3.

Let X be a complete intersection in G(2,5) or G(2,6) with two hyperplanes under the Plücker embedding. Araujo and Castravet proved that X is not 2-Fano, but questioned if it is weak 2-Fano [AC13, Proposition 32 and Questions 39,41]. In [dA15, Corollary 5.1] it is proved that a general such X is not 2-Fano by showing that there exists an effective surface S such that $[i(S)] = \sigma_{1,1}^{\vee}$, where i is the inclusion. In this circumstance we can prove that all the smooth complete intersections of this type are not weak 2-Fano, and this completes the classification given in [AC13, Theorem 3 and 4].

Theorem 1.4. Let Y = G(2,5) or G(2,6), let X be a smooth complete intersection of type (1,1) in Y under the Plücker embedding. Then X is not weak 2-Fano.

These ideas can be improved in three very promising directions: to generalize Tschinkel's example to higher dimensions, to prove the conjecture for some Fano 4-folds of index 1, and to use minimal families of rational curves to prove the conjecture for other 2-Fano's.

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2. General facts about cycles

A variety is a reduced and irreducible algebraic scheme over \mathbb{C} . Throughout this paper we will use the following.

Notation.

- X is a variety of dimension $n \geq 4$.
- k is an integer such that $1 \le k \le n-1$.
- $H_i(X,G)$ and $H^i(X,G)$ are the singular homology and cohomology groups of X for $1 \le i \le 2n$ and coefficients in a group G.
- $b_i(X)$ is the i^{th} Betti number of X for $1 \leq i \leq 2n$, that is the rank of $H_i(X,\mathbb{Z})$ or of $H^i(X,\mathbb{Z})$.
- $-Z_k(X)$ is the group of k-cycles with integer coefficients.
- $\operatorname{Rat}_k(X)$ is the group of k-cycles rationally equivalent to zero.
- $A_k(X)$ is the Chow group of k-cycles on X, that is $A_k(X) = Z_k(X)/Rat_k(X)$.
- $-A_*(X) = \bigoplus_{k=0}^n A_k(X)$ is the Chow ring of X.
- $\operatorname{Alg}_k(X)$ is the group of k-cycles algebraically equivalent to zero.
- $\operatorname{Hom}_k(X)$ is the group of k-cycles homologically equivalent to zero, that is the kernel of the cycle map $cl: Z_k(X) \to H_{2k}(X, \mathbb{Z})$.
- $\operatorname{Num}_k(X)$ is the group of cycles numerically equivalent to zero, that is the group of cycles $\alpha \in Z_k(X)$ such that $P \cdot cl(\alpha) = 0$ for all polynomials P in Chern classes of vector bundles on X.
- $-N_k(X)$ is the quotient group $Z_k(X)/\operatorname{Num}_k(X)$, and $N_k(X)_{\mathbb{R}} := N_k(X) \otimes \mathbb{R}$.
- $\operatorname{Eff}_k(X) \subseteq N_k(X)_{\mathbb{R}}$ is the cone generated by numerical classes of effective k-cycles.
- Let $s \ge 1$ be an integer. The s^{th} Chern character of X, $ch_s(X)$, is the homogeneous part of degree s of the total Chern character of X. For example, if $c_i(X)$ are the Chern classes of X, then $ch_1(X) = c_1(X)$, $ch_2(X) = \frac{1}{2}(c_1^2(X) 2c_2(X))$, $ch_3(X) = \frac{1}{6}(c_1^3(X) 4c_1(X)c_2(X) + 3c_3(X))$

We will often use the following well-known facts:

Remark 2.1. There is a chain of inclusions [Ful84, p.374]

$$\operatorname{Rat}_k(X) \subseteq \operatorname{Alg}_k(X) \subseteq \operatorname{Hom}_k(X) \subseteq \operatorname{Num}_k(X) \subseteq Z_k(X)$$

that gives rise to a diagram

$$(2.1) A_k(X) \longrightarrow Z_k(X)/\operatorname{Alg}_k(X) \longrightarrow Z_k(X)/\operatorname{Hom}_k(X) \xrightarrow{\pi_k} N_k(X)$$

$$\downarrow \qquad \qquad \qquad \qquad H_{2k}(X, \mathbb{Z})$$

We set

$$(2.2) \pi_{k,\mathbb{R}}: Z_k(X)/_{\mathrm{Hom}_k(X)} \otimes \mathbb{R} \to N_k(X)_{\mathbb{R}}$$

the tensor product of π_k and $id_{\mathbb{R}}$.

Remark 2.2. By linearity of the intersection product, $N_k(X)$ is torsion free. When X is smooth, the intersection product gives a perfect pairing [Ful84, Definition 19.1]

$$N_k(X)_{\mathbb{R}} \otimes N_{n-k}(X)_{\mathbb{R}} \to \mathbb{R}.$$

Definition 2.3. Let X be a smooth variety. A class $\alpha \in N_k(X)_{\mathbb{R}}$ is *positive* if $\alpha \cdot \beta > 0$ for every $\beta \in \overline{\mathrm{Eff}}_{n-k}(X) \setminus \{0\}$, and it is *nef* if $\alpha \cdot \beta \geq 0$ for every $\beta \in \overline{\mathrm{Eff}}_{n-k}(X)$. The cone generated by nef classes of k-cycles is $\mathrm{Nef}_k(X)$.

Kleiman's criterion for amplitude [Laz04a, Theorem 1.4.29] states that the cone of positive (n-1)-cycles is exactly the cone of numerical classes of ample divisors.

Lemma 2.4. Let X be a projective variety. Then

- (1) If either $\operatorname{rk} A_k(X) = 1$ or $b_{2k}(X) = 1$, then $\overline{\operatorname{Eff}}_k(X)$ is a half-line.
- (2) If either $\operatorname{rk} A_k(X) = 2$ or $b_{2k}(X) = 2$, then $\overline{\operatorname{Eff}}_k(X)$ is either a half-line or it is spanned by two extremal rays.

Proof. In the first case, by diagram (2.1), we have a surjection $\mathbb{Z} \to N_k(X)$ and, as $N_k(X)$ is torsion-free, it must be $N_k(X) \cong \mathbb{Z}$. In the second case, again by diagram (2.1), there is a surjection $\mathbb{Z}^2 \to N_k(X)$ and then either $N_k(X) \cong \mathbb{Z}$ or $N_k(X) \cong \mathbb{Z}^2$. Since $\overline{\mathrm{Eff}}_k(X)$ generates $N_k(X)_{\mathbb{R}}$, it is either a half-line or it is spanned by two extremal rays, depending on the rank of $N_k(X)_{\mathbb{R}}$.

Remark 2.5. In a general, a variety X with $ch_k(X)$ positive may not be k-Fano. For example, in [Mum79] Mumford found a smooth surface S of general type with $ch_2(S) = \frac{3}{2}$.

3. Cycles on Fano Varieties

We study here the pseudoeffective cones of k-cycles on some well-known classes of Fano varieties.

3.1. Weighted projective spaces. Let $\mathbb{P}(\mathbf{w})$ be the weighted projective space where $\mathbf{w} = (w_0, ..., w_n) \in \mathbb{N}_0^n$.

Proposition 3.1. Let X be a n-dimensional smooth complete intersection in a weighted projective space. If $k \neq \frac{n}{2}$ then $b_{2k}(X) = 1$. In particular $\overline{\mathrm{Eff}}_k(X)$ is polyhedral.

Proof. Recall [Dim92, B13] that dim $H^{2i}(\mathbb{P}(\mathbf{w}), \mathbb{Q}) = 1$ for every $0 \le i \le \dim \mathbb{P}(\mathbf{w})$. By Lefschetz's Hyperplane Theorem [Dim92, B22] we have that $H^{2k}(X, \mathbb{Q}) \cong H^{2k}(\mathbb{P}(\mathbf{w}), \mathbb{Q})$ for 2k < n, then $b_{2k}(X) = 1$ for $k < \frac{n}{2}$. But $b_{2n-2k}(X) = b_{2k}(X)$, then it follows that, for $k \ne \frac{n}{2}$, $b_{2k}(X) = 1$ and by Lemma 2.4 that $\overline{\mathrm{Eff}}_k(X)$ is a half-line.

Furthermore, if X is a k-Fano complete intersection in a projective space, then we can solve Conjecture 1.2, even for weak Fano.

Theorem 3.2. Let X be a n-dimensional weak k-Fano complete intersection in a projective space. If $1 \le s \le k$, then $b_{2s}(X) \le 2$. In particular $\overline{\operatorname{Eff}}_s(X)$ is polyhedral.

Proof. Let X be of type $(d_1, ..., d_c)$ in \mathbb{P}^{n+c} , with $d_i \geq 2$ for $1 \leq i \leq c$. By Proposition 3.1, we can suppose n even and $s = \frac{n}{2}$. We know from [AC13, 3.3.1] that $ch_{\frac{n}{2}}(X)$ is nef if and only if $d_1^{\frac{n}{2}} + ... + d_c^{\frac{n}{2}} \leq n + c + 1$. Since $n \geq 4$, it follows easily that c = 1. On the other hand $d_1^{\frac{n}{2}} \leq n + 2$ is possible only for $d_1 = 2$, that is X is an n-dimensional quadric. But $b_n(X) = 2$ [Rei72, p.20] and the theorem follows by Lemma 2.4.

3.2. Rational homogeneous varieties. Let G be a reductive linear algebraic group defined over \mathbb{C} , B a Borel subgroup of G. We consider the set of simple B-positive roots and denote by S the corresponding set of reflections in the Weyl group W. Then the pair (W, S) is a Coxeter system in the sense of [Bou68, Chapitre IV, Définition 3]. Let $l: W \to \mathbb{N}_0$ be the length function relative to the system S of generators of W.

Furthermore we fix a subset Θ of S and denote by W_{Θ} the subgroup of W generated by Θ and by P a subgroup of G associated to Θ . Then the quotient G/P is a projective variety, which is called a rational homogeneous variety. Any rational homogeneous variety is a Fano variety [BH58], and the action of G on G/P by left multiplication is transitive. Let w_0 (respectively, w_{θ}) be the unique element of maximal length of W (respectively, W_{Θ}). A simple calculation shows that $\dim G/P = l(w_0) - l(w_{\theta})$. The element w_0 and w_{θ} are characterized by the property [Bou68, Chapitre IV, Exercise 22]

$$(3.1) l(ww_0) = l(w_0) - l(w), \forall w \in W$$

$$(3.2) l(ww_{\theta}) = l(w_{\theta}) - l(w), \forall w \in W_{\Theta}$$

that imply immediately $w_0^2=1$ and $w_\theta^2=1$. It follows that, for every $w\in W$

$$l(w_0w) = l((w_0w)^{-1}) = l(w^{-1}w_0^{-1}) = l(w^{-1}w_0) = l(w_0) - l(w^{-1}) = l(w_0) - l(w).$$

Furthermore, set $W^{\Theta} = \{w \in W/l(ws) = l(w) + 1 \ \forall s \in \Theta\}$. We have, for every $(w, \bar{w}) \in W^{\Theta} \times W_{\Theta}$,

$$(3.3) l(w\bar{w}) = l(w) + l(\bar{w}).$$

Proposition 3.3. Let X be a smooth n-dimensional variety and let G be an affine group which acts transitively on X. Suppose that, for every k = 1, ..., n-1, there exists a finite family of subvarieties $\{\Omega_a\}_{a \in I_k}$ of dimension k such that

- (1) $\langle \{ [\Omega_a] / a \in I_k \} \rangle = H_{2k}(X, \mathbb{Z}) \text{ or } A_k(X), \text{ and }$
- (2) $\forall a \in I_k, \exists b \in I_{n-k} \text{ such that } \Omega_c \cdot \Omega_b = \delta_{a,c} \ \forall c \in I_k.$

Then $\operatorname{Nef}_k(X) = \overline{\operatorname{Eff}}_k(X) = \operatorname{Eff}_k(X)$ is polyhedral and simplicial.

Proof. We will suppose that the classes of the subvarieties $\{\Omega_a\}_{a\in I_k}$ generate $H_{2k}(X,\mathbb{Z})$, the case $A_k(X)$ being similar. Let ω_a be the class of Ω_a in $N_k(X)$. Let $\gamma\in\operatorname{Nef}_k(X)$. By (2.2) there is a class $\beta\in Z_k(X)/\operatorname{Hom}_k(X)\otimes\mathbb{R}\subseteq H_{2k}(X,\mathbb{R})$ such that $\pi_{k,\mathbb{R}}(\beta)=\gamma$. By (1) we have that $\beta=\sum_{a\in I_k}\gamma_a[\Omega_a]$ and then $\gamma=\sum\gamma_a\pi_k([\Omega_a])=\sum\gamma_a\omega_a$. Let $a\in I_k$ and let $b\in I_{n-k}$ be as in (2). Then $\gamma\cdot\omega_b=\gamma_a\geq 0$ because γ is nef and ω_b is effective. Therefore $\gamma\in\operatorname{Eff}_k(X)$, then $\operatorname{Nef}_k(X)\subseteq\operatorname{Eff}_k(X)$. Furthermore, from $\omega_c\cdot\omega_a=\delta_{a,c}$ it follows that the system $\{\omega_a\}_{a\in I_k}$ is linearly independent. Let A a subvariety of X of dimension k, and let B be a subvariety of X of codimension k. By Kleiman's Theorem [Kle74] there is an element $g\in G$ such that gA is rationally equivalent to A and generically transverse to B. Then $A\cdot B=(gA)\cdot B=\#((gA)\cap B)\geq 0$, so $\operatorname{Eff}_k(X)\subseteq\operatorname{Nef}_k(X)$. It is clear that $\operatorname{Nef}_k(X)$ is generated by $\{\omega_a/a\in I_k\}$. Since $\operatorname{Nef}_k(X)$ is closed and, as seen above, generated by the ω_a , we get that $\operatorname{Nef}_k(X)=\overline{\operatorname{Eff}_k}(X)$ is polyhedral.

Proposition 3.4. Let X be a rational homogeneous variety. Then $\operatorname{Nef}_k(X) = \overline{\operatorname{Eff}}_k(X) = \operatorname{Eff}_k(X)$ is polyhedral.

Proof. The description of the Chow ring of any rational homogeneous variety given in [Köc91, Corollary(1.5)] is

$$A_*(X) = \bigoplus_{w \in W^{\Theta}} \mathbb{Z}[X_w]$$

where X_w is the closure of the set BwP/P, with dimension l(w) [Köc91, Proposition(1.3)]. Let $I_k = \{w \in W^{\Theta}/l(w) = k\}$. Given $w \in W^{\Theta}$ we claim that

 $w_0ww_\theta \in I_{\dim X-k}$. Indeed for all $s \in \Theta$, using (3.1) and (3.3), we have

$$l(w_0ww_\theta s) = l(w_0) - l(ww_\theta s) = l(w_0) - l(w) - l(w_\theta s)$$

= $l(w_0) - l(w) - l(w_\theta) + l(s) = l(w_0) - l(ww_\theta) + 1 = l(w_0ww_\theta) + 1$

Similarly we can prove that $l(w_0ww_\theta) = l(w_0) - l(w_\theta) - l(w)$. Now given $w \in I_k$ we have, by [Köc91, Proposition(1.4)], that (2) of Proposition 3.3 is satisfied.

The pseudoeffective cone is also polyhedral in the case when the action of G on X has finitely many orbits, see [FMSS95, Corollary p.2].

Among the rational homogeneous varieties, the following are particularly interesting.

Definition 3.5. Let r, s be two integers such that $2 \le r \le \frac{s}{2}$. The Grassmann variety of r-planes G(r, s) is the scheme of r-dimensional subspaces of \mathbb{C}^s . Let ω be a non-degenerate symmetric bilinear form on \mathbb{C}^s . The orthogonal Grassmannian of isotropic r-planes OG(r, s) is the scheme of r-dimensional subspaces of \mathbb{C}^s isotropic with respect to ω . The scheme OG(r, 2m) has two isomorphic connected components if r = m or m - 1. In these two cases, we will denote by $OG_+(r, 2m)$ a connected component of OG(r, 2m). Let σ be a non-degenerate symplectic bilinear form on \mathbb{C}^s . The symplectic Grassmannian of isotropic r-planes SG(r, s) is the scheme of r-dimensional subspaces of \mathbb{C}^s isotropic with respect to σ .

Remark 3.6. Let S be the universal subbundle of G(r,s). The Plücker embedding is the embedding given by the very ample line bundle $\wedge^r S^{\vee}$. The varieties OG(r,s) and SG(r,s) can be embedded in G(r,s) as zero sections of, respectively, $Sym^2 S^{\vee}$ and $\wedge^2 S^{\vee}$.

3.2.1. Complete intersection of rational homogeneous varieties.

Remark 3.7. In [AC13, Proposition 34], it is stated that the smooth complete intersection of $OG_+(k, 2k)$ of type (2, 2) under the Plücker embedding is a weak 2-Fano variety. This should be read as (2).

Remark 3.8. We introduce the following notation: the group $H^4(G(r,s),\mathbb{Z})$ is generated by $\{\sigma_2,\sigma_{1,1}\}$, while $H^{r(s-r)-4}(G(r,s),\mathbb{Z})$ is generated by a basis $\{\sigma_2^{\vee},\sigma_{1,1}^{\vee}\}$ dual to $\{\sigma_2,\sigma_{1,1}\}$.

Remark 3.9. Let X be a smooth complete intersection of G(2,5) of type (1,1) under the Plücker embedding, let Z be the variety of lines through a general point of X. [AC13, Example 30] says that Z has homology class equal to $\sigma_2^{\vee} + \sigma_{1,1}^{\vee}$. This should be read as $2\sigma_{1,1}^{\vee} + \sigma_2^{\vee}$.

Remark 3.10. By Serre duality $\chi(\Omega_{G(2,5)}^p(-m)) = \chi(\Omega_{G(2,5)}^{6-p}(m))$, and for m=1,2,3 we have $\chi(\Omega_{G(2,5)}(-m)) = \chi(\Omega_{G(2,5)}^5(m)) = 0$ because all the groups $H^p(G(2,5),\Omega_{G(2,5)}^5(m))$ are zero by [Sno86, Theorem p. 171(3)]. If m=1,2 we have $\chi(\Omega_{G(2,5)}^2(-m)) = \chi(\Omega_{G(2,5)}^4(m)) = 0$, because $\forall p \geq 0$ $H^p(G(2,5),\Omega_{G(2,5)}^4(m)) = 0$ by [Sno86, Theorem p.p. 165,169]. It can easily be seen that $\chi(\Omega_{G(2,5)}) = -1$ and $\chi(\Omega_{G(2,5)}^2) = 2$.

Lemma 3.11. Let X be a smooth complete intersection of type (1,1) in a Grassmann variety G(2,5) under the Plücker embedding. Then $b_4(X) = 2$.

Proof. By [Laz04b, Example 7.1.5], all rows of the Hodge Diamond of X, except the middle row, are equal to those of the Hodge Diamond of G = G(2,5). Since X is Fano, $h^{0,4}(X) = 0$ then

$$\chi(\Omega_X) = -1 - h^{1,3}(X)$$

$$\chi(\Omega_X^2) = h^{2,2}(X)$$

(3.6)
$$b_4(X) = h^{2,2}(X) + 2h^{1,3}(X)$$

Note that by Serre duality and adjunction formula, for any integer m

$$h^4(\mathcal{O}_X(-m)) = h^0(\mathcal{O}_X(m) \otimes \mathcal{O}_G(2-5)|_X) = h^0(\mathcal{O}_X(m-3))$$

then by Kodaira Vanishing Theorem, $\chi(\mathcal{O}_X(-1)) = \chi(\mathcal{O}_X(-2)) = 0$. Take the Koszul resolution of the sheaf \mathcal{O}_X

$$(3.7) 0 \to \mathcal{O}_G(-2) \to \mathcal{O}_G(-1)^{\oplus 2} \to \mathcal{O}_G \to \mathcal{O}_X \to 0$$

and tensor it by Ω_G

$$(3.8) 0 \to \Omega_G(-2) \to \Omega_G(-1)^{\oplus 2} \to \Omega_G \to \Omega_{G|X} \to 0$$

then, by Remark 3.10,

$$\chi(\Omega_{G|X}) = \chi(\Omega_G(-2)) - 2\chi(\Omega_G(-1)) + \chi(\Omega_G) = -1$$

If we tensor (3.8) by $\mathcal{O}_G(-1)$ we have

$$\chi(\Omega_{G|X}(-1)) = \chi(\Omega_{G}(-3)) - 2\chi(\Omega_{G}(-2)) + \chi(\Omega_{G}(-1)) = 0$$

From the canonical sequence

$$(3.9) 0 \to \mathcal{O}_X(-1)^{\oplus 2} \to \Omega_{G|X} \to \Omega_X \to 0$$

we get $\chi(\Omega_X) = \chi(\Omega_{G|X}) - 2\chi(\mathcal{O}_X(-1)) = -1$, then $h^{1,3}(X) = 0$ by (3.4). If, instead, we tensor (3.7) by Ω_G^2 , that is

$$0 \to \Omega^2_G(-2) \to \Omega^2_G(-1)^{\oplus 2} \to \Omega^2_G \to \Omega^2_{G|X} \to 0$$

we get, by Remark 3.10,

$$\chi(\Omega_{G|X}^2) = \chi(\Omega_G^2(-2)) - 2\chi(\Omega_G^2(-1)) + \chi(\Omega_G^2) = 2$$

By [Har77, Exercise II.5.16d] and (3.9) we get

$$\chi(\Omega_X^2) = \chi(\Omega_{G|X}^2) - 2\chi(\Omega_{G|X}(-1)) - 3\chi(\mathcal{O}_X(-2)) = 2$$

Then by (3.5) and (3.6) we get $h^{2,2}(X) = 2$ and $b_4(X) = 2$.

Proposition 3.12. Let X be a n-dimensional weak 2-Fano complete intersection in a Grassmann variety G(r,s) under the Plücker embedding. Then, $b_4(X) \leq 2$. In particular $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.

Proof. Assume that X is of type $(d_1, ..., d_c)$. If n > 4, by [Laz04b, Theorem 7.1.1], we have $b_4(X) = b_4(G(r,s)) \le 2$ and we can apply Lemma 2.4. If n = 4, using [AC13, Proposition 31], we have the following conditions: c = r(s-r) - 4 and $\sum_{i=1}^{c} d_i \le s - 1$. It is easy to see that this leads to the following cases

| G(r,s) | Type | | G(r,s) | Type |
|--------|---------------|--|--------|--------|
| G(2,7) | (1,1,1,1,1,1) | | G(2,5) | (1, 1) |
| G(3,6) | (1,1,1,1,1) | | | (1,2) |
| G(2,6) | (1, 1, 1, 1) | | | (1, 3) |
| | (1, 1, 1, 2) | | | (2, 2) |

None of them is weak 2-Fano by [AC13, Proposition 31 and 32(iv)], and Theorem 1.4. $\hfill\Box$

Now we can prove Theorem 1.4.

Proof. Let $\mathcal{O}_Y(1)$ be the Plücker line bundle and let

$$\mathcal{U} \subseteq \mathbb{P}(H^0(Y, \mathcal{O}_Y(1))) \times \mathbb{P}(H^0(Y, \mathcal{O}_Y(1)))$$

be the open set parametrizing the smooth complete intersections in Y of bidegree (1,1). For $t \in \mathcal{U}$, we denote by X_t the corresponding variety. Let $\mathcal{X} := \{(x,t) \in Y \times \mathcal{U} : x \in X_t\}$ and consider the family

$$\mathcal{X} \xrightarrow{pr_1} Y$$

$$\downarrow^{pr_2}$$

$$\mathcal{U}$$

Suppose Y = G(2,5). Let $i: X_t \to Y$ be the inclusion, the map $i^*: H^4(Y,\mathbb{Z}) \to H^4(X_t,\mathbb{Z})$ is injective with torsion free cokernel by [Laz04b, Theorem 7.1.1 and Example 7.1.2], since $b_4(Y) = b_4(X_t) = 2$ by Lemma 3.11, we have that $i^*: H^4(Y,\mathbb{Z}) \to H^4(X_t,\mathbb{Z})$ is an isomorphism. By [dA15, Corollary 5.1], for a general t there exists a surface S_t such that $[i(S_t)] = \sigma_{1,1}^{\vee}$. Then there exist $a_t, b_t \in \mathbb{Z}$ such that $S_t = a_t \sigma_{2|X_t} + b_t \sigma_{1,1|X_t}$. Since

$$\begin{split} (\sigma_{2|X_t})^2 &= (\sigma_2^2) \cdot \sigma_1^2 = (\sigma_{3,1} + \sigma_{2,2}) \cdot \sigma_1^2 = 2 \\ (\sigma_{1,1|X_t})^2 &= (\sigma_{1,1}^2) \cdot \sigma_1^2 = \sigma_{2,2} \cdot \sigma_1^2 = 1 \\ \sigma_{2|X_t} \cdot \sigma_{1,1|X_t} &= (\sigma_2 \cdot \sigma_{1,1}) \cdot \sigma_1^2 = \sigma_{3,1} \cdot \sigma_1^2 = 1 \end{split}$$

Using the condition $[i(S_t)] = \sigma_{1,1}^{\vee} = \sigma_{2,2}$, we have

$$0 = \sigma_{2,2} \cdot \sigma_2 = S_t \cdot \sigma_{2|X_t} = 2a_t + b_t$$

$$1 = \sigma_{2,2} \cdot \sigma_{1,1} = S_t \cdot \sigma_{1,1|X_t} = a_t + b_t$$

then $a_t = -1$ and $b_t = 2$. Let $S := pr_1^*(-\sigma_2 + 2\sigma_{1,1})$, then the surface $S_{|X_t}$ is such that $[S_t] = [S_{|X_t}]$, and since we see that it is effective for a general t, hence it is effective for all t. Let $t \in \mathcal{U}$, then X_t is not weak 2-Fano since using [AC13, Proposition 32]

¹This is a well-known fact for experts. A good reference is [Ott15, Proposition 3].

$$ch_2(X_t) \cdot S_{|X_t} = \frac{1}{2} (\sigma_{2|X_t} - \sigma_{1,1|X_t}) \cdot (-\sigma_{2|X_t} + 2\sigma_{1,1|X_t}) = -\frac{1}{2}.$$

Suppose Y = G(2,6). By [Laz04b, Theorem 7.1.1] we have that $H^4(Y,\mathbb{Z}) \cong H^4(X_t,\mathbb{Z})$, then $b_8(X_t) = b_4(X_t) = 2$. Now consider $i^* : H^8(Y,\mathbb{Z}) \to H^8(X_t,\mathbb{Z})$, where $i : X_t \to Y$ is the inclusion. From

$$\begin{split} \sigma_{4|X_{t}} \cdot \sigma_{2|X_{t}} &= (\sigma_{4} \cdot \sigma_{2}) \cdot \sigma_{1}^{2} = \sigma_{4,2} \cdot \sigma_{1}^{2} = \sigma_{4,3} \cdot \sigma_{1} = 1 \\ \sigma_{2,2|X_{t}} \cdot \sigma_{2|X_{t}} &= (\sigma_{2,2} \cdot \sigma_{2}) \cdot \sigma_{1}^{2} = \sigma_{4,2} \cdot \sigma_{1}^{2} = \sigma_{4,3} \cdot \sigma_{1} = 1 \\ \sigma_{4|X_{t}} \cdot \sigma_{1,1|X_{t}} &= (\sigma_{4} \cdot \sigma_{1,1}) \cdot \sigma_{1}^{2} = 0 \cdot \sigma_{1}^{2} = 0 \\ \sigma_{2,2|X_{t}} \cdot \sigma_{1,1|X_{t}} &= (\sigma_{2,2} \cdot \sigma_{1,1}) \cdot \sigma_{1}^{2} = (\sigma_{2,2} \cdot (\sigma_{1}^{2} - \sigma_{2})) \cdot \sigma_{1}^{2} \\ &= (\sigma_{2,2} \cdot \sigma_{1}^{2} - \sigma_{2,2} \cdot \sigma_{2}) \cdot \sigma_{1}^{2} = (\sigma_{3,2} \cdot \sigma_{1} - \sigma_{4,2}) \cdot \sigma_{1}^{2} \\ &= (\sigma_{4,2} + \sigma_{3,3} - \sigma_{4,2}) \cdot \sigma_{1}^{2} = (\sigma_{3,3}) \cdot \sigma_{1}^{2} = \sigma_{4,3} \cdot \sigma_{1} = 1 \end{split}$$

Hence $\sigma_{4|X_t}$ and $\sigma_{2,2|X_t}$ are a basis of $H^8(X_t,\mathbb{Z})$, since that group is torsion free (see Remark 3.13). Then $[S_t] = a_t \sigma_{4|X_t} + b_t \sigma_{2,2|X_t}$, where as before S_t is the surface described in [dA15, Corollary 5.1] for general $t \in \mathcal{U}$. Using the condition $[i(S_t)] = \sigma_{1,1}^{\vee} = \sigma_{3,3}$, we have

$$0 = \sigma_{3,3} \cdot \sigma_2 = S_t \cdot \sigma_{2|X_t} = a_t + b_t$$

$$1 = \sigma_{3,3} \cdot \sigma_{1,1} = S_t \cdot \sigma_{1,1|X_t} = b_t$$

then $a_t = -1$ and $b_t = 1$. Let $S := pr_1^*(-\sigma_4 + \sigma_{2,2})$, then $[S_t] = [S_{|X_t}]$, that is $S_{|X_t}$ is effective for all t. Let $t \in \mathcal{U}$, then X_t is not weak 2-Fano since using [AC13, Proposition 32]

$$ch_2(X_t) \cdot S_{|X_t} = (\sigma_{2|X_t} - \sigma_{1,1|X_t}) \cdot (-\sigma_{4|X_t} + \sigma_{2,2|X_t}) = -1.$$

Remark 3.13. By [Laz04b, Theorem 7.1.1] we have $H^5(X_t, \mathbb{Z}) = 0$. By [Hat02, Corollary 3.3] $H_4(X_t, \mathbb{Z})$ is torsion free, then also $H^8(X_t, \mathbb{Z})$ is torsion free by Poincaré duality.

We now deal with complete intersections in orthogonal Grassmannians, so let us recall the useful notation in [Cos11]. Given a connected component $X \subseteq OG(r,s)$, we will write $s = 2m + 1 - \epsilon$ with $\epsilon \in \{0,1\}$ and $2 \le r \le m$. Let t be an integer such that $0 \le t \le r$, and $t \equiv m \pmod{2}$ if 2r = s. Given a sequence of integers $\lambda = (\lambda_1, ..., \lambda_t)$ of length t such that

$$m - \epsilon \ge \lambda_1 > \dots > \lambda_t > -\epsilon$$
.

Let $\tilde{\lambda} = (\tilde{\lambda}_{t+1}, ..., \tilde{\lambda}_m)$ be the unique sequence of length m-t such that

- $m-1 \geq \tilde{\lambda}_{t+1} > \dots > \tilde{\lambda}_m \geq 0$,
- $\tilde{\lambda}_i + \lambda_i \neq m \epsilon$ for every i = 1, ..., t and j = t + 1, ..., m.

The Schubert varieties in X are parametrized by pairs (λ, μ) , where μ is any subsequence of $\tilde{\lambda}$ of length r-t. Given an isotropic flag of subvector spaces F_{\bullet}

$$0\subseteq F_1\subseteq F_2\subseteq \ldots \subseteq F_m\subseteq F_{m-1}^\perp\subseteq F_{m-2}^\perp\subseteq \ldots \subseteq F_1^\perp\subseteq \mathbb{C}^s,$$

 $\Omega_{(\lambda,\mu)}(F_{\bullet})$ is defined as the closure of the locus

$$\{[W] \in X/\dim(W \cap F_{m+1-\epsilon-\lambda_i}) = i \text{ for } 1 \le i \le t; \\ \dim(W \cap F_{\mu_j}^{\perp}) = j \text{ for } t < j \le r \}.$$

Let us define another sequence λ' of length t' in this way:

- $\lambda' = \lambda$ if either $\epsilon = 0$ or $\epsilon = 1$ and $t \equiv m \pmod{2}$; otherwise
- $\lambda' = \lambda \cup \{b\}$ where $b = \min\{a \in \mathbb{N}/0 \le a \le m-1, a \notin \lambda, a + \mu_j \ne m-1 \,\forall j = m-1 \}$

Let $\tilde{\lambda}'$ be the unique sequence associated to λ' as above. Then the pair (λ, μ) is a subsequence of (λ', λ') . Suppose $(\lambda, \mu) = (\lambda'_{i_1}, ..., \lambda'_{i_t}, \lambda'_{i_{t+1}}, ..., \lambda'_{i_r})$ and let the discrepancy of λ and μ be the non-negative number

$$dis(\lambda, \mu) = \sum_{i=1}^{r} (m - r + j - i_j).$$

Then the codimension of a Schubert cycle $\Omega_{(\lambda,\mu)}(F_{\bullet})$ is (see [Cos11, p.2448])

$$\operatorname{codim}(\Omega_{(\lambda,\mu)}(F_{\bullet})) = \sum_{i=1}^{t'} \lambda'_i + \operatorname{dis}(\lambda,\mu).$$

Let $\Omega_{(\lambda,\mu)}(F_{\bullet})$ be of codimension k and set $\sigma_{(\lambda,\mu)} = [\Omega_{(\lambda,\mu)}(F_{\bullet})] \in H^{2k}(X,\mathbb{Z})$. The set of all $\sigma_{(\lambda,\mu)}$ of codimension k is a basis of $H^{2k}(X,\mathbb{Z})$ (by the Ehresmann's Theorem [Ehr34]).

Lemma 3.14. Let X be a connected component of OG(r,s), $2 \le r \le m = \left\lceil \frac{s}{2} \right\rceil$, we have

$$b_4(X) = \begin{cases} 1 & r = m \\ 3 & 1 \le m - r \le 2, s even \\ 2 & otherwise \end{cases}$$

Proof. We have to count the number of sequences (λ, μ) such that

$$\sum_{i=1}^{t'} \lambda_i' + dis(\lambda, \mu) = 2.$$

For $1 \leq j \leq r$ let $c_j = m - r + j - i_j$. It can easily be seen that

$$m - r \ge c_1 \ge c_2 \ge \dots \ge c_r \ge 0$$

and we can write

$$dis(\lambda, \mu) = \sum_{i=1}^{r} c_j.$$

We are in one of the following cases:

- $\begin{array}{ll} (1) \ \sum_{i=1}^{t'} \lambda_i' = 0 \ \text{and} \ dis(\lambda,\mu) = 2, \ \text{or} \\ (2) \ \sum_{i=1}^{t'} \lambda_i' = 1 \ \text{and} \ dis(\lambda,\mu) = 1, \ \text{or} \\ (3) \ \sum_{i=1}^{t'} \lambda_i' = 2 \ \text{and} \ dis(\lambda,\mu) = 0. \end{array}$

Let s be odd. Then

Case (1) t must be 0. If $m-r \ge 1$ then $c_1 = c_2 = 1$, and, if m-r > 1, we have also the possibility $c_1 = 2$. These cases correspond to

$$(\lambda, \mu) = \begin{cases} (\emptyset, (r, r - 1, r - 3, ...,)) \\ (\emptyset, (r + 1, r - 2, r - 3, ...,)). \end{cases}$$

- Case (2) Only one possibility if m-r=1, that is $\lambda=(1)$ and $c_2=1$. This case corresponds to $(\lambda,\mu)=((1),(r-2,r-3,...,))$. No other possibilities if $m-r\neq 1$.
- Case (3) It must be $\lambda = (2)$, then $i_1 = 1$ and since $c_j = 0 \ \forall j \ge 1, c_1 = m r + 1 1 = 0$ implies m = r. This is the case $(\lambda, \mu) = ((2), (m 1, m 3, ...))$.

Let s be even. If s = 2r, then the discrepancy is 0 because $c_j \le m - r \ \forall j \ge 1$, then it is possible only the case 3, that is

$$(\lambda, \mu) = \begin{cases} ((2), (m-1, m-2, m-4)) & m \text{ odd} \\ ((2, 0), (m-2, m-4)) & m \text{ even.} \end{cases}$$

Suppose m > r. Let m be even, then

Case (1) It must be $\lambda' = \emptyset$, then $\lambda = \lambda' = \emptyset$ and $\tilde{\lambda} = (m-1, m-2, m-3, m-4, ...)$. If $m-r \geq 1$ then $c_1 = c_2 = 1$, and, if $m-r \geq 2$, we have also the possibility $c_1 = 2$. These cases corresponds to

$$(\lambda, \mu) = \begin{cases} (\emptyset, (r, r - 1, r - 3, ...,)) \\ (\emptyset, (r + 1, r - 2, r - 3, ...,)). \end{cases}$$

Case (2) It must be $\lambda' = (1,0)$, then we can have $\lambda = (0)$ or $\lambda = (1)$.

Suppose $\lambda = (0)$, $\lambda = (m-2, m-3, ...)$, and we have to choose a μ such that b=1 in order to have $\lambda' = \lambda \cup \{1\}$ which implies $\tilde{\lambda}' = (m-3, m-4, ...)$. This can happen only if $m-2 \notin \mu$, that is, it is enough to choose μ as a subsequence of (m-3, m-4, ...). This case implies that $i_1 = 2$, then $c_1 = m-r+1-2 = m-r-1$, then it must be m-r=2. Since $c_j = 0 \ \forall j \geq 2$, that corresponds to the case

$$(\lambda, \mu) = ((0), (m-4, m-5, ...,)).$$

Suppose $\lambda = (1)$, $\tilde{\lambda} = (m-1, m-3, ...)$, and we have to choose a μ such that b = 0 in order to have $\lambda' = \lambda \cup \{0\}$ which implies $\tilde{\lambda}' = (m-3, m-4, ...)$. This can happen only if $m-1 \notin \mu$, that is, it is enough to choose μ as a subsequence of (m-3, m-4, ...). This case implies that $i_1 = 1$, then $c_1 = m-r+1-1 = m-r$, then it must be m-r=1. Since $c_j = 0 \ \forall j \geq 2$, that corresponds to the case

$$(\lambda, \mu) = ((1), (m-3, m-4, m-5, ...,)).$$

Case (3) It must be $\lambda' = (2,0)$, then we can have $\lambda = (0)$ or $\lambda = (2)$.

If $\lambda = (2)$, then $c_1 = m - r$, then the discrepancy is not 0.

So $\lambda = (0)$, $\lambda = (m-2, m-3, ...)$, $c_j = 0 \ \forall j \geq 1$, and we have to choose a μ such that b = 2 in order to have $\lambda' = \lambda \cup \{2\}$ which implies $\tilde{\lambda}' = (m-2, m-4, ...)$. This can happen only if $m-2 \in \mu$ and $m-3 \notin \mu$. That is, the sequence

$$((0), \mu) = ((0), (\tilde{\lambda'}_{i_1}, ..., \tilde{\lambda'}_{i_r}))$$

seen as a subsequence of $((2,0), (m-2, m-4, ...)) = (\lambda', \tilde{\lambda}')$ must satisfy $i_1 = 2$. The condition $c_j = 0$ implies $i_j = m - r + j$, then $i_1 = m - r + 1 = 2$ implies m - r = 1. Then, if m - r = 1, we have the sequence

$$(\lambda, \mu) = ((0), (m-2, m-4, ...)).$$

Let m be odd, then

Case (1) It must be $\lambda' = (0)$, then we can have $\lambda = \lambda' = (0)$ or $\lambda = \emptyset$.

Suppose $\lambda = \lambda' = (0)$, this implies $\tilde{\lambda} = (m-2, m-3, m-4, ...)$ and $c_1 = m-r$. Then

-if $m-r \geq 3$, then this case in not possible since the first summand of the discrepancy (which it must be 2) is m-r,

-if m-r=2, then $c_j=0$ for $j\geq 2$, that is $i_j=m-r+j$ for $j\geq 2$, then

$$(\lambda, \mu) = ((0), (\tilde{\lambda}_{m-r+2}, \tilde{\lambda}_{m-r+3}, ...,)) = ((0), (r-2, r-3, ...)),$$

-if m-r=1, then $c_j=0$ for $j\geq 3$ and $c_2=1$, that is

$$(\lambda, \mu) = ((0), (\tilde{\lambda}_{m-r+1}, \tilde{\lambda}_{m-r+3}, ...,)) = ((0), (r-1, r-3, ...)).$$

Suppose $\lambda = \emptyset$, $\tilde{\lambda} = (m-1, m-2, ...)$, and we have to choose a μ such that b = 0 in order to have $\lambda' = \lambda \cup \{0\}$ which implies $\tilde{\lambda}' = (m-2, m-3, m-4, ...)$. This can happen only if $m-1 \notin \mu$, that is, it is enough to choose μ as a subsequence of (m-2, m-3, m-4, ...). If $m-r \geq 1$ we have $c_1 = c_2 = 1$, that corresponds to the case

$$(\lambda, \mu) = (\emptyset, (r, r - 1, r - 3, ...,)).$$

But, in order to make $m-1 \notin \mu$, we must have $r \neq m-1$, then this case only happen if $m-r \geq 2$. If $m-r \geq 2$, we have also the possibility $c_1 = 2$, that corresponds to the case

$$(\lambda, \mu) = (\emptyset, (r+1, r-2, r-3, ...,)).$$

But, in order to make $m-1 \notin \mu$, $r+1 \neq m-1$, then this case only happen if $m-r \geq 3$.

Case (2) It must be $\lambda' = (1)$, then we can have $\lambda = \lambda' = (1)$ or $\lambda = \emptyset$.

Suppose $\lambda = \lambda' = (1)$, then $\tilde{\lambda} = (m-1, m-3, m-4, ...)$, $c_1 = m-r$, and $c_j = 0$ for $j \geq 2$. So, if m-r=1, we have the sequence

$$(\lambda, \mu) = ((1), (\tilde{\lambda}_{m-r+2}, \tilde{\lambda}_{m-r+3}, ...,)) = ((1), (m-3, m-4, ...)).$$

Suppose $\lambda = \emptyset$, $\tilde{\lambda} = (m-1, m-2, ...)$, $c_1 = 1$, $c_j = 0 \ \forall j \geq 2$, and we have to choose a μ such that b = 1 in order to have $\lambda' = \lambda \cup \{1\}$ which implies

$$\tilde{\lambda}' = (m-1, m-3, m-4, ...).$$

This can happen only if $m-1 \in \mu$ and $m-2 \notin \mu$. That is, the sequence

$$(\emptyset, \mu) = (\emptyset, (\tilde{\lambda'}_{i_1}, ..., \tilde{\lambda'}_{i_r}))$$

seen as a subsequence of

$$((1), (m-1, m-3, m-4, ...)) = (\lambda', \tilde{\lambda'})$$

must satisfy $i_1 = 2$. The condition $c_1 = 1$ implies $1 = m - r + 1 - i_1$, then 1 = m - r + 1 - 2 that is m - r = 2, while the condition $c_j = 0 \,\forall j \geq 2$ implies $i_j = m - r + j$. Then, if m - r = 2, we have the sequence

$$(\lambda, \mu) = ((\emptyset), (m-1, m-4, m-5, ...)).$$

Case (3) It must be $\lambda' = (2)$, then we can have $\lambda = \lambda' = (1)$ or $\lambda = \emptyset$.

If $\lambda = (2)$, then $c_1 = m - r$, then the discrepancy is not 0.

So $\lambda = \emptyset$, $\lambda = (m-1, m-2, ...)$, $c_j = 0 \ \forall j \geq 1$, and we have to choose a μ such that b=2 in order to have $\lambda'=\lambda\cup\{2\}$ which implies

$$\tilde{\lambda}' = (m-1, m-2, m-4, ...).$$

This can happen only if $m-1, m-2 \in \mu$ and $m-3 \notin \mu$. That is, the sequence

$$(\emptyset,\mu)=(\emptyset,(\tilde{\lambda'}_{i_1},...,\tilde{\lambda'}_{i_r}))$$

seen as a subsequence of

$$((2), (m-1, m-2, m-4, ...)) = (\lambda', \tilde{\lambda'})$$

must satisfy $i_1 = 2$ and $i_2 = 3$. The condition $c_j = 0$ implies $i_j = m - r + j$, then $i_1 = m - r + 1 = 2$ and $i_2 = m - r + 2 = 3$ imply m - r = 1. Then, if m - r = 1, we have the sequence

$$(\lambda, \mu) = ((\emptyset), (m-1, m-2, m-4, ...)).$$

Lemma 3.15. $b_6(OG_+(r,2r)) = 2$.

Proof. We have to calculate the number of Schubert cycles of dimension 6, that is the number of sequences $r-1 \ge \lambda_1 > ... > \lambda_t \ge 0$ such that $\sum_{i=1}^t \lambda_i = 3$, $t \equiv r \pmod{2}$. We get

- If r is odd, $\lambda = (3)$ and $\lambda = (2, 1, 0)$;
- If r is even, $\lambda = (3,0)$ and $\lambda = (2,1)$.

We now deal with complete intersections in symplectic Grassmannians SG(r,s) with $2 \le r \le m = \frac{s}{2}$. Let us recall the useful notation in [Cos18]. Let t be an integer such that $0 \le t \le r$. Given a sequence of integers $\lambda = (\lambda_1, ..., \lambda_t)$ of length t such that

$$m \ge \lambda_1 > \dots > \lambda_t > 0$$

let $\tilde{\lambda} = (\tilde{\lambda}_{t+1}, ..., \tilde{\lambda}_m)$ be the unique sequence of length m-t such that

- $$\begin{split} \bullet & \ m-1 \geq \tilde{\lambda}_{t+1} > ... > \tilde{\lambda}_m \geq 0, \\ \bullet & \ \tilde{\lambda}_j + \lambda_i \neq m \text{ for every } i=1,..,t \text{ and } j=t+1,...,m. \end{split}$$

The Schubert varieties in SG(r,s) are parametrized by pairs (λ,μ) , where μ is any subsequence of λ of length r-t. Given an isotropic flag of subvector spaces F_{\bullet}

$$0\subseteq F_1\subseteq F_2\subseteq \ldots \subseteq F_m\subseteq F_{m-1}^\perp\subseteq F_{m-2}^\perp\subseteq \ldots\subseteq F_1^\perp\subseteq \mathbb{C}^s$$

 $\Omega_{(\lambda,\mu)}(F_{\bullet})$ is defined as the closure of the locus

$$\{[W] \in SG(r,s)/\dim(W \cap F_{m+1-\lambda_i}) = i \text{ for } 1 \le i \le t; \\ \dim(W \cap F_{\mu_j}^{\perp}) = j \text{ for } t < j \le r\}.$$

Suppose $(\lambda, \mu) = (\lambda_1, ..., \lambda_t, \tilde{\lambda}_{i_{t+1}}, ..., \tilde{\lambda}_{i_r})$, the codimension of $\Omega_{(\lambda, \mu)}(F_{\bullet})$ is (see [Cos18, p. 57])

$$\operatorname{codim}(\Omega_{(\lambda,\mu)}(F_{\bullet})) = \sum_{i=1}^{t} \lambda_i + \operatorname{dis}(\lambda,\mu).$$

The set all $\sigma_{(\lambda,\mu)} = [\Omega_{(\lambda,\mu)}(F_{\bullet})]$ of codimension k is a basis of $H^{2k}(SG(r,s),\mathbb{Z})$ by Ehresmann's Theorem. The proof of the following lemma is the same of the case of OG(r,2m+1).

Lemma 3.16. Let $2 \le r \le m = \frac{s}{2}$, then

$$b_4(SG(r,s)) = \begin{cases} 2 & m-r \ge 1\\ 1 & r=m \end{cases}$$

3.3. Other examples.

Proposition 3.17. Let s, r be positive integers such that $2 \le r \le \left[\frac{s}{2}\right]$, and $\left[\frac{s}{2}\right] - r \ne 1, 2$ if s is even. Let $s \ne 2r$ (respectively, s = 2r), let X be a n-dimensional weak 2-Fano complete intersection in a connected component of the orthogonal Grassmann variety OG(r,s) under the Plücker (respectively, half-spinor) embedding, with X very general if $X \subseteq OG(2,7)$. Then $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.

Proof. Assume that X is of type $(d_1, ..., d_c)$. If n > 4, by [Laz04b, Theorem 7.1.1] and Lemma 3.14, we have $b_4(X) \le 2$ and we can apply Lemma 2.4. Then we have n = 4 and $c = \frac{r(2s-3r-1)}{2} - 4$. If 2r = s, by [AC13, Proposition 34] and Remark 3.7, we see that X is weak 2-Fano if and only if either $d_i = 1$ and $c \le 4$, or X of type (2). Therefore we get r = 4 and X of type (1,1). By [AC13, Proposition 34] we have that $K_X = -c_1(X) = -4H$, where H is the half-spinor embedding. But then, by [KO73, Corollary p.37], X is a smooth quadric in \mathbb{P}^5 and then $b_4(X) = 2$ by [Rei72, p.20], so we apply by Lemma 2.4.

If $2r \neq s$, since $c_1(OG(r,s)) = (s-r-1)\sigma_1$ we get that $\sum_{i=1}^{c} d_i \leq s-r-2$. It is easy to see that this leads to the following cases

| OG(r,s) | Type | |
|---------------|-----------|--|
| OG(3,7) | (1,1) | |
| OG(2,7) | (1, 1, 1) | |
| $OG_{+}(2,6)$ | (2) | |
| $OO_{+}(2,0)$ | (1) | |

But $OG(3,7) \cong OG_+(4,8)$, then the first case is a quadric. Let X_{111} be the variety (1,1,1) in OG(2,7). This is the variety (b8) in the classification given in [K95]. Indeed, for the reader's convenience, we point out that X_{111} is the zero-locus of a global section of the bundle

$$\left(\wedge^2 S^{\vee}\right)^{\oplus 3} \oplus Sym^2 S^{\vee}$$

where S^{\vee} is (1,0;0,0,0,0,0) in Küchle's notation (see [K95, Section 2.5]). So $h^{1,3}(X_{111}) > 0$ by [K95, Theorem 4.8]. Now apply [Spa96, Theorem 2] to conclude that the space of algebraic cycles of X_{111} is induced by the space of algebraic cycles of OG(2,7). Then

$$Z_2(X_{111})/\operatorname{Alg}_2(X_{111}) \otimes \mathbb{R}$$

is at most 2-dimensional. Hence $\overline{\mathrm{Eff}}_2(X_{111})$ is polyhedral by (2.1) and Lemma 2.4. The last two varieties do not satisfy the condition $\left[\frac{s}{2}\right] - r \neq 1, 2$, anyway, they are not weak 2-Fano by [AC13, Example 21]. Indeed, $OG_+(2,6)$ is the zero section of the bundle $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ [Kuz15, Proposition 2.1], and it can easily be seen that the Plücker embedding is given by the divisor (1,1), then the two varieties are isomorphic to, respectively, a complete intersection of type (1,1) and (1,2) in $\mathbb{P}^3 \times \mathbb{P}^3$ under the embedding given by $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$.

Proposition 3.18. Let X be a smooth n-dimensional weak 2-Fano complete intersection in a symplectic Grassmann variety SG(r,s) under the Plücker embedding. Then, $b_4(X) \leq 2$. In particular $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.

Proof. Assume that X is of type $(d_1, ..., d_c)$. If n > 4, by [Laz04b, Theorem 7.1.1] and Lemma 3.16, we have $b_4(X) = b_4(SG(r,s)) \le 2$ and we can apply Lemma 2.4. If n = 4, since $c_1(SG(r,s)) = (s-r+1)\sigma_1$ we have the following conditions: $c = \frac{r(2s-3r+1)}{2} - 4$ and $\sum_{i=1}^{c} d_i \le s - r$. It is easy to see that this leads to the following cases:

| SG(r,s) | Type |
|---------------------|-----------|
| SG(3, 6) | (1,1) |
| 50(0,0) | (1, 2) |
| SG(2,6) | (1, 1, 1) |
| $\mathcal{DG}(2,0)$ | (1, 1, 2) |

The variety SG(2,6) is a section of $\wedge^2(S^{\vee}) = \mathcal{O}_{G(2,6)}(1)$, as we said in Remark 3.6. Thus the last two case are, respectively, (1,1,1,1) and (1,1,1,2) in G(2,6). The first two cases are not weak 2-Fano by [AC13, Proposition 36], the last two by [AC13, Proposition 32(i)].

4. Fano manifolds of dimension n and index $i_X > n-3$

A very important invariant of a Fano variety X is its index: this is the maximal integer i_X such that $-K_X$ is divisible by i_X in Pic(X).

Fano varieties of high index have been classified: [KO73] proved that $i_X \leq n+1$, $i_X = n+1$ if and only if $X = \mathbb{P}^n$, and $i_X = n$ if and only if $X \subset \mathbb{P}^{n+1}$ is a smooth hyperquadric. Furthermore the case $i_X = n-1$ (the so called Del Pezzo varieties) has been classified by Fujita in [Fuj82a, Fuj82b], and the case $i_X = n-2$ (the so called Mukai varieties) by Mukai (see [Muk89] and [IP99]).

Araujo and Castravet [AC13, Theorem 3] succeeded to classify 2-Fano Del Pezzo and Mukai varieties. They proved:

Theorem 4.1. Let X be a 2-Fano variety of dimension $n \geq 3$ and index $i_X \geq n-2$. Then X is isomorphic to one of the following.

- \bullet \mathbb{P}^n
- Complete intersection in projective spaces:
 - Quadric hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with n > 2;
 - Complete intersections of type (2,2) in \mathbb{P}^{n+2} with n > 5;
 - Cubic hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with n > 7;
 - Quartic hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with n > 15;
 - Complete intersections of type (2,3) in \mathbb{P}^{n+2} with n > 11;
 - Complete intersections of type (2,2,2) in \mathbb{P}^{n+3} with n>9.
- Complete intersection in weighted projective spaces:
 - Degree 4 hypersurfaces in $\mathbb{P}(2,1,...,1)$ with n > 11;
 - Degree 6 hypersurfaces in $\mathbb{P}(3,2,1,...,1)$ with n > 23;
 - Degree 6 hypersurfaces in $\mathbb{P}(3,1,...,1)$ with n > 26;
 - Complete intersections of type (2,2) in $\mathbb{P}(2,1,...,1)$ with n > 14.
- G(2,5).
- $OG_{+}(5,10)$ and its linear sections of codimension c < 4.
- SG(3,6).
- G_2/P_2 .

Here G_2/P_2 is a 5-dimensional homogeneous variety for a group of type G_2 . Using the results in the previous sections we can prove Theorem 1.3.

Proof. For $\overline{\mathrm{Eff}}_2(X)$: In the case \mathbb{P}^n and its complete intersections, we can invoke Theorem 3.2. Since none of the complete intersections in $\mathbb{P}(\mathbf{w})$ of the list has dimension 4, we can use Proposition 3.1. Also G(2,5), $OG_+(5,10)$, SG(3,6) and G_2/P_2 are rational homogeneous varieties, then their cone of pseudoeffective 2-cycles is polyhedral by Proposition 3.4. Whereas the complete intersections of $OG_+(5,10)$ have polyhedral cone of pseudoeffective 2-cycles by Proposition 3.17.

For Eff₃(X): In Theorem 4.1, the only complete intersections of dimension 6 in a weighted projective space are the one of type (2,2) in \mathbb{P}^8 and the smooth quadric $Q \subseteq \mathbb{P}^7$. The first one is not weak 3-Fano since by [AC13, Equation (3.1)], $ch_3(X) = -\frac{7}{6}h_{|X}^3$ where h is the class of an hyperplane in \mathbb{P}^8 . Then $h_{|X}^3$ is effective, and $ch_3(X) \cdot h_{|X}^3 = -\frac{7}{6}h_{|X}^6 < 0$. For the quadric, by [Rei72, p.20] $b_6(Q) = 2$, then $\overline{\text{Eff}}_3(X)$ is polyhedral by Lemma 2.4. For the other complete intersections we can use Proposition 3.1, whilst for the rational homogeneous varieties we can use Proposition 3.4. Also for the complete intersections in $OG_+(5,10)$ we have $b_6(X) = 2$, because $b_6(OG_+(5,10)) = 2$ by Lemma 3.15 and we can use [Laz04b, Theorem 7.1.1].

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