On the Dimension of Finite Point Sets II. "Das Budapester Programm"

György Elekes

Abstract

1 Introduction

1.1 History: "Das Erlanger Programm" of Felix Klein

"Klein's synthesis of geometry as the study of the properties of a space that are invariant under a given group of transformations, known as the Erlanger Programme (1872), profoundly influenced mathematical development. The Erlanger Programme gave a unified approach to geometry which is now the standard accepted view." (Article by: J. J. O'Connor and E. F. Robertson)
www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Klein.html

1.2 A modest sub-program.

We are going to use the nickname "Das Budapester Programm" for a large class of finite combinatorial problems to be posed below. Following Klein's original idea, they concern various dimensional Euclidean spaces and several groups of transformations.

Of course, we do not expect unifying any theory, we just hope that the questions may be interesting on their own right. The one dimensional special cases (which are already solved) have non-trivial applications both to algebraic and geometric problems (see Section 1.3). That is why we believe that also the higher dimensional versions — of which we solve one here — may be useful.

Given a group G of transformations of \mathbb{R}^d and a finite point set $\mathcal{P} \subset \mathbb{R}^d$ we shall be interested in the number of transformations $\varphi \in G$ which map many points of \mathcal{P} to some other points of \mathcal{P} :

$$\#\{\varphi \in G : \varphi(\mathcal{P}) \cap \mathcal{P} \text{ is "not too small"}\}.$$

Usually, "not too small" will mean $|\varphi(\mathcal{P}) \cap \mathcal{P}| \geq k$ for a given (large) $k \leq |\mathcal{P}|$. However, sometimes we shall even require that $\varphi(\mathcal{P}) \cap \mathcal{P}$ be "proper d–dimensional" in some sense, introduced in Part I [Ele05].

Definition 1.1 A set of N points is proper d-dimensional up to a constant factor C, (for short, "proper d-D") if it can be cut into singletons by at most $C\sqrt[d]{N}$ appropriate hyperplanes.

The ultimate goal of our "Budapester Programm" is to find sharp upper bounds (as a function of N, for various dimensions d and groups G) for

$$f_G^d(N,k) \stackrel{\mathrm{def}}{=} \max_{\substack{|\mathcal{P}|=N\\ \mathcal{P} \subset \mathbb{R}^d\\ \text{proper } d\text{-}\mathrm{dim}}} \#\{\varphi \in G \; ; \; \varphi(\mathcal{P}) \cap \mathcal{P} \text{ proper } d\text{-}\mathrm{dim}, \, |\varphi(\mathcal{P}) \cap \mathcal{P}| \geq k\}.$$

Problem 1.2 ("Meta-problem") Is it true that, for any "reasonable Euclidean" group G which acts on \mathbb{R}^d ,

$$f_G^d(N,k) = O\left(\frac{N^{\alpha}}{k^{\alpha-1}}\right),$$

for an $\alpha = \alpha(G) > 1$, independent from N and k?

1.3 More history: old results in \mathbb{R} .

In one dimension, the assumption of being proper 1D only means that the points of \mathcal{P} are all distinct.

Proposition 1.3 For the group of non-constant affine transforms of \mathbb{R} , i.e., for $G = \{ \varphi : \mathbb{R} \to \mathbb{R} : \varphi(x) = mx + b, \ m \neq 0 \}$, we have

$$\#\{\varphi:\mathbb{R}\to\mathbb{R} \text{ affine } ; |\varphi(\mathcal{P})\cap\mathcal{P}|\geq k\} \leq C\cdot|\mathcal{P}|^4/k^3,$$

for an absolute constant C and any $2 \le k \le |\mathcal{P}|$.

Proof: See [Ele97] and Section 4.1 for some remarks; also [Ele02] for a survey.

Proposition 1.4 For the group of non-constant projective transforms of \mathbb{R} , i.e., for $G = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi(x) = (ax + b)/(cx + d), \ ac - bd \neq 0\}$, we have

$$\#\{\varphi:\mathbb{R}\to\mathbb{R} \text{ projective } ; |\varphi(\mathcal{P})\cap\mathcal{P}|\geq k\} \leq C\cdot|\mathcal{P}|^6/k^5,$$

for an(other) absolute constant C and any $3 \le k \le |\mathcal{P}|$.

For any positive integer r and for the group G_r of non-constant rational transforms of \mathbb{R} of total degree $\leq r$, i.e., for $G = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi(x) = p(x)/q(x), p,q \in \mathbb{R}[x], \deg(p) + \deg(q) \leq r\}$, we have

$$\#\{\varphi \in G_r : |\varphi(\mathcal{P}) \cap \mathcal{P}| \ge k\} \le C \cdot |\mathcal{P}|^{2r+2}/k^{2r+1}$$

for a constant C = C(r, c).

Proof: See [EK01], [ER00] and Section 4.1 for some remarks.

The upper bound in Proposition 1.3 gives the best possible order of magnitude. However, this is unknown for those in Proposition 1.4; perhaps they can be improved for proper (i.e., non-affine) rational or even projective transforms.

Remark 1.5 It is worth noting that, for any fixed 0 < c < 1 and $k = c|\mathcal{P}|$, the bounds in Propositions 1.3 and 1.4 are linear in $|\mathcal{P}|$, i.e., $\#\{\varphi : \mathbb{R} \to \mathbb{R} \}$ affine or projective or rational; $|\varphi(\mathcal{P}) \cap \mathcal{P}| \geq c|\mathcal{P}| \leq C \cdot |\mathcal{P}|$. Though this only is a very special case, bounds like this were crucial in finding Freiman–Ruzsa type structure results for small composition sets of affine or projective transforms of \mathbb{R} in [Ele98, EK01].

2 Some problems and an affine result in \mathbb{R}^2 .

2.1 Isometries.

The first interesting case is the group of isometries of the Euclidean plane, i.e., $G = \{\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : \varphi \text{ preserves distances} \}$. Even the following is unknown.

Conjecture 2.1 There is an absolute constant C such that

$$\#\{\varphi: \mathbb{R}^2 \to \mathbb{R}^2 \text{ isometry}; |\varphi(\mathcal{P}) \cap \mathcal{P}| \ge 2\} \le C \cdot |\mathcal{P}|^3.$$

This, if true, could provide a missing link to the "distinct distances" problem of Erdős.

Remark 2.2 If we restrict ourselves to translations (=shifts), then it is not difficult to show — even in arbitrary dimension — that

$$\#\{\varphi: \mathbb{R}^d \to \mathbb{R}^d \text{ translation} ; |\varphi(\mathcal{P}) \cap \mathcal{P}| \ge k\} \le C \cdot |\mathcal{P}|^2/k,$$

for an absolute constant C and any $1 \leq k \leq |\mathcal{P}|$. (Just draw the — at least k — "arrows" which indicate which point is mapped to which other point.) Moreover, this order of magnitude is best possible, as shown by Example 4.1.

2.2 The affine group of \mathbb{R}^2 .

Our principal result Theorem 2.3 concerns the affine group of \mathbb{R}^2 , i.e. that of mappings $\varphi(x,y)=(a_1x+b_1y+c_1,\ a_2x+b_2y+c_2)$ where $a_1b_2-a_2b_1\neq 0$. Equivalently — using projective coordinates — this is the group of non-singular matrices

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

We shall restrict ourselves to the case $|\varphi(\mathcal{P}) \cap \mathcal{P}| \geq k = c|\mathcal{P}|$. According to Remark 1.5, we may expect a bound linear in $|\mathcal{P}|$, even here in the two dimensional plane.

However, we must face several degenerate situations. First, if \mathcal{P} is collinear then there are infinitely many affine mappings φ with $\varphi(\mathcal{P}) = \mathcal{P}$ — whence $|\varphi(\mathcal{P}) \cap \mathcal{P}| = |\mathcal{P}|$. Also requiring that $\varphi(\mathcal{P}) \cap \mathcal{P}$ be non-collinear — and, of course, $|\varphi(\mathcal{P}) \cap \mathcal{P}| \geq c|\mathcal{P}|$ — will not help: there still are point sets with a super-linear (actually, quadratic) number of such mappings (see Example 4.2).

That is why in our main result we must assume that $\varphi(\mathcal{P}) \cap \mathcal{P}$ is proper 2–dimensional. (Note that a positive proportion, i.e., cN points of a proper 2–dimensional set is still proper 2–dimensional — up to another (larger) constant factor C/\sqrt{c} .)

Theorem 2.3 (Main Theorem) Let $\mathcal{P} \subset \mathbb{R}^2$ be proper 2-dimensional up to a constant factor C. Moreover, let 0 < c < 1 be arbitrary. Then

$$\#\{\varphi: \mathbb{R}^2 \to \mathbb{R}^2 \text{ affine } ; |\varphi(\mathcal{P}) \cap \mathcal{P}| \ge c|\mathcal{P}|\} \le C^* \cdot |\mathcal{P}|,$$

where $C^* = C^*(C, c)$ does not depend on $|\mathcal{P}|$.

The following questions remain open.

Conjecture 2.4 For any $\mathcal{P} \subset \mathbb{R}^2$, any $3 \leq k \leq |\mathcal{P}|$ and any C > 0,

$$\#\{\varphi: \mathbb{R}^2 \to \mathbb{R}^2 \text{ affine } ; \varphi(\mathcal{P}) \cap \mathcal{P} \text{ proper 2D up to } C, \text{ and } |\varphi(\mathcal{P}) \cap \mathcal{P}| \ge k\}$$

 $< C^* \cdot |\mathcal{P}|^6/k^5.$

If true, this order of magnitude (as a function of $|\mathcal{P}|$ and k) is best possible, as shown by Example 4.3.

Problem 2.5 Does Theorem 2.3 hold if, instead of \mathcal{P} being proper 2D, we only require that

$$\frac{|\mathit{maximal\ collinear\ subset\ of\ } \mathcal{P}|}{|\mathcal{P}|} \to 0 \quad \mathit{while\ } |\mathcal{P}| \to \infty?$$

3 Higher dimensions

Conjecture 3.1 In \mathbb{R}^d , for the group G of non-degenerate (bijective) affine mappings,

$$f_G^d(N,k) = O\left(\frac{N^{2d+2}}{k^{2d+1}}\right).$$

The case d=2 was our Main Theorem 2.3. We mention without detailed proof that it can also be extended to d=3 (see Remark 5.13). However, we know nothing about $d \geq 4$.

We believe that the main feature of our Main Theorem 2.3 is that we could prove a sharp bound on the number of certain mappings even without an incidence bound at hand. Of course, life would be much easier if we had such bounds, e.g. the following.

Conjecture 3.2 Let C > 0 and $\mathcal{P} \subset \mathbb{R}^D$ proper D-dimensional up to C. We consider affine subspaces $S^{(r)} \subset \mathbb{R}^D$ of dimension r < D.

$$\#\{S^{(r)} \subset \mathbb{R}^D \; ; \; S^{(r)} \cap \mathcal{P} \; is \; r-dim \; up \; to \; C \; and \; |S^{(r)} \cap \mathcal{P}| \geq k\} \; \leq \; C^* \frac{|\mathcal{P}|^{r+1}}{k^{D+1}}.$$

(Part I, Problem 4.1 is the special case r = 1.) This bound can, again, be attained, see Example 4.4.

Remark 3.3 This conjecture would immediately imply the previous one. (Let $\mathcal{P}_0 \subset \mathbb{R}^d$ be an N-element set which attains $f_G^d(N,k)$ and apply Conjecture 3.2 with D=2d and r=d to the N^2 -element set $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_0 \times \mathcal{P}_0$ and the graphs — as subsets of $\mathbb{R}^d \times \mathbb{R}^d$ — of the non-degenerate affine mappings $\varphi: \mathcal{P}_0 \to \mathcal{P}_0$.)

4 Fine points and point sets

4.1 Proof of Propositions 1.3 and 1.4

As for Proposition 1.3, it is enough to consider the Cartesian product $\mathcal{P} \times \mathcal{P} \subset \mathbb{R}^2$ of $n = |\mathcal{P}| \times |\mathcal{P}|$ points and use an incidence bound of Szemerédi–Trotter [ST83] which states that, for any n points of \mathbb{R}^2 and any $2 \le k \le \sqrt{n}$, at most $C \cdot n^2/k^3$ straight lines can pass through $\ge k$ points each.

The proof of Proposition 1.4 is based upon the same idea, using (a special case of) a result of Pach and Sharir [PS98] which states that, for any n points of \mathbb{R}^2 and any $3 \leq k \leq \sqrt{n}$, at most $C \cdot n^s/k^{2s-1}$ members of a family of curves of s degrees of freedom can pass through $\geq k$ points each.

On the one hand, for hyperbolas y = (ax + b)/(cx + d) — which form a family of three degrees of freedom — this gives a bound of $C \cdot n^3/k^5 = C \cdot |\mathcal{P}|^6/k^5$. On the other hand, the degree of freedom for rational functions of total degree r is s = r + 1, resulting in an upper estimate $C \cdot n^{r+1}/k^{2r+1} = C \cdot |\mathcal{P}|^{2r+2}/k^{2r+1}$.

4.2 Examples in \mathbb{R}^2

Example 4.1 (Our original argument was probabilistic; here we present a simplified version of a construction by Géza Tóth.)

First we select a set $Y = \{y_1, \dots, y_t\}$ of t := n/(2k) reals — considered as points on the y-axis — such that the differences $y_i - y_j$ are all distinct for $i \neq j$. Then we define our point set as the $2k \times t$ Cartesian product

$$\mathcal{P} := \{1, 2, \dots, 2k\} \times Y \subset \mathbb{R}^2.$$

Now, for each pair $i \neq j$, we have at least k distinct translations which map k or more points at level y_i to points at level y_j , yielding a total of

$$t(t-1) \cdot k \approx \left(\frac{n}{2k}\right)^2 \cdot k = \frac{1}{4} \cdot \frac{n^2}{k}.$$

Example 4.2 Let \mathcal{P} consist of N=2n points: n of them on the x-axis (but otherwise arbitrary) plus n other points $(0,y_i)$ $(1 \le i \le n)$ such that no y_i is 0 and the quotients y_i/y_j $(i \ne j)$ are all distinct.

Then each of the affine mappings $(x,y) \mapsto (x,\lambda y)$ $(\lambda = y_i/y_j; i \neq j)$ map all n points on the x-axis to themselves plus exactly one point on the y-axis to another such point — a total of n+1 > N/2. Moreover, the number of such mappings is n(n-1) = N(N-2)/4, indeed quadratic in $N = |\mathcal{P}|$.

Example 4.3 Let $k \leq n/4$ and put $t = \sqrt{k}$. Define the $t \times (n/t)$ Cartesian product $\mathcal{P} \stackrel{\text{def}}{=} \{1 \dots t\} \times \{1 \dots n/t\} \subset \mathbb{R}^2$, a set of n elements. Then there are $\sim n^6/k^5$ non-degenerate affine mappings φ for which $\varphi(\mathcal{P}) \cap \mathcal{P}$ contains a parallelogram-lattice of $k = t \times t$ points.

Proof: $\mathcal{P} \times \mathcal{P} \subset \mathbb{R}^4$ is a $t \times (n/t) \times t \times (n/t)$ Cartesian product. Using (x, y, z, w) as coordinates, we describe some planes in \mathbb{R}^4 , parameterized by x and z (NOT y!), and we shall make sure that these planes be graphs of non-degenerate (bijective) affine mappings $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$.

Let the planes be determined by the equations

$$y = a_1x + b_1z + c_1$$

 $w = a_2x + b_2z + c_2$,

such that $a_1b_2 - a_2b_1 \neq 0$ and

$$a_1, a_2, b_1, b_2 \in \left\{1, 2, \dots, \frac{1}{3} \cdot \frac{n}{t^2}\right\};$$

$$c_1, c_2 \in \left\{1, 2, \dots, \frac{1}{3} \cdot \frac{n}{t}\right\}.$$

Then we have

$$\left(\frac{1}{3} \cdot \frac{n}{t^2}\right)^4 \cdot \left(\frac{1}{3} \cdot \frac{n}{t}\right)^2 = \frac{1}{729} \cdot \frac{n^6}{t^{10}}$$

such planes, of which, given any triple (a_1, b_1, a_2) , at most one b_2 can violate $a_1b_2 - a_2b_1 \neq 0$. Hence, at least

$$\frac{1}{729} \cdot \frac{n^6}{t^{10}} \cdot \left(1 - \frac{3t^2}{n}\right) \ge \frac{1}{729} \cdot \frac{n^6}{t^{10}} \cdot \frac{1}{4} > \frac{1}{3000} \cdot \frac{n^6}{t^{10}}$$

planes satisfy the extra condition. Moreover, each of these contains a $k = t \times t$ parallelogram lattice, e.g., the points which correspond to the values $x, z \in$

 $\{1,\ldots,t\}$, since then

$$y, w = a_i x + b_i y + c_i \le \left(\frac{1}{3} \cdot \frac{n}{t^2}\right) \cdot t + \left(\frac{1}{3} \cdot \frac{n}{t^2}\right) \cdot t + \frac{1}{3} \cdot \frac{n}{t} = \frac{n}{t}$$

We are left to show that these planes are graphs of bijective affine mappings. This follows from re-writing the equations as

$$z = -\frac{a_1}{b_1} \cdot x + \frac{1}{b_1} \cdot y - \frac{c_1}{b_1}$$

$$w = \frac{a_2b_1 - a_1b_2}{b_1} \cdot x + \frac{b_2}{b_1} \cdot y + \frac{c_2b_1 - c_1b_2}{b_1}$$

whose determinant is

$$\det = -\frac{a_1}{b_1} \cdot \frac{b_2}{b_1} - \frac{1}{b_1} \cdot \frac{a_2b_1 - a_1b_2}{b_1} = \frac{a_2}{b_1} \neq 0,$$

since $a_2 \neq 0$. We conclude that we have $(1/3000)n^6/k^5$ non-degenerate affine mappings φ for which $\varphi(\mathcal{P}) \cap \mathcal{P}$ is proper 2-dimensional, as required, since it contains a parallelogram-lattice of $k = t \times t$ points.

4.3 An example in \mathbb{R}^D

Example 4.4 Let r < D and $k \le N^{r/D}$ be given and assume (at the cost of a constant factor) that $k = t^r$ for a positive integer $t \le N^{1/D}$. Define $\mathcal{P} \subset \mathbb{R}^D$ as a cube lattice of size

$$t \times t \times \ldots \times t \times \left(\frac{N}{k}\right)^{\frac{1}{D-r}} \times \ldots \times \left(\frac{N}{k}\right)^{\frac{1}{D-r}}$$

a product of r + (D - r) terms. Consider the r-dimensional affine subspaces determined by the systems of equations

$$x_{r+1} = a_1^{(r+1)} x_1 + \ldots + a_r^{(r+1)} x_r + c^{(r+1)}$$

$$\vdots$$

$$x_D = a_1^{(D)} x_1 + \ldots + a_r^{(D)} x_r + c^{(D)},$$

for

$$a_i^{(.)} = 1, 2, \dots, \frac{1}{t(r+1)} \cdot \left(\frac{N}{k}\right)^{\frac{1}{D-r}}$$
 and $c^{(.)} = 1, 2, \dots, \frac{1}{r+1} \cdot \left(\frac{N}{k}\right)^{\frac{1}{D-r}}$.

Then each such affine subspace contains $k = t \times t \times ... \times t$ points of \mathcal{P} , i.e. those for $x_1, x_2, ..., x_r \in \{1, 2, ..., t\}$. Moreover, the number of them is at least

$$\begin{split} & \geq \left[\frac{1}{t(r+1)} \cdot \left(\frac{N}{k}\right)^{\frac{1}{D-r}}\right]^{(D-r) \cdot r} \cdot \left[\frac{1}{r+1} \cdot \left(\frac{N}{k}\right)^{\frac{1}{D-r}}\right]^{(D-r)} & \approx \\ & \approx c(r,D) \cdot \left(\frac{1}{k}\right)^{D-r} \cdot \left(\frac{N}{k}\right)^{r+1} = c(r,D) \cdot \frac{N^{r+1}}{k^{D+1}}, \end{split}$$

where c(r, D) > 0 does not depend on N or k.

5 Some Lemmata and the main proof

5.1 Arrangements of straight lines in \mathbb{R}^2 .

Let H be a (finite) set of n planes in \mathbb{R}^2 . They cut the plane into at most $\binom{n}{2} + \binom{n}{1} + \binom{n}{0} \sim n^2$ open convex cells, with equality iff H is in general position, i.e., if any two intersect but no three do.

The set of cells, together with their vertices and edges, is called the *arrangement* defined by H. We shall denote it by $\mathcal{A}(H)$.

For two cells $C_i, C_i \in \mathcal{A}(H)$, a natural notion of distance is

$$\operatorname{dist}(\mathcal{C}_i, \mathcal{C}_j) \stackrel{\operatorname{def}}{=} \#\{h \in H ; h \text{ separates } \mathcal{C}_i \text{ and } \mathcal{C}_j\}.$$

It is easy to see that "dist" is a metric, i.e. it satisfies the triangle inequality.

The result presented below bounds — from above, in terms of |H| — the number of pairs (C_i, C_j) whose distance is at most a given $\varrho > 0$. To this end, we also defined in Part I [Ele05] the $\leq \varrho$ -neighborhood of a cell C_j as

$$B_{\varrho}(\mathcal{C}_{j}) = \{ \mathcal{C}_{i} \in \mathcal{A}(H) ; \operatorname{dist}(\mathcal{C}_{i}, \mathcal{C}_{j}) \leq \varrho \},$$

and the number of " ϱ -close pairs" mentioned above equals $\sum |B_{\varrho}(\mathcal{C}_{j})|$.

Proposition 5.1 $\sum_{\mathcal{C}_j \in \mathcal{A}(H)} |B_{\varrho}(\mathcal{C}_j)| = O(\varrho^2 |H|^2)$ in \mathbb{R}^2 and $= O(\varrho^3 |H|^3)$ in \mathbb{R}^3 .

Proof: See Part I [Ele05], Corollary 3.3 (??) and Lemma 3.7 (??).

As for the second moment $\sum |B_{\varrho}(\mathcal{C}_j)|^2$, it may not always be bounded by a quadratic function of |H| (e.g., if the lines all surround a regular polygon).

Problem 5.2 Let $\mathcal{A}(H)$ be a simple arrangement in \mathbb{R}^2 . Is it true that it can be refined to an $\mathcal{A}(H^+)$ by adding O(|H|) new straight lines such that $\sum_{\mathcal{C}_j \in \mathcal{A}(H^+)} |B_{\varrho}(\mathcal{C}_j)|^2 = O(\varrho^4 |H|^2)$?

It may well be true that one can even force the stronger upper bound $|B_{\varrho}(\mathcal{C}_j)| = O(\varrho^2)$ for each $\mathcal{C}_j \in \mathcal{A}(H^+)$ — but it is "even more" unknown. However, the following weaker version is an easy consequence of Proposition 5.1.

Corollary 5.3 There is an absolute constant K_1 with the following property. Let $\mathcal{A}(H)$ be an arrangement, $\varrho \leq |H|$ and c > 0 arbitrary. Given any subset $A_0 \subset \mathcal{A}(H)$ of $\geq c|H|^2$ cells, there is a sub-subset $A_1 \subset A_0$ of at least $(c/2)|H|^2$ cells (i.e., half of them) which satisfy

$$|B_{\varrho}(\mathcal{C}_j)| \leq \frac{K_1}{c} \varrho^2 \text{ for each } \mathcal{C}_j \in A_1.$$

Proof: Let K_0 be the absolute constant hidden in the right hand side of Proposition 5.1, i.e., $\sum |B_{\varrho}(C_j)| \leq K_0 \varrho^2 |H|^2$. Then $K_1 = 2K_0$ satisfies the requirement of our Corollary, since otherwise more than $(c/2)|H|^2$ cells with ϱ -neighborhoods larger than $(2K_0/c)\varrho^2$ would give rise to a total strictly more than allowed.

Remark 5.4 The previous Corollary holds for $c|H|^3$ cells in \mathbb{R}^3 , too, with an upper bound $|B_{\rho}(\mathcal{C}_i)| \leq (K_1/c)\varrho^3$ for another (larger) K_1 .

Unfortunately, neither Proposition 5.1, nor Corollary 5.3 is known in dimensions higher than three. That is the main reason why we cannot extend Theorem 2.3 to such generality.

5.2 The triangle selection lemma

Proposition 5.5 Let A(H) be a simple arrangement (i.e. the lines of H are in general position). Then for any $C_i \in A(H)$ and any $\varrho \leq |H|$ we have

$$|B_{\varrho}(\mathcal{C}_j)| > \frac{\varrho^2}{32}.$$

Proof: We demonstrate the statement in two steps.

Step 1. $B_{\rho}(\mathcal{C}_j)$ contains at least $\varrho^2/8$ vertices.

We copy the proof from [Wel92]. Draw a generic straight line which enters C_j and select, on one of its two rays, the $\varrho/2$ lines $h \in H$ which are closest to C_j . On each such line walk $\varrho/2$ steps in any direction, reaching at least $(\varrho/2) \cdot (\varrho/2) = \varrho^2/4$ vertices of $\mathcal{A}(H)$. Each of these is counted at most twice (since \mathcal{A} is simple), whence the required inequality.

Step 2. Using the bound found in Step 1, we want to show $|B_{\varrho}(\mathcal{C}_j)| \geq \varrho^2/32$. To this end, consider the edges of $B_{\varrho}(\mathcal{C}_j)$ as a graph on its vertex set. Denote by v_4 the number of vertices of degree exactly 4 and note that this set coincides with the vertex set of $B_{\varrho-1}(\mathcal{C}_j)$. Beyond them, there usually exist some other vertices along the "outer boundary", each of degree exactly 2. If there is such an "outer" vertex, we delete it and glue together the two original edges incident on it to form one new edge. (This changes neither v_4 nor the number of faces.) We repeat until only v_4 vertices remain, each of degree 4. Then, using Euler's relation,

$$v_4 + f = e + 2 = \frac{4v_4}{2} + 2 = 2v_4 + 2,$$

whence $f = v_4 + 2 \ge (\varrho - 1)^2/8 + 2$, by the lower bound in Step 1. Thus

$$|B_{\varrho}(C_j)| = \# \text{ of bounded faces } = f - 1 \ge \frac{(\varrho - 1)^2}{8} + 1 > \frac{\varrho^2}{32},$$

since the inequality is obvious for $\varrho < 2$ and $\varrho - 1 \ge \varrho/2$ for $\varrho \ge 2$.

Remark 5.6 Proposition 5.5 can be extended to \mathbb{R}^d as $|B_{\varrho}(\mathcal{C}_j)| > c_d \varrho^d$, for some $c_d > 0$, independent of ϱ .

Lemma 5.7 (Triangle–selection Lemma) Let there be given a set \mathcal{P} of m points in \mathbb{R}^2 which lie in distinct cells of a simple arrangement of Km cells. (K is fixed while m is large.)

If $\varrho \geq \varrho_0(K) = 2048K + 1024$, then there exist at least m/6 triangles $U_1U_2U_3$, each spanned by three given points, such that $dist(U_iU_j) \leq \varrho$ for $1 \leq i, j \leq 3$.

Proof: If a point P_j in cell C_j is not the vertex of any such "small" triangle then $B_{\rho/2}(C_j) \cap \mathcal{P}$ consists of collinear points and thus

$$|B_{\varrho/2}(\mathcal{C}_j) \cap \mathcal{P}| \le \varrho + 1 \le 2\varrho.$$

Consider the even smaller neighborhood $B_{\varrho/4}(\mathcal{C}_j)$. By Proposition 5.5, it contains

$$|E| \ge |B_{\varrho/4}(\mathcal{C}_j)| - |B_{\varrho/2}(\mathcal{C}_j) \cap \mathcal{P}| \ge \frac{\varrho^2}{512} - 2\varrho^2$$

empty cells. Moreover, each such empty cell C_i was counted at most

$$|B_{\varrho/4}(\mathcal{C}_i) \cap \mathcal{P}| \le |B_{\varrho/2}(\mathcal{C}_j) \cap \mathcal{P}| \le 2\varrho$$

times, giving a total of $\leq K \cdot m \cdot 2\varrho$. Thus the number of points which are incident upon no good triangle is

$$\leq \frac{K \cdot m \cdot 2\varrho}{\varrho^2/512 - 2\varrho} = \frac{K \cdot m \cdot 2}{\varrho/512 - 2} = \frac{1024Km}{\varrho - 1024} \leq \frac{m}{2},$$

if $\varrho \ge \varrho_0(K) = 2048K + 1024$. Therefore, at least m/2 points ARE incident upon at least one good "small" triangle, giving

$$\geq \frac{m/2}{3} = \frac{m}{6}$$

distinct such triangles.

Remark 5.8 In \mathbb{R}^d , if $\varrho \geq \varrho_0^{(d)}(K)$, then it is possible to select $c_d m$ "small" simlpices (where $c_d > 0$ and $\varrho_0^{(d)}(K)$ do not depend on m).

5.3 The "Average-Forcing Lemma"

We start with a simple observation.

Let \mathcal{U}_1 , \mathcal{U}_2 , \mathcal{U}_3 be disjoint finite sets and Δ a system of (ordered) triples (U_1, U_2, U_3) such that $U_i \in \mathcal{U}_i$ for i = 1, 2, 3. As usual, we call the number of triples which contain a given element U_i the degree of U_i and denote it by $\deg_{\Delta}(U_i)$. We shall also use the average degree in \mathcal{U}_i which, of course, is $|\Delta|/|\mathcal{U}_i|$.

Proposition 5.9 (Folklore) Given a triple system Δ as above, there always exist subsets $\mathcal{U}'_i \subset \mathcal{U}_i$ (for i = 1, 2, 3) such that the subsystem

$$\Delta' \stackrel{\text{def}}{=} \{ (U_1, U_2, U_3) \in \Delta \; ; \; U_i \in \mathcal{U}'_i \quad (for \; i = 1, 2, 3) \}$$

satisfies

(i) Each new degree is at least one quarter of the corresponding original average degree, i.e., for each $U_i \in \mathcal{U}'_i$ we have

$$\deg_{\Delta'} U_i \ge \frac{1}{4} \cdot \frac{|\Delta|}{|\mathcal{U}_i|};$$

(ii) At least one quarter of the original triples are preserved, i.e.,

$$|\Delta'| \ge \frac{1}{4} \cdot ||\Delta|.$$

Proof: Repeatedly delete those elements whose degree is less than required. You cannot delete more than $3(|\Delta|/4)$ triples.

The setting of the following Lemma is "2/3 Euclidean" while "1/3 abstract". We start with two (disjoint) point sets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$ and another (abstract) set \mathcal{S} — that is why the Lemma is "1/3 abstract". Moreover, let Δ be a triple system on these three sets, as in the previous Proposition, with the additional requirement that

any pair (S, P_i) is contained in at most one triple of Δ .

We assume that $|\mathcal{P}_i| \leq N$ for i = 1, 2 while $|\mathcal{S}| = \lambda N$ and each $S \in \mathcal{S}$ has $\deg_{\Delta}(S) \geq cN$, for a fixed c > 0. Finally, let H_1 and H_2 be two systems of straight lines in \mathbb{R}^2 in general position, such that, for i = 1, 2, all $P_i \in \mathcal{P}_i$ lie in distinct cells of $\mathcal{A}(H_i)$ while $|H_i| \leq C\sqrt{N}$, for a fixed C > 0.

Lemma 5.10 (Average–Forcing Lemma) Beyond the setting outlined above, let there be given an arbitrary $\varrho > 0$. Then there exist $\mathcal{P}_1^* \subset \mathcal{P}_1$, $\mathcal{P}_2^* \subset \mathcal{P}_2$, and $\mathcal{S}^* \subset \mathcal{S}$ which, together with the corresponding subsystem Δ^* , satisfy

- (i) $|\mathcal{S}^*| \ge c^* |\mathcal{S}| = c^* \lambda N$;
- (ii) for all $S \in \mathcal{S}^*$ we have $\deg_{\Delta^*} S \geq c^* N$;
- (iii) consequently $|\mathcal{P}_i^*| \geq c^* N$, for i = 1, 2;
- (iv) for any $P_i \in \mathcal{P}_i^*$ and the cell $C_i \in \mathcal{A}(H_i)$ which contains P_i , we have

$$|B_{\varrho}(\mathcal{C}_i)| \leq C^* \varrho^2;$$

where $c^* = c^*(c, C)$ and $C^* = C^*(c, C)$ do not depend on N, ϱ , or λ .

Proof: In order to fulfill (iv) we shall, of course, use Corollary 5.3 twice; once for \mathcal{P}_1 and once for \mathcal{P}_2 . However, at each step, we must make sure that the subsets selected still participate in many triples of Δ . This will be achieved by always applying Proposition 5.9 before each such selection — and also at the very end, to guarantee (ii).

Though this brief outline is certainly enough for the experienced reader, we also describe all details as follows.

Step 1. First, using Proposition 5.9, we select $\mathcal{P}'_1 \subset \mathcal{P}_1$, $\mathcal{P}'_2 \subset \mathcal{P}_2$, $\mathcal{S}' \subset \mathcal{S}$ and the corresponding subsystem Δ' , such that $|\Delta'| \geq (1/4)|\Delta| \geq (1/4)c\lambda N^2$, whence

$$|\mathcal{P}_i'| \ge \frac{(1/4)c\lambda N^2}{\lambda N} = \frac{1}{4}cN = \frac{c}{4C^2} \left(C\sqrt{N}\right)^2 \quad \text{and} \quad |\mathcal{S}'| \ge \frac{|\Delta'|}{|\mathcal{P}_i'|} \ge \frac{(1/4)c\lambda N^2}{N} = \frac{1}{4}c\lambda N.$$

Moreover, for all $P_i \in \mathcal{P}'_i$,

$$\deg_{\Delta'} P_i \ge \frac{1}{4} \cdot \frac{c\lambda N^2}{N} = \frac{1}{4}c\lambda N. \tag{1}$$

Step 2. Using Corollary 5.3 with $c/(4C^2)$ in place of c there, we can select a $\mathcal{P}_1'' \subset \mathcal{P}_1'$ of at least (1/8)cN elements, such that the cells $\mathcal{C}_j \in \mathcal{A}(H_1)$ which contain its points, satisfy

$$|B_{\varrho}(\mathcal{C}_j)| \le K_1 \frac{4C^2}{c} \varrho^2.$$

By (1), these $P_1 \in \mathcal{P}_1''$ participate in a total of

$$\geq \left(\frac{1}{8}cN\right) \cdot \left(\frac{1}{4}c\lambda N\right) = \frac{c^2}{32}\lambda N^2$$

triples. We denote the set of these by Δ'' while preserving $\mathcal{P}_2'' = \mathcal{P}_2'$ and $\mathcal{S}'' = \mathcal{S}'$. At the moment we have

$$\frac{c}{8}N \le |\mathcal{P}_1''| \le N;$$

$$\frac{c}{4}N \le |\mathcal{P}_2''| \le N;$$

$$\frac{1}{4}c\lambda N \le |\mathcal{S}''| \le \lambda N;$$

$$|\Delta''| \ge \frac{c^2}{32}\lambda N^2,$$

whence the average degree in S'' is at least $(c^2/32)N$.

Step 3. As in Step 1, we use Proposition 5.9 to select $\mathcal{P}_1''' \subset \mathcal{P}_1''$, $\mathcal{P}_2''' \subset \mathcal{P}_2''$, $\mathcal{S}''' \subset \mathcal{S}''$ and the corresponding subsystem Δ''' , such that $|\Delta'''| \geq (1/4)|\Delta''| \geq (c^2/128)\lambda N^2$, whence

$$|\mathcal{P}_{i}^{"'}| \ge \frac{(c^2/128)\lambda N^2}{\lambda N} = \frac{c^2}{128}N = \frac{c^2}{128C^2}(C\sqrt{N})^2$$
 and $|\mathcal{S}^{"'}| \ge \frac{|\Delta^{"'}|}{|\mathcal{P}_{i}^{"'}|} \ge \frac{(c^2/128)\lambda N^2}{N} = \frac{c^2}{128}\lambda N.$

Moreover, for all $P_i \in \mathcal{P}_i^{\prime\prime\prime}$,

$$\deg_{\Delta'''} P_i \ge \frac{1}{4} \frac{(c^2/32)\lambda N^2}{N} = \frac{c^2}{128} \lambda N.$$
 (2)

Step 4. Using Corollary 5.3 again, this time with $c^2/(128C^2)$ in place of c there, we can select a $\mathcal{P}_2'''' \subset \mathcal{P}_2'''$ of at least $(1/2)|\mathcal{P}_2'''| \geq (c^2/256)N$ elements, such that the cells $\mathcal{C}_j \in \mathcal{A}(H_2)$ which contain its points, satisfy

$$|B_{\varrho}(\mathcal{C}_j)| \le K_1 \frac{128C^2}{c^2} \varrho^2.$$

By (2), these $P_2 \in \mathcal{P}_2''''$ participate in a total of

$$\geq \left(\frac{c^2}{256}N\right) \cdot \left(\frac{c^2}{128}\lambda N\right) = \frac{c^4}{2^{15}}\lambda N^2$$

triples. We denote the set of these by Δ'''' while preserving $\mathcal{P}_1'''' = \mathcal{P}_1'''$ and $\mathcal{S}'''' = \mathcal{S}'''$.

At this moment we have

$$\begin{split} \frac{c^2}{128} N \leq & |\mathcal{P}_1''''| \leq N; \\ \frac{c^2}{256} N \leq & |\mathcal{P}_2''''| \leq N; \\ \frac{c^2}{128} \lambda N \leq & |\mathcal{S}''''| \leq \lambda N; \\ & |\Delta''''| \geq \frac{c^4}{215} \lambda N^2, \end{split}$$

whence the average degree in S'''' is at least $(c^4/2^{15})N$ and at least $(c^4/2^{15})\lambda N$ in the \mathcal{P}_i'''' .

Step 5. Finally, we use Proposition 5.9 for the third time to select subsets $\mathcal{P}_1^* \subset \mathcal{P}_1''''$, $\mathcal{P}_2^* \subset \mathcal{P}_2''''$ and $\mathcal{S}^* \subset \mathcal{S}''''$ together with the corresponding subsystem $\Delta^* \subset \Delta''''$, which already satisfy the requirements of Lemma 5.10 with $c^* \stackrel{\text{def}}{=} c^4/2^{16}$ and $C^* \stackrel{\text{def}}{=} 128K_1C^2/c^2$.

Remark 5.11 Lemma 5.10 can be extended to \mathbb{R}^3 with $c^*\varrho^3$ on the right hand side of (iv). As for higher dimensions, no such result is known, due to the lack of suitable generalizations of Proposition 5.1 and Corollary 5.3.

5.4 Proof of the Main Theorem 2.3

The graph of an affine mapping φ is a plane $S = S_{\varphi}$ in $\mathbb{R}^2 \times \mathbb{R}^2$. The assumption of $|\varphi(\mathcal{P}) \cap \mathcal{P}| \geq c|\mathcal{P}|$ is equivalent to $|S_{\varphi} \cap (\mathcal{P} \times \mathcal{P})| \geq c|\mathcal{P}|$. In such a case we shall say that the plane is " $c|\mathcal{P}|$ -rich". Thus it suffices to prove the following.

Theorem 5.12 Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^2$ with $|\mathcal{P}_1| = |\mathcal{P}_2| = N$ be proper 2-dimensional point sets up to a constant factor C. Moreover, let 0 < c < 1 be arbitrary. Then

$$\#\{planes\ S\subset\mathbb{R}^4\ ;\ |S\cap(\mathcal{P}_1\times\mathcal{P}_2)|\geq cN\}\ \leq\ C_1\cdot N,$$

where $C_1 = C_1(C, c)$ does not depend on $N = |\mathcal{P}_i|$.

Proof: Denote the set of cN-rich planes by S and let $|S| = \lambda N$, for some positive λ . We want to bound λ from above by a constant C_1 . To this end we shall use Lemmata 5.7 and 5.10 for a value of $\varrho = \varrho(C, c)$, to be specified later in terms of $c^* = c^*(c, C)$ and $C^* = C^*(c, C)$, given in Lemma 5.10.

We shall call the planes $\{(x, y, 0, 0) ; x, y \in \mathbb{R}\}$ and $\{(0, 0, z, w) ; z, w \in \mathbb{R}\}$ in \mathbb{R}^4 — which contain the copies of the \mathcal{P}_i generating $\mathcal{P}_1 \times \mathcal{P}_2$ — the first and second coordinate plane, respectively.

First we define the triple system

$$\Delta \stackrel{\text{def}}{=} \{ (P_1, P_2, S) ; P_i \in \mathcal{P}_i, S \in \mathcal{S}, \{P_1\} \times \{P_2\} \in S \}$$

and apply Lemma 5.10 to Δ and the two systems H_1 , H_2 of straight lines which cut the \mathcal{P}_i into singletons, according to the "proper 2D" assumption. Thus we can find subsets $\mathcal{P}_1^* \subset \mathcal{P}_1$, $\mathcal{P}_2^* \subset \mathcal{P}_2$, and $\mathcal{S}^* \subset \mathcal{S}$ together with the corresponding subsystem Δ^* , with the following properties.

- (a) $|\mathcal{S}^*| \ge c^* |\mathcal{S}| = c^* \lambda N$ and for all $S \in \mathcal{S}^*$ we have $\deg_{\Delta^*}(S) \ge c^* N$. For these S, we introduce
 - (i) the notation $m_{\mathcal{S}} \stackrel{\text{def}}{=} |S \cap (\mathcal{P}_1^* \times \mathcal{P}_2^*)| \geq c^* N;$
 - (ii) and also a new arrangement \mathcal{A}_S defined by the union of projections (to S) of H_1 and H_2 (considered in the first and second coordinate plane and cutting \mathcal{P}_1 and \mathcal{P}_2 , respectively, into singletons).

Thus

$$|\mathcal{A}_S| \le (2C\sqrt{N})^2 = 4C^2N = \frac{4C^2}{c^*} \cdot c^*N \le \frac{4C^2}{c^*} \cdot m_S.$$

Putting $K = 4C^2/c^*$ and defining $\varrho = \varrho(K) = 2048K + 1024$ we can apply Lemma 5.7, to find in each $S \in \mathcal{S}^*$ at least $m/6 \ge c^*N/6$ distinct "small" triangles whose vertices are in $\mathcal{P}_1^* \times \mathcal{P}_2^*$ and any pair of vertices has dist $\le \varrho$ as measured in \mathcal{A}_S .

(b) On the other hand, for i=1,2, each $P_i \in \mathcal{P}_i^*$ lies in an original cell $\mathcal{C}_i \in \mathcal{A}(H_i)$ with $|B_{\varrho}(\mathcal{C}_i)| \leq C^* \varrho^2$. Therefore, the number of small triangles which have P_i as a vertex, cannot exceed $|B_{\varrho}(\mathcal{C}_i)|^2 \leq (C^*)^2 \varrho^4$. (Note that the distance measured in \mathcal{A}_S is at least as much as if measured in any $\mathcal{A}(H_i)$.) Hence the total number of such triangles is $\leq (C^*)^2 \varrho^4 |\mathcal{P}_1^*| |\mathcal{P}_2^*| = (C^*)^2 \varrho^4 N^2$.

Putting (a) and (b) together for the already defined ϱ , we have

$$(c^*\lambda N)\cdot (c^*N/6) \le \#$$
 of small triangles $\le (C^*)^2\varrho^4N^2$,

whence

$$\lambda \leq \frac{6(C^*)^2}{(c^*)^2} \cdot \varrho^4 \stackrel{\text{def}}{=} C_1(C, c),$$

a constant in terms of the parameters C and c.

Remark 5.13 Essentially the same idea works in \mathbb{R}^3 , with Remarks 5.8 and 5.11 in place of Lemmata 5.7 and 5.10, respectively.

Remark 5.14 An affirmative answer to Problem 5.2 could substitute the very technical and complicated Lemma 5.10 in the proof of Theorem 2.3 as follows. We first refine H_1 and H_2 , by adding $O(\sqrt{N})$ lines, to get some H_1^+ and H_2^+ with small ϱ -neighborhoods. Then we use the Triangle-Selection Lemma 5.7 for the corresponding \mathcal{A}_S^+ and finish as in Step (b).

This could even extend to higher dimensions, provided that Problem $5.2~\mathrm{can}$.

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