Multiple Dedekind Zeta Values are Periods of Mixed Tate Motives

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Abstract

Recently, the author defined multiple Dedekind zeta values [5] associated to a number K field and a cone C. These objects are number theoretic analogues of multiple zeta values. In this paper we prove that every multiple Dedekind zeta value over any number field K is a period of a mixed Tate motive. Moreover, if K is a totally real number field, then we can choose a cone C so that every multiple Dedekind zeta associated to the pair (K;C) is unramified over the ring of algebraic integers in K. In [7], the author proves similar statements in the special case of a real quadratic fields for a particular type of a multiple Dedekind zeta values.

The mixed motives are defined over K in terms of a the Deligne-Mumford compactification of the moduli space of curves of genus zero with n marked points.

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1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}$$

is widely used in number theory, algebraic geometry and quantum field theory. Euler's multiple zeta values

$$\zeta(s_1, \dots, s_m) = \sum_{0 < n_1 < \dots < n_m} \frac{1}{n_1^{s_1} \dots n_m^{s_m}},$$

where s_1, \ldots, s_m are positive integers and $s_m \geq 2$, appear as values of some Feynman amplitudes, and in algebraic geometry, as periods of mixed Tate motives over $Spec(\mathbb{Z})$ (see [4], [3], [1], [8]).

Dedekind zeta values

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N(\mathfrak{a})^s},$$

are a generalization of the Riemann zeta function to a number field K. In some Feynman amplitudes one of the summands is $\log(1+\sqrt{2})$ or $\log\left(\frac{1+\sqrt{5}}{2}\right)$. These values are essentially the residues at s=1 of Dedekind zeta functions over $\mathbb{Q}(\sqrt{2})$ and over $\mathbb{Q}(\sqrt{5})$, respectively. For $s=2,3,4,\ldots$ the values $\zeta_K(s)$ are periods of mixed Tate motives over the ring of algebraic integers in K with ramification only at the discriminant of K (see [2]).

In [5], the author has constructed multiple Dedekind zeta values, which are a generalization of Euler's multiple zeta values to number fields in the same way as Dedekind zeta values generalizes Riemann zeta values. For a quadratic number field K, the key examples of multiple Dedekind zeta values are

$$\zeta_{K;C}(s_1,\dots,s_m) = \sum_{\beta_1,\dots,\beta_m \in C} \frac{1}{N(\beta_1)^{s_1} N(\beta_1 + \beta_2)^{s_2} \cdots N(\beta_1 + \dots + \beta_m)^{s_m}}, \quad (1)$$

where s_1, \ldots, s_m are positive integers and $s_m \geq 2$ and C is a cone generated by totally positive algebraic integers e_1, \ldots, e_n in K defined by

$$C = \mathbb{N}\{e_1, \dots, e_n\} = \{\gamma \in K \mid \gamma = a_1 e_1 + \dots a_i e_i, \text{ for positive integers } a_i\}.$$

Similar types of cones were considered by Zagier in [9] and [10].

In [5], the author has proven that multiple Dedekind zeta values can be interpolated to multiple Dedekind zeta functions, which have meromorphic continuation to all complex values of the variables s_1, \ldots, s_m .

In this paper we prove the following two theorems.

Theorem 1 Every multiple Dedekind zeta over any number field K is a period of a mixed Tate motive over K.

Theorem 2 If K is a totally real field, then we can find a cone C such that every multiple Dedekind zeta $\zeta_{K;C}(s_1,\ldots,s_m)$ is a period of mixed Tate motive, which is unramified over any prime.

2 Background

2.1 Multiple zeta values and iterated integrals

The Riemann zeta function at the value s=2 can be expressed in term of an iterated integral in the following way

$$\int_0^1 \left(\int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} = \int_0^1 \left(\int_0^y (1+x+x^2+x^3\dots) dx \right) \frac{dy}{y}$$

$$= \int_0^1 \left(y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \dots \right) \frac{dy}{y} = y + \frac{y^2}{2^2} + \frac{y^3}{3^2} + \frac{y^4}{4^2} \dots \Big|_{y=0}^{y=1}$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots = \zeta(2).$$

Let us examine the domain of integration of the iterated integral. Note that 0 < x < y and 0 < y < 1. We can put both inequalities together. Then we obtain the domain 0 < x < y < 1, which is a simplex. Thus, we can express the iterated integral as

$$\zeta(2) = \int_0^1 \left(\int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} = \int_{0 < x < y < 1} \frac{dx}{1-x} \wedge \frac{dy}{y}.$$

Moreover, Goncharov and Manin [4] have expressed all multiple zeta values of weight m as periods of motives related to the moduli space of curves of genus zero with m+3 marked points, $\mathcal{M}_{0,m+3}$. In particular, $\zeta(2)$ can be expressed as a period of the motive $H^2(\overline{\mathcal{M}}_{0,5}-A,B-A\cap B)$ by pairing of $[\Omega_A]\in Gr_4^WH^2(\overline{\mathcal{M}}_{0,5}-A)$ for $\Omega_A=\frac{dx}{1-x}\wedge\frac{dy}{y}$, with $[\Delta_B]\in \left(Gr_0^WH^2(\overline{\mathcal{M}}_{0,5}-B)\right)^\vee$. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,5}$ of the moduli space $\mathcal{M}_{0,5}$ can be obtained by three blow-ups of $\mathbb{P}^1\times\mathbb{P}^1$ at the points (0,0), (1,1) and (∞,∞) . Let us name the exceptional divisors at the three points by E_0 , E_1 and E_∞ , respectively. Then $A=(x=1)\cup(y=0)\cup(x=\infty)\cup(y=\infty)\cup E_\infty$ and $B=(x=0)\cup(x=y)\cup(y=1)\cup E_0\cup E_1$.

Similarly, one can express $\zeta(3)$ and $\zeta(1,2)$ as iterated integrals

$$\zeta(3) = \int_0^1 \left(\int_0^z \left(\int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} \right) \frac{dz}{z} = \int_{0 < x < y < z < 1} \frac{dx}{1-x} \wedge \frac{dy}{y} \wedge \frac{dz}{z},$$

$$\zeta(1,2) = \int_0^1 \left(\int_0^z \left(\int_0^y \frac{dx}{1-x} \right) \frac{dy}{1-y} \right) \frac{dz}{z} = \int_{0 < x < y < z < 1} \frac{dx}{1-x} \wedge \frac{dy}{1-y} \wedge \frac{dz}{z}.$$

Again, $\zeta(3)$ and $\zeta(1,2)$ can be expressed as periods of motives related to $\mathcal{M}_{0,6}$. In the same paper, Goncharov and Manin prove that the motives associated to multiple zeta values (MZVs) are mixed Tate motives unramified over $Spec(\mathbb{Z})$.

A few years later, Francis Brown [1] proved that periods of mixed Tate motives unramified over $Spec(\mathbb{Z})$ can be expressed as a \mathbb{Q} -linear combination of MZVs times an integer power of $2\pi i$.

2.2 Multiple Dedekind zeta values (MDZVs) and iterated integrals on membranes

We recall the construction of MDZVs. Let \mathcal{O}_K be the ring of integers in a number field K of degree n over \mathbb{Q} .

Denote by $\sigma_1, \ldots, \sigma_n$ all the embedding of K into the complex numbers \mathbb{C} . And let e_1, \ldots, e_n be elements of \mathcal{O}_K such that

- 1. $e_i \in \mathcal{O}_K$ for all i
- 2. (e_1,\ldots,e_n) forms a basis of K over \mathbb{Q}
- 3. $Re(\sigma_i(e_i) \geq 0 \text{ for all } i \text{ and } j$

Let C be the cone defined as N-linear combinations of e_1, \ldots, e_2 , that is,

$$C = \{ \alpha \in \mathcal{O}_K \mid \gamma = a_1 e_1 + \dots + a_n e_n, \text{ for } a_1, \dots, a_n \in \mathbb{N} \}.$$

Let

$$f_0(C; t_1, \dots, t_n) = \sum_{\alpha \in C} \exp \left[-\sum_{i=1}^n t_j \sigma_j(\alpha) \right].$$

We express $\zeta_{K;C}(2)$, $\zeta_{K;C}(3)$ and $\zeta_{K;C}(1,2)$ as iterated integrals on a membrane. See [5] and [6], for more examples and properties of these constructions. We have

$$\int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(\int_{u_{1}}^{\infty} \cdots \int_{u_{n}}^{\infty} f_{0}(C; t_{1}, \dots, t_{n}) dt_{1} \dots dt_{n} \right) du_{1} \dots du_{n} \tag{2}$$

$$= \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(\int_{u_{1}}^{\infty} \cdots \int_{u_{n}}^{\infty} \sum_{\alpha \in C} \exp \left[-\sum_{j=1}^{n} t_{j} \sigma_{j}(\alpha) \right] dt_{1} \dots dt_{n} \right) du_{1} \dots du_{n}$$

$$= \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(\sum_{\alpha \in C} \left(\prod_{j=1}^{n} \exp \left[-t_{j} \sigma_{j}(\alpha) \right] \right) dt_{1} \dots dt_{n} \right) du_{1} \dots du_{n}$$

$$= \int_{0}^{\infty} \cdots \int_{0}^{\infty} \sum_{\alpha \in C} \prod_{j=1}^{n} \frac{\exp \left[-u_{j} \sigma_{j}(\alpha) \right]}{\prod_{j} \sigma_{j}(\alpha)} du_{1} \dots du_{n}$$

$$= \sum_{\alpha \in C} \frac{1}{(\prod_{j} \sigma_{j}(\alpha))^{2}} = \sum_{\gamma \in C} \frac{1}{N(\alpha)^{2}}$$

$$= \zeta_{K;C}(2).$$

Similarly,

$$\begin{split} &\int_0^\infty \dots \int_0^\infty \left(\int_{v_1}^\infty \dots \int_{v_n}^\infty \left(\int_{u_1}^\infty \dots \int_{u_n}^\infty \right. \right. \\ &\left. f_0(C;t_1,\dots,t_n) dt_1 \cdots dt_n \right) du_1 \cdots du_n \right) dv_1 \cdots dv_n \\ &= \int_0^\infty \dots \int_0^\infty \left(\int_{v_1}^\infty \dots \int_{v_n}^\infty \left(\int_{u_1}^\infty \dots \int_{u_n}^\infty \right. \right. \\ &\left. \sum_{\alpha \in C} \exp \left[-\sum_{j=1}^n t_j \sigma_j(\alpha) \right] dt_1 \dots dt_n \right) du_1 \dots du_n \right) dv_1 \cdots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \dots \int_0^\infty \left(\int_{v_1}^\infty \dots \int_{v_n}^\infty \prod_{j=1}^n \frac{\exp \left[-u_j \sigma_j(\alpha) \right]}{\sigma_j(\alpha)} du_1 \dots du_n \right) dv_1 \dots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha) \right]}{\sigma_j(\alpha)^2} dv_1 \dots dv_n = \sum_{\alpha \in C} \frac{1}{N(\alpha)^3} \\ &= \zeta_{K;C}(3). \end{split}$$

We recall the simplest type of multiple Dedekind zeta value

$$\begin{split} &\int_0^\infty \cdots \int_0^\infty \left(\int_{v_1}^\infty \cdots \int_{v_n}^\infty \left(\int_{u_1}^\infty \cdots \int_{u_n}^\infty \right. \right. \\ &\left. f_0(C;t_1,\ldots,t_n) dt_1 \cdots dt_n \right) f_0(C;u_1,\ldots,u_n) du_1 \cdots du_n \right) dv_1 \cdots dv_n \\ &= \int_0^\infty \cdots \int_0^\infty \left(\int_{v_1}^\infty \cdots \int_{v_n}^\infty \left(\int_{u_1}^\infty \cdots \int_{u_n}^\infty \right. \right. \\ &\left. \sum_{\alpha \in C} \exp \left[-\sum_{j=1}^n t_j \sigma_j(\alpha) \right] dt_1 \ldots dt_n \right) \sum_{\beta \in C} \exp \left[-\sum_{j=1}^n u_j \sigma_j(\beta) \right] du_1 \ldots du_n \right) dv_1 \cdots dv_n \\ &= \sum_{\alpha,\beta \in C} \int_0^\infty \cdots \int_0^\infty \left(\int_{v_1}^\infty \cdots \int_{v_n}^\infty \prod_{j=1}^n \frac{\exp \left[-u_j \sigma_j(\alpha) \right]}{\sigma_j(\alpha)} \exp \left[-u_j \sigma_j(\beta) \right] du_1 \ldots du_n \right) dv_1 \ldots dv_n \\ &= \sum_{\alpha,\beta \in C} \int_0^\infty \cdots \int_0^\infty \left(\int_{v_1}^\infty \cdots \int_{v_n}^\infty \prod_{j=1}^n \frac{\exp \left[-u_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha)} du_1 \ldots du_n \right) dv_1 \ldots dv_n \\ &= \sum_{\alpha,\beta \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_1 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j(\alpha + \beta) \right]}{\sigma_j(\alpha) \sigma_j(\alpha + \beta)} dv_2 \ldots dv_n \\ &= \sum_{\alpha \in C} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n \frac{\exp \left[-v_j \sigma_j$$

3 Transition to Algebraic Geometry

We may write the infinite sum in the definition of f_0 as a product of n geometric series as follows.

Lemma 3 Let
$$x_i = e^{-t_i}$$
 for $i = 1, 2, ..., n$. Then $e^{-t_j \sigma_j(e_i)} = x_j^{\sigma_j(e_i)}$ and

$$f_0(C; t_1, \dots, t_n) = \prod_{i=1}^n \left(\frac{\prod_{j=1}^n x_j^{\sigma_j(e_i)}}{1 - \prod_{j=1}^n x_j^{\sigma_j(e_i)}} \right)$$
(3)

Proof. We simplify the function f_0 by expressing it in terms of products:

$$f_0(C; t_1, \dots, t_n) = \sum_{\alpha \in C} \exp\left[-\sum_{j=1}^n \sigma_j(\alpha)t_j\right]$$

$$= \sum_{a_1=1}^\infty \dots \sum_{a_n=1}^\infty \exp\left[-\sum_{j=1}^n t_j[a_1\sigma_j(e_1) + \dots + a_n\sigma_j(e_n)]\right]$$

$$= \sum_{a_1=1}^\infty \dots \sum_{a_n=1}^\infty \exp[-a_1[t_1\sigma_1(e_1) + \dots + t_n\sigma_n(e_1)]] \times \dots \times \exp[-a_n[t_1\sigma_1(e_n) + \dots + t_n\sigma_n(e_n)]]$$

$$= \frac{\exp\left[-[t_1\sigma_1(e_1) + \dots + t_n\sigma_n(e_1)]\right]}{1 - \exp\left[-[t_1\sigma_1(e_1) + \dots + t_n\sigma_n(e_n)]\right]} \times \dots \times \frac{\exp\left[-[t_1\sigma_1(e_n) + \dots + t_n\sigma_n(e_n)]\right]}{1 - \exp\left[-[t_1\sigma_1(e_n) + \dots + t_n\sigma_n(e_n)]\right]} =$$

$$= \prod_{i=1}^n \frac{\exp\left[-\sum_{j=1}^n t_j\sigma_j(e_i)\right]}{1 - \exp\left[-\sum_{j=1}^n t_j\sigma_j(e_i)\right]}$$

$$= \prod_{i=1}^n \frac{\prod_{j=1}^n \exp\left[-t_j\sigma_j(e_i)\right]}{1 - \prod_{j=1}^n \exp\left[-t_j\sigma_j(e_i)\right]}. \quad \Box$$

3.1 The Algebraic Exponent

We are going to define new variables x_{ij} , so that when we express $f_0(C; t_1, \ldots, t_n)$ in terms of x_{ij} , then we obtain a rational function. Intuitively $x_{ij} = x_j^{\sigma_j(e_i)}$. To achieve that, we need to define algebraically $f^{\gamma}(x) = x^{\gamma}$ where $\gamma \in \mathcal{O}_K$. We follow similar ideas as in the announcement [7]. Let $\mathcal{O}_K = \mathbb{Z}\{\mu_1, \mu_2, \ldots, \mu_n\}$ as a \mathbb{Z} -module, where $\mu_1 = 1$. Let (c_{ij}) by the $n \times n$ -matrix associated to γ in the basis (μ_1, \ldots, μ_n) , that is,

$$\gamma \mu_i = \sum_{k=1}^n c_{ik} \mu_k.$$

We define a function f^{γ} corresponding to raising to a power γ by sending monomials in the variables y_1, \ldots, y_n to monomials in the same variables. Let

$$f^{\gamma}(y_i) = \prod_{k=1}^n y_k^{c_{ik}}.$$

Lemma 4 Iterated application of the above definition of exponentiation has the following property:

$$f^{\beta}f^{\alpha}(y_i) = f^{\alpha\beta}(y_i).$$

Proof: It follows from the fact that $\gamma \mapsto (c_{ik})$ is a representation of the ring \mathcal{O}_K as an endomorphism of \mathbb{Z}^n . More precisely, let $\alpha \mapsto (a_{ij})$, $\beta \mapsto (b_{jk})$ and $\gamma \mapsto (c_{ik})$. If $\alpha\beta = \gamma$ then $\sum_j a_{ij}b_{jk} = c_{ik}$. Thus,

$$f^{\beta} f^{\alpha}(y_i) = f^{\beta} \left(\prod_j y_j^{a_{ij}} \right) = \prod_{j,k} y_k^{a_{ij}b_{jk}} = \prod_k y_k^{\sum_j a_{ij}b_{jk}} = \prod_{j,k} y_k^{c_{ik}} = f^{\alpha\beta}(y_i). \ \Box$$

Now that we have defined an algebraic power of a variable, we return to expressing f_0 as a rational function. Let

$$x_{ij} = f^{\sigma(e_i)}(x_j).$$

Intuitively, $x_{ij} = x_i^{\sigma_j(e_i)}$.

Then

$$f_0(C; t_1, \dots, t_n) = \prod_{i=1}^n \left(\frac{\prod_{j=1}^n x_j^{\sigma_j(e_i)}}{1 - \prod_{j=1}^n x_j^{\sigma_j(e_i)}} \right)$$

can be written formally as

$$f_0(C; t_1, \dots, t_n) = \prod_{i=1}^n \left(\frac{\prod_{j=1}^n f^{\sigma_j(e_i)}(x_j)}{1 - \prod_{j=1}^n f^{\sigma_j(e_i)}(x_j)} \right) = \prod_{i=1}^n \left(\frac{\prod_{j=1}^n x_{ij}}{1 - \prod_{j=1}^n x_{ij}} \right).$$
(4)

4 Multiple Dedekind Zeta Values, Differential Forms and Rational Functions

Let us recall the definition of a multiple Dedekind zeta value (see [5]). Let

$$\alpha_0(t_1, \dots, t_n) = dt_1 \wedge \dots \wedge dt_n$$

$$\alpha_1(t_1, \dots, t_n) = f_0(t_1, \dots, t_n) dt_1 \wedge \dots \wedge dt_n.$$

The definition of a multiple Dedekind zeta value (1) is as follows.

$$\zeta_{K;C}(s_1,\ldots,s_m) = \int_{\Delta} \bigwedge_{k=1}^{M} \alpha_{\epsilon_k}(t_{1,k},\cdots,t_{n,k}), \tag{5}$$

where

- 1. $M = s_1 + \cdots + s_d$;
- 2. $\Delta = \Delta_1 \times \cdots \times \Delta_n$, is an *n*-fold product of *M*-simplices $\Delta_1, \dots, \Delta_n$
- 3. Δ_j is a simplex consisting of points $(t_{j,1}, t_{j,2}, \dots, t_{j,M})$ such that $t_{j,1} > t_{j,2} > \dots > t_{j,M} > 0$;
- 4. the indecies ϵ_k have values 0 or 1 and

$$\epsilon_1 = 1, \qquad \epsilon_1 = \dots = \epsilon_{s_1} = 0$$

$$\epsilon_{s_1+1} = 1, \qquad \epsilon_{s_1+2} = \dots = \epsilon_{s_1+s_2} = 0$$

$$\dots$$

$$\epsilon_{s_1+\dots+s_{d-1}+1} = 1, \quad \epsilon_{s_1+\dots+s_{d-1}+2} = \dots = \epsilon_{s_1+\dots+s_d} = 0$$

Definition. Let $z_i = \prod_{j=1}^n x_{ij}$. We define the differential forms ω_0 and ω_1 by

$$\omega_0(z_1, \dots, z_n) = \bigwedge_{i=1}^n \frac{dz_i}{z_i} \tag{6}$$

$$\omega_1(z_1, \dots, z_n) = \bigwedge_{i=1}^n \frac{dz_i}{1 - z_i} \tag{7}$$

Proposition 5 Evaluate x_{ij} at $e^{-t_j^{\sigma_{e_i}}}$. Then

$$\omega_0(z_1, \dots, z_n) = \sqrt{\Delta} \cdot \alpha_0(t_1, \dots, t_n)$$

$$\omega_1(z_1, \dots, z_n) = \sqrt{D} \cdot \alpha_1(t_1, \dots, t_n),$$

where $\sqrt{D} = \det(\sigma_j(e_i))$ and D is an integer multiple of the discriminant.

Proof. If we evaluate x_{ij} at $e^{-t_j\sigma_j(e_i)}$, then $z_i = \prod_j x_{ij} = \prod_j e^{-t_j\sigma_j(e_i)} = e^{-\sum_j t_j\sigma_j(e_i)}$. Then

$$\omega_0(z_1, \dots, z_n) = \bigwedge_i \frac{dz_i}{z_i} = \det(\sigma_j(e_i)) \bigwedge dt_i =$$

$$= \sqrt{D} \cdot \alpha_0(t_1, \dots, t_n).$$
(8)

In that case, we also have

$$f_0(C; t_1, \dots, t_n) = \prod_i \frac{z_i}{1 - z_i}.$$

Therefore,

$$\omega_{1}(z_{1},\ldots,z_{n}) = \bigwedge_{i} \frac{dz_{i}}{1-z_{i}} = \left(\prod_{i} \frac{z_{i}}{1-z_{i}}\right) \bigwedge_{i} \frac{dz_{i}}{z_{i}} =$$

$$= \det(\sigma_{j}(e_{i})) f_{0}(t_{1},\ldots,t_{n}) \bigwedge dt_{i} =$$

$$= \sqrt{D} \cdot \alpha_{1}(t_{1},\ldots,t_{n}).$$
(9)

Proposition 6 We have the following relation between multiple Dedekind zeta values of the differential forms ω_0 and ω_1

$$\int_{\Delta} \bigwedge_{k=1}^{M} \omega_{\epsilon_k}(t_{1,k}, \cdots, t_{n,k}) = \left(\sqrt{D}\right)^{M} \zeta_{K;C}(s_1, \dots, s_m).$$

Proof. It follows directly from Equations (5), (8) and (9). \square

5 Tangential base points

Let $y = e^{-bt}$ and $z = e^{-ct}$, where b and c are complex numbers such that Re(b) > 0 and Re(c) > 0, and $|b| \neq |c|$. Then

$$\lim_{t\to +\infty}\frac{dy}{dz}=\lim_{t\to +\infty}\frac{de^{-bt}}{de^{-ct}}=\lim_{t\to +\infty}\frac{de^{ct}}{de^{bt}}=q,$$

where

$$p = \left\{ \begin{array}{cc} +\infty & \text{or } [0:1] & \text{if } b < c \\ 0 & \text{or } [0:1] & \text{if } b > c \end{array} \right.$$

Also

$$\lim_{t \to 0} \frac{dy}{dz} = \lim_{t \to 0} \frac{de^{-bt}}{de^{-ct}} = \lim_{t \to 0} \frac{-be^{-bt}}{-ce^{-ct}} = \frac{b}{c}$$

Let $\gamma:[0,1]\to\mathcal{M}_5$, by sending t to (y,z), where $y=e^{-bt}$ and $z=e^{-ct}$. For a vector v=(a,b), consider [v]=[a:b] as an element of \mathbb{P}^1 .

We have proven the following lemma.

Lemma 7 (a)

$$\lim_{t \to \infty} \left[\frac{d\gamma}{dt} \right] = \left\{ \begin{array}{ll} [0:1] & \text{if } b < c \\ [1:0] & \text{if } b > c \end{array} \right.$$

Moreover, the limit is well defined for for any distinct positive real numbers b and c.

(b)

$$\lim_{t \to 0} \left[\frac{d\gamma}{dt} \right] = [b:c].$$

Let $t = t_j$. Let also $b = \sigma_j(e_i)$, $c = \sigma_j(e_k)$. Then

$$\lim_{t_j \to 0} \left[\frac{dx_{i_1,j}}{dx_{i_2,j}} \right] = [\sigma_j(e_i) : \sigma_j(e_k)].$$

We define

$$[q(i,k)] = [\sigma_j(e_i) : \sigma_j(e_k)].$$

And

$$\lim_{t_j \to \infty} \left[\frac{dx_{i,j}}{dx_{k,j}} \right] = \begin{cases} [0:1] & \text{if } \sigma_j(e_i) < \sigma_j(e_k) \\ [1:0] & \text{if } \sigma_j(e_i) > \sigma_j(e_k) \end{cases}$$

Let

$$[p(i,k)] = \begin{cases} [0:1] & \text{if } \sigma_j(e_i) < \sigma_j(e_k) \\ [1:0] & \text{if } \sigma_j(e_i) > \sigma_j(e_k) \end{cases}$$

More generally, let $[p(i_0, \ldots, i_r)] = [a_0 : \cdots : a_r]$ be a point on $\mathbb{P}^r(\mathbb{Q})$ with a_i being 0 or 1, where all a_i 's are zero except the one whose index c, for a_c , is such that $|\sigma_j(e_c)|$ is a maximal value among the elements $|\sigma_j(e_0)|, \ldots, |\sigma_j(e_r)|$.

6 Multiple Dedekind Zeta Values and the Moduli Space $\mathcal{M}_{0,m\cdot n^2+3}$

Let $z_{ik} = \prod_{j=1}^k x_{ij}$. there are n^2 such variables. If we consider multiple Dedekind zeta value of depth m then we need $m \cdot n^2$ variables $(z_{ikd})_{i,k,d=1,1,1}^{n,n,m} \in \mathcal{M}_{0,m \cdot n^2+3}$. The dimension of the differential form $\Omega(A)$ is mn. Let ϵ_d be 0 or 1 for $d=1,2,\ldots,m$. Let also $\epsilon_1=1$ and $\epsilon_m=0$.

$$\Omega(A) = \bigwedge_{d=1}^{m} \omega_{\epsilon_d}.$$

where

$$A = (z_{ind} = \epsilon_d)_{i,d=1,1}^{n,m} \cup (z_{ind} = \infty)_{i,d=1,1}^{n,m}$$

and $B = B_1 \cap B_2 \cap \cdots \cap B_n$, where $codim B_r = r$ contains the divisors

$$B_1 = (z_{i,1,1} = 0)_{i=1}^n \cup (z_{i,j,d} = z_{i,j,d+1})_{i,j,d=1,1,1}^{n,n,m-1} \cup (z_{i,1,m} = 1)_{i=1}^n$$

together with the intersection of boundary components of $\overline{\mathcal{M}}_{0,m\cdot n^2+3} - \mathcal{M}_{0,m\cdot n^2+3}$ containing the same variable or the same constant 0 or 1. Besides codimension 1 components, B also contains a codimension 2 components.

Let B_{i_1,i_2}^\prime be a quiasi-subvariety in the boundary of the Deligne-Mumford compactification that has coordinates with

$$[z_{i_1,1,1}:z_{i_2,1,1}]=[p(i_1,i_2)].$$

in the blow-up of the intersection $(z_{i_1,j,1}=0) \cap (z_{i_2,j,1}=0)$. Let B''_{i_1,i_2} be a cycle in the boundary of the Deligne-Mumford compactification above the intersection $(z_{i,1,m}=1) \cap (z_{k,j,m}=1)$ such that the coordinated of B'_{i_1,i_2} in the blowup are

$$[1 - z_{i_1,1,1} : 1 - z_{1_2,1,1}] = [\sigma_j(e_{i_1}) : \sigma_j(e_{i_2})].$$

The codimension 2 components of B are the union of all B'_{i_1,i_2} and B''_{i_1,i_2} . That is

$$B_2 = \bigcup_{i_1 < i_2} \left(B'_{i_1, i_2} \cup B''_{i_1, i_2} \right).$$

Now let us write $\omega_0(x_1, x_2)$ and $\omega_1(x_1, x_2)$, when we want to specify the dependence on the variables. In fact, both forms depend also on y_1 and y_2 ; however, we will take care of that by choosing a region of integration together with tangential base points.

Theorem 8 (a) Every multiple Dedekind zeta value over a field K times a suitable multiple of a power of the discriminant of K is a period of a mixed Tate motive over K. More precisely,

$$\left(\sqrt{D}\right)^{M} \zeta_{K;C}(s_{1},\ldots,s_{m}) = \int_{\Delta} \bigwedge_{k=1}^{M} \omega_{\epsilon_{k}}(t_{1,k},\cdots,t_{n,k})$$

is a period of

$$H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3}-A,B-(A\cap B)),$$

 $\overline{\mathcal{M}}_{0,n^2M+3}$ is the Delingne-Mumford compactification of the moduli space of curves of genus zero with n^2M+3 marked points, and A and B that consist of a union of lower dimensional moduli spaces of curves of genus zero with marked points.

(b) For any field K of degree n over \mathbb{Q} , with the property that K has n units e_1, \ldots, e_n that are linearly independent over \mathbb{Q} and $|\sigma_j(e_i) \neq |\sigma_j(e_k)|$, we have the following stronger statement. In particular, when K is a totally real number field, we choose a cone $C = \mathbb{N}\{e_1, \ldots, e_n\}$. Then for any positive integers s_1, \ldots, s_m with $s_m \geq 2$, we have that

$$\left(\sqrt{D}\right)^{M}\zeta_{K;C}(s_{1},\ldots,s_{m})$$

is a period of an unramified mixed Tate motive over the ring of algebraic integers \mathcal{O}_K in the field K.

Proof: In this proof we are going to follow closely the paper by Goncharov and Manin [4]. The period will be a pairing between $[\Omega_A] \in Gr_{2nM}^W H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3} - A)$ and $[\Delta_B] \in (Gr_0^W H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3} - B))^{\vee}$ associated to a mixed Tate motive $H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3} - A; B - A \cap B)$.

We have that A and B_1 are defined over \mathbb{Z} . Moreover, any component and any intersection of components of A and B are isomorphic to a moduli space $\mathcal{M}_{0,N}$ for some N. The component B_2, B_3, \ldots are defined over the field K. Moreover, any intersection is isomorphic to $\mathcal{M}_{0,N/K}$ for some N. Thus all multiple Dedekind zeta values are mixed Tate motives over the field of definition K.

If e_1, \ldots, e_n are unit in \mathcal{O}_K , which are linearly independent over \mathbb{Q} , then all $[q(i_1, i_2, j)]$, $[q(i_1, i_2, i_3, j])$ etc., have coordinates 0 or units. Then, the component B_2, B_3, \ldots are defined over the ring \mathcal{O}_K . Moreover, any intersection is isomorphic to $\mathcal{M}_{0,N/\mathcal{O}_K}$ for some N.

We have that $H^i(\overline{\mathcal{M}}_{0,N})$ is a mixed Tate motive over $Spec(\mathbb{Z})$. This implies that $H^i(\overline{\mathcal{M}}_{0,N})$ is a mixed Tate motive over $Spec(\mathcal{O}_K)$. we obtain that the motivic cohomology of the components of B are mixed Tate motives. Using Proposition 1.7 from Deligne and Goncharov, [3], we conclude that for $l \neq char(\nu)$ the l-adic cohomology of the reduction of B_j modulo ν of the motive $H^i(B_j)$ is unramified for any component B_j of B, since B_j is isomorphic to $\overline{\mathcal{M}}_{0,N}$ over $Spec(\mathcal{O}_K)$ for some N. We conclude that for $l \neq char(\nu)$ the l-adic cohomology of the reduction modulo any $\nu \in Spec(\mathcal{O}_K)$ of the motive $H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3}-A;B-A\cap B)$ is unramified. Thus, $H^{nM}(\overline{\mathcal{M}}_{0,n^2M+3}-A;B-A\cap B)$ is a mixed Tate motive unramified over $Spec(\mathcal{O}_K)$.

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