On hypergraph cliques with chromatic number 3 and a given number of vertices*

D.D. Cherkashin, A.B. Kulikov, A.M. Raigorodskii

In 1973 P. Erdős and L. Lovász noticed that any hypergraph whose edges are pairwise intersecting has chromatic number 2 or 3. In the first case, such hypergraph may have any number of edges. However, Erdős and Lovász proved that in the second case, the number of edges is bounded from above. For example, if a hypergraph is n-uniform, has pairwise intersecting edges, and has chromatic number 3, then the number of its edges does not exceed n^n . Recently D.D. Cherkashin improved this bound (see [2]). In this paper, we further improve it in the case when the number of vertices of an n-uniform hypergraph is bounded from above by n^m with some m = m(n).

1 Introduction and formulation of the main result

This work is devoted to a problem in extremal hypergraph theory, which goes back to P. Erdős and L. Lovász (see [3]). Before giving an exact statement of the problem, we recall some definitions and introduce some notation.

Let H = (V, E) be a hypergraph without multiple edges. We call it n-uniform, if any of its edges has cardinality n: for every $e \in E$, we have |e| = n. By the chromatic number of a hypergraph H = (V, E) we mean the minimum number $\chi(H)$ of colors needed to color all the vertices in V so that any edge $e \in E$ contains at least two vertices of some different colors. Finally, a hypergraph is said to form a clique, if its edges are pairwise intersecting.

In 1973 Erdős and Lovász noticed that if an n-uniform hypergraph H = (V, E) forms a clique, then $\chi(H) \in \{2, 3\}$. They also observed that in the case of $\chi(H) = 3$, one certainly has $|E| \leq n^n$ (see [3]). Thus, the following definition has been motivated: $M(n) = \max\{|E|: \exists \text{ an } n - \text{ uniform clique } H = (V, E) \text{ with } \chi(H) = 3\}$.

Obviously such definition has no sense in the case of $\chi(H) = 2$.

$$M(n) = \max\{|E|: \exists \text{ an } n - \text{uniform clique } H = (V, E) \text{ with } \chi(H) = 3\}.$$

Obviously such definition has no sense in the case of $\chi(H)=2$.

Theorem 1 (P. Erdős, L. Lovász, [3]). The inequalities hold

$$n!\left(\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \leqslant M(n) \leqslant n^n.$$

Almost nothing better has been done during the last 35 years. In the book [5] the estimate $M(n) \leq$ $(1-\frac{1}{\epsilon}) n^n$ is mentioned as "to appear". However, we have not succeeded in finding the corresponding paper.

At the same time, another quantity r(n) was introduced in [6]:

$$r(n) = \max\{|E|: \exists \text{ an } n - \text{uniform clique } H = (V, E) \text{ s.t. } \tau(H) = n\},$$

^{*}This work was supported by the grant of RFBR N 12-01-00683, the grant of the Russian President N MD-8390.2010.1, and the grant NSh-8784.2010.1.

where $\tau(H)$ is the covering number of H, i.e.,

$$\tau(H) = \min\{|f|: \ f \subset V, \ \forall \ e \in E \ \ f \cap e \neq \emptyset\}.$$

Clearly, for any *n*-uniform clique H, we have $\tau(H) \leq n$ (since every edge forms a cover), and if $\chi(H) = 3$, then $\tau(H) = n$. Thus, $M(n) \leq r(n)$. Lovász noticed that for r(n) the same estimates as in Theorem 1 apply and conjectured that the lower estimate is best possible. In 1996 P. Frankl, K. Ota, and N. Tokushige (see [4]) disproved this conjecture and showed that $r(n) \geq \left(\frac{n}{2}\right)^{n-1}$.

In [2] D.D. Cherkashin discovered a new upper bound for the initial value M(n) which is actually true for r(n) as well.

Theorem 2 (D.D. Cherkashin, [2]). There exists a constant c > 0 such that

$$M(n) \leqslant c n^{n - \frac{1}{2}} \ln n.$$

To formulate the main result of this paper we take any natural numbers $n, m \ge 2$ and put $q(n, m) = \left[\frac{n}{2m}\right]$,

$$A(n,m) = \sum_{i=0}^{2q(n,m)} \binom{n^m}{i}.$$

We note that of course

$$A(n,m) \leqslant \left(\frac{n}{m}+1\right) \binom{n^m}{2q(n,m)} \leqslant \left(\frac{n}{m}+1\right) \left(\frac{en^m}{2q(n,m)}\right)^{n/m} = n^n \cdot A'(n,m),$$

where

$$A'(n,m) = \left(\frac{n}{m} + 1\right) \left(\frac{e}{2q(n,m)}\right)^{n/m}.$$

Obviously, if m is a function of n, which is o(n) as $n \to \infty$, then

$$A'(n,m) = \frac{m}{n\omega(n)},$$

where $\omega(n) \to \infty$ as $n \to \infty$. Thus, $A(n, m) = o(mn^{n-1})$.

Theorem 3. Let $m \ge 2$ be any function of $n \in \mathbb{N}$ which is o(n) as $n \to \infty$; moreover, $m(n) \le \frac{n}{2}$. For any $n \ge 4$ and any n-uniform clique H = (V, E) with $\chi(H) = 3$ and $|V| \le n^{m(n)}$, we have

$$|E| \le 4m(n)n^{n-1} + A(n, m(n)) = (4 + o(1))m(n)n^{n-1}$$

Clearly, if $m(n) \leq c\sqrt{n} \ln n$ with some constant c > 0, then the bound in Theorem 3 is stronger than the bound in Theorem 2. Note that the number of vertices in any n-uniform clique with chromatic number 3 does not exceed 4^n (see [3]). Unfortunately, $n^{\sqrt{n} \ln n} = e^{o(n)}$, so that Theorem 3 does not cover all possible values of |V|.

2 Proof of Theorem 3

Fix an $n \ge 4$ and put m = m(n), q = q(n, m), A = A(n, m). Fix an n-uniform clique H = (V, E) with $\chi(H) = 3$ and $|V| \le n^m$. For any set $W \subseteq V$, denote by E(W) the set of all edges $B \in E$ such that $W \subseteq B$. Also denote by E_W the set of all edges $B \in E$ such that $W \cap B \ne \emptyset$. Clearly $E(W) \subseteq E_W$. Let

$$Q = \{1, 2, 3, \dots, q\} \cup \{n - q + 1, n - q + 2, \dots, n\}.$$

The two parts which form the set Q do not intersect and do not cover the whole set $\{1, \ldots, n\}$, since $m \ge 2$. Moreover, Q is not empty, since $m \le \frac{n}{2}$ and so $q \ge 1$.

Lemma 1. Let $W \subseteq V$, i = |W|. Either there exists a vertex $x \in W$ such that $\deg x \geqslant \frac{|E|-A}{i}$, or there exist two edges $B_1, B_2 \in E$ such that $B_1, B_2 \notin E_W$ and $|B_1 \cap B_2| \notin Q$.

Proof of Lemma 1. If there exists a vertex $x \in W$ such that deg $x \geqslant \frac{|E|-A}{i}$, then we are done. If there are no such vertices, then

$$|E_W| \leqslant \sum_{x \in W} \deg x < |E| - A.$$

Therefore, $|E \setminus E_W| > A$. We have to show that there exist two edges $B_1, B_2 \in E \setminus E_W$ with $|B_1 \cap B_2| \notin Q$. Suppose to the contrary that for any $B_1, B_2 \in E \setminus E_W$, we have $|B_1 \cap B_2| \in Q$. We shall consequently prove that $|E \setminus E_W| \leq A$ obtaining a contradiction and thus completing the proof of Lemma 1.

In principle, it is possible just to cite the paper [8]. We use instead a version of the linear algebra method in combinatorics (see [1] and [7]). To any edge B from $E \setminus E_W$ we assign a vector $\mathbf{x} = (x_1, \dots, x_v) \in \{0, 1\}^v$, where $v = |V| \leq n^m$ and $x_\nu = 1$, if and only if $\nu \in B$. In particular, $x_1 + \dots + x_v = n$. Let $E \setminus E_W \to \{\mathbf{x}_1, \dots, \mathbf{x}_s\}$.

Denote by (\mathbf{x}, \mathbf{y}) the Euclidean inner product of vectors \mathbf{x}, \mathbf{y} . Note that if $B, B' \in E \setminus E_W$ and \mathbf{x}, \mathbf{x}' are the corresponding vectors, then $|B \cap B'| = (\mathbf{x}, \mathbf{x}')$.

Take an arbitrary vector \mathbf{x}_{ν} , $\nu \in \{1, \dots, s\}$, and consider the polynomial

$$F_{\mathbf{x}_{\nu}}(\mathbf{y}) = \prod_{j \in Q \setminus \{n\}} (j - (\mathbf{x}_{\nu}, \mathbf{y})) \in \mathbb{R}[y_1, \dots, y_v].$$

Eventually, we get s polynomials $F_{\mathbf{x}_1}, \dots, F_{\mathbf{x}_s}$. All of them depend on v variables and have degree not exceeding the quantity $|Q| \leq 2q$. Of course any such polynomial is a linear combination of some monomials which are of type

1,
$$y_{\nu_1}^{\alpha_{\nu_1}} \cdot \ldots \cdot y_{\nu_r}^{\alpha_{\nu_r}}$$
, $\alpha_{\nu_1}, \ldots, \alpha_{\nu_r} \geqslant 1$, $\alpha_{\nu_1} + \ldots + \alpha_{\nu_r} \leqslant |Q| \leqslant 2q$.

Replace each monomial of this type by $y_{\nu_1} \cdot \ldots \cdot y_{\nu_r}$. Denote by $F'_{\mathbf{x}_1}, \ldots, F'_{\mathbf{x}_s}$ the resulting polynomials. They also depend on v variables and have degree not exceeding the quantity $|Q| \leq 2q$. Moreover, they span a linear space whose dimension is less then or equal to

$$\sum_{r=0}^{2q} \binom{v}{r} \leqslant \sum_{r=0}^{2q} \binom{n^m}{r} = A.$$

At the same time $F'_{\mathbf{x}_{\nu}}(\mathbf{y}) = F_{\mathbf{x}_{\nu}}(\mathbf{y})$, provided $\mathbf{y} \in \{0, 1\}^{v}$ and $\nu \in \{1, \dots, s\}$.

To show that $s = |E \setminus E_W| \leq A$ (which we need to complete the proof) it suffices to establish the linear independence of the polynomials $F'_{\mathbf{x}_1}, \dots, F'_{\mathbf{x}_s}$ over \mathbb{R} . Assume that

$$c_1 F'_{\mathbf{x}_1}(\mathbf{y}) + \ldots + c_s F'_{\mathbf{x}_s}(\mathbf{y}) = 0.$$

Let $\mathbf{y} = \mathbf{x}_{\nu}, \ \nu \in \{1, \dots, s\}$. Then $(\mathbf{x}_{\nu}, \mathbf{y}) = (\mathbf{x}_{\nu}, \mathbf{x}_{\nu}) = n$ and

$$F'_{\mathbf{x}_{\nu}}(\mathbf{y}) = F_{\mathbf{x}_{\nu}}(\mathbf{y}) = F_{\mathbf{x}_{\nu}}(\mathbf{x}_{\nu}) \neq 0.$$

However, if $\mu \neq \nu$, then $(\mathbf{x}_{\mu}, \mathbf{y}) = (\mathbf{x}_{\mu}, \mathbf{x}_{\nu}) \in Q \setminus \{n\}$, that is,

$$F'_{\mathbf{x}_{\mu}}(\mathbf{y}) = F_{\mathbf{x}_{\mu}}(\mathbf{y}) = F_{\mathbf{x}_{\mu}}(\mathbf{x}_{\nu}) = 0.$$

Therefore, $c_{\nu} = 0$ for every ν , and we are done. Lemma 1 is proved.

Lemma 2. Let $W \subseteq V$, i = |W|, j = |E(W)|. Assume that there exist two edges $B_1, B_2 \in E \setminus E_W$ such that $|B_1 \cap B_2| \notin Q$. Put $\tau = 1 + \frac{1}{4m}$. Either there exists an $x \notin W$ such that $|E(W \cup \{x\})| \geqslant \frac{j\tau}{n}$, or there exist $x, y \notin W$ such that $|E(W \cup \{x, y\})| \geqslant \frac{j\tau^2}{n^2}$.

Proof of Lemma 2. Let $l = |B_1 \cap B_2| \notin Q$. Consider the set E(W). Since H is a clique, any edge $B \in E(W)$ intersects both B_1 and B_2 . Either B intersects the set $B_1 \cap B_2$, or it has common vertices with both $B_1 \setminus (B_1 \cap B_2)$ and $B_2 \setminus (B_1 \cap B_2)$. Denote by E_1 the set of edges of the first type; $E_2 = E(W) \setminus E_1$. By pigeon-hole principle, there is an $x \in B_1 \cap B_2$ such that x belongs to at least $\frac{|E_1|}{l}$ edges from E_1 ; also

there are $x \in B_1 \setminus (B_1 \cap B_2)$ and $y \in B_2 \setminus (B_1 \cap B_2)$ such that the set $\{x, y\}$ belongs to at least $\frac{|E_2|}{(n-l)^2}$ edges from E_2 . It remains to show that for any partition $E(W) = E_1 \cup E_2$, we have

either
$$\frac{|E_1|}{l} \geqslant \frac{j\tau}{n}$$
, or $\frac{|E_2|}{(n-l)^2} \geqslant \frac{j\tau^2}{n^2}$,

which is equivalent to

$$\max\left\{\frac{|E_1|^2}{j^2l^2}, \frac{|E_2|}{j(n-l)^2}\right\} \geqslant \frac{\tau^2}{n^2}.$$

Here the worst case is that of $\frac{|E_1|^2}{j^2l^2} = \frac{|E_2|}{j(n-l)^2}$. Let $a = |E_1|$. Then $|E_2| = j-a$ and we have $\frac{a^2}{j^2l^2} = \frac{j-a}{j(n-l)^2}$. Solving this equation we get

$$a = \frac{jl^2 + \sqrt{(jl^2)^2 + 4j^2l^2(n-l)^2}}{2(n-l)^2}.$$

Of course the value of $|E_1|$ (which is integer) may differ from the real number a. However, we do certainly know that

$$\max\left\{\frac{|E_1|^2}{j^2l^2}, \frac{|E_2|}{j(n-l)^2}\right\} \geqslant \frac{a^2}{j^2l^2}.$$

Thus, we need to prove that $\frac{a}{jl} \ge \frac{\tau}{n}$ or that $\frac{an}{jl} \ge \tau$. We have

$$\frac{an}{jl} = \frac{l + \sqrt{l^2 + 4(n-l)^2}}{2(n-l)^2} n = \frac{ln}{2(n-l)^2} + \sqrt{\left(\frac{ln}{2(n-l)^2}\right)^2 + \frac{n^2}{(n-l)^2}} \geqslant 1 + \frac{ln}{2(n-l)^2}.$$

The function $\frac{ln}{2(n-l)^2}$ is monotone increasing in l. Since $l \notin Q$, we may use the bound $l \geqslant \frac{n}{2m}$. Consequently,

$$\frac{an}{jl} \geqslant 1 + \frac{ln}{2(n-l)^2} \geqslant 1 + \frac{n^2}{4m(n-l)^2} \geqslant 1 + \frac{1}{4m} = \tau.$$

Lemma 2 is proved.

Completion of the proof of Theorem 3. Let

$$k = \min \left\{ |W| : W \subseteq V, \exists x \in W \text{ deg } x \geqslant \frac{|E| - A}{|W|} \right\}.$$

The quantity k is well-defined. Indeed, take any edge $W \in E$. Since H is a clique, W intersects all the edges from E and so there exists an $x \in W$ with deg $x \geqslant \frac{|E|}{|W|} > \frac{|E|-A}{|W|}$.

Let W_0 be a set on which the value of k is attained. Take a vertex $x \in W_0$ that has deg $x \geqslant \frac{|E|-A}{k}$. The last inequality can be rewritten as $|E(\{x\})| \geqslant \frac{|E|-A}{k}$. If $k \geqslant 2$, we may apply Lemmas 1 and 2 to $W = \{x\}$. Thus, we obtain either a set W' of two elements with $|E(W')| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau}{n}$ or a set W'' of three elements with $|E(W'')| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau^2}{n^2}$. We continue this process until we get a set W with |W| = k and $|E(W)| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau^{k-1}}{n^{k-1}}$ (even if k = 1, we do have such a set).

 $|E(W)| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau^{k-1}}{n^{k-1}} \text{ (even if } k = 1, \text{ we do have such a set)}.$ In [3] Erdős and Lovász proved that for any n-uniform clique H = (V, E) with chromatic number 3, if $W \subseteq V$ is of cardinality k, then $|E(W)| \leqslant n^{n-k}$. In our case, we have $\frac{|E|-A}{k} \cdot \frac{\tau^{k-1}}{n^{k-1}} \leqslant n^{n-k}$. Therefore,

$$|E| \le k \cdot n^{n-k} \cdot \frac{n^{k-1}}{\tau^{k-1}} + A = k \frac{n^{n-1}}{\tau^{k-1}} + A.$$

To complete the proof of Theorem 3 it remains to show that for any k, $\frac{k}{\tau^{k-1}} \leqslant 4m$. It is very easy to see that the maximum value of the quantity $\frac{k}{\tau^{k-1}}$ is attained on k=4m, and we are done.

3 A refinement in Theorem 3

For m=2, one can prove a simple result, which is however substantially better than that of Theorem 3.

Theorem 4. Let H=(V,E) be any n-uniform clique with $\chi(H)=3$. Put v=|V|. Assume that $v\leqslant \frac{n^2}{c}$, where c may be any function of n such that $c(n)\in (1,n)$. Put

$$d = ce^{\frac{2}{ec} - 1}.$$

Then

$$|E| \le (1 + o(1)) \frac{e^{3/2}}{\sqrt{c}} (n/d)^n.$$

If c is a constant, then we get an exponential improvement for the Erdős and Lovász bound by n^n . Otherwise, the improvement is even more considerable.

Proof of Theorem 4. Take an arbitrary integer $a \in (1, n)$ and consider all the a-element subsets of V. The number of such subsets is $\binom{v}{a}$. On the one hand, any edge from E contains exactly $\binom{n}{a}$ subsets. On the other hand, any subset is contained in at most n^{n-a} edges (see [3]). So the number of edges does not exceed the quantity $\frac{n^{n-a}\binom{v}{a}}{\binom{n}{a}}$. To estimate this quantity we use the bound $\binom{v}{a} \leqslant \frac{v^a}{a!}$ and the Stirling formula. Hence,

$$\frac{n^{n-a}\binom{v}{a}}{\binom{n}{a}} \leqslant \frac{n^{n-a}v^a}{a!\binom{n}{a}} \leqslant \frac{n^{n+a}}{c^a\frac{n!}{(n-a)!}}.$$

Now put

$$a = \left[\left(1 - \frac{1}{ec} \right) n \right] + 1.$$

Then $n - a \leqslant \frac{n}{ec}$, so that

$$\frac{n!}{(n-a)!} \sim \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\sqrt{2\pi (n-a)} \left(\frac{n-a}{e}\right)^{n-a}} \geqslant \sqrt{ec} \cdot n^a (e^2 c)^{n-a} e^{-n}$$

and

$$|E| \leqslant (1+o(1)) \frac{n^{n+a}}{c^a \sqrt{ec} \cdot n^a (e^2 c)^{n-a} e^{-n}} = (1+o(1)) \frac{n^n}{\sqrt{ec} \cdot c^n e^{-n} e^{2n-2a}} \leqslant$$

$$\leqslant (1+o(1)) \frac{n^n}{\sqrt{ec} \cdot c^n e^{-n} e^{\frac{2n}{ec}-2}} = (1+o(1)) \frac{e^{3/2}}{\sqrt{c}} (n/d)^n.$$

Theorem 4 is proved.

Note that for constant values of c, the choice of a in the proof was nearly optimal.

References

- [1] L. Babai, P. Frankl, *Linear algebra methods in combinatorics*, Part 1, Department of Computer Science, The University of Chicago, Preliminary version 2, September 1992.
- [2] D.D. Cherkashin, On hypergraph cliques with chromatic number 3, Moscow J. of Combinatorics and Number Th. 1 (2011), N3.
- [3] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolyai, North Holland, 10 (1975), 609 627.
- [4] P. Frankl, K. Ota, N. Tokushige, Covers in uniform intersecting families and a counterexample to a conjecture of Lovász, Journal of Combin. Th., Ser. A 74 (1996), 33 42.
- [5] T. Jensen, B. Toft, Graph coloring problems, New York: Wiley Interscience, 1995.
- [6] L. Lovász, On minimax theorems of combinatorics, Math. Lapok 26 (1975), 209 264 (in Hungarian).
- [7] A.M. Raigorodskii, *The linear algebra method in combinatorics*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2007 (book in Russian).
- [8] D.K. Ray-Chaudhury, R.M. Wilson, On t-designs, Osaka J. Math. 12 (1975), 735 744.