THE GAUSS MAP ON TRANSLATIONAL RIEMANNIAN MANIFOLDS AND THE TOPOLOGY OF HYPERSURFACES

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ABSTRACT. We introduce the notion of translational Riemannian manifolds and define a Gauss map for orientable immersed hypersurfaces lying in these ambients, an associated translational curvature and prove a Gauss-Bonnet theorem. We also use this Gauss map to prove that if M^n is a compact, connected and oriented immersed hypersurface of the unit sphere \mathbb{S}^{n+1} $(n \geq 2)$ contained in a geodesic ball of radius R and whose principal curvatures are strictly bigger than $\tan{(R/2)}$, then M is diffeomorphic to \mathbb{S}^n . Additionally, we show that for any $\varepsilon \in (0, \sqrt{2} - 1)$ there exists a compact, connected and oriented immersed hypersurface M_{ε} of \mathbb{S}^{n+1} whose principal curvatures are strictly bigger than $\varepsilon \tan{(R/2)}$ but M_{ε} is not homeomorphic to a sphere. Finally, using this previous result, we reobtain a theorem of Qiaoling Wang and Changyu Xia ([4]) which asserts that if a compact and oriented hypersurface of \mathbb{S}^{n+1} is contained in an open hemisphere and has nowhere zero Gauss-Kronecker curvature, then it is diffeomorphic to \mathbb{S}^n .

1. Introduction

In this paper we define a Gauss map for an orientable hypersurface of a Riemannian manifold by extending the notion of translation of the Euclidean space — used to define the Euclidean Gauss map — to more general ambient spaces. We then use this map to obtain results on the topology of the hypersurface.

The main objects we are interested in are translational manifolds, defined as follows:

Definition 1.1. A translational Riemannian manifold is a pair (\overline{M}, Γ) , where \overline{M} is an (n+1)-dimensional Riemannian manifold, $\Gamma: T\overline{M} \to \overline{M} \times V$ is a smooth vector bundle map, and V is an (n+1)-dimensional real vector space with an inner product such that the map $\Gamma_p: T_p\overline{M} \to V$ implicitly defined by

$$v \mapsto \Gamma(p,v) = (p,\Gamma_p(v))$$

is an isometry for every point $p \in \overline{M}$. The manifold \overline{M} is said to be equipped with a translational structure.

The maps Γ_p are to be thought as translations, as means of identifying the tangent spaces to \overline{M} with the vector space V. Notice that any translational Riemannian manifold is parallelisable, that is, it admits as many linearly independent vector fields as its dimension. Conversely, any paralellisable manifold has infinitely many translational structures (see next section). Natural examples of such manifolds are obtained by considering left translation on Lie groups with a left invariant metric and parallel transport

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to a fixed point, defined on the manifold minus the cut locus of the point (Examples 2.5 and 2.6). The former case was studied in [2], and in the present paper we investigate the latter when \overline{M} is an Euclidean sphere. Both scenarios reduce to the usual Euclidean translations when $\overline{M} = \mathbb{R}^{n+1}$.

On a translational Riemannian manifold (\overline{M}, Γ) one can naturally define a Gauss map for an orientable hypersurface M^n of \overline{M} . Indeed, being η a unit normal vector field along M, the Gauss map $\gamma: M \to \mathbb{S}^n$ of M is defined by $\gamma(p) = \Gamma_p(\eta(p))$, where $\mathbb{S}^n \subset V$ is the unit sphere centered at the origin of V. It is easy to see that $\Gamma_p^{-1} \circ D\gamma(p)$ is a linear map on T_pM and it is shown (Proposition 2.8) that $\Gamma_p^{-1} \circ D\gamma(p) = -(A_p + \alpha_p)$, where A_p is the shape operator of M and α_p is a translational shape operator that depends essentially on Γ . The translational curvature κ_{Γ} of M is defined by $\kappa_{\Gamma}(p) = \det\left(\Gamma_p^{-1} \circ D\gamma(p)\right)$. Observe that κ_{Γ} is the Gauss-Kronecker curvature of M when M is a hypersurface of \mathbb{R}^{n+1} and γ is the usual Gauss map associated to the translation $\Gamma_p(v) = v$.

Our first result is a Gauss-Bonnet theorem:

Theorem 1.2. Let (\overline{M}, Γ) be a translational Riemannian manifold and M^n a compact, connected and orientable immersed hypersurface of even dimension of \overline{M} , and denote by ω the volume element of M induced by the metric of \overline{M} . Then

$$\int_{M} \kappa_{\Gamma} \, \omega = \frac{c_n}{2} \chi(M),$$

where c_n is the volume of $\mathbb{S}^n \subset V$ and $\chi(M)$ is the Euler characteristic of M.

If M is compact and its translational curvature is everywhere nonzero, then $\gamma: M \to \mathbb{S}^n$ is a local diffeomorphism, and hence a global diffeomorphism, since \mathbb{S}^n is simply connected (for $n \geq 2$). An instance which follows from this remark and from the above mentioned formula $D\gamma(p) = -\Gamma_p \circ (A_p + \alpha_p)$ is that if M is a compact, connected and orientable hypersurface of a Lie group with a bi-invariant metric and its principal curvatures have the same sign, then γ is a diffeomorphism (see Theorem 9 of [2]). Also using this remark, we prove:

Theorem 1.3. Let M^n be a compact, connected and oriented immersed hypersurface of \mathbb{S}^{n+1} , $n \geq 2$, and let R be the radius of the smallest geodesic ball containing M. If the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M satisfy

$$|\lambda_i(p)| > \tan\left(\frac{R}{2}\right), \quad \forall p \in M, \ \forall i \in \{1, \dots, n\},$$

then M is diffeomorphic to \mathbb{S}^n . Moreover, for any $\varepsilon \in (0, \sqrt{2} - 1)$ there exists a compact, connected and oriented immersed hypersurface M_{ε} of \mathbb{S}^{n+1} whose principal curvatures satisfy

(1)
$$|\lambda_i(p)| > \varepsilon \tan\left(\frac{R}{2}\right), \quad \forall p \in M_\varepsilon, \ \forall i \in \{1, \dots, n\},$$

but M_{ε} is not homeomorphic to a sphere.

Lastly, we provide an alternative proof for Theorem 1.1 of [4]:

Theorem 1.4. Let M^n be a compact, connected and oriented immersed hypersurface of \mathbb{S}^{n+1} , $n \geq 2$, with non-vanishing Gauss-Kronecker curvature. If M is contained in an open hemisphere, then M is diffeomorphic to \mathbb{S}^n .

2. Gauss map

Let \overline{M}^{n+1} be a Riemannian manifold. As stated in the Introduction, \overline{M} admits a translational structure Γ if and only if it is parallelisable, for if Γ is given, choose a basis $\{v_1,..,v_{n+1}\}$ of V and define $V_i(p)=\Gamma_p^{-1}(v_i)$. Conversely, if \overline{M} is parallelisable we may consider, after an orthonormalisation process, vector fields V_i such that $\langle V_i,V_j\rangle=\delta_{ij},$ $1\leq i,j\leq n+1$. Then, choose a point p_0 in \overline{M} , set $V=T_{p_0}\overline{M}$ and define

$$\Gamma_p(v) = \sum_{i=1}^{n+1} \langle v, V_i(p) \rangle V_i(p_0), \quad p \in \overline{M}, v \in T_p \overline{M}.$$

Let (\overline{M}, Γ) be a translational Riemannian manifold and $f: M^n \to \overline{M}$ an immersion of an orientable manifold M into \overline{M} . The following constructions are purely local, so we identify small neighbourhoods of M with their images via f, and the tangent spaces to M with their images via Df. Let $\eta: M \to T\overline{M}$ be a unit normal vector field along f, and let \mathbb{S}^n be the unit sphere of V.

Definition 2.1. The Gauss map $\gamma: M \to \mathbb{S}^n$ associated to the normal vector field η is given by

$$\gamma(p) = \Gamma_p(\eta(p)), \quad p \in M.$$

The tangent space of V at any point is canonically isomorphic to V, and via this isomorphism the tangent space of \mathbb{S}^n at a point x is just $\{x\}^{\perp}$. Thus, the derivative $D\gamma(p)$ maps T_pM into $T_{\gamma(p)}\mathbb{S}^n=\{\gamma(p)\}^{\perp}$ and Γ_p^{-1} maps the latter back into T_pM . This makes possible the following:

Definition 2.2. The Γ -curvature of M is the map $\kappa_{\Gamma}: M \to \mathbb{R}$ given by

$$\kappa_{\Gamma}(p) = \det \left(\Gamma_p^{-1} \circ D\gamma(p) \right), \quad p \in M.$$

Next, we define a special type of vector field that will play an important role.

Definition 2.3. Given a vector $X \in T_p \overline{M}$, the vector field $\widetilde{X} \in \mathfrak{X}(\overline{M})$ defined by

$$\widetilde{X}(q) = \left(\Gamma_q^{-1} \circ \Gamma_p\right)(X), \quad q \in \overline{M}$$

is called the Γ -invariant (or simply invariant) vector field of \overline{M} associated with X.

Example 2.4 (The Euclidean translation). If $\overline{M} = \mathbb{R}^{n+1}$ and $\Gamma : T\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is the identity, then the Gauss map γ for an orientable hypersurface M is the ordinary one. The invariant vector fields of \mathbb{R}^{n+1} are the constant vector fields and κ_{Γ} is the Gauss-Kronecker curvature of M.

Example 2.5 (Left translation on Lie groups). More generally, let $\overline{M} = G$ be a Lie group and $V = \mathfrak{g}$ be the Lie algebra of G, considered as the tangent space of G at the identity. Choose a left invariant metric for G and define $\Gamma: TG \to G \times \mathfrak{g}$ by

$$\Gamma(g,v) = (g, DL_{g-1}(g) \cdot v), \quad (g,v) \in TG,$$

where $L_x: y \mapsto xy$ is the left translation. Here, the Γ -invariant vector fields are the left invariant vector fields of G. This is the setting studied in [2].

Example 2.6 (Parallel transport). Assume \overline{M} is a Cartan-Hadamard manifold, that is, a complete, connected and simply connected Riemannian manifold with nonpositive sectional curvature. Given a point $p_0 \in \overline{M}$, the exponential map at p_0 is, by Hadamard's Theorem, a diffeomorphism from $T_{p_0}\overline{M}$ onto \overline{M} , so that every point p can be joined to p_0 by a unique geodesic. Setting $V = T_{p_0}\overline{M}$, we may then define $\Gamma_p : T_p\overline{M} \to V$ by choosing $\Gamma_p(v)$ as being the parallel transport of $v \in T_p\overline{M}$ to $T_{p_0}\overline{M}$ along this geodesic. Thus, the invariant vector fields here are the parallel vector fields along the geodesic rays issuing from p_0 .

More generally, given any complete Riemannian manifold \overline{M} and a point p_0 in \overline{M} , we can define the parallel transport to $T_{p_0}\overline{M}$ on $\overline{M}\setminus C_{p_0}$ as above, where C_{p_0} is the cut locus of p_0 ([5], Chapter I). We study this case in detail on the sphere (Section 3).

We next describe the geometry of the Gauss map. Let $\overline{\nabla}$ be the Levi-Civita connection of \overline{M} . Recall that the shape operator of M is the section A of the vector bundle $\operatorname{End}(TM)$ of endomorphisms of TM given by

$$A_p(X) = -\overline{\nabla}_X \eta, \quad p \in M, X \in T_p M.$$

Similarly, we define another section of $\operatorname{End}(TM)$, which depends additionally on the choice of the translation Γ .

Definition 2.7. The invariant shape operator of M is the section α of the bundle $\operatorname{End}(TM)$ given by

$$\alpha_p(X) = \overline{\nabla}_X \widetilde{\eta(p)}, \quad p \in M, X \in T_pM.$$

The proposition below establishes a relationship between γ and the extrinsic geometry of M.

Proposition 2.8. For any $p \in M$, the following identity holds:

$$\Gamma_p^{-1} \circ D\gamma(p) = -(A_p + \alpha_p).$$

Proof. Fix $p \in M$ and an orthonormal basis $\{X_1, \ldots, X_{n+1}\}$ of $T_p\overline{M}$ such that $X_{n+1} = \eta(p)$. The vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_{n+1}$ form a global orthonormal referential of $T\overline{M}$, so that we can write

(2)
$$\eta = \sum_{i=1}^{n+1} a_i \widetilde{X}_i$$

for certain functions $a_i \in C^{\infty}(M)$. Notice that $a_i(p) = 0$ for $i \in \{1, ..., n\}$ and $a_{n+1}(p) = 1$.

For $y \in M$ we have

$$\gamma(y) = \Gamma_y(\eta(y)) = \Gamma_y\left(\sum_{i=1}^{n+1} a_i(y)\widetilde{X}_i(y)\right) = \sum_{i=1}^{n+1} a_i(y)\Gamma_p(X_i).$$

Therefore, if $X \in T_pM$,

(3)
$$\Gamma_p^{-1}(D\gamma(p) \cdot X) = \Gamma_p^{-1}\left(\sum_{i=1}^{n+1} X(a_i)\Gamma_p(X_i)\right) = \sum_{i=1}^{n+1} X(a_i)X_i.$$

From (2) and (3) we obtain

$$-A_p(X) = \overline{\nabla}_X \eta = \sum_{i=1}^{n+1} \overline{\nabla}_X (a_i \widetilde{X}_i) = \sum_{i=1}^{n+1} \left[a_i(p) \overline{\nabla}_X \widetilde{X}_i + X(a_i) \widetilde{X}_i(p) \right]$$
$$= \overline{\nabla}_X \widetilde{X}_{n+1} + \sum_{i=1}^{n+1} X(a_i) X_i = \alpha_p(X) + \Gamma_p^{-1}(D\gamma(p) \cdot X),$$

which gives the desired result.

We now provide the proof for our Gauss-Bonnet theorem.

Proof of Theorem 1.2. Let σ be the volume form of \mathbb{S}^n induced by the metric on V. From the fact that Γ restricts to isometries in each fibre and from the definition of κ_{Γ} , it follows that $\gamma^*\sigma = \kappa_{\Gamma} \omega$. Then, the change of variables formula yields

$$\int_{M} \kappa_{\Gamma} \omega = \int_{M} \gamma^* \sigma = \deg(\gamma) \int_{\mathbb{S}^n} \sigma = c_n \deg(\gamma).$$

It remains to show that $\deg(\gamma) = \frac{1}{2}\chi(M)$. For this, define $\widetilde{\Gamma}: TM \to T\mathbb{S}^n$ by

$$\widetilde{\Gamma}(p,v) = \left(\gamma(p), \Gamma_p(v)\right), \quad (p,v) \in TM.$$

We have the following vector bundle map

$$TM \xrightarrow{\tilde{\Gamma}} T\mathbb{S}^r$$

$$\downarrow \qquad \qquad \downarrow$$

$$M \xrightarrow{\gamma} \mathbb{S}^n$$

Therefore, if e(M) and $e(\mathbb{S}^n)$ are the Euler classes of M and \mathbb{S}^n , and since n is even, we obtain

$$\chi(M) = (e(M), [M]) = (\gamma^*(e(\mathbb{S}^n)), [M])$$

$$= (e(\mathbb{S}^n), \gamma^*([M])) = (e(\mathbb{S}^n), \deg(\gamma)[\mathbb{S}^n])$$

$$= \deg(\gamma) (e(\mathbb{S}^n), [\mathbb{S}^n]) = \deg(\gamma) \chi(\mathbb{S}^n) = 2 \deg(\gamma),$$

where $[\cdot]$ indicates fundamental class in homology and (\cdot, \cdot) the duality between homology and cohomology.

3. Topological rigidity of hypersurfaces of the sphere

In this section we will investigate the earlier constructions in the following situation. Let \overline{M} be the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ with a point $-p_0$ deleted, which we will denote by $\mathbb{S}^{n+1}_{-p_0}$, and let V be the tangent space of the sphere at p_0 . The metrics of \mathbb{S}^{n+1} and V are those induced from \mathbb{R}^{n+2} . Given two non-antipodal points p,q in the sphere, let $\tau_p^q:T_p\mathbb{S}^{n+1}\to T_q\mathbb{S}^{n+1}$ be the parallel transport along the unique geodesic joining p to q (we agree that τ_p^p is the identity of $T_p\mathbb{S}^{n+1}$). Since this map is a linear isometry, we define $\Gamma:T\mathbb{S}^{n+1}_{-p_0}\to\mathbb{S}^{n+1}_{-p_0}\times V$ by

$$\Gamma(p,v) = \left(p, \tau_p^{p_0}(v)\right), \quad (p,v) \in T\mathbb{S}_{-p_0}^{n+1}.$$

If M^n is an orientable immersed hypersurface of \mathbb{S}^{n+1} not containing $-p_0$ and $\eta: M \to \mathbb{R}^{n+2}$ is a unit normal vector field (tangent to the sphere), the Gauss map $\gamma: M \to \mathbb{S}^n$ calculated at a point p is just the parallel transport of the normal $\eta(p)$ to M along the geodesic joining p to p_0 . The next proposition contains the relevant information we will need.

Proposition 3.1. Let p and q be non-antipodal points in \mathbb{S}^{n+1} . With the above notations, the following formulae hold:

(i)

$$\tau_p^q(v) = -\left[\frac{\langle v, q \rangle}{1 + \langle q, p \rangle}\right] (q+p) + v, \quad v \in T_p \mathbb{S}^{n+1}.$$

(ii)

$$\gamma(p) = -\left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}\right] (p + p_0) + \eta(p).$$

(iii)

$$\alpha_p(X) = \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle} \right] X, \quad X \in T_p M.$$

Proof. To prove the first two items, let $\beta:[0,t_q]\to\mathbb{S}^{n+1}$ be the unit speed geodesic joining p to q, given by

$$\beta(t) = (\cos t)p + (\sin t)\overline{q}, \quad t \in [0, t_a],$$

where

$$\overline{q} = \frac{q - \langle q, p \rangle p}{\|q - \langle q, p \rangle p\|} = \frac{q - \langle q, p \rangle p}{\sqrt{1 - \langle q, p \rangle^2}}.$$

For fixed $v \in T_p \mathbb{S}^{n+1}$, let $X : [0, t_q] \to \mathbb{R}^{n+2}$ be the parallel vector field along β and tangent to the sphere with prescribed initial value X(0) = v. Differentiating $\langle X, \beta \rangle \equiv 0$, we obtain

$$-\langle X', \beta \rangle \equiv \langle X, \beta' \rangle,$$

and since X and β' are parallel along β , $\langle X, \beta' \rangle$ is constant, equal to $C \in \mathbb{R}$, say, with

$$C = \langle X(0), \beta'(0) \rangle = \langle v, \overline{q} \rangle = \frac{\langle v, q \rangle}{\sqrt{1 - \langle q, p \rangle^2}}$$

The equation for X to be a parallel vector field is $X' - \langle X', \beta \rangle \beta \equiv 0$. Writing $X = (x_1, \ldots, x_{n+2})$, using (4) and the expression for β , we have

$$X'(t) = -C\left[(\cos t)p + (\sin t)\overline{q}\right], \quad t \in [0, t_q].$$

The solution satisfying X(0) = v is then

$$X(t) = C \left[(\cos t - 1)\overline{q} - (\sin t)p \right] + v, \quad t \in [0, t_q].$$

Noticing that $\cos t_q = \langle q, p \rangle$ and $\sin t_q = \sqrt{1 - \langle q, p \rangle^2}$, we finally obtain

$$\tau_p^q(v) = X(t_p) = -\left[\frac{\langle v, q \rangle}{1 + \langle q, p \rangle}\right](q+p) + v$$

and

$$\gamma(p) = \tau_p^{p_0}(\eta(p)) = -\left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}\right](p + p_0) + \eta(p),$$

as required.

For the last item, let $v = \gamma(p)$. Recall that $\widetilde{v} \in \mathfrak{X}\left(\mathbb{S}^{n+1}_{-p_0}\right)$ is the invariant vector field associated with v, and $\widetilde{v} = \widetilde{\eta(p)}$. From (i) we have

(5)
$$\widetilde{v}(q) = \tau_{p_0}^q(v) = -\left[\frac{\langle v, q \rangle}{1 + \langle q, p_0 \rangle}\right] (q + p_0) + v, \quad q \in \mathbb{S}_{-p_0}^{n+1}.$$

If $\widetilde{\nabla}$ denotes the Riemannian connection of \mathbb{R}^{n+2} , then

$$\alpha_p(X) = \overline{\nabla}_X \widetilde{v} = \widetilde{\nabla}_X \widetilde{v} - \langle \widetilde{\nabla}_X \widetilde{v}, p \rangle p, \quad X \in T_p M.$$

A straightforward calculation shows that

(6)
$$\widetilde{\nabla}_X \widetilde{v} = \left[\frac{-\langle v, X \rangle (1 + \langle p, p_0 \rangle) + \langle v, p \rangle \langle X, p_0 \rangle}{(1 + \langle p, p_0 \rangle)^2} \right] (p + p_0) - \left[\frac{\langle v, p \rangle}{1 + \langle p, p_0 \rangle} \right] X.$$

Notice that $\langle X, \widetilde{v}(p) \rangle = \langle X, \eta(p) \rangle = 0$, since $X \in T_pM$. Using the expression of \widetilde{v} in (5), we have

$$-\langle v, X \rangle (1 + \langle p, p_0 \rangle) + \langle v, p \rangle \langle X, p_0 \rangle = 0.$$

Substituting this in (6), we obtain

$$\widetilde{\nabla}_X \widetilde{v} = -\left[\frac{\langle v, p \rangle}{1 + \langle p, p_0 \rangle}\right] X.$$

Then, using formula (ii) for $\gamma(p)$,

$$\langle v, p \rangle = \langle \gamma(p), p \rangle = -\left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}\right] \langle p + p_0, p \rangle + \langle \eta(p), p \rangle = -\langle \eta(p), p_0 \rangle.$$

Hence.

(7)
$$\alpha_p(X) = \widetilde{\nabla}_X \widetilde{v} = \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle} \right] X, \quad X \in T_p M.$$

since $\widetilde{\nabla}_X \widetilde{v}$ is already tangent to the sphere at p.

Before we prove Theorem 1.3, we need the following lemma:

Lemma 3.2. For a parameter $r \in (0,1)$, let

$$M_r = \mathbb{S}^1(r) \times \mathbb{S}^{n-1}(s) = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^n : ||x|| = r, ||y|| = s\} \subset \mathbb{S}^{n+1},$$

where $s = \sqrt{1 - r^2}$. If R is the radius of the largest open geodesic ball of \mathbb{S}^{n+1} which does not intersect M_r , then

$$\cos R = \min\{r, s\}.$$

Proof. Recall that the distance between two points p,q in the sphere \mathbb{S}^{n+1} is given by $\arccos\langle p,q\rangle$, so that

$$\cos R = \inf \left\{ \sup \{ \langle p, q \rangle : q \in M_r \} : p \in \mathbb{S}^{n+1} \right\}.$$

Writing $p = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^n$, we have

$$\sup\{\langle p, q \rangle : q \in M_r\} = \sup\{\langle x, u \rangle + \langle y, v \rangle : (u, v) \in M_r\}$$
$$= r||x|| + s||y||.$$

Thus,

$$\cos R = \inf \left\{ r \|x\| + s \|y\| : (x,y) \in \mathbb{S}^{n+1} \right\} = \min \{r,s\}.$$

Proof of Theorem 1.3. Let $\eta: M \to \mathbb{R}^{n+2}$ be the unit normal vector field which gives rise to the orientation of M and let p_0 be the center of a geodesic ball of radius R containing M. Define a function $c: M \to \mathbb{R}$ by

$$c(p) = \frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}, \quad p \in M$$

and a vector field $E \in \mathfrak{X}(\mathbb{S}^{n+1})$ by

$$E(p) = p_0 - \langle p, p_0 \rangle p, \quad p \in M.$$

Notice that $\langle \eta(p), E(p) \rangle = \langle \eta(p), p_0 \rangle$ for p in M. Then, using Cauchy-Schwarz inequality, we have the following estimate for c:

$$|c(p)| \leq \frac{\|\eta(p)\| \|E(p)\|}{1 + \langle p, p_0 \rangle} = \frac{\sqrt{1 - \langle p, p_0 \rangle^2}}{1 + \langle p, p_0 \rangle} = \sqrt{\frac{1 - \langle p, p_0 \rangle}{1 + \langle p, p_0 \rangle}}, \quad \forall \, p \in M.$$

Thus,

$$|c(p)| \le \sqrt{\frac{1 - \cos d(p, p_0)}{1 + \cos d(p, p_0)}} = \tan \left(\frac{d(p, p_0)}{2}\right) \le \tan \left(\frac{R}{2}\right), \quad \forall p \in M.$$

Let $p \in M$. Choosing an orthonormal basis of T_pM that diagonalises the shape operator A_p , the matrix of $\Gamma_p^{-1} \circ D\gamma(p)$ with respect to this basis is diagonal with entries $\lambda_i(p) + c(p) \neq 0$. Therefore, this map is an isomorphism for each $p \in M$, and so is $D\gamma(p)$. Since M is compact, γ is a covering map, and since M is connected with $n \geq 2$, γ is a diffeomorphism.

For the second part, let $\varepsilon \in (0, \sqrt{2} - 1)$. We will show that it is possible to choose $r \in I = \left(0, \frac{1}{\sqrt{2}}\right]$ so that the principal curvatures of the hypersurface $M_r \subset \mathbb{S}^{n+1}$ from Lemma 3.2 satisfy (1).

For any $r \in (0,1)$, the principal curvatures λ_i of M_r are constant, with

$$\lambda_1 = -\frac{\sqrt{1 - r^2}}{r}$$

and

$$\lambda_2 = \dots = \lambda_n = \frac{r}{\sqrt{1 - r^2}}.$$

If $r \in I$ then $r \leq \sqrt{1-r^2}$ and, according to Lemma 3.2, $\cos R = r$. A simple calculation then shows that (1) holds if and only if $r \in J_{\varepsilon} = \left(\frac{\varepsilon}{1-\varepsilon}, \frac{1}{1+\varepsilon}\right)$. Since $\varepsilon \in (0, \sqrt{2}-1)$, we have $J_{\varepsilon} \neq \emptyset$ and $I \cap J_{\varepsilon} \neq \emptyset$. Thus, any r in this intersection is suitable for our purposes.

In order to prove Theorem 1.4, we start by introducing some ingredients and notations. Let p_0 be the north pole of \mathbb{S}^{n+1} and let \mathbb{S}^{n+1}_+ be the open hemisphere centered at p_0 . The Beltrami map $B: S^{n+1}_+ \to \mathbb{R}^{n+1} \approx T_{p_0} \mathbb{S}^{n+1}$ is the diffeomorphism obtained by central projection. Explicitly, it is given by

$$B(p) = \left(\frac{p_1}{p_{n+2}}, \dots, \frac{p_{n+1}}{p_{n+2}}\right), \quad p = (p_1, \dots, p_{n+2}) \in S_+^{n+1}.$$

For t > 0, let $H_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homothety $x \mapsto tx$. The map we are interested in is $C_t = B^{-1} \circ H_t \circ B$. It can be shown that

$$C_t(p) = \frac{m_t(p)}{\|m_t(p)\|}, \quad p \in S_+^{n+1},$$

where $m_t: S^{n+1}_+ \to \mathbb{R}^{n+2} \setminus \{0\}$ is defined by

(8)
$$m_t(p) = \left(p_1, \dots, p_{n+1}, \frac{p_{n+2}}{t}\right), \quad p = (p_1, \dots, p_{n+2}) \in S_+^{n+1}.$$

Some long but easy calculations yield

$$DC_{t}(p) \cdot v = \frac{1}{\|m_{t}(p)\|} \left\{ \left[\frac{(t-1)\langle v, p_{0} \rangle}{t^{2} \|m_{t}(p)\|^{2}} \right] \left[(t+1)\langle p, p_{0} \rangle p - t p_{0} \right] + v \right\},\,$$

for $(p, v) \in TS_{+}^{n+1}$.

Let M be an oriented hypersurface of \mathbb{S}^{n+1} with unit normal vector field $\eta: M \to \mathbb{R}^{n+2}$. Recall that the second fundamental form of M at a point p (in the direction of η) is the quadratic form $\Pi_p: T_pM \to \mathbb{R}$ induced by the shape operator A_p , that is,

$$II_p(v) = \langle A_p(v), v \rangle, \quad v \in T_pM.$$

Alternatively, if $\alpha:(-\varepsilon,\varepsilon)\to M$ is a curve with $\alpha(0)=p$ and $\alpha'(0)=v$, then

$$II_p(v) = \langle \alpha''(0), \eta(p) \rangle,$$

where the double prime indicates the usual second derivative, regarding α as a curve in \mathbb{R}^{n+2}

Proof of Theorem 1.4. After a rotation, we may suppose M is contained in \mathbb{S}^{n+1}_+ . By Theorem 1.3 (with $R = \frac{\pi}{2}$), M would be diffeomorphic to \mathbb{S}^n if all its principal curvatures were bigger that 1 in absolute value. This is not necessarily true. However, defining $M_t = C_t(M)$, we will show that if t is sufficiently small, then this bound on the principal curvatures holds for M_t .

Let $\eta: M \to \mathbb{R}^{n+2}$ be the unit normal vector field (tangent do the sphere) that induces the orientation of M. One can directly check that the vector field $\eta_t: M_t \to \mathbb{R}^{n+2}$ given by

(9)
$$\eta_t(C_t(p)) = \frac{\eta(p) + (t-1)\langle \eta(p), p_0 \rangle p_0}{\sqrt{1 + (t^2 - 1)\langle \eta(p), p_0 \rangle^2}}, \quad p \in M,$$

is normal to M_t ; it has unit length because the denominator is the norm of the numerator. We will establish a relationship between the second fundamental forms II and II^t of M and M_t with respect to the normals η and η_t . Let $\alpha: (-\varepsilon, \varepsilon) \to M$ be a curve with $\alpha(0) = p$ and $\alpha'(0) = v$, with ||v|| = 1. Consider $\beta = C_t \circ \alpha$ the corresponding curve in M_t , with $\beta(0) = q$ and $\beta'(0) = w$.

Introducing the functions $y_t, z_t : M \to \mathbb{R}$ given by

$$y_t(p) = \frac{(t+1)\langle p, p_0 \rangle}{t || m_t(p)||}, \quad p \in M$$

and

$$z_t(p) = \frac{1}{\|m_t(p)\|}, \quad p \in M,$$

one has, after rearranging,

$$\beta'(s) = z_t(\alpha(s)) \left\{ \left\lceil \frac{(t-1)\langle \alpha'(s), p_0 \rangle}{t} \right\rceil \left[y_t(\alpha(s))\beta(s) - p_0 \right] + \alpha'(s) \right\}.$$

Differentiating this expression and evaluating at s=0, we obtain

$$\beta''(0) = (Dz_t(p) \cdot v) \|m_t(p)\|w + z_t(p) \left\{ \left[\frac{(t-1)\langle \alpha''(0), p_0 \rangle}{t} \right] [y_t(p)q - p_0] + \left[\frac{(t-1)\langle v, p_0 \rangle}{t} \right] [(Dy_t(p) \cdot v) q + y_t(p)w] + \alpha''(0) \right\}.$$

Since $\langle q, \eta_t(q) \rangle = \langle w, \eta_t(q) \rangle = 0$, we have

$$\langle \beta''(0), \eta_t(q) \rangle = z_t(p) \left[\langle \alpha''(0), \eta_t(q) \rangle - \frac{(t-1)\langle \alpha''(0), p_0 \rangle \langle \eta_t(q), p_0 \rangle}{t} \right].$$

Using expression (9) for η_t we arrive at

$$\Pi_q^t(w) = \langle \beta''(0), \eta_t(q) \rangle = \frac{\Pi_p(v)}{\|m_t(p)\| \left[1 + (t^2 - 1)\langle \eta(p), p_0 \rangle^2\right]^{1/2}}.$$

Furthermore,

$$\|w\|^2 = \frac{1}{\|m_t(p)\|^2} \left[\frac{(1-t^2)(\langle p, p_0 \rangle^2 + \langle v, p_0 \rangle^2) + t^2}{t^2 \|m_t(p)\|^2} \right].$$

Thus, these two last equations and the value of $||m_t(p)||$ obtainable from (8) yield the desired relationship between Π_p and Π_q^t :

$$\Pi_q^t \left(\frac{w}{\|w\|} \right) = F_t(p, v) \Pi_p(v),$$

where

$$F_t(p,v) = \frac{\left[(1-t^2)\langle p, p_0 \rangle^2 + t^2 \right]^{3/2}}{t \left[(1-t^2)(\langle p, p_0 \rangle^2 + \langle v, p_0 \rangle^2) + t^2 \right] \left[1 + (t^2-1)\langle \eta(p), p_0 \rangle^2 \right]^{1/2}}.$$

Since M is compact and contained in \mathbb{S}^{n+1}_+ we may choose $h, \varepsilon \in (0,1)$ such that $\langle x, p_0 \rangle^2 \geq h$ and $\langle \eta(x), p_0 \rangle^2 < 1 - \varepsilon^2$ for all $x \in M$. We have the following estimates if $0 < t < \frac{1}{\sqrt{2}}$:

$$(1 - t^{2})\langle p, p_{0}\rangle^{2} + t^{2} \ge \frac{h}{2}$$
$$(1 - t^{2})(\langle p, p_{0}\rangle^{2} + \langle v, p_{0}\rangle^{2}) + t^{2} \le 3$$
$$1 + (t^{2} - 1)\langle \eta(p), p_{0}\rangle^{2} \le 1.$$

This way,

$$F_t(p,v) \ge \frac{K}{t}, \quad \forall p \in M, \ \forall v \in T_p M, \ ||v|| = 1,$$

for
$$K = \frac{h^{3/2}}{6\sqrt{2}}$$
.

for $K = \frac{h^{3/2}}{6\sqrt{2}}$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_{1,t} \leq \cdots \leq \mu_{n,t}$ be the principal curvatures of M and M_t , respectively. The variational principle for eigenvalues gives

$$\lambda_j(p) = \min\left\{\max\left\{ \mathrm{II}_p(v) : v \in V, \, \|v\| = 1 \right\} : V \subseteq T_pM, \, \dim V = j \right\}$$

and

$$\mu_{j,t}(C_t(p)) = \min \left\{ \max \left\{ F_t(p, v) \coprod_p (v) : v \in V, \|v\| = 1 \right\} : V \subseteq T_p M, \dim V = j \right\}.$$

Notice that M must contain an elliptic point, that is, a point where all principal curvatures have the same sign, which we assume to be positive. The connectedness of M and the fact that its Gauss-Kronecker curvature is nowhere zero implies all principal curvatures are everywhere positive. So, for every point $p \in M$ and subspace V of T_pM , we have

$$\max\left\{F_t(p,v)\Pi_p(v):v\in V,\,\|v\|=1\right\}\geq F_t(p,v(V))\,\Pi_p(v(V))\geq \frac{K}{t}\,\Pi_p(v(V)),$$
 where $v(V)\in V$ satisfies $\|v(V)\|=1$ and

$$II_n(v(V)) = \max\{II_n(v) : v \in V, ||v|| = 1\} > 0.$$

Hence,

$$\mu_{j,t}(C_t(p)) \ge \min\left\{\frac{K}{t}\operatorname{II}_p(v(V)) : V \subseteq T_pM, \dim V = j\right\}$$

$$= \frac{K}{t}\min\left\{\max\left\{\operatorname{II}_p(v) : v \in V, \|v\| = 1\right\} : V \subseteq T_pM, \dim V = j\right\}$$

$$= \frac{K}{t}\lambda_j(p).$$

Setting

$$\lambda = \min \{ \lambda_j(p) : p \in M, j \in \{1, ..., n\} \} > 0,$$

we have

$$\mu_{j,t}(C_t(p)) \ge \frac{K}{t}\lambda$$

for every $p \in M$ and $0 < t < \frac{1}{\sqrt{2}}$. Thus, provided that t is sufficiently small, all the principal curvatures of M_t are bigger than 1 in absolute value, as we wanted.

Remark 3.3. We observe that the same constructions done in the sphere can be done in the hyperbolic space using the Lorentzian model. In particular, one can prove using the same technique that a compact hypersurface of the hyperbolic space having everywhere nonzero Gauss-Kronecker curvature is diffeomorphic to a sphere. However, in the hyperbolic space, if the Gauss-Kronecker curvature is nowhere zero then necessarily the principal curvatures of the hypersurface have the same sign, and then the result follows from the Proposition of [3].

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