Coloring graphs of various maximum degree from random lists

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Abstract. Let G = G(n) be a graph on n vertices with maximum degree $\Delta = \Delta(n)$. Assign to each vertex v of G a list L(v) of colors by choosing each list independently and uniformly at random from all k-subsets of a color set C of size $\sigma = \sigma(n)$. Such a list assignment is called a $random\ (k,C)$ -list assignment. In this paper, we are interested in determining the asymptotic probability (as $n \to \infty$) of the existence of a proper coloring φ of G, such that $\varphi(v) \in L(v)$ for every vertex v of G, a so-called L-coloring. We give various lower bounds on σ , in terms of n, k and Δ , which ensures that with probability tending to 1 as $n \to \infty$ there is an L-coloring of G. In particular, we show, for all fixed k and growing n, that if $\sigma(n) = \omega(n^{1/k^2} \Delta^{1/k})$ and $\Delta = O\left(n^{\frac{k-1}{k(k^3+2k^2-k+1)}}\right)$, then the probability that G has an L-coloring tends to 1 as $n \to \infty$. If $k \ge 2$ and $\Delta = \Omega(n^{1/2})$, then the same conclusion holds provided that $\sigma = \omega(\Delta)$. We also give related results for other bounds on Δ , when k is constant or a strictly increasing function of n.

Keywords: list coloring, random list, coloring from random lists

1 Introduction

Given a graph G, assign to each vertex v of G a set L(v) of colors (positive integers). Such an assignment L is called a *list assignment* for G and the sets L(v) are referred to as *lists* or color lists. If all lists have equal size k, then L is called a k-list assignment. We then want to find a proper vertex coloring φ of G, such that $\varphi(v) \in L(v)$ for all $v \in V(G)$. If such a coloring φ exists then G is L-colorable and φ is called an L-coloring. Furthermore, G is called G-colorable if it is G-colorable for every G-list assignment G-colorable.

This particular variant of vertex coloring is known as *list coloring* and was introduced by Vizing [11] and independently by Erdős et al. [8].

In this paper we continue the study of the problem of coloring graphs from random lists introduced by Krivelevich and Nachmias [9, 10]: Assign lists of colors to the vertices of a graph G = G(n) with n vertices by choosing for each vertex v its list L(v) independently and uniformly at

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random from all k-subsets of a color set $C = \{1, 2, ..., \sigma\}$. Such a list assignment is called a random (k, C)-list assignment for G. Intuitively it should hold that the larger σ is, the more spread are the colors chosen for the lists and thus the more likely it is that we can find a proper coloring of G with colors from the lists. The question that we address in this paper is how large $\sigma = \sigma(n)$ should be in order to guarantee that with high probability¹ (as $n \to \infty$) there is a proper coloring of the vertices of G with colors from the random list assignment.

This problem was first studied by Krivelevich and Nachmias [9, 10] for the case of powers of cycles and the case of complete bipartite graphs with parts of equal size n. In the latter case they showed that for all fixed $k \geq 2$, the property of having a proper coloring from a random (k, \mathcal{C}) -list assignment exhibits a sharp threshold, and that the location of that threshold is exactly $\sigma(n) = 2n$ for k = 2. In [4], we generalized the second part of this result and showed that for a complete multipartite graph with s parts (fixed $s \geq 3$) of equal size n, the property of having a proper coloring from a random $(2, \mathcal{C})$ -list assignment, has a sharp threshold at $\sigma(n) = 2(s-1)n$.

Let C_n^r be the rth power of a cycle with n vertices. For powers of cycles, Krivelevich and Nachmias proved the following theorem establishing a coarse threshold for the property of being colorable from a random list assignment.

Theorem 1.1. [9] Assume r, k are fixed integers satisfying $r \ge k$ and let L be a random (k, \mathcal{C}) -list assignment for C_n^r . If we denote by $p_C(n)$ the probability that C_n^r is L-colorable, then

$$p_C(n) = \begin{cases} o(1), & \sigma(n) = o(n^{1/k^2}), \\ 1 - o(1), & \sigma(n) = \omega(n^{1/k^2}). \end{cases}$$

In [5, 6] we generalized Theorem 1.1:

Theorem 1.2. [5, 6] Let G = G(n) be a graph on n vertices with maximum degree bounded by some absolute constant, k a fixed positive integer, and L a random (k, \mathcal{C}) -list assignment for G. If $\sigma(n) = \omega(n^{1/k^2})$, then **whp** G is L-colorable.

Note that Theorem 1.2 is best possible for graphs with bounded maximum degree. Further results on the problem of coloring graphs from random lists appears in [2, 7].

In this paper we consider random (k, \mathcal{C}) -list assignments for graphs G = G(n) on n vertices whose maximum degree $\Delta = \Delta(n)$ is an increasing function of n. We would like to suggest the following:

Conjecture 1.3. Let G = G(n) be a graph on n vertices with maximum degree at most $\Delta = \Delta(n)$, $k \geq 2$ a fixed positive integer, and L a random (k, \mathcal{C}) -list assignment for G.

(i) If
$$\Delta = o\left(n^{\frac{1}{k^2-k}}\right)$$
 and $\sigma = \omega\left(n^{1/k^2}\Delta^{1/k}\right)$, then whp G is L-colorable.

(ii) If
$$\Delta = \Omega\left(n^{\frac{1}{k^2-k}}\right)$$
 and $\sigma = \omega(\Delta)$, then **whp** G is L-colorable.

Note that when Δ is bounded then part (i) of the conjecture reduces to Theorem 1.2. Moreover, for the case k=1, it is easily seen that for a graph G=G(n) on n vertices with strictly increasing maximum degree $\Delta=\Delta(n)$, the coarse threshold for colorability from a random $(1,\mathcal{C})$ -list assignment trivially occurs at $\sigma=n\Delta$.

We prove that part (i) of Conjecture 1.3 is true for a slightly more restrictive bound on Δ :

An event A_n occurs with high probability (abbreviated whp) if $\lim_{n\to\infty} \mathbb{P}[A_n] = 1$.

Theorem 1.4. Let G = G(n) be a graph on n vertices with maximum degree at most $\Delta = \Delta(n)$, k a fixed positive integer, and L a random (k, \mathcal{C}) -list assignment for G. If $\sigma(n) = \omega\left(n^{1/k^2}\Delta^{1/k}\right)$ and $\Delta = O\left(n^{\frac{k-1}{k(k^3+2k^2-k+1)}}\right)$, then **whp** G is L-colorable.

For the case k=2 we prove that both part (i) and (ii) of Conjecture 1.3 is true:

Theorem 1.5. Let G = G(n) be a graph on n vertices with maximum degree at most $\Delta = \Delta(n)$, and L a random $(2, \mathcal{C})$ -list assignment for G. If

(i)
$$\Delta = o(n^{1/2})$$
 and $\sigma = \omega(n^{1/4}\Delta^{1/2})$, or

(ii)
$$\Delta = \Omega(n^{1/2})$$
 and $\sigma = \omega(\Delta)$,

then whp G is L-colorable.

For a complete graph K_n on n vertices, the property of being colorable from a random $(2, \mathcal{C})$ -list assignment has a sharp threshold at $\sigma(n) = 2n$ [5], and for $k \geq 3$, K_n is **whp** colorable from a random (k, \mathcal{C}) -list assignment if $\sigma \geq 1.223n$ [6]. Thus Theorem 1.5 yields a better bound on $\sigma(n)$ for graphs G with maximum degree o(n).

In Section 2 we shall prove Theorem 1.4 and also give an example which shows that part (i) of Conjecture 1.3 (and therefore also Theorem 1.4) is sharp in the "coarse threshold sense": for each integer n and each integer-valued function $\Delta = \Delta(n)$ satisfying $\Delta = O(n^{\frac{1}{k^2-k}})$, there is a graph H = H(n) with maximum degree Δ such that if $\sigma(n) = o(n^{1/k^2} \Delta^{1/k})$ and L is a random (k, \mathcal{C}) -list assignment for H, then **whp** H is not L-colorable. In Section 2 we also prove some related results for graphs with other bounds on the maximum degree.

Section 3 contains the proof of Theorem 1.5. Note also that part (ii) of the conjecture (and thus Theorem 1.5) is best possible in the same sense as Theorem 1.4, since a clique on Δ vertices requires at least Δ colors for a proper coloring.

In Section 4 we prove some related results for graphs with fixed girth greater than 3, and we also prove a result for graphs with girth $\Omega(\log \log n)$: for each constant C > 0, there are constants $k_0 = k_0(C)$ and B = B(C), such that if G is a graph with n vertices, maximum degree at most Δ , and girth $g \geq C \log \log n$, $k \geq k_0$ and $\sigma(n) > B\Delta$, then **whp** G is colorable from a random (k, \mathcal{C}) -list assignment. (For instance when C = 1, then $k_0 = 9$ and $B = 81e^{7/2}$.)

In Section 5 we consider random list assignments where the uniform list size k is a strictly increasing function of n. In particular, we present an analogue of Theorem 1.4 for lists of nonconstant size, and we prove that for any constant C > 0, there is a constant A = A(C), such that if $k \ge C \log n$ and $\sigma \ge A\Delta \log n$, then G is **whp** colorable from a random (k, \mathcal{C}) -list assignment.

2 Proof of Theorem 1.4

In this section we prove Theorem 1.4 and some related results. Our basic method in this paper is rather similar to the proof of the main result of [6], but we need to refine the method introduced in that paper, and use sharper estimates in many counting arguments.

Let H be a graph and L a list assignment for H. If H is not L-colorable, but removing any vertex from H yields an L-colorable graph, then H is L-vertex-critical (or just L-critical). Obviously, if L is a list assignment for a graph G, and G is not L-colorable, then G contains a connected induced L-critical subgraph.

Suppose now that $H - w_1$ is L-colorable, where w_1 is some vertex of H. Given an L-coloring φ of $H - w_1$, a path $P = w_1 w_2 \dots w_t$ in H is called (φ, L) -alternating if there are colors c_2, c_3, \dots, c_t such that $\varphi(w_i) = c_i$ and $c_i \in L(w_{i-1}), i = 2, \dots, t$. We allow such a path to have length 0 and thus only consist of w_1 . The set of vertices which are adjacent to a vertex x in a graph G is denoted by $N_G(x)$. The following lemma was proved in [6].

Lemma 2.1. Let F be a graph and L a list assignment for F. If F is L-critical, then for any vertex $v_1 \in V(F)$, $F - v_1$ has an L-coloring φ that satisfies the following conditions:

- (i) all vertices in F lie on (φ, L) -alternating paths with origin at v_1 ;
- (ii) for each color $c \in L(v_1)$, there is a vertex $w \in N_F(v_1)$, such that $\varphi(w) = c$.
- (iii) Define a rank function $R: V(F) \to \{0, 1, ..., |V(F)| 1\}$ on the vertices of F by setting R(u) = j if a shortest (φ, L) -alternating path from v_1 to u has length j. Then for every vertex x of $F v_1$ and every color $c \in L(x) \setminus \{\varphi(x)\}$, there is either
 - (a) a vertex $y \in N_F(x)$ colored $\varphi(y) = c$ or
 - (b) a vertex $z \in N_F(x)$ such that $c \in L(z)$ and R(z) < R(x).

For a rank function R defined as in part (iii) of Lemma 2.1, we say that R is the rank function on V(F) induced by L and φ .

Let F be a connected induced subgraph of a graph G, v_1 a fixed vertex of F and $R:V(F) \to \{0,1,\ldots,|V(F)|-1\}$ a rank function on the vertices of F. The triple (F,v_1,R) is proper, if $R(v_1)=0$ and R(u)>0 for each vertex $u\in V(F)\setminus \{v_1\}$, and if R(u)=s, then there is a vertex $x\in N_F(u)$ such that R(x)=s-1. (This definition of proper triple is slightly different from the one used in [6].) We also say that (F,v_1,R) is a proper triple of G. Note that if F,v_1 and F0 satisfies the conditions of Lemma 2.1 for some choice of F1 and F2, then F3, F4 is proper. The next lemma gives an upper bound on the number of proper triples in a graph.

Lemma 2.2. Let G be a graph on n vertices whose maximum degree is at most Δ . The number of proper triples (F, v_1, R) , such that F is a subgraph of G with m vertices does not exceed

$$n\Delta^{m-1}(m-1)!.$$

Proof. If (F, v_1, R) is a proper triple, then by removing some edges of F we can construct a tree T with root v_1 , such that if $u \in V(F)$ has rank R(u) = r $(1 \le r \le m - 1)$, then u is adjacent to a vertex x in T with rank R(x) = r - 1, and v_1 is the unique vertex of rank $R(v_1) = 0$.

Moreover, given such a tree T in G with root v_1 and with a rank function $R:V(T)\to \{0,1,\ldots,m-1\}$ satisfying these conditions, there is a uniquely determined proper triple (F,v_1,R) with V(F)=V(T), because F is an induced subgraph of G. Hence, the number of proper triples of G is bounded by the number of such trees in G together with a rank function R. This latter quantity is bounded by

$$n\Delta(2\Delta)(3\Delta)\dots((m-1)\Delta) = n\Delta^{m-1}(m-1)!;$$

because there are n ways of selecting v_1 , then we have Δ choices for a neighbor of v_1 as the next vertex v_2 of T; thereafter, there are at most (2Δ) ways of choosing an edge incident with v_1 or v_2 that connects one of these vertices with the next vertex of T, etc.

Given a proper triple (F, v_1, R) of a graph G and a list assignment L for G such that F is not L-colorable, we say that (F, v_1, R) is L-bad (or just bad) if there is an L-coloring φ of $F - v_1$, such that F, v_1, R, L and φ satisfy conditions (i)-(iii) of Lemma 2.1. In particular, R is the rank function on V(F) induced by L and φ .

Consider a random (k, \mathcal{C}) -list assignment for a graph G, where $\mathcal{C} = \{1, 2, \dots, \sigma\}$. The following lemma was proved in [6].

Lemma 2.3. Let L be a random (k, \mathcal{C}) -list assignment for a graph G with maximum degree at most Δ . If (F, v_1, R) is a proper triple of G with m = |V(F)|, then

$$\mathbb{P}[(F, v_1, R) \text{ is } L\text{-bad}] \leq \frac{\sigma^{m-1} {\binom{\Delta}{k}} {\binom{\Delta k}{k-1}}^{m-1}}{{\binom{\sigma}{k}}^m}.$$

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let G = G(n) be a graph on n vertices with maximum degree at most Δ and let L be a random (k, \mathcal{C}) -list assignment for G, where k is a fixed positive integer, and suppose further that $\Delta = O(n^{\frac{k-1}{k(k^3+2k^2-k+1)}})$ and $\sigma(n) = \omega(n^{1/k^2}\Delta^{1/k})$. As pointed out above, the theorem is trivially true in the case when k = 1. So in the following we shall assume that $k \geq 2$. We will show that **whp** G has no connected induced L-critical subgraph. This suffices for proving the theorem. By Lemma 2.1, this means that we need to prove that if $\sigma(n) = \omega(n^{1/k^2}\Delta^{1/k})$, then **whp** G does not contain an L-bad proper triple (F, v_1, R) . We will use first moment calculations.

If (F, v_1, R) is L-bad, then F has at least k + 1 vertices. We first consider the case when F has exactly k + 1 vertices. It is not hard to see that if F is not L-colorable and |V(F)| = k + 1, then F is a (k + 1)-clique where all vertices have identical lists. The number of (k + 1)-cliques in G is at most $n\Delta^k$. Thus the expected number of (k + 1)-cliques where the vertices get identical lists is at most

$$n\Delta^k \binom{\sigma}{k}^{-k},$$
 (1)

which tends to 0 as $n \to \infty$, because $\sigma(n) = \omega(n^{1/k^2} \Delta^{1/k})$. Hence, **whp** there is no bad proper triple (F, v_1, R) in G satisfying that |V(F)| = k + 1.

Let us now consider the case when $|V(F)| \ge k+2$. First we show that if (F, v_1, R) is bad, then whp F contains at most Δ^{k^2+k} vertices. Consider a path P on r vertices in G with origin at some vertex v. The probability that there is an L-coloring φ of P-v, such that P is (φ, L) -alternating is at most

$$\frac{\sigma(\sigma-1)^{r-2}\binom{\sigma-1}{k-1}^2\binom{\sigma-2}{k-2}^{r-2}}{\binom{\sigma}{k}^r} \le \frac{k^{2r}}{\sigma^{r-1}},$$

because there are at most $\sigma(\sigma-1)^{r-2}$ ways of choosing the proper coloring φ and thereafter at most $\binom{\sigma-1}{k-1}^2\binom{\sigma-2}{k-2}^{r-2}$ ways of choosing the list assignment L for P so that φ is an L-coloring of P-v and P is (φ, L) -alternating. Moreover, the number of distinct paths in G on r vertices is at most $n\Delta^{r-1}$. Therefore, the expected number of paths P in G on at least k^2+k+1 vertices, for which there is an L-coloring φ of P-v such that P is (φ, L) -alternating is at most

$$\sum_{r=k^2+k+1}^{n} \frac{n\Delta^{r-1}k^{2r}}{\sigma^{r-1}} = o(1), \tag{2}$$

since $\Delta = O(n^{\frac{k-1}{k(k^3+2k^2-k+1)}})$ and $\sigma(n) = \omega(n^{1/k^2}\Delta^{1/k})$. Hence, by Markov's inequality, **whp** there is no *L*-coloring φ of a subgraph of *G* such that *G* contains a (φ, L) -alternating path of length $k^2 + k + 1$.

Now, by Lemma 2.1, if F is a subgraph of G that belongs to a bad proper triple (F, v_1, R) , then there is an L-coloring φ of $F - v_1$ such that all vertices of F lie on (φ, L) -alternating paths with origin at v_1 . Since **whp** the maximum length of such a path in G is at most $k^2 + k$, the maximum number of vertices in a subgraph of G that is in a bad proper triple is **whp** at most

$$1 + \Delta + \Delta^2 + \dots + \Delta^{k^2 + k - 1} \le \Delta^{k^2 + k}$$

Let X_m be a random variable counting the number of bad proper triples (F, v_1, R) in G such that F has m vertices and set

$$X = \sum_{m=k+2}^{\Delta^{k^2+k}} X_m.$$

Lemma 2.3 gives an upper bound on the probability that a given proper triple of G on m vertices is bad. Additionally, by Lemma 2.2,

$$f(m) \le n\Delta^{m-1}(m-1)!,$$

where f(m) is the number of proper triples (F, v_1, R) in G such that F has m vertices. Let p_m be the least number such that

$$\mathbb{P}[(F, v_1, R) \text{ is } L\text{-bad}] \leq p_m,$$

whenever (F, v_1, R) is a proper triple in G and F has m vertices. Since such a subgraph F whp has at most Δ^{k^2+k} vertices if (F, v_1, R) is L-bad, we conclude from Lemmas 2.2 and 2.3 that

 $\mathbb{P}[G \text{ contains an } L\text{-bad proper triple}] \leq \mathbb{E}[X] + o(1)$

$$\leq \sum_{m=k+2}^{\Delta^{k^2+k}} f(m)p_m + o(1)
\leq \sum_{m=k+2}^{\Delta^{k^2+k}} n\Delta^{2m-2}(m-1)! \frac{\sigma^{m-1}\binom{\Delta}{k}\binom{\Delta k}{k-1}^{m-1}}{\binom{\sigma}{k}^m} + o(1)
= O\left(\frac{n}{\sigma\Delta}\right) \sum_{m=k+2}^{\Delta^{k^2+k}} \left(\frac{mk^k\Delta^k}{\sigma^{k-1}}\right)^m
= O\left(\frac{n\Delta^{k(k+2)}}{\sigma^{(k-1)(k+2)+1}}\right) \sum_{m=0}^{\infty} \left(\frac{k^k\Delta^{k^2+2k}}{\sigma^{k-1}}\right)^m
= o(1).$$

provided that
$$\sigma(n) = \omega(n^{1/k^2} \Delta^{1/k}), \ \Delta = O(n^{\frac{k-1}{k(k^3+2k^2-k+1)}})$$
 and $k \ge 2$.

We now show that the bound on σ in part (i) of Conjecture 1.3 (and also Theorem 1.4) is best possible in the "coarse threshold sense". We will show that for positive integers $k \geq 2$ and $n \geq k+1$ (large enough), and each non-constant increasing integer-valued function $\Delta = O(n^{\frac{1}{k^2-k}})$, there is a

graph $H = H(n, \Delta)$ with n vertices and maximum degree Δ such that if $\sigma(n) = o(n^{1/k^2} \Delta^{1/k})$ and L is a random (k, \mathcal{C}) -list assignment for H, then **whp** H is not L-colorable.

So fix k, and let n be a positive integer satisfying $n \geq k+1$ and assume that Δ satisfies $\Delta = O(n^{\frac{1}{k^2-k}})$. We set $n_{\Delta} = \lfloor \frac{n}{\Delta+1} \rfloor$, and let H be a graph on n vertices which is the disjoint union of n_{Δ} complete graphs, each of which has $\Delta + 1$ vertices, and possibly some isolated vertices. Let $J_1, \ldots, J_{n_{\Delta}}$ be the non-trivial components of H and let L be a random (k, \mathcal{C}) -list assignment for H. We will prove that **whp** there is at least one (k+1)-clique in H where all vertices have identical lists, which means that **whp** H is not L-colorable. Note that we may assume that $\Delta \leq \sigma$, since otherwise, trivially there is no L-coloring of H.

Let X be a random variable counting the number of (k+1)-cliques in H where all vertices have identical lists. Then

$$\mathbb{E}[X] = \left\lfloor \frac{n}{\Delta + 1} \right\rfloor {\Delta + 1 \choose k + 1} {\sigma \choose k}^{-k} = \Theta(n(\Delta + 1)^k \sigma^{-k^2}). \tag{3}$$

To prove that $\mathbb{P}[X > 0] = 1 - o(1)$ we use the second moment method with the inequality due to Chebyshev in the following form:

$$\mathbb{P}[Y=0] \le \frac{\operatorname{Var}[Y]}{\mathbb{E}[Y]^2},\tag{4}$$

valid for all non-negative random variables Y. Since X is a sum of indicator random variables, we can use the following approach from [1].

Let $X = X_1 + \cdots + X_d$, where each X_i is the indicator random variable for the event that the vertices of a (k+1)-clique gets identical lists. Let A_i be the event corresponding to X_i , that is, $X_i = 1$ if A_i occurs and $X_i = 0$ otherwise. For indices i, j we write $i \sim j$ if $i \neq j$ and the events A_i, A_j are not independent. Set

$$\Pi = \sum_{i \sim j} \mathbb{P}[A_i \wedge A_j].$$

When $i \sim j$, we have

$$Cov[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] \le \mathbb{E}[X_i X_j] = \mathbb{P}[A_i \wedge A_j]$$

and when $i \neq j$ and not $i \sim j$ then $Cov[X_i, X_j] = 0$. Thus

$$Var[X] \le \mathbb{E}[X] + \Pi$$

and the following proposition follows from (4).

Claim 2.4. If
$$\mathbb{E}[X] \to \infty$$
 and $\Pi = o(\mathbb{E}[X]^2)$, then $\mathbb{P}[X > 0] = 1 - o(1)$.

It is clear from (3) that $\mathbb{E}[X] \to \infty$ if $\sigma(n) = o(n^{1/k^2} \Delta^{1/k})$. We now show that the second criterion of Claim 2.4 is satisfied. If $i \sim j$, then clearly A_i and A_j are events for cliques which are in the same component of H, and all vertices in these cliques have identical lists. Since H has n_{Δ} components and two distinct (k+1)-cliques have at most k vertices in common we have

$$\Pi = O(n_{\Delta}) \sum_{l=1}^{k} {\Delta+1 \choose k+1+l} {\sigma \choose k}^{-(k+l)} = O\left(n(\Delta+1)^{k+2} \sigma^{-k(k+1)}\right),$$

and thus $\Pi = o(\mathbb{E}[X]^2)$ as required. We conclude that Theorem 1.4 is best possible in the "coarse threshold sense".

Next, we will prove the following two propositions which show that weaker versions of Conjecture 1.3 hold for larger Δ . As usual, G = G(n) is a graph on n vertices with maximum degree at most $\Delta = \Delta(n)$, and L is a random (k, \mathcal{C}) -list assignment for G, where $k \geq 2$ is a fixed positive integer.

Proposition 2.5. Suppose that α and s are constants satisfying $1 \le \alpha \le 3$ and and $s \ge 2 + \frac{2}{k-1}$, $\Delta = O(n^{1/k^{\alpha}})$, and $\sigma(n) = \omega\left(n^{1/k^{\frac{\alpha+1}{2}}}\Delta^{s}\right)$. Then whp G is L-colorable.

Proposition 2.6. If $\Delta = \Omega(n^{1/k})$ and $\sigma(n) = \omega(n^{\frac{1}{k}}\Delta)$, then whp G is L-colorable.

We first prove Proposition 2.5. The proof is similar to the proof of Theorem 1.4 and therefore the proof will not be given in full detail.

Proof of Proposition 2.5 (sketch). Let G = G(n) be a graph on n vertices with maximum degree at most $\Delta = O(n^{1/k^{\alpha}})$, where α is a constant satisfying $1 \le \alpha \le 3$, and $k \ge 2$ is a fixed positive integer. Assume further that s is a constant satisfying $s \ge 2 + \frac{2}{k-1}$, and $\sigma = \omega\left(n^{1/k^{\frac{\alpha+1}{2}}}\Delta^s\right)$. Note that the condition on s implies that

$$sk \ge 2k + s. \tag{5}$$

As in the proof of Theorem 1.4, we shall prove that G whp has no bad proper triple.

Proceeding as in the proof of Theorem 1.4, one may first deduce that any (φ, L) -alternating path in G whp has at most $k^{(\alpha+1)/2}$ vertices, and thus if (F, v, R) is a bad proper triple in G then whp F has at most $\Delta^{k^{(\alpha+1)/2}}$ vertices. Using Lemmas 2.2 and 2.3 it now follows, as in the proof of Theorem 1.4, that the probability that G has a bad proper triple tends to zero if the sum

$$O\left(\frac{n}{\sigma\Delta}\right) \sum_{m=k+1}^{\Delta^{k} \frac{\alpha+1}{2}} \left(\frac{mk^{k}\Delta^{k}}{\sigma^{k-1}}\right)^{m}$$

tends to 0 as $n \to \infty$. Rewriting this sum yields that it is at most

$$O\left(\frac{n\Delta^{k(k+1)}}{\sigma^{k^2}\Delta}\right) \sum_{m=0}^{\infty} \left(\frac{\Delta^{k^{\frac{\alpha+1}{2}}} k^k \Delta^k \sigma}{\sigma^k}\right)^m.$$
 (6)

Setting $\sigma = n^{1/k^{\frac{\alpha+1}{2}}} \Delta^s$, we have that the ratio of the geometric sum in (6) is at most

$$\frac{\Delta^k \Delta^s}{\Delta^{(s-1)k}} \frac{\Delta^{k^{\frac{\alpha+1}{2}}} k^k n^{1/k^{\frac{\alpha+1}{2}}}}{n^{1/k^{\frac{\alpha-1}{2}}} \Delta^k}.$$

The first factor in this expression is bounded by (5). As regards the second factor, this quantity is maximum when $\Delta = \Theta(n^{1/k^{\alpha}})$, and using the fact that $\alpha - 1 \le \alpha/2 + 1/2$, when $\alpha \le 3$, it follows that this factor is bounded as well.

It follows that if $\sigma = \omega \left(n^{1/k^{\frac{\alpha+1}{2}}} \Delta^s \right)$, then the ratio of the geometric sum in (6) tends to zero as $n \to \infty$, and using this fact it is straightforward to verify that the expression (6) tends to zero.

Let us now prove Proposition 2.6.

Proof of Proposition 2.6 (sketch). Let $k \geq 2$ be a positive integer and G = G(n) be a graph on n vertices with maximum degree at most $\Delta = \Omega(n^{1/k})$. We have to prove that if $\sigma(n) = \omega(n^{1/k}\Delta)$ and L is a random (k, \mathcal{C}) -list assignment for G, then whp G is L-colorable.

We will show that **whp** G contains no connected induced L-critical subgraph. By Lemma 2.1 it suffices to prove that **whp** G contains no vertex v such that there are k neighbors u_1, \ldots, u_k of v such that $c_i \in L(u_i)$, for $i = 1, \ldots, k$, where $L(v) = \{c_1, \ldots, c_k\}$. The expected number of such vertices v in G is at most

$$n\binom{\Delta}{k} \frac{\binom{\sigma}{k} k! \binom{\sigma-1}{k-1}^k}{\binom{\sigma}{k}^{k+1}} = O\left(\frac{n\Delta^k}{\sigma^k}\right),$$

which tends to 0 as $n \to \infty$, so the desired result follows from Markov's inequality.

3 Lists of size 2

In this section we prove Theorem 1.5. The proof is not very different from the proofs in the preceding section, but we use a somewhat different technique employed in [5] for proving results on random $(2, \mathcal{C})$ -list assignments. Below we introduce some terminology and auxiliary results from that paper.

Let $P = v_1 e_1 v_2 \dots e_{d-1} v_d$ be a path. Then the sequence $C = v_1 e_1 v_2 \dots v_d e_d v_1$ is an ordered cycle if $e_d = v_d v_1$. Similarly, the sequence $D = v_1 e_1 v_2 \dots v_d e_d v_j$ is called an ordered lollipop if $e_d = v_d v_j$ and $j \in \{2, \dots, d-2\}$. Note that an ordered cycle C and an ordered lollipop D is uniquely determined by a sequence of vertices (as is also a path). We may thus write $C = v_1 \dots v_d v_1$ for ordered cycles C, and similarly $D = u_1 \dots u_j \dots u_d u_j$, for ordered lollipops D. Paths, ordered cycles and ordered lollipops will usually be referred to as sequences of vertices; however, sometimes we will refer to such sequences as graphs and then mean the graph consisting of the vertices and edges of the sequence. In particular, if C is an ordered cycle or lollipop, then V(C) and E(C) are the sets of all vertices and edges in C, respectively.

Let L be a 2-list assignment for a graph G and let $C = v_1 \dots v_d v_1$ be an ordered cycle of G. Suppose that there are colors c_1, \dots, c_{d-1} such that $c_1 \in L(v_1)$, $L(v_i) = \{c_{i-1}, c_i\}$, $i = 2, \dots, d-1$ and $L(v_d) = \{c_{d-1}, c_1\}$. Then C is L-alternating. Similarly, an ordered lollipop $D = u_1 \dots u_j \dots u_d u_j$ in G is L-alternating if there are colors c_1, \dots, c_{d-1} such that $c_1 \in L(u_1)$ and $L(u_i) = \{c_{i-1}, c_i\}$, $i = 2, \dots, d-1$ and $L(u_d) = \{c_{d-1}, c_j\}$. For an L-alternating ordered cycle or ordered lollipop D, the common color c_1 of the lists of the first two vertices of D will be referred to as the first color of D.

The following lemma was proved in [5].

Lemma 3.1. Let G be a graph and L a 2-list assignment for G. If G is not L-colorable, then there are subgraphs H_1 and H_2 of G, such that for i = 1, 2:

- (i) H_i is either an L-alternating ordered cycle or an L-alternating ordered lollipop;
- (ii) there is a vertex v of G with $L(v) = \{c_1, c_2\}$, such that the first vertex of H_i is v and the first color is c_i :
- (iii) the second vertex of H_1 and the second vertex of H_2 are distinct.

Consider a graph G with a 2-list assignment L. The pair $F = (H_1, H_2)$ is then called a *proper* pair if H_1 is an ordered cycle or ordered lollipop in G, H_2 is an ordered cycle or ordered lollipop in G and H_1 and H_2 have a common first vertex. Moreover, a proper pair $F = (H_1, H_2)$ is (L, 2)-bad (or just 2-bad) if H_1 , H_2 and L satisfy the conditions (i)-(iii) of Lemma 3.1. Note that if $F = (H_1, H_2)$ is a 2-bad proper pair, then F is not L-colorable.

By slight abuse of terminology we will sometimes refer to proper pairs $F = (H_1, H_2)$ in G as subgraphs of G. For such a proper pair $F = (H_1, H_2)$, $|V(H_1) \cup V(H_2)|$ is the number of vertices of F.

Remark 3.2. If $F = (H_1, H_2)$ is a 2-bad proper pair with a common first vertex v, then trivially there is a subgraph J of F such that J - v has an L-coloring φ . Moreover, if R is the rank function induced by L and φ as in Section 2, then (J, v, R) is a bad proper triple. So for a graph with a 2-list assignment L, Lemma 3.1 provides a stronger characterization than Lemma 2.1 of which 2-list assignments do not contain a proper coloring of the graph.

Suppose that H_1 is an ordered cycle or lollipop and H_2 is an ordered cycle or lollipop. Assume further that H_1 contains the vertices v, v_2, \ldots, v_{d_1} and that the vertices lie in this order along H_1 . Suppose that $v_i \in V(H_1) \cap V(H_2)$, $v_i \neq v$ and let u be the vertex that precedes v_i along H_2 . If $uv_i \notin E(H_1)$, then v_i is called a non-consecutive common vertex of H_1 and H_2 . Otherwise, if $uv_i \in H_1$ then v_i is called a consecutive common vertex.

The following two lemmas are essentially variants of Lemmas 13 and 14 in [5], respectively; since we have not found a way to deduce them directly from those lemmas, we provide brief sketches of the proofs (for details see [5]).

Lemma 3.3. Let G be a graph and let L be a random $(2, \mathcal{C})$ -list assignment for G. Suppose that $F = (H_1, H_2)$ is a proper pair in G on l vertices and r non-consecutive common vertices. Then

$$\mathbb{P}[F \text{ is 2-bad}] \le \frac{2^{l+2r}}{\sigma^{l-1}(\sigma-1)^{2+r}}.$$

Proof (sketch). Let $l_1 = |V(H_1)|$ and $l_2 = |V(H_2) \setminus V(H_1)|$. We will prove the lemma assuming that $l_2 > 0$. If $l_2 = 0$, then a similar argument applies. Suppose that H_1 is an ordered cycle or an ordered lollipop on the vertices $v, v_2, v_3, \ldots, v_{l_1}$ and that the vertices lie in that order along H_1 . Assume further that there are r_1 non-consecutive common vertices u of H_1 and H_2 such that the predecessor of u along H_2 is in $V(H_1)$. Denote the set of these vertices by R_1 .

There are $\sigma(\sigma-1)$ ways of choosing the list L(v) and selecting one of the colors of L(v) as the first color of H_1 . Then there are at most $(\sigma-1)^{l_1-r_1-2}2^{2r_1}$ ways of choosing the lists for H_1-v , so that H_1 is L-alternating, and so that the list of every vertex of R_1 has a color in common with the list of its predecessor along H_2 ; we choose the lists for the vertices $v_2, v_3, \ldots, v_{l_1}$ sequentially except that for any pair of vertices v_i and v_j (i < j) that are adjacent on H_2 and satisfying that one of v_i and v_j is a non-consecutive common vertex, we fix one color of $L(v_j)$ immediately after choosing the colors for $L(v_i)$.

So in total we have at most $\sigma(\sigma-1)(\sigma-1)^{l_1-r_1-2}2^{2r_1}$ choices for the lists of H_1 .

Since there are $r-r_1$ non-consecutive common vertices of H_1 and H_2 that are not in R_1 , there are thereafter at most $(\sigma-1)^{l_2-1-(r-r_1)}2^{r-r_1}$ ways of choosing the lists of the vertices in $V(H_2)\setminus V(H_1)$ so that H_2 is L-alternating as well. Since, in total, there are $\binom{\sigma}{2}^l$ ways of choosing the restriction of L to the vertices in F, the result follows.

Lemma 3.4. Let G be a graph on n vertices with maximum degree Δ . The number of proper pairs $F = (H_1, H_2)$ on l vertices in G with r non-consecutive vertices is at most $n\Delta^{l-1+r}2^l$.

Proof (sketch). Let $|V(H_1)| = l_1$ and $|V(H_2) \setminus V(H_1)| = l_2$. The first vertex v of H_1 and H_2 can be chosen in at most n ways. After that, there are at most Δ^{l_1-1} choices for the rest of H_1 . Thereafter, there are at most $2^{l_1}\Delta^{l_2+r}$ choices for the vertices of H_2-v , because there are at most l_1 vertices of H_1 which might be consecutive common vertices of H_1 and H_2 , and H_2 are the vertices of H_2 . \square

Proof of Theorem 1.5. Let G = G(n) be a graph on n vertices with maximum degree at most Δ and let L be a random $(2, \mathcal{C})$ -list assignment for G. By Lemma 3.1, it suffices to prove that if either $\Delta = \Omega(n^{1/2})$ and $\sigma = \omega(\Delta)$ or $\Delta = o(n^{1/2})$ and $\sigma = \omega(n^{1/2})$, then **whp** G does not contain a 2-bad proper pair. We will use easy first moment calculations.

Any 2-bad proper pair contains at least 3 vertices. For $l \in \{3, ..., n\}$ and $r \in \{1, ..., l\}$, let $X_{l,r}$ be a random variable counting the number of 2-bad proper pairs with l vertices and r non-consecutive common vertices in G, and let

$$X = \sum_{l=3}^{n} \sum_{r=0}^{l} X_{l,r}.$$

By Lemma 3.4, $f(l,r) \leq n\Delta^{l-1+r}2^l$, where f(l,r) is the number of proper pairs on l vertices and r non-consecutive common vertices in G. Let $p_{l,r}$ be the least number such that $\mathbb{P}[F_{l,r}]$ is 2-bad] $\leq p_{l,r}$, whenever $F_{l,r}$ is a proper pair on altogether l vertices and r non-consecutive common vertices in G. By Lemmas 3.3 and 3.4 we have

$$\begin{split} \mathbb{P}[G \text{ contains a 2-bad proper pair}] &\leq \mathbb{E}[X] \leq \sum_{l=3}^n \sum_{r=0}^l f(l,r) p_{l,r} \\ &\leq \sum_{l=3}^n \sum_{r=0}^l n \Delta^{l-1+r} 2^l \frac{2^{l+2r}}{\sigma^{l-1}(\sigma-1)^{2+r}} \\ &= O\left(\frac{n\Delta^2}{\sigma^4}\right) \sum_{l=0}^n \left(\frac{4\Delta}{\sigma}\right)^l \\ &= o(1), \end{split}$$

provided that
$$\Delta = o(n^{1/2})$$
 and $\sigma(n) = \omega(n^{1/4}\Delta^{1/2})$, or $\Delta = \Omega(n^{1/2})$ and $\sigma = \omega(\Delta)$.

Consider a random $(2, \mathcal{C})$ -list assignment for a graph G = G(n) on n vertices with maximum degree at most $\Delta = \Delta(n)$ and girth g. A 2-bad proper pair in G has at least g vertices, and by proceeding precisely as in the proof of Theorem 1.5 we can prove the following result which yields a better bound than the one of Proposition 2.6 if $k \leq g$.

Proposition 3.5. Let G = G(n) be a graph on n vertices with maximum degree at most $\Delta = \Delta(n)$ and girth at least g, where g is a fixed positive integer, and let L be a random $(2, \mathcal{C})$ -list assignment for G. If $\sigma(n) = \omega(n^{\frac{1}{g+1}}\Delta)$, then **whp** G is L-colorable.

In the next section we shall prove several other results for graphs with girth greater than 3.

4 Graphs with girth greater than three

For the graph H in the example in Section 2 showing that Theorem 1.4 is best possible, the threshold for the property of being colorable from a random (k, \mathcal{C}) -list assignment coincides with

the threshold for disappearence of (k+1)-cliques where each vertex has the same list; that is, when $\sigma(n) = \omega(n^{1/k^2}\Delta^{1/k})$, then **whp** H has no such cliques, and when $\sigma(n) = o(n^{1/k^2}\Delta^{1/k})$, then **whp** H contains a (k+1)-clique where the vertices have identical lists.

As in [5, 6], for a graph G = G(n) on n vertices with girth $g \ge 4$ (and thus with no (k+1)-cliques if $k \ge 2$) it is possible to establish a better bound on $\sigma(n)$ than those given by Theorem 1.4, Propositions 2.5 and 2.6, which implies list colorability of G from a random (k, \mathcal{C}) -list assignment. Indeed, the coarse threshold $n^{1/k^2}\Delta^{1/k}$ in Theorem 1.4 is essentially due to the fact that the probability that a (k+1)-clique has a list assignment where all lists are equal is $\binom{\sigma}{k}^{-k}$ and the maximum number of (k+1)-cliques in a graph with maximum degree Δ is roughly $n\Delta^k$, if Δ is sufficiently small compared to n.

If L is a k-list assignment for a graph G, then any L-critical subgraph of G which is not a (k+1)-clique has strictly more than k+1 vertices, and employing this fact we can use the very same methods as in Section 2 and 3, to prove better bounds on σ which ensures that **whp** G is colorable from a random (k, \mathcal{C}) -list assignment. Considering graphs with large girth is one way of increasing the minimum number of vertices in an L-critical graph - which is the most relevant parameter - and one can of course derive corresponding results for other families of graphs.

We emphasize that the results in this section are probably not best possible for any values of Δ or k. To prove sharp results, a first step would be to investigate how many vertices the smallest non-k-choosable graph with girth g has, and then estimate the number of k-list assignments that do not contain a proper coloring of such a graph.

We will proceed as in the proof of Theorem 1.4 and use first moment calculations to show that if L is a random (k, \mathcal{C}) -list assignment for a graph G = G(n) on n vertices with girth g, where k and g are fixed integers satisfying $k \geq 3$ and g > 3, respectively, $\Delta = \Delta(n)$ is sufficiently small, and $\sigma = \sigma(n)$ is large enough, then **whp** G has no connected induced L-critical subgraph and thus is **whp** L-colorable. (The case k = 2 was considered in the previous section.)

Now, any L-critical subgraph of G has minimum degree k+1 (which follows from the list-coloring version of Brooks' theorem since $k \geq 3$), and as pointed out in [6] (see also [3]), such a graph contains at least Q(k+1) vertices, where Q(k) = Q(k,g) satisfies

$$Q(k) = \begin{cases} 1 + k \left(1 + (k-1) + (k-1)^2 + \dots + (k-1)^{\frac{g-3}{2}} \right), & \text{if } g \text{ is odd,} \\ 2 \left(1 + (k-1) + (k-1)^2 + \dots + (k-1)^{\frac{g-2}{2}} \right), & \text{if } g \text{ is even,} \end{cases}$$
(7)

and thus

$$Q(k) = \begin{cases} 1 + \frac{k}{k-2} ((k-1)^{(g-1)/2} - 1) & \text{if } g \text{ is odd,} \\ \frac{2}{k-2} ((k-1)^{g/2} - 1) & \text{if } g \text{ is even.} \end{cases}$$

Suppose that $\sigma = \omega\left(n^{\frac{1}{(k-1)Q(k+1)+1}}\Delta^s\right)$, where s is a fixed integer satisfying $s \geq 1 + \frac{1}{k-1}$ and $\Delta = O\left(n^{\frac{1}{kQ^2(k+1)}}\right)$. By proceeding as in the proof of Theorem 1.4, it is not hard to show that if F is L-critical and thus by Lemma 2.1 belongs to an L-bad proper triple (F, v, R) of G, then **whp** F contains at most $\Delta^{(k-1)Q(k+1)+1}$ vertices. Hence, by Lemma 2.1, it suffices to prove that **whp** G contains no bad proper triple (F, v, R) such that

$$Q(k+1) \le |V(F)| \le \Delta^{(k-1)Q(k+1)+1}$$
.

Set $h(k) = \Delta^{(k-1)Q(k+1)+1}$, and let Z_m be a random variable counting the number of L-bad

proper triples (F, v, R) in G such that F has m vertices, and set

$$Z = \sum_{m=Q(k+1)}^{h(k)} Z_m.$$

Using Lemmas 2.2 and 2.3 we may now conclude that

 $\mathbb{P}[G \text{ contains an } L\text{-bad proper triple}] \leq \mathbb{E}[Z] + o(1)$

$$\leq \sum_{m=Q(k+1)}^{h(k)} n\Delta^{m-1}(m-1)! \frac{\sigma^{m-1}\binom{\Delta}{k}\binom{\Delta k}{k-1}^{m-1}}{\binom{\sigma}{k}^{m}} + o(1)$$

$$= O\left(\frac{n}{\sigma\Delta}\right) \sum_{m=Q(k+1)}^{h(k)} \left(\frac{mk^k \Delta^k}{\sigma^{k-1}}\right)^m$$

$$= O\left(\frac{n\Delta^{kQ(k+1)}}{\sigma^{(k-1)Q(k+1)+1}}\right) \sum_{m=0}^{\infty} \left(\frac{k^k \Delta^{(k-1)Q(k+1)+k+1}}{\sigma^{k-1}}\right)^m$$

$$= o(1),$$

provided that $\sigma(n) = \omega\left(n^{\frac{1}{(k-1)Q(k+1)+1}}\Delta^s\right)$, $s \ge 1 + 1/(k-1)$ and $\Delta = O\left(n^{\frac{1}{kQ^2(k+1)}}\right)$. Let us determine (k-1)Q(k+1) + 1 explicitly for some small values of g. When g = 4, then if

$$\sigma(n) = \omega(n^{\frac{1}{2k^2 - 1}} \Delta^s) \text{ and } \Delta = O\left(n^{\frac{1}{4(k^3 + 2k^2 + k)}}\right),$$

then **whp** there is an L-coloring of G, and when g = 5, then it suffices to require that

$$\sigma(n) = \omega(n^{1/(k^3 + k^2 - 1)} \Delta^s) \text{ and } \Delta = O\left(n^{\frac{1}{k^5 + 4k^4 + 8k^3 + 8k^2 + 4k}}\right).$$

We collect these results in the following proposition.

Proposition 4.1. Let G = G(n) be a graph on n vertices with maximum degree Δ and girth at least g, where g is a fixed positive integer. Suppose that k is a fixed integer satisfying $k \geq 3$ and that L is a random (k, \mathcal{C}) -list assignment for G. Moreover, let s be a fixed integer satisfying $s \geq 1 + \frac{1}{k-1}$

- (i) If g = 4, $\sigma(n) = \omega\left(n^{\frac{1}{2k^2-1}}\Delta^s\right)$, and $\Delta = O\left(n^{\frac{1}{4k^3+8k^2+4k}}\right)$, then whp G has an L-coloring.
- (ii) If g = 5, $\sigma(n) = \omega\left(n^{1/(k^3 + k^2 1)}\Delta^s\right)$, and $\Delta = O\left(n^{\frac{1}{k^5 + 4k^4 + 8k^3 + 8k^2 + 4k}}\right)$, then **whp** G has an L-coloring.
- (iii) If g > 5, then there are polynomials P(k) and R(k) in k of degree $\lceil g/2 \rceil$ and $2\lceil g/2 \rceil 1$, respectively, such that if $\sigma(n) = \omega\left(n^{1/P(k)}\Delta^s\right)$, and $\Delta = O(n^{1/R(k)})$, then **whp** G is L-colorable. Moreover,

$$P(k) = (k-1)Q(k+1) + 1$$
 and $R(k) = kQ^{2}(k+1)$,

where Q(k) is given by (7).

For graphs with no restriction on the maximum degree, we shall prove the following, which yields a slightly weaker bound on σ .

Proposition 4.2. Let G = G(n) be a graph on n vertices with maximum degree $\Delta = \Delta(n)$ and girth at least g, where g > 3 is a fixed positive integer. Suppose further that k is a fixed integer satisfying $k \geq 3$ and that L is a random (k, \mathcal{C}) -list assignment for G. If $\sigma = \omega\left(n^{\frac{1}{Q(k)-1}}\Delta\right)$, where Q(k) is given by (7), then whp G is L-colorable.

For the proof of the above result we need some new tools. The distance between two vertices in a graph G is the number of edges in a shortest path in G between them.

Given a graph G with girth g, where g is a positive integer, an odd rooted k-proper tree in G is a rooted tree T with root v such that v has k neighbors in T, and for each $i=1,\ldots,\lfloor (g-3)/2\rfloor$, each vertex at distance i from v has exactly k-1 neighbors at distance i+1 from v, and no vertex of distance $\lfloor (g-1)/2 \rfloor$ from v has a neighbor at greater distance from v. Note that the subgraph of G induced by all vertices at distance at most $\lfloor (g-3)/2 \rfloor$ from v in T is a tree, and that no two vertices in G at distance $\lfloor (g-3)/2 \rfloor$ from v has a common neighbor at distance $\lfloor (g-1)/2 \rfloor$ from v, because G has girth g.

For an even integer g, and a graph G with girth g we define the concept of an even k-proper tree as follows: let u and v be adjacent vertices and let T_u and T_v be two vertex-disjoint odd rooted k-proper trees in G with roots u and v, respectively, except for the fact that u and v has exactly k-1 neighbors in T_u and T_v , respectively. An even rooted k-proper tree with root v and semiroot v is the union of T_v along with the edge v.

It is straightforward to prove that for odd g, an odd k-proper rooted tree has exactly Q(k) vertices, and that for even g, an even k-proper rooted tree has exactly Q(k) vertices, where Q(k) is given by (7). (Indeed, this formula can be proved by considering a breadth-first search tree with root at some vertex in a graph with minimum degree k.)

Suppose now that L is a k-list assignment for G, and that T is an odd rooted k-proper tree T in G with root v. Then T is (tree, L)-bad (or just tree-bad) if there is an L-coloring φ of T - v such that

- (i) $L(v) = \{ \varphi(x) : x \in N_T(v) \}$
- (ii) for every $i = 1, ..., \lfloor (g 3)/2 \rfloor$, every vertex $x \in V(T)$ at distance i from v satisfies that $L(x) \setminus \{\varphi(x)\} = \{\varphi(y) : y \in N_T(x) \text{ and the distance between } y \text{ and } v \text{ is } i + 1\}.$

Suppose now that g is even and that T is an even rooted k-proper tree with root v and semiroot u. Then T is (tree, L)-bad (or just tree-bad) if there is an L-coloring φ of T - v such that

- (iii) $L(v) = \{ \varphi(x) : x \in N_T(v) \}$
- (iv) for every i = 1, ..., (g 4)/2, every vertex $x \in V(T)$ at distance i from v satisfies that $L(x) \setminus \{\varphi(x)\} = \{\varphi(y) : y \in N_T(x) \text{ and the distance between } y \text{ and } v \text{ is } i + 1\}$, and for every i = 1, ..., (g 4)/2, every vertex $z \in V(T)$ at distance i from u satisfies that $L(z) \setminus \{\varphi(z)\} = \{\varphi(w) : w \in N_T(z) \text{ and the distance between } w \text{ and } u \text{ is } i + 1\}$.

Lemma 4.3. Let G be a graph with girth g and L a k-list assignment for G, where g and k are positive integers. If G is not L-colorable and g is odd (even), then there is a tree-bad odd (even) rooted k-proper tree T in G.

Proof. We shall prove the lemma in the case when g is odd. The case when g is even can be done similarly. If G is not L-colorable, then it contains a vertex-minimal connected induced L-critical subgraph H. Let v be a vertex of H; then H - v has an L-coloring θ . For such an L-coloring ψ of H - v, let W^{ψ} be the set of vertices u in H for which there is an (ψ, L) -alternating path of length at most $\frac{g-1}{2}$ with origin at v and terminus u. For a vertex u in $H[W^{\theta}]$ we say that u is at level r if the distance between u and v is r. We shall prove by contradiction that $H[W^{\theta}]$ contains an tree-bad odd rooted k-proper tree T.

If ψ is an L-coloring of H - v, then the subgraph of $H[W^{\psi}]$ induced by all vertices at level at most (g-3)/2 in $H[W^{\psi}]$ is a tree and no two vertices at level (g-3)/2 has a common neighbor at level (g-1)/2, because G has girth g.

Now consider the subgraph $H[W^{\theta}]$. For each color $c \in L(v)$, there is clearly a neighbor of v that is colored c under θ (since otherwise H is L-colorable). Suppose that $H[W^{\theta}]$ does not contain a tree-bad odd rooted k-proper tree. We shall prove that this implies that H is L-colorable, establishing the required contradiction. Let \mathcal{T} be the set of all subgraphs T_{θ} of $H[W^{\theta}]$ such that

- (a) $v \in V(T_{\theta})$,
- (b) either 0 or exactly k neighbors of v are in T_{θ} , and in the latter case $L(v) = \{\theta(x) : x \in N_{T_{\theta}}(v)\},\$
- (c) for every vertex u of T_{θ} at level q, where $1 \leq q \leq \frac{g-3}{2}$, T_{θ} contains exactly 0 or k-1 neighbors of u at level q+1,
- (d) for every vertex $u \in V(T_{\theta})$ and every color $c \in L(u) \setminus \{\theta(u)\}$, if u is at level q, where $1 \leq q \leq \frac{g-3}{2}$, and u has k-1 neighbors in T_{θ} at level q+1, then there is a neighbor x of u at level q+1 with $\theta(x)=c$.

We choose an element T_{θ} from \mathcal{T} such that the shortest maximal path with origin at v in T_{θ} has maximum length. If \mathcal{T} contains several elements with shortest maximal paths of equal length, then we choose T_{θ} to be an element of \mathcal{T} with the minimum number of such paths. We say that such a subgraph of $H[W^{\theta}]$ is $path{-maximal}$.

If each maximal path in T_{θ} with origin at v has length at least $\frac{g-1}{2}$, then $H[W^{\theta}]$ clearly contains a tree-bad odd rooted k-proper tree; so suppose that there is some maximal path in T_{θ} with origin at v of length strictly less than $\frac{g-1}{2}$. Let $P = vw_1w_2 \dots w_j$ be such a shortest path in T_{θ} and suppose further that $\theta(w_i) = c_i, i = 1, \dots, j$.

Now, if $v = w_j$, then H is L-colorable; a contradiction and the desired result follows. So suppose that $v \neq w_j$. Since T_θ is path-maximal, there is a color $c_{j+1} \in L(w_j) \setminus \{\theta(w_j)\}$ such that no vertex in $H[W^\theta]$ at level j+1 is adjacent to w_j and colored c_{j+1} . We call such a color a *free color* of w_j . In fact, since $T_\theta \in \mathcal{T}$ and T_θ is path-maximal, there is some color $c \in L(w_{j-1}) \setminus \{\theta(w_{j-1})\}$ such that each neighbor of w_{j-1} at level j in $H[W^\theta]$ colored c has a free color.

Let H' be the subgraph of $H[W^{\theta}]$ consisting of all vertices u for which there is a (θ, L) -alternating path of length at most j with origin at v, terminus at u and whose second vertex is colored c_1 . Note that H' is a tree and denote by $\{a_1, \ldots, a_r\}$ the set of neighbors of v in H'. For $i = 1, \ldots, r$, denote by H'_i the subgraph of H' consisting of all vertices u for which there is a (θ, L) -alternating path of length at most j with origin at v, terminus at u and whose second vertex is a_i .

For i = 1, ..., r, let T_i be a path-maximal tree in H'_i satisfying (a) and (c)-(d) (with T_i in place of T_{θ}) and the additional condition that $a_i \in V(T_i)$. Since $T_{\theta} \in \mathcal{T}$ and T_{θ} is path-maximal, it follows

that each T_i contains a maximal path $P_i = vz_1^{(i)} \dots z_{j_i}^{(i)}$ of length at most j such that $a_i = z_1^{(i)}$, and $z_{j_i}^{(i)}$ has a free color; and, more generally, there is a color $c^{(i)} \in L(z_{j_i-1}^{(i)}) \setminus \{\theta(z_{j_i-1}^{(i)})\}$, such that any vertex at level j_i in H_i' that is adjacent to $z_{j_i-1}^{(i)}$ and colored $c^{(i)}$ has a free color. We call such a path in T_i good.

We shall now prove that there is an L-coloring φ of H - v such that v is not adjacent to any vertex colored c_1 , and thus there is an L-coloring of H.

For $i \in \{1, ..., r\}$, a colorful tree J with root a_i is a tree in $H'_i - v$ such that:

- $a_i \in V(J)$ and all vertices of J lie on (θ, L) -alternating paths with origin at v, and
- if a vertex $u \in V(J)$ at level q, $1 \le q \le j-1$ has a neighbor in J at level q+1 colored c under θ , then J contains all neighbors of u at level q+1 with color c under θ ; and, moreover, all neighbors of u in J at level q+1 is colored c under θ .

A colorful tree J with root a_i in H' is *correct* if for each maximal path P' in J with origin at a_i , the terminus of P' has a free color.

Claim 4.4. For i = 1, ..., r, the graph H'_i contains a correct colorful tree.

This claim can easily be proved by using the fact that each T_i contains a good path. Since suppose that there is no correct colorful tree in H'_i for some $i \in \{1, ..., r\}$. Then every colorful tree in H'_i is not correct, which contradicts that T_i satisfies (a), (c), (d), is path-maximal, and has a good path.

By the claim above, for i = 1, ..., r, H'_i contain a correct colorful tree J_i . By recoloring all leaves of each J_i with its free color, recoloring all other vertices of J_i with the (unique) color of its children, and retaining the color of every other vertex of H - v, we obtain an L-coloring φ of H - v such that v is not adjacent to any vertex colored c_1 , implying the required contradiction.

Remark 4.5. If T is a rooted k-proper tree with root v in a graph G, and L is a list assignment for G such that T is tree-bad, and we define R(x) to be distance between v and x for any vertex x of T, then (T, v, R) is an L-bad proper triple according to the definition in Section 2. So for graphs with girth g > 3, Lemma 4.3 provides a stronger characterization of which list assignments do not contain a proper coloring of the graph compared to Lemma 2.1.

Let us now prove Proposition 4.2.

Proof of Proposition 4.2. Let G = G(n) be a graph on n vertices with maximum degree $\Delta = \Delta(n)$ and girth at least g, where g is a fixed positive integer. Suppose further that k is a fixed integer satisfying $k \geq 3$ and that L is a random (k, \mathcal{C}) -list assignment for G. We need to prove that if $\sigma = \omega\left(n^{\frac{1}{Q(k)-1}}\Delta\right)$, where Q(k) is given by (7), then **whp** G is L-colorable. By Lemma 4.3, it suffices to prove that **whp** G does not contain a tree-bad odd (even) rooted k-proper tree if g is odd (even).

We will prove the proposition in the case when g is odd; the case when g is even is similar.

As pointed out above, a rooted k-proper tree has Q(k) vertices, where Q(k) is given by (7). Hence, the number of odd rooted k-proper trees in G is at most $n\Delta^{Q(k)-1}$. Given such a tree T with root v, the number of ways of choosing the list assignment L for T such that T is (L, tree)-bad is at most

$$\sigma^{Q(k)-1} \binom{\sigma-1}{k-1}^{k(k-1)^{\frac{g-3}{2}}},$$

because there are less than $\sigma^{Q(k)-1}$ ways of choosing a proper coloring φ of T-v, and there are thereafter at most $\binom{\sigma-1}{k-1}^{k(k-1)^{\frac{g-3}{2}}}$ ways of choosing the remaining colors of the lists for the vertices at distance $\frac{g-1}{2}$ from v. Note that by conditions (i) and (ii) above, the list of every vertex at distance at most $\frac{g-3}{2}$ from v in T is determined, as soon as we have chosen the coloring φ of T-v.

Denote by X a random variable counting the number of tree-bad odd rooted k-proper trees in G. By the preceding paragraphs we have

$$\mathbb{E}[X] \le \frac{n\Delta^{Q(k)-1}\sigma^{Q(k)-1}\binom{\sigma-1}{k-1}^{k(k-1)^{\frac{g-3}{2}}}}{\binom{\sigma}{k}^{Q(k)}} \\ \le A\frac{n\Delta^{Q(k)-1}k^{2Q(k)}}{\sigma^{Q(k)-1}} \\ = o(1),$$

where A is some constant independent of n, because k is a fixed positive integer and $\sigma = \omega \left(n^{\frac{1}{Q(k)-1}} \Delta \right)$.

Finally, we have the following result for graphs with large girth.

Proposition 4.6. Let G = G(n) be a graph on n vertices with maximum degree at most Δ (where Δ is either constant or an increasing function of n) and girth g = g(n) and let L be a random (k, \mathcal{C}) -list assignment for G. Then

- (i) if k = 2, $g \ge C \log n$, where C > 0 is a fixed constant, and $\sigma(n) \ge A\Delta \log n$, where A = A(C) is a fixed constant satisfying $A > 4e^{1/C}$, then whp G is L-colorable;
- (ii) for each constant $C_2 > 0$, there are constants $k_0 = \lceil \exp(2/C_2) + 1 \rceil$ and $B_0 = \exp(\frac{k_0 2}{2})k_0^2$ such that if $g \ge C_2 \log \log n$, $k \ge k_0$, $B > B_0$ and $\sigma(n) \ge B\Delta$, then **whp** G is L-colorable.

Proof (sketch). We first prove part (i). We shall prove that **whp** G has no L-coloring φ such that it contains a (φ, L) -alternating path of length at least g. Since an L-critical induced subgraph of G has minimum degree 2, by Lemma 2.1, this means that G has no L-critical induced subgraph, which implies the desired result.

As in the proof of Theorem 1.4 the expected number of paths P in G on at least g vertices, for which there is an L-coloring φ of P-v, where v is the origin of P, such that P is (φ, L) -alternating is at most

$$\sum_{r=g}^{n} \frac{n\Delta^{r-1}k^{2r}}{\sigma^{r-1}} = o(1), \tag{8}$$

provided that $k=2, g \geq C \log n$ and $\sigma(n) \geq A\Delta \log n$, where $A > 4e^{1/C}$.

Let us now prove part (ii). Evidently, it suffices to prove the theorem in the case when $k = k_0$. The proof is almost identical to the proof Proposition 4.2. Arguing as in that proof it suffices to prove that the expression

$$\frac{n\Delta^{Q(k)-1}\sigma^{Q(k)-1}\binom{\sigma-1}{k-1}^{k(k-1)^{\frac{g-3}{2}}}}{\binom{\sigma}{k}^{Q(k)}} \leq \frac{n\Delta^{Q(k)-1}k^{2Q(k)}}{\sigma^{Q(k)-1}} \tag{9}$$

tends to 0 as $n \to \infty$. Noting that for large g, $Q(k) \ge \frac{2}{k-2}((k-1)^{g/2}-1)$, it is easily verified that (9) tends to 0 as $n \to \infty$ provided that $k \ge \exp(2/C_2) + 1$, $\sigma(n) \ge B\Delta$ and $g \ge C_2 \log \log n$, where $B > B_0 = \exp(\frac{k_0-2}{2})k_0^2$.

5 Random lists of non-constant size

In this section we demonstrate how the methods from Section 2 and 3 can be used for proving some analogous results on list coloring when the size k of the lists in a random (k, \mathcal{C}) -list assignment is a (slowly) increasing function of n. We shall derive such analogues of several of the results in the preceding sections.

There are some previous results on coloring graphs from random lists of non-constant size in the literature: In [2] it is proved that there is a constant c > 0 such that if L is a random $(k, \{1, ..., n\})$ list, assignment for $\mathcal{L}(K_{n,n})$, where $\mathcal{L}(K_{n,n})$ is the line graph of the balanced complete bipartite graph on n + n vertices and k > (1 - c)n, then **whp** there is an L-coloring of $\mathcal{L}(K_{n,n})$. Note that in [2] this result is formulated in the language of arrays and Latin squares.

In [7] it is proved that for the complete graph K_n on n vertices the property of being colorable from a random from a random $(k, \{1, \ldots, n\})$ -list assignment has a sharp threshold at at $k = \log n$. Moreover, a similar result for the line graph of the complete bipartite graph $K_{m,n}$ with parts of size m and n, where $m = o(\sqrt{n})$ is also proved.

Let us now prove an analogue of Theorem 1.4. The proofs of all of the results in this section follow proofs in Section 2-4 quite closely, so in general, we omit proofs or just provide brief sketches. Througout this section we assume that G = G(n) is a graph on n vertices with maximum degree at most Δ , and L a random (k, \mathcal{C}) -list assignment for G, where k = k(n) and $\Delta = \Delta(n)$ are non-constant increasing integer-valued functions of n satisfying $k < \Delta$.

Theorem 5.1. Suppose that $k = O(\log^{1/4} n)$ and $\Delta = O(n^{1/3k^3})$. For any $\epsilon > 0$ the following holds:

- (i) If $k = o(\log \Delta)$ and $\sigma(n) \ge (1 + \epsilon)n^{1/k^2} \Delta^{1/k} k$, then **whp** G is L-colorable;
- (ii) if $k = C \log \Delta$ and $\sigma(n) \ge (1 + \epsilon)n^{1/k^2} \exp(1/C)k$, then whp G is L-colorable;
- (iii) if $k = \omega(\log \Delta)$ and $\sigma(n) \ge (1 + \epsilon)n^{1/k^2}k$, then **whp** G is L-colorable.

The proof of the above theorem is almost identical to the proof of Theorem 1.4. Since k is relatively small compared to σ and n, the same arguments and calculations as in the proof of Theorem 1.4 yield the required conclusions, given that k, Δ and σ satisfy the above conditions. The details are omitted.

Note further that the example after Thereom 1.4, shows that the bound on σ in the above theorem is best possible up to the multiplicative factor k, provided that $k = o(\Delta^{1/2})$.

Next, we have the following analogue of Proposition 2.5 for random lists of non-constant size.

Proposition 5.2. Let α and s be constants satisfying $1 < \alpha \le 3$ and $s \ge 2 + \gamma$, for some $\gamma > 0$. For any $\epsilon > 0$, if

- $k = O\left(\log^{1/\alpha \delta} n\right)$ for some $\delta > 0$,
- $\Delta = O\left(n^{1/k^{\alpha}}\right)$, and

•
$$\sigma(n) \ge (1+\epsilon)n^{1/k^{\frac{\alpha+1}{2}}} \Delta^s k$$
,

then G is L-colorable.

Again, the proof is virtually identical to the proof of Proposition 2.5 and is therefore omitted. For lists of greater size we have the following which is valid for all Δ :

Proposition 5.3.

- (i) If $k = o(\log n)$, then for any $\epsilon > 0$, if $\sigma(n) \ge (1 + \epsilon)n^{1/k}\Delta k$, then whp G is L-colorable.
- (ii) If $k \ge C \log n$, where C is some constant, then for any $\epsilon > 0$, if $\sigma \ge (1+\epsilon)C \exp(1/C)\Delta \log n$, then **whp** G is L-colorable.

Proof (sketch). The proofs of both part (i) and (ii) are similar to the proof of Proposition 2.6. Arguing as in that proof, it suffices to verify that the expression

$$n\binom{\Delta}{k} \frac{\binom{\sigma}{k} k! \binom{\sigma-1}{k-1}^k}{\binom{\sigma}{k}^{k+1}} = O\left(\frac{n\Delta^k k^k}{\sigma^k}\right)$$

tends to 0 as $n \to \infty$.

Finally, we remark that it is possible to derive corresponding results for lists of non-constant size of Propositions 4.1 and 4.2 by proceeding as in the proofs of these propositions. Proposition 4.1 is valid for non-constant k under the additional assumption that

$$k = O\left(\log^{\frac{1}{g}-\delta} n\right)$$
, for arbitrarily small $\delta > 0$,

and provided that σ satisfies

$$\sigma(n) \geq (1+\epsilon)n^{\frac{1}{(k-1)Q(k+1)+1}}\Delta^{1+\gamma}k$$
, for arbitrarily small $\gamma > 0$ and $\epsilon > 0$.

Similarly, Proposition 4.2 is valid for non-constant k provided that

- $k = o\left(\log^{\left\lceil \frac{2}{g-2} \right\rceil} n\right)$ and $\sigma(n) \ge (1+\epsilon)n^{\frac{1}{Q(k)-1}} \Delta k^2$, or
- $k \ge C \log^{\lceil \frac{2}{g-2} \rceil} n$, where C is some constant, and $\sigma(n) \ge A \Delta \log^{\lceil \frac{4}{g-2} \rceil} n$, where A = A(C) is a suitably chosen constant.

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