CANONICAL WEIERSTRASS REPRESENTATIONS FOR MINIMAL SURFACES IN EUCLIDEAN 4-SPACE

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ABSTRACT. Minimal surfaces of general type in Euclidean 4-space are characterized with the conditions that the ellipse of curvature at any point is centered at this point and has two different principal axes. Any minimal surface of general type locally admits geometrically determined parameters - canonical parameters. In such parameters the Gauss curvature and the normal curvature satisfy a system of two natural partial differential equations and determine the surface up to a motion. For any minimal surface parametrized by canonical parameters we obtain Weierstrass representations - canonical Weierstrass representations. These Weierstrass formulas allow us to solve explicitly the system of natural partial differential equations and to establish geometric correspondence between minimal surfaces of general type, the solutions to the system of natural equations and pairs of holomorphic functions in the Gauss plane. On the base of these correspondences we obtain that any minimal surface of general type in Euclidean 4-space determines locally a pair of two minimal surfaces in Euclidean 3-space and vice versa. Finally some applications of this phenomenon are given.

1. Introduction

Analytic methods to study surfaces and their properties are of essential importance in differential geometry. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces, which is one of the most powerful instruments for constructing new surfaces.

In this paper we study minimal surfaces in the four-dimensional Euclidean space \mathbb{R}^4 . For any surface \mathcal{M} in \mathbb{R}^4 we denote by K, \varkappa and H the Gauss curvature, the normal curvature and the mean curvature, respectively. These three invariants satisfy the following inequality [12]

$$K + |\varkappa| \le ||H||^2.$$

A surface \mathcal{M} is said to be minimal if H = 0, which means geometrically that the ellipse of curvature at any point is centered at this point. Therefore any minimal surface satisfies the inequality

$$K^2 - \varkappa^2 \ge 0,$$

which divides the minimal surfaces into two classes:

- the class of minimal super-conformal surfaces characterized by $K^2 \varkappa^2 = 0$;
- the class of minimal surfaces of general type characterized by $K^2 \varkappa^2 > 0$.

Geometrically, any superconformal surface is characterized by the condition that its ellipse of curvature is a circle. Minimal superconformal surfaces in \mathbb{R}^4 were described geometrically in [1] (see also [10]).

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Here we consider minimal surfaces in \mathbb{R}^4 of general type. At any point of such a surface the ellipse of curvature has two different principal axes.

Next we describe our scheme of investigation.

The leading idea is to study surfaces in \mathbb{R}^3 or in \mathbb{R}^4 with respect to special geometrically determined parameters - canonical parameters [6]. With respect to such parameters all coefficients of the first and the second fundamental form are expressed by the invariants of the surface.

Any minimal surface in \mathbb{R}^4 of general type admits special isothermal parameters - canonical parameters (of the first type or of the second type) (cf [9]). To endow locally the minimal surface under consideration with these parameters means that the tangent to any parametric line is transformed by the second fundamental tensor in a normal, which is collinear to a principal axis of the ellipse of curvature. Further we obtain Weierstrass representation formulas with respect to canonical parameters which describe locally all minimal surfaces in terms of two holomorphic functions. Introducing canonical parameters on a minimal surface, one obtains the system of natural PDE's of minimal surfaces and a Bonnet type fundamental theorem for minimal surfaces of general type [2]. The canonical Weierstrass formulas allow us to obtain explicitly the solutions to the system of natural PDE's of minimal surfaces [5].

We consider the set $\mathbf{MS_4}$ of equivalence classes of minimal surfaces containing a fixed point, the set $\mathbf{SNE_4}$ of equivalence classes of solutions to the system of natural PDE's and the set $\mathbf{H^2}$ of equivalent pairs of holomorphic functions. Our main result is that any two of these sets $\{\mathbf{MS_4}, \mathbf{SNE_4}, \mathbf{H^2}\}$ are in a natural one-to-one correspondence.

This result leads to a natural correspondence between the minimal surfaces in \mathbb{R}^4 and pairs of minimal surfaces in \mathbb{R}^3 , which is a base of a systematical study of minimal surfaces in \mathbb{R}^4 having in mind the well developed theory of minimal surfaces in \mathbb{R}^3 .

2. Preliminaries

Let \mathcal{M} be a two-dimensional Riemannian manifold and $\mathbf{x}: \mathcal{M} \to \mathbb{R}^n$ be an isometric immersion of \mathcal{M} into \mathbb{R}^n . Then we say that $(\mathcal{M}, \mathbf{x})$ (or \mathcal{M}) is a regular surface in \mathbb{R}^n . If $\mathbf{x}: (u, v) \to \mathbf{x}(u, v) \in \mathbb{R}^n$; $(u, v) \in \mathcal{D} \subset \mathbb{R}^2$ is a parametrization of \mathcal{M} , then the coefficients of the first fundamental form are $E = \mathbf{x}_u^2$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$ and $G = \mathbf{x}_v^2$. Without loss of generality we can assume that the parameters (u, v) are isothermal local coordinates, i.e. E = G and F = 0.

In addition to the real coordinates (u, v) we also consider the complex coordinate t = u + iv, identifying the coordinate plane \mathbb{R}^2 with the complex plane \mathbb{C} . Thus all functions defined on the surface can be considered as functions of the complex variable t.

We denote by $T_p(\mathcal{M})$ the tangent plane of \mathcal{M} at a point $p \in \mathcal{M}$, which is identified with the corresponding plane in \mathbb{R}^n . The normal space $N_p(\mathcal{M})$ at p is the normal complement of $T_p(\mathcal{M})$ in \mathbb{R}^n . Using the standard imbedding of \mathbb{R}^n into \mathbb{C}^n we consider the complexified tangent space $T_{p,C}(\mathcal{M})$ to \mathcal{M} at the point p as a subspace of \mathbb{C}^n , which is the linear span of $T_p(\mathcal{M})$ in \mathbb{C}^n . In a similar way the complexified normal space $N_{p,C}(\mathcal{M})$ to \mathcal{M} is identified with the corresponding subspace of \mathbb{C}^n , which is the linear span of $N_p(\mathcal{M})$ in \mathbb{C}^n .

If a and b are two vectors in \mathbb{C}^n , then $a \cdot b$ (or ab) denotes the bilinear dot product

$$a \cdot b = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

and the dot product of the vector a with itself is

$$a^2 = a \cdot a = a_1^2 + a_2^2 + \dots + a_n^2$$
.

The Hermitian dot product of a and b is given by the formula

$$\mathbf{a} \cdot \bar{\mathbf{b}} = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n$$

and the norm of the vector a with respect to the Hermitian dot product is

$$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \bar{\mathbf{a}} = |a_1|^2 + |a_2|^2 + \dots + |a_n|^2$$
.

Since $T_{p,C}(\mathcal{M})$ and $N_{p,C}(\mathcal{M})$ are generated by the real spaces $T_p(\mathcal{M})$ and $N_p(\mathcal{M})$ respectively, they are closed under the complex conjugation. They are mutually orthogonal with respect to the bilinear or Hermitian dot product. Therefore we have the following orthogonal decomposition:

$$\mathbb{C}^n = T_{p,C}(\mathcal{M}) \oplus N_{p,C}(\mathcal{M}) .$$

For a given vector a in \mathbb{C}^n a^{\top} and a^{\perp} denote the orthogonal projections of a into $T_{p,C}(\mathcal{M})$ and $N_{p,C}(\mathcal{M})$, respectively. For any vector we have:

$$a = a^{T} + a^{\perp}$$
.

This decomposition is valid with respect to both dot products in \mathbb{C}^n .

The second fundamental form σ of \mathcal{M} , is given by:

$$\sigma(X, Y) = (\nabla_X Y)^{\perp},$$

where $X, Y \in T(\mathcal{M})$, and ∇ is the canonical linear connection in \mathbb{R}^n .

Let X_1 and X_2 be the unit coordinate vector fields on \mathcal{M} of the same direction as \mathbf{x}_u and \mathbf{x}_v , respectively, i.e.

$$X_1 = \frac{X_u}{\|X_u\|} = \frac{X_u}{\sqrt{E}}; \quad X_2 = \frac{X_v}{\|X_v\|} = \frac{X_v}{\sqrt{G}} = \frac{X_v}{\sqrt{E}}.$$

As usual, H will denote the mean curvature vector field of \mathcal{M} :

$$H = \frac{1}{2} \operatorname{trace} \sigma = \frac{1}{2} (\sigma(X_1, X_1) + \sigma(X_2, X_2)).$$

Any regular surface with zero mean curvature vector field is said to be a minimal surface.

3. The function
$$\Phi(t)$$

Let $\mathcal{M}: \mathbf{x} = \mathbf{x}(u, v)$; $(u, v) \in \mathcal{D} \subset \mathbb{R}^2$ be a regular surface in \mathbb{R}^n . The complex vector-valued function $\Phi(t)$ is defined by the equality:

(1)
$$\Phi(t) = 2\frac{\partial \mathbf{x}}{\partial t} = \mathbf{x}_u - i\mathbf{x}_v.$$

The defining equality implies immediately that

$$\Phi^2 = 0 \ \Leftrightarrow \ \begin{array}{ll} \mathbf{x}_u^2 - \mathbf{x}_v^2 = 0 \\ \mathbf{x}_u \mathbf{x}_v = 0 \end{array} \ \Leftrightarrow \ \begin{array}{ll} E = \mathbf{x}_u^2 = \mathbf{x}_v^2 = G \\ F = 0 \end{array} .$$

Hence, the parameters (u, v) are isothermal if and only if

$$\Phi^2 = 0$$

The norm of Φ satisfies the following equalities:

$$\|\Phi\|^2 = \Phi\bar{\Phi} = \mathbf{x}_u^2 + \mathbf{x}_v^2 = E + G = 2E = 2G.$$

The coefficients of the first fundamental form of \mathcal{M} are given in terms of Φ as follows:

(3)
$$E = G = \frac{1}{2} ||\Phi||^2; \quad F = 0.$$

Denoting by I the first fundamental form, then we have:

(4)
$$\mathbf{I} = \frac{1}{2} \|\Phi\|^2 (du^2 + dv^2) = \frac{1}{2} \|\Phi\|^2 |dt|^2.$$

It follows that the function Φ satisfies the condition:

Differentiating (1) and taking into account the equality $\frac{\partial}{\partial t} \frac{\partial}{\partial t} = \frac{1}{4}\Delta$, we find:

(6)
$$\frac{\partial \Phi}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}} \left(2 \frac{\partial \mathbf{x}}{\partial t} \right) = \frac{1}{2} \Delta \mathbf{x} ,$$

where Δ is the Laplace operator.

The last formula implies that $\frac{\partial \Phi}{\partial \bar{t}}$ is a real vector-valued function, i.e.

(7)
$$\frac{\partial \Phi}{\partial \bar{t}} = \frac{\partial \bar{\Phi}}{\partial t} .$$

Thus, any function Φ given by (1) satisfies the conditions: (2), (5) and (7).

Conversely, any function Φ satisfying these three conditions determines locally the surface up to a translation.

The last assertion follows immediately from the fact that the condition

(8)
$$\frac{\partial \Phi}{\partial \bar{t}} = \frac{\partial \bar{\Phi}}{\partial t}$$

is the integrability condition for the system

(9)
$$x_u = \operatorname{Re}(\Phi)$$
$$x_v = -\operatorname{Im}(\Phi).$$

Further we express the components of the second fundamental form σ by the function Φ . Taking into account (6), we find:

$$\left(\frac{\partial \Phi}{\partial \bar{t}}\right)^{\perp} = \left(\frac{1}{2}\Delta \mathbf{x}\right)^{\perp} = \frac{1}{2}(\mathbf{x}_{uu}^{\perp} + \mathbf{x}_{vv}^{\perp}) = \frac{1}{2}(\nabla_{\mathbf{x}_u}^{\perp} \mathbf{x}_u + \nabla_{\mathbf{x}_v}^{\perp} \mathbf{x}_v) = \frac{1}{2}(\sigma(\mathbf{x}_u, \mathbf{x}_u) + \sigma(\mathbf{x}_v, \mathbf{x}_v)).$$

Differentiating (1) with respect to t we get:

(10)
$$\frac{\partial \Phi}{\partial t} = \frac{1}{2} (\mathbf{x}_{uu} - \mathbf{x}_{vv}) - i\mathbf{x}_{uv}.$$

Therefore

(11)
$$\left(\frac{\partial \Phi}{\partial t}\right)^{\perp} = \frac{1}{2} (\sigma(\mathbf{x}_u, \mathbf{x}_u) - \sigma(\mathbf{x}_v, \mathbf{x}_v)) - i\sigma(\mathbf{x}_u, \mathbf{x}_v).$$

Hence

(12)
$$\sigma(\mathbf{x}_{u}, \mathbf{x}_{u}) = \operatorname{Re}\left(\frac{\partial \Phi}{\partial \overline{t}}\right)^{\perp} + \operatorname{Re}\left(\frac{\partial \Phi}{\partial t}\right)^{\perp};$$

$$\sigma(\mathbf{x}_{v}, \mathbf{x}_{v}) = \operatorname{Re}\left(\frac{\partial \Phi}{\partial \overline{t}}\right)^{\perp} - \operatorname{Re}\left(\frac{\partial \Phi}{\partial t}\right)^{\perp};$$

$$\sigma(\mathbf{x}_{u}, \mathbf{x}_{v}) = -\operatorname{Im}\left(\frac{\partial \Phi}{\partial t}\right)^{\perp}.$$

Next we find how the function Φ is being transformed under a change of the isothermal coordinates and under a motion of the surface $(\mathcal{M}, \mathbf{x})$ in \mathbb{R}^n .

Let us consider a change of the isothermal coordinates which in complex form is given by: t = t(s). Since the isothermal coordinates are preserved, then the transformation t = t(s) is either holomorphic or antiholomorphic. Denote by $\tilde{\Phi}(s)$ the function with respect to the new coordinates s.

The holomorphic case. Using the definition (1) we have:

$$\tilde{\Phi}(s) = 2\frac{\partial \mathbf{x}}{\partial s} = 2\frac{\partial \mathbf{x}}{\partial t}\frac{\partial t}{\partial s} .$$

This means that under a holomorphic change of the coordinates t = t(s) we have:

(13)
$$\tilde{\Phi}(s) = \Phi(t(s)) \frac{\partial t}{\partial s} .$$

The antiholomorphic case: As in the above we find:

$$\tilde{\Phi}(s) = 2\frac{\partial \mathbf{x}}{\partial s} = 2\frac{\partial \mathbf{x}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial s}.$$

Therefore under an antiholomorphic change of the coordinates t = t(s) we have:

(14)
$$\tilde{\Phi}(s) = \bar{\Phi}(t(s)) \frac{\partial \bar{t}}{\partial s} .$$

In particular, under the change $t = \bar{s}$, the function Φ is transformed in the following way:

(15)
$$\tilde{\Phi}(s) = \bar{\Phi}(\bar{s}) .$$

Now, let us consider two surfaces $(\mathcal{M}, \mathbf{x})$ and $(\hat{\mathcal{M}}, \hat{\mathbf{x}})$ in \mathbb{R}^n , parameterized by isothermal coordinates t = u + iv defined in one and the same domain $\mathcal{D} \subset \mathbb{C}$. Suppose that $(\hat{\mathcal{M}}, \hat{\mathbf{x}})$ is obtained from $(\mathcal{M}, \mathbf{x})$ via a motion in \mathbb{R}^n by the formula:

(16)
$$\hat{\mathbf{x}}(t) = A\mathbf{x}(t) + b; \qquad A \in \mathbf{O}(n, \mathbb{R}), \ b \in \mathbb{R}^n.$$

Differentiating (16) we get the relation between Φ and $\hat{\Phi}$:

(17)
$$\hat{\Phi}(t) = A\Phi(t); \qquad A \in \mathbf{O}(n, \mathbb{R}) .$$

Conversely, if Φ and $\hat{\Phi}$ are related by (17), then we have $\hat{\mathbf{x}}_u = A\mathbf{x}_u$ and $\hat{\mathbf{x}}_v = A\mathbf{x}_v$ which imply (16) and (17) are equivalent.

4. Characterizing of minimal surfaces in \mathbb{R}^n by Φ

Let \mathcal{M} be a surface in \mathbb{R}^n , parameterized by isothermal coordinates and Φ is the function given by (1).

Differentiating (2), we find:

(18)
$$\Phi \cdot \frac{\partial \Phi}{\partial \bar{t}} = 0 .$$

Since $\frac{\partial \Phi}{\partial \bar{t}}$ is real, then it follows that

(19)
$$\bar{\Phi} \cdot \frac{\partial \Phi}{\partial \bar{t}} = 0 .$$

Equalities (18) and (19) imply that $\frac{\partial \Phi}{\partial \overline{t}}$ is orthogonal to $T(\mathcal{M})$ and $\frac{\partial \Phi}{\partial \overline{t}} \in N(\mathcal{M})$.

Taking into account the last property and (6) we calculate:

$$\frac{\partial \Phi}{\partial \overline{t}} = \left(\frac{\partial \Phi}{\partial \overline{t}}\right)^{\perp} = \frac{1}{2}(\Delta \mathbf{x})^{\perp} = \frac{1}{2}(\mathbf{x}_{uu} + \mathbf{x}_{vv})^{\perp} = \frac{1}{2}(\nabla_{\mathbf{x}_u}\mathbf{x}_u + \nabla_{\mathbf{x}_v}\mathbf{x}_v)^{\perp}$$

$$= \frac{1}{2}(\sigma(\mathbf{x}_u, \mathbf{x}_u) + \sigma(\mathbf{x}_v, \mathbf{x}_v)) = E \frac{1}{2}(\sigma(\mathbf{X}_1, \mathbf{X}_1) + \sigma(\mathbf{X}_2, \mathbf{X}_2)) = E\mathbf{H}.$$

Thus we have:

(20)
$$\frac{\partial \Phi}{\partial \bar{t}} = \frac{1}{2} \Delta \mathbf{x} = E \mathbf{H} .$$

These equalities imply the following statement.

Proposition 4.1. Let $(\mathcal{M}: (u,v) \to \mathbf{x}(u,v); (u,v) \in \mathcal{D})$ be a surface in \mathbb{R}^n parameterized by isothermal coordinates and $\Phi(t)$ be the complex function:

$$\Phi(t) = 2 \frac{\partial \mathbf{x}}{\partial t} = \mathbf{x}_u - i\mathbf{x}_v; \quad t = u + iv.$$

The following conditions are equivalent:

- (1) the function $\Phi(t)$ is holomorphic $\left(\frac{\partial \Phi}{\partial \overline{t}} = 0\right)$;
- (2) the function x(u, v) is harmonic $(\Delta x = 0)$;
- (3) $(\mathcal{M}, \mathbf{x})$ is a minimal surface in \mathbb{R}^n $(\mathbf{H} = 0)$.

Let (\mathcal{M}, x) be a minimal surface. We can introduce the harmonic conjugate function y to x determined by the conditions:

$$y_u = -x_v; \quad y_v = x_u.$$

Then the function

$$\Psi = x + iv$$
.

is holomorphic and

$$\mathbf{x} = \operatorname{Re} \Psi; \qquad \Phi = \mathbf{x}_u - \mathrm{i} \mathbf{x}_v = \mathbf{x}_u + \mathrm{i} \mathbf{y}_u = \frac{\partial \Psi}{\partial u} = \Psi'.$$

Since H = 0, we have:

(21)
$$\sigma(X_2, X_2) = -\sigma(X_1, X_1).$$

and

$$\sigma(\mathbf{x}_v, \mathbf{x}_v) = E\sigma(\mathbf{X}_2, \mathbf{X}_2) = -E\sigma(\mathbf{X}_1, \mathbf{X}_1) = -\sigma(\mathbf{x}_u, \mathbf{x}_u).$$

Then the formulas (10) and (11) for the derivative Φ' of Φ and its orthogonal projection on $N_{p,C}(\mathcal{M})$ become

(22)
$$\Phi' = \frac{\partial \Phi}{\partial u} = \mathbf{x}_{uu} - i\mathbf{x}_{uv}; \quad \Phi'^{\perp} = \mathbf{x}_{uu}^{\perp} - i\mathbf{x}_{uv}^{\perp} = \sigma(\mathbf{x}_u, \mathbf{x}_u) - i\sigma(\mathbf{x}_u, \mathbf{x}_v).$$

Taking into account (12), we express $\sigma(\mathbf{x}_u, \mathbf{x}_u)$, $\sigma(\mathbf{x}_v, \mathbf{x}_v)$ and $\sigma(\mathbf{x}_u, \mathbf{x}_v)$ by means of Φ :

$$\sigma(\mathbf{x}_{u}, \mathbf{x}_{u}) = \operatorname{Re}(\Phi'^{\perp}) = \frac{1}{2} (\Phi'^{\perp} + \overline{\Phi'^{\perp}}) = \frac{1}{2} (\Phi'^{\perp} + \overline{\Phi'^{\perp}})$$

$$\sigma(\mathbf{x}_{v}, \mathbf{x}_{v}) = -\operatorname{Re}(\Phi'^{\perp}) = -\frac{1}{2} (\Phi'^{\perp} + \overline{\Phi'^{\perp}}) = -\frac{1}{2} (\Phi'^{\perp} + \overline{\Phi'^{\perp}})$$

$$\sigma(\mathbf{x}_{u}, \mathbf{x}_{v}) = -\operatorname{Im}(\Phi'^{\perp}) = \frac{-1}{2i} (\Phi'^{\perp} - \overline{\Phi'^{\perp}}) = \frac{i}{2} (\Phi'^{\perp} - \overline{\Phi'^{\perp}}).$$

5. Formulas for the Gauss curvature and the normal curvature

Let $\mathcal{M}: (u,v) \to \mathbf{x}(u,v)$; $(u,v) \in \mathcal{D}$ be a minimal surface in \mathbb{R}^4 parameterized by isothermal coordinates. Suppose that \mathbf{n}_1 , \mathbf{n}_2 be an orthonormal pair of normal vector fields of \mathcal{M} , so that the quadruple $\{X_1, X_2, \mathbf{n}_1, \mathbf{n}_2\}$ is right oriented in \mathbb{R}^4 . For any normal vector \mathbf{n} we denote by $A_\mathbf{n}$ the Weingarten operator in $T(\mathcal{M})$. This operator is connected with the second fundamental form σ by means of the equality: $A_\mathbf{n}\mathbf{X}\cdot\mathbf{Y}=\sigma(\mathbf{X},\mathbf{Y})\cdot\mathbf{n}$. The condition $\mathbf{H}=0$ implies that trace $A_\mathbf{n}=0$ for any normal \mathbf{n} . Then the matrix representation of the operators $A_{\mathbf{n}_1}$ and $A_{\mathbf{n}_2}$ has the following form:

(24)
$$A_{n_1} = \begin{pmatrix} \nu & \lambda \\ \lambda & -\nu \end{pmatrix}; \qquad A_{n_2} = \begin{pmatrix} \rho & \mu \\ \mu & -\rho \end{pmatrix}$$

Therefore

(25)
$$\sigma(X_1, X_1) = (\sigma(X_1, X_1) \cdot n_1)n_1 + (\sigma(X_1, X_1) \cdot n_2)n_2 = \nu n_1 + \rho n_2,$$

$$\sigma(X_1, X_2) = (\sigma(X_1, X_2) \cdot n_1)n_1 + (\sigma(X_1, X_2) \cdot n_2)n_2 = \lambda n_1 + \mu n_2,$$

$$\sigma(X_2, X_2) = -\sigma(X_1, X_1) = -\nu n_1 - \rho n_2$$

Denoting by R the curvature tensor of the surface \mathcal{M} , the Gauss equation and (21) imply that the Gauss curvature K of \mathcal{M} is given by

(26)
$$K = R(X_1, X_2, X_1, X_2) = R(X_1, X_2)X_2 \cdot X_1 = -\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2).$$

On the other hand (25) and (26) imply the formula

(27)
$$K = -(\nu^2 + \rho^2) - (\lambda^2 + \mu^2) = -\nu^2 - \lambda^2 - \rho^2 - \mu^2 = \det(A_{n_1}) + \det(A_{n_2}).$$

In view of (22) we find the relation:

(28)
$$\Phi'^{\perp} = E(\sigma(X_1, X_1) - i\sigma(X_1, X_2)) .$$

Thus we have:

$$\|\Phi'^{\perp}\|^2 = \Phi'^{\perp} \cdot \overline{\Phi'^{\perp}} = E^2(\sigma^2(X_1, X_1) + \sigma^2(X_1, X_2)).$$

The last formula and (3) give that

(29)
$$\sigma^{2}(X_{1}, X_{1}) + \sigma^{2}(X_{1}, X_{2}) = \frac{\|\Phi'^{\perp}\|^{2}}{E^{2}} = \frac{4\|\Phi'^{\perp}\|^{2}}{\|\Phi\|^{4}}.$$

Now (26) and (29) imply that

(30)
$$K = \frac{-4\|\Phi'^{\perp}\|^2}{\|\Phi\|^4}.$$

We shall give to (30) another useful form. First we note that the vector functions Φ and $\bar{\Phi}$ are orthogonal with respect to the Hermitian dot product in \mathbb{C}^4 and form an orthogonal tangential basis. Therefore the tangential component of Φ' is given by

$$\Phi'^{\top} = \frac{\Phi'^{\top} \cdot \bar{\Phi}}{\|\Phi\|^2} \Phi + \frac{\Phi'^{\top} \cdot \Phi}{\|\bar{\Phi}\|^2} \bar{\Phi} = \frac{\Phi' \cdot \bar{\Phi}}{\|\Phi\|^2} \Phi + \frac{\Phi' \cdot \Phi}{\|\bar{\Phi}\|^2} \bar{\Phi}.$$

Differentiating $\Phi^2 = 0$, we find $\Phi \cdot \Phi' = 0$. Then we obtain for the projections of Φ' the following expression:

(31)
$$\Phi'^{\top} = \frac{\Phi' \cdot \bar{\Phi}}{\|\Phi\|^2} \Phi; \qquad \Phi'^{\perp} = \Phi' - \Phi'^{\top} = \Phi' - \frac{\Phi' \cdot \bar{\Phi}}{\|\Phi\|^2} \Phi.$$

Using a complex conjugation in (31) we get:

$$\|\Phi'^{\perp}\|^2 = \Phi'^{\perp} \cdot \overline{\Phi'^{\perp}} = \frac{\|\Phi\|^2 \|\Phi'\|^2 - |\bar{\Phi} \cdot \Phi'|^2}{\|\Phi\|^2}.$$

Since the bi-vector $\Phi \wedge \Phi'$ satisfies the equality

$$\|\Phi \wedge \Phi'\|^2 = \|\Phi\|^2 \|\Phi'\|^2 - |\bar{\Phi} \cdot \Phi'|^2$$

then we have:

$$\|\Phi'^{\perp}\|^2 = \frac{\|\Phi\|^2 \|\Phi'\|^2 - |\bar{\Phi} \cdot \Phi'|^2}{\|\Phi\|^2} = \frac{\|\Phi \wedge \Phi'\|^2}{\|\Phi\|^2} \ .$$

Replacing into (30) we obtain:

(32)
$$K = \frac{-4\|\Phi'^{\perp}\|^2}{\|\Phi\|^4} = \frac{-4\|\Phi \wedge \Phi'\|^2}{\|\Phi\|^6} .$$

Further we find a similar formula for the normal curvature \varkappa of \mathcal{M} .

Denoting by \mathbb{R}^N the curvature tensor of the normal connection on \mathcal{M} we have:

(33)
$$\varkappa = R^{N}(X_{1}, X_{2}, n_{1}, n_{2}) = A_{n_{1}}X_{1} \cdot A_{n_{2}}X_{2} - A_{n_{2}}X_{1} \cdot A_{n_{1}}X_{2}$$
$$= 2\nu\mu - 2\rho\lambda .$$

Let us denote by det(a, b, c, d) the determinant formed by the coordinates of the four vectors a, b, c and d, with respect to the standard basis in \mathbb{C}^4 . Using (25) we get

$$\det(\mathbf{x}_{u}, \mathbf{x}_{v}, \sigma(\mathbf{x}_{u}, \mathbf{x}_{u}), \sigma(\mathbf{x}_{u}, \mathbf{x}_{v})) = E^{3} \det(\mathbf{X}_{1}, \mathbf{X}_{2}, \sigma(\mathbf{X}_{1}, \mathbf{X}_{1}), \sigma(\mathbf{X}_{1}, \mathbf{X}_{2}))$$

$$= E^{3} \det(\mathbf{X}_{1}, \mathbf{X}_{2}, \nu \mathbf{n}_{1}, \mu \mathbf{n}_{2}) + E^{3} \det(\mathbf{X}_{1}, \mathbf{X}_{2}, \rho \mathbf{n}_{2}, \lambda \mathbf{n}_{1})$$

$$= E^{3}(\nu \mu - \rho \lambda) \det(\mathbf{X}_{1}, \mathbf{X}_{2}, \mathbf{n}_{1}, \mathbf{n}_{2}) = E^{3}(\nu \mu - \rho \lambda)$$

Hence

(34)
$$\nu\mu - \rho\lambda = \frac{1}{F^3} \det(\mathbf{x}_u, \mathbf{x}_v, \sigma(\mathbf{x}_u, \mathbf{x}_u), \sigma(\mathbf{x}_u, \mathbf{x}_v)).$$

In the last equality we replace x_u and x_v taking into account (9) and find:

(35)
$$\det(\mathbf{x}_{u}, \mathbf{x}_{v}, \sigma(\mathbf{x}_{u}, \mathbf{x}_{u}), \sigma(\mathbf{x}_{u}, \mathbf{x}_{v})) = \frac{i}{4} \det(\Phi + \bar{\Phi}, \Phi - \bar{\Phi}, \sigma(\mathbf{x}_{u}, \mathbf{x}_{u}), \sigma(\mathbf{x}_{u}, \mathbf{x}_{v}))$$
$$= -\frac{i}{2} \det(\Phi, \bar{\Phi}, \sigma(\mathbf{x}_{u}, \mathbf{x}_{u}), \sigma(\mathbf{x}_{u}, \mathbf{x}_{v})).$$

In view of (23), replacing $\sigma(\mathbf{x}_u, \mathbf{x}_u)$ and $\sigma(\mathbf{x}_u, \mathbf{x}_v)$ we have:

(36)
$$\det(\Phi, \bar{\Phi}, \sigma(\mathbf{x}_u, \mathbf{x}_u), \sigma(\mathbf{x}_u, \mathbf{x}_v)) = -\frac{\mathrm{i}}{2} \det(\Phi, \bar{\Phi}, \Phi'^{\perp}, \overline{\Phi'}^{\perp}).$$

Now (36) and (35) imply that:

(37)
$$\det(\mathbf{x}_u, \mathbf{x}_v, \sigma(\mathbf{x}_u, \mathbf{x}_u), \sigma(\mathbf{x}_u, \mathbf{x}_v)) = -\frac{1}{4} \det(\Phi, \bar{\Phi}, \Phi'^{\perp}, \overline{\Phi'}^{\perp}) = -\frac{1}{4} \det(\Phi, \bar{\Phi}, \Phi', \overline{\Phi'}).$$

Now (33), (34) and (37) give:

$$\varkappa = 2\nu\mu - 2\rho\lambda = \frac{2}{E^3}\det(\mathbf{x}_u, \mathbf{x}_v, \sigma(\mathbf{x}_u, \mathbf{x}_u), \sigma(\mathbf{x}_u, \mathbf{x}_v)) = -\frac{1}{2E^3}\det(\Phi, \bar{\Phi}, \Phi', \overline{\Phi'}).$$

Finally, in view of (3) we obtain the following formula for \varkappa :

(38)
$$\varkappa = -\frac{4}{\|\Phi\|^6} \det(\Phi, \bar{\Phi}, \Phi', \overline{\Phi'}).$$

Thus we obtained the following statement:

Theorem 5.1. The Gauss curvature K and the normal curvature \varkappa of any minimal surface $(\mathcal{M}, \mathbf{x})$ in \mathbb{R}^4 parameterized by isothermal coordinates, are given by the following formulas:

(39)
$$K = \frac{-4\|\Phi'^{\perp}\|^{2}}{\|\Phi\|^{4}} = \frac{-4\|\Phi \wedge \Phi'\|^{2}}{\|\Phi\|^{6}}; \qquad \varkappa = -\frac{4}{\|\Phi\|^{6}} \det(\Phi, \bar{\Phi}, \Phi', \overline{\Phi'}).$$

6. CANONICAL COORDINATES ON MINIMAL SURFACES IN \mathbb{R}^4 .

Let \mathcal{M} be a surface in \mathbb{R}^4 . A point $p \in \mathcal{M}$ is said to be super-conformal if the ellipse of curvature of \mathcal{M} at the point p is a circle.

Now let $(\mathcal{M}, \mathbf{x} = \operatorname{Re} \Psi)$ be a minimal surface in \mathbb{R}^4 parameterized by isothermal coordinates (u, v). A point $p \in \mathcal{M}$ is superconformal if

(40)
$$\sigma(X_1, X_1) \perp \sigma(X_1, X_2)$$
$$\sigma^2(X_1, X_1) = \sigma^2(X_1, X_2)$$

Next we express the condition (40) by means of the function Φ . Taking the square in (28), we find:

(41)
$$\Phi'^{\perp 2} = E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) - i 2E^2\sigma(X_1, X_1)\sigma(X_1, X_2).$$

Comparing (40) with (41) we get the equivalence

(42)
$$\sigma(X_1, X_1) \perp \sigma(X_1, X_2) \\ \sigma^2(X_1, X_1) = \sigma^2(X_1, X_2) \qquad \Leftrightarrow \quad \Phi'^{\perp 2} = 0$$

Squaring the second equality of (31), we find:

$$\Phi'^{\perp 2} = \Phi'^2 - 2\Phi' \frac{\Phi' \cdot \bar{\Phi}}{\|\Phi\|^2} \Phi + \left(\frac{\Phi' \cdot \bar{\Phi}}{\|\Phi\|^2}\right)^2 \Phi^2.$$

Taking into account $\Phi^2 = 0$ and $\Phi \cdot \Phi' = 0$, we obtain:

$$\Phi'^{\perp 2} = \Phi'^2 .$$

Thus we obtained the following proposition.

Proposition 6.1. A point $p \in \mathcal{M}$ is superconformal if and only if $\Phi'^2 = 0$.

Now the fact that Φ'^2 is holomorphic implies the following assertion.

Theorem 6.2. If \mathcal{M} is a connected minimal surface in \mathbb{R}^4 , then the set of the superconformal points of \mathcal{M} is either \mathcal{M} or mostly a countable set without limit points.

Further we only consider minimal surfaces in \mathbb{R}^4 without superconformal points and call them minimal surfaces of general type. Any minimal surface of general type admits special isothermal coordinates [9, 2, 7], such that the coordinate vectors $\sigma(X_1, X_1)$ and $\sigma(X_1, X_2)$ are directed along the principal axes of the ellipse of curvature at the corresponding point. This means that $\sigma(X_1, X_1) \perp \sigma(X_1, X_2)$. These coordinates become uniquely determined adding the normalizing condition $E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) = \pm 1$. The sign "+" in the last formula corresponds to the case when $\sigma^2(X_1, X_1)$ is directed along the major axis, while the sign "-" corresponds to the case when $\sigma^2(X_1, X_1)$ is directed along the minor axis of the ellipse. We call the so described special isothermal coordinates briefly canonical coordinates of the first type and canonical coordinates of the second type.

In view of (41) we conclude that the isothermal coordinates (u, v) are canonical of the first kind if and only if

(44)
$$\sigma(X_1, X_1) \perp \sigma(X_1, X_2) \\ E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) = 1 \qquad \Leftrightarrow \quad \Phi'^2 = \Phi'^{\perp 2} = 1$$

The isothermal coordinates (u, v) are canonical of the second type if and only if

(45)
$$\sigma(X_1, X_1) \perp \sigma(X_1, X_2) \\ E^2(\sigma^2(X_1, X_2) - \sigma^2(X_1, X_1)) = 1 \qquad \Leftrightarrow \quad \Phi'^2 = \Phi'^{\perp 2} = -1$$

Using the properties of the function Φ , we shall show that any minimal surface of general type in \mathbb{R}^4 carries locally canonical coordinates of both types.

Let (u, v) be isothermal coordinates on \mathcal{M} and denote t = u + vi. Consider the change $t = t(\tilde{t})$, where \tilde{t} is a new complex coordinate. Next we find the conditions under which the change $t(\tilde{t})$ gives canonical coordinates. Firstly, the new coordinates \tilde{t} have to be isothermal. Therefore the transformation $t = t(\tilde{t})$ is conformal in \mathbb{C} , which means that $t(\tilde{t})$ is either a holomorphic or an antiholomorphic function. It is enough to consider only the case of a holomorphic change $t = t(\tilde{t})$.

Let $\tilde{\Psi}$ be the holomorphic function representing \mathcal{M} with respect to the new coordinates, and $\tilde{\Phi}$ be its derivative. Then we have:

(46)
$$\tilde{\Phi} = \tilde{\Psi}'_{\tilde{t}} = \Psi'_{t}t' = \Phi t'$$

Further we find: $\tilde{\Phi}'_{\tilde{t}} = \Phi'_t t'^2 + \Phi t''$. Since Φ is tangent to \mathcal{M} , then $\Phi^{\perp} = 0$ and therefore:

(47)
$$\tilde{\Phi}_{\tilde{t}}^{\prime\perp} = (\Phi_{t}^{\prime}t^{\prime2} + \Phi t^{\prime\prime})^{\perp} = \Phi_{t}^{\prime\perp}t^{\prime2}; \\
\tilde{\Phi}_{\tilde{t}}^{\prime\perp} = \Phi_{t}^{\prime\perp^{2}}t^{\prime4}.$$

According to (44) and (45) the change \tilde{t} determines canonical coordinates if $\tilde{\Phi}_{\tilde{t}}^{\prime \perp^2} = \pm 1$. Equalities (47) imply that if $\Phi_t^{\prime \perp^2} = 0$, then $\tilde{\Phi}_{\tilde{t}}^{\prime \perp^2} = 0$, which is the condition \mathcal{M} to be superconformal. Hence, there do not exist canonical coordinates on a superconformal surface.

If \mathcal{M} is a minimal surface of general type, i.e. $\Phi'^{\perp 2} \neq 0$, then \tilde{t} determines canonical coordinates if $\Phi'^{\perp 2}_t t'^4 = \pm 1$. Thus the function $t(\tilde{t})$ satisfies the following first order ordinary differential equation:

$$\sqrt[4]{\pm \Phi_t^{\prime \perp^2}} \, dt = d\tilde{t}$$

According to (43) the left hand side of (48) is holomorphic and after integrating of (48) we obtain \tilde{t} as a holomorphic function of t.

The condition $\Phi_t'^{\perp^2} \neq 0$ means that $\tilde{t}' \neq 0$ and therefore the correspondence between \tilde{t} and t is one to one. Hence \tilde{t} determines new isothermal coordinates satisfying $\tilde{\Phi}_{\tilde{t}}'^{\perp^2} = \pm 1$, i.e. the new coordinates are canonical.

Thus we proved the following assertion.

Proposition 6.3. Any minimal surface \mathcal{M} in \mathbb{R}^4 of general type admits locally canonical coordinates of the first or of the second type.

Next we consider the question of uniqueness of the canonical coordinates.

Let us assume that t and \tilde{t} are canonical coordinates on \mathcal{M} of one and the same type. Then $t = t(\tilde{t})$ is either holomorphic or antiholomorphic function. First we consider the holomorphic case. Then the conditions (44), (45) and (47) imply the equalities:

$$\pm 1 = \tilde{\Phi}_{\tilde{t}}^{\prime \perp}{}^2 = \Phi_{t}^{\prime \perp}{}^2 t^{\prime 4} = \pm 1 t^{\prime 4} = \pm t^{\prime 4}$$

Therefore $t'^4 = 1$ and hence $t' = \pm 1$; $\pm i$. This implies that t and \tilde{t} satisfy one of the following relations: $t = \pm \tilde{t} + c$; $\pm i\tilde{t} + c$, where c = const.

The antiholomorphic case reduces to the previous case by the change $\tilde{t} = \bar{s}$. From the last equality it follows that $t = \pm \bar{t} + c$; $\pm i\bar{t} + c$. The last eight relations mean that the canonical coordinates of one and the same type are unique up to numbering and change of the direction of the coordinate lines.

Finally, let us consider the relation between the canonical coordinates of different type. Let t = u + vi be canonical coordinates of the first type an let us introduce new coordinates by means of $t = e^{\frac{\pi i}{4}}\tilde{t}$. We find from here that $t'^4 = -1$. Taking into account (47) we obtain that $\tilde{\Phi}'_{\tilde{t}}^{\perp 2} = -1$ and hence \tilde{t} determines canonical coordinates of the second type. Geometrically this means that the canonical coordinates of both types are related to each other by a rotation to an angle $\frac{\pi}{4}$ in the coordinate plane (u, v)

Let (\mathcal{M}, x) be a minimal surface of general type in \mathbb{R}^4 parameterized by canonical coordinates of the first type. Up to now the vectors n_1 and n_2 were only an orthonormal pair in $N(\mathcal{M})$. If the coordinates are canonical, then $\sigma(X_1, X_1) \perp \sigma(X_1, X_2)$. Therefore we can choose n_1 and n_2 to have the directions of $\sigma(X_1, X_1)$ and $\sigma(X_1, X_2)$, i.e. along the principal axes of the ellipse of curvature at the corresponding point. More precisely, let n_1 be the unit normal vector with the direction of $\sigma(X_1, X_1)$, and n_2 be the unit normal vector such that the quadruple (X_1, X_2, n_1, n_2) determine a positive oriented orthonormal basis in \mathbb{R}^4 . Then n_2 is collinear with $\sigma(X_1, X_2)$. Under these conditions formulas (25) become

(49)
$$\begin{aligned}
\sigma(X_1, X_1) &= \nu n_1 \\
\sigma(X_1, X_2) &= \mu n_2 ; \quad \nu > 0 . \\
\sigma(X_2, X_2) &= -\nu n_1
\end{aligned}$$

which means that $\lambda = 0$, $\rho = 0$ and formulas (24) become as follows:

(50)
$$A_{\mathbf{n}_1} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}, \qquad A_{\mathbf{n}_2} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}.$$

The functions ν and μ satisfy the following relations:

(51)
$$\begin{array}{rcl}
\nu &= \|\sigma(X_1, X_1)\| \\
|\mu| &= \|\sigma(X_1, X_2)\|
\end{array}; \quad \nu > |\mu|.$$

These functions do not depend on the canonical coordinates and are invariants of a minimal surface in \mathbb{R}^4 [7]. According to (49), these functions determine completely the second fundamental form of \mathcal{M} . The second condition in (44) implies that the first fundamental form is also completely determined by the formula:

(52)
$$E = G = \frac{1}{\sqrt{\nu^2 - \mu^2}} \,.$$

Next we obtain explicit formulas expressing the pair (ν, μ) by the pair (K, \varkappa) and vice versa. Under the condition $\lambda = 0$ and $\rho = 0$ formulas (27) have the following form [7]:

(53)
$$K = -\nu^2 - \mu^2 < 0 \; ; \qquad \varkappa = 2\nu\mu; \qquad -K > |\varkappa|.$$

Therefore

(54)
$$\nu = \frac{1}{2}(\sqrt{-K + \varkappa} + \sqrt{-K - \varkappa}), \quad \mu = \frac{1}{2}(\sqrt{-K + \varkappa} - \sqrt{-K - \varkappa}).$$

Further we give formulas for ν , μ and \varkappa , with respect to canonical coordinates of the first type.

Taking into account (44) and (3) we have:

(55)
$$\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2) = \frac{1}{E^2} = \frac{4}{\|\Phi\|^4}.$$

From here and (29) we get:

(56)
$$\nu^{2} + \mu^{2} = \frac{4\|\Phi'^{\perp}\|^{2}}{\|\Phi\|^{4}} \Leftrightarrow \nu^{2} = \frac{2(\|\Phi'^{\perp}\|^{2} + 1)}{\|\Phi\|^{4}} \\ \nu^{2} - \mu^{2} = \frac{4}{\|\Phi\|^{4}} \qquad \qquad \mu^{2} = \frac{2(\|\Phi'^{\perp}\|^{2} - 1)}{\|\Phi\|^{4}}.$$

In view of (56) we find

(57)
$$|\varkappa| = |2\nu\mu| = \frac{4\sqrt{\|\Phi'^{\perp}\|^4 - 1}}{\|\Phi\|^4}.$$

7. Weierstrass representations for minimal surfaces in \mathbb{R}^4 .

First we give some Weierstrass representations for minimal surfaces of general type parameterized by isothermal coordinates. Such kind of formulas have been written by a number of mathematicians: e.g. Eisenhart [3], Hoffman and Osserman [8].

Let $(\mathcal{M}, \mathbf{x})$: $\mathbf{x} = \operatorname{Re} \Psi$ be a minimal surfaces in \mathbb{R}^4 , parameterized by isothermal coordinates and let $\Phi = \Psi'$. If $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, then the condition for isothermal coordinates $\Phi^2 = 0$ has the form:

(58)
$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 0.$$

This equality can be "parameterized" in different ways by means of three holomorphic functions.

First we shall find a representation of Φ by means of trigonometric functions. Equality (58) is equivalent to one of the following equalities:

$$\phi_1^2 + \phi_2^2 = -\phi_3^2 - \phi_4^2; \quad \phi_1^2 + \phi_3^2 = -\phi_2^2 - \phi_4^2; \quad \phi_1^2 + \phi_4^2 = -\phi_2^2 - \phi_3^2.$$

At least one of the functions $\phi_1^2 + \phi_2^2$, $\phi_1^2 + \phi_3^2$ and $\phi_1^2 + \phi_4^2$ has to be different from zero. (The inverse leads by means of (58) to $\phi_1^2 = \phi_2^2 = \phi_3^2 = \phi_4^2 = 0$, which is impossible.) Without loss of generality we can assume that $\phi_1^2 + \phi_2^2 \neq 0$. Therefore, there exists a holomorphic function $f \neq 0$, such that:

(59)
$$f^2 = \phi_1^2 + \phi_2^2 = -\phi_3^2 - \phi_4^2$$

The last equality is equivalent to

(60)
$$\left(\frac{\phi_1}{f}\right)^2 + \left(\frac{\phi_2}{f}\right)^2 = \left(\frac{\phi_3}{if}\right)^2 + \left(\frac{\phi_4}{if}\right)^2 = 1.$$

It follows from here that there exist holomorphic functions h_1 and h_2 , such that

$$\frac{\phi_1}{f} = \cos h_1; \quad \frac{\phi_2}{f} = \sin h_1; \quad \frac{\phi_3}{if} = \cos h_2; \quad \frac{\phi_4}{if} = \sin h_2.$$

Thus we obtain the following representation of the vector function Φ :

(61)
$$\Phi:
\begin{aligned}
\phi_1 &= f \cos h_1, \\
\phi_2 &= f \sin h_1, \\
\phi_3 &= if \cos h_2, \\
\phi_4 &= if \sin h_2.
\end{aligned}$$

Hence, any minimal surface M in \mathbb{R}^4 , parameterized by isothermal parameters has a Weierstrass representation of the type (61).

Conversely, for any three holomorphic functions $(f \neq 0, h_1, h_2)$ determined in a region $D \subset \mathbb{C}$, formulas (61) generate a holomorphic function Φ with values in \mathbb{C}^4 . The condition $f \neq 0$ gives $\Phi \neq 0$. By direct calculations, formulas (61) imply (58), which is $\Phi^2 = 0$. Determining Ψ by the condition $\Psi' = \Phi$ and defining $M : x = \text{Re}(\Psi)$, we obtain a minimal surface M in \mathbb{R}^4 , parameterized by isothermal coordinates.

Hence, any triplet of holomorphic functions $(f \neq 0, h_1, h_2)$ generates a minimal surface in \mathbb{R}^4 via formulas (61).

Finally, we shall establish to what extent the triple $(f \neq 0, h_1, h_2)$ is determined by Φ . For that purpose, let us assume that one and the same function Φ is represented by (61) via two different triplets $(f \neq 0, h_1, h_2)$ and $(\hat{f} \neq 0, \hat{h}_1, \hat{h}_2)$. It is seen from (59) that, f is determined by Φ up to a sign. Therefore, two cases are possible. If $\hat{f} = f$, then \hat{h}_1 and \hat{h}_2 differ from h_1 and h_2 by constants even multiples to π . If $\hat{f} = -f$, then \hat{h}_1 and \hat{h}_2 differ from h_1 and h_2 by constants odd multiples to π . Thus we have:

$$\begin{array}{lll} \hat{f} &= f & \qquad & \hat{f} &= -f \\ \hat{h}_1 &= h_1 + 2k_1\pi & \text{or} & \hat{h}_1 &= h_1 + (2k_1 + 1)\pi \\ \hat{h}_2 &= h_2 + 2k_2\pi & & \hat{h}_2 &= h_2 + (2k_2 + 1)\pi \end{array} \quad \begin{array}{ll} k_1 = \text{const} \\ k_2 = \text{const} \end{array}$$

Using (61) we can obtain another forms of the Weierstrass representation for minimal surfaces applying different replacements.

In order to obtain the Weierstrass representation by means of hyperbolic functions, we make the following replacements in (61):

$$f \to if; \quad h_1 \to -ih_1; \quad h_2 \to \pi + ih_2.$$

Thus we obtain the following Weierstrass representation by means of hyperbolic functions:

(62)
$$\Phi:
 \begin{cases}
 \phi_1 &= if \cosh h_1, \\
 \phi_2 &= f \sinh h_1, \\
 \phi_3 &= f \cosh h_2, \\
 \phi_4 &= if \sinh h_2.
 \end{cases}$$

Let us introduce the functions $w_1 \bowtie w_2$ instead of $h_1 \bowtie h_2$ in (62) as follows:

Thus we obtain Weierstrass representation of the following type:

$$\phi_{1} = if \cosh \frac{w_{1} + w_{2}}{2},$$

$$\phi_{2} = f \sinh \frac{w_{1} + w_{2}}{2},$$

$$\phi_{3} = f \cosh \frac{w_{1} - w_{2}}{2},$$

$$\phi_{4} = if \sinh \frac{w_{1} - w_{2}}{2}.$$

Further, let us introduce the functions g_1 and g_2 by the following formulas:

$$(65) g_1 = e^{w_1}; g_2 = e^{w_2}.$$

With the aid of these functions, in view of (64), we obtain Weierstrass representation, which is a natural analogue of the classical Weierstrass representation for minimal surfaces in \mathbb{R}^3 . First, we calculate ϕ_1 :

$$\phi_1 = \frac{if}{2} \left(e^{\frac{w_1 + w_2}{2}} + e^{-\frac{w_1 + w_2}{2}} \right) = \frac{if}{2} e^{-\frac{w_1}{2}} e^{-\frac{w_2}{2}} \left(e^{w_1 + w_2} + 1 \right)$$
$$= \frac{if}{2\sqrt{g_1 g_2}} \left(e^{w_1} e^{w_2} + 1 \right) = \frac{if}{2\sqrt{g_1 g_2}} \left(g_1 g_2 + 1 \right).$$

Analogously to the above, we compute ϕ_2 :

$$\phi_2 = \frac{f}{2} \left(e^{\frac{w_1 + w_2}{2}} - e^{-\frac{w_1 + w_2}{2}} \right) = \frac{f}{2\sqrt{g_1 g_2}} (g_1 g_2 - 1).$$

In a similar way we find ϕ_3 :

$$\phi_3 = \frac{f}{2} \left(e^{\frac{w_1 - w_2}{2}} + e^{-\frac{w_1 - w_2}{2}} \right) = \frac{f}{2} e^{-\frac{w_1}{2}} e^{-\frac{w_2}{2}} \left(e^{w_1} + e^{w_2} \right)$$
$$= \frac{f}{2\sqrt{g_1 g_2}} \left(e^{w_1} + e^{w_2} \right) = \frac{f}{2\sqrt{g_1 g_2}} (g_1 + g_2).$$

Finally we calculate ϕ_4 :

$$\phi_4 = \frac{if}{2} \left(e^{\frac{w_1 - w_2}{2}} - e^{-\frac{w_1 - w_2}{2}} \right) = \frac{if}{2\sqrt{q_1 q_2}} (g_1 - g_2).$$

In the last four formulas, we make the change:

$$(66) f \to f2\sqrt{g_1g_2}$$

and obtain the following polynomial Weierstrass representation:

(67)
$$\Phi:
\begin{aligned}
\phi_1 &= if(g_1g_2 + 1), \\
\phi_2 &= f(g_1g_2 - 1), \\
\phi_3 &= f(g_1 + g_2), \\
\phi_4 &= if(g_1 - g_2).
\end{aligned}$$

Conversely, if $(f \neq 0, g_1, g_2)$ are three holomorphic functions, determined in a region in \mathbb{C} , then by virtue of (67) we obtain a holomorphic function Φ with values in \mathbb{C}^4 . It follows from $f \neq 0$ that $\Phi \neq 0$. It is easy to see by direct calculations that (67) implies (58), which is $\Phi^2 = 0$. Therefore, if we define Ψ by the equality $\Psi' = \Phi$, then the surface $M: x = \text{Re}(\Psi)$,

will be a minimal surface in \mathbb{R}^4 , parameterized by isothermal coordinates. Hence any triplet of holomorphic functions $(f \neq 0, g_1, g_2)$ generates a minimal surface in \mathbb{R}^4 via formulas (67).

Finally, we shall obtain that the triplet $(f \neq 0, g_1, g_2)$ is determined uniquely by Φ . For that purpose we express the functions f, g_1 Π g_2 explicitly by Φ . As an immediate consequence of (67), we find:

$$i\phi_1 + \phi_2 = -f(g_1g_2 + 1) + f(g_1g_2 - 1) = -2f,$$

 $\phi_3 + i\phi_4 = f(g_1 + g_2) - f(g_1 - g_2) = 2fg_2,$
 $\phi_3 - i\phi_4 = f(g_1 + g_2) + f(g_1 - g_2) = 2fg_1.$

The above equalities imply the following formulas for f, $g_1 \bowtie g_2$:

(68)
$$f = -\frac{1}{2}(i\phi_1 + \phi_2); \quad g_1 = -\frac{\phi_3 - i\phi_4}{i\phi_1 + \phi_2}; \quad g_2 = -\frac{\phi_3 + i\phi_4}{i\phi_1 + \phi_2}.$$

8. Canonical Weierstrass representations of minimal surfaces

A Weierstrass representation with respect to isothermal coordinates is said to be *canonical* of the first or the second type if the coordinates are in addition canonical of the first or the second type, respectively. In this section we shall only consider canonical Weierstrass representations of the first type.

8.1. **Preliminary calculations.** In order to obtain canonical Weierstrass representations of minimal surfaces in \mathbb{R}^4 we give first some relations between the functions f, h_1 and h_2 , that are used in the Weierstrass representation of minimal surfaces.

Here we prefer to use the representation (62) via hyperbolic functions. From now on we use the scalar holomorphic functions w_1 and w_2 , defined by (63) and the vector holomorphic function a defined in the following way:

(69)
$$a = \frac{\Phi}{f}$$

Taking into account (62) and (69) we get the following formulas for the functions a, \bar{a}, a' and \bar{a}' :

(70)
$$a = (i \cosh h_1, \sinh h_1, \cosh h_2, i \sinh h_2),
\bar{a} = (-i \cosh \bar{h}_1, \sinh \bar{h}_1, \cosh \bar{h}_2, -i \sinh \bar{h}_2),
\underline{a'} = (i h'_1 \sinh h_1, h'_1 \cosh h_1, h'_2 \sinh h_2, i h'_2 \cosh h_2),
\bar{a'} = (-i h'_1 \sinh \bar{h}_1, \bar{h}'_1 \cosh \bar{h}_1, \bar{h}'_2 \sinh \bar{h}_2, -i \bar{h}'_2 \cosh \bar{h}_2).$$

Now we can find the inner products between a, \bar{a} , a' and $\bar{a'}$.

The condition $\Phi^2 = 0$, implies that $a^2 = 0$. By means of differentiation and complex conjugation we get

(71)
$$a^2 = aa' = \bar{a}^2 = \bar{a}\bar{a}' = 0$$

Multiplying equations (70) we also find

(72)
$$||a||^2 = a\bar{a} = \cosh h_1 \cosh \bar{h}_1 + \sinh h_1 \sinh \bar{h}_1 + \cosh h_2 \cosh \bar{h}_2 + \sinh h_2 \sinh \bar{h}_2$$
$$= 2 \cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2);$$

(73)
$$a\bar{a'} = \bar{h'_1} \cosh h_1 \sinh \bar{h}_1 + \bar{h'_1} \sinh h_1 \cosh \bar{h}_1 + \bar{h'_2} \cosh h_2 \sinh \bar{h}_2 + \bar{h'_2} \sinh h_2 \cosh \bar{h}_2 = \bar{h'_1} \sinh(2\operatorname{Re}h_1) + \bar{h'_2} \sinh(2\operatorname{Re}h_2);$$

(74)
$$\bar{a}a' = \overline{a\bar{a'}} = h'_1 \sinh(2\operatorname{Re}h_1) + h'_2 \sinh(2\operatorname{Re}h_2);$$

(75)
$$a'^{2} = -h'^{2}_{1} \sinh^{2} h_{1} + h'^{2}_{1} \cosh^{2} h_{1} + h'^{2}_{2} \sinh^{2} h_{2} - h'^{2}_{2} \cosh^{2} h_{2} = h'^{2}_{1} - h'^{2}_{2} = w'_{1} w'_{2};$$

(76)
$$||a'||^{2} = a'\bar{a'} = |h'_{1}|^{2} \sinh h_{1} \sinh \bar{h}_{1} + |h'_{1}|^{2} \cosh h_{1} \cosh \bar{h}_{1} + |h'_{2}|^{2} \sinh h_{2} \sinh \bar{h}_{2} + |h'_{2}|^{2} \cosh h_{2} \cosh \bar{h}_{2} = |h'_{1}|^{2} \cosh(2\operatorname{Re}h_{1}) + |h'_{2}|^{2} \cosh(2\operatorname{Re}h_{2}).$$

Next we find formulas for a'^{\perp} , a'^{\perp^2} and $\|a'^{\perp}\|^2$ expressed by h_1 , h_2 and by w_1 u w_2 , respectively. We have $a'^{\perp} = a' - a'^{\top}$. The equality $a^2 = 0$ means that the vectors a and \bar{a} are mutually orthogonal with respect to the Hermitian inner product in \mathbb{C}^4 . Therefore a'^{\top} being tangent to M, can be represented with respect to the orthogonal basis (a, \bar{a}) in the following way:

$$a'^{\top} = \frac{a'^{\top} \cdot \bar{a}}{\|a\|^2} a + \frac{a'^{\top} \cdot a}{\|\bar{a}\|^2} \bar{a} = \frac{a' \cdot \bar{a}}{\|a\|^2} a + \frac{a' \cdot a}{\|\bar{a}\|^2} \bar{a}.$$

In view of (71) we get $a' \cdot a = 0$. Hence

(77)
$$a'^{\top} = \frac{a' \cdot \bar{a}}{\|a\|^2} a; \qquad a'^{\perp} = a' - a'^{\top} = a' - \frac{a' \cdot \bar{a}}{\|a\|^2} a.$$

Squaring the second equality of (77) we find

$$a'^{\perp 2} = a'^2 - 2a' \frac{a' \cdot \bar{a}}{\|a\|^2} a + \left(\frac{a' \cdot \bar{a}}{\|a\|^2}\right)^2 a^2.$$

According to (71) $a' \cdot a = 0$ and $a^2 = 0$. Therefore we have ${a'}^{\perp 2} = {a'}^2$. By means of (75) we obtain the required expression for the function ${a'}^{\perp 2}$:

(78)
$$a'^{\perp 2} = a'^2 = h_1'^2 - h_2'^2 = w_1' w_2'$$

By means of complex conjugation in (77) we find the following formula for $\|a'^{\perp}\|^2$:

$$||a'^{\perp}||^{2} = a'^{\perp} \cdot \overline{a'^{\perp}} = \left(a' - \frac{a' \cdot \overline{a}}{\|a\|^{2}} a\right) \left(\overline{a'} - \frac{\overline{a'} \cdot a}{\|a\|^{2}} \overline{a}\right)$$

$$= ||a'||^{2} - \frac{|\overline{a'} \cdot a|^{2}}{\|a\|^{2}} - \frac{|a' \cdot \overline{a}|^{2}}{\|a\|^{2}} + \frac{|a' \cdot \overline{a}|^{2}}{\|a\|^{4}} ||a||^{2} = ||a'||^{2} - \frac{|\overline{a'} \cdot a|^{2}}{\|a\|^{2}}$$

$$= \frac{||a||^{2} ||a'||^{2} - |\overline{a} \cdot a'|^{2}}{\|a\|^{2}}$$

$$= \frac{||a||^{2} ||a'||^{2} - |\overline{a} \cdot a'|^{2}}{\|a\|^{2}}$$

Let us denote the numerator in formula (79) by k_1 . Applying formulas (72), (74) and (76) after the corresponding simplification we find:

(80)
$$k_{1} = ||a||^{2}||a'||^{2} - |\bar{a} \cdot a'|^{2}$$

$$= (|h'_{1}|^{2} + |h'_{2}|^{2})(1 + \cosh(2\operatorname{Re}h_{1})\cosh(2\operatorname{Re}h_{2}))$$

$$-2\operatorname{Re}(h'_{1}\bar{h}'_{2})\sinh(2\operatorname{Re}h_{1})\sinh(2\operatorname{Re}h_{2})$$

Denoting the determinant of the vectors a, \bar{a} , a' and $\bar{a'}$ by $-k_2$, by direct calculations we find that

(81)
$$k_2 = -\det(a, \bar{a}, a', \bar{a'}) \\ = 2\operatorname{Re}(h'_1 \bar{h}'_2)(1 + \cosh(2\operatorname{Re}h_1)\cosh(2\operatorname{Re}h_2)) \\ -(|h'_1|^2 + |h'_2|^2)\sinh(2\operatorname{Re}h_1)\sinh(2\operatorname{Re}h_2)$$

Adding and subtracting equalities (80) and (81) we obtain:

(82)
$$k_1 + k_2 = 2|h'_1 + h'_2|^2 \cosh^2(\operatorname{Re} h_1 - \operatorname{Re} h_2).$$

(83)
$$k_1 - k_2 = 2|h'_1 - h'_2|^2 \cosh^2(\operatorname{Re} h_1 + \operatorname{Re} h_2).$$

Equalities (82) and (83) give the following expressions for k_1 and k_2 :

(84)
$$k_1 = |h'_1 + h'_2|^2 \cosh^2(\operatorname{Re} h_1 - \operatorname{Re} h_2) + |h'_1 - h'_2|^2 \cosh^2(\operatorname{Re} h_1 + \operatorname{Re} h_2) k_2 = |h'_1 + h'_2|^2 \cosh^2(\operatorname{Re} h_1 - \operatorname{Re} h_2) - |h'_1 - h'_2|^2 \cosh^2(\operatorname{Re} h_1 + \operatorname{Re} h_2)$$

Replacing h_1 and h_2 by w_1 and w_2 , respectively, we get:

(85)
$$k_1 = |w_1'|^2 \cosh^2(\operatorname{Re} w_2) + |w_2'|^2 \cosh^2(\operatorname{Re} w_1) k_2 = |w_1'|^2 \cosh^2(\operatorname{Re} w_2) - |w_2'|^2 \cosh^2(\operatorname{Re} w_1)$$

8.2. Canonical Weierstrass representations of minimal surfaces in \mathbb{R}^4 . Let the minimal surface of general type \mathcal{M} in \mathbb{R}^4 be parameterized by canonical coordinates of the first type and assume that \mathcal{M} is given by the representation (62) by means of hyperbolic functions. The condition (44) for the coordinates to be canonical implies a relation between the three functions f, h_1 and h_2 . In order to obtain this relation, we express the condition $\Phi'^{\perp 2} = 1$ via f, h_1 and h_2 . In view of (69) we have $\Phi = f$ a and therefore $\Phi' = f'$ a + fa'. Since the vector a is tangent to \mathcal{M} , then we have

(86)
$$\Phi'^{\perp} = (f'a + fa')^{\perp} = fa'^{\perp}; \qquad \Phi'^{\perp 2} = f^2a'^{\perp 2}.$$

Taking into account (78) we have ${a'}^{\perp 2} = {h'_1}^2 - {h'_2}^2$ and therefore ${\Phi'}^{\perp 2} = f^2({h'_1}^2 - {h'_2}^2)$. Thus we obtain that the minimal surface \mathcal{M} in \mathbb{R}^4 represented by (62) is parameterized by canonical coordinates of the first type if and only if

$$(87) f^2(h_1^{\prime 2} - h_2^{\prime 2}) = 1$$

The last formula implies that the surface \mathcal{M} parameterized by canonical coordinates of the first type has the following canonical Weierstass representation:

$$\phi_{1} = i \frac{\cosh h_{1}}{\sqrt{h'_{1}^{2} - h'_{2}^{2}}}$$

$$\phi_{2} = \frac{\sinh h_{1}}{\sqrt{h'_{1}^{2} - h'_{2}^{2}}}$$

$$\phi_{3} = \frac{\cosh h_{2}}{\sqrt{h'_{1}^{2} - h'_{2}^{2}}}$$

$$\phi_{4} = i \frac{\sinh h_{2}}{\sqrt{h'_{1}^{2} - h'_{2}^{2}}}$$

Conversely, if the pair (h_1, h_2) of holomorphic functions, determined in a domain in \mathbb{C} satisfy the condition ${h'_1}^2 \neq {h'_2}^2$, then formulas (88) give a minimal surface of general type in \mathbb{R}^4 parameterized by canonical coordinates of the first type.

If we use the functions w_1 and w_2 given by (63), then the condition (87) gets the form:

$$(89) f^2w_1'w_2' = 1$$

Substituting h_1 and h_2 in (88) by w_1 and w_2 , respectively, we obtain the following canonical Weierstrass representation of \mathcal{M} :

(90)
$$\phi_{1} = \frac{i}{\sqrt{w'_{1}w'_{2}}} \cosh \frac{w_{1} + w_{2}}{2}$$

$$\phi_{2} = \frac{1}{\sqrt{w'_{1}w'_{2}}} \sinh \frac{w_{1} + w_{2}}{2}$$

$$\phi_{3} = \frac{1}{\sqrt{w'_{1}w'_{2}}} \cosh \frac{w_{1} - w_{2}}{2}$$

$$\phi_{4} = \frac{i}{\sqrt{w'_{1}w'_{2}}} \sinh \frac{w_{1} - w_{2}}{2}$$

Conversely, if (w_1, w_2) is a pair of holomorphic functions determined in a domain in \mathbb{C} , satisfying the condition $w'_1w'_2 \neq 0$, then formulas (90) give a minimal surface of general type in \mathbb{R}^4 parameterized by canonical coordinates of the first type.

Finally we obtain a canonical Weierstrass representation of the type (67). For this aim we use the functions g_1 and g_2 given by (65). After a differentiation of (65) we get

(91)
$$g_1' = e^{w_1} w_1' = g_1 w_1'; \quad g_2' = e^{w_2} w_2' = g_2 w_2'.$$

From the above we have

(92)
$$w_1' = \frac{g_1'}{g_1}; \quad w_2' = \frac{g_2'}{g_2}.$$

Applying (66) in (89) and (92), we get $(f2\sqrt{g_1g_2})^2 \frac{g_1'}{g_1} \frac{g_2'}{g_2} = 1$.

The condition for canonical coordinates of the first type in the Weierstrass representation (67) gets the form:

$$4f^2g_1'g_2' = 1$$

We find f from (93), replace it into (67) and find the following canonical representation of a minimal surface of general type:

$$\phi_{1} = \frac{i}{2} \frac{g_{1}g_{2} + 1}{\sqrt{g'_{1}g'_{2}}}$$

$$\phi_{2} = \frac{1}{2} \frac{g_{1}g_{2} - 1}{\sqrt{g'_{1}g'_{2}}}$$

$$\phi_{3} = \frac{1}{2} \frac{g_{1} + g_{2}}{\sqrt{g'_{1}g'_{2}}}$$

$$\phi_{4} = \frac{i}{2} \frac{g_{1} - g_{2}}{\sqrt{g'_{1}g'_{2}}}$$

Conversely, if (g_1, g_2) is a pair of holomorphic functions defined in a domain in \mathbb{C} satisfying the condition $g'_1g'_2 \neq 0$, then formulas (94) give a minimal surface of general type in \mathbb{R}^4 parameterized by canonical parameters of the first type.

9. Formulas for K and \varkappa in a general Weierstrass representation

Let \mathcal{M} be a minimal surface in \mathbb{R}^4 , parameterized by isothermal coordinates. First we assume that \mathcal{M} is given by the representation (64). In order to obtain formula for E, we use equalities (3), (69) and (72) and find

(95)
$$E = |f|^2 \cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2).$$

Further we express $\cosh(\operatorname{Re} w_j)$ by means of $g_j, j = 1, 2$ in view of (65):

(96)
$$\cosh(\operatorname{Re} w_j) = \frac{e^{\operatorname{Re} w_j} + e^{-\operatorname{Re} w_j}}{2} = \frac{e^{2\operatorname{Re} w_j} + 1}{2e^{\operatorname{Re} w_j}} = \frac{|e^{w_j}|^2 + 1}{2|e^{w_j}|} = \frac{|g_j|^2 + 1}{2|q_j|}.$$

Now making the change (66) we get

(97)
$$E = |f|^2 (|g_1|^2 + 1)(|g_2|^2 + 1).$$

Let us consider the formula (39). Expressing Φ'^{\perp} by means of (86) we get:

$$K = \frac{-4\|\Phi'^{\perp}\|^{2}}{\|\Phi\|^{4}} = \frac{-4\|fa'^{\perp}\|^{2}}{\|fa\|^{4}} = \frac{-4|f|^{2}\|a'^{\perp}\|^{2}}{|f|^{4}\|a\|^{4}} = \frac{-4\|a'^{\perp}\|^{2}}{|f|^{2}\|a\|^{4}}.$$

Now using (79) and (80), we find:

$$K = \frac{-4(\|\mathbf{a}\|^2 \|\mathbf{a}'\|^2 - |\bar{\mathbf{a}} \cdot \mathbf{a}'|^2)}{|f|^2 \|\mathbf{a}\|^6} = \frac{-4k_1}{|f|^2 \|\mathbf{a}\|^6}.$$

In order to obtain a similar formula for \varkappa we use (39). We express Φ by means of f and a and taking into account (81), we find:

$$\varkappa = -\frac{4}{\|\Phi\|^6} \det(\Phi, \bar{\Phi}, \Phi', \bar{\Phi'}) = -\frac{4}{\|f\mathbf{a}\|^6} \det(f\mathbf{a}, \bar{f}\bar{\mathbf{a}}, f'\mathbf{a} + f\mathbf{a}', \bar{f}'\bar{\mathbf{a}} + \bar{f}\bar{\mathbf{a}'})
= -\frac{4}{\|f\mathbf{a}\|^6} \det(f\mathbf{a}, \bar{f}\bar{\mathbf{a}}, f\mathbf{a}', \bar{f}\bar{\mathbf{a}'}) = -\frac{4|f|^4}{\|f\mathbf{a}\|^6} \det(\mathbf{a}, \bar{\mathbf{a}}, \mathbf{a}', \bar{\mathbf{a}'}) = \frac{4k_2}{\|f\mathbf{a}\|^6}.$$

Thus we obtained the following formulas for K and \varkappa :

(98)
$$K = \frac{-4k_1}{|f|^2 ||\mathbf{a}||^6}; \quad \varkappa = \frac{4k_2}{|f|^2 ||\mathbf{a}||^6}.$$

Now using (72) and (85) we get:

$$K = \frac{-4(|w_1'|^2 \cosh^2(\operatorname{Re} w_2) + |w_2'|^2 \cosh^2(\operatorname{Re} w_1))}{|f|^2 8 \cosh^3(\operatorname{Re} w_1) \cosh^3(\operatorname{Re} w_2)}$$

$$\varkappa = \frac{4(|w_1'|^2 \cosh^2(\operatorname{Re} w_2) - |w_2'|^2 \cosh^2(\operatorname{Re} w_1))}{|f|^2 8 \cosh^3(\operatorname{Re} w_1) \cosh^3(\operatorname{Re} w_2)}$$

From here we find the following formulas for K and \varkappa with respect to the representation (64):

(99)
$$K = \frac{-1}{2|f|^2 \cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)} \left(\frac{|w_1'|^2}{\cosh^2(\operatorname{Re} w_1)} + \frac{|w_2'|^2}{\cosh^2(\operatorname{Re} w_2)} \right)$$

$$\varkappa = \frac{1}{2|f|^2 \cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)} \left(\frac{|w_1'|^2}{\cosh^2(\operatorname{Re} w_1)} - \frac{|w_2'|^2}{\cosh^2(\operatorname{Re} w_2)} \right)$$

In order to obtain analogous formulas by means of the functions g_j , j = 1; 2, first we note that (96) and (92) imply:

(100)
$$\frac{|w_j'|^2}{\cosh^2(\operatorname{Re} w_j)} = \frac{4|g_j'|^2}{(|g_j|^2 + 1)^2} \qquad j = 1; 2.$$

Now taking into account the change (66) and equality (100) we obtain from (99) the following formulas for K and \varkappa with respect to the representation (67):

(101)
$$K = \frac{-2}{|f|^2(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} + \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right) \\ \varkappa = \frac{2}{|f|^2(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} - \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right).$$

10. Formulas for the curvatures K, \varkappa , ν and μ in canonical Weierstrass representation

Let \mathcal{M} be a minimal surface of general type in \mathbb{R}^4 , prameterized by canonical coordinates of the first type. First we obtain the coefficient E of the first fundamental form in the canonical Weierstrass representation (90). In the general form (95) we express f under the condition (89) that the coordinates are canonical of the first type and find the following formula:

(102)
$$E = \frac{\cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)}{|w_1'w_2'|}.$$

In a similar way, we find a formula for E in the case when \mathcal{M} is given by the representation (94). In view of (93) we find from the general formula (97):

(103)
$$E = \frac{(|g_1|^2 + 1)(|g_2|^2 + 1)}{4|g_1'g_2'|}.$$

To obtain formulas for K and \varkappa , first let \mathcal{M} be given by means of the representation (90). We find f from the condition (89) and replace it into (99). Thus we obtain the following formulas for K and \varkappa in canonical coordinates, with respect to the representation (90):

(104)
$$K = \frac{-|w_1'w_2'|}{2\cosh(\operatorname{Re} w_1)\cosh(\operatorname{Re} w_2)} \left(\frac{|w_1'|^2}{\cosh^2(\operatorname{Re} w_1)} + \frac{|w_2'|^2}{\cosh^2(\operatorname{Re} w_2)}\right)$$
$$\varkappa = \frac{|w_1'w_2'|}{2\cosh(\operatorname{Re} w_1)\cosh(\operatorname{Re} w_2)} \left(\frac{|w_1'|^2}{\cosh^2(\operatorname{Re} w_1)} - \frac{|w_2'|^2}{\cosh^2(\operatorname{Re} w_2)}\right).$$

Now let \mathcal{M} be given by the representation (94). We find the function f from (93) and replace it into the general formulas (101). Thus we obtain the following formulas for K and \varkappa in canonical coordinates with respect to the representation (94):

(105)
$$K = \frac{-8|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} + \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right) \\ \varkappa = \frac{8|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} - \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right).$$

Next we find the corresponding formulas for the invariants ν and μ . Taking into account (104) we find

(106)
$$-K + \varkappa = \frac{|w'_1|^3 |w'_2|}{\cosh^3(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)} -K - \varkappa = \frac{|w'_1||w'_2|^3}{\cosh(\operatorname{Re} w_1) \cosh^3(\operatorname{Re} w_2)}.$$

Replacing (106) into (54), we obtain formulas for the curvatures ν and μ for a minimal surface of general type given by the representation (90):

(107)
$$\mu = \frac{1}{2} \sqrt{\frac{|w_1'w_2'|}{\cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)}} \left(\frac{|w_1'|}{\cosh(\operatorname{Re} w_1)} + \frac{|w_2'|}{\cosh(\operatorname{Re} w_2)} \right)$$

$$\mu = \frac{1}{2} \sqrt{\frac{|w_1'w_2'|}{\cosh(\operatorname{Re} w_1) \cosh(\operatorname{Re} w_2)}} \left(\frac{|w_1'|}{\cosh(\operatorname{Re} w_1)} - \frac{|w_2'|}{\cosh(\operatorname{Re} w_2)} \right).$$

Taking into account (105), we get:

$$(108) -K + \varkappa = \frac{16|g_1'|^3|g_2'|}{(|g_1|^2 + 1)^3(|g_2|^2 + 1)} -K - \varkappa = \frac{16|g_1'||g_2'|^3}{(|g_1|^2 + 1)(|g_2|^2 + 1)^3}.$$

Replacing (108) into (54) we obtain formulas for the curvatures ν and μ for a minimal surface of general type given by the representation (94):

(109)
$$\mu = 2\sqrt{\frac{|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)}} \left(\frac{|g_1'|}{|g_1|^2+1} + \frac{|g_2'|}{|g_2|^2+1}\right)$$

$$\mu = 2\sqrt{\frac{|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)}} \left(\frac{|g_1'|}{|g_1|^2+1} - \frac{|g_2'|}{|g_2|^2+1}\right).$$

With the help of (105) we can find transformation formulas for the pair of functions (g_1, g_2) under a motion of the minimal surface \mathcal{M} of general type in \mathbb{R}^4 .

Let $\hat{\mathcal{M}}$ be another minimal surface of general type in \mathbb{R}^4 , given by the representation (94) by means of the pair of functions (\hat{g}_1, \hat{g}_2) . Both surfaces \mathcal{M} and $\hat{\mathcal{M}}$ are related by a motion from $\mathbf{SO}(4,\mathbb{R})$ if and only if they have one and the same curvatures K and \varkappa , calculated with respect to canonical coordinates of te same type. We note that formulas (105) coincide with formulas (2) of [5]. Applying Theorem 1 and Theorem 2 in [5] to the curvatures K and \varkappa , we obtain that the surfaces \mathcal{M} and $\hat{\mathcal{M}}$ are related by a motion from $\mathbf{SO}(4,\mathbb{R})$, if and only if the functions g_j and \hat{g}_j , j=1,2 are related by linear fractional transformations from $\mathbf{SU}(2,\mathbb{C})$:

(110)
$$\hat{g}_j = \frac{-\bar{b}_j + \bar{a}_j g_j}{a_j + b_j g_j}, \quad a_j = \text{const}, \ b_j = \text{const}, \ |a_j|^2 + |b_j|^2 = 1; \ (j = 1; 2).$$

Replacing g_i with e^{w_i} , we obtain transformation formulas for the pair of functions (w_1, w_2) :

(111)
$$e^{\hat{w}_j} = \frac{-\bar{b}_j + \bar{a}_j e^{w_j}}{a_j + b_j e^{w_j}}, \quad a_j = \text{const}, \ b_j = \text{const}, \ |a_j|^2 + |b_j|^2 = 1; \ (j = 1; 2).$$

- 11. Geometric correspondence between minimal surfaces in \mathbb{R}^4 , pairs of solutions to the system of natural equations and pairs of holomorphic functions
- 11.1. Equivalent minimal surfaces in \mathbb{R}^4 . In this section we fix a coordinate system $O(e_1, e_2, e_3, e_4)$ in \mathbb{R}^4 , where $\{e_1, e_2, e_3, e_4\}$ is a positive oriented orthonormal quadruple. We suppose that any minimal surface $(\mathcal{M}, \mathbf{x})$ of general type

$$\mathcal{M}: (u,v) \to \mathbf{x}(u,v); (u,v) \in \mathcal{D}$$

is defined in a disc \mathcal{D} with center (0,0) in $\mathbb{R}^2 \equiv \mathbb{C}$ and passes through the point $O: \mathbf{x}(0,0) = (0,0,0,0)$. The parameters (u,v) are always supposed to be canonical.

Two minimal surfaces (\mathcal{M}, x) and (\mathcal{M}, \hat{x}) of the above type are said to be equivalent if there exists a disc \mathcal{D}_0 , such that

$$\hat{\mathbf{x}} = A\mathbf{x}, \quad A \in \mathbf{SO}(4, \mathbb{R}).$$

We denote by $\mathbf{MS_4}$ the set of equivalence classes of minimal surfaces of general type in \mathbb{R}^4 .

11.2. Equivalent solutions to the system of natural equations of minimal surfaces of general type in \mathbb{R}^4 . The system of natural equations of minimal surfaces of general type in \mathbb{R}^4 is the following:

$$\frac{(K^2 - \varkappa^2)^{\frac{1}{4}} \Delta \ln |\varkappa - K| = 2(2K - \varkappa)}{(K^2 - \varkappa^2)^{\frac{1}{4}} \Delta \ln |\varkappa + K| = 2(2K + \varkappa)}; \qquad K < 0$$

Two pairs of solutions (K, \varkappa) and $(\hat{K}, \hat{\varkappa})$ to the above system are said to be equivalent if there exists a disc \mathcal{D}_0 , centered at (0,0), such that $K = \hat{K}, \ \varkappa = \hat{\varkappa}$ in \mathcal{D}_0 .

We denote by $\mathbf{SNE_4}$ the set of equivalence classes of pairs of solutions to the system of natural equations.

11.3. Equivalent pairs of holomorphic functions in \mathbb{C} . Let $g_k : \mathcal{D} \to \mathbb{C}$ and $\hat{g_k} : \hat{\mathcal{D}} \to \mathbb{C}$, k = 1; 2 be two pairs of holomorphic functions such that $g'_k \neq 0$ in $\hat{g}'_k \neq 0$, k = 1, 2.

The two pairs $\{g_1, g_2\}$ and $\{\hat{g}_1, \hat{g}_2\}$ are said to be equivalent if there exists a disc \mathcal{D}_0 such that

$$\hat{g}_k = \frac{-\bar{b}_k + \bar{a}_k g_k}{a_k + b_k q_k}, \quad a_k, b_k = \text{const}, \ |a_k|^2 + |b_k|^2 = 1, \quad k = 1, 2; \ u + iv \in \mathcal{D}_0.$$

We denote by \mathbf{H}^2 the set of equivalence classes of pairs of holomorphic functions.

11.4. Correspondences between the equivalence classes. Let (\mathcal{M}, x) be a minimal surface in MS_4 with Gauss curvature K and normal curvature \varkappa .

Then the correspondence $(\mathcal{M}, \mathbf{x}) \to (K, \varkappa)$ generates a correspondence $\mathbf{MS_4} \to \mathbf{SNE_4}$.

This correspondence was obtained by de Azevero Tribuzy and Guadalupe [2].

Further, let $g_k: \mathcal{D} \to \mathbb{C}, \ k=1,2$ be two holomorphic functions such that $g_k' \neq 0, \ k=1,2$.

Denote by Φ the vector holomorphic function $\Phi: \mathcal{D} \to \mathbb{C}^4$ defined by the canonical Weierstrass representation

$$\Phi = \left(\frac{\mathrm{i}}{2} \frac{g_1 g_2 + 1}{\sqrt{g_1' g_2'}} , \frac{1}{2} \frac{g_1 g_2 - 1}{\sqrt{g_1' g_2'}} , \frac{1}{2} \frac{g_1 + g_2}{\sqrt{g_1' g_2'}} , \frac{\mathrm{i}}{2} \frac{g_1 - g_2}{\sqrt{g_1' g_2'}}\right).$$

Integrating the equality $\Psi' = \Phi$ we find the function $\Psi : \mathcal{D} \to \mathbb{C}^4$ satisfying the condition $\Psi(0,0) = (0,0,0,0)$. Then $\mathbf{x} = \operatorname{Re} \Psi$ gives a minimal surface (\mathcal{M},\mathbf{x}) is a minimal surface in \mathbb{R}^4 .

Hence the correspondence $(g_1, g_2) \to (\mathcal{M}, \mathbf{x})$ generates a correspondence $\mathbf{H^2} \to \mathbf{MS_4}$.

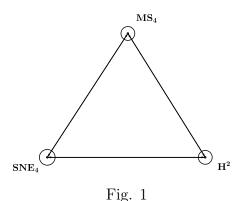
Now, let $g_k : \mathcal{D} \to \mathbb{C}$, k = 1, 2 be two holomorphic functions satisfying the condition $g'_k \neq 0$, k = 1, 2. Then we find the functions (K, \varkappa) in \mathcal{D} from

$$K = \frac{-8|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} + \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right),$$

$$\varkappa = \frac{8|g_1'g_2'|}{(|g_1|^2+1)(|g_2|^2+1)} \left(\frac{|g_1'|^2}{(|g_1|^2+1)^2} - \frac{|g_2'|^2}{(|g_2|^2+1)^2}\right).$$

Thus the correspondence $(g_1, g_2) \to (K, \varkappa)$ generates a correspondence $\mathbf{H^2} \to \mathbf{SNE_4}$. This correspondence was obtained by Ganchev and Kanchev in [5]. Summarizing, we have the following statement:

Theorem 11.1. The triangle diagram (Fig.1) is commutative.



Finally we shall give a correspondence between minimal surfaces in \mathbb{R}^4 and pairs of minimal surfaces in \mathbb{R}^3 .

First we recall the correspondence between minimal surfaces, solutions of the natural equation of minimal surfaces and holomorphic functions.

- 12. Geometric correspondence between minimal surfaces in \mathbb{R}^3 , solutions to the natural equation and holomorphic functions
- 12.1. Equivalent minimal surfaces in \mathbb{R}^3 . As in \mathbb{R}^4 we fix a coordinate system $O(e_1, e_2, e_3)$ in \mathbb{R}^3 , where $\{e_1, e_2, e_3\}$ is a positive oriented orthonormal triple. Let

$$\mathcal{M}: (u,v) \to \mathbf{x}(u,v); (u,v) \in \mathcal{D}$$

be a minimal surface in \mathbb{R}^3 free of umbilical points defined in a disc \mathcal{D} with center (0,0) in $\mathbb{R}^2 \equiv \mathbb{C}$. We consider minimal surfaces passing through the point O, so that $\mathbf{x}(0,0) = (0,0,0)$. Parameters (u,v) are supposed to be canonical, i.e. principal and isothermal [4].

If ν is the positive principal curvature, then the first and the second fundamental form are given as follows:

$$\mathbf{I} = \frac{1}{\nu} (du^2 + dv^2); \qquad \mathbf{II} = du^2 - dv^2.$$

Two minimal surfaces (\mathcal{M}, x) and $(\hat{\mathcal{M}}, \hat{x})$ of the above type are said to be equivalent if there exists a disc \mathcal{D}_0 (with center (0,0)), such that

$$\hat{\mathbf{x}} = A\mathbf{x}, \quad A \in \mathbf{SO}(3, \mathbb{R}).$$

We denote by MS_3 the set of equivalence classes of minimal surfaces in \mathbb{R}^3 .

12.2. Equivalent solutions to the natural equation of minimal surfaces in \mathbb{R}^3 . The natural equation of minimal surfaces in \mathbb{R}^3 is the following:

$$\Delta \ln \nu + 2\nu = 0.$$

Any solution to the natural equation determines a unique minimal surface in MS_3 .

Two solutions of the natural equation (112) are said to be equivalent if they coincide in a disc \mathcal{D}_0 in \mathbb{C} .

We denote by $\mathbf{SNE_3}$ the set of equivalence classes of solutions to the natural equation (112).

12.3. Equivalent holomorphic functions in \mathbb{C} . Let $g: \mathcal{D} \to \mathbb{C}$ and $\hat{g}: \mathcal{D} \to \mathbb{C}$, be two holomorphic functions such that $g' \neq 0$ and $\hat{g}' \neq 0$. Two holomorphic functions g and \hat{g} generate one and the same minimal surface in \mathbb{R}^3 if and only if [11], [5]:

(113)
$$\hat{g} = \frac{-\bar{b} + \bar{a} g}{a + b g}, \quad a, b = \text{const}, \ |a|^2 + |b|^2 = 1, \quad u + iv \in \mathcal{D}_0.$$

The two holomorphic functions g and \hat{g} are said to be equivalent if they satisfy (113). We denote by **H** the set of equivalence classes of holomorphic functions.

12.4. Correspondence between the equivalence classes. Let (\mathcal{M}, x) be a minimal surfaces in \mathbb{R}^3 , parameterized by canonical parameters. If ν is the normal curvature of \mathcal{M} , then the correspondence $\mathcal{M} \to \nu$ generates a correspondence $MS_3 \to SNE_3$.

Further, let $g: \mathcal{D} \to \mathbb{C}$ be a holomorphic function defined in the disc \mathcal{D} satisfying the condition $g' \neq 0$. Using the canonical Weierstrass representation [4]

condition
$$g' \neq 0$$
. Using the canonical Weierstrass representation [4]
$$\Phi = \left(\frac{1}{2} \frac{g^2 - 1}{g'}, -\frac{i}{2} \frac{g^2 + 1}{g'}, -\frac{g}{g'}\right)$$

we find the vector holomorphic function $\Psi : \mathcal{D} \to \mathbb{C}^3$ from the equality $\Psi' = \Phi$ and the condition $\Psi(0,0) = (0,0,0)$. Then (\mathcal{M}, x) , where $x = \text{Re } \Psi$, is a minimal surface in \mathbb{R}^3 .

The correspondence $g \to \mathcal{M}$ generates a correspondence $H \to MS_3$.

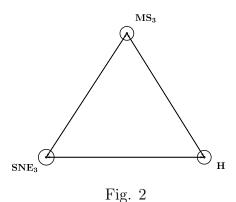
Now let $g: \mathcal{D} \to \mathbb{C}$ be a holomorphic function satisfying the condition $g' \neq 0$. This function generates a solution $\nu: \mathcal{D} \to \mathbb{R}$ to the natural equation by means of the formula [4]

(114)
$$\nu = \frac{4|g'|^2}{(|g|^2 + 1)^2}.$$

The correspondence $g \to \nu$ determines a correspondence $\mathbf{H} \to \mathbf{SNE_3}$.

Thus we obtained correspondences between MS_3 , SNE_3 and H:

The triangle diagram (Fig. 2) is commutative and the three sidelines of the triangle are bijections.



13. A GEOMETRIC CORRESPONDENCE BETWEEN THE CLASSES MS_4 AND $MS_3 \times MS_3$ Let us consider formulas (105) and (114). Putting

(115)
$$\nu_i = \frac{4|g_i'|^2}{(|g_i|^2 + 1)^2}, \quad i = 1, 2.$$

we can write the functions K and \varkappa in the form:

$$K = -\frac{1}{2}\sqrt{\nu_1 \nu_2} (\nu_1 + \nu_2), \quad \varkappa = \frac{1}{2}\sqrt{\nu_1 \nu_2} (\nu_1 - \nu_2).$$

Thus we obtain the statement:

Theorem 13.1.

$$SNE_4 \Leftrightarrow SNE_3 \times SNE_3$$
.

$$MS_4 \Leftrightarrow MS_3 \times MS_3$$
.

14. Some applications

Let us take the holomorphic functions: $g_1 = e^{-k_1 az}$ and $g_2 = e^{-k_2 az}$, where $k_1 \neq k_2$ are positive constants, $a = \cos \alpha + i \sin \alpha$, $\alpha = \text{const} \in [0, \pi/4]$ and z = u + iv. Replacing g_1 and g_2 into (94) we find a family of minimal surfaces $\mathcal{M}(k_1, k_2; \alpha)$:

$$z_{1} = \frac{1}{k'\sqrt{k_{1}k_{2}}} (\sin 2\alpha \sinh k'p \cos k'q - \cos 2\alpha \cosh k'p \sin k'q),$$

$$z_{2} = \frac{1}{k'\sqrt{k_{1}k_{2}}} (-\cos 2\alpha \cosh k'p \cos k'q - \sin 2\alpha \sinh k'p \sin k'q),$$

$$z_{3} = \frac{1}{k''\sqrt{k_{1}k_{2}}} (\cos 2\alpha \sinh k''p \cos k''q + \sin 2\alpha \cosh k''p \sin k''q),$$

$$z_{4} = \frac{1}{k''\sqrt{k_{1}k_{2}}} (-\sin 2\alpha \cosh k''p \cos k''q + \cos 2\alpha \sinh k''p \sin k''q),$$

where $p = u \cos \alpha - v \sin \alpha$; $q = u \sin \alpha + v \cos \alpha$ and $k' = \frac{k_1 + k_2}{2}$, $k'' = \frac{k_1 - k_2}{2}$. Let us fix k_1 and k_2 . Then we obtain a one-parameter family $\mathcal{M}(\alpha)$.

- $\mathcal{M}(0)$ gives the two-parameter family of catenoids in \mathbb{R}^4 .
- $\mathcal{M}(\pi/4)$ gives the two-parameter family of helicoids in \mathbb{R}^4 .

• All minimal surfaces $\mathcal{M}(\alpha)$ have the same $K(\alpha) = K(0)$ and $\varkappa(\alpha) = \varkappa(0)$. This implies that any $\mathcal{M}(\alpha)$ is isometric to $\mathcal{M}(0)$.

Remark 14.1. The family $\mathcal{M}(\alpha)$ is the family of the associated with $\mathcal{M}(0)$ minimal surfaces in \mathbb{R}^4 . In some questions in \mathbb{R}^4 the analogue of an isometry in \mathbb{R}^3 is the notion of a *strong* isometry, i.e. a deformation of a surface, preserving both K and \varkappa .

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