# VOLUME COMPARISON WITH RESPECT TO SCALAR CURVATURE

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ABSTRACT. In this article, we investigate the volume comparison with respect to scalar curvature. In particular, we show volume comparison hold for small geodesic balls of metrics near V-static metrics. As for global results, we give volume comparison for metrics near Einstein metrics with certain restrictions. As an application, we recover a volume comparison result of compact hyperbolic manifolds due to Besson-Courtois-Gallot, which provides a partial answer to a conjecture of Schoen on volume of hyperbolic manifolds.

### 1. Introduction

Volume comparison is a fundamental result in Riemannian geometry. It is a very powerful tool in geometric analysis and being used frequently in solving many problems.

Generically speaking, scalar curvature is not sufficient in controlling the volume. This is not possible even for compactly supported deformations of a generic domain which increases scalar curvature strictly inside due to a result of Corvino, Eichmair and Miao ([7]). In order to state their result, we need the following fundamental concept, which was introduced by Miao and Tam in [10]:

**Definition.** Let  $(M, \bar{g})$  be a Riemannian manifold. We say  $\bar{g}$  is a V-static metric if there is a smooth function f and a  $\kappa \in \mathbb{R}$  solves the following V-static equation:

(1.1) 
$$\gamma_{\bar{g}}^* f = \nabla_{\bar{g}}^2 f - \bar{g} \Delta_{\bar{g}} f - f Ric_{\bar{g}} = \kappa \bar{g},$$

where  $\gamma_{\bar{g}}^*: C^{\infty}(M) \to S_2(M)$  is the formal  $L^2$ -adjoint of  $\gamma_{\bar{g}} := DR_{\bar{g}}$ , the linearization of scalar curvature at  $\bar{g}$ . We will also refer a quadruple  $(M, \bar{g}, f, \kappa)$  as a V-static space.

Remark 1.1. An essential property of a V-static metric is that its scalar curvature  $R_{\bar{g}}$  is a constant (see Proposition 2.1 in [7]). Another one is that f also satisfies the linear equation

(1.2) 
$$\Delta_{\bar{g}}f + \frac{R_{\bar{g}}}{n-1}f + \frac{n\kappa}{n-1} = 0,$$

which can be derived easily by taking the trace of equation (1.1).

Typical examples of V-static metrics are space forms. In fact, the classification problem for V-static metrics is very interesting and important in understanding the interplay between scalar curvature and volume. For more results, please refer to [1, 2, 7, 10, 11].

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Now we state the deformation result with respect to scalar curvature and volume. Note that the original statement is much more stronger, but we adapt it here for our purpose.

**Theorem** (Corvino-Eichmair-Miao [7]). Let  $(M, \bar{g})$  be a Riemannian manifold and  $\Omega \subset M$ be a pre-compact domain with smooth boundary. Suppose  $(\Omega, \bar{g})$  is not V-static, i.e the V-static equation (1.1) only admits trivial solutions in  $C^{\infty}(\Omega) \times \mathbb{R}$ . Then there exists an  $\delta_0 > 0$  such that for any  $(\rho, V) \in C^{\infty}(M) \times \mathbb{R}$  with  $supp(R_{\bar{q}} - \rho) \subset \Omega$  and

$$||R_{\bar{q}} - \rho||_{C^1(\Omega,\bar{q})} + |Vol_{\Omega}(\bar{q}) - V| < \delta_0,$$

there exists a metric g on M such that  $supp(g - \bar{g}) \subset \Omega$ ,  $R_q = \rho$  and  $Vol_{\Omega}(g) = V$ .

This deformation result suggests that for a non-V-static domain, the information of scalar curvature is not sufficient in giving volume comparison not even after fixing the geometry outside. To be precise, we can take  $\rho > R_{\bar{q}}$  inside  $\Omega$  and either  $V > Vol_{\Omega}(g)$  or  $V < Vol_{\Omega}(g)$ . In either case, we can find a metric g realizing  $(\rho, V)$  on  $\Omega$  and this shows that no volume comparison holds in this case.

However, the volume comparison can be obtained for some special metrics. For instance, Miao and Tam proved a rigidity result for upper hemisphere with respect to non-decreasing scalar curvature and volume (see [11]). In the same article, they also showed a similar result holds for the Euclidean background metric.

Note that all space forms are V-static, it is natural to ask that whether all V-static metric support such a volume comparison result. Inspired by the work of Brendle and Marques on rigidity of geodesic balls in upper hemisphere (see [5]) and related work due to Miao and Tam (cf. [11]), we can show this is actually true. To be precise, we obtained the following result:

**Theorem A.** For  $n \geq 3$ , suppose  $(M^n, \bar{g}, f, \kappa)$  is a V-static space. For any  $p \in M$  with f(p) > 0, there exist constants  $r_0 > 0$  and  $\varepsilon_0 > 0$  such that for any geodesic ball  $B_r(p) \subset M$ with radius  $0 < r < r_0$  and metric g on  $B_r(p)$  satisfies

- $R_g \geq R_{\bar{g}}$  in  $B_r(p)$
- $H_g \ge H_{\bar{g}}$  on  $\partial B_r(p)$  g and  $\bar{g}$  induce the same metric on  $\partial B_r(p)$
- $||g \bar{g}||_{C^2(B_r(p),\bar{g})} < \varepsilon_0$ ,

the following volume comparison hold:

• if  $\kappa < 0$ , then

$$Vol_{\Omega}(q) \leq Vol_{\Omega}(\bar{q});$$

• if  $\kappa > 0$ , then

$$Vol_{\Omega}(g) \geq Vol_{\Omega}(\bar{g});$$

with equality holds in either case if and only if the metric q is isometric to  $\bar{q}$ .

Remark 1.2. By replacing  $(f, \kappa)$  with  $(-f, -\kappa)$ , we only need to consider the case f(p) > 0.

Remark 1.3. If  $\kappa = 0$ , V-static metrics reduce to vacuum static metrics. Under same assumptions on q, Qing and the author showed that q is isometric to  $\bar{q}$  (see [14]). This rigidity result suggests that the borderline case  $\kappa = 0$  is not necessary to be considered. On the other hand, we can view this volume comparison theorem as an extension of the rigidity result of vacuum static metrics in [14].

In general, the function f may change its sign in large scale. Thus we do not expect the volume comparison still holds for generically large domains. However, in some special situations, we can get global volume comparison. For example, we proved the following volume comparison result for closed non-Ricci flat Einstein manifolds. Here throughout this article, we refer a manifold to be closed, if it is compact without boundary.

**Theorem B.** Suppose  $(M, \bar{g})$  is a closed Einstein manifold satisfies

$$Ric_{\bar{q}} = (n-1)\lambda \bar{g}$$

with  $\lambda \neq 0$ . Moreover, if  $\lambda < 0$ , we assume its Weyl tensor satisfies

$$||W||_{L^{\infty}(M,\bar{q})} < \alpha(n,\lambda) := -(3n-4)\lambda.$$

Then there exists a constant  $\varepsilon_0 > 0$  such that for any metric g on M satisfies

$$R_q \ge n(n-1)\lambda$$

and

$$||g - \bar{g}||_{C^2(M,\bar{g})} < \varepsilon_0,$$

the following volume comparison hold:

• if  $\lambda > 0$ , then

$$Vol_M(q) \leq Vol_M(\bar{q});$$

• if  $\lambda < 0$ , then

$$Vol_M(g) \ge Vol_M(\bar{g});$$

with equality holds in either case if and only if the metric g is isometric to  $\bar{g}$ .

Remark 1.4. This volume comparison does not hold for Ricci flat metrics. This is easy to see by taking  $g := c\bar{g}$  for a constant c > 0. Clearly, the scalar curvature  $R_g = R_{\bar{g}} = 0$ , but the volume  $Vol_M(g)$  can be either larger or smaller than  $Vol_M(\bar{g})$  depending on c < 1 or c > 1.

As a special case of positive Einstein manifolds, we achieve the volume comparison for round spheres  $\mathbb{S}^n$ :

Corollary A. For  $n \geq 3$ , let  $(\mathbb{S}^n, g_{\mathbb{S}^n})$  be the unit round sphere. There exists a constant  $\varepsilon_0 > 0$  such that for any metric g on  $\mathbb{S}^n$  with

$$R_g \ge n(n-1)$$

and

$$||g - \bar{g}||_{C^2(\mathbb{S}^n, g_{\mathbb{S}^n})} < \varepsilon_0,$$

we have

$$Vol_{\mathbb{S}^n}(g) \leq Vol_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

with equality holds if and only if the metric g is isometric to  $\bar{g}$ .

Remark 1.5. The closeness of the metric g to  $\bar{g}$  is necessary. For example, Corvino, Eichmair and Miao constructed a metric on upper hemisphere satisfies the scalar comparison but has arbitrarily large volume (see Proposition 6.2 in [7]). In fact, by gluing a lower hemisphere, we can get a metric on the whole sphere with scalar curvature no less than n(n-1) but has larger volume.

Unlike the situation of round sphere, it is conjectured that volume comparison hold for closed hyperbolic manifolds regardless of the distance between g and the hyperbolic metric  $\bar{g}$ . This is in fact equivalent to say that the Yamabe invariant of a compact hyperbolic manifold is achieved by its hyperbolic metric, which is conjectured to be true by Schoen (cf. [15]), thus it is also referred as Schoen's conjecture:

**Schoen's Conjecture.** For  $n \geq 3$ , let  $(M^n, \bar{g})$  be a closed hyperbolic manifold. Then for any metric g on M with

$$R_g \geq R_{\bar{g}},$$

we have the volume comparison

$$Vol_M(g) \ge Vol_M(\bar{g}).$$

This conjecture can be shown to be true for three dimensional hyperbolic manifolds due to works of Perelman about geometrization of 3-manifolds ([12, 13]). For higher dimensions, by studying the minimal entropy of compact hyperbolic manifolds, Besson, Courtois and Gallot verified the conjecture for metrics  $C^2$ -closed to the hyperbolic one (cf. [3]). For global results in the sense of space of all metrics, they proved the volume comparison by replacing the assumption on scalar curvature with Ricci curvature (see [4]). However the original conjecture in higher dimensions is still open so far. As an application of our volume comparison result on negative Einstein manifolds, we can recover the local volume comparison result for compact hyperbolic manifolds in [3]:

**Corollary B.** For  $n \geq 3$ , let  $(M^n, \bar{g})$  be a closed hyperbolic manifold. There exists a constant  $\varepsilon_0 > 0$  such that for any metric g on M with

$$R_g \ge R_{\bar{g}}$$

and

$$||g - \bar{g}||_{C^2(M,\bar{g})} < \varepsilon_0,$$

we have

$$Vol_M(g) \ge Vol_M(\bar{g})$$

with equality holds if and only if the metric g is isometric to  $\bar{g}$ .

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# 2. Volume comparison for V-static spaces

In this section, we will investigate the volume comparison for geodesic balls in generic Vstatic spaces. The proof follows from an adapted idea inspired by [5], thus some calculations
are similar to [5, 9, 11, 14]. It would be interesting for readers to compare the original idea
and this adapted one.

First we recall the following well-known formulae for variations of scalar curvature (cf. [9]). For detailed calculations, please refer to [16].

Lemma 2.1. Let h be a symmetric 2-tensor. The first variation of scalar curvature is

(2.1) 
$$DR_{\bar{g}} \cdot h = -\Delta_{\bar{g}}(tr_{\bar{g}}h) + \delta_{\bar{q}}^2 h - Ric_{\bar{g}} \cdot h,$$

and the second variation is given by

(2.2) 
$$D^{2}R_{\bar{g}} \cdot (h,h) = -2\gamma_{\bar{g}}(h^{2}) - \Delta_{\bar{g}}(|h|_{\bar{g}}^{2}) - \frac{1}{2}|\nabla_{\bar{g}}h|_{\bar{g}}^{2} - \frac{1}{2}|d(tr_{\bar{g}}h)|_{\bar{g}}^{2} + 2\langle h, \nabla_{\bar{g}}^{2}(tr_{\bar{g}}h)\rangle_{\bar{g}} - 2\langle \delta_{\bar{g}}h, d(tr_{\bar{g}}h)\rangle_{\bar{g}} + \nabla_{\alpha}h_{\beta\gamma}\nabla^{\beta}h^{\alpha\gamma},$$

where  $(h^2)_{\alpha\beta} = \bar{g}^{\gamma\delta}h_{\alpha\gamma}h_{\delta\beta}$  and  $(\delta_{\bar{g}}h)_{\beta} = -\nabla^{\alpha}h_{\alpha\beta}$  is the formal  $L^2$ -adjoint of the covariant differentiation  $\nabla_{\bar{g}}$ . Here  $\alpha, \beta, \gamma, \delta = 1, 2, \cdots, n$ .

Now for simplicity, we will omit subscriptions  $\bar{g}$  and do all calculations under the metric  $\bar{g}$  unless we point it out in particular. We also make conventions that greek indices run through  $1, 2, \dots, n$  and latin indices run through  $1, 2, \dots, n-1$ .

We denote  $\Sigma := \partial \Omega$  to be the boundary of a pre-compact domain  $\Omega$ . Let  $\{e_1, \dots, e_{n-1}, e_n = \bar{\nu}\}$  be an orthonormal frame on  $\Sigma$  such that  $e_i$  is tangent to  $\Sigma$  and  $\bar{\nu}$  is the outward normal vector field of  $\Sigma$  with respect to the metric  $\bar{g}$ . We also denote the induced connection on  $\Sigma$  by  $\nabla^{\Sigma}$ . From now on, we assume that  $h|_{T\Sigma} = 0$ . That means,  $h_{ij} = 0$  on  $\Sigma$  for any  $i, j \in \{1, 2, \dots, n-1\}$ .

The following variations formulae for mean curvature play an important role in our argument:

**Lemma 2.2** (Brendle and Marques [5]). Suppose  $h|_{T\Sigma} = 0$ , then

(2.3) 
$$DH_{\bar{g}} \cdot h = \frac{1}{2} h_{nn} H_{\bar{g}} - \nabla_i h_n^{\ i} + \frac{1}{2} \nabla_n h_i^{\ i}$$

and

(2.4) 
$$D^{2}H_{\bar{g}}\cdot(h,h) = \left(-\frac{1}{4}h_{nn}^{2} + \sum_{i=1}^{n-1}h_{in}^{2}\right)H_{\bar{g}} + h_{nn}\left(\nabla_{i}h_{n}^{i} - \frac{1}{2}\nabla_{n}h_{i}^{i}\right).$$

The last one we need is variations formulae for the volume functional:

**Lemma 2.3.** For any  $h \in S_2(\Omega)$ , we have

(2.5) 
$$DVol_{\Omega,\bar{g}} \cdot h = \frac{1}{2} \int_{\Omega} (trh) dv_{\bar{g}}$$

and

(2.6) 
$$D^{2}Vol_{\Omega,\bar{g}} \cdot (h,h) = \frac{1}{4} \int_{\Omega} \left[ (trh)^{2} - 2|h|^{2} \right] dv_{\bar{g}}.$$

*Proof.* We recall the following fact from linear algebra first:

Let A be an  $n \times n$  symmetric matrix, then the characteristic polynomial of A is given by

$$p_{A}(\lambda) = \det(\lambda I - A)$$

$$= \sum_{k=0}^{n} (-1)^{k} \sigma_{k}(A) \lambda^{n-k}$$

$$= \lambda^{n} - (trA) \lambda^{n-1} + \frac{1}{2} ((trA)^{2} - trA^{2}) \lambda^{n-2} + \sum_{k=3}^{n} (-1)^{k} \sigma_{k}(A) \lambda^{n-k},$$

where  $\sigma_k(A)$  is the  $k^{th}$ -elementary polynomial associated to eigenvalues of the matrix A.

Choose a normal coordinates around any  $x \in \Omega$  with respect to  $\bar{g}$ , then the metric  $\bar{g}$  is the identity matrix at x. From the linear algebra fact above, we have the expansion

$$\det(\bar{g} + h) = 1 + (trh) + \frac{1}{2}((trh)^2 - |h|^2) + O(|h|^3)$$

and hence

$$\sqrt{\det(\bar{g}+h)} = 1 + \frac{1}{2}(trh) + \frac{1}{8}((trh)^2 - 2|h|^2) + O(|h|^3).$$

Therefore,

$$DVol_{\Omega,\bar{g}} \cdot h = \frac{1}{2} \int_{\Omega} (trh) dv_{\bar{g}}$$

and

$$D^{2}Vol_{\Omega,\bar{g}}\cdot(h,h) = \frac{1}{4}\int_{\Omega} \left( (trh)^{2} - 2|h|^{2} \right) dv_{\bar{g}}.$$

Let  $(\Omega, \bar{g}, f)$  be a V-static space. We consider the functional

(2.7) 
$$\mathscr{F}_{\Omega,\bar{g}}[g] = \int_{\Omega} R(g) f dv_{\bar{g}} + 2 \int_{\Sigma} H(g) f d\sigma_{\bar{g}} - 2\kappa Vol_{\Omega}(g).$$

This functional is designed particularly for V-static metrics, since these metrics can be characterized as its critical points.

**Proposition 2.4.** The metric  $\bar{g}$  is a critical point of the functional  $\mathscr{F}_{\Omega,\bar{g}}[g]$ .

*Proof.* Applying lemma 2.1 and integrating by parts, we have

$$\begin{split} &\int_{\Omega} \left(DR_{\bar{g}} \cdot h\right) f dv_{\bar{g}} = \int_{\Omega} (\gamma_{\bar{g}} h) f dv_{\bar{g}} \\ &= \int_{\Omega} \left(-\Delta (trh) + \delta^2 h - Ric \cdot h\right) f dv_{\bar{g}} \\ &= \int_{\Omega} \langle h, \gamma_{\bar{g}}^* f \rangle dv_{\bar{g}} + \int_{\Sigma} \left[-(\partial_{\bar{\nu}} (trh) + \langle \delta h, \bar{\nu} \rangle) f + (trh) \partial_{\bar{\nu}} f - h(\bar{\nu}, \nabla f)\right] d\sigma_{\bar{g}} \\ &= \int_{\Omega} \langle h, \gamma_{\bar{g}}^* f \rangle dv_{\bar{g}} + \int_{\Sigma} \left[-(\partial_{n} (trh) + (\delta h)_{n}) f - h_{n}^{\ i} \partial_{i} f\right] d\sigma_{\bar{g}}, \end{split}$$

where we used the fact  $trh = h_{nn}$  on  $\Sigma$  for the last step. Thus, together with Lemmas 2.2 and 2.3,

$$\begin{split} D\mathscr{F}_{\Omega,\bar{g}} \cdot h &= \int_{\Omega} (DR_{\bar{g}} \cdot h) f dv_{\bar{g}} + 2 \int_{\Sigma} (DH_{\bar{g}} \cdot h) f d\sigma_{\bar{g}} - 2\kappa \left( DVol_{\Omega,\bar{g}} \cdot h \right) \\ &= \int_{\Omega} \left[ \langle h, \gamma_{\bar{g}}^* f \rangle - \kappa (trh) \right] dv_{\bar{g}} \\ &+ \int_{\Sigma} \left[ -(\partial_n (trh) + (\delta h)_n + 2\nabla_i h_n^{\ i} - \nabla_n h_i^{\ i} - h_{nn} H_{\bar{g}}) f - h_n^{\ i} \partial_i f \right] d\sigma_{\bar{g}}. \end{split}$$

On the other hand,

$$\nabla_i h_n^{\ i} = \partial_i h_n^{\ i} + \Gamma_{i\alpha}^i h_n^{\ \alpha} - \Gamma_{in}^\alpha h_\alpha^{\ i} = \nabla_i^\Sigma h_n^{\ i} + H_{\bar{g}} h_{nn}$$

thus

$$(\delta h)_n = -\nabla_\alpha h_n^{\ \alpha} = -\nabla_i^\Sigma h_n^{\ i} - \nabla_n h_{nn} - H_{\bar{g}} h_{nn}.$$

Therefore, we have

$$D\mathscr{F}_{\Omega,\bar{g}}\cdot h = \int_{\Omega} \langle h, \gamma_{\bar{g}}^* f - \kappa \bar{g} \rangle dv_{\bar{g}} - \int_{\Sigma} \left[ \nabla_i^{\Sigma} h_n^{\ i} f + h_n^{\ i} \partial_i f \right] d\sigma_{\bar{g}} = - \int_{\Sigma} \nabla_i^{\Sigma} (h_n^{\ i} f) d\sigma_{\bar{g}} = 0.$$

i.e.  $\bar{g}$  is a critical point of  $\mathscr{F}_{\Omega,\bar{g}}[g]$ .

When dealing with a geometric problem, it is important to consider actions of diffeomorphism group. In order to fix this gauge, we need the following slice lemma proved in [5], which is a generalization of the well-known *Ebin's Slice Lemma* (see [8]):

**Lemma 2.5.** ([5, Proposition 11]) Suppose that  $\Omega$  is a domain in a Riemannian manifold  $(M^n, \bar{g})$ . Fix a real number p > n, there exists an  $\varepsilon > 0$ , such that for a metric g on  $\Omega$  with

$$||g - \bar{g}||_{W^{2,p}(\Omega,\bar{g})} < \varepsilon,$$

there exists a diffeomorphism  $\varphi: \Omega \to \Omega$  such that  $\varphi|_{\partial\Omega} = id$  and  $h = \varphi^*g - \bar{g}$  is divergence-free in  $\Omega$  with respect to  $\bar{g}$ . Moreover,

$$||h||_{W^{2,p}(\Omega,\bar{g})} \le N||g-\bar{g}||_{W^{2,p}(\Omega,\bar{g})},$$

for some constant N > 0 that only depends on  $\Omega$ .

From now on, we assume g is  $C^2$ -closed to  $\bar{g}$ . Thus by Lemma 2.5, there is a diffeomorphism  $\varphi: \Omega \to \Omega$  such that  $\varphi|_{\partial\Omega} = id$  and

$$\delta_{\bar{g}}h = 0,$$

where  $h = \varphi^* g - \bar{g}$ . Easy to see,  $h|_{T\Sigma} = 0$ .

Furthermore, we assume the metric g satisfies that

- $R_g \geq R_{\bar{g}}$ ;
- $\bullet \ H_g \ge H_{\bar{g}}.$

Since  $R_{\bar{g}}$  is a constant on  $\Omega$  and  $\varphi|_{\partial\Omega} = id$ , these assumptions can be preserved under the diffeomorphism  $\varphi$ . That is,

- $R_{\varphi^*g} = R_g \circ \varphi \ge R_{\bar{g}};$
- $\bullet \ H_{\varphi^*g} = H_g \circ \varphi = H_g \ge H_{\bar{g}}.$

By considering expansions of scalar curvature and mean curvature at  $\bar{g}$ , we get

(2.8) 
$$R_{\varphi^*g} = R_{\bar{g}} + DR_{\bar{g}} \cdot h + \frac{1}{2} D^2 R_{\bar{g}} \cdot (h, h) + E_{\bar{g}}(h)$$

and

(2.9) 
$$H_{\varphi^*g} = H_{\bar{g}} + DH_{\bar{g}} \cdot h + \frac{1}{2}D^2H_{\bar{g}} \cdot (h, h) + F_{\bar{g}}(h).$$

Here

$$|E_{\bar{g}}(h)| \leq C \int_{\Omega} |h| \left( |\nabla h|^2 + |h|^2 \right) dv_{\bar{g}}$$

and

$$|F_{\bar{g}}(h)| \le C \int_{\Sigma} |h|^2 (|\nabla h| + |h|) d\sigma_{\bar{g}}$$

for some positive constant  $C = C(\Omega, \bar{g})$ .

By Proposition 2.4,

$$D\mathscr{F}_{\Omega,\bar{g}}\cdot h = \int_{\Omega} (DR_{\bar{g}}\cdot h)fdv_{\bar{g}} + 2\int_{\Sigma} (DH_{\bar{g}}\cdot h)fd\sigma_{\bar{g}} - 2\kappa \left(DVol_{\Omega,\bar{g}}\cdot h\right) = 0.$$

Thus, if we assume

(2.10) 
$$\kappa(Vol_{\Omega}(g) - Vol_{\Omega}(\bar{g})) \le 0,$$

then together with assumptions  $R_q \geq R_{\bar{q}}$  on  $\Omega$  and  $H_q \geq H_{\bar{q}}$  on  $\Sigma$ , we get

$$DR_{\bar{g}} \cdot h = \gamma_{\bar{g}}h = 0$$

and

$$DH_{\bar{g}} \cdot h = 0.$$

That is,

(2.11) 
$$\Delta(trh) + Ric \cdot h = 0$$

and

(2.12) 
$$h_{nn}H_{\bar{g}} - 2\nabla_i h_n^{\ i} + \nabla_n h_i^{\ i} = 0,$$

where we used the gauge fixing condition  $\delta h = 0$  for the first equation.

For simplicity, we use notations  $\langle Rm_{\bar{g}}\cdot h, h\rangle := R_{\alpha\beta\gamma\delta}h^{\alpha\delta}h^{\beta\gamma}$  and  $\langle W_{\bar{g}}\cdot h, h\rangle := W_{\alpha\beta\gamma\delta}h^{\alpha\delta}h^{\beta\gamma}$ .

**Lemma 2.6.** Assume  $\delta h = 0$  and  $DH_{\bar{g}} \cdot h = 0$ , then

$$\begin{split} & \int_{\Omega} \left( D^2 R_{\bar{g}} \cdot (h,h) \right) f dv_{\bar{g}} \\ & = -\frac{1}{2} \int_{\Omega} \left[ \left( |\nabla h|^2 + |d(trh)|^2 - 2 \mathcal{R}_{\bar{g}}(h,h) \right) f + 2 \kappa \left( |h|^2 + \frac{2}{n-1} (trh)^2 \right) \right] dv_{\bar{g}} \\ & - \int_{\Sigma} \left[ \left( A^{ij} h_{in} h_{jn} + \left( h_{nn}^2 + 3 \sum_{i=1}^{n-1} h_{in}^2 \right) H_{\bar{g}} \right) f + \left( 2 h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) \partial_n f + 2 h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \right] d\sigma_{\bar{g}}, \end{split}$$

where

$$\mathscr{R}_{\bar{g}}(h,h) := \langle Rm_{\bar{g}} \cdot h, h \rangle + 2(trh)Ric_{\bar{g}} \cdot h - \frac{2R_{\bar{g}}}{n-1}(trh)^2$$

and  $A_{ij} = \frac{1}{2} \partial_{\bar{\nu}} \bar{g}_{ij}$  is the second fundamental form of  $\Sigma$  with respect to  $\bar{g}$ .

*Proof.* Applying Lemma 2.1 with  $\delta h = 0$ , we have

$$\begin{split} &\int_{\Omega} \left( D^2 R_{\bar{g}} \cdot (h,h) \right) f dv_{\bar{g}} \\ &= \int_{\Omega} \left[ -2 \gamma_{\bar{g}}(h^2) - \Delta (|h|^2) - \frac{1}{2} |\nabla h|^2 - \frac{1}{2} |d(trh)|_{\bar{g}}^2 + 2 \langle h, \nabla^2(trh) \rangle + \nabla_{\alpha} h_{\beta \gamma} \nabla^{\beta} h^{\alpha \gamma} \right] f dv_{\bar{g}}. \end{split}$$

From integration by parts, we have

$$\begin{split} &-2\int_{\Omega}(\gamma_{\bar{g}}(h^2))fdv_{\bar{g}}\\ &=-2\int_{\Omega}(-\Delta(tr(h^2))+\delta^2(h^2)-Ric\cdot h^2)fdv_{\bar{g}}\\ &=-2\int_{\Omega}\langle h^2,\gamma_{\bar{g}}^*f\rangle dv_{\bar{g}}-2\int_{\Sigma}\left[-(\partial_{\bar{\nu}}(tr(h^2))+\langle\delta(h^2),\bar{\nu}\rangle)f+(tr(h^2))\partial_{\bar{\nu}}f-h^2(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}\\ &=-2\int_{\Omega}\langle h^2,\gamma_{\bar{g}}^*f\rangle dv_{\bar{g}}+2\int_{\Sigma}\left[(\partial_{\bar{\nu}}(|h|^2)+\langle\delta(h^2),\bar{\nu}\rangle)f+h^2(\bar{\nu},\nabla f)-|h|^2\partial_{\bar{\nu}}f\right]d\sigma_{\bar{g}}. \end{split}$$

Similarly,

$$-\int_{\Omega} f\Delta(|h|^2) dv_{\bar{g}} = -\int_{\Omega} |h|^2 \Delta f dv_{\bar{g}} - \int_{\Sigma} \left[ f\partial_{\bar{\nu}}(|h|^2) - |h|^2 \partial_{\bar{\nu}} f \right] d\sigma_{\bar{g}}$$

and

$$\begin{split} &2\int_{\Omega}\langle h,\nabla^{2}(trh)\rangle fdv_{\bar{g}}\\ =&2\int_{\Omega}\left[\langle \delta h,d(trh)\rangle f-h(\nabla(trh),\nabla f)\right]dv_{\bar{g}}+2\int_{\Sigma}h(\bar{\nu},\nabla(trh))fd\sigma_{\bar{g}}\\ =&2\int_{\Omega}\left[\langle \delta h,d(trh)\rangle f+(trh)(\langle \delta h,df\rangle+\langle h,\nabla^{2}f\rangle)\right]dv_{\bar{g}}+2\int_{\Sigma}\left[h(\bar{\nu},\nabla(trh))f-(trh)h(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}\\ =&2\int_{\Omega}(trh)\langle h,\gamma_{\bar{g}}^{*}f+\bar{g}\Delta f+fRic\rangle dv_{\bar{g}}+2\int_{\Sigma}\left[h(\bar{\nu},\nabla(trh))f-(trh)h(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}. \end{split}$$

The last one is

$$\begin{split} &\int_{\Omega} \left[ \nabla_{\alpha} h_{\beta\gamma} \nabla^{\beta} h^{\alpha\gamma} \right] f dv_{\bar{g}} \\ &= -\int_{\Omega} h_{\gamma}^{\beta} \left[ \nabla_{\alpha} \nabla_{\beta} h^{\alpha\gamma} f + \nabla_{\beta} h^{\alpha\gamma} \nabla_{\alpha} f \right] dv_{\bar{g}} + \int_{\Sigma} \left[ h_{\beta\gamma} \nabla^{\beta} h^{\alpha\gamma} \bar{\nu}_{\alpha} \right] f d\sigma_{\bar{g}} \\ &= -\int_{\Omega} h_{\gamma}^{\beta} \left[ (\nabla_{\beta} \nabla_{\alpha} h^{\alpha\gamma} + R_{\alpha\beta\delta}^{\alpha} h^{\delta\gamma} + R_{\alpha\beta\delta}^{\gamma} h^{\alpha\delta}) f + \nabla_{\beta} h^{\alpha\gamma} \nabla_{\alpha} f \right] dv_{\bar{g}} + \int_{\Sigma} \left[ h_{\beta\gamma} \nabla^{\beta} h^{\alpha\gamma} \bar{\nu}_{\alpha} \right] f d\sigma_{\bar{g}} \\ &= -\int_{\Omega} \left[ (h_{\gamma}^{\beta} \nabla_{\beta} \nabla_{\alpha} h^{\alpha\gamma} + R_{\beta\delta} h^{\delta\gamma} h_{\gamma}^{\beta} - R_{\alpha\beta\gamma\delta} h^{\alpha\delta} h^{\beta\gamma}) f + h_{\gamma}^{\beta} \nabla_{\beta} h^{\alpha\gamma} \nabla_{\alpha} f \right] dv_{\bar{g}} + \int_{\Sigma} \left[ h_{\beta\gamma} \nabla^{\beta} h^{\alpha\gamma} \bar{\nu}_{\alpha} \right] f d\sigma_{\bar{g}} \end{split}$$

$$\begin{split} &=-\int_{\Omega}\left[-\nabla_{\beta}h_{\gamma}^{\beta}\nabla_{\alpha}h^{\alpha\gamma}f-2h_{\gamma}^{\beta}\nabla_{\alpha}h^{\alpha\gamma}\nabla_{\beta}f-h_{\gamma}^{\beta}h^{\alpha\gamma}\nabla_{\beta}\nabla_{\alpha}f+(\langle Ric,h^{2}\rangle-\langle Rm\cdot h,h\rangle)f\right]dv_{\bar{g}}\\ &+\int_{\Sigma}\left[(h_{\beta\gamma}\nabla^{\beta}h^{\alpha\gamma}\bar{\nu}_{\alpha}-h_{\gamma}^{\beta}\bar{\nu}_{\beta}\nabla_{\alpha}h^{\alpha\gamma})f-h_{\gamma}^{\beta}h^{\alpha\gamma}\bar{\nu}_{\beta}\nabla_{\alpha}f\right]d\sigma_{\bar{g}}\\ &=\int_{\Omega}\left[(\delta h)^{2}f-2h(\delta h,df)+\langle\nabla^{2}f,h^{2}\rangle-(\langle Ric,h^{2}\rangle-\langle Rm\cdot h,h\rangle)f\right]dv_{\bar{g}}\\ &+\int_{\Sigma}\left[(-\langle\delta(h^{2}),\bar{\nu}\rangle+2h(\bar{\nu},\delta h))f-h^{2}(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}\\ &=\int_{\Omega}\left[(\delta h)^{2}f-2h(\delta h,df)+\langle\nabla^{2}f-fRic,h^{2}\rangle+\langle Rm\cdot h,h\rangle f\right]dv_{\bar{g}}\\ &-\int_{\Sigma}\left[(\langle\delta(h^{2}),\bar{\nu}\rangle-2h(\bar{\nu},\delta h))f+h^{2}(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}\\ &=\int_{\Omega}\left[\langle\gamma_{\bar{g}}^{*}f+\bar{g}\Delta f,h^{2}\rangle+\langle Rm\cdot h,h\rangle f\right]dv_{\bar{g}}-\int_{\Sigma}\left[\langle\delta(h^{2}),\bar{\nu}\rangle f+h^{2}(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}.\\ &\text{Combining them, we get}\\ &\int_{\Omega}\left(D^{2}R_{\bar{g}}\cdot(h,h)\right)fdv_{\bar{g}}\\ &=-\frac{1}{2}\int_{\Omega}\left[(|\nabla h|^{2}+|d(trh)|^{2}-2\langle Rm\cdot h,h\rangle)f-4(trh)\langle\gamma_{\bar{g}}^{*}f+\bar{g}\Delta f+fRic,h\rangle+2\langle\gamma_{\bar{g}}^{*}f,h^{2}\rangle\right]dv_{\bar{g}}\\ &+\int_{\Sigma}\left[(\partial_{\bar{\nu}}(|h|^{2})+\langle\delta(h^{2}),\bar{\nu}\rangle+2h(\nabla(trh),\bar{\nu}))f-|h|^{2}\partial_{\bar{\nu}}f+h^{2}(\bar{\nu},\nabla f)-2(trh)h(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}\\ &=-\frac{2}{2}\int_{\Omega}\left[|\nabla h|^{2}+|d(trh)|^{2}-2\langle Rm\cdot h,h\rangle-4(trh)Ric\cdot h+\frac{4R_{\bar{g}}}{n-1}(trh)^{2}\right]fdv_{\bar{g}}\\ &-\frac{2\kappa}{n-1}\int_{\Omega}\left(\frac{n-1}{2}|h|^{2}+(trh)^{2}\right)dv_{\bar{g}}\\ &+\int_{\Sigma}\left[(\partial_{\bar{\nu}}(|h|^{2})+\langle\delta(h^{2}),\bar{\nu}\rangle+2h(\nabla(trh),\bar{\nu}))f-|h|^{2}\partial_{\bar{\nu}}f+h^{2}(\bar{\nu},\nabla f)-2(trh)h(\bar{\nu},\nabla f)\right]d\sigma_{\bar{g}}, \end{split}$$

where we used the V-static equation (1.1) and its trace equation (1.2). That is,

$$\begin{split} &\int_{\Omega} \left( D^2 R_{\bar{g}} \cdot (h,h) \right) f dv_{\bar{g}} \\ &= -\frac{1}{2} \int_{\Omega} \left[ \left( |\nabla h|^2 + |d(trh)|^2 - 2 \mathscr{R}_{\bar{g}}(h,h) \right) f + 2 \kappa \left( |h|^2 + \frac{2}{n-1} (trh)^2 \right) \right] dv_{\bar{g}} \\ &+ \int_{\Sigma} \left[ \left( \partial_{\bar{\nu}} (|h|^2) + \langle \delta(h^2), \bar{\nu} \rangle + 2 h (\nabla(trh), \bar{\nu}) \right) f - |h|^2 \partial_{\bar{\nu}} f + h^2(\bar{\nu}, \nabla f) - 2 (trh) h(\bar{\nu}, \nabla f) \right] d\sigma_{\bar{g}}, \end{split}$$

where we denote

$$\mathscr{R}_{\bar{g}}(h,h) := \langle Rm_{\bar{g}} \cdot h, h \rangle + 2(trh)Ric_{\bar{g}} \cdot h - \frac{2R_{\bar{g}}}{n-1}(trh)^2.$$

Now we rewrite the boundary integral in terms of the special orthonormal frame adapted to the boundary.

Since

$$\delta h = 0$$

and

$$h_{ij} = 0, \quad i, j = 1, \cdots, n-1$$

then

$$\langle \delta(h^2), \bar{\nu} \rangle = (\delta(h^2))_n = -\nabla_{\alpha}(h_{\beta}^{\ \alpha}h_n^{\ \beta}) = -h_{\beta}^{\ \alpha}\nabla_{\alpha}h_n^{\ \beta} = -h_{nn}\nabla_n h_{nn} - h_n^{\ i}\nabla_i h_{nn} - h_n^{\ i}\nabla_n h_{in}$$
 and

$$\partial_{\bar{\nu}}(|h|^2) = \nabla_n(|h|^2) = 2h_{nn}\nabla_n h_{nn} + 4h_n^i \nabla_n h_{in}$$

on  $\Sigma$ . Thus we have

$$\partial_{\bar{\nu}}(|h|^2) + \langle \delta(h^2), \bar{\nu} \rangle + 2h(\nabla(trh), \bar{\nu})$$

$$= h_{nn} \nabla_n h_{nn} + 3h_n^i \nabla_n h_{in} - h_n^i \nabla_i h_{nn} + 2h_{nn} \nabla_n (trh) + 2h_n^i \nabla_i (trh)$$

$$= 3h_{nn} \nabla_n h_{nn} + 3h_n^i \nabla_n h_{in} - h_n^i \nabla_i h_{nn} + 2h_{nn} \nabla_n h_i^i + 2h_n^i \nabla_i^{\Sigma} h_{nn}$$

$$= -3h_{nn} \nabla_i h_n^i - 3h_n^i \nabla_i h_i^j - h_n^i \nabla_i h_{nn} + 2h_{nn} \nabla_n h_i^i + 2h_n^i \nabla_i^{\Sigma} h_{nn},$$

where we used the fact

$$\nabla_n h_{n\alpha} = -(\delta h)_{\alpha} - \nabla_i h_{\alpha}^{\ i} = -\nabla_i h_{\alpha}^{\ i}.$$

Moreover, since

$$\nabla_j h_i^{\ j} = \partial_j h_i^{\ j} + \Gamma_{j\alpha}^j h_i^{\ \alpha} - \Gamma_{ji}^{\alpha} h_{\alpha}^{\ j} = A_{ij} h_n^{\ j} + H_{\bar{g}} h_{in}$$

and

$$\nabla_i h_{nn} = \partial_i h_{nn} - 2\Gamma_{in}^{\alpha} h_{\alpha n} = \nabla_i^{\Sigma} h_{nn} - 2A_{ij} h_n^j,$$

we obtain

$$\partial_{\bar{\nu}}(|h|^{2}) + \langle \delta(h^{2}), \bar{\nu} \rangle + 2h(\nabla(trh), \bar{\nu})$$

$$= -A^{ij}h_{in}h_{jn} - 3H_{\bar{g}}\sum_{i=1}^{n-1}h_{in}^{2} + h_{n}^{i}\nabla_{i}^{\Sigma}h_{nn} - 3h_{nn}\nabla_{i}h_{n}^{i} + 2h_{nn}\nabla_{n}h_{i}^{i}.$$

On the other hand,

$$-|h|^2 \partial_{\bar{\nu}} f + h^2(\bar{\nu}, \nabla f) - 2(trh)h(\bar{\nu}, \nabla f)$$
$$= -\left(2h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2\right) \partial_n f - h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f.$$

Hence using integration by parts,

$$\int_{\Sigma} \left[ \left( \partial_{\bar{\nu}}(|h|^2) + \langle \delta(h^2), \bar{\nu} \rangle + 2h(\nabla(trh), \bar{\nu}) \right) f - |h|^2 \partial_{\bar{\nu}} f + h^2(\bar{\nu}, \nabla f) - 2(trh)h(\bar{\nu}, \nabla f) \right] d\sigma_{\bar{g}}$$

$$= \int_{\Sigma} \left[ \left( -A^{ij}h_{in}h_{jn} - 3H_{\bar{g}} \sum_{i=1}^{n-1} h_{in}^2 \right) f - \left( 2h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) \partial_n f - 2h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \right] d\sigma_{\bar{g}}$$

$$+ \int_{\Sigma} \left( -h_{nn} \nabla_i^{\Sigma} h_n^i - 3h_{nn} \nabla_i h_n^i + 2h_{nn} \nabla_n h_i^i \right) f d\sigma_{\bar{g}}.$$

Note that

$$\nabla_i h_n^{\ i} = \partial_i h_n^{\ i} + \Gamma_{i\alpha}^i h_n^{\ \alpha} - \Gamma_{in}^\alpha h_\alpha^{\ i} = \nabla_i^\Sigma h_n^{\ i} + H_{\bar{g}} h_{nn}$$

and applying equation (2.12), we get

$$\int_{\Sigma} \left( -h_{nn} \nabla_{i}^{\Sigma} h_{n}^{i} - 3h_{nn} \nabla_{i} h_{n}^{i} + 2h_{nn} \nabla_{n} h_{i}^{i} \right) f d\sigma_{\bar{g}}$$

$$= \int_{\Sigma} \left( -4 \nabla_{i} h_{n}^{i} + 2 \nabla_{n} h_{i}^{i} + H_{\bar{g}} h_{nn} \right) h_{nn} f d\sigma_{\bar{g}}$$

$$= -\int_{\Sigma} H_{\bar{g}} h_{nn}^{2} f d\sigma_{\bar{g}}.$$

Combining all these calculations, we have

$$\int_{\Omega} \left( D^{2} R_{\bar{g}} \cdot (h, h) \right) f dv_{\bar{g}}$$

$$= -\frac{1}{2} \int_{\Omega} \left[ \left( |\nabla h|^{2} + |d(trh)|^{2} - 2 \mathcal{R}_{\bar{g}}(h, h) \right) f + 2 \kappa \left( |h|^{2} + \frac{2}{n-1} (trh)^{2} \right) \right] dv_{\bar{g}}$$

$$- \int_{\Sigma} \left[ \left( A^{ij} h_{in} h_{jn} + \left( h_{nn}^{2} + 3 \sum_{i=1}^{n-1} h_{in}^{2} \right) H_{\bar{g}} \right) f + \left( 2 h_{nn}^{2} + \sum_{i=1}^{n-1} h_{in}^{2} \right) \partial_{n} f + 2 h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_{i} f \right] d\sigma_{\bar{g}}.$$

**Lemma 2.7.** Suppose  $DH_{\bar{g}} \cdot h = 0$ , then we have

$$\int_{\Sigma} \left( D^2 H_{\bar{g}} \cdot (h, h) \right) f d\sigma_{\bar{g}} = \int_{\Sigma} \left( \frac{1}{4} h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) H_{\bar{g}} f d\sigma_{\bar{g}}.$$

*Proof.* By Lemma 2.2 and equation (2.12),

$$\begin{split} & \int_{\Sigma} (D^2 H_{\bar{g}} \cdot (h,h)) f d\sigma_{\bar{g}} \\ &= \int_{\Sigma} \left[ \left( -\frac{1}{4} h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) H_{\bar{g}} + h_{nn} \left( \nabla_i h_n^{\ i} - \frac{1}{2} \nabla_n h_i^{\ i} \right) \right] f d\sigma_{\bar{g}} \\ &= \int_{\Sigma} \left[ \left( -\frac{1}{4} h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) H_{\bar{g}} + \frac{1}{2} h_{nn}^2 H_{\bar{g}} \right] f d\sigma_{\bar{g}} \\ &= \int_{\Sigma} \left( \frac{1}{4} h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) H_{\bar{g}} f d\sigma_{\bar{g}}. \end{split}$$

Combining Lemmas 2.3, 2.6 and 2.7, we get the second variation of the functional  $\mathscr{F}_{\Omega,\bar{g}}$  at metric  $\bar{g}$ .

**Proposition 2.8.** Assume  $\delta h = 0$  and  $DH_{\bar{g}} \cdot h = 0$ , then

$$\begin{split} &D^2\mathscr{F}_{\Omega,\bar{g}}\cdot(h,h)\\ &=-\frac{1}{2}\int_{\Omega}\left[\left(|\nabla h|^2+|d(trh)|^2-2\mathscr{R}_{\bar{g}}(h,h)\right)f+\frac{n+3}{n-1}(trh)^2\kappa\right]dv_{\bar{g}}\\ &-\int_{\Sigma}\left[\left(A^{ij}h_{in}h_{jn}+\left(\frac{1}{2}h_{nn}^2+\sum_{i=1}^{n-1}h_{in}^2\right)H_{\bar{g}}\right)f+\left(2h_{nn}^2+\sum_{i=1}^{n-1}h_{in}^2\right)\partial_nf+2h_{nn}\sum_{i=1}^{n-1}h_{in}\partial_if\right]d\sigma_{\bar{g}}. \end{split}$$

Denote

$$I_{\Omega} := -\frac{1}{2} \int_{\Omega} \left[ \left( |\nabla h|^2 + |d(trh)|^2 - 2\mathscr{R}_{\bar{g}}(h,h) \right) f + \frac{n+3}{n-1} (trh)^2 \kappa \right] dv_{\bar{g}}$$

and

$$I_{\Sigma} := -\int_{\Sigma} \left[ \left( A^{ij} h_{in} h_{jn} + \left( \sum_{i=1}^{n-1} h_{in}^2 + \frac{1}{2} h_{nn}^2 \right) H_{\bar{g}} \right) f + \left( 2h_{nn}^2 + \sum_{i=1}^{n-1} h_{in}^2 \right) \partial_n f + 2h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_i f \right] d\sigma_{\bar{g}}.$$

Now we take  $\Omega$  to be a geodesic ball  $B_r(p)$  centered at p with radius r > 0. By continuity, we can choose a constant  $r_1 > 0$  such that f(x) > 0 for any  $x \in B_r(p)$ , since f(p) > 0.

We will show the non-positivity of  $I_{\partial B_r(p)}$  and  $I_{B_r(p)}$  for r sufficiently small.

**Lemma 2.9.** There exists a constant  $r_2 < r_1$  such that for any  $0 < r < r_2$ ,

$$(2.13) I_{\partial B_r(p)} \le 0.$$

*Proof.* For a geodesic ball  $B_r(p)$  with r > 0 small, we have

$$A_{ij} = \frac{1}{r}\bar{g}_{ij} + O(r)$$

and

$$H_{\bar{g}} = \frac{n-1}{r} + O(r).$$

Then

$$I_{\partial B_{r}(p)} = -\int_{\partial B_{r}(p)} \left[ \left( A^{ij} h_{in} h_{jn} + \left( \frac{1}{2} h_{nn}^{2} + \sum_{i=1}^{n-1} h_{in}^{2} \right) H_{\bar{g}} \right) f + \left( 2h_{nn}^{2} + \sum_{i=1}^{n-1} h_{in}^{2} \right) \partial_{n} f + 2h_{nn} \sum_{i=1}^{n-1} h_{in} \partial_{i} f \right] d\sigma_{\bar{g}}$$

$$= -\int_{\partial B_{r}(p)} \left[ \frac{1}{2r} \left( (n-1)h_{nn}^{2} + 2n\sum_{i=1}^{n-1} h_{in}^{2} \right) f + \left( 2h_{nn}^{2} + \sum_{i=1}^{n-1} h_{in}^{2} \right) \partial_{n} f + 2h_{nn}\sum_{i=1}^{n-1} h_{in} \partial_{i} f \right] d\sigma_{\bar{g}} + O(r) \cdot \int_{\partial B_{r}(p)} |h|^{2} d\sigma_{\bar{g}}$$

$$\leq -\int_{\partial B_{r}(p)} \left[ \frac{1}{2r} \left( (n-1)h_{nn}^{2} + 2n\sum_{i=1}^{n-1}h_{in}^{2} \right) f - \left( 3h_{nn}^{2} + n\sum_{i=1}^{n-1}h_{in}^{2} \right) |\nabla f| \right] d\sigma_{\bar{g}} + O(r||h||_{L^{2}(\partial B_{r}(p),\bar{g})}^{2})$$

$$= -\int_{\partial B_{r}(p)} \left[ \left( \frac{n-1}{2r} - 3\frac{|\nabla f|}{f} \right) h_{nn}^{2} + n\left( \frac{1}{r} - \frac{|\nabla f|}{f} \right) \sum_{i=1}^{n-1}h_{in}^{2} \right] f d\sigma_{\bar{g}} + O(r||h||_{L^{2}(\partial B_{r}(p),\bar{g})}^{2}).$$

Since f(p) > 0 and  $|\nabla f|$  is bounded on  $B_r(p)$  for any  $r < r_1$ , we can choose an  $r_2 < r_1$  such that

$$I_{\partial B_r(p)} \le 0$$

for any 
$$0 < r < r_2$$
.

In order to estimate the interior term, we need to study the eigenvalue problem of Laplacian operator acting on symmetric 2-tensors:

$$\mu(\Omega, \bar{g}) = \inf \left\{ \frac{\int_{\Omega} |\nabla h|^2 d\mathrm{vol}_{\bar{g}}}{\int_{\Omega} |h|^2 d\mathrm{vol}_{\bar{g}}} : h \not\equiv 0 \text{ and } h_{T\partial\Omega} = 0 \right\}$$

We recall the following estimate on this type of eigenvalue.

**Lemma 2.10.** ([14, Lemma 3.7]) Suppose  $(M^n, \bar{g})$  is a Riemannian manifold with  $n \geq 3$  and that  $B_r(p)$  is a geodesic ball of radius r centered at any  $p \in M$ . Then, there are positive constants  $r_0$  and  $c_0$  such that

for all  $0 < r < r_0$ .

From this, we have

**Lemma 2.11.** There exists a constant  $r_3 < r_1$  such that for any  $0 < r < r_3$ ,

$$(2.15) I_{B_r(p)} \le -\frac{1}{4} \left( \inf_{B_r(p)} f \right) ||h||_{W^{1,2}(B_r(p),\bar{g})}^2.$$

*Proof.* Since

$$|\mathscr{R}_{\bar{g}}(h,h)| = \left| \langle Rm_{\bar{g}} \cdot h, h \rangle + 2(trh)Ric_{\bar{g}} \cdot h - \frac{2R_{\bar{g}}}{n-1}(trh)^2 \right| \le \Lambda |h|^2,$$

where  $\Lambda = \Lambda(n, \bar{g}, |Rm_{\bar{q}}|, B_r(p))$  is a constant.

Thus,

$$\begin{split} I_{B_{r}(p)} &= -\frac{1}{2} \int_{B_{r}(p)} \left[ \left( |\nabla h|^{2} + |d(trh)|^{2} - 2\mathcal{R}_{\bar{g}}(h,h) \right) f + \frac{n+3}{n-1} (trh)^{2} \kappa \right] dv_{\bar{g}} \\ &\leq -\frac{1}{2} \int_{B_{r}(p)} \left[ \left( |\nabla h|^{2} - 2|\mathcal{R}_{\bar{g}}(h,h)| \right) f - 3n|\kappa||h|^{2} \right] dv_{\bar{g}} \\ &\leq -\frac{1}{2} \int_{B_{r}(p)} \left[ \left( \inf_{B_{r}(p)} f \right) |\nabla h|^{2} - \left( 2\Lambda \left( \sup_{B_{r}(p)} f \right) + 3n|\kappa| \right) |h|^{2} \right] dv_{\bar{g}} \\ &= -\frac{1}{4} \int_{B_{r}(p)} \left[ \left( \inf_{B_{r}(p)} f \right) |\nabla h|^{2} - \left( 4\Lambda \left( \sup_{B_{r}(p)} f \right) + \left( \inf_{B_{r}(p)} f \right) + 6n|\kappa| \right) |h|^{2} \right] dv_{\bar{g}} \\ &- \frac{1}{4} \left( \inf_{B_{r}(p)} f \right) \int_{B_{r}(p)} [|\nabla h|^{2} + |h|^{2}] dv_{\bar{g}} \\ &= -\frac{1}{4} \left( \inf_{B_{r}(p)} f \right) ||h||^{2}_{W^{1,2}(B_{r}(p),\bar{g})} - \frac{1}{4} \left( \inf_{B_{r}(p)} f \right) \int_{B_{r}(p)} [|\nabla h|^{2} - \mu_{0}|h|^{2}] dv_{\bar{g}}, \end{split}$$

where

$$\mu_0 := \frac{4\Lambda \left(\sup_{B_r(p)} f\right) + \left(\inf_{B_r(p)} f\right) + 6n|\kappa|}{\inf_{B_r(p)} f} > 0.$$

Since f(p) > 0, by Lemma 2.10, we can choose  $r_3 < r_1$  sufficiently small such that for any  $0 < r < r_3$ ,

$$\int_{B_r(p)} |\nabla h|^2 dv_{\bar{g}} \ge \mu_0 \int_{B_r(p)} |h|^2 dv_{\bar{g}}.$$

Hence we have

$$I_{B_r(p)} \le -\frac{1}{4} \left( \inf_{B_r(p)} f \right) ||h||_{W^{1,2}(B_r(p),\bar{g})}^2$$

for any  $0 < r < r_3$ .

Now we take  $r_0 := \min\{r_2, r_3\}$  and consider all geodesic ball  $B_r(p)$  with  $0 < r < r_0$ .

**Proposition 2.12.** Suppose  $R_g \geq R_{\bar{g}}$  on  $B_r(p)$  and  $H_g \geq H_{\bar{g}}$  on  $\partial B_r(p)$ . Moreover, assume  $\kappa(Vol_{\Omega}(g) - Vol_{\Omega}(\bar{g})) \leq 0$ ,

then the metric g is isometric to  $\bar{g}$ .

*Proof.* By assumptions, we have

$$\mathscr{F}_{B_r(p),\bar{g}}[\varphi^*g] \ge \mathscr{F}_{B_r(p),\bar{g}}[\bar{g}].$$

From Proposition 2.4,

$$D\mathscr{F}_{B_r(p),\bar{g}} \cdot h = 0.$$

Thus the inequality passes to the second order:

$$D^2 \mathscr{F}_{B_r(p),\bar{g}} \cdot (h,h) \ge 0.$$

On the other hand,

$$D^{2}\mathscr{F}_{B_{r}(p),\bar{g}}\cdot(h,h) = I_{B_{r}(p)} + I_{\partial B_{r}(p)} \le -\frac{1}{4} \left(\inf_{B_{r}(p)} f\right) ||h||_{W^{1,2}(B_{r}(p),\bar{g})}^{2}$$

by Lemmas 2.9 and 2.11.

Therefore

$$||h||_{W^{1,2}(B_r(p),\bar{q})}^2 = 0,$$

which implies  $h = \varphi^* g - \bar{g} \equiv 0$  on  $B_r(p)$ . Hence  $\varphi^* g = \bar{g}$ . i.e.  $\varphi : B_r(p) \to B_r(p)$  is an isometry.

We finish this section by giving the proof of our main theorem:

**Theorem A.** Suppose  $(M^n, \bar{g}, f, \kappa)$  is a V-static space with  $n \geq 3$ . For any  $p \in M$  with f(p) > 0, there exist constants  $r_0 > 0$  and  $\varepsilon_0 > 0$  such that for any geodesic ball  $B_r(p) \subset M$  with radius  $0 < r < r_0$  and metric g on  $B_r(p)$  satisfies

- $R(g) \geq R(\bar{g})$  in  $B_r(p)$ ;
- $H(g) \ge H(\bar{g})$  on  $\partial B_r(p)$ ;
- g and  $\bar{g}$  induce the same metrics on  $\partial B_r(p)$ ;
- $\bullet ||g \bar{g}||_{C^2(B_r(p),\bar{g})} < \varepsilon_0,$

the following volume comparison hold:

• if  $\kappa < 0$ , then

$$Vol_{\Omega}(g) \leq Vol_{\Omega}(\bar{g});$$

• if  $\kappa > 0$ , then

$$Vol_{\Omega}(q) > Vol_{\Omega}(\bar{q});$$

with equality holds in either case if and only if the metric g is isometric to  $\bar{q}$ .

*Proof.* Suppose the volume comparison is not true, then

$$\kappa(Vol_{\Omega}(g) - Vol_{\Omega}(\bar{g})) < 0.$$

This would imply the metric g is isometric to  $\bar{g}$  by Proposition 2.12 and hence

$$Vol_{\Omega}(q) = Vol_{\Omega}(\bar{q}).$$

But this is a contradiction. Therefore the volume comparison holds. Applying Proposition 2.12 again with

$$Vol_{\Omega}(g) = Vol_{\Omega}(\bar{g}),$$

we conclude the metric g has to be isometric to  $\bar{g}$ .

#### 3. Volume comparison for Einstein Manifolds

By taking the function  $f \equiv 1$  in the V-static equation (1.1), we get

$$\gamma_{\bar{g}}^* 1 = -Ric_{\bar{g}} = \kappa \bar{g}.$$

This means,  $(1, \kappa)$  is a solution to V-static equation (1.1) if and only if the metric  $\bar{g}$  is an Einstein metric with  $(-\kappa)$  as its Einstein constant.

From this simple observation, we will investigate the volume comparison with respect to a closed Einstein manifold  $(M, \bar{g})$ . By a closed manifold here, we mean a compact manifold without boundary.

For a Einstein manifold with

$$Ric_{\bar{q}} = (n-1)\lambda \bar{q}$$

its Riemann curvature tensor is given by

$$R_{ijkl} = W_{ijkl} + \lambda (\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).$$

Thus, we have

(3.1) 
$$\mathscr{R}_{\bar{g}}(h,h) = \langle Rm_{\bar{g}} \cdot h, h \rangle + 2(trh)Ric_{\bar{g}} \cdot h - \frac{2R_{\bar{g}}}{n-1}(trh)^2$$
$$= \langle W \cdot h, h \rangle - \lambda(|h|^2 + (trh)^2).$$

Similar to the situation of a generic V-static domain, we consider the functional

(3.2) 
$$\mathscr{F}_{M,\bar{g}}[g] = \int_{M} R(g)dv_{\bar{g}} + 2(n-1)\lambda Vol_{M}(g).$$

By Proposition 2.4 or a simple calculation, we have

**Proposition 3.1.** The metric  $\bar{g}$  is a critical point of the functional  $\mathscr{F}_{M,\bar{g}}[g]$ .

*Proof.* For any  $h \in S_2(M)$ ,

$$\begin{split} D\mathscr{F}_{M,\bar{g}} \cdot h &= \int_{M} (DR_{\bar{g}} \cdot h) dv_{\bar{g}} + 2(n-1)\lambda \left( DVol_{\bar{g}} \cdot h \right) \\ &= \int_{M} (-\Delta(trh) + \delta^{2}h - Ric_{\bar{g}} \cdot h) dv_{\bar{g}} + (n-1)\lambda \int_{M} (trh) dv_{\bar{g}} \\ &= -\int_{M} \langle h, Ric_{\bar{g}} - (n-1)\lambda \bar{g} \rangle dv_{\bar{g}} \\ &= 0 \end{split}$$

In the rest part of this section, we will consider a metric g on M sufficiently  $C^2$ -closed to the Einstein metric  $\bar{g}$ . Like what we did for generic V-static metrics, we need to fix the gauge when considering such a deformation problem. Applying Lemma 2.5 on closed manifolds or simply Ebin's  $Slice\ Theorem(cf.\ [8])$ , we can find a diffeomorphism  $\varphi: M \to M$  such that  $h := \varphi^* g - \bar{g}$  satisfies

$$\delta_{\bar{g}}h = 0.$$

Now the volume comparison can be obtained if we have some informations on the first order expansion of scalar curvature:

**Proposition 3.2.** Suppose  $R_g \geq R_{\bar{g}}$  and  $\gamma_{\bar{g}}h \not\equiv 0$  on M, then following conclusions hold:

• If  $\lambda > 0$ , then

$$Vol_M(g) < Vol_M(\bar{g});$$

• If  $\lambda < 0$ , then

$$Vol_M(q) > Vol_M(\bar{q}).$$

*Proof.* By the assumption  $R_g \geq R_{\bar{g}}$ ,

$$R_{\varphi^*g} = R_g \circ \varphi \ge R_{\bar{g}}$$

on M. Thus  $DR_{\bar{g}} \cdot h = \gamma_{\bar{g}} h \geq 0$ . Since it is not vanishing identically, we have

$$\int_{M} (\gamma_{\bar{g}} h) dv_{\bar{g}} > 0.$$

On the other hand, by Proposition 3.1,  $\bar{g}$  is a critical point of  $\mathscr{F}_{M,\bar{q}}[g]$ . Then

$$2(n-1)\lambda \left(DVol_{M,\bar{g}}\cdot h\right) = -\int_{M} (\gamma_{\bar{g}}h)dv_{\bar{g}} < 0.$$

Therefore,

$$Vol_M(g) = Vol_M(\varphi^*g) = Vol_M(\bar{g}) + DVol_{M,\bar{g}} \cdot (\xi h) < Vol_M(\bar{g}),$$

for some  $\xi \in (0,1)$ , if  $\lambda > 0$ . Similarly,

$$Vol_M(g) > Vol_M(\bar{g}),$$

if  $\lambda < 0$ .

Now we consider the case when lacking of the first order information. First, recalling the following classic eigenvalue estimate for Laplacian operator acting on functions (cf. Theorem 9 on P.82 in [6]):

**Lemma 3.3** (Lichnerowicz-Obata's eigenvalue estimate). Suppose  $\lambda > 0$ , then for any  $u \in C^{\infty}(M)$  with

$$\int_{M} u dv_{\bar{g}} = 0,$$

we have

(3.3) 
$$\int_{M} |du|^{2} dv_{\bar{g}} \ge n\lambda \int_{M} u^{2} dv_{\bar{g}}.$$

From this, we can get

**Lemma 3.4.** Suppose  $\lambda \neq 0$  and  $\gamma_{\bar{q}}h \equiv 0$ , then

$$trh = 0$$

on M.

*Proof.* Since  $\delta h = 0$ ,

$$\gamma_{\bar{q}}h = -\Delta(trh) - Ric \cdot h = -\Delta(trh) - (n-1)\lambda(trh) = 0.$$

Then

$$0 = \int_{M} \left[ -(trh)\Delta(trh) - (n-1)\lambda(trh)^{2} \right] dv_{\bar{g}}$$
$$= \int_{M} \left[ |d(trh)|^{2} - (n-1)\lambda(trh)^{2} \right] dv_{\bar{g}}.$$

Thus, trh vanishes identically on M, if  $\lambda < 0$ . Otherwise, we have  $\lambda > 0$  and

$$\int_{M} (trh)dv_{\bar{g}} = -\frac{1}{(n-1)\lambda} \int_{M} \Delta(trh)dv_{\bar{g}} = 0.$$

Hence

$$0 = \int_{M} \left[ |dtrh|^{2} - (n-1)\lambda(trh)^{2} \right] dv_{\bar{g}} \ge \lambda \int_{M} (trh)^{2} dv_{\bar{g}},$$

by Lichnerowicz-Obata's eigenvalue estimate (Lemma 3.3), which implies trh vanishes on M.

We can easily get the following volume comparison:

**Proposition 3.5.** Suppose  $\lambda \neq 0$  and  $\gamma_{\bar{q}}h \equiv 0$  on M, then

$$Vol_M(g) \leq Vol_M(\bar{g})$$

and equality holds if and only if the metric g is isometric to  $\bar{g}$ .

*Proof.* By Lemmas 2.3 and 3.4,

$$DVol_{M,\bar{g}} \cdot h = \frac{1}{2} \int_{M} (trh) dv_{\bar{g}} = 0$$

and

$$D^{2}Vol_{M,\bar{g}}\cdot(h,h) = \frac{1}{4}\int_{M} \left[ (trh)^{2} - 2|h|^{2} \right] dv_{\bar{g}} = -\frac{1}{2}\int_{M} |\mathring{h}|^{2} dv_{\bar{g}} \le 0,$$

where  $\overset{\circ}{h} := h - \frac{1}{n}(trh)\bar{g}$  is the traceless part of the tensor h. Then

$$Vol_{M}(g) = Vol_{M}(\varphi^{*}g)$$

$$= Vol_{M}(\bar{g}) + DVol_{M,\bar{g}} \cdot h + D^{2}Vol_{M,\bar{g}} \cdot (\xi h, \xi h)$$

$$= Vol_{M}(\bar{g}) + \xi^{2} \left( D^{2}Vol_{M,\bar{g}} \cdot (h, h) \right)$$

$$\leq Vol_{M}(\bar{g}),$$

for some  $\xi \in (0,1)$ . Clearly, the equality holds if and only if  $\overset{\circ}{h} = 0$ , *i.e.* h = 0, since trh = 0.

Remark 3.6. Note that there is no assumption on the comparison of scalar curvature in this proposition. i.e. we do not need to assume  $R_g \geq R_{\bar{g}}$  on M.

Together with Proposition 3.2, we get

Corollary 3.7. Suppose  $\lambda > 0$ , then

$$Vol_M(g) \leq Vol_M(\bar{g})$$

and equality holds if and only if the metric g is isometric to  $\bar{g}$ .

According to Proposition 3.5, it seems the volume comparison does not hold for the case  $\lambda < 0$  due to a different inequality given by Proposition 3.2. However, we will show that the strict inequality in Proposition 3.5 does not occur in case  $\lambda < 0$  provided we pose certain restrictions on Weyl tensor. In order to justify this claim, we use the trick which involving calculating the second variation of total scalar curvature with a fixed volume form:

**Proposition 3.8.** Suppose  $\gamma_{\bar{q}}h \equiv 0$  on M, then

$$\int_{M} \left( D^{2} R_{\bar{g}} \cdot (h, h) \right) dv_{\bar{g}} = -\frac{1}{2} \int_{M} \left( |\nabla \mathring{h}|^{2} - 2 \langle W \cdot \mathring{h}, \mathring{h} \rangle - 2(n-2) \lambda |\mathring{h}|^{2} \right) dv_{\bar{g}}.$$

*Proof.* By Lemma 2.6 and Proposition 3.4,

$$\begin{split} \int_{M} \left( D^{2}R_{\bar{g}} \cdot (h,h) \right) dv_{\bar{g}} &= -\frac{1}{2} \int_{M} \left( |\nabla \overset{\circ}{h}|^{2} - 2\mathscr{R}_{\bar{g}} (\overset{\circ}{h},\overset{\circ}{h}) + 2\kappa |\overset{\circ}{h}|^{2} \right) dv_{\bar{g}} \\ &= -\frac{1}{2} \int_{M} \left( |\nabla \overset{\circ}{h}|^{2} - 2\langle W \cdot \overset{\circ}{h},\overset{\circ}{h} \rangle - 2(n-2)\lambda |\overset{\circ}{h}|^{2} \right) dv_{\bar{g}}, \end{split}$$

where we used the fact  $\kappa = -(n-1)\lambda$ .

Unlike the corresponding argument for generic V-static metric, Lemma 2.10 is not applicable in this setting. We have to seek a different way to estimate the eigenvalue of Laplacian operator acting on symmetric 2-tensors for  $(M, \bar{q})$ .

**Proposition 3.9.** Suppose  $(M, \bar{g})$  is a closed Einstein manifold with

$$Ric_{\bar{g}} = (n-1)\lambda \bar{g}.$$

Then for any  $h \in S_2(M)$  and  $\theta \in \mathbb{R}$ ,

$$(3.4) \qquad \int_{M} |\nabla h|^{2} dv_{\bar{g}} \geq -\frac{2\theta}{1+\theta^{2}} \int_{M} \left( |\delta h|^{2} + \langle W \cdot \overset{\circ}{h}, \overset{\circ}{h} \rangle \right) dv_{\bar{g}} + \frac{2n\theta\lambda}{1+\theta^{2}} \int_{M} |\overset{\circ}{h}|^{2} dv_{\bar{g}}.$$

*Proof.* We have

$$\begin{split} &\int_{M} \nabla_{\alpha} h_{\beta\gamma} \nabla^{\beta} h^{\alpha\gamma} dv_{\bar{g}} \\ &= -\int_{M} \nabla_{\beta} \nabla_{\alpha} h_{\gamma}^{\ \beta} h^{\alpha\gamma} dv_{\bar{g}} \\ &= -\int_{M} \left( \nabla_{\alpha} \nabla_{\beta} h_{\gamma}^{\ \beta} + R_{\beta\alpha\delta}^{\beta} h_{\gamma}^{\ \delta} - R_{\beta\alpha\gamma}^{\delta} h_{\delta}^{\ \beta} \right) h^{\alpha\gamma} dv_{\bar{g}} \\ &= -\int_{M} \left( -\nabla_{\alpha} (\delta h)_{\gamma} + R_{\alpha\delta} h_{\gamma}^{\ \delta} - R_{\beta\alpha\gamma\delta} h^{\beta\delta} \right) h^{\alpha\gamma} dv_{\bar{g}} \\ &= -\int_{M} \left[ -|\delta h|^{2} + ((n-1)\lambda \bar{g}_{\alpha\delta} h_{\gamma}^{\ \delta} - W_{\beta\alpha\gamma\delta} h^{\beta\delta} - \lambda (\bar{g}_{\beta\delta} \bar{g}_{\alpha\gamma} - \bar{g}_{\beta\gamma} \bar{g}_{\alpha\delta}) h^{\beta\delta}) h^{\alpha\gamma} \right] dv_{\bar{g}} \\ &= \int_{M} \left[ |\delta h|^{2} + \langle W \cdot h, h \rangle + \lambda ((trh)^{2} - n|h|^{2}) \right] dv_{\bar{g}} \\ &= \int_{M} \left[ |\delta h|^{2} + \langle W \cdot h, h \rangle - n\lambda |\mathring{h}|^{2} \right] dv_{\bar{g}}. \end{split}$$

Thus for any  $\theta \in \mathbb{R}$ ,

$$\begin{split} 0 & \leq \int_{M} |\nabla_{\alpha}h_{\beta\gamma} + \theta\nabla_{\beta}h_{\alpha\gamma}|^{2}dv_{\bar{g}} \\ & = \int_{M} \left[ (1+\theta^{2})|\nabla h|^{2} + 2\theta\nabla_{\alpha}h_{\beta\gamma}\nabla^{\beta}h^{\alpha\gamma} \right]dv_{\bar{g}} \\ & = \int_{M} \left[ (1+\theta^{2})|\nabla h|^{2} + 2\theta(|\delta h|^{2} + \langle W \cdot \overset{\circ}{h}, \overset{\circ}{h} \rangle - n\lambda |\overset{\circ}{h}|^{2}) \right]dv_{\bar{g}}. \end{split}$$

That is,

$$\int_{M} |\nabla h|^2 dv_{\bar{g}} \geq -\frac{2\theta}{1+\theta^2} \int_{M} \left( |\delta h|^2 + \langle W \cdot \overset{\circ}{h}, \overset{\circ}{h} \rangle \right) dv_{\bar{g}} + \frac{2n\theta\lambda}{1+\theta^2} \int_{M} |\overset{\circ}{h}|^2 dv_{\bar{g}}.$$

From this estimate, we have

# **Proposition 3.10.** Suppose $\lambda < 0$ and

$$||W||_{L^{\infty}(M,\bar{g})} < \alpha(n,\lambda) := -(3n-4)\lambda,$$

then there exists a constant  $\eta > 0$  such that

$$\int_{M} \left( D^{2} R_{\bar{g}} \cdot (h, h) \right) dv_{\bar{g}} \le -\eta \int_{M} |\mathring{h}|^{2} dv_{\bar{g}}.$$

*Proof.* From Propositions 3.8 and 3.9, we have

$$\begin{split} \int_{M} \left( D^{2} R_{\bar{g}} \cdot (h,h) \right) dv_{\bar{g}} &= -\frac{1}{2} \int_{M} \left( |\nabla \mathring{h}|^{2} - 2 \langle W \cdot \mathring{h}, \mathring{h} \rangle - 2(n-2) \lambda |\mathring{h}|^{2} \right) dv_{\bar{g}} \\ &\leq \frac{1 + \theta + \theta^{2}}{1 + \theta^{2}} \int_{M} \langle W \cdot \mathring{h}, \mathring{h} \rangle dv_{\bar{g}} + \frac{[(n-2)\theta^{2} - n\theta + (n-2)]\lambda}{1 + \theta^{2}} \int_{M} |\mathring{h}|^{2} dv_{\bar{g}} \\ &\leq \left( \frac{1 + \theta + \theta^{2}}{1 + \theta^{2}} ||W||_{L^{\infty}(M,\bar{g})} + \frac{[(n-2)\theta^{2} - n\theta + (n-2)]\lambda}{1 + \theta^{2}} \right) \int_{M} |\mathring{h}|^{2} dv_{\bar{g}}. \end{split}$$

Taking  $\theta = -1$ ,

$$\int_{M} (D^{2}R_{\bar{g}} \cdot (h,h)) dv_{\bar{g}} \leq \frac{1}{2} (||W||_{L^{\infty}(M,\bar{g})} + (3n-4)\lambda) \int_{M} |\mathring{h}|^{2} dv_{\bar{g}} 
= -\frac{1}{2} (\alpha(n,\lambda) - ||W||_{L^{\infty}(M,\bar{g})}) \int_{M} |\mathring{h}|^{2} dv_{\bar{g}}.$$

The conclusion follows if we take

$$\eta := \frac{1}{2} (\alpha(n, \lambda) - ||W||_{L^{\infty}(M, \bar{g})}) > 0.$$

Remark 3.11. It is well-known that on an Einstein manifold  $(M, \bar{g})$  with

$$Ric_{\bar{g}} = (n-1)\lambda \bar{g},$$

its Weyl tensor satisfies the following equation:

$$(3.5) \qquad \Delta W - 2(n-1)\lambda W - 2\mathcal{Q}(W) = 0,$$

where  $Q(W) := B_{ijkl} - B_{jikl} + B_{ikjl} - B_{jkil}$  is a quadratic combination of Weyl tensors and  $B_{ijkl} := g^{pq}g^{rs}W_{pijr}W_{qkls}$ . Thus by applying the standard  $De\ Giorgi-Nash-Moser\ estimate$  on this equation, we can replace the assumption on  $||W||_{L^{\infty}(M,\bar{g})}$  by the corresponding one on  $||W||_{L^{\frac{n}{2}}(M,\bar{g})}$ .

**Proposition 3.12.** Suppose  $\lambda < 0$  and  $R_g \geq R_{\bar{g}}$  on M. Moreover, assume  $\gamma_{\bar{g}}h \equiv 0$  and  $||W||_{L^{\infty}(M,\bar{q})} < \alpha(n,\lambda),$ 

then the metric g is isometric to  $\bar{g}$ .

*Proof.* By assumptions,

$$\int_{M} R_{\varphi^*g} dv_{\bar{g}} \ge \int_{M} R_{\bar{g}} dv_{\bar{g}}$$

and

$$\int_{M} (DR_{\bar{g}} \cdot h) dv_{\bar{g}} = \int_{M} (\gamma_{\bar{g}} h) dv_{\bar{g}} = 0.$$

Thus there is a constant  $\xi \in (0,1)$  such that

$$\frac{1}{2}\int_{M}\left(D^{2}R_{\bar{g}}\cdot(\xi h,\xi h)\right)dv_{\bar{g}}=\int_{M}R_{\varphi^{*}g}dv_{\bar{g}}-\int_{M}R_{\bar{g}}dv_{\bar{g}}-\int_{M}(DR_{\bar{g}}\cdot h)dv_{\bar{g}}\geq0$$

and hence

$$\int_{M} \left( D^{2} R_{\bar{g}} \cdot (h, h) \right) dv_{\bar{g}} \ge 0.$$

On the other hand, by Proposition 3.10,

$$\int_{M} \left( D^{2} R_{\bar{g}} \cdot (h, h) \right) dv_{\bar{g}} \le -\eta \int_{M} |\mathring{h}|^{2} dv_{\bar{g}},$$

for a constant  $\eta > 0$ . This implies

$$\overset{\circ}{h} = 0$$

on M. Together with Proposition 3.4, we have h vanishes identically on M, which means  $\varphi^*g=\bar{g}$ . i.e.  $\varphi:M\to M$  is an isometry.

Now we can prove the main theorem in this section:

**Theorem B.** Suppose  $(M^n, \bar{g})$  is a closed Einstein manifold satisfies

$$Ric_{\bar{g}} = (n-1)\lambda \bar{g}$$

with  $\lambda \neq 0$ . Moreover, if  $\lambda < 0$ , we assume its Weyl tensor satisfies

$$||W||_{L^{\infty}(M,\bar{g})} < \alpha(n,\lambda) := -(3n-4)\lambda.$$

Then there exists a constant  $\varepsilon_0 > 0$  such that for any metric g on M satisfies

$$R_q \ge n(n-1)\lambda$$

and

$$||g - \bar{g}||_{C^2(M,\bar{q})} < \varepsilon_0,$$

the following volume comparison hold:

• if  $\lambda > 0$ , then

$$Vol_M(g) \leq Vol_M(\bar{g});$$

• if  $\lambda < 0$ , then

$$Vol_M(g) \ge Vol_M(\bar{g});$$

with equality holds in either case if and only if the metric g is isometric to  $\bar{g}$ .

*Proof.* For the case  $\lambda > 0$ , the conclusion follows from Corollary 3.7. Now we consider the case  $\lambda < 0$ .

If  $\gamma_{\bar{g}}h \not\equiv 0$  on M, then

$$Vol_M(g) > Vol_M(\bar{g})$$

by Proposition 3.2. Otherwise, we have  $\gamma_{\bar{g}}h \equiv 0$  on M. According to Proposition 3.12, we get the metric g is isometric to  $\bar{g}$  and hence

$$Vol_M(g) = Vol_M(\bar{g}).$$

On the other hand, this justifies that the equality can only be achieved when g is isometric to the Einstein metric  $\bar{g}$ .

By taking Weyl tensor to be identically zero in Theorem B, we achieve volume comparison for round spheres and hyperbolic manifolds:

**Corollary A.** For  $n \geq 3$ , let  $(\mathbb{S}^n, g_{\mathbb{S}^n})$  be the unit round sphere. There exists a constant  $\varepsilon_0 > 0$  such that for any metric g on  $\mathbb{S}^n$  with

$$R_q \ge n(n-1)$$

and

$$||g - \bar{g}||_{C^2(\mathbb{S}^n, q_{\mathbb{S}^n})} < \varepsilon_0,$$

we have

$$Vol_M(g) \leq Vol_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

with equality holds if and only if the metric g is isometric to  $\bar{g}$ .

Corollary B. For  $n \geq 3$ , let  $(M^n, \bar{g})$  be a closed hyperbolic manifold. There exists a constant  $\varepsilon_0 > 0$  such that for any metric g on M with

$$R_g \ge R_{\bar{g}}$$

and

$$||g - \bar{g}||_{C^2(M,\bar{g})} < \varepsilon_0,$$

we have

$$Vol_M(g) \ge Vol_M(\bar{g})$$

with equality holds if and only if the metric g is isometric to  $\bar{g}$ .

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