REGULARITY OF VISCOUS SOLUTIONS FOR A DEGENERATE NON-LINEAR CAUCHY PROBLEM

ERIC HERNÁNDEZ SASTOQUE¹, CHRISTIAN KLINGENBERG², LEONARDO RENDÓN³, AND JUAN C. JUAJIBIOY⁴

ABSTRACT. We consider the Cauchy problem for a class of nonlinear degenerate parabolic equation with forcing. By using the vanishing viscosity method we obtain generalized solutions. We prove some regularity results about this generalized solutions.

1. Introduction

We consider the Cauchy problem for the following nonlinear degenerate parabolic equation with forcing

(1)
$$u_t = u\Delta u - \gamma |\nabla u|^2 + f(t, u), \ (x, t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

(2)
$$u(x,0) = u_0(x) \in C(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N),$$

where γ is a non-negative constant. Equation (1) arisen in severals applications of biology and phisycs, see [15], [12]. Equation (1) is of degenerate parabolic type: parabolicity it is loss at points where u = 0, see [15], [1] for a most datailed description. In [11] a weak solution for the homogeneous equation (1) is constructed by using the vanishing viscosity method, this method was introduced by Lions and Crandall [10], when they studied the existence of solutions to Hamilton-Jacobi equations

$$u_t + H(x, t, u, Du) = 0$$

and consists in view the equation (1) as the limit for $\epsilon \to 0$ of the equation

(3)
$$u_t = \epsilon \Delta u + u \Delta u - \gamma |\nabla u|^2 + f(t, u),$$

where ϵ is a small positive number. The regularity of the weak solutions for the homogeneous Cauchy problem (1),(2) was studied by the author in [9].In this paper we extend the above results for the inhomogeneous case, this extension is interesting, from physical viewpoint, since the equation (1) is related with non-equilibrium process in poros media due to external forces. We obtain the following main theorem.

Theorem 1.1. If $\gamma \geq \sqrt{2N} - 1$, $|\nabla (u_0^{1+\frac{\alpha}{2}})| \leq M$, where M is a positive constant such as

$$\alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \le 0,$$

then the viscosity solutions of the Cauchy problem (1), (2) satisfies

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 $^{^{1}\}mathrm{Departamento}$ de Matemáticas, Universidad de Magdalena, Santa Marta, Colombia

²DEPARTMENT OF MATHEMATICS, WÄRZBURG UNIVERSITY, GERMANY

³DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL DE COLOMBIA, BOGOTÁ

⁴ DEPARTAMENTO DE CIENCIAS NATURALES Y EXÁCTAS, FUNDACIÓN UNIVERSIDAD AUTONOMA DE COLOMBIA, BOGOTÁ E-mail addresses: ¹eric.hernandez@unimagdalena.edu.co, ²klingenberg@mathematik.uni-wuerzburg.de,

 $^{^3 {\}tt lrendona@unal.eu.co}$, $^4 {\tt jcjuajibioyo@unal.edu.co}$.

2. Preliminaries

Definition 2.1. A function $u \in L^{\infty}(\Omega) \cap L^{2}_{Loc}([0,+\infty); H^{1}_{Loc}(\mathbb{R}^{N}))$, is called a weak solution of (1),(2) if it satisfies the following conditions:

- (i) $u(x,t) \ge 0$, a.e in Ω .
- (ii) u(x,t) satisfies the following relation

(5)
$$\int_{\mathbb{R}^N} u_0 \psi(x,0) \, dx + \iint_{\Omega} \left(u \psi_t - u \nabla u \cdot \nabla \psi - (1+\gamma) |\nabla u|^2 \psi - f(t,u) \psi \right) dx \, dt = 0,$$

for any $\psi \in C^{1,1}(\overline{\Omega})$ with compact support in $\overline{\Omega}$.

For the construction of a weak solution to the Cauchy problem (1),(2), we use the viscosity method: we add the term $\epsilon \Delta u$ in the equation (1) and we consider the following Cauchy problem

(6)
$$u_t = u\Delta u - \gamma |\nabla u|^2 + f(t, u) + \epsilon \Delta u, \ u \in \Omega,$$

(7)
$$u(x,0) = u_0(x), x \in \mathbb{R}^N$$

where $\gamma \geq 0$, the existence of solutions is garanteed by the Maximum principle and then we investigate the convergence of the solutions when $\epsilon \to 0$, in fact, we will show that when $\epsilon \to 0$, u^{ϵ} converges to the weak solution of (1),(2), but to cost of the loss of the uniqueness.

Definition 2.2. The weak solution for the Cauchy problem (1),(2) constructed by the vanishing viscosity method is called viscosity solution.

3. Estimates of Hölder

In this section we begin by collecting some a priori estimates for the function u.

Theorem 3.1. If $\gamma \geq \sqrt{2N} - 1$, the initial data (2) satisfies $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, where M is a positive constant, $\alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0$ and $f \in C^1(\mathbb{R}^+ \times \mathbb{R})$ satisfies, $f \geq 0$, $f_u \leq 0$, then the viscosity solution u(x,t) of Cauchy problem (1),(2) satisfies

$$|\nabla(u^{1+\frac{\alpha}{2}})| \le M$$
, in $\overline{\Omega}$.

Proof. Let

(8)
$$w = \frac{1}{2} \sum_{i=1}^{N} u_{x_i}^2.$$

Deriving with respect t in (8) and replacing in (1) we have

$$w_{t} = \sum_{i=1}^{N} u_{x_{i}} \left[u_{x_{i}} \Delta u + u \left(\sum_{j=1}^{N} u_{x_{i} x_{j} x_{j}} \right) - 2\gamma w_{x_{i}} + f_{u} u_{x_{i}} \right].$$

By other hand

$$\Delta w = \frac{1}{2} \sum_{j=1}^{N} \left(\sum_{i=1}^{N} u_{x_{i}}^{2} \right)_{x_{j}x_{j}}$$

$$= \frac{1}{2} \left[\sum_{j=1}^{N} (2u_{x_{1}}u_{x_{1}x_{j}})_{x_{j}} + \sum_{j=1}^{N} (2u_{x_{2}}u_{x_{2}x_{j}})_{x_{j}} + \dots + \sum_{j=1}^{N} (2u_{x_{N}}u_{x_{N}x_{j}})_{x_{j}} \right]$$

(9)
$$\Delta w = \sum_{i,j=1}^{N} u_{x_i x_j}^2 + \sum_{i,j=1}^{N} u_{x_i} u_{x_i x_j x_j},$$

thereby,

$$(10) w_t = 2w\Delta u + u\Delta w - u\sum_{i,j=1}^N u_{x_ix_j}^2 - 2\gamma\sum_{i=1}^N u_{x_i}w_{x_i} + 2f_uw.$$

Set,

$$(11) z = g(u)w.$$

Deriving two times with respect x_i in (11) we have

$$(12) w_{x_i} = (g^{-1})_{x_i} z + g^{-1} z_{x_i}$$

(13)
$$w_{x_i x_i} = (g^{-1})_{x_i x_i} z + 2(g^{-1})_{x_i} z_{x_i} + g^{-1} z_{x_i x_i}.$$

From equations (9),(12), (13) we have that,

$$\Delta w = \sum_{i=1}^{N} w_{x_i x_i} = \sum_{i=1}^{N} \left[(g^{-1})_{x_i x_i} z + 2(g^{-1})_{x_i} z_{x_i} + g^{-1} z_{x_i x_i} \right],$$

Deriving two times with respect x_i in (11) we have

$$(14) (g^{-1}(u))_{x_i} = -g^{-2}g'u_{x_i}$$

(15)
$$(g^{-1}(u))_{x_i x_i} = \left(\frac{2g^{'2} - gg^{''}}{g_4}\right) g u_{x_i}^2 - \frac{g^{'}}{g^2} u_{x_i x_i},$$

then,

$$\Delta w = \left(\frac{2g^{'2} - gg^{''}}{g^4}\right) g \sum_{i=1}^N u_{x_i}^2 z - \frac{g^{'}}{g^2} \sum_{i=1}^N u_{x_i x_i} z - 2g^{-2} g^{'} \sum_{i=1}^N u_{x_i} z_{x_i} + g^{-1} \sum_{i=1}^N z_{x_i x_i}$$

$$= g^{-1} \sum_{i=1}^N z_{x_i x_i} - 2g^{-2} g^{'} \sum_{i=1}^N u_{x_i} z_{x_i} + 2\left(\frac{2g^{'2} - gg^{''}}{g^4}\right) gwz - \frac{g^{'}}{g^2} z \sum_{i=1}^N u_{x_i x_i}$$

$$\Delta w = g^{-1} \Delta z - 2g^{-2} g^{'} \sum_{i=1}^N u_{x_i} z_{x_i} + 2\left(\frac{2g^{'2} - gg^{''}}{g^4}\right) z^2 - \frac{g^{'}}{g^2} z \Delta u.$$

From (10), (11), (12), (16), we obtain

(17)
$$z_{t} = u\Delta z - (2g^{-1}ug' + 2\gamma) \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + (2f_{u} + g'g^{-1}f(t, u))z + \left(\frac{4ug'^{2}}{g^{3}} - \frac{2ug''}{g^{2}} + \frac{2\gamma g'}{g^{2}}\right) z^{2} + 2z\Delta u - ug(u) \sum_{i,j=1}^{N} u_{x_{i}x_{j}}^{2}.$$

By choosing $g(u) = u^{\alpha}$, and since

(18)
$$\sum_{i,j=1}^{N} u_{x_i x_j}^2 \ge \frac{1}{N} (\Delta u)^2,$$

replacing g in (17),(18) we have

(19)
$$z_{t} \leq u\Delta z - 2(\alpha + \gamma) \sum_{i=1}^{N} u_{x_{i}} z_{x_{i}} + (2f_{u} + \alpha u^{-1} f(t, u)) z + 2\alpha(\alpha + 1 + \gamma) u^{-\alpha - 1} z^{2} + 2z\Delta u - \frac{u^{\alpha + 1}}{N} (\Delta u)^{2}.$$

For $\gamma \geq \sqrt{2N} - 1$, if α satisfies

(20)
$$\alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \le 0,$$

where $\alpha^2 + (\gamma + 1)\alpha \le -\frac{N}{2}$, then,

(21)
$$2\alpha(\alpha + \gamma + 1)u^{-\alpha - 1}z^2 + 2z\Delta u - \frac{u^{\alpha + 1}}{N}(\Delta u)^2 \le 0.$$

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Therefore from (19) and (21) we have

(22)
$$z_t \le u\Delta z - 2(\alpha + \gamma) \sum_{i=1}^N u_{x_i} z_{x_i} + (2f_u + \alpha u^{-1} f(t, u)) z.$$

By an application of the maximum principle in (22) we have

$$|z|_{\infty} \leq |z_0|_{\infty}$$
.

Now, from (8), (11), with $g(u) = u^{\alpha}$, since the initial data (2) satisfies

$$|\nabla (u_0^{1+\frac{\alpha}{2}})| \le M,$$

with M a positive constant and α satisfies (20), we have

$$|\nabla(u^{1+\frac{\alpha}{2}})|^{2} = \left|\sum_{i=1}^{N} (u^{1+\frac{\alpha}{2}})_{x_{i}} e_{i}\right|^{2}$$

$$= \sum_{i=1}^{N} \left[(u^{1+\frac{\alpha}{2}})_{x_{i}} \right]^{2}$$

$$= \sum_{i=1}^{N} \left[\left(1 + \frac{\alpha}{2} \right) u^{\frac{\alpha}{2}} u_{x_{i}} \right]^{2}$$

$$= \left(1 + \frac{\alpha}{2} \right)^{2} u^{\alpha} \sum_{i=1}^{N} u_{x_{i}}^{2}$$

$$= 2 \left(1 + \frac{\alpha}{2} \right)^{2} u^{\alpha} w$$

$$= 2 \left(1 + \frac{\alpha}{2} \right)^{2} z,$$

therefore

4. HÖLDER CONTINUITY OF u(x,t)

 $|\nabla (u^{1+\frac{\alpha}{2}})| \le M.$

Now, using Theorem 3.1, we have the following corollary about the regularity of the viscosity solution u(x,t) to the Cauchy problem (1),(2).

Corollary 4.1. Let f be a continuous fuctions such that

$$|f(t,w)| \leq k|w|^m$$
,

where w is a real value function and m, k non-negative constants. Under conditions of the Theorem 3.1 the viscosity solution u(x,t) of the Cauchy problem (1), (2) is Lipschitz continuous with respect to x and locally Hölder continuous with exponent $\frac{1}{2}$ with respect to t in $\overline{\Omega}$.

Proof. From Theorem 3.1 there exists $\alpha \in \mathbb{R}$ with $\alpha^2 + (\gamma + 1)\alpha + \frac{N}{2} \leq 0$, with $\alpha < 0$, or,

$$-\frac{\sqrt{(\gamma+1)^2-2N}}{2} - \frac{\gamma+1}{2} \leq \alpha \leq -\frac{\gamma+1}{2} + \frac{\sqrt{(\gamma+1)^2-2N}}{2} < 0.$$

Since $\alpha < 0$, taking $\alpha \neq -2$, we have the estimate,

$$|\nabla(u^{1+\frac{\alpha}{2}})| = \left| (1+\frac{\alpha}{2})u^{\frac{\alpha}{2}}\nabla u \right|$$
$$= \left| 1+\frac{\alpha}{2} |u^{\frac{\alpha}{2}}|\nabla u| \le M.$$

Now, as $u \ge 0$, we have that

(23)
$$|\nabla u| \le \left| 1 + \frac{\alpha}{2} \right|^{-1} u^{-\frac{\alpha}{2}} M \le M_1 \text{ in } \overline{\Omega},$$

since u is bounded.

Using the value mean theorem we have

(24)
$$u(x_1,t) - u(x_2,t) = \nabla u(x_1 + \theta(x_2 - x_1), t) \cdot (x_1 - x_2),$$

for any $\theta \in (0,1)$. From (23), (24) we have,

$$|u(x_1,t)-u(x_2,t)| \le |\nabla u(x_1+\theta(x_2-x_1),t)||x_1-x_2||$$

$$\leq M_1|x_1-x_2|, \quad \forall (x_1,t), (x_2,t) \in \Omega.$$

Therefore u(x,t) is a Lipschitz continuous with respect to the spatial variable.

For Hölder continuity of u(x,t) with respect to the temporary variable, we are going to use the ideas developed in [5]. Let $u_{\epsilon}(x,t) \in C^{2.1}(\Omega) \cap C(\overline{\Omega}) \cap L^{\infty}(\Omega)$ the classical solution to the Cauchy problem problem (1), (2), namely,

$$\begin{cases} u_t = u\Delta u - \gamma |\nabla u|^2 + f(t, u) & \text{in } \Omega \\ u(x, 0) = u_0(x) + \epsilon & \text{on } \mathbb{R}^N, \end{cases}$$

We have that

$$\left| \nabla (u_0 + \epsilon)^{1 + \frac{\alpha}{2}} \right| = \left| \left(1 + \frac{\alpha}{2} \right) (u_0 + \epsilon)^{\frac{\alpha}{2}} \nabla u_0 \right|$$

$$\leq \left| 1 + \frac{\alpha}{2} \right| (u_0)^{\frac{\alpha}{2}} |\nabla u_0|$$

$$= \left| \nabla \left(u_0^{1 + \frac{\alpha}{2}} \right) \right|.$$

$$\leq M,$$

Then, the conditions of Theorem 3.1 holds. Thereby

$$\left|\nabla (u_0 + \epsilon)^{1 + \frac{\alpha}{2}}\right| \le M.$$

Since u_{ϵ} is a classical solution, u is also a weak solution of the Cauchy problem (6), (7). Hence, using the same arguments in the proof of Theorem 3.1, we have that u_{ϵ} is a Lipschitz continuous with respect to the spatial variable, with constant M, namely

$$(25) |u_{\epsilon}(x_1,t) - u_{\epsilon}(x_2,t)| \le M|x_1 - x_2| \ \forall \ (x_1,t), (x_2,t) \in \Omega.$$

Now, let $z = u_{\epsilon}$ be, then we have,

$$z_t = u_{\epsilon_t} = u_{\epsilon} \Delta u_{\epsilon} - \gamma |\nabla u_{\epsilon}|^2 + f(t, u_{\epsilon})$$

or,

(26)
$$u_{\epsilon} \Delta z - z_{t} = \gamma |\nabla u_{\epsilon}|^{2} - f(t, u_{\epsilon}) \text{ in } \Omega.$$

Using (26) we have that for all T > 0, R > 0, z satisfies the equation

(27)
$$u_{\epsilon} \Delta z - z_t = \gamma |\nabla u_{\epsilon}|^2 - f(t, u_{\epsilon}) \text{ in } B_{2R}(0) \times (0, T],$$

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where $B_{2R}(0)$ is the open ball centered in 0, with radius 2R in \mathbb{R}^N . Noticing that $u_{\epsilon} \in C^{2.1}(B_{2R}(0)) \times (0,T]$.

Now, since u_{ϵ} and ∇u_{ϵ} are bounded in $\overline{B_{2R}(0)} \times (0,T]$, there exists a constant $\mu > 0$, such that

$$\sum_{i=1}^{N} u_{\epsilon}(x,t) = Nu_{\epsilon}(x,t) \le \mu,$$

$$\gamma |\nabla u_{\epsilon}(x,t)| \le \mu, \qquad \forall (x,t) \in B_{2R}(0) \times (0,T],$$

and

$$f(t, u_{\epsilon}) \leq \mu$$
.

From (25), we have also

$$|z(x_1,t)-z(x_2,t)| \le M|x_1-x_2| \qquad \forall (x,t) \in B_{2R}(0)) \times (0,T].$$

In according with [5], there exists a positive constant δ (which depends only of μ and R) and a positive constant K, which depends only of μ , R and M, such that

$$|z(x,t)-z(x,t_0)| \le K|t-t_0|^{\frac{1}{2}},$$

for all $(x,t), (x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$.

That is,

$$|u_{\epsilon}(x,t) - u_{\epsilon}(x,t_0)| \le K|t - t_0|^{\frac{1}{2}},$$

for all $(x,t), (x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$.

Whenever K is independent of ϵ , taken $\epsilon \searrow 0$, we obtain

$$|u(x,t) - u(x,t_0)| \le K|t - t_0|^{\frac{1}{2}},$$

for all $(x,t), (x,t_0) \in B_R(0) \times (0,T]$ with $|t-t_0| < \delta$.

References

- 1. Emmanuele DiBenedetto, Degenerate parabolic equations, Springer-Verlag, New York, Heidelberg, Berlin, 1993.
- 2. Lawrence C. Evans, Partial differential equations, American Mathematical Society, Graduate Studies In Mathematics. Rhode Island,, 1998.
- 3. Avner Friedman, Partial differential equations of parabolic type, Englewood Cliffs, N.J., Prentice-Hall Inc, 1964.
- 4. John Fritz, Differential equations, Springer-Verlag, New York, Heidelberg, Berlin, 1978.
- 5. B.H. Gilding, Hölder continuity of solutions of parabolic equations, J. Landon Math. Soc. 13, 103-106, 1976.
- 6. S. Kesavan, Topics in functional analysis and applications, John Wiley & Sons. New York, 1989.
- O.A. Ladysenskaya, V.A. Solonnikov, and Ural´ceva N.N, Linear and quasilinear equations of parabolic type, Amer.Math.Soc. Transl, 1968.
- 8. Yun-Guang Lu, Hölder estimates of solutions to some doubly nonlinear degenerate parabolic equations, Comm. Partial Differential Equations 24, no. 5-6, 895–913.5656, 1999.
- Yun Guang Lu and Liwen Qian, Regularity of viscosity solutions of a degenerate parabolic equation, American Mathematical Society, volume 130, number 4. Pages 999-1004, 2001.
- 10. Pierre-Louis Lions Michael G. Crandall, Viscosity solutions of hamilton-jacobi equations, Transactions of the American Mathematical Society (1983).
- Maura Ughi Michiel Bertsch, Roberta Dal Passo, Discontinuous viscosity solutions of a degenerate parabolic equation, Trans Amer. Math. Soc. 320, no. 2, 779-798, 1990.

- 12. A. Mikelic M.S Espedal, A. Fasano, Filtration in porous media and industrial applications, Springer-Verlag, New York, Heidelberg, Berlin, 2000.
- 13. Murray H. Protter and Hans F.W. Weinberger, Maximum principles in differential equations, Springer-Verlag, New York, Heidelberg, Berlin, 1984.
- 14. Liwen Quian and Wentao Fan, Hölder estimate of solutions of some degenerate parabolic equations, Acta Math. Sci. (English Ed.) 19, no. 4, 463–468, 1999.
- 15. Juan Luis Vazquéz, The porous medium equation, mathematical theory, Oxford Science Publications, 2007.