A Source-Channel Separation Theorem with Application to the Source Broadcast Problem

Kia Khezeli and Jun Chen

Abstract—A converse method is developed for the source broadcast problem. Specifically, it is shown that the separation architecture is optimal for a variant of the source broadcast problem and the associated source-channel separation theorem can be leveraged, via a reduction argument, to establish a necessary condition for the original problem, which unifies several existing results in the literature. Somewhat surprisingly, this method, albeit based on the source-channel separation theorem, can be used to prove the optimality of non-separation based schemes and determine the performance limits in certain scenarios where the separation architecture is suboptimal.

Index Terms—Bandwidth mismatch, broadcast channel, capacity region, joint source-channel coding, separation theorem, side information.

I. INTRODUCTION

In the source broadcast problem, a source is sent over a broadcast channel through suitable encoding and decoding so that the reconstructions at the receivers satisfy the prescribed constraints. The special case of sending a Gaussian source over a Gaussian broadcast channel has received particular attention. For this special case, it is known that source-channel separation is in general suboptimal [1] and hybrid digital-analog coding schemes can outperform pure digital/analog schemes [2]–[5]. The extension of the hybrid coding architecture to the non-Gaussian setting can be found in [6].

In contrast, the progress on the converse side is still somewhat limited. To the best of our knowledge, the first nontrivial result in this direction was obtained by Reznic *et al.* [3] for the scalar version of the aforementioned Gaussian case. The converse argument in [3] involves an auxiliary random variable, which is generated by the source via an additive Gaussian noise channel. This auxiliary random variable is constructed in exactly the same manner as the one in Ozarow's celebrated work on the Gaussian multiple description problem [7]. However, this resemblance is, in a certain sense, rather superficial. Indeed, on a more technical level, the auxiliary random variable introduced by Ozarow (as elucidated in [8]–[11]) plays the role of exploiting an implicit conditional independence structure whereas the role of the auxiliary random

variable in [3] is apparently different and still largely elusive. Recent years have seen several new converse results [12]–[14] for the source broadcast problem. These results are based on arguments similar to the original one by Reznic *et al.*, especially in terms of the way the auxiliary random variables are constructed and exploited. It is worth noting that such arguments can only handle a restricted class of auxiliary random variables (essentially those that can be generated by the source via certain additive noise channels); this restriction typically leads to certain constraints on the set of sources, channels, or distortion measures that can be analyzed.

The present paper is, to a certain extent, an outcome of our effort in seeking a conceptual understanding of the converse argument by Reznic *et al.* in general and the role of the associated auxiliary random variable in particular. We shall show that one can establish a source-channel separation theorem for a variant of the source broadcast problem and leverage it to derive a necessary condition for the original problem. This necessary condition, when specialized to the case of sending a scalar Gaussian source over a Gaussian broadcast channel, recovers the corresponding result by Reznic *et al.* [3]; moreover, in this way, the converse argument in [3] finds a simple interpretation, and the associated auxiliary random variable acquires an operational meaning. It should be pointed out that, in our approach, the auxiliary random variable can be generated by the source in an arbitrary manner. Therefore, the restriction imposed in the existing arguments [12]–[14] is in fact unnecessary. On the other hand, the problem of identifying the optimal auxiliary random variable naturally arises due to this additional freedom. It will be seen that the analytical solutions for this problem can be found in some special cases; interestingly, these solutions indicate that the specific choices of auxiliary random variables in [3], [13] are actually optimal in their respective contexts.

Our work is also partly motivated by the problem of sending a bivariate Gaussian source over a Gaussian broadcast channel first studied by Bross *et al.* [15]. For this problem, it is known that the achievable distortion region of a certain hybrid digital-analog coding scheme [16] matches the outer bound in [15] whereas separate source-channel coding is in general suboptimal [16], [17]. An alternative proof of the outer bound in [15] was recently obtained by Song *et al.* [18]. This new proof [18] bears some similarity to the aforementioned converse argument by Reznic *et al.* [3]. We will clarify their connection by giving a unified proof for the vector Gaussian case, which implies, among other things, that the outer bound in [15] can be deduced from the general necessary condition for the source broadcast problem found in the present paper. Therefore, our converse method, albeit based on the

K. Khezeli was with the Department of Electrical and Computer Engineering, McMaster University, Hamilton, ON L8S 4K1, Canada. He is now with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14853, USA (email: kk839@cornell.edu).

J. Chen is with the Department of Electrical and Computer Engineering, McMaster University, Hamilton, ON L8S 4K1, Canada (email: junchen@ece.mcmaster.ca).
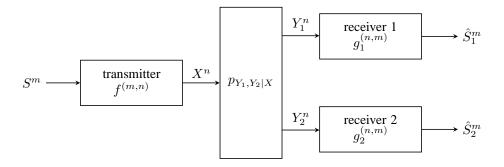
Fig. 1. System Π

source-channel separation theorem, can be used to prove the optimality of non-separation based schemes and determine the performance limits in certain scenarios where the separation architecture is suboptimal.

The rest of this paper is organized as follows. We present the problem setup in Section II and the relevant capacity results for broadcast channels with receiver side information in Section III. We establish a source-channel separation theorem for a variant of the source broadcast problem in Section IV. It is shown in Section V that this separation theorem can be used in conjunction with a simple reduction argument to derive a necessary condition for the original source broadcast problem; moreover, this necessary condition is evaluated for the special case of the binary uniform source with the Hamming distortion measure. The quadratic Gaussian case is treated in Section VI. We conclude the paper in Section VII.

Throughout this paper, the binary entropy function and its inverse are denoted by $H_b(\cdot)$ and $H_b^{-1}(\cdot)$, respectively. For any $a, b \in [0,1]$, we define $a * b = a(1-b) + (1-a)b$. The logarithm function is assumed to be base 2 unless specified otherwise.

II. Problem Setup

The source broadcast system (System Π) consists of the following components (see Fig. 1):

- an i.i.d source $\{S(t)\}_{t=1}^{\infty}$ with marginal distribution p_S over alphabet $\mathcal{S}$,
- a discrete memoryless broadcast channel $p_{Y_1, Y_2|X}$ with input alphabet $\mathcal{X}$ and output alphabets $\mathcal{Y}_i$, $i = 1, 2$,
- a transmitter, which is equipped with an encoding function $f^{(m,n)} : \mathcal{S}^m \to \mathcal{X}^n$ that maps a block of source samples $S^m \triangleq (S(1), \cdots, S(m))$ of length m to a channel input block $X^n \triangleq (X(1), \cdots, X(n))$ of length n (the number of channel uses per source sample, i.e., $\frac{n}{m}$, is referred to as the bandwidth expansion ratio),
- two receivers, where receiver i is equipped with a decoding function $g_i^{(n,m)} : \mathcal{Y}_i^n \to \hat{\mathcal{S}}_i^m$ that maps the channel output block $Y_i^n \triangleq (Y_i(1), \cdots, Y_i(n))$ generated by X^n to a source reconstruction block $\hat{S}_i^m \triangleq (\hat{S}_i(1), \cdots, \hat{S}_i(m))$, $i = 1, 2$.

Unless stated otherwise, we assume that $\mathcal{S}$, $\hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_2$, $\mathcal{X}$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$ are finite sets.

Let $\mathcal{P}_{\mathcal{S} \times \hat{\mathcal{S}}_i}(p_S)$ denote the set of joint distributions over $\mathcal{S} \times \hat{\mathcal{S}}_i$ with the marginal distribution on $\mathcal{S}$ fixed to be p_S, $i = 1, 2$.

Definition 1: Let κ be a non-negative number and $\mathcal{Q}_i$ be a non-empty compact subset of $\mathcal{P}_{\mathcal{S} \times \hat{\mathcal{S}}_i}(p_S)$, $i = 1, 2$. We say $(\kappa, \mathcal{Q}_1, \mathcal{Q}_2)$ is achievable for System Π if, for every $\epsilon > 0$, there exist encoding function $f^{(m,n)} : \mathcal{S}^m \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \hat{\mathcal{S}}_i^m$, $i = 1, 2$, such that

$$\frac{n}{m} \leq \kappa + \epsilon, \tag{1}$$

$$\min_{q_i \in \mathcal{Q}_i} \left\| \frac{1}{m} \sum_{t=1}^{m} p_{S(t), \hat{S}_i(t)} - q_i \right\| \leq \epsilon, \quad i = 1, 2, \tag{2}$$

where $\|\cdot\|$ is the 1-norm. The set of all achievable $(\kappa, \mathcal{Q}_1, \mathcal{Q}_2)$ for System Π is denoted by Γ.

Remark: It is easy to verify that

$$\frac{1}{m} \sum_{t=1}^{m} p_{S(t), \hat{S}_i(t)} \in \mathcal{P}_{\mathcal{S} \times \hat{\mathcal{S}}_i}(p_S), \quad i = 1, 2.$$

Now consider the following more conventional definition.

Definition 2: Let $w_i : \mathcal{S} \times \hat{\mathcal{S}}_i \to [0, \infty)$ be two distortion measures. For non-negative numbers κ, d_1, and d_2, we say (κ, d_1, d_2) is achievable for System Π under distortion measures w_1 and w_2 if, for every $\epsilon > 0$, there exist encoding function $f^{(m,n)} : \mathcal{S}^m \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \hat{\mathcal{S}}_i^m$, $i = 1, 2$, such that

$$\frac{n}{m} \leq \kappa + \epsilon,$$

$$\frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[w_i(S(t), \hat{S}_i(t))] \leq d_i + \epsilon, \quad i = 1, 2. \tag{3}$$

The following result shows that Definition 1 is more general than Definition 2.

Proposition 1: (κ, d_1, d_2) is achievable for System Π under distortion measures w_1 and w_2 if and only if $(\kappa, \mathcal{Q}(w_1, d_1), \mathcal{Q}(w_2, d_2)) \in \Gamma$, where $\mathcal{Q}(w_i, d_i) = \{p_{S, \hat{S}_i} \in \mathcal{P}_{\mathcal{S} \times \hat{\mathcal{S}}_i}(p_S) : \mathbb{E}[w_i(S, \hat{S}_i)] \leq d_i\}$, $i = 1, 2$.

Proof: Let T be a random variable independent of $(S^m, \hat{S}_1^m, \hat{S}_2^m)$ and uniformly distributed over $\{1, \cdots, m\}$. It is easy to verify that (2) can be written equivalently as

$$\min_{q_i \in \mathcal{Q}_i} \left\| p_{S(T), \hat{S}_i(T)} - q_i \right\| \leq \epsilon, \quad i = 1, 2,$$

and (3) can be written equivalently as

$$\mathbb{E}[w_i(S(T), \hat{S}_i(T))] \leq d_i + \epsilon, \quad i = 1, 2.$$
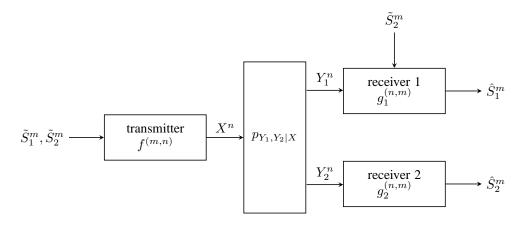
Fig. 2.   System $\tilde{\Pi}$

Note that

$$
\mathbb{E}[w_i(S(T), \hat{S}_i(T))]
$$

$$
= \sum_{s \in \mathcal{S}, \hat{s}_i \in \hat{\mathcal{S}}_i} p_{S(T), \hat{S}_i(T)}(s, \hat{s}_i) w_i(s, \hat{s}_i)
$$

$$
\leq \sum_{s \in \mathcal{S}, \hat{s}_i \in \hat{\mathcal{S}}_i} q_i(s, \hat{s}_i) w_i(s, \hat{s}_i)
$$

$$
+ \sum_{s \in \mathcal{S}, \hat{s}_i \in \hat{\mathcal{S}}_i} |p_{S(T), \hat{S}_i(T)}(s, \hat{s}_i) - q_i(s, \hat{s}_i)| w_i(s, \hat{s}_i)
$$

$$
\leq d_i + \|p_{S(T), \hat{S}_i(T)} - q_i\| \max_{s \in \mathcal{S}, \hat{s}_i \in \hat{\mathcal{S}}_i} w_i(s, \hat{s}_i)
$$

for any $q_i \in \mathcal{Q}_i(w_i, d_i)$, $i = 1, 2$. Therefore, we have

$$
\mathbb{E}[w_i(S(T), \hat{S}_i(T))]
$$
$$
\leq d_i + \min_{q_i \in \mathcal{Q}_i(w_i, d_i)} \|p_{S(T), \hat{S}_i(T)} - q_i\| \max_{s \in \mathcal{S}, \hat{s}_i \in \hat{\mathcal{S}}_i} w_i(s, \hat{s}_i),
$$
$$
i = 1, 2,
$$

from which the "if" part follows immediately.

Now we proceed to prove the "only if" part. Assume that (κ, d_1, d_2) is achievable for System Π under distortion measures w_1 and w_2. For every $\epsilon > 0$, according to Definition 2, we can find encoding function $f^{(m,n)} : \mathcal{S}^m \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \hat{\mathcal{S}}_i^m$, $i = 1, 2$, satisfying $\frac{n}{m} \leq \kappa + \epsilon$ and $\mathbb{E}[w_i(S(T), \hat{S}_i(T))] \leq d_i + \epsilon$, $i = 1, 2$. We shall denote $S(T)$ simply by S since the distribution of $S(T)$ is p_S, and denote $\hat{S}_1$ and $\hat{S}_2$ by $\hat{S}_1^{(\epsilon)}$ and $\hat{S}_2^{(\epsilon)}$, respectively, to stress their dependence on ϵ. Note that $\{p_{S, \hat{S}_1^{(\epsilon)}, \hat{S}_2^{(\epsilon)}} : \epsilon > 0\}$ is contained in a compact set and $\mathbb{E}[w_i(S, \hat{S}_i^{(\epsilon)})] \leq d_i + \epsilon$ for every $\epsilon > 0$, $i = 1, 2$. Therefore, one can find a sequence $\epsilon_1, \epsilon_2, \cdots$ converging to zero such that

$$
\lim_{k \to \infty} p_{S, \hat{S}_1^{(\epsilon_k)}, \hat{S}_2^{(\epsilon_k)}} = p_{S, \hat{S}_1, \hat{S}_2}
$$

for some $p_{S, \hat{S}_1, \hat{S}_2}$ with $p_{S, \hat{S}_i} \in \mathcal{Q}_i(w_i, d_i)$, $i = 1, 2$. This completes the proof of the "only if" part. ∎

Source-channel separation is known to incur a performance loss for System Π in general. However, it turns out that, for the following variant of System Π (see Fig. 2), separate source-channel coding is in fact optimal. This system (System $\tilde{\Pi}$) is the same as System Π except for two differences.

1) The source is an i.i.d. vector process $\{(\tilde{S}_1(t), \tilde{S}_2(t))\}_{t=1}^{\infty}$ with marginal distribution $p_{\tilde{S}_1, \tilde{S}_2}$ over finite alphabet $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$.
2) $\tilde{S}_2^m$ is available at receiver 1 and can be used together with Y_1^n to construct $\hat{S}_1^m$.

Let $\mathcal{P}_{\tilde{S}_1 \times \tilde{S}_2 \times \hat{S}_1}(p_{\tilde{S}_1, \tilde{S}_2})$ denote the set of joint distributions over $\tilde{S}_1 \times \tilde{S}_2 \times \hat{S}_1$ with the marginal distribution on $\tilde{S}_1 \times \tilde{S}_2$ fixed to be $p_{\tilde{S}_1, \tilde{S}_2}$. Moreover, let $\mathcal{P}_{\tilde{S}_2 \times \hat{S}_2}(p_{\tilde{S}_2})$ denote the set of joint distributions over $\tilde{S}_2 \times \hat{S}_2$ with the marginal distribution on $\tilde{S}_2$ fixed to be $p_{\tilde{S}_2}$.

Definition 3: Let $\tilde{\kappa}$ be a non-negative number, $\tilde{\mathcal{Q}}_1$ be a non-empty compact subset of $\mathcal{P}_{\tilde{S}_1 \times \tilde{S}_2 \times \hat{S}_1}(p_{\tilde{S}_1, \tilde{S}_2})$, and $\tilde{\mathcal{Q}}_2$ be a non-empty compact subset of $\mathcal{P}_{\tilde{S}_2 \times \hat{S}_2}(p_{\tilde{S}_2})$. We say $(\tilde{\kappa}, \tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2)$ is achievable for System $\tilde{\Pi}$ if, for every $\epsilon > 0$, there exist encoding function $f^{(m,n)} : \tilde{\mathcal{S}}_1^m \times \tilde{\mathcal{S}}_2^m \to \mathcal{X}^n$ as well as decoding functions $g_1^{(n,m)} : \mathcal{Y}_1^n \times \tilde{\mathcal{S}}_2^m \to \hat{\mathcal{S}}_1^m$ and $g_2^{(n,m)} : \mathcal{Y}_2^n \to \hat{\mathcal{S}}_2^m$ such that

$$
\frac{n}{m} \leq \tilde{\kappa} + \epsilon, \tag{4}
$$

$$
\min_{\tilde{q}_1 \in \tilde{\mathcal{Q}}_1} \left\| \frac{1}{m} \sum_{t=1}^{m} p_{\tilde{S}_1(t), \tilde{S}_2(t), \hat{S}_1(t)} - \tilde{q}_1 \right\| \leq \epsilon, \tag{5}
$$

$$
\min_{\tilde{q}_2 \in \tilde{\mathcal{Q}}_2} \left\| \frac{1}{m} \sum_{t=1}^{m} p_{\tilde{S}_2(t), \hat{S}_2(t)} - \tilde{q}_2 \right\| \leq \epsilon. \tag{6}
$$

The set of all achievable $(\tilde{\kappa}, \tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2)$ for System $\tilde{\Pi}$ is denoted by $\tilde{\Gamma}$.

Remark: For the ease of subsequent applications, here we allow $f^{(m,n)}$, $g_1^{(n,m)}$, and $g_2^{(n,m)}$ to be non-deterministic functions as long as the Markov chains $(\tilde{S}_1^m, \tilde{S}_2^m) \leftrightarrow X^n \leftrightarrow (Y_1^n, Y_2^n)$, $\tilde{S}_1^m \leftrightarrow (Y_1^n, \tilde{S}_2^m) \leftrightarrow \hat{S}_1^m$, and $\tilde{S}_2^m \leftrightarrow Y_2^n \leftrightarrow \hat{S}_2^m$ are preserved. It will be clear that such a relaxation does not affect $\tilde{\Gamma}$.

To discuss source-channel separation for System $\tilde{\Pi}$, we need to specify the source coding component and the channel coding component. It will be seen that the source coding part is the conventional lossy source coding scheme. The channel coding part is more involved and is described in the next section.
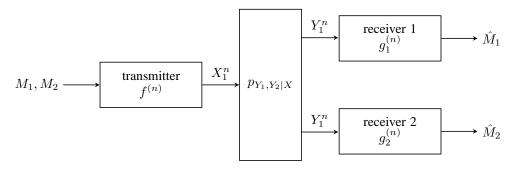
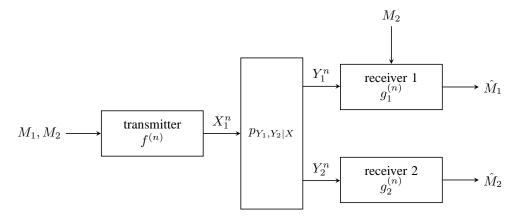Fig. 3. Broadcast channel with two private messages

Fig. 4. Broadcast channel with receiver side information

III. BROADCAST CHANNELS WITH RECEIVER SIDE INFORMATION

A. Definitions

Let $p_{Y_1,Y_2|X}$ be a discrete memoryless broadcast channel with input alphabet $\mathcal{X}$ and output alphabets $\mathcal{Y}_i$, $i = 1, 2$. A length-n coding scheme (see Fig. 3) for $p_{Y_1,Y_2|X}$ consists of

- two private messages M_1 and M_2, where (M_1, M_2) is uniformly distributed over $\mathcal{M}_1 \times \mathcal{M}_2$,
- an encoding function $f^{(n)} : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ that maps (M_1, M_2) to a channel input block X^n,
- two decoding functions $g_i^{(n)} : \mathcal{Y}_i^n \to \mathcal{M}_i$, $i = 1, 2$, where $g_i^{(n)}$ maps the channel output block at receiver i, i.e., Y_i^n, to $\hat{M}_i$, $i = 1, 2$.

Definition 4: A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable for broadcast channel $p_{Y_1,Y_2|X}$ if there exists a sequence of encoding functions $f^{(n)} : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ with $\frac{1}{n} \log |\mathcal{M}_i| \geq R_i$, $i = 1, 2$, and decoding functions $g_i^{(n)} : \mathcal{Y}_i^n \to \mathcal{M}_i$, $i = 1, 2$, such that

$$\lim_{n \to \infty} \Pr\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\} = 0.$$

The private-message capacity region $\mathcal{C}(p_{Y_1,Y_2|X})$ is the closure of the set of all achievable (R_1, R_2) for broadcast channel $p_{Y_1,Y_2|X}$.

A computable characterization of $\mathcal{C}(p_{Y_1,Y_2|X})$ is still largely unknown. Interestingly, the problem becomes significantly simpler if message M_2 is available at receiver 1 or message M_1 is available at receiver 2; in fact, this is the setting that is most relevant to the present work. Specifically, consider the scenario where two private messages M_1 and M_2 need to be sent over broadcast channel $p_{Y_1,Y_2|X}$ to receiver 1 and receiver 2, respectively, and M_2 is available at receiver 1. In this case, a length-n coding scheme (see Fig. 4) consists of

- two private messages M_i, $i = 1, 2$, where (M_1, M_2) is uniformly distributed over $\mathcal{M}_1 \times \mathcal{M}_2$,
- an encoding function $f^{(n)} : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ that maps (M_1, M_2) to a channel input block X^n,
- two decoding functions $g_1^{(n)} : \mathcal{Y}_1^n \times \mathcal{M}_2 \to \mathcal{M}_1$ and $g_2^{(n)} : \mathcal{Y}_2^n \to \mathcal{M}_2$, where $g_1^{(n)}$ maps (Y_1^n, M_2) to $\hat{M}_1$, and $g_2^{(n)}$ maps Y_2^n to $\hat{M}_2$.

Definition 5: A rate pair (R_1, R_2) is said to be achievable for broadcast channel $p_{Y_1,Y_2|X}$ with message M_2 available at receiver 1 if there exists a sequence of encoding functions $f^{(n)} : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$ with $\frac{1}{n} \log |\mathcal{M}_i| \geq R_i$, $i = 1, 2$, as well as decoding functions $g_1^{(n)} : \mathcal{Y}_1^n \times \mathcal{M}_2 \to \mathcal{M}_1$ and $g_2^{(n)} : \mathcal{Y}_2^n \to \mathcal{M}_2$ such that

$$\lim_{n \to \infty} \Pr\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\} = 0.$$

The capacity region $\mathcal{C}_1(p_{Y_1,Y_2|X})$ is the closure of the set of all such achievable (R_1, R_2). The capacity region $\mathcal{C}_2(p_{Y_1,Y_2|X})$ for broadcast channel $p_{Y_1,Y_2|X}$ with message M_1 available at receiver 2 can be defined in an analogous manner.

B. Capacity Results

It is known [19, Theorem 3] that $\mathcal{C}_1(p_{Y_1,Y_2|X})$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq I(X; Y_1), \tag{7}$$

$$R_2 \le I(V; Y_2), \tag{8}$$
$$R_1 + R_2 \le I(X; Y_1|V) + I(V; Y_2) \tag{9}$$

for some $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$; moreover, it suffices to assume that $|\mathcal{V}| \le |\mathcal{X}|+1$. By symmetry, $\mathcal{C}_2(p_{Y_1,Y_2|X})$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le I(V; Y_1), \tag{10}$$
$$R_2 \le I(X; Y_2), \tag{11}$$
$$R_1 + R_2 \le I(V; Y_1) + I(X; Y_2|V) \tag{12}$$

for some $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$; again, it suffices to assume that $|\mathcal{V}| \le |\mathcal{X}| + 1$.

A class of distributions $\mathcal{P}$ on the input alphabet $\mathcal{X}$ is said to be a sufficient class of distributions [20, Definition 1] for broadcast channel $p_{Y_1,Y_2|X}$ if, for any $p_{V_1,V_2,X,Y_1,Y_2} = p_{V_1,V_2,X}p_{Y_1,Y_2|X}$, there exists $p_{\tilde{V}_1,\tilde{V}_2,\tilde{X},\tilde{Y}_1,\tilde{Y}_2} = p_{\tilde{V}_1,\tilde{V}_2,\tilde{X}}p_{\tilde{Y}_1,\tilde{Y}_2|\tilde{X}}$ with $p_{\tilde{X}} \in \mathcal{P}$ and $p_{\tilde{Y}_1,\tilde{Y}_2|\tilde{X}} = p_{Y_1,Y_2|X}$ such that[1]

$$I(V_1; Y_1) \le I(\tilde{V}_1; \tilde{Y}_1),$$
$$I(V_2; Y_2) \le I(\tilde{V}_2; \tilde{Y}_2),$$
$$I(V_1; Y_1) + I(X; Y_2|V_1) \le I(\tilde{V}_1; \tilde{Y}_1) + I(\tilde{X}; \tilde{Y}_2|\tilde{V}_1),$$
$$I(X; Y_1|V_2) + I(V_2; Y_2) \le I(\tilde{X}; \tilde{Y}_1|\tilde{V}_2) + I(\tilde{V}_2; \tilde{Y}_2).$$

For broadcast channel $p_{Y_1,Y_2|X}$, we say that $p_{Y_1|X}$ is essentially less noisy than $p_{Y_2|X}$ if there exists a sufficient class of distributions $\mathcal{P}$ such that $I(V; Y_1) \ge I(V; Y_2)$ for any $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$ [20, Definition 2], and simply say that $p_{Y_1|X}$ is less noisy than $p_{Y_2|X}$ if $\mathcal{P}$ can be chosen to be the set of all distributions on $\mathcal{X}$; similarly, we say that $p_{Y_1|X}$ is essentially more capable than $p_{Y_2|X}$ if there exists a sufficient class of distributions $\mathcal{P}$ such that $I(X; Y_1|V) \ge I(X; Y_2|V)$ for any $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$ [20, Definition 3], and simply say that $p_{Y_1|X}$ is more capable than $p_{Y_2|X}$ if $\mathcal{P}$ can be chosen to be the set of all distributions on $\mathcal{X}$. It is known that "less noisy" ("more capable") implies "essentially less noisy" ("essentially more capable"), and "less noisy" implies "more capable", but the converses are not true in general.

Proposition 2: If $p_{Y_1|X}$ is essentially less noisy than $p_{Y_2|X}$, then $\mathcal{C}_1(p_{Y_1,Y_2|X}) = \mathcal{C}(p_{Y_1,Y_2|X})$.

Proof: To compute $\mathcal{C}_1(p_{Y_1,Y_2|X})$ defined by (7)-(9), it suffices to consider those p_X in a sufficient class $\mathcal{P}$. It is easy to see that

$$I(X; Y_1|V) + I(V; Y_2) \le I(X; Y_1|V) + I(V; Y_1) \tag{13}$$
$$= I(X; Y_1)$$

for any $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$, where (13) is due to the fact that $p_{Y_1|X}$ is essentially less noisy than $p_{Y_2|X}$. Therefore, (7) is redundant if p_X is restricted to $\mathcal{P}$. Note that the rate region defined by (8) and (9) for $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$ is exactly $\mathcal{C}(p_{Y_1,Y_2|X})$ [20, Theorem 1]. This completes the proof of Proposition 2. ∎

[1]Setting $V_1 = X$, one can readily verify that $I(X; Y_1) = I(V_1; Y_1) \le I(\tilde{V}_1; \tilde{Y}_1) \le I(\tilde{X}; \tilde{Y}_1)$. Similarly, one can obtain $I(X; Y_2) \le I(\tilde{X}; \tilde{Y}_2)$ by setting $V_2 = X$.

Proposition 3: If $p_{Y_1|X}$ is essentially more capable than $p_{Y_2|X}$, then $\mathcal{C}_2(p_{Y_1,Y_2|X})$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_2 \le I(X; Y_2),$$
$$R_1 + R_2 \le I(X; Y_1)$$

for some $p_{X,Y_1,Y_2} = p_X p_{Y_1,Y_2|X}$.

Proof: To compute $\mathcal{C}_2(p_{Y_1,Y_2|X})$ defined by (10)-(12), it suffices to consider those p_X in a sufficiently class $\mathcal{P}$. Note that

$$I(V; Y_1) + I(X; Y_2|V) \le I(V; Y_1) + I(X; Y_1|V) \tag{14}$$
$$= I(X; Y_1)$$

for any $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$, where (14) is due to the fact that $p_{Y_1|X}$ is essentially more capable than $p_{Y_2|X}$. Therefore, given $p_X \in \mathcal{P}$, the right-hand side of inequality (12) attains its maximum value $I(X; Y_1)$ when $V = X$. Clearly, given p_X, the right-hand side of inequality (10) also attains its maximum value $I(X; Y_1)$ when $V = X$. As a consequence, $\mathcal{C}_2(p_{Y_1,Y_2|X})$ can be expressed as the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_2 \le I(X; Y_2),$$
$$R_1 + R_2 \le I(X; Y_1)$$

for some $p_{X,Y_1,Y_2} = p_X p_{Y_1,Y_2|X}$ with $p_X \in \mathcal{P}$. Removing the redundant constraint $p_X \in \mathcal{P}$ completes the proof of Proposition 3. ∎

C. Examples

Consider a broadcast channel $p_{Y_1,Y_2|X}$ with $\mathcal{X} = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$, where $p_{Y_i|X}$ is a binary symmetric channel with crossover probability p_i, $i = 1, 2$; such a channel will be denoted by BS-BC(p_1, p_2). Without loss of generality, we shall assume $0 \le p_1 \le p_2 \le \frac{1}{2}$. It is well known that $\mathcal{C}(\text{BS}(p_1, p_2))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le H_b(\alpha * p_1) - H_b(p_1),$$
$$R_2 \le 1 - H_b(\alpha * p_2)$$

for some $\alpha \in [0, \frac{1}{2}]$. Next consider a broadcast channel $p_{Y_1,Y_2|X}$ with $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y}_i = \{0, 1, e\}$, $i = 1, 2$, where $p_{Y_i|X}$ is a binary erasure channel with erasure probability ϵ_i, $i = 1, 2$; such a channel will be denoted by BE-BC(ϵ_1, ϵ_2). Without loss of generality, we shall assume $0 \le \epsilon_1 \le \epsilon_2 \le 1$. It is well known that $\mathcal{C}(\text{BE-BC}(\epsilon_1, \epsilon_2))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le \beta(1 - \epsilon_1), \tag{15}$$
$$R_2 \le (1 - \beta)(1 - \epsilon_2) \tag{16}$$

for some $\beta \in [0, 1]$.

The following results are simple consequences of Proposition 2 and Proposition 3.

Proposition 4: For BS-BC(p_1, p_2) with $0 \le p_1 \le p_2 \le \frac{1}{2}$,

$$\mathcal{C}_1(\text{BS-BC}(p_1, p_2)) = \mathcal{C}(\text{BS-BC}(p_1, p_2)),$$
$$\mathcal{C}_2(\text{BS-BC}(p_1, p_2)) = \{(R_1, R_2) \in \mathbb{R}_+^2 : R_2 \le 1 - H_b(p_2),$$
$$R_1 + R_2 \le 1 - H_b(p_1)\}.$$

Proposition 5: For BE-BC(ϵ_1, ϵ_2) with $0 \leq \epsilon_1 \leq \epsilon_2 \leq 1$,

$$\mathcal{C}_1(\text{BE-BC}(\epsilon_1, \epsilon_2)) = \mathcal{C}(\text{BE-BC}(\epsilon_1, \epsilon_2)),$$
$$\mathcal{C}_2(\text{BE-BC}(\epsilon_1, \epsilon_2)) = \{(R_1, R_2) \in \mathbb{R}_+^2 : R_2 \leq 1 - \epsilon_2,$$
$$R_1 + R_2 \leq 1 - \epsilon_1\}.$$

Now consider a broadcast channel $p_{Y_1, Y_2 | X}$ with $\mathcal{X} = \mathcal{Y}_1 = \{0, 1\}$ and $\mathcal{Y}_2 = \{0, 1, e\}$, where $p_{Y_1 | X}$ is a binary symmetric channel with crossover probability p, and $p_{Y_2 | X}$ is a binary erasure channel with erasure probability ϵ; such a channel will be denoted by BSC(p)&BEC(ϵ). Without loss of generality, we shall assume $p \in [0, \frac{1}{2}]$ and $\epsilon \in [0, 1]$. One can obtain the following explicit characterization of $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ [20, Theorem 4].

1) $\epsilon \in [0, 4p(1 - p)]$: $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq (1 - \epsilon) H_b(\alpha)$$

for some $\alpha \in [0, \frac{1}{2}]$.

2) $\epsilon \in (4p(1 - p), H_b(p))$: $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq (1 - \epsilon) H_b(\alpha)$$

for some $\alpha \in [0, \hat{\alpha}]$, or

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq H_b(\alpha * p) - \epsilon$$

for some $\alpha \in (\hat{\alpha}, \frac{1}{2}]$, where $\hat{\alpha}$ is the unique number in $(0, \frac{1}{2})$ satisfying

$$1 - H_b(\hat{\alpha} * p) + (1 - \epsilon) H_b(\hat{\alpha}) = 1 - \epsilon.$$

3) $\epsilon \in [H_b(p), 1]$: $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq \beta[1 - H_b(p)],$$
$$R_2 \leq (1 - \beta)(1 - \epsilon)$$

for some $\beta \in [0, 1]$.

Proposition 6: $\mathcal{C}_1(\text{BSC}(p)\&\text{BEC}(\epsilon))$ has the following explicit characterization.

1) $\epsilon \in [0, H_b(p)]$:

$$\mathcal{C}_1(\text{BSC}(p)\&\text{BEC}(\epsilon)) = \{(R_1, R_2) \in \mathbb{R}_+^2 :$$
$$R_1 \leq 1 - H_b(p), R_1 + R_2 \leq 1 - \epsilon\}.$$

2) $\epsilon \in (H_b(p), 1]$:

$$\mathcal{C}_1(\text{BSC}(p)\&\text{BEC}(\epsilon)) = \mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon)).$$

Proof: According to [20, Theorem 3], BEC(ϵ) is more capable than BSC(p) when $\epsilon \in [0, H_b(p)]$. Therefore, one can readily prove Part 1) by invoking Proposition 3 as well as the fact that $I(X; Y_1)$ and $I(X; Y_2)$ are simultaneously maximized when $p_X(0) = p_X(1) = \frac{1}{2}$. Part 2) follows from Proposition 2 and the fact that BSC(p) is essentially less noisy than BEC(ϵ) when $\epsilon \in (H_b(p), 1]$ [20, Theorem 3]. ∎

Proposition 7: $\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon))$ has the following explicit characterization.

1) $\epsilon \in [0, 4p(1 - p)]$:

$$\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon)) = \mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon)).$$

2) $\epsilon \in (4p(1 - p), 1)$ and $p \neq 0$: $\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq (1 - \epsilon) H_b(\alpha)$$

for some $\alpha \in [0, \tilde{\alpha}]$, or

$$R_1 \leq 1 - H_b(\tilde{\alpha} * p),$$
$$R_2 \leq 1 - \epsilon,$$
$$R_1 + R_2 \leq 1 - H_b(\tilde{\alpha} * p) + (1 - \epsilon) H_b(\tilde{\alpha})$$

for some $\alpha \in (\tilde{\alpha}, \frac{1}{2}]$, where $\tilde{\alpha}$ is the unique number in $(0, \frac{1}{2})$ satisfying

$$(1 - 2p) \log \left(\frac{1 - \tilde{\alpha} * p}{\tilde{\alpha} * p} \right) = (1 - \epsilon) \log \left(\frac{1 - \tilde{\alpha}}{\tilde{\alpha}} \right).$$

3) $\epsilon = 1$ or $p = 0$:

$$\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon)) = \{(R_1, R_2) \in \mathbb{R}_+^2 :$$
$$R_2 \leq 1 - \epsilon, R_1 + R_2 \leq 1 - H_b(p)\}.$$

Proof: Part 1) follows from Proposition 2 and the fact that BEC(ϵ) is less noisy than BSC(p) when $\epsilon \in [0, 4p(1-p)]$ [20, Theorem 3]. Part 3) is trivial. For Part 2), one can readily show that $\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq 1 - \epsilon,$$
$$R_1 + R_2 \leq 1 - H_b(\alpha * p) + (1 - \epsilon) H_b(\alpha)$$

for some $\alpha \in [0, \frac{1}{2}]$ by following the proof of [20, Claim 2 and Claim 3]. In light of [11, Lemma 6], when $\epsilon \in (4p(1 - p), 1)$ and $p \neq 0$, the following optimization problem

$$\max_{\alpha \in [0, \frac{1}{2}]} 1 - H_b(\alpha * p) + (1 - \epsilon) H_b(\alpha)$$

has a unique maximizer at $\alpha = \tilde{\alpha}$. This completes the proof of Proposition 7. ∎

Remark: It might be tempting to conjecture that Proposition 2 continues to hold if "essentially less noisy" is replaced by "essentially more capable". However, this conjecture turns out to be false. Indeed, for BSC(p)&BEC(ϵ), it is known [20, Theorem 3] that BEC(ϵ) is more capable (but not less noisy) than BSC(p) when $\epsilon \in (4p(1 - p), H_b(p)]$, yet Part 2) of Proposition 7 indicates that in this case $\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is strictly larger than $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ (see Fig. 5). Analogously, Proposition 3 is not true in general if "essentially more capable" is replaced by "essentially less noisy". For example, according to [20, Theorem 3], BSC(p) is essentially less noisy than BEC(ϵ) when $\epsilon \in [H_b(p), 1)$ and $p \neq 0$, but Part 2) of Proposition 7 shows that in this case $\mathcal{C}_2(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is strictly larger than $\{(R_1, R_2) \in \mathbb{R}_+^2 : R_2 \leq 1 - \epsilon, R_1 + R_2 \leq 1 - H_b(p)\}$ (see Fig. 6).
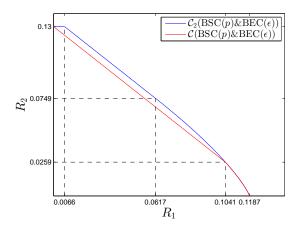
Fig. 5. $\mathcal{C}_2(\mathrm{BSC}(p)\&\mathrm{BEC}(\epsilon))$ vs. $\mathcal{C}(\mathrm{BEC}(\epsilon)\&\mathrm{BSC}(p))$ with $p = 0.3$ and $\epsilon = 0.87$
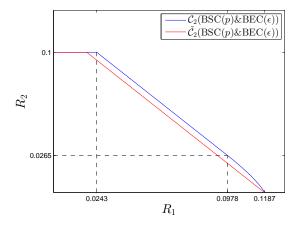

Fig. 6. $\mathcal{C}_2(\mathrm{BSC}(p)\&\mathrm{BEC}(\epsilon))$ vs. $\tilde{\mathcal{C}}_2(\mathrm{BSC}(p)\&\mathrm{BEC}(\epsilon)) \triangleq \{(R_1, R_2) \in \mathbb{R}_+^2 : R_2 \le 1 - \epsilon, R_1 + R_2 \le 1 - H_b(p)\}$ with $p = 0.3$ and $\epsilon = 0.9$

Finally consider the case where $p_{Y_1,Y_2|X}$ is a scalar Gaussian broadcast channel with power constraint P and noise variances N_1 and N_2 ($0 < N_1 \le N_2$); such a channel will be denoted by G-BC(P, N_1, N_2). It is well known that $\mathcal{C}(\mathrm{G\text{-}BC}(P, N_1, N_2))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le \frac{1}{2} \log \left(\frac{\beta P + N_1}{N_1} \right),$$
$$R_2 \le \frac{1}{2} \log \left(\frac{P + N_2}{\beta P + N_2} \right)$$

for some $\beta \in [0, 1]$. One can readily prove the following result by adapting Proposition 2 and Proposition 3 to this channel model.

Proposition 8: For G-BC(P, N_1, N_2) with $0 < N_1 \le N_2$,

$$\mathcal{C}_1(\mathrm{G\text{-}BC}(P, N_1, N_2)) = \mathcal{C}(\mathrm{G\text{-}BC}(P, N_1, N_2)),$$
$$\mathcal{C}_2(\mathrm{G\text{-}BC}(P, N_1, N_2)) = \Big\{ (R_1, R_2) \in \mathbb{R}_+^2 :$$
$$R_2 \le \frac{1}{2} \log \left(\frac{P + N_2}{N_2} \right), R_1 + R_2 \le \frac{1}{2} \log \left(\frac{P + N_1}{N_1} \right) \Big\}.$$

IV. Optimality of Source-Channel Separation for System $\tilde{\Pi}$

Now we are in a position to state the following source-channel separation theorem, which shows that a separation-based scheme that consists of lossy source coding and broadcast channel coding (see Fig. 4 and the associated description) is optimal for System $\tilde{\Pi}$. This result can be viewed as an extension of [17, Lemma 3] from degraded broadcast channels to general broadcast channels.

Theorem 1: $(\tilde{\kappa}, \tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2) \in \tilde{\Gamma}$ if and only if $(R_{\tilde{S}_1|\tilde{S}_2}(\tilde{\mathcal{Q}}_1), R_{\tilde{S}_2}(\tilde{\mathcal{Q}}_2)) \in \tilde{\kappa}\mathcal{C}_1(p_{Y_1,Y_2|X})$, where

$$R_{\tilde{S}_1|\tilde{S}_2}(\tilde{\mathcal{Q}}_1) = \min_{p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1} \in \tilde{\mathcal{Q}}_1} I(\tilde{S}_1; \hat{S}_1|\tilde{S}_2),$$
$$R_{\tilde{S}_2}(\tilde{\mathcal{Q}}_2) = \min_{p_{\tilde{S}_2,\hat{S}_2} \in \tilde{\mathcal{Q}}_2} I(\tilde{S}_2; \hat{S}_2).$$

Proof: The proof of the "if" part hinges on a separation-based scheme. We shall only give a sketch here since the argument only involves standard techniques. Let $\hat{S}_1$ be jointly distributed with $(\tilde{S}_1, \tilde{S}_2)$ such that $p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1} \in \tilde{\mathcal{Q}}_1$ and $I(\tilde{S}_1; \hat{S}_1|\tilde{S}_2) = R_{\tilde{S}_1|\tilde{S}_2}(\tilde{\mathcal{Q}}_1)$. Let $\hat{S}_2$ be jointly distributed with $\tilde{S}_2$ such that $p_{\tilde{S}_2,\hat{S}_2} \in \tilde{\mathcal{Q}}_2$ and $I(\tilde{S}_2; \hat{S}_2) = R_{\tilde{S}_2}(\tilde{\mathcal{Q}}_2)$. By the functional representation lemma [21, p. 626] (see also [22, Lemma 1]), we can find a random variable W of cardinality $|\mathcal{W}| \le |\tilde{S}_2|(|\hat{S}_1| - 1) + 1$ with the following properties:

- W is independent of $\tilde{S}_2$;
- $\hat{S}_1 = \psi(\tilde{S}_2, W)$ for some deterministic function $\psi : \tilde{S}_2 \times \mathcal{W} \to \hat{S}_1$;
- $\tilde{S}_1 \leftrightarrow (\tilde{S}_2, \hat{S}_1) \leftrightarrow W$ form a Markov chain.

It is easy to see that

$$I(\tilde{S}_1; \hat{S}_1|\tilde{S}_2) = I(\tilde{S}_1; W|\tilde{S}_2)$$
$$= I(\tilde{S}_1, \tilde{S}_2; W).$$

For any $\delta > 0$, let $R_1 = (1 + \delta)I(\tilde{S}_1; \hat{S}_1|\tilde{S}_2)$ and $R_2 = (1 + \delta)I(\tilde{S}_2; \hat{S}_2)$. We independently generate 2^{mR_1} codewords $W^m(m_1)$, $m_1 = 1, \cdots, 2^{mR_1}$, each according to $\prod_{t=1}^m p_W$, and independently generate 2^{mR_2} codewords $\hat{S}_2^m(m_2)$, $m_2 = 1, \cdots, 2^{mR_2}$, each according to $\prod_{t=1}^m p_{\hat{S}_2}$. Codebooks $\{W^m(m_1)\}_{m_1=1}^{2^{mR_1}}$ and $\{\hat{S}_2^m(m_2)\}_{m_2=1}^{2^{mR_2}}$ are revealed to the transmitter and the receivers. It can be shown that, given $(\tilde{S}_1^m, \tilde{S}_2^m)$, with high probability one can find an index M_1 such that $(\tilde{S}_1^m, \tilde{S}_2^m, W^m(M_1))$ are jointly typical with respect to $p_{\tilde{S}_1,\tilde{S}_2,W}$ when m is large enough (see [21] for the definition of typical sequences and the related properties). Similarly, given $\tilde{S}_2^m$, with high probability one can find an index M_2 such that $(\tilde{S}_2^m, \hat{S}_2^m(M_2))$ are jointly typical with respect to $p_{\tilde{S}_2,\hat{S}_2}$. If there is more than one such M_1 (or M_2), we choose the smallest index among them; if no such M_1 (or M_2) exists, we set $M_1 = 1$ (or $M_2 = 1$). Now a length-n coding scheme is used to send messages M_1 and M_2 over broadcast channel $p_{Y_1,Y_2|X}$ to receiver 1 and receiver 2, respectively. Given $\tilde{S}_2^m$, receiver 1 can recover M_2 and use it together with Y_1^n to produce an estimate $\hat{M}_1$. Receiver 2 can use Y_2^n to produce an estimate $\hat{M}_2$. We assume that this length-n coding scheme is good in the sense that $M_i = \hat{M}_i$, $i = 1, 2$, with high probability. Note that the existence of such

a good length-n coding scheme is guaranteed by Definition 5 when $\frac{n}{m} \geq \tilde{\kappa}(1 + 2\delta)$ and n is large enough. Receiver 1 then constructs $\hat{S}_1^m$ with

$$\hat{S}_1(t) = \psi(\tilde{S}_2(t), W(\hat{M}_1, t)), \quad t = 1, \cdots, m,$$

where $W(\hat{M}_1, t)$ is the t-th entry of $W_1^m(\hat{M}_1)$. Receiver 2 sets $\hat{S}_2^m = \hat{S}_2^m(\hat{M}_2)$. It is easy to show that $(\tilde{S}_1^m, \tilde{S}_2^m, \hat{S}_1^m)$ are jointly typical with respect to $p_{\tilde{S}_1, \tilde{S}_2, \hat{S}_1}$ with high probability, and $(\tilde{S}_2^m, \hat{S}_2^m)$ are jointly typical with respect to $p_{\tilde{S}_2, \hat{S}_2}$ with high probability. This completes the proof of the "if" part.

Now we proceed to prove the "only if" part. Consider an arbitrary tuple $(\tilde{\kappa}, \tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2) \in \tilde{\Gamma}$. Given any $\epsilon > 0$, according to Definition 3, we can find encoding function $f^{(m,n)} : \tilde{\mathcal{S}}_1^m \times \tilde{\mathcal{S}}_2^m \to \mathcal{X}^n$ as well as decoding functions $g_1^{(n,m)} : \mathcal{Y}_1^n \times \tilde{\mathcal{S}}_2^m \to \hat{\mathcal{S}}_1^m$ and $g_2^{(n,m)} : \mathcal{Y}_2^n \to \hat{\mathcal{S}}_2^m$ such that (4)-(6) are satisfied. Let Q be a random variable independent of $(\tilde{S}_1^m, \tilde{S}_2^m, X^n, Y_1^n, Y_2^n)$ and uniformly distributed over $\{1, \cdots, n\}$. Define $X = X(Q)$, $Y_i = Y_i(Q)$, $i = 1, 2$, and $V = (V(Q), Q)$, where $V(t) = (Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m)$ for all t. It is easy to verify that $V \leftrightarrow X \leftrightarrow (Y_1, Y_2)$ form a Markov chain. Note that

$$
\begin{aligned}
I(\tilde{S}_1^m; \hat{S}_1^m | \tilde{S}_2^m) &\leq I(\tilde{S}_1^m; Y_1^n | \tilde{S}_2^m) \\
&\leq I(\tilde{S}_1^m, \tilde{S}_2^m; Y_1^n) \\
&\leq I(X^n; Y_1^n) \\
&= \sum_{t=1}^n I(X^n; Y_1(t) | Y_1^{t-1}) \\
&\leq \sum_{t=1}^n I(X^n, Y_1^{t-1}; Y_1(t)) \\
&= \sum_{t=1}^n I(X(t); Y_1(t)) \\
&= nI(X(Q); Y_1(Q) | Q) \\
&\leq n(Q, X(Q); Y_1(Q)) \\
&= nI(X(Q); Y_1(Q)) \\
&= nI(X; Y_1) \tag{17}
\end{aligned}
$$

and

$$
\begin{aligned}
I(\tilde{S}_2^m; \hat{S}_2^m) &\leq I(\tilde{S}_2^m; Y_2^n) \\
&= \sum_{t=1}^n I(\tilde{S}_2^m; Y_2(t) | Y_{2,t+1}^n) \\
&\leq \sum_{t=1}^n I(Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m; Y_2(t)) \\
&= \sum_{t=1}^n I(V(t); Y_2(t)) \\
&= nI(V(Q); Y_2(Q) | Q) \\
&\leq nI(V(Q), Q; Y_2(Q)) \\
&= nI(V; Y_2). \tag{18}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&I(\tilde{S}_1^m; \hat{S}_1^m | \tilde{S}_2^m) + I(\tilde{S}_2^m; \hat{S}_2^m) \\
&\leq I(\tilde{S}_1^m; Y_1^n | \tilde{S}_2^m) + I(\tilde{S}_2^m; Y_2^n) \\
&= \sum_{t=1}^n [I(\tilde{S}_1^m; Y_1(t) | Y_1^{t-1}, \tilde{S}_2^m) + I(\tilde{S}_2^m; Y_2(t) | Y_{2,t+1}^n)] \\
&\leq \sum_{t=1}^n [I(X(t); Y_1(t) | Y_1^{t-1}, \tilde{S}_2^m) + I(\tilde{S}_2^m; Y_2(t) | Y_{2,t+1}^n)] \\
&\leq \sum_{t=1}^n [I(X(t), Y_{2,t+1}^n; Y_1(t) | Y_1^{t-1}, \tilde{S}_2^m) \\
&\qquad + I(Y_{2,t+1}^n, \tilde{S}_2^m; Y_2(t))] \\
&= \sum_{t=1}^n [I(X(t); Y_1(t) | Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m) \\
&\qquad + I(Y_{2,t+1}^n; Y_1(t) | Y_1^{t-1}, \tilde{S}_2^m) + I(Y_{2,t+1}^n, \tilde{S}_2^m; Y_2(t))] \\
&= \sum_{t=1}^n [I(X(t); Y_1(t) | Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m) \\
&\qquad + I(Y_1^{t-1}; Y_2(t) | Y_{2,t+1}^n, \tilde{S}_2^m) + I(Y_{2,t+1}^n, \tilde{S}_2^m; Y_2(t))] \tag{19} \\
&= \sum_{t=1}^n [I(X(t); Y_1(t) | Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m) \\
&\qquad + I(Y_1^{t-1}, Y_{2,t+1}^n, \tilde{S}_2^m; Y_2(t))] \\
&= \sum_{t=1}^n [I(X(t); Y_1(t) | V(t)) + I(V(t); Y_2(t))] \\
&= n[I(X(Q); Y_1(Q) | V(Q), Q) + I(V(Q); Y_2(Q) | Q)] \\
&\leq n[I(X(Q); Y_1(Q) | V(Q), Q) + I(V(Q), Q; Y_2(Q))] \\
&= nI(X; Y_1 | V) + nI(V; Y_2), \tag{20}
\end{aligned}
$$

where (19) follows by the Csiszár sum identity [21, p. 25]. Let T be a random variable independent of $(\tilde{S}_1^m, \tilde{S}_2^m, \hat{S}_1^m, \hat{S}_2^m)$ and uniformly distributed over $\{1, \cdots, m\}$. Define $\tilde{S}_i = \tilde{S}_i(T)$ and $\hat{S}_i^{(\epsilon)} = \hat{S}_i(T)$, $i = 1, 2$. Note that

$$p_{\tilde{S}_1, \tilde{S}_2, \hat{S}_1^{(\epsilon)}, \hat{S}_2^{(\epsilon)}} = \frac{1}{m} \sum_{t=1}^m p_{\tilde{S}_1(t), \tilde{S}_2(t), \hat{S}_1(t), \hat{S}_2(t)}.$$

Moreover, we have

$$
\begin{aligned}
I(\tilde{S}_1^m; \hat{S}_1^m | \tilde{S}_2^m) &= \sum_{t=1}^m I(\tilde{S}_1(t); \hat{S}_1^m | \tilde{S}_1^{t-1}, \tilde{S}_2^m) \\
&= \sum_{t=1}^m I(\tilde{S}_1(t); \hat{S}_1^m, \tilde{S}_1^{t-1}, \tilde{S}_2^{t-1}, \tilde{S}_{2,t+1}^n | \tilde{S}_2(t)) \\
&\geq \sum_{t=1}^m I(\tilde{S}_1(t); \hat{S}_1(t) | \tilde{S}_2(t)) \\
&= mI(\tilde{S}_1(T); \hat{S}_1(T) | \tilde{S}_2(T), T) \\
&= mI(\tilde{S}_1(T); \hat{S}_1(T), T | \tilde{S}_2(T)) \\
&\geq mI(\tilde{S}_1(T); \hat{S}_1(T) | \tilde{S}_2(T)) \\
&= mI(\tilde{S}_1; \hat{S}_1^{(\epsilon)} | \tilde{S}_2) \tag{21}
\end{aligned}
$$

and

$$I(\tilde{S}_2^m; \hat{S}_2^m) = \sum_{t=1}^{m} I(\tilde{S}_2(t); \hat{S}_2^m | \tilde{S}_2^{t-1})$$
$$= \sum_{t=1}^{m} I(\tilde{S}_2(t); \hat{S}_2^m, \tilde{S}_2^{t-1})$$
$$\geq \sum_{t=1}^{m} I(\tilde{S}_2(t); \hat{S}_2(t))$$
$$= m I(\tilde{S}_2(T); \hat{S}_2(T) | T)$$
$$= m I(\tilde{S}_2(T); \hat{S}_2(T), T)$$
$$\geq m I(\tilde{S}_2(T); \hat{S}_2(T))$$
$$= m I(\tilde{S}_2; \hat{S}_2^{(\epsilon)}). \tag{22}$$

It follows by (17), (18), (20), (21), and (22) that

$$(I(\tilde{S}_1; \hat{S}_1^{(\epsilon)} | \tilde{S}_2), I(\tilde{S}_2; \hat{S}_2^{(\epsilon)})) \in \frac{n}{m} \mathcal{C}_1(p_{Y_1,Y_2|X}).$$

Since $\{p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1^{(\epsilon)},\hat{S}_2^{(\epsilon)}} : \epsilon > 0\}$ is contained in a compact set and

$$\min_{\tilde{q}_1 \in \tilde{\mathcal{Q}}_1} \|p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1^{(\epsilon)}} - \tilde{q}_1\| \leq \epsilon,$$
$$\min_{\tilde{q}_2 \in \tilde{\mathcal{Q}}_2} \|p_{\tilde{S}_2,\hat{S}_2^{(\epsilon)}} - \tilde{q}_2\| \leq \epsilon$$

for every $\epsilon > 0$, $i = 1, 2$, one can find a sequence $\epsilon_1, \epsilon_2, \cdots$ converging to zero such that

$$\lim_{k \to \infty} p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1^{(\epsilon_k)},\hat{S}_2^{(\epsilon_k)}} = p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1,\hat{S}_2}$$

for some $p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1,\hat{S}_2}$ with $p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1} \in \tilde{\mathcal{Q}}_1$ and $p_{\tilde{S}_2,\hat{S}_2} \in \tilde{\mathcal{Q}}_2$. It is clear that

$$I(\tilde{S}_1; \hat{S}_1 | \tilde{S}_2) \geq R_{\tilde{S}_1 | \tilde{S}_2}(\tilde{\mathcal{Q}}_1),$$
$$I(\tilde{S}_2; \hat{S}_2) \geq R_{\tilde{S}_2}(\tilde{\mathcal{Q}}_2).$$

Now the proof can be completed via a simple limiting argument. ∎

V. A NECESSARY CONDITION FOR THE SOURCE BROADCAST PROBLEM

A. Necessary Condition

We shall show that the source-channel separation theorem for System $\tilde{\Pi}$ (i.e., Theorem 1) can be leveraged to establish a necessary condition for System Π via a simple reduction argument. Let $\mathcal{R}_1(p_{S,\hat{S}_1,\hat{S}_2})$ denote the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq I(S; \hat{S}_1 | U),$$
$$R_2 \leq I(U; \hat{S}_2)$$

for some $p_{U,S,\hat{S}_1,\hat{S}_2} = p_{U|S} p_{S,\hat{S}_1,\hat{S}_2}$. Similarly, let $\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2})$ denote the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq I(U; \hat{S}_1),$$
$$R_2 \leq I(S; \hat{S}_2 | U)$$

for some $p_{U,S,\hat{S}_1,\hat{S}_2} = p_{U|S} p_{S,\hat{S}_1,\hat{S}_2}$.

Theorem 2: For any $(\kappa, \mathcal{Q}_1, \mathcal{Q}_2) \in \Gamma$, there exists $p_{S,\hat{S}_1,\hat{S}_2}$ with $p_{S,\hat{S}_i} \in \mathcal{Q}_i$, $i = 1, 2$, such that

$$\mathcal{R}_i(p_{S,\hat{S}_1,\hat{S}_2}) \subseteq \kappa \mathcal{C}_i(p_{Y_1,Y_2|X}), \quad i = 1, 2. \tag{23}$$

Proof: By symmetry, it suffices to prove (23) for $i = 1$. We augment the probability space by introducing a remote source $\{(\tilde{S}_1(t), \tilde{S}_2(t))\}_{t=1}^{\infty}$ such that $(\tilde{S}_1(t), \tilde{S}_2(t), S(t))$, $t = 1, 2, \cdots$, are independent and identically distributed over finite alphabet $\tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 \times \mathcal{S}$. Consider an arbitrary tuple $(\kappa, \mathcal{Q}_1, \mathcal{Q}_2) \in \Gamma$. Given any $\epsilon > 0$, according to Definition 1, we can find encoding function $f^{(m,n)} : \mathcal{S}^m \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \hat{\mathcal{S}}_i^m$, $i = 1, 2$, satisfying (1) and (2). Let T be a random variable independent of $(\tilde{S}_1^m, \tilde{S}_2^m, S^m, \hat{S}_1^m, \hat{S}_2^m)$ and uniformly distributed over $\{1, \cdots, m\}$. Define $\tilde{S}_i = \tilde{S}_i(T)$, $i = 1, 2$, $S = S(T)$, and $\hat{S}_i^{(\epsilon)} = \hat{S}_i(T)$, $i = 1, 2$. It is clear that the distribution of $(\tilde{S}_1, \tilde{S}_2, S)$ is identical with that of $(\tilde{S}_1(t), \tilde{S}_2(t), S(t))$ for every t, and $(\tilde{S}_1, \tilde{S}_2) \leftrightarrow S \leftrightarrow (\hat{S}_1^{(\epsilon)}, \hat{S}_2^{(\epsilon)})$ form a Markov chain. Moreover, we have

$$\frac{1}{m} \sum_{t=1}^{m} p_{\tilde{S}_1(t),\tilde{S}_2(t),S(t),\hat{S}_1(t),\hat{S}_2(t)} = p_{\tilde{S}_1,\tilde{S}_2,S,\hat{S}_1^{(\epsilon)},\hat{S}_2^{(\epsilon)}}.$$

Since $\min_{q_i \in \mathcal{Q}_i} \|p_{S,\hat{S}_i^{(\epsilon)}} - q_i\| \leq \epsilon$ for every $\epsilon > 0$, $i = 1, 2$, one can find a sequence $\epsilon_1, \epsilon_2, \cdots$ converging to zero such that

$$\lim_{k \to \infty} p_{\tilde{S}_1,\tilde{S}_2,S,\hat{S}_1^{(\epsilon_k)},\hat{S}_2^{(\epsilon_k)}} = p_{\tilde{S}_1,\tilde{S}_2,S,\hat{S}_1,\hat{S}_2} \tag{24}$$

for some $p_{\tilde{S}_1,\tilde{S}_2,S,\hat{S}_1,\hat{S}_2}$ with $p_{S,\hat{S}_i} \in \mathcal{Q}_i$, $i = 1, 2$. Note that (24) implies $(\kappa, \{p_{\tilde{S}_1,\tilde{S}_2,\hat{S}_1}\}, \{p_{\tilde{S}_2,\hat{S}_2}\}) \in \tilde{\Gamma}$. Therefore, it follows from Theorem 1 that

$$(I(\tilde{S}_1; \hat{S}_1 | \tilde{S}_2), I(\tilde{S}_2; \hat{S}_2)) \in \kappa \mathcal{C}_1(p_{Y_1,Y_2|X}).$$

Here one can fix $p_{S,\hat{S}_1,\hat{S}_2}$ and choose $p_{\tilde{S}_1,\tilde{S}_2|S}$ arbitrarily. Since $I(\tilde{S}_1; \hat{S}_1 | \tilde{S}_2) \leq I(S; \hat{S}_1 | \tilde{S}_2)$, there is no loss of generality in setting $\tilde{S}_1 = S$. Denoting $\tilde{S}_2$ by U completes the proof of Theorem 2. ∎

Remark: Since $\mathcal{C}_1(p_{Y_1,Y_2|X})$ and $\mathcal{C}_2(p_{Y_1,Y_2|X})$ are convex sets, it follows that (23) holds if and only if $\kappa \mathcal{C}_i(p_{Y_1,Y_2|X})$ contains all the extreme points of $\mathcal{R}_i(p_{S,\hat{S}_1,\hat{S}_2})$, $i = 1, 2$. One can show via a standard application of the support lemma [21, p. 631] that, in contrast with the cardinality bound $|\mathcal{U}| \leq |\mathcal{S}| + 1$ for preserving $\mathcal{R}_i(p_{S,\hat{S}_1,\hat{S}_2})$, $i = 1, 2$, it suffices to have $|\mathcal{U}| \leq |\mathcal{S}|$ for the purpose of realizing all their extreme points.

B. The Binary Uniform Source with the Hamming Distortion Measure

In this subsection we set $\mathcal{S} = \hat{\mathcal{S}}_1 = \hat{\mathcal{S}}_2 = \{0, 1\}$, $p_S(0) = p_S(1) = \frac{1}{2}$, and $w_1 = w_2 = w_H$, where w_H is the Hamming distortion measure, i.e.,

$$w_H(s, \hat{s}) = \begin{cases} 0, & s = \hat{s} \\ 1, & \text{otherwise} \end{cases}.$$

The problem is trivial[2] when $d_1 = \frac{1}{2}$ or $d_2 = \frac{1}{2}$. Therefore, we shall focus on the non-degenerate case $d_i \in [0, \frac{1}{2})$, $i = 1, 2$,

---

[2]In fact, it reduces to a point-to-point problem.

and assume

$$C(p_{Y_i|X}) \triangleq \max_{p_X} I(X;Y_i) > 0, \quad i = 1,2,$$

correspondingly.

Proposition 9: If $p_{S,\hat{S}_1,\hat{S}_2}$ is such that $\mathbb{E}[w_H(S,\hat{S}_i)] \leq d_i$, $i = 1,2$, with $d_1 \leq d_2$, then

$$\mathcal{R}_1(p_{S,\hat{S}_1,\hat{S}_2}) \supseteq \mathcal{C}(\text{BS-BC}(d_1,d_2)), \tag{25}$$

$$\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2}) \supseteq \tilde{\mathcal{C}}(\text{BS-BC}(d_1,d_2)), \tag{26}$$

where $\mathcal{C}(\text{BS-BC}(d_1,d_2))$ (see Section III-C for its definition) is given by the set of $(R_1, R_2) \in \mathbb{R}_2^+$ satisfying

$$R_1 \leq H_b(\alpha * d_1) - H_b(d_1),$$
$$R_2 \leq 1 - H_b(\alpha * d_2)$$

for some $\alpha \in [0, \frac{1}{2}]$, and $\tilde{\mathcal{C}}(\text{BS-BC}(d_1,d_2))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_2^+$ satisfying

$$R_1 \leq \beta[1 - H_b(d_1)],$$
$$R_2 \leq (1 - \beta)[1 - H_b(d_2)]$$

for some $\beta \in [0,1]$. Moreover,

$$\mathcal{R}_1(p_{S,\hat{S}_1,\hat{S}_2}) = \mathcal{C}(\text{BS-BC}(d_1,d_2)), \tag{27}$$

$$\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2}) = \tilde{\mathcal{C}}(\text{BS-BC}(d_1,d_2)) \tag{28}$$

when $p_{\hat{S}_1,\hat{S}_2|S}$ is a BS-BC(d_1,d_2) with $d_1 \leq d_2$.

Proof: Let $p_{U,S,\hat{S}_1,\hat{S}_2} = p_{U|S}p_{S,\hat{S}_1,\hat{S}_2}$, where $p_{U|S}$ is a BSC(α) with $\alpha \in [0, \frac{1}{2}]$. We have

$$\min_{p_{\hat{S}_1|S}:\mathbb{E}[w_H(S,\hat{S}_1)]\leq d_1} I(S;\hat{S}_1|U)$$

$$= \min_{p_{\hat{S}_1|S}:\mathbb{E}[w_H(S,\hat{S}_1)]\leq d_1} I(S;\hat{S}_1) - I(U;\hat{S}_1)$$

$$= \min_{p_{\hat{S}_1|S}:\mathbb{E}[w_H(S,\hat{S}_1)]\leq d_1} H(U|\hat{S}_1) - H(S|\hat{S}_1) \tag{29}$$

$$= \min_{d_1' \in [0,d_1]} H_b(\alpha * d_1') - H_b(d_1') \tag{30}$$

$$= H_b(\alpha * d_1) - H_b(d_1), \tag{31}$$

where (29) follows since $H(S) = H(U) = 1$, (30) follows from [11, Lemma 2], and (31) is due to the fact that $H_b(\alpha * d_1') - H_b(d_1')$ is a monotonically decreasing function of d_1' for $d_1' \in [0, \frac{1}{2}]$. Similarly, it can be shown that

$$\min_{p_{\hat{S}_2|S}:\mathbb{E}[w_H(S,\hat{S}_2)]\leq d_2} I(U;\hat{S}_2) = 1 - H_b(\alpha * d_2). \tag{32}$$

Combining (31) and (32) proves (25).

It is easy to see that $(I(S;\hat{S}_1),0)$ and $(0, I(S;\hat{S}_2))$ are contained in $\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2})$. Note that

$$I(S;\hat{S}_i) \geq 1 - H_b(d_i)$$

if $\mathbb{E}[w_H(S,\hat{S}_i)] \leq d_i$, $i = 1,2$. Now one can readily prove (26) by invoking the fact that $\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2})$ is a convex set.

Since (27) is obviously true, only (28) remains to be proved. If $p_{\hat{S}_1,\hat{S}_2|S}$ is a BS-BC(d_1,d_2) with $d_1 \leq d_2$, then, for any $\lambda \in [0,1]$,

$$\lambda I(U;\hat{S}_1) + (1-\lambda)I(S;\hat{S}_2|U)$$

$$= \lambda(1 - H(\hat{S}_1|U)) + (1-\lambda)[H(\hat{S}_2|U) - H_b(d_2)]$$

$$\leq \max_{u \in \mathcal{U}} \lambda(1 - H(\hat{S}_1|U = u))$$
$$\quad + (1-\lambda)[H(\hat{S}_2|U = u) - H_b(d_2)]$$

$$\leq \max_{\alpha \in [0,\frac{1}{2}]} \lambda(1 - H_b(\alpha * d_1))$$
$$\quad + (1-\lambda)[H_b(\alpha * d_2) - H_b(d_2)].$$

Define $v = H_b(\alpha * d_1)$, which is a monotonically increasing function of α. Note that

$$\lambda(1 - H_b(\alpha * d_1)) + (1-\lambda)[H_b(\alpha * d_2) - H_b(d_2)]$$
$$= \lambda(1 - v) + (1-\lambda)[H_b(H_b^{-1}(v) * d) - H_b(d_2)],$$

where $d = \frac{d_2-d_1}{1-2d_1}$. It follows by the convexity of $H_b(H_b^{-1}(v) * d)$ in v [23, Lemma 2] that

$$\max_{\alpha \in [0,\frac{1}{2}]} \lambda(1 - H_b(\alpha * d_1)) + (1-\lambda)[H_b(\alpha * d_2) - H_b(d_2)]$$

$$= \max_{\alpha \in \{0,\frac{1}{2}\}} \lambda(1 - H_b(\alpha * d_1))$$
$$\quad + (1-\lambda)[H_b(\alpha * d_2) - H_b(d_2)]. \tag{33}$$

Therefore, we must have $\mathcal{R}_2(p_{S,\hat{S}_1,\hat{S}_2}) \subseteq \tilde{\mathcal{C}}(\text{BS-BC}(d_1,d_2))$, which together with (26), proves (28). ∎

Remark: The proof of Proposition 9 indicates that, for the binary uniform source with the Hamming distortion measure, there is no loss of optimality (as far as Theorem 2 is concerned) in restricting $p_{U|S}$ to be a binary symmetric channel, which provides a certain justification for the choice of the auxiliary random variable in [13].

Note that the rate pairs $(C(p_{Y_1|X}),0)$ and $(0, C(p_{Y_2|X}))$ are contained in both $\mathcal{C}_1(p_{Y_1,Y_2|X})$ and $\mathcal{C}_2(p_{Y_1,Y_2|X})$. It is easy to see that $\mathcal{C}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_1(p_{Y_1,Y_2|X})$ implies

$$1 - H_b(d_i) \leq \kappa C(p_{Y_i|X}), \quad i = 1,2,$$

which further implies $\tilde{\mathcal{C}}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_2(p_{Y_1,Y_2|X})$ when $d_1 \leq d_2$. This observation, together with Proposition 9, shows that, for the binary uniform source with the Hamming distortion measure, Theorem 2 is equivalent to the following more explicit result.

Theorem 3: For any $(\kappa, \mathcal{Q}(w_H,d_1), \mathcal{Q}(w_H,d_2)) \in \Gamma$ with $d_1 \leq d_2$,

$$\mathcal{C}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_1(p_{Y_1,Y_2|X}).$$

By symmetry, for any $(\kappa, \mathcal{Q}(w_H,d_1), \mathcal{Q}(w_H,d_2)) \in \Gamma$ with $d_1 \geq d_2$,

$$\mathcal{C}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_2(p_{Y_1,Y_2|X}).$$

Define $\kappa^\star = \min\{\kappa \geq 0 : \mathcal{C}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_1(p_{Y_1,Y_2|X})\}$ if $d_1 \leq d_2$, and $\kappa^\star = \min\{\kappa \geq 0 : \mathcal{C}(\text{BS-BC}(d_1,d_2)) \subseteq \kappa\mathcal{C}_2(p_{Y_1,Y_2|X})\}$ if $d_1 \geq d_2$. It is obvious that

$$\kappa^\star \geq \kappa^\dagger \triangleq \max\left\{\frac{1 - H_b(d_1)}{C(p_{Y_1|X})}, \frac{1 - H_b(d_2)}{C(p_{Y_2|X})}\right\}, \tag{34}$$

i.e., the necessary condition stated in Theorem 3 is at least as strong as the one implied by the source-channel separation theorem for point-to-point communication systems. We shall show that in some cases it is possible to determine whether $\kappa^\star$ is equal to or strictly greater than $\kappa^\dagger$ without an explicit characterization of $\mathcal{C}_i(p_{Y_1,Y_2|X})$, $i = 1, 2$.

Recall that $\mathcal{C}(\text{BS-BC}(d_1, d_2))$ with $d_1 \leq d_2$ is given by the the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq R_1(\alpha) \triangleq H_b(\alpha * d_1) - H_b(d_1), \tag{35}$$

$$R_2 \leq R_2(\alpha) \triangleq 1 - H_b(\alpha * d_2) \tag{36}$$

for some $\alpha \in [0, \frac{1}{2}]$. It can be verified that[3]

$$\left.\frac{\mathrm{d}R_2(\alpha)}{\mathrm{d}R_1(\alpha)}\right|_{\alpha=0} = -\frac{(1 - 2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1 - 2d_1)\log\left(\frac{1-d_1}{d_1}\right)}, \tag{37}$$

$$\left.\frac{\mathrm{d}R_2(\alpha)}{\mathrm{d}R_1(\alpha)}\right|_{\alpha=\frac{1}{2}} = -\frac{(1 - 2d_2)^2}{(1 - 2d_1)^2}. \tag{38}$$

In view of the fact that $\frac{\mathrm{d}R_2(\alpha)}{\mathrm{d}R_1(\alpha)}$ is a monotonically decreasing function of α for $\alpha \in [0, \frac{1}{2}]$, it is clear that

$$\mathcal{C}(\text{BS-BC}(d_1, d_2))$$
$$\subseteq \kappa\left\{(R_1, R_2) \in \mathbb{R}_+^2 : \frac{R_1}{C(p_{Y_1|X})} + \frac{R_2}{C(p_{Y_2|X})} \leq 1\right\}$$

if one of the following conditions are satisfied:

1) $1 - H_b(d_1) \leq \kappa C(p_{Y_1|X})$ and $\frac{(1-2d_1)^2}{(1-2d_2)^2} \geq \frac{C(p_{Y_1|X})}{C(p_{Y_2|X})}$,

2) $1 - H_b(d_2) \leq \kappa C(p_{Y_2|X})$ and $\frac{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)}{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)} \leq \frac{C(p_{Y_1|X})}{C(p_{Y_2|X})}$.

This observation, together with (34) as well as the fact that

$$\left\{(R_1, R_2) \in \mathbb{R}_+^2 : \frac{R_1}{C(p_{Y_1|X})} + \frac{R_2}{C(p_{Y_2|X})} \leq 1\right\}$$
$$\subseteq \mathcal{C}_1(p_{Y_1,Y_2|X}),$$

yields the following result.

Proposition 10: If $d_1 \leq d_2$, then

$$\kappa^\star = \kappa^\dagger = \begin{cases} \frac{1-H_b(d_1)}{C(p_{Y_1|X})}, & \frac{(1-2d_1)^2}{(1-2d_2)^2} \geq \frac{C(p_{Y_1|X})}{C(p_{Y_2|X})} \\ \frac{1-H_b(d_2)}{C(p_{Y_2|X})}, & \frac{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)}{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)} \leq \frac{C(p_{Y_1|X})}{C(p_{Y_2|X})} \end{cases}.$$

By symmetry, if $d_1 \geq d_2$, then

$$\kappa^\star = \kappa^\dagger = \begin{cases} \frac{1-H_b(d_2)}{C(p_{Y_2|X})}, & \frac{(1-2d_2)^2}{(1-2d_1)^2} \geq \frac{C(p_{Y_2|X})}{C(p_{Y_1|X})} \\ \frac{1-H_b(d_1)}{C(p_{Y_1|X})}, & \frac{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)} \leq \frac{C(p_{Y_2|X})}{C(p_{Y_1|X})} \end{cases}.$$

Remark: A simple sufficient condition for $(\kappa, \mathcal{Q}(w_H, d_1), \mathcal{Q}(w_H, d_2)) \in \Gamma$ is that

$$\max\{1 - H_b(d_1), 1 - H_b(d_2)\} \leq \kappa C(p_{Y_1|X}, p_{Y_2|X}),$$

---

[3]We set $\frac{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)} = 1$ when $d_1 = d_2 = 0$.

where $C(p_{Y_1|X}, p_{Y_2|X}) \triangleq \max_{p_X} \min\{I(X; Y_1), I(X; Y_2)\}$ is the capacity of the compound channel $\{p_{Y_1|X}, p_{Y_2|X}\}$. Proposition 10 indicates that this sufficient condition is also necessary when $C(p_{Y_1|X}, p_{Y_2|X}) = C(p_{Y_1|X})$ and $d_1 \leq d_2$ (or $C(p_{Y_1|X}, p_{Y_2|X}) = C(p_{Y_2|X})$ and $d_1 \geq d_2$). For the special case $d_1 = d_2 = d$, it can be shown that $(\kappa, \mathcal{Q}(w_H, d), \mathcal{Q}(w_H, d)) \in \Gamma$ if and only if

$$1 - H_b(d) \leq \kappa C(p_{Y_1|X}, p_{Y_2|X}).$$

On the other hand, for this special case, Proposition 10 gives

$$\kappa^\star = \kappa^\dagger = \max\left\{\frac{1 - H_b(d)}{C(p_{Y_1|X})}, \frac{1 - H_b(d)}{C(p_{Y_2|X})}\right\}.$$

Since $C(p_{Y_1|X}, p_{Y_2|X})$ can be strictly smaller than $\min\{C(p_{Y_1|X}), C(p_{Y_2|X})\}$, the necessary condition stated in Theorem 3 is not sufficient in general.

For every $R_1 \in [0, C(p_{Y_1|X})]$, we set

$$\phi(R_1) = \max\{R_2 : (R_1, R_2) \in \mathcal{C}_1(p_{Y_1,Y_2|X})\}.$$

Note that $\phi : [0, C(p_{Y_1|X})] \to [0, C(p_{Y_2|X})]$ is monotonically decreasing and concave. Define

$$\phi_+'(0) = \lim_{R_1 \downarrow 0} \frac{C(p_{Y_2|X}) - \phi(R_1)}{R_1},$$

$$\phi_-'(C(p_{Y_1|X})) = \lim_{R_1 \uparrow C(p_{Y_1|X})} \frac{\phi(R_1)}{C(p_{Y_1|X}) - R_1}.$$

Similarly, we set

$$\varphi(R_2) = \max\{R_1 : (R_1, R_2) \in \mathcal{C}_2(p_{Y_1,Y_2|X})\}$$

for every $R_2 \in [0, C(p_{Y_2|X})]$, and define

$$\varphi_+'(0) = \lim_{R_2 \downarrow 0} \frac{C(p_{Y_1|X}) - \varphi(R_2)}{R_2},$$

$$\varphi_-'(C(p_{Y_2|X})) = \lim_{R_2 \uparrow C(p_{Y_2|X})} \frac{\varphi(R_2)}{C(p_{Y_2|X}) - R_2}.$$

Now consider the case $d_1 \leq d_2$. It is clear that we must have $1 - H_b(d_1) < \kappa^\star C(p_{Y_1|X})$ if

$$\frac{(1 - 2d_2)^2}{(1 - 2d_1)^2} > \phi_-'(C(p_{Y_1|X})); \tag{39}$$

similarly, we must have $1 - H_b(d_2) < \kappa^\star C(p_{Y_2|X})$ if

$$\frac{(1 - 2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1 - 2d_1)\log\left(\frac{1-d_1}{d_1}\right)} < \phi_+'(0); \tag{40}$$

moreover, since $\phi_+'(0) \leq \phi_-'(C(p_{Y_1|X}))$, it follows that (39) and (40) cannot be satisfied simultaneously when $d_1 = d_2$. The following result is a simple consequence of this observation.

Proposition 11: When $d_1 < d_2$, we have $\kappa^\star > \kappa^\dagger$ if

$$\frac{(1 - 2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1 - 2d_1)\log\left(\frac{1-d_1}{d_1}\right)} < \phi_+'(0),$$

$$\frac{(1 - 2d_2)^2}{(1 - 2d_1)^2} > \phi_-'(C(p_{Y_1|X})).$$

By symmetry, when $d_1 > d_2$, we have $\kappa^\star > \kappa^\dagger$ if

$$\frac{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)}{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)} < \varphi'_+(0),$$

$$\frac{(1-2d_1)^2}{(1-2d_2)^2} > \varphi'_-(C(p_{Y_2|X})).$$

A channel $p_{Y|X} : \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X} = \{0, 1, \cdots, M-1\}$ for some integer $M \geq 2$ is said to be circularly symmetric [24, Definition 1] (see also [20, Definition 4]) if there exists a bijective function $\mu : \mathcal{Y} \to \mathcal{Y}$ such that $\mu^M(y) = y$ and $p_{Y|X}(\mu^x(y)|x) = p_{Y|X}(y|0)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, where μ^k denotes the k-times self-composition of μ (with μ^0 being the identity function). Note that the binary symmetric channel is circularly symmetric with $\mu : \{0, 1\} \to \{0, 1\}$ given by

$$\mu(y) = \begin{cases} 1, & y = 0 \\ 0, & y = 1 \end{cases};$$

the binary erasure channel is also circularly symmetric, and the associated $\mu : \{0, 1, e\} \to \{0, 1, e\}$ is given by

$$\mu(y) = \begin{cases} 1, & y = 0 \\ 0, & y = 1 \\ e, & y = e \end{cases}.$$

Proposition 12: If both $p_{Y_1|X}$ and $p_{Y_2|X}$ are circularly symmetric, then

$$\kappa^\star = \min\{\kappa \geq 0 : \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(p_{Y_1,Y_2|X})\}.$$

Proof: By symmetry, it suffices to consider the case $d_1 \leq d_2$. Let $\mathcal{C}_{\text{sc}}(p_{Y_1,Y_2|X})$ denote the superposition coding inner bound of $\mathcal{C}(p_{Y_1,Y_2|X})$, i.e., the set of $(R_1, R_2) \in \mathbb{R}^2_+$ satisfying

$$R_2 \leq I(V; Y_2),$$
$$R_1 + R_2 \leq I(X; Y_1|V) + I(V; Y_2),$$
$$R_1 + R_2 \leq I(X; Y_1)$$

for some $p_{V,X,Y_1,Y_2} = p_{V,X}p_{Y_1,Y_2|X}$. In light of [20, Lemma 2], the uniform distribution on $\mathcal{X}$ forms a sufficient class of distributions for broadcast channel $p_{Y_1,Y_2|X}$ if both $p_{Y_1|X}$ and $p_{Y_2|X}$ are circularly symmetric. As a consequence, one can readily show that

$$\mathcal{C}_{\text{sc}}(p_{Y_1,Y_2|X})$$
$$= \mathcal{C}_1(p_{Y_1,Y_2|X}) \cap \{(R_1, R_2) : R_1 + R_2 \leq C(p_{Y_1|X})\}.$$

Note that, if $\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}_1(p_{Y_1,Y_2|X})$, then we must have

$$1 - H_b(d_1) \leq \kappa C(p_{Y_1|X}),$$

which, together with the fact that $\frac{\mathrm{d}R_2(\alpha)}{\mathrm{d}R_1(\alpha)} \in [-1, 0]$ for $\alpha \in [0, \frac{1}{2}]$, implies

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\{(R_1, R_2) : R_1 + R_2 \leq C(p_{Y_1|X})\}.$$

Therefore,

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}_1(p_{Y_1,Y_2|X})$$
$$\Rightarrow \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}_{\text{sc}}(p_{Y_1,Y_2|X}).$$

Since $\mathcal{C}_{\text{sc}}(p_{Y_1,Y_2|X}) \subseteq \mathcal{C}(p_{Y_1,Y_2|X}) \subseteq \mathcal{C}_1(p_{Y_1,Y_2|X})$, the proof is complete. ∎

Now we proceed to consider several concrete examples.

1) BS-BC(p_1, p_2): First consider the case where $p_{Y_1,Y_2|X}$ is a BS-BC(p_1, p_2) with $0 \leq p_1 \leq p_2 < \frac{1}{2}$. Without loss of generality, we shall assume $d_1 \leq d_2$. By Theorem 3 and Proposition 4 (or by Theorem 3 and Proposition 12), if $(\kappa, \mathcal{Q}(w_H, d_1), \mathcal{Q}(w_H, d_2)) \in \Gamma$, then

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(\text{BS-BC}(p_1, p_2)). \quad (41)$$

On the other hand, the necessary condition implied by the source-channel separation theorem for point-to-point communication systems is

$$1 - H_b(d_i) \leq \kappa[1 - H_b(p_i)], \quad i = 1, 2. \quad (42)$$

For the special case $\kappa = 1$, both (41) and (42) reduce to

$$d_i \geq p_i, \quad i = 1, 2,$$

which is achievable by the uncoded scheme.

In view of Proposition 4 as well as (37) and (38), we have

$$\phi'_+(0) = \frac{(1-2p_2)\log\left(\frac{1-p_2}{p_2}\right)}{(1-2p_1)\log\left(\frac{1-p_1}{p_1}\right)},$$

$$\phi'_-(C(p_{Y_1|X})) = \frac{(1-2p_2)^2}{(1-2p_1)^2}.$$

Hence, it follows from Proposition 11 that $\kappa^\star > \kappa^\dagger$ if

$$\frac{(1-2d_2)\log\left(\frac{1-d_2}{d_2}\right)}{(1-2d_1)\log\left(\frac{1-d_1}{d_1}\right)} < \frac{(1-2p_2)\log\left(\frac{1-p_2}{p_2}\right)}{(1-2p_1)\log\left(\frac{1-p_1}{p_1}\right)}, \quad (43)$$

$$\frac{(1-2d_2)^2}{(1-2d_1)^2} > \frac{(1-2p_2)^2}{(1-2p_1)^2}. \quad (44)$$

For example, (43) and (44) are satisfied when $d_1 = 0.035$, $d_2 = 0.095$, $p_1 = 0.15$, and $p_2 = 0.2$.

2) BE-BC(ϵ_1, ϵ_2): Next consider the case where $p_{Y_1,Y_2|X}$ is a BE-BC(ϵ_1, ϵ_2) with $0 \leq \epsilon_1 \leq \epsilon_2 < 1$. Without loss of generality, we shall assume $d_1 \leq d_2$. By Proposition 5 (or by Proposition 12),

$$\kappa^\star = \min\{\kappa \geq 0 : \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(\text{BE-BC}(\epsilon_1, \epsilon_2))\},$$

where the expressions of $\mathcal{C}(\text{BS-BC}(d_1, d_2))$ and $\mathcal{C}(\text{BE-BC}(\epsilon_1, \epsilon_2))$ can be found in (35)-(36) and (15)-(16), respectively. It is clear that, for any $\alpha \in [0, \frac{1}{2}]$, there exists $\beta \in [0, 1]$ such that

$$H_b(\alpha * d_1) - H_b(d_1) \leq \kappa^\star\beta(1 - \epsilon_1), \quad (45)$$
$$1 - H_b(\alpha * d_2) \leq \kappa^\star(1 - \beta)(1 - \epsilon_2), \quad (46)$$

which implies

$$\kappa^\star \geq \frac{H_b(\alpha * d_1) - H_b(d_1)}{1 - \epsilon_1} + \frac{1 - H_b(\alpha * d_2)}{1 - \epsilon_2} \quad (47)$$

for any $\alpha \in [0, \frac{1}{2}]$. Moreover, the equalities must hold in (45) and (46) for some $\alpha \in [0, \frac{1}{2}]$ and $\beta \in [0, 1]$; as a consequence, the equality must hold in (47) for some $\alpha \in [0, \frac{1}{2}]$. Therefore, we have

$$\kappa^\star = \max_{\alpha \in [0, \frac{1}{2}]} \frac{H_b(\alpha * d_1) - H_b(d_1)}{1 - \epsilon_1} + \frac{1 - H_b(\alpha * d_2)}{1 - \epsilon_2}. \quad (48)$$

from which one can readily recover [13, Theorem 1] by invoking Theorem 3. In light of [11, Lemma 2], for the optimization problem in (48), the maximum value is not attained at $\alpha = 0$ or $\alpha = \frac{1}{2}$ if and only if

$$\frac{(1 - 2d_2) \log \left(\frac{1 - d_2}{d_2} \right)}{(1 - 2d_1) \log \left(\frac{1 - d_1}{d_1} \right)} < \frac{1 - \epsilon_2}{1 - \epsilon_1} < \frac{(1 - 2d_2)^2}{(1 - 2d_1)^2},$$

which gives the necessary and sufficient condition for $\kappa^\star > \kappa^\dagger$ to hold. The same condition can be obtained through Proposition 10 and Proposition 11.

3) BSC(p)&BEC(ϵ): Finally consider the case where $p_{Y_1, Y_2 | X}$ is a BSC(p)&BEC(ϵ) with $p \in [0, \frac{1}{2})$ and $\epsilon \in [0, 1)$. By Proposition 12,

$$\kappa^\star = \min\{\kappa \geq 0 : \mathcal{C}(\text{BS-BC}(d_1, d_2))$$
$$\subseteq \kappa\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))\}. \quad (49)$$

Note that

$$\kappa^\star \geq \kappa^\dagger = \max\left\{ \frac{1 - H_b(d_1)}{1 - H_b(p)}, \frac{1 - H_b(d_2)}{1 - \epsilon} \right\}.$$

For the case $d_1 \leq d_2$, in view of the expression of $\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ (see Section III-C) and the fact that $\frac{dR_2(\alpha)}{dR_1(\alpha)} \in [-1, 0]$ for $\alpha \in [0, \frac{1}{2}]$, one can readily verify that

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$$
$$\Leftrightarrow \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(\text{BE-BC}(H_b(p), \epsilon));$$

as a consequence,

$$\kappa^\star = \max_{\alpha \in [0, \frac{1}{2}]} \frac{H_b(\alpha * d_1) - H_b(d_1)}{1 - H_b(p)} + \frac{1 - H_b(\alpha * d_2)}{1 - \epsilon},$$

and we have $\kappa^\star > \kappa^\dagger$ if and only if

$$\frac{(1 - 2d_2) \log \left(\frac{1 - d_2}{d_2} \right)}{(1 - 2d_1) \log \left(\frac{1 - d_1}{d_1} \right)} < \frac{1 - \epsilon}{1 - H_b(p)} < \frac{(1 - 2d_2)^2}{(1 - 2d_1)^2}.$$

For the case $d_1 \geq d_2$, we shall show that

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$$
$$\Leftrightarrow \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\tilde{\mathcal{C}}(\text{BSC}(p)\&\text{BEC}(\epsilon)), \quad (50)$$

where $\tilde{\mathcal{C}}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is given by the set[4] of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \leq 1 - H_b(\alpha * p),$$
$$R_2 \leq (1 - \epsilon)H_b(\alpha)$$

for some $\epsilon \in [0, \frac{1}{2}]$. It is easy to see that (50) is true when $\epsilon \in [H_b(p), 1)$; moreover,

$$\mathcal{C}(\text{BSC}(p)\&\text{BEC}(\epsilon))$$
$$= \tilde{\mathcal{C}}(\text{BSC}(p)\&\text{BEC}(\epsilon)) \cap \{(R_1, R_2) : R_1 + R_2 \leq 1 - \epsilon\}$$

[4]It follows from [23, Lemma 2] that $\tilde{\mathcal{C}}(\text{BSC}(p)\&\text{BEC}(\epsilon))$ is a convex set.

when $\epsilon \in [0, H_b(p))$. Combining this observation with the fact that

$$\mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\tilde{\mathcal{C}}(\text{BSC}(p)\&\text{BEC}(\epsilon))$$
$$\Rightarrow 1 - H_b(d_2) \leq \kappa(1 - \epsilon)$$
$$\overset{d_1 \geq d_2}{\Rightarrow} \mathcal{C}(\text{BS-BC}(d_1, d_2)) \subseteq \kappa\{(R_1, R_2) : R_1 + R_2 \leq 1 - \epsilon\}$$

proves (50). Now we proceed to show that[5] $\kappa^\star = \kappa^\dagger$ if $\kappa^\dagger \geq 1$. In view of (49) and (50), it suffices to show that, if $\kappa^\dagger \geq 1$, then

$$1 - H_b(\alpha * d_1) \leq \kappa^\dagger[1 - H_b(\alpha * p)], \quad (51)$$
$$H_b(\alpha * d_2) - H_b(d_2) \leq \kappa^\dagger(1 - \epsilon)H_b(\alpha) \quad (52)$$

for any $\alpha \in [0, \frac{1}{2}]$. Note that (51) and (52) hold when $\alpha = 0$ or $\alpha = \frac{1}{2}$. Moreover, $\kappa^\dagger \geq 1$ implies $p \geq d_1$. Therefore, an argument similar to that for (33) can be used here to finish the proof.

VI. THE QUADRATIC GAUSSIAN CASE

Let $\{S(t)\}_{t=1}^\infty$ in System Π be an i.i.d. vector Gaussian process, where each $S(t)$ is an $\ell \times 1$ zero-mean Gaussian random vector with positive definite covariance matrix Σ_S. The following definition is the quadratic Gaussian counterpart of Definition 1.

Definition 6: Let κ be a non-negative number and $\mathcal{D}_i$ be a non-empty compact set of $\ell \times \ell$ positive semi-definite matrices, $i = 1, 2$. We say $(\kappa, \mathcal{D}_1, \mathcal{D}_2)$ is achievable for System Π if, for every $\epsilon > 0$, there exist encoding function $f^{(m,n)} : \mathbb{R}^{\ell \times m} \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \mathbb{R}^{\ell \times m}$, $i = 1, 2$, such that

$$\frac{n}{m} \leq \kappa + \epsilon,$$
$$\min_{D_i \in \mathcal{D}_i} \left\| \frac{1}{m} \sum_{t=1}^m \mathbb{E}[(S(t) - \hat{S}_i(t))(S(t) - \hat{S}_i(t))^T] - D_i \right\| \leq \epsilon,$$
$$i = 1, 2.$$

The set of all achievable $(\kappa, \mathcal{D}_1, \mathcal{D}_2)$ for System Π is denoted by Γ_G.

Remark: It is clear that $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$ if and only if $(\kappa, \bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) \in \Gamma_G$, where

$$\bar{\mathcal{D}}_i = \bigcup_{D_i \in \mathcal{D}_i} \{D_i' : 0 \preceq D_i' \preceq D_i\}, \quad i = 1, 2.$$

Furthermore, to determine whether or not $(\kappa, \bar{\mathcal{D}}_1, \bar{\mathcal{D}}_2) \in \Gamma_G$, there is no loss of generality in setting $\hat{S}_i^m = \mathbb{E}[S^m | Y_i^n]$, $i = 1, 2$, for which we have

$$\frac{1}{m} \sum_{t=1}^m \mathbb{E}[(S(t) - \hat{S}_i(t))(S(t) - \hat{S}_i(t))^T] \preceq \Sigma_S, \quad i = 1, 2.$$

Therefore, it suffices to consider those $\mathcal{D}_1$ and $\mathcal{D}_2$ with the property that

$$\mathcal{D}_i = \bar{\mathcal{D}}_i \cap \{D : 0 \preceq D \preceq \Sigma_S\}, \quad i = 1, 2. \quad (53)$$

Henceforth we shall implicitly assume that (53) is satisfied.

[5]This result is not implied by Proposition 10.

Now we proceed to introduce the corresponding System $\tilde{\Pi}$ in the quadratic Gaussian setting and establish its associated source-channel separation theorem. Let $\tilde{S} \triangleq (\tilde{S}_1^T, \tilde{S}_2^T)^T$ be an $\tilde{\ell} \times 1$ zero-mean Gaussian random vector with positive definite covariance matrix $\Sigma_{\tilde{S}}$, where $\tilde{S}_i$ is an $\tilde{\ell}_i \times 1$ random vector, and its covariance matrix is denoted by $\Sigma_{\tilde{S}_i}$, $i = 1, 2$. Let $\{(\tilde{S}_1(t), \tilde{S}_2(t))\}_{t=1}^{\infty}$ be i.i.d. copies of $(\tilde{S}_1, \tilde{S}_2)$, and define $\tilde{S}(t) = (\tilde{S}_1^T(t), \tilde{S}_2^T(t))^T$, $t = 1, 2, \cdots$.

Definition 7: Let $\tilde{\kappa}$ be a non-negative number, $\tilde{\mathcal{D}}_1$ be a non-empty compact subset of $\{\tilde{D}_1 : 0 \preceq \tilde{D}_1 \preceq \Sigma_{\tilde{S}}\}$, and $\tilde{\mathcal{D}}_2$ be a non-empty compact subset of $\{\tilde{D}_2 : 0 \preceq \tilde{D}_2 \preceq \Sigma_{\tilde{S}_2}\}$. We say $(\tilde{\kappa}, \tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2)$ is achievable for System $\tilde{\Pi}$ if, for every $\epsilon > 0$, there exist an encoding function $f^{(m,n)} : \mathbb{R}^{\tilde{\ell}_1 \times m} \times \mathbb{R}^{\tilde{\ell}_2 \times m} \to \mathcal{X}^n$ as well as decoding functions $g_1^{(n,m)} : \mathcal{Y}_1^n \times \mathbb{R}^{\tilde{\ell}_2 \times m} \to \mathbb{R}^{\tilde{\ell} \times m}$ and $g_2^{(n,m)} : \mathcal{Y}_2^n \to \mathbb{R}^{\tilde{\ell}_2 \times m}$ such that

$$\frac{n}{m} \le \tilde{\kappa} + \epsilon,$$

$$\min_{\tilde{D}_1 \in \tilde{\mathcal{D}}_1} \left\| \sum_{t=1}^m \mathbb{E}[(\tilde{S}(t) - \hat{S}_1(t))(\tilde{S}(t) - \hat{S}_1(t))^T] - \tilde{D}_1 \right\| \le \epsilon,$$

$$\min_{\tilde{D}_2 \in \tilde{\mathcal{D}}_2} \left\| \sum_{t=1}^m \mathbb{E}[(\tilde{S}_2(t) - \hat{S}_2(t))(\tilde{S}_2(t) - \hat{S}_2(t))^T] - \tilde{D}_2 \right\| \le \epsilon.$$

The set of all achievable $(\tilde{\kappa}, \tilde{D}_1, \tilde{D}_2)$ for System $\tilde{\Pi}$ is denoted by $\tilde{\Gamma}_G$.

Remark: Here we allow $f^{(m,n)}$, $g_1^{(n,m)}$, and $g_2^{(n,m)}$ to be non-deterministic functions as long as the Markov chains $(\tilde{S}_1^m, \tilde{S}_2^m) \leftrightarrow X^n \leftrightarrow (Y_1^n, Y_2^n)$, $\tilde{S}_1^m \leftrightarrow (Y_1^n, \tilde{S}_2^m) \leftrightarrow \hat{S}_1^m$, and $\tilde{S}_2^m \leftrightarrow Y_2^n \leftrightarrow \hat{S}_2^m$ are preserved.

Note that

$$\Sigma_{\tilde{S}} = \begin{pmatrix} \Sigma_{\tilde{S}_1} & \Sigma_{\tilde{S}_1, \tilde{S}_2} \\ \Sigma_{\tilde{S}_2, \tilde{S}_1} & \Sigma_{\tilde{S}_2} \end{pmatrix},$$

where $\Sigma_{\tilde{S}_1, \tilde{S}_2} = \mathbb{E}[\tilde{S}_1 \tilde{S}_2^T]$ and $\Sigma_{\tilde{S}_2, \tilde{S}_1} = \mathbb{E}[\tilde{S}_2 \tilde{S}_1^T]$. Moreover, we write

$$\tilde{D}_1 = \begin{pmatrix} \tilde{D}_{1,1} & \tilde{D}_{1,2} \\ \tilde{D}_{2,1} & \tilde{D}_{2,2} \end{pmatrix}$$

for any $\tilde{D}_1 \in \tilde{\mathcal{D}}_1$, where $\tilde{D}_{i,i}$ is an $\tilde{\ell}_i \times \tilde{\ell}_i$ matrix, $i = 1, 2$. The following source-channel separation theorem is a simple translation of Theorem 1 to the quadratic Gaussian setting. Its proof is omitted.

Theorem 4: $(\tilde{\kappa}, \tilde{D}_1, \tilde{D}_2) \in \tilde{\Gamma}_G$ if and only if $(R_{\tilde{S}_1 | \tilde{S}_2}(\tilde{\mathcal{D}}_1), R_{\tilde{S}_2}(\tilde{\mathcal{D}}_2)) \in \tilde{\kappa} \mathcal{C}_1(p_{Y_1, Y_2 | X})$, where

$$R_{\tilde{S}_1 | \tilde{S}_2}(\tilde{\mathcal{D}}_1) = \min_{\tilde{D}_1 \in \tilde{\mathcal{D}}_1} \frac{1}{2} \log \left(\frac{|\Sigma_{\tilde{S}_1} - \Sigma_{\tilde{S}_1, \tilde{S}_2} \Sigma_{\tilde{S}_2}^{-1} \Sigma_{\tilde{S}_2, \tilde{S}_1}|}{|\tilde{D}_{1,1} - K \tilde{D}_{2,1}|} \right),$$

$$R_{\tilde{S}_2}(\tilde{\mathcal{D}}_2) = \min_{\tilde{D}_2 \in \tilde{\mathcal{D}}_2} \frac{1}{2} \log \left(\frac{|\Sigma_{\tilde{S}_2}|}{|\tilde{D}_2|} \right)$$

with K being any solution[6] of $K \tilde{D}_{2,2} = \tilde{D}_{1,2}$.

Remark: It can be verified that

$$R_{\tilde{S}_1 | \tilde{S}_2}(\tilde{\mathcal{D}}_1) = \min_{p_{\hat{S}_1 | \tilde{S}} : \mathbb{E}[(\tilde{S} - \hat{S}_1)(\tilde{S} - \hat{S}_1)^T] \in \tilde{\mathcal{D}}_1} I(\tilde{S}_1; \hat{S}_1 | \tilde{S}_2),$$

$$R_{\tilde{S}_2}(\tilde{\mathcal{D}}_2) = \min_{p_{\hat{S}_2 | \tilde{S}_2} : \mathbb{E}[(\tilde{S}_2 - \hat{S}_2)(\tilde{S}_2 - \hat{S}_2)^T] \in \tilde{\mathcal{D}}_2} I(\tilde{S}_2; \hat{S}_2).$$

[6] If $\tilde{D}_{2,2}$ is invertible, then $K = \tilde{D}_{1,2} \tilde{D}_{2,2}^{-1}$.

which highlights the similarity between Theorem 1 and Theorem 4.

Again, in the quadratic Gaussian setting, the source-channel separation theorem for System $\tilde{\Pi}$ can be leveraged to derive a necessary condition for System Π. For any $D_i \in \mathcal{D}_i$, $i = 1, 2$, let $\mathcal{R}_1(\Sigma_S, D_1, D_2)$ denote the convex closure of the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le \frac{1}{2} \log \left(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||\Sigma_S + \Sigma_Z|} \right),$$

$$R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|} \right)$$

for some $\Sigma_Z \succ 0$, and let $\mathcal{R}_2(\Sigma_S, D_1, D_2)$ denote the convex closure of the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_1 \le \frac{1}{2} \log \left(\frac{|\Sigma_S + \Sigma_Z|}{|D_1 + \Sigma_Z|} \right),$$

$$R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_S||D_2 + \Sigma_Z|}{|D_2||\Sigma_S + \Sigma_Z|} \right)$$

for some $\Sigma_Z \succ 0$. By setting $\Sigma_U = \Sigma_S (\Sigma_S + \Sigma_Z)^{-1} \Sigma_S$, we can write $\mathcal{R}_1(\Sigma_S, D_1, D_2)$ equivalently as the convex hull of the set of $(R_1, R_2) \in \mathbb{R}_+^2$ such that

$$R_1 \le \frac{1}{2} \log \left(\frac{|\Sigma_U \Sigma_S^{-1} D_1 + \Sigma_S - \Sigma_U|}{|D_1|} \right),$$

$$R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|\Sigma_U \Sigma_S^{-1} D_2 + \Sigma_S - \Sigma_U|} \right)$$

for some Σ_U satisfying $0 \preceq \Sigma_U \preceq \Sigma_S$; similarly, $\mathcal{R}_2(\Sigma_S, D_1, D_2)$ can be written equivalently as the convex hull of the set of $(R_1, R_2) \in \mathbb{R}_+^2$ such that

$$R_1 \le \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|\Sigma_U \Sigma_S^{-1} D_1 + \Sigma_S - \Sigma_U|} \right),$$

$$R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_U \Sigma_S^{-1} D_2 + \Sigma_S - \Sigma_U|}{|D_2|} \right)$$

for some Σ_U satisfying $0 \preceq \Sigma_U \preceq \Sigma_S$.

Let S be an $\ell \times 1$ zero-mean Gaussian random vector with positive definite covariance matrix Σ_S. Recall the definition of $\mathcal{R}_i(p_{S, \hat{S}_1, \hat{S}_2})$, $i = 1, 2$ in Section V. The following result provides a connection between $\mathcal{R}_i(\Sigma_S, D_1, D_2)$ and $\mathcal{R}_i(p_{S, \hat{S}_1, \hat{S}_2})$, $i = 1, 2$.

Proposition 13: If $\mathbb{E}[(S - \hat{S}_i)(S - \hat{S}_i)^T] = D_i \in \mathcal{D}_i$, $i = 1, 2$, then

$$\mathcal{R}_i(p_{S, \hat{S}_1, \hat{S}_2}) \supseteq \mathcal{R}_i(\Sigma_S, D_1, D_2), \quad i = 1, 2. \tag{54}$$

Moreover, if $S - \hat{S}_i$ and $\hat{S}_i$ are independent zero-mean Gaussian random vectors with covariance matrices D_i and $\Sigma_S - D_i$, respectively, $i = 1, 2$, where $0 \preceq D_1 \preceq D_2 \preceq \Sigma_S$, then

$$\mathcal{R}_1(p_{S, \hat{S}_1, \hat{S}_2}) = \mathcal{R}_1(\Sigma_S, D_1, D_2), \tag{55}$$

$$\mathcal{R}_2(p_{S, \hat{S}_1, \hat{S}_2}) \subseteq \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|D_2|} \right), \right.$$

$$\left. R_1 + R_2 \le \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|D_1|} \right) \right\}. \tag{56}$$

Proof: By symmetry, it suffices to prove (54) for $i = 1$. Given any Σ_U satisfying $0 \preceq \Sigma_U \preceq \Sigma_S$, we can find U jointly distributed with S such that U and $S - U$ are independent zero-mean Gaussian random vectors with covariance matrices Σ_U and $\Sigma_S - \Sigma_U$, respectively. Note that for any $(\hat{S}_1, \hat{S}_2)$ jointly distributed with such (U, S) subject to the constraints that $\mathbb{E}[(S - \hat{S}_i)(S - \hat{S}_i)^T] = D_i \in \mathcal{D}_i$, $i = 1, 2$, and that $U \leftrightarrow S \leftrightarrow (\hat{S}_1, \hat{S}_2)$ form a Markov chain, we have

$$I(S; \hat{S}_1 | U) \geq \frac{1}{2} \log \left(\frac{|\Sigma_U \Sigma_S^{-1} D_1 + \Sigma_S - \Sigma_U|}{|D_1|} \right), \quad (57)$$

$$I(U; \hat{S}_2) \geq \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|\Sigma_U \Sigma_S^{-1} D_2 + \Sigma_S - \Sigma_U|} \right), \quad (58)$$

where the equalities in (57) and (58) hold when $S - \hat{S}_i$ and $\hat{S}_i$ are independent zero-mean Gaussian random vectors with covariance matrices D_i and $\Sigma_S - D_i$, respectively, $i = 1, 2$. Now the desired result follows by the convexity of $\mathcal{R}_1(p_{S, \hat{S}_1, \hat{S}_2})$.

To prove (55), it suffices to consider the non-degenerate case $0 \prec D_1 \preceq D_2 \prec \Sigma_S$; the general case $0 \preceq D_1 \preceq D_2 \preceq \Sigma_S$ can be proved via a simple limiting argument. Let O_i be a zero-mean Gaussian random vector, independent of (U, S), with covariance matrix $\Sigma_{O_i} = (D_i^{-1} - \Sigma_S^{-1})^{-1}$, $i = 1, 2$. It is clear that

$$I(S; \hat{S}_1 | U) = I(S; S + O_1 | U),$$
$$I(U; \hat{S}_2) = I(U; S + O_2).$$

For any $\lambda \in [0, 1]$,

$$\begin{aligned}
&\max_{(R_1, R_2) \in \mathcal{R}_1(p_{S, \hat{S}_1, \hat{S}_2})} \lambda R_1 + (1 - \lambda) R_2 \\
&= \max_{p_{U|S}} \lambda I(S; \hat{S}_1 | U) + (1 - \lambda) I(U; \hat{S}_2) \\
&= \max_{p_{U|S}} \lambda I(S; S + O_1 | U) + (1 - \lambda) I(U; S + O_2) \\
&= \max_{0 \preceq \Sigma_U \preceq \Sigma_S} \frac{\lambda}{2} \log \left(\frac{|\Sigma_S - \Sigma_U + \Sigma_{O_1}|}{|\Sigma_{O_1}|} \right) \\
&\quad + \frac{1 - \lambda}{2} \log \left(\frac{|\Sigma_S + \Sigma_{O_2}|}{|\Sigma_S - \Sigma_U + \Sigma_{O_2}|} \right) \quad (59) \\
&= \max_{0 \preceq \Sigma_U \preceq \Sigma_S} \frac{\lambda}{2} \log \left(\frac{|\Sigma_U \Sigma_S^{-1} D_1 + \Sigma_S - \Sigma_U|}{|D_1|} \right) \\
&\quad + \frac{1 - \lambda}{2} \log \left(\frac{|\Sigma_S|}{|\Sigma_U \Sigma_S^{-1} D_2 + \Sigma_S - \Sigma_U|} \right) \\
&= \max_{(R_1, R_2) \in \mathcal{R}_1(\Sigma_S, D_1, D_2)} \lambda R_1 + (1 - \lambda) R_2,
\end{aligned}$$

where (59) is due to the conditional version of [25, Corollary 4]. This together with the convexity of $\mathcal{R}_1(p_{S, \hat{S}_1, \hat{S}_2})$ and $\mathcal{R}_1(\Sigma_S, D_1, D_2)$ proves (55). It can be verified that

$$\begin{aligned}
I(S; \hat{S}_2 | U) &\leq I(S; \hat{S}_2) \\
&\leq \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|D_2|} \right)
\end{aligned}$$

and

$$\begin{aligned}
I(U; \hat{S}_1) + I(S; \hat{S}_2 | U) &\leq I(U; \hat{S}_1) + I(S; \hat{S}_1 | U) \\
&= I(S; \hat{S}_1) \\
&= \frac{1}{2} \log \left(\frac{|\Sigma_S|}{|D_1|} \right),
\end{aligned}$$

from which (56) follows immediately. ∎

Theorem 5: For any $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$, there exist $D_i \in \mathcal{D}_i$, $i = 1, 2$, such that

$$\mathcal{R}_i(\Sigma_S, D_1, D_2) \subseteq \kappa \mathcal{C}_i(p_{Y_1, Y_2 | X}), \quad i = 1, 2. \quad (60)$$

Proof: By symmetry, it suffices to prove (60) for $i = 1$. Let $\{Z(t)\}_{t=1}^{\infty}$ be an i.i.d. vector Gaussian process, independent of $\{S(t)\}_{t=1}^{\infty}$, where each $Z(t)$ is an $\ell \times 1$ zero-mean Gaussian random vector with positive definite covariance matrix Σ_Z. Define $\tilde{S}_1(t) = S(t)$ and $\tilde{S}_2(t) = S(t) + Z(t)$ for $t = 1, 2, \cdots$. Now consider an arbitrary tuple $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$. Given any $\epsilon > 0$, according to Definition 6, there exist encoding function $f^{(m,n)} : \mathbb{R}^{\ell \times m} \to \mathcal{X}^n$ and decoding functions $g_i^{(n,m)} : \mathcal{Y}_i^n \to \mathbb{R}^{\ell \times m}$, $i = 1, 2$, satisfying[7]

$$\frac{n}{m} \leq \kappa + \epsilon,$$

$$\min_{D_i \in \mathcal{D}_i} \left\| \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(S(t) - \hat{S}_i^{(\epsilon)}(t))(S(t) - \hat{S}_i^{(\epsilon)}(t))^T] - D_i \right\|$$
$$\leq \epsilon, \quad i = 1, 2.$$

Therefore, one can find a sequence $\epsilon_1, \epsilon_2, \cdots$ converging to zero such that

$$\lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(S(t) - \hat{S}_i^{(\epsilon_k)}(t))(S(t) - \hat{S}_i^{(\epsilon_k)}(t))^T] = D_i \quad (61)$$

for some $D_i \in \mathcal{D}_i$, $i = 1, 2$. Note that

$$\begin{aligned}
&\lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(\tilde{S}_1(t) - \hat{S}_1^{(\epsilon_k)}(t))(\tilde{S}_1(t) - \hat{S}_1^{(\epsilon_k)}(t))^T] \\
&= \lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(\tilde{S}_1(t) - \hat{S}_1^{(\epsilon_k)}(t))(\tilde{S}_2(t) - \hat{S}_1^{(\epsilon_k)}(t))^T] \\
&= \lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(\tilde{S}_2(t) - \hat{S}_1^{(\epsilon_k)}(t))(\tilde{S}_1(t) - \hat{S}_1^{(\epsilon_k)}(t))^T] \\
&= D_1, \\
&\lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(\tilde{S}_2(t) - \hat{S}_1^{(\epsilon_k)}(t))(\tilde{S}_2(t) - \hat{S}_1^{(\epsilon_k)}(t))^T] \\
&= D_1 + \Sigma_Z, \\
&\lim_{k \to \infty} \frac{1}{m} \sum_{t=1}^{m} \mathbb{E}[(\tilde{S}_2(t) - \hat{S}_2^{(\epsilon_k)}(t))(\tilde{S}_2(t) - \hat{S}_2^{(\epsilon_k)}(t))^T] \\
&= \tilde{D}_2 \triangleq D_2 + \Sigma_Z.
\end{aligned}$$

As a consequence, we must have $(\kappa, \{\tilde{D}_1\}, \{\tilde{D}_2\}) \in \tilde{\Gamma}_G$, where

$$\tilde{D}_1 = \begin{pmatrix} D_1 & D_1 \\ D_1 & D_1 + \Sigma_Z \end{pmatrix}.$$

It then follows from Theorem 4 that

$$\left(\frac{1}{2} \log \left(\frac{|\Sigma_S - \Sigma_S(\Sigma_S + \Sigma_Z)^{-1} \Sigma_S|}{|D_1 - D_1(D_1 + \Sigma_Z)^{-1} D_1|} \right), \frac{1}{2} \log \left(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|} \right) \right)$$
$$\in \kappa \mathcal{C}_1(p_{Y_1, Y_2 | X}).$$

---

[7] We have denoted $\hat{S}_i(t)$ by $\hat{S}_i^{(\epsilon)}(t)$ to stress its dependence on ϵ

Here one can fix (D_1, D_2) and choose the positive definite covariance matrix Σ_Z arbitrarily; moreover, it can be verified that

$$\frac{|\Sigma_S - \Sigma_S(\Sigma_S + \Sigma_Z)^{-1}\Sigma_S|}{|D_1 - D_1(D_1 + \Sigma_Z)^{-1}D_1|} = \frac{|D_1^{-1} + \Sigma_Z^{-1}|}{|\Sigma_S^{-1} + \Sigma_Z^{-1}|}$$
$$= \frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||\Sigma_S + \Sigma_Z|}.$$

This completes the proof of Theorem 5. ∎

Note that $\mathcal{R}_1(\Sigma_S, D_1, D_2)$ coincides with the capacity region of vector Gaussian broadcast channel with covariance power constraint Σ_S and noise covariances $\Delta_i \triangleq (D_i^{-1} - \Sigma_S^{-1})^{-1}$, $i = 1, 2$, when $0 \prec D_1 \preceq D_2 \prec \Sigma_S$. For this reason, we shall denote $\mathcal{R}_1(\Sigma_S, D_1, D_2)$ alternatively by $\mathcal{C}(\text{G-BC}(\Sigma_S, \Delta_1, \Delta_2))$ (even when Δ_1 and Δ_2 are not well-defined). One can obtain the following refined necessary condition for the case where $p_{Y_1, Y_2|X}$ is a scalar Gaussian broadcast channel.

Theorem 6: If $p_{Y_1, Y_2|X}$ is a G-BC(P, N_1, N_2) with $0 < N_1 \leq N_2$, then, for any $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$, there exist $D_i \in \mathcal{D}_i$, $i = 1, 2$, with $D_1 \preceq D_2$ such that

$$\mathcal{C}(\text{G-BC}(\Sigma_S, \Delta_1, \Delta_2)) \subseteq \kappa\mathcal{C}(\text{G-BC}(P, N_1, N_2)).$$

Proof: According to the remark after Definition 6, there is no loss of generality in setting $\hat{S}_i^m = \mathbb{E}[S^m|Y_i^n]$, $i = 1, 2$. As a consequence, in (61) we must have $D_1 \preceq D_2$ if $p_{Y_2|X}$ is degraded with respect to $p_{Y_1|X}$. Now one can readily adapt the proof of Theorem 5 to the current setting to show that, for any $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$, there exist $D_i \in \mathcal{D}_i$, $i = 1, 2$, with $D_1 \preceq D_2$, such that

$$\mathcal{R}_i(\Sigma_S, D_1, D_2) \subseteq \kappa\mathcal{C}_i(\text{G-BC}(P, N_1, N_2)), \quad i = 1, 2. \quad (62)$$

It follows from Proposition 8 that $\mathcal{C}_1(\text{G-BC}(P, N_1, N_2)) = \mathcal{C}(\text{G-BC}(P, N_1, N_2))$, and $\mathcal{C}_2(\text{G-BC}(P, N_1, N_2))$ is given by the set of $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_2 \leq \frac{1}{2}\log\left(\frac{P + N_2}{N_2}\right),$$
$$R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{P + N_1}{N_1}\right).$$

Note that $\mathcal{R}_1(\Sigma_S, D_1, D_2) \subseteq \kappa\mathcal{C}_1(\text{G-BC}(P, N_1, N_2))$ implies

$$\frac{1}{2}\log\left(\frac{|\Sigma_S|}{|D_i|}\right) \leq \frac{\kappa}{2}\log\left(\frac{P + N_i}{N_i}\right), \quad i = 1, 2.$$

Moreover, in view of (54) and (56) in Proposition 13, we have

$$\mathcal{R}_2(\Sigma_S, D_1, D_2) \subseteq \Big\{(R_1, R_2) \in \mathbb{R}_+^2 : R_2 \leq \frac{1}{2}\log\left(\frac{|\Sigma_S|}{|D_2|}\right),$$
$$R_1 + R_2 \leq \frac{1}{2}\log\left(\frac{|\Sigma_S|}{|D_1|}\right)\Big\}.$$

Therefore,

$$\mathcal{R}_1(\Sigma_S, D_1, D_2) \subseteq \kappa\mathcal{C}_1(\text{G-BC}(P, N_1, N_2))$$
$$\Rightarrow \mathcal{R}_2(\Sigma_S, D_1, D_2) \subseteq \kappa\mathcal{C}_2(\text{G-BC}(P, N_1, N_2))$$

when $0 \preceq D_1 \preceq D_2 \preceq \Sigma_S$. This completes the proof of Theorem 6. ∎

For the case $0 \preceq D_1 \preceq D_2 \preceq \Sigma_S$, one can show by leveraging Proposition 13 that (62) is equivalent to the

existence of $(\hat{S}_1, \hat{S}_2)$ with $\mathbb{E}[(S - \hat{S}_i)(S - \hat{S}_i)^T] = D_i \in \mathcal{D}_i$, $i = 1, 2$, such that

$$\mathcal{R}_i(p_{S, \hat{S}_1, \hat{S}_2}) \subseteq \kappa\mathcal{C}_i(\text{G-BC}(P, N_1, N_2)), \quad i = 1, 2;$$

in fact, there is no loss of generality in assuming that $S - \hat{S}_i$ and $\hat{S}_i$ are independent zero-mean Gaussian random vectors with covariance matrices D_i and $\Sigma_S - D_i$, respectively, $i = 1, 2$. Note that U is not restricted to the form $U = S + Z$ (or equivalently $U = \mathbb{E}[S|S + Z]$) in the definition of $\mathcal{R}_i(p_{S, \hat{S}_1, \hat{S}_2})$, $i = 1, 2$, where Z is a zero-mean Gaussian random vector independent of S. Therefore, removing this restriction does not lead to a stronger necessary condition. This provides a certain justification for the choice of the auxiliary random variable in [3].

With no essential loss of generality, henceforth we focus on the non-degenerate case $\kappa > 0$. Define

$$P^\star = \min\{P \geq 0 : \mathcal{C}(\text{G-BC}(\Sigma_S, \Delta_1, \Delta_2))$$
$$\subseteq \kappa\mathcal{C}(\text{G-BC}(P, N_1, N_2))\}.$$

It is clear that, for any $\Sigma_Z \succ 0$, there exists $\beta \in [0, 1]$ such that

$$\frac{1}{2}\log\left(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||\Sigma_S + \Sigma_Z|}\right) \leq \frac{\kappa}{2}\log\left(\frac{\beta P^\star + N_1}{N_1}\right),$$
$$\frac{1}{2}\log\left(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|}\right) \leq \frac{\kappa}{2}\log\left(\frac{P^\star + N_2}{\beta P^\star + N_2}\right),$$

which can be rewritten as

$$\beta P^\star \geq N_1\left(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||\Sigma_S + \Sigma_Z|}\right)^{\frac{1}{\kappa}} - N_1,$$
$$\beta P^\star \leq (P^\star + N_2)\left(\frac{|D_2 + \Sigma_Z|}{|\Sigma_S + \Sigma_Z|}\right)^{\frac{1}{\kappa}} - N_2.$$

Hence, for any $\Sigma_Z \succ 0$, we have

$$(P^\star + N_2)\left(\frac{|D_2 + \Sigma_Z|}{|\Sigma_S + \Sigma_Z|}\right)^{\frac{1}{\kappa}} - N_2$$
$$\geq N_1\left(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||\Sigma_S + \Sigma_Z|}\right)^{\frac{1}{\kappa}} - N_1,$$

i.e.,

$$P^\star \geq N_1\left(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||D_2 + \Sigma_Z|}\right)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\left(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|}\right)^{\frac{1}{\kappa}} - N_2. \quad (63)$$

Moreover, there must exist some $\beta \in [0, 1]$ and a sequence of positive definite matrices $\Sigma_Z^{(k)}$, $k = 1, 2, \cdots$, such that

$$\lim_{k \to \infty} \frac{1}{2}\log\left(\frac{|\Sigma_S||D_1 + \Sigma_Z^{(k)}|}{|D_1||\Sigma_S + \Sigma_Z^{(k)}|}\right) = \frac{\kappa}{2}\log\left(\frac{\beta P^\star + N_1}{N_1}\right),$$
$$\lim_{k \to \infty} \frac{1}{2}\log\left(\frac{|\Sigma_S + \Sigma_Z^{(k)}|}{|D_2 + \Sigma_Z^{(k)}|}\right) = \frac{\kappa}{2}\log\left(\frac{P^\star + N_2}{\beta P^\star + N_2}\right),$$

which implies

$$P^\star = \lim_{k \to \infty} N_1\left(\frac{|\Sigma_S||D_1 + \Sigma_Z^{(k)}|}{|D_1||D_2 + \Sigma_Z^{(k)}|}\right)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\left(\frac{|\Sigma_S + \Sigma_Z^{(k)}|}{|D_2 + \Sigma_Z^{(k)}|}\right)^{\frac{1}{\kappa}} - N_2. \quad (64)$$

Combining (63) and (64) gives

$$P^\star = \sup_{\Sigma_Z \succ 0} N_1\Big(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}} - N_2. \quad (65)$$

Therefore, by Theorem 6, if $(\kappa, \mathcal{D}_1, \mathcal{D}_2) \in \Gamma_G$, then

$$P \geq \inf_{D_1, D_2} \sup_{\Sigma_Z \succ 0} N_1\Big(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}} - N_2, \quad (66)$$

where the infimum is over D_1 and D_2 subject to the constraints $D_i \in \mathcal{D}_i$, $i = 1, 2$, and $D_1 \preceq D_2$. For the case where $\mathcal{D}_i = \{D_i : 0 \preceq D_i \preceq \Theta_i\}$, $i = 1, 2$, for some Θ_1 and Θ_2 satisfying $0 \prec \Theta_1 \preceq \Theta_2 \preceq \Sigma_S$, we can simplify (66) to

$$P \geq \sup_{\Sigma_Z \succ 0} N_1\Big(\frac{|\Sigma_S||\Theta_1 + \Sigma_Z|}{|\Theta_1||\Theta_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_S + \Sigma_Z|}{|\Theta_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}} - N_2,$$

from which one can readily recover [3, Theorem 1] by setting $\ell = 1$.

Now partition $S(t)$ to the form $S(t) = (S_1^T(t), S_2^T(t))^T$, $t = 1, 2, \cdots$, where each $S_i(t)$ is an $\ell_i \times 1$ zero-mean Gaussian random vector with positive definite covariance matrix Σ_{S_i}, $i = 1, 2$. We require that $\{S_i(t)\}_{t=1}^{\infty}$ be reconstructed at receiver i subject to positive definite covariance distortion constraint Λ_i, $i = 1, 2$. This corresponds to the case where $\mathcal{D}_i = \mathcal{D}_i(\Lambda_i) \triangleq \{D_i : 0 \preceq D_i \preceq \Sigma_S, D_{i,i} \preceq \Lambda_i\}$ with D_i partitioned to the form

$$D_i = \begin{pmatrix} D_{i,1} & \# \\ \# & D_{i,2} \end{pmatrix}, \quad i = 1, 2.$$

Therefore, the lower bound in (66) is also applicable here. By restricting Σ_Z to a special block diagonal form[8]

$$\Sigma_Z = \begin{pmatrix} \lambda I & 0 \\ 0 & \Sigma_{Z_2} \end{pmatrix},$$

one can deduce from (66)

$$P \geq \inf_{D_1, D_2} \sup_{\Sigma_{Z_2} \succ 0} \lim_{\lambda \to \infty} N_1\Big(\frac{|\Sigma_S||D_1 + \Sigma_Z|}{|D_1||D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_S + \Sigma_Z|}{|D_2 + \Sigma_Z|}\Big)^{\frac{1}{\kappa}} - N_2$$
$$= \inf_{D_1, D_2} \sup_{\Sigma_{Z_2} \succ 0} N_1\Big(\frac{|\Sigma_S||D_{1,2} + \Sigma_{Z_2}|}{|D_1||D_{2,2} + \Sigma_{Z_2}|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_{S_2} + \Sigma_{Z_2}|}{|D_{2,2} + \Sigma_{Z_2}|}\Big)^{\frac{1}{\kappa}} - N_2, \quad (67)$$

where the infimum is over D_1 and D_2 subject to the constraints $D_i \in \mathcal{D}_i(\Lambda_i)$, $i = 1, 2$, and $D_1 \preceq D_2$. This potentially weakened lower bound, when specialized to the case $\kappa = 1$, is at least as tight as [18, Theorem 1]. Note that, for any

----

[8]Here I is an $\ell_1 \times \ell_1$ identity matrix

$D_i \in \mathcal{D}_i(\Lambda_i)$, $i = 1, 2$, and any positive definite matrix Σ_Z partitioned to the form

$$\Sigma_Z = \begin{pmatrix} \Sigma_{Z_1} & \# \\ \# & \Sigma_{Z_2} \end{pmatrix}, \quad (68)$$

we have

$$\frac{|\Sigma_S||D_{1,2} + \Sigma_{Z_2}|}{|D_1||D_{2,2} + \Sigma_{Z_2}|} \geq \frac{|\Sigma_S + \Sigma_Z||D_{1,2} + \Sigma_{Z_2}|}{|D_1 + \Sigma_Z||D_{2,2} + \Sigma_{Z_2}|}$$
$$\geq \frac{|\Sigma_S + \Sigma_Z|}{|D_{1,1} + \Sigma_{Z_1}||D_{2,2} + \Sigma_{Z_2}|}$$
$$\geq \frac{|\Sigma_S + \Sigma_Z|}{|\Lambda_1 + \Sigma_{Z_1}||\Lambda_2 + \Sigma_{Z_2}|} \quad (69)$$

and

$$\frac{|\Sigma_{S_2} + \Sigma_{Z_2}|}{|D_{2,2} + \Sigma_{Z_2}|} \geq \frac{|\Sigma_{S_2} + \Sigma_{Z_2}|}{|\Lambda_2 + \Sigma_{Z_2}|}. \quad (70)$$

Substituting (69) and (70) into (67) gives

$$P \geq \sup_{\Sigma_Z \succ 0} N_1\Big(\frac{|\Sigma_S + \Sigma_Z|}{|\Lambda_1 + \Sigma_{Z_1}||\Lambda_2 + \Sigma_{Z_2}|}\Big)^{\frac{1}{\kappa}}$$
$$+ (N_2 - N_1)\Big(\frac{|\Sigma_{S_2} + \Sigma_{Z_2}|}{|\Lambda_2 + \Sigma_{Z_2}|}\Big)^{\frac{1}{\kappa}} - N_2, \quad (71)$$

where Σ_Z is partitioned to the form in (68). Setting $\kappa = 1$ in (71) recovers [18, Corollary 1]. An equivalent form of the lower bound in (71) was first obtained by Bross *et al.* [15] via a different approach for the special case $\kappa = \ell_1 = \ell_2 = 1$. It is worth mentioning that source-channel separation is known to be suboptimal in general for this problem [16], [17]. Somewhat surprisingly, the lower bound in (71), derived with the aid of a source-channel separation theorem (i.e., Theorem 4), turns out to be tight when $\kappa = \ell_2 = 1$ [18, Theorem 2] and is achievable by a class of hybrid digital-analog coding schemes[9] [18, Section IV.B]. Therefore, the application of source-channel separation theorems is not restricted to the relatively limited scenarios where the separation architecture is optimal; they can also be used to prove the optimality of non-separation based schemes and determine the performance limits in certain scenarios where the separation architecture is suboptimal.

VII. CONCLUSION

We have established a source-channel separation theorem, which is further leveraged to derive a general necessary condition for the source broadcast problem. It is intriguing to note that, in certain cases (see, e.g., Theorem 3 and Theorem 6), this necessary condition takes the form of comparison of two capacity regions. This is by no means a coincidence. In fact, it suggests a new direction that can be explored to establish stronger converse results for the source broadcast problem [26].

----

[9]The hybrid scheme in [16] can be viewed as an extremal case of this class of schemes.

REFERENCES

[1] T. J. Goblick, Jr., "Theoretical limitations on the transmission of data from analog sources," *IEEE Trans. Inf. Theory*, vol. IT-11, no. 4, pp. 558–567, Oct. 1965.

[2] U. Mittal and N. Phamdo, "Hybrid digital-analog (HDA) joint source-channel codes for broadcasting and robust communications," *IEEE Trans. Inf. Theory*, vol. 48, no. 5, pp. 1082–1102, May 2002.

[3] Z. Reznic, M. Feder, and R. Zamir, "Distortion bounds for broadcasting with bandwidth expansion," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3778–3788, Aug. 2006.

[4] K. Narayanan, G. Caire, and M. Wilson, "Duality between broadcasting with bandwidth expansion and bandwidth compression," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, Nice, France, Jun. 2007, pp. 1161–1165.

[5] V. M. Prabhakaran, R. Puri, and K. Ramchandran, "Hybrid digital-analog codes for source-channel broadcast of Gaussian sources over Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 57, no. 7, pp. 4573–4588, Jul. 2011.

[6] P. Minero, S. H. Lim, and Y.-H. Kim, "A unified approach to hybrid coding," *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1509–1523, Apr. 2015.

[7] L. Ozarow, "On a source coding problem with two channels and three receivers," *Bell Syst. Tech. J.*, vol. 59, no. 10, pp. 1909–1921, Dec. 1980.

[8] H. Wang and P. Viswanath, "Vector Gaussian multiple description with individual and central receivers," *IEEE Trans. Inf. Theory*, vol. 53, no. 6, pp. 2133–2153, Jun. 2007.

[9] H. Wang and P. Viswanath, "Vector Gaussian multiple description with two levels of receivers," *IEEE Trans. Inf. Theory*, vol. 55, no. 1, pp. 401–410, Jan. 2009.

[10] J. Chen, "Rate region of Gaussian multiple description coding with individual and central distortion constraints," *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 3991–4005, Sep. 2009.

[11] L. Song, S. Shao, and J. Chen, "On the sum rate of multiple description coding with symmetric distortion constraints," *IEEE Trans. Inf. Theory*, submitted for publication.

[12] C. Tian, S. Diggavi, and S. Shamai, "Approximate characterizations for the Gaussian source broadcast distortion region," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 124–136, Jan. 2011.

[13] L. Tan, A. Khisti, and E. Soljanin, "Distortion bounds for broadcasting a binary source over binary erasure channels," in *Proc. 13th Canadian Workshop on Information Theory*, Toronto, ON, Canada, Jun. 18 - 21, 2013, pp. 49–54.

[14] C. Tian, J. Chen, S. Diggavi, and S. Shamai (Shitz), "Optimality and approximate optimality of source-channel separation in networks," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 904–918, Feb. 2014.

[15] S. Bross, A. Lapidoth, and S. Tinguely, "Broadcasting correlated Gaussians," *IEEE Trans. Inf. Theory*, vol. 56, no. 7, pp. 3057–3068, Jul. 2010.

[16] C. Tian, S. Diggavi, and S. Shamai (Shitz), "The achievable distortion region of sending a bivariate Gaussian source on the Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6419–6427, Oct. 2011.

[17] Y. Gao and E. Tuncel, "Separate source-channel coding for transmitting correlated Gaussian sources over degraded broadcast channels," *IEEE Trans. Inf. Theory*, vol. 59, no. 6, pp. 3619–3634, Jun. 2013.

[18] L. Song, J. Chen, and C. Tian, "Broadcasting Correlated Vector Gaussians," *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2465–2477, May 2015.

[19] G. Kramer and S. Shamai (Shitz), "Capacity for classes of broadcast channels with receiver side information," in *Proc. IEEE Inf. Theory Workshop*, Lake Tahoe, CA, Sep. 2 - 6, 2007, pp. 313–318.

[20] C. Nair, "Capacity regions of two new classes of two-receiver broadcast channels," *IEEE Trans. Inf. Theory*, vol. 56,, no. 9, pp. 4207–4214, Sep. 2010.

[21] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge, U.K.: Cambridge Univ. Press, 2011.

[22] J. Wang, J. Chen, L. Zhao, P. Cuff, and H. Permuter, "On the role of the refinement layer in multiple description coding and scalable coding," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp. 1443-1456, Mar. 2011.

[23] A. D. Wyner and J. Ziv, "A theorem on the entropy of certain binary sequences and applications: Part I," *IEEE Trans. Inf. Theory*, vol. IT-19, no. 6, pp. 769–772, Nov. 1973.

[24] C. C. Wang, S. R. Kulkarni, and H. V. Poor, "Finite-dimensional bounds on $\mathbb{Z}_m$ and binary LDPC codes with belief propagation decoders," *IEEE Trans. Inf. Theory*, vol. 53, no. 1, pp. 56–81, Jan. 2007.

[25] T. Liu and P. Viswanath, "An extremal inequality motivated by multi-terminal information-theoretic problems," *IEEE Trans. Inf. Theory*, vol. 53, no. 5, pp. 1839–1851, May 2007.

[26] K. Khezeli and J. Chen, "Outer bounds on the admissible source region for broadcast channels with correlated sources," *IEEE Trans. Inf. Theory*, vol. 61, no. 9, pp. 4616–4629, Sep. 2015.