ASYMPTOTIC ANALYSIS OF VLASOV-TYPE EQUATIONS UNDER STRONG LOCAL ALIGNMENT REGIME

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ABSTRACT. We consider the hydrodynamic limit of a collisionless and non-diffusive kinetic equation under strong local alignment regime. The local alignment is first considered by Karper, Mellet and Trivisa in [24], as a singular limit of an alignment force proposed by Motsch and Tadmor in [32]. As the local alignment strongly dominate, a weak solution to the kinetic equation under consideration converges to the local equilibrium, which has the form of mono-kinetic distribution. We use the relative entropy method and weak compactness to rigorously justify the weak convergence of our kinetic equation to the pressureless Euler system.

1. INTRODUCTION

Recently, a variety of mathematical models capturing the emergent phenomena of such as a flock of birds, a swarm of bacteria or a school of fish have received lots of attention extensively in the mathematical community as well as the physics, biology, engineering and social science, etc. (See for instance [7, 12, 33, 42] and the refereces therein.) In particular, the flocking model introduced by Cucker and Smale in [13] has received a considerable attention. (See for instance [9, 19, 20, 35].) In [32], Motsch and Tadmor improved the Cucker-Smale model by considering new interaction, which is non-local and non-symmetric alignment. Recently, in [25], Karper, Mellet and Trivisa introduced a strong local alignment interaction as the singular limit of the alignment proposed by Motsch-Tadmor.

In this paper, we consider the Vlasov-type equation with the local alignment interaction, which is described by

$$(1.1) \qquad \partial_t f + v \cdot \nabla_x f - \lambda \nabla_v \cdot (vf) + \nabla_v \cdot ((u - v)f) = 0.$$

Here, $f = f(t, x, v)$ is the one-particle distribution function at position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time $t > 0$, and $\lambda \geq 0$ is a frictional coefficient. In the force term $\nabla_v \cdot ((u - v)f)$, u denotes the averaged local velocity defined by

$$u = \frac{\int_{\mathbb{R}^d} vf \, dv}{\int_{\mathbb{R}^d} f \, dv},$$

thus the force term governs the local alignment interaction.

In [23, 25], the authors studied the kinetic Cucker-Smale model with the local alignment

term in (1.1). The local alignment force can be regarded as the singular limit of the following kinetic Motsch-Tadmor model [32]:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = 0,$$

(1.2)
$$L[f](x,v) = \frac{\int_{\mathbb{R}^{2d}} K^r(x-y)f(y,w)(w-v)dwdy}{\int_{\mathbb{R}^{2d}} K^r(x-y)f(y,w)dwdy}.$$

Here, the kernel K^r is a communication weight and r denotes the radius of influence of K^r. We know that the non-local alignment $L[f]$ is not symmetric interaction.

In [25], the authors considered the case that the communication weight is extremely concentrated nearby each agent. That is, they rigorously justified the singular limit $r \to 0$, i.e., as K^r converges to the Dirac distribution δ_0, the Motsch-Tadmor kernel K^r converges to a local alignment term:

$$L[f](x,v) \to \frac{\int_{\mathbb{R}^d} f(x,w)(w-v)dw}{\int_{\mathbb{R}^d} f(x,w)dw} =: u - v.$$

We here address a natural question on the asymptotic regime corresponding to a strong local alignment force in the revised Motsch-Tadmor model (1.1). More precisely, we aim to rigorously justify the hydrodynamic limit of the singular equation:

(1.3)
$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \lambda \nabla_v \cdot (v f^\varepsilon) + \frac{1}{\varepsilon} \nabla_v \cdot ((u^\varepsilon - v) f^\varepsilon) = 0, \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0,$$
$$f^\varepsilon(0, x, v) = f_0^\varepsilon(x, v),$$

where $u^\varepsilon = \dfrac{\int_{\mathbb{R}^d} v f^\varepsilon dv}{\int_{\mathbb{R}^d} f^\varepsilon dv}$ is the averaged local velocity.

The analytical difficulty for such issue is mainly due to the singularity of the local equilibrium. More precisely, the local equilibrium for (1.3) has the form of mono-kinetic distribution as

(1.4)
$$\rho(x, t)\delta(v - u(x, t)).$$

For this reason, we will consider the hydrodynamic limit in the sense of distributions. Under this hydrodynamic regime, the associated limit system is the well-known damped pressureless Euler system:

(1.5)
$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = -\lambda \rho u, \quad x \in \mathbb{R}^d, \; t > 0,$$
$$\rho|_{t=0} = \rho_0, \quad u|_{t=0} = u_0.$$

When there is no friction, i.e., $\lambda = 0$, the system (1.5) simply becomes the pressureless Euler system, which serve as a mathematical model for the formation of large-scale structures in astrophysics and the aggregation of sticky particles [36, 43]. We refer to [5, 6, 21, 41] as the study on the existence and uniqueness of weak solutions to the pressureless Euler system in one space dimention, and the δ-shock formation was studied in [10]. Concerning the study on the multidimensional pressureless Euler system with non-local alignment force capturing flocking behavior, we refer to [17, 32]. The macroscopic models studied in [17, 18, 32] was formally derived from kinetic Cucker-Smale model [20] and Motsch-Tadmor model [32] under the mono-kinetic ansatz (1.4), respectively.

If we consider the stochastic particles as the Brownian motion in (1.3), the kinetic equation (1.3) becomes Vlasov-Fokker-Planck type equation, in which the Laplacian term with respect to v is added as follows.

$$(1.6) \qquad \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \lambda \nabla_v \cdot (v f^\varepsilon) + \frac{1}{\varepsilon} \nabla_v \cdot ((u^\varepsilon - v) f^\varepsilon) - \frac{1}{\varepsilon} \Delta_v f^\varepsilon = 0.$$

This type model have been studied in [8, 23, 24, 25]. In [24], the authors justified the hydrodynamic limit of (1.6) with the non-local alignment term due to Cucker-Smale model, under the strong local alignment and strong noise regime. As a mathematical tool, they used the relative entropy method with strictly convex entropy to show the strong convergence from (1.6) to the isothermal Euler system with non-local alignment. Indeed, the solution f of (1.6) strongly converges to the local Maxwellian $\rho \exp(-\frac{|v-u|^2}{2})$ as $\varepsilon \to 0$, thanks to the existence of strictly convex entropy for the isothermal Euler system.

On the other hand, for the pressureless Euler system (1.5), we have a convex entropy given by

$$(1.7) \qquad \mathcal{E}(\rho, u) = \rho \frac{|u|^2}{2},$$

which is not strictly convex with respect to ρ. Notice that (1.7) is physically the kinetic energy, but regarded as an entropy in the theory of conservation laws. We will use the relative entropy method with (1.7) in the weak sense, compared to the previous results based on the strictly convex entropy. (See for example [8, 24, 31].)

Our main tool based on the relative entropy method follows a program initiated in [27, 28, 37, 38, 39] for the study on the stability of inviscid shocks for the scalar or system of conservation laws verifying a certain entropy condition. That has been used as an important tool in the study of asymptotic limits to conservation laws as well. (See for instance [1, 2, 3, 4, 11, 16, 15, 26, 29, 30, 31, 34, 40].)

The rest of this paper is organized as follows. In Section 2, we first present the existence results of the kinetic equation (1.3) and the asymptotic system (1.5), then the main theorem of this article. In Section 3 is devoted to the proof of Theorem 2.1.

2. Preliminaries

In this section, we present our main result on the hydrodynamic limit from a weak solution f^ε of the kinetic equation(1.3) to a strong solution (ρ, u) of the asymptotic system (1.5). For that, we first need to present the existence result for the weak solution of (1.3) and the classical solution of (1.5).

2.1. **Existence of weak solutions to** (1.3). We here say that f^ε is a weak solution of (1.3) if for any $T > 0$,

$$(2.1) \qquad \begin{aligned} &f^\varepsilon \in C(0, T; L^1(\mathbb{R}^{2d})) \cap L^\infty((0, T) \times \mathbb{R}^{2d}), \\ &|v|^2 f \in L^\infty(0, T; L^1(\mathbb{R}^{2d})), \end{aligned}$$

and (1.3) holds in the sense of distribution, that is, for any $\psi \in C_c^\infty([0, T) \times \mathbb{R}^{2d})$, the weak formulation holds

(2.2)
$$\int_0^t \int_{\mathbb{R}^{2d}} f^\varepsilon [\partial_t \psi + v \cdot \nabla_x \psi - \lambda v \cdot \nabla_v \psi + \frac{1}{\varepsilon}(u^\varepsilon - v) \cdot \nabla_v \psi] dv dx ds$$
$$+ \int_{\mathbb{R}^{2d}} f_0^\varepsilon \psi(0, \cdot) dv dx = 0.$$

Before stating the existence of weak solution to (1.3), we define a kinetic entropy $\mathcal{F}(f^\varepsilon)$ mathematically and dissipation $\mathcal{D}_\varepsilon(f^\varepsilon)$ for (1.3) by

$$\mathcal{F}(f^\varepsilon) := \int_{\mathbb{R}^d} \frac{|v|^2}{2} f^\varepsilon dv,$$

$$\mathcal{D}_\varepsilon(f^\varepsilon) := \int_{\mathbb{R}^{2d}} f^\varepsilon |u^\varepsilon - v|^2 dx dv.$$

Proposition 2.1. *For any $\varepsilon > 0$, assume that f_0^ε satisfies*

(2.3)
$$f_0^\varepsilon \geq 0, \quad f_0^\varepsilon \in L^1 \cap L^\infty(\mathbb{R}^{2d}), \quad |v|^2 f_0^\varepsilon \in L^1(\mathbb{R}^{2d}).$$

Then there exists a weak solution $f^\varepsilon \geq 0$ of (1.3) in the sense of (2.1) and (2.2). Moreover, f^ε satisfies the mass conservation $\|f^\varepsilon(t)\|_{L^1(\mathbb{R}^{2d})} = \|f_0^\varepsilon\|_{L^1(\mathbb{R}^{2d})}$ for all $t > 0$, and the entropy inequality

(2.4)
$$\int_{\mathbb{R}^d} \mathcal{F}(f^\varepsilon)(t) dx + \frac{1}{\varepsilon} \int_0^t \mathcal{D}_\varepsilon(f^\varepsilon)(s) ds + \lambda \int_0^t \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dv dx ds \leq \int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx.$$

We make use of the main result in [23] to prove this result. In [23], the authors showed the existence of weak solutions to the kinetic Cucker-Smale model with local alignment, noise, self-propulsion and friction:

(2.5)
$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) + \nabla_v \cdot ((u - v)f)$$
$$= \mu \Delta_v f - \nabla_v \cdot ((a - b|v|^2)vf),$$
$$L[f] = \int_{\mathbb{R}^{2d}} \psi(x - y) f(y, w)(w - v) dw dy, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0,$$

where the kernel ψ is a smooth symmetric, μ, a, b are nonnegative constants.
From their analysis in [23], we notice that all terms except the local alignment term $\nabla_v \cdot ((u - v)f)$ are not crucial in the existence theory of weak solutions to (2.5). Therefore the same existence theory can be applied to our case (1.3), so we refer [23] for its proof.
On the other hand, we notice that the entropy inequality (2.4) for (1.3) is simply reduced more than that for (2.5) with $\mu > 0$. Indeed, at least formally, multiplying (1.3) by $|v|^2$ and integrating this over the phase space $\mathbb{R}^{2d}$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} \frac{|v|^2}{2} f^\varepsilon dv dx + \lambda \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dv dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^{2d}} f^\varepsilon |u^\varepsilon - v|^2 dx dv = 0.$$

This justifies the inequality (2.4) rigorously by the standard method in [23].

2.2. **Existence of classical solutions to** (1.5)**.** We here present the local existence of classical solution to the pressureless Euler system (1.5) as the following Proposition.

Proposition 2.2. *Assume that*

(2.6) $\rho_0 > 0$ *in* $\mathbb{R}^d$ *and* $(\rho_0, u_0) \in H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$ *for* $s > \dfrac{d}{2} + 1$.

Then, there exists $T_* > 0$ *such that* (1.5) *has a unique classical solution* $(\rho, u) \in \mathcal{X}$ *where* $\mathcal{X}$ *is the solution space defined by*

$$\mathcal{X} := \{(\rho, u) : \rho \in C^0([0, T_*]; H^s(\mathbb{R}^d)) \cap C^1([0, T_*]; H^{s-1}(\mathbb{R}^d)),$$
$$u \in C^0([0, T_*]; H^{s+1}(\mathbb{R}^d)) \cap C^1([0, T_*]; H^s(\mathbb{R}^d))\}.$$

Moreover, the smooth solution (ρ, u) *satisfies the entropy equality*

$$\frac{d}{dt}\mathcal{E}(\rho, u) + 2\lambda\mathcal{E}(\rho, u) = 0,$$

where $\mathcal{E}(\rho, u)$ *is an entropy for* (1.5) *given by* (1.7).

Remark 2.1. *For* $s > \frac{d}{2} + 1$, *by Sobolev inequality, the solution* (ρ, u) *obtained from the theorem above is indeed a classical solution, i.e,* $(\rho, u) \in \mathcal{C}^1(\mathbb{R}^d \times [0, T_*])$.

Proposition 2.2 is the well-known result. We refer to [17] and [18] for its proof. In particular, the pressureless Euler system (1.5) with damping $\lambda > 0$ has a global-in-time classical solution, provided the initial data is small enough.

2.3. **Main result.** We here state our result on the hydrodynamic limit of (1.3). For any fixed initial data (ρ_0, u_0) satisfying (2.6), we will consider a proper initial data f_0^ε satisfying (2.3) and

- ($\mathcal{A}1$): $\int_{\mathbb{T}^d}(\mathcal{F}(f_0^\varepsilon) - \mathcal{E}(\rho_0, u_0))dx = \mathcal{O}(\sqrt{\varepsilon})$,

- ($\mathcal{A}2$): $\rho_0^\varepsilon - \rho_0 = \mathcal{O}(\sqrt{\varepsilon})$ in $L^\infty(\mathbb{R}^d)$,

- ($\mathcal{A}3$): $u_0^\varepsilon - u_0 = \mathcal{O}(\sqrt{\varepsilon})$ in $L^\infty(\mathbb{R}^d)$,

where ρ_0^ε and u_0^ε are defined by

$$\rho_0^\varepsilon = \int_{\mathbb{R}^d} f_0^\varepsilon dv, \quad u^\varepsilon = \frac{1}{\rho_0^\varepsilon}\int_{\mathbb{R}^d} v f_0^\varepsilon dv.$$

Theorem 2.1. *Assume that the initial datas* f_0^ε *and* (ρ_0, u_0) *satisfy* (2.3), (2.6) *and* ($\mathcal{A}1$)-($\mathcal{A}3$). *Let* f^ε *be the weak solution to* (1.3) *satisfying* (2.4), *and* (ρ, u) *be the classical solution to* (1.5). *Then, we have*

(2.7) $$\int_{\mathbb{R}^d} \rho^\varepsilon(t)|(u^\varepsilon - u)(t)|^2 dx \leq C(T_*)\sqrt{\varepsilon}, \quad t \leq T_*,$$

where $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon dv$, $\rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^d} v f^\varepsilon dv$ *and the constant* $C(T_*)$ *depends on* T_*. *Therefore, we have the following convergences*

(2.8)
$$\rho^\varepsilon \rightharpoonup \rho \quad weakly \quad in \ \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$
$$\rho^\varepsilon u^\varepsilon \rightharpoonup \rho u \quad weakly \quad in \ \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$
$$\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \rightharpoonup \rho u \otimes u \quad weakly \quad in \ \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$
$$\int_{\mathbb{R}^d} f^\varepsilon |v|^2 dv \rightharpoonup \rho|u|^2 \quad weakly \quad in \ \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$

where $\mathcal{M}([0,T_*]\times\mathbb{R}^d)$ is the space of nonnegative Radon measures on $[0,T_*]\times\mathbb{R}^d$. Moreover, for any $\psi\in C^1(\mathbb{R}^d)$ with $\nabla\psi\in L^\infty(\mathbb{R}^d)$,

$$(2.9) \qquad \int_{\mathbb{R}^d} f^\varepsilon\psi(v)dv \rightharpoonup \rho\psi(u) \quad \text{weakly} \quad \text{in } \mathcal{M}([0,T_*]\times\mathbb{R}^d).$$

3. Proof of Theorem 2.1

In this section, we present the proof of the Theorem 2.1. We first use the relative entropy method to derive a relative entropy inequality (2.7), which underlies the proof of the convergence (2.8). To use the relative entropy method, we consider the entropy $\mathcal{E}(\rho,u) = \rho\frac{|u|^2}{2}$, as mentioned in Introduction. Since $\mathcal{E}(\rho,u)$ is not strictly convex with respect to the density ρ, we will not be able to get the strong convergence of ρ by the relative entropy method with the entropy (1.7). Thus, after obtaining (2.7), we will use the mass conservation law and (2.7) to get the weak convergence of ρ.

3.1. **Relative entropy estimates.** In this part, we show the inequality (2.7). We begin by defining the following notations:

$$P = \rho u, \quad U = \begin{pmatrix}\rho\\P\end{pmatrix}, \quad A(U) = \begin{pmatrix}P^T\\\frac{P\otimes P}{\rho}\end{pmatrix}, \quad F(U) = \begin{pmatrix}0\\-P\end{pmatrix},$$

Then, we can rewrite (1.5) as a simpler form, that is the system of conservation laws of the form

$$(3.10) \qquad \partial_t U + \text{div}_x A(U) = \lambda F(U).$$

By the theory of conservation laws, the system (3.10) has a convex entropy $\mathcal{E}(\rho,u) = \rho\frac{|u|^2}{2}$ with entropy flux G given as a solution of

$$(3.11) \qquad \partial_{U_i} G_j(U) = \sum_{k=1}^{d+1}\partial_{U_k}\mathcal{E}(U)\partial_{U_i}A_{kj}(U), \quad 1\le i\le d+1, \quad 1\le j\le d.$$

introduce the relative entropy and relative flux:

$$(3.12) \qquad \begin{aligned} \mathcal{E}(V|U) &= \mathcal{E}(V) - \mathcal{E}(U) - D\mathcal{E}(U)\cdot(V-U),\\ A(V|U) &= A(V) - A(U) - DA(U)\cdot(V-U), \end{aligned}$$

where $D\mathcal{E}(U)$ and $DA(U)$ denote the gradients with respect to U, and $DA(U)\cdot(V-U)$ is a matrix defined as

$$(DA(U)\cdot(V-U))_{ij} = \sum_{k=1}^{d+1}\partial_{U_k}A_{ij}(U)(V_k-U_k), \quad 1\le i\le d+1, \quad 1\le j\le d.$$

Since $\mathcal{E}(U) = \frac{|P|^2}{2\rho}$, we have

$$D\mathcal{E}(U) = \begin{pmatrix}-\frac{|P|^2}{2\rho^2}\\\frac{P}{\rho}\end{pmatrix} = \begin{pmatrix}-\frac{|u|^2}{2}\\u\end{pmatrix}.$$

If we consider $V = \binom{q}{qw}$, then we have

$$
\begin{aligned}
\mathcal{E}(V|U) &= \frac{q}{2}|w|^2 - \frac{\rho}{2}|u|^2 + \frac{|u|^2}{2}(q - \rho) - u(qw - \rho u) \\
&= \frac{q}{2}|u - w|^2.
\end{aligned}
$$

(3.13)

The next proposition gives a key formulation to get the relative entropy inequality for the system of conservation laws (3.10).

Proposition 3.1. *Let U be a strong solution of* (3.10) *and V be a smooth function. Then, the following equality holds*

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} \mathcal{E}(V|U)dx = &\frac{d}{dt}\int_{\mathbb{R}^d} \mathcal{E}(V)dx - \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(V|U)dx \\
&- \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t V + div_x A(V) - \lambda F(V)]dx \\
&- \lambda \int_{\mathbb{R}^d} (D^2\mathcal{E}(U)F(U) \cdot (V - U) + D\mathcal{E}(U) \cdot F(V))dx
\end{aligned}
$$

(3.14)

The derivation for (3.14) can be found in [24, 31]. We present its proof in Appendix for the reader's convenience.

We now apply (3.14) to our issue together with the following macroscopic quantities corresponding to the weak solution f^ε obtained in previous section:

$$
P^\varepsilon = \rho^\varepsilon u^\varepsilon, \quad U^\varepsilon = \begin{pmatrix} \rho^\varepsilon \\ P^\varepsilon \end{pmatrix}
$$

where $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon dv$, $\rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^d} v f^\varepsilon dv$.

Lemma 3.1. *Assume that the initial datas U_0^ε and U_0 satisfy* (2.3), (2.6) *and* ($\mathcal{A}$1)-($\mathcal{A}$3). *Let U^ε be a weak solution of* (1.3) *satisfying* (2.4) *and U the classical solution of* (1.5). *Then, the following inequality holds*

$$
\begin{aligned}
\int_{\mathbb{R}^d} &\mathcal{E}(U^\varepsilon|U)(t)dx \\
&\leq C\sqrt{\varepsilon} + C\int_0^t \int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon|U)dxds + \int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon))dx \\
&- \int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + div_x A(U^\varepsilon) - \lambda F(U^\varepsilon)]dxds \\
&- \lambda \int_0^t \int_{\mathbb{R}^d} (D^2\mathcal{E}(U)F(U) \cdot (U^\varepsilon - U) + D\mathcal{E}(U) \cdot F(U^\varepsilon))dxds,
\end{aligned}
$$

(3.15)

where the integrand $\partial_t U^\varepsilon + div_x A(U^\varepsilon) - \lambda F(U^\varepsilon)$ is regarded in the sense of distributions.

Proof. First of all, it follows from (3.14) that

$$\int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon|U)(t)dx \leq \int_{\mathbb{R}^d} \mathcal{E}(U_0^\varepsilon|U_0)dx + \int_{\mathbb{R}^d} (\mathcal{E}(U^\varepsilon)(t) - \mathcal{E}(U_0^\varepsilon))dx$$

$$- \int_0^t \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(U^\varepsilon|U)dxds$$

(3.16)
$$- \int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + \text{div}_x A(U^\varepsilon) - \lambda F(U^\varepsilon)]dxds$$

$$- \lambda \int_0^t \int_{\mathbb{R}^d} (D^2\mathcal{E}(U)F(U) \cdot (U^\varepsilon - U) + D\mathcal{E}(U) \cdot F(U^\varepsilon))dxds$$

$$:= \sum_{k=1}^5 \mathcal{I}_k.$$

We estimate the three terms $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ above as follows.

• **Estimate of $\mathcal{I}_1$** We use (3.13), assumption ($\mathcal{A}3$) and the mass conservation to get

$$\mathcal{I}_1 = \frac{1}{2} \int_{\mathbb{R}^d} \rho_0^\varepsilon |u_0^\varepsilon - u_0|^2 dx \leq \mathcal{O}(\varepsilon) \int_{\mathbb{R}^d} \rho_0^\varepsilon dx \leq C\varepsilon.$$

• **Estimate of $\mathcal{I}_2$** We decompose $\mathcal{I}_2$ as

$$\mathcal{I}_2 = \int_{\mathbb{R}^d} (\mathcal{E}(U^\varepsilon)(t) - \mathcal{F}(f^\varepsilon)(t))dx + \int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon))dx$$

(3.17)
$$+ \int_{\mathbb{R}^d} (\mathcal{F}(f_0^\varepsilon) - \mathcal{E}(U_0))dx + \int_{\mathbb{R}^d} (\mathcal{E}(U_0) - \mathcal{E}(U_0^\varepsilon))dx$$

$$:= \sum_{k=1}^4 \mathcal{I}_2^k.$$

We first notice that Hölder's inequality yields

(3.18)
$$\rho^\varepsilon |u^\varepsilon|^2 = \frac{|\rho^\varepsilon u^\varepsilon|^2}{\rho^\varepsilon} = \frac{\left|\int_{\mathbb{R}^d} v f^\varepsilon dv\right|^2}{\int_{\mathbb{R}^d} f^\varepsilon dv} \leq \int_{\mathbb{R}^d} |v|^2 f^\varepsilon dv.$$

Thanks to (3.18), we have the minimization property

(3.19)
$$\mathcal{E}(U^\varepsilon) \leq \mathcal{F}(f^\varepsilon),$$

which yields $\mathcal{I}_2^1 \leq 0$.

The assumption ($\mathcal{A}1$) directly gives $\mathcal{I}_2^3 = \mathcal{O}(\sqrt{\varepsilon})$.

We use the assumptions ($\mathcal{A}2$) and ($\mathcal{A}3$) to get

$$\mathcal{I}_2^4 = \frac{1}{2} \int_{\mathbb{R}^d} \rho_0(|u_0|^2 - |u_0^\varepsilon|^2) + (\rho_0 - \rho_0^\varepsilon)|u_0^\varepsilon|^2 dx \leq C\sqrt{\varepsilon},$$

where we have used $u_0^\varepsilon = \mathcal{O}(1)$ due to ($\mathcal{A}2$). Thus we have

$$\mathcal{I}_2 \leq C\sqrt{\varepsilon} + \int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon))dx$$

• **Estimate of $\mathcal{I}_3$** We first compute the relative flux as follows. Since

$$A(U^\varepsilon|U) = A(U^\varepsilon) - A(U) - DA(U) \cdot (U^\varepsilon - U),$$

and

$$DA(U) \cdot (U^\varepsilon - U) = D_\rho A(U)(\rho^\varepsilon - \rho) + D_{P_i} A(U)(P_i^\varepsilon - P_i)$$
$$= \begin{pmatrix} (P^\varepsilon - P)^T \\ -\frac{\rho^\varepsilon - \rho}{\rho^2} P \otimes P + \frac{1}{\rho} P \otimes (P^\varepsilon - P) + \frac{1}{\rho}(P^\varepsilon - P) \otimes P \end{pmatrix},$$

we have

$$A(U^\varepsilon | U) = \begin{pmatrix} 0 \\ \frac{1}{\rho^\varepsilon} P^\varepsilon \otimes P^\varepsilon - \frac{1}{\rho} P \otimes P + \frac{\rho^\varepsilon - \rho}{\rho^2} P \otimes P - \frac{1}{\rho} P \otimes (P^\varepsilon - P) - \frac{1}{\rho}(P^\varepsilon - P) \otimes P \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \rho^\varepsilon (u^\varepsilon - u) \otimes (u^\varepsilon - u) \end{pmatrix}.$$

Then, by using $D\mathcal{E}(U) = \begin{pmatrix} -\frac{|u|^2}{2} \\ u \end{pmatrix}$, we have

$$\mathcal{I}_3 = \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon (u^\varepsilon - u) \otimes (u^\varepsilon - u) : \nabla_x u \, dx ds$$

$$\leq C \|u\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{R}^d))} \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds.$$

Therefore, by $\mathcal{E}(U^\varepsilon | U) = \frac{\rho^\varepsilon}{2} |u^\varepsilon - u|^2$, we have

$$\mathcal{I}_3 \leq C \|u\|_{L^\infty(0,T_*;W^{1,\infty}(\mathbb{R}^d))} \int_0^t \int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon | U) dx ds.$$

$\square$

To get the relative entropy inequality (2.7) from Lemma 3.1, we need to estimate the last three terms in the right hand side of (3.15) as follows.

Lemma 3.2. *Under the same hypotheses as Lemma 3.1, we have*

(3.20)
$$\left| \int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + div_x A(U^\varepsilon) - \lambda F(U^\varepsilon)] dx ds \right| \leq C\sqrt{\varepsilon},$$

and
(3.21)
$$\int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon)) dx - \lambda \int_0^t \int_{\mathbb{R}^d} (D^2\mathcal{E}(U)F(U)(U^\varepsilon - U) + D\mathcal{E}(U)F(U^\varepsilon)) dx ds \leq 0.$$

Proof. • **Estimate of** (3.20) First of all, it follows from (2.2) that

$$\partial_t \rho^\varepsilon + \nabla_x \cdot (\rho^\varepsilon u^\varepsilon) = 0,$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \nabla_x \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \lambda \rho^\varepsilon u^\varepsilon = \nabla_x \cdot \int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v) f^\varepsilon dv,$$

in the distributional sense. This implies that

$$\int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + div_x A(U^\varepsilon) - F(U^\varepsilon)] dx ds$$

$$= \int_0^t \int_{\mathbb{R}^d} D_P\mathcal{E}(U) \cdot div_x \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v) f^\varepsilon dv \right) dx ds$$

$$= -\int_0^t \int_{\mathbb{R}^d} \nabla_x D_P\mathcal{E}(U) : \left(\int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v) f^\varepsilon dv \right) dx ds,$$

Since $D_P\mathcal{E}(U) = u$ and $u^\varepsilon \otimes u^\varepsilon - v \otimes v = u^\varepsilon \otimes (u^\varepsilon - v) + (u^\varepsilon - v) \otimes v$, we use Hölder's inequality to estimate

$$\Big| \int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + \mathrm{div}_x A(U^\varepsilon) - F(U^\varepsilon)] dx ds \Big|$$

$$\leq \int_0^t \int_{\mathbb{R}^d} |\nabla_x u| \Big| \int_{\mathbb{R}^d} (u^\varepsilon \otimes u^\varepsilon - v \otimes v) f^\varepsilon dv \Big| dx ds$$

$$\leq \|u\|_{L^\infty(0,T_*;W^{1,\infty})} \int_0^t \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} (u^\varepsilon \otimes (u^\varepsilon - v) + (u^\varepsilon - v) \otimes v) f^\varepsilon dv \Big| dx ds$$

$$\leq C \int_0^t \int_{\mathbb{R}^{2d}} (|u^\varepsilon| + |v|)|u^\varepsilon - v| f^\varepsilon dv dx ds$$

$$\leq C \int_0^t \Big(\underbrace{\int_{\mathbb{R}^{2d}} (|u^\varepsilon|^2 + |v|^2) f^\varepsilon dv dx}_{J} \Big)^{1/2} \Big(\int_{\mathbb{R}^{2d}} |u^\varepsilon - v|^2 f^\varepsilon dv dx \Big)^{1/2} ds.$$

By the minimization property (3.19), since

$$J = \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon|^2 dx + \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dv dx \leq 2 \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dv dx,$$

the Hölder's inequality and entropy inequality (2.4) yield

$$\Big| \int_0^t \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t U^\varepsilon + \mathrm{div}_x A(U^\varepsilon) - F(U^\varepsilon)] dx ds \Big|$$

$$\leq \Big(\int_0^t \int_{\mathbb{R}^d} \mathcal{F}(f^\varepsilon)(t) dx ds \Big)^{1/2} \Big(\int_0^t \mathcal{D}_\varepsilon(f^\varepsilon) ds \Big)^{1/2}$$

$$\leq \Big(t \int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx \Big)^{1/2} \Big(\int_0^t \mathcal{D}_\varepsilon(f^\varepsilon) ds \Big)^{1/2}$$

$$\leq C\sqrt{T_*} \Big(\int_0^t \mathcal{D}_\varepsilon(f^\varepsilon) ds \Big)^{1/2}, \quad t \leq T_*,$$

where $\int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx$ is bounded w.r.t. ε by the assumption $(\mathcal{A}1)$. We use (2.4) to estimate

$$(3.22) \qquad \int_0^t \mathcal{D}_\varepsilon(f^\varepsilon)(s) ds \leq \varepsilon \int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx \leq C\varepsilon,$$

which complete (3.20).
• **Estimate of** (3.21) Since

$$D_P D_\rho \mathcal{E}(U) = -\frac{P}{\rho^2} = -\frac{u}{\rho} \quad \text{and} \quad D_P D_P \mathcal{E}(U) = \frac{1}{\rho} I,$$

we have

$$-\lambda \int_0^t \int_{\mathbb{R}^d} (D^2 \mathcal{E}(U) F(U)(U^\varepsilon - U) + D\mathcal{E}(U) F(U^\varepsilon)) dx ds$$

$$= -\lambda \int_0^t \int_{\mathbb{R}^d} (|u|^2 (\rho^\varepsilon - \rho) - u(\rho^\varepsilon u^\varepsilon - \rho u) - \rho^\varepsilon u u^\varepsilon) dx ds$$

$$= -\lambda \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u - u^\varepsilon|^2 dx ds + \lambda \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon|^2 dx ds$$

$$\leq \lambda \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon|^2 dx ds$$

$$\leq \lambda \int_0^t \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dx dv ds$$

where the last inequality is due to minimization property (3.18).
Thus, we have

$$\int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon)) dx - \lambda \int_0^t \int_{\mathbb{R}^d} (D^2 \mathcal{E}(U) F(U)(U^\varepsilon - U) + D\mathcal{E}(U) F(U^\varepsilon)) dx ds$$

$$\leq \int_{\mathbb{R}^d} (\mathcal{F}(f^\varepsilon)(t) - \mathcal{F}(f_0^\varepsilon)) dx + \lambda \int_0^t \int_{\mathbb{R}^d} \mathcal{F}(f^\varepsilon)(s) dx ds.$$

Since the kinetic entropy inequality (2.4) yields

$$\int_{\mathbb{R}^d} \mathcal{F}(f^\varepsilon)(t) dx + \lambda \int_0^t \int_{\mathbb{R}^{2d}} |v|^2 f^\varepsilon dx dv ds \leq \int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx,$$

we have (3.21). $\square$

By Lemma 3.1 and 3.2, we have the Gronwall type inequality

$$\int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon | U)(t) dx \leq C\sqrt{\varepsilon} + C \int_0^t \int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon | U) dx ds.$$

This implies

$$\int_{\mathbb{R}^d} \mathcal{E}(U^\varepsilon | U)(t) dx \leq C\sqrt{\varepsilon}.$$

Since

(3.23)
$$\mathcal{E}(U^\varepsilon | U)(t) = \frac{1}{2} \rho^\varepsilon(t) |(u^\varepsilon - u)(t)|^2,$$

we complete the relative entropy inequality (2.7).

3.2. **Convergence.** In this part, we use (2.7) to show the convergence (2.8), thus the convergence of the kinetic equation (1.3) to the pressureless Euler system (3.10).

3.2.1. *Convergence of ρ^ε and $\rho^\varepsilon u^\varepsilon$.* First of all, by Proposition 2.1, we have the conservation of mass

$$\|\rho^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_0^\varepsilon\|_{L^1(\mathbb{R}^d)}, \quad t > 0.$$

Notice that by the assumption ($\mathcal{A}1$), $\{\rho_0^\varepsilon\}_{\varepsilon > 0}$ is uniformly bounded in $L^\infty(0, T_*; L^1(\mathbb{R}^d))$. Thus there exists $\tilde{\rho}$ such that

(3.24)
$$\rho^\varepsilon \rightharpoonup \tilde{\rho} \quad \text{weak-} * \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$

where $\mathcal{M}([0, T_*] \times \mathbb{R}^d)$ is the space of nonnegative Radon measures on $[0, T_*] \times \mathbb{R}^d$. We now claim that

$$(3.25) \qquad \tilde{\rho} = \rho \text{ on } [0, T_*] \times \mathbb{R}^d.$$

We start with the fact that for any function $\phi \in C_c^\infty([0, T_*] \times \mathbb{R}^d)$, taking $\psi = \phi$ as a test function in (2.2), we find

$$(3.26) \qquad \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon \partial_t \phi + \rho^\varepsilon u^\varepsilon \cdot \nabla_x \phi) dx ds + \int_{\mathbb{R}^d} \rho_0^\varepsilon \phi(0, x) dx = 0.$$

From (3.26), we will derive the following equation of $\tilde{\rho}$:

$$(3.27) \qquad \int_0^t \int_{\mathbb{R}^d} (\tilde{\rho} \partial_t \phi + \tilde{\rho} u \cdot \nabla_x \phi) dx ds + \int_{\mathbb{R}^d} \rho_0 \phi(0, x) dx = 0.$$

The weak convergence (3.24) yields

$$(3.28) \qquad \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon \partial_t \phi \, dx ds \quad \rightarrow \quad \int_0^t \int_{\mathbb{R}^d} \tilde{\rho} \partial_t \phi \, dx ds \quad \text{as } \varepsilon \to 0.$$

For the convergence of second term in (3.26), we consider

$$(3.29) \qquad \begin{aligned} &\int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon - \tilde{\rho} u) \cdot \nabla_x \phi \, dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla_x \phi \, dx ds + \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon - \tilde{\rho}) u \cdot \nabla_x \phi \, dx ds \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

We use the Hölder's inequality and the conservation of mass to estimate

$$\begin{aligned} \mathcal{I}_1 &\leq \|\nabla_x \phi\|_\infty \int_0^t \left(\int_{\mathbb{R}^d} \rho^\varepsilon dx \right)^{1/2} \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} ds \\ &\leq C T_* \|\rho_0^\varepsilon\|_{L^1(\mathbb{R}^d)}^{1/2} \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2}. \end{aligned}$$

Then by the assumption $(\mathcal{A}2)$ and the relative entropy inequality (2.7), we have

$$\mathcal{I}_1 \leq C \varepsilon^{1/4} \to 0 \quad \text{as } \varepsilon \to 0.$$

Since u is smooth, (3.24) yields

$$\mathcal{I}_2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we have

$$(3.30) \qquad \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon u^\varepsilon \cdot \nabla_x \phi \, dx ds \to \int_0^t \int_{\mathbb{R}^d} \tilde{\rho} u \cdot \nabla_x \phi \, dx ds \quad \text{as } \varepsilon \to 0.$$

It obviously follows from the assumption $(\mathcal{A}2)$ that

$$(3.31) \qquad \int_{\mathbb{R}^d} \rho_0^\varepsilon \phi(0, x) dx \to \int_{\mathbb{R}^d} \rho_0 \phi(0, x) dx \quad \text{as } \varepsilon \to 0.$$

Therefore, by (3.26), (3.28), (3.30) and (3.31), we have shown the equation (3.27). On the other hand, since (ρ, u) is a classical solution to the continuity equation

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

we can find

(3.32)
$$\int_0^t \int_{\mathbb{R}^d} (\rho \partial_t \phi + \rho u \cdot \nabla_x \phi) dx ds + \int_{\mathbb{R}^d} \rho_0 \phi(0, x) dx = 0.$$

We now use (3.27) and (3.32) to complete the claim (3.25).
For any function $g \in C_c^\infty([0, T_*] \times \mathbb{R}^d)$, there is a smooth solution $\phi \in C_c^\infty([0, T_*] \times \mathbb{R}^d)$ of

$$\partial_t \phi + u \cdot \nabla_x \phi = g,$$

thanks to the smoothness of u.
Then, by (3.27) and (3.32), we have

$$\int_0^t \int_{\mathbb{R}^d} (\tilde{\rho} - \rho) g \, dx ds = \int_0^t \int_{\mathbb{R}^d} (\tilde{\rho} - \rho)(\partial_t \phi + u \cdot \nabla_x \phi) dx ds$$
$$= 0,$$

which implies the claim (3.25).
Therefore, by (3.24), (3.25) and (3.29), we have shown

(3.33)
$$\rho^\varepsilon \rightharpoonup \rho \quad \text{weak-} * \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d),$$
$$\rho^\varepsilon u^\varepsilon \rightharpoonup \rho u \quad \text{weak-} * \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d).$$

3.2.2. *Convergence of $\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon$.* For any vector-valued function $\Phi \in C_c^\infty([0, T_*] \times \mathbb{R}^d)$, we consider

(3.34)
$$\int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u) : \Phi \, dx ds$$
$$= \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon (u^\varepsilon - u) \otimes u^\varepsilon) : \Phi \, dx ds + \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon u \otimes (u^\varepsilon - u)) : \Phi \, dx d$$
$$+ \int_0^t \int_{\mathbb{R}^d} ((\rho^\varepsilon - \rho) u \otimes u) : \Phi \, dx d$$
$$:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.$$

For $\mathcal{J}_1$, we use the Hölder's inequality and minimization property (3.19) to get

$$\mathcal{J}_1 \leq \|\Phi\|_{L^\infty} \int_0^t \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \right)^{1/2} \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon|^2 dx \right)^{1/2} ds$$
$$\leq C \int_0^t \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx ds \right)^{1/2} \left(\int_{\mathbb{R}^d} \mathcal{F}(f^\varepsilon)(t) dx \right)^{1/2} ds.$$

Then, by (2.4), (2.7) and the assumption ($\mathcal{A}1$), we have

$$\mathcal{J}_1 \leq C T_* \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^d} \mathcal{F}(f_0^\varepsilon) dx \right)^{1/2} \to 0 \quad \text{as } \varepsilon \to 0.$$

For $\mathcal{J}_2$, we use the Hölder's inequality and the conservation of mass to estimate

$$\mathcal{J}_2 \leq \|\Phi\|_{L^\infty} \|u\|_{L^\infty} \int_0^t \left(\int_{\mathbb{R}^d} \rho^\varepsilon dx \right)^{1/2} \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} ds$$
$$\leq C T_* \|\rho_0^\varepsilon\|_{L^1(\mathbb{R}^d)}^{1/2} \left(\int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 dx \right)^{1/2} \to 0 \quad \text{as } \varepsilon \to 0.$$

Taking $u \otimes u : \nabla_x \Phi$ as a test function due to regularity of u, it follows from the weak convergence of mass $(3.33)_1$ that

$$\mathcal{J}_3 \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we have

$$\int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon - \rho u \otimes u) : \Phi dx ds \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, we have shown

$$\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \rightharpoonup \rho u \otimes u \quad \text{weak-} * \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d).$$

3.2.3. *Convergence of $\int_{\mathbb{R}^d} f^\varepsilon |v|^2 dv$.* For any $\phi \in C_c^\infty([0, T_*) \times \mathbb{R}^d)$, we have

(3.35)
$$\int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f^\varepsilon |v|^2 dv - \rho |u|^2 \right) \phi dx ds$$
$$= \int_0^t \int_{\mathbb{R}^{2d}} f^\varepsilon |v - u^\varepsilon|^2 \phi dv dx ds + \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon |u^\varepsilon|^2 - \rho |u|^2) \phi dx dt$$
$$= I_1 + I_2.$$

By (3.22), we have

$$I_1 \le \|\phi\|_{L^\infty} \int_0^t \mathcal{D}(f^\varepsilon) ds \to 0 \quad \text{as } \varepsilon \to 0.$$

We use (2.7) and (3.33) to get

$$I_2 = \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon |u^\varepsilon - u|^2 \phi dx dt + 2 \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon - \rho u) \cdot u \phi dx dt$$
$$- \int_0^t \int_{\mathbb{R}^d} (\rho^\varepsilon - \rho) |u|^2 \phi dx dt$$
$$\to 0.$$

Therefore, we have

$$\int_{\mathbb{R}^d} f^\varepsilon |v|^2 dv \rightharpoonup \rho |u|^2 \quad \text{weak-} * \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d).$$

This complete the proof of theorem.

3.2.4. *Convergence of $\int_{\mathbb{R}^d} f^\varepsilon \psi(v) dv$ in (2.9).* We first use (2.7) and (3.22) to estimate

(3.36)
$$\int_0^t \int_{\mathbb{R}^{2d}} f^\varepsilon |v - u|^2 dv dx ds \le 2 \int_0^t \int_{\mathbb{R}^{2d}} f^\varepsilon (|v - u^\varepsilon|^2 + |u^\varepsilon - u|^2) dv dx ds$$
$$\le C(\varepsilon + T^* \sqrt{\varepsilon}) \le C\sqrt{\varepsilon}.$$

For any $\phi \in C_c^\infty([0, T_*) \times \mathbb{R}^d)$ and $\psi \in C^1(\mathbb{R}^d)$ with $\nabla\psi \in L^\infty(\mathbb{R}^d)$, by using (3.36) and mean-value theorem, we have

$$
\int_0^t \int_{\mathbb{R}^{2d}} \phi(t,x) f^\varepsilon (\psi(v) - \psi(u)) dv dx ds
$$
$$
= \int_{|v-u| \leq \varepsilon^{1/4}} \phi(t,x) f^\varepsilon (\psi(v) - \psi(u)) dv dx ds + \int_{|v-u| > \varepsilon^{1/4}} \phi(t,x) f^\varepsilon (\psi(v) - \psi(u)) dv dx ds
$$
$$
\leq \|\phi\|_\infty \|\nabla\psi\|_\infty \left(\int_{|v-u| \leq \varepsilon^{1/4}} f^\varepsilon |v-u| dv dx ds + \int_{|v-u| > \varepsilon^{1/4}} f^\varepsilon |v-u| dv dx ds \right)
$$
$$
\leq \|\phi\|_\infty \|\nabla\psi\|_\infty \left(\varepsilon^{1/4} T^* \|f^\varepsilon\|_{L^1(\mathbb{R}^{2d})} + \varepsilon^{-1/4} \int_0^t \int_{\mathbb{R}^{2d}} f^\varepsilon |v-u|^2 dv dx ds \right)
$$
$$
\leq C\varepsilon^{1/4},
$$

which yields

$$
\int_{\mathbb{R}^d} f^\varepsilon \psi(v) dv \rightharpoonup \rho^\varepsilon \psi(u) \quad \text{weakly} \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d).
$$

Therefore, by the convergence of mass $(3.33)_1$, we complete

$$
\int_{\mathbb{R}^d} f^\varepsilon \psi(v) dv \rightharpoonup \rho \psi(u) \quad \text{weakly} \quad \text{in } \mathcal{M}([0, T_*] \times \mathbb{R}^d).
$$

APPENDIX A. PROOF OF PROPOSITION 3.1

First of all, by the definition of the relative entropy $(3.12)_1$, we have

$$
\frac{d}{dt} \int_{\mathbb{R}^d} \mathcal{E}(V|U) dx = \int_{\mathbb{R}^d} \partial_t \mathcal{E}(V) dx - \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot \partial_t U dx
$$
$$
- \int_{\mathbb{R}^d} D^2\mathcal{E}(U) \partial_t U \cdot (V - U) dx - \int_{\mathbb{R}^d} D\mathcal{E}(U) \partial_t (V - U) dx
$$
$$
= \int_{\mathbb{R}^d} \partial_t \mathcal{E}(V) dx - \int_{\mathbb{R}^d} D^2\mathcal{E}(U) \partial_t U \cdot (V - U) dx - \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot \partial_t V dx
$$
$$
=: \sum_{l=1}^3 \mathcal{I}_K.
$$

By (3.10), we rewrite $\mathcal{I}_2$ as

$$
\mathcal{I}_2 = \int_{\mathbb{R}^d} D^2\mathcal{E}(U)(\text{div}_x A(U) - \lambda F(U)) \cdot (V - U) dx
$$
$$
= \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : DA(U) \cdot (V - U) dx - \lambda \int_{\mathbb{R}^d} D^2\mathcal{E}(U) F(U) \cdot (V - U) dx,
$$

where we have used the following formula on integration by parts: (See [4, 24] for its derivation.)

$$
\int_{\mathbb{R}^d} D^2\mathcal{E}(U) \text{div}_x A(U) \cdot (V - U) dx = \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : DA(U) \cdot (V - U) dx.
$$

By adding and subtracting, $\mathcal{I}_3$ can be written as

$$\mathcal{I}_3 = -\int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot (\partial_t V + \mathrm{div}_x A(V) - \lambda F(V))dx$$

$$+ \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot \mathrm{div}_x A(V)dx - \lambda \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot F(V)dx$$

$$= -\int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot (\partial_t V + \mathrm{div}_x A(V) - \lambda F(V))dx$$

$$- \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(V)dx - \lambda \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot F(V)dx$$

We use the relative flux $(3.12)_2$ to get

$$\mathcal{I}_2 + \mathcal{I}_3 = -\int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(V|U)dx - \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot [\partial_t V + \mathrm{div}_x A(V) - \lambda F(V)]dx$$

$$- \lambda \int_{\mathbb{R}^d} (D^2\mathcal{E}(U)F(U) \cdot (V-U) + D\mathcal{E}(U) \cdot F(V))dx - \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(U).$$

Thanks to (3.11), the last term vanish as follows.

$$- \int_{\mathbb{R}^d} \nabla_x D\mathcal{E}(U) : A(U) = \int_{\mathbb{R}^d} D\mathcal{E}(U) \cdot \mathrm{div}_x A(U)dx = \int_{\mathbb{R}^d} \mathrm{div}_x G(U)dx = 0,$$

which complete the proof.

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