# ASYMPTOTIC SYZYGIES GROW EXPONENTIALLY: A REMARK ON A PAPER OF EIN-LAZARSFELD

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ABSTRACT. For all  $2 \le q \le \dim(X)$  and most relevant p values, the dimension of the asymptotic Koszul cohomology group  $K_{p,q}(X,B;L_d)$  grows exponentially with d.

While interest in the syzygies of a projective variety X goes back to Hilbert, it was Mark L. Green's seminal paper [5] that popularized the notion that the syzygies of X can be viewed as a type of cohomology, called Koszul cohomology, attached to the pair (X, L) where L is a line bundle on X. Since then, chiefly under the leadership of Robert Lazarsfeld and Claire Voisin, numerous results have been established on the computation or nonvanishing of syzygies in many cases. Recently particular attention has been focused on the case of *asymptotic* Koszul cohomology groups  $K_{p,q}(X, B; L_d)$  where  $L_d$  is sufficiently ample [2]. Some of the results are stated in [3].

Larzarsfeld and Ein [2] have shown, for a large range of p and  $q \ge 2$  that these groups don't vanish. Our purpose here is to show that the construction of [2] can be pushed to yield lower bounds on the dimension of these groups which are exponential in d. Our argument differs from [2] mainly in the elimination of the induction, i.e. 'drilling down to the base case'.

After this was written, Lazarsfeld communicated that he and Ein had been aware in a general way of similar bounds, based on an argument different from the one given here, which was never written down. Nonetheless, the asymptotic behavior of the dimension of the  $K_{p,q}(X, B; L_d)$  is stated an an open problem in [2]. See [1] for further related results and problems.

#### NOTATIONS AND CONVENTIONS

Let L be a line bundle and B a coherent sheaf on a variety X. A linear system on X, belonging to L, is by definition a vector space V endowed with a map, not necessarily injective,  $V \to H^0(X, L)$ . This induces a map of coherent sheaves  $V \otimes O_X \to L$ . In our applications, we will initially take V complete, i.e.  $V = H^0(X, L)$ , but various associated linear systems, e.g. restrictions on subvarieties, need not be complete.

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Given these data, the associated Koszul complexes are by definition the complexes

$$(1) \hspace{1cm} C_{p,c}(X,B,L,V): \hspace{0.2cm} \bigwedge^{p} V \otimes (B+(c+1)L) \rightarrow \bigwedge^{p-1} V \otimes (B+(c+2)L) \rightarrow \dots$$

where we use customary additive notation:  $B + aL := B \otimes L^{\otimes a}$ . We will often suppress X, L when understood and denote this by  $C_{p,c}(B, V)$  or  $C_p(B, V)$  when c = 0. When V is basepoint free, the complex (1) admits local homotopy operators, hence it is locally acyclic, i.e. exact in strictly positive degrees. We will denote the kernel of (1), i.e.  $\mathcal{H}^0(C_p(B, V))$ , by  $M_p(B, V)$ . In our applications, L will have the form

$$L = L_d := dA + P$$

with *A* ample and *d* as large as desired. Then the members of the Koszul complex are themselves acyclic sheaves thanks to Serre vanishing. Whenever the Koszul complex is locally acyclic and consists of acyclic sheaves, we have

(2) 
$$\mathbb{H}^{i}(C_{p,c}(B,V)) = H^{i}(M_{p,c}(B,V)) = K_{p-i,c+1+i}(X,L,B,V), i > 0.$$

It will be convenient to use the following terminology: a nonnegative complex  $C^{\bullet}$  of sheaves is *globally acyclic* if it is locally acyclic and consists of acyclic sheaves. In this case, the hypercohomology can be identified with either the cohomology of the  $\mathcal{H}^0$  sheaf, or with the cohomology of the associated  $H^0$  complex. Thus, we have

(3) 
$$\mathbb{H}^{i}(C^{\bullet}) = H^{i}(\mathcal{H}^{0}(C^{\bullet})) = H^{i}(H^{0}(C^{\bullet})).$$

If  $C^{\bullet}$  is globally acyclic and these groups are moreover trivial for all i > 0,  $C^{\bullet}$  is said to be *hyper-acyclic*.

We will often use  $H^{\bullet}$  rather than  $\mathbb{H}^{\bullet}$  to denote hypercohomology of a complex of sheaves, when there is no confusion.

### EXPONENETIAL EIN-LAZARSFELD

Let  $Y \subset X$  be a smooth subvariety of dimension m = n - c and consider a subspace  $V_0 \subset V_0^+ := V \cap H^0(T_Y(L))$  whose zero-locus is m-dimensional. In our application, we will take  $V = H^0(X, L)$ . Let  $V_Y = V/V_0$ , which admits a natural map to  $L_Y := L.O_Y$ . Suppose  $p \ge v_Y := \dim(V_Y)$ . Then the complex  $C_{v_Y}(Y, B_Y, L_Y, V_Y)$  is exact, hence so is  $C_{v_Y}(Y, B_Y, L_Y, V_Y) \otimes \wedge^{p-v_Y} V_0$ . Now there is a natural quotient map

$$\wedge^{i}V \to \wedge^{i-\nu_{Y}}V_{0} \otimes \wedge^{\nu_{Y}}V_{Y}$$

(where the target is taken to be  $\wedge^{v_Y} V_Y$  if  $i = v_Y$  and zero if  $i < v_Y$ ). This induces a natural termwise surjective map

$$C_p(X, B, L, V) \rightarrow C_{\nu_Y}(Y, B_Y, L_Y, V_Y) \otimes \wedge^{p-\nu_Y} V_0.$$

The kernel of the latter map, denoted  $C_p^0(X, B, L, V)$ , is hence a subcomplex quasi-isomorphic to  $C_p(X, B, L, V)$ . We call it the *adapted* Koszul complex (with respect to Y). Note that there is a natural map

(4) 
$$C_p^0(X, B, L, V) \to \wedge^{\nu_Y} V_Y \otimes \wedge^{p-\nu_Y} V_0 \otimes \mathcal{I}_{Y/X}(B+L).$$

In the sequel we will use for *Y* the subvariety used by Ein-Lazarsfeld [2], which in turn is based on a construction of Eisenbud et al. [4]. For a very ample divisor *H* fixed independently of *d*, this *Y* is of type

$$H_1 \cap ... \cap H_{c-2} \cap D_1 \cap D_2, H_i \in |H|, D_1 \in |H + B - K_X|, D_2 \in |L - (c - 1)H|.$$

*Y* has dimension m = q - 2 and satisfies

(5) 
$$h^{q-1}(\mathcal{I}_{Y/X}(B+L)) = 1; h^{q-1+i}(\mathcal{I}_{Y/X}(B+L+kH)) = 0, k > i \ge 0.$$

Now let  $\bar{X}$  be a smooth subvariety of X transverse to (or disjoint from) Y and let  $\bar{L}$ ,  $\bar{B}$  etc denote restrictions on  $\bar{X}$ . Then for any divisor B' we have an analogous map

(6) 
$$C_p^0(\bar{X}, \bar{B}', \bar{L}, V) \to \wedge^{\nu_Y} V_Y \otimes \wedge^{p-\nu_Y} V_0 \otimes \mathcal{I}_{\bar{Y}/\bar{X}}(\bar{B}' + \bar{L}).$$

Let  $V_K$ ,  $\bar{V}$  denote respectively the kernel and image of the restriction map  $V \to H^0(\bar{L})$ . Then we have an exact diagram

Now take  $\bar{X}$  general of the form

$$\bar{X}=H_1'\cap\ldots\cap H_{m+1}',H_i'\in |H|$$

where H is a fixed very ample divisor as above. Note that a Koszul resolution of  $O_{\bar{X}}$  on X remains exact upon tensoring with  $I_Y$  and its locally free twists, and yields a torsion-free (not locally free) Koszul resolution of  $O_{\bar{X}}$ , since  $\bar{Y} = Y \cap \bar{X} = \emptyset$  by transversality. Thus we have exact

(8) 
$$I_{Y/X}(B+L) \to (m+1)I_{Y/X}(B+L+H) \to \dots \to I_{Y/X}(B+L+(m+1)H) \to O_{\bar{X}}(B+L+(m+1)H).$$

Then setting B' = B + (m+1)H, consider an analogous torsion-free Koszul resolution of  $\wedge^{p-\nu_Y}V_Y \otimes \mathcal{I}_{\bar{Y}/\bar{X}}(\bar{B}' + \bar{L}) = \wedge^{p-\nu_Y}V_Y \otimes \mathcal{O}_{\bar{X}}(\bar{B} + \bar{L})$ , together with a compatible one for  $C^0_p(\bar{X},\bar{B}',\bar{L},V) = C_p(\bar{X},\bar{B}',\bar{L},V)$ . This takes the form of a morphism of complexes

$$C_{p}^{0}(X,B,L,V) \rightarrow \dots \rightarrow (m+1)C_{p}^{0}(X,B+L+(m+1)H) \rightarrow C_{p}(\bar{X},\bar{B}',\bar{L},V)$$

$$(9) \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\wedge^{\nu_{Y}}V_{Y} \otimes \wedge^{p-\nu_{Y}}V_{0}\bar{I}_{Y/X}(B+L) \rightarrow \dots \rightarrow (m+1)\wedge^{\nu_{Y}}V_{Y}\wedge^{p-\nu_{Y}}V_{0} \otimes \bar{I}_{Y/X}(B+L+(m+1)H) \rightarrow \wedge^{\nu_{Y}}V_{Y}\wedge^{p-\nu_{Y}}V_{0} \otimes O_{\bar{X}}(\bar{B}'+\bar{L})$$

Pick a natural number  $\bar{p} \leq v_Y$  independent of d, e.g.  $\bar{p} = 1$ , and choose  $V_0$  so that  $\dim(\bar{V}_Y) = \bar{p}$ . For example, we can choose  $V_0$  generated by n general elements of  $V_0^+$  (which cut out Y), plus the inverse image in  $V_0^+$  of a general codimension- $(\bar{p}+n)$  subspace of the image of  $V_0^+ \to \bar{V}$  (cf. (7)). Note that in that case  $\dim(\bar{V}_0)$  is asymptotically  $d^m$  for large d. Then the rightmost column of (9) fits in a diagram, in which the upper horizontal arrows represent a direct summand inclusion (with a suitable shift):

(10)

By [2], Prop. 4.6, the upper left arrow in (10) induces a surjection on  $H^0$  if d is large enough. Now considering the  $E_1$  spectral sequence associated to the resolution (9), we see from the vanishings of [2] (cf. (5)) that  $H^{q-1}(\mathcal{I}_{Y/X}(B+L))$ , which is 1-dimensional, injects into  $H^0(O_{\bar{X}}(\bar{B}'+\bar{L}))$ . Therefore the image of the left vertical map in (9) on  $H^{q-1}$  must contain  $\wedge^{\nu_Y}V_Y\otimes \wedge^{p-\nu_Y}(V_0\cap V_K)H^{q-1}(\mathcal{I}_{Y/X}(B+L))$ . Consequently,

(11) 
$$\dim K_{p,q}(X,B,L) \ge \binom{\dim(V_0 \cap V_K)}{p - v_Y}.$$

Choosing  $p > v_Y \sim d^{q-1}$ , hence  $\dim(V_0) \sim d^n$ , we can arrange things so that  $\dim(V_0 \cap V_K) \sim d^n$  while  $p - v_Y \sim d$ , hence the above lower bound is exponential in d. On the other hand by the very definition of Koszul cohomology, its dimension is at most exponential in d. Hence we conclude:

**Theorem 1.** For  $n = \dim(X) \ge 2$ ,  $q \ge 2$  and p asymptotically between  $d^{q-1}$  and  $d^n$ , the dimension of  $K_{p,q}(X,B;L_d)$  grows exponentially with d; in particular,

$$\lim_{d\to\infty} \dim K_{p,q}(X,B,L_d) = \infty.$$

This sheds some light on Problem 7.3 in [2].

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