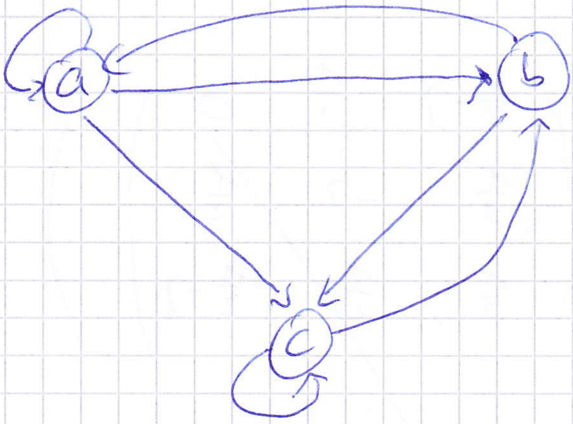
Exercise 2



a)

$$M = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$N = 3$

$$r^{(0)} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\varepsilon = \frac{1}{12}$$

$$M \cdot r^{(0)} = M \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{18} \\ \frac{5}{18} \\ \frac{4}{9} \end{pmatrix} = r^{(1)}$$

$$|r^{(1)} - r^{(0)}|_1 = \frac{2}{9} > \varepsilon$$

$$M \cdot r^{(1)} = M \begin{pmatrix} \frac{5}{18} \\ \frac{5}{18} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{25}{108} \\ \frac{17}{54} \\ \frac{49}{108} \end{pmatrix} = r^{(2)}$$

$$|r^{(2)} - r^{(1)}|_1 = \frac{5}{108} + \frac{1}{27} + \frac{1}{108} = \frac{5}{54} > \varepsilon$$

$$M \cdot r^{(2)} = M \begin{pmatrix} \frac{25}{108} \\ \frac{17}{54} \\ \frac{48}{108} \end{pmatrix} = \begin{pmatrix} \frac{19}{81} \\ \frac{197}{648} \\ \frac{299}{648} \end{pmatrix} = r^{(3)}$$

$$|r^{(3)} - r^{(2)}|_1 = \frac{1}{324} + \frac{7}{648} + \frac{5}{648} = \frac{7}{324} < \varepsilon \Rightarrow \text{END}$$

b) Find eigenvector with eigenvalue 1.

$$(M - \lambda \mathbb{I})\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{3}{4} & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad -\frac{2}{3}a + \frac{1}{2}b = 0$$

$$-\frac{3}{4}b + \frac{1}{2}c = 0$$

$$\Rightarrow \quad b = \frac{4}{6}c \qquad \Rightarrow \begin{pmatrix} \frac{c}{2} \\ \frac{2c}{3} \\ c \end{pmatrix}, \; c \in \mathbb{R}$$

$$a = \frac{1}{2}c$$

If we constrain additionally with

$$a + b + c = 1$$

we get

$$\frac{c}{2} + \frac{2c}{3} + c = 1$$

$$\Rightarrow \quad c = \frac{6}{13}$$

$$\Rightarrow \text{ eigenvector: } \begin{pmatrix} \frac{3}{13} \\ \frac{4}{13} \\ \frac{6}{13} \end{pmatrix}$$

c)

$$A = \beta M + (1-\beta)\left[\frac{1}{N}\right]_{N \times N} \quad , \beta = 0.8$$

$$= \begin{pmatrix} \frac{4}{15} & \frac{2}{5} & 0 \\ \frac{4}{15} & 0 & \frac{2}{5} \\ \frac{4}{15} & \frac{2}{5} & \frac{2}{5} \end{pmatrix} + \begin{pmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{7}{15} & \frac{1}{15} \\ \frac{1}{3} & \frac{1}{15} & \frac{7}{15} \\ \frac{1}{3} & \frac{7}{15} & \frac{7}{15} \end{pmatrix}$$

$$A \cdot r^{(0)} = A \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{13}{45} \\ \frac{13}{45} \\ \frac{19}{45} \end{pmatrix} = r^{(1)}$$

$$\left| r^{(1)} - r^{(0)} \right|_1 = \frac{2}{45} + \frac{2}{45} + \frac{4}{45} = \frac{8}{45} > \varepsilon$$

$$A \cdot r^{(1)} = A \cdot \begin{pmatrix} \frac{13}{45} \\ \frac{13}{45} \\ \frac{19}{45} \end{pmatrix} = \begin{pmatrix} \frac{7}{27} \\ \frac{211}{675} \\ \frac{289}{675} \end{pmatrix} = r^{(2)}$$

$$\left| r^{(2)} - r^{(1)} \right|_1 = \frac{4}{135} + \frac{16}{675} + \frac{4}{675} = \frac{8}{135} < \varepsilon$$

$$\Rightarrow \text{ENDE}$$

d)

$$(M - 2\mathbb{I})\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{7}{15} & \frac{1}{15} \\ \frac{1}{3} & -\frac{14}{15} & \frac{7}{15} \\ \frac{1}{3} & \frac{7}{15} & -\frac{8}{15} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{7}{15} & \frac{1}{15} \\ 0 & -\frac{7}{10} & \frac{1}{2} \\ 0 & \frac{7}{10} & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{2}{3} & \frac{7}{15} & \frac{1}{15} \\ 0 & -\frac{7}{10} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow -\frac{7}{10} b + \frac{1}{2} c = 0$$

$$-\frac{2}{3} a + \frac{7}{15} b + \frac{1}{15} c = 0$$

$$\Rightarrow b = \frac{10}{15} c \qquad \Rightarrow \begin{pmatrix} \frac{6}{10} c \\ \frac{10}{15} c \\ c \end{pmatrix}, \ c \in \mathbb{R}$$

$$a = \frac{6}{10} c$$

If we additionally constrain by $a+b+c=1$ we get

$$\frac{6}{10} c + \frac{10}{15} c + c = 1$$

$$\Rightarrow c = \frac{35}{81}$$

$\Rightarrow$ Eigenvector: $\underline{\underline{\begin{pmatrix} \frac{7}{27} \\ \frac{25}{81} \\ \frac{35}{81} \end{pmatrix}}}$

Exercise 3

If we follow a random teleport link at dead-ends, the general stochastic adjacency matrix would look like this:

$$A = \begin{pmatrix} \frac{(1-\beta)}{n+1} & \frac{\beta}{n} + \frac{(1-\beta)}{n+1} & \cdots & \frac{\beta}{n} + \frac{(1-\beta)}{n+1} & \frac{1}{n+1} \\ \frac{\beta}{n} + \frac{(1-\beta)}{n+1} & \frac{(1-\beta)}{n+1} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & \frac{(1-\beta)}{n+1} & \\ \frac{\beta}{n} + \frac{(1-\beta)}{n+1} & \cdots & & \frac{\beta}{n} + \frac{(1-\beta)}{n+1} & \frac{1}{n+1} \end{pmatrix}$$

The last column ensures that we follow a random link with probability 1.0 once we get to the dead-end.
The other entries are a result of using the Google Matrix $A = \beta M + (1-\beta)\left[\frac{1}{N}\right]_{N\times N}$ where in this case $N = n+1$ (members of the clique) and

$$M = \begin{pmatrix} 0 & \frac{1}{n} & \cdots & & \frac{1}{n+1} \\ \frac{1}{n} & 0 & \frac{1}{n} & \cdots & \vdots \\ \vdots & \frac{1}{n} & & \ddots & \vdots \\ \vdots & \vdots & & & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n+1} \end{pmatrix}$$

So we pre-processed M to include the teleport from the dead-end ~~last rather we explicitly changed the last column~~.

If we calculate the eigenvector for the eigenvalue 1 with the matrix A, we get:

$$\text{PageRank} = \begin{pmatrix} \frac{n}{n+\beta} \\ \vdots \\ \frac{n}{n+\beta} \\ 1 \end{pmatrix}$$