Project Euler #69: Totient maximum

This problem is a programming version of Problem 69 from projecteuler.net

Euler's Totient function, $\phi(n)$ [sometimes called the phi function], is used to determine the number of numbers less than n which are relatively prime to n. For example, as $1, 2, 4, 5, 7,$ and 8, are all less than nine and relatively prime to nine, $\phi(9) = 6$.

n	*Relatively Prime*	$\phi(n)$	$n/\phi(n)$
2	1	1	2
3	1, 2	2	1.5
4	1, 3	2	2
5	1, 2, 3, 4	4	1.25
6	1, 5	2	3
7	1, 2, 3, 4, 5, 6	6	1.1666...
8	1, 3, 5, 7	4	2
9	1, 2, 4, 5, 7, 8	6	1.5
10	1, 3, 7, 9	4	2.5

It can be seen that $n = 6$ produces a maximum $n/\phi(n)$ for $n < 10$. Find the value of $n < N$ for which $n/\phi(n)$ is maximum. In case of multiple answers, print the minimum.

Input Format

First line contains T, denoting number of test cases. T lines follow
Each line contains N

Constraints

$1 \le T \le 1000$
$3 \le N \le 10^{18}$

Output Format

Print the answer corresponding to each testcase on a new line.

Sample Input

```
2
3
10
```

Sample Output

```
2
6
```

Solution:

Project Euler # 69 (Hackerrank): Totient Maximum

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(*) Euler Totient Function:

- Number of Coprimes of $n = \varphi(n) = n \cdot \underbrace{\prod_{p|n} \left(1 - \frac{1}{p}\right)}_{K \text{ times}}$

where p is prime factor of n

(*) Optimization function:

- $\text{ratio} = \frac{n}{\varphi(n)} = \frac{1}{\prod_{p|n}\left(1 - \frac{1}{p}\right)}$

- $\frac{n}{\varphi(n)}$ max $\Leftrightarrow \prod_{p|n}\left(1 - \frac{1}{p}\right)$ min $\Leftrightarrow \frac{1}{p}$ max and K max $\Leftrightarrow$ $\boxed{p \text{ min and } K \text{ max}}$

(*) Algorithm:

- Create all primes list < Lim, e.g. Lim = 100 for this problem
- For prime in AllPrimesList:

  If $n \cdot p \geqslant N \Rightarrow$ return n and ratio

  Else:

  $n = n \cdot p$ ($n = 1$ at the start of loop)

  Calculate ratio $= \frac{n}{\varphi(n)}$

(*) How it works:

- AllPrimesList = [2, 3, 5, 7, ..., Lim]
- Take $p = 2 \Rightarrow p$ is min and $K = 1$
- $n = p = 2$. (n is still smaller than $N \Rightarrow$ continue with $p = 3$)
- Take $p = 3 \Rightarrow p$ is also min and $K = 2$
- $n = p_1 \cdot p_2 = 2 \cdot 3 = 6$ (n has prime factors 2 and 3)

⋮

- Break the loop when $n \geqslant N$