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Estimation of the parameters of log-gamma distribution using order statistics

N. V. Sreekumar · P. Yageen Thomas

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Abstract In this work we propose a technique of estimating the location parameter μ and scale parameter σ of log-gamma distribution by U-statistics constructed by taking best linear functions of order statistics as kernels. The efficiency comparison of the proposed estimators with respect to maximum likelihood estimators is also made.

Keywords Order statistics · Best linear unbiased estimators based on order statistics · U-statistics · Log-gamma distribution · Maximum likelihood estimators

1 Introduction

A continuous random variable X is said to have log-gamma distribution if its probability density function (pdf) is given by

$$f(x;\,\mu,\sigma,\kappa) = \frac{1}{\sigma\Gamma(\kappa)}\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)+\kappa\left(\frac{x-\mu}{\sigma}\right)\right],$$
$$-\infty < x < \infty,\ \ -\infty < \mu < \infty,\ \ \sigma,\kappa > 0, \tag{1.1}$$

where μ is the location parameter, σ is the scale parameter and κ is the shape parameter. This distribution finds several applications in life-testing

N. V. Sreekumar · P. Y. Thomas (✉)
Department of Statistics, University of Kerala, Trivandrum 695 581, India
e-mail: yageenthomas@rediffmail.com

N. V. Sreekumar
e-mail: nvsreekumar@gmail.com

(see, Lawless 1982). Prentice (1974) and Lawless (1980) have discussed the maximum likelihood estimation of the parameters of (1.1). The log-gamma regression model was studied by Young and Bakir (1987).

Balakrishnan and Chan (1998) have determined the means, variances and covariances of order statistics from standard log-gamma distribution and used these quantities to obtain the best linear unbiased estimators (BLUEs) of μ and σ in (1.1) based on complete as well as Type-II censored samples. They have also discussed the linear estimation of μ and σ based on k optimally selected order statistics. Balakrishnan and Chan (1994) have discussed the asymptotic approximations to the BLUEs of μ and σ based on Type-II censored samples.

Eventhough best linear unbiased estimation of location and scale parameters using order statistics (see Lloyd 1952) is a widely accepted method of estimation, one serious problem involved in the application of this method is that, in order to obtain these estimators one requires the values of means, variances and covariances of the entire order statistics of a random sample of size n arising from the corresponding standard distribution. Thus the results of Balakrishnan and Chan (1998) cannot help one to obtain the BLUEs of μ and σ for larger values of n. However if one uses the BLUEs of μ and σ based on order statistics of a small or moderate sample of size m and use this as kernel of degree m to construct appropriate U-statistics to estimate μ and σ, then these U-statistics are highly useful as they estimate the parameters explicitly. Moreover these estimators are highly preferred as they utilize the optimality conditions of BLUE as well as U-statistics. Nair (1936) has used the U-statistic based on order statistics, viz. Gini's mean difference to estimate the scale parameter of any distribution. Samuel and Thomas (2003) have used an estimator based on Gini's mean difference to estimate the scale parameter of triangular distribution. Thomas and Sreekumar (2004) have extended the concept to develop U-statistics by taking best linear unbiased estimator based on the order statistics of a random sample of size 2 as kernel of degree 2 to estimate the scale parameter of generalized exponential distribution.

The aim of this work is to generalize further the results of Thomas and Sreekumar (2004) to generate U-statistics of μ and σ involved in (1.1) by taking best linear functions of order statistics of a sample of size $m < n$ as kernels. The usual mathematical intractability in obtaining the variance of the U-statistics based on order statistics is routed out by a mathematical formula derived in this work.

2 Some basic results

2.1 BLUEs of location and scale parameters

Let $X = (X_{1:m}, X_{2:m}, \ldots, X_{m:m})'$ be the vector of order statistics of a random sample of size m drawn from (1.1). Define $Y_{r:m} = (X_{r:m} - \mu)/\sigma, r = 1, 2, \ldots, m$. Then $Y_{r:m}, r = 1, 2, \ldots, m$ are distributed as the order statistics of a random sample of size m drawn from the standard form of (1.1) given by the pdf $f(x; 0, 1, \kappa)$.

Then the BLUEs of μ and σ are given by (see, Lloyd 1952),

$$\mu^* = \frac{-\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}(\mathbf{1}\,\boldsymbol{\alpha}'_{\mathbf{m}} - \boldsymbol{\alpha}_{\mathbf{m}}\mathbf{1}')V_{\mathrm{m}}^{-1}}{(\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\boldsymbol{\alpha}_{\mathbf{m}})(\mathbf{1}'\,V_{\mathrm{m}}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\mathbf{1})^2} X, \tag{2.1}$$

$$\sigma^* = \frac{\mathbf{1}'\,V_{\mathrm{m}}^{-1}(\mathbf{1}\boldsymbol{\alpha}'_{\mathbf{m}} - \boldsymbol{\alpha}_{\mathbf{m}}\mathbf{1}')\,V_{\mathrm{m}}^{-1}}{(\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\boldsymbol{\alpha}_{\mathbf{m}})(\mathbf{1}'\,V_{\mathrm{m}}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\mathbf{1})^2} X, \tag{2.2}$$

with variances given by

$$\mathrm{Var}(\mu^*) = \frac{\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\boldsymbol{\alpha}'_{\mathbf{m}}}{(\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\boldsymbol{\alpha}_{\mathbf{m}})(\mathbf{1}'\,V_{\mathrm{m}}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\mathbf{1})^2}\sigma^2, \tag{2.3}$$

$$\mathrm{Var}(\sigma^*) = \frac{\mathbf{1}'\,V_{\mathrm{m}}^{-1}\mathbf{1}'}{(\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\boldsymbol{\alpha}_{\mathbf{m}})(\mathbf{1}'\,V_{\mathrm{m}}^{-1}\mathbf{1}) - (\boldsymbol{\alpha}'_{\mathbf{m}} V_{\mathrm{m}}^{-1}\mathbf{1})^2}\sigma^2, \tag{2.4}$$

where $\boldsymbol{\alpha}_{\mathbf{m}}$ denotes the vector of means, V_{m} the matrix of variances and covariances of the order statistics of a random sample of size m arising from (1.1) and $\mathbf{1}$ is a column vector of 1's of the same dimension as X. For an extensive literature on best linear unbiased estimation of the parameters of a distribution using order statistics see, David and Nagaraja (2003), Sarhan and Greenberg (1962), Balakrishnan and Cohen (1991) and Balakrishnan and Rao (1998).

2.2 U-statistics

Let $X_1, X_2, \ldots, X_n$ be independent observations drawn from a population with cumulative distribution function $F(x;\theta)$. The U-statistic for the estimable parameter θ with the symmetric kernel $h(\cdot)$ of degree m is given by,

$$U(X_1, X_2, \ldots, X_n) = \frac{1}{\binom{n}{m}} \sum_{\beta \in B} h\left(X_{\beta_1}, X_{\beta_2}, \ldots, X_{\beta_{\mathrm{m}}}\right), \tag{2.5}$$

where $B = \{\beta | \beta = (\beta_1, \ldots, \beta_{\mathrm{m}}), \beta_1 < \beta_2 < \cdots < \beta_{\mathrm{m}}$ is one of the $\binom{n}{m}$ combinations of m integers chosen without replacement from the set $(1, 2, \ldots, n)\}$. Suppose that

$$E[h(X_1, X_2, \ldots, X_{\mathrm{m}})] = \theta \text{ and } E[h^2(X_1, X_2, \ldots, X_{\mathrm{m}})] < \infty.$$

Let $h\,(X_1, X_2, \ldots, X_c, X_{c+1}, \ldots, X_{\mathrm{m}})$ and $h(X_1, X_2, \ldots, X_c, X_{\mathrm{m}+1}, \ldots, X_{2\mathrm{m}-c})$ be two random variables having exactly c sample observations in common, $c = 1, 2, \ldots, m$. Let $\xi_c^{(m)}$ be the covariance between these two random variables.

Then Hoeffding (1948) has obtained the variance of the U-statistic given in (2.5) as

$$\mathrm{Var}[U(X_1, X_2, \ldots, X_n)] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \xi_c^{(m)}. \tag{2.6}$$

Clearly the U-statistic defined in (2.5) is an unbiased estimator of θ. For the optimal properties of U-statistic see, Serfling (1980).

2.3 U-Statistics with kernels based on best linear functions of order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from (1.1). Let the BLUE of μ as given in (2.1) can be written as

$$h_1(X_1, X_2, \ldots, X_{\mathrm{m}}) = a_1 X_{1:\mathrm{m}} + a_2 X_{2:\mathrm{m}} + \cdots + a_{\mathrm{m}} X_{\mathrm{m:m}} \tag{2.7}$$

and that of σ as given in (2.2) be written as

$$h_2(X_1, X_2, \ldots, X_{\mathrm{m}}) = d_1 X_{1:\mathrm{m}} + d_2 X_{2:\mathrm{m}} + \cdots + d_{\mathrm{m}} X_{\mathrm{m:m}}, \tag{2.8}$$

where $a_1, a_2, \ldots, a_{\mathrm{m}}$ and $d_1, d_2, \ldots, d_{\mathrm{m}}$ are constants. Now we can easily write

$$U_{1;n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} a_{i+1} \right] X_{r:n} \tag{2.9}$$

as the U-statistic for estimating μ based on kernel (2.7) and

$$U_{2;n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} d_{i+1} \right] X_{r:n} \tag{2.10}$$

as the U-statistic for estimating σ based on kernel (2.8), where we define $\binom{r-1}{i} = 0$ for i $\geq$ r and $\binom{n-r}{m-1-i} = 0 \quad$ for $n - r < m - 1 - i$.

Blom (1980) has introduced linear U-estimates of location and scale parameters of a distribution using linear estimates based on the order statistics of smaller sample sizes by defining suitable generating polynomials. Though our estimator $U_{1;n}^{(m)}$ for μ and $U_{2;n}^{(m)}$ for σ are in agreement with the results of Blom (1980), the expression involved in our estimates is explicit and simple. In Blom (1980), nothing is mentioned in obtaining the variance of these linear

U-estimates. However in this paper we describe below an algebraic method to evaluate the variances of our estimators $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ explicitly. The results given in equations (2.9) and (2.10) are in strong agreement with the results of Blom (1980), where the generating polynomial technique has been used to find the coefficients of $X_{r:n}$ involved in the respective U-statistics based on kernels as BLUEs. Also the expression for the variance of the U-statistics defined in equations (2.9) and (2.10) given by Blom (1980) is based on variance covariance matrix of the entire order statistics of a random sample of size n arising from the population, which is computationally not practical as for the log-gamma distribution no explicit expressions for the moments exists. Hence we derive a new method for the computation of the variance of the U-statistics.
If we write

$$\begin{aligned}\xi_c^{(m)} &= \mathrm{Cov}[h_1(X_1, X_2, \ldots, X_c, X_{c+1}, \ldots, X_\mathrm{m}),\\ &\qquad h_1(X_1, X_2, \ldots, X_c, X_{\mathrm{m}+1}, \ldots, X_{2\mathrm{m}-c})], \qquad (2.11)\\ \zeta_c^{(m)} &= \mathrm{Cov}[h_2(X_1, X_2, \ldots, X_c, X_{c+1}, \ldots, X_\mathrm{m}),\\ &\qquad h_2(X_1, X_2, \ldots, X_c, X_{\mathrm{m}+1}, \ldots, X_{2\mathrm{m}-c})], \qquad (2.12)\end{aligned}$$

for $c = 1, 2, \ldots, m$, then the variances of $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ are given by

$$\mathrm{Var}\left[U_{1;n}^{(m)}\right] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c} \xi_c^{(m)}, \qquad (2.13)$$

$$\mathrm{Var}\left[U_{2;n}^{(m)}\right] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c}\binom{n-m}{m-c} \zeta_c^{(m)}. \qquad (2.14)$$

Clearly $\xi_\mathrm{m}^{(m)} = \mathrm{Var}[h_1(X_1, X_{2,\cdots,} X_\mathrm{m})]$ and $\zeta_\mathrm{m}^{(m)} = \mathrm{Var}[h_2(X_1, X_{2,\cdots,} X_\mathrm{m})]$ and are given by (2.3) and (2.4), respectively.

Clearly to obtain $\mathrm{Var}[U_{1;n}^{(m)}]$ and $\mathrm{Var}[U_{2;n}^{(m)}]$, we have to obtain the values of $\xi_c^{(m)}, \zeta_c^{(m)}$, $c = 1, 2, \ldots, m-1$. If we put $n = m + k$ in (2.13), we have the following.

$$\begin{aligned}\mathrm{Var}\left[U_{1;\mathrm{m}+k}^{(m)}\right] = \frac{1}{\binom{m+k}{m}} &\left[\binom{m}{m-k}\binom{k}{k}\xi_{\mathrm{m}-k}^{(m)} + \binom{m}{m-k+1}\right.\\ &\left.\times \binom{k}{k-1}\xi_{\mathrm{m}-k+1}^{(m)} + \cdots + \binom{m}{m}\binom{k}{0}\xi_\mathrm{m}^{(m)}\right] \qquad (2.15)\end{aligned}$$

for $k = 1, 2, \ldots, m-1$. The expression for $U^{(m)}_{1;\mathrm{m}+k}$ for $k = 1, 2, \ldots, m-1$ is given by (putting $n = m + k$ in (2.9)).

$$U^{(m)}_{1;\mathrm{m}+k} = \frac{1}{\binom{m+k}{m}} \left\{ \left[\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i} \binom{0}{i} a_{i+1} \right] X_{1:\mathrm{m}+k} \right.$$
$$+ \left[\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i} \binom{1}{i} a_{i+1} \right] X_{2:\mathrm{m}+k} + \cdots$$
$$\left. + \left[\sum_{i=0}^{m-1} \binom{0}{m-1-i} \binom{m+k-1}{i} a_{i+1} \right] X_{\mathrm{m}+k:\mathrm{m}+k} \right\}.$$

The above expression can be also written as

$$U^{(m)}_{1;\mathrm{m}+k} = b'_{\mathrm{m}+k} X_{\mathrm{m}+k}, \tag{2.16}$$

where vector $b_{\mathrm{m}+k}$ given by

$$b'_{\mathrm{m}+k} = \left[\frac{\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i} \binom{0}{i} a_{i+1}}{\binom{m+k}{m}}, \frac{\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i} \binom{1}{i} a_{i+1}}{\binom{m+k}{m}}, \ldots, \right.$$
$$\left. \times \frac{\sum_{i=0}^{m-1} \binom{0}{m-1-i} \binom{m+k-1}{i} a_{i+1}}{\binom{m+k}{m}} \right], \tag{2.17}$$

and

$$X_{\mathrm{m}+k} = \left(X_{1:\mathrm{m}+k}, X_{2:\mathrm{m}+k}, \ldots, X_{\mathrm{m}+k:\mathrm{m}+k}\right)'.$$

Hence

$$\mathrm{Var}\left[U^{(m)}_{1;\mathrm{m}+k}\right] = \left(b'_{\mathrm{m}+k} V_{\mathrm{m}+k} b_{\mathrm{m}+k}\right) \sigma^2, \tag{2.18}$$

where $V_{\mathrm{m}+k}$ is the variance covariance matrix of the vector of order statistics of random sample of size $m + k$ drawn from the distribution with pdf $f(x; 0, 1)$. Equations (2.15) and (2.18) are identically equal and consequently we can write

$$\binom{m}{m-k}\binom{k}{k}\xi^{(m)}_{\mathrm{m}-k}+\binom{m}{m-k+1}\binom{k}{k-1}\xi^{(m)}_{\mathrm{m}-k+1}+\cdots$$
$$+\binom{m}{m-1}\binom{k}{1}\xi^{(m)}_{\mathrm{m}-1}=\binom{m+k}{m}\left(b'_{\mathrm{m}+k}V_{\mathrm{m}+k}\,b_{\mathrm{m}+k}\right)\sigma^2-\xi^{(m)}_{\mathrm{m}},\tag{2.19}$$

$k=1,2,\ldots,m-1$. The above system of equations can be written by the following matrix equation.

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & \cdots & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & \cdots & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix}$$
$$\times\begin{bmatrix}\xi^{(m)}_1\\ \xi^{(m)}_2\\ \vdots\\ \xi^{(m)}_{\mathrm{m}-1}\end{bmatrix}=\begin{bmatrix}w_1\\ w_2\\ \vdots\\ w_{\mathrm{m}-1}\end{bmatrix},\tag{2.20}$$

where $w_k=\binom{m+k}{m}\left(b'_{\mathrm{m}+k}V_{\mathrm{m}+k}\,b_{\mathrm{m}+k}\right)\sigma^2-\xi^{(m)}_{\mathrm{m}}, k=1,2,\ldots,m-1$. If we write H to denote the coefficient matrix on the left side of (2.20) and W to denote the vector in the right side of (2.20) then we have

$$\begin{bmatrix}\xi^{(m)}_1\\ \xi^{(m)}_2\\ \vdots\\ \xi^{(m)}_{\mathrm{m}-1}\end{bmatrix}=H^{-1}W.$$

Similarly the values of $\zeta_c^{(m)}, c=1,2,\ldots,m-1$ can be obtained as

$$\begin{bmatrix}\zeta^{(m)}_1\\ \zeta^{(m)}_2\\ \vdots\\ \zeta^{(m)}_{\mathrm{m}-1}\end{bmatrix}=H^{-1}Z,$$

where $Z' = (z_1, z_2, \ldots, z_{\mathrm{m}-1})$ with $z_k = \binom{m+k}{m} \left(g'_{\mathrm{m}+k} V_{\mathrm{m}+k} g_{m+k}\right) \sigma^2 - \zeta_{\mathrm{m}}^{(m)}$ and $g_{\mathrm{m}+k}$ is the same as $b_{\mathrm{m}+k}$ defined in (2.17) in which each a_i is replaced by $d_i, i = 1, 2, \ldots, m$. Once we obtain the values of $\xi_c^{(m)}, \zeta_c^{(m)}, c = 1, 2, \ldots, m-1$, then the exact variances of the U-statistics for estimating μ and σ based on any sample of size n can be obtained using (2.13) and (2.14) without any further direct evaluation of moments of order statistics.

The main advantage of this method is that if one uses the BLUE based on a sample of size m as the kernel, then the evaluation of variances and covariances of order statistics of sample sizes up to $2m-1$ arising from (1.1) alone are necessary to obtain the explicit variance of the U-statistics $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ whatever is the sample size.

3 Estimation of the parameters of log-gamma distribution using *U*-statistics

Though Balakrishnan and Chan (1998) have evaluated the means, variances and covariances of standard log-gamma order statistics, coefficients of BLUEs and the variances of the BLUEs of μ and σ, we have also evaluated those values using Mathcad software and found agreeing with their values. Using these values and procedure suggested in the previous section we have computed $\xi_c^{(m)}$ and $\zeta_c^{(m)}$ for $c = 1, 2 \ldots, m-1$; m = 2, 3, 4, 5 and are given in Table 1 for κ = 2.0 (2.0) 10.0. Using these values we have obtained the variances of $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ for n = 5 (5) 20(10) 40 (20) 100; m = 2-5 and for κ = 2.0 (2.0) 10.0 and are given in Tables 2 and 3.

4 Comparison of the *U*-statistics with maximum likelihood estimators

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from (1.1).The log-likelihood function L of the random sample is given by,

$$L = -n \log \Gamma(\kappa) - n \log \sigma - \kappa \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right) - \sum_{i=1}^{n} \exp\left[\exp\left(-\left(\frac{x_i - \mu}{\sigma}\right)\right)\right]. \tag{4.1}$$

Then the asymptotic variance of the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ are given by the diagonal elements of the matrix I^{-1} where,

$$\mathrm{I} = -\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \mu^2}\right) E\left(\frac{\partial^2 L}{\partial \mu\, \partial \sigma}\right) \\ E\left(\frac{\partial^2 L}{\partial \mu\, \partial \sigma}\right) E\left(\frac{\partial^2 L}{\partial \sigma^2}\right) \end{bmatrix}, \tag{4.2}$$

Table 1 Values of $\xi_c^{(m)}, \zeta_c^{(m)}$ for $c = 1, 2, \ldots, m$ and $m = 2\ (1)\ 5$

m	c	$\kappa = 2$		$\kappa = 4$		$\kappa = 6$		$\kappa = 8$		$\kappa = 10$	
		$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$	$\sigma^{-2}\xi_c^{(m)}$	$\sigma^{-2}\zeta_c^{(m)}$
2	1	0.25466	0.16539	0.38844	0.14622	0.52863	0.13991	0.65545	0.13682	0.76919	0.13495
	2	0.56496	0.64206	1.27074	0.60541	1.97045	0.59351	2.58682	0.58768	3.13217	0.13495
3	1	0.11142	0.06896	0.16874	0.06288	0.23077	0.06085	0.28731	0.05987	0.33792	0.05923
	2	0.22871	0.17076	0.39107	0.15972	0.56128	0.15598	0.71415	0.15410	0.85117	0.15304
	3	0.35388	0.31144	0.67196	0.29332	0.99742	0.28728	1.28733	0.28428	1.54534	0.28250
4	1	0.06219	0.03763	0.09382	0.03479	0.12877	0.03393	0.16003	0.03333	0.18934	0.03318
	2	0.12590	0.08355	0.20111	0.07807	0.28193	0.07618	0.35526	0.07527	0.42075	0.07465
	3	0.19146	0.13853	0.32252	0.13024	0.46119	0.12742	0.58609	0.12599	0.69805	0.12517
	4	0.25924	0.20386	0.45928	0.19199	0.66778	0.18798	0.85466	0.18598	1.02149	0.18478
5	1	0.03965	0.02377	0.05969	0.02202	0.08095	0.02106	0.10273	0.02145	0.12224	0.02139
	2	0.07981	0.05037	0.12415	0.04716	0.17250	0.04606	0.21623	0.04544	0.25563	0.04510
	3	0.12068	0.08030	0.19373	0.07545	0.27250	0.07380	0.34378	0.07295	0.40786	0.07245
	4	0.16230	0.11373	0.26865	0.10708	0.38203	0.10480	0.48442	0.10366	0.57612	0.10297
	5	0.20478	0.15100	0.34925	0.14224	0.50170	0.13926	0.63888	0.13777	0.76160	0.13687

Table 2 Variances of $U_{1;n}^{(m)}$ and their relative efficiencies

κ	n	$\frac{Var(\hat{\mu})}{\sigma^2}$	$\frac{Var\left(U_{1;n}^{(2)}\right)}{\sigma^2}$	$\frac{Var\left(U_{1;n}^{(3)}\right)}{\sigma^2}$	$\frac{Var\left(U_{1;n}^{(4)}\right)}{\sigma^2}$	$\frac{Var\left(U_{1;n}^{(5)}\right)}{\sigma^2}$	$e\left(U_{1;n}^{(2)}\|\hat{\mu}\right)$	$e\left(U_{1;n}^{(3)}\|\hat{\mu}\right)$	$e\left(U_{1;n}^{(4)}\|\hat{\mu}\right)$	$e\left(U_{1;n}^{(5)}\|\hat{\mu}\right)$
2.0	5	0.19515	0.20929	0.20604	0.20502	0.20478	0.93243	0.94715	0.95188	0.95297
	10	0.09758	0.10310	0.10147	0.10076	0.10041	0.94641	0.96163	0.96835	0.97180
	15	0.06505	0.06844	0.06736	0.06687	0.06661	0.95048	0.96571	0.97280	0.97658
	20	0.04879	0.05122	0.05042	0.05004	0.04985	0.95242	0.96764	0.97488	0.97873
	30	0.03253	0.03408	0.03355	0.03330	0.03316	0.95430	0.96951	0.97687	0.98074
	40	0.02439	0.02554	0.02514	0.02495	0.02485	0.95522	0.97042	0.97784	0.98169
	60	0.01626	0.01701	0.01674	0.01662	0.01655	0.95612	0.97131	0.97878	0.98262
	80	0.01220	0.01275	0.01255	0.01246	0.01241	0.95657	0.97175	0.97925	0.98306
	100	0.00976	0.01020	0.01004	0.00996	0.00992	0.95684	0.97201	0.97952	0.98333
4.0	5	0.29064	0.36014	0.35246	0.34987	0.34925	0.80702	0.82460	0.83070	0.83218
	10	0.14532	0.16635	0.16263	0.16098	0.16010	0.87358	0.89359	0.90274	0.90767
	15	0.09688	0.10829	0.10585	0.10472	0.10410	0.89466	0.91527	0.92517	0.93064
	20	0.07266	0.08029	0.07848	0.07762	0.07715	0.90500	0.92589	0.93613	0.94176
	30	0.04844	0.05293	0.05173	0.05115	0.05085	0.91522	0.93636	0.94693	0.95267
	40	0.03633	0.03948	0.03859	0.03815	0.03792	0.92028	0.94155	0.95227	0.95804
	60	0.02422	0.02618	0.02558	0.02529	0.02514	0.92531	0.94670	0.95758	0.96335
	80	0.01817	0.01958	0.01914	0.01892	0.01880	0.92781	0.94926	0.96022	0.96599
	100	0.01453	0.01564	0.01528	0.01511	0.01502	0.92931	0.95079	0.96180	0.96757
6.0	5	0.39844	0.51422	0.50574	0.50251	0.50170	0.77484	0.78783	0.79290	0.79418
	10	0.19922	0.23175	0.22769	0.22577	0.22451	0.85965	0.87496	0.88240	0.88736
	15	0.13281	0.14967	0.14702	0.14578	0.14460	0.88740	0.90334	0.91107	0.91850
	20	0.09961	0.11053	0.10858	0.10766	0.10659	0.90118	0.91742	0.92521	0.93447
	30	0.06641	0.07258	0.07130	0.07070	0.06985	0.91490	0.93142	0.93924	0.95077
	40	0.04980	0.05403	0.05307	0.05264	0.05193	0.92173	0.93839	0.94620	0.95904
	60	0.03320	0.03576	0.03512	0.03484	0.03432	0.92855	0.94534	0.95314	0.96740
	80	0.02490	0.02672	0.02625	0.02603	0.02563	0.93196	0.94881	0.95660	0.97162
	100	0.01992	0.02133	0.02095	0.02078	0.02045	0.93400	0.95090	0.95867	0.97416
8.0	5	0.49738	0.65195	0.64342	0.63980	0.63888	0.76292	0.77304	0.77740	0.77852
	10	0.24869	0.29053	0.28654	0.28427	0.28301	0.85598	0.86791	0.87485	0.87874
	15	0.16579	0.18694	0.18436	0.18278	0.18212	0.88690	0.89930	0.90706	0.91036
	20	0.12435	0.13781	0.13590	0.13470	0.13435	0.90233	0.91495	0.92314	0.92556
	30	0.08290	0.09033	0.08908	0.08826	0.08816	0.91775	0.93058	0.93920	0.94029
	40	0.06217	0.06718	0.06626	0.06564	0.06562	0.92546	0.93838	0.94722	0.94748
	60	0.04145	0.04442	0.04381	0.04339	0.04342	0.93316	0.94618	0.95524	0.95455
	80	0.03109	0.03318	0.03272	0.03241	0.03245	0.93701	0.95008	0.95925	0.95804
	100	0.02487	0.02648	0.02611	0.02586	0.02590	0.93932	0.95242	0.96165	0.96012
10.0	5	0.58677	0.77473	0.76661	0.76274	0.76160	0.75739	0.76541	0.76930	0.77045
	10	0.29339	0.34309	0.33924	0.33709	0.33559	0.85512	0.86483	0.87034	0.87423
	15	0.19559	0.22030	0.21779	0.21652	0.21586	0.88785	0.89806	0.90334	0.90612
	20	0.14669	0.16223	0.16037	0.15950	0.15929	0.90425	0.91469	0.91973	0.92092
	30	0.09780	0.10622	0.10500	0.10448	0.10461	0.92066	0.93134	0.93604	0.93486
	40	0.07335	0.07896	0.07806	0.07768	0.07790	0.92888	0.93967	0.94417	0.94149
	60	0.04890	0.05218	0.05158	0.05135	0.05159	0.93710	0.94800	0.95228	0.94790
	80	0.03667	0.03896	0.03852	0.03835	0.03856	0.94121	0.95217	0.95633	0.95102
	100	0.02934	0.03109	0.03073	0.03060	0.03079	0.94368	0.95467	0.95875	0.95286

Table 3 Variances of $U_{2;n}^{(m)}$ and their relative efficiencies

κ	n	$\frac{Var(\hat{\sigma})}{\sigma^2}$	$\frac{Var\left(U_{2;n}^{(2)}\right)}{\sigma^2}$	$\frac{Var\left(U_{2;n}^{(3)}\right)}{\sigma^2}$	$\frac{Var\left(U_{2;n}^{(4)}\right)}{\sigma^2}$	$\frac{Var\left(U_{2;n}^{(5)}\right)}{\sigma^2}$	$e\left(U_{2;n}^{(2)}\,\lvert\hat{\sigma}\right)$	$e\left(U_{2;n}^{(.3)}\,\lvert\hat{\sigma}\right)$	$e\left(U_{2;n}^{(4)}\,\lvert\hat{\sigma}\right)$	$e\left(U_{2;n}^{(5)}\,\lvert\hat{\sigma}\right)$
2.0	5	0.11174	0.16344	0.15429	0.15160	0.15100	0.68368	0.72423	0.73709	0.74000
	10	0.05587	0.07307	0.06868	0.06695	0.06609	0.76457	0.81346	0.83456	0.84532
	15	0.03725	0.04707	0.04420	0.04301	0.04242	0.79133	0.84261	0.86603	0.87813
	20	0.02794	0.03472	0.03259	0.03169	0.03124	0.80466	0.85709	0.88163	0.89409
	30	0.01862	0.02277	0.02137	0.02076	0.02047	0.81798	0.87151	0.89714	0.90975
	40	0.01397	0.01694	0.01590	0.01544	0.01522	0.82462	0.87871	0.90487	0.91746
	60	0.00931	0.01120	0.01051	0.01020	0.01007	0.83126	0.88588	0.91258	0.92511
	80	0.00698	0.00837	0.00785	0.00762	0.00752	0.83458	0.88947	0.91643	0.92890
	100	0.00559	0.00668	0.00627	0.00608	0.00600	0.83657	0.89162	0.91874	0.93117
4.0	5	0.10608	0.14827	0.14403	0.14259	0.14224	0.71546	0.73655	0.74398	0.74581
	10	0.05304	0.06544	0.06341	0.06251	0.06203	0.81051	0.83653	0.84853	0.85515
	15	0.03536	0.04197	0.04065	0.04003	0.03968	0.84248	0.87000	0.88327	0.89114
	20	0.02652	0.03089	0.02991	0.02945	0.02917	0.85853	0.88678	0.90066	0.90917
	30	0.01768	0.02022	0.01957	0.01926	0.01907	0.87461	0.90358	0.91805	0.92724
	40	0.01326	0.01502	0.01454	0.01431	0.01416	0.88267	0.91199	0.92676	0.93629
	60	0.00884	0.00992	0.00960	0.00945	0.00935	0.89073	0.92041	0.93547	0.94535
	80	0.00663	0.00741	0.00717	0.00705	0.00698	0.89477	0.92462	0.93982	0.94989
	100	0.00530	0.00591	0.00572	0.00563	0.00557	0.89719	0.92715	0.94244	0.95261
6.0	5	0.10410	0.14330	0.14057	0.13953	0.13926	0.72645	0.74054	0.74605	0.74751
	10	0.05205	0.06293	0.06164	0.06103	0.06060	0.82703	0.84445	0.85282	0.85886
	15	0.03470	0.04030	0.03945	0.03907	0.03862	0.86109	0.87952	0.88818	0.89849
	20	0.02602	0.02963	0.02901	0.02873	0.02830	0.87823	0.89715	0.90584	0.91952
	30	0.01735	0.01938	0.01896	0.01879	0.01843	0.89543	0.91484	0.92348	0.94150
	40	0.01301	0.01439	0.01409	0.01396	0.01366	0.90406	0.92371	0.93230	0.95286
	60	0.00867	0.00950	0.00930	0.00922	0.00899	0.91270	0.93259	0.94112	0.96449
	80	0.00651	0.00709	0.00694	0.00688	0.00670	0.91703	0.93704	0.94553	0.97041
	100	0.00520	0.00566	0.00554	0.00549	0.00534	0.91963	0.93971	0.94818	0.97400
8.0	5	0.10309	0.14086	0.13885	0.13799	0.13777	0.73185	0.74245	0.74708	0.74826
	10	0.05154	0.06171	0.06077	0.06024	0.05994	0.83531	0.84821	0.85564	0.85995
	15	0.03436	0.03948	0.03887	0.03851	0.03836	0.87047	0.88403	0.89230	0.89577
	20	0.02577	0.02902	0.02857	0.02830	0.02823	0.88817	0.90205	0.91074	0.91297
	30	0.01718	0.01896	0.01867	0.01849	0.01848	0.90597	0.92015	0.92926	0.92963
	40	0.01289	0.01408	0.01387	0.01373	0.01374	0.91490	0.92923	0.93855	0.93775
	60	0.00859	0.00930	0.00916	0.00906	0.00908	0.92385	0.93833	0.94786	0.94571
	80	0.00644	0.00694	0.00683	0.00676	0.00678	0.92834	0.94289	0.95253	0.94964
	100	0.00515	0.00554	0.00545	0.00540	0.00541	0.93103	0.94563	0.95533	0.95198
10.0	5	0.10248	0.13939	0.13784	0.13709	0.13687	0.73516	0.74343	0.74750	0.74871
	10	0.05124	0.06097	0.06023	0.05982	0.05953	0.84045	0.85068	0.85657	0.86075
	15	0.03416	0.03898	0.03850	0.03826	0.03812	0.87631	0.88714	0.89279	0.89614
	20	0.02562	0.02864	0.02829	0.02813	0.02807	0.89438	0.90551	0.91089	0.91284
	30	0.01708	0.01872	0.01848	0.01839	0.01839	0.91256	0.92397	0.92898	0.92877
	40	0.01281	0.01390	0.01373	0.01366	0.01368	0.92168	0.93324	0.93802	0.93643
	60	0.00854	0.00917	0.00906	0.00902	0.00905	0.93083	0.94253	0.94707	0.94389
	80	0.00640	0.00685	0.00676	0.00673	0.00676	0.93541	0.94719	0.95159	0.94754
	100	0.00512	0.00546	0.00539	0.00537	0.00540	0.93817	0.94998	0.95430	0.94970

Table 4 Asymptotic Relative Efficiencies of $U_{1;n}^{(m)}$ and $U_{2;n}^{(m)}$ as estimators of μ and σ

κ	$E_1\left(U_{1;n}^{(2)}\,\vert\hat{\mu}\right)$	$E_1\left(U_{1;n}^{(3)}\,\vert\hat{\mu}\right)$	$E_1\left(U_{1;n}^{(4)}\,\vert\hat{\mu}\right)$	$E_1\left(U_{1;n}^{(5)}\,\vert\hat{\mu}\right)$	$E_2\left(U_{2;n}^{(2)}\,\vert\hat{\sigma}\right)$	$E_2\left(U_{2;n}^{(3)}\,\vert\hat{\sigma}\right)$	$E_2\left(U_{2;n}^{(4)}\,\vert\hat{\sigma}\right)$	$E_2\left(U_{2;n}^{(5)}\,\vert\hat{\sigma}\right)$
2	0.95789	0.97304	0.98061	0.98436	0.84452	0.90020	0.92795	0.94018
4	0.93528	0.95690	0.96808	0.97383	0.90689	0.93727	0.95289	0.96352
6	0.94215	0.95920	0.96693	0.98441	0.93004	0.95041	0.95876	0.98858
8	0.94855	0.96176	0.97127	0.96833	0.94182	0.95659	0.96655	0.96119
10	0.95356	0.96468	0.96845	0.96003	0.94920	0.96119	0.96515	0.95817

with

$$-E\left[\frac{\partial^2 L}{\partial \mu^2}\right] = \frac{n\kappa}{\sigma^2},$$
$$-E\left[\frac{\partial^2 L}{\partial \mu\,\partial \sigma}\right] = \frac{n\kappa}{\sigma^2}\psi(\kappa+1),$$
$$-E\left[\frac{\partial^2 L}{\partial \sigma^2}\right] = \frac{n}{\sigma^2} + \frac{n\kappa}{\sigma^2}\left\{\psi'(\kappa+1) + \left[\psi(\kappa+1)\right]^2\right\}.$$

Using the above equations in (4.2), the variances of $\hat{\mu}$ and $\hat{\sigma}$ are numerically computed for $\kappa = 2.0$ (2.0) 10.0 and $n = 5$ (5) 20 (10) 40 (20) 100. The efficiency $e\left(U_{1;n}^{(m)}\,\vert\hat{\mu}\right)$ of $U_{1;n}^{(m)}$ relative to $\hat{\mu}$ for $m = 2, 3, 4, 5$; $n = 5$ (5) 20 (10) 40 (20) 100 and are given in Table 2. The efficiency $e\left(U_{2;n}^{(m)}\,\vert\hat{\sigma}\right)$of $U_{2;n}^{(m)}$relative to $\hat{\sigma}$ are also calculated for $m = 2, 3, 4, 5$; $n = 5$ (5) 20 (10) 40 (20) 100 and are given in table 3. From these tables it is clear that the efficiency of our estimators $U_{1;n}^{(m)}$ for μ and $U_{2;n}^{(m)}$ for σ relative to their respective maximum likelihood estimators increases as sample size increases. We have also observe that $U_{1;n}^{(m)}$ estimates μ more efficiently than do $U_{2;n}^{(m)}$ estimates σ for all $\kappa = 2.0$ (2.0) 10.0.

The asymptotic relative efficiencies $E_1\left(U_{1;n}^{(m)}\,\vert\hat{\mu}\right)$ of $U_{1;n}^{(m)}$ relative to $\hat{\mu}$ and $E_2\left(U_{2;n}^{(m)}\,\vert\hat{\sigma}\right)$ of $U_{2;n}^{(m)}$ relative to $\hat{\sigma}$ are given by

$$E_1\left(U_{1;n}^{(m)}\,\vert\hat{\mu}\right) = \lim_{n\to\infty}\left[\frac{\mathrm{Var}\left(\hat{\mu}\right)}{\mathrm{Var}\left(U_{1;n}^{(m)}\right)}\right] = \frac{1+\kappa\left\{\psi'(\kappa+1) + [\psi(\kappa+1)]^2\right\}}{\kappa[1+\psi'(\kappa+1)]m^2\xi_1^{(m)}},$$

$$E_2\left(U_{2;n}^{(m)}\,\vert\hat{\sigma}\right) = \lim_{n\to\infty}\left[\frac{\mathrm{Var}(\hat{\sigma})}{\mathrm{Var}\left(U_{2;n}^{(n)}\right)}\right] = \frac{1}{\kappa[1+\psi'(\kappa+1)]m^2\zeta_1^{(m)}}.$$

Consequently the numerical values of $E_1\left(U_{1;n}^{(m)}\,\vert\hat{\mu}\right)$ and that of $E_2\left(U_{2;n}^{(m)}\,\vert\hat{\sigma}\right)$ for $m = 2, 3, 4, 5$ are calculated for $\kappa = 2.0$ (2.0) 10.0 and are given in table 4

From the table it is clear that the asymptotic relative efficiencies in all cases considered exceeds 96% when the U-statistic constructed for μ is based on BLUE μ^* based on order statistics of a sample of size 5. Similarly the asymptotic relative efficiencies in all cases considered exceeds 94% when the U-statistic constructed for σ is based on BLUE σ * based on order statistics of a sample of size 5.

Remarks It is to be noted that though we have compared the efficiency of the U-statistics developed in this paper which have explicit expression in terms of order statistics, with the maximum likelihood estimators, we do not have explicit expressions for the maximum likelihood estimators.

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