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On the usage of ideal lattices in post-quantum cryptography

Bachelor's Project Mathematics

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Contents

1 Introduction

1.1 Actual introduction here with some funny name

1.2 Motivation

include the table from page 16 of [Ber09] of systems broken by quantum computers

quote from [Ber09]: “As it turns out, number theoretic problems are also the main place where quantum computers have been shown to have exponential speedups. Examples of such problems include factoring and discrete log [38], Pell’s equation [18], and computing the unit group and class group of a number field [17, 37]. The existence of these algorithms implies that a quantum computer could break RSA, Diffie-Hellman and elliptic curve cryptography, which are currently used".

On the 18^{th} of November 2022, a document was issued on migrating to post-quantum cryptography.

1.3 Main results

I imagine this will be filled last.

1.4 Outline of the paper

2 Background

2.1 Notation

Most of the notation is "standard" in the field of number theory and cryptography. Nonetheless, to avoid any misunderstandings and simplify some statements, we will present the notation used throughout the text. Anything that is not mentioned here shall be defined "on the go" with definitions and such.

Any scalars $a, t, \beta, \ldots$ are represented by non-bold, latin or greek letters.
Bold symbols $\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{s}, \ldots$ will denote vectors.
Algorithms will be represented by lmodern font names such as Encrypt or Evaluate.
Traditionally, the symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ shall represent the sets of integer, rational, real and complex numbers, respectivelly.
Additionally, for any real $m \geq 0$, $\lfloor m \rfloor$ shall denote greatest integer not exceeding m, $\lfloor m \rceil = \lfloor m + \frac{1}{2} \rfloor$ and $[m]$ is the set $\{1, 2, \ldots, \lfloor m \rfloor\}$.

2.2 Lattices

Basic Definitions

We define a *lattice* as a discrete additive subgroup of $\mathbb{R}^n$. Once we fix a basis $\boldsymbol{B} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^n$ we can then describe the lattice as

$$\Lambda = \mathcal{L}(\boldsymbol{B}) = \Big\{ \sum_i z_i \boldsymbol{b}_i : z_i \in \mathbb{Z} \Big\}.$$

There are many bases for a lattice (actually, for $n \geq 2$, there are infinitely many as can be proven using a diagonalization argument), some "better" than others. This will be the foundation for some of the cryptosystems later like the GGH.

Example 2.1. The simplest example of a lattice is the $\mathbb{Z}^n$ itself. Taking the standard basis $\boldsymbol{B}_1 = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ we obtain

$$\mathcal{L}(\boldsymbol{B}_1) = \Big\{ \sum_i z_i \boldsymbol{e}_i : z_i \in \mathbb{Z} \Big\} = \mathbb{Z}^n.$$

More generally, Λ is a lattice of rank m in $\mathbb{R}^n$ if it is a rank m free abelian group. Recall that we call a group *free abelian group* of rank m if it can be written as $\Lambda = \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_m$ with $\beta_1, \ldots, \beta_m$ linearly independent over $\mathbb{R}$ where $\oplus$ represents the direct sum. In this paper we will only consider lattices of full rank n.

Remark 2.2. We can also view the vectors $\boldsymbol{b}_i$ as the columns of the matrix $\boldsymbol{B} \in \mathbb{R}^n \times \mathbb{R}^n$ in which case, our definition becomes:

$$\Lambda = \mathcal{L}(B) = \{\boldsymbol{Bz} : \boldsymbol{z} \in \mathbb{Z}^n\}.$$

Reciprocally, any matrix $\boldsymbol{B} \in GL_n(\mathbb{R})$ spans a lattice: the set of all integer linear combinations of its rows.

Example 2.3. 1. $\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in which case $\boldsymbol{b}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\boldsymbol{b}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$

2. $\mathcal{L} = \{(z_1, z_2) : z_1 + z_2 \text{ is even}\}$

3. $\mathcal{L} = \begin{pmatrix} 13 & 21 \\ 21 & 34 \end{pmatrix}$

As noted before, the basis of a lattice is not unique. There is one that is particularly interesting to us, namely, the *Hermite Normal Form* (HNF). A basis $\boldsymbol{B}$ is in HNF if it is upper triangular (or lower triangular - does not matter as long as one is consistent), all elements on the diagonal are strictly positive and any other element $\boldsymbol{b}_{i,j}$ satisfies $0 \leq \boldsymbol{b}_{i,j} < \boldsymbol{b}_{i,i}$.

Fundamental Domain

Definition 2.1 (Fundamental Domain)**.** Let $\mathcal{L}$ be a lattice of dimension n and let $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ be a basis for $\mathcal{L}$. The *fundamental domain* (or *fundamental parallelepiped*) for $\mathcal{L}$ corresponding to this basis is the set

$$\mathcal{F}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) = \{t_1\boldsymbol{b}_1 + \cdots + t_n\boldsymbol{b}_n : 0 \leq t_i < 1\}.$$

We define the *volume* of $\mathcal{F}(\boldsymbol{B})$ as the volume of the corresponding parallelepiped in $\mathbb{R}^n$. The *volume* - closely connected to the determinant - plays a very important role in our study which will become evident in later chapters. One of the advantages, of defining the fundamental domain, is that we can formalize the notion of area (or the determinant) of any given lattice. Recall that a lattice is just a countable collection of points and therefore has no volume by itself. This, however, is resolved by introducing the following.

Definition 2.2. Let $\mathcal{L}$ be a lattice of dimension n and let $\mathcal{F}(\boldsymbol{B})$ be a fundamental domain for $\mathcal{L}$ over some basis $\boldsymbol{B}$. We define the *determinant* of that lattice as

$$\det(\mathcal{L}) = \mathrm{Vol}(\mathcal{F}(\boldsymbol{B})) = |\det(\boldsymbol{B})|$$

The next two propositions are *de facto* foundation for lattice based cryptography. The first one states that the $\det(\mathcal{L})$ does not depend on the choice of the basis for that lattice. The second, that our whole ambient space $\mathbb{R}^n$ can be described using only vectors from the lattice and the fundamental domain. We will only give an outline of the proofs for the sake of keeping this section compact. Full proofs, however, can be found in [HPS14], chapter 6.4.

Proposition 2.3. *The* $\det(\mathcal{L})$ *of an n-dimensional lattice is invariant under the choice of the basis.*

Outline of the proof. Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be two bases for a lattice $\mathcal{L}$. The crucial part of the proof is to note that any two bases are related by some unimodular matrix U (i.e. a matrix with the determinant of ± 1) s.t. $\boldsymbol{B}_1 = U\boldsymbol{B}_2$. It now easily follows to compute $|\det(\boldsymbol{B}_1)| = \det(\mathcal{L}) = |\det(U \cdot \boldsymbol{B}_2)| = |\det(U)| \cdot |\det(\boldsymbol{B}_2)| = |\det(\boldsymbol{B}_2)|$ □

From now on we will write $\mathcal{F}$ to denote the fundamental domain of the lattice without specifying the basis.

Proposition 2.4. *Let $\mathcal{L} \subset \mathbb{R}^n$ be a lattice of dimension n and let $\mathcal{F}$ be a fundamental domain for $\mathcal{L}$. Then every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written in the form*

$$\boldsymbol{v} = \boldsymbol{f} + \boldsymbol{t}$$

for $\boldsymbol{f} \in \mathcal{F}$ and $\boldsymbol{t} \in \mathcal{L}$ both unique and associated to the original $\boldsymbol{v}$.

Equivalently, the space $\mathbb{R}^n$ is spanned exactly (without overlap) by shifting the fundamental domain by the vectors from our lattice.

$$\mathbb{R}^n = \bigcup_{\boldsymbol{t} \in \mathcal{L}} \{\boldsymbol{f} : \boldsymbol{f} \in \mathcal{F}\}$$

Remark 2.4. Sometimes the *fundamental domain* is refered to as a parallelepiped or parallelotope and denoted by caligraphic $\mathcal{P}$. If we take a matrix $\boldsymbol{B}$ to represent our lattice $\mathcal{L}$, then $\mathcal{P}_{1/2}(\boldsymbol{B}) = \{\boldsymbol{x}\boldsymbol{B}, \boldsymbol{x} \in [-1/2, 1/2]^n\}$ can also represent the (shifted by a half) fundamental domain of $\mathcal{L}$ (like for example in [Gen09b]).

We will now present two results that give us an upper bound on the length of the shortest vector in a lattice. This will later on be useful to determine the security and/or correctness of our schemes. These theorems are due to Hermite (1822 - 1901) and Minkowski (1864 - 1909).

Theorem 2.5 (Hermite's Theorem)**.** *Every lattice $\mathcal{L}$ of dimension n contains a nonzero vector $v \in \mathcal{L}$ satisfying*

$$\|v\| \leq \sqrt{n} \det(\mathcal{L})^{\frac{1}{n}}.$$

[Krzys: add minkowski's theorem]

We can also need some notion of the shortest possible vector of a lattice.

Definition 2.6. For a n-dimensional lattice $\mathcal{L}$, we denote its shortest nonzero vector (also called the *minimum distance*) by $\lambda_1(\mathcal{L})$. Formally

$$\lambda_1(\mathcal{L}) := \min_{0 \neq \boldsymbol{v} \in \mathcal{L}} ||\boldsymbol{v}||$$

More generally, for $1 \leq i \leq n$, the ith sucessive minimum of $\mathcal{L}$ is

$$\lambda_i(\mathcal{L}) := \inf\{r : \mathcal{L} \text{ has } i \text{ linearly independent vectors of length at most } r\}.$$

For example, in order to be able to go "away" from the lattice $\mathcal{L}$ "out" into the ambient space $\mathbb{R}^n$ and still be able to return to the original vector, we need to make sure the distance d we travel is at most $\lambda_1(\mathcal{L})/2$. This is for example a requirement in the BDD problem where the decoding is *promised*. For the exact definition see Section 2.4.

2.3 Algebraic Number Theory

Algebraic number theory is the study of *number fields*, *rings of integers* and *finite fields*. In this section we will provide all the necessary background needed to understand and verify the results presented in the cryptographic schemes later in the text. Most results will be stated without proof however all of them can be found in the book **Number Fields** by **Daniel A. Marcus** [Mar77] after which this sections is modelled.

Number Fields

A *number field* is defined as a subfield of $\overline{\mathbb{Q}}$ having finite dimension as a vector space over the rationals $\mathbb{Q}$. The *degree* of a number field K is defined as the dimension of K over $\mathbb{Q}$. There exists a monic irreducible polynomial[1] $f \in \mathbb{Q}[x]$ such that $K \cong \mathbb{Q}[x]/\langle f \rangle$. In fact, every monic and irreducible polynomial in $\mathbb{Q}[x]$ defines a number field via such isomorphism.

Definition 2.7 (Algebraic integer)**.** An element $\alpha \in \mathbb{C}$ is an *algebraic integer* if and only if, it is a root of some monic polynomial in $\mathbb{Z}[x]$.

In fact, the set of *algebraic integers* forms a ring.

Definition 2.8 (Ring of Integers)**.** We define the *ring of integers* (sometimes also called *maximal order*) $\mathcal{O}_K$ of a number field K as the intersection:

$$\mathcal{O}_K = K \cap \overline{\mathbb{Z}} = \{x \in K : \ x \text{ is an algebraic integer}\}.$$

Example 2.5. The field $K = \mathbb{Q}$ is a number field of degree 1. Its ring of integers is, as one can guess, the ordinary integers $\mathbb{Z}$.

Example 2.6. The ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ is the ring of integers of $K = \mathbb{Q}(\sqrt{-1}) = \{a+b\sqrt{-1} \ : \ a, b \in \mathbb{Q}\}$ which has degree 2 since x^2+1 is the minimal (and irreducible) polynomial of $\sqrt{-1}$ over $\mathbb{Q}$.

As another example, we can make a following statement about the ring of integers of a quadratic extension of rationals (real quadratic field).

Lemma 2.9. *Let $d \in \mathbb{Z}$ be a square-free integer. For the field $K = \mathbb{Q}(\sqrt{d})$, its ring of integers is*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \textit{if } d \equiv 2, 3 \textit{ mod } 4, \\ \mathbb{Z}[(1+\sqrt{d})/2] & \textit{otherwise.} \end{cases}$$

Proof. Take $d \equiv 1$ mod 4 square-free. □

Example 2.7. For $K = \mathbb{Q}(\sqrt{5})$ the ring of integers is $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{5})/2]$.

[1] Recall that we call polynomial monic if its leading coefficient is 1. It is an irreducible polynomial if it is irreducible as an element of the polynomial ring $\mathbb{Q}[x]$.

Cyclotomic fields

Definition 2.10 (Roots of unity)**.** Given a field K and a positive integer n, an element $\zeta \in K$ is called *primitive n-th root of unity* if ζ has order n in the multiplicative group $K^{\times}$. (In other words, $\zeta^n = 1$ and $\zeta^m \neq 1$ for $1 \leq m < n$).

The minimal polynomial Φ_n of ζ over $\mathbb{Q}$ is called the n-th cyclotomic polynomial. Formally we define it as

$$\Phi_n(x) =$$

It can be shown that every such ζ is equal to precisely $e^{2\pi\sqrt{-1}/n}$ and $\zeta^i = \zeta^j$ if and only if $i = j$.

What is worth noting here, is that for a number field $\mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{C}$, the ring of integers is not necessarily the $\mathbb{Z}[\alpha]$. Instead, $\mathbb{Z}[\alpha]$ is what's called an *order* in $\mathcal{O}_K$. We will not consider them in general here because they are not relevant for our study. However, one very useful feature of cyclotomic fields is that their ring of integers is actually just $\mathbb{Z}[\zeta]$. That is - $\mathcal{O}_K = \mathbb{Z}[\zeta]$ for $K = \mathbb{Q}(\zeta)$ and ζ is some n-th root of unity. This greately simplifies the approach in proving some of the results later in this paper. In general, a field $K = \mathbb{Q}(\alpha)$ such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$ is called a *monogenic* field. For more details on orders, look at for example Chapter 5 of [Ste04].

Proposition 2.11. *The ring of integers of a cyclotomic number field $K = \mathbb{Q}(\zeta)$ is $\mathcal{O}_K = \mathbb{Z}[\zeta]$.*

Embeddings in $\mathbb{C}$

Let $K = \mathbb{Q}(\alpha)$ be a number field of degree n for some α. Then there are exactly n embeddings (injective ring homomorphisms) of K in $\mathbb{C}$. These are easily described by observing that α can be sent to any one of its n conjugates over $\mathbb{Q}$. Each conjugate β determines a unique embedding ($\sigma_i : K \to \mathbb{C}$ and every embedding must arise in this way since α must be sent to one of its conjugates).

Example 2.8. The quadratic field $\mathbb{Q}[\sqrt{d}]$, d squarefree, has two embeddings in $\mathbb{C}$: The identity mapping, and also the one which sends $a + b\sqrt{d}$ to $a - b\sqrt{d}$ (a, $b \in \mathbb{Q}$), since $\sqrt{d}$ and $-\sqrt{d}$ are the two conjugates of $\sqrt{d}$. The n-th cyclotomic field has $\varphi(n)$ embeddings in $\mathbb{C}$, the $\varphi(n)$ automorphisms where $\sigma_i(\zeta) = \zeta^i$.

Definition 2.12 (Canonical embedding)**.**

[Pinar: what do we want to consider as a cyclotomic poly? is $n = 2^a$? is n a prime? and why?]

[Pinar: Maximal orders (ring of integers) are dedekind domains, embedding of $\mathbb{Q}(\alpha)$ to $\mathbb{C}$ hence embedding of the ideals. Properties of ideals in dedekind domains, operations, unique factorization and so on. all the necessary info.]

2.4 Complexity Theory and Hard Problems

When talking about cryptography, we cannot avoid talking about algorithms as an encryption scheme is simply a instruction on how to encode sensitive data. We will thus require some terminology from computational complexity theory - a field on the overlap of computer science and mathematics. The problems it is concerned are those of the time and space required for solving computational problems. Most of the time, the goal of the study is to prove the lower bound on the resources required to solve a problem using the best know algorithm. For example, how long does it take to find a factorization of a large composite number. The goal of this section is to give an overview of few parts of the field that we will be using throughout the paper.

We begin with necessary terminology. Throughout this section, n shall denote the "size" of the input to the algorithm. Here the word size could mean many things like for example the bit-length of a number or its value in the base 10 representation.

In order to represent the time or space required to solve the problem, most of the time we will use the "big-O" notation which, for any function f and g of some variable x, is defined as $f(x) \in O(g(x))$ if there exists some constant M and x_0 such that $f(x) \leq Mg(x)$ for all $x \geq x_0$.

[Pinar: will wait] mention "efficient", "negligible", "hard" and define reductions. show some problems like svp, cvp, bdd In this section we will briefly introduce what it means for a problem to be considered *hard* and provide couple of examples We define a `negligible` amount in n as an amount that is asymptotically smaller than n^{-c} for any constant $c > 0$. More precisely,

Definition 2.13. $f(n)$ is a `negligible` function in n if $\lim_{n\to\infty} n^{-c} f(n) = 0$ for any $c > 0$.

The best know examples `FACTORIZE`

Quantum computations

I'm not sure if that is supposed to be in a section about preliminaries but I also don't want to include it in the introduction coz its a bit long

3 Homomorphic Encryption

Fully Homomorphic Encryption (FHE) has been referred as the "holy grail" of modern cryptography as it was one of the most sought goals for the past couple of decades. First formally introduced by Rivest, Adleman and Dertouzos in [RAD78] (at the time called "privacy homomorphism"), shortly after the discovery of public key cryptography, it has been an open and elusive problem. Only "recently", in 2009, Craig Gentry proposed first FHE in his PhD thesis [Gen09a]. Since then, there has been a lot of development in the area like for example [Krzys: TODO: finish developments of fhe].

Simply stated, in homomorphic encryption we want our data to be secure but we also want to perform calculations on it. This is useful when you need a third party (e.g. someone with more computational power) to perform operations on your data while still retaining privacy. Alice can store her data somewhere on external server (the cloud) and ask to perform computations on it. For example query searches without the engine knowing what is actually being searched for.

In other words, we would like our encryption scheme to satisfy the following. Say the ciphertexts c_i's decrypt to messages m_i's. Then we want

$$\mathsf{Decrypt}_{\mathcal{E}}(c_1 + c_2) = m_1 + m_2, \qquad \mathsf{Decrypt}_{\mathcal{E}}(c_1 * c_2) = m_1 * m_2$$

Equivalently, we want Decrypt to be a ring homomorphism with respect to both addition and multiplication (of the given ring). $\mathcal{E}$ is *fully homomorphic* means that whenever f is a composition of **arbitrily many** additions and multiplications, then $\mathsf{Decrypt}_{\mathcal{E}}(f(c_1,\dots,c_n)) = f(m_1,\dots,m_n)$[2] which is also refered to as the *correctness* of the scheme.

Remark 3.1. Typically, an encryption scheme $\mathcal{E}$ is a tuple $\mathsf{KeyGen}_{\mathcal{E}}$, $\mathsf{Encrypt}_{\mathcal{E}}$ and $\mathsf{Decrypt}_{\mathcal{E}}$ (representing the key-generation, encryption and decryption respectively), all of which we require to be *efficient* - i.e. run in time $\mathrm{poly}(\lambda)$ - polynomial in the security parameter λ that represents the bit-length of the keys (see for example [KL14] or [HPS14] for more details). A homomorphic encryption scheme has a fourth algorithm - $\mathsf{Evaluate}_{\mathcal{E}}$ which we associate with some set of *permitted functions*. In our case this will simply be $\mathsf{Add}_{\mathcal{E}}$ and $\mathsf{Mult}_{\mathcal{E}}$ which we will introduce in further sections. Adopting the notation from [Gen10] we will denote by $\mathcal{F}_{\mathcal{E}}$, the generalized set of such functions.

3.1 Somewhat Homomorphic Encryption

Before we introduce the solution on to how to construct such FHE presented by Gentry, we will start with something slightly simpler, introduced in [Dij+10] by van Dijk et al. Their scheme works over the integers rather than lattices but relies on a similar assumption. Namely, that finding the greatest common divisor of many "noisy" multiples of a number is computationally difficult. We will come back to this problem later. To keep the exposition compact, we will avoid specifying most parameter choices.

Symmetric Key Scheme

We begin with the symmetric key scheme. We take our message to be a bit $m \in \{0,1\}$. The private key is an odd integer chosen from some interval $p \in [2^{\eta-1}, 2^{\eta})$. To encrypt our message m, we choose integers q and r at random (from some other interval and such that the magnitude of $2r$ is smaller than $p/2$). We obtain the ciphertext c by computing:

$$c = pq + 2r + m. \tag{3.1}$$

If we now want to decrypt our message, simply compute (c mod p) mod 2.
Let's say we have two messages c_1 and c_2. Then we can compute:

$$c_1 + c_2 = m_1 + m_2 + 2(r_1 + r_2) + p(q_1 + q_2),$$

$$c_1 * c_2 = m_1 * m_2 + 2(m_1r_2 + m_2r_1 + 2r_1r_2) + p(m_1q_2 + m_2q_1 + 2(r_1q_2 + r_2q_1) + pq_1q_2)$$

where we can see that the noise grows with each operation and the message becomes impossible to decrypt after we do too many of them. If we can assure that $2(m_1r_2 + m_2r_1 + 2r_1r_2)$ is small enough - i.e. smaller than p[3] - then we can assure that $\mathsf{Decrypt}(c_1 * c_2)$ evaluates correctly to the starting $m_1 * m_2$. Notice that $\mathsf{Decrypt}$ removes all the noise. This will be useful later for bootstrapping.

This simple encryption scheme is thus somewhat homomorphic as per definition by Gentry in [Gen09a] – namely, it can be used to evaluate low-degree polynomials over encrypted data.

[2] There are two more technical requitements, namely *compactness of the ciphertexts* and *efficiency* but we will not consider them in this paper.

[3] When $2r > p$ then it might be the case that $2r = 1 \bmod p$ and so $pq + 2r + m \bmod p = 1 + m \neq m$.

Further on in the Section 6 of [Dij+10], van Dijk et al. use the techniques (called bootstrapping and squashing) to lift it to a Fully Homomorphic Scheme.

Public Key Scheme

The public key scheme is build very similarly. The private key p stays the same. For the public key, sample $x_i = pq_i + 2r_i$ for $i = 0, 1, \dots, t$ where the q_i and r_i stay as before. The x_i may be viewed as encryption of 0 under the symmetric key scheme. The x_i are now taken s.t. x_0 is the largest, odd and $x_0 \bmod p$ is even.

To now encrypt a message $m \in \{0, 1\}$, chose a random subset $S \subseteq \{1, 2, \dots, t\}$ and a random integer r, and output

$$c = (m + 2r + 2\sum_{i \in S} x_i) \bmod x_0. \tag{3.2}$$

To decrypt, we again output $m = (c \bmod p) \bmod 2$.

The security of this preliminary SH scheme relies on the *Approximate GCD Problem*[4]. In the simplest case, Euclid has shown us, that given two integers c_1 and c_2, it is easy to compute their gcd. However, suppose now that $c_1 = p \cdot q_1 + r_1$ and $c_2 = p \cdot q_2 + r_2$ are "near" multiples of p, where r_1 and r_2 is some small noise sampled at random. This turns out to be much more difficult. In fact, if we pick our values appropriately (see [Gen10] Section 3.4 and [Dij+10] Section 3 for details) we do not know any efficient (running in polynomial time) algorithm even if we are given arbitrarily many samples $c_i = r_i + p \cdot q_i$.

However, this comfortable security comes at great cost because, as shown in [Dij+10], the parameters chosen to assure the secrecy, yield a scheme that has complexity of $\tilde{O}(\lambda^{10})$ where λ is our security parameter (the greater it is the more secure message). As a small example, consider $\lambda = 10$ as the (small) key size. To now encrypt a single(!) bit, it will take approximately 10^{10} operations. On a modern laptop this would take a little less than 5 seconds. To send the message 'hello', we need to use 5 letters $\cdot$ 16-bits per letter $= 5 \cdot 16 \cdot 5 = 650$ seconds which is almost 11 minutes! As one can imagine, this is completely impractical for most applications.

3.2 Fully Homomorphic Encryption

We will now present the main idea introduced in Gentry's PhD thesis [Gen09a]. Namely, the initial *bootstrapping* result, our SHE from previous section and a technique to *squash* the decryption circuit to allow bootstrapping. At the end, we will finally be left with *Fully Homomorphic Encryption* scheme. As a basis, we will be using the SHE scheme from previous section.

Boostrapping

We are faced with a problem. Because our method relies on some error being added to the message, it builds up after we perform operations on our data. The scheme $\mathcal{E}$ can handle functions in a limited set $\mathcal{F}_{\mathcal{E}}$ until the noise becomes too large.

[4] Later in [CS15], a reduction was constructed to LWE. This means, that under few more assumptions, this problem (and by extension any scheme based on it) is as secure as one based on LWE.

The basic idea behind bootstrapping is to use a homomorphic decryption circuit to "refresh" the ciphertext, so that it can be used for further homomorphic operations. The process involves encrypting the decryption circuit itself, evaluating it on the ciphertext, and then re-encrypting the result to obtain a new ciphertext that can be used for further homomorphic operations.

Remark 3.2. In cryptography, instead of general functions, one often considers a *circuit* instead.

To simplify our construction assume that our scheme $\mathcal{E}$ is "circularly secure". This means that we can encrypt the secret key using itself under the same scheme and no information is leaked to third parties. This is in practice very difficult to prove and has to be explicitly assumed.

4 Lattice based cryptography

Gentry's work was a true breakthrough. It not only presented the first, fully homomorphic encryption scheme, but also gave researchers a very powerful tool, the *bootstrapping*. From now on, all we need to construct another FHE scheme, is some suitable (one requirement would be to use a scheme based on ring rather than a group) SHE method, apply appropriate "squashing" to obtain the bootstrapping and we are done. In the following years this is exactly what happened in academia and the industry.

This section will mostly serve as a survey of the main developments towards more efficient fully homomorphic encryption using (ideal) lattices and their security based on computational hardness of the underlying problems. We adopt chronological narrative of the sections, starting with the oldest, the GGH scheme from 1997, progressing through works on (ring-)LWE and eventually arriving at the work of Gentry [Gen09a] on ideal lattices and further developments on FHE. For a good survey on the lattice based cryptography, see for example [NS01], [HPS14] chapter 6 or [Pei16].

4.1 The GGH public key cryptosystem

We will start this section with a somewhat simpler cryptosystem that was developed by Goldreich, Goldwasser and Halevi and presented in 1997 [GGH97], called the GGH cryptosystem. This scheme, rather than using ideal lattices (i.e. lattices that are also ideals in the ring of integers), relies on general properties of lattices. Namely, the hardness of the SVP and CVP (see section 2.4).

Idea behind the scheme

The basic GGH cryptosystem, as mentioned before, is based on the problem of finding the closest vector in the lattice $\mathcal{L}$ to a given point in the ambient space $\mathbb{R}^n$. We are given two bases, call them $\boldsymbol{B}_{good}$ and $\boldsymbol{B}_{bad}$. The $\boldsymbol{B}_{bad}$ will be our public key and $\boldsymbol{B}_{good}$ the secret key. The $\boldsymbol{B}_{bad}$ consists of long and highly non-orthogonal vectors, as opposed to $\boldsymbol{B}_{good}$. Our secret message $\boldsymbol{m}$ is represented as a binary vector which we will use to form a linear combination $\boldsymbol{s} = \sum m_i \boldsymbol{v}_i^{bad} \in \mathcal{L}$ of the vectors in $\boldsymbol{B}_{bad}$. We now add some small and random error $\boldsymbol{e} \in \mathbb{R}^n$ to obtain the ciphertext $\boldsymbol{c} = \boldsymbol{s} + \boldsymbol{e} = \sum m_i \boldsymbol{v}_i^{bad} + \boldsymbol{e} \in \mathbb{R}^n$ - some point that is not in the lattice, but rather, very close to a point in it.

To decrypt, we can use our good basis $\boldsymbol{B}_{good}$ to represent $\boldsymbol{c}$ and, for example Babai's algorithm[5] to find $\boldsymbol{v}$ and represent it in terms of the basis $\boldsymbol{B}_{good}$ to recover $\boldsymbol{m}$. On the other hand, any eavesdropping adversary that is trying to learn our secret, is left with some bad basis that will be of no help in solving the CVP.

GGH construction - concretely

The encryption algorithm can be seen in the Table 1.

KeyGen:

- Pick a basis $(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n) \subset \mathbb{Z}^n$ such that they are reasonably orthogonal to one another - i.e. with small Hadamard ratio. We will associatie the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ as the n-by-n matrix $\boldsymbol{V}$ and let $\boldsymbol{\mathcal{L}}$ be the lattice generated by these vectors. This is our good basis $\boldsymbol{B}_{good}$ - the **private key**.
- Pick an n-by-n matrix $\boldsymbol{U}$ with integer coefficients and determinant ±1 and compute $\boldsymbol{W} = \boldsymbol{U}\boldsymbol{V}$. The column vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n$ of $\boldsymbol{W}$ are the bad basis $\boldsymbol{B}_{bad}$ of $\boldsymbol{\mathcal{L}}$ - the **public key**[a].

Encrypt:
To encrypt a message $\boldsymbol{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, choose random small vector $\boldsymbol{r} \in \mathbb{R}^n$ and compute $\boldsymbol{e} = \boldsymbol{m}\boldsymbol{V} + \boldsymbol{r}$.

Decrypt:
Use Babai's algorithm to compute the vector $\boldsymbol{v} \in \boldsymbol{\mathcal{L}}$ closest to $\boldsymbol{e}$. Finally, compute the $\boldsymbol{v}\boldsymbol{W}^{\boldsymbol{-1}}$ to recover $\boldsymbol{m}$.

[a] As an alternative, in [Mic01a], Micciancio suggested to use the Hermite Normal Form (HNF) of $\boldsymbol{B}_{good}$ which essentially provides the worst possible lattice choice from both cryptoanalitical and efficiency point of view.

Table 1: GGH encryption algorithm

The greatest drawback of GGH is that there were no proofs of security presented along the algorithm, only heuristic assumptions. This motivated researchers to look for possible exploits beased on the choice of parameters. Indeed, this scheme turned out to be insecure for most practical choices of the security parameter only 2 years later, in [Ngu99] and broken completely in [NR06]. Nonetheless, the ideas presented there have served as a basis for many schemes that are proven to be secure, like for example LWE, and has led to a plethora of applications.

4.2 Learning With Errors

Let us now begin with what went wrong in GGH. Namely, first prove the hardness of a problem, then use it to construct a secure and efficient cryptosystem. In this section we introduce *Learning With Errors* (LWE) problem and the cryptosystem introduced by Oded Regev in [Reg05] (he

[5] Simply stated, if the vectors of the basis are sufficiently orthogonal to one another, then this algorithm solves `approxCVP`. However, if the Hadamard ratio is too small, the algorithm fails to find the closest vector - [HPS14].

won the 2018 Gödel Prize for this work). This very important work in the field of lattice based cryptography is, up to the date, one of the most efficient schemes with an actual proof of security. It has served as a foundation for countless subsequent works in the field. [Krzys: provide validation for this statement]

LWE problem

There are multiple equivalent definitions of this problem. We adopt the notation and approach introduced in the original paper by Regev. In this section, we will mainly focus on the parts that are the most relevant for our study of ring-LWE such as 4.12 and 4.15.

The problem is parametrized by positive integers n, m[6] and prime q, as well as a discrete error distibution χ over $\mathbb{Z}_q$. It is now defined as follows. We are given m equations of the form $(\boldsymbol{a}_i, b_i = \langle \boldsymbol{a}_i, \boldsymbol{s} \rangle + e_i)$ and are asked to find the vector $\boldsymbol{s} \in \mathbb{Z}_q^n$. Here, $\boldsymbol{a}_i$ are chosen uniformly and independently from $\mathbb{Z}_q^n$, $b_i \in \mathbb{Z}_q$ and $\langle \cdot, \cdot \rangle$ denotes the usual dot product. The errors e_i are obtained by sampling independently from the probability distribution $\chi : \mathbb{Z}_q \to \mathbb{R}_{>0}$ on $\mathbb{Z}_q$. We will denote the problem of recovering $\boldsymbol{s}$ from such equations, by $\mathrm{LWE}_{q,\chi}$ (learning with errors).

Example 4.1. Given as input this set of equations, LWE asks us to recover the vector $\boldsymbol{s} = (s_1, s_2, s_3, s_4) \in \mathbb{Z}_{17}^4$. In this case $n = 4$, $q = 17$ and the error distribution is giving us $e_i < 1$ for each $i \in [m]$.

$$
\begin{aligned}
0s_1 + 11s_2 + 8s_3 + 2s_4 &\approx_{e_1} 3 \ (\bmod 17) \\
9s_1 + 3s_2 + 7s_3 + 0s_4 &\approx_{e_2} 16 \\
1s_1 + 15s_2 + 9s_3 + 5s_4 &\approx_{e_3} 16 \\
0s_1 + 0s_2 + 13s_3 + 5s_4 &\approx_{e_4} 1
\end{aligned}
$$

In this case, $\boldsymbol{s} = (7, 13, 12, 16)$. Note that if not for the error, the secret would be very easy to find. Given about n equations, we could recover $\boldsymbol{s}$ in an efficient way using Gaussian elimination. Inducing the error is what seems to render the problem untraceable for modern day algorithms.

The central part of [Reg05] revolves around proving the hardness of LWE. Specifically, that for appropriately chosen q and χ, a *quantum*[7] reduction algorithm exists that approximates worst-case lattice problems. The following is the main result presented in the paper.

Theorem 4.1 ([Reg05], Theorem 1.1)**.** *Let n, q be positive integers and $\alpha \in (0, 1)$ be such that $\alpha q > 2\sqrt{n}$. If there exists an efficient algorithm that solves LWE_{q,Ψ_α}, then there exists an efficient quantum algorithm that approximates the decision version of the shortest vector problem ($\mathtt{GapSVP}_\gamma$) and the shortest independent vectors problem ($\mathtt{SIVP}_\gamma$) to within $\gamma = \tilde{O}(n/\alpha)$ in the worst case on any lattice of dimension n.*

Let us unwrap this statement. As said before, we need an appropriate choice of paramenters to obtain our results and $\alpha > 2\sqrt{n}/q$ is one of those choices (and requirements). It specifies the shape of the Ψ_α distribution. This one is almost identical to the discrete Gaussian distribution

[6] As will be seen later, m plays virtually no role in the problem definition and is usually omitted.

[7] In fact, quantum reduction is used only in the small part of the whole proof.

over $\mathbb{Z}_q$ that is centered around 0 with standard deviation αq[8]. It will be used to sample the errors with some nice and predictable properties. Later, by the end of this section, we will also show that in fact, there is sort of an equivalence between drawing them from continuous and discrete distributions.

The theorem can be rephrased as follows. Imagine that we have an efficient algorithm that solves the $\mathrm{LWE}_{q,\Psi_\alpha}$. Then, there exists a quantum solution to worst-case lattice problems, namely `GapSVP` and `SIVP`. Since we strongly believe that `GapSVP` and `SIVP` are difficult to solve ([Mic01b], [Mic08], [Din+03], [MG02]) we are left with a difficult, yet efficient way to share secrets. Oded Regev proceeds to prove this using various lemmas and results from few areas of mathematics like probability, lattice theory and quantum computing. We will now present selected proofs and outline of the whole the approach.

4.2.1 Background II

Here we will define essential and convenient definitions and lemmas concerning Gaussian distributions lattices and lattice-related problems. Note that no algebraic number theory is needed for this section as we work exclusively over $\mathbb{R}^n$ and $\mathbb{Z}_q^n$ for q an integer (it is only required for q to be prime in Theorem 4.15, otherwise no assumptions are made).

Gaussians The ideas presented here heavily rely on the distribution of errors. The most important distribution we will be using is the *Gaussian distribution* (also called the normal distribution). For some vector $\boldsymbol{x}$ and $r > 0$, we define the Gaussian function as

$$\rho_r(\boldsymbol{x}) := \exp(-\pi||\boldsymbol{x}/r||^2)$$

where $\exp(y)$ stands for e^y. We will also denote ρ_1 by ρ. By normalizing it, we obtain the n-dimensional Gaussian distribution

$$\mathcal{D}_r = \rho_r \Big/ \Big(\int_{\boldsymbol{s}\in\mathbb{R}^n} \rho_r(\boldsymbol{s}) d\boldsymbol{s} \Big) = \rho_r \cdot r^{-n}.$$

We can also define the *discrete Gaussian distribution* $\mathcal{D}_{\mathcal{L},r}$ over a lattice $\mathcal{L}$ (or any countable set A for that matter) by

$$\mathcal{D}_{\mathcal{L},r}(\boldsymbol{x}) = \frac{\rho_r(\boldsymbol{x})}{\rho_r(\mathcal{L})} \tag{4.1}$$

where $\rho_{A,r}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in A} \rho_r(\boldsymbol{x})$ is an extension of ρ_r to a countable set.

[Krzys: i dont think ill include this part about Psi beta at all] In order to avoid making our proofs depend on the specific choice of modulus q, we now define the family of Gaussian distributions over $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$ - a periodization of the normal distribution. More specifically, following [Reg05], for $\beta \in \mathbb{R}_{>0}$ define Ψ_β as the distribution on $\mathbb{T}$ obtained by sampling from a normal variable with mean 0 and standard deviation $\frac{\beta}{\sqrt{2\pi}}$ and reducing the result modulo 1,

$$\forall r \in [0, 1), \Psi_\beta(r) := \sum_{k=-\infty}^{\infty} \frac{1}{\beta} \cdot \exp\Big(-\pi \Big(\frac{r-k}{\beta}\Big)^2 \Big). \tag{4.2}$$

[8] A comment from [Pei16]: Originally, Regev considered the continuous Gaussian and rounded the result to the nearest integer. This does not exactly yield the discrete distribution but thanks to [Pei10] we know how the problem can be fixed.

[Krzys: also, it turns out, that for whatever value of r we pick, the result of that sum is always $|\beta|/\beta = 1$?????] We will, for the sake of clearer notation, be a bit vague about the doman of Ψ_β. This is because sampling from Ψ_β with parameter β, that is defined over $\mathbb{T} = [0, 1)$ is equivalent to sampling from $\Psi_{q\beta}$ over $\mathbb{Z}_q$. [Krzys: i am very hesitant about this statement - need to check if thats true]

We also need to define *distretization* of a probability density function for our cryptosystem introduced later. For an arbitrary probability density function $\phi\ \mathbb{T} \ \rightarrow \mathbb{R}_{>0}$ and an integer $q \geq 2$, we define its discretization $\bar{\phi} : \mathbb{Z}_q \rightarrow R_{>0}$ as the discrete probability distribution obtained by sampling from Ψ, multiplying by q, and rounding to the closest integer modulo q. More formally,

$$\bar{\phi}(i) = \int_{(i-1/2)/q}^{i+1/2)/q} \phi(x)dx \tag{4.3}$$

Lastly, following the definition given in [MG02], we define the *statistical distance* as follows.

Definition 4.2 (Statistical distance)**.** Let X, Y be two random variables over a countable set A. The statistical distance between them is defined as

$$\Delta(X, Y) = \sum_{a \in A} \Big| \Pr\{X = a\} - \Pr\{Y = a\} \Big|.$$

Similarly, if φ_X, φ_Y are the *probability density* functions over $\mathbb{R}^n$ of X and Y respectively, then

$$\Delta(\varphi_X, \varphi_Y) = \int_{\boldsymbol{s} \in \mathbb{R}^n} \Big| \varphi_X(\boldsymbol{s}) - \varphi_Y(\boldsymbol{s}) \Big| d\boldsymbol{s}.$$

It can be interpreted as a measure of how much apart two distributions are from each other. It is indeed not hard to show that this definition satisfies the properties of a distance function. We will be using this tool to measure how our elements are behaving, for example, in Lemma 4.15.

Lattices II

Definition 4.3 (Dual)**.** For a lattice $\mathcal{L} \subset \mathbb{R}^n$ its $\mathbb{Z}$-*dual* is

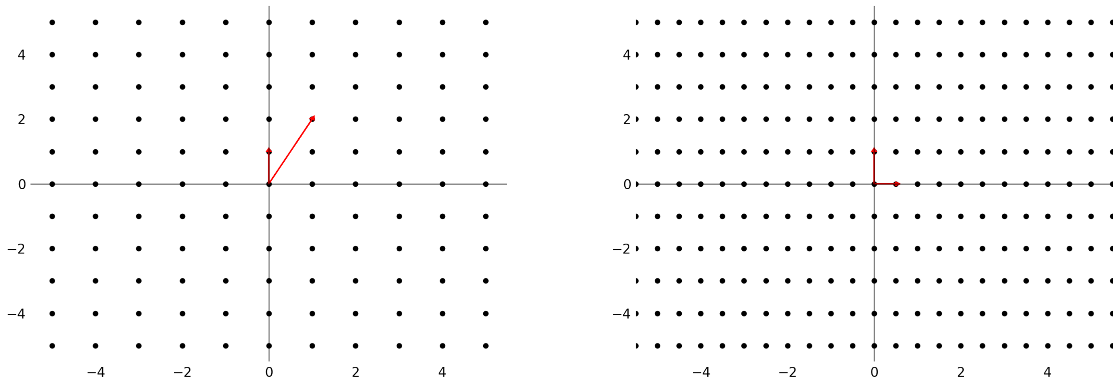
$$\mathcal{L}^\vee = \{y \in \mathbb{R}^n : \langle y, \mathcal{L} \rangle \subseteq \mathbb{Z}\}.$$

We simply require that the elements of the dual to be precisely those vectors that yield an integer when "multiplied" with an element of our lattice. Note that this is different from our standard definition of a dual. Namely, it is not the orthogonal compliment of our starting space, i.e. not all of the elements of the dual have 0 dot product against the vectors of the lattice.

Example 4.2. Take $\mathcal{L} = \mathbb{Z}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) + \mathbb{Z}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. To calculate the dual of $\mathcal{L}$ we need our $y = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ elements to satisfy $a + 2b \in \mathbb{Z}$ and $b \in \mathbb{Z}$ which is equivalent to asking $a \in (1/2)\mathbb{Z}$ and so $\mathcal{L}^\vee = \left(\begin{smallmatrix}1/2\\0\end{smallmatrix}\right)\mathbb{Z} + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\mathbb{Z}$ See Figure 1 where the red vectors represent the basis.

Note that $\mathcal{L}^\vee$ is itself a lattice of the same dimension.

Figure 1: Lattice $\mathcal{L}$ from example 4.2 and its dual.



(a) $\mathcal{L}$

(b) $\mathcal{L}^\vee$

Smoothing parameter One of the most important concepts invoked in the paper is the so called *smoothing parameter* η of a lattice $\mathcal{L}$. It was first introduced in [MR07] and has proven useful over the course of years as we will note later. The definition is quite technical as it makes use of the dual rather than the lattice itself. The intuition behind it could be obtained by imagining the following. Pick an element from your favourite lattice $\mathcal{L}$. If one now picks a "noise" vector from a Gaussian distribution with width at least as large as the smoothing parameter, the resulting distribution is very close to uniform. The precise definition is as follows.

Definition 4.4 (Smoothing parameter)**.** For an n-dimensional lattice $\mathcal{L}$ and positive real $\epsilon > 0$, we define the smoothing parameter $\eta_\epsilon(\mathcal{L})$ to be the smallest s such that $\rho_{1/s}(\mathcal{L}^\vee \backslash \{\mathbf{0}\}) \leq \epsilon$.

In other words, $\eta_\epsilon(\mathcal{L})$ is the smallest s such that a Gaussian measure scaled by $1/s$ on the dual lattice $\mathcal{L}^\vee$ gives all but $\epsilon/(1+\epsilon)$ of its weight to the origin - [Reg05]. ϵ is our desired tolerance and as the name suggests, it is usually taken to be very small (in our cases, a negligible function of n). It is also worth noting that $\eta_\epsilon(\mathcal{L})$ is a continuous function of ϵ.

In the Figure 2, there are plotted four Gaussians. The first one has very small deviation and assigns most of its values to the origin. The second one, with the deviation $\sqrt{10}$ times the size of the first one, looks a bit more *smooth* and continuous. Finally, the last distribution is almost uniform or "flat". For an appropriate choice of ϵ, the smoothing parameter will lay somewhere between those four values of r.

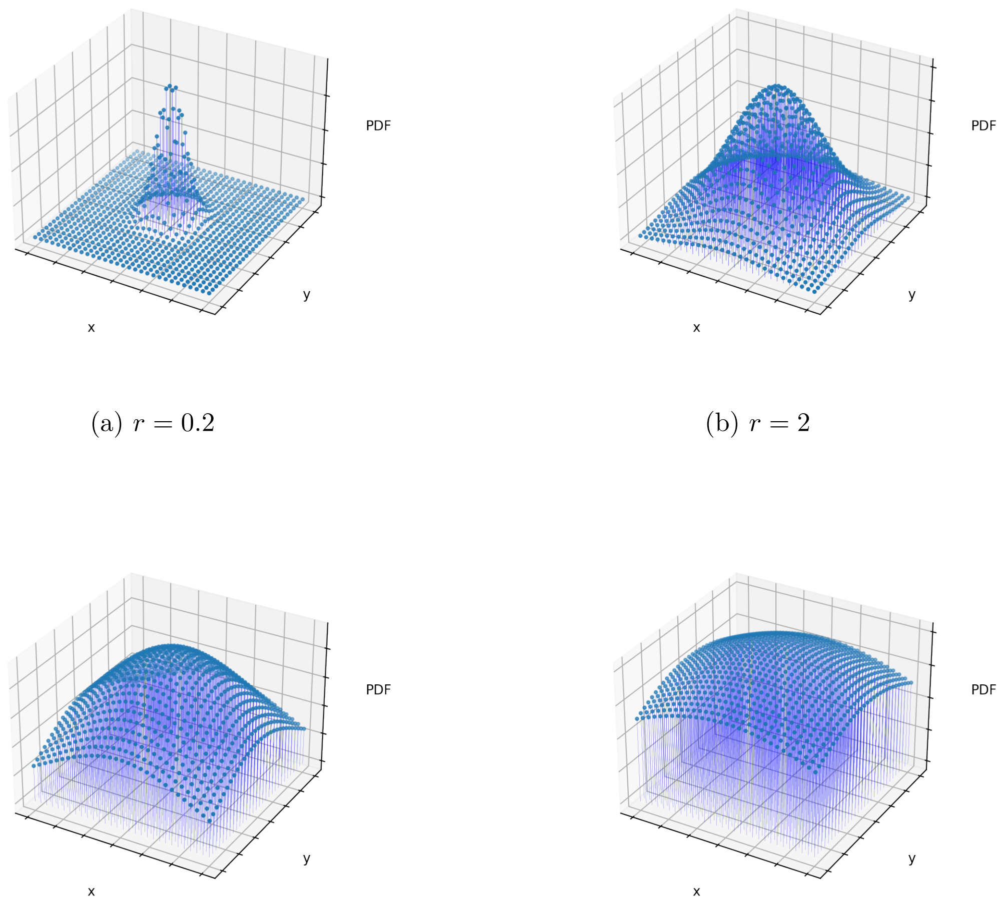
Proposition 4.5 (Sum of Gaussians)**.** *If $X_1, X_2, \cdots, X_n$ are mutually independent normal random variables with means $\mu_1, \mu_2, \cdots, \mu_n$ and variances $\sigma_1, \sigma_2, \cdots, \sigma_n$, then the linear combination*

$$Y = \sum_{i \in [n]} c_i X_i$$

follows the normal distribution

$$\mathcal{N}\Big(\sum_{i \in [n]} c_i \mu_i, \sum_{i \in [n]} c_i^2 \sigma_i^2\Big).$$

Figure 2: 2-dimensional Gaussian with standard deviation r. The z-axis represents the probability.



(a) $r = 0.2$

(b) $r = 2$

(a) $r = 7$

(b) $r = 20$

Problems We will now define problems related specifically to the LWE. First, let us define the distribution induced by LWE.

Definition 4.6 (LWE distribution)**.** Let $q \geq 2$ be some integer, and let $\chi : \mathbb{Z}_q \to \mathbb{R}_{>0}$ be some probability distribution on $\mathbb{Z}_q$. Let n be an integer and let $\boldsymbol{s} \in \mathbb{Z}_n^q$ be a vector. We define $A_{\boldsymbol{s},\chi}$ as the distribution on $\mathbb{Z}_n^q \times \mathbb{Z}_q$ obtained by choosing a vector $\boldsymbol{a} \in \mathbb{Z}_q^n$ uniformly at random, choosing $e \in \mathbb{Z}_q$ according to χ, and outputting $(\boldsymbol{a}, \langle \boldsymbol{a}, \boldsymbol{s} \rangle + e)$, where additions are performed in $\mathbb{Z}_q$, i.e., modulo q. In parallel, we define U as the uniform distribution on $\mathbb{Z}_q^n \times \mathbb{Z}_q$.

There are two types of problems we can define over LWE - the *search* and *decision*. We will prove at the end of this section that they are in fact equivalent, i.e. there exists a reduction between them. Search-LWE is asking us to find $\boldsymbol{s}$ from a list of samples distributed according to $A_{\boldsymbol{s},\chi}$ for some unknown χ. Similarly, the *decision* asks us to distinguish samples $(\boldsymbol{a}, b)$ that were draw from either $A_{\boldsymbol{s},\chi}$ or the uniform distribution. Although not present in the original paper, we define them formally here.

Definition 4.7 (Search-LWE)**.** Given access to arbitrarily many samples from the $A_{\boldsymbol{s},\chi}$, find $\boldsymbol{s}$.

Definition 4.8 (Decision-LWE)**.** Given access to arbitrarily many samples from the $A_{\boldsymbol{s},\chi}$ and the same amount of samples from the uniform distribution, distinguish (with non-negligible advantage) between them.

The following is used as an intermediate step in our reduction to SIVP. The acronym DGS stands for *discrete Gaussian sampling* problem and is concerned exactly with that - to find a sample from the normal distribution over our lattice. Formally:

Definition 4.9 (DGS)**.** Given an n-dimensional lattice $\mathcal{L}$ and a number $r \geq \sqrt{2}n \cdot \eta_\epsilon(\mathcal{L})/\alpha$, output a sample from $\mathcal{D}_{\mathcal{L},r}$

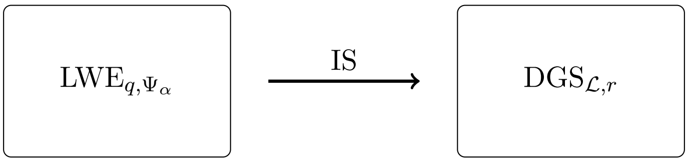
4.2.2 Hardness of search LWE

In this section we will focus on proving the main statement of [Reg05] following the steps presented in the paper. We will put an emphasis on the parts that require the most adjustment when trying to prove the same results for ring-LWE in the next section.

Recall what we want to prove, that being able to solve $\text{LWE}_{q,\chi}$ implies that we are able to solve standard worst-case lattice problems like SIVP. To achieve this, we need to proceed in steps, in other words, we perform reductions from one problem to another. From this point on, a probabilistic algorithm which solves a given $\text{LWE}_{q,\chi}$ instance, will be called an LWE-*oracle* (or, when there is no confusion, simply an oracle).

The high-level version of the proof is as follows. Let us assume that we have such an oracle that solves $\text{LWE}_{q,\chi}$ on a lattice $\mathcal{L}$ just like in the assumption. The procedure is based on the so called *iterative step* (IS). It is repeatedly used to reduce our LWE problem to the problem of sampling from the discrete Gaussian distribution on $\mathcal{L}$. As it turns out, it is indeed sufficient to solve the DGS problem (Definition 4.9). Intuitively, if we have enough samples from the Gaussian distribution of some small radius r around our lattice, we can use it to obtain short lattice vectors. Indeed, if we call the DGS oracle enough times we can prove, that with very high probability, there are n linearly independent vectors among the samples returned by oracle - thus, a solution to SIVP.

On each of the iterations, the IS is using the oracle to produce discrete Gaussian samples of smaller and smaller radius around our desired *closest vector*. Once we have a sample with radius small enough, we can use that to solve DGS and use one more step which is the reduction from DGS to the GapSVP and SIVP as required in Theorem 4.1.



The algorithm can be divided into two basic steps, the *classical* step and the *quantum* step. First, given some polynomial number n^c (where c is some positive integer) of samples from a discrete Gaussian on $\mathcal{L}$ with some (large enough) parameter r, we us the first step to obtain a solution to a CVP on the dual lattice $\mathcal{L}^\vee$ to within some (smaller than the initial r) distance. We then use this solution along with the second step - the quantum algorithm - to obtain n^c samples from a dicrete Gaussian with parameter $r' < r$. Iterating these steps polynomial amount of times, we finally get n^c samples from $\mathrm{D}_{\mathcal{L},r''}$ where $r'' \ll r$. We simply pick one of those samples to solve the $\mathtt{DGS}_{\mathcal{L},r}$ and we are done.

The full picture now stands as follows. We begin with a assumed solution to LWE and we end up with a solution to some (conjured) hard lattice problems like $\mathtt{SIVP}$ to within some tolerance $\gamma = \tilde{O}(n/\alpha)$.

$$\begin{tikzcd}
\boxed{\mathrm{LWE}_{q,\Psi_\alpha}} \arrow[r, "\mathrm{IS}"] & \boxed{\mathrm{DGS}_{\mathcal{L},r}} \arrow[r, "n^2 \text{ samples}"] & \boxed{\mathrm{SIVP}_\gamma}
\end{tikzcd}$$

We will now focus on the first step of the reduction from LWE to $\mathtt{DGS}$ as it is what is altered the most in the ring-LWE hardness proof. This, however, will follow in the next section.

The following statement (Theorem 3.1 in [Reg05]) is the core of the hardness results of (search) LWE. Given an oracle for LWE, there exists an algorithm that gives us samples from discrete Gaussian distribution which in turn can yield us a solution to the $\mathtt{CVP}$. For example, later we will show the equivalence between search-LWE and decision-LWE (Theorem 4.15). This is an important result as it is usually much easier to construct cryptographic schemes based on some *decision* version of a problem rather than the *search*.

Theorem 4.10. *Let $\epsilon = \epsilon(n)$ be some negligible function of n. Also, let $q = q(n)$ be some integer and $\alpha = \alpha(n) \in (0,1)$ be such that $\alpha q > 2\sqrt{n}$. Assume that we have access to an oracle W that solves LWE_{q,Ψ_α} given a polynomial number of samples. Then there exists an efficient quantum algorithm for $DGS_{\sqrt{2n}\cdot\eta_\epsilon(\mathcal{L})/\alpha}$.*

Proof. The proof follows in a straight-forward manner by the Lemma 4.11. We sample a polynomial amount of elements from the $\mathcal{D}_{\mathcal{L},r}$ for r large enough[9]. We then apply the IS to obtain $\mathrm{D}_{\mathcal{L},r'}$ for $r' \le r/2$. Applying this procedure enough times (turns out only about $3n$ iterations are needed). At the end of the loop, we are left with the same amount of samples but each within sufficient distance $d \le \sqrt{2}n \cdot \eta_\epsilon(\mathcal{L})/\alpha$ and we complete the algorithm by simply outputting the first one. □

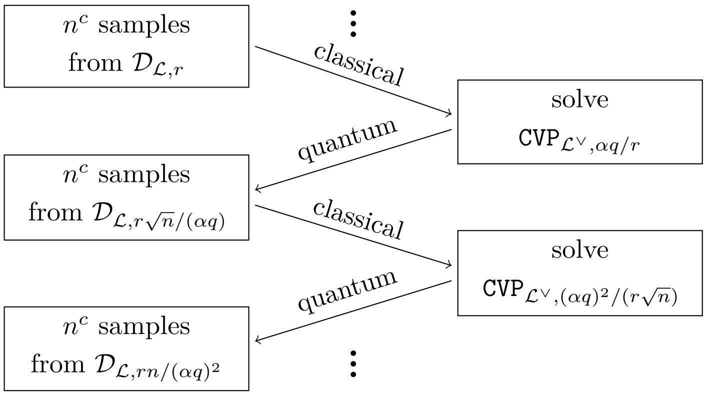
The iterative step The proof of Theorem 4.10 relies mainly on the following algorithm. This is slightly altered Lemma 3.3 in [Reg05].

Lemma 4.11 (Iterative Step)**.** *Let $\epsilon = \epsilon(n)$ be a negligible function, $\alpha > 0$ real, and $q \ge 2$ be an integer. Assume that we have access to an oracle that solves $LWE_{q,\Psi_{\le\alpha}}$ given a polynomial number of samples. Then, there exists an efficient quantum algorithm that, given any n-dimensional lattice $\mathcal{L}$, a number $r > \sqrt{2}q\eta_\epsilon(\mathcal{L})$, and a polynomial list of samples from the discrete Gaussian distribution $\mathcal{D}_{\mathcal{L},r}$, produces a sample from $\mathcal{D}_{\mathcal{L},r\sqrt{n}/(\alpha q)}$.*

Proof. As mentioned before, the algorithm consists of two parts. The first part is presented in 4.12, where, given an oracle W and samples from $\mathcal{D}_{\mathcal{L},r}$, solves $\mathtt{CVP}_{\mathcal{L}^\vee,\alpha q/(\sqrt{2}r)}$. The second part - the quantum algorithm 4.13 - when given an oracle that solves $\mathtt{CVP}_{\mathcal{L}^\vee,\alpha q/(\sqrt{2}r)}$, yields us a sample from $\mathcal{D}_{\mathcal{L},r\sqrt{n}/(\alpha q)}$. □

[9] By Claim 2.13 and Lemma 3.2 of [Reg05], we can efficiently draw samples that are exponentially close to $\mathcal{D}_{\mathcal{L},r}$ if $r > 2^{2n}\lambda_n(\mathcal{L})$.

Figure 4: Two iterations of the iterative step


n^c samples from $\mathcal{D}_{\mathcal{L},r}$
classical
solve $\texttt{CVP}_{\mathcal{L}^\vee,\alpha q/r}$
quantum
n^c samples from $\mathcal{D}_{\mathcal{L},r\sqrt{n}/(\alpha q)}$
classical
solve $\texttt{CVP}_{\mathcal{L}^\vee,(\alpha q)^2/(r\sqrt{n})}$
quantum
n^c samples from $\mathcal{D}_{\mathcal{L},rn/(\alpha q)^2}$

Pictographically, two iterations of the algorithm are presented in Figure 4.

We now present heavily simplified version of the algorithm that, when given a polynomial amount of samples from the discrete Gaussian, gives us a solution to the $\texttt{CVP}$ to within error of $\alpha q/(\sqrt{2}r)$ to the dual lattice $\mathcal{L}^\vee$.

Lemma 4.12 (Step 1 - classical)**.** *Let $\epsilon = \epsilon(n)$ be a negligible function, $\alpha > 0$ real, and $q \geq 2$ be an integer. Assume that we have access to an oracle that solves $LWE_{q,\Psi_{\leq\alpha}}$ given a polynomial number of samples. Then, there exists an efficient algorithm that, given an n-dimensional lattice $\mathcal{L}$, a number $r > \sqrt{2}q\eta_\epsilon(\mathcal{L})$, and a polynomial list of samples from $D_{\mathcal{L},r}$, solves $\texttt{CVP}_{\mathcal{L}^\vee,\alpha q/(\sqrt{2}r)}$.*

Our goal is, given a point $\boldsymbol{x}$ close enough (precisely within $\alpha q/(\sqrt{2}r)$) to $\mathcal{L}^\vee$, construct an instance of a $A_{\boldsymbol{s},\Psi_{\leq\alpha}}$ where the $\boldsymbol{s}$ will depend on $\boldsymbol{x}$. After polynomial amount of such samples, we can then use our oracle to recover $\boldsymbol{s}$ and consecutively $\boldsymbol{x}$. We say an algorithm solves $\texttt{CVP}_{\mathcal{L},d}$ if, given any point $\boldsymbol{x} \in \mathbb{R}^n$ within distance d of $\mathcal{L}$ it gives $\boldsymbol{y} \bmod q \in \mathbb{Z}_q^n$, the coefficient vector of the closest vector to $\boldsymbol{x}$ reduced modulo q[10].

The following proof is a simplified version of the proof of Lemma 3.11 from [Reg05] where we skip few technical details. For all of them, the reader is redirected to the original.

Proof. We start with a sample vector $\boldsymbol{v} \in \mathcal{L}$ from $\mathrm{D}_{\mathcal{L},r}$ and set $\boldsymbol{a} = \mathcal{L}^{-1}\boldsymbol{v} \bmod q$ - its coefficient vector (notice that we are slightly abusing the notation here using $\mathcal{L}$ as the matrix representing the lattice). We note at this point that it is in fact enough to find solution modulo q as there exists an algorithm iterating over the coefficients along with (for example) Babai's nearest plane algorithm (see [Bab86]) to obtain the full, unreduced solution. For the details, see [Reg05] Lemma 3.5. We now output

$$(\boldsymbol{a}, \langle \boldsymbol{x}, \boldsymbol{v} \rangle + e' \bmod q) \tag{4.4}$$

where the $e' \in \mathbb{R}^n$ is chosen from a continuous normal distribution with deviation $\alpha/(\sqrt{2}\pi)$. We claim now that our output is within negligible statistical distance of $A_{\boldsymbol{s},\Psi_{\leq\alpha}}$ for $\boldsymbol{s} = (\mathcal{L}^\vee)^{-1}\boldsymbol{y} \bmod q$.

To see this, first note that $\boldsymbol{a}$ is a uniform sample ([Reg05], Claim 3.8). This is because $r \geq \eta_\epsilon(\mathcal{L})$ and by our discussion in 4.4, the $\mathcal{D}_{\mathcal{L},r}$ will behave like a uniform distribution. Let us

[10] This is technically a solution to $\texttt{CVP}^{(q)}_{\mathcal{L},d}$ - regular $\texttt{CVP}$ but reduced modulo q. However as we will see later in the proof, there exists a reduction from one to another.

now fix $\boldsymbol{a}$ and consider $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e}$. Then

$$\langle \boldsymbol{x}, \boldsymbol{v} \rangle + e' \bmod q = \langle \boldsymbol{e}, \boldsymbol{v} \rangle + \langle \boldsymbol{y}, \boldsymbol{v} \rangle + e' \bmod q.$$

Focusing now on the second term, we observe that

$$\langle \boldsymbol{y}, \boldsymbol{v} \rangle = (\mathcal{L}^\vee)^{-1} \langle \boldsymbol{y}, \boldsymbol{v} \rangle \mathcal{L}^\vee = \langle (\mathcal{L}^\vee)^{-1} \boldsymbol{y}, \mathcal{L}^{-1} \boldsymbol{v} \rangle = \langle \boldsymbol{s}, \boldsymbol{a} \rangle$$

since $\mathcal{L}^{-1} = (\mathcal{L}^\vee)^T$.

To finish the proof we need to show that the remaining term $\langle \boldsymbol{e}, \boldsymbol{v} \rangle + e'$ is distributed within negligible statistical distance of $\Psi_{\leq \alpha}$. This is obtained by noting that summing two normal distributions with the standard deviation less than the smoothing parameter, gives us a distribution that is indeed close enough to Ψ_β for some $\beta \leq \alpha$ as desired. This is Claim 3.10 in the original paper by Regev. □

Proposition 4.13 (Step 2 - quantum)**.** *There exists an efficient quantum algorithm that, given any n-dimensional lattice $\mathcal{L}$, a number $d < \lambda_n(\mathcal{L}^\vee)/2$, and an oracle that solves $\mathtt{CVP}_{\mathcal{L}^\vee, d}$, outputs a sample from $D_{\mathcal{L}, \sqrt{n}/(\sqrt{2}d)}$.*

Lastly, we can finish the proof of Theorem 4.1 by presenting the following proposition that provides reduction from `SIVP` and `GapSVP` to `DGS`.

Proposition 4.14. *Let $\mathcal{L}$ be an n-dimensional lattice and let r be such that $r \geq \sqrt{2}\eta_\epsilon(\mathcal{L})$ where $\epsilon \leq \frac{1}{10}$. Then, the probability that a set of n^2 vectors chosen independently from $D_{\mathcal{L},r}$ contains no n linearly independent vectors is exponentially small.*

4.2.3 Pseudorandomness of LWE

In this section we will prove that the search version of LWE - Definition 4.7 - to the decision version - Definition 4.8. More precisely, if our sample is of the form $(\boldsymbol{a}, b)$, there is no way for us to tell if b is uniform or actually $b = \langle \boldsymbol{a}, \boldsymbol{s} \rangle + e$ for some secret $\boldsymbol{s}$ and small error e.

In general, the decision version is considered more suitable for cryptographic purposes because it is often easier to analyze and prove security guarantees for. For many cryptographic problems, it is computationally infeasible to find the solution to the search version, but it is possible to determine the correct answer to the decision version with high probability. Thus, cryptographic schemes are often designed based on decision versions of hard computational problems.

The statement can be phrased as follows.

Theorem 4.15 ([Reg05], 4.2 - Decision to Search)**.** *Let $n \geq 1$ be some integer, $2 \leq q \leq \mathsf{poly}(n)$ be a prime, and χ be some distribution on $\mathbb{Z}_q$. Assume that we have access to procedure `W` that for all $\boldsymbol{s}$ accepts with probability exponentially close to 1 on inputs from $A_{\boldsymbol{s},\chi}$ and rejects with probability exponentially close to 1 on inputs from U. Then, there exists an efficient algorithm `W'` that, given samples from $A_{\boldsymbol{s},\chi}$ for some s, outputs $\boldsymbol{s}$ with probability exponentially close to 1.*

In other words, if we have an oracle (called procedure in this theorem) that solves the decision-LWE, then there is an efficient (running in polynomial time) algorithm that outputs us the secret $\boldsymbol{s}$ used for the generation of the sample (if the sample was indeed taken from $A_{\boldsymbol{s},\chi}$).

Proof. We are given a sample $(\boldsymbol{a}, b)$ and an oracle that tells us whether this element was chosen uniformly at random, or whether b is related to $\boldsymbol{a}$ via $b = \langle \boldsymbol{a}, \boldsymbol{s} + e$. Note that by definition in both cases $\boldsymbol{a}$ is chosen uniformly from $\mathbb{Z}_q^n$. The idea to obtain the secret $\boldsymbol{s}$ is relatively simple. We pick some element $k \in \mathbb{Z}_q^n$ and check coordinate by coordinate if this is the respective coordinate of our key $\boldsymbol{s}$. The procedure is identical for each coordinate hence we will only consider the case of the first one. Pick any k as before and consider the transformation $(\boldsymbol{a} + (l, 0, \cdots, 0), b + k \cdot l)$ for $l \in Z_q^n$ chosen uniformly at random. We have now three cases to consider.

1. Our original sample was in fact taken from uniform distribution. Then it is easy to see that the transformation maps it again to uniform and nothing is changed. In this case, there is no $\boldsymbol{s}$ to be found and we terminate.

2. The sample was taken from the $A_{\boldsymbol{s},\chi}$ distribution. Then either:

   (a) $k = s_1$. In this case, the sample is mapped back to $A_{\boldsymbol{s},\chi}$ because

   $$\langle (a_1 + l, a_2, \cdots, a_n), \boldsymbol{s} \rangle + e = \langle \boldsymbol{a}, \boldsymbol{s} \rangle + l \cdot s_1 + e = b + k \cdot l.$$

   We can now proceed with s_2 in a similar manner.

   (b) Or $k \neq s_1$ in which case the sample is mapped to a uniform distribution and oracle tells us we picked wrong. Note that this requires our q to be prime because otherwise it might happen that $a_1 \cdot k = a_1 \cdot s_1$ as either of the three could be a zero divisor. We have therefore picked wrong k and we need to pick another one.

Note that by the choice of our $q \leq \mathsf{poly}(n)$, we can try all of them. □

4.2.4 LWE cryptosystem

Now that we have a solid hardness assumptions, we can attempt to construct a cryptosystem that employs those results. The following public key cryptosystem was presented in the same paper. To keep the notation consistent with previous section, we will slightly deviate from the original.

We begin by specifying our parameters. Let us denote by n our security parameter. As before, the scheme is characterized by two integers m and q and a probability distribution χ over $\mathbb{Z}_q$. To now make the scheme secure and correct, we should choose q prime between n^2 and $2n^2$, $m = (1 + \epsilon)(n + 1) \log q$ for some arbitrary constant $\epsilon > 0$. We define the distribution χ to be $\bar{\Psi}_{\alpha(n)}$ (the discretized Gaussian distribution Ψ_α) where $\alpha(n) = o(1/(\sqrt{n} \log n))$ (recall from Section 2.4 that it means $\lim_{n\to\infty} \alpha(n) \cdot \sqrt{n} \log n = 0$). The scheme is presented in Table 2.

Example 4.3. Almost exactly like in the Example 4.1, set $n = 4$, $q = 17$ and $m = 4$. We pick our **secret key** $\boldsymbol{s} = (2, 1, 3, 7)$ and artificially (i.e. by design and not uniformly at random) pick

$$\boldsymbol{A} = [\boldsymbol{a}_1\, \boldsymbol{a}_2\, \boldsymbol{a}_3\, \boldsymbol{a}_4] = \begin{pmatrix} 1 & 16 & 4 & 5 \\ 16 & 4 & 5 & 1 \\ 4 & 5 & 1 & 16 \\ 5 & 1 & 16 & 4 \end{pmatrix}.$$

KeyGen:

- Choose $\boldsymbol{s} \in \mathbb{Z}_q^n$ uniformly at random. This is the **private key**.
- For $i = 1, \ldots, m$ choose m vectors $\boldsymbol{a}_i \in \mathbb{Z}_q^n$ independent from the uniform distribution. Additionally choose m elements $e_i \in \mathbb{Z}_q$ independently according to χ. The **public key** is the array of m vectors of the form $(\boldsymbol{a}_i, b_i)$ where each b_i is given by $b_i = \langle \boldsymbol{a}_i, \boldsymbol{s} \rangle + e_i$.

Encrypt:

To encrypt a single bit we choose a random set S uniformly among all 2^m subsets of $\{1, \ldots, m\}$. The encryption is $(\sum_{i \in S} \boldsymbol{a}_i, \sum_{i \in S} b_i)$ if the bit is 0, and $(\sum_{i \in S} \boldsymbol{a}_i, \lfloor q/2 \rfloor + \sum_{i \in S} b_i)$ otherwise.

Decrypt:

The decryption of a pair $(\boldsymbol{a}, b)$ is 0 if $b - \langle \boldsymbol{a}, \boldsymbol{s} \rangle$ is closer to 0 than to $\lfloor q/2 \rfloor$ modulo q.

Table 2: LWE encryption scheme

Take $e_1 = e_2 = 1$ and $e_3 = e_4 = -1$. Now we can compute the **public key**

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \langle \boldsymbol{a}_i, \boldsymbol{s} \rangle + e_i \end{bmatrix} = \begin{pmatrix} 1 & 16 & 4 & 5 \\ 16 & 4 & 5 & 1 \\ 4 & 5 & 1 & 16 \\ 5 & 1 & 16 & 4 \\ 15 & 8 & 8 & 11 \end{pmatrix}.$$

To now encrypt a bit 1, we can take as our S the set $\{1, 2, 4\}$ and output the encryption as

$$(\boldsymbol{a}, b)^T = \begin{pmatrix} 1 + 16 + 5 \\ 16 + 4 + 1 \\ 4 + 5 + 16 \\ 5 + 1 + 4 \\ \lfloor 17/2 \rfloor + 15 + 8 + 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 8 \\ 10 \\ 0 \end{pmatrix}.$$

Decryption is just another simple computation, we first compute $\langle \boldsymbol{a}, \boldsymbol{s} \rangle = 108 \equiv 6 \bmod 17$ which gives us $b - \langle \boldsymbol{a}, \boldsymbol{s} \rangle = 0 - 6 \equiv 11 \bmod 17$. Let us compare the distances to 0 and $\lfloor q/2 \rfloor = 8$. $|11 - 17| = 6 > 3 = |11 - 8|$. Hence, our decryption worked correctly as indeed, the result is closer to $\lfloor q/2 \rfloor$ than to 0. Finally, we output 1 as the decryption of our message and we are done.

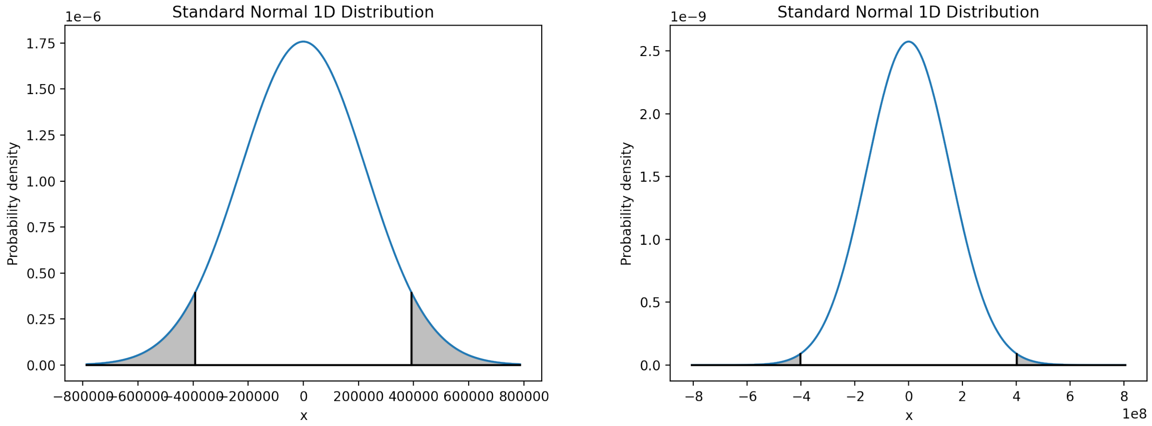
Analysis Now, that we have finally defined a cryptographic scheme we need to verify it. The two remaining questions we now have are first, is this scheme correct? That is, does the decryption algorithm correctly evaluate back to the original message? This is much more difficult to prove compared to the scheme over the integers presented in 3.1. The following is a somewhat simpler version of Claim 5.2 in [Reg05].

Lemma 4.16 (Correctness)**.** *For the above choice of parameters and e following the $\bar{\Psi}_\alpha$ distribution we have*

$$\Pr_{e \sim \bar{\Psi}_\alpha} \left[|e| < \left\lfloor \frac{q}{2} \right\rfloor / 2 \right] > 1 - \delta(n) \tag{4.5}$$

for some negligible function $\delta(n)$.

Figure 5: 1 dimensional Gaussian with standard deviation $\sigma^2 = q/\log n$ for:



(a) $n = 1024$

(b) $n = 32768$

Proof. Note that if not for the error term, the decryption would always be correct. Hence we need to pay attention to the case where the error term is greater than $q/4$. Since there are m such terms, each with standard deviation of $\sigma^2 = \alpha q$, we can use Proposition 4.5 to show that if $X_i \sim \mathcal{N}(0, \alpha q)$ are i.i.d. for $i \in [m]$, then the sum

$$\sum_{i \in [m]} X_i \sim \mathcal{N}(0, \sum_{i \in [m]} \sigma^2) = \mathcal{N}(0, m \cdot \sigma^2)$$

is distributed with standard deviation equal to $\sqrt{m\alpha q} < q/\log(n)$ and the probability that such sample is greater than $q/4$ is negligible. $\square$

An example of a normal distribution with standard deviation $\sigma^2 = q/\log n$ and mean 0 can be seen in Figure 5. The shaded region corresponds to $|e| > q/4$. At the begining, for n not as big, we can notice that the area does not necessarily correspond to an intuitive notion of "negligible". However, the definition concerns the asymptotic behavior of n and indeed, we can see the area becomes quite small. One might even call it "negligible".

This, in turn, implies that (this is Lemma 5.1)

Lemma 4.17. *The decryption is correct with probability* $1 - \delta(n)$ *where the* $\delta(n)$ *is some negligible function.*

Proof. Consider first the encryption of 0. It is given by $(\boldsymbol{a}, b)$ with $\boldsymbol{a} = \sum_{i \in S} \boldsymbol{a}_i$ and $b = \sum_{i \in S} b_i = \sum_{i \in S} \langle \boldsymbol{a}_i, \boldsymbol{s} \rangle + e_i$. Then the decryption gives us precisely $b - \langle \boldsymbol{a}, \boldsymbol{s} \rangle = \sum_{i \in S} e_i$. By our assumption, $\left| \sum_{i \in S} e_i \right| < \lfloor \frac{q}{2} \rfloor / 2$ with probability at least $1 - \delta(n)$. In that case, it is closer to 0 than $\lfloor \frac{q}{2} \rfloor$ and thus correctly decrypts to 0. The case for the encryption of 1 is similar. $\square$

Note that it seems almost trivial that we decrypt correctly, the scheme was designed in that way. This is only the case when we know the secret key $\boldsymbol{s}$ that is definitely not know to the public. This ties closely to the second and last question, that is, how secure the scheme is? We have established hardness based on average and worst-case lattice problems. However, it might be the case that our choice of parameters required for correctness, hinders on the security. This is resolved with the following theorem. Let us first define some required terminology.

Proposition 4.18 ([Reg05], Lemma 5.4)**.** *For any $\epsilon > 0$ and $m \geq (1+\epsilon)(n+1)\log q$, if there exists a polynomial time algorithm W that distinguishes between encryptions of 0 and 1 then there exists a distinguisher Z that distinguishes between $A_{\boldsymbol{s},\chi}$ and U for a non-negligible fraction of all possible $\boldsymbol{s}$.*

Note that this closely relates to the last lemma from the previous section on the hardness of LWE. For a thorough and less technical analysis than the one given in the original paper, the reader is encouraged to look into section 5.4 in [MR09].

Epilogue

[Krzys: do last if theres time] Until the day of writing this paper, LWE is one of the most influential schemes that can be used for post-quantum cryptographic schemes. It was used as a basis for schemes like the one introduced in [PVW07] (along with an oblivious transfer protocol), [LP10] or [Pei09]. However, arguably the most important contribution, was that of laying groundwork for the *ring*-LWE scheme introduced in the next section. We will now present few alternatives to the results and proofs presented here.

4.3 Ring-LWE

One of the recurring problems in lattice-based cryptography is the key-size and general efficiency. In the GGH cryptosystem, the key-size is $\tilde{O}(n^4)$. In the system based on the hardness of LWE presented in the previous section, the size is in the range of $\tilde{O}(n^2)$[11]. As we will also see later, there is some minimal efficiency needed for the scheme in order to enable the boostrapping (for FHE). Unfortunately, none of the schemes presented so far satisfy those criterions and so, we need to look for something better.

One idea to improve the efficiency, is to assume some underlying structure of the space we are performing computations in. For example, we can assume that the $\boldsymbol{a}$ vectors from previous section are given to us in block of n samples $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathbb{Z}_q^n$ where all of the elements are related. Namely, $\boldsymbol{a}_1 = (a_1, \ldots, a_n)$ is again chosen uniformly but each $\boldsymbol{a}_i = (a_i, \ldots, a_n, -a_1, \ldots, -a_{i-1})$ is a "anti-cyclic" of the initial $\boldsymbol{a}_1$. This choice seems rather arbitrary however we will show how it is a natural consequence of everything we did so far and yields arguably the best results. For example if $n = 4$ and $q = 17$ and $\boldsymbol{a}_1 = (1, 16, 4, 5)$ as before, then $\boldsymbol{a}_3$ has the form $(4, 5, -1, -16) = (4, 5, 16, 1)$. Note that representing n vectors now takes only $O(n)$ elements from $\mathbb{Z}_q$ rather than $O(n^2)$. The underlying structure is a ring, hence the name ring-LWE (or R-LWE), that is, we replace the group $\mathbb{Z}_q^n$ by picking some ring R of degree n over $\mathbb{Z}$ and a positive modulus q defining the quotient ring $R_q := R/qR$. In most of the cases (as well as in this exposition), R is taken to be a *cyclotomic* ring - i.e. $R_q = \mathbb{Z}_q[x]/\langle x^n + 1\rangle$ for $n = 2^k$ which turns out to yield much simpler proofs for slightly weaker results.

In the year 2010, Vadim Lyubashevski, Chris Peikert and Oded Regev presented their paper "On Ideal Lattices and Learning With Errors Over Rings" [LPR12]. The main purpose of the paper was to "translate" the LWE problem onto a ring as was done with the SIS problem (mainly

[11] There are m samples of length n. Turns out that for $m > n$, the problem can become only easier, but the same holds for $m \ll n$. Therefore, in most applications, m is chosen to be roughly the size of n.

by Micciancio [Mic07] that was followed up by other works but these results are not presented in this paper) and followed the heuristic approach behind the NTRU[12] cryptosystem [Buh98]. This in particular means first, defining the ring-LWE and later proving the hardness based on some difficult lattice problems like `SVP` along with pseudorandomness of the ring-LWE distribution (analogous to 4.15 whose definition will appear later). The second issue turned out to be quite nontrivial and required good insight in the algebraic number theory as well as Gaussian measures and distributions.

Somewhat analogous to the previous section, this one is also split into two main parts. First part - Section 4.3.2 - focuses on the hardness of the search version of the RLWE. The approach is identical to the one presented for the standard LWE. However, one needs to pay attention to details that are implied by the shift to R like for example the shape of the error distribution under the canonical embedding. Fortunately for us, the quantum part of the reduction can be adopted almost as is and so, we will only mention it briefly. The second part - Section 4.3.3 - deals with pseudorandomness of the RLWE distribution and thus proving the equivalence between the search and decision versions. This one will require much more insight into the algebraic number theory compared to any other part of this paper and so, we will discuss it in much more detail. Before we dwell any further, we need to establish some terminology and useful lemmas. This is done in the following section.

4.3.1 Background III

We begin this section by specifying some details on the underlying ring along with the definitions of the RLWE distribution analogous to the section on standard LWE. We could not find one single source that would cover all of the details necessary, however a good choice would be [Mar77], [Ste04] or [Mil20]. For a good discussion on the different idal, see Keith Conrads paper.

In this exposition, we will simplify few details of the proofs and definitions by taking harder assumptions on the underlying structures. For example, in places, our underlying number field will be cyclotomic, i.e. a field extension $K = \mathbb{Q}(\zeta_m)$ obtained by adjoining an element ζ_m of order m (a primitive mth root of unity) to the rationals. For such field, its ring of integers happens to be $R := \mathcal{O}_K = \mathbb{Z}[\zeta_m] \cong \mathbb{Z}[X]/\Phi_m(X)$ and so, it is generated by the set $\{\zeta_m^j\}_{j\in[n]}$ (also called the power basis). Furthermore, we fix $m = 2^k$ for some $k \geq 1$ and motivate this choice in 4.3.1.

Algebraic Number Theory II

Definition 4.19 (Trace)**.**

Definition 4.20 (Norm)**.**

rings of integers

Definition 4.21 (The space H)**.**

Definition 4.22 (Canonical embedding)**.**

Definition 4.23 (Dual)**.**

[Pinar: For cyclotomic fields, how does this dual look like?]

[12] As mentioned by Peikert in his survey: “The meaning of the acronym NTRU is somewhat mysterious; plausible candidates include "*N*th degree *tru*ncated polynomial ring" and "Number Theorists ’R’ Us."” - [Pei16]

Why cyclotomic? As mentioned earlier, we wish to fix the degree of our cyclotomic polynomial to a power-of-two. This leads to greatly simplified proofs of some results like for example search-to-decision reduction in section 4.3.3. Nonetheless, in places where it is not necessary, we will be using arbitrary number fields, sometimes not even cyclotomic. We will now list few desirable properties of cyclotomic rings (also those of power of two) that will be useful for us later on. Let us fix m, n and k to some positive integers.

- Under the canonical embedding, the cyclotomic ring $R = \mathcal{O}_K$ for $\Phi_m(x) = x^{\phi(m)} - 1$ embeds as a lattice which in general, is not self-dual (this is only the case for $\mathbb{Z}^n$). Instead, its dual lattice corresponds to a fractional ideal $R^\vee \subset K$ such that $R \subseteq R^\vee \subseteq m^{-1}R$. In the case where $n = \varphi(m) = 2^{k-1}$ these two are actually equivalent, namely $R^\vee = n^{-1}R$.
- Polynomial arithmetic modulo $\Phi_m(X)$ can be performed very efficiently using slightly adjusted, classical n-dimensional FFT - [LPR13], [Lyu+08].
- In general, the smaller the shortest vector $\lambda_1(\mathcal{L})$ is, the less secure the system. Cyclotomic fields have relatively large $\lambda_1(R^\vee)$ - [PRS17].
- In fact, for $n = 2^k$, we have $\lambda_n(R^\vee) = 1/\sqrt{n}$

Unfortunately, cyclotomic fields with degree of a power-of-two are quite rare and restrictive. Imagine that our system is deemed insecure for some large $n = 2^k$. It might so happen that the next power of two is completely impractical when implemented. We should be able to find something inbetween the two instead. Cyclotomic number fields have *Galois* group

The RLWE distribution [Pinar: Add explicit examples?? Fix a quadratic field $\mathbb{Q}(\zeta_3)$ or $\mathbb{Q}(\sqrt{-1})$ and so on.]

Definition 4.24 (Ring-LWE Distribution)**.** For $s \in R_q^\vee$ (the “secret”) and an error distribution ψ over $K_\mathbb{R}$, a sample from the ring-LWE distribution $A_{s,\psi}$ over $R_q \times \mathbb{T}$ is generated by choosing $a \leftarrow R_q$ uniformly at random, choosing $e \leftarrow \psi$, and outputting $(a, b = (a \cdot s)/q + e \bmod R^\vee)$.

Definition 4.25 (Ring-LWE, Search))**.** Let Ψ be a family of distributions over $K_\mathbb{R}$. The search version of the ring-LWE problem, denoted R-LWE$_{q,\Psi}$, is defined as follows: given access to arbitrarily many independent samples from $A_{s,\psi}$ for some arbitrary $s \in R_q^\vee$ and $\psi \in \Psi$, find s.

From this point onwards, for $\alpha > 0$, we will denote by $\Psi_{\leq\alpha}$ the set of all elliptical Gaussian distributions $\mathcal{D}_{\boldsymbol{r}}$ (over $K_\mathbb{R}$) where each parameter $r_i \leq \alpha$.

4.3.2 Hardness of search Ring-LWE

We can finally discuss the main results from the [LPR12]. This section focuses on the quantum reduction from R-LWE$_{q,\Psi_{\leq\alpha}}$ to the K-DGS$_\gamma$ - the approximate (to within γ) discrete Gaussian sampling problem (the definition for an arbitrary number field K is created by simply replacing the lattice $\mathcal{L}$ with an ideal $\mathcal{I}$ in Definition 4.9, and so, the samples are defined to be from $\mathrm{D}_{\mathcal{I},r}$.). Note that all the results from this section apply to general number fields, not only cyclotomic.

The following is an equivalent of Theorem 4.10 of the classical LWE. Statement is almost identical however the proof turns out to be much more difficult to achieve than it may look at the first glimpse.

Theorem 4.26. *Let K be an arbitrary number field of degree n and $R = \mathcal{O}_K$. Let $\alpha = \alpha(n) > 0$, and let $q = q(n) \geq 2$ be such that $\alpha q > 2\sqrt{n}$. For some negligible $\epsilon = \epsilon(n)$, there is a probabilistic polynomial-time quantum reduction from K-DGS$_\gamma$ to R-LWE$_{q,\Psi_{\leq\alpha}}$ where $\gamma > 0$ is usually taken*[13] *to be $\gamma = \omega(\log n)$.*

And just as before, the proof relies on the (adjusted) *iterative step* (IS) in parallel with the previous section. It presents as follows.

Lemma 4.27 (Iterative Step)**.** *Let $\epsilon = \epsilon(n)$ be a negligible function, $\alpha > 0$ real, and $q \geq 2$ be an integer. Assume that we have access to an oracle that solves LWE$_{q,\Psi_{\leq\alpha}}$ given a polynomial number of samples. Then, there exists an efficient quantum algorithm that, given a fractional ideal $\mathcal{I}$ in K, a number $r > \sqrt{2}q \cdot \eta_\epsilon(\mathcal{I})$ and a (polynomial) list of samples from the discrete Gaussian distribution $D_{\mathcal{I},r}$, produces a sample from $D_{\mathcal{I},r\cdot\gamma/(\alpha q)}$.*

It is easy to notice that the statement is almost identical to the Lemma 4.11. The place of a n-dimensional lattice $\mathcal{L}$ has been taken up by a fractional ideal $\mathcal{I}$ (which, when embedded, lives in $\mathbb{C}^n$ as an n-dimensional lattice). The proof is also almost intact. We will use Lemma 4.28 and Lemma **??** to create a sequence of discrete Gaussians with decreasing radii to obtain the solution to K-DGS$_\gamma$.

Lemma 4.28 (Step 1 - classical)**.** *Let $\epsilon = \epsilon(n)$ be a negligible function, $\alpha > 0$ real, and $q \geq 2$ be an integer with known factorization. Let $\mathcal{I}$ be a fractional ideal in K, and let $r \geq \sqrt{2}q\cdot\eta_\epsilon(\mathcal{I})$. Given polynomial list of samples from the discrete Gaussian distribution $D_{\mathcal{I},r}$, there is a probabilistic polynomial-time (classical) reduction from $\texttt{CVP}_{\mathcal{I}^\vee,d}$ to R-LWE$_{q,\Psi_{\leq\alpha}}$, where $d = \alpha q/(\sqrt{2}r)$.*

Remark 4.4. In the original statement in [LPR12], `CVP` is replaced by `BDD` to within the same distance. These statements are very similar with a simple difference that `BDD` is concerned with finding *any* lattice vector within d whereas `CVP` wants to find the *closest* vector. Since the distance d is less than $\lambda_1(\mathcal{L})/2$, `BDD` will find the closest, unique vector to $\mathcal{L}$. Hence, in this case, they are equivalent and we use `CVP` to keep it consistent with previous section.

We will proof the lemma using three steps.

Proof. Just as before, it is enough to show reduction to $\texttt{CVP}^{(q)}_{\mathcal{I},d}$ by a similar argument. The high-lever reduction presents now as follows. We are given a $\texttt{CVP}_{\mathcal{I},d}$ instance $y = x + e$ where $x \in \mathcal{I}^\vee$ and $\|e\| \leq d$. We are also granted access to (as many as necessary) samples from the discrete Gaussian over $\mathcal{I}$ and standard deviation r as well as an oracle for R-LWE [Krzys: why do they not require an oracle for LWE in the assumptions?][Krzys: also, why do we need the theta u if the z is already taken from ideal I?] □

4.3.3 Pseudorandomness of Ring-LWE

Just like in the case of classical LWE, we want to show that the ring-LWE distribution is pseudorandom - i.e. samples from the RLWE distribution are indistinguishable from truly random (uniform) ones. This is encapsulated in the following theorem 4.29. We should fix some terminology first.[Krzys: finish preliminaries]

[13] See the discussion just before Section 4.1 in [LPR12].

Additionally, we will present a proof of a slightly simpler version of the result. Namely, instead of using a family of distributions over the Gaussian distributions which is required in a general case, we shall make one more assumption. This is, we will assume that the number of R-LWE samples given (for example to an adversary) is bounded and we will denote it by $l \geq 1$.

Theorem 4.29. *Let R and q be as above and let $\alpha q \geq \eta_\epsilon(R^\vee)$ for some negligible $\epsilon = \epsilon(n)$. Then there is a reduction from R-LWE$_{q,\Psi}$ to R-DLWE$_{q,\mathcal{D}_\xi}$ given only l samples, where $\xi = \alpha \cdot (nl/\log(nl))^{1/4}$.*

The reduction follows from search-R-LWE to decision-R-LWE and in high-level, it consists of 4 steps presented in Figure 6 and described in more detail below.

1. LWE$_{q,\Psi}$ to $\mathfrak{q}_i$-LWE$_{q,\Psi}$.

$$\begin{tikzcd}
\boxed{\mathrm{LWE}_{q,\Psi}} \arrow[r, "(1)"] & \boxed{\mathfrak{q}_i\text{-}\mathrm{LWE}_{q,\Psi}} \arrow[r, "(2)"] & \boxed{\mathrm{WDLWE}^i_{q,\Psi}} \arrow[r, "(3)"] & \boxed{\mathrm{DLWE}^i_{q,\mathcal{D}_\xi}} \arrow[r, "(4)"] & \boxed{\mathrm{DLWE}_{q,\mathcal{D}_\xi}}
\end{tikzcd}$$

Figure 6: Schematic of reductions from R-LWE to R-DLWE

Search to Worst-Case Decision We begin with the reduction of the LWE to $\mathfrak{q}_i$-LWE distribution relative to only one $\mathfrak{q}_i$ arbitrary prime ideal.

Definition 4.30. The $\mathfrak{q}_i$-LWE$_{q,\Psi}$ problem is: given access to $\mathrm{A}_{s,\Psi}$ for some arbitrary $s \in R_q^\vee$ and $\psi \in \Psi$, find s mod $\mathfrak{q}_i R$.

Lemma 4.31 (LWE to $\mathfrak{q}_i$-LWE, Lemma 5.5 [LPR12])**.** *Suppose that the family Ψ is closed under all the automorphisms of K, i.e., $\psi \in \Psi \Rightarrow \tau_k(\psi) \in \Psi$ for every $k \in \mathbb{Z}_m^*$. Then for every $i \in \mathbb{Z}_m^*$, there is a deterministic polynomial-time reduction from LWE$_{q,\Psi}$ to $\mathfrak{q}_i$-LWE$_{q,\Psi}$.*

Proof. By assumptions, we are given access to an $\mathfrak{q}_i$-LWE oracle along with n field automorphisms τ_k that "act transitively" on the prime ideals $\mathfrak{q}_i$ thanks to the underlying field being a cyclotomic number field. The idea is to use the oracle to recover the value of s relative to every $\mathfrak{q}_j R^\vee$ using the automorphisms. Once we have that, we can (efficiently) recover the s using the Chinese Remainder Theorem.

The reduction to find s mod $\mathfrak{q}_j R^\vee$ works as follows: transform each given sample $(a, b) \leftarrow \mathrm{A}_{s,\psi}$ to $(\tau_k(a), \tau_k(b)) \in R_q \times \mathbb{T}$ where $k = j/i \in \mathbb{Z}_m^*$ which gives $\tau_k(\mathfrak{q}_j) = \mathfrak{q}_i$. Give the transformed sample to the oracle which outputs some $t \in R^\vee/\mathfrak{q}_i R^\vee$. Since τ_k is a bijection, we can compute $\tau_k^{-1}(t) \in R^\vee/\mathfrak{q}_j R^\vee$.

We now need to verify that this output is actually equal to s mod $\mathfrak{q}_j R^\vee$. This is equivalent to asking if $(\tau_k(a), \tau_k(b))$ is distributed according to $\mathrm{A}_{\tau_k(s),\psi'}$ for $\psi' = \tau_k(\psi)$. First, note that the automorphisms fix the underlying structure. Namely, $\tau_k(R^\vee) = R^\vee$ and $\tau_k(\mathbb{T}) = \mathbb{T} = K_\mathbb{R}/R^\vee$ and in particular, $\tau_k(q) = q$ for any $k \in \mathbb{Z}_m^*$. We therefore have

$$\tau_k(b) = \tau_k(as)/q + \tau_k(e) \bmod R^\vee.$$

If a was uniformly distributed, $\tau_k(a)$ will be as well. The pairs are therefore distributed according to $\mathrm{A}_{\tau_k(s),\psi'}$ for $\psi' = \tau_k(\psi) \in \Psi$. The t returned by an oracle must therefore be $t = \tau_k(s)$ mod $\mathfrak{q}_i R^\vee$ and so $\tau_k^{-1}(t) = s$ mod $\tau_k^{-1}(\mathfrak{q}_i R^\vee) = s$ mod $\mathfrak{q}_j R^\vee$ as required.

All we need to show now is that the automorphisms preserve the distribution. The proof is short and omited from here but can be found as Lemma 5.6 in [LPR12]. As an outline, recall that the automorphisms simply permute the coordinates of the canonical embedding. So for any $\mathcal{D}_{\boldsymbol{r}}$ with each $r_i \leq \alpha$, it is sent to $\mathcal{D}_{\boldsymbol{r'}}$ where $\boldsymbol{r'}$ is simply the permutation of coefficients of the vector $\boldsymbol{r}$ and so, each $r'_i \leq \alpha$. Thus $\tau_k(\mathcal{D}_{\boldsymbol{r}}) \in \Psi_{\leq \alpha}$ as required. □

Chinese remainder theorem for rings –> Thm II.4.12 in Top's lecture notes.
For instance, they have unique factorization of ideals, and their fractional ideals form a multiplicative group; in general, neither property holds in $\mathbb{Z}[x]/\langle f(x)\rangle$ for monic irreducible $f(x)$, as demonstrated by the ring $\mathbb{Z}[x]/\langle x^2+3\rangle = \mathbb{Z}[\sqrt{-3}]$. (For example, in this ring $4 = 22 = (1+\sqrt{-3})(1-\sqrt{-3})$, but 2, $1+\sqrt{-3}$, and $1-\sqrt{-3}$ are all irreducible.) Toward basing fully homomorphic encryption on worst-case hardness
One of the applications is [Alk+19] signature scheme.

4.4 Fully Homomorphic Encryption Using Ideal Lattices

three "generations" of fhe schemes, first original gentry, smart and explain here how we can construct a really nice homomorphic encryption scheme using ideal lattices [Gen09b]. present the

On Ideal Lattices and Learning With Errors Over Rings

this is somewhat too difficult for me i think so ill just present main findings without proofs and details [Reg05],
First explain what lattices are.
How do lattices relate to LWE? The secret key is associated with a random vector.
then show how ring-lwe satisfies both of our requirements [LPR12], namely, the believed hardness for quantum computers (SVP or approximate SVP) and FHE. Show also the problem with ring-LWE because the lattices that are used there are ideal lattices which obviously possess more structure than "normal" lattices.

5 Conclusions

a quote from [JS16]: "In real world scenarios, cryptosystems based on N P-hard or N P-complete problems tend to rely on a particular subclass of problems, either to achieve efficiency or to allow the creation of a trapdoor. When this is done, there is always the possibility that some special property of the chosen subclass of problems makes them easier to solve than the general case"

Remark 5.1. We are still faced with a problem that is inherent to all of modern-day cryptography. That is, we are assuming the hardness of the problem based on our inability to efficiently solve it. As correctly trivialized by Daniel J. Bernstein [Ber09]: "nobody has figured out an attack so we conjecture that no attack exists". It might so happen that tomorrow someone finds an efficient (polynomial time) algorithm to find the shortest vector in a given lattice and our secrets are compromised. This is exactly what happened in the case of RSA cryptosystem when Shor found such efficient algorithm for integer factorization. There is not much we can do about it at least with our current approach to cryptography which is based on very precise complex-theoretic assumptions. This is because complexity theory does not provide any tools to prove that an efficient algorithm does not exist for any given problem. This puts cryptography as a empirical study (like physics)

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