University of Manitoba
Department of Statistics

STAT 2400

Introduction to Probability I

Supplementary Problems: Module 2

A. Problems from the textbook

From Chapter 3: 3, 4, 11, 15, 17, 19, 21, 30, 35, 36, 41, 42, 62, 63, 75, 76, 91, 93, 97, 100, 104, 105

B. Other problems

Question 2.1:

A committee is to be formed by selecting 3 of 13 individuals (one of whom will be Chair). Brad, David and John are among these 13 individuals. How many possible committees can be formed if

(A) there are no restrictions;

(B) John has to chair the committee;

(C) David would serve only if he does not chair the committee;

(D) Brad and John serve together or not at all;

(E) Brad and David refuse to serve together.

Question 2.2:

What is the probability that a hand of five-card draw poker contains at least one ace? To solve this, we define the events

$$A_i : \text{the hand contains } \textit{exactly } i \text{ aces}, \qquad \text{for } i = 0, 1, 2, 3, 4,$$

so that the actual event of interest is simply given by $\bigcup_{i=1}^{4} A_i$.

(A) A direct approach is to use the fact that

$$\mathbb{P}\left(\bigcup_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} \mathbb{P}(A_i).$$

Explain why this equation holds and find the wanted probability using it.

(B) It is actually simpler here to work using the complementation rule. To see this, find the wanted probability by instead calculating $\mathbb{P}(A_0)$ and then, by using the complementation rule.

Question 2.3:

Suppose that a person sequentially chooses at random a letter from the word S T A T I S T I C S and one from the word M A T H E M A T I C S. What is the probability that the same letter is chosen?

Question 2.4:

Two fair dice are rolled n times in succession.

(A) What is the probability that a double six is obtained at least once?

(B) How large should n be for this probability to be at least 1/2?

Question 2.5:

Suppose you have a key ring with N keys, one of which is your house key. Further suppose that you get home after dark and can't see the keys on the ring. You are to randomly try one key at a time until you get the correct key. Find the probability that you get the correct key on your n^{th} try

(A) if you are careful not to mix the keys you have tried with those you have not;

(B) if you carelessly put back the keys you have tried with the ones you have not.

Question 2.6:

At the game of bridge, an entire deck of 52 playing cards is distributed among four players so that each player is dealt a hand of 13 cards. Cards with a denomination of ten or higher (*i.e.* ten, jack, queen, king or ace) are referred to as *honour cards*. Finally, some cards are given a high card point (HCP) value:

Ace	4 pts,
King	3 pts,
Queen	2 pts,
Jack	1 pt.

(A) What is the probability that a bridge hand contains no aces?

(B) The second Earl of Yarborough is reported to have bet, at odds of 1000 to 1, that a bridge hand would contain at least one honour card. Nowadays, a bridge hand that contains no honour cards (*i.e.* no cards higher than nine) is called a *Yarborough*. What is the probability that a bridge hand is a Yarborough?

(C) What is the probability that a bridge hand contains only honour cards?

(D) What is the most likely number of honour cards in a bridge hand, and what is the probability of having a bridge hand with that many honour cards?

(E) What is the probability of being dealt a hand with zero HCPs in total?

(F) What is the probability of being dealt a hand with exactly 4 HCPs in total?

Hint: You might have two queens with eleven other cards contributing zero HCPs, etc.

(G) What is the probability that a bridge hand is void of at least one suit? Note that

$$\mathbb{P}(\text{“Bridge hand is void of at least one suit”}) \neq \frac{\binom{4}{1}\binom{39}{13}}{\binom{52}{13}}.$$

Why not?

Hint: To correctly calculate this probability, use the inclusion-exclusion principle.

Question 2.7:

Show the following equivalent to Proposition 2.8 (B) from the course notes:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

C. Challenges

Question 2.8:

The following equality holds for all integers $n \geq 1$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{1}$$

We here consider different proofs of this result.

(A) Prove (1) using mathematical induction.

(B) Prove (1) by considering the number of possible committees that can be formed from a group of n people.

Hint: The size of the committee is not specified. Think of using the result of Problem 3.21 from the textbook.

(C) Using (1), prove that for $n \geq 1$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}. \tag{2}$$

(D) Prove (2) by considering the number of possible committees with a chairperson that can be formed from a group of n people.

Question 2.9:

Vandermonde's identity states that

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k},$$

for integers k, n and m such that $0 < k \leq \min(n, m)$.

Prove Vandermonde's identity by considering the number of possible committees that can be formed from a group of people that includes n male and m female individuals.

Question 2.10:

Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \qquad \text{for all integers } n \geq 1.$$

Hint: Use Vandermonde's identity.

Question 2.11:

The following equality holds for all integers $1 \leq k \leq n$:

$$\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k} = \binom{n}{k} 2^{n-k}. \tag{3}$$

We here consider different proofs of this result.

(A) Prove (3) directly by using (1).

(B) Prove (3) by considering the number of possible committees that can be formed from a group of n people, but where a subcommittee of k people also needs to be identified.

Hint: Again, the size of the committee is not specified (but it has to be at least k).

Obviously, mathematical induction could also be considered.

(C) Prove that for $1 \leq k \leq n$:

$$\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k}(-1)^{n-i} = 0.$$

Hint: Have a look at Example 2.19 from the notes.

D. Answers

Textbook Problems:

3.4. 640 224 000

3.30. a) 161 700 b) 970 200

3.36. a) 40 b) 624 c) 3744 d) 5 108
e) 10 200 f) 54 912 g) 123 552 h) 1 098 240

3.42. a) 3.855×10^{26} b) 2.138×10^{25} c) 1.264×10^{14}

3.62. a) $\simeq 0.2143$ b) $\simeq 0.1244$

3.76. $\frac{2(n-k-1)}{n(n-1)}$

3.100. a) $\simeq 0.1541$ b) $\simeq 0.1389$

3.104. a) $\simeq 0.0769$

Other Problems from the List:

2.1.	a) 858	b) 66	c) 792	d) 528	e) 825
2.2.	0.3412				
2.3.	7/55				
2.4.	a) $1 - (35/36)^n$		b) 25		
2.5.	a) $1/N$		b) $(N-1)^{n-1}/N^n$		
2.6.	a) 0.3038	b) 5.47×10^{-4}	c) 1.22×10^{-7}	d) 5; 0.2568	e) 3.64×10^{-3}
	f) 0.03845	g) 0.05107			

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