University of Manitoba
Department of Statistics

STAT 2400

Introduction to Probability I

Solutions to Sample Test #1 (C)

Question 1: (15 marks: 2.5 each)

How many different student lineups are there if

(A) the students are lined up completely at random?

Solution: $N(\cdot) = 20!$

(B) the smallest student and tallest student of the class necessarily come first and last, respectively?

Solution: $N(\cdot) = 18!$

(C) Andrew, Carrie and Kathy will not be separated in the line?

Solution: $N(\cdot) = 18! \times 3!$

(D) Andrew, Carrie and Kathy will not be separated in the line, but Andrew is also the tallest student in the class and has to be last in line?

Solution: $N(\cdot) = 17! \times 2$

(E) Andrew and Carrie have been fighting and so, won't be next to each other in line;

Solution: $N(\cdot) = 20! - 19! \times 2$

(F) boys and girls should alternate in the line (assuming the class has 10 boys and 10 girls).

Solution: $N(\cdot) = (10!)^2 \times 2$

Question 2: (5 marks: 2.5 each)

Let A and B be events of Ω such that

$$\mathbb{P}(A) = 0.2, \qquad \mathbb{P}(A \cup B) = 0.6 \qquad \text{and} \qquad \mathbb{P}(A \cap B) = 0.1.$$

(A) Determine $\mathbb{P}(B)$.

Solution: From the general addition rule we have that

$$\mathbb{P}(B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) + \mathbb{P}(A \cap B) = 0.5.$$

(B) Determine $\mathbb{P}(A \cup B^c)$.

Solution: Again, using the general addition rule,

$$\begin{aligned}\mathbb{P}(A \cup B^c) &= \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c) \\ &= \mathbb{P}(A) + \mathbb{P}(B^c) - \big[\mathbb{P}(A) - \mathbb{P}(A \cap B)\big] = \mathbb{P}(B^c) + \mathbb{P}(A \cap B) = 0.6.\end{aligned}$$

Question 3: (4 marks)

What is the probability the word ABRACADABRA is formed?

Solution: There are here

$$\binom{11}{5,2,1,1,2} = \frac{11!}{5!(2!)^2} = 83{,}160$$

equally likely words that can be formed when drawing the balls. Hence $\mathbb{P}(\cdot) = 1/83{,}160$.

Question 4: (4 marks)

Find $\mathbb{P}(A_i)$ for $i = 4, 5, \dots, 10$.

Solution: In this case,

$$\mathbb{P}(A_i) = \frac{\binom{i-1}{3}(4)_4(6)_{i-4}}{(10)_i},$$

with the numerator coming from the fact that there are $i-1$ positions to choose from for the first three red balls (the fourth red ball needs to be in position i) and that 4 red and $i-4$ black balls need to be selected. This formula leads to the following table of probabilities:

i	4	5	6	7	8	9	10
$\mathbb{P}(A_i)$	1/210	2/105	1/21	2/21	1/6	4/15	2/5
	0.0048	0.0190	0.0476	0.0952	0.1667	0.2667	0.4000

Actually, a much easier approach is to write this as

$$\mathbb{P}(A_i) = \frac{\binom{i-1}{3}}{\binom{10}{4}},$$

which comes from selecting the positions of the red balls when drawing all 10 balls. For the numerator, we need a red ball in position i and the three other reds in the first $i-1$ positions.

Question 5: (4 marks)

What is the probability that a double six is obtained at least once?

Solution: This uses the complementation rule

$$\mathbb{P}() = 1 - \mathbb{P}(\text{“no double six”}) = 1 - \left(\frac{35}{36}\right)^n,$$

since

$$\mathbb{P}(\text{“no double six”}) = \frac{N(\text{“no double six”})}{N(\Omega)} = \frac{35^n}{36^n}.$$

Question 6: (4 marks)

According to Barbara, what is the probability that she will be hired?

Solution: Let H_J , H_M and H_B denote the events that John, Martin and Barbara are hired, respectively. Let also $\mathbb{P}(H_J) = \mathbb{P}(H_M) = p$. Then, it should be clear that

$$\mathbb{P}(H_J) + \mathbb{P}(H_M) + \mathbb{P}(H_B) = 1, \quad \text{and that} \quad \mathbb{P}(H_B) = \frac{3}{2}p.$$

From these, we have that

$$1 = \frac{7}{2}p, \qquad \text{or } p = \frac{2}{7} \simeq 0.289.$$

Hence, for Barbara,

$$\mathbb{P}(H_B) = \frac{3}{2}p = \frac{3}{7} \simeq 0.429.$$

Question 7: (4 marks)

State Kolmogorov's axioms of probability.

Solution: The three axioms are:

1) The nonnegativity axiom: for any event $E \subset \Omega$,

$$\mathbb{P}(E) \geq 0.$$

2) The certainty axiom:

$$\mathbb{P}(\Omega) = 1.$$

3) The countable additivity axiom: for $A_1, A_2, \ldots$ disjoint events of Ω,

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Question 8: (8 marks: 4 each)

(A) Prove Bonferroni's inequality.

Solution: We start with the general addition rule, which implies that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1,$$

since $\mathbb{P}(A \cup B) \leq 1$.

(B) Show that $\mathbb{P}(A \cap B) = \mathbb{P}(A \cup B) = 1$.

Solution: Firstly, $A \subset A \cup B$, so that the domination principle implies

$$1 = \mathbb{P}(A) \leq \mathbb{P}(A \cup B),$$

implying, in turn, that $\mathbb{P}(A \cup B) = 1$.

Secondly, using Bonferroni's equality,

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = 1,$$

implying $\mathbb{P}(A \cap B) = 1$.

<u>**Question 9:**</u> (4 marks)

Prove that for $0 \leq i \leq k \leq m \leq n$,

$$\binom{n}{m}\binom{m}{k}\binom{k}{i} = \binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{m-k}.$$

Solution: Recall that

$$\binom{m'}{r}\binom{r}{k'} = \binom{m'}{k'}\binom{m'-k'}{r-k'}.$$

It suffices to use this equality twice to get the result. Indeed, applying this with $m' = n$, $r = m$ and $k' = k$, we have

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k},$$

so that

$$\binom{n}{m}\binom{m}{k}\binom{k}{i} = \binom{n}{k}\binom{k}{i}\binom{n-k}{m-k}.$$

Using the same argument with $m' = n$, $r = k$ and $k' = i$ leads to the wanted result.

A proof based on a combinatorial argument is also acceptable here.