University of Manitoba
Department of Statistics

STAT 2400

Introduction to Probability

Solutions to Sample Test #1 (A)

Question 1:

(A) What is the appropriate sample space Ω for this random experiment?

Solution:

$$\Omega = \{(i, j) : i, j \in \mathbb{N}, 1 \leq i \leq 6 \text{ and } 1 \leq j \leq 52\}.$$

(B) What is the probability that the die and card show the same denomination?

Solution:

$$\mathbb{P}(\text{“same denomination”}) = \frac{6 \times 4}{6 \times 52} = \frac{1}{13}.$$

Question 2:

Let S: the patient is referred to a specialist,
L: the patient is referred for lab work.

Then, we are given that $\mathbb{P}(S) = 0.25$, $\mathbb{P}(L) = 0.35$ amd $\mathbb{P}(S^c \cap L^c) = 0.45$.

(A) Patient is referred to a specialist and for lab work.

Solution: First note that, from De Morgan's law,

$$\mathbb{P}(S^c \cap L^c) = \mathbb{P}\big((S \cup L)^c\big) = 1 - \mathbb{P}(S \cup L),$$

so that $\mathbb{P}(S \cup L) = 1 - 0.45 = 0.55$. Using the general addition rule, we can now find

$$\mathbb{P}(S \cap L) = \mathbb{P}(S) + \mathbb{P}(L) - \mathbb{P}(S \cup L) = 0.05.$$

(B) Patient is referred to a specialist but not for lab work.

Solution: Using the result from part (A), we have

$$\mathbb{P}(S \cap L^c) = \mathbb{P}(S) - P(S \cap L) = 0.2.$$

Question 3:

(A) The three prizes are identical:

Solution:

$$\binom{53}{3} = 23426.$$

(B) The three prizes are identical, but at least one prize goes to a student from Science:

Solution:

$$\binom{53}{3} - \binom{30}{3} = 23426 - 4060 = 19366.$$

(C) The three prizes are different:

Solution:

$$(53)_3 = \binom{53}{3}3! = 140556.$$

(D) There is a first prize and two identical runner-up prizes:

Solution:

$$\binom{53}{1}\binom{52}{2} = 53(1326) = 70278.$$

(E) One prize is specifically for Science students, the other two are are identical and can be awarded to any student (including students from Science):

Solution:

$$\binom{23}{1}\binom{52}{2} = 23(1326) = 30498.$$

(F) One prize is specifically for Science students, one is for Management students and one is for Engineering students:

Solution:

$$\binom{23}{1}\binom{12}{1}\binom{10}{1} = 23(12)(10) = 2760.$$

Question 4:

(A) Show that $1/2 \leq \mathbb{P}(A \cup B) \leq 3/4$.

Solution: As we have seen in class,

$$\mathbb{P}(A \cup B) \geq \mathbb{P}(B) = 1/2.$$

Also, we have that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB) \leq \mathbb{P}(A) + \mathbb{P}(B) = 3/4.$$

(B) Show that $\mathbb{P}(A \cap B) \leq 1/4$.

Solution: Also a straightforward application of something we have seen in class,

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 1/4.$$

(C) Show that $\mathbb{P}(A \triangle B) \geq 1/4$.

Solution:

$$\mathbb{P}(A \triangle B) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) \geq 1/2 - 1/4 = 1/4.$$

Question 5:

Using mathematical induction, prove the generalized version of the second distributive law.

Solution:

We know that

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i),$$

holds for the case of $n = 2$ since (1) is true. Now, we assume that the previous property is also satisfied for some integer $n = N$ and want to show it also holds for $n = N + 1$. First note that

$$A \cup \left(\bigcap_{i=1}^{N+1} B_i \right) = A \cup (C_1 \cap C_2),$$

where

$$C_1 = \bigcap_{i=1}^{N} B_i \qquad \text{and} \qquad C_2 = B_{N+1}.$$

Hence, from (1), we have that

$$A \cup \left(\bigcap_{i=1}^{N+1} B_i \right) = A \cup (C_1 \cap C_2) = (A \cup C_1) \cap (A \cup C_2).$$

Now notice that

$$A \cup C_1 = A \cup \left(\bigcap_{i=1}^{N} B_i \right) = \bigcap_{i=1}^{N} (A \cup B_i)$$

since the property is assumed to hold for $n = N$. Hence, we have

$$\begin{aligned} A \cup \left(\bigcap_{i=1}^{N+1} B_i \right) = (A \cup C_1) \cap (A \cup C_2) &= \left[\bigcap_{i=1}^{N} (A \cup B_i) \right] \cap (A \cup B_{N+1}) \\ &= \bigcap_{i=1}^{N+1} (A \cup B_i) \end{aligned}$$

and the property is also true for $n = N + 1$. This completes the proof by induction and the property is true for all $n \geq 2$.

Question 6:

Assume that $B_1, B_2, \ldots, B_n$ are events that form a partition of Ω.

(A) Clearly (and precisely) explain what the previous assumption means.

Solution: There are two important things to mention here:

i. the events $B_1, B_2, \ldots, B_n$ are pairwise disjoint, i.e. $B_i \cap B_j = \emptyset$ for all $i \neq j$,

ii. together, these events make up the whole sample space, i.e.

$$\bigcup_{i=1}^{n} B_i = \Omega.$$

(B) Show that, for any event $A \subset \Omega$, we have that

$$\mathbb{P}(A^c) = 1 - \sum_{i=1}^{n} \mathbb{P}(A \cap B_i).$$

Solution: There are many ways to do this, the simplest being to use the result of Question 1.5 from the supplementary problems. Indeed, letting

$$E_i = A \cap B_i \qquad \text{for } i = 1, 2, \dots n \qquad \text{and } E_{n+1} = A^c,$$

then $E_1, E_2, \dots, E_n, E_{n+1}$ also form a partition of Ω. From this,

$$1 = \mathbb{P}(\Omega) = \mathbb{P}\Big(\bigcup_{i=1}^{n+1} E_i\Big) = \sum_{i=1}^{n+1} \mathbb{P}(E_i),$$

so that

$$\mathbb{P}(A^c) = \mathbb{P}(E_{n+1}) = 1 - \sum_{i=1}^{n} \mathbb{P}(E_i) = 1 - \sum_{i=1}^{n} \mathbb{P}(A \cap B_i).$$