MATH 2080 2025F Assignment 4 Solutions

1. (2 points) Use an ε-δ argument to show that $\lim_{x\to 0}\sqrt{2x+1} = 1$.

Solution:

The domain of $\sqrt{2x+1}$ is $A = [-\frac{1}{2}, \infty)$. when $x \in A$, we have

$$|\sqrt{2x+1} - 1| = \left|\frac{2x}{\sqrt{2x+1}+1}\right| < 2|x|.$$

Given $\varepsilon > 0$, we choose $\delta = \frac{1}{2}\varepsilon$. Then,

whenever $x \in A$ and $|x - 0| = |x| < \delta$, we have

$$|\sqrt{2x+1} - 1| < 2|x| < 2\delta = \varepsilon.$$

by definition, $\lim_{x\to 0}\sqrt{2x+1} = 1$.

2. (2 points) Show that $\lim_{x\to 1}\sin(\frac{1}{\sqrt{x}-1})$ does not exist.

Solution:

Take $x_n = (\frac{1}{2n\pi} + 1)^2$ and $y_n = (\frac{1}{2n\pi + \frac{\pi}{2}} + 1)^2$. Then $x_n \to 1$ and $y_n \to 1$. But

$$\frac{1}{\sqrt{x_n} - 1} = 2n\pi \quad \text{and} \quad \frac{1}{\sqrt{y_n} - 1} = 2n\pi + \frac{\pi}{2},$$

and hence

$$\lim_{n\to\infty}\sin(\frac{1}{\sqrt{x_n}-1}) = \lim_{n\to\infty}\sin(2n\pi) = 0 \quad \text{and}$$
$$\lim_{n\to\infty}\sin(\frac{1}{\sqrt{y_n}-1}) = \lim_{n\to\infty}\sin(2n\pi + \frac{\pi}{2}) = 1.$$

Since $0 \neq 1$, by the divergence criterion for functional limit, we conclude that $\lim_{x\to 1}\sin(\frac{1}{\sqrt{x}-1})$ does not exist.

3. (2 points) Show that $y = \sqrt[3]{x}$ is uniformly continuous on $(1, \infty)$.

Solution:

For $x, y \in (1, \infty)$,

$$|\sqrt[3]{x} - \sqrt[3]{y}| = \left| \frac{x - y}{(x)^{\frac{2}{3}} + (xy)^{\frac{1}{3}} + (y)^{\frac{2}{3}}} \right| < \frac{1}{3}|x - y|.$$

Given $\varepsilon > 0$, we take $\delta = 3\varepsilon$. Then

$$|\sqrt[3]{x} - \sqrt[3]{y}| < \frac{1}{3}|x - y| < \frac{1}{3}\delta = \varepsilon$$

whenever $x, y \in (1, \infty)$ and $|x - y| < \delta$.

By definition, $y = \sqrt[3]{x}$ is uniformly continuous on $(1, \infty)$.

4. (2 points) Let $(f_n(x))$ be a sequence of functions on a set A, and let $f(x)$ be a function on A. Suppose that $f_n \to f$ uniformly on A. Show that $f_n + |f_n| \to f + |f|$ uniformly on A as well.

Solution:

$$|(f_n + |f_n|) - (f + |f|)| \leq |f_n - f| + |\,|f_n| - |f|\,| \leq 2|f_n - f|.$$

Since $f_n \to f$ uniformly on A, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{for all } x \in A$$

whenever $n \geq N$. So

$$|(f_n(x) + |f_n(x)|) - (f(x) + |f(x)|)| \leq 2|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in A$$

whenever $n \geq N$.

By definition, $f_n + |f_n| \to f + |f|$ uniformly on A.