EXERCISES 2.1 Basic Exercises

2.1 Construct a Venn diagram representing each event and interpret the event in words.
a) E^c **b)** $A \cup B$ **c)** $A \cap B$ **d)** $A \cap B \cap C$ **e)** $A \cup B \cup C$ **f)** $A^c \cap B$

2.2 Answer true or false to each statement and give reasons for your answers.
a) If event A and event B are mutually exclusive, so are events A, B, and C for every event C.
b) If event A and event B are not mutually exclusive, neither are events A, B, and C for every event C.

2.3 Suppose that, in the petri-dish illustration of Example 2.6 on page 28, you can't observe the location of the spot but you can observe its distance from the center of the petri dish.
a) Determine the sample space for this random experiment.
b) Find, as a subset of the sample space, the event that the spot is between 1/4 and 1/2 unit, inclusive, from the center of the petri dish.
c) Describe, in words, the event $[0, 1/3]$.

2.4 Suppose that one die is rolled and that you observe the number of dots facing up.
a) Obtain a sample space for this random experiment whose elements are integers.
b) Determine as a subset of the sample space each of the events A = die comes up even, B = die comes up at least 4, C = die comes up at most 2, and D = die comes up 3.
c) Determine as a subset of the sample space and describe in words each of the events A^c, $A \cap B$, and $B \cup C$.
d) Determine which of the following collections of events are mutually exclusive and explain your answers: A and B; B and C; A, C, and D.
e) Are there three mutually exclusive events among A, B, C, and D? four?
f) Describe in words each of the events $\{5\}$, $\{1, 3, 5\}$, and $\{1, 2, 3, 4\}$.

2.5 Recent research by Schidt et al. (*African Entomology*, 1999, 7, pp. 107–112) describes the effectiveness of a seed-eating weevil on the population control of a nonnative, invasive species of tree in South Africa, called *Paraserianates lophantha*. A frequency distribution of percent seed damage caused by the weevil for 39 trees is provided in the following table. (The notation $a \lessdot b$ is shorthand for "a up to, but not including b.")

Percent seed damage	Number of trees	Percent seed damage	Number of trees
$0 \lessdot 10$	19	$40 \lessdot 50$	6
$10 \lessdot 20$	2	$50 \lessdot 60$	2
$20 \lessdot 30$	5	$60 \lessdot 70$	2
$30 \lessdot 40$	3		

Suppose that one of these 39 trees is selected at random. Let A = event that the tree has less than 40% seed damage, B = event that the tree has at least 20% seed damage, C = event that the tree has at least 30% but less than 60% seed damage, and D = event that the tree has at least 50% seed damage.
a) What is the sample space for this random experiment?

Describe each of the following events in words and determine the number of outcomes (trees) that comprise each event.
b) B^c **c)** $C \cap D$ **d)** $A \cup D$ **e)** C^c **f)** $A \cap D$
g) Among the events A, B, C, and D, identify the collections that are mutually exclusive.

2.6 An urn contains 10 balls, numbered 0, 1, 2, ..., 9. Three balls are removed, one at a time, without replacement.
a) Obtain the sample space for this random experiment.
b) Determine, as a subset of the sample space, the event that an even number of odd-numbered balls are removed from the urn.

2.7 Refer to Example 2.3 on page 27 where two dice are rolled, one black and one gray. For $i = 2, 3, \ldots, 12$, determine explicitly as a subset of the sample space the event A_i that the sum of the faces is i.

2.8 Consider the following random experiment: First a die is rolled and you observe the number of dots facing up; then a coin is tossed the number of times that the die shows and you observe the total number of heads.
a) Determine the sample space for this random experiment.
b) Determine the event that the total number of heads is even.

2.9 George and Laura take turns tossing a coin. The first person to get a tail wins. George goes first. *Note:* You may assume that eventually a tail will be tossed.
a) Describe the sample space for this random experiment.
b) Determine, as a subset of the sample space, the event that Laura wins.

2.10 This exercise considers two random experiments involving the repeated tossing of a coin. *Note:* You may assume that eventually a head will be tossed.
a) If the coin is tossed until the first time a head appears, find the sample space.
b) If the coin is tossed until the second time a head appears, find the sample space.
c) For the experiment in part (a), express the event that the coin is tossed exactly six times in the form $\{\ldots\}$, where in place of "..." you list all of the outcomes in that event.
d) Repeat part (c) for the experiment described in part (b).

2.11 From 10 men and 8 women in a pool of potential jurors, 12 are chosen at random to constitute a jury. Suppose that you observe the number of men who are chosen for the jury. Let A be the event that at least half of the 12 jurors are men and let B be the event that at least half of the 8 women are on the jury.
a) Determine the sample space for this random experiment.
b) Find $A \cup B$, $A \cap B$, and $A \cap B^c$, listing all the outcomes for each of those three events.
c) Are A and B mutually exclusive? A and B^c? A^c and B^c? Explain your answers.

2.12 Let A and B be events of a sample space.
a) Show that, if A and B^c are mutually exclusive, then B occurs whenever A occurs.
b) Show that, if B occurs whenever A occurs, then A and B^c are mutually exclusive.

2.13 Let A, B, and C be events of a sample space. Write a mathematical expression for each of the following events.
a) A occurs, but B doesn't occur.
b) Exactly one of A and B occurs.
c) Exactly one of A, B, and C occurs.
d) At most two of A, B, and C occur.

2.14 Refer to Example 2.17 on page 34, but now suppose that two cards are selected at random, one after the other, without replacement.
a) What is Ω for this random experiment?
b) Let A be the event that at least one of the cards is a face card and let B be the event that at least one of the cards is an ace. Are A and B mutually exclusive? Why or why not?

a) Which of assignments #1–#6 are legitimate probability assignments?
b) Determine the probability of obtaining exactly two heads by using each of the legitimate probability assignments.

Solution a) In view of Proposition 2.3, we need only check each assignment for nonnegativity and summing to 1. Doing so, we find that assignments #1–#4 are legitimate probability assignments. Assignment #5 consists of nonnegative numbers, but those numbers don't sum to 1; therefore this assignment is not legitimate. Regarding assignment #6, although the numbers sum to 1, not all of them are nonnegative, so this assignment is also not legitimate. (Again, just because assignments #1–#4 are each legitimate from a probabilistic point of view, we can't conclude that each assignment is reasonable from a practical point of view. At most one—and perhaps none—of those assignments reflects the true nature of the coin and the random experiment.)

b) The event of obtaining exactly two heads is $E = \{\text{HHT, HTH, THH}\}$. Applying Proposition 2.2 on page 42, we have

$$P(E) = P(\{\text{HHT, HTH, THH}\}) = P(\{\text{HHT}\}) + P(\{\text{HTH}\}) + P(\{\text{THH}\}).$$

Referring to Table 2.4, we can now determine the probability of event E by using each of the legitimate probability assignments—namely, assignments #1–#4. We have for those four assignments, respectively,

$$P(E) = 0.125 + 0.125 + 0.125 = 0.375,$$
$$P(E) = 0 + 0 + 0 = 0,$$
$$P(E) = 0.032 + 0.032 + 0.032 = 0.096, \text{ and}$$
$$P(E) = 0.220 + 0.050 + 0.110 = 0.380.$$

As expected, the probability of event E depends on the probability measure being employed. ■

EXERCISES 2.2 Basic Exercises

2.20 The U.S. Coast Guard maintains a database of the number, source, and location of oil spills in U.S. navigable and territorial waters. According to the *Statistical Abstract of the United States,* a probability distribution for location of oil spill events is as follows.

Location	Probability	Location	Probability	Location	Probability
Atlantic Ocean	0.011	Great Lakes	0.018	Bays and sounds	0.094
Pacific Ocean	0.059	Other lakes	0.003	Harbors	0.099
Gulf of Mexico	0.271	Rivers and canals	0.211	Other	0.234

a) Explain why this probability assignment is legitimate from a probabilistic point of view.
Determine the probability that an oil spill in U.S. navigable and territorial waters
b) occurs in an ocean. c) occurs in a lake or harbor.
d) doesn't occur in a lake, ocean, river, or canal.

2.21 Consider the experiment of tossing a coin once and observing whether it comes up a head (H) or a tail (T). The sample space for this random experiment is $\Omega = \{\text{H,T}\}$. Let p be a real number with $0 \le p \le 1$.

2.24 Let Ω be the sample space for a random experiment and let P be a probability measure on Ω. Use the Kolmogorov axioms to verify the following for events A and B.

a) $P(B) = P(B \cap A) + P(B \cap A^c)$

b) $P(A \cup B) = P(A) + P(B \cap A^c)$

c) Suppose that at least one of events A and B must occur—that is, $A \cup B = \Omega$. Show that the probability that both events occur is $P(A) + P(B) - 1$.

2.25 Consider the experiment of rolling two dice. The possible outcomes are shown in Figure 2.1 on page 27.

a) Assign each outcome a probability of 1/36. Show that this probability assignment is legitimate.

b) Based on the probability assignment in part (a), determine the probability of the event A_i that the sum of the faces is i, for each $i = 2, 3, \ldots, 12$.

c) Provide another probabilistically legitimate assignment to the 36 possible outcomes, and then repeat part (b) for that assignment.

d) Assuming that the die is balanced, is your probability assignment in part (c) reasonable? What about the one in part (a)? Explain your answers.

2.26 A number is chosen at random from the integers 1, 2, ..., 100. The sample space is the set $\Omega = \{1, 2, \ldots, 100\}$, and each outcome is assigned probability 0.01.

a) Show that this probability assignment is legitimate.

b) Let A be the event that the number chosen is even, B be the event that the number chosen is at most 10π, and C be the event that the number chosen is prime. Determine the probabilities of events A, B, and C.

2.27 An urn contains four balls numbered 1, 2, 3, and 4. A ball is chosen at random, its number noted, and the ball is replaced in the urn. This process is repeated one more time.

a) Determine the sample space Ω.

b) If each outcome is assigned the same probability, what is that common probability?

c) Using the probability assignment in part (b), find the probability that the two numbers chosen are different.

2.28 Suppose that Ω is a finite sample space—say, with N possible outcomes. Further suppose that those N possible outcomes are equally likely.

a) What common probability should be assigned to each possible outcome?

b) Determine the probability of an event that consists of m outcomes.

Theory Exercises

2.29 A special case of a relation called *Boole's inequality* is that, for each positive integer n,

$$P(A_1 \cup \ldots \cup A_n) \leq P(A_1) + \cdots + P(A_n), \qquad (*)$$

for all events $A_1, \ldots, A_n$ of a sample space.

a) If $P(A_i) = 1/6$ for each i and $n = 10$, Relation $(*)$ is trivial. However, if $P(A_i) = 1/60$ for each i and $n = 10$, or if $P(A_i) = 1/6$ for each i and $n = 4$, Relation $(*)$ conveys more substantial information than it does in the trivial case. Explain the difference between the trivial case and the two cases in which more substantial information is conveyed. Why is one case "trivial" while the other two are "more substantial"?

b) Prove Relation $(*)$. *Hint:* Use Exercise 2.24, the nonnegativity axiom, and mathematical induction.

The probability that a head will eventually be tossed is 1. In other words, we will eventually get a head when we repeatedly toss a balanced coin. As we demonstrate in Example 4.17, this result holds regardless of whether the coin is balanced, provided only that the probability is not 0 of getting a head when the coin is tossed once. ■

EXERCISES 2.4 Basic Exercises

2.58 Let A and B be events of a sample space. Provide an example where, as sets, A is a proper subset of B, but $P(A) = P(B)$.

2.59 Give an example to show that the converse of the domination principle fails.

2.60 A person is selected at random from among the inhabitants of a state. Which is more probable: that the person so chosen is a lawyer, or that the person so chosen is a Republican lawyer? Explain your answer.

2.61 Refer to Exercise 2.41 on page 62. Use the complementation rule to find the probability that at least one Republican will be on the subcommittee. Why would use of the complementation rule for this problem make things easier than if that rule weren't used?

2.62 Refer to Exercise 2.20 on page 45. Determine the probability that an oil spill in U.S. navigable and territorial waters doesn't occur in the Gulf of Mexico
a) without use of the complementation rule **b)** by using the complementation rule.
c) Compare the work done in your solutions in parts (a) and (b).

2.63 Refer to Exercise 2.39 on page 61. Suppose that a player on the New England Patriots is selected at random. Determine the probability that the player obtained
a) has at least 1 year of experience. **b)** weighs at most 300 lb.
c) is either a rookie or weighs more than 300 lb. Solve this problem both with and without use of the general addition rule and compare your work.

2.64 If a point is selected at random from the unit square $\{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, find the probability that the magnitude of the difference between the x and y coordinates of the point obtained is at most 1/4. Solve this problem both with and without use of the complementation rule and compare your work.

2.65 According to *Current Population Reports,* published by the U.S. Bureau of the Census, 51.0% of U.S. adults are female, 7.1% are divorced, and 4.1% are divorced females. Determine the probability that a U.S. adult selected at random is
a) either female or divorced. **b)** a male.
c) a female but not divorced. **d)** a divorced male.

2.66 Let A and B be events such that $P(A) = 1/4$, $P(B) = 1/3$, and $P(A \cup B) = 1/2$.
a) Are events A and B mutually exclusive? Explain your answer.
b) Determine $P(A \cap B)$.

2.67 Let A and B be events such that $P(A) = 1/3$, $P(A \cup B) = 5/8$, and $P(A \cap B) = 1/10$. Determine
a) $P(B)$. **b)** $P(A \cap B^c)$. **c)** $P(A \cup B^c)$. **d)** $P(A^c \cup B^c)$.

2.68 Gerald Kushel, Ed.D., was interviewed by *Bottom Line/Personal* on the secrets of successful people. To study success, Kushel questioned 1200 people, among whom were lawyers, artists, teachers, and students. He found that 15% enjoy neither their jobs nor their personal lives, 80% enjoy their jobs but not their personal lives, and 4% enjoy both their jobs and their personal lives. Obtain the percentage of the 1200 people interviewed who enjoy

a) either their jobs or their personal lives. **b)** their jobs.
c) their personal lives but not their jobs. **d)** their personal lives.

2.69 Refer to Example 2.31 on page 73. Determine the probability that a household selected at random gets

a) either the *Times* or the *Herald*, but not both.
b) exactly one of the three newspapers.
c) none of the three newspapers.
d) the *Times* and the *Herald*, but not the *Examiner*.
e) exactly two of the three newspapers.

2.70 This exercise deals with the relationships among the general addition rule, the inclusion–exclusion principle, and the additivity axiom.

a) Show that, for two events, the inclusion–exclusion principle reduces to the general addition rule.
b) Show that the general addition rule is consistent with the additivity axiom—that is, for two mutually exclusive events, the general addition rule reduces to the additivity axiom when applied to two events.
c) More generally than in part (b), show that the inclusion–exclusion principle is consistent with the additivity axiom—that is, for N mutually exclusive events, the inclusion–exclusion principle reduces to the additivity axiom when applied to N events.

2.71 A quiz was administered to four students. Somehow the quizzes got shuffled, and the one at the top of the stack was returned to the first student, the one below it was returned to the second student, and so on. For $i \in \{1, 2, 3, 4\}$, let A_i be the event that the ith student got his own quiz back. Without trying to evaluate the result, use the inclusion–exclusion principle to write an expression for the probability that at least one student got the right quiz back. *Hint:* The expression you get should have 15 terms.

Theory Exercises

2.72 Use mathematical induction to prove the inclusion–exclusion principle.

2.73 In this exercise, you are to prove the continuity properties of a probability measure, Proposition 2.11 on page 74. Let $A_1, A_2, \ldots$ be events of a sample space Ω. Suppose first that $A_1 \subset A_2 \subset \cdots$. Let $B_1 = A_1$ and, for $n \geq 2$, let $B_n = A_n \cap A_{n-1}^c$.

a) Interpret B_n in words.
b) Prove that $B_1, B_2, \ldots$ are mutually exclusive and that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.
c) Prove that, for $n \geq 2$, $P(B_n) = P(A_n) - P(A_{n-1})$.
d) Show that $P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(A_n)$.
e) Deduce that Proposition 2.11(a) holds.
f) Use Proposition 2.11(a) to deduce Proposition 2.11(b). *Hint:* Consider complements.

2.74 Bonferroni's inequality: A useful relation in probability theory is *Bonferroni's inequality*, named for its discoverer Carlo Emilio Bonferroni (1892–1960). This inequality provides a lower bound on the probability of the simultaneous occurrence of a finite number of events in terms of the individual probabilities of the events. Specifically, it states that, for each positive integer N and events $A_1, A_2, \ldots, A_N$,

$$P\left(\bigcap_{n=1}^{N} A_n\right) \geq \sum_{n=1}^{N} P(A_n) - (N-1).$$

the number of motor vehicles in use in North America by country and type of vehicle. Frequencies are in thousands.

Vehicle type	U.S C_1	Canada C_2	Mexico C_3
Automobiles V_1	129,728	13,138	8,607
Motorcycles V_2	3,871	320	270
Trucks V_3	75,940	6,933	4,287

Obtain the row and column totals and then determine how many vehicles are

a) not automobiles. **b)** Canadian. **c)** motorcycles.
d) Canadian motorcycles. **e)** either Canadian or motorcycles.

Suppose that a North American vehicle is selected at random.

f) Describe the events C_1, V_3, and $C_1 \cap V_3$ in words.
g) Compute the probability of each event in part (f).
h) Compute $P(C_1 \cup V_3)$ directly from the table.
i) Compute $P(C_1 \cup V_3)$, using the general addition rule and your answers from part (g).
j) Construct a joint probability distribution.

2.98 An urn contains three balls, numbered 1, 2, and 3. Three balls are drawn at random without replacement. For $i = 1, 2,$ and 3, let A_i be the event that ball i occurs on draw i.
a) Find $P(A_i)$ for $i = 1, 2,$ and 3.
b) Are A_1, A_2 and A_3 mutually exclusive? Explain.

2.99 A balanced coin is tossed four times. What is the probability that
a) the first tail is followed by two consecutive heads?
b) a run of three or more heads occurs?

2.100 Three couples are paired at random on the dance floor, with each pair consisting of one man and one woman. Determine the probability of each of the following events.
a) Each wife dances with her own husband.
b) No wife dances with her own husband.
c) At least one wife dances with her own husband.

2.101 A customer in an appliance store will purchase a washer with probability 0.4, a dryer with probability 0.3, and an iron with probability 0.23. He will purchase both a washer and dryer with probability 0.15, both a washer and an iron with probability 0.13, both a dryer and an iron with probability 0.09, and all three items with probability 0.05. Determine the probability that the customer will purchase
a) none of the items. **b)** two or more of the items. **c)** exactly one of the items.

2.102 Suppose that you classify each e-mail message as nonlegitimate (e.g., unwanted or unsolicited) or legitimate. Further suppose that, during any given hour, the probability of getting a nonlegitimate e-mail message is 0.5, the probability of getting a legitimate e-mail message is 0.7, and the probability of getting some of each is 0.4. What is the probability of receiving no e-mail message during a given hour?

Theory Exercises

2.103 Use mathematical induction to prove that Equation (2.4) on page 40 implies finite additivity; that is, it implies that Equation (2.5) on page 41 holds for all finite collections of mutually exclusive events. Why then is it necessary to assume Equation (2.5) instead of simply Equation (2.4)?

2.104 Let A and B be events of a sample space.

a) Prove that, if A and B both have probability 0, so does $A \cup B$.

b) Prove that, if A and B both have probability 1, so does $A \cap B$.

c) Prove or give a counterexample: Parts (a) and (b) hold for countably many events.

d) Prove or give a counterexample: Parts (a) and (b) hold for uncountably many events.

Advanced Exercises

2.105 Two cards are drawn at random from an ordinary deck of 52 playing cards. Determine the probability that the second card drawn is an ace

a) if the first card is replaced in the deck before the second card is drawn.

b) if the first card is not replaced in the deck before the second card is drawn.

2.106 Relative complement of events: Suppose that A and B are events of a sample space.

a) Interpret the event $A \setminus B$. *Note:* Refer to Exercise 1.62 on page 23.

b) Prove that $P(A \setminus B) = P(A) - P(A \cap B)$.

c) Prove that, if $B \subset A$, then $P(A \setminus B) = P(A) - P(B)$.

2.107 Symmetric difference of events: Suppose that A and B are events of a sample space.

a) Interpret the event $A \,\Delta\, B$. *Note:* Refer to Exercise 1.63 on page 23.

b) Prove that $P(A \,\Delta\, B) = P(A) + P(B) - 2P(A \cap B)$.

2.108 Interpret, in words, each of the following events.

a) $\displaystyle\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$ **b)** $\displaystyle\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$

c) What is the relationship between the events in parts (a) and (b)? Explain your answer.

2.109 A balanced die is thrown eight times. Your naive colleague wrote

$$\begin{aligned}&P(\text{“5” on first throw OR “5” on second throw OR} \ldots \text{ OR “5” on eighth throw})\\ &= P(\text{“5” on first throw}) + P(\text{“5” on second throw}) + \cdots + P(\text{“5” on eighth throw})\\ &= 1/6 + 1/6 + \cdots + 1/6 = 8/6 = 1.333\ldots\end{aligned}$$

Explain why this result is impossible and what's wrong with your colleague's reasoning.

2.110 A balanced die is thrown three times. Using the inclusion–exclusion principle, find

$$P(\text{“5” on first throw OR “5” on second throw OR “5” on third throw}).$$

2.111 Tom, Dick, and Harry alternate tossing a balanced die—first Tom, next Dick, then Harry, then Tom, then Dick, then Harry, and so on. The game stops when the first six appears, the winner being the person who tossed the six.

a) Obtain a sample space for this random experiment.

b) Assign probabilities to the individual outcomes. *Hint:* When a die is tossed n times, there are 6^n outcomes possible. Of those 6^n possible outcomes, 5^{n-1} have the property that the first $n-1$ tosses are not six and the nth toss is six.

c) Determine the probability that, eventually, the game stops.

d) Determine the probability that the winner is Tom; Dick; Harry.