Matrix methods – Absorbing Markov chains

MATH 2740 – Mathematics of Data Science – Lecture 14

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

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A simple absorbing chain

We create a Markov chain by repeatedly mating a female offspring with a **fixed Orange Male ($X^O Y$).**

The state of our chain is the genotype of the female offspring from the previous generation.

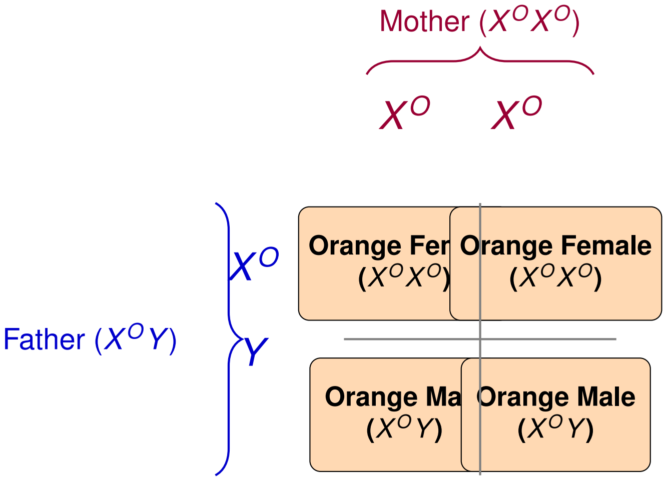
There are 3 possible states for the female offspring:

- S_1: $X^O X^O$ (Pure Orange female)
- S_2: $X^o X^o$ (Pure Black female)
- S_3: $X^O X^o$ (Tortoiseshell female)

We will examine the transition probabilities $\mathbb{P}(S_i \to S_j)$.

State 1: Orange female ($X^O X^O$)

Current state is S_1. We mate this $X^O X^O$ female with our fixed $X^O Y$ male.

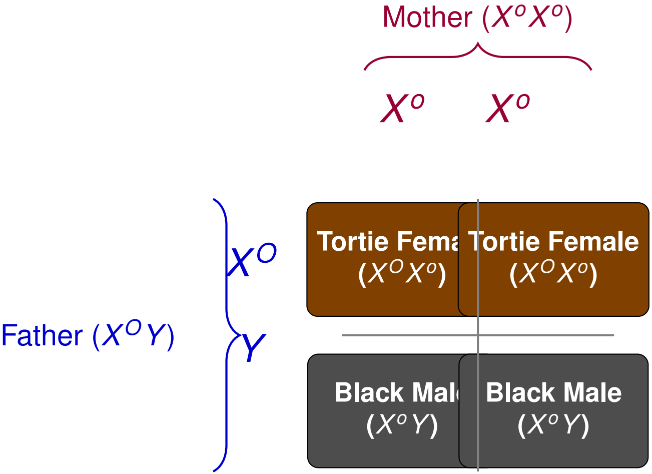


The female offspring are:

- 100% $X^O X^O$ (State S_1)

State 2: Black female (X^oX^o)

Current state is S_2. We mate this X^oX^o female with our fixed X^OY male.

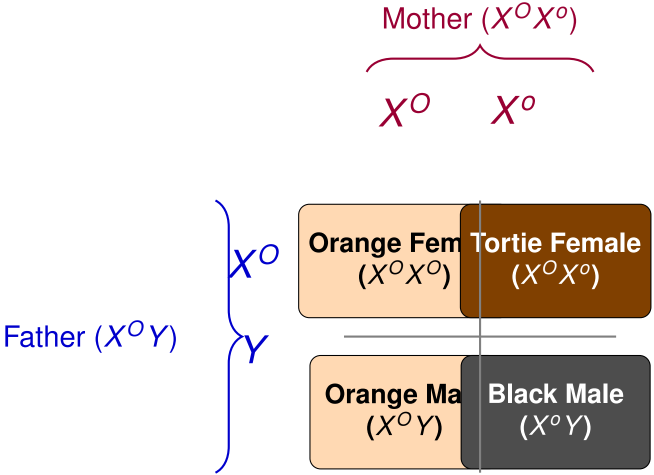


The female offspring are:

- 100% X^OX^o (State S_3) $\implies \mathbb{P}(S_2 \to S_3) = 1$

State 3: Tortoiseshell female ($X^O X^o$)

Current state is S_3. We mate this $X^O X^o$ female with our fixed $X^O Y$ male.



The female offspring have a 50/50 chance:

- 50% $X^O X^O$ (State S_1) $\implies \mathbb{P}(S_3 \to S_1) = 1/2$

Summary of the chain

The transition probability matrix $P = [\mathbb{P}(S_i \to S_j)]$ for states $\{S_1, S_2, S_3\}$ is:

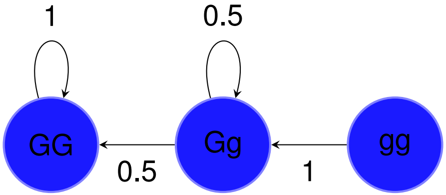
$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

- S_1 $(X^O X^O)$ is an **absorbing state.**
- S_2 $(X^o X^o)$ and S_3 $(X^O X^o)$ are **transient states.**
- From S_2, you are forced to S_3.
- From S_3, you will (with probability 1) eventually be absorbed into state S_1.

This system fulfils the criteria for an absorbing Markov chain

Changing the setting of the genetic experiment

Suppose now the same type of experiment, but mate each new generation with a GG individual instead of a Gg individual



$\swarrow$	GG	Gg	gg
GG	1	0.5	0
Gg	0	0.5	1
gg	0	0	0

$$P = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- leave gg after 1 iteration and can never return
- when we leave Gg, we can never return
- we can never leave GG when we get there

Absorbing Markov chains

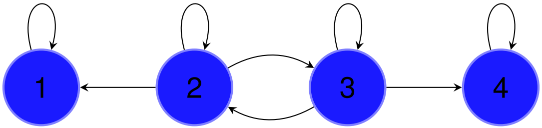
Definition 1 (Absorbing state)

A state S_i in a Markov chain is **absorbing** if whenever it occurs on the t^{th} generation of the experiment, it then occurs on every subsequent step. In other words, S_i is absorbing if $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$

Definition 2 (Absorbing chain)

A Markov chain is **absorbing** if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state. In an absorbing Markov chain, a state that is not absorbing is called **transient**

Suppose we have a chain like the following



1. Does the process eventually reach an absorbing state?
2. What is the average number of steps spent in a transient state, if starting in a transient state?
3. What is the average number of steps before entering an absorbing state?
4. What is the probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?

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The answer to the first question ("Does the process eventually reach an absorbing state?") is given by the following result

Theorem 3

In an absorbing Markov chain, the probability of reaching an absorbing state is 1

To answer the other questions, write the transition matrix in **standard** form

For an absorbing chain with k absorbing states and $r-k$ transient states, write transition matrix as

$$P = \begin{pmatrix} \mathbb{I}_k & R \\ \mathbf{0} & Q \end{pmatrix}$$

with following meaning

	Absorbing states	Transient states
Absorbing states	$\mathbb{I}_k$	R
Transient states	$\mathbf{0}$	Q

with $\mathbb{I}_k$ the $k \times k$ identity matrix, $\mathbf{0}$ an $(r-k) \times k$ matrix of zeros, R an $k \times (r-k)$ matrix and Q an $(r-k) \times (r-k)$ matrix. The matrix $\mathbb{I}_{r-k} - Q$ is invertible. Let

- $N = (\mathbb{I}_{r-k} - Q)^{-1}$ the **fundamental matrix** of the MC
- T_i sum of the entries on row i of N
- $B = RN$

Answers to our remaining questions:

2. N_{ij} average number of times the process is in the *j*th transient state if it starts in the *i*th transient state

3. T_i average number of steps before the process enters an absorbing state if it starts in the *i*th transient state

4. B_{ij} probability of eventually entering the *i*th absorbing state if the process starts in the *j*th transient state

Back to the genetic example

The matrix is already in standard form

$$P = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_1 & R \\ \mathbf{0} & Q \end{pmatrix}$$

with $\mathbb{I}_1 = 1$, $\mathbf{0} = (0 \quad 0)^T$ and

$$R = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}$$

We have

$$\mathbb{I}_2 - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix}$$

so

$$N = (\mathbb{I}_2 - Q)^{-1} = 2 \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

We have

$$N = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

So

$$T = N\mathbb{1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$B = RN = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

2. N_{ij} average number of times the process is in the jth transient state if it starts in the ith transient state

$$N = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

3. T_i average number of steps before the process enters an absorbing state if it starts in the ith transient state

$$T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

4. B_{ij} probability of eventually entering the ith absorbing state if the process starts in the jth transient state

$$B = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

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