Name: ______________________________ Student ID: ____________________

Math 2740 – Fall 2025
Sample final examination (Variant 3) – SOLUTIONS
2 hours

Instructions

- This examination has **8 exercises**.
- Show all your work. Correct answers without justification will receive little or no credit.
- You may use the back of pages if needed.
- No electronic devices (including calculators) are permitted.
- The exam is out of 120 points.

Exercise 1. **[Definitions and Core Results – 15 points]**

State the definition or theorem for each of the following. Be precise and complete.

1. [**4 pts**] Define the eigenpairs of a matrix $A \in \mathcal{M}_n$.
2. [**4 pts**] Define linear independence of a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors.
3. [**4 pts**] Give a necessary and sufficient condition for two vectors to be orthogonal.
4. [**3 pts**] Define the *principal components* of a centered data matrix.

Solution of Exercise 1.

1. An eigenpair of $A \in \mathcal{M}_n$ is a pair $(\lambda, \mathbf{v})$ where $\lambda \in \mathbb{R}$ or $\mathbb{C}$ is an eigenvalue and $\mathbf{v} \neq \mathbf{0}$ is an eigenvector, with both satisfying $A\mathbf{v} = \lambda\mathbf{v}$.
2. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors is linearly independent if the only solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ is $c_1 = c_2 = \cdots = c_k = 0$.
3. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their inner product is zero: $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ (equivalently, $\mathbf{u}^T\mathbf{v} = 0$).
4. The principal components of a centered data matrix $\tilde{X}$ are the eigenvectors of the covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$, ordered by decreasing eigenvalues.

Exercise 2. **[Gram–Schmidt Orthonormalization – 20 points]**

Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

1. **[6 pts]** Apply the Gram–Schmidt procedure to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to obtain an *orthogonal* set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
2. **[6 pts]** Normalize your vectors to obtain an *orthonormal* set $\{q_1, q_2, q_3\}$.
3. **[4 pts]** Verify orthonormality by computing the inner products $\langle q_i, q_j \rangle$ for all i, j and by checking $\|q_i\| = 1$.
4. **[4 pts]** Form the matrix $Q = [q_1 \ q_2 \ q_3]$ and state whether Q is orthogonal (justify your answer).

Solution of Exercise 2.

1. Apply Gram–Schmidt:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 = (2, -1, 0)^T \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = (1, 1, 1)^T - \frac{1}{5}(2, -1, 0)^T = \left(\frac{3}{5}, \frac{6}{5}, 1\right)^T \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2
\end{aligned}$$

   With $\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = -1$, $\|\mathbf{u}_1\|^2 = 5$, $\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = 1$, $\|\mathbf{u}_2\|^2 = 2$, we get $\mathbf{u}_3 = \left(\frac{1}{2}, \frac{3}{2}, -1\right)^T$ (after simplification).

2. Normalize:

$$\begin{aligned}
q_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{5}}(2, -1, 0)^T \\
q_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}}\left(\frac{3}{5}, \frac{6}{5}, 1\right)^T \\
q_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}
\end{aligned}$$

3. Verify: $\langle q_i, q_j \rangle = 0$ for $i \neq j$ and $\|q_i\| = 1$ for all i.
4. Since Q has orthonormal columns, $Q^T Q = I$, so Q is orthogonal.

Exercise 3. **[Least Squares via QR – 15 points]**

Let $A \in \mathbb{R}^{m\times n}$ have full column rank and let $A = QR$ be its *reduced* QR decomposition, where $Q \in \mathbb{R}^{m\times n}$ has orthonormal columns and $R \in \mathbb{R}^{n\times n}$ is upper triangular.

1. **[8 pts]** Using an *important theorem*, prove that the least-squares solution to $A\mathbf{x} = \mathbf{b}$ is $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$.

   Important Theorem 1 (Least Squares via QR)**.** Let $A = QR$ be a reduced QR decomposition with $Q^TQ = I$ and R upper triangular. Then the least-squares solution to $A\mathbf{x} = \mathbf{b}$ satisfies $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$ and the residual is orthogonal to $\mathrm{col}(A)$.

2. **[7 pts]** For

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \end{pmatrix},$$

   compute the reduced QR decomposition $A = QR$ (you may use Gram–Schmidt on the columns) and find $\tilde{\mathbf{x}}$.

Solution of Exercise 3.

1. The normal equations are $A^TA\tilde{\mathbf{x}} = A^T\mathbf{b}$. With $A = QR$, we have $A^TA = R^TQ^TQR = R^TR$ (since $Q^TQ = I$). Thus $R^TR\tilde{\mathbf{x}} = R^TQ^T\mathbf{b}$. Since R is invertible (full rank), multiply by $(R^T)^{-1}$ to get $R\tilde{\mathbf{x}} = Q^T\mathbf{b}$, so $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$.
2. Apply Gram–Schmidt to columns of A. Let $\mathbf{a}_1 = (2, 1, 0, 1)^T$, $\mathbf{a}_2 = (0, 1, 1, -2)^T$.

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{a}_1 = (2,1,0,1)^T, \quad \|\mathbf{u}_1\| = \sqrt{6} \\ \mathbf{u}_2 &= \mathbf{a}_2 - \frac{\langle \mathbf{a}_2, \mathbf{u}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = (0,1,1,-2)^T - \frac{-1}{6}(2,1,0,1)^T = \left(\frac{1}{3}, \frac{7}{6}, 1, -\frac{11}{6}\right)^T \end{aligned}$$

   Then $q_1 = \mathbf{u}_1/\|\mathbf{u}_1\|$ and $q_2 = \mathbf{u}_2/\|\mathbf{u}_2\|$. The R matrix has entries $R_{11} = \|\mathbf{u}_1\|$, $R_{12} = \langle \mathbf{a}_2, q_1\rangle$, $R_{22} = \|\mathbf{u}_2\|$.
   Computing: $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$.

Exercise 4. **[15 points]**

Consider

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. **[6 pts]** Compute the singular values and singular vectors of B.
2. **[5 pts]** Is B invertible?
3. **[4 pts]** Compute the pseudo-inverse of B.

Solution of Exercise 4.

1. B is diagonal, so its singular values are the absolute values of its diagonal entries (reordered): $\sigma_1 = 3$, $\sigma_2 = 1$, $\sigma_3 = 0$. The singular vectors are the standard basis vectors permuted accordingly: $\mathbf{v}_1 = (0,1,0)^T$, $\mathbf{v}_2 = (0,0,1)^T$, $\mathbf{v}_3 = (1,0,0)^T$ for V, and $\mathbf{u}_1 = (0,1,0)^T$, $\mathbf{u}_2 = (0,0,1)^T$, $\mathbf{u}_3 = (1,0,0)^T$ for U.
2. No, B is not invertible because it has a zero singular value (equivalently, $\det(B) = 0$ or $\operatorname{rank}(B) = 2 < 3$).
3. The pseudo-inverse is $B^+ = V\Sigma^+U^T$ where Σ^+ has $1/\sigma_i$ for nonzero σ_i and 0 otherwise:

$$B^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 5. **[PCA on Centered Data – 10 points]**

Let the centered data matrix be

$$\tilde{X} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 0 & -1 \\ 2 & 0 \end{pmatrix}.$$

1. **[6 pts]** Compute the covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$ and its eigenvalues/eigenvectors.
2. **[4 pts]** Identify the first principal component and the variance explained by it.

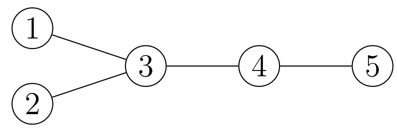
Solution of Exercise 5.

1. With $n = 4$:
$$\tilde{X}^T\tilde{X} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}, \quad S = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$
Eigenvalues: $\det(S - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda_1 = 3$, $\lambda_2 = 1$.
Eigenvectors: For $\lambda_1 = 3$: $(S - 3I)\mathbf{v} = 0 \Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}}(1,-1)^T$. For $\lambda_2 = 1$: $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1,1)^T$.
2. The first principal component is $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1,-1)^T$, and the variance explained by it is $\lambda_1 = 3$.

Exercise 6. **[Graph Measures I – 12 points]**

Consider the simple undirected graph G on vertices $V = \{1, 2, 3, 4, 5\}$ with edge set

$$E = \{\{3, 1\}, \{3, 2\}, \{3, 4\}, \{4, 5\}\}.$$



1. **[4 pts]** Compute the degree $\deg(i)$ of each vertex and give the degree sequence in nonincreasing order.
2. **[4 pts]** Compute the density of G, defined as $\delta(G) = \dfrac{2|E|}{|V|(|V| - 1)}$.
3. **[4 pts]** Compute the local clustering coefficient C_i for each vertex with $\deg(i) \geq 2$ and state the average clustering coefficient.

Solution of Exercise 6.

1. Degrees: $\deg(1) = 1$, $\deg(2) = 1$, $\deg(3) = 3$, $\deg(4) = 2$, $\deg(5) = 1$. Degree sequence in nonincreasing order: $(3, 2, 1, 1, 1)$.
2. Density: $\delta(G) = \frac{2|E|}{|V|(|V|-1)} = \frac{2 \cdot 4}{5 \cdot 4} = \frac{8}{20} = 0.4$.
3. For vertex 3 ($\deg(3) = 3$): neighbors are $\{1, 2, 4\}$. No edges among neighbors, so $e_3 = 0$ and $C_3 = \frac{2 \cdot 0}{3 \cdot 2} = 0$.
   For vertex 4 ($\deg(4) = 2$): neighbors are $\{3, 5\}$. No edge between 3 and 5, so $C_4 = 0$.
   Average clustering coefficient: $\frac{C_3 + C_4}{2} = 0$.

Exercise 7. **[Graph Measures II – 13 points]**

For the same graph G as in Exercise 6:

1. **[5 pts]** Compute the graph diameter and the average shortest-path length $\ell(G)$.
2. **[4 pts]** Compute the (normalized) degree centrality of each vertex, $C_D(i) = \deg(i)/(n-1)$ where $n = |V|$.
3. **[4 pts]** Compute the closeness centrality of each vertex, $C_C(i) = \dfrac{n-1}{\sum_{j\neq i} d(i,j)}$.

Solution of Exercise 7.

1. Shortest paths: $d(1,2) = 2$, $d(1,3) = 1$, $d(1,4) = 2$, $d(1,5) = 3$, $d(2,3) = 1$, $d(2,4) = 2$, $d(2,5) = 3$, $d(3,4) = 1$, $d(3,5) = 2$, $d(4,5) = 1$.
   Diameter: $\max d(i,j) = 3$ (e.g., $d(1,5) = d(2,5) = 3$).
   Average shortest-path length: $\ell(G) = \frac{1}{\binom{5}{2}} \sum_{\{i,j\}} d(i,j) = \frac{18}{10} = 1.8$.
2. Degree centrality: $C_D(1) = 1/4 = 0.25$, $C_D(2) = 1/4 = 0.25$, $C_D(3) = 3/4 = 0.75$, $C_D(4) = 2/4 = 0.5$, $C_D(5) = 1/4 = 0.25$.
3. Closeness centrality:

$$
\begin{aligned}
C_C(1) &= \frac{4}{1+2+2+3} = \frac{4}{8} = 0.5 \\
C_C(2) &= \frac{4}{2+1+2+3} = \frac{4}{8} = 0.5 \\
C_C(3) &= \frac{4}{1+1+1+2} = \frac{4}{5} = 0.8 \\
C_C(4) &= \frac{4}{2+2+1+1} = \frac{4}{6} = 2/3 \approx 0.667 \\
C_C(5) &= \frac{4}{3+3+2+1} = \frac{4}{9} \approx 0.444
\end{aligned}
$$

Exercise 8. **[Markov Chains – 20 points]**

Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and column-stochastic transition matrix:

$$P = \begin{pmatrix} 1 & 1/4 & 1/3 & 0 \\ 0 & 1/2 & 1/3 & 1/2 \\ 0 & 1/4 & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where P_{ij} is the probability of moving from state j to state i.

1. [**4 pts**] Determine whether this Markov chain is regular or absorbing. Justify your answer.
2. [**8 pts**] If the chain is regular, find the limiting distribution $\boldsymbol{\pi}$ by solving $P\boldsymbol{\pi} = \boldsymbol{\pi}$ with $\sum_i \pi_i = 1$. If the chain is absorbing, reorder the states to write P in canonical form

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

   and identify the matrices I, R, and Q.
3. [**8 pts**] If the chain is absorbing, compute the fundamental matrix $N = (I - Q)^{-1}$ and interpret what the entries represent. If the chain is regular, explain why the limiting distribution is independent of the initial state.

Solution of Exercise 8.

1. State 1 is absorbing: $P_{11} = 1$ and all other entries in column 1 are 0. State 4 is also absorbing (actually a "death state" with no outgoing transitions). States 2 and 3 are transient (can reach absorbing states but cannot return).
   This is an **absorbing** Markov chain because it has absorbing states and all transient states can reach at least one absorbing state.
   It is **not regular** because not all entries of P^k can be positive (absorbing states create permanent zeros).
2. Reorder states as $(1, 4, 2, 3)$ to get canonical form:

$$P' = \begin{pmatrix} 1 & 0 & 1/4 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/3 \\ 0 & 1/2 & 1/4 & 1/3 \end{pmatrix} = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

   where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/3 \\ 1/4 & 1/3 \end{pmatrix}.$$

   Note: State 4 with $P_{44} = 0$ is a death state. For proper canonical form with absorbing states, we should only include state 1 as absorbing.
3. Compute $I - Q = \begin{pmatrix} 1/2 & -1/3 \\ -1/4 & 2/3 \end{pmatrix}$.
   The fundamental matrix is:

$$N = (I - Q)^{-1} = \begin{pmatrix} 8/5 & 4/5 \\ 3/5 & 6/5 \end{pmatrix}.$$

   Interpretation: N_{ij} represents the expected number of times the chain visits transient state j before absorption, starting from transient state i.

END OF EXAMINATION