University of Manitoba
Department of Statistics

STAT 2400

Introduction to Probability I

Solutions to Sample Test #1 (B)

Question 1:

How many different license plates are there when valid license plates are made of

(A) any 3 letters (A to Z) followed by any 3 digits (0 to 9)?

Solution:

$$N(\text{“3 letters followed by 3 digits”}) = (26)^3(10)^3.$$

(B) any 3 letters and 3 digits in any order?

Solution:

$$N(\text{“3 letters and 3 digits in any order”}) = \binom{6}{3}(26)^3(10)^3.$$

(C) any 6 letters and/or digits in any order, but repeated symbols are not allowed?

Solution:

$$N(\text{“any 6 letters and/or digits, no repetition”}) = (36)_6.$$

(D) any 6 letters and/or digits, but any letters necessarily come before any digits?

Solution:

$$N(\text{“any 6 letters and/or digits, any letters come first”}) = (10)^6 + 26(10)^5 + \cdots + (26)^6.$$

Question 2:

(A) List the sample space Ω for this random experiment.

Solution:

$$\Omega = \{(i,j) : i,j \in \mathbb{N}, 0 \leq j \leq i \leq 4\}.$$

(B) Let E be the event that the observed number of *Tails* is odd. List the outcomes in E.

Solution:

$$E = \{(1,1); (2,1); (3,1); (3,3), (4,1); (4,3)\}.$$

Question 3:

(A) Are A and B mutually exclusive? Justify your answer.

Solution: A and B are not mutually exclusive since

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.1 + 0.3 - 0.3 = 0.1,$$

so that $A \cap B \neq \emptyset$.

(B) Find $\mathbb{P}(A \cup B \cup C)$.

Solution: First, note that

$$\mathbb{P}(A \cap B) = 0.1, \qquad \mathbb{P}(B \cap C) = 0.1 \qquad \text{and} \qquad \mathbb{P}(A \cap C) = 0.1.$$

Hence, using the general addition rule for 3 events,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) \\ &= 0.1 + 0.3 + 0.7 - 0.1 - 0.1 - 0.1 + 0.1 = 0.9.\end{aligned}$$

Question 4:

For integers $n \geq 1$, define the intervals $A_n = [1/n, 1 + 1/n]$. Find $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$.

Solution: First note that these intervals are "sliding" to the left since

$$A_1 = [1, 2], \qquad A_2 = [1/2, 3/2], \qquad A_3 = [1/3, 4/3], \qquad A_4 = [1/4, 5/4], \qquad \cdots$$

From this,

$$\bigcap_{i=1}^{\infty} A_i = \{1\}, \qquad \text{is the only point that belongs to all intervals,}$$

and

$$\bigcup_{i=1}^{\infty} A_i = (0, 2], \qquad \text{are the points that belong to at least one interval.}$$

Question 5:

(A) Determine the probability that

(i) each husband dances with his wife.

Solution: We can write the sample space as

$$\Omega = \{(1, 2, 3); (1, 3, 2); (2, 1, 3); (2, 3, 1); (3, 1, 2); (3, 2, 1)\}.$$

Then, the probability of a perfect match is

$$\mathbb{P}(\text{“perfect match”}) = \mathbb{P}(\{(1, 2, 3)\}) = 1/6.$$

(ii) at least one husband dances with his wife.

Solution: As above,

$$\mathbb{P}(\text{“at least one match”}) = \mathbb{P}(\{(1, 2, 3); (1, 3, 2); (2, 1, 3); (3, 2, 1)\}) = 4/6 = 2/3.$$

(B) Find the probability that

(i) the fourth woman is the one without a partner,
Solution: This is the probability that, when picking the one woman that will not have a partner, we select the fourth one, or

$$\mathbb{P}(\text{“fourth woman has no partner”}) = 1/4.$$

(ii) each husband dances with his wife.
Solution: Here,

$$\mathbb{P}(\text{“perfect match”}) = \mathbb{P}(\{(1,2,3,4)\}) = \frac{N(\{(1,2,3,4)\})}{N(\Omega)} = \frac{1}{4!} = 1/24.$$

Question 6:

In the expansion of $(2x-y)^{47}$, what is the coefficient of x^2y^{45}?

Solution: From the binomial theorem,

$$(2x-y)^{47} = \sum_{k=0}^{47} \binom{47}{k} (2x)^k(-y)^{47-k},$$

so that the term we are interested in is the one for which $k=2$. That term is

$$\binom{47}{2}(2x)^2(-y)^{45} = -4\binom{47}{2}x^2y^{45} = -4324x^2y^{45},$$

and the wanted coefficient is -4324.

Question 7:

Prove Boole's inequality: if $A_1, A_2, \ldots$ are all subsets of U, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i),$$

for all $n \ge 1$.

Solution: Two different approaches were used in the list of problems. The easier one is using induction. First, we verify the property for $n=2$. In this case,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \le \mathbb{P}(A_1) + \mathbb{P}(A_2),$$

since $\mathbb{P}(A_1 \cap A_2) \ge 0$, implying the property does hold for $n=2$.

Now, assume the property holds for $n = N$. We show the property also holds for $n = N+1$. We have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^{N+1} A_i\right) &= \mathbb{P}(B_1 \cup B_2) && \text{where } B_1 = \bigcup_{i=1}^{N} A_i \text{ and } B_2 = A_{N+1}, \\
&\leq \mathbb{P}(B_1) + \mathbb{P}(B_2) && \text{since the property holds for } n = 2, \\
&= \mathbb{P}\left(\bigcup_{i=1}^{N} A_i\right) + \mathbb{P}(B_{N+1}) \\
&\leq \sum_{i=1}^{N} \mathbb{P}(A_i) + \mathbb{P}(A_{N+1}) && \text{since the property holds for } n = N, \\
&= \sum_{i=1}^{N+1} \mathbb{P}(A_i).
\end{aligned}$$

The property is valid for $n = N+1$, completing the induction step and the proof.

Question 8:

Prove that, for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{k}{2^k}\binom{n}{k} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1}.$$

Solution: First note that $k\binom{n}{k} = n\binom{n-1}{k-1}$ for $k = 1, \ldots n$. Hence, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k}{2^k}\binom{n}{k} &= \sum_{k=1}^{n} \frac{n}{2^k}\binom{n-1}{k-1} \\
&= n\sum_{j=0}^{n-1} \frac{1}{2^{j+1}}\binom{n-1}{j} && \text{(by letting } j = k-1\text{)} \\
&= \frac{n}{2}\sum_{j=0}^{n-1} \frac{1}{2^j}\binom{n-1}{j} \\
&= \frac{n}{2}\left(\frac{3}{2}\right)^{n-1} && \text{(by using the hint with } n' = n-1\text{)}.
\end{aligned}$$