Name: ______________________________ Student ID: ____________________

Math 2740 – Fall 2025
Sample midterm examination
2 hours

Instructions

- This examination has **7 exercises**.
- Show all your work. Correct answers without justification will receive little or no credit.
- You may use the back of pages if needed.
- No electronic devices (including calculators) are permitted.
- The exam is out of 100 points.

Exercise 1. [**Definitions and Theorems – 15 points**]

State the definition or theorem for each of the following. Be precise and complete.

1. [**3 pts**] Define the *singular values* of a matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$.
2. [**4 pts**] State the *Best approximation theorem.*
3. [**4 pts**] State the *Least squares theorem.*
4. [**4 pts**] State the *Singular value decomposition (SVD) theorem.*

Exercise 2. **[Linear Least Squares – 15 points]**

Consider the over-determined system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

1. [**5 pts**] Set up the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ by computing $A^T A$ and $A^T\mathbf{b}$.
2. [**5 pts**] Solve the normal equation to find the least squares solution $\tilde{\mathbf{x}}$.
3. [**5 pts**] Compute the residual $\mathbf{b} - A\tilde{\mathbf{x}}$ and its norm $\|\mathbf{b} - A\tilde{\mathbf{x}}\|$.

Exercise 3. **[Singular Value Decomposition – 20 points]**

Consider the matrix

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$$

1. [**6 pts**] Compute $A^T A$ and find its eigenvalues.
2. [**4 pts**] Determine the singular values of A.
3. [**5 pts**] Find the right singular vectors (eigenvectors of $A^T A$) and construct the matrix V.
4. [**5 pts**] Construct the matrices Σ and U to complete the SVD $A = U\Sigma V^T$. (You may verify your answer by computing the product.)

Exercise 4. [Principal Component Analysis – 15 points]

Consider a dataset with the following data matrix (each row is an observation):

$$X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

1. [**4 pts**] Compute the mean of each variable and center the data matrix to obtain $\tilde{X}$.
2. [**6 pts**] Compute the sample covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$ where $n = 3$.
3. [**5 pts**] Find the eigenvalues of the covariance matrix. Which eigenvalue corresponds to the first principal component?

Exercise 5. **[Proof – 15 points]**

Let $A \in \mathcal{M}_{mn}(\mathbb{R})$. Prove that for any nonzero eigenvalue λ of $A^T A$, we have $\lambda > 0$.

Hint: Use the definition of eigenvalue and properties of the inner product.

Exercise 6. **[Markov Chains – 20 points]**

Consider a Markov chain with three states and transition matrix given in the column-stochastic convention (columns sum to 1):

$$P = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.7 \end{pmatrix},$$

where P_{ij} denotes the probability of moving from state j to state i.

1. **[6 pts]** Determine whether this chain is regular or absorbing. Give a brief justification.
2. **[10 pts]** If the chain is regular, compute the equilibrium (stationary) distribution π satisfying $P\pi = \pi$ and $\sum_i \pi_i = 1$. If the chain is absorbing, describe the fundamental matrix $N = (I - Q)^{-1}$ and the vectors that characterise absorption probabilities and expected times (you do not need to compute them numerically).
3. **[4 pts]** Suppose the initial distribution is $\pi^{(0)} = (1, 0, 0)^T$ (certainly in state 1). Compute the distribution after two steps, $\pi^{(2)} = P^2\pi^{(0)}$ (give the vector explicitly).

Exercise 7. **[10 points]**

What does the following mystery function do? Explain your answer and justify it by describing a representative sample run (you do not need to carry out the run numerically, but indicate the steps and expected outcome).

```
mystery_function <- function(vecs, tol = 1e-10) {
  if (!is.list(vecs)) stop("vecs␣must␣be␣a␣list␣of␣numeric␣vectors")

  n <- length(vecs[[1]])
  m <- length(vecs)
  if (m > n) {
    return(list(success = FALSE,
                message = sprintf("Cannot␣be␣independent:␣%d␣vectors␣in␣%d-dimensional␣space
    ", m, n),
                Q = NULL))
  }

  A <- do.call(cbind, vecs)
  R <- pracma::rref(A)
  rownorms <- apply(abs(R), 1, max)
  rank_est <- sum(rownorms > tol)

  if (rank_est < m) {
    return(list(success = FALSE,
                message = sprintf("Vectors␣are␣linearly␣dependent␣(estimated␣rank␣%d␣<␣%d)",
     rank_est, m),
                Q = NULL))
  }

  Q <- matrix(0, nrow = n, ncol = m)
  for (j in seq_len(m)) {
    v <- A[, j]
    if (j > 1) {
      for (i in seq_len(j - 1)) {
        proj <- sum(Q[, i] * v) * Q[, i]
        v <- v - proj
      }
    }
    normv <- sqrt(sum(v^2))
    if (normv < tol) {
      return(list(success = FALSE,
                  message = sprintf("Numerical␣breakdown:␣vector␣%d␣became␣(near)␣zero␣
    during␣orthogonalisation", j),
                  Q = NULL))
    }
    Q[, j] <- v / normv
  }

  return(list(success = TRUE, Q = Q))
}
```

END OF EXAMINATION