MATH 2080 Introductory Analysis

Chapter 3 Basic Topology of $\mathbb{R}$

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2025F A01

The Cantor set

The Cantor set plays a very important role in solving many difficult problems in morden analysis. Here we only introduce the construction of the set. It is a subset of $[0, 1]$, obtained by removing some (infinitely many) open subintervals from $[0, 1]$. The precise process is as follows.

- Denote $C_0 = [0, 1]$.
- Devide C_0 into three subintervals of equal length. Then remove the middle open interval. The remainder set is denoted by C_1. Precisely, the removed open interval is $(1/3, 2/3)$ of length $1/3$, and C_1 consists of two closed interval components $[0, 1/3]$ and $[2/3, 1]$.
- Devide each component of C_1 into three subintervals of equal length. Then remove the open middle third. The remainder set is denoted by C_2. In this step, 2 open intervals are removed, each of length $1/3^2$., and C_2 consists of $2^2 = 4$ closed interval components.

The Cantor set - continued

Continue this "removing open middle third" process inductively. The remainder set after the nth stage is denoted by C_n.

1. C_n has 2^n closed interval components, each of length $1/3^n$.
2. To get C_{n+1}, one needs to remove (from C_n) 2^n open middle thirds, each of length $1/3^{n+1}$. The total length removed is $2^n \times 1/3^{n+1} = 1/3 \times (2/3)^n$.
3. After the whole process, the final remainder set is $C = \cap_{n=0}^{\infty} C_n$, called the Cantor set.
4. It is not hard to see $C \neq \emptyset$. One can also prove that C has cardinality equal to that of $\mathbb{R}$. So C is uncountable and "**big**".
5. On the other hand, the total length of the open intervals removed from $[0, 1]$ after all stages is $\sum_{n=0}^{\infty} 1/3 \times (2/3)^n = 1$. Since the length of the mother set $[0, 1]$ is 1, the "total length measure" of C is 0. In this sense C is very **small**, often referred to as Cantor "dust".

Open sets in $\mathbb{R}$

Definition 3.1

- *Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$ is called the ε-neighborhood of a.*
- *A subset O of $\mathbb{R}$ is called an open set if for every point $a \in O$ there is $\varepsilon > 0$ such that $V_\varepsilon(a) \subset O$.*

Simple examples of open sets include $\mathbb{R}$ itself and all (bounded or unbounded) open intervals. Empty set $\emptyset$ is also regarded as an open set.

Theorem 3.2

1. *If $\{O_1, O_2, \ldots, O_n\}$ is a finite collection of open sets, then $O = \cap_{k=1}^{n} O_k$ is an open set.*
2. *If $\{O_\alpha : \alpha \in \Lambda\}$ is an arbitrary collection of open sets, then $U = \cup_{\alpha \in \Lambda} O_\alpha$ is an open set.*

Limit points of a set

Let $V_\varepsilon(a)$ be the ε-neighborhood of a. Remove a from $V_\varepsilon(a)$. The result set is called the deleted ε-neighborhood of a, denoted by $\mathring{V}_\varepsilon(a)$. Indeed, $\mathring{V}_\varepsilon(a) = \{x \in \mathbb{R} : 0 < |x - a| < \varepsilon\}$.

Definition 3.3

Given a set $A \subset \mathbb{R}$, a point $p \in \mathbb{R}$ is called a limit point of A if, for every $\varepsilon > 0$, $\mathring{V}_\varepsilon(p) \cap A \neq \emptyset$. Limit points are also called accumulation points.

Note $\mathring{V}_\varepsilon(p) \cap A \neq \emptyset$ really means that $V_\varepsilon(p)$ contains at least one point of A other than p. A limit point itself may not belong to A.

Example 1

1. *Let $A = (a, b)$. Then every point of $[a, b]$ is a limit point of A.*
2. *Let $A = \{1/n : n \in \mathbb{N}\}$. Then 0 is the only limit point of A.*
3. *If $A = \mathbb{Q}$, then all real numbers are limit points of A. Similarly, the set of all limit points of $\mathbb{I}$ is $\mathbb{R}$. While $\mathbb{N}$ has no limit point.*

Isolated points

Remark

- *If p is a limit point of A, then, $\forall \varepsilon > 0$, $V_\varepsilon(p) \cap A$ is an infinite set.*
- *The limit point p of A may not belong to A, and a point $p \notin A$ is a limit point of A iff, $\forall\ \varepsilon > 0$, $V_\varepsilon(p) \cap A \neq \emptyset$.*
- *Denote by A' the set of all limit points of A. Then $A' \subset B'$ if $A \subset B$.*

From the definition of a limit point, a point $p \in \mathbb{R}$ is **not** a limit point of A iff there is an $\varepsilon > 0$ such that $\mathring{V}_\varepsilon(p) \cap A = \emptyset$, meaning either $V_\varepsilon(p)$ contains no point of A or p is the only point of A inside $V_\varepsilon(p)$.

Let $p \in A$. If there is $\varepsilon > 0$ such that p is the only point of A inside $V_\varepsilon(p)$, then we call p an isolated point of A. In other words, p is an isolated point of A iff $p \in A$ and p is not a limit point of A.

Example 2

Let $A = \{1/n : n \in \mathbb{N}\}$. Then every point $\frac{1}{n}$ is an isolated point of A.

Limit Characterization of limit points

Theorem 3.4

A point $p \in \mathbb{R}$ is a limit point of A if and only if there is a sequence $(a_n) \subset A$ such that $a_n \neq p$ for all n and $a_n \to p$.

Proof.

$\Rightarrow$: let p be a limit point of A. Then $\mathring{V}_{\frac{1}{n}}(p) \cap A \neq \emptyset$ for each $n \in \mathbb{N}$. So we may pick up a_n from $\mathring{V}_{\frac{1}{n}}(p) \cap A$ to form $(a_n) \subset A$. Clearly $a_n \neq p$, and $a_n \to p$ since $|a_n - p| < \frac{1}{n} \to 0$.
$\Leftarrow$: If $(a_n) \subset A$ such that $a_n \neq p$ and $a_n \to p$, then for each $\varepsilon > 0$, there is N such that $a_n \in V_\varepsilon(p)$ for $n \geq N$. Of course, such a_n belongs to $\mathring{V}_\varepsilon(p) \cap A$. Thus $\mathring{V}_\varepsilon(p) \cap A \neq \emptyset$ for every $\varepsilon > 0$. $\square$

Theorem 3.4 reveals the meaning for p being a limit point of A.: It means that p is the limit of a **non-constant sequence** from A.

Closed sets in $\mathbb{R}$

Definition 3.5

A set $F \subset \mathbb{R}$ is called a closed set if all limit points of F are included in F.

Simple examples of closed sets include $\mathbb{R}$ and all closed intervals.

Theorem 3.6

A set $F \subset \mathbb{R}$ is closed if and only if every Cauchy sequence in F converges to a point of F.

Remark

From the Cauchy Criterion, a Cauchy sequence always converges to a point in $\mathbb{R}$. But this point may not be a point of F. The above theorem asserts that one can guarantee a Cauchy sequence taken from F converges to a point that still belongs to F only if F is a closed set.

Relation between closed and open

Theorem 3.7

A set $F \subset \mathbb{R}$ is closed if and only if its complement F^C is open.

Proof.

F is closed $\Leftrightarrow$ all limit points of F are in F $\Leftrightarrow$ every $x \in F^C$ is not a limit point of F $\Leftrightarrow$ every point $x \in F^C$ has a neighborhood $V_\varepsilon(x)$ that contains no point of F $\Leftrightarrow$ every point $x \in F^C$ has a neighborhood $V_\varepsilon(x)$ that is contained in F^C $\Leftrightarrow$ F^C is open. $\square$

Properties of closed sets

Using Theorems 3.2 and 3.7, we easily obtain the following result.

Theorem 3.8

1. *If* $\{F_1, F_2, \dots, F_n\}$ *is a finite collection of closed sets, then* $F = \cup_{k=1}^{n} F_k$ *is a closed set.*
2. *If* $\{F_\alpha : \alpha \in \Lambda\}$ *is an arbitrary collection of closed sets, then* $V = \cap_{\alpha\in\Lambda} F_\alpha$ *is a closed set.*

Proof.

Sketch: If F_k is closed, then F_k^C is open. Note $F^C = \cap_{k=1}^{n} F_k^C$ by the De Morgan's Law. So F^C is open by Theorem 3.2. Thus F is closed by Theorem 3.7.

Similarly, $V^C = \cup_{\alpha\in\Lambda} F_\alpha^C$ is open and hence V is closed. □

Closure

Definition 3.9

Let $A \subset \mathbb{R}$ and let A' be the set of all limit points of A. We call the union $A \cup A'$ the closure of A, denoted by $\overline{A}$.

Example 3

1. *If $A = (a, b)$, then $\overline{A} = [a, b]$.*
2. *If $A = \{1/n : n \in \mathbb{N}\}$, then $\overline{A} = A \cup \{0\}$.*
3. *$\overline{\mathbb{Q}} = \mathbb{R}$, and also $\overline{\mathbb{I}} = \mathbb{R}$.*

Remark

1. *It is clear from the definition that $\overline{A} = A$ if A is closed.*
2. *If $A \subset B$, then $\overline{A} \subset \overline{B}$.In particular, if B is closed and $A \subset B$, then $\overline{A} \subset B$.*

Closure-continued

Theorem 3.10

Let $A \subset \mathbb{R}$. Then $\overline{A}$ is a closed set. Moreover, $\overline{A}$ is the smallest closed set containing A.

Proof.

It suffices to show that $\overline{A}^C$ is open. For this we only need to show that $\forall\ p \in \overline{A}^C$, $\exists\ \varepsilon > 0$ such that $V_\varepsilon(p) \subset \overline{A}^C$.

If $p \in \overline{A}^C$, then $p \notin A$ and p is not a limit point of A. So $\exists\ \varepsilon > 0$ such that $V_\varepsilon(p) \cap A = \emptyset$. We claim further $V_\varepsilon(p) \cap A' = \emptyset$.

Proof by contridiction: Let $q \in V_\varepsilon(p) \cap A'$. Then there is $\varepsilon' > 0$ such that $V_{\varepsilon'}(q) \subset V_\varepsilon(p)$ since $V_\varepsilon(p)$ is open. So $V_{\varepsilon'}(q) \cap A = \emptyset$ since $V_\varepsilon(p) \cap A = \emptyset$. Then $q \notin A'$, contradicting to $q \in V_\varepsilon(p) \cap A'$.

Therefore, the claim is true. Then $V_\varepsilon(p) \cap \overline{A} = \emptyset$. So $V_\varepsilon(p) \subset \overline{A}^C$.

$\overline{A}$ is the smallest since $\overline{A} \subset B$ for every closed set B containing A. $\square$

Introduction to compact sets

Definition 3.11

A set $K \subset \mathbb{R}$ is called compact if every sequence in K has a subsequence that converges to a point in K.

We are not going to discuss anything deep concerning compact sets. We only introduce the following characterization theorem.

Theorem 3.12

a set $K \subset \mathbb{R}$ is compact if and only if it is bounded and closed.

Example 4

1. *Every bounded closed interval $[a, b]$ is compact.*
2. *Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\overline{A} = A \cup \{0\}$ is compact.*
3. *In general, if $B \subset \mathbb{R}$ is bounded, then $\overline{B}$ is compact.*

Nested compact sets

Theorem 3.13 (Nested Compact Set Property)

Let $\{K_n : n \in \mathbb{N}\}$ *be a sequence of non-enpty compact sets such that* $K_n \supset K_{n+1}$ *for all* $n \in \mathbb{N}$*. Then* $K = \cap_{n=1}^{\infty} K_n \neq \emptyset$ *and is compact.*

Proof.

To show $K \neq \emptyset$, $\forall n \in \mathbb{N}$, we take a point $a_n \in K_n$. Then, from the nested assumption, $(a_n) \subset K_1$. Moreover, $(a_k)_{k=n}^{\infty} \subset K_n$.
Since K_1 is compact, There is a subsequence $(a_{n_i})_{i=1}^{\infty}$ that converges to some $p \in K_1$. We claim $p \in K_m$ for all $m \in \mathbb{N}$.
Given $m \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $n_j \geq m$. Then $(a_{n_i})_{i=j}^{\infty} \subset (a_k)_{k=m}^{\infty} \subset K_m$. We have $p = \lim_{i \to \infty} a_{n_i} \in K_m$ since K_m is closed. The claim is proved.
Therefore, $p \in \cap_{m=1}^{\infty} K_m = K$. So $K \neq \emptyset$.
K is clearly closed and bounded (as the intersection of compact sets). So K is compact. $\square$