Module 1: Some Basic Concepts

1.1 Review of Set Theory and Notation (*Weiss* §*1.2*)

▶ Some Notation and Definitions:

$\in$ ← element of.

- A **set** is a collection of *distinct* elements (numbers, individuals, objects, ...).

  $A = \{1, 2, 4\}$ $1 \in A$. $3 \notin A$

- If A is a set and x is an element of A, we write: $x \in A$. →
- If y is **not** an element of A, we write $y \notin A$. →
- $\emptyset$ denotes the **empty set** and U the **Universal set**, i.e. the set of all elements of interest.

  $\emptyset$, $\mathcal{U}$

- The **complement** of A, denoted A^c is

$$A^c = \{x \in U : x \notin A\}.$$

  $A^c : \{x \in \mathcal{U} : x \notin A\}$

  ↑ all of those elements that exist, such that they do not belongs to A.

  This is sometimes also denoted A' or $\bar{A}$.

- The set A is a subset of B, denoted $A \subset B$, if all elements of A also belong to B.
  In other words, if $x \in A \implies x \in B$, then $A \subset B$.

  $A = \{1, 2, 4\}$, $B = \{1, 2, 3, 4\} \Rightarrow A \subset B$

  all elements of A are also in B.

  if $x \in A \Rightarrow x \in B$

- Two sets A and B are equal ($A = B$)

$$\iff A \subset B \text{ and } B \subset A.$$

- The **union** of A and B is

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\},$$

  i.e. it is the set of all elements belonging to *at least one* of the two sets.
  As a result: $A \cup \emptyset = A$ and $A \cup A^c = U$.

- The **intersection** of A and B is

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\} = AB,$$

  i.e. it is the set of all elements belonging to *both* sets.
  As a result: $A \cap \emptyset = \emptyset$.

- The sets A and B are **disjoint** or **mutually exclusive** if $A \cap B = AB = \emptyset$
  i.e. if they have no elements in common.

Union

$A \cup B = \{ x \in \mathcal{U} : x \in A \text{ or } x \in B \}$

"OR" → belongs to at least one of A or B.

Note: $A \cup \emptyset = A$, $A \cup A^c = \mathcal{U}$

→ union makes resulting set bigger

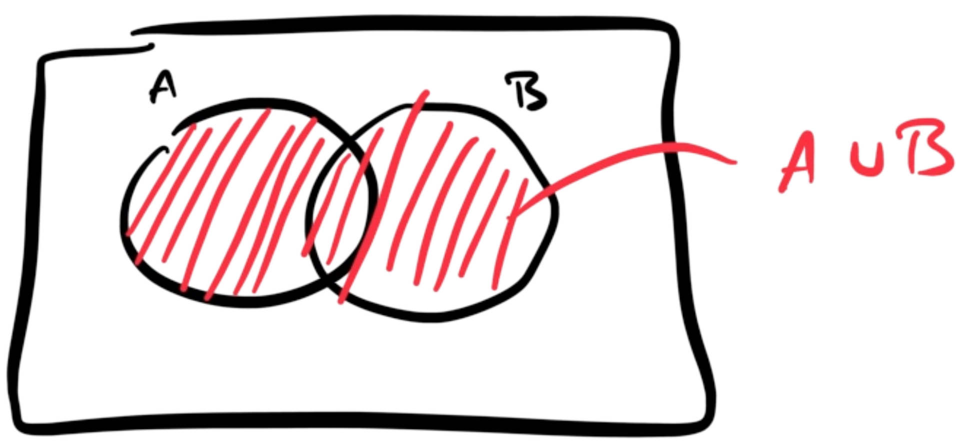
→ behaves with an analogy to addition.

$A = \{1, 2, 4\}$, $B = \{1, 2, 3, 4\}$, $D = \{5\}$

$\Rightarrow A \cup B = \{1, 2, 3, 4\} = B$

$A \cup D = \{1, 2, 4, 5\}$

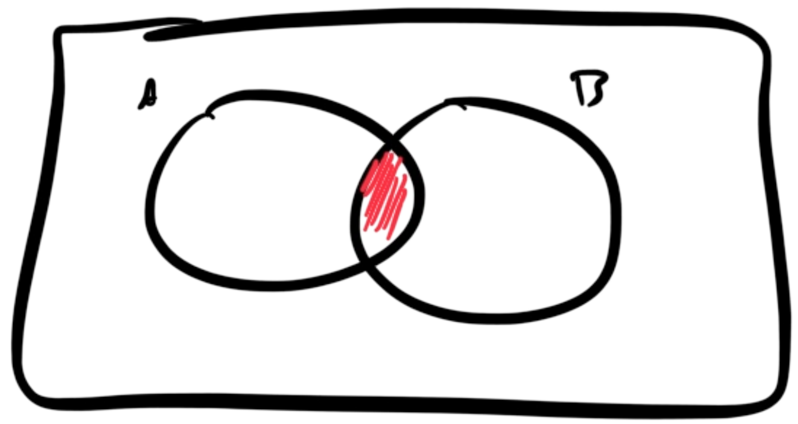
in general:



Intersection

$A \cap B = \{ x \in \mathcal{U} : x \in A \text{ and } x \in B \}$

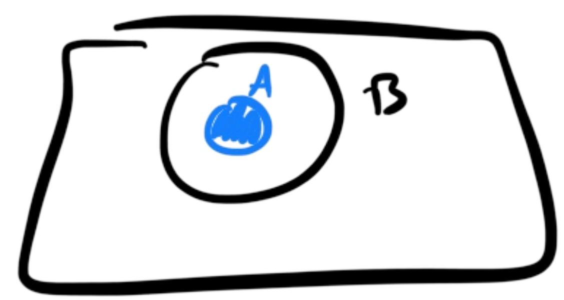
→ elements belonging to both sets, at the same time.



$A \cap \emptyset = \emptyset$

$A = \{1, 2, 4\}$, $B = \{1, 2, 3, 4\}$ $\quad A \subset B$

$A \cap B = A = \{1, 2, 4\}$



Disjoint

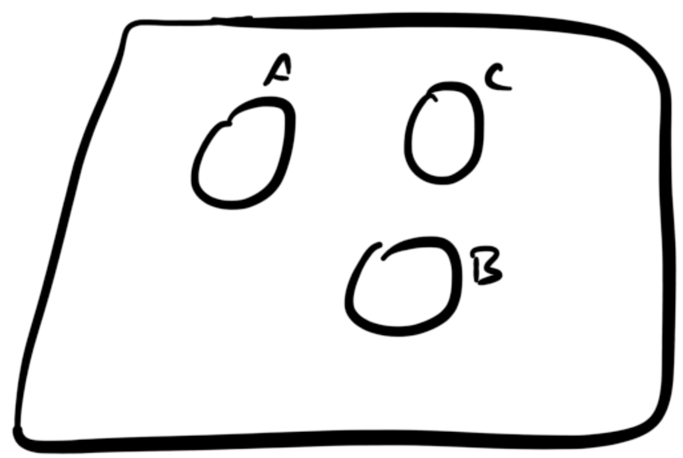
$A \cap B = \phi$

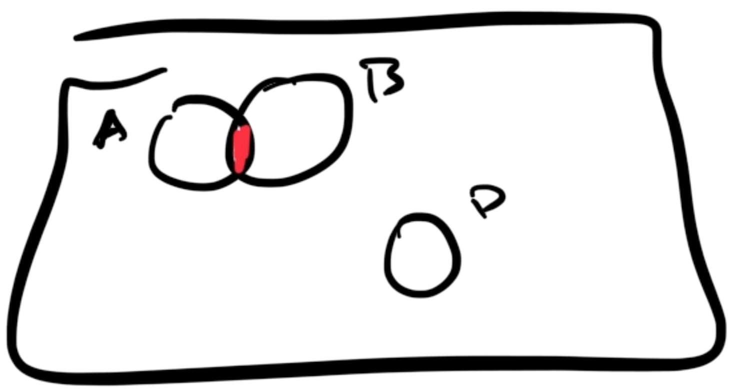
→ no elements in common.

Generally, A, B, C. They are "pairwise disjoint"

if

$$A \cap B = \phi$$
$$A \cap C = \phi$$
$$B \cap C = \phi$$





$$\left.\begin{aligned} A \cap D &= \phi \\ B \cap D &= \phi \end{aligned}\right\} \text{disjoint}$$

$A \cap B \neq \phi$ → not disjoint.

A, B, D ⇒ not pairwise disjoint.

$A \cap B = AB$

Proposition 1.1. Commutative Laws.

Let A and B be subsets of U.

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$ or $AB = BA$

order in which we apply union / intersection for A, B doesn't matter

Proposition 1.2. Associative Laws.

Let A, B and C be subsets of U.

1. $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
2. $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$ or $A \cap (BC) = (AB) \cap C = ABC$

Proposition 1.3. Distributive Laws.

Let A, B and C be subsets of U.

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

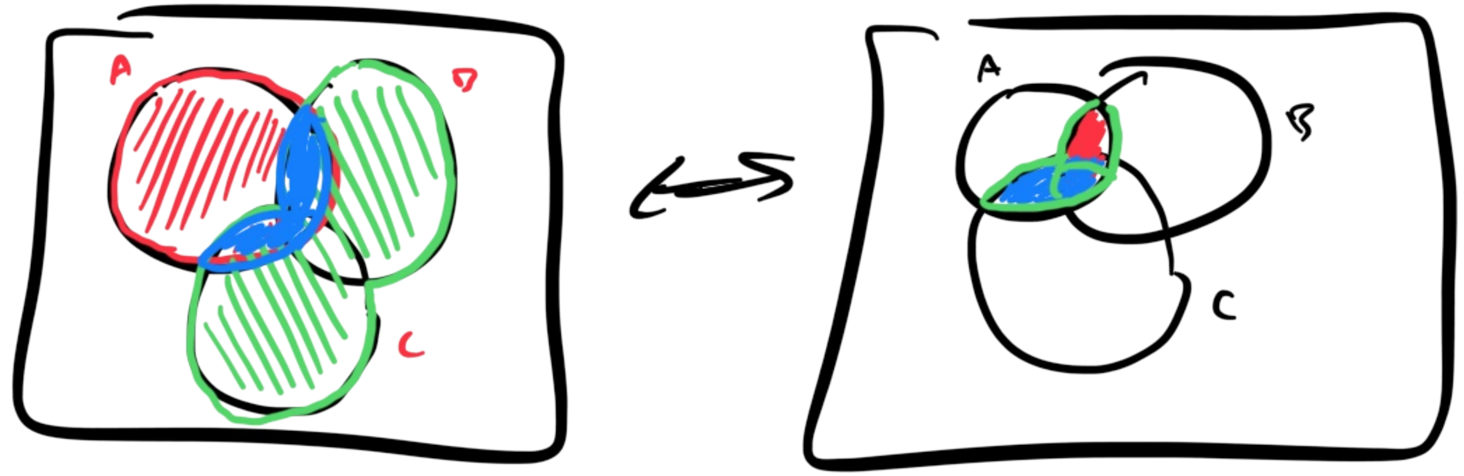
Proposition 1.4. De Morgan's Laws.

Let A and B be subsets of U.

1. $(A \cup B)^c = A^c \cap B^c$ or $(A \cup B)^c = A^c B^c$
2. $(A \cap B)^c = A^c \cup B^c$ or $(AB)^c = A^c \cup B^c$

1. complement of the union is intersection of complements.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



► Some more notation and definitions:

Let $A_1, A_2, \ldots, A_n$ be subsets of U.

- Then, we write

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \left\{x \in U : x \in A_i \text{ for some } i = 1, 2 \ldots, n\right\},$$

$\bigcup_{i=1}^{n} A_i$

i.e. the set of x's belonging to *at least one* of $A_1, A_2, \ldots, A_n$.

- We also write

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \left\{x \in U : x \in A_i \text{ for all } i = 1, 2 \ldots, n\right\},$$

$\bigcap_{i=1}^{n} A_i$

i.e. the set of x's belonging to *all* of $A_1, A_2, \ldots, A_n$.

De Morgan's Laws Basic Intuition:

$$(\underline{A \cap B})^c = A^c \cup B^c$$

"it is not true that both A and B occur"

is the same as

"At least one of A or B does not occur"

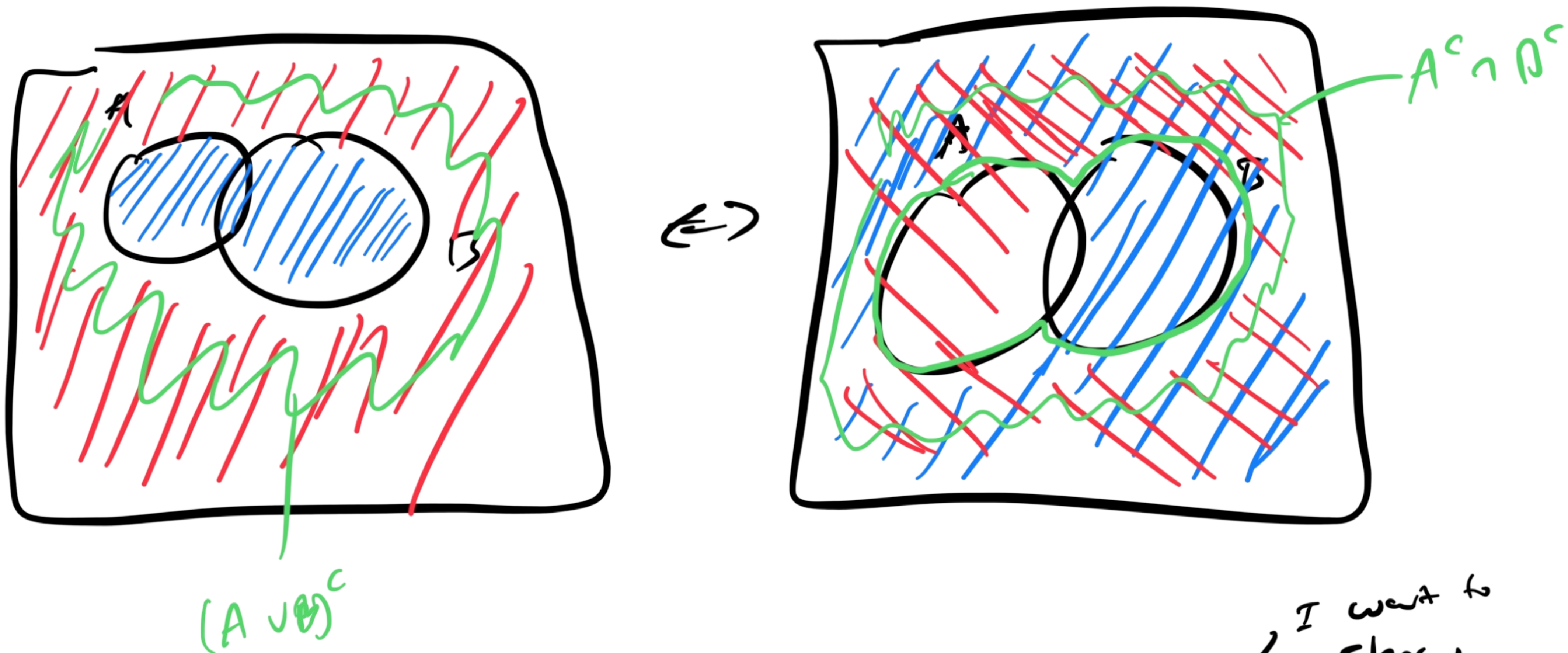
and similarly,

$$(A \cup B)^c = A^c \cap B^c$$

"it is not true that A or B occurs"

$\Rightarrow$ "Both A and B do not occur"

1st $(A \cup B)^c = A^c \cap B^c$



Prove 2nd BML: $(A \cap B)^c = A^c \cup B^c$ ← I want to show

Start from 1st law: $(A \cup B)^c = A^c \cap B^c$ ← use this.

$$A \to A^c, \quad B \to B^c$$

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c$$

$$: (A^c \cup B^c)^c = A \cap B$$

apply ^{c} to both sides.

$$\Rightarrow : A^c \cup B^c = (A \cap B)^c$$

↪ 2nd BML.

- The sets $A_1, A_2, \ldots, A_n$ are said to be **pairwise disjoint** or **mutually exclusive** if

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j.$$

Proposition 1.5. Generalized Distributive Laws. $n=3$

Let A and $B_1, B_2, \ldots$ be subsets of U. $\rightarrow (A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_n)$

1. $A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} AB_i \qquad \text{for all } n \geq 2$
2. $A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i) \qquad \text{for all } n \geq 2$

Proposition 1.6. Generalized De Morgan's Laws.

Let $A_1, A_2, \ldots$ be subsets of U.

1. $\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c \qquad \text{for all } n \geq 2$
2. $\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c \qquad \text{for all } n \geq 2$

▶ One last definition:

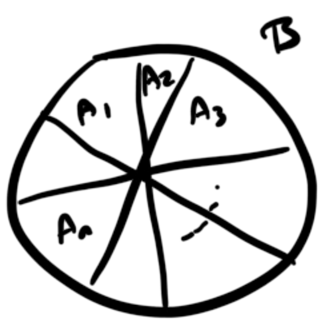
- The sets $A_1, A_2, \ldots, A_n$ form a **partition** of B if
  1. $A_i A_j = \emptyset$ for all $i \neq j$, (i.e. they are mutually exclusive),
  2. $\bigcup_{i=1}^{n} A_i = B$.

  Essentially, each A_i is a distinct "piece" of B.
- A and A^c always form a partition of U!

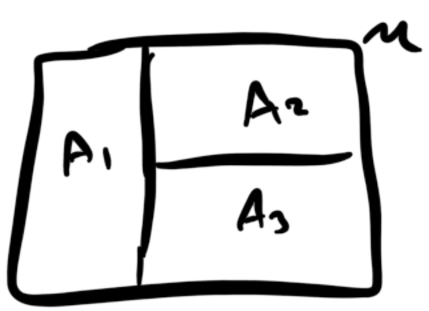
Essentially, each A_i is a distinct "piece" of B

→ non-overlapping and all pieces together ($\bigcup_{i=1}^{n}$) form the whole set B.

→ physical restriction



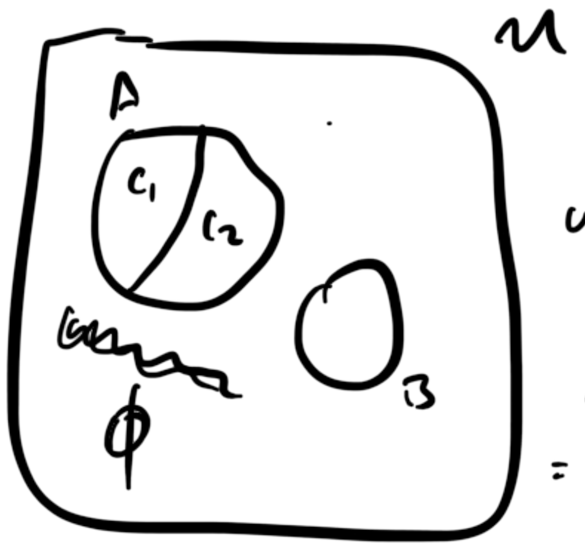
$A_1 \cap A_4 = \emptyset$ any $i \neq j$, $A_i \cap A_j = \emptyset$



$\left.\begin{array}{l} A_1 \cap A_2 = \emptyset \\ A_2 \cap A_3 = \emptyset \\ A_1 \cap A_3 = \emptyset \end{array}\right\}!$ $\quad A_1 \cup A_2 \cup A_3 = U. \}?$

$\Rightarrow$ partition for U.

A_1, A_2, A_3 forms a partition for U.



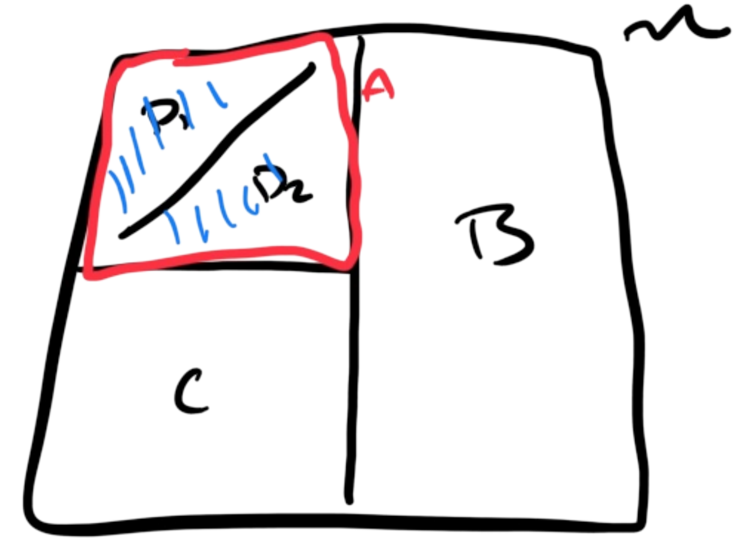
do A, C_1, C_2, B form a partition for $\mathcal{U}$?

$A \cup C_1 \cup C_2 \cup B = \mathcal{U}$

$= A \cup C_1 \cup C_2 \cup B \cup \phi = \mathcal{U}$

problem is not with the union but with the intersections.

$A \cap C_1 = C_1$
$A \cap C_2 = C_2$
not ϕ.

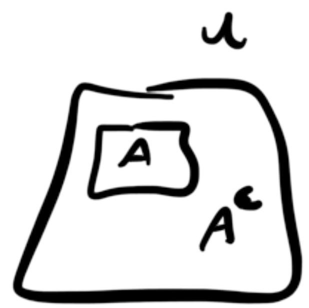


Do A, D_1, D_2, C, B form a partition for $\mathcal{U}$?

A and A^c form a partition always

1. $A \cap A^c = \phi$ yes!

2. $A \cup A^c = \mathcal{U}$ yes!



1.2 Sample Space and Events (*Weiss §2.1*)

- A few definitions:

  - A **random experiment** is an action whose outcome cannot be predicted with certainty.

    Think of:
    – flipping a coin,
    – rolling a die,
    – arrival of patients at the ER in a hospital,
    – playing the lotto 6/49,
    – etc.

  - The set of all possible outcomes for a random experiment is called the **sample space**. It is denoted Ω ($\mathcal{S}$ is also used sometimes).

    To denote an individual outcome, we use ω.

  - An **event** is a subset of the sample space.

    Events are denoted by capital letters (they are sets).

  - A **simple event** is an event consisting of only one outcome.

  - An event E is said to occur when the outcome of the random experiment is an element of E.

Example 1.1. Flipping a card from a regular 52 card deck (*cf.* Figure 2.2 from Weiss)

Describe the sample space Ω for this random experiment, and consider the following three events:

D: the card is a diamond,
H: the card is a "high" card (i.e. 10, J, Q, K or A),
F: the card is a "face" card (i.e. J, Q or K).

Also, find at least three partitions of Ω.

Example 1.2. Rolling two dice (*cf.* Figure 2.1 from Weiss)

Describe the sample space of this random experiment.

Are the following events mutually exclusive?

A: the sum is at most 4,
B: at least one result of 4 or more,
C: the difference between the two dice is 3,
D: the sum of the dice is 7.

→ randomness has to deal with long run patterns.

→ flip a coin

T T H T T H H

Ω

→ all possible outcomes.

$\Omega = \{H, T\}$.

↑ ↑ outcomes to your experiment.

ω

ω

roll a 6-sided die

$\Omega : \{1, 2, 3, 4, 5, 6\}$ → $E : \{6\}$

$E = \{2, 4, 6\}$

→ events are sets
→ should be possible.
→ made up of "possible outcomes"

event that I roll a even number

$\Omega : \{1, 2, 3, 4, 5, 6\}$

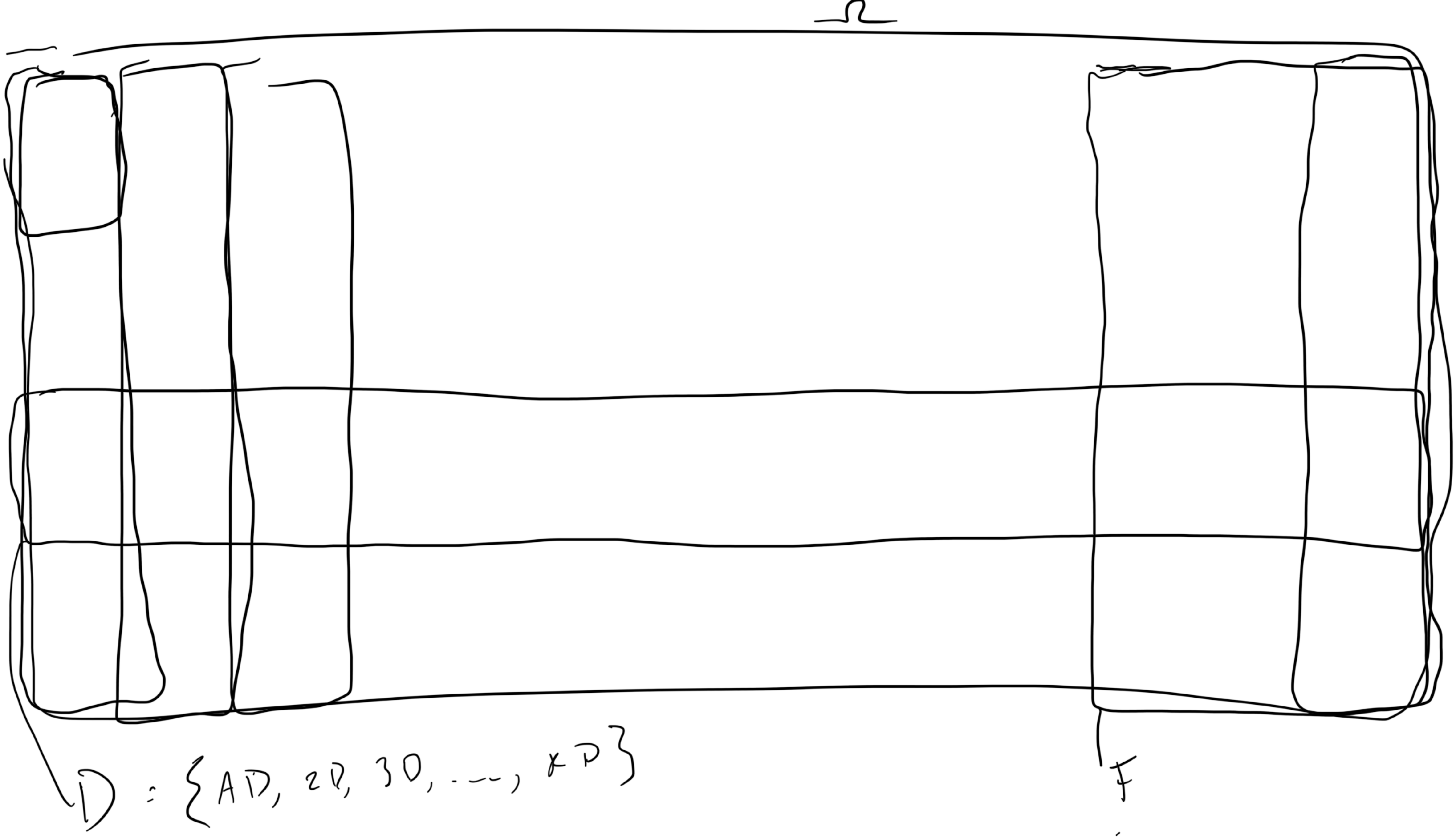
A: roll even # : $\{2, 4, 6\}$ $A \subset \Omega$

B: roll odd # = $\{1, 3, 5\}$ $B \subset \Omega$

C: roll a # greater than 4 = $\{5, 6\}$

Say you roll a 6

- A has occured
- C has occured
- $A \cap C$ has occured $A \cap C = \{6\}$



Ω
D = {1D, 2D, 3D, ..., kD}
F

1.3 Axioms of Probability (*Weiss* §2.2, 2.3)

$\mathbb{P}(E)$:

Definition 1.7. Kolmogorov's Axioms

Let Ω be the sample space of some random experiment.

A function $\mathbb{P}$ defined on the events of Ω is called a **probability measure** if it satisfies the following three properties.

1. The **nonnegativity axiom**:
$$\mathbb{P}(E) \geq 0 \qquad \text{for all events } E \subset \Omega.$$
2. The **certainty axiom**:
$$\mathbb{P}(\Omega) = 1,$$
i.e. one of the possible outcomes will occur.
3. The **additivity axiom**: (infinite additivity.)
If $A_1, A_2, \ldots$ are mutually exclusive subsets of Ω, then
disjoint
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$
Ω.

• for disjoint groups we can add probabilities.

Proposition 1.8. Two consequences of Kolmogorov's axioms

1. For the empty set, we have $\mathbb{P}(\emptyset) = 0$.
2. Finite additivity:
Let $A_1, A_2, \ldots$ be mutually exclusive events of Ω. Then
$A_1, \ldots, A_n$
$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) \qquad \text{for all } n \in \mathbb{N}.$$

Prop. 1.8.1 $P(\emptyset) = 0$ ← show.

→ probability that nothing occurs under random experiment is 0.
→ you are guaranteed to see a result from Ω.

introduce a sequence of events.

$$\left.\begin{aligned} A_1 &= \Omega \\ A_2 &= \emptyset \\ A_3 &= \emptyset \\ A_4 &= \emptyset \\ &\vdots \end{aligned}\right\} \quad A_1, A_2, A_3, \dots \text{ form a partition of } \Omega.$$

- union is Ω
- A_i's are disjoint.

Note: $$\underbrace{1 = P(\Omega)}_{\text{Ax. 2}} = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \underbrace{\sum_{i=1}^{\infty} P(A_i)}_{\text{Ax. 3}}$$

$$\Rightarrow \quad 1 = P(A_1) + P(A_2) + P(A_3) + \dots$$

$$\Rightarrow \quad 1 = P(\underset{\Omega}{A_1}) + \sum_{i=2}^{\infty} P(A_i)$$

$$\Rightarrow \quad 1 = 1 + \underbrace{\sum_{i=2}^{\infty} P(\emptyset)}_{=0}$$

$$\rightarrow \quad \sum_{i=2}^{\infty} P(\emptyset) = 0$$

- the sum is $= 0$ and by Ax 1 we can't have neg. probs.

$$\Rightarrow \underline{P(\emptyset) = 0}$$

by using Ax 1, Ax 2, Ax 3, and also a smart partition of Ω.

Prop. 1.8.2

Show: if $A_1, \dots, A_n$ are mutually disjoint events of Ω.

then $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad \text{for } n \geq 1.$$

Let $$\left.\begin{aligned} B_1 &= A_1 & B_{n+1} &= \emptyset \\ B_2 &= A_2 & B_{n+2} &= \emptyset \\ B_3 &= A_3 & &\vdots \\ &\vdots & & \\ B_n &= A_n & & \end{aligned}\right\}$$ B_j's are disjoint
→ because A_i's are disjoint and additional $B_{n+1} \dots$ are $\emptyset$.

→ now, we can use infinite additivity.

Ax 3. $$P\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} P(B_j)$$

because B_j are disjoint ✓

$$\bigcup_{j=1}^{\infty} B_j = \underbrace{\left(\bigcup_{j=1}^{n} B_j\right)}_{\text{finite part}} \cup \underbrace{\left(\bigcup_{j=n+1}^{\infty} B_j\right)}_{\text{infinite part}}^{\emptyset}$$

$$= \bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j$$

$P(B_j) = P(A_j)$ $\quad i = 1, \dots, n$
$P(B_j) = 0$ $\quad n+1, n+2, \dots$ $\quad (P(\emptyset) = 0)$

$$\sum_{j=1}^{\infty} P(B_j) = \sum_{i=1}^{n} P(B_j) + \sum_{j=n+1}^{\infty} P(B_j)$$
$$= \sum_{j=1}^{n} P(A_j) + 0$$

$$\underbrace{P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} P(A_j)}_{\text{finite additivity.}}$$

∎

Proposition 1.9. Finite or infinite countable sample space

For any event $E \subset \Omega$,

$$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\{\omega\}),$$

i.e. $\mathbb{P}(E)$ is equal to the sum of the probabilities of each outcome in E. Also,

$$\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 1,$$

i.e. the probabilities of all the outcomes in Ω sum to one.

$\Omega = \{1, 2, 3, 4, 5, 6\}$, ω

$P(\{\omega\}) = \frac{1}{6}$

$P(E$ = even number$)$

$E = \{2, 4, 6\}$

$P(\{2\} \cup \{4\} \cup \{6\}) = \Sigma\ \{2\} \ldots$

$P(E) = P(\{2\}) + P(\{4\}) + P(\{6\})$

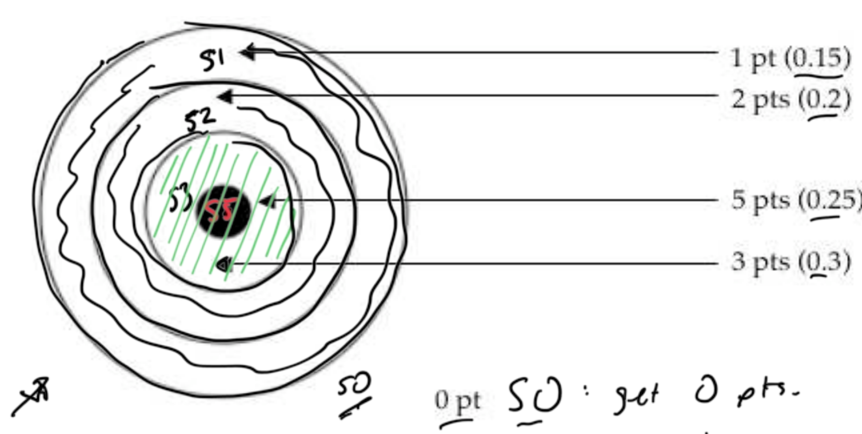
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$

► Conclusion:

We can assign probabilities to each outcome and calculate the probabilities of events by adding the probabilities of the individual outcomes!

Example 1.3. Archery

An archer fires at a target.



S1: get one point. $P(S1) = 0.15$.

S2: get 2 pts. $P(S2) = 0.2$.

S5: get 5 pts. $P(S5) = 0.25$.

S3: get 3 pts. $P(S3) = 0.3$.

S0: get 0 pts. $P(S0) = 0.1$

$= 1$

Important: $(S0 \cup S1 \cup S2 \cup S3 \cup S5) = \Omega$ & $S_i \cap S_j = \emptyset$ $i \neq j$ ⇒ partition!

Define the following events:

- A: the arrow lands inside the inner circle (i.e. the shot scores 3 pts or more),
- M: the arrow misses the target completely. $M: S_0$

Find $\mathbb{P}(A)$ and $\mathbb{P}(M)$.

$A: S3 \cup S5$

$P(A) = P(S3 \cup S5) = P(S3) + P(S5)$

$= 0.3 + 0.25 = 0.55$

$P(S0) =$

$1 = P(\Omega) = P(S0 \cup S1 \cup S2 \cup S3 \cup S5)$

$1 = P(S0) + P(S1) + P(S2) + P(S3) + P(S5)$

$\Rightarrow P(S0) = 1 - P(S1) - \ldots - P(S5) = 0.1$

Proposition 1.10. Equal-likelihood model or classical probability model

Let Ω be a finite sample space with equally likely outcomes. Then, for any event $E \subset \Omega$, we have

$$\mathbb{P}(E) = \frac{N(E)}{N(\Omega)} = \frac{\text{nb. of outcomes in } E}{\text{nb. of outcomes in } \Omega}.$$

► Note:

The previous result implies that in the case of a finite sample space,

$$0 \leq \mathbb{P}(E) \leq 1 \quad \text{for any } E \subset \Omega, \qquad \text{and} \qquad \mathbb{P}(\emptyset) = 0.$$

Example 1.4. (*cf.* Example 1.2 on page 4)

When rolling two dice,

$$\Omega = \Big\{(x, y) : x, y \in \{1, 2, \ldots, 6\}\Big\}.$$

Assume that the $N(\Omega) = 36$ outcomes are equally likely. Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(C)$ and $\mathbb{P}(D)$.

Example 1.5. Flipping a coin 3 times

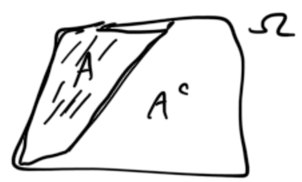
What is the probability of seeing a sequence of at least 2 consecutive identical results?

1.4 General Properties of Probability (*Weiss* §2.4)

Proposition 1.11. Complementation Rule

Let $A \subset \Omega$. Then,

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$



Proposition 1.12. Domination Principle

If $A \subset B \subset \Omega$ are any events, then

$$\mathbb{P}(A) \leq \mathbb{P}(B).$$

► Two consequences of the domination principle:

- For any $E \subset \Omega$, we have that $\mathbb{P}(E) \leq 1$.
- For any $A, B \subset \Omega$,

$$\mathbb{P}(AB) \leq \mathbb{P}(A) \leq \mathbb{P}(A \cup B) \qquad \text{and} \qquad \mathbb{P}(AB) \leq \mathbb{P}(B) \leq \mathbb{P}(A \cup B).$$

Example 1.6. (*cf.* Example 1.4)

Argue that $D \subset B$ and verify that $\mathbb{P}(D) \leq \mathbb{P}(B)$. Also, find the probability that the sum of the dice is not 7.

Ex 1.5

· assuming you have a fair coin.

· Flip a coin 3 times.

P("at least consecutive 2 identical results")

: $P(A) = \dfrac{N(A)}{N(\Omega)}$

Ω: "all possible outcomes of a sequence of 3 flips"

: {HHH, HHT, HTH, THH, TTT, TTH, THT, HTT} ⇒ $N(\Omega) = 8$

$2^3 = 8$

A = {HHH, HHT, THH, TTT, TTH, HTT}

$N(A) = 6$.

$\Rightarrow P(A) = \dfrac{N(A)}{N(\Omega)} = \dfrac{6}{8} = 0.75$

Proposition 1.11 (complement).

Let $A \subset \Omega$. Show $P(A^c) \stackrel{!}{=} 1 - P(A)$

we know A and A^c always form a partition for Ω.

$$A \cap A^c = \emptyset$$
$$A \cup A^c = \Omega$$

$$1 = P(\Omega) \quad \text{ax. 2.}$$
$$= P(A \cup A^c)$$

disjoint.

$$1 = P(A) + P(A^c) \quad \text{(finite additivity).}$$

$$\Rightarrow P(A^c) = 1 - P(A) \quad \square$$

Prop. 1.12 (Domination)

If $A \subset B$ are events in Ω.
then $P(A) \stackrel{.}{\leq} P(B)$

A and $A^c \cap B$ forms a partition for B.

$$P(B) = P(A \cup (A^c \cap B))$$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0} \quad \text{Ax. 1.}$$

$$\Rightarrow P(B) \geq P(A) \quad \square$$

since adding a nonnegative number to $P(A)$ can not make the result smaller than $P(A)$
$\Rightarrow$ establish the lower bound.

Proposition 1.13. General Addition Rule for two events

Let $A, B \subset \Omega$. Then,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(AB).$$

► Proving the above property, we have obtained the following result that is often useful:

$$\mathbb{P}(A^c B) = \mathbb{P}(B) - \mathbb{P}(AB).$$

Example 1.7. Defective CD's

Of all CD's produced by a manufacturer,

- 3% have a surface defect,
- 8% have a balance defect,
- 91% are defect-free.

Find the probability a randomly selected CD has:

- at least one type of defect,
- both defects,
- only a surface defect,
- only a balance defect.

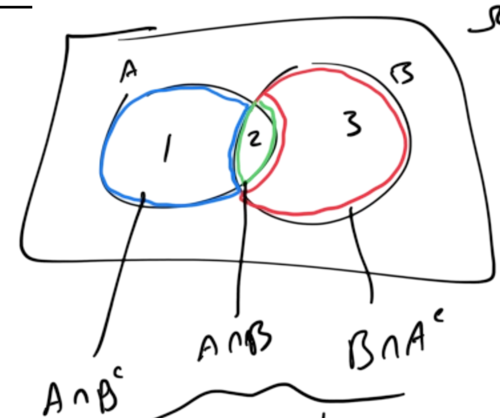
Proposition 1.14. Inclusion-exclusion principle

Let $A_1, A_2, \ldots$ be events (and subsets of Ω). Then, for all $n \geq 2$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum\sum_{i_1<i_2} \mathbb{P}(A_{i_1}A_{i_2}) + \sum\sum\sum_{i_1<i_2<i_3} \mathbb{P}(A_{i_1}A_{i_2}A_{i_3}) + \cdots$$
$$+ (-1)^{k+1} \sum\sum\sum_{i_1<i_2<\cdots<i_k} \mathbb{P}(A_{i_1}A_{i_2}\cdots A_{i_k}) + \cdots + (-1)^{n+1}\mathbb{P}(A_1A_2\cdots A_n).$$

In other words, the probability of the union is equal to

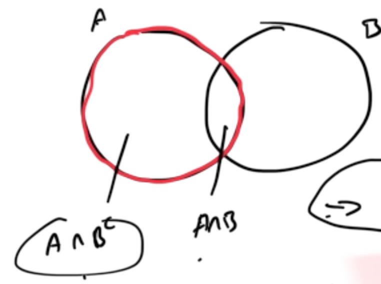
- the sum of the probabilities of each event,
- $-$ the sum of the probabilities of all 2×2 (pairwise) intersections,
- $+$ the sum of the probabilities of the 3×3 intersections,
- etc.



$P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

disjoint and union is $(A\cup B)$ ⇒ partition.

$P(A\cup B) = P(A\cap B^c) + P(A\cap B) + P(B\cap A^c)$



$(A\cap B^c) \cap (A\cap B) = \emptyset$.

$(A\cap B^c) \cup (A\cap B) = A$.

$\Rightarrow P(A) = P(A\cap B^c) + P(A\cap B)$

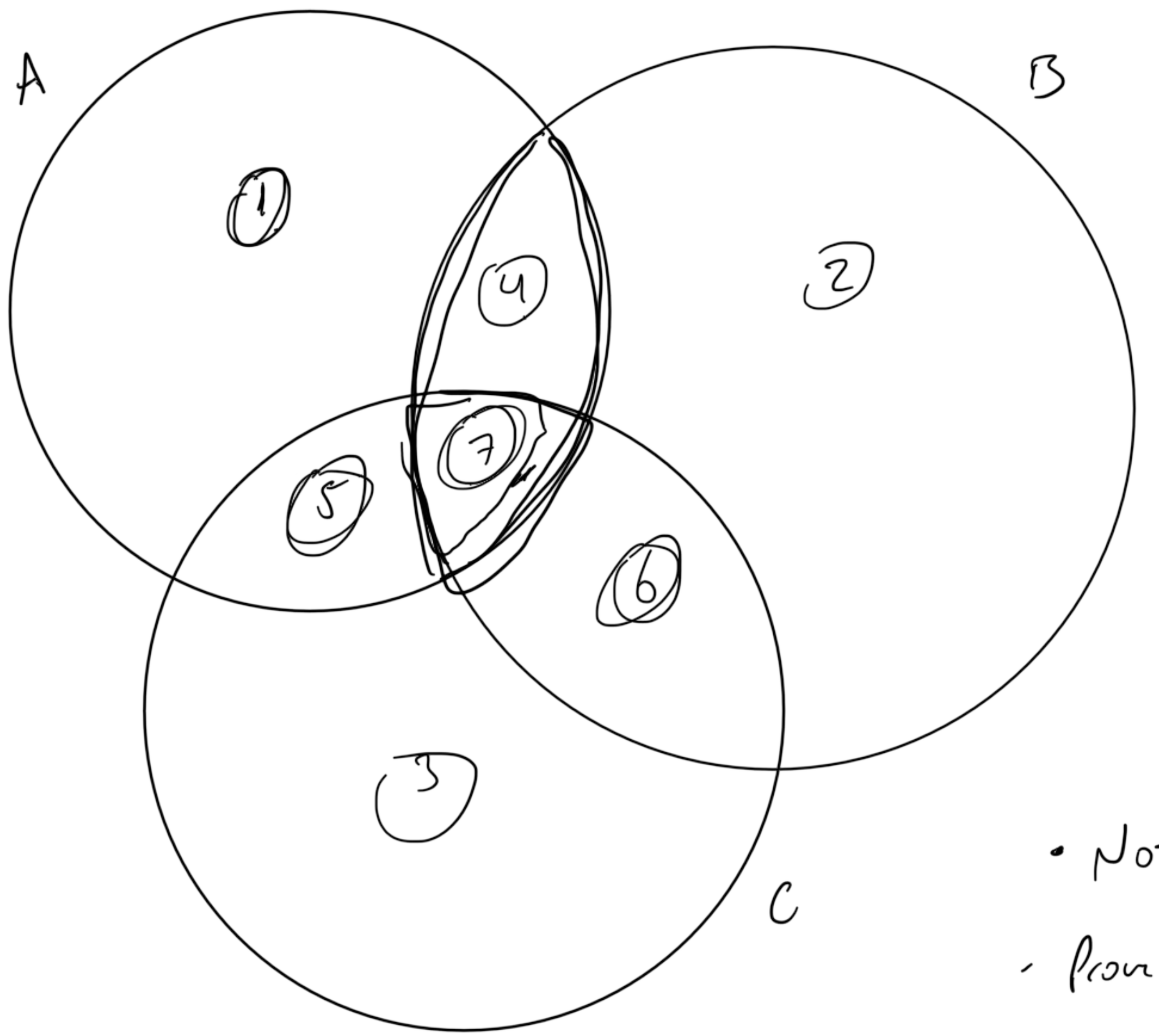
$\Rightarrow P(A\cap B^c) = P(A) - P(A\cap B)$

Similarly: $P(B) = P(B\cap A^c) + P(A\cap B)$

$\Rightarrow P(B\cap A^c) = P(B) - P(A\cap B)$

$= P(A) - \cancel{P(A\cap B)} + \cancel{P(A\cap B)} + P(B) - P(A\cap B)$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$ □



$$P(A \cup B \cup C)$$

$$= \overset{1\,4\,5\,7}{P(A)} + \overset{2\,4\,6\,7}{P(B)} + \overset{3\,5\,6\,7}{P(C)}$$

$$- \overset{4\,7}{P(A \cap B)} - \overset{5\,7}{P(A \cap C)} - \overset{6\,7}{P(B \cap C)}$$

$$+ \overset{7}{P(A \cap B \cap C)}$$

- Not a proof.
- Prove the result for $n=3$
  - → general addition
  - → distributivity.

Example 1.8. (*cf.* Example 1.4 on page 7)

We have defined:

A: the sum is at most 4,
D: the sum of the dice is 7,
E: there is at least one result of 3.

Calculate $\mathbb{P}(A \cup D \cup E)$.

Example 1.9. Wrong numbers?

We are told that, in a study of 1000 subscribers to a certain magazine, the following data was reported:

P:	Professionals,	$N(P) = 312$,	$N(MP) = 86$,	$N(MPC) = 25$.
M:	Married,	$N(M) = 470$,	$N(PC) = 42$,	
C:	College graduates,	$N(C) = 525$,	$N(MC) = 147$,	

How can we tell that there is an error in the reported data?