Name: ______________________________ Student ID: ____________________

Math 2740 – Fall 2025
Sample final examination – Variant 1
2 hours

Instructions

- This examination has **8 exercises**.
- Show all your work. Correct answers without justification will receive little or no credit.
- You may use the back of pages if needed.
- No electronic devices (including calculators) are permitted.
- The exam is out of 100 points.

Exercise 1. [**Definitions and Theorems – 15 points**]

State the definition or theorem for each of the following. Be precise and complete.

1. [**3 pts**] Define the *singular values* of a matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$.
2. [**4 pts**] State the *Singular value decomposition (SVD) theorem.*
3. [**4 pts**] Given a matrix $A \in \mathcal{M}_n$, define its eigenpairs.
4. [**4 pts**] State the *Least squares theorem.*

Exercise 2. **[Linear Least Squares – 15 points]**

Consider the over-determined system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

1. **[5 pts]** Set up the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ by computing $A^T A$ and $A^T\mathbf{b}$.
2. **[5 pts]** Solve the normal equation to find the least squares solution $\tilde{\mathbf{x}}$.
3. **[5 pts]** Compute the residual $\mathbf{b} - A\tilde{\mathbf{x}}$ and its norm $\|\mathbf{b} - A\tilde{\mathbf{x}}\|$.

Exercise 3. [Singular Value Decomposition – 20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$$

1. [**6 pts**] Compute $A^T A$ and find its eigenvalues.
2. [**4 pts**] Determine the singular values of A.
3. [**5 pts**] Find the right singular vectors (eigenvectors of $A^T A$) and construct the matrix V.
4. [**5 pts**] Construct the matrices Σ and U to complete the SVD $A = U\Sigma V^T$. (You may verify your answer by computing the product.)

Exercise 4. **[Principal Component Analysis – 15 points]**

Consider a dataset with the following data matrix (each row is an observation):

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

1. **[4 pts]** Compute the mean of each variable and center the data matrix to obtain $\tilde{X}$.
2. **[6 pts]** Compute the sample covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$ where $n = 3$.
3. **[5 pts]** Find the eigenvalues of the covariance matrix. Which eigenvalue corresponds to the first principal component?

Exercise 5. [Proof – 15 points]

Let $A \in \mathcal{M}_{mn}(\mathbb{R})$. Prove that for any nonzero eigenvalue λ of $A^T A$, we have $\lambda > 0$.

Hint: Use the definition of eigenvalue and properties of the inner product.

Exercise 6. **[Graph Measures I – 10 points]**
Consider the simple undirected graph G on vertices $V = \{1, 2, 3, 4, 5\}$ with edge set

$$E = \{\{1, 2\},\ \{1, 3\},\ \{2, 3\},\ \{2, 4\},\ \{3, 4\},\ \{4, 5\}\}.$$

1. [**4 pts**] Compute the degree $\deg(i)$ of each vertex and give the degree sequence in nonincreasing order.
2. [**3 pts**] Compute the density of G, defined as $\delta(G) = \dfrac{2|E|}{|V|(|V|-1)}$.
3. [**3 pts**] For each vertex with $\deg(i) \geq 2$, compute its local clustering coefficient $C_i = \dfrac{2e_i}{\deg(i)(\deg(i)-1)}$, where e_i is the number of edges between neighbors of i. State the average (mean) clustering coefficient of G.

Exercise 7. **[Graph Measures II – 10 points]**

For the same graph G as in Exercise 6:

1. **[4 pts]** Compute the graph diameter (the maximum shortest-path distance between any two distinct vertices) and the average shortest-path length $\ell(G)$ over all unordered vertex pairs.
2. **[3 pts]** Compute the (normalized) degree centrality of each vertex, defined as $C_D(i) = \deg(i)/(n-1)$ where $n = |V|$.
3. **[3 pts]** Compute the closeness centrality of each vertex, defined for connected graphs as $C_C(i) = \dfrac{n-1}{\sum_{j \neq i} d(i,j)}$, where $d(i,j)$ is the shortest-path distance.

Exercise 8. **[Regular Markov Chains – 10 points]**

Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/3 \\ 1/2 & 0 & 1/4 & 1/3 \\ 0 & 1/2 & 1/4 & 1/3 \end{pmatrix}$$

1. **[3 pts]** Verify that this Markov chain is regular.
2. **[7 pts]** Find the limiting distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^T$ by solving the system $P\boldsymbol{\pi} = \boldsymbol{\pi}$ with the constraint $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

END OF EXAMINATION