# MATH 2080 2025F Assignment 2 Solutions

1. [2] Suppose that $a_n \to 2$. Show that $a_n^2 \to 4$ using an $\varepsilon$-$N$ argument. (Hint: The sequence $(a_n)$ must be bounded—why?)

   **Solution.**

   Since a convergent sequence is bounded, there is $M > 0$ such that

   $$|a_n| \leq M \quad \text{for all } n.$$

   Given $\varepsilon > 0$, since $a_n \to 2$, there is $N \in \mathbb{N}$ such that

   $$|a_n - 2| < \frac{\varepsilon}{M+2} \quad \text{whenever } n \geq N.$$

   Then, for $n \geq N$ we have

   $$|a_n^2 - 4| = |(a_n + 2)(a_n - 2)| \leq (M+2)|a_n - 2| < \varepsilon.$$

   By definition, $a_n^2 \to 4$. $\square$

2. [3] Let $(x_n)$ be defined by

   $$x_1 = 2, \quad x_{n+1} = 2 - \frac{1}{x_n} \text{ for } n \geq 1.$$

   Suppose that it is known that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. Show that $(x_n)$ converges and find its limit. (Hint: Use MCT and induction.)

   **Solution.**

   Use induction to show that $(x_n)$ is decreasing, i.e.

   $$x_n \geq x_{n+1} \quad \text{for all } n. \tag{1}$$

   For $n = 1$: $x_1 = 2$ and $x_2 = 2 - \frac{1}{x_1} = \frac{3}{2}$. So $x_1 \geq x_2$. (1) holds.

   Assume that (1) holds for $n = k$, i.e. $x_k \geq x_{k+1}$.

   Then for $n = k+1$ we show $x_{k+2} \geq x_{k+1}$, which means (1) holds also for $n = k+1$.

   From the assumption and the given condition $1 \leq x_n \leq 2$, we have

   $$x_k \geq x_{k+1} \geq 1 > 0.$$

   So

   $$\frac{1}{x_k} \leq \frac{1}{x_{k+1}}.$$

Then

$$x_{k+2} = 2 - \frac{1}{x_{k+1}} \le 2 - \frac{1}{x_k} = x_{k+1}.$$

Thus, (1) holds for $n = k+1$. By induction, (1) holds for all $n$. So $(x_n)$ is a bounded inctreasing sequence.

By the MCT, $(x_n)$ converges.

Let $x_n \to x$. Then $x_{n+1} \to x$ and $x \ge 1$. From the identity $x_{n+1} = 2 - \frac{1}{x_n}$, letting $n \to \infty$, we obtain

$$x = 2 - \frac{1}{x}.$$

Solve for $x$. We get $x = 1$. So the limit of $(x_n)$ is 1. □

3. [3] Determine whether the series

$$\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$$

converges or diverges. Justify your answer.

**Solution.**

It diverges.

Consider the $n$th partial sum

$$s_n = \sum_{k=1}^{n} \ln \frac{k}{k+1}.$$

Use the identity $\ln \frac{a}{b} = \ln a - \ln b$ for $a, b > 0$. We have

$$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln n - \ln(n+1)) = -\ln(n+1)$$

after cancellation. This leads to,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n+1) = -\infty.$$

Therefore, $(s_n)$ diverges.

By definition, $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ diverges. □