University of Manitoba
Department of Statistics

STAT 2400

Introduction to Probability I

Supplementary Problems: Module 1

A. Problems from the textbook

From Chapter 1: 18, 19, 50, 52, 53, 55

From Chapter 2: 4, 8, 11, 13, 20, 27, 62, 66, 67, 68, 100, 101, 106, 107, 109, 110

B. Other problems

Question 1.1:

A bag of marbles contains 10 marbles: 4 red, 3 green, 2 blue and 1 yellow. Consider the simple experiment that consists of taking a marble at random from the bag, then replacing it in the bag and drawing a second marble, each time noting the colour of the selected marble. Determine the sample space Ω_1 of this first experiment.

Is it reasonable here to think that the outcomes $\omega \in \Omega_1$ could be equally likely?

What if the second marble is instead drawn without first replacing the first marble? What is the sample space Ω_2 of this new experiment?

Question 1.2:

Assume two events A and B are such that $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 3/8$. Show that

$$\mathbb{P}(A \cup B) \geq 3/4 \qquad \text{and} \qquad 1/8 \leq \mathbb{P}(A \cap B) \leq 3/8.$$

Question 1.3:

Suppose a random experiment is such that $\Omega = \mathbb{N}$.

(A) Explain why the outcomes $k \in \mathbb{N}$ cannot be assigned an equal probability.

(B) Is it possible to assign the probabilities so that $\mathbb{P}(\{k\}) > 0$ for all $k \in \mathbb{N}$? If so, find one such assignment of the probabilities. If not, formally show it.

Question 1.4:

Let A and B be subsets of Ω. The *relative complement* (sometimes also called the *set difference*) of B with respect to A, denoted $A \setminus B$, is defined as the set of all elements belonging to A that do not belong to B.

(A) Express $A \setminus B$ in terms of intersections and complements.

(B) Express B^c as a relative complement.

(C) Show that $(A \setminus B)^c = A^c \cup B$.

The *symmetric difference* of A and B, denoted $A \,\Delta\, B$, is the set of all elements that belong to either A or B, but not both.

(D) Express $A \,\Delta\, B$ in terms of unions and relative complements.

(E) Express $A \,\Delta\, B$ in terms of unions and intersections.

(F) Show the following properties of symmetric differences:

- $A \,\Delta\, A = \emptyset$
- $A \,\Delta\, \emptyset = A$
- $A \,\Delta\, \Omega = A^c$
- Associativity: $A \,\Delta\, (B \,\Delta\, C) = (A \,\Delta\, B) \,\Delta\, C$
- Distributivity of the intersection: $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Question 1.5:

Assume the sets A and $B_1, B_2, \ldots, B_n$ are subsets of U. Assume also $B_1, B_2, \ldots, B_n$ altogether form a partition of U. Now, define

$$E_i = A \cap B_i \quad \text{for } i = 1, 2, \ldots, n \qquad \text{and } E_{n+1} = A^c.$$

Show that $E_1, E_2, \ldots, E_{n+1}$ also form a partition of U.

C. CHALLENGES

Question 1.6:

Prove the second generalized distributive law: if A and $B_1, B_2, \ldots$ are subsets of U, then

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i) \qquad \text{for all } n \geq 2.$$

Try (1) by induction and (2) using the generalized De Morgan Laws.

Question 1.7:

Prove the second generalized De Morgan law: if $A_1, A_2, \ldots, A_n$ are subsets of U, then, for all $n \geq 2$, we have that

$$\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c.$$

Do this (1) by induction and (2) using the first generalized De Morgan law.

Question 1.8:

Assume the sets $A_1, A_2, \ldots, A_n$ are subsets (not necessarily disjoint) of U such that

$$\bigcup_{i=1}^{n} A_i = U.$$

Also, define the sets $B_1 = A_1$ and $B_i = A_i \cap \left(\bigcup_{k=1}^{i-1} A_k \right)^c$ for $i = 2, \ldots, n$.

Show that $B_1, B_2, \ldots B_n$ form a partition of U.

Question 1.9:

Assume that $A_1, A_2, \ldots$ are subsets of U.

Then, Boole's inequality (named after George Boole, 1815-1864) states that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) \qquad \text{for all } n \geq 1.$$

(A) Prove Boole's inequality by using mathematical induction.

(B) Prove Boole's inequality without using mathematical induction, but rather by making use of the result of Question 1.8.

Bonferronni's inequality (named after Carlo Emilio Bonferronni, 1892-1960), on the other hand, states that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i \right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - (n-1) \qquad \text{for all } n \geq 1.$$

(C) Prove Bonferronni's inequality by using mathematical induction.

(D) Derive Bonferronni's inequality by using Boole's inequality and the De Morgan laws.

D. Answers

Textbook Problems:

From Chapter 1:

1.18. a) $[0, 1/4]$ and $[0, 1]$ b) $\{0\}$ and $[0, 1]$
1.50. a) 0, 1, 4 b) $[0, 4)$
1.52. A

From Chapter 2:

2.4. a) $\Omega = \{1, 2, 3, 4, 5, 6\}$

b) $A = \{2, 4, 6\}$, $B = \{4, 5, 6\}$
$C = \{1, 2\}$ and $D = \{3\}$

c) $A^c = \{1, 3, 5\}$, $AB = \{4, 6\}$,
$B \cup C = \{1, 2, 4, 5, 6\}$

d) B and C

e) B, C and D are mutually exclusive

f) $\{5\}$: result is five
$\{1, 3, 5\}$: result is odd
$\{1, 2, 3, 4\}$: result is four or less

2.8. a) $\Omega = \{(i, j) : 1 \leq i \leq 6, 0 \leq j \leq i\}$
b) $\big\{(i, j) : 1 \leq i \leq 6, 0 \leq j \leq i \text{ and } j \in \{0, 2, 4, 6\}\big\}$

2.20. a) Probabilities are nonnegative and sum to 1 b) 0.07
c) 0.12 d) 0.698

2.62. a) 0.729 b) 0.729

2.66. a) No b) 1/12

2.68. a) 0.85 b) 0.84
c) 0.01 d) 0.05

2.100. a) 1/6 b) 1/3
c) 2/3

2.106. $A \setminus B$: A occurs but B does not

2.110. 91/216

Other Problems from the List:

1.1. a) $\Omega_1 = \{(R, R); (R, G); (R, B); (R, Y); (G, R), (G, G); (G, B); (G, Y);$
$(B, R); (B, G); (B, B); (B, Y); (Y, R); (Y, G); (Y, B); (Y, Y)\}$
b) $\Omega_2 = \Omega_1 \backslash \{(Y, Y)\}$

1.3. b) Yes: let $\mathbb{P}(\{k\}) = 1/2^k$ for $k = 1, 2, \ldots$

1.4. a) $A \cap B^c$ b) $\Omega \setminus B$ d) $(A \setminus B) \cup (B \setminus A)$ e) $(A \cap B^c) \cup (A^c \cap B)$