University of Manitoba

Department of Mathematics

MATH 2740

Mathematics of Data Science

Exercises

Julien Arino

Department of Mathematics

julien.arino@umanitoba.ca

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University of Manitoba

Winnipeg, Manitoba, Canada

Foreword

This booklet contains exercises for MATH 2740 (Mathematics of Data Science). Even though there will not be enough time in tutorials to cover all exercises, it is recommenced you try your hand on most of them. Doing so will help you develop the "muscle memory" needed to succeed in the course. Note that some exercises are computer coding related and will not be worked on during tutorials.

Solutions to the exercises are available. Some are "full solutions" with all the details, others just indicate what the solution is. In any case, solutions will be shown after the related tutorial. Indeed, it is best if you work on the exercises to the point where you potentially get stuck on something, then seek clarification during the tutorials, rather than see the answers. It is easy to trick yourself into believing you understand how to do something, but it is only by actually trying that you will work out how to do it.

I will add more questions throughout the term, even in older Exercise groups. Also, I will progressively "unveil" solutions. Therefore, in order to make it easy for you to check if you are using the latest version, I indicate at the bottom of this page when this booklet was generated.

Vectors are typically written as $\mathbf{v} = (v_1, \ldots, v_n)$. If an orientation is needed, then by default, it is assumed that vectors are column vectors. So for instance, the inner product $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle$ is $\mathbf{u}^T\mathbf{v}$. If omitting direction leads to confusions, then a column vector is written as $\mathbf{v} = (v_1, \ldots, v_n)^T$.

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Contents

EXERCISE GROUP 1

Review of first-year linear algebra

Exercises in this chapter are from MATH 1210/1220/1300. It is important that you be comfortable doing them. If you struggle with some of them, you should return to your notes from these courses and ensure you refresh your memory, as we will be building on this material throughout the term.

1.1 Linear systems and basic matrix arithmetic

Exercise 1

Consider the following system of linear equations:

$$\begin{aligned} -3x_2 - 6x_3 + 4x_4 &= 9 \\ -x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ -2x_1 - 3x_2 + 3x_4 &= -1 \\ x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \end{aligned}$$

(**a**) Write the system as an augmented matrix and find its reduced row-echelon form (RREF).

(**b**) Solve the system.

Solution of Exercise 1

(**a**) First, we write the system as an augmented matrix:

$$\left(\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right)$$

Applying row operations to find the RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(b) From the RREF, we can read off the solution. The pivot columns are 1, 2 and 4, so x_1, x_2 and x_4 are basic variables, while x_3 is a free variable. Let $x_3 = t$, with $t \in \mathbb{R}$.
From the RREF:

$$x_1 - 3x_3 = 5 \Rightarrow x_1 = 5 + 3t \tag{1.1}$$
$$x_2 + 2x_3 = -3 \Rightarrow x_2 = -3 - 2t \tag{1.2}$$
$$x_4 = 0 \tag{1.3}$$
$$x_3 = t \tag{1.4}$$

Therefore, the general solution is: $x_1 = 5 + 3t$, $x_2 = -3 - 2t$, $x_3 = t$, $x_4 = 0$, where $t \in \mathbb{R}$.

Exercise 2

Consider a system of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots& \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m.
\end{aligned}$$

(a) If $b_1 = b_2 = b_3 = \ldots = b_m = 0$, then the system is *homogeneous*. Can you find a solution to a homogeneous system without solving it? If so, what is it?

(b) How many solutions can a homogenous system of linear equations have?

Solution of Exercise 2

(a) Yes, $x_1 = x_2 = \cdots = x_n = 0$ is a solution.It is called the *trivial solution*.

(b) Homogeneous systems have either one or infinitely many solutions since they always have the trivial solution.

Exercise 3

Consider the matrices:

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 6 \\ -7 & 8 & 9 \end{pmatrix}.$$

(a) Compute AB

(b) Compute BA.

(c) Find the matrix X such that $4X - 2A = 3B$.

Solution of Exercise 3

(a) Computing AB:

$$AB = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 6 \\ -7 & 8 & 9 \end{pmatrix}$$

Row 1: $(3)(-1) + (2)(-4) + (-1)(-7) = -3 - 8 + 7 = -4$, $(3)(2) + (2)(5) + (-1)(8) = 6 + 10 - 8 = 8$, $(3)(3) + (2)(6) + (-1)(9) = 9 + 12 - 9 = 12$
Row 2: $(0)(-1)+(2)(-4)+(0)(-7) = -8$, $(0)(2)+(2)(5)+(0)(8) = 10$, $(0)(3)+(2)(6)+(0)(9) = 12$
Row 3: All entries are 0 since the third row of A is zero.
Therefore,

$$AB = \begin{pmatrix} -4 & 8 & 12 \\ -8 & 10 & 12 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Computing BA:

$$BA = \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 6 \\ -7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Row 1: $(-1)(3)+(2)(0)+(3)(0) = -3$, $(-1)(2)+(2)(2)+(3)(0) = 2$, $(-1)(-1)+(2)(0)+(3)(0) = 1$
So

$$BA = \begin{pmatrix} -3 & 2 & 1 \\ -12 & 2 & 4 \\ -21 & 2 & 7 \end{pmatrix}.$$

(c) To find X such that $4X - 2A = 3B$: $4X = 3B + 2A$, so $X = \frac{1}{4}(3B + 2A)$

$$3B = 3\begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 6 \\ -7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 9 \\ -12 & 15 & 18 \\ -21 & 24 & 27 \end{pmatrix}$$

$$2A = 2\begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3B + 2A = \begin{pmatrix} 3 & 10 & 7 \\ -12 & 19 & 18 \\ -21 & 24 & 27 \end{pmatrix}$$

Therefore,

$$X = \begin{pmatrix} \frac{3}{4} & \frac{5}{2} & \frac{7}{4} \\ -3 & \frac{19}{4} & \frac{9}{2} \\ \frac{-21}{4} & 6 & \frac{27}{4} \end{pmatrix}.$$

Exercise 4

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix. The *trace of* A, denoted $tr(A)$, is the sum of its diagonal entries. In other words,

$$tr(A) = a_{11} + a_{22} + a_{33} + \ldots + a_{nn}.$$

Find the traces of the matrices AB, BA, $2A^2 - B^2$ and X defined in Exercise 3.

Solution of Exercise 4

$tr(AB) = 6$, $tr(BA) = 6$, $tr(2A^2 - B^2) = -119$, $tr(X) = \dfrac{49}{4}$.

Exercise 5

An $n \times n$ matrix X is *invertible* if there is a matrix Y such that $XY = YX = \mathbb{I}_n$. In this case, we say that $X^{-1} = Y$. Suppose that A, B and $A + B$ are invertible $n \times n$ matrices.

(a) Simplify the expression $A(A^{-1} + B^{-1})B(A + B)^{-1}$.

(b) What does part (a) tell you about the matrix $A^{-1} + B^{-1}$.

Solution of Exercise 5

(a) Let us simplify $A\left(A^{-1} + B^{-1}\right)B(A + B)^{-1}$ step by step:

$$\begin{aligned} A\left(A^{-1} + B^{-1}\right)B(A + B)^{-1} &= AA^{-1}B(A + B)^{-1} + AB^{-1}B(A + B)^{-1} \\ &= \mathbb{I}_n B(A + B)^{-1} + A\mathbb{I}_n(A + B)^{-1} \\ &= B(A + B)^{-1} + A(A + B)^{-1} \\ &= (B + A)(A + B)^{-1} \\ &= (A + B)(A + B)^{-1} \\ &= \mathbb{I}_n \end{aligned}$$

(b) From part (a), we have $A\left(A^{-1} + B^{-1}\right)B(A + B)^{-1} = \mathbb{I}_n$. Right-multiplying both sides by $A + B$, this can be rewritten as $A(A^{-1} + B^{-1})B = A + B$. Multiplying both sides on the left by A^{-1} and on the right by B^{-1} gives $A^{-1}A(A^{-1} + B^{-1})BB^{-1} = A^{-1}(A + B)B^{-1}$, i.e., $(A^{-1} + B^{-1}) = A^{-1}(A + B)B^{-1}$. This means that $A^{-1} + B^{-1}$ is invertible and specifically,

$$(A^{-1} + B^{-1})^{-1} = B(A + B)^{-1}A.$$

We can verify this, by checking that the product of the matrix and its stated inverse gives the identity matrix:

$$\begin{aligned} (A^{-1} + B^{-1})B(A + B)^{-1}A &= A^{-1}(A + B)B^{-1}B(A + B)^{-1}A \\ &= A^{-1}(A + B)(A + B)^{-1}A \\ &= A^{-1}A \\ &= \mathbb{I}_n. \end{aligned}$$

Exercise 6

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 4 & 2 \\ 2 & 5 & -4 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{pmatrix}.$$

(a) Compute A^{-1}, B^{-1} and C^{-1}, if they exist.

(b) What is $(AB)^{-1}$?

Solution of Exercise 6

(a) $A^{-1} = \begin{pmatrix} 13 & -2 & -5 \\ -7 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} \frac{-13}{5} & \frac{3}{2} & \frac{-6}{5} \\ 2 & -1 & 1 \\ \frac{6}{5} & \frac{-1}{2} & \frac{2}{5} \end{pmatrix}$ and C is not invertible.

(b) $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} \frac{-467}{10} & \frac{67}{10} & \frac{187}{10} \\ 35 & -5 & -14 \\ \frac{199}{10} & \frac{-29}{10} & \frac{-79}{10} \end{pmatrix}$

Exercise 7

Compute the following matrix products (Note: These can be done by inspection).

(a) $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} -3 & 2 & 0 & 4 & -4 \\ 1 & -5 & 3 & 0 & 3 \\ -6 & 2 & 2 & 2 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Solution of Exercise 7

(a) $A = \begin{pmatrix} -15 & 10 & 0 & 20 & -20 \\ 2 & -10 & 6 & 0 & 6 \\ 18 & -6 & -6 & -6 & -6 \end{pmatrix}$

(b) $B = \begin{pmatrix} -24 & -10 & 12 \\ 3 & -10 & 0 \\ 60 & 20 & -16 \end{pmatrix}$

Exercise 8

Let

$$A = \begin{pmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 7 \end{pmatrix}.$$

Find all values of a, b and c for which A is symmetric.

Solution of Exercise 8

$a = 11$, $b = -9$ and $c = -13$

Exercise 9

A matrix X is *symmetric* if $X^T = X$. Suppose A is a symmetric $n \times n$ matrix and B is an $n \times m$ matrix. Show that the matrices $B^T B$, BB^T and $B^T AB$ are all symmetric.

Solution of Exercise 9
To show that each matrix is symmetric, we need to prove that each matrix equals its transpose.

For $B^T B$, we have $(B^T B)^T = B^T (B^T)^T = B^T B$. Therefore, $B^T B$ is symmetric.

For BB^T, we have $(BB^T)^T = (B^T)^T B^T = BB^T$. Therefore, BB^T is symmetric.

For $B^T AB$, we have $(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B$. Since A is symmetric, we have $A^T = A$, so $(B^T AB)^T = B^T AB$. Therefore, $B^T AB$ is symmetric.

Exercise 10

Compute the determinants of the following matrices.

(a) $A = \begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

Solution of Exercise 10

(a) $\det(A) = 0$

(b) $\det(B) = 24$

Exercise 11

Let
$$A = \begin{pmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{pmatrix}.$$

Find all values of λ for which $\det(A) = 0$.

Solution of Exercise 11
$\lambda = -2$, $\lambda = 3$ and $\lambda = 4$

Exercise 12

For each matrix A, compute $\det(A)$.

(a) $A = \begin{pmatrix} 0 & 0 & -2 & 4 & 3 \\ 0 & 3 & 1 & 1 & -8 \\ 0 & 0 & 0 & 0 & 4 \\ -1 & 3 & 9 & -7 & 2 \\ 0 & 0 & 0 & 6 & -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix}$

Solution of Exercise 12

(a) -144.

(b) -30.

Exercise 13

Find the inverse of

$$\begin{pmatrix} 1 & 0 \\ i & -i \end{pmatrix},$$

where $i = \sqrt{-1}$.

Solution of Exercise 13
The determinant is $-i$, so the inverse is

$$\frac{1}{i}\begin{pmatrix} -i & 0 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & \frac{1}{i} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -i \end{pmatrix}$$

Exercise 14

Find all values of k such that

$$A = \begin{pmatrix} k & -3 & 0 & 3 \\ 9 & 2 & 2 & 5 \\ 0 & 3 & 0 & 0 \\ 12 & 4 & 0 & k \end{pmatrix}$$

is nonsingular.

Solution of Exercise 14
We have

$$\begin{aligned} \begin{vmatrix} k & -3 & 0 & 3 \\ 9 & 2 & 2 & 5 \\ 0 & 3 & 0 & 0 \\ 12 & 4 & 0 & k \end{vmatrix} &= -2 \begin{vmatrix} k & -3 & 3 \\ 0 & 3 & 0 \\ 12 & 4 & k \end{vmatrix} \\ &= -2\left(3\begin{vmatrix} 0 & 3 \\ 12 & 4 \end{vmatrix} + k\begin{vmatrix} k & -3 \\ 0 & 3 \end{vmatrix}\right) \\ &= -2(3(-36) + k(3k)) = -6(-36 + k^2). \end{aligned}$$

So $|A| \neq 0$, i.e., A is invertible, if and only if $k^2 \neq 36$, i.e., $k \neq \pm 6$.

Exercise 15

Solve the following system by finding the inverse of the coefficient matrix.

$$\begin{array}{ccccccc} x & + & 3y & + & 2z & = & 5 \\ x & + & 3y & + & 3z & = & 7 \\ & & y & & & = & -2 \end{array}$$

Solution of Exercise 15

This system of equations is the same as the matrix equation

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \leftarrow R_2 - R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 3R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

Therefore, $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 & -3 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ and so the solution to the system is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 & -2 & -3 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}. \end{aligned}$$

Thus the solution to the system is

$$(x, y, z) = (7, -2, 2).$$

1.2 Eigenvalues and eigenvectors

Exercise 16

Let

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

(b) For each eigenvalue λ in (a), find the eigenvectors that correspond to λ.

Solution of Exercise 16

(a) -3, -2, -2 and 4

(b) $t(0,0,-1,1)$ (for -3), $s(-1,1,0,0) + t(0,0,-2,1)$ (for -2), and $t(1,1,0,0)$ (for 4)

Exercise 17

Let

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}.$$

(a) Find all of the eigenvalues of B.

(b) For each eigenvalue λ in (a), find an eigenvector $\mathbf{v}$ corresponding to λ.

(c) For any eigenvector $\mathbf{v}$ from part (b), what are $B^2\mathbf{v}$ and $B^3\mathbf{v}$ (Note: you shouldn't need to do any calculations)?

Solution of Exercise 17

(a) 1, 1 and 3

(b) (Lots of possible answers here) $(-1,2,0)$ (for 1) and $(1,0,2)$ (for 3)

(c) $\lambda^2\mathbf{v}$ and $\lambda^3\mathbf{v}$, where λ is the corresponding eigenvalue

Exercise 18

Let

$$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(a) Find the eigenvalues of C.

(b) For each eigenvalue λ in (a), find a unit (i.e. length one) eigenvector $\mathbf{v}$ corresponding to λ.

(c) Let λ_1 and λ_2 be the eigenvalues from part (a) and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the unit eigenvectors from (b). What is $\lambda_1\mathbf{v}_1\mathbf{v}_1^T + \lambda_2\mathbf{v}_2\mathbf{v}_2^T$?

Solution of Exercise 18

(a) 2 and 4

(b) $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (for 2) and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (for 4)

(c) $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Exercise 19

Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Find all eigenvalues of A.

(b) For each eigenvalue λ, find a corresponding eigenvector.

Solution of Exercise 19

(a) The characteristic equation is given by $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (-1)(1) = \lambda^2 + 1 = 0.$$

The eigenvalues are the roots of this equation, which are $\lambda_1 = i$ and $\lambda_2 = -i$.

(b) For $\lambda_1 = i$, we find the eigenvector by solving $(A - iI)\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equation $-ix - y = 0$, or $y = -ix$. A corresponding eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

For $\lambda_2 = -i$, we find the eigenvector by solving $(A + iI)\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equation $ix - y = 0$, or $y = ix$. A corresponding eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Exercise 20

Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{x} \in \mathbb{R}^n$. Show that $A\mathbf{x} = \mathbf{0}$ if and only if $A^T A\mathbf{x} = \mathbf{0}$.

Hint: One direction is straightforward. For the other direction, consider $\mathbf{x}^T A^T A\mathbf{x}$.

Solution of Exercise 20

This is an "if and only if" result and the hint talks about "direction", so we guess that we need to prove both $A\mathbf{x} = \mathbf{0} \implies A^T A\mathbf{x} = \mathbf{0}$ and $A^T A\mathbf{x} = \mathbf{0} \implies A\mathbf{x} = \mathbf{0}$.

- Let us start with the first implication. Assume that $A\mathbf{x} = \mathbf{0}$. Then $A^T A\mathbf{x} = A^T(A\mathbf{x}) = A^T(\mathbf{0}) = \mathbf{0}$.
- Let us now look at the reverse implication. Assume that $A^T A\mathbf{x} = \mathbf{0}$. Following the hint, let us left-multiply by $\mathbf{x}^T$: $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{0}$. The right hand side is the dot (or inner) product of $\mathbf{x}$ and $\mathbf{0}$, i.e., (the scalar) 0. On the left hand side, we have $\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = \|A\mathbf{x}\|^2$. (The first equality comes from properties of the transpose, the second from the definition of a norm.) We thus have $A^T A\mathbf{x} = \mathbf{0} \implies \mathbf{x}^T\mathbf{0} = 0$. Now recall that for any norm and any $\mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{y}\| = 0 \iff \mathbf{y} = \mathbf{0}$. Thus, $A^T A\mathbf{x} = \mathbf{0} \implies \|A\mathbf{x}\| = 0 \implies A\mathbf{x} = \mathbf{0}$.

Exercise 21

Suppose that A is an $m \times n$ matrix whose columns are linearly independent (so that $A^T A$ is invertible) and let C be an invertible $n \times n$ matrix. If $B = AC$, show that

$$A\left(A^T A\right)^{-1} A^T = B\left(B^T B\right)^{-1} B^T.$$

Hint: Note that A and B are not necessarily invertible (they never are if $m \neq n$).

Solution of Exercise 21
We have

$$\begin{aligned}
B\left(B^T B\right)^{-1} B^T &= AC\left((AC)^T AC\right)^{-1} (AC)^T \\
&= AC\left(C^T A^T AC\right)^{-1} C^T A^T \\
&= AC\left(C^T (A^T A) C\right)^{-1} C^T A^T \\
&= AC\left(C^{-1}(A^T A)^{-1}\left(C^T\right)^{-1}\right) C^T A^T \\
&= ACC^{-1}(A^T A)^{-1}\left(C^T\right)^{-1} C^T A^T \\
&= A\mathbb{I}(A^T A)^{-1}\mathbb{I}A^T \\
&= A\left(A^T A\right)^{-1} A^T.
\end{aligned}$$

Exercise 22

Suppose that $\mathbf{x}$ is an eigenvector of A that corresponds to the eigenvalue λ. Show that

$$\frac{\mathbf{x}^T A\mathbf{x}}{\|\mathbf{x}\|^2} = \lambda.$$

Hint: Use the definition of eigenvalues and eigenvectors.

Solution of Exercise 22
Since $(\lambda, \mathbf{x})$ is an eigenpair of A, we have $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Left multiply both sides of the eigenvalue-eigenvector equation by $\mathbf{x}^T$: $\mathbf{x}^T(A\mathbf{x}) = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda(\mathbf{x}^T\mathbf{x})$ since the product of a vector and a scalar commutes. In the right hand side, $\mathbf{x}^T\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ by definition. Furthermore, since it is an eigenvector, $\mathbf{x} \neq \mathbf{0}$, so $\|\mathbf{x}\| \neq 0$ and we can indeed divide both sides by $\|\mathbf{x}\|^2$. This gives the desired result.

Exercise 23

Let $A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -4 \end{pmatrix}$.

(a) Find all of the eigenvalues of A. For each eigenvalue, find a corresponding eigenvector.

(b) Pick an eigenvalue and a corresponding eigenvector and verify the result of Problem 22.

Solution of Exercise 23

(a) $\{1, (0, 1, 0)\}$; $\{2, (-1, 2, 2)\}$; and $\{3, (-1, 1, 1)\}$

EXERCISE GROUP 2

Linear algebra & Multivariable calculus

Here, we collect some exercises on second-year linear algebra and multivariable calculus. This is just enough that you will be able to tackle the problems in the course. Like with the previous Exercise group, you may want to return to this Exercise group later on in the course.

2.1 Vector spaces and subspaces

Exercise 24

Show that the set M_{22} of all real 2×2 matrices, with the standard operations of matrix addition and scalar multiplication, forms a vector space.

Solution of Exercise 24

To show that M_{22} is a vector space, we must verify the ten vector space axioms. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ be matrices in M_{22}, and let α and β be scalars in $\mathbb{R}$.

(1) **Closure under addition:** $A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$. Since this is a 2×2 matrix, it is in M_{22}.

(2) **Closure under scalar multiplication:** $\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$. This is a 2×2 matrix, so it is in M_{22}.

(3) **Associativity of addition:**

$$\begin{aligned} (A + B) + C &= \begin{pmatrix} (a_{11} + b_{11}) + c_{11} & (a_{12} + b_{12}) + c_{12} \\ (a_{21} + b_{21}) + c_{21} & (a_{22} + b_{22}) + c_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + (b_{11} + c_{11}) & a_{12} + (b_{12} + c_{12}) \\ a_{21} + (b_{21} + c_{21}) & a_{22} + (b_{22} + c_{22}) \end{pmatrix} \\ &= A + (B + C). \end{aligned}$$

(4) **Commutativity of addition:**

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{pmatrix} \\ &= B + A \end{aligned}$$

(5) **Zero vector:** The zero vector is the zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We have $A + O = A$.

(6) **Additive inverse:** The additive inverse of A is $-A = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$. We have $A + (-A) = O$.

(7) **Distributivity of scalar multiplication w.r.t. vector addition:**

$$\begin{aligned} \alpha(A + B) &= \alpha \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha(a_{11} + b_{11}) & \alpha(a_{12} + b_{12}) \\ \alpha(a_{21} + b_{21}) & \alpha(a_{22} + b_{22}) \end{pmatrix} \\ &= \begin{pmatrix} \alpha a_{11} + \alpha b_{11} & \alpha a_{12} + \alpha b_{12} \\ \alpha a_{21} + \alpha b_{21} & \alpha a_{22} + \alpha b_{22} \end{pmatrix} \\ &= \alpha A + \alpha B. \end{aligned}$$

(8) **Distributivity of scalar multiplication w.r.t. scalar addition:**

$$\begin{aligned} (\alpha + \beta)A &= (\alpha + \beta) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} (\alpha + \beta)a_{11} & (\alpha + \beta)a_{12} \\ (\alpha + \beta)a_{21} & (\alpha + \beta)a_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha a_{11} + \beta a_{11} & \alpha a_{12} + \beta a_{12} \\ \alpha a_{21} + \beta a_{21} & \alpha a_{22} + \beta a_{22} \end{pmatrix} \\ &= \alpha A + \beta A. \end{aligned}$$

(9) **Associativity of scalar multiplication:** $(\alpha\beta)A = \begin{pmatrix} (\alpha\beta)a_{11} & (\alpha\beta)a_{12} \\ (\alpha\beta)a_{21} & (\alpha\beta)a_{22} \end{pmatrix} = \alpha \begin{pmatrix} \beta a_{11} & \beta a_{12} \\ \beta a_{21} & \beta a_{22} \end{pmatrix} = \alpha(\beta A)$.

(10) **Scalar identity:** $1A = \begin{pmatrix} 1a_{11} & 1a_{12} \\ 1a_{21} & 1a_{22} \end{pmatrix} = A$.

Since all ten axioms hold, M_{22} is a vector space.

Exercise 25

Show that the set U_{22} of all real 2×2 upper triangular matrices is a subspace of the vector space M_{22} of all 2×2 matrices.

Solution of Exercise 25

To show that U_{22} is a subspace of M_{22}, we must verify three properties. Let $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ 0 & h \end{pmatrix}$ be two matrices in U_{22}, and let k be a scalar.

(1) **Contains the zero vector:** The zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is upper triangular, so it is in U_{22}.

(2) **Closure under addition:** $A+B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} e & f \\ 0 & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ 0 & d+h \end{pmatrix}$. The resulting matrix is also upper triangular, so it is in U_{22}.

(3) **Closure under scalar multiplication:** $kA = k\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ 0 & kd \end{pmatrix}$. The resulting matrix is also upper triangular, so it is in U_{22}.

Since all three properties hold, U_{22} is a subspace of M_{22}.

Exercise 26

Let P_2 be the set of all polynomials of degree 2 or less, with real coefficients. Show that P_2 is a vector space under the standard operations of polynomial addition and scalar multiplication.

Solution of Exercise 26

Let $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$ be two polynomials in P_2. Let k be a scalar. We check the vector space axioms.

(1) **Closure under addition:** $p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$. This is a polynomial of degree at most 2, so it is in P_2.

(2) **Commutativity of addition:** $p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) = (b_2 + a_2)x^2 + (b_1 + a_1)x + (b_0 + a_0) = q(x) + p(x)$.

(3) **Associativity of addition:** Let $r(x) = c_2x^2+c_1x+c_0$. Then $(p(x)+q(x))+r(x) = ((a_2+b_2)+c_2)x^2+\ldots = (a_2 + (b_2 + c_2))x^2 + \ldots = p(x) + (q(x) + r(x))$.

(4) **Zero vector:** The zero polynomial is $0(x) = 0x^2 + 0x + 0$. $p(x) + 0(x) = p(x)$.

(5) **Additive inverse:** The additive inverse of $p(x)$ is $-p(x) = -a_2x^2 - a_1x - a_0$. $p(x) + (-p(x)) = 0(x)$.

(6) **Closure under scalar multiplication:** $kp(x) = (ka_2)x^2 + (ka_1)x + (ka_0)$. This is a polynomial of degree at most 2, so it is in P_2.

(7) **Distributivity (vector):** $k(p(x)+q(x)) = k((a_2+b_2)x^2+\ldots) = (k(a_2+b_2))x^2+\ldots = (ka_2+kb_2)x^2+\ldots = kp(x) + kq(x)$.

(8) **Distributivity (scalar):** Let m be another scalar. $(k+m)p(x) = ((k+m)a_2)x^2+\ldots = (ka_2+ma_2)x^2+\ldots = kp(x) + mp(x)$.

(9) **Associativity of scalar multiplication:** $(km)p(x) = ((km)a_2)x^2 + \ldots = k(ma_2)x^2 + \ldots = k(mp(x))$.

(10) **Scalar identity:** $1p(x) = (1a_2)x^2 + (1a_1)x + (1a_0) = p(x)$.

Since all axioms are satisfied, P_2 is a vector space.

2.2 Linear independence, similarity and diagonalization

Exercise 27

Determine whether the following sets of vectors are linearly independent.

(a) The vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

(b) The vectors $u_1 = (1, -2, 1)$, $u_2 = (2, 1, -1)$, and $u_3 = (7, -4, 1)$ in $\mathbb{R}^3$.

(c) The polynomials $p_1(x) = 1$, $p_2(x) = x$, and $p_3(x) = x^2$ in P_2.

Solution of Exercise 27

(a) To check for linear independence, we set a linear combination of the vectors to zero: $c_1v_1+c_2v_2+c_3v_3 = 0$. This gives $c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$, which simplifies to $(c_1, c_2, c_3) = (0, 0, 0)$. Since the only solution is $c_1 = c_2 = c_3 = 0$, the vectors are linearly independent.

(b) We set up the equation $c_1u_1 + c_2u_2 + c_3u_3 = 0$: $c_1(1, -2, 1) + c_2(2, 1, -1) + c_3(7, -4, 1) = (0, 0, 0)$. This yields the system of linear equations: $c_1 + 2c_2 + 7c_3 = 0$, $-2c_1 + c_2 - 4c_3 = 0$, and $c_1 - c_2 + c_3 = 0$. From the third equation, $c_1 = c_2 - c_3$. Adding the second and third equations gives $-c_1 - 3c_3 = 0$, so $c_1 = -3c_3$. Substituting into the third equation: $-3c_3 - c_2 + c_3 = 0$, which gives $c_2 = -2c_3$. Let's check with the first equation: $(-3c_3) + 2(-2c_3) + 7c_3 = -3c_3 - 4c_3 + 7c_3 = 0$. So any choice of c_3 gives a solution. For instance, if $c_3 = 1$, then $c_1 = -3$ and $c_2 = -2$. Since there is a non-trivial solution, the vectors are linearly dependent.

(c) We set $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$ for all x. $c_1(1) + c_2(x) + c_3(x^2) = 0$. A polynomial is zero for all x if and only if all its coefficients are zero. Thus, $c_1 = 0, c_2 = 0, c_3 = 0$. The polynomials are linearly independent.

Exercise 28

Show that the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is similar to the diagonal matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

Solution of Exercise 28

Two matrices A and D are similar if there exists an invertible matrix P such that $D = P^{-1}AP$. If D is a diagonal matrix of eigenvalues, then the columns of P are the corresponding eigenvectors. First, we find the eigenvalues of A. Since A is upper triangular, the eigenvalues are its diagonal entries: $\lambda_1 = 1$ and $\lambda_2 = 3$.

Next, we find the corresponding eigenvectors. For $\lambda_1 = 1$: $(A - 1I)v = 0$ gives $\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This gives $2y = 0$, so $y = 0$. The eigenvector is of the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$. We can choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$: $(A - 3I)v = 0$ gives $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This gives $-2x + 2y = 0$, so $x = y$. The eigenvector is of the form $\begin{pmatrix} x \\ x \end{pmatrix}$. We can choose $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The matrix P is formed by the eigenvectors: $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The inverse is $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Now we check if $D = P^{-1}AP$:

$$P^{-1}AP = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 3\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix} = D.$$

Since we found such a matrix P, A is similar to D.

Exercise 29

Find a matrix P that diagonalizes $A = \begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}$.

Solution of Exercise 29
To diagonalize A, we need to find its eigenvalues and eigenvectors. The characteristic equation is $\Big|\ (\ \Big|A - \lambda I) = 0$. We compute $(3-\lambda)(-\lambda) - (-2)(1) = -3\lambda + \lambda^2 + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

Now, find the eigenvectors. For $\lambda_1 = 1$: $(A - 1I)v = 0$ gives $\begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. This gives $x - y = 0$, so $x = y$. An eigenvector is $v_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$.

For $\lambda_2 = 2$: $(A - 2I)v = 0$ gives $\begin{pmatrix}1 & -2\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$. This gives $x - 2y = 0$, so $x = 2y$. An eigenvector is $v_2 = \begin{pmatrix}2\\ 1\end{pmatrix}$.

The matrix P is constructed from the eigenvectors: $P = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$. This matrix P diagonalizes A, and we would find that $P^{-1}AP = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$.

2.3 Partial derivatives and gradients

Exercise 30

Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions:

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = xy$

(c) $f(x, y) = e^{x+y}$

(d) $f(x, y) = \ln\left(x^2 + y^2\right)$

Solution of Exercise 30

(a) $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$

(b) $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$

(c) $\frac{\partial f}{\partial x} = e^{x+y}$, $\frac{\partial f}{\partial y} = e^{x+y}$

(d) $\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$

Exercise 31

For the function $f(x, y) = x^3y + xy^3 - 2x^2y^2$:

(a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

(b) Find all second-order partial derivatives: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$

(c) Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Solution of Exercise 31

(a) $\frac{\partial f}{\partial x} = 3x^2y + y^3 - 4xy^2$, $\frac{\partial f}{\partial y} = x^3 + 3xy^2 - 4x^2y$

(b) $\frac{\partial^2 f}{\partial x^2} = 6xy - 4y^2$, $\frac{\partial^2 f}{\partial y^2} = 6xy - 4x^2$, $\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 3y^2 - 8xy$

(c) $\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 3y^2 - 8xy = \frac{\partial^2 f}{\partial x \partial y}$

Exercise 32

Find the gradient ∇f for each function:

(a) $f(x, y) = x^2 - y^2$

(b) $f(x, y, z) = xyz$

(c) $f(x, y, z) = x^2 + y^2 + z^2$

Solution of Exercise 32

(a) $\nabla f = (2x, -2y)$

(b) $\nabla f = (yz, xz, xy)$

(c) $\nabla f = (2x, 2y, 2z)$

2.4 Lagrange multipliers

Exercise 33

Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

Solution of Exercise 33

Setting up the Lagrangian: $L(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 1)$

Taking partial derivatives and setting equal to zero: $\frac{\partial L}{\partial x} = y - 2\lambda x = 0$ $\frac{\partial L}{\partial y} = x - 2\lambda y = 0$ $\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$

From the first two equations: $y = 2\lambda x$ and $x = 2\lambda y$ Substituting: $y = 2\lambda(2\lambda y) = 4\lambda^2 y$ If $y \neq 0$: $1 = 4\lambda^2$, so $\lambda = \pm\frac{1}{2}$

Critical points: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
Maximum value: $f = \frac{1}{2}$ at $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$ Minimum value: $f = -\frac{1}{2}$ at $(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$

Exercise 34

Find the points on the curve $x^2 + y^2 = 4$ that are closest to and farthest from the point $(3, 0)$.

Solution of Exercise 34
We want to minimize/maximize $f(x, y) = (x - 3)^2 + y^2$ subject to $g(x, y) = x^2 + y^2 - 4 = 0$.
The Lagrangian takes the form $L(x, y, \lambda) = (x-3)^2 + y^2 - \lambda(x^2 + y^2 - 4)$. Set the partial derivatives to zero:

$$\frac{\partial L}{\partial x} = 2(x - 3) - 2\lambda x = 0 \Rightarrow x - 3 = \lambda x$$
$$\frac{\partial L}{\partial y} = 2y - 2\lambda y = 0 \Rightarrow y(1 - \lambda) = 0$$
$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 4 = 0$$

From the second equation, either $y = 0$ or $\lambda = 1$.

Case 1: $y = 0$ **–** From the constraint, $x^2 = 4$, so $x = \pm 2$. From the first equation, $x - 3 = \lambda x$. For $x = 2$: $\lambda = -\frac{1}{2}$. For $x = -2$: $\lambda = \frac{5}{2}$.
Case 2: $\lambda = 1$ From first equation: $x - 3 = x$, which gives $-3 = 0$ (impossible)
Critical points: $(2, 0)$ and $(-2, 0)$ Distance to $(3, 0)$: $f(2, 0) = 1$, $f(-2, 0) = 25$
Closest point: $(2, 0)$ with distance 1 Farthest point: $(-2, 0)$ with distance 5

Exercise 35

A rectangular box without a top is to be made from 12 square meters of cardboard. Find the dimensions that maximize the volume.

Solution of Exercise 35
Let the box have dimensions $x \times y \times z$ (length × width × height). Volume: $V = xyz$ Constraint (surface area): $xy + 2xz + 2yz = 12$
Lagrangian: $L(x, y, z, \lambda) = xyz - \lambda(xy + 2xz + 2yz - 12)$
Setting partial derivatives to zero: $\frac{\partial L}{\partial x} = yz - \lambda(y + 2z) = 0$ $\frac{\partial L}{\partial y} = xz - \lambda(x + 2z) = 0$ $\frac{\partial L}{\partial z} = xy - \lambda(2x + 2y) = 0$ $\frac{\partial L}{\partial \lambda} = xy + 2xz + 2yz - 12 = 0$
From the first three equations: $yz = \lambda(y + 2z)$ $xz = \lambda(x + 2z)$ $xy = 2\lambda(x + y)$
Dividing the first two: $\frac{y}{x} = \frac{y+2z}{x+2z}$ Cross-multiplying: $y(x + 2z) = x(y + 2z)$ Simplifying: $yx + 2yz = xy + 2xz$ Therefore: $2yz = 2xz$, so $y = x$
From the third equation with $y = x$: $x^2 = 4\lambda x$, so $\lambda = \frac{x}{4}$ From the first equation: $xz = \frac{x}{4}(x + 2z)$ Simplifying: $4z = x + 2z$, so $2z = x$, hence $z = \frac{x}{2}$
From the constraint with $y = x$ and $z = \frac{x}{2}$: $x^2 + 2x \cdot \frac{x}{2} + 2x \cdot \frac{x}{2} = 12$ $x^2 + x^2 + x^2 = 12$ $3x^2 = 12$ $x = 2$
Therefore: $x = y = 2$ and $z = 1$ Maximum volume: $V = 2 \times 2 \times 1 = 4$ cubic meters

EXERCISE GROUP 3

Least squares

Exercise 36

Find a least squares solution of $A\mathbf{x} = \mathbf{b}$ for each of the following:

(a) $A = \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$.

Solution of Exercise 36

(a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(b) $\begin{pmatrix} \frac{4}{3} \\ \frac{-1}{3} \end{pmatrix}$.

(c) $\begin{pmatrix} 3 \\ -5 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$. Have to use normal forms of the equations, not case of invertible $A^T A$.

Exercise 37

Consider the overdetermined system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 8 \\ 7 \end{pmatrix}$.

(a) Compute $A^T A$ and $A^T\mathbf{b}$.

(b) Solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ to find the least squares solution.

(c) Calculate the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ and verify that $A^T\mathbf{r} = \mathbf{0}$.

(d) Compute the sum of squared residuals $\|\mathbf{r}\|^2$.

Solution of Exercise 37

(a) We compute:

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

$$A^T\mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 43 \\ 21 \end{pmatrix}$$

(b) Solving the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$:

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 43 \\ 21 \end{pmatrix}$$

From the second equation: $6x_1 + 3x_2 = 21 \Rightarrow x_2 = 7 - 2x_1$ Substituting into the first: $14x_1 + 6(7 - 2x_1) = 43 \Rightarrow 2x_1 = 1 \Rightarrow x_1 = \frac{1}{2}$ Therefore: $x_2 = 6$, so $\mathbf{x} = \begin{pmatrix} \frac{1}{2} \\ 6 \end{pmatrix}$.

(c) The residual vector is:

$$\mathbf{r} = \mathbf{b} - A\mathbf{x} = \begin{pmatrix} 6 \\ 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 6.5 \\ 7 \\ 7.5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

Verification: $A^T\mathbf{r} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

(d) $\|\mathbf{r}\|^2 = \left(-\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{2}$.

Exercise 38

(a) Prove that if A has linearly independent columns, then $A^T A$ is invertible.

(b) Show that the least squares solution $\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b}$ minimizes $\|A\mathbf{x} - \mathbf{b}\|^2$.

(c) Prove that the projection matrix $P = A(A^T A)^{-1} A^T$ is symmetric and idempotent (i.e., $P^2 = P$).

Solution of Exercise 38

(a) If A has linearly independent columns and $A^TA\mathbf{z} = \mathbf{0}$, then $\|A\mathbf{z}\|^2 = \mathbf{z}^TA^TA\mathbf{z} = \mathbf{z}^T\mathbf{0} = 0$. This implies $A\mathbf{z} = \mathbf{0}$. Since A has linearly independent columns, we must have $\mathbf{z} = \mathbf{0}$. Therefore, A^TA is invertible.

(b) Let $\mathbf{x}^* = (A^TA)^{-1}A^T\mathbf{b}$ be the least squares solution. For any $\mathbf{x}$, we have $\|A\mathbf{x} - \mathbf{b}\|^2 = \|A(\mathbf{x} - \mathbf{x}^*) + (A\mathbf{x}^* - \mathbf{b})\|^2$. Since $A\mathbf{x}^* - \mathbf{b}$ is orthogonal to the column space of A (by the normal equations), and $A(\mathbf{x} - \mathbf{x}^*)$ is in the column space of A, these vectors are orthogonal. By the Pythagorean theorem: $\|A\mathbf{x} - \mathbf{b}\|^2 = \|A(\mathbf{x} - \mathbf{x}^*)\|^2 + \|A\mathbf{x}^* - \mathbf{b}\|^2 \geq \|A\mathbf{x}^* - \mathbf{b}\|^2$.

(c) $P^T = (A(A^TA)^{-1}A^T)^T = A((A^TA)^{-1})^TA^T = A(A^TA)^{-1}A^T = P$ (symmetric). $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T =$ $A(A^TA)^{-1}A^T = P$ (idempotent).

Exercise 39

Consider fitting a line $y = ax + b$ to the data points $(1,2)$, $(2,3)$, $(3,7)$, $(4,8)$.

(a) Set up the matrix equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$.

(b) Find the least squares estimates for a and b.

(c) Calculate the coefficient of determination $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$, where SS_{res} is the sum of squared residuals and SS_{tot} is the total sum of squares.

Solution of Exercise 39

(a) $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 7 \\ 8 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$.

(b) $A^TA = \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix}$, $A^T\mathbf{b} = \begin{pmatrix} 67 \\ 20 \end{pmatrix}$. Solving: $a = 2.1$, $b = -0.5$.

(c) $\bar{y} = 5$, predicted values: $1.6, 3.7, 5.8, 7.9$. $SS_{res} = (2-1.6)^2 + (3-3.7)^2 + (7-5.8)^2 + (8-7.9)^2 = 2.06$. $SS_{tot} = (2-5)^2 + (3-5)^2 + (7-5)^2 + (8-5)^2 = 22$. $R^2 = 1 - \frac{2.06}{22} \approx 0.906$.

Exercise 40

Use R to solve the following least squares problems:

(a) Generate a random 10×3 matrix A and a random vector $\mathbf{b}$ of length 10. Find the least squares solution using both the normal equations and the QR decomposition.

(b) Create synthetic data for polynomial fitting: generate x values from 0 to 2 with 0.1 spacing, and $y = 2x^2 - 3x + 1 + \epsilon$ where $\epsilon \sim N(0, 0.1^2)$. Fit a quadratic polynomial and plot the results.

(c) Compare the condition numbers of A and A^TA for an ill-conditioned matrix. Explain why QR decomposition is preferred for numerical stability.

Solution of Exercise 40

(a)
```
# Set seed for reproducibility
set.seed(123)
```

```
A <- matrix(rnorm(30), nrow=10, ncol=3)
b <- rnorm(10)

# Normal equations
x_normal <- solve(t(A) %*% A) %*% t(A) %*% b

# QR decomposition
qr_decomp <- qr(A)
x_qr <- solve.qr(qr_decomp, b)

# Compare solutions
print(max(abs(x_normal - x_qr)))  # Should be very small
```

(b)

```
x <- seq(0, 2, by=0.1)
set.seed(456)
y <- 2*x^2 - 3*x + 1 + rnorm(length(x), 0, 0.1)

# Design matrix for quadratic fit
A <- cbind(x^2, x, 1)
coeffs <- solve(t(A) %*% A) %*% t(A) %*% y

# Plot
plot(x, y, pch=16)
y_fit <- A %*% coeffs
lines(x, y_fit, col="red", lwd=2)
```

(c)

```
# Create ill-conditioned matrix
A <- matrix(c(1, 1, 1.0001, 1.0001, 1, 1.0001), nrow=3, ncol=2)

cond_A <- kappa(A)
cond_AtA <- kappa(t(A) %*% A)

print(paste("Condition number of A:", cond_A))
print(paste("Condition number of A^T A:", cond_AtA))
# A^T A typically has condition number squared of A
```

Exercise 41

Consider the weighted least squares problem: minimize $\sum_{i=1}^{m} w_i(a_i^T\mathbf{x} - b_i)^2$ where $w_i > 0$ are weights.

(a) Express this as a standard least squares problem by introducing a weight matrix $W = \mathrm{diag}(w_1, w_2, \ldots, w_m)$.

(b) Show that the solution is $\mathbf{x} = (A^TWA)^{-1}A^TW\mathbf{b}$.

(c) If the weights are $w_1 = 4, w_2 = 1, w_3 = 2$ and we want to fit a line through $(1, 3), (2, 5), (3, 4)$, find the weighted least squares solution.

Solution of Exercise 41

(a) The weighted problem is equivalent to minimizing $\|\sqrt{W}A\mathbf{x}-\sqrt{W}\mathbf{b}\|^2$ where $\sqrt{W} = \mathrm{diag}(\sqrt{w_1}, \ldots, \sqrt{w_m})$.

(b) Setting $\tilde{A} = \sqrt{W}A$ and $\tilde{\mathbf{b}} = \sqrt{W}\mathbf{b}$, the normal equations become $\tilde{A}^T\tilde{A}\mathbf{x} = \tilde{A}^T\tilde{\mathbf{b}}$, which gives $(A^TWA)\mathbf{x} = A^TW\mathbf{b}$.

(c) $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$, $W = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. $A^TWA = \begin{pmatrix} 26 & 9 \\ 9 & 7 \end{pmatrix}$, $A^TW\mathbf{b} = \begin{pmatrix} 35 \\ 19 \end{pmatrix}$. Solving: $\mathbf{x} = \begin{pmatrix} 0.4 \\ 3.8 \end{pmatrix}$ (approximately).

EXERCISE GROUP 4

Gram-Schmidt and the QR decomposition

Exercise 42

Let

$$C = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 0 & 3 \\ 2 & 3 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}.$$

Use Exercise **??** from Exercise group 1 to compute $\text{proj}_{\text{Col}(C)}(\mathbf{d})$. Also compute $A\mathbf{x}$ from Problem 1b above. Can you explain why you get the same answer? Why can you not use Exercise **??** from Exercise group 1 for the matrices in Problem 1 above?

Solution of Exercise 42

First, we need to check if the columns of C are orthogonal: $\mathbf{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{c}_2 = \begin{pmatrix} -3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$

$\mathbf{c}_1 \cdot \mathbf{c}_2 = (1)(-3) + (-1)(3) + (0)(3) + (2)(3) = -3 - 3 + 0 + 6 = 0$ ✓
Since the columns are orthogonal, we can use the orthogonal projection formula:

$$\text{proj}_{\text{Col}(C)}(\mathbf{d}) = \frac{\mathbf{d} \cdot \mathbf{c}_1}{\mathbf{c}_1 \cdot \mathbf{c}_1}\mathbf{c}_1 + \frac{\mathbf{d} \cdot \mathbf{c}_2}{\mathbf{c}_2 \cdot \mathbf{c}_2}\mathbf{c}_2$$

Computing the dot products: $\mathbf{d} \cdot \mathbf{c}_1 = (3)(1) + (1)(-1) + (-4)(0) + (2)(2) = 3 - 1 + 0 + 4 = 6$ $\mathbf{c}_1 \cdot \mathbf{c}_1 = 1^2 + (-1)^2 + 0^2 + 2^2 = 6$
$\mathbf{d} \cdot \mathbf{c}_2 = (3)(-3) + (1)(3) + (-4)(3) + (2)(3) = -9 + 3 - 12 + 6 = -12$ $\mathbf{c}_2 \cdot \mathbf{c}_2 = (-3)^2 + 3^2 + 3^2 + 3^2 = 36$
Therefore:

$$\text{proj}_{\text{Col}(C)}(\mathbf{d}) = \frac{6}{6}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \frac{-12}{36}\begin{pmatrix} -3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

From Problem 1b (from worksheet-03-least-squares), we have $A\mathbf{x} = \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix}$, which matches our result.

This happens because both A and C have the same column space, so their orthogonal projections onto their column spaces are identical.

We cannot use the orthogonal projection formula for the matrices in Problem 1 because their columns are not pairwise orthogonal, which is a requirement for the formula to apply directly.

Exercise 43

Suppose that $A \in \mathcal{M}_{mn}$ and $\mathbf{b} \in \mathbb{R}^m$. Verify that if $\mathbf{x} \in \mathbb{R}^n$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ is also a least squares solution.

Solution of Exercise 43
If $\mathbf{x}$ is an exact solution to $A\mathbf{x} = \mathbf{b}$, then the residual vector is:

$$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Since the residual is the zero vector, we have $\|\mathbf{b} - A\mathbf{x}\| = 0$.

For any other vector $\mathbf{y} \in \mathbb{R}^n$, the residual norm is:

$$\|\mathbf{b} - A\mathbf{y}\| \geq 0$$

with equality if and only if $A\mathbf{y} = \mathbf{b}$.

Since $\|\mathbf{b} - A\mathbf{x}\| = 0 \leq \|\mathbf{b} - A\mathbf{y}\|$ for all $\mathbf{y} \in \mathbb{R}^n$, the vector $\mathbf{x}$ minimizes the residual norm $\|\mathbf{b} - A\mathbf{y}\|$.

By definition, a least squares solution is a vector that minimizes $\|\mathbf{b} - A\mathbf{y}\|$ over all $\mathbf{y} \in \mathbb{R}^n$.

Therefore, $\mathbf{x}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$.

This shows that exact solutions are always least squares solutions, which makes intuitive sense since achieving zero error is the best possible outcome for minimizing the squared error.

Exercise 44

Let $A \in \mathcal{M}_{mn}$ and let $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent. If $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$, show that $\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\bar{\mathbf{x}}\|$ for all $\bar{\mathbf{x}} \in \mathbb{R}^n$.

Solution of Exercise 44
By definition, $\hat{\mathbf{x}}$ is a least squares solution if it minimizes the function $f(\mathbf{x}) = \|\mathbf{b} - A\mathbf{x}\|^2$ over all $\mathbf{x} \in \mathbb{R}^n$.

This means that for any $\bar{\mathbf{x}} \in \mathbb{R}^n$:

$$\|\mathbf{b} - A\hat{\mathbf{x}}\|^2 \leq \|\mathbf{b} - A\bar{\mathbf{x}}\|^2$$

Taking square roots of both sides (since both sides are non-negative):

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\bar{\mathbf{x}}\|$$

Alternatively, we can prove this using the orthogonal projection theorem. The least squares solution corresponds to the orthogonal projection of $\mathbf{b}$ onto the column space of A.

Let $\mathbf{p} = A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}(\mathbf{b})$ be this projection.

For any $\bar{\mathbf{x}} \in \mathbb{R}^n$, we have $A\bar{\mathbf{x}} \in \text{Col}(A)$.

By the orthogonal projection theorem, $\mathbf{p}$ is the closest point in $\text{Col}(A)$ to $\mathbf{b}$, so:

$$\|\mathbf{b} - \mathbf{p}\| \leq \|\mathbf{b} - A\bar{\mathbf{x}}\|$$

Since $\mathbf{p} = A\hat{\mathbf{x}}$, this gives us:

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\bar{\mathbf{x}}\|$$

This inequality shows that the least squares solution gives the smallest possible residual norm among all possible solutions, which is why it's called the "least squares" solution.

Exercise 45

The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

can be factored as

$$A = QR = \begin{pmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix},$$

where Q is an orthogonal matrix and R is upper triangular and invertible. In this question you are going to justify this claim (and get a sense of the Gram-Schmidt procedure). Let

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

denote the columns of A.

(a) Set $\mathbf{v}_1 = \mathbf{a}_1$.

(b) Compute $\mathbf{v}_2 = \mathbf{a}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_2)$.

(c) Compute $\mathbf{v}_3 = \mathbf{a}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3)$.

(d) What do you notice about the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$?

(e) Construct Q from $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

(f) Construct R from Q and A.

(g) Verify that $A = QR$.

Solution of Exercise 45

(a) $\mathbf{v}_1 = \mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(b) First compute the projection: $\text{proj}_{\mathbf{v}_1}(\mathbf{a}_2) = \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \frac{3}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{pmatrix}$

Therefore: $\mathbf{v}_2 = \mathbf{a}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_2) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$

(c) First compute the projections: $\text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) = \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{2}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$

$\text{proj}_{\mathbf{v}_2}(\mathbf{a}_3) = \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1/2}{3/8}\begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix} = \frac{4}{3}\begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix} = \begin{pmatrix}-1\\1/3\\1/3\\1/3\end{pmatrix}$

Therefore: $\mathbf{v}_3 = \mathbf{a}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3) = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} - \begin{pmatrix}-1\\1/3\\1/3\\1/3\end{pmatrix} = \begin{pmatrix}1/2\\-5/6\\1/6\\1/6\end{pmatrix}$

(d) The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal. This is the key property of the Gram-Schmidt process.

(e) To construct Q, we normalize each vector: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{3/4}}\begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix} = \frac{-3}{\sqrt{12}}\begin{pmatrix}1\\-1/3\\-1/3\\-1/3\end{pmatrix} = \begin{pmatrix}-3/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\end{pmatrix}$

$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \begin{pmatrix}0\\-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$

So $Q = \begin{pmatrix}1/2 & -3/\sqrt{12} & 0\\1/2 & 1/\sqrt{12} & -2/\sqrt{6}\\1/2 & 1/\sqrt{12} & 1/\sqrt{6}\\1/2 & 1/\sqrt{12} & 1/\sqrt{6}\end{pmatrix}$

(f) $R = Q^T A = \begin{pmatrix}2 & 3/2 & 1\\0 & 3/\sqrt{12} & 2/\sqrt{12}\\0 & 0 & 2/\sqrt{6}\end{pmatrix}$

(g) We can verify by computing QR and showing it equals A.

Exercise 46

Let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be a set of linearly independent vectors of $\mathbb{R}^m$ and define vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as in Exercise 45.

(a) Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

(b) Show that $\mathbf{v}_1$ and $\mathbf{v}_3$ are orthogonal.

(c) Show that $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal.

Solution of Exercise 46

(a) By definition, $\mathbf{v}_1 = \mathbf{a}_1$ and $\mathbf{v}_2 = \mathbf{a}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_2)$.
We need to show $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (\mathbf{a}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_2))$$

$$= \mathbf{v}_1 \cdot \mathbf{a}_2 - \mathbf{v}_1 \cdot \text{proj}_{\mathbf{v}_1}(\mathbf{a}_2)$$

$$= \mathbf{v}_1 \cdot \mathbf{a}_2 - \mathbf{v}_1 \cdot \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$= \mathbf{v}_1 \cdot \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}(\mathbf{v}_1 \cdot \mathbf{v}_1)$$

$$= \mathbf{v}_1 \cdot \mathbf{a}_2 - \mathbf{a}_2 \cdot \mathbf{v}_1 = 0$$

(b) By definition, $\mathbf{v}_3 = \mathbf{a}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3)$.
We need to show $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$:

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot (\mathbf{a}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3))$$

$$= \mathbf{v}_1 \cdot \mathbf{a}_3 - \mathbf{v}_1 \cdot \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \mathbf{v}_1 \cdot \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3)$$

Since $\mathbf{v}_1 \perp \mathbf{v}_2$ (from part a), we have $\mathbf{v}_1 \cdot \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3) = 0$.
For the projection term:

$$\mathbf{v}_1 \cdot \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) = \mathbf{v}_1 \cdot \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}(\mathbf{v}_1 \cdot \mathbf{v}_1) = \mathbf{a}_3 \cdot \mathbf{v}_1$$

Therefore: $\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{a}_3 - \mathbf{a}_3 \cdot \mathbf{v}_1 = 0$

(c) We need to show $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$:

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot (\mathbf{a}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3))$$

$$= \mathbf{v}_2 \cdot \mathbf{a}_3 - \mathbf{v}_2 \cdot \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) - \mathbf{v}_2 \cdot \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3)$$

Since $\mathbf{v}_1 \perp \mathbf{v}_2$, we have $\mathbf{v}_2 \cdot \text{proj}_{\mathbf{v}_1}(\mathbf{a}_3) = 0$.
For the second projection term:

$$\mathbf{v}_2 \cdot \text{proj}_{\mathbf{v}_2}(\mathbf{a}_3) = \mathbf{v}_2 \cdot \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}(\mathbf{v}_2 \cdot \mathbf{v}_2) = \mathbf{a}_3 \cdot \mathbf{v}_2$$

Therefore: $\mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{a}_3 - \mathbf{a}_3 \cdot \mathbf{v}_2 = 0$
This completes the proof that the Gram-Schmidt process produces mutually orthogonal vectors.

Exercise 47

Use the QR factorisation to find a least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ -2 \\ 0 \end{pmatrix}.$$

Solution of Exercise 47

To solve $A\mathbf{x} = \mathbf{b}$ using QR factorization, we first need to find the QR decomposition of A.

Step 1: Apply Gram-Schmidt to the columns of A.

Let $\mathbf{a}_1 = \begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix}2 & 1 & 0 & 1\end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix}2 & 2 & 1 & 2\end{pmatrix}$

$\mathbf{v}_1 = \mathbf{a}_1 = \begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix}$

$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix}2 & 1 & 0 & 1\end{pmatrix} - \frac{2}{4}\begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix} = \begin{pmatrix}3/2 & 3/2 & 1/2 & 1/2\end{pmatrix}$

$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$

After computing: $\mathbf{v}_3 = \begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix}$

Step 2: Normalize to get Q.

$\mathbf{q}_1 = \frac{1}{2}\begin{pmatrix}1 & -1 & -1 & 1\end{pmatrix}$, $\mathbf{q}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}3/2 & 3/2 & 1/2 & 1/2\end{pmatrix}$, $\mathbf{q}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix}$

Step 3: Construct $R = Q^T A$.

$R = \begin{pmatrix}2 & 1 & 3 0 & \sqrt{5} & 2\sqrt{5} 0 & 0 & \sqrt{2}\end{pmatrix}$

Step 4: Solve $R\mathbf{x} = Q^T\mathbf{b}$ for the least squares solution.

$Q^T\mathbf{b} = \begin{pmatrix}-3 & \sqrt{5} & \sqrt{2}\end{pmatrix}$

Solving $R\mathbf{x} = Q^T\mathbf{b}$ by back substitution: $\mathbf{x} = \begin{pmatrix}-1 & 1 & 1\end{pmatrix}$

Therefore, the least squares solution is $\mathbf{x} = \begin{pmatrix}-1 & 1 & 1\end{pmatrix}$.

Exercise 48

Let $A = \begin{pmatrix}3 & -2 \\ 2 & 0 \\ 1 & -1 \\ 0 & 2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-1 \\ 1 \\ 2 \\ -1\end{pmatrix}$.

(a) Find a least squares solution to $A\mathbf{x} = \mathbf{b}$. Any fractions should be left as fractions (Hint: Factoring out the fractions makes the arithmetic easier).

(b) Compute $\text{proj}_{\text{Col}(A)}(\mathbf{b})$. Any fractions should be left as fractions (Hint: Factoring out the fractions makes the arithmetic easier).

Solution of Exercise 48

(a) To find the least squares solution, we solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

First, compute A^T: $A^T = \begin{pmatrix}3 & 2 & 1 & 0 \\ -2 & 0 & -1 & 2\end{pmatrix}$

Next, compute $A^T A$: $A^T A = \begin{pmatrix}3 & 2 & 1 & 0 \\ -2 & 0 & -1 & 2\end{pmatrix}\begin{pmatrix}3 & -2 \\ 2 & 0 \\ 1 & -1 \\ 0 & 2\end{pmatrix} = \begin{pmatrix}14 & -7 \\ -7 & 9\end{pmatrix}$

Compute $A^T\mathbf{b}$: $A^T\mathbf{b} = \begin{pmatrix}3 & 2 & 1 & 0 \\ -2 & 0 & -1 & 2\end{pmatrix}\begin{pmatrix}-1 \\ 1 \\ 2 \\ -1\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}$

Now solve $A^T A\mathbf{x} = A^T\mathbf{b}$: $\begin{pmatrix}14 & -7 \\ -7 & 9\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}$

From the first equation: $14x_1 - 7x_2 = 1$, so $x_2 = 2x_1 - \frac{1}{7}$

From the second equation: $-7x_1 + 9x_2 = 0$, so $x_2 = \frac{7x_1}{9}$

Setting equal: $2x_1 - \frac{1}{7} = \frac{7x_1}{9}$
Solving: $18x_1 - \frac{9}{7} = 7x_1$, so $11x_1 = \frac{9}{7}$, giving $x_1 = \frac{9}{77}$
Then: $x_2 = \frac{7\cdot 9}{9\cdot 77} = \frac{7}{77} = \frac{1}{11}$

Therefore, the least squares solution is $\mathbf{x} = \begin{pmatrix} \frac{9}{77} \\ \frac{1}{11} \end{pmatrix}$.

(b) The projection of **b** onto Col(A) is $A\mathbf{x}$: $\text{proj}_{\text{Col}(A)}(\mathbf{b}) = A\mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{9}{77} \\ \frac{1}{11} \end{pmatrix}$

$$= \begin{pmatrix} 3\cdot\frac{9}{77} - 2\cdot\frac{1}{11} \\ 2\cdot\frac{9}{77} \\ \frac{9}{77} - \frac{1}{11} \\ 2\cdot\frac{1}{11} \end{pmatrix} = \begin{pmatrix} \frac{27}{77} - \frac{14}{77} \\ \frac{18}{77} \\ \frac{9}{77} - \frac{7}{77} \\ \frac{2}{11} \end{pmatrix} = \begin{pmatrix} \frac{13}{77} \\ \frac{18}{77} \\ \frac{2}{77} \\ \frac{2}{11} \end{pmatrix}$$

Converting to common denominator: $\text{proj}_{\text{Col}(A)}(\mathbf{b}) = \begin{pmatrix} \frac{13}{77} \\ \frac{18}{77} \\ \frac{2}{77} \\ \frac{14}{77} \end{pmatrix}$

Exercise 49

Bothan Motor Works has released a new car. The company has collected the following data where w_i is the number of weeks since the introduction of the car and s_i is the total sales of the car in week w_i (in millions of dollars):

w_i	1	2	3	4	5	6	7	8	9	10
s_i	0.8	0.5	3.2	4.3	4.0	5.1	4.3	3.8	1.2	0.8

Experience suggests that a quadratic polynomial $s(w) = a + bw + cw^2$ is good model for this data. Describe how you would find such a model. That is, write down any relevant matrices/vectors/equations and describe how you would use the solution(s) of the equations (You do NOT need to perform any calculations).

Exercise 50

Let

$$A = \begin{pmatrix} 1 & -1 & -4 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 35 \end{pmatrix}.$$

(a) Find a QR factorization of A. Any radicals should be left as radicals.

(b) Use your answer from (a) to find a least squares solution to $A\mathbf{x} = \mathbf{b}$. Any radicals should be left as radicals.

Exercise 51

Apply the Gram-Schmidt process to the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find the orthogonal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

(b) Normalize these vectors to obtain an orthonormal basis.

(c) Verify that your orthonormal vectors are indeed orthogonal and have unit length.

Solution of Exercise 51

(a) Step 1: $\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Step 2: $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$

Step 3: $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix} -1/3 \\ 1/3 \\ 2/3 \end{pmatrix}$

(b) $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{q}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{q}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

(c) Check: $\mathbf{q}_i \cdot \mathbf{q}_j = 0$ for $i \neq j$ and $\|\mathbf{q}_i\| = 1$ for all i.

Exercise 52

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Find the QR decomposition of A.

(b) Use the QR decomposition to solve the least squares problem $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

(c) Compare your solution with the normal equations approach.

Solution of Exercise 52

(a) Using Gram-Schmidt: $Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$, $R = \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} \\ 0 & \frac{\sqrt{30}}{6} \end{pmatrix}$

(b) Solving $R\mathbf{x} = Q^T\mathbf{b}$: $\mathbf{x} = \begin{pmatrix} \frac{7}{5} \\ \frac{2}{5} \end{pmatrix}$

(c) Normal equations give the same result: $(A^TA)^{-1}A^T\mathbf{b} = \begin{pmatrix} \frac{7}{5} \\ \frac{2}{5} \end{pmatrix}$

Exercise 53

(a) Prove that if $A = QR$ is a QR decomposition where Q has orthonormal columns and R is upper triangular with positive diagonal entries, then this decomposition is unique.

(b) Show that if A has linearly independent columns, then $Q^T Q = I$ where Q is the orthogonal factor in the QR decomposition.

(c) Prove that the QR decomposition preserves the column space: $\mathrm{Col}(A) = \mathrm{Col}(Q)$.

Solution of Exercise 53

(a) Suppose $A = Q_1 R_1 = Q_2 R_2$ where both are valid QR decompositions. Then $Q_1 R_1 = Q_2 R_2$, so $Q_2^T Q_1 = R_2 R_1^{-1}$. The left side is orthogonal, the right side is upper triangular, so both must be diagonal. Since diagonal entries of R_1 and R_2 are positive, the diagonal entries of $R_2 R_1^{-1}$ are positive, making it the identity matrix. Hence $Q_1 = Q_2$ and $R_1 = R_2$.

(b) By construction in Gram-Schmidt, the columns of Q are orthonormal, so $Q^T Q = I$.

(c) Since $A = QR$ and R is invertible (when A has linearly independent columns), we have $\mathrm{Col}(A) = \mathrm{Col}(QR) = \mathrm{Col}(Q)$.

Exercise 54

Use R to explore QR decomposition:

(a) Generate a random 5×3 matrix A with linearly independent columns. Compute its QR decomposition using R's built-in function and verify the factorization.

(b) Compare the computational efficiency of solving least squares problems using QR decomposition versus normal equations for ill-conditioned matrices.

(c) Implement the modified Gram-Schmidt algorithm and compare it with the classical Gram-Schmidt process for numerical stability.

Solution of Exercise 54

(a)
```
set.seed(123)
A <- matrix(rnorm(15), nrow=5, ncol=3)
qr_result <- qr(A)
Q <- qr.Q(qr_result)
R <- qr.R(qr_result)

# Verify factorization
print(max(abs(A - Q %*% R)))  # Should be very small
print(max(abs(t(Q) %*% Q - diag(3))))  # Should be very small
```

(b)
```
# Create ill-conditioned matrix
A <- matrix(c(1, 1, 1.0001, 1.0001, 1, 1.0001), nrow=3, ncol=2)
b <- c(1, 2, 3)

# QR approach
system.time(x_qr <- solve.qr(qr(A), b))

# Normal equations
```

```
system.time(x_normal <- solve(t(A) %*% A) %*% t(A) %*% b)

print(max(abs(x_qr - x_normal)))
```

(c)
```
# Modified Gram-Schmidt is more numerically stable
# Classical GS: u_k = v_k - sum(proj_u_j(v_k))
# Modified GS: u_k = (...((v_k - proj_u_1(v_k)) - proj_u_2(...)) ...)
```

Exercise 55

Consider the least squares problem for polynomial fitting: fit a quadratic polynomial $p(x) = a + bx + cx^2$ to the data points $(0, 1)$, $(1, 2)$, $(2, 5)$, $(3, 10)$.

(a) Set up the matrix equation $A\mathbf{x} = \mathbf{b}$ for this problem.

(b) Find the QR decomposition of A.

(c) Solve for the coefficients using the QR decomposition.

(d) Calculate the coefficient of determination R^2 for your fitted polynomial.

Solution of Exercise 55

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 10 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(b) $Q = \begin{pmatrix} 1/2 & -3/\sqrt{20} & 1/\sqrt{5} \\ 1/2 & -1/\sqrt{20} & -1/\sqrt{5} \\ 1/2 & 1/\sqrt{20} & -1/\sqrt{5} \\ 1/2 & 3/\sqrt{20} & 1/\sqrt{5} \end{pmatrix}$, $R = \begin{pmatrix} 2 & 3 & 7 \\ 0 & \sqrt{5} & 2\sqrt{5} \\ 0 & 0 & \sqrt{5} \end{pmatrix}$

(c) Solving $R\mathbf{x} = Q^T\mathbf{b}$: $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, so $p(x) = 1 + x^2$

(d) $R^2 = 1$ (perfect fit since the data follows $y = 1 + x^2$ exactly)

Exercise 56

Let A be an $m \times n$ matrix with linearly independent columns, and let $A = QR$ be its QR decomposition.

(a) Show that the projection matrix onto the column space of A can be written as $P = QQ^T$.

(b) Prove that P is symmetric and idempotent.

(c) If $\mathbf{b} \in \mathbb{R}^m$, show that the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$ is $QQ^T\mathbf{b}$.

(d) Demonstrate that the least squares solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = R^{-1}Q^T\mathbf{b}$.

Solution of Exercise 56

(a) Since $A = QR$ with R invertible, the projection matrix is $P = A(A^TA)^{-1}A^T = QR(R^TQ^TQR)^{-1}R^TQ^T = QR(R^TR)^{-1}R^TQ^T = QQ^T$.

(b) $P^T = (QQ^T)^T = QQ^T = P$ (symmetric). $P^2 = QQ^TQQ^T = QQ^T = P$ (idempotent since $Q^TQ = I$).

(c) The orthogonal projection of **b** onto $\mathrm{Col}(A)$ is $P\mathbf{b} = QQ^T\mathbf{b}$.

(d) From the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$, we get $R^TQ^TQR\mathbf{x} = R^TQ^T\mathbf{b}$, which simplifies to $R^TR\mathbf{x} = R^TQ^T\mathbf{b}$. Multiplying by $(R^T)^{-1}$: $R\mathbf{x} = Q^T\mathbf{b}$, so $\mathbf{x} = R^{-1}Q^T\mathbf{b}$.

EXERCISE GROUP 5

Singular values and the SVD

Exercise 57

Compute the full SVD of the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of $A^T A$.

(b) Find the singular values of A.

(c) Construct the matrix V from the eigenvectors of $A^T A$.

(d) Compute the left singular vectors using $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$.

(e) Write the complete SVD as $A = U\Sigma V^T$.

Solution of Exercise 57

(a) $A^T A = \begin{pmatrix} 11 & 7 \\ 7 & 11 \end{pmatrix}$ has eigenvalues $\lambda_1 = 18$, $\lambda_2 = 4$ with eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, $\sigma_2 = \sqrt{4} = 2$.

(c) $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(d) $\mathbf{u}_1 = \frac{1}{3\sqrt{2}} A\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

(e) $A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T$.

Exercise 58

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Find the reduced SVD of A.

(b) Compute the Moore-Penrose pseudoinverse A^+ using the SVD.

(c) Verify that $AA^+A = A$ and $A^+AA^+ = A^+$.

Solution of Exercise 58

(a) $A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has eigenvalues $1, 1$ with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So $V = I_2$, $\sigma_1 = \sigma_2 = 1$, and $U = A$.

Reduced SVD: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$.

(b) $A^+ = V\Sigma^+U^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) Direct verification shows both properties hold.

Exercise 59

(a) Prove that for any matrix A, the rank of A equals the number of nonzero singular values.

(b) Show that for any matrix A, $\|A\|_2 = \sigma_1$ where σ_1 is the largest singular value.

(c) Prove that if $A = U\Sigma V^T$ is the SVD of A, then $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2$ where $\|\cdot\|_F$ is the Frobenius norm.

Solution of Exercise 59

(a) Since $A = U\Sigma V^T$ and U, V are orthogonal (hence full rank), $\text{rank}(A) = \text{rank}(\Sigma)$ = number of nonzero diagonal entries = number of nonzero singular values.

(b) $\|A\|_2 = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|U\Sigma V^T\mathbf{x}\| = \max_{\|\mathbf{y}\|=1} \|\Sigma\mathbf{y}\| = \sigma_1$ (where $\mathbf{y} = V^T\mathbf{x}$).

(c) $\|A\|_F^2 = \text{tr}(A^TA) = \text{tr}(V\Sigma^2V^T) = \text{tr}(\Sigma^2) = \sum_{i=1}^r \sigma_i^2$.

Exercise 60

Consider the data matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$ representing two variables measured on three subjects.

(a) Find the SVD of A.

(b) What is the rank of this data matrix? What does this tell you about the relationship between the two variables?

(c) Compute the best rank-1 approximation to A using SVD.

(d) Calculate the approximation error $\|A - A_1\|_F$ where A_1 is the rank-1 approximation.

Solution of Exercise 60

(a) $A^T A = \begin{pmatrix} 14 & 28 \\ 28 & 56 \end{pmatrix}$ has eigenvalues 70, 0 with eigenvectors $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. SVD: $A = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\left(\sqrt{70}\right)\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \end{pmatrix}$.

(b) Rank = 1. The second variable is exactly twice the first variable.

(c) $A_1 = A$ (since A already has rank 1).

(d) $\|A - A_1\|_F = 0$ (perfect approximation).

Exercise 61

Use R to explore SVD applications:

(a) Generate a random 4×3 matrix and compute its SVD. Verify the decomposition and explore the properties of U, Σ, and V.

(b) Create a noisy version of a rank-2 matrix and use SVD to denoise it by keeping only the top 2 singular values.

(c) Implement image compression using SVD: load a grayscale image, compute its SVD, and reconstruct the image using different numbers of singular values.

Solution of Exercise 61

(a)

```
set.seed(123)
A <- matrix(rnorm(12), nrow=4, ncol=3)
svd_result <- svd(A)
U <- svd_result$u
sigma <- svd_result$d
V <- svd_result$v

# Verify decomposition
A_reconstructed <- U %*% diag(sigma) %*% t(V)
print(max(abs(A - A_reconstructed)))

# Check orthogonality
print(max(abs(t(U) %*% U - diag(4))))
print(max(abs(t(V) %*% V - diag(3))))
```

(b)

```
# Create rank-2 matrix
true_A <- matrix(rnorm(20), 5, 4) %*% matrix(rnorm(16), 4, 4)[1:2,]
# Add noise
noisy_A <- true_A + 0.1 * matrix(rnorm(20), 5, 4)

# Denoise using SVD
svd_noisy <- svd(noisy_A)
denoised_A <- svd_noisy$u[,1:2] %*% diag(svd_noisy$d[1:2]) %*% t(svd_noisy$v[,1:2])
```

(c)

```
# Load image (assuming grayscale matrix)
# img <- as.matrix(read.table("image.txt"))
# svd_img <- svd(img)
```

```
#
# # Compress with k singular values
# compress_image <- function(k) {
#   svd_img$u[,1:k] %*% diag(svd_img$d[1:k]) %*% t(svd_img$v[,1:k])
# }
```

Exercise 62

Let A be an $m \times n$ matrix with SVD $A = U\Sigma V^T$.

(a) Show that the best rank-k approximation to A in the Frobenius norm is given by $A_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

(b) Prove that $\|A - A_k\|_F^2 = \sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots + \sigma_r^2$.

(c) Demonstrate that the optimal rank-k approximation is unique if $\sigma_k > \sigma_{k+1}$.

Solution of Exercise 62

(a) This follows from the Eckart-Young-Mirsky theorem. The SVD provides the optimal low-rank approximation: $A_k = U_k \Sigma_k V_k^T$ where U_k, V_k contain the first k columns of U, V and Σ_k contains the first k singular values.

(b) $\|A - A_k\|_F^2 = \| \sum_{i=k+1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$ since the terms $\mathbf{u}_i \mathbf{v}_i^T$ are orthogonal in the Frobenius inner product.

(c) If $\sigma_k > \sigma_{k+1}$, then the gap between the k-th and $(k+1)$-th singular values ensures uniqueness of the optimal rank-k subspace, making the approximation unique.

Exercise 63

Consider the least squares problem $A\mathbf{x} = \mathbf{b}$ where A has more rows than columns but may not have full column rank.

(a) Using the SVD $A = U\Sigma V^T$, derive the general solution using the Moore-Penrose pseudoinverse.

(b) Show that when A has full column rank, the SVD solution reduces to the normal equations solution.

(c) For the underdetermined case (more columns than rows), show that the SVD gives the minimum-norm solution.

Solution of Exercise 63

(a) With $A = U\Sigma V^T$, the Moore-Penrose pseudoinverse is $A^+ = V\Sigma^+ U^T$ where Σ^+ has $1/\sigma_i$ for nonzero σ_i and 0 elsewhere. The least squares solution is $\mathbf{x} = A^+\mathbf{b} + (I - A^+A)\mathbf{w}$ for any $\mathbf{w}$.

(b) When A has full column rank, $A^+A = I$, so the solution is unique: $\mathbf{x} = A^+\mathbf{b} = V\Sigma^{-1}U^T\mathbf{b} = (A^TA)^{-1}A^T\mathbf{b}$.

(c) For underdetermined systems, among all solutions to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} = A^+\mathbf{b}$ has minimum norm because A^+ projects onto the row space of A.

Exercise 64

Find singular values of the following matrices.

(a) $A = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

(d) $D = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

Exercise 65

Suppose that A is $m \times n$ matrix with rank k and a singular value decomposition $A = USV^T$, where $U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{pmatrix}$. Recall that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are pairwise orthonormal eigenvectors of $A^T A$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i,$$

where $\sigma_i = \sqrt{\lambda_i}$ for all $i \in [k]$.

(a) Show that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are all unit vectors.

(b) Show that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are pairwise orthogonal.

(c) Show that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are eigenvectors of AA^T. What are the corresponding eigenvalues?

(d) Find a singular value decomposition of A^T.

(e) Suppose that A is square ($m = n$) and invertible. Find a singular value decomposition of A^{-1} (Hint: What do you know about the eigenvalues/singular values of A?).

Solution of Exercise 65

(a) Compute $\|\mathbf{u}_i\|^2$

(b) Compute $\mathbf{u}_i \cdot \mathbf{u}_j$

(c) Compute $AA^T\mathbf{u}_i$. Eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_k$

(d) $A^T = VS^TU^T$

(e) $A^{-1} = VS^{-1}U^T$ (Note that $U^{-1} = U^T$ and $V^{-1} = V^T$)

Exercise 66

Suppose that $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ and consider the system of linear equations $A\mathbf{x} = \mathbf{b}$.

(a) Suppose that $AA^+\mathbf{b} = \mathbf{b}$. Show that $A\mathbf{x} = \mathbf{b}$ is consistent.

(b) Suppose that $A\mathbf{x} = \mathbf{b}$ is consistent (i.e. there is a $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{z} = \mathbf{b}$). Show that $AA^+\mathbf{b} = \mathbf{b}$.Parts **??** and (b) show that $\mathbf{x}^* = A^+\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

Solution of Exercise 66

(a) Hint: If you stare at it for long enough, you'll get it.

(b) Hint: Use $A = AA^+A$.

Exercise 67

Suppose that $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ and consider the system of linear equations $A\mathbf{x} = \mathbf{b}$.

(a) Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} = (I - A^+A)\mathbf{w}$. Show that $A\mathbf{v} = \mathbf{0}$.

(b) Let $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$. Show that $\mathbf{v} = (I - A^+A)\mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^n$.
Parts **??** and (b), together with Problem 66, show that all solutions to $A\mathbf{x} = \mathbf{b}$ are described by $\hat{\mathbf{x}} = A^+\mathbf{b} + (I - A^+A)\mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^n$.

Solution of Exercise 67

(a) Hint: Use $A = AA^+A$.

(b) Hint: Let $\mathbf{w} = \mathbf{v}$.

Exercise 68

Let $A\mathbf{x} = \mathbf{b}$ be the system of linear equations where

$$A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.$$

(a) Compute A^+ (Hint: It might be easier to work with A^T instead).

(b) Use A^+ to find all least squares solutions to $A\mathbf{x} = \mathbf{b}$.

Solution of Exercise 68

(a) $A^+ = \begin{pmatrix} \frac{4}{25} & \frac{2}{25} & 0 \\ \frac{-2}{25} & \frac{-1}{25} & 0 \end{pmatrix}$

(b) $\hat{\mathbf{x}} = \begin{pmatrix} \frac{14}{25} \\ \frac{-7}{25} \end{pmatrix} + \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^2$.

Exercise 69

Let $A\mathbf{x} = \mathbf{b}$ be the system of linear equations where

$$A = \begin{pmatrix} 12 & 0 & 6 \\ 4 & -8 & 10 \\ 4 & -8 & 10 \\ 12 & 0 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 18 \\ 6 \\ 6 \\ 18 \end{pmatrix}.$$

(a) Compute A^+.

(b) Use A^+ to find all solutions to $A\mathbf{x} = \mathbf{b}$.

Solution of Exercise 69

(a) $A^+ = \begin{pmatrix} \frac{1}{24} & \frac{-1}{72} & \frac{-1}{72} & \frac{1}{24} \\ \frac{1}{48} & \frac{-5}{144} & \frac{-5}{144} & \frac{1}{48} \\ 0 & \frac{1}{36} & \frac{1}{36} & 0 \end{pmatrix}$

(b) $\hat{\mathbf{x}} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{9} & \frac{-2}{9} & \frac{-2}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix} \mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^3$.

EXERCISE GROUP 6

Change of basis

Exercise 70

Let $\mathbf{w} = (1, 2, -6, 2)^T$ in the standard basis (coordinate system). Find the coordinates $\mathbf{w}_{\text{new}}$ of $\mathbf{w}$ in the basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Solution of Exercise 70

$\mathbf{w}_{\text{new}} = \begin{pmatrix} 3 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$

Hint: If $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$, then $\mathbf{w}_{\text{new}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$

Exercise 71

Let $\mathbf{w} = 3 - 2t + 4t^2$ under the standard basis. Find the coordinates of $\mathbf{w}$ using the basis

$$\mathbf{v}_1 = 1 + t, \quad \mathbf{v}_2 = 1 + t^2, \quad \mathbf{v}_3 = 1 - t + t^2$$

Solution of Exercise 71

$\mathbf{w} = -(1 + t) + 3(1 + t^2) + (1 - t + t^2)$ so the new coordinates are $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}_{new}$ (Hint: The "standard" basis in this context is $\{1, t, t^2\}$, so the coordinates are $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$)

Exercise 72

Let the standard basis vectors be $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ and let

$$\mathbf{v}_1 = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}.$$

In this question, we consider the coordinate system based on $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) What are $[\mathbf{v}_1]_{\mathcal{B}}$, $[\mathbf{v}_2]_{\mathcal{B}}$, and $[\mathbf{v}_3]_{\mathcal{B}}$ (i.e. the coordinates of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ under $\mathcal{B}$)?

(b) The change of coordinates matrix P satisfies $P[\mathbf{v}_1]_{\mathcal{B}} = \mathbf{v}_1$, $P[\mathbf{v}_2]_{\mathcal{B}} = \mathbf{v}_2$, and $P[\mathbf{v}_3]_{\mathcal{B}} = \mathbf{v}_3$. Find P.

(c) Find $[\mathbf{e}_1]_{\mathcal{B}}$, $[\mathbf{e}_2]_{\mathcal{B}}$, and $[\mathbf{e}_3]_{\mathcal{B}}$ (i.e. Write each of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$).

(d) Compute P^{-1}. What do you notice?

(e) Let $\mathbf{w} = \begin{pmatrix} 4 \\ -9 \\ 5 \end{pmatrix}$. Find $[\mathbf{w}]_{\mathcal{B}}$

Solution of Exercise 72

(a) $[\mathbf{v}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[\mathbf{v}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $[\mathbf{v}_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(b) $P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$

(c) $[\mathbf{e}_1]_{\mathcal{B}} = \begin{pmatrix} \frac{17}{6} \\ \frac{-3}{2} \\ 2 \end{pmatrix}$, $[\mathbf{e}_2]_{\mathcal{B}} = \begin{pmatrix} \frac{-7}{6} \\ \frac{1}{2} \\ 1 \end{pmatrix}$, and $[\mathbf{e}_3]_{\mathcal{B}} = \begin{pmatrix} \frac{-25}{6} \\ \frac{5}{2} \\ 3 \end{pmatrix}$

(d) $P^{-1} = \begin{pmatrix} [\mathbf{e}_1]_{\mathcal{B}} & [\mathbf{e}_2]_{\mathcal{B}} & [\mathbf{e}_3]_{\mathcal{B}} \end{pmatrix}$

(e) $\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2 - 2\mathbf{v}_3$ so $[\mathbf{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ OR $[\mathbf{w}]_{\mathcal{B}} = P^{-1}\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Exercise 73

Let

$$A = \begin{pmatrix} 4 & 2 \\ 0 & -1 \end{pmatrix}, C = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and $\mathcal{B}$ the standard basis of $\mathcal{M}_2$.

(a) Find the coordinate vectors $[A]_{\mathcal{B}}$ and $[A]_C$ with respect to the bases $\mathcal{B}$ and C, respectively.

(b) Find the change of basis matrix $P_{C \leftarrow \mathcal{B}}$ from $\mathcal{B}$ to C.

(c) Find the change of basis matrix $P_{\mathcal{B} \leftarrow C}$ from C to $\mathcal{B}$.

Solution of Exercise 73

(a)

$$[A]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ -1 \end{pmatrix}, [A]_{C} = \begin{pmatrix} 5/2 \\ 0 \\ -3 \\ 9/2 \end{pmatrix}$$

(b)

$$P_{C\leftarrow\mathcal{B}} = \begin{pmatrix} 1/2 & 0 & -1 & -1/2 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 3/2 & -1 & -2 & -1/2 \end{pmatrix}$$

(c)

$$P_{\mathcal{B}\leftarrow C} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

Exercise 74

Let $\mathbf{x} \in \mathbb{R}^2$.

(a) Find a matrix R such that $R\mathbf{x} \in \mathbb{R}^2$ is a counter clockwise rotation of $\mathbf{x}$ by $\frac{\pi}{6}$.

(b) Find a matrix S such that $S\mathbf{x} \in \mathbb{R}^2$ is a counter clockwise rotation of $\mathbf{x}$ by θ.

Solution of Exercise 74

(a) $R = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (Hint: Where do the standard basis vectors go?)

(b) $S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (Hint: Same idea as 1a)

Exercise 75

Let $\mathbf{x} \in \mathbb{R}^n$ and let $P \in \mathcal{M}_n$ such that P has pairwise orthonormal columns. Show that $\|P\mathbf{x}\| = \|\mathbf{x}\|$.

Solution of Exercise 75

A classic. Compute $\|P\mathbf{x}\|^2$. Actually, I cannot remember if I have not done it in a lecture already :) Not bad if I have.

Exercise 76

Let $X \in \mathcal{M}_{mn}$ be a data matrix in mean-deviation form (i.e., each variable is scaled to its mean) and let $P \in \mathcal{M}_m$ such that P has pairwise orthonormal columns. Show that $Y = P^T X$ is in mean-deviation form.

Solution of Exercise 76

Let $\mathbf{e}_k = (1, \ldots, 1)^T \in \mathbb{R}^k$ be the vector of all 1's. If X is in mean-deviation form, this means that

$$\mathbf{e}_m^T X = \mathbf{0}_m$$

Compute $Y\mathbf{e}_n$.

Exercise 77

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$, and let $C = \{\mathbf{w}_1, \mathbf{w}_2\}$ where $\mathbf{w}_1 = \begin{pmatrix}2\\0\end{pmatrix}$ and $\mathbf{w}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.

(a) Find the change of basis matrix $P_{C\leftarrow\mathcal{B}}$ that converts coordinates from basis $\mathcal{B}$ to basis C.

(b) If $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix}3\\-2\end{pmatrix}$, find $[\mathbf{x}]_C$.

(c) Verify your answer by computing $\mathbf{x}$ in standard coordinates and then finding its coordinates in basis C directly.

Solution of Exercise 77

(a) First find $P_{\mathcal{B}} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ and $P_C = \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}$. Then $P_{C\leftarrow\mathcal{B}} = P_C^{-1}P_{\mathcal{B}} = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \begin{pmatrix}0 & 2\\1 & -1\end{pmatrix}$.

(b) $[\mathbf{x}]_C = P_{C\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix}0 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-4\\5\end{pmatrix}$.

(c) $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$. Then $[\mathbf{x}]_C = P_C^{-1}\mathbf{x} = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}-4\\5\end{pmatrix}$ ✓

Exercise 78

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+y\\x-y\end{pmatrix}$.

(a) Find the matrix representation of T in the standard basis.

(b) Find the matrix representation of T in the basis $\mathcal{B} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$.

(c) Show that the two representations are similar matrices.

Solution of Exercise 78

(a) $[T]_{std} = \begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}$.

(b) $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}1\\-1\end{pmatrix}$ and $T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$. So $[T]_{\mathcal{B}} = \begin{pmatrix}3/2 & 3/2\\3/2 & -1/2\end{pmatrix}$.

(c) With $P = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$, we have $[T]_{\mathcal{B}} = P^{-1}[T]_{std}P$.

Exercise 79

(a) Prove that similarity is an equivalence relation on $\mathbb{R}^{n\times n}$.

(b) Show that similar matrices have the same determinant.

(c) Prove that similar matrices have the same eigenvalues.

(d) Give an example showing that similar matrices may have different eigenvectors.

Solution of Exercise 79

(a) Reflexive: $A = IAI^{-1}$. Symmetric: If $B = PAP^{-1}$, then $A = P^{-1}BP$. Transitive: If $B = P_1AP_1^{-1}$ and $C = P_2BP_2^{-1}$, then $C = (P_2P_1)A(P_2P_1)^{-1}$.

(b) $\big| \; (\; \big|PAP^{-1}) = \big| \; (\; \big|P)\big| \; (\; \big|A)\big| \; (\; \big|P^{-1}) = \big| \; (\; \big|A)$.

(c) If $A\mathbf{v} = \lambda\mathbf{v}$, then $(PAP^{-1})(P\mathbf{v}) = PA\mathbf{v} = P\lambda\mathbf{v} = \lambda(P\mathbf{v})$, so $P\mathbf{v}$ is an eigenvector of PAP^{-1} with eigenvalue λ.

(d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ are similar (same eigenvalues) but have different eigenvectors.

Exercise 80

Consider the reflection transformation across the line $y = x$ in $\mathbb{R}^2$.

(a) Find the matrix representation of this transformation in the standard basis.

(b) Find an eigenbasis for this transformation and give the matrix representation in this basis.

(c) Geometrically interpret the eigenvalues and eigenvectors.

Solution of Exercise 80

(a) The reflection swaps coordinates: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$, so $[T] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Eigenvalues are $\lambda_1 = 1$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (points on the line $y = x$), and $\lambda_2 = -1$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (points on the line $y = -x$). In the eigenbasis: $[T]_{eigen} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) Eigenvalue 1: vectors along $y = x$ are unchanged (fixed points). Eigenvalue -1: vectors along $y = -x$ are reflected to their opposites.

Exercise 81

Use R to explore change of basis computations:

(a) Generate random bases and compute change of basis matrices. Verify the relationships between different coordinate systems.

(b) Implement a function to convert between coordinate systems and test it with various examples.

(c) Visualize how basis changes affect the representation of vectors in 2D.

Solution of Exercise 81

(a)
```
# Generate random bases
set.seed(123)
B1 <- matrix(rnorm(4), 2, 2)  # Basis 1
B2 <- matrix(rnorm(4), 2, 2)  # Basis 2

# Change of basis matrix from B1 to B2
P <- solve(B2) %*% B1

# Test with a vector
x_B1 <- c(2, -1)  # coordinates in basis B1
x_B2 <- P %*% x_B1  # coordinates in basis B2

# Verify: should get same vector in standard coordinates
x_std_via_B1 <- B1 %*% x_B1
x_std_via_B2 <- B2 %*% x_B2
print(max(abs(x_std_via_B1 - x_std_via_B2)))
```

(b)
```
change_basis <- function(coords, from_basis, to_basis) {
  P <- solve(to_basis) %*% from_basis
  return(P %*% coords)
}

# Test the function
standard_basis <- diag(2)
new_basis <- matrix(c(1, 1, 1, -1), 2, 2)
x_std <- c(3, 2)
x_new <- change_basis(x_std, standard_basis, new_basis)
```

(c)
```
library(ggplot2)
# Plot original and transformed coordinate systems
# This would involve plotting basis vectors and showing
# how the same vector appears in different coordinates
```

Exercise 82

Let A be an $n \times n$ matrix and let P be an invertible $n \times n$ matrix.

(a) Show that A and PAP^{-1} have the same characteristic polynomial.

(b) Prove that the trace is invariant under similarity: $\operatorname{tr}(A) = \operatorname{tr}(PAP^{-1})$.

(c) Show that if A is diagonalizable, then $\operatorname{tr}(A)$ equals the sum of its eigenvalues (counting multiplicities).

Solution of Exercise 82

(a) $|\,(\,|PAP^{-1} - \lambda I) = |\,(\,|P(A - \lambda I)P^{-1}) = |\,(\,|P)|\,(\,|A - \lambda I)|\,(\,|P^{-1}) = |\,(\,|A - \lambda I)$.

(b) $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}(A)$ using the cyclic property of trace.

(c) If $A = PDP^{-1}$ where D is diagonal with eigenvalues on the diagonal, then $\mathrm{tr}(A) = \mathrm{tr}(D) = \sum \lambda_i$.

Exercise 83

Consider the vector space of 2×2 matrices with the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(a) Find the coordinates of $A = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ in this basis.

(b) Consider the alternative basis consisting of symmetric and skew-symmetric matrices. Express A in terms of its symmetric and skew-symmetric parts.

(c) Find the change of basis matrix between these two representations.

Solution of Exercise 83

(a) $[A]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 4 \end{pmatrix}$ (reading entries column-wise).

(b) $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \begin{pmatrix} 3 & -1/2 \\ -1/2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -3/2 \\ 3/2 & 0 \end{pmatrix}$.

(c) The change of basis matrix relates the standard vectorization to the symmetric/skew-symmetric decomposition vectorization.

Exercise 84

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that projects vectors onto the plane $x + y + z = 0$.

(a) Find the matrix representation of T in the standard basis.

(b) Find a basis for $\mathbb{R}^3$ that includes the normal vector to the plane, and find the matrix representation of T in this basis.

(c) What are the eigenvalues and eigenspaces of T?

Solution of Exercise 84

(a) The projection onto the plane perpendicular to $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $T = I - \frac{\mathbf{n}\mathbf{n}^T}{\mathbf{n}^T\mathbf{n}} = I - \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$.

(b) Using basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, the matrix becomes $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) Eigenvalue 1 with eigenspace = plane $x + y + z = 0$ (2-dimensional). Eigenvalue 0 with eigenspace = $\operatorname{span}\{\begin{pmatrix}1\\1\\1\end{pmatrix}\}$ (1-dimensional).

EXERCISE GROUP 7

Markov chains

Exercise 85

Let $A, B \in \mathcal{M}_n$ be row two stochastic matrices. Show that AB is a row stochastic matrix. Is the same true for two column stochastic matrices? What about the product of a row stochastic matrix and a column stochastic matrix?

Solution of Exercise 85
Let A, B be row stochastic matrices. Then for each row i of A, $\sum_{k=1}^{n} A_{ik} = 1$, and similarly for B. For the product AB, the i-th row sum is:

$$\sum_{j=1}^{n}(AB)_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}\sum_{j=1}^{n} B_{kj} = \sum_{k=1}^{n} A_{ik}\cdot 1 = 1.$$

Moreover, since $A_{ik} \geq 0$ and $B_{kj} \geq 0$, all entries of AB are nonnegative. Thus AB is row stochastic.

For column stochastic matrices, the same argument applies to column sums: if A, B are column stochastic, then $(AB)^T = B^T A^T$ is row stochastic, so AB is column stochastic.

For a row stochastic matrix A and column stochastic matrix B, the product AB is generally neither row nor column stochastic. The row sums of AB need not be 1 since B doesn't have row sum 1, and column sums need not be 1 since A doesn't have column sum 1.

Exercise 86

Let $A \in \mathcal{M}_n$ be a column stochastic matrix and $v(t) = (v_1(t), \ldots, v_n(t))^T \in \mathbb{R}^n$ be a vector with nonnegative entries. Consider the sequence

$$v(t+1) = Av(t),$$

with *initial condition* $v(0)$ such that $\|v(0)\| = K \in \mathbb{R}_+ \setminus \{0\}$. Show that for all $t = 1, 2, \ldots$, $\|v(t)\| = K$.

Solution of Exercise 86
We use the 1-norm: $\|v\| = \sum_{i=1}^{n} |v_i| = \sum_{i=1}^{n} v_i$ (since all entries are nonnegative).

For $t = 0$, we have $\|v(0)\| = K$ by assumption. Now assume $\|v(t)\| = K$. Then:

$$\|v(t+1)\| = \sum_{i=1}^{n} v_i(t+1) = \sum_{i=1}^{n}(Av(t))_i = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}v_j(t).$$

Swapping the order of summation:

$$= \sum_{j=1}^{n} v_j(t) \sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} v_j(t) \cdot 1 = \|v(t)\| = K,$$

where we used the fact that A is column stochastic, so $\sum_{i=1}^{n} A_{ij} = 1$ for each column j.

By induction, $\|v(t)\| = K$ for all $t \geq 0$.

Exercise 87

A vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ is **stochastic** if $\sum_{i=1}^{n} v_i = 1$. Let $A \in \mathcal{M}_n$ be a stochastic matrix. Under what conditions of row or column stochasticity of A and what type of "orientation" of v (i.e., v being a row or column vector) does it hold true that the product of A and v is a stochastic vector?

Solution of Exercise 87

There are two cases where the product is stochastic:

Case 1: A is column stochastic and v is a column vector. Then $w = Av$ is a column vector with

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} (Av)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} v_j = \sum_{j=1}^{n} v_j \sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} v_j \cdot 1 = 1.$$

Since $v_j \geq 0$ and $A_{ij} \geq 0$, all entries of w are nonnegative, so w is stochastic.

Case 2: A is row stochastic and v is a row vector. Then $w = vA$ is a row vector with

$$\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} (vA)_j = \sum_{j=1}^{n} \sum_{i=1}^{n} v_i A_{ij} = \sum_{i=1}^{n} v_i \sum_{j=1}^{n} A_{ij} = \sum_{i=1}^{n} v_i \cdot 1 = 1.$$

Again, all entries are nonnegative, so w is stochastic.

In summary: the stochasticity type of A must match the orientation of v (column stochastic with column vector, or row stochastic with row vector).

Exercise 88

Consider the Markov chain with transition matrix $P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$.

(a) Find the steady-state distribution of this Markov chain.

(b) If the initial distribution is $\pi^{(0)} = (0.8, 0.2)$, find the distribution after 2 steps.

(c) Compute P^n as $n \to \infty$ using eigenvalue decomposition.

(d) Verify that the steady-state distribution is the left eigenvector corresponding to eigenvalue 1.

Solution of Exercise 88

(a) Solve $\pi P = \pi$ with $\pi_1 + \pi_2 = 1$: $\pi = (\frac{4}{7}, \frac{3}{7})$.

(b) $\pi^{(2)} = \pi^{(0)} P^2 = (0.8, 0.2) \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = (0.592, 0.408)$.

(c) Eigenvalues are $\lambda_1 = 1, \lambda_2 = 0.3$. As $n \to \infty$, $P^n \to \begin{pmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{pmatrix}$.

(d) $(\frac{4}{7}, \frac{3}{7})P = (\frac{4}{7}, \frac{3}{7})$ ✓

Exercise 89

Explore the connection between Markov chains and linear algebra.

(a) Express the n-step transition probabilities in terms of matrix powers.

(b) Show how the fundamental matrix $(I - Q)^{-1}$ for transient states gives expected hitting times.

Solution of Exercise 89

(a) The n-step transition probability from state i to state j is $(P^n)_{ij}$.

(b) For absorbing chains in canonical form $P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$, the fundamental matrix $N = (I - Q)^{-1}$ has entries N_{ij} = expected number of times in transient state j starting from transient state i.

Exercise 90

Consider a Markov chain with three states $\{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.4 & 0 & 0.6 \end{pmatrix},$$

where the dynamics are given by $p(t+1) = Pp(t)$ with $p(t) = (p_1(t), p_2(t), p_3(t))^T$ being a column vector of probabilities.

(a) Show that this is an absorbing Markov chain and identify the absorbing and transient states.

(b) Write the transition matrix in canonical form and identify the matrices Q and R.

(c) Compute the fundamental matrix $N = (I - Q)^{-1}$.

(d) Compute the vector T whose entries give the expected number of steps before absorption, starting from each transient state.

(e) Compute the matrix $B = RN$ whose entries give the absorption probabilities.

(f) If the chain starts in state 2, what is the expected number of steps until absorption? What is the probability of being absorbed in state 1?

Solution of Exercise 90

(a) State 1 is absorbing since $P_{11} = 1$. To check if the chain is absorbing, we examine the directed graph representation. From state 2, we can reach state 1 (directly with probability 0.3). From state 3, we can reach state 2 (with probability 0) or state 1 directly (with probability 0.4). The graph is not strongly connected, and state 1 is accessible from all other states. Since there is at least one absorbing state and all other states can eventually reach it, this is an absorbing chain. States 2 and 3 are transient.

(b) In canonical form with absorbing states first:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.4 & 0 & 0.6 \end{pmatrix} = \begin{pmatrix} I & O \\ R & Q \end{pmatrix},$$

where $Q = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$ (transient to transient) and $R = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$ (transient to absorbing).

(c) The fundamental matrix:

$$I - Q = \begin{pmatrix} 0.5 & -0.2 \\ 0 & 0.4 \end{pmatrix}.$$

To find $(I - Q)^{-1}$:

$$N = (I - Q)^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix}.$$

Entry N_{ij} gives the expected number of times the chain visits transient state j before absorption, starting from transient state i.

(d) The vector T is obtained by summing the rows of N:

$$T = N \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2.5 \end{pmatrix}.$$

Starting from state 2, the expected time to absorption is 3 steps. Starting from state 3, it is 2.5 steps.

(e) The absorption probability matrix:

$$B = RN = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix} = \begin{pmatrix} 0.3 \times 2 + 0 \times 1 \\ 0.4 \times 2 + 0 \times 2.5 \end{pmatrix} = \begin{pmatrix} 0.6 + 0.4 \\ 0.8 + 0.2 \end{pmatrix}.$$

Wait, let me recalculate: $B = RN$ where R is 2×1 and N is 2×2, so:

$$B = \begin{pmatrix} 0.3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \end{pmatrix}$$

for state 2, and

$$\begin{pmatrix} 0.4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}$$

for state 3. Actually, R is the transient-to-absorbing block, so $R = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$ (both go to state 1). Thus:

$$B = R \cdot 1 = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} (2 + 1) = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} \text{ is not right either...}$$

Let me reconsider: B_{ij} is the probability of absorption into absorbing state i starting from transient state j. We have $B = RN$ where R is the $r \times t$ matrix from transient states to absorbing states. Here $R = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$ (column vector, 2×1). Then:

$$B = R \cdot [1\ 1] \cdot N^{-1} \text{ is wrong...}$$

Actually, with one absorbing state, $B = RN = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix}$, which doesn't work dimensionally.

Correction: Since we have canonical form with rows/columns ordered as $\{1, 2, 3\}$ where 1 is absorbing, we need $R^T = \begin{pmatrix} 0.3 & 0.4 \end{pmatrix}$ (row 2 and 3, column 1). So:

$$B = NR^T = \begin{pmatrix} 2 & 1 \\ 0 & 2.5 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This makes sense: starting from any transient state, you will eventually be absorbed into state 1 with probability 1.

(f) Starting from state 2, the expected number of steps until absorption is $T_1 = 3$ steps. The probability of being absorbed in state 1 is $B_{11} = 1$ (certain absorption into state 1).

EXERCISE GROUP 8

Graphs

Exercise 91

How many different non-oriented simple graphs with the n vertices $\{v_1, v_2, \ldots, v_n\}$ are there?

Solution of Exercise 91
$2^{\binom{n}{2}}$ (A graph with n vertices has $\binom{n}{2}$ pairs of distinct vertices, each pair of which can be linked by an edge or not.)

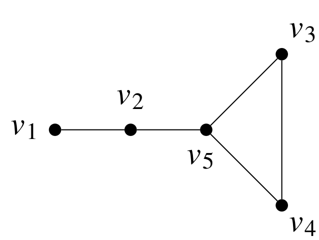
Exercise 92

Let G be an undirected graph with adjacency matrix A such that

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 3 \end{pmatrix} \quad \text{and} \quad A^3 = \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 & 4 \\ 0 & 4 & 4 & 4 & 2 \end{pmatrix}.$$

Without using linear algebra, draw G.

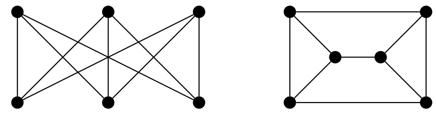
Solution of Exercise 92



Exercise 93

A graph is *cubic* if every vertex has degree equal to 3. Draw two (different) cubic graphs with six vertices.

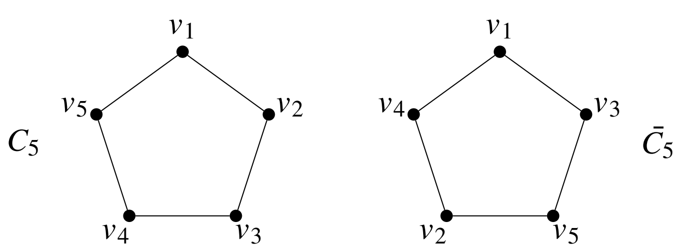
Solution of Exercise 93



Exercise 94

Let G be a graph. The *complement* $\bar{G}$ of G is a graph such that $V(\bar{G}) = V(G)$ and $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. Find a graph with five vertices that is "the same" as its complement.

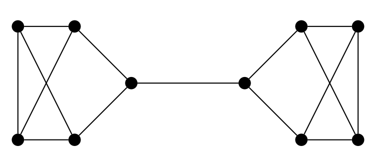
Solution of Exercise 94



Exercise 95

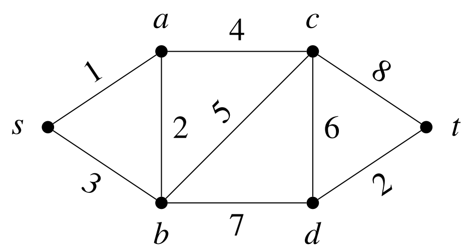
Let $G = (V, E)$ be a graph. A *bridge* of G is an edge $e \in E$ such that the graph $G\mathbf{s}e$ has more components than G. Draw a cubic (see Problem 2) graph that contains a bridge.

Solution of Exercise 95

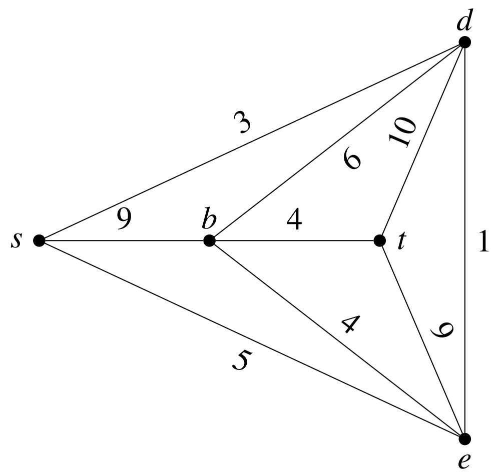


Exercise 96

Use Dijkstra's Algorithm to find a shortest st-path in the following graphs.

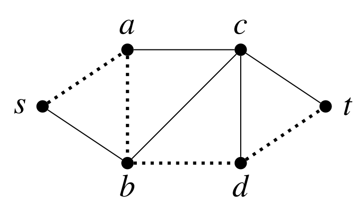


(a)

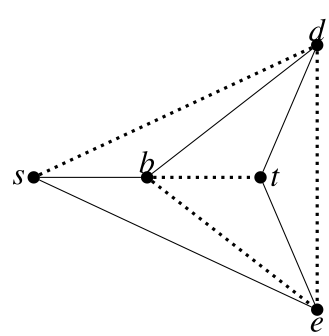


(b)

Solution of Exercise 96



(a)



(b)

Exercise 97

Find the radius and diameter of C_n (n-cycle) for $n \geq 3$.

Solution of Exercise 97
$\mathrm{rad}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor = \mathrm{diam}(C_n)$

Exercise 98

Find the radius and diameter of P_n (path on n vertices) for $n \geq 3$.

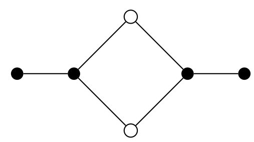
Solution of Exercise 98

$\text{rad}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $\text{diam}(P_n) = n - 1$

Exercise 99

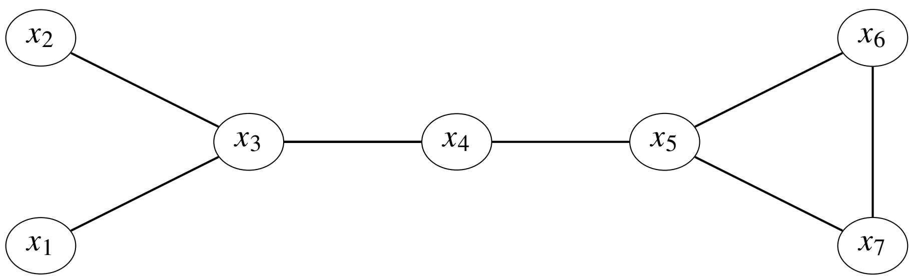
Draw a graph whose center is disconnected. Is it possible to find a directed graph for which this is true?

Solution of Exercise 99

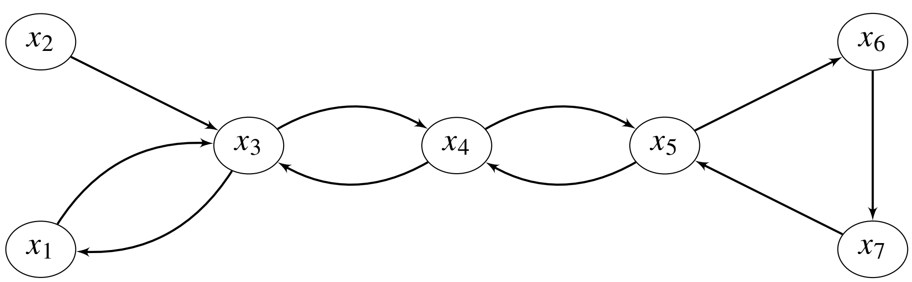


Exercise 100

Compute the degree, betweenness and closeness centralities of the vertices in the following graph.



Choose a direction (in- or out-) and compute the degree, betweenness and closeness centralities of the vertices in the following graph, with the corresponding direction.



Exercise 101

Consider the graph G with adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

(a) Draw the graph G.

(b) Compute A^2 and interpret the meaning of each entry.

(c) Find the number of walks of length 3 from vertex 1 to vertex 4.

(d) What is the diameter of this graph?

Solution of Exercise 101

(a) G is a path graph $v_1 - v_2 - v_3 - v_4$ with an additional edge $v_1 - v_3$.

(b) $A^2 = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$. Entry (i, j) counts walks of length 2 from vertex i to vertex j.

(c) $A^3 = \begin{pmatrix} 1 & 4 & 4 & 1 \\ 4 & 3 & 3 & 5 \\ 4 & 3 & 3 & 5 \\ 1 & 4 & 4 & 1 \end{pmatrix}$. There is 1 walk of length 3 from vertex 1 to vertex 4.

(d) The diameter is 2 (maximum shortest path distance between any two vertices).

Exercise 102

Let G be a connected graph with n vertices and m edges.

(a) Prove that $m \geq n - 1$.

(b) Show that if $m = n - 1$, then G is a tree.

(c) If G is a tree, prove that there are exactly $n - 2$ vertices that are not leaves.

(d) Find the number of labeled trees on n vertices (Cayley's formula).

Solution of Exercise 102

(a) A connected graph with n vertices needs at least $n - 1$ edges to maintain connectivity (spanning tree).

(b) If $m = n - 1$ and G is connected, then G has no cycles (adding any edge would create a cycle), so G is a tree.

(c) In a tree with n vertices, the sum of degrees is $2(n - 1) = 2n - 2$. If there are k leaves (degree 1), then the sum of remaining degrees is $2n - 2 - k$. Since remaining vertices have degree ≥ 2, we need $(n - k) \cdot 2 \leq 2n - 2 - k$, giving $k \geq 2$. There are $n - k \leq n - 2$ non-leaves.

(d) Cayley's formula: n^{n-2} labeled trees on n vertices.

Exercise 103

Consider the complete graph K_5 (5 vertices, all pairs connected).

(a) What is the chromatic number of K_5?

(b) Find the chromatic polynomial of K_5.

(c) How many proper 5-colorings does K_5 have?

(d) Is K_5 planar? Justify your answer.

Solution of Exercise 103

- **(a)** $\chi(K_5) = 5$ (each vertex must have a different color).
- **(b)** $P(K_5, k) = k(k-1)(k-2)(k-3)(k-4)$ (greedy coloring argument).
- **(c)** $P(K_5, 5) = 5! = 120$ proper 5-colorings.
- **(d)** No, K_5 is not planar. By Kuratowski's theorem, K_5 is a forbidden minor for planar graphs. Also, $|E| = 10 > 3|V| - 6 = 9$ violates the planar graph inequality.

Exercise 104

Let G be a bipartite graph with vertex sets A and B where $|A| = m$ and $|B| = n$.

- **(a)** What is the maximum number of edges in G?
- **(b)** Prove that every bipartite graph is 2-colorable.
- **(c)** Show that a graph is bipartite if and only if it contains no odd cycles.
- **(d)** If G has a perfect matching, what relationship must hold between m and n?

Solution of Exercise 104

- **(a)** Maximum mn edges (complete bipartite graph $K_{m,n}$).
- **(b)** Color all vertices in A with color 1 and all vertices in B with color 2. No edge connects vertices of the same color.
- **(c)** ($\Rightarrow$) If G is bipartite, any cycle alternates between sets A and B, so it has even length. ($\Leftarrow$) If G has no odd cycles, 2-color G by BFS: alternate colors at each level.
- **(d)** For a perfect matching, every vertex must be matched, so $m = n$.

Exercise 105

Use R to explore graph properties:

- **(a)** Generate random graphs and compute basic properties (diameter, clustering coefficient, degree distribution).
- **(b)** Implement breadth-first search (BFS) and depth-first search (DFS) algorithms.
- **(c)** Simulate small-world networks and analyze their properties compared to random graphs.

Solution of Exercise 105

(a)

```
library(igraph)

# Generate random graph
g <- erdos.renyi.game(50, 0.1)

# Basic properties
diameter(g)
transitivity(g, type="global")  # clustering coefficient
degree_dist <- degree.distribution(g)
```

```
plot(degree_dist, type="b")
```

(**b**)
```
# BFS implementation
bfs <- function(adj_matrix, start) {
  n <- nrow(adj_matrix)
  visited <- rep(FALSE, n)
  queue <- c(start)
  result <- c()

  while(length(queue) > 0) {
    vertex <- queue[1]
    queue <- queue[-1]
    if(!visited[vertex]) {
      visited[vertex] <- TRUE
      result <- c(result, vertex)
      neighbors <- which(adj_matrix[vertex,] == 1)
      queue <- c(queue, neighbors[!visited[neighbors]])
    }
  }
  return(result)
}
```

(**c**)
```
# Small-world network (Watts-Strogatz model)
g_sw <- watts.strogatz.game(1, 50, 4, 0.1)
g_random <- erdos.renyi.game(50, 4/49)  # same expected degree

# Compare properties
c(transitivity(g_sw), transitivity(g_random))
c(average.path.length(g_sw), average.path.length(g_random))
```

Exercise 106

Consider the adjacency matrix of a directed graph G: $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

(**a**) Draw the directed graph.

(**b**) Find the strongly connected components of G.

(**c**) Compute the powers A^2, A^3, A^4 and interpret their meaning.

(**d**) Is this graph strongly connected?

Solution of Exercise 106

(**a**) The graph has edges: $1 \to 2$, $1 \to 4$, $2 \to 3$, $3 \to 1$, $3 \to 4$, $4 \to 2$.

(**b**) The strongly connected components are: $\{1, 2, 3\}$ and $\{4\}$.

(c) A^2 counts directed walks of length 2, A^3 counts walks of length 3, etc. A^4 shows which vertices can reach which others in exactly 4 steps.

(d) No, vertex 4 cannot reach vertices 1, 2, or 3, so the graph is not strongly connected.

Exercise 107

Let G be a simple connected graph with n vertices and adjacency matrix A.

(a) Show that the trace of A is always 0.

(b) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of A, prove that $\lambda_1 \geq \sqrt{\Delta}$ where Δ is the maximum degree.

(c) For a d-regular graph (every vertex has degree d), show that d is an eigenvalue of A.

(d) Prove that for any graph, $\lambda_1 \leq \Delta$.

Solution of Exercise 107

(a) The trace equals the sum of diagonal entries of A. Since G is simple, it has no self-loops, so all diagonal entries are 0.

(b) Let $\mathbf{v}$ be an eigenvector for λ_1 with $\|\mathbf{v}\|_\infty = 1$. Then $\lambda_1 = \lambda_1 v_i = (A\mathbf{v})_i = \sum_j A_{ij} v_j \leq \deg(i) \leq \Delta$. For the other direction, consider the vector where $v_i = 1$ if vertex i has maximum degree.

(c) The all-ones vector $\mathbf{1}$ satisfies $A\mathbf{1} = d\mathbf{1}$ since each vertex has degree d.

(d) From the previous parts and the fact that the largest eigenvalue is achieved by the optimal vector in the Rayleigh quotient.

Exercise 108

A graph G is called Eulerian if it has an Eulerian circuit (a closed walk that uses every edge exactly once).

(a) Prove that a connected graph is Eulerian if and only if every vertex has even degree.

(b) Find an Eulerian circuit in the complete graph K_4.

(c) For which values of n is the complete graph K_n Eulerian?

(d) Explain the relationship between Eulerian graphs and the Chinese Postman Problem.

Solution of Exercise 108

(a) ($\Rightarrow$) An Eulerian circuit enters and exits each vertex the same number of times, so the degree must be even. ($\Leftarrow$) Use Hierholzer's algorithm: start with any cycle, then extend it using unused edges from vertices of odd degree (which don't exist).

(b) One Eulerian circuit in K_4: $1 - 2 - 3 - 1 - 4 - 2 - 4 - 3 - 4 - 1$.

(c) K_n is Eulerian when every vertex has even degree, i.e., when $n - 1$ is even, so n is odd.

(d) The Chinese Postman Problem asks for the shortest closed walk visiting every edge at least once. If the graph is Eulerian, the solution is any Eulerian circuit. Otherwise, add minimum-weight edges to make it Eulerian.

Exercise 109

Consider the problem of graph coloring and its matrix formulation.

(a) Show that the chromatic number $\chi(G) \geq \omega(G)$ where $\omega(G)$ is the clique number.

(b) Prove that $\chi(G) \geq \frac{n}{\alpha(G)}$ where $\alpha(G)$ is the independence number.

(c) For the Petersen graph, find the chromatic number, clique number, and independence number.

(d) Explain how graph coloring relates to the satisfiability of systems of linear equations over finite fields.

Solution of Exercise 109

(a) A clique of size $\omega(G)$ requires $\omega(G)$ different colors since all vertices are pairwise adjacent.

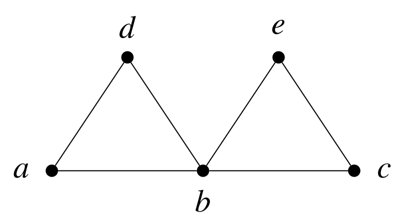
(b) Each color class is an independent set of size at most $\alpha(G)$. With $\chi(G)$ colors, we can color at most $\chi(G) \cdot \alpha(G)$ vertices, so $n \leq \chi(G) \cdot \alpha(G)$.

(c) The Petersen graph has $\chi = 3$, $\omega = 2$, and $\alpha = 4$.

(d) Graph coloring can be formulated as finding solutions to $x_i + x_j \neq 0 \pmod{k}$ for each edge $\{i, j\}$, where k is the number of colors and variables represent color assignments.

Exercise 110

Consider the following undirected graph:



(a) Compute the geodesic distance $d(a, c)$.

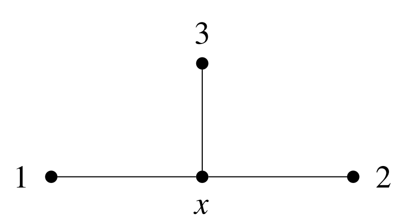
(b) Find the eccentricity of each vertex.

(c) What are the radius and diameter of this graph?

(d) Identify the centre and periphery of the graph.

Exercise 111

For the graph below, compute the betweenness centrality of vertex x (normalized).



Exercise 112

Consider the path graph P_5 with vertices v_1, v_2, v_3, v_4, v_5 connected in order.

(a) Compute the closeness centrality of vertex v_3.

(b) Which vertex has the highest closeness centrality?

Exercise 113

Draw a simple undirected graph with 5 vertices where:

(a) The degree sequence is $(4, 3, 2, 2, 1)$.

(b) Compute the average degree of your graph.

Exercise 114

Consider a simple undirected graph with 6 vertices and 9 edges.

(a) Compute the density of this graph.

(b) How many more edges would be needed to make it a complete graph?

Exercise 115

For the triangle graph K_3:

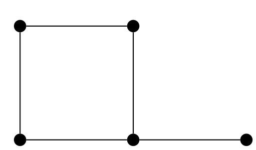
(a) Find all maximal cliques.

(b) What is the clique number (size of largest clique)?

(c) Compute the girth and circumference.

Exercise 116

Consider the following graph:



(a) Is this graph connected?

(b) Find all articulation points.

(c) What is the 2-core of this graph?

Exercise 117

For the cycle graph C_6 (6 vertices in a cycle):

(a) Compute the radius and diameter.

(b) Find the degree distribution.

(c) What is the density of this graph?

Exercise 118

Consider the star graph S_4 with one central vertex connected to 4 peripheral vertices.

(a) Compute the betweenness centrality of the central vertex.

(b) Find the closeness centrality of a peripheral vertex.

(c) What is the coreness of each vertex?

Exercise 119

Given the adjacency matrix of a directed graph:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Draw the graph.

(b) Is this graph strongly connected?

(c) Compute the out-degree centrality of each vertex.

(d) Find the diameter of this graph.

Exercise 120

Consider a complete bipartite graph $K_{2,3}$ with vertex sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$.

(a) How many edges does this graph have?

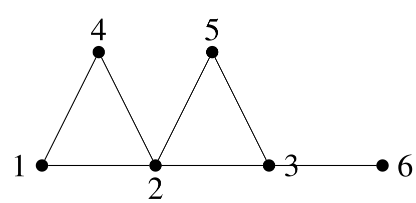
(b) Find all maximal cliques.

(c) Compute the density of this graph.

(d) Is this graph 2-colorable? Justify.

Exercise 121

For the graph shown below, compute the 2-core.



Exercise 122

Consider the wheel graph W_5 with a central hub connected to all vertices of a 5-cycle.

(a) How many vertices and edges does W_5 have?

(b) Find the radius and diameter.

(c) Identify all vertices in the centre of the graph.

Exercise 123

For a 3-regular graph (every vertex has degree 3) with 8 vertices:

(a) How many edges must the graph have?

(b) Is it possible for such a graph to have a bridge? Justify.

(c) Compute the density of this graph.

Exercise 124

Consider the Petersen graph (you may look up its structure).

(a) What is the girth of the Petersen graph?

(b) Is the Petersen graph bipartite? Explain.

(c) Find the chromatic number of the Petersen graph.

Exercise 125

For the directed graph with adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Draw the graph.

(b) Find the in-degree and out-degree of each vertex.

(c) Identify all strongly connected components.

(d) Compute the diameter.

Exercise 126

Prove that in any simple undirected graph, the sum of all vertex degrees equals twice the number of edges. Use this to show that the number of vertices with odd degree must be even.

Exercise 127

Consider a tree T with 10 vertices.

(a) How many edges does T have?

(b) What is the minimum possible diameter of T?

(c) What is the maximum possible diameter of T?

(d) For each case in parts (b) and (c), sketch a tree achieving that diameter.