EXERCISES 1.2 Basic Exercises

1.14 In Example 1.5(d) on page 16, we showed that it is possible for the sets in a collection to not be pairwise disjoint even though the intersection of all the sets is empty. If the sets in a collection are pairwise disjoint, must the intersection of all the sets be empty?

1.15 Draw a Venn diagram showing three subsets, A, B, and C, such that no two are disjoint but that $A \cap B \cap C = \emptyset$.

1.16 Make a Venn diagram with four subsets, A, B, C, and D, such that $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, $C \cap D \neq \emptyset$, and $A \cap D \neq \emptyset$ but that $A \cap B \cap C \cap D = \emptyset$.

1.17 Give an example of a collection of sets satisfying the following properties: The collection contains at least four sets, the sets are not pairwise disjoint, and every three sets have an empty intersection.

1.18 Let $U = \mathcal{R}$ and, for each $n \in \mathcal{N}$, define $A_n = [0, 1/n]$.

a) Determine $\bigcap_{n=1}^{4} A_n$ and $\bigcup_{n=1}^{4} A_n$. **b)** Determine $\bigcap_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} A_n$.

1.19 Express each of the following sets in a simple form in which "$\bigcup$" and "$\bigcap$" don't occur.

a) $\bigcup_{n=1}^{\infty}[1 + 1/n,\ 2 - 1/n]$ **b)** $\bigcup_{n=1}^{\infty}[1,\ 2 - 1/n]$ **c)** $\bigcap_{n=1}^{\infty}(1 - 1/n,\ 2 + 1/n)$
d) $\bigcap_{n=1}^{\infty}(3 - 1/n,\ 3 + 1/n)$ **e)** $\bigcap_{n=1}^{\infty}(n, \infty)$ **f)** $\bigcap_{n=1}^{\infty}(5 - 1/n,\ 5)$
g) $\bigcap_{n=1}^{\infty}(5 - 1/n,\ 6)$

1.20 Let the universal set U be $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$.

a) List the members of $\{1, 2, 3\} \times \{3, 4, 5\}$.
b) List the members of $(\{1, 2, 3\} \times \{3, 4, 5\}) \cup (\{3, 4, 5\} \times \{1, 2, 3\})$.
c) List the members of $A = \big((\{1, 2, 3\} \times \{3, 4, 5\}) \cup (\{3, 4, 5\} \times \{1, 2, 3\})\big)^c$.
d) Write the set A in part (c) as the union of two Cartesian products—that is, in the form (some set $\times$ some set) $\cup$ (some set $\times$ some set).

1.21 List the members of each of the following sets.

a) $\{0, 1\}^3 = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ **b)** $\{0, 1\} \times \{0, 1\} \times \{1, 2\}$
c) $(\{a, b\} \cup \{c, d, e\}) \times \{f, g, h\}$ **d)** $(\{a, b\} \times \{f, g, h\}) \cup (\{c, d, e\} \times \{f, g, h\})$

1.22 Simplify the expression $([0, 2] \times [0, 2]) \cap ([1, 3] \times [1, 3])$, writing your answer in the form (some set $\times$ some set).

1.23 Find a countable collection of subintervals $I_1, I_2, \ldots$ of the interval $[0, 1]$ such that $I_j \cap I_k \neq \emptyset$ for all j and k, but $\bigcap_{i=1}^{\infty} I_i = \emptyset$.

1.24 Find a countable collection of intervals $I_1, I_2, \ldots$ of $\mathcal{R}$ such that $I_j \cap I_k \neq \emptyset$ for all j and k, but $\bigcap_{i=1}^{\infty} I_i$ consists of a single point (i.e., is a singleton set).

1.25 Express $\mathcal{R}$ as a countably infinite union of pairwise disjoint intervals, each of length 1.

Theory Exercises

1.26 Refer to De Morgan's laws, as given in Proposition 1.1 on page 11.

a) Verify part (b) of De Morgan's laws by using Venn diagrams.
b) Prove part (b) of De Morgan's laws mathematically in a manner similar to that done for part (a) of De Morgan's laws on page 11.
c) Prove part (b) of De Morgan's laws by using part (a) of De Morgan's laws.

1.27 Refer to the distributive laws, as given in Proposition 1.2 on page 12.

a) Verify parts (a) and (b) by using Venn diagrams.
b) Prove parts (a) and (b) mathematically.

If the die is rolled 10,000 times, roughly how many times will it come up
c) 3? **d)** 3 or more?

1.49 Suppose that 60% of the voters in a population will vote *yes* on a particular proposition. A political opinion poll chooses voters one at a time from the population with replacement. Let n be the number of such randomly chosen voters who have been polled so far and let $n(Y)$ be the number of those sampled voters who will vote *yes*. For large n, express $n(Y)$ approximately as a function of n.

1.50 Solve each of the following problems.
a) List explicitly the elements of the set $\{ x^2 : x \in \{-2, -1, 0, 1, 2\} \}$.
b) Express $\{ x^2 : -2 < x < 2 \}$ as a bounded half-open interval.

1.51 Explain how the eight members of $\{0, 1\}^3$ can be regarded as the outcomes of the experiment of tossing a coin three times. Consider those eight members a finite population in the sense of Section 1.1. By examining the list of all eight members of the population (and not by some other method), find the probability that you get two heads and one tail when you toss a balanced coin three times.

1.52 Simplify $(A \cap B) \cup (A \cap B^c)$.

1.53 Solve each of the following problems.
a) Express $\bigcup_{n=1}^{\infty} (1/(n+1), 1/n]$ as a bounded half-open interval.
b) Express $\bigcup_{n=1}^{\infty} [1/(n+1), 1/n]$ as a bounded half-open interval.
c) One of parts (a) and (b) involves a union of pairwise disjoint sets; the other involves a union of sets that are not pairwise disjoint but whose intersection is empty. Which is which? Explain your answer.

1.54 Classify each of the following sets as finite, countably infinite, or uncountable.
a) $\{1, 2, 3\} \times \{2, 3, 4\}$ **b)** $\{1, 2, 3\} \times [2, 4]$ **c)** $[2, 4] \times \{1, 2, 3\}$
d) $\{1, 2, 3\} \times \{1, 2, 3, \ldots\}$ **e)** $[1, 3] \times [2, 4]$ **f)** $\bigcup_{n=1}^{\infty} \{n, n+1, n+2\}$
g) $\bigcap_{n=1}^{\infty} \{n, n+1, n+2\}$

1.55 Suppose that the universal set is $\mathcal{Z}$, the collection of all integers.
a) Use De Morgan's laws to simplify $(\{3, 4\}^c \cap \{4, 5\}^c)^c$ without explicitly mentioning any infinite sets.
b) Simplify the set in part (a) without De Morgan's laws by first finding $\{3, 4\}^c$ and $\{4, 5\}^c$, next finding their intersection, and then finding its complement.

Theory Exercises

1.56 Let $A_1, A_2, \ldots$ be a countably infinite collection of sets.
a) For each positive integer n, determine pairwise disjoint sets, $B_1, B_2, \ldots, B_n$, such that $\bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j$.
b) Find pairwise disjoint sets, $B_1, B_2, \ldots$, such that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

1.57 Let A, B, and C be subsets of a set U.
a) Show that $(A \cup B) \cap C \subset A \cup (B \cap C)$.
b) Give an example of three sets, A, B, and C, such that $(A \cup B) \cap C \neq A \cup (B \cap C)$.
c) Prove the *modular law:* If $A \subset C$, then $(A \cup B) \cap C = A \cup (B \cap C)$.
d) Show that, if $(A \cup B) \cap C = A \cup (B \cap C)$, then $A \subset C$.

1.58 Let A and B be subsets of a set U.
a) Show that $A \cup B$ is the intersection of all sets $C \subset U$ such that $A \subset C$ and $B \subset C$.
b) If $A \subset D$ and $B \subset D$, what is the relationship between D and $A \cup B$? Interpret your answer.