MATH 2080 Introductory Analysis

Chapter 1 The Real Numbers

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- the natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- the integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- the rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

One can perform addition, substraction, mutiplication and division to rational numbers. The results will still be rationals. This means $\mathbb{Q}$ is closed under these operations.

Is $\mathbb{Q}$ big enough for us to play “real” math? Is there a sort of a “real” number that is not rational?

The answer is affirmation. Such numbers are called irrationals.

Example 1

We define $\sqrt{2}$ to be the positive number x that satisfies the equation $x^2 = 2$. This is a well-defined number: consider the square with side length equal to 1. Then the Pythagras theorem tells us that the length of the diagonal shall be $\sqrt{2}$.

Theorem -6.1

$\sqrt{2}$ is irrational.

Siminally, you may prove that $\sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots$ are irrationals. In fact, for any prime number p, $\sqrt{p}$ must be irrational.

On a straight line, we choose a point O and call it the origin. We call this line the real line, and call the points on the line the real numbers.

Indeed, if a point x on the line is on the right side of O, then x represents the number given by the length ℓ_x of the line segment $\overline{Ox}$. If x is located to the left of O, then it represents the number $-\ell_x$.

- the real numbers: $\mathbb{R} = \{\text{all real numberts}\}$

A useful technical result

- the absolute value: If $x \in \mathbb{R}$, then $|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$
- $|x| = 0$ if and only if $x = 0$.
- triangle inequlity: $|x \pm y| \leq |x| + |y|$.

Theorem -6.2

Let $a, b \in \mathbb{R}$. Then $a = b$ if and only if for every $\varepsilon > 0$ we have $|a - b| < \varepsilon$.

Proof.

$\Longrightarrow$ (necessity): If $a = b$, then $a - b = 0$. So $|a - b| = 0 < \varepsilon$ for every $\varepsilon > 0$.
$\Longleftarrow$ (sufficiency): Suppose that $|a - b| < \varepsilon$ for all $\varepsilon > 0$. We claim that $a = b$. If not, then $|a - b| = \ell > 0$. Take $\varepsilon = \ell/2$. We have $\varepsilon > 0$, but obviously $|a - b| \not< \varepsilon$. This contradicts the assumption that we started from. So the claim $a = b$ must be true. □

Sets

A set is a collection of objects. Members are called elements or points of the set. If s is an element of a set S, we write $s \in S$. Otherwise, we write $s \notin S$. If a set A has no elements, we call it an empty set, denoted by $A = \emptyset$.

In this course, we only concern sets of real numbers.
To define a set, we may

1. list the elements, like $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$, and $\mathbb{E} = \{2, 4, 6, 8, \ldots\}$;
2. give the defining condition, like $\mathbb{Z} = \{n : \ n \in \mathbb{N}, \text{or } -n \in \mathbb{N}, \text{ or } n = 0\}$, $\mathbb{E} = \{m : \ m = 2n \text{ where } n \in \mathbb{N}\}$, $\mathbb{Q} = \{\frac{p}{q} : \ p, q \in \mathbb{Z}, q \neq 0\}$ and $\mathbb{Q}_2 = \{r \in \mathbb{Q} : \ r^2 < 2\}$.

Note: $\mathbb{Q}_2$ is a "small" subset of $\mathbb{R}_2 = \{r \in \mathbb{R} : \ r^2 < 2\}$. The latter is indeed the interval $(-\sqrt{2}, \sqrt{2})$, which contains a lot of irrationals.

Set operations

- Union: $A \cup B = \{x : \ x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x : \ x \in A \text{ and } x \in B\}$

Given a sequence of sets $A_1, A_2, A_3, \ldots$, we can have the union and the intersection of the sequence:

$$\bigcup_{n=1}^{\infty} A_n = \{x : \ x \in A_n \text{ for some } n \in \mathbb{N}\}$$
$$\bigcap_{n=1}^{\infty} A_n = \{x : \ x \in A_n \text{ for all } n \in \mathbb{N}\}$$

- set minus: $A \setminus B = \{x : \ x \in A \text{ but } \ x \notin B\}$
- complement: Let $A \subset \mathbb{R}$. Then $A^c = \mathbb{R} \setminus A = \{x \in \mathbb{R} : \ x \notin A\}$
- De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$
- In general,

$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty} A_n^c \ \text{ and } \ (\cap_{n=1}^{\infty} A_n)^c = \cup_{n=1}^{\infty} A_n^c$$

Upper and lower bounds

- A number $a \in \mathbb{R}$ is positive (resp. negative) if, on the real line, a is on the right (resp. left) side of O.
- The relation $a > b$ (or equivalently $b < a$) means $a - b$ is positive. While $a \geq b$ (or $b \leq a$) means either $a > b$ or $a = b$.
- $\mathbb{R}$ is totally ordered: Given any $a, b \in \mathbb{R}$, one (and only one) of the following holds, (1) $a > b$, (2) $a < b$, (3) $a = b$.
- A set $A \subset \mathbb{R}$ is called bounded above if there is $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Such b is called an upper bound of A. Similarly, A is bounded below if there is $c \in \mathbb{R}$ such that $a \geq c$ for all $a \in A$. Such c is called a lower bound of A.

Sup and inf

- $s \in \mathbb{R}$ is the least upper bound of the set A (also called the supremum of A) if it is an upper bound of A and for any upper bound b of A we have $b \geq s$. In the case we write $s = \sup A$.
- $t \in \mathbb{R}$ is the greatest lower bound of A (also called the infimum of A) if it is a low bound of A and for any lower bound c of A we have $c \leq t$. In the case we write $t = \inf A$.

Let $c \in \mathbb{R}$ and $A \subset \mathbb{R}$. Then $c + A$ denotes the set

$$c + A = \{x = c + a : \ a \in A\}.$$

It is very easy to prove the following.

- If A is bounded above, then so is $c + A$. Moreover, $\sup(c + A) = c + \sup A$

Completeness of $\mathbb{R}$

To determine the sup, the following lemma is useful.

Lemma -6.3

Let $A \subset \mathbb{R}$ be bounded above. Then a number $s \in \mathbb{R}$ is the least upper bound of A (i.e. $s = \sup A$) iff s is an upper bound of A and, for every $\varepsilon > 0$, there is an $a \in A$ such that $s - \varepsilon < a$.

The notation iff stands for "if and only if". Of course, we also have a similar result for inf. You should try to state and prove it.

Axiom of Completeness (AoC). If $A \subset \mathbb{R}$ is nonempty and bounded above, then $\sup A$ exists. Similarly, if A is bounded below, then $\inf A$ exists.

An axiom is an unprovable statement which we simply accept as a fact.

Some important consequences

Theorem -6.4 (Nested Interval property)

Let $I_n = [a_n, b_n]$, $n = 1, 2, \ldots$, be a sequence of bounded closed intervals such that $I_{n+1} \subset I_n$ for each n. Then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof.

Consider $A = \{a_1, a_2, \ldots\}$. It is easy to see that A is bounded above and each b_n is an upper bound of A.
By the Axiom of Completeness, $a = \sup A$ exists. So $a_n \leq a$ for all n. We have further that $a \leq b_n$ for all n (think of the reason). So $a_n \leq a \leq b_n$, i.e. $a \in I_n$ for all n Thus, $a \in \cap_{n=1}^{\infty} I_n$. □

Important consequences - continued

Theorem -6.5 (the Archimedean Property)

Let $x \in \mathbb{R}$. Then following hold.

1. *there is an $n \in \mathbb{N}$ such that $x < n$;*
2. *if $x > 0$, then there is an $n \in \mathbb{N}$ such that $x > 1/n$.*

Theorem -6.5 may look obvious. But it is not trivial as you might have thought.

Proof.

For (1): If n does not exist, then x is an upper bound of $\mathbb{N}$. By the AoC, $s = \sup \mathbb{N}$ exists. By definition of $\sup \mathbb{N}$, $s - 1$ is no longer an upper bound of $\mathbb{N}$. So there is $m \in \mathbb{N}$ such that $s - 1 < m$. Then $s < m + 1$. But $m + 1 \in \mathbb{N}$ and hence $m + 1 \leq \sup \mathbb{N} = s$, a contradiction. Thus n must exists.

For (2): Consider $y = \frac{1}{x}$. Apply (1) to y. We have $y < n$ for some $n \in \mathbb{N}$. Therefore, $\frac{1}{x} < n$ and hence $x > \frac{1}{n}$. The proof is complete. $\square$

Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem -6.6

For each nonempty open interval $(a, b) \subset \mathbb{R}$, there is a rational number $\frac{p}{q}$ such that $\frac{p}{q} \in (a, b)$.

Proof.

Sketch of the proof for reading:
After shifting the interval by a natual number if necessary, we may assume $a \geq 0$. We need to show $\exists m, n \in \mathbb{N}$ such that $a < \frac{m}{n} < b$.
Choose $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$ (why we can?). So $na + 1 < nb$.
Choose the smallest $m \in \mathbb{N}$ s.t. $na < m$. (why we can?). So $m - 1 \leq na < m$.
Then $na < m \leq na + 1 < nb$. So $a < \frac{m}{n} < b$. □

Theorem -6.6 means $\mathbb{Q}$ is dense in $\mathbb{R}$. Find the places in the proof where we have used Archimedean properties!

Cardinality

Remark

Let $\mathbb{I}$ denote the set of all irrational numbers. Then, applying Theorem -6.6, one can easily show that $\mathbb{I}$ is also dense in $\mathbb{R}$.

Actually, as a set, $\mathbb{I}$ is much big than $\mathbb{Q}$. To see this we need a way to compare the sizes of sets.

Definition -6.7

Two sets A and B are said to have the same cardinality if there is a one-to-one (abbreviated 1-1) and onto mapping $f : A \to B$. In this case, we write $A \sim B$. (Clearly, $\sim$ is an equivalent relation for sets.)

- Two finite sets have the same cardinality iff they have the same number of elements.
- An infinite set A is called countable if $A \sim \mathbb{N}$.
- Any infinite set must have a subset that is countable. (Using induction, we can construct such subset.)

Countable sets

So countable sets are a kind of "smallest" infinite sets.

Example 2

The set $\mathbb{E}$ of all even (natural) numbers is countable. So is the set of all odd numbers. Moreover, $\mathbb{Z}$ is countable.

The following is a surprising fact.

Theorem -6.8

$\mathbb{Q}$ is countable.

Proof.

Sketch: Let $A_1 = \{0\}$ and, for each $n > 1$, let $A_n =$

$$\{p/q : p \neq 0, p \in \mathbb{Z}\ q \in \mathbb{N} \text{ are prime to each other, } |p| + q = n\}.$$

Each A_n is finite and $\mathbb{Q} = \cup_{n=1}^{\infty} A_n$. We then can list elements of $\mathbb{Q}$ through listing elements of A_n one after the other. $\square$

Uncountable sets

The set A being countable really means its elements can be fully listed, i.e. one can write $A = \{a_1, a_2, \ldots\}$.
An infinite set is called uncountable if it is not countable.

Theorem -6.9

$\mathbb{R}$ is uncountable.

Proof.

Sketch: Use contridiction proof. If it were countable, then write $\mathbb{R} = \{r_1, r_2, \ldots\}$. We may use induction to get a sequence $I_1, I_2, \ldots$ of bounded closed intervals, such that, for each n, $I_{n+1} \subset I_n$ and $r_n \notin I_n$. Then $\cap_{n=1}^{\infty} I_n \neq \emptyset$ by the nested interval property. But the intersection contains no r_n by the choice of I_n, a contridition. $\square$

Note $(0, 1) \sim \mathbb{R}$. (consider the mapping $f(x) = \frac{1}{x}(x - \frac{1}{2})$ for $x \in (0, \frac{1}{2}]$, and $= \frac{1}{1-x}(x - \frac{1}{2})$ for $x \in (\frac{1}{2}, 1)$.) Thus, $(0, 1)$ is uncountable. So is any nontrivial interval I by a similar argument.

More on countable sets

Traditionally, a finite set is also regarded as countable.

Theorem -6.10

A subset of a countable set is countable.

Theorem -6.11

A finite union of countable sets is countable.

Theorem -6.12

The union of a countable collection of countable sets is still countable, i.e. $\cup_{n=1}^{\infty} A_n$ is countable if each A_n is countable.

Proof.

Sketch: We may list the elements of the union through diagonal counting. □

Using Theorem -6.12, combined with Theorems -6.8 and -6.9, we may immediately conclude that $\mathbb{I}$ is uncountable.