MATH 2080 Introductory Analysis
Chapter 2 Sequences and Series

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Sequences

An intuitional depiction of a sequence is an infinite list of numbers: $a_1, a_2, \ldots$. We notice that this list really defines a function on $\mathbb{N}$. This observation leads to a formal definition as follows.

Definition 2.1

A numerical sequence is a function $f: \mathbb{N} \to \mathbb{R}$

Note $\mathbb{R}$ may be replaced by $\mathbb{C}$ if one need to consider complex-valued sequences.

We can write a sequence in two ways. (1) List the terms: $(a_1, a_2, \ldots)$; (2) Just give the general term: $(a_n)_{n=1}^{\infty}$, abbreviated (a_n).

We are interested in the behavior of the infinite tail of the sequence, wanting to know whether the terms approach some number or not.

Convergence

Definition 2.2

Let (a_n) be a sequence. We say that the sequence converges to a number a if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

If (a_n) converges to a, we write $\lim_{n \to \infty} a_n = a$, abbreviated $\lim a_n = a$, or simply write $a_n \to a$.

We may also use the notion of neighborhood to define the convergence.

Definition 2.3

Let $a \in \mathbb{R}$. For $\varepsilon > 0$, the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$ is called the ε-neighborhood of a.

Convergence - continued

Definition (2.2')

The sequence (a_n) converges to a if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n \in V_\varepsilon(a)$ for all $n \geq N$.

If a sequence does not converge, we say that it diverges.

Example 1

Show $\lim_{n \to \infty} \frac{n^2 + n}{n^2} = 1$.

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then, for $n \geq N$, we have

$$|\frac{n^2 + n}{n^2} - 1| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

By definition, This means $\lim_{n \to \infty} \frac{n^2 + n}{n^2} = 1$. $\qquad\square$

Note that finding an appropriate N so that the desired inequatity can hold for all $n \geq N$ is the knottiest part of this sort of proof. But the nerve-racking computation is usually done on the scratch paper, which may not be put in the formal proof.

Convergence - continued

Example 2

Show $\lim_{n\to\infty} \frac{2n+100\sqrt{n}\sin n}{n+1} = 2$.

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $N > (\frac{102}{\varepsilon})^2$, which is equivalent to $\frac{102}{\sqrt{N}} < \varepsilon$. Then, for $n \geq N$, we have

$$\left| \frac{2n+100\sqrt{n}\sin n}{n+1} - 2 \right| = \left| \frac{100\sqrt{n}\sin n - 2}{n+1} \right| \leq \left| \frac{100\sqrt{n}\sin n - 2}{n} \right|$$
$$\leq \left| \frac{100\sqrt{n}\sin n}{n} \right| + \frac{2}{n} \leq \frac{100}{\sqrt{n}} + \frac{2}{\sqrt{n}} = \frac{102}{\sqrt{n}} \leq \frac{102}{\sqrt{N}} < \varepsilon.$$

By definition, We can conclude $\lim_{n\to\infty} \frac{2n+100\sqrt{n}\sin n}{n+1} = 2$. $\qquad\square$

Example 3

Prove $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof.

Challenging! Hint: use the formula

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

for $x = \sqrt[n]{n}$ to estimate $\sqrt[n]{n} - 1$. $\qquad\square$

Properties of limits

Theorem 2.4

If a sequence converges, then its limit is unique.

Proof.

Let $\lim a_n = a$ and $\lim a_n = b$. We show $a = b$ by showing $|a - b| < \varepsilon$ for every $\varepsilon > 0$. $\qquad\square$

We call (a_n) bounded if there is $M > 0$ such that $|a_n| \leq M$ for all n.

Theorem 2.5

If the sequence (a_n) converges, then it is bounded.

Proof.

Let $\lim a_n = a$. By defn, $\exists N \in \mathbb{N}$ such that $|a_n - a| < 1$ for $n \geq N$. So $|a_n| < |a| + 1$ for $n \geq N$. Then we have $|a_n| \leq M$ for all n, where $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 1\}$. Thus (a_n) is bounded. $\qquad\square$

Properties of limits - continued

It is trivial that for any constant sequence (c), we have $\lim c = c$.

Theorem 2.6

Suppose $\lim a_n = a$ and $\lim b_n = b$. Then

- $\lim(c_1 a_n + c_2 b_n) = c_1 a + c_2 b$, *where* $c_1, c_2 \in \mathbb{R}$ *are constants;*
- $\lim a_n b_n = ab$; *In particular,* $\lim a_n{}^2 = a^2$;
- $\lim \frac{a_n}{b_n} = \frac{a}{b}$, *if* $b \neq 0$;
- $\lim |a_n| = |a|$;
- $\lim a_n = 0$ *iff* $\lim |a_n| = 0$;
- $a \geq b$ *if* $a_n \geq b_n$ *for all n. In paticular,* $a \geq c$ *if* $a_n \geq c$ *for all n, and* $a \leq c$ *if* $a_n \leq c$ *for all n.*

In the last bullet, the phrase "for all n" may be replaced by "for all sufficiently large n", meaning for all $n \geq N$ with some $N \in \mathbb{N}$.
The proof to each bullet of Theorem 2.6 is very standard. The student should attempt them as exercises.

Monotone sequence

A sequence (a_n) is increasing if $a_n \leq a_{n+1}$ for all n. Similarly, it is decreasing if $a_n \geq a_{n+1}$ for all n. We call (a_n) monotone if it is either increasing or decreasing.

Theorem 2.7 (Monotone Convergence Theorem)

A bounded monotone sequence must converge.

Proof.

Let (a_n) be bounded increasing. From the Axiom of Completeness of $\mathbb{R}$, the boundedness of (a_n) implies $a = \sup(a_n) \in \mathbb{R}$ exists. So we have $a_n \leq a$ for all n and, for every $\varepsilon > 0$, there is a_{n_0} such that $a - \varepsilon < a_{n_0}$. Take $N = n_0$. Since (a_n) is increasing, we have

$$a_n \leq a < a_{n_0} + \varepsilon \leq a_n + \varepsilon$$

for all $n \geq N$. So $|a_n - a| = a - a_n < \varepsilon$ for $n \geq N$.
By definition, (a_n) converges to a. □

Example 4

Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Then (a_n) is increasing and bounded. So, by the MCT, its limit exists.

Proof.

Using induction, we can get $0 < a_n < 2$ and (a_n) is increasing.So $\lim a_n$ exists by the MCT. Moreover, it is not hard to see from the identity $a_{n+1} = \sqrt{2 + a_n}$ that $\lim a_n = 2$. $\qquad\square$

Example 5

Let $a_n = (1 + \frac{1}{n})^n$. Then (a_n) is increasing and bounded. So, by the MCT, its limit exists. The limit is the definition of the number e.

Proof.

This is challenging! Hint: use the binormial formula to expand a_n. $\qquad\square$

Subsequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence, and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers. Then $(a_{n_k})_{k=1}^{\infty}$ is called a subsequence of (a_n).

Theorem 2.8

If (a_n) converges to a, then every subsequence of it converges to a.

Example 6

Suppose $0 < r < 1$. Then $r^n \to 0$.

Proof.

It is easy to see that (r^n) is bounded and decreasing. By the MCT, it converges. Let $r^n \to \ell$. We show $\ell = 0$.

Clearly, $0 \leq \ell < r < 1$. (think of the reason!) Consider the subsequence (r^{2n}). We have $r^{2n} \to \ell$. But $r^{2n} = (r^n)^2 \to \ell^2$. We have $\ell^2 = \ell$. So $\ell = 0$ (why?). $\qquad\square$

Test for divergence

From Theorem 2.8, we immediately derive the following.

Theorem 2.9 (Divergence Criterion)

It there are two subsequences of (a_n) that converge to different limits, then (a_n) diverges.

Example 7

The sequence (a_n), where $a_n = (2 + (-1)^n) \frac{n}{n+1}$, is divergent.

Proof.

Consider subsequences $(a_{2k})_{k=1}^{\infty}$ and $(a_{2k+1})_{k=1}^{\infty}$.
We have $a_{2k} = 3 \cdot \frac{2k}{2k+1} \to 3$, while $a_{2k+1} = \frac{2k+1}{2k+2} \to 1$.
By the Divergence Criterion, (a_n) diverges. $\qquad\square$

Theorem 2.10 (Bolzano-Weierstrass Theorem)

If (a_n) is a bounded sequence, then it has a subseqence that converges.

Proof.

We use the Nested Interval Property (NIP). Choose a bounded closed interval I_1 such that $(a_n) \subset I_1$ (why we can?). Let ℓ be the length of I_1. Use its midpoint to cut I_1 into two subintervals of equal length. One of the subintervals must contain infinite terms of (a_n). Choose such a closed subinterval and denote it by I_2. In general, whence I_k has been chosen, use its midpoint to cut it into two subintervals. One of the subintervals must contain infinite terms of (a_n). Choose it and denote it by I_{k+1}. We then obtain a nested sequence $(I_k)_{k=1}^{\infty}$. Each member contains infinite terms of (a_n) and the length of I_k is $\frac{1}{2^{k-1}}\ell$. The intersection contains a unique point a. (Why?) We then can choose a_{n_k} from I_k to form a subsequence. (How?) We will have $a_{n_k} \to a$. (what is the reason?) □

Cauchy sequence

Definition 2.11
We call (a_n) a Cauchy sequence if, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ whenever $m, n \geq N$.

Theorem 2.12
If (a_n) converges, then it is a Cauchy sequence.

The proof is easy. Simply use the definition of convergence.

Theorem 2.13
If (a_n) is a Cauchy sequence, then it is bounded.

The proof is similar to that for a convergent sequence.

Cauchy Criterion

Theorem 2.14 (Cauchy Criterion)

A sequence (a_n) converges iff it is a Cauchy sequence.

Proof.

$\Rightarrow$ is Theorem 2.12.

$\Leftarrow$: Use Bolzano-Weierstrass Theorem. Then Show that Cauchy + convergence of a subsequence imply convergence of (a_n). $\qquad\square$

Series

Let $(a_n) \subset \mathbb{R}$ be a sequence. The formal summation

$$a_1 + a_2 + a_3 + \dots a_n + \dots$$

is called a series, denoted by $\sum_{n=1}^{\infty} a_n$ (abbreviated $\sum a_n$).
For each $n \in \mathbb{N}$,

$$s_n = a_1 + a_2 + a_3 + \dots a_n = \sum_{k=1}^{n} a_k$$

is called the *n*th partial sum of the series.

Definition 2.15

We say that the series $\sum_{n=1}^{\infty} a_n$ converges (or it is summable) if
$\lim s_n$ *converges. In the case, the limit value $s = \lim s_n$ is called the*
the sum of the series.

If $\lim s_n$ diverges, then the series does not have a sum. In this case
we say that the series diverges.

Partial sum of a series

Example 8

The telescope series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Proof.

$s_n = 1 - \frac{1}{n+1}$. So $\lim s_n = 1$, i.e. , the series converges and the sum is 1. $\qquad\square$

Example 9

A series of the form $\sum_{n=0}^{\infty} r^n$ is called a geometric series, where $r \in \mathbb{R}$ is a constant. It converges if $|r| < 1$, and it diverges if $|r| \geq 1$.

Proof.

$s_n = \frac{1-r^n}{1-r}$ if $r \neq 1$, and $s_n = n$ if $r = 1$. $r^n \to 0$ if $|r| < 1$, and $\lim r^n$ diverges if $|r| > 1$ or $r = -1$. So the series converges iff $|r| < 1$. $\qquad\square$

Properties of series

Examining the partial sum sequence and using the corresponding results for sequences, we immediately obtain the following results.

Theorem 2.16

Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. Then

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B,$$

where $\alpha, \beta \in \mathbb{R}$ are constants.

Theorem 2.17 (Cauchy Criterion for Series)

$\sum_{n=1}^{\infty} a_n$ converges iff, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|\sum_{k=m}^{n} a_k| := |a_m + a_{m+1} + \cdots + a_n| < \varepsilon \quad \text{whenever } n > m \geq N.$$

Application of Cauchy Criterion

Theorem 2.18

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. The series diverges if $a_n \not\to 0$.

Example 10

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ diverges.

Warning

Theorem 2.18 only asserts that $a_n \to 0$ is necessary for $\sum a_n$ to be convergent. It is not sufficient! i.e. even we have $a_n \to 0$, it is still possible that $\sum a_n$ diverges!

Example 11

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (Note we do have $\frac{1}{n} \to 0$.)

Alternating series

A series is called an alternating series if its terms alternate in sign. Precisely, an alternating series is of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$, where $b_n > 0$ for all n.

Theorem 2.19 (Alternating Series Test)

Suppose that (b_n) is a positive and decreasing sequence. If $b_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^n b_n$ (and $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$) converges.

Proof.

Hint: Under the condition, we will have $|\sum_{k=m}^{n}(-1)^k b_k| \leq b_m$. $\qquad\square$

Example 12

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n}$ converges.

Positive series

We call $\sum_{n=1}^{\infty} a_n$ a positive series if $a_n \geq 0$ for all n.

Theorem 2.20 (Easy but useful observation)

Let $\sum_{n=1}^{\infty} a_n$ be a positive series. Then

- ▶ *its partial sum sequnce (s_n) is increasing;*
- ▶ *the series converges if (s_n) is bounded (due to the MCT);*
- ▶ *the series diverges to ∞ (meaning $\lim s_n = \infty$) if (s_n) is unbounded.*

Theorem 2.21 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence and $a_n \geq 0$ for all n. Then the positive series $\sum_{n=1}^{\infty} a_n$ converges iff the $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Example 13

Let $p \in \mathbb{R}$. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Comparision

Theorem 2.22 (Comparision Test)

Let $\sum a_n$ and $\sum b_n$ be two positive series such that $a_n \leq b_n$ for all $n \in \mathbb{N}$.

1. If $\sum b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Remark

In the above theorem, the condition $a_n \leq b_n$ for all $n \in \mathbb{N}$ can be replaced by $a_n \leq b_n$ for all $n \geq N$, where N is a fixed number. Removing finite terms from a series won't change its convergence.

Warning

*The Comparision Test can be used **only for positive series**.*

Comparision - continued

Example 14

The positive series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Proof.

In fact, $n! \geq (n-1)n$. So $0 < \frac{1}{n!} \leq \frac{1}{(n-1)n}$ for $n \geq 2$. We have known that the telecope series $\sum_{n=2}^{\infty} \frac{1}{(n-1)n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. Then, by the CT, $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. $\qquad \square$

Example 15

$\sum_{n=1}^{\infty} \frac{\sqrt{n}+100}{2n^2-n}$ converges, while $\sum_{n=1}^{\infty} \frac{\sqrt{n}+100}{2n+\sqrt{n}}$ diverges.

Absolute convergence

Theorem 2.23 (Absolute Convergence Test)

If $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

Hint for proof: Use the Cauchy Criterion.

Definition 2.24

The series $\sum a_n$ is called *absolute convergent* if its absolute series $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then we call the series $\sum a_n$ *conditionally convergent*.

Example 16

1. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ converges absolutely. So does $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$ for $p > 1$.
2. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges conditionally. So does $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$ for $0 < p < 1$