Module 2: Combinatorial Probability

▶ The *classical probability model*:

In the case of a finite sample space with equally likely outcomes, we use

$$\mathbb{P}(E) = \frac{N(E)}{N(\Omega)} \qquad \text{for any } E \subset \Omega,$$

so that we must be able to count the number of outcomes in the sample space and events.

Listing the outcomes is sometimes enough...but what about difficult problems?

2.1 Counting, Permutations and Combinations (*Weiss* §*3.1, 3.2*)

Example 2.1. Coin and die

A random experiment consists of flipping a coin and then rolling a six-sided die.

What is $N(\Omega)$?

Proposition 2.1. Basic Counting Rule (BCR)

If r experiments are to be performed in such a way that

- there are m_1 outcomes to the 1st experiment,
- for each outcome of experiment 1, there are m_2 outcomes to the 2nd experiment,
- for each outcome of experiments 1 and 2, there are m_3 outcomes to the 3rd experiment,
- etc.

Then, altogether, there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_r = \prod_{i=1}^{r} m_i$$

outcomes to the r experiments.

Example 2.1. (cont'd)

Find $N(\Omega)$ by using the BCR.

Example 2.2. Flipping a coin 3 times (*cf*. Example 1.5)

We have seen that $N(\Omega) = 8$. Find this using the BCR.

Example 2.3. License plates

How many Manitoba license plates of three letters followed by three digits are there?

Example 2.4. 5-digit PINs

A 5-digit PIN is to be selected.

Find the number of ways this can be done if

- there are no restrictions,
- no digit can occur twice,
- adjacent digits cannot be identical.

Definition 2.2. Permutations

A **permutation** of r objects from a collection of m objects is any *ordered* arrangement of r *distinct* objects selected from the m objects.

$(m)_r$ denotes the number of possible permutations of r objects selected from m.

Proposition 2.3. Permutation Rule

$$(m)_r = \frac{m!}{(m-r)!}$$

Recall that $0! = 1$.

► Note:

- $(m)_m$ denotes the number of permutations of m objects among themselves, and $(m)_m = m!$
- Also, $(m)_0 = 1$.

Example 2.4. (cont'd)

How many options are there if no digit can occur twice?

Example 2.5. Four-letter words

How many four-letter words can be made from the english alphabet,

- if repeated letters are allowed,

- if repeated letters are not allowed,
- if these words have to start with a consonant and end with a vowel, and if repetitions are not allowed.

Example 2.6. Books on a shelf

On a shelf, in my office, there are 12 books:

- 3 in applied Statistics,
- 4 in theoretical Statistics,
- 5 in Mathematics (not Statistics).

Find the number of ordered arrangements of these books if

- there are no restrictions,
- the Math books should be together and come first,
- the Math books should be together (not necessarily first),
- books on the same topic should be together.

Definition 2.4. Combinations

A **combination** of r objects from a collection of m objects is any *unordered* arrangement of r *distinct* objects selected from the m objects.

$\binom{m}{r}$: denotes the number of ways r distinct objects can be selected from m objects without regards to order.

$\binom{m}{r}$: "m choose r" is also called a **binomial coefficient**.

Proposition 2.5. Combination Rule

$$\binom{m}{r} = \frac{m!}{r!(m-r)!}$$

Recall that $0! = 1$.

Example 2.7. Lotto 6/49

How many combinations are there at Lotto 6/49?

Example 2.8. 5-card draw Poker

When dealt 5 cards from a regular playing deck of 52 cards, how many different hands are there

- in total,
- that are "pairs",
- that are "two pairs",
- that are "three of a kind".

► Notes:

- For any $m \geq 1$, we have that

$$\binom{m}{0} = 1 \qquad \text{and} \qquad \binom{m}{1} = m.$$

- For any $0 \leq r \leq m$, we have that

$$\binom{m}{m-r} = \binom{m}{r}.$$

Example 2.9. Integer solutions to equations

How many positive integer solutions are there to $x_1 + x_2 + x_3 = 8$?

There are 21 and they are the following:

$(1,1,6)$	$(1,6,1)$	$(6,1,1)$			
$(1,2,5)$	$(1,5,2)$	$(2,1,5)$	$(2,5,1)$	$(5,1,2)$	$(5,2,1)$
$(1,3,4)$	$(1,4,3)$	$(3,1,4)$	$(3,4,1)$	$(4,1,3)$	$(4,3,1)$
$(2,2,4)$	$(2,4,2)$	$(4,2,2)$			
$(2,3,3)$	$(3,2,3)$	$(3,3,2)$			

But, how can we find this result using counting rules?

► In general, there are $\binom{n-1}{k-1}$ positive integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

Example 2.9. (cont'd)

How many nonnegative integer solutions are there to $x_1 + x_2 + x_3 = 8$?

► In general, there are $\binom{n+k-1}{k-1}$ nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

Example 2.10. Investment strategies

You want to invest \$15 000. There are 4 possible types of investments, each one available in units of \$1000.

How many investment strategies are there if

- all your money is to be invested,
- you do not have to invest everything.

Example 2.11. Making up words

How many *different* words can be made from the letters STATISTICS?

Definition 2.6. Ordered partitions

An **ordered partition** of m objects into k distinct groups of size $m_1, m_2, \ldots, m_k$ is any division of the m objects such that

- m_1 objects are in group 1,
- m_2 objects are in group 2,
- etc.

Note that $\sum_{i=1}^{k} m_i = m$.

$\binom{m}{m_1,m_2,\ldots,m_k}$: denotes the number of such ordered partitions.

$\binom{m}{m_1,m_2,\ldots,m_k}$: is called a **multinomial coefficient**.

Proposition 2.7. Rule of Ordered Partitions

$$\binom{m}{m_1, m_2, \ldots, m_k} = \frac{m!}{m_1!m_2!\cdots m_k!}.$$

► Note:

The rule of ordered partitions can be used to find the number of permutations of m objects, among which there are k groups of undistinguishable objects. (*cf*. Example 2.11)

Example 2.12. Group projects

A class of 40 students is to be broken down into 8 groups of 5 to work on projects.

How many ways are there to allocate the workload if

- there are 8 different projects,
- the teams all work on a similar project.

2.2 Probabilities Using Counting Rules (*Weiss* §3.3)

► For the classical probability model, we know that

$$\mathbb{P}(E) = \frac{N(E)}{N(\Omega)} \qquad \text{for any } E \subset \Omega.$$

Example 2.13. Flipping a coin

A fair coin is flipped 10 times.

What is the probability of getting no more than two results of *Heads*?

Example 2.14. Birthday problem

Assuming there are 365 days in every year and that people are equally likely to be born on any day of the year, what is the probability that no two people in this room have their birthday on the same day of the year?

Solve this problem for any possible number $n \geq 2$ of people in the room.

Example 2.15. Lotto 6/49, again

On any given draw of Lotto 6/49, let

A_i : all the selected numbers are between 1 and i

Find $\mathbb{P}(A_i)$ for $1 \leq i \leq 49$.

Example 2.16. Ordering pizza

Four friends sitting at the same table in a pizzeria have each ordered their own individual pizza. When the pizzas are ready, the waiter is to bring them to the table. Having lost track of the orders, the waiter randomly serves the four individuals one of the four prepared pizzas. What is the probability that nobody is served the pizza they ordered if

- the four friends ordered different pizzas,
- two of the four friends ordered the "meat lover's" pizza, but all the ordered pizzas are otherwise different.

2.3 More on Binomial Coefficients

Proposition 2.8. Properties of binomial coefficients

(A) $\binom{m}{m-r} = \binom{m}{r}$.

(B) $\binom{m-1}{r} + \binom{m-1}{r-1} = \binom{m}{r}$.

(C) $\frac{m}{r}\binom{m-1}{r-1} = \binom{m}{r}$ and $r\binom{m}{r} = m\binom{m-1}{r-1}$.

(D) $\binom{m}{r}\binom{r}{k} = \binom{m}{k}\binom{m-k}{r-k}$ for $0 \le k \le r \le m$.

Example 2.17. A simple equality

Verify directly that

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \qquad \text{for any } 2 \le k \le n.$$

Also give a combinatorial argument proving this identity.

Theorem 2.9. Binomial Theorem

For any $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{for } n \ge 1.$$

Example 2.18. Using the Binomial Theorem

Prove that, for all integers $n \ge 1$,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$