Module 2: Combinatorial Probability

► The *classical probability model*:

In the case of a finite sample space with equally likely outcomes, we use

$$\mathbb{P}(E) = \frac{N(E)}{N(\Omega)} \qquad \text{for any } E \subset \Omega,$$

outcomes in E

outcomes in Ω.

so that we must be able to count the number of outcomes in the sample space and events.

Listing the outcomes is sometimes enough... but what about difficult problems?

2.1 Counting, Permutations and Combinations (*Weiss* §*3.1, 3.2*)

Example 2.1. Coin and die

A random experiment consists of flipping a coin and then rolling a six-sided die.

What is $N(\Omega)$?

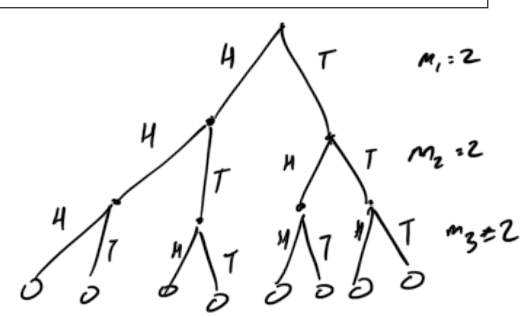
> **Proposition 2.1.** Basic Counting Rule (BCR)
>
> If r experiments are to be performed in such a way that
>
> - there are m_1 outcomes to the 1st experiment,
> - for each outcome of experiment 1, there are m_2 outcomes to the 2nd experiment,
> - for each outcome of experiments 1 and 2, there are m_3 outcomes to the 3rd experiment,
> - etc.
>
> The # of outcomes for each branch of the previous level have to be equal.
>
> Then, altogether, there are
>
> $$m_1 \cdot m_2 \cdot \dots \cdot m_r = \prod_{i=1}^{r} m_i$$
>
> 
>
> outcomes to the r experiments.

Example 2.1. (cont'd)

Find $N(\Omega)$ by using the BCR. $m_1 \times m_2 = 12.$

Example 2.2. Flipping a coin 3 times (*cf.* Example 1.5)

We have seen that $N(\Omega) = 8$. Find this using the BCR.



$2 \times 2 \times 2 = 8.$

Ex 2.1

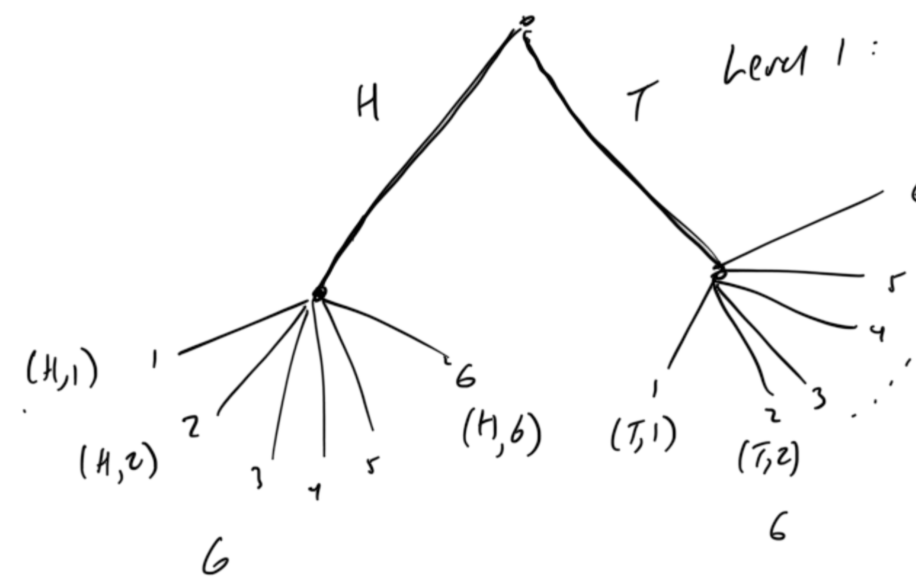
H1	T1
H2	T2
H3	T3
H4	T4
H5	T5
H6	T6
6 elements	6 elements

⇒ 12 outcomes $= N(\Omega)$

$\Omega = \{(H,1), (H,2) \dots (T,6)\}.$

⇕

$\Omega = \{(\omega_1, \omega_2) : \omega_1 \in \{H,T\}, \omega_2 \in \{1,2,\dots,6\}\}$



Level 1: coin flip (2 outcomes).

Level 2: die roll. (6 outcomes)

⇒ 12. $2 \times 6 = 12.$

Level 1: m_1 = # outcomes in level 1 = 2

Level 2: m_2 = # of outcomes for each branch of level 1 = 6

⇒ $m_1 \times m_2 = 12$

⇒ Basic counting rule.

Example 2.3. License plates

How many Manitoba license plates of three letters followed by three digits are there?

Example 2.4. 5-digit PINs

A 5-digit PIN is to be selected.

Find the number of ways this can be done if

- there are no restrictions,
- no digit can occur twice,
- adjacent digits cannot be identical.

Definition 2.2. Permutations

A **permutation** of r objects from a collection of m objects is any *ordered* arrangement of r *distinct* objects selected from the m objects.

$(m)_r$ denotes the number of possible permutations of r objects selected from m.

Proposition 2.3. Permutation Rule

$$(m)_r = \frac{m!}{(m-r)!}$$

Recall that $0! = 1$.

► Note:

- $(m)_m$ denotes the number of permutations of m objects among themselves, and $(m)_m = m!$
- Also, $(m)_0 = 1$.

Example 2.4. (cont'd)

How many options are there if no digit can occur twice?

Example 2.5. Four-letter words

How many four-letter words can be made from the english alphabet,

- if repeated letters are allowed,

Ex. 2.3

$$\underbrace{26}_{m_1} \times \underbrace{26}_{m_2} \times \underbrace{26}_{m_3} \times \underbrace{10}_{m_4} \times \underbrace{10}_{m_5} \times \underbrace{10}_{m_6}$$

m_1 # of options for 1st letter; m_2 2nd letter; m_3 3rd letter; m_4 # of options for digit 1; m_5 2nd digit; m_6 3rd digit.

$N(\Omega) = 26^3 \times 10^3 = 17\ 576\ 000$ ✓

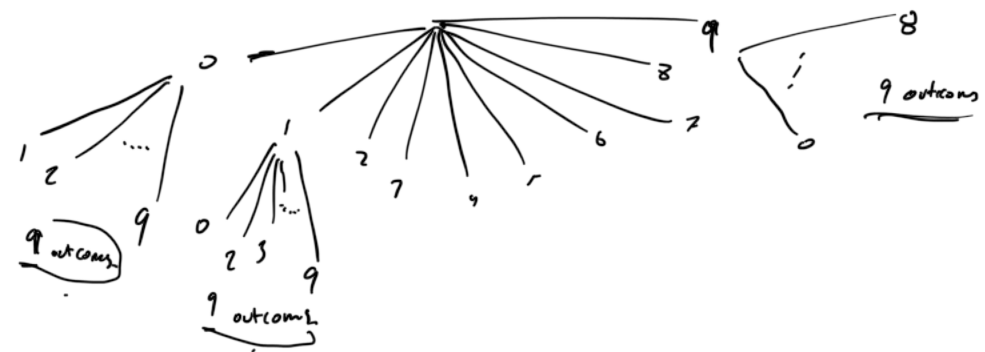
Ex 2.4

A) No restriction.

$10 \times 10 \times 10 \times 10 \times 10 = 10^5$

B) No repeated digits

$$10 \times 9 \times 8 \times 7 \times 6 = \frac{10!}{5!} = \frac{10\times9\times8\times7\times6 \times \cancel{5\times4\times3\times2\times1}}{\cancel{5\times4\times3\times2\times1}}$$



C) Adjacent digits are different

$10 \times 9 \times 9 \times 9 \times 9$

Permutations

\# of ways to select r <u>distinct</u> items from a collection of m items.

Ex. 5 digit pins with no repetition.

$m = 10$ digits

$r = 5$

$$\Rightarrow (10)_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!}$$

$$= \underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6}$$

$$\underbrace{(m)_r} = \frac{m!}{(m-r)!} \qquad \cancel{P^r_m} \qquad \cancel{{}_rP_m}$$

- r first numbers in $m!$

  $m \cdot (m-1) \cdot (m-2) \cdot \ldots (m-r-1)$

Ex. 2.5 Four letter words. 21 consonants
5 vowels.

A) if repeated letters allowed:

$$\underset{1}{\underline{26}} \times \underset{2}{\underline{26}} \times \underset{3}{\underline{26}} \times \underset{4}{\underline{26}} = 26^4$$

B) repeated letters not allowed.

$$\underset{1}{\underline{26}} \times \underset{2}{\underline{25}} \times \underset{3}{\underline{24}} \times \underset{4}{\underline{23}} = (26)_4$$

LINK ⟷ KILN

- Considered different outcomes by a permutation ✓✓✓

c) Start with consonant, end with a vowel, and repetition is not allowed.

$$\underset{1}{\underline{21}} \times \underset{2}{\underline{25}} \times \underset{3}{\underline{24}} \times \underset{4}{\underline{\quad}}$$

consonant

5 vowels
20 cons.

vowel.

Don't know how many vowels I have selected at this point
⇒ BCR fails!

2 scenarios.
1) Vowel in 2nd spot: 20 cons. 4 vowels. } 24
2) cons. in 2nd spot: 19 cons. 5 vowels. } 24

BCR still holds for spot 3

$$\underset{4}{\underline{?}} :$$

(C - C - C) : 5 vowels remain. → 21 × 20 × 19 × 5
+

(C - V - C) or (C - C - V) : 4 vowels remaining →

(C - V - V) : 3 vowels remaining →

⇒ BCR fails.

Idea: Change order of selection!

$$\underbrace{21}_{\text{cons}} \times \underbrace{5}_{\text{vowel}} \times \underbrace{24}_{\text{letter 2}} \times \underbrace{23}_{\text{letter 3}}$$

1st __ 2nd __ 3rd __ 4th __

- if repeated letters are not allowed,
- if these words have to start with a consonant and end with a vowel, and if repetitions are not allowed.

Example 2.6. Books on a shelf

On a shelf, in my office, there are 12 books:

- 3 in applied Statistics,
- 4 in theoretical Statistics,
- 5 in Mathematics (not Statistics).

Find the number of ordered arrangements of these books if

- there are no restrictions,
- the Math books should be together and come first,
- the Math books should be together (not necessarily first),
- books on the same topic should be together.
-

A) no restrictions

$12! = \underset{1}{12} \times \underset{2}{11} \cdots \underset{12}{1}$

B) Math together and come first.

$\underbrace{\underset{1}{5} \times \underset{2}{4} \times \underset{3}{3} \times \underset{4}{2} \times \underset{5}{1}}_{\text{Math}} \; ; \; \underbrace{\underset{6}{7} \times \underset{7}{6} \cdots \underset{12}{1}}_{\text{Rest}}$

Tree with 12 levels.

$= 5! \times 7!$

Equivalently: $\underbrace{5!}_{1} \times \underbrace{7!}_{2}$

arrange Math

arrange the rest.

Tree with 2 levels

> **Definition 2.4.** Combinations
>
> A **combination** of r objects from a collection of m objects is any *unordered* arrangement of r *distinct* objects selected from the m objects.
>
> $\binom{m}{r}$: denotes the number of ways r distinct objects can be selected from m objects without regards to order.
>
> $\binom{m}{r}$: "m choose r" is also called a **binomial coefficient**.
>
> **Proposition 2.5.** Combination Rule
>
> $$\binom{m}{r} = \frac{m!}{r!(m-r)!}$$
>
> Recall that $0! = 1$.

c) Math together but not necessarily first.

$\underset{1}{\_} \times \underset{2}{\_} \times \underset{3}{\_} \times \underset{4}{\_} \times \underset{5}{\_} \times \underset{6}{\_} \times \underset{7}{\_} \times \underset{8}{\_}$

8 items: 3 AS

4 TS

1 "Math block"

$\underbrace{8!}_{1} \times \underbrace{5!}_{2}$

arrangement of Math block.

arrangement of 7 books + Math block

Combination vs. Permutation

4 letters a, b, c, d ← n

→ pick 3 different letters from the 4 : how many outcomes?

(a, b, c)	(a, c, d)	(a, b, d)	(b, c, d)
(a, c, b)	(a, d, c)	(a, d, b)	(b, d, c)
(b, a, c)	(c, a, d)	(b, a, d)	(c, b, d)
(b, c, a)	(c, d, a)	(b, d, a)	(c, d, b)
(c, a, b)	(d, a, c)	(d, a, b)	(d, b, c)
(c, b, a)	(d, c, a)	(d, b, a)	(d, c, b)
without "d"	without "b"	without "c"	without "a"
3! = 6 outcomes	3!	3!	3!

→ 24 outcomes

Scenario 1: order is important → $(4)_3 = 24 = \frac{4!}{(4-3)!}$

Scenario 2: order is not important → $\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$

ordering factor

$$(n)_r = \binom{n}{r} \times r! \iff \binom{n}{r} = \frac{(n)_r}{r!}$$

Example 2.7. Lotto 6/49
How many combinations are there at Lotto 6/49?

Example 2.8. 5-card draw Poker
When dealt 5 cards from a regular playing deck of 52 cards, how many different hands are there

- in total,
- that are "pairs",
- that are "two pairs",
- that are "three of a kind".

► Notes:

- For any $m \geq 1$, we have that

$$\binom{m}{0} = 1 \qquad \text{and} \qquad \binom{m}{1} = m.$$

- For any $0 \leq r \leq m$, we have that

$$\binom{m}{m-r} = \binom{m}{r}.$$

Example 2.9. Integer solutions to equations
How many positive integer solutions are there to $x_1 + x_2 + x_3 = 8$?
There are 21 and they are the following:

(1, 1, 6) (1, 6, 1) (6, 1, 1)
(1, 2, 5) (1, 5, 2) (2, 1, 5) (2, 5, 1) (5, 1, 2) (5, 2, 1)
(1, 3, 4) (1, 4, 3) (3, 1, 4) (3, 4, 1) (4, 1, 3) (4, 3, 1)
(2, 2, 4) (2, 4, 2) (4, 2, 2)
(2, 3, 3) (3, 2, 3) (3, 3, 2)

But, how can we find this result using counting rules?

► In general, there are $\binom{n-1}{k-1}$ positive integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

Example 2.9. (cont'd)
How many nonnegative integer solutions are there to $x_1 + x_2 + x_3 = 8$?

► In general, there are $\binom{n+k-1}{k-1}$ nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

Video on UM Learn!

Ex. 2.7
49 balls 1, 2, ..., 49 numbered.
draw 6 of them (different).
→ order in which they are drawn is not important

→ 7 10 12 13 39 48

Question: how many ways to select the 6 numbers.

$$\binom{49}{6} = \frac{49!}{6!(49-6)!}$$
$$= 13\ 983\ 816$$

$$P(\text{winning}) = \frac{1}{13\ 983\ 816} \approx \frac{1}{14 \text{ million}}$$

Ex. 2.8

- order for cards does not matter

A) In total

$$\binom{52}{5}$$ "pick 5 different cards from 52"

B) hands that have exactly one "pair"

aa b c d

pair

$$\binom{13}{1} \times \binom{4}{2} \times \underbrace{\qquad\qquad}_{\text{complete the hand.}}$$

$\binom{13}{1}$: denomination A,2,3,...,J,Q,K. $\binom{4}{2}$: suits for the pair.

$\binom{48}{3}$ ✗ also allows for another pair to be chosen.

$\binom{50}{3}$ ✗ wrong. because it allows for same denominations as the pair (or another pair).

$$\binom{12}{3} \times \underbrace{\binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}}_{\text{a suit for each of the remaining 3 cards}}$$

$\binom{12}{3}$: other denominations different from the pair.

$$\Rightarrow \binom{13}{1} \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}$$

K   ♡♧   3,7,6   ◇ 3   ♡ 7   ♧ 6

Incorrect: $\binom{13}{4} \times \binom{4}{2} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}$

$\binom{13}{4}$: denomination for pair and 3 others $\binom{4}{2}$: suits for pair $\binom{4}{1}\binom{4}{1}\binom{4}{1}$: suits for the other 3 cards.

Wrong because you select 4 denomination total but you aren't selecting which one belongs to the pair.

$$\binom{13}{1}\binom{12}{3} = \binom{13}{4}\binom{4}{1}$$

$\binom{13}{1}$: denom. for pair $\binom{12}{3}$: denom. for other 3 cards $\binom{13}{4}$: choose 4 denom. $\binom{4}{1}$: assign one of them to the pair.

Equivalent Solutions:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = \binom{13}{4}\binom{4}{1}\binom{4}{2}\binom{4}{1}^3.$$

c) Hands that are exactly 2 pairs.

aa bb c

$$\binom{13}{2}\binom{4}{2}\binom{4}{2} \times \text{complete the hand.}$$

$\binom{11}{1}\binom{4}{1} = 44$ ✓

$\binom{44}{1} = 44$ ✓

$\binom{48}{1}$ wrong.

last card has to have different denomination.

---

aa bb c

$$\binom{13}{1}\binom{4}{2} \times \binom{12}{1}\binom{4}{2} \times \binom{44}{1}$$

$$\begin{cases} aa\ bb\ c \\ bb\ aa\ c \end{cases}$$

includes 2 orderings.

$$\binom{13}{1}\binom{12}{1} = 13 \times 12$$

$$\binom{13}{2} = \frac{13!}{11!\,2!} = \frac{13 \times 12}{2}$$

Ex 2.9

How many integer solutions to $x_1 + x_2 + x_3 = 8$?

in general:

$$x_1 + x_2 + \cdots + x_k = n \ ?$$

$$\underbrace{1+1+1}_{3} + \underbrace{1+1+1}_{3} + \underbrace{1+1}_{2} = 8$$

How many ways to choose + signs

$\hookrightarrow$ 7 of them $\to$ choose 2

$$\binom{7}{2} = \frac{7\times 6}{2} = 21$$

$$\underbrace{1+1+1+\cdots+1}_{} = n$$

$n-1$ "+" signs. $\Rightarrow \binom{n-1}{k-1}$

choose $k-1$ of them.

What if we want to allow some of x_i's to be 0.

1) $\binom{7}{2}$ strictly positive. : 21

2) $(0,0,8), (0,8,0), (8,0,0) \Rightarrow 3$

3) $x_1 = 0$ not x_2 and x_3

$x_2 + x_3 = 8 \Rightarrow \binom{8-1}{2-1} = \binom{7}{1} = 7$

4) $x_2 = 0$, x_1 and x_3 not $0 \Rightarrow 7$

5) similar, $x_3 = 0$. $\Rightarrow 7$

45

$$\begin{array}{rl} & x_1 + x_2 + x_3 = 8 \qquad \geq 0 \\ + & 1 + 1 + 1 = +3 \\ \hline & x_1^* + x_2^* + x_3^* = 11 \end{array}$$

$\leftarrow$ here strictly positive solutions.

$$\binom{11-1}{3-1} = \binom{10}{2} = 45$$

General:

$$\begin{array}{rl} & x_1 + x_2 + \cdots + x_k = n \quad \leftarrow \geq 0 \\ + & 1 + 1 + \cdots + 1 = +k \quad \leftarrow \geq 1 \\ \hline & x_1^* + x_2^* + \cdots + x_k^* = n + k \quad \geq 1 \end{array}$$

$$\binom{n+k-1}{k-1} \checkmark$$

Example 2.10. Investment strategies

You want to invest \$15 000. There are 4 possible types of investments, each one available in units of \$1000.

How many investment strategies are there if

- all your money is to be invested,
- you do not have to invest everything.

Example 2.11. Making up words

How many *different* words can be made from the letters STATISTICS?

> **Definition 2.6.** Ordered partitions
>
> An **ordered partition** of m objects into k distinct groups of size $m_1, m_2, \ldots, m_k$ is any division of the m objects such that
>
> - m_1 objects are in group 1,
> - m_2 objects are in group 2,
> - etc.
>
> S T A
>
> 1 2 1
>
> Note that $\sum_{i=1}^{k} m_i = m$.
>
> $\binom{m}{m_1, m_2, \ldots, m_k}$: denotes the number of such ordered partitions.
>
> $\binom{m}{m_1, m_2, \ldots, m_k}$: is called a **multinomial coefficient**.
>
> **Proposition 2.7.** Rule of Ordered Partitions
>
> total
>
> $$\binom{m}{m_1, m_2, \ldots, m_k} = \frac{m!}{m_1! m_2! \cdots m_k!}. \qquad = \frac{4!}{1!\,2!\,1!}$$
>
> S T A

► Note:

The rule of ordered partitions can be used to find the number of permutations of m objects, among which there are k groups of undistinguishable objects. (*cf.* Example 2.11)

Example 2.12. Group projects

A class of 40 students is to be broken down into 8 groups of 5 to work on projects.

How many ways are there to allocate the workload if

- there are 8 different projects,
- the teams all work on a similar project.

2.11

How many words could be made from the letters in STAR. STAT

STAR, STRA, SART, SATR, SRAT, SRTA | TASR ⋮ | RATS ⋮ | ASTR ⋮ } 24

start with S | T | R | A

STAT → $\frac{24 \text{ unique words}}{2} = 12$ unique.

or $\frac{4!}{2!}$

4! ⇝ rearrangements of 4 letters.

2! → "take away" swaps of R and T ⇒ identical outcomes.

STAT → __ __ __ __ (1, 2, 3, 4)

Consider positions {1,2,3,4}

1) pick a position for S : $\binom{4}{1} = 4$
2) pick a position for T : $\binom{3}{2} = 3$
3) pick remaining position for A : $\binom{1}{1} = 1$

$4 \times 3 \times 1 = 12$

1) T : $\binom{4}{2} = 6$

A : $\binom{2}{1} = 2$

S : $\binom{1}{1} = 1$

$6 \times 2 \times 1 = 12$.

STATISTICS

$A = 1,\ S = 3,\ T = 3,\ C = 1,\ I = 2.$

1) $$\frac{10!}{\underset{A}{1!}\ \underset{S}{3!}\ \underset{T}{3!}\ \underset{C}{1!}\ \underset{I}{2!}} = \_\_\_$$

2) : $$\binom{10}{1}\binom{9}{3}\binom{6}{3}\binom{3}{1}\binom{2}{2} = \_\_\_$$

2.2 Probabilities Using Counting Rules (*Weiss* §3.3)

► For the classical probability model, we know that

$$\mathbb{P}(E) = \frac{N(E)}{N(\Omega)} \qquad \text{for any } E \subset \Omega.$$

Example 2.13. Flipping a coin

A fair coin is flipped 10 times.

What is the probability of getting no more than two results of *Heads*?

Example 2.14. Birthday problem

Assuming there are 365 days in every year and that people are equally likely to be born on any day of the year, what is the probability that no two people in this room have their birthday on the same day of the year?

Solve this problem for any possible number $n \geq 2$ of people in the room.

Example 2.15. Lotto 6/49, again

On any given draw of Lotto 6/49, let

A_i : all the selected numbers are between 1 and i

Find $\mathbb{P}(A_i)$ for $1 \leq i \leq 49$.

Example 2.16. Ordering pizza

Four friends sitting at the same table in a pizzeria have each ordered their own individual pizza. When the pizzas are ready, the waiter is to bring them to the table. Having lost track of the orders, the waiter randomly serves the four individuals one of the four prepared pizzas. What is the probability that nobody is served the pizza they ordered if

- the four friends ordered different pizzas,
- two of the four friends ordered the "meat lover's" pizza, but all the ordered pizzas are otherwise different.

$(H, T, \ldots T)$

$(T, H, \ldots)$

Chapter 2

E = event

$$P(E) = \frac{N(E)}{N(\Omega)}$$

$N(E)$ ← # of outcomes in E

$N(\Omega)$ ← # of outcomes in Ω.

How do we calculate $N(\cdot)$.

→ permutation / combination &

→ ordered partitions ←

Ex. 2.13

$\Omega = \{(\omega_1, \ldots, \omega_{10});\ \omega_i \in \{H, T\}\}$

$P($"getting no more than 2 H"$)$

E_i : exactly i heads are obtained $i = 0, \ldots, 10$.

$= P(0H \text{ or } 1H, \text{ or } 2H)$

disjoint.

by finite additivity the E_i's are disjoint.

$$= P\left(\bigcup_{i=0}^{2} E_i\right) = \sum_{i=0}^{2} P(E_i) = \sum_{i=0}^{2} \frac{N(E_i)}{N(\Omega)}$$

$$= \frac{N(E_0) + N(E_1) + N(E_2)}{N(\Omega)}$$

$N(\Omega) = \underset{1}{2} \times \underset{2}{2} \cdots \times \underset{10}{2} = 2^{10} = 1024$

$N(E_0)$: "no heads" $= 1$

$N(E_1) = \binom{10}{1} = 10$

$N(E_2) = \binom{10}{2} = 45$

$\Rightarrow P(\text{no more than } 2H) = \frac{1 + 10 + 45}{1024}$

$= 0.055$

Ex. 2.14

Assume: – 365 days in a year

– people are equally likely to be born on any day of the year.

If there are $n \geq 2$ people in a room what is the prob. nobody share a bday with anyone else?

D_n: among n people, all birthdays different.

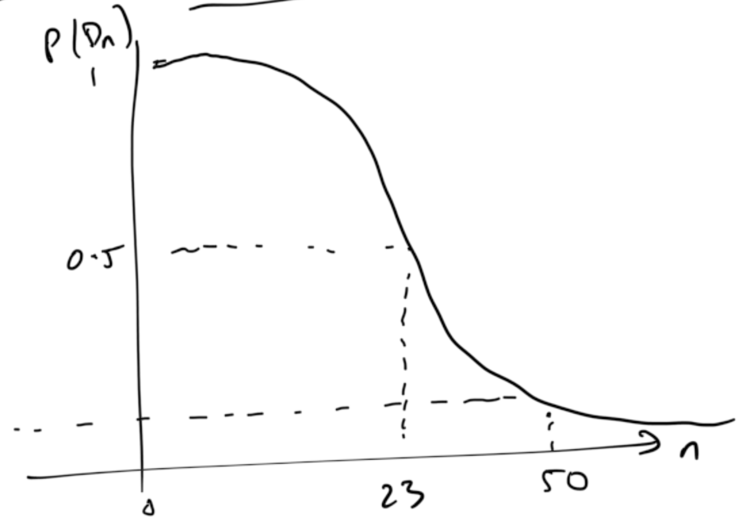
$P(D_n)$

$P(D_{366}) = 0, \quad P(D_1) = 1$

$$P(D_2) = \frac{365 \times 364}{365 \times 365} = 0.999 \approx 1$$

$$P(D_3) = \frac{365 \times 364 \times 363}{365^3} = \frac{(365)_3}{365^3} \approx 1$$

$$\boxed{P(D_n) = \frac{(365)_n}{365^n}} \leftarrow \text{decreasing as } n \uparrow$$



Ex 2.16

· All order different pizzas.

$$\frac{4}{F_1} \times \frac{3}{F_2} \times \frac{2}{F_3} \times \frac{1}{F_4} = 4! = 24$$

$P(\text{"no match"}) = 1 - P(\text{"at least one matches"})$

A_i = person i gets their original pizza. $i = 1, 2, 3, 4$.

$$= P\left(\bigcap_{i=1}^{4} A_i^c\right) = 1 - P\left(\bigcup_{i=1}^{4} A_i\right)$$

inclusion exclusion.

$$= 1 - \Big[\sum_{i=1}^{4} P(A_i) - \text{"all 2×2 intersections" } P(A_i \cap A_j)$$

$$+ \text{"all 3×3 intersections"}$$

$$- \text{"all 4×4 intersections"} \Big]$$

$$P(A_i) = \frac{1}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = \frac{3!}{4!} = \frac{1}{4}$$

$i = 1, 2, 3, 4$

$$P(A_i \cap A_j) = \frac{1}{1} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} = \frac{2!}{4!} = \frac{1}{12}$$

$\underline{P(A_i)} = \frac{1}{4}$, $P(A_i \cap A_j) = \frac{1}{12}$

$P(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times 1 = \frac{1}{4!} = \frac{1}{24}$

$P(A_i \cap A_j \cap A_k \cap A_l) = \frac{1}{24}$

$$P(\text{"no match"}) = 1 - \left[4 \times \frac{1}{4} - \binom{4}{2} \times \frac{1}{12} + \binom{4}{3} \times \frac{1}{24} - \binom{4}{4} \times \frac{1}{24} \right]$$

$$= \frac{9}{24}$$

2.3 More on Binomial Coefficients $\binom{n}{r}$

Proposition 2.8. Properties of binomial coefficients

(A) $\binom{m}{m-r} = \binom{m}{r}$.

(B) $\binom{m-1}{r} + \binom{m-1}{r-1} = \binom{m}{r}$.

(C) $\frac{m}{r}\binom{m-1}{r-1} = \binom{m}{r}$ and $r\binom{m}{r} = m\binom{m-1}{r-1}$.

$\boxed{\text{(D)}}$ $\binom{m}{r}\binom{r}{k} = \binom{m}{k}\binom{m-k}{r-k}$ for $0 \le k \le r \le m$.

Example 2.17. A simple equality

Verify directly that

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \qquad \text{for any } 2 \le k \le n.$$

Also give a combinatorial argument proving this identity.

Theorem 2.9. Binomial Theorem

For any $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{for } n \ge 1.$$

Example 2.18. Using the Binomial Theorem

Prove that, for all integers $n \ge 1$,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

and

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

A) show $\binom{m}{m-r} = \binom{m}{r}$

1st approach: $\binom{m}{m-r} = \frac{m!}{(m-r)!\,(m-(m-r))!}$

$= \frac{m!}{(m-r)!\,r!} = \frac{m!}{r!\,(m-r)!} = \binom{m}{r}$ $\square$

2nd approach: "combinatorial proof"

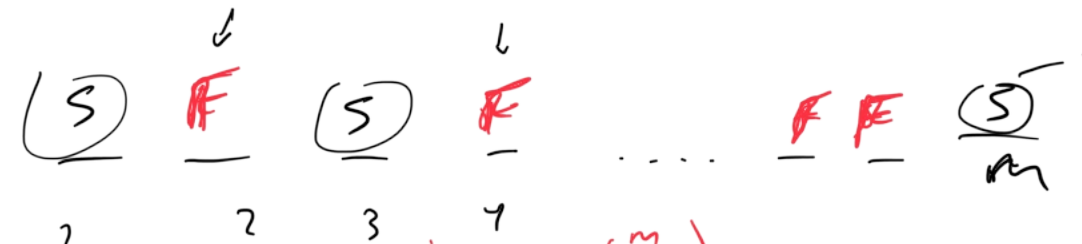
identify a problem with 2 solutions — goal: start with m objects and select r.

1) $\Rightarrow$ # of ways to select r items from a group of m $\Rightarrow \binom{m}{r}$

2) Consider selecting the rejects instead.
$\rightarrow$ select all rejects ($m-r$ of them) and "throw them out"
Left with the designated selection (size r)

$\binom{m}{m-r}$ $\rightarrow$ for each way of selecting the rejects you identify a way of selecting the desired items

R



Select F's $\rightarrow \binom{m}{m-r}$

c) Show $r\binom{m}{r} = m\binom{m-1}{r-1}$

1) $$m\binom{m-1}{r-1} = m \times \frac{(m-1)!}{(r-1)!\,\big((m-\cancel{1}) - (r-\cancel{1})\big)!}$$

$$= \frac{m!}{(r-1)!\,(m-r)!}$$

$$= \frac{r}{r} \times \frac{m!}{(r-1)!\,(m-r)!}$$

$$= r \times \frac{m!}{r!\,(m-r)!} = r \times \binom{m}{r} \qquad \square$$

2) using combination:

$$r\binom{m}{r} = m\binom{m-1}{r-1} \iff \underbrace{\binom{r}{1}\binom{m}{r}} = \underbrace{\binom{m}{1}\binom{m-1}{r-1}}$$

Problem: from m people, pick a committee of r people and choose a leader of the team.

How many ways to do this?

1) LHS: choose the committee first, and then from the committee, choose the leader.

$$\underbrace{\binom{m}{r}}_{\substack{\uparrow \\ \text{team}}} \times \underbrace{\binom{r}{1}}_{\substack{\uparrow \\ \text{leader from the team.}}} \leftarrow \text{BCR.}$$

2) RHS: choose leader first (from m) then complete the team.

$$\binom{m}{1} \times \binom{m-1}{r-1}$$

$\uparrow$ leader first $\qquad \uparrow$ rest of the team.

D) problem: # of ways to select committee of size r, and a subcommittee of size k from those r.

LHS: $\binom{n}{r}\binom{r}{k}$

committee selection

from committee select subcommittee.

RHS: $\binom{n}{k}\binom{n-k}{r-k}$

subcommittee selection.

complete the committee (k already selected)
→ $r-k$ more people from the available $(n-k)$