MATH 2080 Introductory Analysis

Chapter 6 Sequences and Series of Fuctions

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Weierstrass Approximation Theorem

Theorem 6.1 (WAT)

Let $f(x)$ be a continuous function on $[a, b]$. Then, for each $\varepsilon > 0$, there is a polynomial $p_\varepsilon(x)$ such that

$$|f(x) - p_\varepsilon(x)| < \varepsilon \quad \textit{for all } x \in [a, b].$$

The proof of the theorem is a bit complicated. Here we only introduce this important theorem without a proof.

Remark

For each $n \in \mathbb{N}$, if we take $\varepsilon = \frac{1}{n}$ and write $p_n(x)$ for $p_\varepsilon(x)$. The above theorem may be interpreted as follows. There is a sequence $(p_n(x))$ of polynomials such that $p_n(x) \to f(x)$ for all $x \in [a, b]$, provided $f(x)$ is continuous on $[a, b]$.

Pointwise and Uniform Convergence

Definition 6.2

Let (f_n) be a sequence of functions on a set A and f be a function on A. We say that (f_n) converges pointwise to f on A, written as $f_n \to f$ on A, if $f_n(x) \to f(x)$ for each $x \in A$.

Definition 6.3

We say that $(f_n(x))$ converges to $f(x)$ uniformly on A, written as $f_n \to f$ uniforly on A, if $\forall \varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in A$ whenever $n \geq N$.

Example 1

Let $f_n(x) = \frac{x^2}{n(1+x^2)}$. Then $f_n(x) \to 0$ uniformly on $\mathbb{R}$.

The WAT ensures that a continuous function on $[a, b]$ can be approximated by a sequence of polynomials uniformly on $[a, b]$.

Criterion for not being uniformly Convergent

Theorem 6.4

If there exist $\varepsilon_0 > 0$ *and a sequence* $(x_n) \subset A$ *such that*

$$|f_n(x_n) - f(x_n)| \geq \varepsilon_0$$

for all n*, then* $(f_n(x))$ *does not converge to* $f(x)$ *uniformly on* A*.*

Proof.

Use contradiction argument. $\square$

Example 2

Let $f_n(x) = \frac{nx^2+x}{n}$ *and* $f(x) = x^2$ *on* $\mathbb{R}$*. Then* $f_n \to f$ *pointwise on* $\mathbb{R}$ *but not uniformly on* $\mathbb{R}$*.*

Continuity of the limit function

Question:

If $f_n(x) \to f(x)$ on A and if each $f_n(x)$ is continuous on A, must $f(x)$ be continuous on A?

Unfortunately, the answer to the above question is negative if the convergence is merely pointwise convergence

Example 3

Let $f_n(x) = x^n$ on $A = [0,1]$. Then $f_n(x)$ converges to the function

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1), \\ 1 & \text{for } x = 1. \end{cases}$$

Clearly, all $f_n(x)$ are continuous on $[0,1]$ but the limit function f is discontinuous at $x = 1$.

Uniform convergence preserves continuity

Theorem 6.5

Suppose that $(f_n(x))$ is a sequence of continuous functions on $A \subset \mathbb{R}$. If it converges to $f(x)$ uniformly on A, then $f(x)$ is continuous on A.

Remark

Let $f_n(x) \to f(x)$ on A. If we find the limit function $f(x)$ is not continuous at even one point of A, then, by the above theorem, we can conclude that the convergence $f_n(x) \to f(x)$ is not uniform on A.

Example 4

Let $f_n(x) = \frac{x}{1+x^n}$. It converges pointwise but not uniformly on $[0, \infty)$ to

$$f(x) = \begin{cases} x & \text{for } x \in [0,1), \\ 1/2 & \text{for } x = 1, \\ 0 & \text{for } x \in (1, \infty). \end{cases}$$

Series of functions

Definition 6.6

Let $(f_n(x))$ be a sequence of functions on A. If, for each $x \in A$, the numerical series $\sum_{n=1}^{\infty} f_n(x)$ converges to $s(x)$, then we say that the function series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on A. In the case we write $\sum_{n=1}^{\infty} f_n(x) = s(x)$. Indeed, $s(x)$ is a function on A, called the sum function of the series.

The n-th partical sum of $\sum_{n=1}^{\infty} f_n(x)$ is

$$s_n(x) = \sum_{k=1}^{n} f_k(x).$$

By the definition, $\sum_{n=1}^{\infty} f_n(x)$ converging pointwise to $s(x)$ on A means exactly $s_n(x) \to s(x)$ pointwise on A.

Uniformly convergent function series

Definition 6.7

We say that the function series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A if its partial sum sequence $(s_n(x))$ converges uniformly on A.

Theorem 6.8

If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and each $f_n(x)$ is continuous on A, then its sum function $s(x)$ is continuous on A.

Proof.

By definition, $s_n(x) \to s(x)$ uniformly on A. We then can simply apply the corresponding theorem for sequences. $\square$

Criterion for uniform convergence

Theorem 6.9 (The Cauchy Criterion for Uniform Convergence)

1. *The function sequece $(f_n(x))$ converges uniformly on A if and only if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that*
$$|f_n(x) - f_m(x)| < \varepsilon \quad \textit{for all } x \in A,$$
whenever $m, n \geq N$.
2. *The function series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A if and only if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that*
$$\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon \quad \textit{for all } x \in A,$$
whenever $n \geq m \geq N$.

Proof.

Straigntforward. $\square$

Test for uniform convergence

The following is a simple but very useful test for the uniform convergence of a function series.

Theorem 6.10 (Weierstrass *M*-Test)

Suppose that, for each n, $\exists M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$.If the numerical series $\sum_{n=1}^{\infty} M_n$ converges, then the function series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

Proof.

Simply apply the Cauchy Criterion for the function series. □

Example 5

Test the series for uniform convergence.

1. $\sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$ *on* $\mathbb{R}$
2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} x^n$ *on* $[0, 1]$.