MATH 2080 Introductory Analysis

Chapter 4 Fuctional Limits and Continuity

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The ε-δ definition for functional limits

A real-valued function $f(x)$ defined on a set $A \subset \mathbb{R}$ is denoted by $f\colon A \to \mathbb{R}$. The set A is called the domain of f.

Definition 4.1

Let $f\colon A \to \mathbb{R}$ be a function and $c \in \mathbb{R}$ be a limit point of A. We say that the limit of $f(x)$, as x tends to c, is L if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ satisfying $0 < |x - c| < \delta$. In the case we write

$$\lim_{x \to c} f(x) = L.$$

Using neighborhood notations, we may restate the above definition as follows.

Definition

$\lim_{x \to c} f(x) = L$ *if,* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $f(x) \in V_\varepsilon(L)$ *whenever* $x \in \mathring{V}_\delta(c) \cap A$.

Functional limits

Example 1

Show the stated limit.

1. $\lim_{x\to 1}(2x-3) = -1;$
2. $\lim_{x\to 2}(x^2+1) = 5;$
3. $\lim_{x\to 0} x\sin\frac{1}{x} = 0.$

Proof.

(1). $|(2x-3)-(-1)| = |2x-2| = 2|x-1|$

Given $\varepsilon > 0$, let $\delta = \frac{1}{2}\varepsilon$. Then

whenever $0<|x-1|<\delta$, we have

$$|(2x-3)-(-1)| = 2|x-1| < 2\cdot\tfrac{1}{2}\varepsilon = \varepsilon.$$

$\therefore$ By definition,

$$\lim_{x \to 1} 2x - 3 = -1$$ $\square$

(2). $|(x^2+1) - 5| = |x^2 - 4| = |(x+2)(x-2)|$

$= |x+2|\,|x-2|$

$|x+2| \le |x-2| + 4 < \underbrace{1+4}_{=5}$ when $|x-2| < \boxed{1}$

Given $\varepsilon > 0$. choose $\delta = \min\{1, \frac{1}{5}\varepsilon\}$.

Then, whenever $0 < |x-2| < \delta$,

$|(x^2+1) - 5| = |x+2|\,|x-2| < 5\,|x-2|$

$< 5 \cdot \frac{1}{5}\varepsilon = \varepsilon$

By defn, $\lim_{x \to 2} x^2 + 1 = 5$. $\square$

Functional limits

Remark

Let $f: A \to \mathbb{R}$ and c a limit point of A. From the definition of the limit, $\lim_{x\to c} f(x) \neq L$ if and only if there is $\varepsilon_0 > 0$ such that $\forall \delta > 0$ $\exists x \in \mathring{V}_\delta(c) \cap A$ for which $|f(x) - L| \geq \varepsilon_0$.

Theorem 4.2 (Sequential Criterion for Functional Limits)

Let $f: A \to \mathbb{R}$ be a function and c be a limit point of A. Then $\lim_{x\to c} f(x) = L$ if and only if $f(x_n) \to L$ whenever $(x_n) \subset A \setminus \{c\}$ is a sequence such that $x_n \to c$.

Proof $\Rightarrow$ suppose $\lim_{x\to c} f(x) = L$. We need to show $\forall \varepsilon > 0$ $\exists N$ s.t. $|f(x_n) - L| < \varepsilon$ for all $n \geq N$.

Since $\lim_{x\to c} f(x) = L$, for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$ $(x \in A)$.

Since $x_n \to c$, $x_n \neq c$, $\exists N > 0$ s.t.

$0 < |x_n - c| < \delta$ for $n \geq N$.

$\therefore$ $|f(x_n) - L| < \varepsilon$ whenever $n \geq N$.

By defn. $f(x_n) \to L$.

$\Leftarrow$ suppose $f(x_n) \to L$ whenever $(x_n) \subset A \setminus \{c\}$

and $x_n \to c$. We need to show $\lim_{x \to c} f(x) = L$.

i.e. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$ whenever $x \in \overset{\circ}{V}_\delta(c) \cap A$.

If this is untrue, then $\lim_{x \to c} f(x) \neq L$. By the

Remark, $\exists\ \varepsilon_0 > 0$ for which $\forall\ \delta > 0$, $\exists\ x_\delta \in \mathring{V}_\delta(c) \cap A$ s.t. $|f(x_\delta) - L| \geq \varepsilon_0$.

Let $\delta = \frac{1}{n}$, denote $x_n = x_\delta$ for $\delta = \frac{1}{n}$. Then $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$ for all n.

$\therefore$ $x_n \to c$, but $f(x_n) \not\to L$,

contradicting to the assumption.

$\therefore \lim\limits_{x \to c} f(x) = L$. $\square$

Functional limits

Corollary 4.3 (Divergence Criterion for Functional Limits)

Let $f: A \to \mathbb{R}$ be a function and c be a limit point of A. Suppose that $\exists (x_n), (y_n) \subset A \setminus \{c\}$ such that $x_n \to c$, $y_n \to c$, $f(x_n) \to L_1$ and $f(y_n) \to L_2$, but $L_1 \neq L_2$. Then $\lim_{x\to c} f(x)$ does not exist.

Example 2

The following limits do not exist.

1. $\lim_{x\to 0} \sin \frac{1}{x}$;
2. $\lim_{x\to 1} \frac{|x-1|}{x-1}$.

Example 3

Consider the Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$ *Then for any $c \in \mathbb{R}$ the limit $\lim_{x\to c} f(x)$ does not exist.*

Solution of Eg 2.

(1). Note $\sin(2n\pi) = 0$ and $\sin(2n\pi + \frac{\pi}{2}) = 1$.

Choose $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ $(n \in \mathbb{N})$.

Clearly $x_n \to 0$ and $y_n \to 0$.

$$\lim_{n\to\infty} \underline{\sin \frac{1}{x_n}} = \lim_{n\to\infty} \sin(2n\pi) = \underline{0}$$

$$\lim_{n\to\infty} \underline{\sin \frac{1}{y_n}} = \lim_{n\to\infty} \sin(2n\pi + \frac{\pi}{2}) = \underline{1}.$$

Since $0 \neq 1$, by the divergence criterion,

$\lim_{x\to 0} \sin \frac{1}{x}$ does not exist. $\square$

(2). Let $g(x) = \dfrac{|x-1|}{x-1}$.

Choose $x_n = 1+\frac{1}{n}$, $y_n = 1-\frac{1}{n}$. ($x_n \to 1$, and $y_n \to 1$)

$$g(x_n) = \frac{|\frac{1}{n}|}{\frac{1}{n}} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1.$$

$$g(y_n) = \frac{|-\frac{1}{n}|}{-\frac{1}{n}} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1.$$

$\therefore \lim_{n\to\infty} g(x_n) = 1$. $\lim_{n\to\infty} g(y_n) = -1$.

By the Divergence Criterion, we have that

$\lim_{x\to 1} g(x) = \lim_{x\to 1} \dfrac{|x-1|}{x-1}$ does not exist.

$\square$

Eg 3. Solution.

Let c be any number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence $(x_n) \subset \mathbb{Q}$ such that $x_n \to c$.

Similarly, $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, so there is a sequence $(y_n) \subset \mathbb{I}$ such that $y_n \to c$.

$f(x_n) = 1$ since $x_n \in \mathbb{Q}$; and

$f(y_n) = 0$ since $y_n \notin \mathbb{Q}$.

$\therefore \lim_{n\to\infty} f(x_n) = 1, \quad \lim_{n\to\infty} f(y_n) = 0.$

They are different. By the Divergence Criterion, we conclude that $\lim_{x \to c} f(x)$ does not exist. $\square$

Algebraic properties of functional limits

Note: if $\lim_{x\to c} f(x)$ exists, then $\exists\, \delta > 0$ such that $f(x)$ is bounded on $\mathring{V}_\delta(c) \cap A$,($\Leftrightarrow \exists\, M > 0$ such that $|f(x)| \le M$ for all $x \in \mathring{V}_\delta(c) \cap A$.)

Theorem 4.4

Let f and g be two functions on a set A and c be a limit point of A. Suppose that $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = J$. *Then*

1. $\lim_{x\to c} (\alpha f(x) + \beta g(x)) = \alpha L + \beta J$ $(\alpha, \beta \in \mathbb{R})$;
2. $\lim_{x\to c} f(x) g(x) = LJ$;
3. $\lim_{x\to c} f(x)/g(x) = L/J$, *provided* $J \neq 0$.

Example 4

Let $f(x)$ *be a polynomial and* $c \in \mathbb{R}$. *Then* $\lim_{x\to c} f(x) = f(c)$.

$$f = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \xrightarrow[x\to c]{} a_0 c^n + a_1 c^{n-1} + \cdots + a_{n-1} c + a_n = f(c).$$

proof of Thm 4.4. only show (2).

$$|f(x)g(x) - LJ| = |\underline{f(x)g(x) - f(x)J} + f(x)J - LJ|$$

$$\le |f(x)(g(x) - J)| + |J(f(x) - L)|$$

Since $\lim_{x\to c} f(x)$ exists, $\exists\ \delta_1 > 0$ and $M > 0$ s.t.

$$|f(x)| \le M \text{ for } x \in \mathring{V}_{\delta_1}(c) \cap A.$$

Given $\varepsilon > 0$, since $\lim_{x\to c} g(x) = J$ and $\lim_{x\to c} f(x) = L$,

$\exists\ \delta_2, \delta_3 > 0$ s.t.

$$|g(x) - J| < \frac{\varepsilon}{2M} \text{ for } x \in \mathring{V}_{\delta_2}(c) \cap A,$$

and $|f(x) - L| < \dfrac{\varepsilon}{2(|J|+1)}$ for $x \in \mathring{V}_{\delta_3}(c) \cap A$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$

when $\underline{x \in \mathring{V}_\delta(c) \cap A,}$ we have

$$\underline{|f(x)g(x) - LJ|} \le M|g(x) - J| + |J||f(x) - L|$$

$$\le M \cdot \frac{\varepsilon}{2M} + |J| \frac{\varepsilon}{2(|J|+1)}$$

$$\underline{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ By defn. $\lim_{x \to c} f(x)g(x) = LJ$.

$\square$.

Continuous at a point

Definition 4.5

Let $f\colon A \to \mathbb{R}$ be a function and $c \in A$. We say that $f(x)$ is continuous at c if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ and $|x - c| < \delta$.

Remark

We note, if $c \in A$, then either c is an isolated point of A or c is a limit point of A. The above definition clearly implies the following conclusions.

- *If $c \in A$ is an isolated point of A, then every function f on A is continuous at c.*
- *If c is a limit point of A, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.*

Example 5.

$f(0) = 0$. For $x \neq 0$,

$$|f(x) - f(0)| = \left|x \sin\frac{1}{x} - 0\right| = \left|x \sin\frac{1}{x}\right| \leq |x|$$
$$= |x-0|$$

For any given $\varepsilon > 0$, let $\delta = \varepsilon$.

Then, when $|x-0| < \delta$, we have

$$|f(x) - f(0)| \leq |x-0| < \delta = \varepsilon.$$

By defn., $f(x)$ is continuous at $x = 0$. $\square$

Continuity

Example 5

The function $f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ *is continuous at* $x = 0$.

Like for limits, we can also restate the definition of the continuity of f at c using neighborhood pharases as follows.

Definition

Let $c \in A$. *The function* $f : A \to \mathbb{R}$ *is continuous at* c *if,* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $f(x) \in V_\varepsilon(f(c))$ *whenever* $x \in V_\delta(c) \cap A$.

We say that $f(x)$ is discontinuous at c if it is not continuous at c.

Example 6

The function $f(x) = \sqrt{x}$ *is continuous at* 0.

Solution.

Domain of f is $[0, \infty)$.

$f(0) = \sqrt{0} = 0$.

$|f(x) - f(0)| = \sqrt{x} = (x-0)^{\frac{1}{2}}$

$\forall \varepsilon > 0$, let $\delta = \varepsilon^2$. Then,

whenever $x \in [0, \infty)$ and $|x - 0| < \delta$, we

have $|f(x) - f(0)| = (x-0)^{\frac{1}{2}} \leq \delta^{\frac{1}{2}} = (\varepsilon^2)^{\frac{1}{2}} = \varepsilon$.

By defn., $f(x)$ is continuous at $x = 0$. $\square$.

Characterization of continuity

Theorem 4.6 (Sequential Criterion for Continuity)

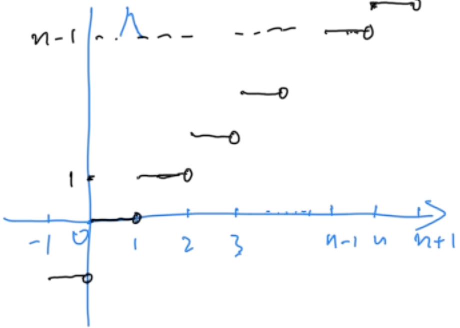
Let $f\colon A \to \mathbb{R}$ be a function and $c \in A$. Then $f(x)$ is continuous at c if and only if $f(x_n) \to f(c)$ whenever $(x_n) \subset A$ is a sequence such that $x_n \to c$.

Corollary 4.7 (Divergence Criterion for Continuity)

Let $f\colon A \to \mathbb{R}$ be a function and $c \in A$. Suppose that $\exists (x_n) \subset A$ such that $x_n \to c$ but $f(x_n) \not\to f(c)$. Then $f(x)$ is discontinuous at c.

Example 7

The function $f(x) = [x]$ is discontinuous at each $n \in \mathbb{Z}$, where $[x]$ represents the largest integer $\leq x$. $\lfloor x \rfloor$ floor function.



For $n \in \mathbb{Z}$, $\quad f(n) = n$.

Let $x_k = n - \frac{1}{k}$. Then $f(x_k) = \lfloor n - \frac{1}{k} \rfloor = \underline{n-1}$

for all $k \geq 1$. $\quad x_k \to n$ for $k \to \infty$.

$$\therefore \lim_{k\to\infty} f(x_k) = n-1 \neq n = f(n).$$

By the sequential criterion, we conclude that

$f(x)$ is discontinuous at $x = n \in \mathbb{Z}$. $\quad$ ☐

Algebraic properties of continuity

Theorem 4.8

let $f(x)$ anf $g(x)$ be two functions on A and $c \in A$. Suppose that both f and g are continuous at c. Then so are the following functions.

1. $\alpha f(x) + \beta g(x)$ *for any* $\alpha, \beta \in \mathbb{R}$ *(linear rule)*
2. $f(x)g(x)$ *(product rule)*
3. $f(x)/g(x)$ *, provided* $g(c) \neq 0$ *(quotient rule)*

If f is continuous at every point of A, we say that f is continuous on A.

Example 8

1. *A polynomial is continuous on* $\mathbb{R}$.
2. *A rational function* $P(x)/Q(x)$ *is continuous on its domain, the set of all real numbers* x *for which* $g(x) \neq 0$.

Composite continuity

Let $f\colon A \to \mathbb{R}$ be a function. Recall that the range of f over A is

$$f(A) = \{f(x) : x \in A\}.$$

If $f(A) \subset B$ and $g\colon B \to \mathbb{R}$, then we can consider the composite function $g \circ f\colon A \to \mathbb{R}$ defined by $g \circ f(x) = g(f(x))$ for $x \in A$.

Theorem 4.9

Suppose that $f\colon A \to \mathbb{R}$, $g\colon B \to \mathbb{R}$, and $f(A) \subset B$. Let $c \in A$ and $b = f(c)$. If f is continuous at c and g is continuous at b. Then the composite function $g \circ f(x) = g(f(x))$ is continuous at c.

Proof. We need to prove that $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ such that $|g\circ f(x) - g\circ f(c)| < \varepsilon$ whenever

$x \in A$ and $|x - c| < \delta$.

Since g is cts. at b. So. for given $\varepsilon > 0$, $\exists$ $\eta > 0$ s.t. $|g(y) - g(b)| < \varepsilon$ when $y \in B$ and $|y - b| < \eta$.

Since f is cts. at c, for this $\eta > 0$ $\exists$ $\delta > 0$ s.t. $|f(x) - f(c)| < \eta$ whenever $x \in A$ and $|x - c| < \delta$.

$\therefore$ $|g(f(x)) - g(f(c))| = |g(f(x)) - g(b)|$

$< \varepsilon$ whenever $x \in A$ and $|x - c| < \delta$.

$\therefore$ By the defn, $g \circ f$ is continuous at c. $\square$

Composite continuity - continued

Remark

Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be continuous functions on their domains, and let $f(A) \subset B$. Then the above theorem ensures that $g \circ f$ is continuous on A.

Example 9

The function $\sqrt{x^2+1}$ is continuous on $\mathbb{R}$.

Proof.

This is the composite function of

$$f(x) = x^2 + 1 \text{ for } x \in \mathbb{R} \quad \text{and} \quad g(x) = \sqrt{x} \text{ for } x \geq 0,$$

i.e. $\sqrt{x^2+1} = g \circ f(x)$ $(x \in \mathbb{R})$. Both f and g are continuous on their domains. So $g \circ f$ is continuous on $\mathbb{R}$. $\square$

On compact sets

The following theorem ensures that a continuous function preserves compactness. In other wards, $f(B)$ is compact whenever B is compact.

Theorem 4.10

Let f be a continuous function on $A \subset \mathbb{R}$. Suppose that $K \subset A$ and K is compact. Then $f(K)$ is compact.

Proof.

For every sequence $(y_n) \subset f(K)$, we show that there is a subsequence (y_{n_k}) that converges to some $y \in f(K)$. $\square$

Indeed, since $y_n \in f(K)$, $\exists\, x_n \in K$ s.t. $y_n = f(x_n)$.

$(x_n) \subset K$. K is compact. So $\exists$ a susequence

(x_{n_k}) s.t. $x_{n_k} \xrightarrow{k\to\infty} \boxed{x_0 \in K}$.

Since $f(x)$ is continuous at x_0, by the sequential criterion of continuity, we have

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$$

i.e. $y_{n_k} \to f(x_0) := y_0 \in f(K)$

By the defn of a cpt set, $f(K)$ is compact.

$\square$

Extreme Values

In Calculus we have learned that a continuous function on a closed interval attains its maximum (resp. minimum) at some point of the interval. This property still holds for a continuous function on a compact set.

Theorem 4.11 (Extreme Value Theorem)

Let $K \subset \mathbb{R}$ be a compact set and let f be a continuous function on K. Then f attains its maximum value at some $k_1 \in K$ and attains its minimum value at some $k_2 \in K$. In other wards, $\exists k_1, k_2 \in K$ such that $f_{\max} = f(k_1)$ and $f_{\min} = f(k_2)$.

Proof.

From Theorem 4.10, $f(K)$ is compact and hence is bounded and closed. So it has a least upper bounded $y_1 \in f(K)$ and a greatest lower bound $y_2 \in f(K)$. Thus $\exists k_1, k_2 \in K$ such that $y_1 = f(k_1)$ and $y_2 = f(k_2)$. This means $f(k_1) = f_{\max}$ and $f(k_2) = f_{\min}$. $\square$

Uniform continuity

Definition 4.12

We say that the function $f\colon A \to \mathbb{R}$ is uniformly continuous on A if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ satisfy $|x - y| < \delta$.

Clearly, if f is uniformly continuous on A, then it is continuous at all points $y = c \in A$. Being uniformly continuous simply means that, for a given $\varepsilon > 0$, we may find $\delta > 0$ that works for all $c \in A$ to ensure $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ and $|x - c| < \delta$.

Example 10

$f(x) = x^2$ is uniformly continuous on $[0, 1]$. However, it is continuous but not uniformly continuous on $\mathbb{R}$.

Proof. $|f(x)-f(y)| = |x^2-y^2| = |x+y||x-y|$

$\leq 2|x-y|$ for $x, y \in [0,1]$.

$\forall \varepsilon > 0$, choose $\delta = \frac{1}{2}\varepsilon$. Then

whenever $x, y \in [0,1]$ and $|x-y| < \delta$, we have $|f(x)-f(y)| \leq 2|x-y| < 2 \cdot \delta = \varepsilon$.

By the defn, $f(x) = x^2$ is uniformly continuous on $[0,1]$.

Now on $\mathbb{R}$: $|f(x)-f(y)| = |x+y||x-y|$.

Consider $\varepsilon = 1$. We show that $\forall \delta > 0$, $\exists\, x_\delta, y_\delta \in \mathbb{R}$ s.t. $|x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq 1$.

In fact, We can choose $x_\delta = \frac{1}{\delta}$ and $y_\delta = \frac{1}{\delta} + \frac{\delta}{2}$.

Then $|x_\delta - y_\delta| = \left|\frac{1}{\delta} - \left(\frac{1}{\delta} + \frac{\delta}{2}\right)\right| = \frac{\delta}{2} < \delta$,

but $|f(x_\delta) - f(y_\delta)| = |x_\delta^2 - y_\delta^2|$

$= |x_\delta + y_\delta|\,|x_\delta - y_\delta| = \left|\frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2}\right| \cdot \left|\frac{\delta}{2}\right|$

$> \frac{2}{\delta} \cdot \frac{\delta}{2} = 1$.

$\therefore$ For $\varepsilon = 1$, there is NO $\delta > 0$ s.t.

$|f(x) - f(y)| < 1$ for all x and y s.t.

$x, y \in \mathbb{R}$ and $|x - y| < \delta$.

$\therefore$ By defn., f is not uniformly continuous on $\mathbb{R}$.

Determine absence of uniform continuity

Observation: From the definition, to say " f is not uniformly continuous on A " is equivalent to say " $\exists\, \varepsilon_0 > 0$ such that no $\delta > 0$ can ensure $|f(x) - f(y)| < \varepsilon_0$ valid for all $x, y \in A$ satisfying $|x - y| < \delta$ ". The latter actually means the following: $\exists\, \varepsilon_0 > 0$ such that for each $\delta > 0$ one can always find $x_\delta, y_\delta \in A$ with $|x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$.

Theorem 4.13 (Sequential Criterion for Non-uniformly Continuous)

The function $f\colon A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exist some $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in A such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

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Proof.

$\Leftarrow$: From the above Observation, this direction is trivial.
$\Rightarrow$: For the ε_0 discribed in the Obeservation, we take $\delta = \frac{1}{n}$ $(n \in \mathbb{N})$ to obtain $x_n, y_n \in A$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$. $\square$

Example 11. Show that $y = \sin\frac{1}{x}$ is not uniformly continuous on $(0,1)$.

proof Let $\varepsilon_0 = 1$. Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ $(n \in \mathbb{N})$. Then $x_n, y_n \in (0,1)$.

Then $\left.\begin{aligned} \sin\frac{1}{x_n} &= \sin(2n\pi) = 0 \\ \sin\frac{1}{y_n} &= \sin\left(2n\pi + \frac{\pi}{2}\right) = 1 \end{aligned}\right\}$ for all n.

$$|x_n - y_n| = \left|\frac{1}{2n\pi} - \frac{1}{2n\pi + \frac{\pi}{2}}\right| = \underline{\frac{\frac{\pi}{2}}{2n\pi\left(2n\pi + \frac{\pi}{2}\right)}}$$

$\to 0$ as $n \to \infty$.

$$\left|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\right| = |0 - 1| = 1 = \varepsilon_0$$

$\therefore$ By the sequential criterion for non-uniform continuity, we have that $y = \sin\frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

$\square$

Uniform continuity - continued

Example 11

$\sin\frac{1}{x}$ *is not uniformly continuous on* $(0, 1)$.

Theorem 4.14

Let f be a continuous function on K. If K is a compact set, then f is uniformly continuous on K.

Proof.

Prove by contridiction: If f were not uniformly continuous on K, then, by the Sequential Criterion, there exist $\varepsilon_0 > 0$ and $(x_n), (y_n) \subset K$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.
Since K is compact, there is a subsequence (x_{n_i}) of (x_n) that converges to some $k \in K$. Then $f(x_{n_i}) \to f(k)$ by the continuity.
Since $|y_{n_i} - k| \leq |y_{n_i} - x_{n_i}| + |x_{n_i} - k| \to 0$, we have $f(y_{n_i}) \to f(k)$ too.
So $|f(x_{n_i}) - f(y_{n_i})| \to 0$, contridicting $|f(x_{n_i}) - f(y_{n_i})| \geq \varepsilon_0$ for all n_i. $\square$