# Math 2740 – Fall 2025
# Sample final examination – Variant 1
# Solutions

**Exercise 1.** **[Definitions and Theorems – 15 points]**

State the definition or theorem for each of the following. Be precise and complete.

1. **[3 pts]** Define the *singular values* of a matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$.
2. **[4 pts]** State the *Singular value decomposition (SVD) theorem.*
3. **[4 pts]** Given a matrix $A \in \mathcal{M}_n$, define its eigenpairs.
4. **[4 pts]** State the *Least squares theorem.*

**Solution of Exercise 1.** **Definitions and Theorems.**

1. **[3 pts]** The singular values of $A \in \mathcal{M}_{mn}(\mathbb{R})$ are the square roots of the eigenvalues of $A^TA$. (When considering *positive* singular values, equivalently, the square roots of the eigenvalues of $AA^T$.)
2. **[4 pts]** SVD theorem: Let $A \in \mathcal{M}_{mn}(\mathbb{R})$. Then $A$ has singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$. Also, there exists $U \in \mathcal{M}_m$ orthogonal, $V \in \mathcal{M}_n$ orthogonal and a block matrix $\Sigma \in \mathcal{M}_{mn}$ taking the form
$$\Sigma = \begin{pmatrix} D & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}$$
where
$$D = \mathsf{diag}(\sigma_1, \ldots, \sigma_r) \in \mathcal{M}_r(\mathbb{R}),$$
such that
$$A = U\Sigma V^T.$$
3. **[4 pts]** An eigenpair of a matrix $A \in \mathcal{M}_n$ is a pair $(\lambda, \mathbf{v})$ where $\lambda \in \mathbb{C}$ is a scalar and $\mathbf{v} \in \mathbb{C}^n$ is a nonzero vector such that $A\mathbf{v} = \lambda\mathbf{v}$. The scalar $\lambda$ is called an eigenvalue and the vector $\mathbf{v}$ is called an eigenvector corresponding to $\lambda$.
4. **[4 pts]** Least squares theorem: Let $A \in \mathcal{M}_{mn}$ and $\mathbf{b} \in \mathbb{R}^m$. Then
    1. $A\mathbf{x} = \mathbf{b}$ always has at least one least squares solution $\tilde{\mathbf{x}}$.
    2. $\tilde{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$ $\iff$ $\tilde{\mathbf{x}}$ is a solution to the normal equations $A^TA\tilde{\mathbf{x}} = A^T\mathbf{b}$.
    3. $A$ has linearly independent columns $\iff$ $A^TA$ invertible.
       In this case, the least squares solution is unique and
$$\tilde{\mathbf{x}} = \left(A^TA\right)^{-1} A^T\mathbf{b}.$$

**Exercise 2.** **[Linear Least Squares – 15 points]**
Consider the over-determined system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

1. **[5 pts]** Set up the normal equation $A^T A\mathbf{x} = A^T\mathbf{b}$ by computing $A^T A$ and $A^T\mathbf{b}$.
2. **[5 pts]** Solve the normal equation to find the least squares solution $\tilde{\mathbf{x}}$.
3. **[5 pts]** Compute the residual $\mathbf{b} - A\tilde{\mathbf{x}}$ and its norm $\|\mathbf{b} - A\tilde{\mathbf{x}}\|$.

**Solution of Exercise 2.** **Linear Least Squares.**
We compute

$$A^T A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix},$$

$$A^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}.$$

The normal equation is $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}$. Row reduce:

$$\begin{pmatrix} 6 & 4 & 14 \\ 4 & 6 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/3 & 7/3 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus $\tilde{\mathbf{x}} = (1, 2)^T$.
The residual is

$$r = \mathbf{b} - A\tilde{\mathbf{x}} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $\|r\| = 0$. (The system is actually consistent!)

**Exercise 3.** **[Singular Value Decomposition – 20 points]**

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$$

1. **[6 pts]** Compute $A^T A$ and find its eigenvalues.
2. **[4 pts]** Determine the singular values of $A$.
3. **[5 pts]** Find the right singular vectors (eigenvectors of $A^T A$) and construct the matrix $V$.
4. **[5 pts]** Construct the matrices $\Sigma$ and $U$ to complete the SVD $A = U\Sigma V^T$. (You may verify your answer by computing the product.)

**Solution of Exercise 3. Singular Value Decomposition.**

We compute

$$A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 6 \end{pmatrix}.$$

Eigenvalues: $\det(A^T A - \lambda I) = \lambda^2 - 11\lambda + 14 = 0$ gives $\lambda = \frac{11 \pm \sqrt{121-56}}{2} = \frac{11 \pm \sqrt{65}}{2}$. So $\lambda_1 = \frac{11+\sqrt{65}}{2} \approx 9.53$ and $\lambda_2 = \frac{11-\sqrt{65}}{2} \approx 1.47$.

Singular values: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{\frac{11+\sqrt{65}}{2}}$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{\frac{11-\sqrt{65}}{2}}$.

For $\lambda_1$: $(A^T A - \lambda_1 I)\mathbf{v} = 0$ gives eigenvector $v_1 = \frac{1}{\sqrt{2}}(1,1)^T$ (after normalization). For $\lambda_2$: eigenvector $v_2 = \frac{1}{\sqrt{2}}(1,-1)^T$. Thus $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

We have $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$ and $U$ is found from $AV = U\Sigma$, giving columns $u_1 = \frac{1}{\sigma_1} A v_1$ and $u_2 = \frac{1}{\sigma_2} A v_2$, with $u_3$ orthogonal to both.

**Exercise 4. [Principal Component Analysis – 15 points]**

Consider a dataset with the following data matrix (each row is an observation):

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

1. [**4 pts**] Compute the mean of each variable and center the data matrix to obtain $\tilde{X}$.
2. [**6 pts**] Compute the sample covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$ where $n = 3$.
3. [**5 pts**] Find the eigenvalues of the covariance matrix. Which eigenvalue corresponds to the first principal component?

**Solution of Exercise 4. Principal Component Analysis.**

Mean of each variable: $\bar{x}_1 = \frac{2+0+1}{3} = 1$, $\bar{x}_2 = \frac{0+2+3}{3} = \frac{5}{3}$.

Centered data matrix:

$$\tilde{X} = \begin{pmatrix} 2-1 & 0-5/3 \\ 0-1 & 2-5/3 \\ 1-1 & 3-5/3 \end{pmatrix} = \begin{pmatrix} 1 & -5/3 \\ -1 & 1/3 \\ 0 & 4/3 \end{pmatrix}.$$

Sample covariance matrix:

$$S = \frac{1}{n-1}\tilde{X}^T\tilde{X} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -5/3 & 1/3 & 4/3 \end{pmatrix}\begin{pmatrix} 1 & -5/3 \\ -1 & 1/3 \\ 0 & 4/3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 26/9+16/9 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 42/9 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 7/3 \end{pmatrix}.$$

Eigenvalues: $\det(S - \lambda I) = \lambda^2 - \frac{10}{3}\lambda + \frac{4}{3} = 0$ gives $\lambda = \frac{10\pm\sqrt{100-48}}{6} = \frac{10\pm\sqrt{52}}{6} = \frac{5\pm\sqrt{13}}{3}$.

So $\lambda_1 = \frac{5+\sqrt{13}}{3} \approx 2.87$ and $\lambda_2 = \frac{5-\sqrt{13}}{3} \approx 0.465$.

The first principal component corresponds to $\lambda_1 = \frac{5+\sqrt{13}}{3}$.

**Exercise 5.** **[Proof – 15 points]**

Let $A \in \mathcal{M}_{mn}(\mathbb{R})$. Prove that for any nonzero eigenvalue $\lambda$ of $A^T A$, we have $\lambda > 0$.

**Hint:** Use the definition of eigenvalue and properties of the inner product.

**Solution of Exercise 5.** **Proof.**

Let $\lambda$ be a nonzero eigenvalue of $A^T A$ with corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$. Then $A^T A\mathbf{v} = \lambda\mathbf{v}$.

Taking the inner product of both sides with $\mathbf{v}$:

$$\langle A^T A\mathbf{v}, \mathbf{v}\rangle = \langle \lambda\mathbf{v}, \mathbf{v}\rangle = \lambda\langle \mathbf{v}, \mathbf{v}\rangle = \lambda\|\mathbf{v}\|^2.$$

By properties of the transpose and inner product:

$$\langle A^T A\mathbf{v}, \mathbf{v}\rangle = \langle A\mathbf{v}, A\mathbf{v}\rangle = \|A\mathbf{v}\|^2.$$

Therefore $\lambda\|\mathbf{v}\|^2 = \|A\mathbf{v}\|^2$.

Since $\mathbf{v} \neq \mathbf{0}$, we have $\|\mathbf{v}\|^2 > 0$. Also, $\|A\mathbf{v}\|^2 \geq 0$. If $\lambda \neq 0$, then from $A^T A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$, we must have $A\mathbf{v} \neq \mathbf{0}$ (otherwise $\lambda\mathbf{v} = \mathbf{0}$ implies $\lambda = 0$), so $\|A\mathbf{v}\|^2 > 0$.

Thus $\lambda = \frac{\|A\mathbf{v}\|^2}{\|\mathbf{v}\|^2} > 0$.

**Exercise 6.** **[Graph Measures I – 10 points]**

Consider the simple undirected graph $G$ on vertices $V = \{1, 2, 3, 4, 5\}$ with edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

1. [**4 pts**] Compute the degree $\deg(i)$ of each vertex and give the degree sequence in nonincreasing order.
2. [**3 pts**] Compute the density of $G$, defined as $\delta(G) = \dfrac{2|E|}{|V|(|V| - 1)}$.
3. [**3 pts**] For each vertex with $\deg(i) \geq 2$, compute its local clustering coefficient $C_i = \dfrac{2e_i}{\deg(i)(\deg(i) - 1)}$, where $e_i$ is the number of edges between neighbors of $i$. State the average (mean) clustering coefficient of $G$.

**Solution of Exercise 6. Graph Measures I.**

Degrees: $\deg(1) = 2$ (edges to 2,3), $\deg(2) = 3$ (edges to 1,3,4), $\deg(3) = 3$ (edges to 1,2,4), $\deg(4) = 3$ (edges to 2,3,5), $\deg(5) = 1$ (edge to 4).

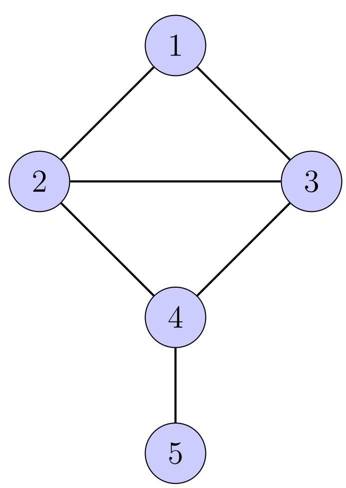
Degree sequence in nonincreasing order: $(3, 3, 3, 2, 1)$.

Density: $\delta(G) = \frac{2|E|}{|V|(|V|-1)} = \frac{2\cdot6}{5\cdot4} = \frac{12}{20} = \frac{3}{5} = 0.6$.

Clustering coefficients for vertices with degree $\geq 2$:

- Vertex 1: neighbors are $\{2, 3\}$; edge $\{2, 3\}$ exists, so $e_1 = 1$. Thus $C_1 = \frac{2\cdot1}{2\cdot1} = 1$.
- Vertex 2: neighbors are $\{1, 3, 4\}$; edges $\{1, 3\}, \{3, 4\}$ exist, so $e_2 = 2$. Thus $C_2 = \frac{2\cdot2}{3\cdot2} = \frac{2}{3}$.
- Vertex 3: neighbors are $\{1, 2, 4\}$; edges $\{1, 2\}, \{2, 4\}$ exist, so $e_3 = 2$. Thus $C_3 = \frac{2\cdot2}{3\cdot2} = \frac{2}{3}$.
- Vertex 4: neighbors are $\{2, 3, 5\}$; edge $\{2, 3\}$ exists, so $e_4 = 1$. Thus $C_4 = \frac{2\cdot1}{3\cdot2} = \frac{1}{3}$.

Average clustering coefficient: $\frac{1+\frac{2}{3}+\frac{2}{3}+\frac{1}{3}}{4} = \frac{\frac{8}{3}}{4} = \frac{2}{3} \approx 0.667$.



**Exercise 7.** **[Graph Measures II – 10 points]**

For the same graph $G$ as in Exercise 6:

1. **[4 pts]** Compute the graph diameter (the maximum shortest-path distance between any two distinct vertices) and the average shortest-path length $\ell(G)$ over all unordered vertex pairs.
2. **[3 pts]** Compute the (normalized) degree centrality of each vertex, defined as $C_D(i) = \deg(i)/(n-1)$ where $n = |V|$.
3. **[3 pts]** Compute the closeness centrality of each vertex, defined for connected graphs as $C_C(i) = \dfrac{n-1}{\sum_{j \neq i} d(i,j)}$, where $d(i,j)$ is the shortest-path distance.

**Solution of Exercise 7. Graph Measures II.**

Shortest-path distances: $d(1,2) = 1$, $d(1,3) = 1$, $d(1,4) = 2$, $d(1,5) = 3$, $d(2,3) = 1$, $d(2,4) = 1$, $d(2,5) = 2$, $d(3,4) = 1$, $d(3,5) = 2$, $d(4,5) = 1$.

Graph diameter: $\max d(i,j) = 3$ (between vertices 1 and 5).

Sum of all shortest paths: $1+1+2+3+1+1+2+1+2+1 = 15$.

Average shortest-path length: $\ell(G) = \frac{15}{\binom{5}{2}} = \frac{15}{10} = 1.5$.

Degree centrality: $C_D(1) = \frac{2}{4} = 0.5$, $C_D(2) = \frac{3}{4} = 0.75$, $C_D(3) = \frac{3}{4} = 0.75$, $C_D(4) = \frac{3}{4} = 0.75$, $C_D(5) = \frac{1}{4} = 0.25$.

Closeness centrality:

- $C_C(1) = \frac{4}{1+1+2+3} = \frac{4}{7} \approx 0.571$
- $C_C(2) = \frac{4}{1+1+1+2} = \frac{4}{5} = 0.8$
- $C_C(3) = \frac{4}{1+1+1+2} = \frac{4}{5} = 0.8$
- $C_C(4) = \frac{4}{2+1+1+1} = \frac{4}{5} = 0.8$
- $C_C(5) = \frac{4}{3+2+2+1} = \frac{4}{8} = 0.5$

**Exercise 8.** **[Regular Markov Chains – 10 points]**

Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/3 \\ 1/2 & 0 & 1/4 & 1/3 \\ 0 & 1/2 & 1/4 & 1/3 \end{pmatrix}$$

1. **[3 pts]** Verify that this Markov chain is regular.
2. **[7 pts]** Find the limiting distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^T$ by solving the system $P\boldsymbol{\pi} = \boldsymbol{\pi}$ with the constraint $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

**Solution of Exercise 8. Regular Markov Chains.**

1. **[3 pts]** To verify regularity, we need to show that some power $P^k$ has all positive entries. Observe that $P$ has several zero entries. However, notice that from state 1, we can reach state 2 or 3 in one step, and from those states we can reach any state in one more step. Similarly, all states communicate. Computing $P^2$ (or analyzing the transition graph), we can verify that all entries become positive, making the chain regular.
2. **[7 pts]** To find the limiting distribution, we solve $P\boldsymbol{\pi} = \boldsymbol{\pi}$, or equivalently $(P - I)\boldsymbol{\pi} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1/2 & 1/4 & 0 \\ 1/2 & -1 & 1/4 & 1/3 \\ 1/2 & 0 & -3/4 & 1/3 \\ 0 & 1/2 & 1/4 & -2/3 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From equation (1): $-\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = 0 \implies \pi_1 = \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3$.
From equation (2): $\frac{1}{2}\pi_1 - \pi_2 + \frac{1}{4}\pi_3 + \frac{1}{3}\pi_4 = 0$.
From equation (3): $\frac{1}{2}\pi_1 - \frac{3}{4}\pi_3 + \frac{1}{3}\pi_4 = 0$.
From equation (4): $\frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 - \frac{2}{3}\pi_4 = 0$.
Substituting equation (1) into equation (3):

$$\frac{1}{2}\left(\frac{1}{2}\pi_2 + \frac{1}{4}\pi_3\right) - \frac{3}{4}\pi_3 + \frac{1}{3}\pi_4 = 0$$

$$\frac{1}{4}\pi_2 + \frac{1}{8}\pi_3 - \frac{3}{4}\pi_3 + \frac{1}{3}\pi_4 = 0$$

$$\frac{1}{4}\pi_2 - \frac{5}{8}\pi_3 + \frac{1}{3}\pi_4 = 0 \quad (5)$$

From equation (4): $\pi_2 = \frac{4}{3}\pi_4 - \frac{1}{2}\pi_3 \quad (6)$
Substituting (6) into (5):

$$\frac{1}{4}\left(\frac{4}{3}\pi_4 - \frac{1}{2}\pi_3\right) - \frac{5}{8}\pi_3 + \frac{1}{3}\pi_4 = 0$$

$$\frac{1}{3}\pi_4 - \frac{1}{8}\pi_3 - \frac{5}{8}\pi_3 + \frac{1}{3}\pi_4 = 0$$

$$\frac{2}{3}\pi_4 - \frac{6}{8}\pi_3 = 0$$

$$\frac{2}{3}\pi_4 = \frac{3}{4}\pi_3 \implies \pi_4 = \frac{9}{8}\pi_3$$

From (6): $\pi_2 = \frac{4}{3} \cdot \frac{9}{8}\pi_3 - \frac{1}{2}\pi_3 = \frac{3}{2}\pi_3 - \frac{1}{2}\pi_3 = \pi_3$

From (1): $\pi_1 = \frac{1}{2}\pi_3 + \frac{1}{4}\pi_3 = \frac{3}{4}\pi_3$
Using normalization $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$:

$$\frac{3}{4}\pi_3 + \pi_3 + \pi_3 + \frac{9}{8}\pi_3 = 1$$

$$\left(\frac{6+8+8+9}{8}\right)\pi_3 = 1$$

$$\frac{31}{8}\pi_3 = 1 \implies \pi_3 = \frac{8}{31}$$

Therefore:

$$\boldsymbol{\pi} = \left(\frac{6}{31}, \frac{8}{31}, \frac{8}{31}, \frac{9}{31}\right)^T$$