# MATH 2080 2025F Assignment 3 Solutions

1. Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

   (a) **[2 points]** $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\ln(n+1)}$. (Hint: You may use the inequality $\ln a < a$ for $a > 0$ when testing for absolute convergence.)

   **Solution:**

   The series converges conditionally.

   $\left|(-1)^n \frac{1}{\ln(n+1)}\right| = \frac{1}{\ln(n+1)} \geq \frac{1}{n+1}$ for all $n$.

   $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent as a harmonic series.

   By the Comparison Test, $\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{\ln(n+1)}\right|$ diverges.

   So $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\ln(n+1)}$ is NOT absolutely convergent.

   The series is an alternating series. $\left(\frac{1}{\ln(n+1)}\right)$ is decreasing and $\frac{1}{\ln(n+1)} \to 0$.

   By the Alternating Series Test, the series converges.

   Therefore, the series converges conditionally.

   (b) **[2 points]** $\sum_{n=1}^{\infty}(-1)^n \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}}$.

   **Solution:**

   The series converges absolutely.

   **Method 1.** Use the Limit Comparison Test.

   Let $a_n = \left|(-1)^n \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}}\right| = \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}}$, and let $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$. We have

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}} \cdot n^{\frac{3}{2}} = \lim_{n\to\infty} \frac{100n^2 + n^{\frac{11}{6}}}{2n^2 - n^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{100 + n^{\frac{-1}{6}}}{2 - n^{\frac{-3}{2}}} = \frac{100}{2} = 50.$$

   By the Limit Comparison Test, the convergence of $\sum a_n$ is the same as that of $\sum b_n$.

   $\sum b_n$ is converges as a $p$-series with $p = \frac{3}{2} > 1$.

   Thus, $\sum a_n$ converges.

   Therefore, the original series converges absolotely.

   **Method 2.** Use the Comparison Test.

   $\left|(-1)^n \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}}\right| = \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}} \leq \frac{101\sqrt{n}}{n^2} = \frac{101}{n^{\frac{3}{2}}}$.

   $\sum_{n=1}^{\infty} \frac{101}{n^{\frac{3}{2}}}$ converges as a $p$-series with $p = \frac{3}{2} > 1$.

   By the Comparison Test, $\sum_{n=1}^{\infty}\left|(-1)^n \frac{100\sqrt{n}+\sqrt[3]{n}}{2n^2-\sqrt{n}}\right|$ converges.

   Therefore, the original series converges absolutely.

(c) **[2 points]** $\sum_{n=1}^{\infty}\left(\frac{2+\sin^2 n}{1+\cos n}\right)^n$.

**Solution:**

The series diverges.

Since $\frac{2+\sin^2 n}{1+\cos n} \geq \frac{2+0}{1+1} = 1$, we have $\left(\frac{2+\sin^2 n}{1+\cos n}\right)^n \geq 1$.

This implies that the general tern cannot not tend to 0.

So the serious diverges.

* One can also use the inequality $\left(\frac{2+\sin^2 n}{1+\cos n}\right)^n \geq 1 = 1^n$ and the Comparison Test, comparing with the geormetric series $\sum r^n$ with $r = 1$. Then use the divergence of $\sum r^n$ when $r \geq 1$ to derive the conclusion.

2. **[1 point]** Suppose that $\sum a_n^2$ and $\sum b_n^2$ converge. Show that $\sum a_n b_n$ also converges. (Hint: Use $|ab| \leq \frac{1}{2}(a^2 + b^2)$.)

**Solution:**

Since $\sum a_n^2$ and $\sum b_n^2$ converge, we have that

$$\sum (a_n^2 + b_n^2) = \sum a_n^2 + \sum b_n^2$$

also converges.

On the other hand, $|a_n b_n| \leq \frac{1}{2}(a_n^2 + b_n^2)$ for all $n$.

By the Comparison Test, $\sum |a_n b_n|$ converges.

Therefore, $\sum a_n b_n$ converges absolutely and thus must converge.

3. **[1 point]** Let $A, B \subset \mathbb{R}$. Suppose that $A$ is open and $B$ is closed. Show that $A \setminus B$ is open. (You may use any theorem covered in the lecture.)

**Solution:**

Since $B$ is closed, $B^C$ must be open.

$A \setminus B = A \cap B^C$.

Therefore $A \setminus B$ is open as the intersection of two open sets.