MATH 2080 Review Questions for the Final Exam

1. Prove the following Squeeze Theorem: Let f, g and h be functions on a set A such that
$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in A.$$
Let c be a limit point of A. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

2. Let $f(x)$ be a function on A and c is a limit point of A. If $\lim_{x\to c} f(x) = L$, show $\lim_{x\to c} |f(x)| = |L|$.

3. Let $(a_n) \subset \mathbb{R}$ be a sequence such that $a_n \to a$. . Show that $\sqrt[3]{|a_n|} \to \sqrt[3]{|a|}$.

4. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and the series $\sum_{n=1}^{\infty} b_n$ converges. Show that $\sum_{n=1}^{\infty} a_n b_n$ must converge absolutely. If $\sum_{n=1}^{\infty} a_n$ is merely conditionally convergent, does the conclusion still hold? Justify your answer.

5. Show that $f(x) = \sqrt{x^2 + 100}$ is uniformly continuous on $\mathbb{R}$.

6. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Use the Cauchy Criterion to show that $\sum_{n=1}^{\infty} (-1)^n (\sin n) a_n$ must converge.

7. Suppose that f is a continuous function on the interval(a, b). Show that the set $E = \{x \in (a, b) : f(x) > 0\}$ is an open set.

8. Let f and g be two functions on A and $c \in A$. If both f and g are continuous at $x = c$, show that fg is continuous at c.

9. Suppose that $f(x)$ is uniformly continuous on a set A and (x_n) is a Cauchy sequence such that $\sqrt{x_n^2 + 1} \in A$ for all n. Show that $(f(\sqrt{x_n^2 + 1}))$ must also be a Cauchy sequence.

10. Let $K \subset \mathbb{R}$ and $K^2 := \{xy : x, y \in K\}$. Suppose that K is compact. Show that K^2 must also be compact.

11. Suppose that it is known that $\sin x$ is continuous on $\mathbb{R}$. Use this fact and the definition of the continuous function to show that $\sin(x^2 + 1)$ must also be a continuous function on $\mathbb{R}$.

12. Recall that the density property of $\mathbb{Q}$ in $\mathbb{R}$ can be stated as follows: For each nonempty open interval (a, b), there is $r \in \mathbb{Q}$ such that $r \in (a, b)$. Use this density property to show that, for every number $c \in \mathbb{R}$, there is a sequence $(r_n) \subset \mathbb{Q}$ such that $r_n \to c$.

13. Use the definition of the limit to show the following limit identity.

    (a) $\lim_{n\to\infty} \frac{n^2 - n \sin n}{n^2 + 1} = 1$

(b) $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

14. Suppose that (a_n) is a bounded sequence. Let $\sum_{n=1}^{\infty} b_n$ be an absolutely convergent series. Show that $\sum_{n=1}^{\infty} a_n b_n$ must converge. If $\sum_{n=1}^{\infty} b_n$ is just conditionally convergent, is the conclusion still true? Justify your answer.

15. Let (a_n) be a convergent sequence. Show that it must be bounded.

16. Determine whether the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$ is convergent. Justify your answer.

17. Let $A, B \subset \mathbb{R}$. Suppose that A and B are both bounded below. Show that $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

18. State the definition of a compact set of real numbers. Show that if $K \subset \mathbb{R}$ is compact, then it must be bounded.

19. Show that the limit $\lim_{x\to 0}\cos\left(\frac{1}{2x}\right)$ does not exist.

20. Let f be a continuous function on a compact set $K \subset \mathbb{R}$. Show that the range $f(K)$ is compact.

21. Is x^2 uniformly continuous on $[0, \infty)$? prove your answer.

22. Let $a, b \in \mathbb{R}$. If for every $\varepsilon > 0$ we have $|a - b| \leq \varepsilon$, show that $a = b$.

23. Let $U \subset \mathbb{R}$ be a closed set. Show that its complement U^C must be an open set.

24. (a) Show that every Cauchy sequence is bounded.

    (b) If (a_n) and (b_n) are Cauchy sequences, show that so is $(a_n b_n)$.

25. Let $f(x)$ be a function on A and c is a limit point of A. If $\lim_{x\to c} f(x) = L$, show $\lim_{x\to c} f^2(x) = L^2$.

26. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent sequences. Show that so is $\sum_{n=1}^{\infty}(a_n + b_n)$. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, is it possible that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges? Justify your answer.

27. Let $A, B \subset \mathbb{R}$. Determine whether the identity $\overline{A \cup B} = \overline{A} \cup \overline{B}$. is true or false. If it is true, give a proof. If it is false, give a counter-example.

28. Use the definition of the continuity to show that the function $\sqrt[3]{2x+1}$ is continuous at $x = 0$.

29. Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

30. Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be uniformly continuous functions on their domains. If $f(A) \subset B$, show that $g \circ f$ is uniformly continuous on A.

31. Recall that the Dirichlet function on $\mathbb{R}$ is defined by $g(x) = 1$ if $x \in \mathbb{Q}$ and $g(x) = 0$ if $x \notin \mathbb{Q}$. Let $\{r_i : i \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$, and let $\mathbb{Q}_n = \{r_1, \ldots, r_n\}$. Define

$$g_n(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}_n \\ 0, & \text{if } x \notin \mathbb{Q}_n. \end{cases}$$

   (a) Show that $g_n \to g$ pointwise on $\mathbb{R}$.

   (b) Show that the convergence is not uniform on $\mathbb{R}$.

32. Let $f_n(x) = \dfrac{nx}{1 + nx^2}$.

   (a) Find the pointwise limit function of (f_n).

   (b) Show that the covergence is uniform on $[1, \infty]$.

   (c) Show that the convergence is not uniform on $(0, 1)$.

33. Assume $f_n \to f$ uniformly on A. Prove the following statements.

   (a) If each f_n is uniformly continuous on A, then so is f.

   (b) If each f_n is bounded on A, then so is f.

   (c) If each f_n has at most a countable number of discontinuities, then so does f.

34. Find the values of x where the series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}}$$

   converges. Show that the sum function is continuous on this set.

35. Show that the series defined function

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

   is continuous on $[0, 1]$.