Name: ______________________________ Student ID: ____________________

Math 2740 – Fall 2025
Sample final examination (Variant 3)
2 hours

Instructions

- This examination has **8 exercises**.
- Show all your work. Correct answers without justification will receive little or no credit.
- You may use the back of pages if needed.
- No electronic devices (including calculators) are permitted.
- The exam is out of 120 points.

Exercise 1. [**Definitions and Core Results – 15 points**]

State the definition or theorem for each of the following. Be precise and complete.

1. [**4 pts**] Define the eigenpairs of a matrix $A \in \mathcal{M}_n$.
2. [**4 pts**] Define linear independence of a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors.
3. [**4 pts**] Give a necessary and sufficient condition for two vectors to be orthogonal.
4. [**3 pts**] Define the *principal components* of a centered data matrix.

Exercise 2. [Gram–Schmidt Orthonormalization – 20 points]
Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

1. **[6 pts]** Apply the Gram–Schmidt procedure to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to obtain an *orthogonal* set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
2. **[6 pts]** Normalize your vectors to obtain an *orthonormal* set $\{q_1, q_2, q_3\}$.
3. **[4 pts]** Verify orthonormality by computing the inner products $\langle q_i, q_j \rangle$ for all i, j and by checking $\|q_i\| = 1$.
4. **[4 pts]** Form the matrix $Q = [q_1 \; q_2 \; q_3]$ and state whether Q is orthogonal (justify your answer).

Exercise 3. **[Least Squares via QR – 15 points]**

Let $A \in \mathbb{R}^{m\times n}$ have full column rank and let $A = QR$ be its *reduced* QR decomposition, where $Q \in \mathbb{R}^{m\times n}$ has orthonormal columns and $R \in \mathbb{R}^{n\times n}$ is upper triangular.

1. [**8 pts**] Using an *important theorem*, prove that the least-squares solution to $A\mathbf{x} = \mathbf{b}$ is $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$.

   Important Theorem 1 (Least Squares via QR)**.** Let $A = QR$ be a reduced QR decomposition with $Q^TQ = I$ and R upper triangular. Then the least-squares solution to $A\mathbf{x} = \mathbf{b}$ satisfies $\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$ and the residual is orthogonal to $\mathrm{col}(A)$.

2. [**7 pts**] For

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ -1 \end{pmatrix},$$

   compute the reduced QR decomposition $A = QR$ (you may use Gram–Schmidt on the columns) and find $\tilde{\mathbf{x}}$.

Exercise 4. **[15 points]**

Consider

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. [**6 pts**] Compute the singular values and singular vectors of B.
2. [**5 pts**] Is B invertible?
3. [**4 pts**] Compute the pseudo-inverse of B.

Exercise 5. **[PCA on Centered Data – 10 points]**

Let the centered data matrix be

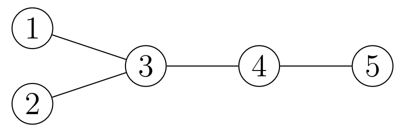
$$\tilde{X} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 0 & -1 \\ 2 & 0 \end{pmatrix}.$$

1. [**6 pts**] Compute the covariance matrix $S = \frac{1}{n-1}\tilde{X}^T\tilde{X}$ and its eigenvalues/eigenvectors.
2. [**4 pts**] Identify the first principal component and the variance explained by it.

Exercise 6. **[Graph Measures I – 12 points]**

Consider the simple undirected graph G on vertices $V = \{1, 2, 3, 4, 5\}$ with edge set

$$E = \{\{3, 1\}, \{3, 2\}, \{3, 4\}, \{4, 5\}\}.$$



1. **[4 pts]** Compute the degree $\deg(i)$ of each vertex and give the degree sequence in nonincreasing order.
2. **[4 pts]** Compute the density of G, defined as $\delta(G) = \dfrac{2|E|}{|V|(|V| - 1)}$.
3. **[4 pts]** Compute the local clustering coefficient C_i for each vertex with $\deg(i) \geq 2$ and state the average clustering coefficient.

Exercise 7. [Graph Measures II – 13 points]

For the same graph G as in Exercise 6:

1. **[5 pts]** Compute the graph diameter and the average shortest-path length $\ell(G)$.
2. **[4 pts]** Compute the (normalized) degree centrality of each vertex, $C_D(i) = \deg(i)/(n-1)$ where $n = |V|$.
3. **[4 pts]** Compute the closeness centrality of each vertex, $C_C(i) = \dfrac{n-1}{\sum_{j \neq i} d(i,j)}$.

Exercise 8. **[Markov Chains – 20 points]**

Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and column-stochastic transition matrix:

$$P = \begin{pmatrix} 1 & 1/4 & 1/3 & 0 \\ 0 & 1/2 & 1/3 & 1/2 \\ 0 & 1/4 & 1/3 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where P_{ij} is the probability of moving from state j to state i.

1. [**4 pts**] Determine whether this Markov chain is regular or absorbing. Justify your answer.
2. [**8 pts**] If the chain is regular, find the limiting distribution $\boldsymbol{\pi}$ by solving $P\boldsymbol{\pi} = \boldsymbol{\pi}$ with $\sum_i \pi_i = 1$. If the chain is absorbing, reorder the states to write P in canonical form

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

   and identify the matrices I, R, and Q.
3. [**8 pts**] If the chain is absorbing, compute the fundamental matrix $N = (I - Q)^{-1}$ and interpret what the entries represent. If the chain is regular, explain why the limiting distribution is independent of the initial state.

END OF EXAMINATION