Math 120 Homework Assignment 2

DUE DATE: SEPT. 19, 2023

1. Find the natural domains of the following functions, writing each domain as a union of intervals.

(a) $\sqrt{1-\sqrt{25-x^2}}$ (b) $\sqrt{\ln\frac{5x-x^2}{4}}+\frac{1}{\ln x}$ (c) $\sqrt{\sin(x)(\frac{1}{2}-\cos(x))}$

2. Suppose that x and y are any real numbers. The goal of this question is to prove the lower bound part of the triangle inequality:

$$\big||x|-|y|\big| \leqslant |x+y|.$$

Throughout the question you are allowed to assume the triangle inequality, as proved in class. Also recall that for any number z and number $A \geqslant 0$, , the inequality $|z| \leqslant A$ is equivalent to the inequalities $-A \leqslant z \leqslant A$.

(a) Explain why the inequality $-A \leqslant z$ is equivalent to the inequality $-z \leqslant A$.

(b) For a number $A \geqslant 0$ and number z, show that $|z| \leqslant A$ is equivalent to the inequalities $z \leqslant A$ and $-z \leqslant A$.

(c) Use the equation $x = (x+y)-y$ and the upper bound triangle inequality (the one from class) to prove that $|x| \leqslant |x+y|+|y|$, and therefore that $|x|-|y| \leqslant |x+y|$.

NOTE: Just because both the triangle inequality from class and this question use symbols called "x" and "y" it doesn't mean that when you go to apply the triangle inequality that you should use the x of the question as the x of the triangle inequality (and the same for the y's).

(d) Similarly, show that $|y|-|x| \leqslant |x+y|$.

(e) Finally (by combining (b), (c), and (d)), prove the lower bound part of the triangle inequality:

$$\big||x|-|y|\big| \leqslant |x+y|.$$

3. If a_1 and a_2 are two positive real numbers, their *Harmonic mean* is defined as

$$H(a_1,a_2) = \frac{2}{\frac{1}{a_1}+\frac{1}{a_2}}.$$

Like the arithmetic mean and the geometric mean, the harmonic mean is a way of combining two numbers to produce a single number which is hopefully representative of the original pair in some way.

One place where the harmonic mean naturally appears is in the theory of electrical circuits. If two resistors with resistance a_1 and a_2 are connected in parallel, the resistance of the total circuit is the harmonic mean of a_1 and a_2.

The goals of this question are (i) to prove the *geometric–harmonic mean inequality*

$$\frac{2}{\frac{1}{a_1}+\frac{1}{a_2}} \leqslant \sqrt{a_1a_2}$$

and (ii) to practice writing down a mathematical argument.

To prove the inequality:

Step 1: First write $\frac{1}{a_1}+\frac{1}{a_2}$ over a common denominator, and then use this expression to write the harmonic mean in a slightly simpler form (one with only a single "fraction").

Step 2: Next write down the inequality you're trying to prove:

$$\begin{pmatrix} \text{Simplified form} \\ \text{from step 1.} \end{pmatrix} \leqslant \sqrt{a_1a_2}.$$

Step 3: Finally, after a little rearranging, you should be able to see that the inequality in step 2 is just the arithmetic–geometric mean inequality we proved in class.

Since we already know this inequality is true, it seems like we're done. Now you just need to write down this argument in a clean way, i.e., by writing down clearly and in the correct order what the logical steps are to your reasoning.

To do this, start by writing:

> *"We want to prove the inequality* $\dfrac{2}{\frac{1}{a_1}+\frac{1}{a_2}} \leqslant \sqrt{a_1a_2}$*."*

Then write

> *" Since* $\dfrac{2}{\frac{1}{a_1}+\frac{1}{a_2}} =$ (*simplified form from step 1*)*, this is the same thing as proving* (*simplified form from step 1*) $\leqslant \sqrt{a_1a_2}$ *"*

Here you should of course replace "simplified form from step 1" by the thing you actually found in step 1.

Next write

> *" Starting with the arithmetic-geometric mean* $\sqrt{a_1a_2} \leqslant \dfrac{a_1+a_2}{2}$ *which we proved in class, if we …*

$$\vdots$$

$$\vdots$$

...we get (simplified form from step 1) $\leqslant \sqrt{a_1 a_2}$, then therefore we've shown that

$$\frac{2}{\frac{1}{a_1} + \frac{1}{a_2}} \leqslant \sqrt{a_1 a_2}. \text{"}$$

In the part that's missing above, you should show how to go from the arithmetic-geometric mean to the inequality that we want to show.

Your final answer will be quite short, but the important thing (as in every proof) is that the explanation be clear and correct.