The triangle inequality it that for all $\vec{v}, \vec{u} \in \mathbb{R}^n$, $||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$ must be true. We can start with the left side of the inequality. Since $||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}}$ we can rewrite left side as:

$$||\vec{u}+\vec{v}|| = \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}$$

This can be expanded, and both sides can be squared to get:

$$||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + 2\vec{u}\cdot\vec{v} + ||\vec{v}||^2$$

Since $2\vec{u}\cdot\vec{v}$ can be either a negative number or a positive number, we know that adding the absolute value of that term instead will result in a number that is either equal (in the positive case) or greater than (in the negative case) than the original equation. So, we can write this equality as an inequality.

$$||\vec{u}+\vec{v}||^2 \leq ||\vec{u}||^2 + 2|\vec{u}\cdot\vec{v}| + ||\vec{v}||^2$$

Using the Cauchy-Shwartz inequality, we can further alter this part of the right side into terms of lengths.

$$||\vec{u}+\vec{v}||^2 \leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2$$

This is just an expanded form of a squared term, so it can be factored as follows.

$$||\vec{u}+\vec{v}||^2 \leq (||\vec{u}|| + ||\vec{v}||)^2$$

We can then apply a square root to both sides to get our inequality. This holds because we know that both terms must result in positive numbers since they are additions of lengths.

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

To show that