Hierarchical Matrices and Inexact GMRES

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Chapter 1

Iterative Methods and Krylov's Subspace

1.1 Iterative Methods and motivation

Iterative methods appear as an alternative to direct solution methods, where the direct solution to a problem usually scales with $\mathcal{O}(m^3)$ complexity, where m is the dimension of the input matrix. Since larger matrices are usually employed in practise, the direct algorithms become inefficient and a more reliable approach is desired.
The idea of the iterable methods is to find, after a certain number of iterations, a sequence x_k that converges to x, the exact solution of the problem 1.1, all while making the large-scale computations faster, i.e. obtaining a complexity smaller than $\mathcal{O}(m^3)$, and keeping a maximum tolerance between the iterable solution and the exact one.

$$x = \lim_{k \to \infty} x_k \tag{1.1}$$

The method stops after k iterations, where x_k is the fist element of the sequence to satisfy the condition 1.2.

$$\frac{||x_k - x||}{||x||} \leq \epsilon \tag{1.2}$$

Where ϵ is a tolerance defined by who is applying the algorithm.
Usually x isn't known, so 1.2 gets modified for 1.3, where A is the system's matrix and b is the RHS(right hand side).

$$\frac{||Ax_k - b||}{||b||} \leq \epsilon \tag{1.3}$$

The first iterative methods used a decomposition of A as a combination of two matrices 1.4, where A_1 isn't singular, and each iteration is defined as 1.5.

$$A = A_1 - A_2 \tag{1.4}$$

$$A_1 x_{k+1} = b + A_2 x_k \tag{1.5}$$

With a substitution of the others x_k, 1.5 gives 1.6, which converges for every initial solution if and only lff $\rho(A_2 A_1^{-1}) < 1$, or $\rho(X)$ is the spectral radius of X [5].

$$x_{k+1} = A_1^{-1}(b + A_2 x_k) = A_1^{-1}(b + A_2 A_1^{-1}(b + A_2 x_{k-1}))... = A_1^{-1}\left[\sum_{i=0}^{k}(A_2 A_1^{-1})^i b\right] \tag{1.6}$$

If $A_1 = I$ and $A_2 = I - A$ in 1.4, the sequence found in 1.6 is: $x_1 = b$,$x_2 = 2b - Ab$, $x_3 = 3b - 3Ab + A^2 b$, ...
Even if the condition $\rho(A - I) \leq 1$ is strong [5], it shows that one approximation x_k could be represented as 1.7.

$$x_k \in span(b, Ab, A^2 b, ..., A^{k-1} b) \tag{1.7}$$

1.2 Krylov's Subspace

Be $A \in \mathbb{K}^{n \times n}$ a matrix and $b \in \mathbb{K}^n$. To each $k \leq n$ the Krylov's Subspace $\mathcal{K}_k = \mathcal{K}_k(A, b)$ associated to A,b is defined as 1.8.

$$\mathcal{K}_k(A, b) = span(b, Ab, A^2 b, \ldots, A^{k-1} b) \tag{1.8}$$

These Subspaces also have the following property: $k < l \rightarrow \mathcal{K}^k \subset \mathcal{K}^l$ [5].
The subspace $\mathcal{K}_k(A, b)$ is also the subspace of all the vectors from $\mathbb{R}^m$ which could be written as $x = p(A)b$, where $p(A)$ is a polynom of degree less than $k - 1$ which $p(0) = 1$.
The problem with using $A^k b, k \in 0, 1, 2, \ldots$ as a base comes from the fact that successive products of A make vectors that are *approximately colinear*, since those are really close of the eigenvector with the largest eigenvalue of A.

1.3 Arnoldi's Method

With the task of obtaining an orthonormal basis to $\mathcal{K}_k(A, b)$, the method searches for a unitary matrix Q for which the expression 1.9 is valid. $H_k = h_{ij}$ is an Hessenberg's matrix.

$$AQ_k = Q_{k+1} H_k \tag{1.9}$$

For each column-vector of Q, q_i, 1.9 could be written as 1.10, where the representation of $\mathcal{K}_k(A, b)$ with an orthonormal basis becomes more evident. In a pratical application, Q is initialized with $q_1 = \frac{b}{||b||}$.

$$Aq_m = h_{1m} q_1 + h_{2m} q_2 + \ldots h_{m+1,m} q_{m+1} \tag{1.10}$$

An algorithm for the method can be found in 1.

Algorithm 1 Arnoldi's iteration

$A \in \mathbb{K}^{n\times n}$ et $b \in \mathbb{K}^n$
$x = 0, \beta = \|b\|, q_1 = \frac{b}{\beta}$
for $j = 1, 2, \dots k$ **do**
    $q_{j+1} = Aq_j$
    for $i = 1, 2, \dots j$ **do**
        $h_{ij} = q_{j+1}^t q_i$
        $q_{j+1} = q_{j+1} - h_{ij} q_i$
    end for
    $h_{j+1,j} = \|q_{j+1}\|$
    $q_{j+1} = \frac{q_{j+1}}{h_{j+1,j}}$
end for

Chapter 2

GMRES

A projection in $\mathcal{K}_k(A, b)$, where the different approximations are taken as in 2.1, where Q_m is the vector in 1.9.

$$x = x_0 + Q_m y \tag{2.1}$$

With 2.1 and 1.9 the residual becomes 2.2, where $x_0 = 0$, $\beta = \|b\|$ and $Q_{m+1}^t b = (\|b\| \, 0 \, 0 \dots)^t$ since the columns of Q_{m+1} are orthonormal vectors and $q_1 = \frac{b}{\|b\|}$.

$$\begin{aligned} r(y) &= \|b - Ax\| \\ &= \|b - A(Q_m y)\| \\ &= \|b - Q_{m+1} H_m y\| \\ &= \left\|Q_{m+1}(Q_{m+1}^t b - H_m y)\right\| \\ &= \|\beta e_1 - H_m y\| \end{aligned} \tag{2.2}$$

Thus, y which appears in 2.1, is found as the solution of the residual's minimisation problem in 2.2.

$$y = min_y \, \|\beta e_1 - H_m y\| \tag{2.3}$$

An initial version of the GMRES is in 2. The lines 4 to 12 bring the Arnoldi's Method presented in 1.
However, 2 doesn't bring an efficient way of finding the residual in each iteration. To solve this problem and also to find a more efficient way of solving the least squares problem in 2.3, a transformation is applied to H_m, turning it into a triangular matrix.

2.1 Givens's Rotation

Givens's operator, $G(i, i+1)$, is an unitary matrix such that the column vector $a = Gb$ has the elements $a(i) = r \in \mathbb{R}$ and $a(i+1) = 0$. It has a structure as in 2.4. The coefficients c_i, s_i only appear in the rows i et $i+1$.

Algorithm 2 Initial GMRES

$A \in \mathbb{K}^{n\times n}$ and $b \in \mathbb{K}^n$
$x = 0, \beta = \|b\|, q_1 = \frac{b}{\beta}$
for $k = 1, 2, \dots$ **do**
  for $j = 1, 2, \dots k$ **do**
    $q_{j+1} = Aq_j$
    for $i = 1, 2, \dots j$ **do**
      $h_{ij} = q_{j+1}^t q_i$
      $q_{j+1} = q_{j+1} - h_{ij} q_i$
    end for
    $h_{j+1,j} = \|q_{j+1}\|$
    $q_{j+1} = \frac{q_{j+1}}{h_{j+1,j}}$
  end for
  Find $y = min_y \|\beta e_1 - H_m y\|$
  $x = Q_k y$
  Stop if the residual is smaller than the tolerance
end for

$$G(i, i+1) = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & c_i & s_i & & & \\ & & & -s_i & c_i & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \tag{2.4}$$

This operator offers a way to transform the columns in H_m, *zeroing* the elements outside the main diagonal. Since a product of unitary operators is still unitary, 2.3 can be written as 2.5, where R_m and g_m are the results from the application of multiple Givens's operators to H_m and βe_1.

$$y = min_y \|\beta e_1 - H_m y\| = min_y \|g_m - R_m y\| \tag{2.5}$$

Thus, the new problem 2.5 can be solved with a simple backwards substitution. If $g_m = [\gamma_1 \dots \gamma_{m+1}]^t$, a $m+1$ column vector, and $\{R_m\}_{ij} = r_{ij}$ a $m+1$ by m upper triangular matrix with $r_{ii} \neq 0$ and its last row filed with zeros, each element of $y_m = [y_1 \dots y_m]$ is given by 2.6.

$$\begin{aligned} \gamma_k &= \sum_{i=k}^{m} r_{ki} y_i \\ y_m &= \frac{\gamma_m}{r_{mm}} \\ y_i &= \frac{1}{r_{ii}} \left(\gamma_i - \sum_{j=i+1}^{m} r_{ij} \gamma_j \right) \end{aligned} \tag{2.6}$$

A simple algorithm to this end can be written as 3.

Algorithm 3 Backwards substitution

1: $A \in \mathbb{K}^{n\times n}, \{A\}_{ij} = a_{ij}$ and $b \in \mathbb{K}^n$
2: **for** $k = n, n-1, \dots$ **do**
3: $\quad y_k = b_k$
4: $\quad$ **for** $j = n, n-1, \dots k+1$ **do**
5: $\quad\quad y_k = y_k - a_{kj}y_j$
6: $\quad$ **end for**
7: $\quad y_k = \frac{y_k}{a_{kk}}$
8: **end for**

It can be shown that g_m also contains the residual of each iteration [7]. Since it's a $m+1$ column vector, we have, with Ω_m being the necessary Givens's Rotations to make H_m upper triangular 2.7.

$$\begin{aligned} \|b - Ax_m\| &= \left\|Q_{m+1}^t(\beta e_1 - H_m y_m)\right\| \\ \|\beta e_1 - H_m y_m\| &= \left\|\Omega_m^t(g_m - R_m y_m)\right\| \end{aligned} \tag{2.7}$$

Since y_m is a solution to the system, the norm in 2.7 is defined by $\|\gamma_{m+1} - (R_m)_{m+1,1:m}y_m\|$. But since the last row in R_m is composed by zeros, we have that $\|b - Ax_m\| = |\gamma_{m+1}|$, which gives a more efficient way to obtain the residuals during each iteration.

2.2 Inexact GMRES

The heaviest part in the code is in the matrix-vector product 1, line 4. Therefore, one approach to accelerate the iterations involves an approximation of Aq, instead of using the exact answer, as shown in 2.8.

$$\mathcal{A}q = (A + E)q \tag{2.8}$$

Where E in 2.8 is a *pertubation matrix* that changes with each iteration and will be written as E_k for iteration k.
When the inexact matrix-vector product is the one being made, the left side of 1.9 must be changed by 2.9.

$$\begin{aligned} [(A+E_1)q_1, (A+E_2)q_2, \dots, (A+E_k)q_k] &= Q_{k+1}H_k \\ (A + \mathcal{E}_k)Q_k = Q_{k+1}H_k, \quad \mathcal{E}_k &= \sum_{i=1}^{k} E_i q_i q_i^t \\ \mathcal{A}Q_k &= W_k \end{aligned} \tag{2.9}$$

Where $W_m = Q_{m+1}H_m$ from this point foward.
Now the subspace spawn by the vectors of Q_k is not the Krylov's subspace $\mathcal{K}_k(A, b)$, but these are still orthonormal. To see what kind of subspace our new Q spams, 2.9 is looked into in 2.10.

$$(A + E_k)q_k = \mathcal{A}_k q_k = h_{1,k}q_1 + h_{2,k}q_2 + \cdots + h_{k+1,k}q_{k+1} \tag{2.10}$$

For $k = 1$, we have that q_2 is a combination of the vectors $\mathcal{A}_1 b$ and b (since $q_1 = \text{b}$). For $k = 2$ we see that q_3 is a combination that involves $\mathcal{A}_2\mathcal{A}_1 b$ and so forth.
Expression 2.9 then shows that Q_k becomes a basis for a new Krylov's subspace, $\mathcal{K}_k(A + \mathcal{E}_k, b)$ $= \text{span}\{b, \mathcal{A}_1 b, \ldots, \mathcal{A}_k \ldots \mathcal{A}_1 b\}$, made by a large pertubation in A, that gets updated in each iteration.
A new distinction should also be made between the two types of residuals appearing in the process: r_k, the exact residual of an iteration, and $\tilde{r}_k$, the one that will really be calculated. A detailed definition for both and a measure of how distant they are is in 2.11.

$$\begin{aligned} r_k &= r_0 - AQ_k y_k \\ &= r_0 - (Q_{k+1}H_k - [E_1 q_1, \ldots, E_k q_k])y_k \\ &= \tilde{r}_k + [E_1 q_1, \ldots, E_k q_k]y_k \\ \rightarrow \delta_k &= \|r_k - \tilde{r}_k\| = \|[E_1 q_1, \ldots, E_k q_k]y_k\| \end{aligned} \tag{2.11}$$

Considering $y_k = [\eta_1^{(k)} \ldots \eta_n^{(k)}]$, upper index to clarify the iteration, an upper bound for δ_k can be found, but before we go through 2.12.

$$\begin{aligned} \|[E_1 q_1, \ldots, E_k q_k]y_k\| &= \left\| \sum_{i=1}^{k} E_i q_i \eta_i^{(k)} \right\| \\ \left\| \sum_{i=1}^{k} E_i q_i \eta_i^{(k)} \right\| &\leq \sum_{i=1}^{k} \|E_i\| \left\| q_i \eta_i^{(k)} \right\| \\ \left\| \sum_{i=1}^{k} E_i q_i \eta_i^{(k)} \right\| &\leq \sum_{i=1}^{k} \|E_i\| \, |\eta_i^{(k)}| \end{aligned} \tag{2.12}$$

Where the fact that q_i are unitary was used between the last two lines. The bound on δ_k is then found in 2.13.

$$\delta_k = \|r_k - \tilde{r}_k\| \leq \sum_{i=1}^{k} \|E_i\| \left\| \eta_i^{(k)} \right\| \tag{2.13}$$

2.13 tells us that in order to keep both residuals close, either the pertubation of A, somewhat measured by $\|E_i\|$, or the elements of y_i should be kept small. Since we expect to use more *relaxed* approximations of A as the iterations go on, a greater tolerance in E_k could be compensated with a sufficiently small y_k.
The problem is y_k is only found after the construction of E_k, so an upper bound must be also found for its value.
Knowing y_k is the solution of the minimization of $\|H_k y_k - e_1 \beta\|$, we consider $\Omega_k = G(k, k+1)G(k-1, k) \ldots G(1, 2)$ where each G represents a Givens rotation as shown in 2.4, so Ω_k is the matrix that

transforms H_k into an upper triangular matrix.
The aplication of Ω_k in either side of $H_k y_k = e_1 \beta$ gives us 2.14.

$$\begin{aligned} \Omega_k H_k y_k &= \Omega_k e_1 \beta \\ R_k y_k &= g_k \\ y_k &= R_k^{-1} g_k \end{aligned} \tag{2.14}$$

Since R_k, the transformation of a Hessenberg matrix by a series of Givens rotations, is upper triangular, then its inverse also is. Being an upper triangular matrix, the first $i-1$ elements of its ith line are zeros, so using Matlab index notation in 2.15

$$(R_k^{-1})_{i,1:k}(g_k)_{1:k} = (R_k^{-1})_{i,i:k}(g_k)_{i:k} \tag{2.15}$$

Using this last result in 2.14 gives 2.16.

$$\begin{aligned} |\eta_i^{(k)}| &= \left\|(R_k^{-1})_{i,i:k}(g_k)_{i:k}\right\| \\ |\eta_i^{(k)}| &\leq \left\|e_k R_k^{-1}\right\| \left\|(g_k)_{i:k}\right\| \\ |\eta_i^{(k)}| &\leq \left\|e_k R_k^{-1}\right\| \left\|(g_k)_{i:k}\right\| \end{aligned} \tag{2.16}$$

Since $\left\|e_k R_k^{-1}\right\| \leq \left\|R_k^{-1}\right\| = \sigma_k(H_k)^{-1}$ and $\|(g_k)_{i:k}\| \leq \|\tilde{r}_{i-1}\|$ [8], the bound is given by 2.17.

$$\left\|\eta_i^{(k)}\right\| \leq \frac{1}{\sigma_k(H_k)} \|\tilde{r}_{i-1}\| \tag{2.17}$$

Putting 2.17 in 2.13 gives the results in 2.18. Setting $\delta_k \leq \epsilon$ and determining a bound for each $\|E_i\|$ gets us 2.19.

$$\delta_k \leq \sum_{i=1}^{k} \frac{\|E_i\|}{\sigma_k(H_k)} \|\tilde{r}_{i-1}\| \tag{2.18}$$

$$\|E_i\| \leq \frac{\sigma_k(H_k)\epsilon}{k \|\tilde{r}_{i-1}\|} \tag{2.19}$$

Since H_k is also one of the matrices being constructed throughout the method, a workaround is necessary to apply find these bounds in a pratical situation. Either using an estimation of $\sigma_k(H_k)$ with the singular values of A or grouping all uncalculated terms in a ℓ_k that will be estimated empirically [8], obtaining 2.20.

$$\|E_i\| \leq \ell_k \frac{1}{\|\tilde{r}_{i-1}\|} \epsilon \tag{2.20}$$

It should be noted [8] that some of the initial bounds found aren't really sharp, mainly 2.12 and 2.17, and further empirical analysis of these bounds could show a better theoretical bound can be found for both. A plot of these bounds for the cavity problem that will be studied is shown later. It should also be noted that $\tilde{r}_i$, after the Givens's Rotations, is also found in the $i+1$'th element of g_m in 2.14. The demonstration follow the same proof as for g_m in 2.5, given that y_m is a solution

to a linear system that involves an upper triangular matrix.
The remaining theory in this report explains the basics of Hierarchical Matrices, the structure that will be used to compress the matrices A that appear in the discretization of integral operators, since these usually appear in large dimensions, but also to construct each E_k. As it will be explained later, at each iteration a E_k will be indirectly constructed during the inexact product $\mathcal{A}_k q$, mainly using the residues that appear during this structure's construction, in the iterations of the $ACAMethod$.

Chapter 3

Hierarchical Matrices and ACA Method

3.1 Low-rank Matrices

In reality, most matrices are large, so storing each element is not efficient, or even possible. If $A \in \mathbb{C}^{n\times m}$ has a rank k such that $k \leq m$ and $k(n+m) < n*m$ (A is low-rank), A can be written in outer product form, as a product between the matrices $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}^{m\times k}$, which can be see in 3.1, where u_i, v_i are the column vectors of U and V.

$$A = UV^H = \sum_{i=1}^{k} u_i v_i^* \tag{3.1}$$

Therefore, storing $k(n+m)$ elements to write A, and not $n\times m$. A matrix A that can be represented as 3.1 is an element of $\mathbb{C}_k^{n\times m}$.
The representation in 3.1 also facilitates other operations with A, like matrix-vector products Ab that are always present in methods like GMRES [4] and different kinds of norms, like $\|A\|_F, \|A\|_2$ [4].
However, even full rank matrices can be approximated by matrices with lower rank. A theorem [4] establishes that the closest matrix from $\mathbb{C}_k^{n\times m}$ of a matrix from $\mathbb{C}^{n\times m}$ can be obtained from the SVD $A = U\Sigma V^H$, where Σ contains the singular valuers $\sigma_1 \geq \sigma_2 \ldots \sigma_m \geq 0$ and U, V are unitary. If A_k is the approximation obtained after taking the first k elements of Σ (creating the matrix Σ_k), the error between A and A_k is 3.2.

$$\|A - A_k\| = \left\|U\Sigma V^H - U^{'}\Sigma_k V_{'H}\right\| = \|\Sigma - \Sigma_k\| \tag{3.2}$$

If the spectral norm , $\|.\|_2$ is used instead, the error in 3.2 is given by σ_{k+1}. For Frobenius's norm, $\|.\|_F$, the error becomes $\sum_{l=k+1}^{n} \sigma_l^2$.
Instead of approximating large matrices entirely, it's better to think in approximations made to each of their blocks. Blocks that appear after the discretization of elliptic operators also have the possibility of being approximated by matrices that decay exponentially with k, S_k, as in 3.3.

$$\|A - S_k\|_2 < q^k \|A\|_2 \tag{3.3}$$

That way, the rank and the precision are related in a logarithmic manner, and the rank required by a certain ϵ is 3.4.

$$k(\epsilon) = min\{k \in \mathbb{N} : \sigma_{k+1} < \epsilon\sigma_1\} \tag{3.4}$$

3.2 ACA Method(Adaptative Cross Approximation)

As shown in the last section, the SVD methods gives us an approximation of A given a certain ϵ, through the relation in 3.2. Nevertheless, this is an expensive method, where the complexity becomes too heavy for some calculations.
The algorithm for the method is in 4, where a_{ij} are the elements of a matrix $A \in \mathbb{R}^{n\times m}$. The main objective is to approximate A as $A = S_k + R_k$, $S_k = \sum_{l=1}^{k} u_l v_l^t$ and R_k is the residual.

Algorithm 4 ACA Method

$k = 1$ et $\mathbf{Z} = \emptyset$
repeat
  TFind i_k
  $\hat{v}_k = a_{i_k,1:m}$
  for $l = 1, \ldots, k-1$ **do**
    $\hat{v}_k = \hat{v}_k - (u_l)_{i_k} v_l$
  end for
  $Z = Z \bigcup \{i_k\}$
  if $\hat{v}_k$ doesn't disappear **then**
    $j_k = argmax_j |(\hat{v}_k)_j|$; $v_k = (\hat{v}_k)_{j_k}^{-1} \hat{v}_k$
    $u_k = a_{1:n,j_k}$
    for $l = 1, \ldots, k-1$ **do**
      $u_k = u_k - (v_l)_{j_k} u_l$
    end for
    $k = k + 1$
  end if
until $\|u_k\| \|v_k\| \leq \epsilon$

Considering $I, J \in \mathbb{N}$ the index set of a given matrix and $\mathbf{T}_{I\times J}$ the cluster block tree that contains an admissible partition P of $I \times J$ in its leaves, $\mathfrak{L}(\mathbf{T}_{I\times J})$. The set of hierarchical matrices in $\mathbf{T}_{I\times J}$ rank k for each block A_b defined in 3.5.

$$\mathfrak{H}(\mathbf{T}_{I\times J}, k) = \left\{A \in \mathbb{C}^{I\times J} : rank A_b \leq k, \forall b \in P\right\} \tag{3.5}$$

In practise, 4 works with ϵ given by the user during the assembling of the Hierarchical Matrix, and the compression acts in each of the admissible blocks that will then be represented as low rank matrices, shown in 3.1. We also store each one of the residuals obtained in the outer loop of 4.
After giving a tolerance σ for an inexact product, the algorithm has, for each admissible block of the cluster tree, use the right amount of columns of the outer product representation of the block as to reach the desired tolerance, what can be infered by the list of residuals of the ACA Method.

So, the different pertubations E_k used in 2.9, are the matrices that when added to A, leave each admissible block with the right amount $k^{'}$ of columns in 3.1 so the product of the admissible block B_k and q be approximated by the tolerance σ. Calling this approximation of each block B_k as $\tilde{B}_k$, we have 3.6.

$$\frac{\left\|B_k q - \tilde{B}_k q\right\|}{\|B_k q\|} \leq \sigma, \qquad \tilde{B}_k = \sum_{i=1}^{k^{'}} u_i v_i^* \tag{3.6}$$

Chapter 4

First results

(Not results in the cluster, left to remake them after finishing the discussions we had about residuals) (I thought about changing the beggining of this chapter as a whole small chapter about BEM)
(Also, I thought about using the graphs with the bounds of each expression in the article, but I'm not shure if we put them here or with the Inexact theory chapter above)
Before using the inexact product in more complex problems, simpler examples are used to validate the approach and fix minor parameters in the scheme.
The two firsts tests evaluate the speedup in the product of a Hierarchical Matrix and a vector and an exection of the Inexact GMRES with few iterations, using the operators obtained through 2nd type Equations of Laplace and Helmholtz 4.1, where the last one is a scaterring problem, where Δ is the Laplace operator and everything is suposed to be solved in two dimentions. The last test will use a cavity problem to test the speedup of the algorithm in a situation with more iterations.

$$\begin{aligned} \Delta u &= 0 \\ \Delta u + k^2 u &= 0 \end{aligned} \tag{4.1}$$

Reformulationg both equations as a Boundary Integral Equation, the simple *direct* formulation is used to write the solution as 4.2, where Γ is boudnary of the domain, S and D are the single and double layer operators, defined as 4.3, and $G(x, y)$ is the fundamental solution of the desired PDE.

$$-\frac{u(x)}{2} + D[u](x) = S[\partial_\nu u](x), \qquad x \in \Gamma \tag{4.2}$$

$$\begin{aligned} S[\sigma](x) &= \int_\Gamma G(x,y)\sigma(y)\, ds(y) \\ D[\sigma](x) &= \int_\Gamma \frac{\partial G}{\partial \nu_y}(x,y)\sigma(y)\, ds(y) \end{aligned} \tag{4.3}$$

For the first to examples, a unit circle around the origin is used as the boundary to generate the operators, with the mesh being created with the Inti library [3].

For the last test, the mesh is made from a cavity *.geo* file avaiable in [1]. A view of the figure can be seen in 4.1. The incident wave' angle is chosen to be $\frac{\pi}{4}$ rad.

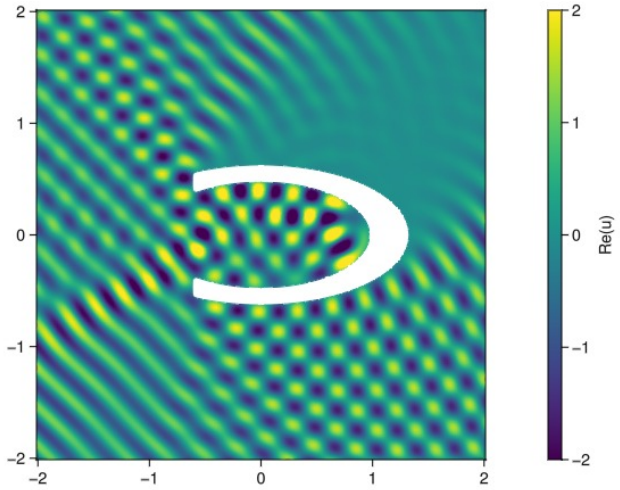


Figure 4.1: Geometry used in the test.

A good way to infer the maximum acceleration possible for the inexact products would be using only admissible rank 1 blocks and measuring its execution time. Although such thing would not happen in a practical situation, it gives a maximum bound for the speed up we should expect. For doing that, the product tolerance is changed to *Infinity*, and the product will be realised with only rank-1 blocks, since it's programmed to get the first aproximation lower than its given tolerance.

4.1 Laplace's results

Setting our product tolerance to *Infinity* and using only rank-1 blocks in the product, we got a () speedup.
All results are contained in 4.2, showing the evolution of the residual with the product tolerance aswell as the speedup to each of these values.

4.2 Helmholtz's results

For a maximum speedup bound in the product, the infinity tolerance brought a () speedup.
For the unitary circle boundary the results can be seen in 4.3.
For the cavity, 4.4.
An evolution of the number of iterations in face of the different tolerances passed to the algorithm is in 4.5.
To start assessing the maximum gain possible, we start by initiating the product tolerance as infitine and seeing the result. Choosing an infinite tolerance grants us the all admissible block used in the products will have rank 1.

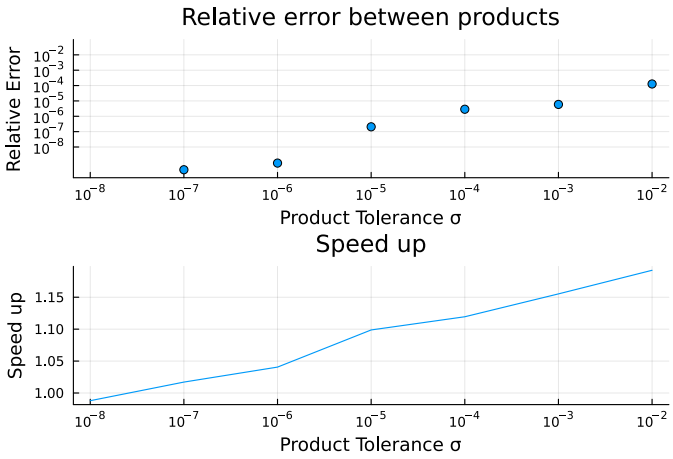
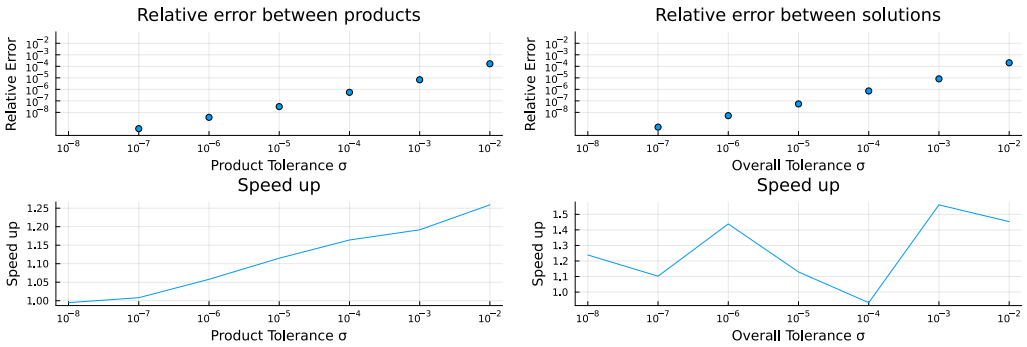


Figure 4.2: Speedup and residual evolution for the product between a 8000x8000 HMatrix and a vector.



(a) Results for the product of a 70000x 70000 HMatrix and a vector.

(b) Results for an initial application of the Inexact GMRES algorithm.

Figure 4.3: Results for the application of the Inexact GMRES algorithm with a 70000x70000 HMatrix.

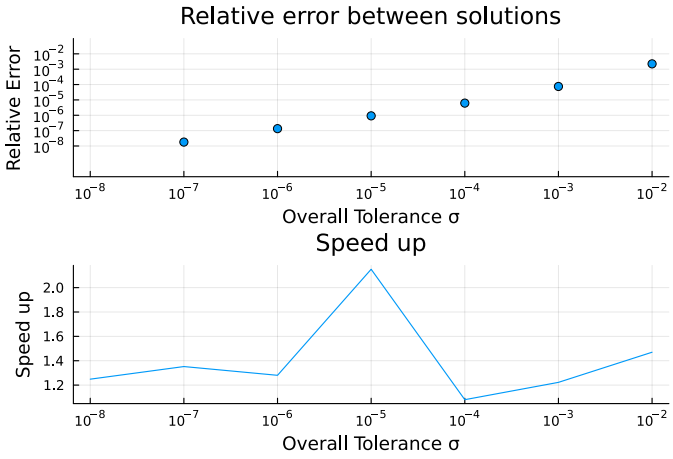


Figure 4.4: Speedup witnessed in the application of the Inexact GMRES in a 50000x50000 matrix.

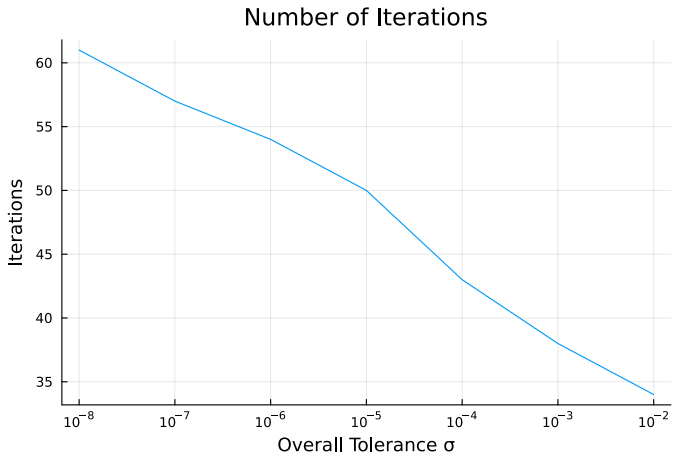


Figure 4.5: Evolution of the quantity of iterations needed for convergence and overall tolerance passed as an argument.

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