Test 2

Tomáš Dudek

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1 Introduction to Harmonic Oscillation

It's a fundamental concept in physics and engineering. It describes an oscillatory motion with a force that reduces the amplitude over time. The equation of motion for a damped harmonic oscillator is:

$$mx'' + cx' + kx = 0$$

where:

- m is the mass
- c is the damping coefficient
- k is the spring constant
- x is displacement
- x' is velocity
- x" is acceleration

The solution for the underdamped case (most common) is:

$$x(t) = Ae^{-\gamma t}cos(\omega' t + \theta)$$

where:

- A is the initial amplitude
- $\gamma = \frac{c}{2m}$ is the damping ratio
- $\omega' = \sqrt{\frac{k}{m} - \gamma^2}$ is the damped angular frequency
- θ is the phase angle

2 Types of damping

There are three main types of damping:

- Underdamped: The system oscillates with decreasing amplitude over time. This occurs when $c^2 < 4mk$. This is the most common case in real systems, like a suspension bridge swaying after being disturbed.
- Critically damped: The system returns to equilibrium as quickly as possible without oscillating. This occurs when $c^2 = 4mk$. This is often desired in systems like door closers.
- Overdamped: The system returns to equilibrium without oscillating, but more slowly than the critically damped case. This occurs when $c^2 > 4mk$. Think of trying to move quickly through a thick fluid.

3 Derivation of the formula

Let's start with Newton's Second Law for a mass on a spring with damping:

$$F = ma$$

$$-kx - cx' = mx''$$

(negative signs because both forces oppose motion)

$$mx'' + cx' + kx = 0$$

To solve this second-order differential equation, we assume a solution of the form:

$$x(t) = e^{rt}$$

Substituting this into our differential equation:

$$mr^2e^{rt} + cre^{rt} + ke^{rt} = 0$$

$$e^{rt}(mr^2 + cr + k) = 0$$

Since $e^{rt} \neq 0$, we have the characteristic equation:

$$mr^2 + cr + k = 0$$

Using the quadratic formula:

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

4 Solving a problem

Problem: A 0.5 kg mass is attached to a spring with spring constant k = 20 N/m. The system has a damping coefficient c = 1.0 Ns/m. If the mass is pulled 0.1 m from equilibrium and released from rest, find:

- Whether the system is underdamped, critically damped, or overdamped
- The position as a function of time
- The time it takes for the amplitude to decrease to half its initial value

Let's solve this step by step:
1. First, let's determine the type of damping: Calculate $c^2 - 4mk$

$$c^2 = 1.0^2 = 1.0N^2s^2/m^2$$

$$4mk = 4(0.5kg)(20N/m) = 40kgN/m$$

Since $1.0 < 40$, the system is underdamped.

2. For the underdamped case:

$$\gamma = \frac{c}{2m} = \frac{1.0}{2(0.5)} = 1.0s^{-1}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.5}} = 6.32rad/s$$

$$\omega' = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{40 - 1} = 6.24rad/s$$

3. The position function is:

$$x(t) = 0.1e^{-t}\cos(6.24t)$$

using initial conditions: $x(0) = 0.1m$, $x'(0) = 0$

4. For half-amplitude:

$$0.1e^{-t_1} = 0.05$$

$$e^{-t_1} = 0.5$$

$$t_1 = ln(2) = 0.693seconds$$

So after about 0.693 seconds, the amplitude will be half of its initial value.

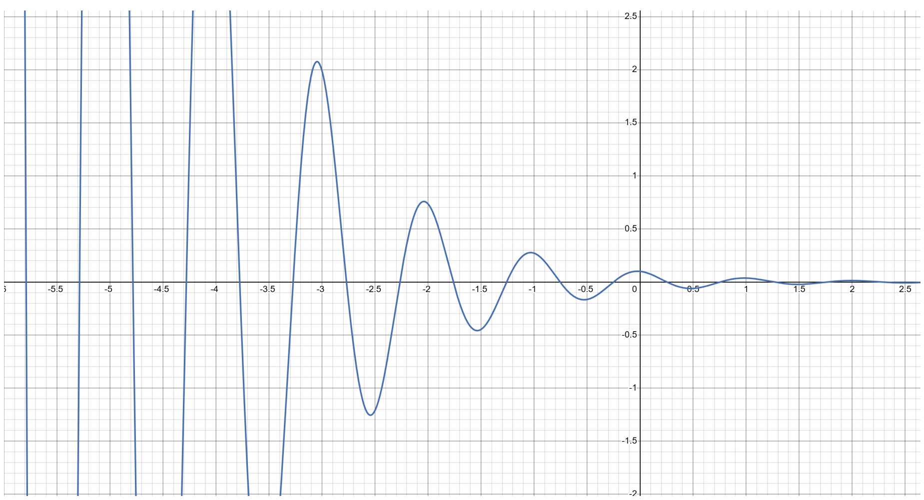

Figure 1: Illustration of the problem in Desmos