$W_1 \cup W_2$?

$\mathbb{R}^2 \qquad W_1 = \mathcal{L}((1,0)) \qquad W_2 = \mathcal{L}((0,1))$

$$(1,0)+(0,1)=(1,1) \notin W_1 \cup W_2$$

Quindi $W_1 \cup W_2$ non é linearmente chiuso.

Possiamo allora considerare il più piccolo sottospazio vett. che contiene $W_1 \cup W_2$: $\quad \mathcal{L}(W_1 \cup W_2)$

__def__

W_1, W_2 sottospaz. vett. di V

La somma di W_1 e W_2 é:

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \wedge w_2 \in W_2 \}$$

vediamo che $W_1 + W_2$ é sott. vett. dimostrando che:

$$W_1 = \mathcal{L}(S_1),\ W_2 = \mathcal{L}(S_2) \implies W_1 + W_2 = \mathcal{L}(S_1 \cup S_2)$$

__Dim__

$W_1 \subseteq W_1 + W_2$: $\quad w_1 \in W_1, \quad w_1 = w_1 + \underbrace{0}_{\in W_2} \in W_1 + W_2$

Analogamente $W_2 \subseteq W_1 + W_2$

$$\left.\begin{array}{l} S_1 \subseteq W_1 \subseteq W_1 + W_2 \\ S_2 \subseteq W_2 \subseteq W_1 + W_2 \end{array}\right) \Longrightarrow S_1 \cup S_2 \subseteq W_1 + W_2$$

"$\subseteq$"

$$u \in W_1 + W_2 \iff \exists\, w_1 \in W_1,\ \exists\, w_2 \in W_2 : u = w_1 + w_2$$

- $w_1 \in W_1 = \mathcal{L}(S_1) \Rightarrow \exists\, u_1, \dots, u_t \in S_1,\ \exists\, \alpha_1, \dots, \alpha_t \in K :$

$$w_1 = \alpha_1 u_1 + \dots\dots + \alpha_t u_t$$

- $w_2 \in W_2 = \mathcal{L}(S_2) \Rightarrow \exists\, v_1, \dots, v_h \in S_2,\ \exists\, \beta_1, \dots, \beta_h \in K :$

$$w_2 = \beta_1 v_1 + \dots\dots + \beta_h v_h$$

Quindi

$$u = w_1 + w_2 = \alpha_1 u_1 + \dots + \alpha_t u_t + \beta_1 v_1 + \dots + \beta_h v_h \in \mathcal{L}(S_1 \cup S_2)$$

"$\supseteq$"

$$v \in \mathcal{L}(S_1 \cup S_2) \Longrightarrow \exists\, u_1', \dots, u_t' \in S_1,\ \exists\, \alpha_1', \dots, \alpha_t' \in K$$
$$\exists\, v_1', \dots, v_h' \in S_2,\ \exists\, \beta_1', \dots, \beta_h' \in K$$

tali che

$$v = \underbrace{\alpha_1' u_1' + \dots + \alpha_t' u_t'}_{\in W_1 \in \mathcal{L}(S_1)} + \underbrace{\beta_1' v_1' + \dots + \beta_h' v_h'}_{\in W_2 \in \mathcal{L}(S_2)} \in W_1 + W_2$$

$\boxtimes$