3. (foglio 6) $\quad V = \mathbb{R}^2[x] \qquad \mathcal{B} = (1+2x,\ 1-x,\ 1-x^2)$

$$\phi_{\mathcal{B}} : \mathbb{R}^2[x] \longrightarrow \mathbb{R}^3$$

$$a_0 + a_1 x + a_2 x^2 \longrightarrow \left(\tfrac{1}{3}a_0 + \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2,\ \tfrac{2}{3}a_0 - \tfrac{1}{3}a_1 + \tfrac{2}{3}a_2,\ -a_2\right)$$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1(1+2x) + \alpha_2(1-x) + \alpha_3(1-x^2) =$$

$$= \alpha_1 + \alpha_2 + \alpha_3 + (2\alpha_1 - \alpha_2)x + (-\alpha_3)x^2$$

$$\Longleftrightarrow \begin{cases} a_0 = \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 = 2\alpha_1 - \alpha_2 \\ a_2 = -\alpha_3 \end{cases} \qquad \begin{cases} a_0 = \alpha_1 + 2\alpha_1 - a_1 - a_3 \Rightarrow \alpha_1 = \frac{1}{3}(a_0 + a_1 + a_2) \\ \alpha_2 = 2\alpha_1 - a_1 \\ \alpha_3 = -a_2 \end{cases}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{5}{3} & -1 \\ \frac{5}{3} & \frac{7}{3} & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} \alpha_1 = \frac{1}{3}(a_0 + a_1 + a_2) \\ \alpha_2 = \frac{2}{3}a_0 + \frac{2}{3}a_1 + \frac{2}{3}a_2 - a_1 = \frac{2}{3}a_0 - \frac{1}{3}a_1 + \frac{2}{3}a_2 \\ \alpha_3 = -a_2 \end{cases}$$

$$\phi_{\mathcal{B}}(1 - x + x^2) = \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{3}} + \tfrac{1}{3},\ \tfrac{2}{3} + \tfrac{1}{3} + \tfrac{2}{3},\ -1\right) = \left(\tfrac{1}{3}, \tfrac{5}{3}, -1\right)$$

$$\phi_{\mathcal{B}}(2 + x + 2x^2) = \left(\tfrac{2}{3} + \tfrac{1}{3} + \tfrac{2}{3},\ \tfrac{4}{3} - \tfrac{1}{3} + \tfrac{4}{3},\ -2\right) = \left(\tfrac{5}{3}, \tfrac{7}{3}, -2\right)$$

$$\phi_{\mathcal{B}}(3x) = (1, -1, 0)$$

$$\begin{pmatrix} 1/3 & 5/3 & -1 \\ 5/3 & 7/3 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underline{a}^2 \longrightarrow \underline{a}^2 - 5\,\underline{a}^1$$

$$\underline{a}^3 \longrightarrow \underline{a}^3 - 3\,\underline{a}^1$$

$$\begin{pmatrix} 1/3 & 5/3 & -1 \\ 0 & -6 & 3 \\ 0 & -6 & 3 \end{pmatrix}$$

$$\underline{a}^3 \longrightarrow \underline{a}^3 - \underline{a}^2$$

$$\begin{pmatrix} 1/3 & 5/3 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

8. (foglio 6) $\mathbb{R}^4$

$W = \mathcal{L}((2,1,2,-1),(1,1,1,1),(0,-1,0,3))$

? U sottosp. vett. di $\mathbb{R}^4$: $W + U = W \oplus U = \mathbb{R}^4$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 0 & -1 & 0 & 3 \end{pmatrix} \quad \underline{a}^2 \longrightarrow \underline{a}^2 - 2\underline{a}^1 \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -1 & 0 & 3 \end{pmatrix} \quad \underline{a}^3 \longrightarrow \underline{a}^3 - \underline{a}^2$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & 1 & 0 \end{matrix}$$

$\dim W = 3$ $\quad \mathcal{B} = \{(2,1,2,-1),(1,1,1,1),(0,-1,0,3)\}$

$U = \mathcal{L}((0,0,1,0))$

$U \cap W = \{\underline{0}\}$ $\quad \dim W + U = 4$

per la relaz. di Grassmann

9. (foglio 6) $\qquad M_2(\mathbb{R}) \simeq \mathbb{R}^3[x]$

$\qquad\qquad\qquad$ dim 4 $\qquad\qquad$ dim 4

$\exists \; T : M_2(\mathbb{R}) \longrightarrow \mathbb{R}^3[x]$

$$\mathcal{B} \quad \begin{array}{lll} e_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \rightsquigarrow & x \\ e_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \rightsquigarrow & 1 \\ e_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \rightsquigarrow & x^3 \\ e_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \rightsquigarrow & x^2 \end{array}$$

$$\bar{\mathcal{B}} = (\underset{\bar{e}_1}{x}, \underset{\bar{e}_2}{1}, \underset{\bar{e}_3}{x^3}, \underset{\bar{e}_4}{x^2})$$

$$M_2(\mathbb{R}) \xrightarrow{\Phi_{\mathcal{B}}} \mathbb{R}^4 \xrightarrow{\Phi_{\bar{\mathcal{B}}}^{-1}} \mathbb{R}^3[x]$$

$$\begin{array}{lllll} e_1 & \rightsquigarrow & (1,0,0,0) & \rightsquigarrow & x \\ e_2 & \rightsquigarrow & (0,1,0,0) & \rightsquigarrow & 1 \\ e_3 & \rightsquigarrow & (0,0,1,0) & \rightsquigarrow & x^3 \\ e_4 & \rightsquigarrow & (0,0,0,1) & \rightsquigarrow & x^2 \end{array}$$

dim Im $= 4$

$T: V \longrightarrow V$ endomorfismo

$\lambda_1, \dots, \lambda_t$ autovalori a due a due distinti.

$v_1, \dots, v_t$ autovettori (non nulli)

$T(v_1) = \lambda_1 v_1, \dots, T(v_t) = \lambda_t v_t$

Th: $\{v_1, \dots, v_t\}$ è lin. indip.

DIM. per induzione sul numero t.

$t = 1$ $\quad v_1 \neq \underline{0}$ $\quad \{v_1\}$ è lin. indip.

$t-1 \Rightarrow t$ $\quad \alpha_1 v_1 + \dots + \alpha_t v_t = \underline{0}_V$ $\quad$ Th: $\alpha_1, \dots, \alpha_t = 0$

$$T(\alpha_1 v_1 + \dots + \alpha_t v_t) = T(\underline{0}_V) = \underline{0}_V$$

$$\alpha_1 T(v_1) + \dots + \alpha_t T(v_t) = \alpha_1 \lambda_1 v_1 + \dots + \alpha_{t-1} \lambda_{t-1} v_{t-1} + \underline{\alpha_t \lambda_t v_t} = 0_V$$

$$\lambda_t(\alpha_1 v_1 + \dots + \alpha_{t-1} v_{t-1} + \alpha_t v_t) = \lambda_t \alpha_1 v_1 + \dots + \lambda_t \alpha_{t-1} v_{t-1} + \underline{\lambda_t \alpha_t v_t} = \underline{0}_V$$

Sottraggo:

$$\alpha_1(\lambda_1 - \lambda_t) v_1 + \dots + \alpha_{t-1}(\lambda_{t-1} - \lambda_t) v_{t-1} + \underline{0}_V = \underline{0}_V$$

$$\begin{cases} \alpha_1 (\underbrace{\lambda_1 - \lambda_t}_{\neq 0}) = 0 \\ \vdots \\ \alpha_{t-1} (\underbrace{\lambda_{t-1} - \lambda_t}_{\neq 0}) = 0 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 0 \\ \vdots \\ \alpha_{t-1} = 0 \end{cases} \qquad \alpha_t v_t = \underline{0}_V \Rightarrow \alpha_t = 0$$

$A \in M_{m\times n}(K)$ $\underline{b} \in M_{m\times 1}(K)$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$Ax = \underline{b}$

$$\begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ a^2_1 & a^2_2 & \cdots & a^2_n \\ \vdots & & & \vdots \\ a^m_1 & a^m_2 & \cdots & a^m_n \end{pmatrix}$$

$$\begin{cases} a^1_1 x_1 + a^1_2 x_2 + \cdots + a^1_n x_n = b_1 \\ a^2_1 x_1 + a^2_2 x_2 + \cdots + a^2_n x_n = b_2 \\ \vdots \\ a^m_1 x_1 + a^m_2 x_2 + \cdots + a^m_n x_n = b_m \end{cases}$$

Consideriamo il caso in cui $\underline{b} = \underline{0}$, $\Sigma_0 : AX = 0$

- L'insieme delle soluzioni S_0 di Σ_0 è un sottosp. vett. di K^n
- Ogni sottosp. vett. di K^n è l'insieme delle soluzioni di qualche sistema lineare omogeneo a coeff. in K in n variabili.

$\nwarrow$ dimostriamo questa affermazione

$$W = \mathcal{L}\left((a_1^1, a_1^2, \dots, a_1^n), (a_2^1, a_2^2, \dots, a_2^n), \dots, (a_h^1, a_h^2, \dots, a_h^n)\right) \subseteq K^n$$

con

$\dim W = h$ (i vettori sono lin. indip. per ipotesi)

$(x_1, \dots, x_n) \in K^n$ quando accade che $(x_1, \dots, x_n) \in W$?

$$\Updownarrow$$

$$\text{rango}\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_h^1 & x_1 \\ a_1^2 & a_2^2 & & a_h^2 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^n & a_2^n & & a_h^n & x_n \end{pmatrix} = h$$

Si riduce a gradini:

$$\begin{pmatrix} b_1^1 & \cdots & & & c_{11}x_1 + \dots + c_{1n}x_n \\ 0 & b_2^2 & & & \vdots \\ \vdots & 0 & \cdots & b_h^h & c_{h1}x_1 + \dots + c_{hn}x_n \\ \vdots & \vdots & & 0 & c_{h+1\,1}x_1 + \dots + c_{h+1\,n}x_n = 0 \\ & & & \vdots & \vdots \\ 0 & 0 & & 0 & c_{n1}x_1 + \dots + c_{nn}x_n = 0 \end{pmatrix}$$

$n-h$ equazione

PONIAMO

Esempio: $W = \mathcal{L}((1,2,0,1),(1,-1,1,0),(0,0,0,1)) \subseteq \mathbb{R}^4$

$$\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 2 & -1 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 1 & 0 & 1 & x_4 \end{pmatrix}$$

deve avere rango 3

$\underline{a}^2 \longrightarrow \underline{a}^2 - 2\underline{a}^1$

$\underline{a}^4 \longrightarrow \underline{a}^4 - \underline{a}^1$

$$\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 0 & -3 & 0 & x_2 - 2x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & -1 & 1 & x_4 - x_1 \end{pmatrix}$$

$\underline{a}^3 \longrightarrow \underline{a}^3 + \frac{1}{3}\underline{a}^2$

$\underline{a}^4 \longrightarrow \underline{a}^4 - \frac{1}{3}\underline{a}^2$

$$\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 0 & -3 & 0 & x_2 - 2x_1 \\ 0 & 0 & 0 & x_3 + \frac{1}{3}x_2 - \frac{2}{3}x_1 \\ 0 & 0 & 1 & x_4 - x_1 - \frac{1}{3}x_2 + \frac{2}{3}x_1 \end{pmatrix}$$

$\underline{a}^3 \longleftrightarrow \underline{a}^4$

$$\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 0 & -3 & 0 & x_2 - 2x_1 \\ 0 & 0 & 1 & x_4 - \frac{1}{3}x_2 - \frac{1}{3}x_1 \\ 0 & 0 & 0 & x_3 + \frac{1}{3}x_2 - \frac{2}{3}x_1 \end{pmatrix}$$

$\left\{ x_3 + \frac{1}{3}x_2 - \frac{2}{3}x_1 = 0 \right.$

$n - h$ equazioni

$\shortparallel$

$4 - 3 = 1$