$$n_{\vec{k}} = \langle \hat{a}^{+}_{\vec{k}} a_{\vec{k}} \rangle = + \lim_{\tau \to 0^-} \langle \hat{T} \hat{a}_k(\tau) \hat{a}^{+}_{\vec{k}} \rangle$$

$$= - \lim_{\tau \to 0^-} G(\vec{k}, \tau)$$

fourier transformation $\cancel{= - \lim_{\tau \to 0^-} \frac{1}{\beta} \sum G(\vec{k}, i\omega_n) e^{-i\omega_n \tau}}$ $(\tau < 0)$

$$= - \lim_{\tau \to 0^-} \frac{1}{\beta} \sum_{i\omega_n} G(\vec{k}, i\omega_n) e^{-i\omega_n \tau} \quad (\tau < 0).$$

$$= - \lim_{\tau \to 0^-} \oint \frac{dz}{2\pi i} G(\vec{k}, i\omega_n) \, e^{-z\tau} n(z)$$

$$\doteq - \lim_{\tau \to 0^-} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon \, n(\varepsilon) \left[G(\vec{k}, \varepsilon + i\eta) - G(\vec{k}, \varepsilon - i\eta) \right] e^{-z\tau}$$

$$= - \lim_{\tau \to 0^-} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon \, n(\varepsilon) \left(2i \, \mathrm{Im} G^R(\vec{k}, \varepsilon) \right) e^{-z\tau}$$

for free boson

$$= - \lim_{\tau \to 0^-} \frac{1}{\pi} \int_{-\infty}^{\infty} d\varepsilon \, n(\varepsilon) \left(-\pi \delta(\omega - \varepsilon_{\vec{k}}) \right) e^{-\varepsilon\tau}$$

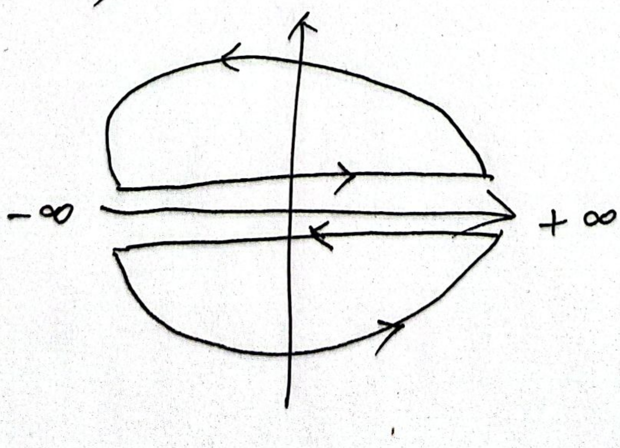
$$G(i\omega_n, \vec{k}) = \frac{1}{i\omega_n - \xi_k}$$

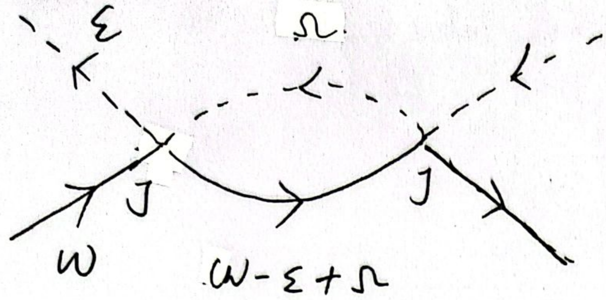
$$= + \lim_{\tau \to 0^-} n(\varepsilon_{\vec{k}}) \, e^{-\varepsilon\tau}$$

$$\Rightarrow G^R(\omega, \vec{k}) = \frac{1}{\omega - \xi_k + i\eta}$$

$$= n(\varepsilon_{\vec{k}}) \quad \ast$$

$$\mathrm{Im} G^R(\omega, \vec{k}) = -\pi \delta(\omega - \xi_k)$$





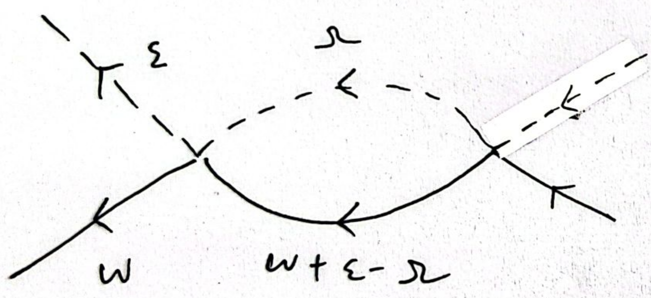
λ ~~Imaginary~~ chemical potential.

$\lambda \to \infty$

$$(-1)\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\frac{1}{\beta}\sum_{i\Omega}\frac{1}{(i\Omega-\varepsilon_{\vec k})(i\omega-i\varepsilon+i\Omega-\lambda)}$$

$$=(-1)\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\left(\frac{f(\varepsilon_{\vec k})}{i\omega-i\varepsilon+\varepsilon_{\vec k}-\lambda}+\frac{f(i\varepsilon+\lambda-i\omega)}{i\varepsilon+\lambda-i\omega-\varepsilon_{\vec k}}\right)$$

$$=(-1)\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\frac{f(\varepsilon_{\vec k})-f(i\varepsilon+\lambda-i\omega)}{i\omega-i\varepsilon+\varepsilon_{\vec k}-\lambda}$$



$$\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\frac{1}{\beta}\sum_{i\Omega}\frac{1}{(i\Omega-\varepsilon_{\vec k})(i\omega+i\varepsilon-i\Omega-\lambda)}$$

$$=\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\left(\frac{f(\varepsilon_{\vec k})}{i\omega+i\varepsilon-\varepsilon_{\vec k}-\lambda}+\frac{f(i\omega+i\varepsilon-\lambda)}{i\omega+i\varepsilon-\varepsilon_{\vec k}-\lambda}\right)$$

$$=\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\frac{f(\varepsilon_{\vec k})+f(i\omega+i\varepsilon-\lambda)}{i\omega+i\varepsilon-\varepsilon_{\vec k}-\lambda}$$

set $i\omega-\lambda\to 0$ $\quad f(\lambda\to\infty)=0$.

$T=0$:

$$\boxed{\frac{J^2}{2}\frac{1}{V}\sum_{\vec k}\left(\frac{\theta(-\varepsilon_{\vec k})}{\varepsilon-\varepsilon_{\vec k}}-\frac{\theta(-\varepsilon_{\vec k})}{\varepsilon+\varepsilon_{\vec k}}\right)}$$